On quotients of ω^* and automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ that preserve or invert the shift

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Abstract

PART ONE of this dissertation concerns the space $\omega^* = \beta\omega \setminus \omega$ and its quotients. A 2002 result from Bella, Dow, Hart, Hrusak, van Mill and Ursino implies that homeomorphisms between 0-dimensional second-countable Hausdorff quotients of ω^* can always be lifted to self-homeomorphisms of ω^*. We show that 0-dimensionality can be dropped and also prove that if the map between the quotients is continuous, but not necessarily a homeomorphism, then it can be lifted to a continuous map from ω^* into itself. We prove that all quotient maps from ω^* onto products of compact metrizable spaces are restrictions of quotient maps from $\beta\omega$ (with the same range). We then defend our choice of hypotheses for this last result by showing that there is a quotient map from ω^* onto the double arrow space (which is separable, 0-dimensional, first-countable, compact and Hausdorff) with no continuous extension to a map from $\beta\omega$.

In PART TWO the focus is on the Boolean algebra $\mathcal{P}(\omega)/\texttt{fin}$ and the automorphism s of $\mathcal{P}(\omega)/\texttt{fin}$ called *the shift*, which is induced by the map $n \mapsto n+1$ on ω. We show that the automorphisms s^m for $m \in \mathbb{Z}$ are the only trivial automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$ which commute with the shift. Together with a 1993 result of Velickovic, this characterizes all automorphisms of $(\mathcal{P}(\omega)/\texttt{fin}, s)$ under OCA+MA$_{\aleph_1}$. We study the algebra $\texttt{Per}$ of all elements of $\mathcal{P}(\omega)/\texttt{fin}$ with finite orbit under the action of the shift, and characterize all the automorphisms of $\texttt{Per}$ which commute with the shift (many of which are not powers of the shift) or conjugate the shift to its inverse. Then, we show that every automorphism of the group of trivial automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$ either preserves or inverts (additively) the index function introduced by van Douwen in 1990. We also construct a set of $2^{\aleph_0}$ many elements of $\mathcal{P}(\omega)/\texttt{fin}$ such that the substructures of $(\mathcal{P}(\omega)/\texttt{fin}, s)$ generated by each of these elements individually are pairwise isomorphic, and a set of $2^{\aleph_0}$ many elements of $\mathcal{P}(\omega)/\texttt{fin}$ such that the substructures of $(\mathcal{P}(\omega)/\texttt{fin}, s)$ generated by each of these elements individually are pairwise non-isomorphic. We finish with a method for constructing automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$ which conjugate the shift to itself or to its inverse on any given countable subalgebra.

About the author

Born in São Paulo and raised in Campinas, Brazil, by the end of my school years at *Escola Comunitária* it was clear to me what I wanted to grow up to be: a physicist, of course. With that in mind I entered, in 2005, the integrated Physics/Mathematics course of the *State University of Campinas (UNICAMP)*, in which students can choose between the two sciences after the initial three semesters, and in 2008 I graduated as a *Bachelor of Science in Mathematics.* During my bachelor studies I did my *Scientific Initiation*, which is a one year undergraduate research program, under the supervision of Prof. Dr. Ary Orozimbo Chiacchio.

After graduation, I started the Master's program at UNICAMP but quit after one semester to accept the offer of a scholarship abroad. So, in September of 2009, I moved to Germany with the clear purpose of becoming a geometer (of course), and started a new Master's program at the *University of Bonn.* In 2012, under the supervision of Prof. Dr. Stefan Geschke, I presented my dissertation in the area of mathematical logic, entitled *Extreme amenability of topological groups*, and received the degree of *Master of Science.*

Since 2012, Prof. Geschke has been supervising my work in the Ph.D. program of the University of Bonn, the result of which I now present. I dare not guess the next paths in my career.

Acknowledgements

> *"If I must, I'd much rather quote a comedian with whose body of work I'm familiar, than some famous poet I've never read"* – Tomás Silveira Salles

> *"Wilde said 'if you want to be a grocer, or a general, or a politician, or a judge, you will invariably become it; that is your punishment' "* – Stephen Fry

I owe a lot of gratitude to Mrs. Bingel. I spent countless hours in her office clueless about the paperwork of which I was supposed to take care, usually embarrassingly close to the deadline, and without her help and infinite patience I would not have made it through the first semester. I thank Prof. Dr. K. P. Hart for a brilliant suggestion which ended up being the core of CHAPTER 5. I thank two friends in particular, who have been with me since the start of my studies in Germany: Tristan, for going through this process before me, and Divy, for going through it together with me. Both of them made the whole experience a lot less scary, and I hope I was able to make their lives easier as well. It is obvious that my family deserves as much credit for this dissertation as I do. They brought me this far because they never expected any less from me, and for this their names should be on the front page right next to mine. I am convinced that a few words here will not suffice to thank my advisor as much as I would like to. He deserves my gratitude for all the usual reasons, such as his questions, solutions, corrections, time and patience, but also for the less conventional reason of his respect for me. Prof. Geschke introduced me to mathematical logic, advised me through my master's studies and now my Ph. D. studies, and treated me from day one as a colleague and an equal. He was interested in my ideas as well as my opinions about his ideas, never made me feel stupid or smart, and earned my absolute respect by never demanding it. Finally, I thank Linn for keeping me in line when I was lazy, for celebrating with me even the smallest achievement, for taking care of the many aspects of my life about which I forgot as I tried to solve my mathematical puzzles, and most of all for thinking that I was a mad genius every time I wrote my formulas on our windows and mirrors – even the wrong ones.

Contents

Introduction

Because this dissertation is separated into two very different topics, and the results in each part are independent and unrelated, I hope this Introduction will clarify what binds this work together, as all the topics and questions involved arise naturally from the study of the algebra $\mathcal{P}(\omega)/\mathtt{fin}$. For the general theory of Boolean algebras and many results about $\mathcal{P}(\omega)/\mathtt{fin}$, I recommend Koppelberg's volume 1 of the *Handbook of Boolean algebras* [Kop89a] (with the following technical remark: Throughout this dissertation, *Boolean algebra* means *non-trivial Boolean algebra*, i.e. we require that $0 \neq 1$).

Definition 0.1. If x and y are sets, we write $x \subseteq^* y$ if the difference $x \setminus y$ is finite. If both $x \subseteq^* y$ and $y \subseteq^* x$ hold, we write $x =^* y$. If $\sim$ is the restriction of the relation $=^*$ to pairs of elements of $\mathcal{P}(\omega)$, (it is an equivalence relation and) the equivalence class of a set $x \subseteq \omega$ is denoted by $[\![x]\!]$. The quotient $\mathcal{P}(\omega)/\sim$ is denoted $\mathcal{P}(\omega)/\mathtt{fin}$ (read $\mathcal{P}(\omega)$ *modulo finite*) as usually found in the literature.

Recall that the *symmetric difference* between two sets x and y, denoted $x \Delta y$, is defined by

$$x \Delta y := (x \setminus y) \cup (y \setminus x)$$

Thus, for $x, y \subseteq \omega$ we have $x =^* y$ if and only if $x \Delta y$ is finite. Yet another formulation would be that $x =^* y$ if and only if their characteristic functions agree on all but finitely many points of ω.

One easily observes that $=^*$ *respects* all the usual Boolean operations of $\mathcal{P}(\omega)$, that is, the formulas

$$[\![x]\!] \vee [\![y]\!] := [\![x \cup y]\!]$$

$$[\![x]\!] \wedge [\![y]\!] := [\![x \cap y]\!]$$

$$\neg [\![x]\!] := [\![\omega \setminus x]\!]$$

define operations on $\mathcal{P}(\omega)/\mathtt{fin}$ well, and give it a natural Boolean algebra structure together with the constants $0 := [\![\emptyset]\!]$ and $1 := [\![\omega]\!]$. In other words, the projection $x \mapsto [\![x]\!]$ becomes a homomorphism from $(\mathcal{P}(\omega), \cup, \cap, \omega \setminus (\cdot), \emptyset, \omega)$ onto $(\mathcal{P}(\omega)/\mathtt{fin}, \vee, \wedge, \neg, 0, 1)$.

As any Boolean algebra, $\mathcal{P}(\omega)/\mathtt{fin}$ is partially ordered by letting $e_0 \leq e_1$ if and only if $e_0 = e_0 \wedge e_1$, and from this we have

$$[\![x]\!] \leq [\![y]\!] \text{ if and only if } x \subseteq^* y$$

In particular, the projection is also increasing.

Stone's Representation Theorem tells us that $\mathcal{P}(\omega)/\mathtt{fin}$ is isomorphic to the algebra of *clopen* (i.e. closed-open) subsets of its *Stone space* $\mathcal{S}(\mathcal{P}(\omega)/\mathtt{fin})$, which is the set of ultrafilters on $\mathcal{P}(\omega)/\mathtt{fin}$, with a basis consisting of all sets $\{\mathcal{F} \in \mathcal{S}(\mathcal{P}(\omega)/\mathtt{fin}) : e \in \mathcal{F}\}$ for $e \in \mathcal{P}(\omega)/\mathtt{fin}$. This means we can learn about this Boolean algebra exploring the topological properties of the corresponding Stone space instead.

In PART ONE we will be concerned with the space $\mathcal{S}(\mathcal{P}(\omega)/\mathtt{fin})$, and more specifically we will study what the quotients of this space are, how the quotient maps can be constructed, and under which assumptions maps between such quotient spaces can be lifted to maps from $\mathcal{S}(\mathcal{P}(\omega)/\mathtt{fin})$ into itself. First, we will consider second countable 0-dimensional Hausdorff quotient spaces, as these correspond (through the Representation Theorem) to the countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$, and there are interesting (known) results in this situation. For example, we know that in this case all continuous maps between quotient spaces can be lifted. Later, however, we will broaden our view by dropping the condition of 0-dimensionality, thus leaving the scope of the Stone duality, and we will see that the mentioned result still holds.

The notation $\mathcal{S}(\mathcal{P}(\omega)/\mathtt{fin})$ will not be used for long in the remaining of this document, for this space will be replaced by the homeomorphic space $\omega^* = \beta\omega \setminus \omega$, which is the Stone-Čech compactification $\beta\omega$ of the natural numbers (equivalently the Stone space of $\mathcal{P}(\omega)$) with the canonical copy of ω removed from it.

The approach in PART TWO to try and understand the algebra $\mathcal{P}(\omega)/\mathtt{fin}$ is to study its automorphisms. The most obvious way to obtain one is to take an automorphism of $\mathcal{P}(\omega)$ and see if it factors through the equivalence relation $=^*$, that is, whether equivalent elements have equivalent images. This is true of *all* automorphisms of $\mathcal{P}(\omega)$, and the reason for it is that the structure of $\mathcal{P}(\omega)$ "recognizes" the finite elements (they are, for example, precisely the disjunctions of atoms, or precisely the elements which cannot start an infinite strictly decreasing chain). On the other hand, we know that the automorphisms of $\mathcal{P}(\omega)$ are simply the permutations of ω applied to sets instead of single points. However, to create automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ we can use maps which are not permutations of ω, but come close enough, since information about finite sets is lost when passing to the quotient anyway. In the literature one usually finds the term *near-bijection* for this purpose. Here we use the concept in slightly generalized form, which will have applications in PART ONE:

Definition 0.2. A *near-surjection* (*of* ω) is a bijection between two subsets of ω whose range is cofinite in ω. The set of all near-surjections is denoted $\mathtt{NS}$. If the domain of a near-surjection is also cofinite in ω, we say it is a *near-bijection* (*of* ω), and the set of all near-bijections is denoted $\mathtt{NB}$.

Near-surjections should really be called "nearly surjective partial injections of ω", a rather impractical term, but which would remind the reader of the importance of injectivity of these maps. Observe that $\mathtt{NS}$ is a monoid with composition as the product, as long as we adjust the domains: If $f : A \to \omega$ and $g : B \to \omega$ are near-surjections, then gf is defined precisely on $f^{-1}[B]$, and is certainly injective. Since $\mathtt{ran}(f)$ is cofinite in ω, we have $B =^* \mathtt{ran}(f) \cap B$, and so $\mathtt{ran}(g) =^* g[\mathtt{ran}(f)] = gf[f^{-1}[B]]$. Since $\mathtt{ran}(g)$ is cofinite in ω it follows that $gf : f^{-1}[B] \to \omega$ is also a near-surjection. Associativity follows easily. The submonoid $\mathtt{NB}$ is not a group, even though its elements are invertible *as functions*, and their inverses are again near-bijections. If we compose a near-bijection with its inverse, the result will generally be the identity on a subset of ω, but not necessarily on all of it. This "problem" will soon be solved.

It is straight-forward to check that any near-surjection maps equivalent subsets of ω (modulo finite) onto equivalent images, which justifies the following:

Definition 0.3. For a near-surjection f, the map $\varphi_f : \mathcal{P}(\omega)/\mathtt{fin} \to \mathcal{P}(\omega)/\mathtt{fin}$ is defined through

$$\varphi_f([\![x]\!]) := [\![f[x]]\!].$$

Lemma 0.4. *If f is a near-surjection, then φ_f is an epimorphism (i.e. a surjective homomorphism) of $\mathcal{P}(\omega)/\mathtt{fin}$. In the affirmative case, φ_f is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ if and only if f is a near-bijection.*

Proof. It follows simply from the fact that f is a function that $\varphi_f(0) = 0$ and

$$\begin{aligned}\varphi_f([\![x]\!] \vee [\![y]\!]) &= \varphi_f([\![x \cup y]\!]) \\ &= [\![f[x \cup y]]\!] \\ &= [\![f[x] \cup f[y]]\!] \\ &= [\![f[x]]\!] \vee [\![f[y]]\!] = \varphi_f([\![x]\!]) \vee \varphi_f([\![y]\!])\end{aligned}$$

To prove that $\varphi_f(\neg[\![x]\!]) = \neg\varphi_f([\![x]\!])$ we need to use the injectivity of f, which implies that $f[\omega \setminus x] = \mathtt{ran}(f) \setminus f[x]$, and the fact that the range of f is cofinite in ω, which means that $\mathtt{ran}(f) \setminus f[x] =^* \omega \setminus f[x]$. This shows that φ_f is a homomorphism. For surjectivity, given $e \in \mathcal{P}(\omega)/\mathtt{fin}$, let y be any representative of e, and let $x := f^{-1}[y]$. It follows that $f[x] = \mathtt{ran}(f) \cap y =^* y$, so $\varphi_f([\![x]\!]) = [\![y]\!] = e$.

Finally, a homomorphism of Boolean algebras is injective if and only if 0 is the only point it maps to 0. In our case, $\varphi_f([\![x]\!]) = 0$ precisely when $f[x]$ is finite, which (since f is injective) happens precisely when $x \cap \mathtt{dom}(f)$ is finite. So φ_f is injective if and only if

$$\forall x \subseteq \omega \ (x \cap \mathtt{dom}(f) \text{ is finite} \Rightarrow x \text{ is finite})$$

and this is clearly equivalent to the domain of f being cofinite in ω. □

Definition 0.5. An epimorphism (respectively an automorphism) φ of $\mathcal{P}(\omega)/\mathtt{fin}$ is called *trivial* if there is a near-surjection (resp. near-bijection) f such that $\varphi = \varphi_f$. We shall say φ is *very trivial* if f can be chosen to be a permutation of ω.

Definition 0.6. The *shift on* ω is the map $S : \omega \to \omega \setminus \{0\}$ which sends n to $n + 1$ (and is clearly a near-bijection). The *shift on* $\mathcal{P}(\omega)/\mathtt{fin}$ (which we will usually call simply *the shift*) is the induced trivial automorphism $s := \varphi_S$.

In PART TWO we will talk more about the importance of the shift, not only but also through analogies with shifts in other structures such as the Calkin algebra and the symmetric group of ω modulo permutations of finite support. The main questions we will try to answer will be what the automorphisms of the structure $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ are, what its substructures are, and how different it is from the structure $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$. This last question might be the most intriguing. The shift on ω is clearly very different from its inverse, which (informally and intuitively) has to do with the fact that ω has a left-endpoint (notice that the shift on $\mathbb{Z}$ can be transformed into its inverse by flipping all integers around 0, which induces an automorphism on $\mathcal{P}(\mathbb{Z})$). Yet, (for ω) the question becomes very difficult in the "modulo finite" setting.

An isomorphism between the structures $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ is an automorphism φ of the algebra $\mathcal{P}(\omega)/\mathtt{fin}$ such that

$$\varphi s = s^{-1}\varphi$$

Since we intuitively think the shift and its inverse are very different on ω, we would not expect such an isomorphism to be induced by a near-bijection of ω, i.e. to be a trivial automorphism, and we shall see (using an index function) that this indeed cannot be the case. We will also see, due to theorems of Rudin [Rud56] and Shelah [She82], that the existence of *non-trivial* automorphisms is independent of ZFC. In particular, it is consistent with ZFC that the structures $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ are not isomorphic (in models which only have trivial automorphisms), but the question remains open whether they can be isomorphic in models where non-trivial automorphisms exist, and I will offer partial results to try and bring us closer to the answer.

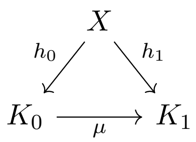
PART ONE

QUOTIENTS OF ω^*

1 STONE-ČECH COMPACTIFICATIONS AND REMAINDERS

This chapter introduces basic definitions and results about compactifications. These can also be found, for example, in Engelking's *General Topology* [Eng89]. The subsequent chapters, however, will mostly be dealing only with the Stone-Čech compactification of the natural numbers.

Definition 1.1. A *compactification* of a topological space X is a pair (K, h), where K is a compact Hausdorff space and $h : X \to K$ is an embedding such that $h[X]$ is dense in K. Given two compactifications (K_0, h_0) and (K_1, h_1) of the same space X, we say that a continuous map $\mu : K_0 \to K_1$ is a *morphism of compactifications* if it closes the diagram



Not every space has a compactification. For example, if X has a compactification, it is homeomorphic to a subset of a compact Hausdorff space, which is therefore a Tychonoff space (that is, a Hausdorff completely regular space), and so X must be a Tychonoff space as well.

Definition 1.2. A compactification (K, h) of a space X is called a *Stone-Čech compactification* of X if it has the following universal property: Given any continuous map $f : X \to K'$ where K' is a compact Hausdorff space, there is a continuous map $g : K \to K'$ such that $f = g \circ h$.

In the universal property above, g is unique, which follows from the fact that the range of h is dense in K. We will sometimes talk about this universal property (formulated exactly as above) for pairs (K, h) where K is not necessarily compact or Hausdorff, and h is not necessarily continuous, or has its range dense in K.

Observe that all Stone-Čech compactifications of a space X are isomorphic: If (K_0, h_0) and (K_1, h_1) are two such compactifications, then the universal property says that for each $i, j \in 2$ there is a unique continuous map $g_{ij} : K_i \to K_j$ such that $h_j = g_{ij} \circ h_i$ (and hence this map is a morphism of compactifications). Clearly the uniqueness part implies that $g_{ii} = \mathtt{id}_{K_i}$ for $i = 0, 1$. On the other hand, for each $i, j \in 2$ we have $g_{ji} \circ g_{ij} : K_i \to K_i$ and

$$h_i = g_{ji} \circ h_j = g_{ji} \circ g_{ij} \circ h_i$$

so it follows that $g_{ji} \circ g_{ij} = g_{ii} = \mathtt{id}_{K_i}$. In particular, g_{01} and g_{10} are each other's inverses. For this reason, we usually refer to any Stone-Čech compactification of X as *the* Stone-Čech compactification of X, and denote the compact space where X is embedded by βX.

Another traditional loosening of notation is to identify the points of X with their images through h given a compactification (K, h), so as to see X as a dense subset of K. I shall use this trick whenever it is unnecessary to talk about the embedding. When this is done, the compactification is considered to be just the space K, and a morphism is a continuous map between compactifications which is the identity on X. Moreover, the universal property of the Stone-Čech compactification under this identification says that maps from X into compact Hausdorff spaces can always be *extended* to maps from βX.

Lemma 1.3. *A topological space has a (Stone-Čech) compactification if and only if it is a Tychonoff space.*

To prove the lemma, we will use a construction involving the space of ultrafilters on X, which is a method usually reserved only for discrete spaces (and the general case indeed requires some modification). A more traditional proof is given in [Eng89], but the one given here will be useful later on. The following material on Stone spaces and Boolean spaces can be found in [Kop89a].

Definition 1.4. For a Boolean algebra $\mathcal{B}$, we define its *Stone space* as the set $\mathcal{S}(\mathcal{B})$ of all ultrafilters on $\mathcal{B}$, with the topology generated by the basis $\{\mathtt{V}(e) : e \in \mathcal{B}\}$, where $\mathtt{V}(e)$ denotes the set of all ultrafilters containing e.

The space $\mathcal{S}(\mathcal{B})$ is often denoted $\mathcal{U}lt(\mathcal{B})$ in the literature. I should remark that my use of the term "filter" is rather incoherent here. For topological purposes, a "filter on a set X" is usually a family of subsets of X (with certain properties), while in the topic of Boolean algebras a "filter on an algebra $\mathcal{B}$" is actually a subset of $\mathcal{B}$ (with certain properties). Unfortunately, I will need both concepts quite often, but they come together nicely considering that a filter on X in the first sense, is just a filter on the algebra $\mathcal{P}(X)$ in the second sense. I believe the context will always make clear what is meant and there should be no confusion. Throughout this document we call a topological space 0-*dimensional* if its topology is generated by a basis of clopen sets.

Lemma 1.5 (Stone)**.** *The Stone space of a Boolean algebra is always compact, Hausdorff and 0-dimensional. The Boolean algebra itself is isomorphic to the algebra of clopen subsets of its Stone space through $e \mapsto \mathtt{V}(e)$. Moreover, every compact Hausdorff 0-dimensional space is homeomorphic to the Stone space of its algebra of clopen subsets, through a canonical homeomorphism defined below.*

Proof. The reader can easily verify that the map $\mathtt{V}$ is indeed a homomorphism of the algebra $\mathcal{B}$ into $\mathcal{P}(\mathcal{S}(\mathcal{B}))$. Since all $\mathtt{V}(e)$'s are open and $\mathcal{S}(\mathcal{B}) \setminus \mathtt{V}(e) = \mathtt{V}(\neg e)$, it follows that these sets are clopen, and thus the Stone space is 0-dimensional. Hausdorffness is just as easy. For compactness, we prove that if $\{F_\alpha\}_\alpha$ is a family of closed sets with the finite intersection property, then $\bigcap_\alpha F_\alpha \neq \emptyset$. Clearly, each F_α is an intersection of the type $\bigcap_\beta \mathtt{V}(e_{\alpha,\beta})$ for certain $e_{\alpha,\beta} \in \mathcal{B}$, and the collection of the $e_{\alpha,\beta}$ for all α and β has what we could call the *finite conjunction property*, meaning the conjunction of any finitely many elements of this collection is different of 0. This implies that $\{e_{\alpha,\beta}\}_{\alpha,\beta}$ can be extended to an ultrafilter $\mathcal{F}$, and so $\mathcal{F} \in \bigcap_{\alpha,\beta} \mathtt{V}(e_{\alpha,\beta}) = \bigcap_\alpha F_\alpha$.

Let us show that $\mathtt{V}$ is injective. Suppose $e_0 \neq e_1$ in $\mathcal{B}$, and observe that either $e_0 \not\leq e_1$ or $e_1 \not\leq e_0$, so we may assume the former. Hence $e_0 \wedge \neg e_1 \neq 0$, showing that there is an ultrafilter containing both e_0 and $\neg e_1$, that is, an ultrafilter in $\mathtt{V}(e_0) \setminus \mathtt{V}(e_1)$. Furthermore, if $C \subseteq \mathcal{S}(\mathcal{B})$ is clopen, then it is a union of certain basic open sets, and also compact, therefore a union of finitely many basic open sets, so $C = \mathtt{V}(e)$ for some $e \in \mathcal{B}$. This proves the second part.

Finally, given X compact Hausdorff 0-dimensional, and denoting by $\mathcal{B}$ its algebra of clopen subsets, we wish to construct a homeomorphism between X and $\mathcal{S}(\mathcal{B})$. Take $x \in X$ and note that the family $\mathcal{B}_x$ of all clopen subsets of X containing x is an ultrafilter on $\mathcal{B}$. We claim that $x \mapsto \mathcal{B}_x$ is the homeomorphism we are looking for. Injectivity follows from the fact that X is 0-dimensional and Hausdorff, so that points are separated by clopen sets. For surjectivity, observe that every ultrafilter $\mathcal{F}$ on $\mathcal{B}$ has the finite intersection property (by definition), and consists of closed sets in the compact space X, hence $\bigcap \mathcal{F} \neq \emptyset$. If $x \in \bigcap \mathcal{F}$, then $\mathcal{F} \subseteq \mathcal{B}_x$, and the maximality of $\mathcal{F}$ implies it is equal to $\mathcal{B}_x$ (and in particular $\bigcap \mathcal{F}$ is a singleton). Continuity is basically a restatement of the definitions involved, and goes through showing that if $e \in \mathcal{B}$, then $\{\mathcal{B}_x : x \in e\} = \mathtt{V}(e)$. We are done, because continuous maps from compact spaces into Hausdorff spaces are always closed maps. □

Due to the lemma above, spaces which are compact, Hausdorff and 0-dimensional are also called *Boolean spaces*. We will go back to the relation between Boolean spaces and Boolean algebras soon.

Given a Tychonoff space X, consider $\mathcal{S}(\mathcal{P}(X))$ and the map $h : X \to \mathcal{S}(\mathcal{P}(X))$ where $h(x)$ is the family of all subsets of X containing x, i.e. the principal filter generated by $\{x\}$. This is similar to what we did in the final part of the previous lemma, but this time all we get is that h is injective. Our goal is to obtain the Stone-Čech compactification of X as a quotient of $\mathcal{S}(\mathcal{P}(X))$, and the embedding as the composition of the quotient map with h. It is already true that $h[X]$ is dense in $\mathcal{S}(\mathcal{P}(X))$: A non-empty open set in $\mathcal{S}(\mathcal{P}(X))$ contains a set of the kind $\mathtt{V}(e)$ for some $e \neq \emptyset$ in $\mathcal{P}(X)$, so it also contains $h(x)$ for every $x \in e$. Hence, the image of the composition of any quotient map with h will also be dense in the quotient space.

It is also true that $(\mathcal{S}(\mathcal{P}(X)), h)$ has the universal property that characterizes the Stone-Čech compactification (even though h is not necessarily an embedding), and even a much stronger version. Given any function $f : X \to K$ into a compact Hausdorff space, if $\mathcal{F} \in \mathcal{S}(\mathcal{P}(X))$, then $\{\overline{f[e]} : e \in \mathcal{F}\}$ has the finite intersection property. Hence $\bigcap_{\mathcal{F}} \overline{f[e]} \neq \emptyset$. If $x \in X$, since $\{x\} \in h(x)$, it is clear that $\bigcap_{h(x)} \overline{f[e]} = \{f(x)\}$. In fact, $\bigcap_{\mathcal{F}} \overline{f[e]}$ is also a singleton for all other $\mathcal{F} \in \mathcal{S}(\mathcal{P}(X))$: If $k_0 \neq k_1$ in K, let U_0 and U_1 be disjoint open neighbourhoods of k_0 and k_1 respectively. Set $e_0 := f^{-1}[U_0]$ and observe that $k_1 \notin \overline{f[e_0]}$, while $k_0 \notin \overline{f[X \setminus e_0]}$, and since either e_0 or $X \setminus e_0$ must be an element of $\mathcal{F}$, we cannot have both k_0 and k_1 in $\bigcap_{\mathcal{F}} \overline{f[e]}$. Letting $g_f : \mathcal{S}(\mathcal{P}(X)) \to K$ be the unique map such that $g_f(\mathcal{F}) \in \bigcap_{\mathcal{F}} \overline{f[e]}$ for all $\mathcal{F}$, it follows at once that $g_f \circ h = f$. For continuity, suppose U is an open neighbourhood of $g_f(\mathcal{F})$ in K, and observe that $\{U\} \cup \{K \setminus \overline{f[e]} : e \in \mathcal{F}\}$ is an open cover for K. Using compactness, and the fact that $\mathcal{F}$ is closed under finite intersections, we see that there is some $e \in \mathcal{F}$ such that $\overline{f[e]} \subseteq U$, and hence $g_f[\mathtt{V}(e)] \subseteq U$.

Now we need an equivalence relation on $\mathcal{S}(\mathcal{P}(X))$ such that the quotient "preserves" the universal property (for continuous maps from X), while making h into an embedding, and the answer is actually quite natural: We identify the points that cannot be distinguished using the universal property (applied to continuous maps from X). In other words, in $\mathcal{S}(\mathcal{P}(X))$ we define that $\mathcal{F} \sim \mathcal{F}'$ if and only if $g_f(\mathcal{F}) = g_f(\mathcal{F}')$ for every continuous map f from X into a compact Hausdorff space. Let $K := \mathcal{S}(\mathcal{P}(X))/\sim$, with the quotient topology, and let q be the quotient map. With our definitions tailored specifically to achieve this, it is not hard to prove that $(K, q \circ h)$ has the universal property as well (this time only for continuous maps from X, as required for the Stone-Čech compactification). More specifically, given $f : X \to K'$ continuous, where K' is compact Hausdorff, our definitions imply that there is a map $\tilde{g}_f : K \to K'$ such that $\tilde{g}_f \circ q = g_f$ (and from this it follows that $\tilde{g}_f$ is continuous, and that $\tilde{g}_f \circ (q \circ h) = f$). Using this fact (slightly stronger than the universal property), we can show that if $\mathcal{F} \not\sim \mathcal{F}'$ in $\mathcal{S}(\mathcal{P}(X))$, then there is a continuous map from K into a Hausdorff space which separates $q(\mathcal{F})$ from $q(\mathcal{F}')$. Thus K is also Hausdorff.

Let us show that $q \circ h$ is continuous. Take $K_0 \subseteq K$ closed, let $e_0 := (q \circ h)^{-1}[K_0] \subseteq X$, and suppose $x \in \overline{e_0}$. Then the family of all neighbourhoods of x together with e_0 has the finite intersection property, and we can choose an ultrafilter $\mathcal{F}$ containing this family, that is, $\mathcal{F} \in \mathtt{V}(e_0)$ and $\mathcal{F} \to x$. Given any basic open neighbourhood $\mathtt{V}(e)$ of $\mathcal{F}$, we have that $e \cap e_0 \neq \emptyset$, and $h[e \cap e_0] \subseteq \mathtt{V}(e) \cap q^{-1}[K_0]$, therefore $\mathcal{F}$ is in $\overline{q^{-1}[K_0]} = q^{-1}[K_0]$ (because q is continuous). It follows that $q(\mathcal{F}) \in K_0$. Since $\mathcal{F}$ converges to x, it is easy to see that for all continuous maps f from X into a compact Hausdorff space we have $g_f(\mathcal{F}) = f(x) = g_f(h(x))$, and so $\mathcal{F} \sim h(x)$. It follows that $q \circ h(x) \in K_0$, so that $x \in e_0$, showing that e_0 is closed, and completing the proof that $q \circ h$ is continuous.

We may finally start using the topology of X. For example, to see that $q \circ h$ is injective we use the fact that X is a Tychonoff space to separate points of X using continuous functions into $[0, 1]$ (and then apply the universal property). This part is simple enough, so we arrive at the last problem, of showing that $q \circ h$ is relatively closed, that is, that it takes closed subsets of X onto subsets of $q \circ h[X]$ which are closed in the subspace topology. Let $e \subseteq X$ be closed, and suppose $k \in q \circ h[X] \setminus q \circ h[e]$. By injectivity we have $q \circ h[X] \setminus q \circ h[e] = q \circ h[X \setminus e]$, so $k = q(h(x))$ for some $x \notin e$. Since X is completely regular, there is a continuous map $f : X \to [0, 1]$ taking e into $\{0\}$ and mapping x to 1, and using the universal property we get $\tilde{g}_f : K \to [0, 1]$ taking $q \circ h[e]$ into $\{0\}$ and mapping k to 1. This shows that k is not in the closure of $q \circ h[e]$ in K, and therefore not in the closure of $q \circ h[e]$ in $q \circ h[X]$. Putting the last few paragraphs together we have proven:

Lemma 1.6. *The pair $(K, q \circ h)$, constructed as above, is the Stone-Čech compactification of X.* □

This lemma also completes the proof of Lemma 1.3. In the last part of the proof, we did not use any specifics of the construction of $(K, q \circ h)$, and in fact we proved something a bit stronger than was required, namely:

Lemma 1.7. *If X is a Tychonoff space, K a topological space, and $h : X \to K$ a function such that (K, h) has the universal property, then h is injective and relatively closed.* □

Corollary 1.8. *If X is a discrete space, then $\mathcal{S}(\mathcal{P}(X)) = \beta X$ (the points of X being identified with the principal filters they generate).* □

We can enter the topic of *remainders* of compactifications now. It will be of central importance for the next chapters. As was said in the INTRODUCTION, we will be interested in the space $\beta\omega \setminus \omega$, which is very intriguing and often counter-intuitive, whereas the space $\beta\omega$ itself is rather well understood already.

Definition 1.9. Let X be a Tychonoff space and (K, h) a compactification of X. The *remainder* of (K, h) is the space $K \setminus h[X]$ (with the subspace topology). The remainder of the Stone-Čech compactification of X will be denoted X^*.

The notation X^* for this purpose is not standard in the literature, but generally accepted in the case of $X = \omega$. As promised, identifying X with its image through the embedding, we get $X^* = \beta X \setminus X$. The remainders of all compactifications of X are empty if X is compact, and non-empty otherwise, and they are also obviously Hausdorff.

Lemma 1.10. *Let X be a Tychonoff space and (K, h) a compactification of X. Then $h[X]$ is open in K (equivalently h is an open map) if and only if X is locally compact.*

Proof. Open subsets of locally compact spaces are always locally compact as subspaces, so if $h[X]$ is open in K we only need to observe that K is locally compact, and X is homeomorphic to $h[X]$. For the other direction, suppose X is locally compact (and therefore so is $h[X]$), and let $k \in h[X]$. Choose a compact neighbourhood C of k in $h[X]$, and let U be its interior in $h[X]$. Of course, C is compact independently of where it is embedded, so it is a closed subset of K, and it follows that $\overline{U} \subseteq C \subseteq h[X]$ (where $\overline{A}$ in this proof always means the closure of A in K). Moreover, by definition of the subspace topology, there is an open set V in K such that $U = V \cap h[X]$.

Let $k' \in V$, and observe that $k' \in K = \overline{h[X]}$, so if V' is a neighbourhood of k' in K, we have $(V' \cap V) \cap h[X] \neq \emptyset$, showing that $k' \in \overline{V \cap h[X]} = \overline{U}$. In particular we get $k \in V \subseteq h[X]$, and so k is in the interior (in K) of $h[X]$. □

Corollary 1.11. *If X is a locally compact Tychonoff space, and it is not compact, then all remainders of compactifications of X are non-empty compact Hausdorff.* □

From now on we specialise to discrete spaces, for the following interesting reason:

Corollary 1.12. *If X is an infinite discrete space, then X^* is a non-empty Boolean space.*

Proof. Clearly X is a locally compact non-compact Tychonoff space, so the previous result tells us that X^* is non-empty compact Hausdorff. We also know that $\beta X = \mathcal{S}(\mathcal{P}(X))$, which is 0-dimensional, and 0-dimensionality is inherited by subspaces, so we are done. □

According to the corollary above, and Lemma 1.5, if X is infinite discrete, X^* is homeomorphic to the Stone space of its algebra of clopen sets, so it would be nice to know more about this algebra, which we shall henceforth denote by $\mathtt{Clop}(X^*)$. Clearly, since $\{\mathtt{V}(e) : e \subseteq X\}$ is a basis of clopen sets for βX, it holds that $\{\mathtt{V}(e) \setminus X : e \subseteq X\}$ is a basis of clopen sets for X^*, and we define for each $e \subseteq X$:

$$\mathtt{W}(e) := \mathtt{V}(e) \setminus X.$$

This notation is based on the one used by Rudin [Rud56], but also not standard in the literature. Note that $\mathtt{V}(e) \cap X = e$, and so $\mathtt{W}(e) = \mathtt{V}(e) \setminus e$. With straight-forward calculations one sees that the map $e \mapsto \mathtt{W}(e)$ is a homomorphism from $\mathcal{P}(X)$ into $\mathtt{Clop}(X^*)$, and just like in the proof of Lemma 1.5, one sees that the $\mathtt{W}(e)$'s are *all* the clopen subsets of X^*. This can be stated in a more general fashion: If a compact space Y has a basis $\mathcal{B}$ of clopen sets, which is closed under taking finite unions, and contains the empty set, then $\mathcal{B}$ contains all clopen subsets of Y. So the map $\mathtt{W} : \mathcal{P}(X) \to \mathtt{Clop}(X^*)$ is an epimorphism. For Boolean algebras, just as (more famously) for groups, we have the following *isomorphism theorem* (for a proof, see [Kop89a]):

Theorem 1.13. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be Boolean algebras and $\varphi : \mathcal{B}_0 \to \mathcal{B}_1$ an epimorphism. Then the relation $\sim$ on $\mathcal{B}_0$ defined by $e \sim e'$ if and only if $\varphi(e) = \varphi(e')$ is an equivalence relation and $\mathcal{B}_0/\sim$ inherits naturally a Boolean algebraic structure. If we denote the quotient map by π, there is a unique map $\Phi : \mathcal{B}_0/\sim \to \mathcal{B}_1$ such that $\Phi \circ \pi = \varphi$. Moreover, Φ is an isomorphism.* □

Thus, the algebra $\mathtt{Clop}(X^*)$ is isomorphic to the quotient $\mathcal{P}(X)/\sim$, where $e \sim e'$ in $\mathcal{P}(X)$ if and only if $\mathtt{W}(e) = \mathtt{W}(e')$. To better understand this equivalence relation we will need some basic facts about ultrafilters.

Lemma 1.14. *For an ultrafilter $\mathcal{F}$ on a set X, the following are equivalent: (a) $\mathcal{F}$ is a principal filter (that is, it has a minimal element). (b) There is some $x \in X$ such that $\{x\} \in \mathcal{F}$. (c) There is a finite set $e \in \mathcal{F}$.*

Proof. (a) implies (b): Let $e \in \mathcal{F}$ be a minimal element. Then, $\mathcal{F}$ consists precisely of all $e' \subseteq X$ such that $e' \supseteq e$. Choose $x \in e$ and observe that the filter generated by x contains $\mathcal{F}$, and must be equal to it because $\mathcal{F}$ is a maximal filter. Thus $\{x\} \in \mathcal{F}$ (and in fact $e = \{x\}$).

(b) implies (c): Obvious.

(c) implies (a): If $\mathcal{F}$ were a free filter (i.e. a non-principal filter), then every element of $\mathcal{F}$ would be the beginning of an infinite strictly decreasing chain in $\mathcal{F}$. Since e is finite, any strictly decreasing chain starting at e must also be finite. □

This lemma tells us, for example, that X^* consists precisely of the free ultrafilters on X, since the elements of X represent the filters generated by singletons.

A useful tool when working with free ultrafilters is the following stronger version of the finite intersection property: A collection of subsets of a given set is said to have the *infinite finite intersection property*, if the intersection of any finitely many of its elements is an infinite set.

Lemma 1.15. *If $\mathcal{C}$ is a collection of subsets of an infinite set X, and has the infinite finite intersection property, then it can be extended to a free ultrafilter.*

Proof. Let $\mathcal{C}_0$ be the collection of all cofinite subsets of X, and observe that $\mathcal{C}_0 \cup \mathcal{C}$ has the finite intersection property, and therefore can be extended to an ultrafilter $\mathcal{F}$. For every $e \subseteq X$ finite, we have $e \notin \mathcal{F}$ because $X \setminus e \in \mathcal{C}_0 \subseteq \mathcal{F}$. By Lemma 1.14, $\mathcal{F}$ must be free. □

Given e and e' subsets of X, we have $\mathtt{W}(e) \setminus \mathtt{W}(e') \neq \emptyset$ if and only if there is a free ultrafilter containing e but not e', that is, a free ultrafilter containing e and $X \setminus e'$. This happens precisely when the collection $\{e, X \setminus e'\}$ has the infinite finite intersection property, that is, when $e \setminus e'$ is infinite. From now on we will reutilize the notation from Definition 0.1, namely we use the analogous definitions but with ω replaced by a generic set X. In this notation we have just proved that $\mathtt{W}(e) \subseteq \mathtt{W}(e')$ if and only if $e \subseteq^* e'$, and hence $\mathtt{W}(e) = \mathtt{W}(e')$ precisely when $e =^* e'$.

Corollary 1.16. *If X is an infinite discrete space, then $\mathcal{P}(X)/\mathtt{fin}$ is isomorphic to $\mathtt{Clop}(X^*)$ through the map $[\![e]\!] \mapsto \mathtt{W}(e)$. Consequently, we have*

$$X^* \simeq \mathcal{S}(\mathcal{P}(X)/\mathtt{fin}).$$

□

For this reason we will now further examine the correspondence between Boolean algebras and their Stone spaces. Some terminology from category theory will be used, for convenience, but it is not of fundamental importance, as long as the constructions are understood.

Definition 1.17. We denote by $\mathtt{BAl}$ the category whose objects are the Boolean algebras, and whose morphisms are homomorphisms of Boolean algebras. The category $\mathtt{BSp}$ is the one whose objects are the Boolean spaces, and whose morphisms are continuous maps.

We have so far defined the map $\mathcal{S}$ from the objects of $\mathtt{BAl}$ into the objects of $\mathtt{BSp}$, and now we shall extend it to a (contravariant fully faithful) functor. Let us fix two Boolean algebras $\mathcal{B}_0$ and $\mathcal{B}_1$ and consider the sets $\mathtt{Mor}(\mathcal{B}_0, \mathcal{B}_1)$ (of homomorphisms $\mathcal{B}_0 \to \mathcal{B}_1$) and $\mathtt{Mor}(\mathcal{S}(\mathcal{B}_1), \mathcal{S}(\mathcal{B}_0))$ (of continuous maps $\mathcal{S}(\mathcal{B}_1) \to \mathcal{S}(\mathcal{B}_0)$). If $\varphi \in \mathtt{Mor}(\mathcal{B}_0, \mathcal{B}_1)$, and $\mathcal{F} \in \mathcal{S}(\mathcal{B}_1)$, then the family $\varphi^{-1}[\mathcal{F}]$ is non-empty (since $\varphi(1_{\mathcal{B}_0}) = 1_{\mathcal{B}_1}$ is in every filter on $\mathcal{B}_1$), it does not contain $0_{\mathcal{B}_0}$, it is closed under finite conjunctions, and it is closed upwards (notice that $e \leq e'$ implies $\varphi(e) \leq \varphi(e')$), therefore $\varphi^{-1}[\mathcal{F}]$ is a filter. To see that it is a maximal filter, observe that for each $e \in \mathcal{B}_0$, because $\mathcal{F}$ is maximal, either $\varphi(e)$ or $\neg\varphi(e)$

must be in $\mathcal{F}$ (that is, either $\varphi(e)$ or $\varphi(\neg e)$), and so either e or $\neg e$ must be in $\varphi^{-1}[\mathcal{F}]$. We let $\mathcal{S}(\varphi) : \mathcal{S}(\mathcal{B}_1) \to \mathcal{S}(\mathcal{B}_0)$ be the mapping $\mathcal{F} \mapsto \varphi^{-1}[\mathcal{F}]$. If $e \in \mathcal{B}_0$, and $\mathcal{F} \in \mathcal{S}(\mathcal{B}_1)$, observe that $\mathcal{S}(\varphi)(\mathcal{F}) \in \mathtt{V}_{\mathcal{B}_0}(e)$ if and only if $\varphi(e) \in \mathcal{F}$, i.e. if and only if $\mathcal{F} \in \mathtt{V}_{\mathcal{B}_1}(\varphi(e))$. In other words, $\mathcal{S}(\varphi)^{-1}[\mathtt{V}_{\mathcal{B}_0}(e)] = \mathtt{V}_{\mathcal{B}_1}(\varphi(e))$, which also shows that $\mathcal{S}(\varphi)$ is continuous, and hence a member of $\mathtt{Mor}(\mathcal{S}(\mathcal{B}_1), \mathcal{S}(\mathcal{B}_0))$. The composition rule $\mathcal{S}(\varphi' \circ \varphi) = \mathcal{S}(\varphi) \circ \mathcal{S}(\varphi')$ is easy to check, as is the equality $\mathcal{S}(\mathtt{id}_{\mathcal{B}}) = \mathtt{id}_{\mathcal{S}(\mathcal{B})}$, and this makes $\mathcal{S}$ into a contravariant functor.

The fact that $\mathtt{V}_{\mathcal{B}_1}$ above is an isomorphism means that we can get φ "back" from $\mathcal{S}(\varphi)$, namely $\varphi(e) = \mathtt{V}_{\mathcal{B}_1}^{-1}(\mathcal{S}(\varphi)^{-1}[\mathtt{V}_{\mathcal{B}_0}(e)])$. This implies that the map $\mathcal{S}$ restricted to $\mathtt{Mor}(\mathcal{B}_0, \mathcal{B}_1)$ is injective, or in category terminology, the functor $\mathcal{S}$ is faithful. On the other hand, if $f : \mathcal{S}(\mathcal{B}_1) \to \mathcal{S}(\mathcal{B}_0)$ is continuous, then the mapping $\mathtt{V}_{\mathcal{B}_0}(e) \mapsto f^{-1}[\mathtt{V}_{\mathcal{B}_0}(e)]$ goes from the algebra of clopen subsets of $\mathcal{S}(\mathcal{B}_0)$ into the algebra of clopen subsets of $\mathcal{S}(\mathcal{B}_1)$, and is clearly a homomorphism. It induces the homomorphism $\varphi : \mathcal{B}_0 \to \mathcal{B}_1$ given by $\varphi(e) := \mathtt{V}_{\mathcal{B}_1}^{-1}(f^{-1}[\mathtt{V}_{\mathcal{B}_0}(e)])$, and from the equality $\mathcal{S}(\varphi)^{-1}[\mathtt{V}_{\mathcal{B}_0}(e)] = f^{-1}[\mathtt{V}_{\mathcal{B}_0}(e)]$ for all $e \in \mathcal{B}_0$ it is straight-forward to prove that $\mathcal{S}(\varphi) = f$. Thus, $\mathcal{S}$ restricted to $\mathtt{Mor}(\mathcal{B}_0, \mathcal{B}_1)$ is also surjective, or in category terminology, the functor $\mathcal{S}$ is full. The following corollary is a simple application of the fact that $\mathcal{S}$ is a fully faithful functor.

Corollary 1.18. *A homomorphism $\varphi : \mathcal{B}_0 \to \mathcal{B}_1$ of Boolean algebras is an isomorphism if and only if $\mathcal{S}(\varphi) : \mathcal{S}(\mathcal{B}_1) \to \mathcal{S}(\mathcal{B}_0)$ is a homeomorphism.* □

The "only if" part of this corollary is actually so intuitive, it was already implicitly used in Corollary 1.16.

Lemma 1.19. *A homomorphism $\varphi : \mathcal{B}_0 \to \mathcal{B}_1$ of Boolean algebras is injective if and only if $\mathcal{S}(\varphi)$ is onto $\mathcal{S}(\mathcal{B}_0)$.*

Proof. Suppose φ is injective and let $\mathcal{F}' \in \mathcal{S}(\mathcal{B}_0)$. It follows that $\varphi[\mathcal{F}']$ is non-empty, does not contain $0_{\mathcal{B}_1}$, and is closed under finite conjunctions (but it needs not be closed upwards). So there is an ultrafilter $\mathcal{F}$ extending $\varphi[\mathcal{F}']$, and hence $\varphi^{-1}[\mathcal{F}]$ is a filter extending $\mathcal{F}'$. By the maximality of $\mathcal{F}'$ we have $\mathcal{F}' = \varphi^{-1}[\mathcal{F}] = \mathcal{S}(\varphi)(\mathcal{F})$, showing that $\mathcal{S}(\varphi)$ is onto $\mathcal{S}(\mathcal{B}_0)$.

Now we assume $\mathcal{S}(\varphi)$ is surjective instead. If $e \neq e'$ in $\mathcal{B}_0$, then either $e \not\leq e'$ or $e' \not\leq e$, and we may assume the former, in which case $e \wedge \neg e' \neq 0_{\mathcal{B}_0}$, so there is an ultrafilter $\mathcal{F}'$ on $\mathcal{B}_0$ such that $e, \neg e' \in \mathcal{F}'$. Let $\mathcal{F} \in \mathcal{S}(\mathcal{B}_1)$ be such that $\mathcal{S}(\varphi)(\mathcal{F}) = \mathcal{F}'$, and observe that $\varphi(e), \varphi(\neg e') \in \mathcal{F}$. Thus $\varphi(e) \wedge \neg\varphi(e') \neq 0_{\mathcal{B}_1}$, showing that $\varphi(e) \neq \varphi(e')$, and proving that φ is injective. □

Note that since Boolean spaces are always compact and Hausdorff, a continuous map between them is surjective precisely if it is a quotient map, so we have established that $\mathcal{S}$ restricted to $\mathtt{Mor}(\mathcal{B}_0, \mathcal{B}_1)$ maps the embeddings precisely onto the quotient maps. The next functor we will construct is much simpler. It will extend our application of the functor $\mathcal{S}$ to get results about Boolean spaces in general.

Let $\mathcal{R}$ be the functor from the category $\mathtt{BSp}$ into itself, whose object function maps a Boolean space X to the space $\mathcal{R}(X) := \mathcal{S}(\mathtt{Clop}(X))$. As we have seen in Lemma 1.5, X is always homeomorphic to $\mathcal{R}(X)$, and there is a canonical homeomorphism between them, namely the map $x \mapsto \{e \in \mathtt{Clop}(X) : x \in e\}$. We shall denote this particular homeomorphism by h_X. To complete the definition of $\mathcal{R}$, given a continuous map $f : X \to Y$ between Boolean spaces, let $\mathcal{R}(f) := h_Y \circ f \circ h_X^{-1}$. It is trivial to check that $\mathcal{R}$ is a covariant (i.e. $\mathcal{R}(f) : \mathcal{R}(X) \to \mathcal{R}(Y)$) fully faithful (i.e. $\mathcal{R}$ restricted to $\mathtt{Mor}(X, Y)$ is a bijection onto $\mathtt{Mor}(\mathcal{R}(X), \mathcal{R}(Y))$) functor.

Fix an infinite set X, and give it the discrete topology. As we have seen in Corollary 1.16, the expression $\mathtt{w}([\![e]\!]) := \mathtt{W}(e)$ defines an isomorphism $\mathtt{w} : \mathcal{P}(X)/\mathtt{fin} \to \mathtt{Clop}(X^*)$, so we have an explicit homeomorphism between X^* and $\mathcal{S}(\mathcal{P}(X)/\mathtt{fin})$ now, namely $\mathcal{S}(\mathtt{w}) \circ h_{X^*}$. We can use the functors $\mathcal{R}$ and $\mathcal{S}$ to study the quotients of X^*, but this method will only work for quotients which are also Boolean spaces, that is, for the Hausdorff 0-dimensional quotients (since all quotients of X^* are compact).

Definition 1.20. Given an infinite discrete space X, let $\mathtt{Quot}_0(X^*)$ be the category whose objects are the pairs (Y, q) where Y is a Boolean space and $q : X^* \to Y$ is a quotient map. If (Y_0, q_0) and (Y_1, q_1) are objects, a morphism from the former to the latter is simply a continuous map $f : Y_0 \to Y_1$.

Our third and last functor, denoted $\mathcal{T}$, will be from $\mathtt{Quot}_0(X^*)$ into $\mathtt{BAl}$, and will map all objects to subalgebras of $\mathcal{P}(X)/\mathtt{fin}$. Given an object (Y, q), we have the map $\mathcal{R}(q) : \mathcal{S}(\mathtt{Clop}(X^*)) \to \mathcal{S}(\mathtt{Clop}(Y))$, and since $\mathcal{S}$ is full, we have $\mathcal{R}(q) = \mathcal{S}(\varphi)$ for some homomorphism $\varphi : \mathtt{Clop}(Y) \to \mathtt{Clop}(X^*)$. Let

$$\mathcal{T}(Y, q) := \mathtt{w}^{-1}[\varphi[\mathtt{Clop}(Y)]]$$

If we want a more explicit formula, we can loosely use the notation $\varphi = \mathcal{S}^{-1}(\mathcal{R}(q))$, where it is understood that we mean the inverse of the restriction

$$\mathcal{S} : \mathtt{Mor}(\mathtt{Clop}(Y), \mathtt{Clop}(X^*)) \to \mathtt{Mor}(\mathcal{S}(\mathtt{Clop}(X^*)), \mathcal{S}(\mathtt{Clop}(Y)))$$

which is indeed a bijection. With this notation we get

$$\mathcal{T}(Y, q) := \mathtt{w}^{-1}[\mathcal{S}^{-1}(\mathcal{R}(q))[\mathtt{Clop}(Y)]]$$

Before we present more of these uninviting formulas to define how $\mathcal{T}$ maps morphisms, let us observe a few things about this object map. Since q is a quotient map, so is $\mathcal{R}(q) = h_Y \circ q \circ h_{X^*}^{-1}$, and therefore $\mathcal{S}^{-1}(\mathcal{R}(q))$ is an embedding. Together with the fact that $\mathtt{w}$ is an isomorphism, this shows that $\mathcal{T}(Y, q)$ is isomorphic to the algebra $\mathtt{Clop}(Y)$. Also important is the fact that the object map $\mathcal{T}$ is onto the collection of all subalgebras of $\mathcal{P}(X)/\mathtt{fin}$. To see this, given $\mathcal{B} \leq \mathcal{P}(X)/\mathtt{fin}$, let $i_{\mathcal{B}}$ denote the inclusion of $\mathcal{B}$ into $\mathcal{P}(X)/\mathtt{fin}$, and consider the pair

$$(\mathcal{S}(\mathcal{B}), \mathcal{S}(\mathtt{w} \circ i_{\mathcal{B}}) \circ h_{X^*})$$

which is easily seen to be an object of $\mathtt{Quot}_0(X^*)$. With a few straight-forward calculations it follows that $h_{\mathcal{S}(\mathcal{B})}$ is simply the mapping $\mathcal{F} \mapsto \mathtt{V}_{\mathcal{B}}[\mathcal{F}]$, and so $h_{\mathcal{S}(\mathcal{B})} = \mathcal{S}(\mathtt{V}_{\mathcal{B}}^{-1})$. From this we get that $\mathcal{R}(\mathcal{S}(\mathtt{w} \circ i_{\mathcal{B}}) \circ h_{X^*}) = \mathcal{S}(\mathtt{w} \circ \mathtt{V}_{\mathcal{B}}^{-1})$, and ultimately that

$$\mathcal{T}(\mathcal{S}(\mathcal{B}), \mathcal{S}(\mathtt{w} \circ i_{\mathcal{B}}) \circ h_{X^*}) = \mathcal{B}$$

Given an object (Y, q), let us temporarily denote the isomorphism $\mathtt{w}^{-1} \circ \mathcal{S}^{-1}(\mathcal{R}(q)) : \mathtt{Clop}(Y) \to \mathcal{T}(Y, q)$ by ψ_q. Let (Y_0, q_0) and (Y_1, q_1) be objects and $f : Y_0 \to Y_1$ a morphism. Then $\mathcal{S}^{-1}(\mathcal{R}(f))$ is a homomorphism from $\mathtt{Clop}(Y_1)$ into $\mathtt{Clop}(Y_0)$, so we can define

$$\mathcal{T}(f) := \psi_{q_0} \circ \mathcal{S}^{-1}(\mathcal{R}(f)) \circ \psi_{q_1}^{-1}$$

which makes $\mathcal{T}(f)$ a homomorphism from $\mathcal{T}(Y_1, q_1)$ into $\mathcal{T}(Y_0, q_0)$.

Lemma 1.21. *As defined above, $\mathcal{T} : \mathtt{Quot}_0(X^*) \to \mathtt{BAl}$ is a contravariant fully faithful functor. Its map of objects takes the Hausdorff 0-dimensional (second-countable) quotients of X^* precisely onto the set of (countable) subalgebras of $\mathcal{P}(X)/\mathtt{fin}$. The map of morphisms takes the quotient maps (resp. homeomorphisms) precisely onto the embeddings (resp. isomorphisms).*

Proof. Most of the proof is straight-forward, and similar to what we have done so far. The part that is new and requires some comment is the correspondence between second-countable quotients and countable algebras.

Fix an object (Y, q). If $\mathcal{T}(Y, q)$ is countable, that means $\mathtt{Clop}(Y)$ is countable, and since Y is 0-dimensional it follows that Y is second-countable. The other direction is less obvious. In general, a space can have a countable basis and still have uncountably many clopen subsets (for example, a countably infinite discrete space), but this is impossible for compact spaces. If Y is second-countable, we fix a countable basis, and write each clopen subset as a union of basic subsets. Since the clopen sets are compact, they can be written as finite unions of basic sets, and since there are only countably many such unions we see that $\mathtt{Clop}(Y)$ (and therefore $\mathcal{T}(Y, q)$) is countable. □

Remark 1.22. At this point I should comment that the functor $\mathcal{T}$ is not as complicated as it looks. It has been defined this way because I wanted to make three clear steps: (a) Show the correspondence between Boolean algebras and their Stone spaces (this is functor $\mathcal{S}$); (b) Show the correspondence between Boolean spaces in general and Stone spaces of Boolean algebras (this is functor $\mathcal{R}$); (c) Use the two previous steps to show the correspondence between the quotients of a particular Boolean space and the subalgebras of a particular Boolean algebra. Indeed, after defining the first two functors, we barely had to prove anything about $\mathcal{T}$, since most of its important properties were easy consequences of things we had already established for $\mathcal{S}$ and $\mathcal{R}$. Nevertheless, if we open up the expressions defining $\mathcal{T}$ they can be simplified: If Y is a Boolean space, and $C \subseteq Y$ is clopen, it has been implicitly shown in Lemma 1.5 that $h_Y[C] = \mathtt{V}_{\mathtt{Clop}(Y)}(C)$. From this, it follows that if $f : Y_0 \to Y_1$ is continuous between the Boolean spaces Y_0 and Y_1, and $C \subseteq Y_1$ is clopen, then $\mathcal{S}^{-1}(\mathcal{R}(f))(C) = f^{-1}[C]$. Consequently, if (Y, q) is an object of $\mathtt{Quot}_0(X^*)$, then:

$$\mathcal{T}(Y, q) = \{\mathtt{w}^{-1}(q^{-1}[C]) \,:\, C \subseteq Y \text{ is clopen}\}$$

Moreover (with some effort), if (Y_0, q_0) and (Y_1, q_1) are objects of $\mathtt{Quot}_0(X^*)$, $f : Y_0 \to Y_1$ is continuous and $e \in \mathcal{T}(Y_1, q_1)$, then:

$$\mathcal{T}(f)(e) = \mathtt{w}^{-1}(q_0^{-1}[f^{-1}[q_1[\mathtt{w}(e)]]])$$

2 THE LIFTING PROBLEM AND THE KEY EXAMPLE

In an article from 2002 [BDH+02], Bella et al. proved the theorem below in the case that φ is an automorphism of $\mathcal{B}$ (though for this case their result was far more general). Here, we use the notion of near-surjections to relax this condition, but the proof will be almost the same.

Theorem 2.1. *Let $\mathcal{B}$ be a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ a homomorphism. Then, φ can be extended to a trivial epimorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. In the case that φ is an embedding, it can be extended to a very trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$.*

To prove this theorem we will need a few more facts about Boolean algebras. The first is the famous *Rasiowa-Sikorski Theorem*, which we will state here without proof, partly because it can be found in every textbook about set theory, but mostly because we will prove a stronger version in PART TWO. In the context of topology, filters are usually filters on a space, or some subalgebra of the power set of a space, and the *stronger* elements of a filter are usually the smaller sets, because their points are "closer together" (in some sense they give more information about the "location" towards which a filter might converge). However, in the context of set theory (especially with forcing methods), we often use filters on collections of partial functions from some set, and try to construct a function on the whole set as the union of a filter, so the *stronger* elements are clearly the bigger partial functions (which contain more information about the resulting global function). In other words, in the one case we want filters to have elements below any two of its own elements, and to be closed under taking bigger elements, while in the other case we want filters to have elements above any two of its own elements, and to be closed under taking smaller elements. For coherence, we should choose one of the definitions, so we stay with the one we have already used in this document (from the topology context), and whenever we need filters of partial functions, for example, we will define the order to be the reversed inclusion.

Definition 2.2. Let $\mathbb{P}$ be a partial order. A *filter* on $\mathbb{P}$ is a subset $\mathcal{F}$ such that: (a) If $p_0, p_1 \in \mathcal{F}$, then there is $p_2 \in \mathcal{F}$ such that $p_2 \leq p_0$ and $p_2 \leq p_1$; (b) If $p_0 \in \mathcal{F}$ and $p_1 \geq p_0$, then $p_1 \in \mathcal{F}$. A subset D of $\mathbb{P}$ is called *dense (in $\mathbb{P}$)* if for every $p_0 \in \mathbb{P}$ there is $p_1 \in D$ such that $p_1 \leq p_0$. If $\mathcal{D}$ is a collection of subsets of $\mathbb{P}$ and $\mathcal{F}$ is a filter, we say $\mathcal{F}$ is *generic for* $\mathcal{D}$, or $\mathcal{D}$*-generic*, if $\mathcal{F} \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Note that there is still a small difference between this definition and what we consider to be filters on topological spaces, namely that in the latter case we require filters not to contain the empty set. To solve this we accept the convention that whenever we speak of filters on a Boolean algebra $\mathcal{B}$ (such as the power set of a topological space), we actually mean filters on $\mathcal{B} \setminus \{0\}$. Of course, this will depend on the context, since a Boolean algebra is also a partial order. In the definition above it is not required that $\mathcal{D}$ be a family of *dense* subsets of $\mathbb{P}$, but that will always be the interesting case.

Theorem 2.3 (Rasiowa-Sikorski)**.** *If $\mathbb{P}$ is a partial order and $\mathcal{D}$ is a countable collection of dense sets in $\mathbb{P}$, then there exists a filter on $\mathbb{P}$ which is generic for $\mathcal{D}$.* □

If X is a subset of a given structure, we use $\langle X \rangle$ to denote the substructure it generates. Also, if R is a binary relation on this structure and a is an element, we use X_{Ra} to denote the set of all $x \in X$ such that xRa. We will use this notation mostly with Boolean algebras and the relations $\leq$, $\geq$, $\not\leq$, and $\not\geq$.

Theorem 2.4. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be Boolean algebras, and $\mathcal{C} \leq \mathcal{B}_0$. Suppose $\varphi : \mathcal{C} \to \mathcal{B}_1$ is a homomorphism, $b_0 \in \mathcal{B}_0$ and $b_1 \in \mathcal{B}_1$. Then, φ can be extended to a homomorphism $\varphi' : \langle \mathcal{C} \cup \{b_0\} \rangle \to \mathcal{B}_1$ which maps b_0 to b_1 if and only if*

$$\varphi[\mathcal{C}_{\leq b_0}] \subseteq \varphi[\mathcal{C}]_{\leq b_1} \text{ and } \varphi[\mathcal{C}_{\geq b_0}] \subseteq \varphi[\mathcal{C}]_{\geq b_1}. \tag{2.5}$$

In the affirmative case, the extension is unique. Moreover, in the case that φ is an embedding, this unique extension is an embedding if and only if

$$\varphi[\mathcal{C}_{\leq b_0}] = \varphi[\mathcal{C}]_{\leq b_1} \text{ and } \varphi[\mathcal{C}_{\geq b_0}] = \varphi[\mathcal{C}]_{\geq b_1}. \tag{2.6}$$

Proof. A proof in full detail here is unnecessary, but a few remarks should make it easier if the reader wants to fill in all the gaps. It is almost immediate that the existence of φ' implies (2.5), and that if φ' is an embedding (2.6) also holds, so let us take care of the proof in the opposite direction. First, one should check the equalities

$$\begin{aligned} e &= (e \wedge e') \vee (e \wedge \neg e') \\ e' &= (1 \wedge e') \vee (0 \wedge \neg e') \\ \neg((e_0 \wedge e') \vee (e_1 \wedge \neg e')) &= (\neg e_0 \wedge e') \vee (\neg e_1 \wedge \neg e') \\ ((e_0 \wedge e') \vee (e_1 \wedge \neg e')) \vee ((e_2 \wedge e') \vee (e_3 \wedge \neg e')) &= ((e_0 \vee e_2) \wedge e') \vee ((e_1 \vee e_3) \wedge \neg e') \end{aligned}$$

for any e, e', e_0, and e_1 in a Boolean algebra. With these equalities it follows that if $\mathcal{B}$ is a subalgebra of a Boolean algebra $\mathcal{B}'$, and $e' \in \mathcal{B}'$, then

$$\langle \mathcal{B} \cup \{e'\} \rangle = \{(e_0 \wedge e') \vee (e_1 \wedge \neg e') : e_0, e_1 \in \mathcal{B}\} \tag{2.7}$$

Next, one should check that for e_0, e_1, e_2, e_3, and e' in a Boolean algebra we have:

$$(e_0 \wedge e') \vee (e_1 \wedge \neg e') = (e_2 \wedge e') \vee (e_3 \wedge \neg e') \text{ if and only if}$$
$$(e_1 \wedge \neg e_3) \vee (\neg e_1 \wedge e_3) \leq e' \leq (e_0 \wedge e_2) \vee (\neg e_0 \wedge \neg e_2)$$

Hence, for c_0, c_1, c_2 and c_3 in $\mathcal{C}$, (2.5) implies:

$$(c_0 \wedge b_0) \vee (c_1 \wedge \neg b_0) = (c_2 \wedge b_0) \vee (c_3 \wedge \neg b_0) \Longrightarrow$$
$$\Longrightarrow (\varphi(c_0) \wedge b_1) \vee (\varphi(c_1) \wedge \neg b_1) = (\varphi(c_2) \wedge b_1) \vee (\varphi(c_3) \wedge \neg b_1) \quad (2.8)$$

whereas (2.6) together with injectivity of φ implies:

$$(c_0 \wedge b_0) \vee (c_1 \wedge \neg b_0) = (c_2 \wedge b_0) \vee (c_3 \wedge \neg b_0) \Longleftrightarrow$$
$$\Longleftrightarrow (\varphi(c_0) \wedge b_1) \vee (\varphi(c_1) \wedge \neg b_1) = (\varphi(c_2) \wedge b_1) \vee (\varphi(c_3) \wedge \neg b_1) \quad (2.9)$$

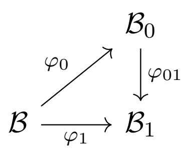
If (2.5) holds, it follows from (2.7) and (2.8) that the expression

$$\varphi'((c_0 \wedge b_0) \vee (c_1 \wedge \neg b_0)) := (\varphi(c_0) \wedge b_1) \vee (\varphi(c_1) \wedge \neg b_1)$$

defines a map $\varphi' : \langle \mathcal{C} \cup \{b_0\} \rangle \to \mathcal{B}_1$ well. The equalities from the beginning of this proof show that φ' extends φ, maps b_0 to b_1, and is a homomorphism. Finally, if φ is injective and (2.6) also holds, then (2.9) implies that φ' is injective. □

The theorem above can be found in [Kop89a]. The lemma below can be alternatively stated as "all countable Boolean algebras are projective", and is found in [Kop89b].

Lemma 2.10. *Let $\mathcal{B}$, $\mathcal{B}_0$ and $\mathcal{B}_1$ be Boolean algebras, and $\varphi_{01} : \mathcal{B}_0 \to \mathcal{B}_1$ and $\varphi_1 : \mathcal{B} \to \mathcal{B}_1$ be homomorphisms, such that $\varphi_1[\mathcal{B}] \subseteq \varphi_{01}[\mathcal{B}_0]$. If $\mathcal{B}$ is countable, then there is a homomorphism $\varphi_0 : \mathcal{B} \to \mathcal{B}_0$ closing the diagram*



Proof. Let $(e_n)_{n<|\mathcal{B}|}$ be an enumeration of $\mathcal{B}$ and for each $\alpha \leq |\mathcal{B}|$ let $\mathcal{C}_\alpha := \langle e_n : n < \alpha \rangle$. We shall inductively define maps $r_\alpha : \mathcal{C}_\alpha \to \mathcal{B}_0$ such that $\varphi_{01} \circ r_\alpha = \varphi_1 \restriction \mathcal{C}_\alpha$, and such that $(r_\alpha)_{\alpha \leq |\mathcal{B}|}$ is an increasing chain. It is clear that there is only one choice for r_0. Given $n < |\mathcal{B}|$, assume we have chosen r_n.

Let e be an arbitrary element in $\varphi_{01}^{-1}(\varphi_1(e_n))$, let $e^{\leq}$ be the greatest element of $\mathcal{C}_n$ below e_n, and let $e^{\geq}$ be the smallest element of $\mathcal{C}_n$ above e_n. Finally, define:

$$e' := (e \vee r_n(e^{\leq})) \wedge r_n(e^{\geq})$$

Clearly $\varphi_{01}(e') = \varphi_1(e_n)$, and using Theorem 2.4 we see that r_n can be uniquely extended to a homomorphism $r_{n+1} : \mathcal{C}_{n+1} \to \mathcal{B}_0$ mapping e_n to e'. It follows that $\varphi_{01} \circ r_{n+1} = \varphi_1 \restriction \mathcal{C}_{n+1}$ as we wanted.

If $\mathcal{B}$ is finite, the induction ends with the definition of $r_{|\mathcal{B}|}$ which is defined on $\mathcal{C}_{|\mathcal{B}|} = \mathcal{B}$, so we let $\varphi_0 := r_{|\mathcal{B}|}$. Otherwise, the induction step defined above creates the maps r_n for $n < \omega$, so we still need to define $\varphi_0 := r_\omega := \bigcup_{n<\omega} r_n$. □

Of course, in the lemma above, if φ_1 is an embedding, φ_0 must be an embedding as well. Applying the lemma to a case of particular interest for us we get:

Corollary 2.11. *Let $\mathcal{B}$ be a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. Then we can choose representatives $r(e) \in e$ for each $e \in \mathcal{B}$ such that the map $r : \mathcal{B} \to \mathcal{P}(\omega)$ is an embedding.* □

The map r in the corollary above is called a *lifting of (the inclusion of) $\mathcal{B}$ (into $\mathcal{P}(\omega)/\mathtt{fin}$)*. We are ready for the theorem with which we started the chapter:

Proof of Theorem 2.1. The algebra $\mathcal{B}' := \langle \mathcal{B} \cup \mathtt{ran}(\varphi)\rangle$ is countable, so there is a lifting r of $\mathcal{B}'$, as in the previous corollary. Consider the partial order

$$\mathbb{P} := \{(f, E) \,:\, f \subseteq \omega \times \omega \text{ is a finite 1-1 map, and } E \subseteq \mathcal{B} \text{ is finite}\}$$

where the order is given by $(f_0, E_0) \geq (f_1, E_1)$ if and only if $f_0 \subseteq f_1$, $E_0 \subseteq E_1$ and for all $e \in E_0$ holds

$$(r(e) \cap \mathtt{dom}(f_1)) \,\Delta\, f_1^{-1}[r\varphi(e)] \subseteq \mathtt{dom}(f_0)$$

This formula is a complicated way to state a very intuitive idea. We want our partial functions to respect the "rule" that precisely the elements of $r(e)$ are mapped to elements of $r\varphi(e)$, since this "rule" will make sure that the resulting global function induces φ on $\mathcal{B}$. So when we try to extended f_0 to a "stronger" map f_1, the formula above says that f_1 will respect the "rule", except perhaps where f_0 was already defined. For each $m, n \in \omega$ and $e \in \mathcal{B}$ we define:

$$\begin{aligned} D_m &:= \{(f, E) \in \mathbb{P} \,:\, m \in \mathtt{dom}(f)\} \\ R_n &:= \{(f, E) \in \mathbb{P} \,:\, n \in \mathtt{ran}(f)\} \\ T_e &:= \{(f, E) \in \mathbb{P} \,:\, e \in E\} \end{aligned}$$

Fix $(f, E) \in \mathbb{P}$. If $e \in \mathcal{B}$, then $(f, E \cup \{e\}) \in \mathbb{P}_{\leq (f,E)} \cap T_e$. This shows that T_e is dense in $\mathbb{P}$. If $n \in \omega \setminus \mathtt{ran}(f)$, let $E' := \{e \in E \,:\, n \in r\varphi(e)\}$. Then n is in the set

$$\bigcap_{e \in E'} r\varphi(e) \cap \left(\omega \setminus \bigcup_{e \in E \setminus E'} r\varphi(e)\right) = r\varphi\left(\bigwedge E' \wedge \neg \bigvee (E \setminus E')\right)$$

The fact that this set is non-empty shows that $\bigwedge E' \wedge \neg \bigvee (E \setminus E') \neq 0$, which in turn implies that $r(\bigwedge E' \wedge \neg \bigvee (E \setminus E'))$ is infinite. Thus we can choose m in

$$\left(\bigcap_{e \in E'} r(e) \cap \left(\omega \setminus \bigcup_{e \in E \setminus E'} r(e)\right)\right) \setminus \mathtt{dom}(f)$$

and it follows that $(f \cup \{(m, n)\}, E) \in \mathbb{P}_{\leq (f,E)} \cap R_n$, showing that R_n is dense in $\mathbb{P}$. Hence, by the Rasiowa-Sikorski Theorem (2.3) there is a filter $\mathcal{F}$ on $\mathbb{P}$ which is generic for $\{T_e \,:\, e \in \mathcal{B}\}$ and for $\{R_n \,:\, n \in \omega\}$. Because $\mathcal{F}$ is a filter, $\sigma := \bigcup_{(f,E) \in \mathcal{F}} f$ is a function, and it is clearly injective. Also, because $\mathcal{F}$ intersects every R_n, we have $\mathtt{ran}(\sigma) = \omega$, so σ is a near-surjection. We must now check that $\varphi_\sigma \restriction \mathcal{B} = \varphi$. For this, given $e \in \mathcal{B}$, since $\mathcal{F}$ intersects T_e we can choose $(f, E) \in \mathcal{F}$ such that $e \in E$. Using the definition of the ordering of $\mathbb{P}$ we see that

$$(r(e) \cap \mathtt{dom}(\sigma)) \,\Delta\, \sigma^{-1}[r\varphi(e)] \subseteq \mathtt{dom}(f)$$

and since $\mathtt{dom}(f)$ is finite it follows that $[\![r(e) \cap \mathtt{dom}(\sigma)]\!] = [\![\sigma^{-1}[r\varphi(e)]]\!]$, and hence $\varphi_\sigma(e) = \varphi_\sigma([\![r(e)]\!]) = [\![\sigma[r(e)]]\!] = [\![r\varphi(e)]\!] = \varphi(e)$. This concludes the first part.

Finally, we assume that φ is an embedding, and show that σ can be chosen to be a permutation of ω. Given $(f, E) \in \mathbb{P}$ and $m \in \omega \setminus \mathtt{dom}(f)$, let $E' := \{e \in E \,:\, m \in r(e)\}$. Then m is in the set

$$\bigcap_{e \in E'} r(e) \cap \left(\omega \setminus \bigcup_{e \in E \setminus E'} r(e)\right) = r\left(\bigwedge E' \wedge \neg \bigvee (E \setminus E')\right)$$

which, as before, implies that $\bigwedge E' \wedge \neg \bigvee(E \setminus E') \neq 0$, and because φ is injective we know that $\varphi(\bigwedge E' \wedge \neg \bigvee(E \setminus E')) \neq 0$, and therefore $r\varphi(\bigwedge E' \wedge \neg \bigvee(E \setminus E'))$ is infinite. So we can choose n in

$$\left(\bigcap_{e \in E'} r\varphi(e) \cap \left(\omega \setminus \bigcup_{e \in E \setminus E'} r\varphi(e) \right) \right) \setminus \mathtt{ran}(f)$$

and it follows that $(f \cup \{(m,n)\}, E) \in \mathbb{P}_{\leq (f,E)} \cap D_m$, showing that D_m is dense in $\mathbb{P}$. This means we can additionally assume that $\mathcal{F}$ is generic for $\{D_m : m \in \omega\}$ above, and then we have $\mathtt{dom}(\sigma) = \omega$. □

At last, Lemma 1.21 together with Theorem 2.1 culminate in the following result which we have been slowly building for quite a while now:

Corollary 2.12. *Let $q : \omega^* \to Y$ be a quotient, where Y is 0-dimensional second-countable Hausdorff, and let $f : \omega^* \to Y$ be a continuous map. Then, there is a continuous map $g : \omega^* \to \omega^*$ lifting f, i.e. closing the diagram:*

$$\begin{array}{ccc} \omega^* & \xrightarrow{\;g\;} & \omega^* \\ & \searrow^{f} & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

Moreover, if f is onto Y, we can choose g to be a homeomorphism.

Proof. If we consider the objects $(\omega^*, \mathtt{id}_{\omega^*})$ and (Y, q) of $\mathtt{Quot}_0(\omega^*)$, then f and q are morphisms from the former to the latter. Also, $\mathcal{T}(\omega^*, \mathtt{id}_{\omega^*}) = \mathcal{P}(\omega)/\mathtt{fin}$ and $\mathcal{T}(q)$ is the inclusion of $\mathcal{T}(Y,q)$ into $\mathcal{P}(\omega)/\mathtt{fin}$. Since $\mathcal{T}(f) : \mathcal{T}(Y,q) \to \mathcal{P}(\omega)/\mathtt{fin}$ is a homomorphism and $\mathcal{T}(Y,q)$ is countable, by Theorem 2.1 there is an epimorphism $\psi : \mathcal{P}(\omega)/\mathtt{fin} \to \mathcal{P}(\omega)/\mathtt{fin}$ extending $\mathcal{T}(f)$, i.e. such that $\psi \circ \mathcal{T}(q) = \mathcal{T}(f)$. The functor $\mathcal{T}$ is full, so there is a continuous $g : \omega^* \to \omega^*$ such that $\mathcal{T}(g) = \psi$, and then we have $\mathcal{T}(q \circ g) = \mathcal{T}(g) \circ \mathcal{T}(q) = \mathcal{T}(f)$, which implies (because $\mathcal{T}$ is faithful) that $q \circ g = f$. Moreover, if f is a quotient onto Y, then $\mathcal{T}(f)$ is an embedding, so we can choose ψ to be an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$, which implies that g is a self-homeomorphism of ω^*. □

We may ask whether the hypotheses of this corollary can be weakened. It is meaningless to consider q non-surjective, because there can only be a lifting for f if $\mathtt{ran}(f) \subseteq \mathtt{ran}(q)$. There is little interest in topological results with non-Hausdorff spaces, so what is left to investigate is the removal of either second-countability or 0-dimensionality.

If Y is not second-countable above, we end up with $\mathcal{T}(Y,q)$ uncountable and the question arises whether there is a homomorphism $\varphi : \mathcal{T}(Y,q) \to \mathcal{P}(\omega)/\mathtt{fin}$ which cannot be extended to an endomorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. A partial answer is found in [BDH$^+$02], where they prove that under the *Continuum Hypothesis (CH)* there is a subalgebra $\mathcal{B}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\mathcal{B} \simeq \mathcal{P}(\omega)/\mathtt{fin}$ and the only automorphism of $\mathcal{B}$ that can be extended to an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ is the identity. While this question is worth looking into, in the following chapters we will be concerned with the remaining alternative: removing 0-dimensionality.

Theorem 2.13. *Let $q : \omega^* \to Y$ be a quotient, where Y is second-countable Hausdorff, and let $f : \omega^* \to Y$ be a continuous map. Then, there is a continuous map $g : \omega^* \to \omega^*$ lifting f, i.e. closing the diagram:*

$$\begin{array}{ccc} \omega^* & \xrightarrow{\;g\;} & \omega^* \\ & \searrow^{f} & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

Moreover, if f is onto Y, we can choose g to be a homeomorphism.

What makes this claim challenging is that it strips us from all the tools we have built to work with ω^*, because quotients which are not 0-dimensional do not correspond to subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. I believe the next example is one of the most powerful to understand the situation described in Theorem 2.13, and should be kept in mind throughout the proof in the next chapter.

Example 2.14 (Key example)**.** Let τ be your favourite bijection from ω onto $[0,2)\cap\mathbb{Q}$, and let $\sigma:\omega\to S^1$ be the mapping

$$\sigma(n):=e^{i\pi\tau(n)}$$

There is a unique continuous map extending σ from $\beta\omega$ into S^1 (universal property of the Stone-Čech compactification), and this map is usually denoted $\beta\sigma$. Since the range of σ is dense in S^1, and the range of $\beta\sigma$ is closed, it follows that $\beta\sigma$ is onto S^1. Now let $\sigma^*:=\beta\sigma\setminus\sigma$, that is, the restriction of $\beta\sigma$ to ω^*. Since $S^1\setminus\mathtt{ran}(\sigma)$ is also dense in S^1, and the range of σ^* is again closed, it follows that $\sigma^*:\omega^*\to S^1$ is a quotient map (onto a space which is second-countable Hausdorff, but not 0-dimensional). If r_α is the anticlockwise rotation on S^1 by an angle $\alpha\in\mathbb{R}$, we ask whether r_α can be lifted to a self-homeomorphism g of ω^*. Of course, the diagram we are trying to close here is

$$\begin{array}{ccc} \omega^* & \xrightarrow{g} & \omega^* \\ {\scriptstyle\sigma^*}\downarrow & & \downarrow{\scriptstyle\sigma^*} \\ S^1 & \xrightarrow[r_\alpha]{} & S^1 \end{array}$$

but it is equivalent to the diagram from Theorem 2.13 if we take $f:=r_\alpha\circ\sigma^*$. The first case we consider is that $\alpha/\pi\in\mathbb{Q}$. In this case, r_α permutes the range of σ, and thus induces a permutation of ω by $n\mapsto\sigma^{-1}r_\alpha\sigma(n)$. Viewing this permutation as a map from ω into $\beta\omega$, there is a unique continuous extension $h:\beta\omega\to\beta\omega$. It is not difficult to check that h is a homeomorphism, that $h[\omega^*]=\omega^*$, and that it closes the diagram

$$\begin{array}{ccc} \beta\omega & \xrightarrow{h} & \beta\omega \\ {\scriptstyle\beta\sigma}\downarrow & & \downarrow{\scriptstyle\beta\sigma} \\ S^1 & \xrightarrow[r_\alpha]{} & S^1 \end{array}$$

Therefore we can simply take $g:=h\restriction\omega^*$. The second case is that $\alpha/\pi\notin\mathbb{Q}$. Note that, given an infinite discrete space X, by Lemma 1.10 we know that X is open in βX, and therefore its points are isolated in βX. On the other hand, the points of X^* are not isolated in βX, since they are accumulation points of X, therefore X is precisely the set of isolated points of βX. It follows that X is invariant under self-homeomorphisms of βX. In the present case, $r_\alpha[\mathtt{ran}(\sigma)]\cap\mathtt{ran}(\sigma)=\emptyset$, so there is no hope of finding a homeomorphism h closing the diagram above (since we would necessarily have $h[\omega]=\omega$). Indeed, this problem will be less obvious to solve.

I should point out that I have not been wasting the reader's time (I hope). Even though Theorem 2.13 is far more general than Corollary 2.12, the preparation for the proof of the latter will still be useful when proving the former (particularly Lemma 1.21 and Theorem 2.1).

3 Lifting without 0-dimensionality

Before jumping into the proof of Theorem 2.13, we need to gather some more knowledge about ω^*. As one would expect, for many of its properties it is easier (or perhaps just more elementary) to prove their counter-parts about $\mathcal{P}(\omega)/\mathtt{fin}$ instead. Some of the facts exposed below may not be used explicitly later, but they provide good intuition about ω^*. Early proofs of many of the following lemmata are given in [Rud56].

Lemma 3.1. *There are no isolated points in ω^*.*

Proof. We have already seen that the points of ω^* are not isolated in $\beta\omega$, but this does not directly imply that they are not isolated in ω^* itself. Recall that for $E \subseteq \omega$, $\mathtt{W}(E) \neq \emptyset$ if and only if E is infinite. In this case, let $E' \subseteq E$ be infinite and such that $E \setminus E'$ is also infinite. Then $\mathtt{W}(E')$ and $\mathtt{W}(E \setminus E')$ are disjoint non-empty subsets of $\mathtt{W}(E)$, showing that $\mathtt{W}(E)$ is not a singleton. □

To avoid using the Hewitt-Marczewski-Pondiczery Theorem, which is much too powerful for our needs here, I offer a short construction of a countable dense subset of the space $T := 2^{2^{\aleph_0}}$ (with the product topology). We use the fact that the Cantor space $2^{\aleph_0}$ is second-countable and Hausdorff, even though its topology is irrelevant for the topology of T. If $\mathcal{B}$ is a countable basis for the Cantor space, we may assume it is closed under finite unions, and it follows that given two disjoint finite sets $F_0, F_1 \subseteq 2^{\aleph_0}$, there is $B \in \mathcal{B}$ such that $F_0 \cap B = \emptyset$ and $F_1 \subseteq B$. Given $B \in \mathcal{B}$, let $f_B : 2^{\aleph_0} \to 2$ be defined by $f_B^{-1}(1) = B$. Then, if U is a non-empty open set in T, there are disjoint finite sets $F_0, F_1 \subseteq 2^{\aleph_0}$ such that $U \supseteq \{f : 2^{\aleph_0} \to 2 : F_0 \subseteq f^{-1}(0) \text{ and } F_1 \subseteq f^{-1}(1)\}$, and hence if we choose $B \in \mathcal{B}$ as above, we have $f_B \in U$. This shows that the set $\{f_B : B \in \mathcal{B}\}$ is dense in T.

Lemma 3.2. *The cardinality of $\beta\omega$ (and consequently of ω^*) is $2^{2^{\aleph_0}}$.*

Proof. Since the space $2^{2^{\aleph_0}}$ is separable, we can map ω onto a dense subset, and use the universal property of the Stone-Čech compactification to obtain a surjective map $\beta\omega \to 2^{2^{\aleph_0}}$, showing that $|\beta\omega| \geq 2^{2^{\aleph_0}}$. The reversed inequality follows from the fact that we have constructed $\beta\omega$ as a subset of $\mathcal{P}(\mathcal{P}(\omega))$. □

Lemma 3.3. *In compact 0-dimensional spaces, disjoint closed sets can be separated by clopen sets.*

Proof. Let F_0 and F_1 be disjoint and closed in a compact 0-dimensional space X. Then $X \setminus F_1$ is a union of clopen sets which cover the compact set F_0, and therefore we can choose finitely many of these clopen sets to cover F_0, and call their union V. It follows that V itself is clopen, and that $F_0 \subseteq V$, and $F_1 \subseteq X \setminus V$ (which is also clopen). □

Lemma 3.4. *Let $(a_n)_{n\in\omega}$ be an increasing chain in $\mathcal{P}(\omega)/\mathtt{fin}$, and $(b_n)_{n\in\omega}$ a decreasing one (these chains are not necessarily strictly increasing or decreasing), and suppose $a_n < b_n$ for every $n \in \omega$. Then, there is $e \in \mathcal{P}(\omega)/\mathtt{fin}$ such that for every $n \in \omega$ holds $a_n < e < b_n$.*

Proof. By Corollary 2.11 there is a lifting r of the algebra $\langle \{a_n\}_{n\in\omega} \cup \{b_n\}_{n\in\omega} \rangle$, and being an embedding it satisfies $r(a_n) \subseteq r(a_{n+1}) \subsetneq r(b_{n+1}) \subseteq r(b_n)$ for all $n \in \omega$. Also, since $[\![r(a_n)]\!] \neq [\![r(b_n)]\!]$, it follows that $r(b_n) \setminus r(a_n)$ is infinite. So we can choose two sequences $(p_n)_{n\in\omega}$ and $(q_n)_{n\in\omega}$ in ω satisfying

$$\begin{aligned} p_n &\in r(b_n) \setminus (r(a_n) \cup \{p_i\}_{i<n} \cup \{q_i\}_{i<n}) \\ \text{and} \quad q_n &\in r(b_n) \setminus (r(a_n) \cup \{p_i\}_{i\leq n} \cup \{q_i\}_{i<n}) \\ \text{and let} \quad E &:= \left(\{p_n\}_{n\in\omega} \cup \bigcup_{n\in\omega} r(a_n) \right) \setminus \{q_n\}_{n\in\omega} \end{aligned}$$

Clearly, for every $n \in \omega$ we have that $r(a_n) \subseteq E \cup \{q_i : i < n\}$, and so $r(a_n) \subseteq^* E$, while $E \setminus r(a_n) \supseteq \{p_m : m \geq n\}$. Letting $e := [\![E]\!]$ we get $a_n < e$. On the other hand, $E \setminus \{p_i : i < n\} \subseteq r(b_n)$, so $E \subseteq^* r(b_n)$, and since $\{q_m : m \geq n\} \subseteq r(b_n) \setminus E$, we have $e < b_n$. □

Lemma 3.5. *Let $U_0, U_1 \subseteq \omega^*$, where U_0 is open F_σ, and U_1 is closed G_δ. If $U_0 \subsetneq U_1$, then there is a clopen set $V \subseteq \omega^*$ such that $U_0 \subsetneq V \subsetneq U_1$.*

Proof. Write $U_0 = \bigcup_{n\in\omega} F_n$ where each F_n is closed, and $U_1 = \bigcap_{n\in\omega} O_n$ where each O_n is open. By Lemma 3.3 there are, for each n, elements $e_{0,n}, e_{1,n} \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $F_n \subseteq \mathtt{w}(e_{0,n}) \subseteq U_0$ and $U_1 \subseteq \mathtt{w}(e_{1,n}) \subseteq O_n$. It follows that $U_0 = \bigcup_{n\in\omega} \mathtt{w}(e_{0,n})$ and $U_1 = \bigcap_{n\in\omega} \mathtt{w}(e_{1,n})$. Now consider the sequences:

$$a_n := e_{0,0} \vee \cdots \vee e_{0,n}$$
$$b_n := e_{1,0} \wedge \cdots \wedge e_{1,n}$$

Clearly, $(a_n)_{n\in\omega}$ is increasing and $(b_n)_{n\in\omega}$ is decreasing. Moreover for each n we have $\mathtt{w}(a_n) \subseteq U_0 \subsetneq U_1 \subseteq \mathtt{w}(b_n)$, hence $a_n < b_n$. By the previous lemma there is $e \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $a_n < e < b_n$ for every n. Then, we can apply the lemma a second time replacing $(a_n)_{n\in\omega}$ with the constant sequence $(e)_{n\in\omega}$ to obtain $e' \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $e < e' < b_n$ for every n. Finally, choosing any e'' such that $e < e'' < e'$ we can define $V := \mathtt{w}(e'')$, and the result follows easily. □

Corollary 3.6. *Disjoint open F_σ subsets of ω^* have disjoint closures.*

Proof. Let U_0 and U_1 be disjoint open F_σ sets in ω^*. If $U_0 \cup U_1 = \omega^*$, then U_0 and U_1 are both closed, so we are done. Otherwise, we have $U_0 \subsetneq \omega^* \setminus U_1$, where $\omega^* \setminus U_1$ is closed G_δ, and the result follows from the previous lemma. □

Lemma 3.7. *In 0-dimensional spaces, every non-empty G_δ set contains a non-empty closed G_δ set.*

Proof. Let U be a G_δ set in a 0-dimensional space X, and suppose $x \in U$. We can write $U = \bigcap_{n\in\omega} O_n$ where each O_n is open, and for each n choose a clopen set V_n such that $x \in V_n \subseteq O_n$. Now $V := \bigcap_{n\in\omega} V_n$ is a closed G_δ set and $x \in V \subseteq U$. □

Corollary 3.8. *Every non-empty G_δ set in ω^* has non-empty interior.*

Proof. Let U be a non-empty G_δ subset of ω^*. The previous lemma shows we can assume that U is closed. Since the empty set is open F_σ, by Lemma 3.5 there is a clopen set V such that $\emptyset \subsetneq V \subsetneq U$, which finishes the proof. □

Lemma 3.9. *There is a collection of $2^{\aleph_0}$ many pairwise disjoint elements of $\mathcal{P}(\omega)/\mathtt{fin}$ (meaning the conjunction of any two of them is 0).*

Proof. We will use a *Cantor scheme* in $\mathcal{P}(\omega)/\mathtt{fin}$, i.e. a map $\xi : 2^{<\omega} \to \mathcal{P}(\omega)/\mathtt{fin}$ such that for all $n \in \omega$ and all $f : n \to 2$ holds $\xi(f) \neq 0$, and if $f_0 := f \cup \{(n, 0)\}$ and $f_1 := f \cup \{(n, 1)\}$, then $\xi(f_0) \vee \xi(f_1) \leq \xi(f)$ and $\xi(f_0) \wedge \xi(f_1) = 0$. This is very easily done by defining ξ on 2^n by induction on n.

Using Lemma 3.4, for each $f \in 2^\omega$ we can find $e_f > 0$ such that $e_f < \xi(f \restriction n)$ for every $n \in \omega$. Given $f, f' \in 2^\omega$ distinct, let $n := \mathtt{min}\{k \in \omega : f(k) \neq f'(k)\}$, and it follows that $e_f \wedge e_{f'} \leq \xi(f \restriction n+1) \wedge \xi(f' \restriction n+1) = 0$ since $f \restriction n = f' \restriction n$. Moreover, since the e_f's are pairwise disjoint and are all different of 0, the map $f \mapsto e_f$ is injective. □

Engelking [Eng89] offers an interesting topological proof of the lemma above. Recall that the topological *weight* of a space is the minimum size of a basis for its topology.

Corollary 3.10. *The weight of ω^* (and consequently of $\beta\omega$) is $2^{\aleph_0}$.*

Proof. The size of the basis $\{V(E) : E \subseteq \omega\}$ for $\beta\omega$ is $2^{\aleph_0}$, so this is an upper bound for its weight. Obviously, the weight of ω^* is at most the weight of $\beta\omega$, so it suffices to show that the weight of ω^* is at least $2^{\aleph_0}$. Let $\{e_f : f \in 2^\omega\}$ be the family constructed in the lemma above. The sets $\mathtt{w}(e_f)$ are open and non-empty, so any basis for ω^* must have at least one non-empty set $B_f \subseteq \mathtt{w}(e_f)$ for each $f \in 2^\omega$, and since the $\mathtt{w}(e_f)$'s are pairwise disjoint, the B_f's are all distinct. □

Putting together Corollaries 1.12, 3.6, 3.8 and 3.10 and Lemma 3.1, we have proven that ω^* is a Parovičenko space (in fact, these results correspond precisely to the definition of a Parovičenko space). Parovičenko proved in [Par63] that the CH implies that ω^* is the only Parovičenko space, up to homeomorphism. Later, van Douwen and van Mill showed in [vDvM78] that the negation of CH implies the existence of other homeomorphism types of Parovičenko spaces. This is not important for the remaining of this document, so we shall not go into further detail. At this point starts the more direct part of the proof of Theorem 2.13. The following lemma is a bit technical, but should make more sense if we keep the diagram from Theorem 2.13 in mind.

Lemma 3.11. *Let $q : \omega^* \to Y$ be a quotient, where Y is Hausdorff. Then there is a set $S_q \subseteq \mathcal{P}(\omega)/\mathtt{fin}$, of cardinality at most $\mathtt{weight}(Y) \cdot \aleph_0$ with the following property: If $f : \omega^* \to Y$, and $g : \omega^* \to \omega^*$ are any functions such that $f[g^{-1}[\mathtt{w}(e)]] \subseteq q[\mathtt{w}(e)]$ for every $e \in S_q$, then $qg = f$.*

Proof. Choose a basis $\mathcal{O}$ for Y and let

$$\mathcal{O}_2 := \{(O_0, O_1) \in \mathcal{O}^2 : \overline{O_0} \cap \overline{O_1} = \emptyset\}$$

For each $p = (O_0, O_1) \in \mathcal{O}_2$ the sets $q^{-1}[\,\overline{O_0}\,]$ and $q^{-1}[\,\overline{O_1}\,]$ are closed and disjoint in ω^*, so by Lemma 3.3 there is some $e_p \in \mathcal{P}(\omega)/\mathtt{fin}$ such that

$$q^{-1}[\,\overline{O_0}\,] \subseteq \mathtt{w}(e_p) \text{ and } q^{-1}[\,\overline{O_1}\,] \subseteq \omega^* \setminus \mathtt{w}(e_p)$$

Let $S_q := \{e_p : p \in \mathcal{O}_2\}$ (so $|S_q| \leq |\mathcal{O}| \cdot \aleph_0$), and take two functions $f : \omega^* \to Y$ and $g : \omega^* \to \omega^*$. Assume $qg \neq f$ and let $\mathcal{F} \in \omega^*$ be such that $qg(\mathcal{F}) \neq f(\mathcal{F})$. Then, since Y is compact and Hausdorff, we can find $p = (O_0, O_1) \in \mathcal{O}_2$ such that $qg(\mathcal{F}) \in O_0$ and $f(\mathcal{F}) \in O_1$. It follows that $g(\mathcal{F}) \in q^{-1}[O_0] \subseteq \mathtt{w}(e_p)$, and so $f(\mathcal{F}) \in f[g^{-1}[\mathtt{w}(e_p)]]$. On the other hand, $f(\mathcal{F}) \in O_1$, so we have $f(\mathcal{F}) \notin q[\mathtt{w}(e_p)]$, and hence $f[g^{-1}[\mathtt{w}(e_p)]] \not\subseteq q[\mathtt{w}(e_p)]$. □

From now on we fix our assumptions as in Theorem 2.13, that is, we fix a quotient $q : \omega^* \to Y$ onto a second-countable Hausdorff space Y, and a continuous map $f : \omega^* \to Y$ (keeping the particular case that f is surjective in mind). We also fix a countable set S_q as in the lemma above. Note that a continuous map $g : \omega^* \to \omega^*$ is an endomorphism of the object $(\omega^*, \mathtt{id}_{\omega^*})$ (in the category $\mathtt{Quot}_0(\omega^*)$) and so $\mathcal{T}(g)$ is an endomorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. Recall the formulas obtained in Remark 1.22, and observe that $\mathcal{T}(g)(e) = \mathtt{w}^{-1}(g^{-1}[\mathtt{w}(e)])$ for $e \in \mathcal{P}(\omega)/\mathtt{fin}$. Hence, the conditions that g must satisfy in Lemma 3.11 are equivalent to $f[\mathtt{w} \circ \mathcal{T}(g)(e)] \subseteq q[\mathtt{w}(e)]$ for every $e \in S_q$. In other words, Theorem 2.13 will be proven if we can show the following:

Lemma 3.12. *There exists a homomorphism $\varphi : \mathcal{P}(\omega)/\mathtt{fin} \to \mathcal{P}(\omega)/\mathtt{fin}$ satisfying*

$$f[\mathtt{w} \circ \varphi(e)] \subseteq q[\mathtt{w}(e)] \tag{3.13}$$

for every $e \in S_q$. Moreover, if f is onto Y we can choose φ to be an automorphism.

Now fix $\mathcal{B} := \langle S_q \rangle$, which is a countable algebra. By Theorem 2.1, the lemma above follows from:

Lemma 3.14. *There exists a homomorphism $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ satisfying (3.13) for every $e \in S_q$. Moreover, if f is onto Y we can choose φ to be an embedding.*

We will prove this only incompletely now, as it depends on yet another technical lemma. The technical lemma will be stated and proved later, after we have seen why it is necessary.

Incomplete proof of Lemma 3.14. We define a partial order

$$\mathbb{P} := \{\varphi : \mathcal{C} \to \mathcal{P}(\omega)/\mathtt{fin} \text{ homomorphism: } \mathcal{C} \text{ is a finite subalgebra of } \mathcal{B} \text{ and (3.13) holds for all } e \in \mathcal{C}\}$$

which is ordered by reversed inclusion. For each $e \in \mathcal{B}$ let $D_e := \{\varphi \in \mathbb{P} : e \in \mathtt{dom}(\varphi)\}$. Clearly, if $\mathcal{G}$ is a filter on $\mathbb{P}$ which is generic for $\{D_e : e \in \mathcal{B}\}$, then $\bigcup \mathcal{G}$ is a homomorphism satisfying the statement of the first part of lemma. Similarly, we define a suborder $\mathbb{P}_{\mathtt{emb}}$ of all injective $\varphi \in \mathbb{P}$ satisfying

$$f[\mathtt{w} \circ \varphi(e)] = q[\mathtt{w}(e)] \tag{3.15}$$

for every $e \in \mathtt{dom}(\varphi)$, and observe that if $\mathcal{G}$ is a filter on $\mathbb{P}_{\mathtt{emb}}$ which is generic for $\{D_e \cap \mathbb{P}_{\mathtt{emb}} : e \in \mathcal{B}\}$, then $\bigcup \mathcal{G}$ is a homomorphism satisfying the statement of the second part of the lemma. Since $\mathcal{B}$ is countable, by the Rasiowa-Sikorski Theorem (2.3) the existence of such $\mathcal{G}$ is guaranteed if we can show that: (a) The partial order $\mathbb{P}$, respectively $\mathbb{P}_{\mathtt{emb}}$, is not empty, and (b) the sets D_e are dense in $\mathbb{P}$, respectively the sets $D_e \cap \mathbb{P}_{\mathtt{emb}}$ are dense in $\mathbb{P}_{\mathtt{emb}}$. For (a), simply consider the minimal homomorphism $\varphi_{\mathtt{min}} : \{0, 1\} \to \mathcal{P}(\omega)/\mathtt{fin}$ which maps 0 to 0 and 1 to 1. Then $\varphi_{\mathtt{min}} \in \mathbb{P}$ in any case, while $\varphi_{\mathtt{min}} \in \mathbb{P}_{\mathtt{emb}}$ if and only if f is onto Y. To prove (b), for the rest of this proof we fix $\varphi : \mathcal{C} \to \mathcal{P}(\omega)/\mathtt{fin}$ in $\mathbb{P}$, and $e \in \mathcal{B} \setminus \mathcal{C}$ and we must show that there is $\varphi' : \langle \mathcal{C} \cup \{e\} \rangle \to \mathcal{P}(\omega)/\mathtt{fin}$ in $\mathbb{P}$ extending φ, and that φ' can be chosen in $\mathbb{P}_{\mathtt{emb}}$ if $\varphi \in \mathbb{P}_{\mathtt{emb}}$ (which already implies that f is surjective). To make this easier, we may assume that there is an atom a of $\mathcal{C}$ such that $0 < e < a$: Otherwise, let $a_0, \dots, a_k$ be the atoms of $\mathcal{C}$, and add the elements $e \wedge a_0, \dots, e \wedge a_k$ to the domain of φ one after the other, so that in the end $e = (e \wedge a_0) \vee \dots \vee (e \wedge a_k)$ will be in the domain of the resulting homomorphism. For some i we may have $e \wedge a_i = 0$ or $e \wedge a_i = a_i$, which need not be added, while for all others we have $0 < e \wedge a_i < a_i$. Observe that once $e \wedge a_0, \dots, e \wedge a_i$ have been included in the domain for some $i < k$, the elements $a_{i+1}, \dots, a_k$ are still atoms of the new domain.

All we need to do is find a suitable image e' for e. By to Theorem 2.4, a homomorphism $\varphi' : \langle \mathcal{C} \cup \{e\} \rangle \to \mathcal{P}(\omega)/\mathtt{fin}$ extending φ and mapping e to e' exists (and is unique) if and only if

$$e' \le \varphi(a) \tag{3.16}$$

and in the case that φ is an embedding, φ' is an embedding if and only if

$$0 < e' < \varphi(a) \tag{3.17}$$

The conditions (3.13) or (3.15) (depending on the case) need only be proven for the atoms of $\langle \mathcal{C} \cup \{e\} \rangle$, and then follow easily for every other element by writing it as a disjunction of some of these atoms. On the other hand, we already assume that φ satisfies these conditions for all elements of $\mathcal{C}$, and the only atoms of $\langle \mathcal{C} \cup \{e\} \rangle$ which are not in $\mathcal{C}$ are e and $a \wedge \neg e$. Thus, in the general case we need (3.16) together with

$$f[\mathtt{w}(e')] \subseteq q[\mathtt{w}(e)] \tag{3.18}$$

$$f[\mathtt{w}(\varphi(a)) \setminus \mathtt{w}(e')] \subseteq q[\mathtt{w}(a \wedge \neg e)] \tag{3.19}$$

and in the case that $\varphi \in \mathbb{P}_{\mathtt{emb}}$ we need (3.17) together with

$$f[\mathtt{w}(e')] = q[\mathtt{w}(e)] \tag{3.20}$$

$$f[\mathtt{w}(\varphi(a)) \setminus \mathtt{w}(e')] = q[\mathtt{w}(a \wedge \neg e)] \tag{3.21}$$

To finish this incomplete proof, we just have to modify these formulas slightly and make them fit the lemma that will follow. First, observe that (3.20) plus the fact that $e \neq 0$ implies that $e' \neq 0$. Similarly, (3.21) plus the fact that $e < a$ implies that $e' \neq \varphi(a)$, therefore even in the case that $\varphi \in \mathbb{P}_{\mathtt{emb}}$ it suffices to use (3.16) instead of (3.17). Second, note that (3.18) and (3.19) obviously follow from the stronger conditions:

$$f[\mathtt{w}(e')] = q[\mathtt{w}(e)] \cap f[\mathtt{w}(\varphi(a))] \tag{3.22}$$

$$f[\mathtt{w}(\varphi(a)) \setminus \mathtt{w}(e')] = q[\mathtt{w}(a \wedge \neg e)] \cap f[\mathtt{w}(\varphi(a))] \tag{3.23}$$

Third, in the case that $\varphi \in \mathbb{P}_{\mathtt{emb}}$, we get that $q[\mathtt{w}(e)] \cup q[\mathtt{w}(a \wedge \neg e)] = q[\mathtt{w}(a)] = f[\mathtt{w}(\varphi(a))]$, and so (3.20) and (3.21) are equivalent to (3.22) and (3.23) respectively. In summary, we are done with both the general case and the case that $\varphi \in \mathbb{P}_{\mathtt{emb}}$ if we can find e' satisfying (3.16), (3.22) and (3.23). The existence of such e' is a consequence of the lemma below, and this finishes this incomplete proof. □

The proof given below is due to Błaszczyk and Szymański and can be found in [BS80].

Lemma 3.24. *Let $V \subseteq \omega^*$ be a clopen set, and suppose $C_0, C_1 \subseteq Y$ are closed and such that $f[V] \subseteq C_0 \cup C_1$. Then, there is a clopen set $V_0 \subseteq V$ such that*

$$f[V_0] = C_0 \cap f[V] \text{ and } f[V \setminus V_0] = C_1 \cap f[V].$$

Proof. We will use Urysohn's Metrization Theorem, namely that T_3 second-countable spaces (such as our space Y) are always metrizable. Besides that, we will use the fact that every open (resp. closed) subset of a metrizable space is F_σ (resp. G_δ).

Consider the closed set $C_0 \cap C_1 \cap f[V]$. By choosing a countable basis for Y, and then one point from each non-empty intersection of $C_0 \cap C_1 \cap f[V]$ with an element of the basis, we can find a countable set D such that

$$\overline{D} = C_0 \cap C_1 \cap f[V]$$

For each $y \in D$, since $\{y\}$ is G_δ, it follows that $f^{-1}(y) \cap V$ is non-empty and G_δ in ω^*, so by Corollary 3.8 there is a non-empty clopen set $V_y \subseteq f^{-1}(y) \cap V$. Let $V_{y,0}$ be a non-empty clopen proper subset of V_y, and let $V_{y,1} := V_y \setminus V_{y,0}$. Then define:

$$U_0 := (f^{-1}[Y \setminus C_1] \cap V) \cup \bigcup_{y \in D} V_{y,0}$$

$$U_1 := (f^{-1}[Y \setminus C_0] \cap V) \cup \bigcup_{y \in D} V_{y,1}$$

Note that $Y \setminus C_0$ and $Y \setminus C_1$ are open, therefore F_σ, from which it easily follows that U_0 and U_1 are open F_σ in ω^*. Using the assumption that $f[V] \subseteq C_0 \cup C_1$, we get that U_0 is disjoint from U_1 and therefore, by Corollary 3.6 together with Lemma 3.3, there is a clopen subset V' of ω^* such that $U_0 \subseteq V'$ and $U_1 \subseteq \omega^* \setminus V'$. Finally, let $V_0 := V \cap V'$. Showing that this choice for V_0 works is mostly straight-forward, and the only interesting part is checking that:

$$C_0 \cap C_1 \cap f[V] \subseteq f[V_0] \cap f[V \setminus V_0]$$

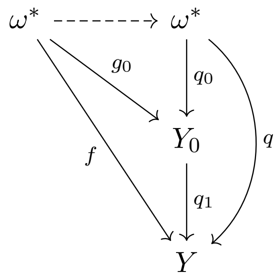
This also becomes easy once we note that we only have to show that $D \subseteq f[V_0] \cap f[V \setminus V_0]$, because f is a closed map. □

As discussed above, this lemma completes the proof of Lemma 3.14, and consequently proves Lemma 3.12, and Theorem 2.13.

Remark 3.25. It might be worth mentioning that we could have used Corollary 2.12 after Lemma 3.14, instead of extending the homomorphism to all of $\mathcal{P}(\omega)/\mathtt{fin}$ as in Lemma 3.12. This proof would go as follows: Let (Y_0, q_0) be an object of $\mathtt{Quot}_0(\omega^*)$ such that $\mathcal{T}(Y_0, q_0) = \mathcal{B}$. If $\mathcal{F}, \mathcal{F}' \in \omega^*$ are such that $q(\mathcal{F}) \neq q(\mathcal{F}')$, then there is $e \in S_q$ such that $\mathcal{F} \in \mathtt{w}(e)$ and $\mathcal{F}' \in \omega^* \setminus \mathtt{w}(e)$ (see Lemma 3.11). Since $e \in \mathcal{T}(Y_0, q_0)$, there is $C \subseteq Y_0$ clopen such that $\mathtt{w}(e) = q_0^{-1}[C]$ (see Remark 1.22), and hence $q_0(\mathcal{F}) \in C$, while $q_0(\mathcal{F}') \notin C$. This shows that for $\mathcal{F}, \mathcal{F}' \in \omega^*$:

$$q_0(\mathcal{F}) = q_0(\mathcal{F}') \Longrightarrow q(\mathcal{F}) = q(\mathcal{F}')$$

which means we have a well-defined map $q_1 : Y_0 \to Y$ given by $q_1(q_0(\mathcal{F})) := q(\mathcal{F})$ for $\mathcal{F} \in \omega^*$. The fact that $q_1 \circ q_0 = q$ implies that q_1 is a quotient map onto Y. Finally, consider the homomorphism $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$, that is $\varphi : \mathcal{T}(Y_0, q_0) \to \mathcal{T}(\omega^*, \mathtt{id}_{\omega^*})$, obtained in Lemma 3.14 and let $g_0 : \omega^* \to Y_0$ be the continuous map $\mathcal{T}^{-1}(\varphi)$. The map g_0 is a quotient in the case that φ is injective, and therefore in the case that f is a quotient. We now have the commutative diagram:



(although proving that $q_1 \circ g_0 = f$ would still require some work). This diagram can be appropriately completed with g at the top using Corollary 2.12 (applied to the upper triangle). Of course, this approach turns out to be longer and more difficult than the proof I chose to present, but it is interesting to look at the "process of lifting f" in the case that Y is only second-countable Hausdorff, as a two-step process, where we first lift f to a map g_0 into a 0-dimensional but still second-countable space, and then lift g_0 to a map into ω^*.

4 Extending quotients of ω^*

My original proof of Lemma 3.24 was very intricate, and at the time it did not seem to solve the general case, but only a specific one. As I said, Example 2.14 (our "key example") was to be kept in mind throughout the entire previous chapter. The reader will notice that in this case we had a powerful tool in our hands: The quotient from ω^* onto S^1 was of the type σ^* for a map $\sigma : \omega \to S^1$. This was unnecessary in the previous chapter, but it was crucial for my original proof, and so it raised the question of whether this really is a special case.

Definition 4.1. Let X be a Tychonoff space, K a compact Hausdorff space and $f : X \to K$ continuous. The unique extension of f to a continuous map from βX into K will be denoted βf. The restriction of βf to X^* will be denoted f^*.

Our goal in this chapter is to prove the following:

Theorem 4.2. *Every quotient map from ω^* onto a second-countable Hausdorff space Y is of the kind σ^* for some map $\sigma : \omega \to Y$.*

Suppose $q : \omega^* \to Y$ is a quotient map, where Y is second-countable Hausdorff, let S_q be the set we get from Lemma 3.11, and apply the lemma with $f = \sigma^*$ for some $\sigma : \omega \to Y$, and $g = \mathtt{id}_{\omega^*}$. It tells us that $\sigma^* = q$ if and only if:

$$\forall e \in S_q \quad \sigma^*[\mathtt{w}(e)] \subseteq q[\mathtt{w}(e)] \tag{4.3}$$

So, understanding the sets of the form $\sigma^*[\mathtt{w}(e)]$ is key for the proof of Theorem 4.2.

Lemma 4.4. *If $E \subseteq \omega$, then $\mathtt{V}(E) = \overline{E}$.*

Proof. We have already observed that $\mathtt{V}(E) \cap \omega = E$. From this it follows that $\overline{E} \subseteq \mathtt{V}(E)$ and $\overline{\omega \setminus E} \subseteq \beta\omega \setminus \mathtt{V}(E)$, and since $\overline{E} \cup \overline{\omega \setminus E} = \overline{\omega} = \beta\omega$, we get $\mathtt{V}(E) = \overline{E}$. □

Definition 4.5. A net $(t_\lambda)_{\lambda\in\Lambda}$ in a space T will be called *eventually constant* if there is some $t \in T$ and $\lambda_0 \in \Lambda$ such that $t_\lambda = t$ for all $\lambda \geq \lambda_0$. On the other hand, the net will be called *eventually different*, if for every $t \in T$ there is some $\lambda_0 \in \Lambda$ such that $t_\lambda \neq t$ for all $\lambda \geq \lambda_0$.

If a net $(t_\lambda)_{\lambda\in\Lambda}$ converges to a point t, we shall write $t_\lambda \xrightarrow{\lambda} t$ instead of simply $t_\lambda \to t$. This helps avoid confusion if other variables are involved. For example, if $t_{\lambda,n}$ is defined for all $\lambda \in \Lambda$ and all $n \in \omega$, then $t_{\lambda,n} \xrightarrow{\lambda} t$ means that n is fixed and the net $(t_{\lambda,n})_{\lambda\in\Lambda}$ converges to t, while $t_{\lambda,n} \xrightarrow{n} t'$ means that λ is fixed and the sequence $(t_{\lambda,n})_{n\in\omega}$ converges to t'.

Lemma 4.6. *Let $(t_\lambda)_{\lambda\in\Lambda}$ be a net in a T_1 space T. If $t_\lambda \xrightarrow{\lambda} t$ and $t_\lambda \neq t$ for all $\lambda \in \Lambda$, then $(t_\lambda)_{\lambda\in\Lambda}$ is eventually different.*

Proof. Let $t' \in T$. If $t' = t$, we already know that $t_\lambda \neq t'$ for every λ. On the other hand, if $t' \neq t$, there is a neighbourhood U of t such that $t' \notin U$, and there is $\lambda_0 \in \Lambda$ such that $t_\lambda \in U$ for all $\lambda \geq \lambda_0$, hence $t_\lambda \neq t'$ for all $\lambda \geq \lambda_0$. □

Lemma 4.7. *No subnet of an eventually different net is eventually constant.*

Proof. Let $(t_\lambda)_{\lambda\in\Lambda}$ be an eventually different net in a space T, and let $(t_{\varphi(\mu)})_{\mu\in M}$ be a subnet. Given $t \in T$ and $\mu_0 \in M$, we need to find $\mu \geq \mu_0$ such that $t_{\varphi(\mu)} \neq t$. There is some $\lambda_0 \in \Lambda$ such that $t_\lambda \neq t$ for all $\lambda \geq \lambda_0$, there is some $\mu_1 \in M$ such that $\varphi(\mu_1) \geq \lambda_0$, and there is some $\mu \in M$ such that $\mu \geq \mu_0, \mu_1$. It follows that $\varphi(\mu) \geq \lambda_0$ and so $t_{\varphi(\mu)} \neq t$ as we wanted. □

Lemma 4.8. *If Y is compact Hausdorff and $\sigma : \omega \to Y$, then for every $E \subseteq \omega$ we have*

$$\sigma^*[\mathtt{W}(E)] = \{y : \textit{there is } (n_\lambda)_{\lambda\in\Lambda} \textit{ in } E, \textit{ eventually different, such that } \sigma(n_\lambda) \xrightarrow{\lambda} y\}.$$

Proof. Take $y \in \sigma^*[\mathtt{W}(E)]$ and $\mathcal{F} \in \mathtt{W}(E)$ such that $y = \sigma^*(\mathcal{F})$. Since $\mathcal{F} \in \overline{E}$ (in $\beta\omega$), there is a net $(n_\lambda)_{\lambda\in\Lambda}$ in E converging to $\mathcal{F}$. Since $\mathcal{F} \notin E$, by Lemma 4.6, $(n_\lambda)_{\lambda\in\Lambda}$ is eventually different. Clearly $\sigma(n_\lambda) \xrightarrow{\lambda} y$ because $\sigma(n_\lambda) = \beta\sigma(n_\lambda)$, and $y = \beta\sigma(\mathcal{F})$, and $\beta\sigma$ is continuous.

On the other hand, if we take $y \in Y$ and an eventually different net $(n_\lambda)_{\lambda\in\Lambda}$ in E such that $\sigma(n_\lambda) \xrightarrow{\lambda} y$, since $\beta\omega$ is compact, there is a subnet $(n_{\varphi(\mu)})_{\mu\in M}$ which converges to some point $\mathcal{F} \in \beta\omega$. It follows that $y = \lim \sigma(n_{\varphi(\mu)}) = \lim \beta\sigma(n_{\varphi(\mu)}) = \beta\sigma(\mathcal{F})$. Clearly $\mathcal{F} \in \overline{E} = \mathtt{V}(E)$. On the other hand, since $(n_\lambda)_{\lambda\in\Lambda}$ is eventually different, by Lemma 4.7, $(n_{\varphi(\mu)})_{\mu\in M}$ is not eventually constant, and therefore $\mathcal{F}$ is not isolated, which implies that $\mathcal{F} \notin \omega$, and hence $\mathcal{F} \in \mathtt{W}(E)$, and $y = \sigma^*(\mathcal{F})$. □

Since we will only consider the case where Y is second-countable, we can use a version of the previous lemma with sequences instead:

Corollary 4.9. *If Y is compact, Hausdorff and first-countable and $\sigma : \omega \to Y$, then for every $E \subseteq \omega$ we have*

$$\sigma^*[\mathtt{W}(E)] = \{y : \text{there is } (n_i)_{i\in\omega} \text{ in } E, \text{ without repetitions, such that } \sigma(n_i) \xrightarrow{i} y\}.$$

Proof. The inclusion "$\supseteq$" is immediate, since a sequence without repetitions is also an eventually different net. For the inclusion "$\subseteq$", given $y \in \sigma^*[\mathtt{W}(E)]$ let $(n_\lambda)_{\lambda\in\Lambda}$ be a net in E which is eventually different, and such that $\sigma(n_\lambda) \xrightarrow{\lambda} y$. Since Y is first-countable, there is a countable local basis $\{U_i\}_{i\in\omega}$ at y, and we may assume that $U_{i+1} \subseteq U_i$ for every $i \in \omega$. The fact that $(n_\lambda)_{\lambda\in\Lambda}$ is eventually different implies that for every $i \in \omega$ there are infinitely many distinct n_λ such that $\sigma(n_\lambda) \in U_i$: Indeed, given a finite set $F \subseteq \omega$, for each $m \in F$ there is λ_m such that $n_\lambda \neq m$ for all $\lambda \geq \lambda_m$. Also, there is λ_{U_i} such that $\sigma(n_\lambda) \in U_i$ for all $\lambda \geq \lambda_{U_i}$. Then, choosing some $\lambda \geq \lambda_{U_i}$ such that $\lambda \geq \lambda_m$ for all $m \in F$, we have $\sigma(n_\lambda) \in U_i$ and $n_\lambda \notin F$, as we wanted to show. Therefore, we can choose a sequence $(n_{\lambda_i})_{i\in\omega}$ by induction, letting λ_i be such that $\sigma(n_{\lambda_i}) \in U_i$ and $n_{\lambda_i} \notin \{n_{\lambda_j} : j < i\}$, and it follows that $(n_{\lambda_i})_{i\in\omega}$ has no repetitions, and $\sigma(n_{\lambda_i}) \xrightarrow{i} y$. □

In the proof above, the sequence $(n_{\lambda_i})_{i\in\omega}$ is not necessarily a subnet of $(n_\lambda)_{\lambda\in\Lambda}$, and it does not converge in $\beta\omega$. In fact:

Lemma 4.10. *The only convergent sequences in $\beta\omega$ are the eventually constant ones.*

Proof. Suppose, for a contradiction, that $(\mathcal{F}_i)_{i\in\omega}$ is a sequence in $\beta\omega$ which is not eventually constant, and converges to a point $\mathcal{F}$. By possibly taking a subsequence, we may assume that $(\mathcal{F}_i)_{i\in\omega}$ has no repetitions, and that $\mathcal{F}_i \neq \mathcal{F}$ for every i. For each $i \in \omega$ we choose disjoint open sets U_i and V_i in $\beta\omega$ such that $\mathcal{F}_i \in U_i$ and $\mathcal{F} \in V_i$. Since cofinitely many of the $\mathcal{F}_j$ are in V_i, we can assume (by possibly shrinking U_i), that $\mathcal{F}_j \notin U_i$ for every $j \neq i$. Inductively, for each $i \in \omega$ we can choose $E_i \subseteq \omega$ such that $\mathcal{F}_i \in \mathtt{V}(E_i) \subseteq U_i \setminus \bigcup_{j<i} \mathtt{V}(E_j)$. Finally, let $E_{\mathtt{even}} := \bigcup_{i\in\omega} E_{2i}$ and $E_{\mathtt{odd}} := \bigcup_{i\in\omega} E_{2i+1}$, and observe that $\mathtt{V}(E_{\mathtt{even}})$ and $\mathtt{V}(E_{\mathtt{odd}})$ are clopen neighbourhoods of $\{\mathcal{F}_{2i}\}_{i\in\omega}$ and $\{\mathcal{F}_{2i+1}\}_{i\in\omega}$ respectively, which would imply that $\mathcal{F} \in \mathtt{V}(E_{\mathtt{even}}) \cap \mathtt{V}(E_{\mathtt{odd}})$, contradicting the fact that $E_{\mathtt{even}} \cap E_{\mathtt{odd}} = \emptyset$. □

The lemma above can also be found in [Eng89].

Proof of Theorem 4.2. Assume a quotient map $q : \omega^* \to Y$ is given, onto a second-countable Hausdorff space Y, and we must find $\sigma : \omega \to Y$ satisfying (4.3). As before, we let $\mathcal{B} := \langle S_q \rangle$, and choose a lifting $r : \mathcal{B} \to \mathcal{P}(\omega)$ and an enumeration $(e_i)_{i\in\omega}$ of $\mathcal{B}$ (possibly with repetitions). Given $n \in \omega$, denote by $\mathcal{B}_n$ the finite algebra $\langle e_i : i < n \rangle$, and observe that $(r(a) : a \text{ is an atom of } \mathcal{B}_n)$ is a partition of ω, hence there is a unique atom a_n of $\mathcal{B}_n$ such that $n \in r(a_n)$. Let $\sigma(n)$ be an arbitrary point of $q[\mathtt{w}(a_n)]$.

If $e \in \mathcal{B}$ and $y \in \sigma^*[\mathtt{w}(e)] = \sigma^*[\mathtt{W}(r(e))]$, by Corollary 4.9, there is a sequence $(n_i)_{i\in\omega}$ in $r(e)$, without repetitions, such that $\sigma(n_i) \xrightarrow{i} y$. For all i we have $n_i \in r(e) \cap r(a_{n_i})$, so $e \wedge a_{n_i} \neq 0$. Since this sequence has no repetitions, it diverges to infinity, and so for i large enough we have $e \in \mathcal{B}_{n_i}$, which implies that $a_{n_i} \leq e$, and therefore $\sigma(n_i) \in q[\mathtt{w}(e)]$. Finally, because $q[\mathtt{w}(e)]$ is closed, we conclude that $y = \lim \sigma(n_i) \in q[\mathtt{w}(e)]$, as we wanted to show. □

Remark 4.11. In the proof above, the condition $\sigma(n) \in q[\mathtt{w}(a_n)]$ was a bit more than necessary. Indeed, all we wanted was to have $\lim \sigma(n_i) \in q[\mathtt{w}(e)]$ in the end, so it suffices if the $\sigma(n_i)$ come closer and closer to $q[\mathtt{w}(e)]$. Formally, observe that Y is metrizable (again by Urysohn's Metrization Theorem), and fix a compatible metric function d. Instead of requiring that $\sigma(n) \in q[\mathtt{w}(a_n)]$, let us require that $d(\sigma(n), q[\mathtt{w}(a_n)]) < \varepsilon_n$, where $(\varepsilon_n)_{n\in\omega}$ is a sequence of positive real numbers converging to 0. In the second part, instead of concluding that $\sigma(n_i) \in q[\mathtt{w}(e)]$, we conclude that $d(\sigma(n_i), q[\mathtt{w}(e)]) < \varepsilon_n$. From this, it follows that $d(\lim \sigma(n_i), q[\mathtt{w}(e)]) = 0$, and since $q[\mathtt{w}(e)]$ is closed, we have $\lim \sigma(n_i) \in q[\mathtt{w}(e)]$, so the proof is complete.

Now, suppose D is a countable dense subset of Y, and $(y_n)_{n\in\omega}$ is an enumeration of D, possibly with repetitions, and let us use a back-and-forth inductive argument to define σ such that $\mathtt{ran}(\sigma) = D$. In the 0-th step we define $\sigma_0 := \emptyset$ and $F_0 := \emptyset$. In the n-th step for $n > 0$ we define a map $\sigma_n : F_n \to D$ satisfying: (i) F_n is a finite subset of ω; (ii) $\sigma_n \supseteq \sigma_{n-1}$; (iii) $n \in F_n$; (iv) $y_n \in \sigma_n[F_n]$; (v) For all $k \in F_n \setminus F_{n-1}$ it holds that $d(\sigma_n(k), q[\mathtt{w}(a_k)]) < 1/n$. If we succeed in defining the maps σ_n, and let $\sigma := \bigcup_{n\in\omega} \sigma_n$, then clearly $\mathtt{dom}(\sigma) = \omega$ and $\mathtt{ran}(\sigma) = D$. For each $k \in \omega$, let $n_k > 0$ be the unique natural number such that $k \in F_{n_k} \setminus F_{n_k-1}$, and then let $\varepsilon_k := 1/n_k$. The sequence $(\varepsilon_k)_{k\in\omega}$ converges to 0 because the sets F_n are finite: Given $\varepsilon > 0$ choose $n \in \omega \setminus \{0\}$ such that $1/n < \varepsilon$, let $k_0 := \mathtt{max}(F_n) + 1$ and it follows that $\varepsilon_k < \varepsilon$ for all $k \geq k_0$. The map σ satisfies $d(\sigma(k), q[\mathtt{w}(a_k)]) < \varepsilon_k$, hence (as discussed above) we have $\sigma^* = q$.

As for the n-th step ($n > 0$) of the induction, we obviously start with $\sigma_n \restriction F_{n-1} := \sigma_{n-1}$. If $n \notin F_{n-1}$, note that the set $B_{1/n}(q[\mathtt{w}(a_n)])$ (of all points of Y with distance less than $1/n$ from $q[\mathtt{w}(a_n)]$) is open and non-empty. Since D is dense in Y, we can choose $\sigma_n(n) \in B_{1/n}(q[\mathtt{w}(a_n)]) \cap D$. If $y_n \notin \sigma_n[F_{n-1} \cup \{n\}]$, we must find $k \notin F_{n-1} \cup \{n\}$ such that $d(y_n, q[\mathtt{w}(a_k)]) < 1/n$. Then, we can define $\sigma_n(k) := y_n$ and the induction step will be done. Using the definition of S_q, it is easy to see that for every $y \in Y \setminus \{y_n\}$ there is some $e \in S_q$ such that $q^{-1}(y_n) \subseteq \mathtt{w}(e)$ and $y \notin q[\mathtt{w}(e)]$. In other words:

$$\bigcap\{q[\mathtt{w}(e)] \,:\, e \in S_q \text{ and } q^{-1}(y_n) \subseteq \mathtt{w}(e)\} = \{y_n\}$$

The sets $q[\mathtt{w}(e)]$ are all closed, and Y is compact, hence there is a finite set $G \subseteq \{e \in S_q : q^{-1}(y_n) \subseteq \mathtt{w}(e)\}$ such that $\bigcap\{q[\mathtt{w}(e)] \,:\, e \in G\} \subseteq B_{1/n}(y_n)$. Let $e' := \bigwedge G$, so that $e' \in \mathcal{B}$ and $q[\mathtt{w}(e')] \subseteq B_{1/n}(y_n)$. It follows that $e' \neq 0$ (because $q^{-1}(y_n) \subseteq \mathtt{w}(e')$), and therefore $r(e')$ is infinite. If we choose $k \in r(e')$ such that $k \notin F_{n-1} \cup \{n\}$ and $e' \in \mathcal{B}_k$, this implies that $a_k \leq e'$, so $q[\mathtt{w}(a_k)] \subseteq B_{1/n}(y_n)$, and in particular $d(y_n, q[\mathtt{w}(a_k)]) < 1/n$ as we wanted.

Corollary 4.12. *Let Y be a second-countable compact Hausdorff space, $q : \omega^* \to Y$ a quotient map, and D a countable dense subset of Y. Then there is a map $\sigma : \omega \to Y$ such that $\sigma^* = q$ and $\mathtt{ran}(\sigma) = D$.* □

In Theorem 4.2 and Corollary 4.12, the restricted map σ^* is onto the quotient space Y, which needs not be the case for arbitrary maps $\sigma : \omega \to Y$. In Example 2.14 we argued that σ^* is onto S^1 because $S^1 \setminus \mathtt{ran}(\sigma)$ is dense in S^1, and σ^* is a closed map. In the general case we have:

Lemma 4.13. *Let Y be a compact Hausdorff space, $\sigma : \omega \to Y$ such that $\mathtt{ran}(\sigma)$ is dense in Y. Then $\mathtt{ran}(\sigma^*) = Y$ if and only if, for every $y \in Y$ isolated, $\sigma^{-1}(y)$ is infinite.*

Proof. Suppose σ^* is onto Y and $y \in Y$ is isolated. By Lemma 4.8, there is an eventually different net $(n_\lambda)_{\lambda\in\Lambda}$ in ω such that $\sigma(n_\lambda) \xrightarrow{\lambda} y$, and hence there is λ_0 such that $\sigma(n_\lambda) = y$ for all $\lambda \geq \lambda_0$. If $m_1, \ldots, m_k \in \omega$, for each i there is λ_i such that $n_\lambda \neq m_i$ whenever $\lambda \geq \lambda_i$. Thus, we can take $\lambda \geq \lambda_0, \ldots, \lambda_k$ and it follows that $n_\lambda \in \sigma^{-1}(y) \setminus \{m_1, \ldots, m_k\}$. This shows that $\sigma^{-1}(y)$ is infinite.

On the other hand, suppose that for every isolated $y \in Y$ we have $\sigma^{-1}(y)$ infinite. If y is isolated, let $(n_i)_{i\in\omega}$ be the strictly increasing enumeration of $\sigma^{-1}(y)$, which is clearly an eventually different net. Since $(\sigma(n_i))_{i\in\omega}$ is the constant sequence $(y)_{i\in\omega}$, it converges to y, and so, by Lemma 4.8, $y \in \sigma^*[\mathtt{W}(\omega)]$. Now, if y is not isolated, it is an accumulation point of $\mathtt{ran}(\sigma)$, so there is a net $(y_\lambda)_{\lambda\in\Lambda}$ in $\mathtt{ran}(\sigma) \setminus \{y\}$ converging to y. By Lemma 4.6 $(y_\lambda)_{\lambda\in\Lambda}$ is eventually different. If we choose $n_\lambda \in \sigma^{-1}(y_\lambda)$ for each $\lambda \in \Lambda$, it follows that $(n_\lambda)_{\lambda\in\Lambda}$ is also an eventually different net, and $\sigma(n_\lambda) = y_\lambda \xrightarrow{\lambda} y$, showing that $y \in \sigma^*[\mathtt{W}(\omega)]$ (again by Lemma 4.8). □

The following corollary is well known, and is an interesting application of what we have seen in this chapter:

Corollary 4.14. *Every separable compact Hausdorff space is a quotient of ω^*.*

Proof. Let $(E_n)_{n\in\omega}$ be a partition of ω into infinitely many infinite sets. Given a separable compact Hausdorff space Y, choose a countable dense subset $D = \{y_n\}_{n\in\omega}$ and let the map $\sigma : \omega \to Y$ be defined by $\sigma[E_n] = \{y_n\}$ for every $n \in \omega$. By the previous lemma, together with the fact that every isolated point of Y must be in D, it follows that $\mathtt{ran}(\sigma^*) = Y$. □

The following remark involves the *network weight* of a topological space, which is not defined here. For the definitions and basic results about it, see [Eng89]. In general, if X and Y are topological spaces, $q : X \to Y$ is a surjective continuous map, and $\mathcal{O}$ is a basis for X, then $\{q[O] : O \in \mathcal{O}\}$ is a network in Y, therefore the network weight of Y is at most the weight of X. If Y is compact Hausdorff, then its network weight is the same as its weight, therefore the weight of Y is at most the weight of X. This shows that the weight of a Hausdorff quotient of $\beta\omega$ or ω^* is at most $2^{\aleph_0}$. Evidently, every separable compact Hausdorff space is a quotient of $\beta\omega$ (simply from the universal property of the Stone-Čech compactification), and has thus weight $\leq 2^{\aleph_0}$. Hence, under CH the corollary above follows from a (ZFC) result of Parovičenko [Par63], which states that every compact Hausdorff space of weight $\leq \aleph_1$ is a quotient of ω^*. Under ¬CH, Parovičenko's theorem does not include $\beta\omega$, for example, which is covered by the corollary above. In both cases there are compact Hausdorff non-separable spaces of weight $\aleph_1$, such as the one-point compactification of a discrete space of size $\aleph_1$ (see the construction in the next chapter or, e.g. in [Eng89]).

Theorem 4.2 can be alternatively formulated by saying that every quotient from ω^* onto a second-countable Hausdorff space Y can be extended to a quotient from $\beta\omega$ onto Y. Indeed, given such a quotient $q : \omega^* \to Y$ the theorem implies that $q = \sigma^*$ for some $\sigma : \omega \to Y$, and so $\beta\sigma : \beta\omega \to Y$ is a continuous extension of q. On the other hand, if a quotient $q : \omega^* \to Y$ has a continuous extension $Q : \beta\omega \to Y$, then we must have $Q = \beta(Q \restriction \omega)$ (by the uniqueness part of the universal property of $\beta\omega$), therefore $q = Q \restriction \omega^* = (Q \restriction \omega)^*$. Moreover, if Y is a second-countable Hausdorff space (not necessarily compact), and $f : \omega^* \to Y$ is continuous (but not necessarily surjective), then f is still a quotient onto its image, so there is a continuous map $F : \beta\omega \to Y$, extending f, and with $\mathtt{ran}(F) = \mathtt{ran}(f)$.

Definition 4.15. We denote by $\mathtt{Ext}(\omega^*, \beta\omega)$ the class of all topological spaces T with the property that every continuous map $f : \omega^* \to T$ has a continuous extension $F : \beta\omega \to T$.

As discussed above, Theorem 4.2 implies that every second-countable Hausdorff space is in $\mathtt{Ext}(\omega^*, \beta\omega)$.

Lemma 4.16. *The class $\mathtt{Ext}(\omega^*, \beta\omega)$ is closed under products.*

Proof. Suppose $\{X_i\}_{i\in I}$ is a subset of $\mathtt{Ext}(\omega^*, \beta\omega)$, let $X := \prod_{i\in I} X_i$, and let $f : \omega^* \to X$ be a continuous map. We use the notation $f = (f_i)_{i\in I}$ where the f_i's are the component functions of f. Then, from the hypothesis, each f_i has a continuous extension $F_i : \beta\omega \to X_i$, and the map $F := (F_i)_{i\in I} : \beta\omega \to X$ is a continuous extension of f. This shows that $X \in \mathtt{Ext}(\omega^*, \beta\omega)$. □

This extends the scope of Theorem 4.2 to far more spaces. Of course, if we wish to extend quotient maps $\omega^* \to X = \prod_{i\in I} X_i$, we should not take I too large, or there might be no such quotient maps at all (recall that the weight of every quotient of ω^* is at most $2^{\aleph_0}$), but if we take $|I| \leq 2^{\aleph_0}$, and take each X_i second-countable compact Hausdorff, then we get X separable compact Hausdorff (therefore a quotient of ω^*) and $X \in \mathtt{Ext}(\omega^*, \beta\omega)$.

5 Singular compactifications

Definition 5.1. Let X be a Tychonoff space. A compactification (K, h) of X is called *singular* if the remainder $K \setminus h[X]$ is a retract of K (i.e. there is some $f : K \to K \setminus h[X]$ continuous such that $f(k) = k$ for all $k \in K \setminus h[X]$). A space Y is called *singular with respect to X* if every compactification of X whose remainder is homeomorphic to Y is singular.

Some restrictions might be appropriate. For example, if (K, h) is singular, then $K \setminus h[X]$ is a continuous image of K, and therefore compact and non-empty, so by Lemma 1.10 it makes sense to require above that X be locally compact Hausdorff non-compact. This, on the other hand, already implies that X is a Tychonoff space (see the construction of the Alexandroff compactification below). Also, in this case, it makes sense to require that Y be compact Hausdorff, since every remainder of every compactification of X is.

Singular compactifications were introduced by Whyburn [Why66], and are a generalization of Alexandroff's one-point compactification. (Of course, the definition provided next is the result of further development by other authors.) Suppose X is a Hausdorff space, Y is a compact Hausdorff space, and $f : X \to Y$ is continuous. Here, X and Y are presumed to be disjoint. The space $X \cup_f Y$ has the union $X \dot{\cup} Y$ as its ground set, and a basis for the topology consists of all open subsets of X, plus all sets of the form $(f^{-1}[U] \setminus K) \cup U$ where $K \subseteq X$ is compact and $U \subseteq Y$ is open. (Of course, if X and Y are not disjoint, the construction can be done using disjoint copies, for example $X \times \{0\} \dot{\cup} Y \times \{1\}$.) With no further assumptions it follows that $X \cup_f Y$ is compact, and that the subspace topologies of both X and Y in $X \cup_f Y$ are their original topologies. If X is locally compact, then $X \cup_f Y$ is Hausdorff. For $X \cup_f Y$ to be a compactification of X, assuming that X is locally compact, all we need is that X be dense in $X \cup_f Y$. The following definition is from Faulkner [Fau88]:

Definition 5.2. A map $f : X \to Y$ between topological spaces X and Y is called *singular* if it is continuous and, for every non-empty open set $U \subseteq Y$, there is no compact subset of X containing $f^{-1}[U]$.

Observe that if $f : X \to Y$ is singular, then for every non-empty open set $U \subseteq Y$ it holds that $f^{-1}[U] \neq \emptyset$. In other words, $f[X]$ is dense in Y. It also follows that X itself is not compact. These maps are called "singular" in contrast to the concept of "compact" maps, for which the preimage of compact sets is compact. The "opposing nature" of these concepts made more sense in early definitions of singular maps.

Lemma 5.3. *Let X be a locally compact Hausdorff space, Y a compact Hausdorff space, and $f : X \to Y$ a singular map. Then, $X \cup_f Y$ is a compactification of X.* □

In the context of the lemma above, the compactification $X \cup_f Y$ is said to be *induced by* f. If X is locally compact Hausdorff non-compact, we may take Y to be a one-point space $\{\infty\}$, and the only map $f : X \to Y$ happens to be singular. The compactification $X \cup_f Y$ obtained in this case is the famous Alexandroff compactification. Finally, the next lemma can be found in [Fau88], and is included here to clarify the origin of the term "singular compactification" (the proof is rather intuitive):

Lemma 5.4. *Let X be a locally compact Hausdorff non-compact space, and (K, h) a compactification of X. Then (K, h) is singular if and only if it is isomorphic to a compactification of X induced by a singular map.* □

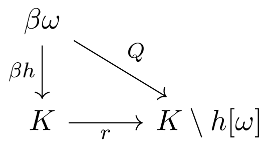
Our goal in this chapter is to show that the hypotheses in Theorem 4.2 are not too far from optimal. Though they can be weakened, such as using Lemma 4.16, we will see that there is a separable 0-dimensional first-countable compact Hausdorff space which is not in $\mathtt{Ext}(\omega^*, \beta\omega)$.

Lemma 5.5. *Every element of* $\mathtt{Ext}(\omega^*, \beta\omega)$ *is singular with respect to* ω.

Proof. Let (K, h) be a compactification of ω such that $K \setminus h[\omega] \in \mathtt{Ext}(\omega^*, \beta\omega)$. Since the range of $h : \omega \to K$ is dense in K, the map $\beta h : \beta\omega \to K$ is surjective. Clearly, $K \setminus h[\omega] \subseteq h^*[\omega^*]$. Since $h[\omega]$ is open in K, it follows that the points of $h[\omega]$ are isolated in K, and since h is injective, these points have each only a singleton as preimage, thus, by Lemma 4.8, $h[\omega] \cap h^*[\omega^*] = \emptyset$. This shows that $h^* : \omega^* \to K \setminus h[\omega]$ is a quotient map. Because $K \setminus h[\omega] \in \mathtt{Ext}(\omega^*, \beta\omega)$, the map h^* has a continuous extension $Q : \beta\omega \to K \setminus h[\omega]$, and we may define a map $r : K \to K \setminus h[\omega]$ through:

$$r(k) := \begin{cases} k & \text{if } k \in K \setminus h[\omega] \\ Q(h^{-1}(k)) & \text{if } k \in h[\omega] \end{cases}$$

We will be done if we can prove that r is continuous. This is an easy consequence of the fact that the following diagram commutes (together with the fact that Q is continuous and βh is a quotient map):



□

Putting together Theorem 4.2 and Lemmata 4.16 and 5.5 we get an alternative proof of the following corollary, which can also be found in [ACFV96].

Corollary 5.6. *Every product of second-countable compact Hausdorff spaces is singular with respect to* ω. □

Of course, in their article, they proved a much more general version, in which ω is replaced by an arbitrary 0-dimensional locally compact Hausdorff space (and they prefer the formulation "compact metrizable" instead of "second-countable compact Hausdorff").

In order to give an example of a (nice) space not in $\mathtt{Ext}(\omega^*, \beta\omega)$, Lemma 5.5 shows that it suffices to find a non-singular compactification of ω (with a nice remainder). The example given in this chapter is based on the $X(\mathtt{FLIP})$ construction from Watson and Weiss [WW88], and the idea to use this construction I owe, very gratefully, to K. P. Hart. As usual, if L is a linear order, the topology *induced by the ordering of* L is the one generated by the subbasis $\{L_{<l} : l \in L\} \cup \{L_{>l} : l \in L\}$, and it is always Hausdorff (see [Eng89]). Moreover, if L and M are linear orders, the *lexicographic ordering of* $L \times M$ is given by $(l_0, m_0) \leq (l_1, m_1)$ if and only if ($l_0 < l_1$ or ($l_0 = l_1$ and $m_0 \leq m_1$)), and it is also linear. To avoid confusion, through the rest of this chapter: (x, y) will always denote an ordered pair (not an open interval); $[x, y]_L$, $]x, y]_L$, $[x, y[_L$ and $]x, y[_L$ will always denote closed, left-open right-closed, left-closed right-open, and open intervals in a linear order L respectively, and if the subscript is omitted the linear order is assumed to be the real line.

Definition 5.7. The *double arrow space* is the subspace $\mathbb{D} := ([0, 1] \times 2) \setminus \{(0, 0), (1, 1)\}$ of the space $[0, 1] \times 2$, which has the topology induced by the lexicographic ordering.

Technically, the topology of $\mathbb{D}$ is not defined as the topology induced by the ordering of $\mathbb{D}$, but rather as the subspace topology, where the topology of $[0,1]\times 2$ is induced by its ordering. The two topologies coincide in this case, because $\mathbb{D}$ is an interval in $[0,1]\times 2$, but this is not the case for every subset of a linear order. For example, if $I := [0,1]\times\{0\}$, the subset $[0,1/2]\times\{0\}$ is not open in the topology induced by its ordering, but it is open in the subspace topology, since it consists of all points of I strictly below $(1/2,1)$.

Lemma 5.8. *The double arrow space $\mathbb{D}$ is separable 0-dimensional first-countable compact Hausdorff. Given $b\in\,]0,1]$, the family of all intervals $[(a,1),(b,0)]_{\mathbb{D}}$ where $a\in[0,b[$ is a local basis at $(b,0)$. Similarly, given $a\in[0,1[$, the family of all intervals $[(a,1),(b,0)]_{\mathbb{D}}$ where $b\in\,]a,1]$ is a local basis at $(a,1)$.*

Proof. As mentioned above, $\mathbb{D}$ is Hausdorff because its topology is induced by a linear ordering. Checking that the families described above are indeed local bases at $(b,0)$ and $(a,1)$ respectively is straight-forward, and very useful to prove the remaining properties of $\mathbb{D}$. For example, notice that if $0\le a<b\le 1$, then there is $q\in\,]a,b[\,\cap\,\mathbb{Q}$, and therefore $(q,0),(q,1)\in[(a,1),(b,0)]_{\mathbb{D}}$, showing that $(\,]0,1[\,\cap\,\mathbb{Q})\times 2$ is a countable dense subset of $\mathbb{D}$. Moreover, this shows that the family of all intervals $[(q,1),(b,0)]_{\mathbb{D}}$ with $q\in[0,b[\,\cap\,\mathbb{Q}$ is a local basis at $(b,0)$ and the family of all intervals $[(a,1),(q,0)]_{\mathbb{D}}$ with $q\in\,]a,1]\cap\mathbb{Q}$ is a local basis at $(a,1)$, proving that $\mathbb{D}$ is first-countable. It is also clear that the intervals $[(a,1),(b,0)]_{\mathbb{D}}$ are clopen, so that $\mathbb{D}$ is 0-dimensional.

It remains to prove compactness, so suppose $\mathcal{O}$ is an open cover of $\mathbb{D}$. Given $x\in\,]0,1]$, there is $O_{(x,0)}\in\mathcal{O}$ containing $(x,0)$, and given $x\in[0,1[$, there is $O_{(x,1)}\in\mathcal{O}$ containing $(x,1)$. Thus, for $x\in\,]0,1]$ there is $a_x\in[0,x[$ such that $[(a_x,1),(x,0)]_{\mathbb{D}}\subseteq O_{(x,0)}$, and for $x\in[0,1[$ there is $b_x\in\,]x,1]$ such that $[(x,1),(b_x,0)]_{\mathbb{D}}\subseteq O_{(x,1)}$. The collection

$$\{[0,b_0[\,,\,]a_1,1]\}\cup\{\,]a_x,b_x[\,:\,x\in\,]0,1[\,\}$$

is an open cover of $[0,1]$, which is compact, hence there is a finite set $F\subseteq\,]0,1[$ such that the collection $\{[0,b_0[\,,\,]a_1,1]\}\cup\{\,]a_x,b_x[\,:\,x\in F\}$ also covers $[0,1]$. It is easy to see that the finite family $\{O_{(0,1)},O_{(1,0)}\}\cup\{O_{(x,i)}\,:\,x\in F\,,\,i\in 2\}\subseteq\mathcal{O}$ is a cover of $\mathbb{D}$. □

We now begin the construction of a compactification of ω that is not singular and whose remainder is $\mathbb{D}$. First, let $Q := \,]0,1[\,\cap\,\mathbb{Q}$. For the ground set of the compactification, take $K := \mathbb{D}\,\dot\cup\,Q$. For the embedding of the compactification, take an arbitrary bijection $h:\omega\to Q$. The topology of K will be chosen later. Let T be the set of all topologies τ on K with the following properties:

- (P1) Q is dense in (K,τ)
- (P2) The points of Q are isolated in (K,τ).
- (P3) The subspace topology on $\mathbb{D}$ coincides with its original topology.
- (P4) (K,τ) is Hausdorff.
- (P5) (K,τ) is compact.
- (P6) For all $b\in\,]0,1]$, if U is an open neighbourhood of b in $[0,1]$, there is an open neighbourhood U_0 of $(b,0)$ in (K,τ) such that $U_0\cap Q\subseteq U$.

Lemma 5.9. *If $\tau\in T$, then (K,τ) is first-countable.*

Proof. The points of Q are isolated, so all we need to prove is that every point $d\in\mathbb{D}$ has a countable local basis in K. Let $\mathcal{B}$ be a countable local basis at d in $\mathbb{D}$. If $B\in\mathcal{B}$, there is a neighbourhood V_B of d in K such that $B=V_B\cap\mathbb{D}$. If $q\in Q$, then $V_B\setminus\{q\}$ is still a neighbourhood of d, and since K is locally compact, there is a compact neighbourhood $K_{B,q}$ of d in K, such that $K_{B,q} \subseteq V_B \setminus \{q\}$. Observe that $\bigcap\{K_{B,q} : (B,q) \in \mathcal{B} \times Q\} = \{d\}$, so if $U \subseteq K$ is open and $d \in U$, the collection $\{U\} \cup \{K \setminus K_{B,q} : (B,q) \in \mathcal{B} \times Q\}$ is an open cover of K. By compactness, there is a finite set $F \subseteq \mathcal{B} \times Q$ such that $\bigcap_{(B,q)\in F} K_{B,q} \subseteq U$. This shows that the collection of all finite intersections of sets of the form $K_{B,q}$ is a local basis at d in K, and it is easy to see that this collection is countable. □

In the proof above we used properties (P2) to (P5). It is also clear that if τ is a topology on K satisfying (P1) to (P5), then $((K,\tau),h)$ is a compactification of ω whose remainder is $\mathbb{D}$. Let

$$R := \{r : K \to \mathbb{D} \,:\, r \restriction \mathbb{D} = \mathtt{id}_{\mathbb{D}} \text{ and there is } \tau \in T \text{ which makes } r \text{ continuous}\}$$

The elements of R are the "threats" we need to avoid. These are the candidates to being retractions of K onto $\mathbb{D}$, and our goal is to find $\tau \in T$ which makes none of the elements of R continuous. Note that $|R| \leq |\mathbb{D}|^{|Q|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, so there is an injection $\sigma : R \to]0,1[$.

Lemma 5.10. *There is a map $p : R \times \omega \to Q$ such that for each fixed $r \in R$ the sequence $(p(r,i))_{i\in\omega}$ has no repetitions, $p(r,i) \xrightarrow{i} \sigma(r)$ in $[0,1]$, and $r(p(r,i)) \xrightarrow{i} (\sigma(r),0)$ in $\mathbb{D}$.*

Proof. Given $r \in R$, let $\tau \in T$ be a topology on K which makes r continuous. Since $\sigma(r) \neq 0$, we have $(\sigma(r),0) \in \mathbb{D}$. By (P1) together with Lemma 5.9, there is a sequence $(q_i)_{i\in\omega}$ in Q converging to $(\sigma(r),0)$ in (K,τ), and by Lemma 4.6 this sequence must be eventually different. Therefore we can choose $(p(r,i))_{i\in\omega}$ to be a subsequence of $(q_i)_{i\in\omega}$ without repetitions. Property (P6) was tailored precisely to make sure that $p(r,i) \xrightarrow{i} \sigma(r)$ in $[0,1]$. Finally, since r is continuous, $r(p(r,i)) \xrightarrow{i} r((\sigma(r),0)) = (\sigma(r),0)$. □

For each $x \in \sigma[R]$ let $E_x := \{p(\sigma^{-1}(x),2i)\}_{i\in\omega}$ and $O_x := \{p(\sigma^{-1}(x),2i+1)\}_{i\in\omega}$. For all $x \in [0,1] \setminus \sigma[R]$ let $E_x := O_x := \emptyset$. We are now ready to choose a topology $\tau \in T$. By (P2), for every $q \in Q$, the set $\mathcal{B}_q := \{\{q\}\}$ must be a local basis at q, so all we need to decide is what the local bases at the points of $\mathbb{D}$ should look like. For $b \in]0,1]$ let

$$\mathcal{B}_{(b,0)} := \{\,](a,1),(b,0)]_{\mathbb{D}} \cup ((Q \setminus O_b) \cap]a,b[\,) \cup (E_b \cap]b,b+\varepsilon[\,) \;:\; a \in [0,b[\,,\ \varepsilon \in]0,\infty[\,\}$$

and for $a \in [0,1[$ let

$$\mathcal{B}_{(a,1)} := \{[(a,1),(b,0)[_{\mathbb{D}} \cup ((Q \setminus E_a) \cap]a,b[\,) \cup (O_a \cap]a-\varepsilon,a[\,) \;:\; b \in]a,1]\,,\ \varepsilon \in]0,\infty[\,\}$$

These definitions need not make sense to the reader immediately, but hopefully by the end of this chapter they will feel like the logical and obvious choice.

Lemma 5.11. *There is a unique topology τ on K such that for every $k \in K$ the family $\mathcal{B}_k$ is a local basis at k in (K,τ).*

Proof. Most of the details are trivial to check. The fact that the families $\mathcal{B}_d$ for $d \in \mathbb{D}$ are closed under finite intersections is perhaps not trivial, but still very easy to check. The only part of the proof that requires some work is showing the following: If $d \in \mathbb{D}$, $B \in \mathcal{B}_d$ and $d' \in B \cap \mathbb{D} \setminus \{d\}$, there is $B' \in \mathcal{B}_{d'}$ such that $B' \subseteq B$.

Let us begin with the case $d = (b,0)$ for some $b \in]0,1]$,

$$B = \,](a,1),(b,0)]_{\mathbb{D}} \cup ((Q \setminus O_b) \cap]a,b[\,) \cup (E_b \cap]b,b+\varepsilon[\,)$$

for some $a \in [0, b[$ and some $\varepsilon > 0$, and $d' = (b', 0)$ for some $b' \in]a, b[$. If $b \in \sigma[R]$, note that $p(\sigma^{-1}(b), 2i+1) \xrightarrow{i} b$ in $[0,1]$, therefore there is $\delta > 0$ such that

$$O_b \cap]b' - \delta, b' + \delta[\subseteq \{b'\}$$

On the other hand, if $b \notin \sigma[R]$, we have the inequality above with any $\delta > 0$. In both cases we may choose δ small enough that $b' - \delta \geq a$ and $b' + \delta \leq b$, and it follows that $Q \cap ( ]b' - \delta, b'[\cup]b', b' + \delta[ ) \subseteq (Q \setminus O_b) \cap]a, b[$, hence

$$B' :=](b' - \delta, 1), (b', 0)]_{\mathbb{D}} \cup ((Q \setminus O_{b'}) \cap]b' - \delta, b'[ ) \cup (E_{b'} \cap]b', b' + \delta[ )$$

is in $\mathcal{B}_{(b',0)}$ and $B' \subseteq B$ as desired. The other three cases are very similar. The case $d = (b, 0), d' = (a', 1)$ is the reason why the $\mathbb{D}$-intervals were chosen left-open in the definition of $\mathcal{B}_{(b,0)}$ (to make sure that $a' > a$), and similarly for the case $d = (a, 1), d' = (b', 0)$. □

Lemma 5.12. *The topology τ (from the previous lemma) is in T.*

Proof. (P1): It suffices to show that whenever $0 \leq a < b \leq 1$, the sets $(Q \setminus O_b) \cap]a, b[$ and $(Q \setminus E_a) \cap]a, b[$ are non-empty. For the first set, if $b \in \sigma[R]$, then $p(\sigma^{-1}(b), i) \xrightarrow{i} b$, so the intersection $O_b \cap]a, (a+b)/2[$ is finite. On the other hand, if $b \notin \sigma[R]$, it follows that $O_b \cap]a, (a+b)/2[= \emptyset$. Of course, $Q \cap]a, (a+b)/2[$ is infinite, therefore in either case $(Q \setminus O_b) \cap]a, (a+b)/2[\neq \emptyset$. The set $(Q \setminus E_a) \cap]a, b[$ is non-empty for a similar reason.

(P2) and (P3): Clear.

(P4): This proof is also straight-forward. Most of the work is done by choosing neighbourhoods as in the definition of the families $\mathcal{B}_{(b,0)}$ and $\mathcal{B}_{(a,1)}$ while making sure that $b - a$ and ε are small enough. Two cases should be pointed out, however, as they further explain why these local bases were defined as they were: First, to separate a point $q \in Q$ from a point $(q, i) \in \mathbb{D}$ we can use the neighbourhood $\{q\}$ together with any element of $\mathcal{B}_{(q,i)}$, since q was purposely excluded from these. Second, for $x \in]0, 1[$, to separate $(x, 0)$ from $(x, 1)$ we can use any element of $\mathcal{B}_{(x,0)}$ together with any element of $\mathcal{B}_{(x,1)}$, as these were designed to be disjoint from each other.

(P5): Let $\mathcal{O}$ be an open cover of K. For each $d \in \mathbb{D}$ there some $U_d \in \mathcal{O}$ containing d, and there is some $B_d \in \mathcal{B}_d$ such that $B_d \subseteq U_d$. If $x \in]0, 1]$, we may assume that

$$B_{(x,0)} =](x - \varepsilon_{(x,0)}, 1), (x, 0)]_{\mathbb{D}} \cup ((Q \setminus O_x) \cap]x - \varepsilon_{(x,0)}, x[ ) \cup (E_x \cap]x, x + \varepsilon_{(x,0)}[ )$$

where $0 < \varepsilon_{(x,0)} \leq x$. Similarly, if $x \in [0, 1[$, we may assume that

$$B_{(x,1)} = [(x, 1), (x + \varepsilon_{(x,1)}, 0)[_{\mathbb{D}} \cup ((Q \setminus E_x) \cap]x, x + \varepsilon_{(x,1)}[ ) \cup (O_x \cap]x - \varepsilon_{(x,1)}, x[ )$$

where $0 < \varepsilon_{(x,1)} \leq 1 - x$. For $x \in]0, 1[$, by shrinking $B_{(x,0)}$ or $B_{(x,1)}$, we may assume that $\varepsilon_{(x,0)} = \varepsilon_{(x,1)}$. So let $\varepsilon_0 := \varepsilon_{(0,1)}$, $\varepsilon_1 := \varepsilon_{(1,0)}$, and for all $x \in]0, 1[$ let $\varepsilon_x := \varepsilon_{(x,0)} = \varepsilon_{(x,1)}$. The sets B_d for $d \in \mathbb{D}$ form an open cover of $\mathbb{D}$ in K, and by (P3) $\mathbb{D}$ is a compact subset of K, so there is a finite set $F \subseteq \mathbb{D}$ such that $\mathbb{D} \subseteq \bigcup_{d \in F} B_d$. Let

$$F_0 := F \cup \{(x, 1 - i) : (x, i) \in F \text{ and } x \in]0, 1[\} \text{ and}$$
$$F_1 := Q \cap \{x : \exists i \ ((x, i) \in F)\}$$

For each $q \in F_1$ let $U_q \in \mathcal{O}$ be such that $q \in U_q$, and finally let $\mathcal{O}' := \{U_k : k \in F_0 \cup F_1\}$, which is a finite subfamily of $\mathcal{O}$. It is clear that $\mathcal{O}'$ covers $\mathbb{D}$, as well as F_1, so take $q \in Q \setminus F_1$. Recall that $0 \notin Q$, so $(q, 0) \in \mathbb{D}$, which means there is some $(x, i) \in F$ such that $(q, 0) \in B_{(x,i)}$. If $x = 0$, then $i = 1$ and $(q, 0) \in [(0, 1), (\varepsilon_0, 0)[_{\mathbb{D}}$, therefore $0 < q < \varepsilon_0$. Note that $B_{(0,1)} \supseteq (Q \setminus E_0) \cap]0, \varepsilon_0[= Q \cap]0, \varepsilon_0[\ni q$. If $x = 1$, then $i = 0$ and $(q, 0) \in](1 - \varepsilon_1, 1), (1, 0)]_{\mathbb{D}}$, hence (recall that $1 \notin Q$) $1 - \varepsilon_1 < q < 1$. Note that

$B_{(1,0)} \supseteq (Q \setminus O_1) \cap]1-\varepsilon_1, 1[= Q \cap]1-\varepsilon_1, 1[\ni q$. Finally, if $x \in]0,1[$, then $(x,0)$ and $(x,1)$ are both in F_0. We have either $(q,0) \in](x-\varepsilon_x, 1), (x,0)]_{\mathbb{D}}$ or $(q,0) \in [(x,1), (x+\varepsilon_x, 0)[_{\mathbb{D}}$, so in any case it holds that $x - \varepsilon_x < q < x + \varepsilon_x$. Moreover, since $q \notin F_1$, we have $q \neq x$. Note that

$$\begin{aligned} B_{(x,0)} \cup B_{(x,1)} &\supseteq ((Q \setminus O_x) \cap]x-\varepsilon_x, x[\,) \cup (E_x \cap]x, x+\varepsilon_x[\,) \\ &\quad \cup ((Q \setminus E_x) \cap]x, x+\varepsilon_x[\,) \cup (O_x \cap]x-\varepsilon_x, x[\,) \\ &= Q \cap (\,]x-\varepsilon_x, x[\cup]x, x+\varepsilon_x[\,) \ni q \end{aligned}$$

In all cases we have shown that $q \in B_d \subseteq U_d$ for some $d \in F_0$, hence $\mathcal{O}'$ is a cover of K.

(P6): Given $b \in]0,1]$ and $U =]b-\delta, b+\delta[\cap [0,1]$ for some $\delta > 0$, let U_0 be the basic neighbourhood in $\mathcal{B}_{(b,0)}$ given by $a := \mathtt{max}\{0, b-\delta\}$ and $\varepsilon := \delta$, and this concludes the proof. □

Lemma 5.13. *(K,h) is non-singular.*

Proof. If there were a retraction $r : K \to \mathbb{D}$, where K has topology τ, by the previous lemma we would have $r \in R$. So it suffices to show that none of the elements of R is continuous when K has topology τ.

Suppose $r \in R$ and note that $r(p(r, 2i+1)) \xrightarrow{i} (\sigma(r), 0)$. If $B \in \mathcal{B}_{(\sigma(r),1)}$, say

$$B = [(\sigma(r),1), (b,0)[_{\mathbb{D}} \cup ((Q \setminus E_{\sigma(r)}) \cap]\sigma(r), b[\,) \cup (O_{\sigma(r)} \cap]\sigma(r)-\varepsilon, \sigma(r)[\,)$$

for some $b \in]\sigma(r), 1]$ and some $\varepsilon > 0$, letting $\delta := \mathtt{min}\{b - \sigma(r), \varepsilon\} > 0$ we get that $O_{\sigma(r)} \cap (\,]\sigma(r)-\delta, \sigma(r)[\cup]\sigma(r), \sigma(r)+\delta[\,) \subseteq B$ (note that $E_{\sigma(r)}$ and $O_{\sigma(r)}$ are disjoint because $(p(r,i))_{i\in\omega}$ has no repetitions). Since $p(r, 2i+1) \xrightarrow{i} \sigma(r)$ in $[0,1]$ and this sequence can only take the value $\sigma(r)$ once, it follows that $p(r, 2i+1) \in B$ for i large enough. This shows that $p(r, 2i+1) \xrightarrow{i} (\sigma(r), 1)$ in K. Thus, if r were continuous, we would have $r(p(r, 2i+1)) \xrightarrow{i} r((\sigma(r),1)) = (\sigma(r),1)$, which cannot happen because $\mathbb{D}$ is Hausdorff. □

Together with Lemma 5.5 we conclude:

Corollary 5.14. $\mathbb{D} \notin \mathtt{Ext}(\omega^*, \beta\omega)$. □

Remark 5.15. In Definition 4.15 we require that every continuous map from ω^* *into* (instead of *onto*) T have a continuous extension from $\beta\omega$ into T. The choice of *into* instead of *onto* makes the requirement stronger, which in turn strengthens the statement that certain spaces *are* in $\mathtt{Ext}(\omega^*, \beta\omega)$. However, with this definition, the previous corollary only shows that there is $f : \omega^* \to \mathbb{D}$ which cannot be extended to a continuous map $F : \beta\omega \to \mathbb{D}$, but does not guarantee that f can be chosen to be onto $\mathbb{D}$. To see that this can be achieved, note that in the proof of Lemma 5.5 the hypothesis is only used to extend the map h^*, which is onto the remainder $K \setminus h[\omega]$. In other words, if (K,h) is the non-singular compactification of ω constructed above, we can choose $f := h^*$ and we have $\mathtt{ran}(f) = \mathbb{D}$.

PART TWO

The shift and its inverse

6 Why the shift?

As the entire second part of my thesis will be dedicated to the study of a particular automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$, namely the shift s as defined in the INTRODUCTION, I would like to provide a few results and examples to justify this very specific choice.

Definition 6.1. The *index on* $\mathtt{NB}$ is the map $\mathtt{IND} : \mathtt{NB} \to \mathbb{Z}$ defined through

$$\mathtt{IND}(f) := |\omega \setminus \mathtt{dom}(f)| - |\omega \setminus \mathtt{ran}(f)|.$$

Remark 6.2. The definition above is due to van Douwen [vD90] up to a minus sign (that is, his index function was given by $\mathtt{IND}(f) = |\omega \setminus \mathtt{ran}(f)| - |\omega \setminus \mathtt{dom}(f)|$). This change makes absolutely no meaningful difference in the theory derived from the existence of the index function. One of the goals in this chapter is to present an embedding of $\mathtt{NB}$ into the monoid of Fredholm operators on the separable infinite-dimensional Hilbert space. There is already a well established index function on these operators, namely the *Fredholm index*, and the modification was made so that the two indices agree on the range of our embedding.

Lemma 6.3. *The index on* $\mathtt{NB}$ *is a homomorphism of monoids (with addition as the operation on* $\mathbb{Z}$*).*

Proof. Let $f, g \in \mathtt{NB}$, and recall that their product in $\mathtt{NB}$ was defined as the near-bijection $f \circ g : g^{-1}[\mathtt{dom}(f)] \to \omega$. Observe that $\mathtt{dom}(f \circ g) = \mathtt{dom}(g) \setminus g^{-1}[\mathtt{ran}(g) \setminus \mathtt{dom}(f)]$, therefore $\omega \setminus \mathtt{dom}(f \circ g) = (\omega \setminus \mathtt{dom}(g)) \dot\cup g^{-1}[\mathtt{ran}(g) \setminus \mathtt{dom}(f)]$. On the other hand, we have $\mathtt{ran}(f \circ g) = \mathtt{ran}(f) \setminus f[\mathtt{dom}(f) \setminus \mathtt{ran}(g)]$, therefore $\omega \setminus \mathtt{ran}(f \circ g) = (\omega \setminus \mathtt{ran}(f)) \dot\cup f[\mathtt{dom}(f) \setminus \mathtt{ran}(g)]$. Since f and g are injective it follows:

$$\begin{aligned}
\mathtt{IND}(f \circ g) &= |\omega \setminus \mathtt{dom}(g)| + |\mathtt{ran}(g) \setminus \mathtt{dom}(f)| - |\omega \setminus \mathtt{ran}(f)| - |\mathtt{dom}(f) \setminus \mathtt{ran}(g)| \\
&= |\omega \setminus \mathtt{dom}(g)| + (|\omega \setminus \mathtt{dom}(f)| - |\omega \setminus (\mathtt{ran}(g) \cup \mathtt{dom}(f))|) \\
&\quad - |\omega \setminus \mathtt{ran}(f)| - (|\omega \setminus \mathtt{ran}(g)| - |\omega \setminus (\mathtt{dom}(f) \cup \mathtt{ran}(g))|) \\
&= |\omega \setminus \mathtt{dom}(f)| - |\omega \setminus \mathtt{ran}(f)| + |\omega \setminus \mathtt{dom}(g)| - |\omega \setminus \mathtt{ran}(g)| \\
&= \mathtt{IND}(f) + \mathtt{IND}(g)
\end{aligned}$$

□

We can use the notation $=^*$ to compare near-bijections as well, simply by viewing them as subsets of ω^2. If $f, g \in \mathtt{NB}$, then $\mathtt{dom}(f) \,\Delta\, \mathtt{dom}(g)$ is always finite, so it is easy to see that $f =^* g$ if and only if $\{n \in \mathtt{dom}(f) \cap \mathtt{dom}(g) \,:\, f(n) \neq g(n)\}$ is finite.

Lemma 6.4. *If $f, g \in \mathtt{NB}$ and $f =^* g$, then $\mathtt{IND}(f) = \mathtt{IND}(g)$.*

Proof. Let

$$\begin{aligned} E_0 &:= \omega \setminus (\mathtt{dom}(f) \cup \mathtt{dom}(g)) \\ F_0 &:= g^{-1}[\mathtt{ran}(f)] \setminus \mathtt{dom}(f) \\ F_1 &:= \mathtt{dom}(g) \setminus (g^{-1}[\mathtt{ran}(f)] \cup \mathtt{dom}(f)) \end{aligned}$$

and observe that $\omega \setminus \mathtt{dom}(f) = E_0 \dot\cup F_0 \dot\cup F_1$. Similarly let

$$\begin{aligned} G_0 &:= f^{-1}[\mathtt{ran}(g)] \setminus \mathtt{dom}(g) \\ G_1 &:= \mathtt{dom}(f) \setminus (f^{-1}[\mathtt{ran}(g)] \cup \mathtt{dom}(g)) \end{aligned}$$

and observe that $\omega \setminus \mathtt{dom}(g) = E_0 \dot\cup G_0 \dot\cup G_1$. For the "range" side, let

$$\begin{aligned} E_1 &:= \omega \setminus (\mathtt{ran}(f) \cup \mathtt{ran}(g)) \\ F_2 &:= g[\mathtt{dom}(f)] \setminus \mathtt{ran}(f) \\ F_3 &:= \mathtt{ran}(g) \setminus (g[\mathtt{dom}(f)] \cup \mathtt{ran}(f)) \\ G_2 &:= f[\mathtt{dom}(g)] \setminus \mathtt{ran}(g) \\ G_3 &:= \mathtt{ran}(f) \setminus (f[\mathtt{dom}(g)] \cup \mathtt{ran}(g)) \end{aligned}$$

and it follows that $\omega \setminus \mathtt{ran}(f) = E_1 \dot\cup F_2 \dot\cup F_3$ and $\omega \setminus \mathtt{ran}(g) = E_1 \dot\cup G_2 \dot\cup G_3$. Finally, let

$$\begin{aligned} E_2 &:= \{n \in \mathtt{dom}(f) \cap \mathtt{dom}(g) \;:\; f(n) \neq g(n)\} \\ E_3 &:= \{n \in \mathtt{ran}(f) \cap \mathtt{ran}(g) \;:\; f^{-1}(n) \neq g^{-1}(n)\} \end{aligned}$$

It is straight-forward to prove that $f[E_2 \dot\cup G_0] = E_3 \dot\cup G_2$. By symmetry, it holds that $g[E_2 \dot\cup F_0] = E_3 \dot\cup F_2$. Hence $|G_0| - |G_2| = |E_3| - |E_2| = |F_0| - |F_2|$. Finally, note that $g[F_1] = F_3$ and $f[G_1] = G_3$, which implies that $|F_1| = |F_3|$ and $|G_1| = |G_3|$. We have:

$$\begin{aligned} \mathtt{IND}(f) &= (|E_0| + |F_0| + |F_1|) - (|E_1| + |F_2| + |F_3|) \\ &= (|E_0| - |E_1|) + (|F_0| - |F_2|) + (|F_1| - |F_3|) \\ &= (|E_0| - |E_1|) + (|G_0| - |G_2|) + (|G_1| - |G_3|) \\ &= (|E_0| + |G_0| + |G_1|) - (|E_1| + |G_2| + |G_3|) = \mathtt{IND}(g) \end{aligned}$$

□

It is noteworthy that the hypothesis of the lemma was used, however discretely, to go from $|E_2| + |G_0| = |E_3| + |G_2|$ to the equality $|E_3| - |E_2| = |G_0| - |G_2|$ (and similarly with F_i's instead of G_i's), which was only possible because E_2 and E_3 are finite sets. In the next lemma the symbol $\neq^*$ will be used, which can be confusing. The formula $f \neq^* g$ is simply the negation of $f =^* g$. (One could *falsely* interpret the formula as "$f(n) \neq g(n)$ for cofinitely many n". In other words, let us agree that * takes precedence over negation.)

Lemma 6.5. *Let $f, g \in \mathtt{NB}$. Then, $\varphi_f = \varphi_g$ if and only if $f =^* g$.*

Proof. Showing that $\varphi_f = \varphi_g$ whenever $f =^* g$ is easy. The other direction is more interesting. Suppose $f \neq^* g$ and take $n_0 \in \mathtt{dom}(f) \cap \mathtt{dom}(g)$ with $f(n_0) \neq g(n_0)$. Assume we have chosen distinct points $n_0, \ldots, n_k \in \mathtt{dom}(f) \cap \mathtt{dom}(g)$ such that

$$f[\{n_0, \ldots, n_k\}] \cap g[\{n_0, \ldots, n_k\}] = \emptyset$$

Since $\{n \in \mathtt{dom}(f) \cap \mathtt{dom}(g) \;:\; f(n) \neq g(n)\}$ is infinite, we can choose $n_{k+1} \in \mathtt{dom}(f) \cap \mathtt{dom}(g)$ such that $n_{k+1} \notin \{n_0, \ldots, n_k\} \cup f^{-1}[g[\{n_0, \ldots, n_k\}]] \cup g^{-1}[f[\{n_0, \ldots, n_k\}]]$ and $f(n_{k+1}) \neq g(n_{k+1})$, which completes the induction step. Now, if we let $x := \{n_i \;:\; i \in \omega\}$, then x is infinite, and we have $f[x] \cap g[x] = \emptyset$, therefore $\varphi_f([\![x]\!]) \neq \varphi_g([\![x]\!])$. □

Definition 6.6. The set of all trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$, that is, the set of all φ_f for $f \in \mathtt{NB}$, is denoted by $\mathtt{Triv}$. The subset of very trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$, that is, the set of all φ_f such that $f \in \mathtt{Sym}(\omega)$, is denoted by $\mathtt{VTriv}$.

It is immediate to see that the map $f \mapsto \varphi_f$ is a homomorphism. Moreover, for every $f \in \mathtt{NB}$ it holds that $f \circ f^{-1} =^* \mathtt{id}_\omega$ and so $\varphi_{f\circ f^{-1}} = \varphi_{\mathtt{id}_\omega} = \mathtt{id}_{\mathcal{P}(\omega)/\mathtt{fin}}$. This shows that $\varphi_{f^{-1}} = \varphi_f^{-1}$. It follows that $\mathtt{Triv}$ and $\mathtt{VTriv}$ are subgroups of $\mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$. Lemma 6.5, together with the fact that the map $f \mapsto \varphi_f$ restricted to $\mathtt{Sym}(\omega)$ is a group homomorphism onto $\mathtt{VTriv}$, implies that $\mathtt{VTriv} \simeq \mathtt{Sym}(\omega)/=^*$. On the other hand if $f, g \in \mathtt{Sym}(\omega)$, then $f =^* g$ precisely when the *support* of $g^{-1} \circ f$ (i.e. the set of its non-fixed points) is finite. The set of all permutations of ω with finite support, denoted $\mathtt{FS}$, is a normal subgroup of $\mathtt{Sym}(\omega)$, and thus we have shown:

$$\mathtt{VTriv} \simeq \mathtt{Sym}(\omega)/\mathtt{FS}$$

Lemmata 6.4 and 6.5 combined justify the following:

Definition 6.7. The *index on* $\mathtt{Triv}$ is the map $\mathtt{ind} : \mathtt{Triv} \to \mathbb{Z}$ defined through

$$\mathtt{ind}(\varphi_f) := \mathtt{IND}(f)$$

for every $f \in \mathtt{NB}$.

Clearly the index on $\mathtt{Triv}$ is a homomorphism of groups.

Lemma 6.8. $\mathtt{VTriv} = \mathtt{ind}^{-1}(0)$.

Proof. If $\varphi \in \mathtt{VTriv}$, there is $f \in \mathtt{Sym}(\omega)$ such that $\varphi = \varphi_f$, and it is obvious that $\mathtt{ind}(\varphi) = \mathtt{IND}(f) = 0$. On the other hand, if $\varphi \in \mathtt{Triv}$ and $\mathtt{ind}(\varphi) = 0$, let $f \in \mathtt{NB}$ be such that $\varphi = \varphi_f$ (and thus $\mathtt{IND}(f) = 0$). Then $|\omega \setminus \mathtt{dom}(f)| = |\omega \setminus \mathtt{ran}(f)|$, so we can choose a bijection $\sigma : \omega \setminus \mathtt{dom}(f) \to \omega \setminus \mathtt{ran}(f)$. It follows that $f \cup \sigma \in \mathtt{Sym}(\omega)$, and since $f \cup \sigma =^* f$ we have $\varphi = \varphi_{f\cup\sigma} \in \mathtt{VTriv}$. □

Corollary 6.9 (van Douwen)**.** $\mathtt{VTriv}$ *is a normal subgroup of* $\mathtt{Triv}$*. Moreover,* $\mathtt{Triv}/\mathtt{VTriv}$ *is isomorphic to* $(\mathbb{Z}, +)$ *and generated by* $s \circ \mathtt{VTriv}$*.*

Proof. Follows from Lemma 6.8 and the observation that $\mathtt{ind}(s) = -1$. □

Evidently, the group $\mathtt{Sym}(\omega)/\mathtt{FS}$ and the shift are key pieces in the study of the trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. Of course, we could have chosen any trivial automorphism with index 1 or -1 instead of the shift, but precisely because any such automorphism would suffice, it seems wise to work with the simplest one we can define.

The group $\mathtt{Sym}(\omega)$ has only two proper normal subgroups other than $\{\mathtt{id}_\omega\}$, namely $\mathtt{FS}$ and $\mathtt{Alt}(\omega)$, the *alternating group*, consisting of all even permutations (see, e.g. [Sco64]). Since $\mathtt{Alt}(\omega) \subseteq \mathtt{FS}$, it follows that $\mathtt{Sym}(\omega)/\mathtt{FS}$ is simple, i.e. its only proper normal subgroup is $\{\mathtt{FS}\}$. Alperin, Covington and Macpherson [ACM96] characterized the automorphism group of $\mathtt{Sym}(\omega)/\mathtt{FS}$, and what follows is their result translated into the context of automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$: Let us adopt the convention that whenever u is an invertible element in a monoid M, the *conjugation by* u (that is, the map $x \mapsto uxu^{-1}$) will be denoted by $C(u)$. This map is always an automorphism of M, and this holds even with some additional structure, such as if M is a ring or a complex algebra (and the monoid operation considered above is the multiplication). Moreover, the map C from the group of invertible elements of M into the group of automorphisms of M is a homomorphism. The kernel of C consists of the central invertible elements of M (that is, the invertible elements which commute with every element of M). The automorphisms of the form $C(u)$ with $u \in M$ are called *inner*, and in contrast all others are called *outer* automorphisms of M. Since VTriv is a normal subgroup of Triv, it follows that all conjugations by trivial automorphisms induce automorphisms of VTriv. In their article, Alperin, Covington and Macpherson proved that these are all the automorphisms of VTriv, that is:

$$\mathtt{Aut}(\mathtt{VTriv}) = \{C(\varphi) \restriction \mathtt{VTriv} : \varphi \in \mathtt{Triv}\}$$

Lemma 6.10. *The map* $\varphi \mapsto (C(\varphi) \restriction \mathtt{VTriv})$ *is injective on* Triv*.*

Proof. Suppose φ_0 and φ_1 are distinct trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$, induced by near-bijections f_0 and f_1 respectively. Clearly $f_0^{-1} \neq^* f_1^{-1}$, so (as in the proof of Lemma 6.5) we can find an infinite set $x \subseteq \omega$ such that $f_0^{-1}[x] \cap f_1^{-1}[x] = \emptyset$. Let σ be a permutation of ω consisting of one infinite cycle on $f_0^{-1}[x]$ and leaving every other point fixed. Then, $f_0\sigma f_0^{-1}$ moves cofinitely many points of x, while $f_1\sigma f_1^{-1}$ fixes cofinitely many points of x. This shows that $f_0\sigma f_0^{-1} \neq^* f_1\sigma f_1^{-1}$, and by Lemma 6.5 we have $C(\varphi_0)(\varphi_\sigma) \neq C(\varphi_1)(\varphi_\sigma)$. □

Corollary 6.11. $\mathtt{Aut}(\mathtt{VTriv}) \simeq \mathtt{Triv}$*.* □

A context which holds many similarities to the study of automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$, is the study of automorphisms of the Calkin algebra. Since this is only a parallel example for motivation and intuition, I will not go through the definitions comprehensively, and will omit the proofs. For a complete and careful construction of the Calkin algebra I recommend Conway's book on functional analysis [Con90]. For a much more profound and detailed comparison between $\mathcal{P}(\omega)/\mathtt{fin}$ and the Calkin algebra (with proofs) I recommend Farah's and Wofsey's lecture notes on operator algebras [FW13].

The infinite-dimensional separable Hilbert space. By l^2, we denote the subset of $\mathbb{C}^\omega$ consisting of all sequences $z : \omega \to \mathbb{C}$ such that $\sum_{i\in\omega} |z(i)|^2 < \infty$. This is made into a complex vector space with pointwise addition, and pointwise scalar multiplication. The inner product

$$\langle z, w\rangle := \sum_{i\in\omega} z(i)\overline{w(i)}$$

makes l^2 into a *Hilbert space*, meaning that the metric induced by this inner product is complete. For each $n \in \omega$, let $e_n \in l^2$ be the sequence whose n-th coordinate is 1 and all other coordinates are 0. Then the family $\{e_n : n \in \omega\}$ is an *orthonormal basis* for l^2 (an orthonormal family whose linear span is dense in l^2), showing that its (*Hilbert-*)dimension is $\aleph_0$. Hilbert spaces can be determined up to isometric isomorphisms by their dimension, and they are separable if and only if their dimension is countable. Therefore, l^2 is (essentially) the unique infinite-dimensional separable Hilbert space.

The collection $\overline{\mathtt{Sub}}(l^2)$ of closed subspaces of l^2 is partially ordered by inclusion. We can embed $\mathcal{P}(\omega)$ into this partial order with the map $x \mapsto M_x$ where $M_x := \overline{\mathtt{span}\{e_n : n \in x\}}$ for each $x \subseteq \omega$. This embedding has very nice properties:

- $M_\emptyset = \{\vec{0}\}$. In other words, $\mathtt{min}(\mathcal{P}(\omega))$ is mapped to $\mathtt{min}(\overline{\mathtt{Sub}}(l^2))$.
- $M_\omega = l^2$. In other words, $\mathtt{max}(\mathcal{P}(\omega))$ is mapped to $\mathtt{max}(\overline{\mathtt{Sub}}(l^2))$.
- $M_{x\cup y} = \overline{M_x + M_y}$. Thus, $\mathtt{sup}_{\mathcal{P}(\omega)}\{x, y\}$ is mapped to $\mathtt{sup}_{\overline{\mathtt{Sub}}(l^2)}\{M_x, M_y\}$.
- $M_{x\cap y} = M_x \cap M_y$. Thus, $\mathtt{inf}_{\mathcal{P}(\omega)}\{x, y\}$ is mapped to $\mathtt{inf}_{\overline{\mathtt{Sub}}(l^2)}\{M_x, M_y\}$.
- $M_{\omega\setminus x} = M_x^\perp$.

Continuous linear operators. By $\mathcal{B}(l^2)$, we denote the set of all continuous linear operators on l^2. The letter $\mathcal{B}$ stands for "bounded", because linear operators on a Hilbert space are continuous if and only if they map bounded sets into bounded sets. The set $\mathcal{B}(l^2)$ can be made into a complex algebra with pointwise addition, pointwise scalar multiplication, and composition as the vector multiplication. A norm can be defined in $\mathcal{B}(l^2)$ through

$$\|L\| := \mathtt{sup}\{\|L(z)\| \,:\, \|z\| \leq 1\}$$

and this norm makes $\mathcal{B}(l^2)$ into a Banach algebra, i.e. the induced metric is complete and for all $L_0, L_1 \in \mathcal{B}(l^2)$ it holds that $\|L_0 L_1\| \leq \|L_0\|\|L_1\|$. Every $L \in \mathcal{B}(l^2)$ has an *adjoint* operator $L^* \in \mathcal{B}(l^2)$ which is the unique operator satisfying

$$\langle L(z), w\rangle = \langle z, L^*(w)\rangle$$

for all $z, w \in l^2$. The operation $L \mapsto L^*$ is an *involution*, that is, it satisfies the properties $(L_0 + L_1)^* = L_0^* + L_1^*$, $(\lambda L)^* = \overline{\lambda} L^*$, $(L_0 L_1)^* = L_1^* L_0^*$, and $(L^*)^* = L$. This involution also satisfies $\|L^* L\| = \|L\|^2$, and consequently $\mathcal{B}(l^2)$ is called a *C*-algebra*.

Given a closed subspace M of l^2, let $\mathtt{proj}_M$ be the orthogonal projection onto M. The map $M \mapsto \mathtt{proj}_M$ is a bijection between the set of closed subspaces of l^2 and the set of orthogonal projections (that is, the set of $P \in \mathcal{B}(l^2)$ such that $P^2 = P$ and $P^* = P$). The order induced on the orthogonal projections ($\mathtt{proj}_M \leq \mathtt{proj}_{M'}$ if and only if $M \subseteq M'$) can be characterized by $P \leq P'$ if and only if $PP' = P$. For each $x \subseteq \omega$ let P_x be the orthogonal projection onto M_x. The properties of the embedding $x \mapsto M_x$ listed above translate to the following:

- $P_\emptyset$ is the constant map $\vec{0}$.
- $P_\omega = \mathtt{id}_{l^2}$.
- $P_{x\cup y} = P_x + P_y - P_x P_y$.
- $P_{x\cap y} = P_x P_y$.
- $P_{\omega\setminus x} = \mathtt{id}_{l^2} - P_x$.

A near-bijection f induces a continuous operator $T_f \in \mathcal{B}(l^2)$ by letting

$$T_f(e_n) := \begin{cases} e_{f(n)} & \text{if } n \in \mathtt{dom}(f) \\ 0 & \text{otherwise} \end{cases}$$

If we take $\mathcal{B}(l^2)$ as a monoid with the operation of composition, then the map $f \mapsto T_f$ is an embedding of monoids. It holds that:

- If $x \subseteq \omega$, then $T_f[M_x] = M_{f[x]}$, and $T_f P_x T_{f^{-1}} = P_{f[x]}$.
- $T_f T_{f^{-1}} = P_{\mathtt{ran}(f)}$ and $T_{f^{-1}} T_f = P_{\mathtt{dom}(f)}$.
- T_f is invertible if and only if $f \in \mathtt{Sym}(\omega)$.
- If $x \subseteq \omega$ is cofinite, then $T_{\mathtt{id}_x} = P_x$.
- $T_f^* = T_{f^{-1}}$.

In particular, if $x \subseteq \omega$ and $f \in \mathtt{Sym}(\omega)$, then $C(T_f)(P_x) = P_{f[x]}$ (recall that if $U \in \mathcal{B}(l^2)$ is invertible, $C(U)$ denotes conjugation by U). It was already mentioned that $C(U)$ is an automorphism of the *complex algebra* $\mathcal{B}(l^2)$ whenever U is invertible. However, it needs not be an automorphism if we consider the extended structure that includes the norm $\|\cdot\|$ and the involution *. An operator $U \in \mathcal{B}(l^2)$ is called *unitary* if it is invertible and $U^{-1} = U^*$. The unitary operators in $\mathcal{B}(l^2)$ are precisely the automorphisms of l^2. Moreover, if U is unitary, then $C(U)$ is an automorphism of $\mathcal{B}(l^2)$ (as a C*-algebra). For this reason, the term *inner automorphism* in the context of C*-algebras is reserved for the automorphisms of the kind $C(U)$ where U is unitary.

In the case that $f \in \mathtt{Sym}(\omega)$, it is clear that $T_f^{-1} = T_f^*$, so the map $f \mapsto C(T_f)$ is a homomorphism from $\mathtt{Sym}(\omega)$ into $\mathtt{Aut}(\mathcal{B}(l^2))$. The equalities $C(T_f)(P_{\{n\}}) = P_{f[\{n\}]}$ for $n \in \omega$ clearly imply that this map is an embedding. If we let f'' denote the map on $\mathcal{P}(\omega)$ which takes $x \subseteq \omega$ to $f[x]$, it is easy to see that $\mathtt{Aut}(\mathcal{P}(\omega)) = \{f'' : f \in \mathtt{Sym}(\omega)\}$. This way, we obtain an embedding of $\mathtt{Aut}(\mathcal{P}(\omega))$ into $\mathtt{Aut}(\mathcal{B}(l^2))$ which is compatible with our embedding $x \mapsto P_x$ of $\mathcal{P}(\omega)$ into the partial order of orthogonal projections in $\mathcal{B}(l^2)$. It turns out that $\mathcal{B}(l^2)$ only has inner automorphisms, which corresponds (in this analogy) to the fact that all automorphisms of $\mathcal{P}(\omega)$ are "trivial" (in the sense that they are induced by permutations of ω).

The Calkin algebra. An element of $\mathcal{B}(l^2)$ is called *compact* if it maps bounded sets into sets whose closures are compact. The set $\mathcal{K}(l^2)$ of all compact operators is a closed ideal of $\mathcal{B}(l^2)$, which implies that the quotient

$$\mathcal{C}(l^2) := \mathcal{B}(l^2)/\mathcal{K}(l^2)$$

inherits a natural C*-algebra structure, the *quotient norm* being given by $\|L + \mathcal{K}(l^2)\| := \mathtt{inf}\{\|L + L_0\| : L_0 \in \mathcal{K}(l^2)\}$. This quotient is known as the *Calkin algebra*.

The fact that the closed unit ball in a Hilbert space is compact if and only if the dimension is finite easily implies that P_x is compact if and only if x is finite. Consequently, for $x, y \subseteq \omega$ we have $|x\Delta y| < \infty$ if and only if $P_{x\Delta y} = P_x + P_y - 2P_xP_y = (P_x - P_y)^2$ is compact (for this computation, use the fact that $P_xP_y = P_{x\cap y} = P_{y\cap x} = P_yP_x$). This happens precisely when $P_x - P_y$ is compact (for the non-obvious direction note that $P_x - P_y = (P_x - P_y)^3$), which shows that the map $[\![x]\!] \mapsto p_{[\![x]\!]} := P_x + \mathcal{K}(l^2)$ is well-defined on $\mathcal{P}(\omega)/\mathtt{fin}$ and injective. As with $\mathcal{B}(l^2)$, the orthogonal projections in the Calkin algebra are the elements $p \in \mathcal{C}(l^2)$ such that $p^2 = p = p^*$, and they are partially ordered by the rule $p \leq p'$ if and only if $pp' = p$. With this, it is easy to see that the map $e \mapsto p_e$ is an embedding of $\mathcal{P}(\omega)/\mathtt{fin}$ into the set of orthogonal projections in $\mathcal{C}(l^2)$, both seen as partial orders.

For $f, g \in \mathtt{NB}$, it holds that $f =^* g$ if and only if $T_f - T_g$ is compact. This fact is by no means trivial, but might be intuitively expected. As a consequence, given $\varphi \in \mathtt{Triv}$, $\varphi = \varphi_f$, the expression $t_\varphi := T_f + \mathcal{K}(l^2)$ is independent of the choice of f. It is easy to see that the map $\varphi \mapsto t_\varphi$ is an embedding of $\mathtt{Triv}$ into the group of unitaries of $\mathcal{C}(l^2)$. Moreover, for every $\varphi \in \mathtt{Triv}$ and $e \in \mathcal{P}(\omega)/\mathtt{fin}$ we have $C(t_\varphi)(p_e) = p_{\varphi(e)}$. As before, it follows that the map $\varphi \mapsto C(t_\varphi)$ is an embedding of $\mathtt{Triv}$ into $\mathtt{Aut}(\mathcal{C}(l^2))$ which is compatible with the embedding $e \mapsto p_e$ of $\mathcal{P}(\omega)/\mathtt{fin}$ into the set of orthogonal projections in $\mathcal{C}(l^2)$.

The same cannot be done with non-trivial automorphisms: It can be shown that if φ is a non-trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$, then there is no automorphism Φ of $\mathcal{C}(l^2)$ such that $\Phi(p_e) = p_{\varphi(e)}$ for all $e \in \mathcal{P}(\omega)/\mathtt{fin}$. (This is a deep result and the proof requires, for example, Alperin, Covington and Macpherson's results about $\mathtt{Aut}(\mathtt{Sym}(\omega)/\mathtt{FS})$ [ACM96].) Evidently, this is one of the points where this analogy breaks down, at least in models of ZFC in which there are non-trivial automorphisms (the existence of such models will be proven at the end of this chapter).

The Fredholm index. An operator $L \in \mathcal{B}(l^2)$ is called a *Fredholm operator* if its class $L + \mathcal{K}(l^2)$ is invertible in $\mathcal{C}(l^2)$. Clearly, these operators form a submonoid of $\mathcal{B}(l^2)$ (with composition). There is an index function defined on the set of Fredholm operators as follows: If $L \in \mathcal{B}(l^2)$ is a Fredholm operator, then both $L^{-1}(\vec{0})$ and $\mathtt{ran}(L)^\perp$ are finite dimensional, and the *Fredholm index* of L is defined as

$$\texttt{F-IND}(L) := \texttt{dim}(L^{-1}(\vec{0})) - \texttt{dim}(\texttt{ran}(L)^{\perp}).$$

The Fredholm index is a homomorphism of monoids and factors down to the group of invertible elements of the Calkin algebra, that is, all representatives of a given invertible $u \in \mathcal{C}(l^2)$ have the same Fredholm index. Therefore, if $u = L + \mathcal{K}(l^2) \in \mathcal{C}(l^2)$ is invertible, we can define the *Fredholm index of* u as $\texttt{F-ind}(u) := \texttt{F-IND}(L)$. For every $f \in \texttt{NB}$ it holds that $\texttt{F-IND}(T_f) = \texttt{IND}(f)$, and consequently $\texttt{F-ind}(t_\varphi) = \texttt{ind}(\varphi)$ for each $\varphi \in \texttt{Triv}$.

The shift s on $\mathcal{P}(\omega)/\texttt{fin}$ corresponds to the element t_s in the Calkin algebra, and it is an open question whether there is an automorphism Φ of the Calkin algebra such that $\Phi(t_s) = t_s^{-1}$. Clearly, inner automorphisms preserve the Fredholm index of invertible elements, and since $\texttt{F-ind}(t_s) \neq \texttt{F-ind}(t_s^{-1})$, there is no inner automorphism mapping t_s to t_s^{-1}. In fact, the following question asked by Brown, Douglas and Fillmore in 1977 [BDF77] is still open: Is there is an automorphism of the Calkin algebra which does not agree with inner automorphisms on every separable subalgebra? The previous observation shows that an automorphism mapping t_s to t_s^{-1} would provide a positive answer. Similarly:

Lemma 6.12. *There is no trivial isomorphism between* $(\mathcal{P}(\omega)/\texttt{fin}, s)$ *and* $(\mathcal{P}(\omega)/\texttt{fin}, s^{-1})$.

Proof. Clearly, every isomorphism $\varphi : (\mathcal{P}(\omega)/\texttt{fin}, s) \to (\mathcal{P}(\omega)/\texttt{fin}, s^{-1})$ is an automorphism of $\mathcal{P}(\omega)/\texttt{fin}$ satisfying $\varphi s = s^{-1}\varphi$. However, for all $\varphi \in \texttt{Triv}$ we have $\texttt{ind}(\varphi s) = \texttt{ind}(\varphi) - 1$ while $\texttt{ind}(s^{-1}\varphi) = \texttt{ind}(\varphi) + 1$. □

Outer automorphisms. These observations raise two questions: Are there outer automorphisms of the Calkin algebra? And are there non-trivial automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$? As it turns out, neither question is determined by ZFC. Rudin showed in 1956 [Rud56] that CH implies the existence of non-trivial automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$ (a proof is provided below). In 1982, Shelah [She82] used the oracle chain condition to find a model of ZFC in which all automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$ are trivial. His proof was improved upon by himself and Steprāns in 1988 [SS88], showing that the same conclusion follows from the *Proper Forcing Axiom (PFA)*. (Technically, PFA requires the existence of large cardinals. However, in their article the authors explain how the proof can be modified to obtain the conclusion $\texttt{Aut}(\mathcal{P}(\omega)/\texttt{fin}) = \texttt{Triv}$ without the need for large cardinals to exist.) Finally, in 1993, Velickovic [Vel93] showed that all automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$ are trivial under the *Open Coloring Axiom (OCA)* together with *Martin's Axiom for* $\aleph_1$ *many dense sets* $(MA_{\aleph_1})$. An immediate consequence is:

Corollary 6.13. *OCA+*$MA_{\aleph_1}$ *implies that the structure* $(\mathcal{P}(\omega)/\texttt{fin}, s)$ *is not isomorphic to* $(\mathcal{P}(\omega)/\texttt{fin}, s^{-1})$. □

This corollary, and later a similar one about automorphisms of $(\mathcal{P}(\omega)/\texttt{fin}, s)$, are the only conclusions we will draw from Velickovic's theorem, so we will not get into detail about the proof, or about the forcing axioms involved. Similarly, for the Calkin algebra Phillips and Weaver proved in 2007 [PW07] that the CH implies the existence of outer automorphisms of the Calkin algebra, and in 2011, Farah [Far11] showed that the OCA implies that all automorphisms of the Calkin algebra are inner, while also providing a simpler proof of Phillips and Weaver's result.

Farah's result gives a partial negative answer to Brown, Douglas and Fillmore's question mentioned above. On the other hand, Theorem 2.1 implies the analogous question has a negative answer in general for automorphisms of $\mathcal{P}(\omega)/\texttt{fin}$, namely, their restrictions to countable subalgebras of $\mathcal{P}(\omega)/\texttt{fin}$ always agree with trivial automorphisms. However, we will also see that it is possible to conjugate the shift to its inverse by trivial automorphisms *on countable subalgebras of* $\mathcal{P}(\omega)/\mathtt{fin}$. This shows that certain aspects of this analogy are weakened by the fact that s is a map on $\mathcal{P}(\omega)/\mathtt{fin}$, while t_s is an element in $\mathcal{C}(l^2)$.

To finish this chapter I would like to prove:

Theorem 6.14 (Rudin)**.** *CH implies the existence of precisely* $2^{\aleph_1}$ *automorphisms of* $\mathcal{P}(\omega)/\mathtt{fin}$*, and consequently the existence of non-trivial automorphisms.*

The second part follows from the first part because there are only $2^{\aleph_0}$ trivial automorphisms, which under CH is less that $2^{\aleph_1}$. The fact that there can be at most $2^{\aleph_1}$ automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ is also clear, since under CH we have $2^{\aleph_1} = \aleph_1^{\aleph_1} = (2^{\aleph_0})^{2^{\aleph_0}} = |(\mathcal{P}(\omega)/\mathtt{fin})^{\mathcal{P}(\omega)/\mathtt{fin}}|$.

Definition 6.15. Let $\mathbb{P}$ be a partial order. Two elements p_0 and p_1 are said to be *compatible* if they have a common lower bound, that is, if there is $p_2 \in \mathbb{P}$ such that $p_2 \leq p_0$ and $p_2 \leq p_1$. Otherwise p_0 and p_1 are said to be *incompatible*. Two filters $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{P}$ are called *compatible* if every element of $\mathcal{F}$ is compatible with every element of $\mathcal{F}'$. Otherwise $\mathcal{F}$ and $\mathcal{F}'$ are called *incompatible*.

Clearly, $\mathcal{F}$ and $\mathcal{F}'$ are compatible if there is a filter containing $\mathcal{F} \cup \mathcal{F}'$. The converse needs not be true without some additional hypothesis (e.g. that every pair of compatible elements of $\mathbb{P}$ has an infimum).

Definition 6.16. Let $\mathbb{P}$ be a partial order. If κ is a cardinal, we say that $\mathbb{P}$ is κ*-closed* if every decreasing chain in $\mathbb{P}$ of length less than κ has a lower bound. An *antichain* in $\mathbb{P}$ is a subset whose elements are pairwise incompatible. If μ is a cardinal, we say that $\mathbb{P}$ is μ*-splitting* if every element of $\mathbb{P}$ has an antichain of size μ below it.

Theorem 6.17 (Generalized Rasiowa-Sikorski)**.** *Let* κ *and* μ *be cardinals,* $\kappa \geq \aleph_0$*, and let* $\mathbb{P}$ *be a* κ*-closed,* μ*-splitting partial order. Suppose* $\tilde{p} \in \mathbb{P}$ *and* $\mathcal{D}$ *is a family of at most* κ *many dense subsets of* $\mathbb{P}$*. Then, there are at least* μ^κ *pairwise incompatible* $\mathcal{D}$*-generic filters on* $\mathbb{P}$ *containing* $\tilde{p}$*.*

Proof. Let C be the set of all decreasing chains in $\mathbb{P}$ of length less than κ and choose a function $f : C \times \mu \to \mathbb{P}$ such that, for each fixed $c \in C$, the set $\{f(c, \alpha) : \alpha \in \mu\}$ is an antichain consisting of lower bounds for c. Let $(D_\beta)_{\beta\in\kappa}$ be an enumeration of $\mathcal{D}$ (possibly with repetitions) and for each $\beta \in \kappa$ let $g_\beta : \mathbb{P} \to D_\beta$ be such that $g_\beta(p) \leq p$ for every $p \in \mathbb{P}$ (using the fact that D_β is dense).

Given $h : \kappa \to \mu$, we can use transfinite induction to define a decreasing chain $(p(h)_\beta)_{\beta\in\kappa}$ in $\mathbb{P}$: At the first step let $p(h)_0 := g_0 \circ f(\tilde{p}, h(0))$ (where $\tilde{p}$ represents the chain of length 1 whose single entry is $\tilde{p}$). Then, let $p(h)_\beta := g_\beta \circ f((p(h)_\gamma)_{\gamma\in\beta}, h(\beta))$ for every $\beta \in \kappa \setminus \{0\}$. Clearly, $\mathcal{F}(h) := \{p \in \mathbb{P} : \exists\beta \in \kappa\, (p(h)_\beta \leq p)\}$ is a $\mathcal{D}$-generic filter and $\tilde{p} \in \mathcal{F}(h)$.

Finally, if $h_0, h_1 : \kappa \to \mu$ are distinct, let β be the smallest ordinal at which they disagree and observe that $p(h_0)_\beta$ is incompatible with $p(h_1)_\beta$. This shows that $\mathcal{F}(h_0)$ and $\mathcal{F}(h_1)$ are incompatible filters. □

In the theorem below, the symbol $\not\subseteq^*$ will appear. Again, we agree that * takes precedence over negation, so $x \not\subseteq^* y$ is simply the negation of $x \subseteq^* y$.

Theorem 6.18. *Suppose* $\mathcal{B}$ *is a countable subalgebra of* $\mathcal{P}(\omega)/\mathtt{fin}$*, and* $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ *is an embedding. Then, for every* $b_0 \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \mathcal{B}$ *there are precisely* $2^{\aleph_0}$ *embeddings* $\varphi' : \langle\mathcal{B} \cup \{b_0\}\rangle \to \mathcal{P}(\omega)/\mathtt{fin}$ *extending* φ*.*

Proof. It is clear that $2^{\aleph_0}$ is the maximum number of such extensions of φ, since they are determined by the image of b_0. The following is an adaptation of the proof of Theorem 2.1. Let r be a lifting of $\mathtt{ran}(\varphi)$, and

$$\mathbb{P} := \{(x, y, E) : x, y \subseteq \omega \text{ are finite and disjoint, and } E \subseteq \mathcal{B} \text{ is finite}\}$$

In $\mathbb{P}$ we define: $(x_0, y_0, E_0) \geq (x_1, y_1, E_1)$ if and only if $x_0 \subseteq x_1$, $y_0 \subseteq y_1$, $E_0 \subseteq E_1$ and for all $e \in E_0$ it holds that

$$\begin{aligned} \text{if } e \leq b_0, \text{ then } \ & r\varphi(e) \cap y_1 \subseteq y_0 \\ \text{and if } e \geq b_0, \text{ then } \ & x_1 \setminus x_0 \subseteq r\varphi(e). \end{aligned}$$

It is easy to see that $\mathbb{P}$ is a partial order. The idea behind these definitions is that we are looking for a suitable image $\varphi'(b_0)$, and the x's in the triples (x, y, E) are finite approximations of this image, while the y's are finite approximations of its complement. For each $n \in \omega$ and $e \in \mathcal{B}$ let

$$\begin{aligned} U_n &:= \{(x, y, E) \in \mathbb{P} : n \in x \cup y\} \\ T_e &:= \{(x, y, E) \in \mathbb{P} : e \in E\} \end{aligned}$$

Take $(x, y, E) \in \mathbb{P}$. If $e \in \mathcal{B}$, it is clear that $(x, y, E \cup \{e\}) \in \mathbb{P}_{\leq(x,y,E)} \cap T_e$. This shows that T_e is dense in $\mathbb{P}$. Let $e' := \bigvee E_{\leq b_0}$ and $e'' := \bigwedge E_{\geq b_0}$. It follows that $e', e'' \in \mathcal{B}$ and $e' \leq b_0 \leq e''$. Since $b_0 \notin \mathcal{B}$, we have $e' < e''$, and hence $\varphi(e') < \varphi(e'')$. If $n \in r\varphi(e'') \setminus (x \cup y)$, it is easy to see that $(x \cup \{n\}, y, E) \in \mathbb{P}_{\leq(x,y,E)} \cap U_n$. On the other hand, if $n \in \omega \setminus (r\varphi(e') \cup x \cup y)$, it follows easily that $(x, y \cup \{n\}, E) \in \mathbb{P}_{\leq(x,y,E)} \cap U_n$. This shows two things: First, since $\omega = r\varphi(e'') \cup (\omega \setminus r\varphi(e'))$, it shows that U_n is dense in $\mathbb{P}$ for all $n \in \omega$. Second, since $r\varphi(e'') \setminus r\varphi(e')$ is infinite, we can always find $n \in (r\varphi(e'') \setminus r\varphi(e')) \setminus (x \cup y)$, which implies that the incompatible elements $(x \cup \{n\}, y, E)$ and $(x, y \cup \{n\}, E)$ are both below (x, y, E). This proves that $\mathbb{P}$ is 2-splitting. (In fact, if a partial order is 2-splitting, then it is at least $\aleph_0$-splitting, but this will not be necessary here.)

Suppose $\mathcal{F}$ is a filter on $\mathbb{P}$ which is generic for $\{U_n : n \in \omega\}$ and for $\{T_e : e \in \mathcal{B}\}$, let $x(\mathcal{F}) := \bigcup\{x : \exists y, E\, ((x, y, E) \in \mathcal{F})\}$, and finally $b_1(\mathcal{F}) := [\![x(\mathcal{F})]\!]$. We shall see that $\varphi[\mathcal{B}_{\leq b_0}] \subseteq \varphi[\mathcal{B}]_{\leq b_1(\mathcal{F})}$ and $\varphi[\mathcal{B}_{\geq b_0}] \subseteq \varphi[\mathcal{B}]_{\geq b_1(\mathcal{F})}$, which by Theorem 2.4 proves the existence of a homomorphism $\varphi' : \langle \mathcal{B} \cup \{b_0\} \rangle \to \mathcal{P}(\omega)/\mathtt{fin}$ extending φ and mapping b_0 to $b_1(\mathcal{F})$. We are not yet concerned with the injectivity of φ'.

Given $e \in \mathcal{B}_{\leq b_0}$, there is $(x_0, y_0, E_0) \in \mathcal{F} \cap T_e$. For every $n \in r\varphi(e) \setminus y_0$ there is some $(x_1, y_1, E_1) \in \mathcal{F}$ such that $n \in x_1 \cup y_1$, and since $\mathcal{F}$ is a filter we may assume that $(x_1, y_1, E_1) \leq (x_0, y_0, E_0)$. Since $r\varphi(e) \cap y_1 \subseteq y_0$, it follows that $n \notin y_1$, and therefore $n \in x_1 \subseteq x(\mathcal{F})$. This implies that $r\varphi(e) \subseteq^* x(\mathcal{F})$, and so $\varphi(e) \leq b_1(\mathcal{F})$. On the other hand, given $e \in \mathcal{B}_{\geq b_0}$, again there is $(x_0, y_0, E_0) \in \mathcal{F} \cap T_e$. For every $n \in x(\mathcal{F}) \setminus x_0$ there is $(x_1, y_1, E_1) \in \mathcal{F}$ such that $n \in x_1$, and we assume $(x_1, y_1, E_1) \leq (x_0, y_0, E_0)$. It follows that $n \in x_1 \setminus x_0 \subseteq r\varphi(e)$, which shows that $x(\mathcal{F}) \subseteq^* r\varphi(e)$, and consequently $b_1(\mathcal{F}) \leq \varphi(e)$, as we wanted to prove.

To make sure that φ' is injective, we need the inclusions $\varphi[\mathcal{B}_{\leq b_0}] \supseteq \varphi[\mathcal{B}]_{\leq b_1(\mathcal{F})}$ and $\varphi[\mathcal{B}_{\geq b_0}] \supseteq \varphi[\mathcal{B}]_{\geq b_1(\mathcal{F})}$, and for this we will need to extend our list of dense sets. For each $e \in \mathcal{B}_{\not\leq b_0}$ and each $k \in \omega$, let

$$T^{\not\leq}_{e,k} := \{(x, y, E) \in \mathbb{P} : r\varphi(e) \cap y \not\subseteq k\}$$

and for each $e \in \mathcal{B}_{\not\geq b_0}$ and each $k \in \omega$, let

$$T^{\not\geq}_{e,k} := \{(x, y, E) \in \mathbb{P} : x \setminus r\varphi(e) \not\subseteq k\}$$

Given $(x, y, E) \in \mathbb{P}$, let e' and e'' be defined as before. If $e \in \mathcal{B}_{\not\geq b_0}$, then $e \not\leq e'$. Hence, $r\varphi(e) \not\subseteq^* r\varphi(e')$, that is, $r\varphi(e) \setminus r\varphi(e')$ is infinite. If $k \in \omega$, choose $n \in r\varphi(e) \setminus (r\varphi(e') \cup x \cup y)$ such that $n \geq k$, and it follows that $(x, y \cup \{n\}, E) \in \mathbb{P}_{\leq(x,y,E)} \cap T^{\not\leq}_{e,k}$, showing that $T^{\not\leq}_{e,k}$ is dense in $\mathbb{P}$. Similarly, if $e \in \mathcal{B}_{\not\geq b_0}$, then $e \not\geq e''$. Thus, $r\varphi(e) \not\supseteq^* r\varphi(e'')$, that is, $r\varphi(e'') \setminus r\varphi(e)$ is infinite. If $k \in \omega$, choose $n \in r\varphi(e'') \setminus (r\varphi(e) \cup x \cup y)$ such that $n \geq k$, and it follows that $(x \cup \{n\}, y, E) \in \mathbb{P}_{\leq(x,y,E)} \cap T^{\not\geq}_{e,k}$, showing that $T^{\not\geq}_{e,k}$ is also dense in $\mathbb{P}$.

If we assume that the filter $\mathcal{F}$ above is also generic for $\{T^{\not\leq}_{e,k} : e \in \mathcal{B}_{\not\leq b_0}, k \in \omega\}$, then for each $e \in \mathcal{B}_{\not\leq b_0}$ the set $r\varphi(e) \setminus x(\mathcal{F})$ is infinite, and therefore $\varphi(e) \not\leq b_1(\mathcal{F})$. It follows that $\varphi[\mathcal{B}_{\leq b_0}] \supseteq \varphi[\mathcal{B}]_{\leq b_1(\mathcal{F})}$. On the other hand, if $\mathcal{F}$ is generic for $\{T^{\not\geq}_{e,k} : e \in \mathcal{B}_{\not\geq b_0}, k \in \omega\}$, then for each $e \in \mathcal{B}_{\not\geq b_0}$ the set $x(\mathcal{F}) \setminus r\varphi(e)$ is infinite, and therefore $b_1(\mathcal{F}) \not\leq \varphi(e)$. Consequently, $\varphi[\mathcal{B}_{\geq b_0}] \supseteq \varphi[\mathcal{B}]_{\geq b_1(\mathcal{F})}$. In short, we have shown that if $\mathcal{F}$ is generic for the family

$$\mathcal{D} := \{U_n : n \in \omega\} \cup \{T_e : e \in \mathcal{B}\} \cup \{T^{\not\leq}_{e,k} : e \in \mathcal{B}_{\not\leq b_0}, k \in \omega\} \cup \{T^{\not\geq}_{e,k} : e \in \mathcal{B}_{\not\geq b_0}, k \in \omega\}$$

then there is a unique embedding $\varphi' : \langle \mathcal{B} \cup \{b_0\} \rangle \to \mathcal{P}(\omega)/\mathtt{fin}$ extending φ and mapping b_0 to $b_1(\mathcal{F})$. Since $\mathcal{D}$ is countable and $\mathbb{P}$ is 2-splitting (and every partial order is $\aleph_0$-closed), the Generalized Rasiowa-Sikorski Theorem (6.17) implies the existence of a collection $\mathcal{C}$ of $2^{\aleph_0}$ pairwise incompatible $\mathcal{D}$-generic filters on $\mathbb{P}$. To prove that these filters induce $2^{\aleph_0}$ distinct embeddings, we need to show that $|\{b_1(\mathcal{F}) : \mathcal{F} \in \mathcal{C}\}| = 2^{\aleph_0}$. For this, it suffices to show that $|\{x(\mathcal{F}) : \mathcal{F} \in \mathcal{C}\}| = 2^{\aleph_0}$ because each class $b_1(\mathcal{F})$ has only $\aleph_0$ representatives. It remains to prove that if $\mathcal{F}_0$ and $\mathcal{F}_1$ are incompatible $\mathcal{D}$-generic filters on $\mathbb{P}$, then $x(\mathcal{F}_0) \neq x(\mathcal{F}_1)$.

Indeed, take $(x_0, y_0, E_0) \in \mathcal{F}_0$ and $(x_1, y_1, E_1) \in \mathcal{F}_1$ incompatible and let $x_2 := x_0 \cup x_1$, $y_2 := y_0 \cup y_1$, and $E_2 := E_0 \cup E_1$. If $(x_2, y_2, E_2) \notin \mathbb{P}$, then $x_2 \cap y_2 \neq \emptyset$, so either $x_0 \cap y_1 \neq \emptyset$ or $x_1 \cap y_0 \neq \emptyset$. In the first case we have $x(\mathcal{F}_0) \setminus x(\mathcal{F}_1) \neq \emptyset$ while in the second case we have $x(\mathcal{F}_1) \setminus x(\mathcal{F}_0) \neq \emptyset$, so in both cases we are done. On the other hand, if $(x_2, y_2, E_2) \in \mathbb{P}$, then either $(x_2, y_2, E_2) \not\leq (x_0, y_0, E_0)$ or $(x_2, y_2, E_2) \not\leq (x_1, y_1, E_1)$, and without loss of generality we may assume the former. Hence, there is some $e \in E_0$ such that $e \leq b_0$ and $r\varphi(e) \cap y_2 \not\subseteq y_0$, or $e \geq b_0$ and $x_2 \setminus x_0 \not\subseteq r\varphi(e)$. The case $e \leq b_0$ implies that there exists $n \in r\varphi(e) \cap (y_1 \setminus y_0)$. We can choose $(x_3, y_3, E_3) \in \mathcal{F}_0 \cap U_n$ such that $(x_3, y_3, E_3) \leq (x_0, y_0, E_0)$, and it follows that $r\varphi(e) \cap y_3 \subseteq y_0$, thus $n \notin y_3$. This implies that $n \in x_3 \cap y_1$, and so $n \in x(\mathcal{F}_0) \setminus x(\mathcal{F}_1)$. Finally, the case $e \geq b_0$ implies that there exists $n \in (x_1 \setminus x_0) \setminus r\varphi(e)$. Again, we choose $(x_3, y_3, E_3) \in \mathcal{F}_0 \cap U_n$ such that $(x_3, y_3, E_3) \leq (x_0, y_0, E_0)$, and it follows that $x_3 \setminus x_0 \subseteq r\varphi(e)$, thus $n \notin x_3$. This implies that $n \in x_1 \cap y_3$, and so $n \in x(\mathcal{F}_1) \setminus x(\mathcal{F}_0)$. □

Corollary 6.19. *Suppose $\mathcal{B}$ is a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$, and $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ is an embedding. Then, there are precisely $2^{\aleph_0}$ (very) trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ extending φ.*

Proof. Since $\mathcal{B}$ is countable, we can choose $b_0 \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \mathcal{B}$. The previous theorem implies the existence of $2^{\aleph_0}$ embeddings of $\langle \mathcal{B} \cup \{b_0\} \rangle$ into $\mathcal{P}(\omega)/\mathtt{fin}$, all of which extend φ. By Theorem 2.1, each of these extensions can be further extended to very trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. Since only $2^{\aleph_0}$ trivial automorphisms exist, the proof is complete. □

Proof of Theorem 6.14. Let $\mathbb{P}$ be the set of all embeddings of countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ into $\mathcal{P}(\omega)/\mathtt{fin}$ itself. This set becomes a partial order with reversed inclusion, that is, $\varphi_0 \leq \varphi_1$ if and only if $\varphi_0 \supseteq \varphi_1$. It is easy to see that $\mathbb{P}$ is $\aleph_1$-closed, and Theorem 6.18 implies that $\mathbb{P}$ is $2^{\aleph_0}$-splitting. For each $e \in \mathcal{P}(\omega)/\mathtt{fin}$ let

$$D_e := \{\varphi \in \mathbb{P} : e \in \mathtt{dom}(\varphi) \cap \mathtt{ran}(\varphi)\}$$

and it follows from Theorem 2.1 that D_e is dense in $\mathbb{P}$. Let $\mathcal{D} := \{D_e : e \in \mathcal{P}(\omega)/\mathtt{fin}\}$.

Under CH it holds that $\mathbb{P}$ is $\aleph_1$-closed and $\aleph_1$-splitting, and $\mathcal{D}$ is a collection of $\aleph_1$ many dense subsets of $\mathbb{P}$, so by the Generalized Rasiowa-Sikorski Theorem (6.17) there are $\aleph_1^{\aleph_1} = 2^{\aleph_1}$ pairwise incompatible $\mathcal{D}$-generic filters on $\mathbb{P}$. It is easy to see that if $\mathcal{F}$ is a $\mathcal{D}$-generic filter on $\mathbb{P}$, then $\varphi(\mathcal{F}) := \bigcup \mathcal{F}$ is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. Suppose $\mathcal{F}_0$ and $\mathcal{F}_1$ are incompatible $\mathcal{D}$-generic filters on $\mathbb{P}$, and take incompatible elements $\varphi_0 \in \mathcal{F}_0$ and $\varphi_1 \in \mathcal{F}_1$. Let $\mathcal{B} := \langle \mathtt{dom}(\varphi_0) \cup \mathtt{dom}(\varphi_1) \rangle$, which is countable, and consider $\varphi_2 := \varphi(\mathcal{F}_0) \restriction \mathcal{B}$. Then $\varphi_2 \in \mathbb{P}_{\leq \varphi_0}$, and consequently $\varphi_2 \not\leq \varphi_1$. Together with the fact that $\mathtt{dom}(\varphi_2) \supseteq \mathtt{dom}(\varphi_1)$, it follows that $\varphi_2 \restriction \mathtt{dom}(\varphi_1) \neq \varphi_1$, that is,

$$\varphi(\mathcal{F}_0) \restriction \mathtt{dom}(\varphi_1) \neq \varphi(\mathcal{F}_1) \restriction \mathtt{dom}(\varphi_1)$$

Thus, the $2^{\aleph_1}$ pairwise incompatible $\mathcal{D}$-generic filters induce $2^{\aleph_1}$ pairwise distinct automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. □

Remark 6.20. In this proof we still did not use the full strength of the Generalized Rasiowa-Sikorski Theorem. The part that was not used was the arbitrarily chosen point $\tilde{p}$ in the partial order which can be required to be in all of the resulting filters. Using this additional requirement in the proof above would show the following strengthening: *(CH) If $\mathcal{B}$ is a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ is an embedding, then there are precisely $2^{\aleph_1}$ automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ extending φ.*

7 Preserving and inverting the shift

Recall that an automorphism of $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ is an automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\varphi s = s\varphi$, or equivalently $C(\varphi)(s) = s$. Similarly, an isomorphism from $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ onto $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ is an automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\varphi s = s^{-1}\varphi$, or equivalently $C(\varphi)(s) = s^{-1}$. In the remaining of this dissertation we often work with *shift-invariant* subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$, and this terminology deserves a short comment: If f is a function and A is a subset of its domain, we shall say that A is *f-closed* if $f[A] \subseteq A$, and that A is *f-invariant* if $f[A] = A$.

Definition 7.1. Suppose $\mathcal{B}$ is a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$, let $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ be a homomorphism and $\mathcal{C}$ a shift-closed subalgebra of $\mathcal{B}$. If $\varphi s \restriction \mathcal{C} = s\varphi \restriction \mathcal{C}$, we say that φ *preserves the shift on $\mathcal{C}$*. On the other hand, if $\varphi s \restriction \mathcal{C} = s^{-1}\varphi \restriction \mathcal{C}$, we say that φ *inverts the shift on $\mathcal{C}$*. Furthermore, we simply say that φ *preserves the shift* (respectively *inverts the shift*) if $\mathcal{B}$ is shift-closed and φ preserves (respectively inverts) the shift on all of $\mathcal{B}$.

So far we have seen that no trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ inverts the shift. On the other hand, for every $m \in \mathbb{Z}$ the trivial automorphism s^m obviously preserves the shift.

Theorem 7.2. *The maps s^m for $m \in \mathbb{Z}$ are the only trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ which preserve the shift.*

Proof. Suppose $\varphi \in \mathtt{Triv}$ preserves the shift, $\varphi = \varphi_f$ for some near-bijection f. Since the domain of f is cofinite in ω, by removing finitely many points we may assume that it is a final segment of ω, that is, $\mathtt{dom}(f) = \{n \in \omega : n \geq \mathtt{min}(\mathtt{dom}(f))\}$. Take the set $D := \{n \in \mathtt{dom}(f) : f(n+1) < f(n)\}$ and suppose, for a contradiction, that D is infinite. Choose $n_0 \in D$ arbitrary. Inductively, having chosen n_k for some $k \in \omega$, take $n_{k+1} \in D$ such that $f(n_{k+1}+1) > f(n_k)+1$. Finally, let $e := [\![\{f(n_k) : k \in \omega\}]\!]$ and observe that $s(e) = [\![\{f(n_k)+1 : k \in \omega\}]\!]$, while $\varphi s \varphi^{-1}(e) = [\![\{f(n_k+1) : k \in \omega\}]\!]$. For every $k \in \omega$ we have the inequalities

$$f(n_k)+1 < f(n_{k+1}+1) < f(n_{k+1})+1$$

which clearly show that $\varphi s \varphi^{-1}(e) \neq s(e)$, contradicting the fact that φ preserves the shift. We conclude that D is finite, and so there is $N \geq \mathtt{min}(\mathtt{dom}(f))$ such that $f(n+1) > f(n)$ for all $n \geq N$. Let $A := \{n \in \omega : n \geq N\}$ and $B := f[A]$. Then, $f \restriction A$ is an isomorphism between the well-orders A and B. Since B is cofinite in ω, there is some $M' \in B$ such that $B' := \{n \in \omega : n \geq M'\} \subseteq B$. Let $N' := f^{-1}(M')$ and $A' := f^{-1}[B']$. Since f maps A isomorphically onto B and A is a final segment of ω, it follows that $A' = \{n \in \omega : n \geq N'\}$. Note that $n \mapsto S^{M'-N'}(n) = n + M' - N'$ gives an isomorphism from A' onto B', and since these are well-orders, this is the only isomorphism between them (see, e.g. [Jec03]). It follows that $f \restriction A' = S^{M'-N'} \restriction A'$, and therefore $\varphi = s^{M'-N'}$. □

As promised, we use Velickovic's result [Vel93] a second time:

Corollary 7.3. *OCA+MA$_{\aleph_1}$ implies that* $\mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin}, s) = \{s^m : m \in \mathbb{Z}\}$. □

It is an open question whether it is consistent with ZFC that there are non-trivial automorphisms which preserve the shift.

Lemma 7.4. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be shift-invariant subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi : \mathcal{B}_0 \to \mathcal{B}_1$ an isomorphism. If φ inverts the shift, then so does φ^{-1}.*

Proof. From the equality $\varphi s \restriction \mathcal{B}_0 = s^{-1}\varphi$ we obtain $s\varphi^{-1} = \varphi^{-1}s^{-1} \restriction \mathcal{B}_1$. Thus,

$$\varphi^{-1}s \restriction \mathcal{B}_1 = s^{-1}(s\varphi^{-1})s \restriction \mathcal{B}_1 = s^{-1}(\varphi^{-1}s^{-1})s \restriction \mathcal{B}_1 = s^{-1}\varphi^{-1}$$

□

Corollary 7.5. *Let $\mathcal{B}$ be a shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and φ an automorphism of $\mathcal{B}$. If φ inverts the shift, then φ^2 preserves the shift.*

Proof. From the equality $\varphi^{-1}s \restriction \mathcal{B} = s^{-1}\varphi^{-1}$ (which holds by the previous lemma) we obtain $s\varphi = \varphi s^{-1} \restriction \mathcal{B}$. Thus,

$$\varphi^2 s \restriction \mathcal{B} = \varphi(\varphi s) \restriction \mathcal{B} = \varphi(s^{-1}\varphi) = (\varphi s^{-1})\varphi = (s\varphi)\varphi = s\varphi^2$$

□

Definition 7.6. For $e \in \mathcal{P}(\omega)/\mathtt{fin}$, the *shift-orbit* of e is the set $s^{\mathbb{Z}}(e) = \{s^m(e) : m \in \mathbb{Z}\}$. The collection of all shift-orbits in $\mathcal{P}(\omega)/\mathtt{fin}$ is denoted $\mathtt{Orb}$. The *shift-period* of an element of $\mathcal{P}(\omega)/\mathtt{fin}$ is the cardinality of its shift-orbit. Finally, an element of $\mathcal{P}(\omega)/\mathtt{fin}$ is called *shift-periodic* if its shift-period is finite.

It is clear that the shift-orbits define a partition of $\mathcal{P}(\omega)/\mathtt{fin}$, that is, for all e_0 and e_1 in $\mathcal{P}(\omega)/\mathtt{fin}$ we have either $s^{\mathbb{Z}}(e_0) = s^{\mathbb{Z}}(e_1)$ or $s^{\mathbb{Z}}(e_0) \cap s^{\mathbb{Z}}(e_1) = \emptyset$. Moreover, if $\mathcal{B}$ is a shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ is an embedding which preserves or inverts the shift, it is easy to see that $\varphi[s^{\mathbb{Z}}(e)] = s^{\mathbb{Z}}(\varphi(e))$ for all $e \in \mathcal{B}$. In particular, for every $e \in \mathcal{B}$ the shift-period of $\varphi(e)$ is equal to the shift-period of e. Another consequence is that if φ is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ which preserves or inverts the shift, then it induces a permutation of $\mathtt{Orb}$. The following definition is due to Geschke and can be found in [Ges10], along with several of the subsequent results.

Definition 7.7. For each natural number $k \geq 1$, the set of all $e \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $s^k(e) = e$ is denoted $\mathtt{Per}_k$. The set of all $e \in \mathcal{P}(\omega)/\mathtt{fin}$ such that there exists $k \geq 1$ for which $s^k(e) = e$ is denoted $\mathtt{Per}$. In other words, $\mathtt{Per} := \bigcup_{k \geq 1} \mathtt{Per}_k$.

Lemma 7.8. *For each $k \geq 1$, the set $\mathtt{Per}_k$ is a shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. An element $e \in \mathcal{P}(\omega)/\mathtt{fin}$ is in $\mathtt{Per}_k$ if and only if e is shift-periodic and the shift-period of e divides k. If $k, l \geq 1$, then $\mathtt{Per}_k \leq \mathtt{Per}_l$ if and only if $k \mid l$. The set $\mathtt{Per}$ is also a shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and is precisely the set of all shift-periodic elements. Finally, if $e \in \mathtt{Per}$, then the shift-period of e is $\mathtt{min}\{k \geq 1 : e \in \mathtt{Per}_k\}$.*

Proof. Let $e \in \mathcal{P}(\omega)/\mathtt{fin}$. The set $\{m \in \mathbb{Z} : s^m(e) = e\}$ is easily seen to be a subgroup of $\mathbb{Z}$, and therefore of the form $p\mathbb{Z}$ for a unique $p \in \omega$. It follows that for $m_0, m_1 \in \mathbb{Z}$ we have $s^{m_0}(e) = s^{m_1}(e)$ if and only if $m_0 - m_1 \in p\mathbb{Z}$. If $p = 0$, the shift-orbit of e is clearly infinite, so e is not shift-periodic. If $p \geq 1$, the shift-orbit of e clearly consists of the pairwise distinct elements $e, s(e), \ldots, s^{p-1}(e)$, so e is shift-periodic with shift-period p.

With this knowledge the proof of the lemma is straight-forward. Only two points still deserve a comment: First, if $k, l \geq 1$ and $\mathtt{Per}_k \leq \mathtt{Per}_l$, to prove that $k \mid l$ we need the existence of an element of shift-period precisely k. It is easy to check that this is the case for $[\![k\mathbb{N}]\!]$. Second, to prove that $\mathtt{Per}$ is closed under disjunctions and conjunctions, we can use the fact that the set $\{\mathtt{Per}_k : k \geq 1\}$ is directed, namely, given $k, l \geq 1$ we have $\mathtt{Per}_k \cup \mathtt{Per}_l \subseteq \mathtt{Per}_{kl}$. □

Given $k \geq 1$, let $\mu_k := [\![k\mathbb{N}]\!] \in \mathcal{P}(\omega)/\mathtt{fin}$.

Lemma 7.9. *For each $k \geq 1$, $\mathtt{Per}_k$ is finite and its atoms are $\mu_k, s(\mu_k), \ldots, s^{k-1}(\mu_k)$. The algebras $\mathtt{Per}_k$ are the only finite shift-closed subalgebras $\mathcal{P}(\omega)/\mathtt{fin}$. The algebra $\mathtt{Per}$ is countably infinite and atomless.*

Proof. As observed in the proof of Lemma 7.8, μ_k is shift-periodic with shift-period k, therefore the elements $\mu_k, s(\mu_k), \ldots, s^{k-1}(\mu_k)$ are pairwise distinct and constitute the entire shift-orbit of μ_k. To see that μ_k is an atom of $\mathtt{Per}_k$, suppose $e \in \mathtt{Per}_k$ and $0 < e \leq \mu_k$. Then, there is a representative $x \in e$ such that $x \subseteq k\mathbb{N}$, that is, $x = ky$ for some infinite set $y \subseteq \mathbb{N}$. The equality $s^k(e) = e$ implies that $S^k[x] =^* x$, and consequently that $S[y] =^* y$. Thus, we can choose $n_0 \in y$ large enough such that $S[y \setminus n_0] \subseteq y$. A simple induction argument shows that $\mathbb{N}_{\geq n_0} \subseteq y$ and it follows that $x = ky =^* k\mathbb{N}$. This proves that $e = \mu_k$, as desired. To see that all shifts of μ_k are also atoms, suppose $m \in \mathbb{Z}$ and $e \in \mathtt{Per}_k$ is such that $0 < e \leq s^m(\mu_k)$. Then, $s^{-m}(e) \in \mathtt{Per}_k$ and $0 < s^{-m}(e) \leq \mu_k$, which implies that $s^{-m}(e) = \mu_k$ and consequently $e = s^m(\mu_k)$. Finally, to see that the shifts of μ_k are all the atoms of $\mathtt{Per}_k$ and that $\mathtt{Per}_k$ is finite, it suffices to observe that $\mu_k \vee s(\mu_k) \vee \cdots \vee s^{k-1}(\mu_k) = 1$.

Suppose $\mathcal{B}$ is a finite shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. Since s is injective and $s[\mathcal{B}] \subseteq \mathcal{B}$, we have $s[\mathcal{B}] = \mathcal{B}$, that is, $\mathcal{B}$ is shift-invariant. In particular, all elements of $\mathcal{B}$ are shift-periodic (because $\mathcal{B}$ contains their entire shift-orbits). Let a be an atom of $\mathcal{B}$ and let $k \geq 1$ be its shift-period. Just as above, all shifts of a are also atoms of $\mathcal{B}$. We know that $a \in \mathtt{Per}_k$, so $s^i(\mu_k) \leq a$ for some $i \in k$. This implies that $a \vee s(a) \vee \cdots \vee s^{k-1}(a) = 1$, and it follows that the shifts of a are all the atoms of $\mathcal{B}$. On the one hand, we get $\mathcal{B} \subseteq \mathtt{Per}_k$. On the other, we get that $|\mathcal{B}| = 2^k = |\mathtt{Per}_k|$, and consequently $\mathcal{B} = \mathtt{Per}_k$.

Evidently, $\mathtt{Per} = \bigcup_{k \geq 1} \mathtt{Per}_k$ is countably infinite because $|\mathtt{Per}_k| = 2^k$ for $k \geq 1$. To see that it is atomless it suffices to show that each μ_k is not an atom in $\mathtt{Per}$. Indeed, given $k \geq 1$ we have $0 < \mu_{2k} < \mu_k$, which concludes the proof. □

Corollary 7.10. *Let $\mathcal{B}$ be a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi : \mathcal{B} \to \mathcal{P}(\omega)/\mathtt{fin}$ an embedding. If $k \geq 1$ is such that $\mathtt{Per}_k \subseteq \mathcal{B}$ and φ preserves or inverts the shift on $\mathtt{Per}_k$, then $\varphi[\mathtt{Per}_k] = \mathtt{Per}_k$. If $\mathtt{Per} \subseteq \mathcal{B}$ and φ preserves or inverts the shift on $\mathtt{Per}$, then $\varphi[\mathtt{Per}] = \mathtt{Per}$.* □

Lemma 7.11. *Let φ be an automorphism of $\mathtt{Per}$ which preserves the shift. Then, there is a unique sequence $(i_k)_{k \geq 1}$ such that:*

(a) $\forall k \geq 1\,(i_k \in k)$

(b) $\forall k, l \geq 1\,(k \mid l \Rightarrow i_k \equiv i_l \pmod{k})$

(c) $\forall k \geq 1\,(\varphi \restriction \mathtt{Per}_k = s^{i_k} \restriction \mathtt{Per}_k)$

Conversely, if $(i_k)_{k \geq 1}$ is a sequence of integers satisfying (b), then there is a unique shift-preserving automorphism φ of $\mathtt{Per}$ satisfying (c).

Proof. For the first part, suppose $\varphi \in \mathtt{Aut}(\mathtt{Per})$ preserves the shift. Given $k \geq 1$ we know that $\varphi[\mathtt{Per}_k] = \mathtt{Per}_k$, and since μ_k is an atom of $\mathtt{Per}_k$, it follows that $\varphi(\mu_k)$ is also an atom of $\mathtt{Per}_k$. Hence, there is $i_k \in k$ such that $\varphi(\mu_k) = s^{i_k}(\mu_k)$. The fact that φ preserves the shift then implies that $\varphi(s^j(\mu_k)) = s^j(\varphi(\mu_k)) = s^j(s^{i_k}(\mu_k)) = s^{i_k}(s^j(\mu_k))$ for all $j \in k$. Therefore, φ agrees with s^{i_k} at every atom of $\mathtt{Per}_k$, implying that they agree on all of $\mathtt{Per}_k$. Finally, if $l \geq 1$ and $k \mid l$, then $\mu_k \in \mathtt{Per}_l$ and so we have $s^{i_k}(\mu_k) = \varphi(\mu_k) = s^{i_l}(\mu_k)$. This implies that $i_k \equiv i_l \,(\mathtt{mod}\ k)$ because k is the shift-period of μ_k. The uniqueness of the sequence is the only reason for including condition (a): If $j \in k$ and $\varphi \restriction \mathtt{Per}_k = s^j \restriction \mathtt{Per}_k$ for some $k \geq 1$, then $s^{i_k}(\mu_k) = s^j(\mu_k)$, which implies that $k \mid (i_k - j)$. Since $-k < i_k - j < k$, we have $j = i_k$.

For the second part, suppose $(i_k)_{k\geq 1}$ is a sequence of integers which satisfies (b). If $k, l \geq 1$ and $k \mid l$, condition (b) clearly implies that s^{i_k} and s^{i_l} agree on $\mathtt{Per}_k$. In the general case that $k, l \geq 1$ but k does not necessarily divide l, for all $e \in \mathtt{Per}_k \cap \mathtt{Per}_l$ we have $s^{i_k}(e) = s^{i_{kl}}(e) = s^{i_l}(e)$. Thus, (c) defines a unique map φ on $\bigcup_{k\geq 1} \mathtt{Per}_k = \mathtt{Per}$. It is easy to verify that φ is indeed an automorphism of $\mathtt{Per}$ and preserves the shift. □

Corollary 7.12. *The maps $s^m \restriction \mathtt{Per}$ with $m \in \mathbb{Z}$ are not the only automorphisms of* $\mathtt{Per}$ *which preserve the shift.*

Proof. Let $i_1 := 1$. Given i_k for some $k \geq 1$, let $i_{k+1} := i_k + (i_k!)$. This defines a strictly increasing sequence $(i_k)_{k\geq 1}$ in ω with the property that

$$\forall k, j \geq 1 \, (k \leq i_j \Rightarrow i_j \equiv i_{j+1} \,(\mathtt{mod}\ k)).$$

With a simple induction, together with the fact that $(i_k)_{k\geq 1}$ is increasing, we get

$$\forall k, j, l \geq 1 \, ((k \leq i_j \text{ and } j \leq l) \Rightarrow i_j \equiv i_l \,(\mathtt{mod}\ k)) \tag{7.13}$$

With another inductive argument one sees that $k \leq i_k$ for every $k \geq 1$. Applying the formula above with $j = k$ we get

$$\forall k, l \geq 1 \, (k \leq l \Rightarrow i_k \equiv i_l \,(\mathtt{mod}\ k))$$

which is even stronger than condition (b) from the previous lemma. Let φ be the shift-preserving automorphism of $\mathtt{Per}$ given by condition (c). For a contradiction, suppose there is $m \in \mathbb{Z}$ such that $\varphi = s^m \restriction \mathtt{Per}$. Since $i_3 = 4 \not\equiv 0 \,(\mathtt{mod}\ 3)$, we have $\varphi \neq \mathtt{id}_{\mathtt{Per}}$, and thus $m \neq 0$. Given $p \geq 1$, if we apply (7.13) with the triple $(k, j, l) := (i_p, p, i_p)$ we get:

$$i_p \equiv i_{i_p} \,(\mathtt{mod}\ i_p)$$

Therefore, $i_{i_p} \equiv 0 \,(\mathtt{mod}\ i_p)$ for all $p \geq 1$. Choose p large enough so that $i_p > |m| > 0$ and let $k := i_p$ once again. This implies that $k \nmid m$ and hence $s^m \restriction \mathtt{Per}_k \neq \mathtt{id}_{\mathtt{Per}_k}$. On the other hand, by condition (c), $\varphi \restriction \mathtt{Per}_k = s^{i_k} \restriction \mathtt{Per}_k = \mathtt{id}_{\mathtt{Per}_k}$ because $i_k \equiv 0 \,(\mathtt{mod}\ k)$, which contradicts $\varphi = s^m \restriction \mathtt{Per}$. □

Lemma 7.14 (Geschke)**.** *There is an automorphism of* $\mathtt{Per}$ *which inverts the shift.*

Proof. For each $k \geq 1$ consider the permutation ν_k of the atoms of $\mathtt{Per}_k$ given by

$$\nu_k(s^i(\mu_k)) := s^{-i}(\mu_k)$$

for each $i \in k$. It is easy to see that the equality then holds for all $i \in \mathbb{Z}$. The map ν_k can be uniquely extended to an automorphism of $\mathtt{Per}_k$. If $i \in k$, note that $\nu_k s(s^i(\mu_k)) = \nu_k(s^{i+1}(\mu_k)) = s^{-i-1}(\mu_k) = s^{-1}(s^{-i}(\mu_k)) = s^{-1}\nu_k(s^i(\mu_k))$. This shows that $\nu_k s = s^{-1}\nu_k$ on the set of atoms of $\mathtt{Per}_k$, and consequently ν_k inverts the shift on all of $\mathtt{Per}_k$.

If $k, l \geq 1$, we can write μ_k in terms of shifts of μ_{kl}: Note that $\mathbb{N} = \bigcup_{i \in l}(l\mathbb{N}+i)$, therefore $k\mathbb{N} = k(\bigcup_{i \in l}(l\mathbb{N}+i)) = \bigcup_{i \in l}(kl\mathbb{N}+ki)$. It follows that $\mu_k = \bigvee_{i \in l} s^{ki}(\mu_{kl})$. Thus,

$$\begin{aligned}
\nu_{kl}(\mu_k) &= \bigvee\{\nu_{kl}(s^{ki}(\mu_{kl})) \,:\, i \in l\} \\
&= \bigvee\{s^{-ki}(\mu_{kl}) \,:\, i \in l\} \\
&= \bigvee\{s^{kl-ki}(\mu_{kl}) \,:\, i \in l\} \\
&= s^{kl}(\mu_{kl}) \vee \bigvee\{s^{k(l-i)}(\mu_{kl}) \,:\, 0 < i < l\} \\
&= \mu_{kl} \vee \bigvee\{s^{ki}(\mu_{kl}) \,:\, 0 < i < l\} \\
&= \bigvee\{s^{ki}(\mu_{kl}) \,:\, i \in l\} = \mu_k
\end{aligned}$$

From this it easily follows that $\nu_{kl} \restriction \texttt{Per}_k = \nu_k$. Consequently, $\nu := \bigcup_{k \geq 1} \nu_k$ is an automorphism of $\texttt{Per}$ which inverts the shift. ☐

The construction in the proof above is particularly interesting because of its simplicity, and because the algebra $\texttt{Per}$ plays such an important role in the study of the shift in $\mathcal{P}(\omega)/\texttt{fin}$. However, the lemma itself will later become obsolete since we will develop a method to invert the shift on any countable shift-invariant subalgebra of $\mathcal{P}(\omega)/\texttt{fin}$. In the next lemma we continue to use the maps ν_k and ν defined above.

Corollary 7.15. *Let φ be an automorphism of $\texttt{Per}$ which inverts the shift. Then, there is a unique sequence $(i_k)_{k \geq 1}$ such that:*

(a) $\forall k \geq 1\, (i_k \in k)$

(b) $\forall k, l \geq 1\, (k \mid l \Rightarrow i_k \equiv i_l \,(\texttt{mod}\ k))$

(c') $\forall k \geq 1\, (\varphi \restriction \texttt{Per}_k = \nu_k s^{i_k} \restriction \texttt{Per}_k)$

Conversely, if $(i_k)_{k \geq 1}$ is a sequence of integers satisfying (b), then there is a unique shift-inverting automorphism φ of $\texttt{Per}$ satisfying (c').

Proof. Given φ which inverts the shift on $\texttt{Per}$, simply observe that $\nu^{-1}\varphi$ is a shift-preserving automorphism of $\texttt{Per}$ and then apply Lemma 7.11. Similarly, given a sequence of integers $(i_k)_{k \geq 1}$ satisfying (b), apply Lemma 7.11 to find an automorphism ψ of $\texttt{Per}$ such that $\psi \restriction \texttt{Per}_k = s^{i_k} \restriction \texttt{Per}_k$. Then, let $\varphi := \nu\psi$. ☐

Corollary 7.16. *If $\varphi \in \texttt{Aut}(\texttt{Per})$ inverts the shift, then $\varphi^2 = \texttt{id}_{\texttt{Per}}$.*

Proof. Let $(i_k)_{k \geq 1}$ be the unique sequence satisfying (a), (b) and (c') in the previous lemma. Then, for each $k \geq 1$ we have:

$$\begin{aligned}
\varphi^2 \restriction \texttt{Per}_k &= \nu_k s^{i_k} \nu_k s^{i_k} \restriction \texttt{Per}_k \\
&= \nu_k s^{i_k} s^{-i_k} \nu_k \\
&= \nu_k^2
\end{aligned}$$

It is easy to see that $\nu_k^2 = \texttt{id}_{\texttt{Per}_k}$. ☐

Automorphisms of Triv. In this section we show how Theorem 7.2 can be used to study the structure of the group Triv. The fact that the map $\mathtt{ind} : \mathtt{Triv} \to \mathbb{Z}$ is a surjective homomorphism gives us a lot of information: For example, we have already seen that VTriv is a normal subgroup as a consequence of it being precisely the kernel of this homomorphism. The following lemma is a basic result in group theory and can be found in any good textbook (see [Rot95], for example).

Lemma 7.17. *Let G_0 and G_1 be groups and $f : G_0 \to G_1$ a surjective homomorphism. Then, the map*

$$\{N : N \trianglelefteq G_1\} \to \{N' : \mathtt{Ker}(f) \subseteq N' \trianglelefteq G_0\}$$
$$N \mapsto f^{-1}[N]$$

is a bijection. □

We know that the subgroups of $\mathbb{Z}$ are precisely the sets $k\mathbb{Z}$ for $k \in \omega$, and since $\mathbb{Z}$ is abelian, all its subgroups are normal.

Corollary 7.18. *The sets $\mathtt{ind}^{-1}[k\mathbb{Z}]$ for $k \in \omega$ are precisely the normal subgroups of* Triv *which contain* VTriv. □

Our next goal is to strengthen this corollary by showing that every normal subgroup of Triv other than $\{\mathtt{id}\}$ in fact contains VTriv. This was first proven by van Douwen in [vD90], but our proof here will be different, particularly because of our focus on the shift.

Lemma 7.19. *If $N \trianglelefteq \mathtt{Triv}$ and $N \subseteq \langle s \rangle$, then $N = \{\mathtt{id}\}$.*

Proof. Suppose, for a contradiction, that $N \neq \{\mathtt{id}\}$, in which case there is $m > 0$ such that $s^m \in N$. Take $k > m$ and consider the permutation $\sigma \in \mathtt{Sym}(\omega)$ given by

$$\sigma(n) := \begin{cases} n+1 & \text{if } n \not\equiv -1 \pmod{k} \\ n+1-k & \text{if } n \equiv -1 \pmod{k} \end{cases}$$

We shall see that $\varphi_\sigma s^m \varphi_\sigma^{-1} \notin \langle s \rangle$, contradicting the fact that N is normal and completing the proof. To see this, observe that if $\varphi_\sigma s^m \varphi_\sigma^{-1}$ were some power of s, it would have to be s^m because of its index. We have

$$\begin{aligned} \varphi_\sigma s^m \varphi_\sigma^{-1}(\mu_{2k}) &= \varphi_\sigma s^m \varphi_\sigma^{-1}(\llbracket 2k\mathbb{N} \rrbracket) \\ &= \varphi_\sigma s^m(\llbracket 2k\mathbb{N} + k - 1 \rrbracket) \\ &= \varphi_\sigma(\llbracket 2k\mathbb{N} + k - 1 + m \rrbracket) \end{aligned}$$

Since $0 < m < k$, for all $n \in \omega$ it holds that $2kn + k - 1 + m \not\equiv -1 \pmod{k}$. Therefore, $\varphi_\sigma s^m \varphi_\sigma^{-1}(\mu_{2k}) = \llbracket 2k\mathbb{N} + k + m \rrbracket = s^{k+m}(\mu_{2k})$. Finally, we have $s^{k+m}(\mu_{2k}) \neq s^m(\mu_{2k})$ because $k + m \not\equiv m \pmod{2k}$. □

Lemma 7.20. *If $N \trianglelefteq \mathtt{Triv}$ and $N \neq \{\mathtt{id}\}$, then $\mathtt{VTriv} \subseteq N$.*

Proof. By the previous lemma, if $N \trianglelefteq \mathtt{Triv}$ and $N \neq \{\mathtt{id}\}$, then there is $\varphi \in N \setminus \langle s \rangle$. We know from Theorem 7.2 that $\varphi s \neq s \varphi$. So if $\psi := s\varphi s^{-1}$, we have $\psi \neq \varphi$, $\mathtt{ind}(\psi) = \mathtt{ind}(\varphi)$ and $\psi \in N$ (because N is normal). Consequently, $\varphi\psi^{-1} \in N \cap \mathtt{VTriv}$ and $\varphi\psi^{-1} \neq \mathtt{id}$. The equality $N \cap \mathtt{VTriv} = \mathtt{VTriv}$ now follows from the fact that VTriv is simple, which was already pointed out in CHAPTER 6. □

Corollary 7.21 (van Douwen)**.** *The sets $\mathtt{ind}^{-1}[k\mathbb{Z}]$ for $k \in \omega$, plus $\{\mathtt{id}\}$, are all the normal subgroups of* Triv. *Moreover, this list has no repetitions.* □

Given $\Phi \in \mathtt{Aut}(\mathtt{Triv})$, let $\delta_\Phi := -\mathtt{ind}(\Phi(s))$.

Theorem 7.22. *If Φ is an automorphism of* Triv*, then $\delta_\Phi = \pm 1$, and for all $\psi \in \mathtt{Triv}$ it holds that $\mathtt{ind}(\Phi(\psi)) = \delta_\Phi \cdot \mathtt{ind}(\psi)$.*

Proof. By Lemma 7.20, we have $\mathtt{VTriv} = \bigcap\{N \trianglelefteq \mathtt{Triv} : N \neq \{\mathtt{id}\}\}$. Thus, VTriv can be "recovered" from the group structure of Triv, and it follows that $\Phi[\mathtt{VTriv}] = \mathtt{VTriv}$. Given $\psi \in \mathtt{Triv}$, we can write $\psi = \psi_0 s^{-\mathtt{ind}(\psi)}$ where $\psi_0 \in \mathtt{VTriv}$ and then:

$$\begin{aligned}\mathtt{ind}(\Phi(\psi)) &= \mathtt{ind}(\Phi(\psi_0)\Phi(s)^{-\mathtt{ind}(\psi)})\\ &= \mathtt{ind}(\Phi(\psi_0)) - \mathtt{ind}(\psi) \cdot \mathtt{ind}(\Phi(s))\\ &= 0 + (-\mathtt{ind}(\Phi(s))) \cdot \mathtt{ind}(\psi)\\ &= \delta_\Phi \cdot \mathtt{ind}(\psi)\end{aligned}$$

Moreover, we have $-1 = \mathtt{ind}(\Phi(\Phi^{-1}(s))) = \delta_\Phi \cdot \mathtt{ind}(\Phi^{-1}(s))$, so $-1 \in \delta_\Phi\mathbb{Z}$. This proves that $\delta_\Phi = \pm 1$. □

Similarly, if Φ is an automorphism of the Calkin algebra, then there is $\delta_\Phi \in \{-1, 1\}$ such that for every invertible element $u \in \mathcal{C}(l^2)$ we have $\mathtt{F\text{-}ind}(\Phi(u)) = \delta_\Phi \cdot \mathtt{F\text{-}ind}(u)$ (see [BDF73]). However, while it is obvious that the group of invertible elements is invariant under every automorphism of the Calkin algebra, the corresponding statement with respect to $\mathcal{P}(\omega)/\mathtt{fin}$ is consistently false, as shown below. The following result was proven by Fuchino in his doctoral thesis [Fuc88] (an alternative proof can be found in [ŠR89]):

Theorem 7.23 (Fuchino)**.** *CH implies that the group $\mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ is simple.* □

In particular, since we know that Triv is not simple, this result gives an alternative proof of the existence of non-trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ under CH.

Corollary 7.24. *CH implies that there is an automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $C(\varphi)[\mathtt{Triv}] \nsubseteq \mathtt{Triv}$.* □

Consider the vocabulary $L := \{0, 1, \vee, \wedge, \neg, \leq, f\}$ which consists of the usual symbols for Boolean algebras together with a unary function symbol f.

Corollary 7.25. *CH implies that there are a trivial automorphism ψ and a non-trivial automorphism ψ' of $\mathcal{P}(\omega)/\mathtt{fin}$ such that the L-structures $(\mathcal{P}(\omega)/\mathtt{fin}, \psi)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, \psi')$ are isomorphic. In particular, under CH there is no first-order L-theory T such that for every automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ we have $(\mathcal{P}(\omega)/\mathtt{fin}, \varphi) \models T$ if and only if φ is trivial.*

Proof. The second part clearly follows from the first, since the structures $(\mathcal{P}(\omega)/\mathtt{fin}, \psi)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, \psi')$ would satisfy precisely the same first-order sentences. To prove the first statement, assume CH. The previous corollary implies that there are $\varphi \in \mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ and $\psi \in \mathtt{Triv}$ such that $\psi' := C(\varphi)(\psi) \notin \mathtt{Triv}$. Since φ is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi\psi = \psi'\varphi$, it follows that φ is also an isomorphism between the structures $(\mathcal{P}(\omega)/\mathtt{fin}, \psi)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, \psi')$. □

Several questions remain open in the current topic. For example, is there $\Phi \in \mathtt{Aut}(\mathtt{Triv})$ such that $\delta_\Phi = -1$? Or the more specific question: Is there $\varphi \in \mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ such that $C(\varphi)[\mathtt{Triv}] = \mathtt{Triv}$ and $\delta_{C(\varphi)\restriction\mathtt{Triv}} = -1$? This second question obviously has a negative answer in models where all automorphisms are trivial. The corresponding question for the Calkin algebra (asked by Brown, Douglas and Fillmore in [BDF73]) is also open, namely: Is there $\Phi \in \mathtt{Aut}(\mathcal{C}(l^2))$ such that for all invertible elements $u \in \mathcal{C}(l^2)$ we have $\mathtt{F\text{-}ind}(\Phi(u)) = -\mathtt{F\text{-}ind}(u)$? This question obviously has a negative answer in models where all automorphisms are inner. Finally, is there a model of ZFC in which non-trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ exist and $\mathtt{Triv}$ is a normal subgroup of $\mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$?

We finish this chapter with an application of Theorem 7.23. The method we saw in the proof of Theorem 6.14 (i.e. extending countable isomorphisms one element at a time using finite approximations for the image of the new element) is enough to show the existence of non-trivial automorphisms, but very difficult to adapt to construct non-trivial automorphisms with specific desired properties. Even simple conditions such as $\varphi^2 = \mathtt{id}$ are very hard to obtain using that construction. The method used below is easily adaptable.

Lemma 7.26. *CH implies the existence of a non-trivial automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\varphi^2 = \mathtt{id}$.*

Proof. Assume the CH. Theorem 7.23 implies that $\mathtt{VTriv}$ is not normal in $\mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$, hence there is an automorphism φ_0 of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $C(\varphi_0)[\mathtt{VTriv}] \nsubseteq \mathtt{VTriv}$. Let X be the set of all $\psi \in \mathtt{VTriv}$ such that $\psi^2 = \mathtt{id}$. Note that for all $\eta \in \mathtt{VTriv}$ it holds that $\eta X \eta^{-1} \subseteq X$, therefore $\langle X \rangle$ is a normal subgroup of $\mathtt{VTriv}$. Since $\mathtt{VTriv}$ is simple we conclude that $\langle X \rangle = \mathtt{VTriv}$. Hence, $C(\varphi_0)[\langle X \rangle] \nsubseteq \mathtt{VTriv}$, which clearly implies that $C(\varphi_0)[X] \nsubseteq \mathtt{VTriv}$. Take $\psi \in X$ such that $\varphi := C(\varphi_0)(\psi) \notin \mathtt{VTriv}$. Clearly, $\varphi^2 = \mathtt{id}$ as desired. Note that if φ were trivial we would have $0 = \mathtt{ind}(\varphi^2) = 2 \cdot \mathtt{ind}(\varphi)$, and consequently $\mathtt{ind}(\varphi) = 0$, contradicting the fact that $\varphi \notin \mathtt{VTriv}$. □

8 Gap sequences

One important common property of automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ which preserve the shift and automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ which invert the shift is that they all induce permutations on $\mathtt{Orb}$ (the set of all shift-orbits). In the current chapter we are interested (implicitly) in subsets of $\mathtt{Orb}$ which are invariant under all such permutations. Explicitly, we give several examples of properties P, applicable to elements of $\mathcal{P}(\omega)/\mathtt{fin}$, with the following two conditions: First, if $e \in \mathcal{P}(\omega)/\mathtt{fin}$ satisfies P, then so does every element of its shift-orbit. Second, there is an analogous property P' such that if $e \in \mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi \in \mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ either preserves or inverts the shift, then e satisfies P if and only if $\varphi(e)$ satisfies P'.

Definition 8.1. Given an infinite subset x of ω, there is a unique strictly increasing enumeration $(x_n)_{n\in\omega}$ of x. The *gap sequence* of x, denoted $\mathtt{GS}(x)$, is the sequence $(x_{n+1} - x_n)_{n\in\omega}$.

Definition 8.2. A *final segment* of a sequence $(\alpha_n)_{n\in\omega}$ is a sequence of the form $(\alpha_{n_0+n})_{n\in\omega}$ for some $n_0 \in \omega$. We define an equivalence relation on the set $(\omega \setminus \{0\})^\omega$ as follows: Two sequences $\alpha, \beta \in (\omega \setminus \{0\})^\omega$ are called equivalent, denoted $\alpha \approx \beta$, if they have a common final segment (that is, if there are $n_0, n_1 \in \omega$ such that $\alpha_{n_0+n} = \beta_{n_1+n}$ for every $n \in \omega$).

If $x, y \subseteq \omega$ are infinite and $x =^* y$, then $\mathtt{GS}(x) \approx \mathtt{GS}(y)$, which justifies the following:

Definition 8.3. The *gap sequence* of an element $e \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$, denoted $\mathtt{gs}(e)$, is the equivalence class of $\mathtt{GS}(x)$ in $(\omega \setminus \{0\})^\omega$ for any representative x of e.

Clearly, $\mathtt{GS}$ maps the collection of infinite subsets of ω onto the collection of all sequences in $\omega \setminus \{0\}$, and $\mathtt{gs}$ maps $(\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ onto $(\omega \setminus \{0\})^\omega / \approx$.

Lemma 8.4. *Let $e_0, e_1 \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$. Then, e_0 and e_1 are in the same shift-orbit if and only if $\mathtt{gs}(e_0) = \mathtt{gs}(e_1)$.*

Proof. If $e_1 = s^m(e_0)$ for some $m \in \mathbb{Z}$, let x be a representative of e_0 and observe that $S^m[x]$ is a representative of e_1. We may assume $m \geq 0$ (otherwise we switch the roles of e_0 and e_1). It follows that $\mathtt{GS}(x) = \mathtt{GS}(S^m[x])$, and consequently $\mathtt{gs}(e_0) = \mathtt{gs}(e_1)$.

On the other hand, if $\mathtt{gs}(e_0) = \mathtt{gs}(e_1)$, let x and y be representatives of e_0 and e_1 respectively, and let $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ respectively be their increasing enumerations. Since $\mathtt{GS}(x) \approx \mathtt{GS}(y)$, there are $n_0, n_1 \in \omega$ such that $x_{n_0+n+1} - x_{n_0+n} = y_{n_1+n+1} - y_{n_1+n}$ for all $n \in \omega$. Let $m := y_{n_1} - x_{n_0}$, and let us assume that $m \geq 0$, otherwise we switch the roles of e_0 and e_1. By an easy inductive argument we see that $y_{n_1+n} = x_{n_0+n} + m$ for all $n \in \omega$, which shows that $S^m[x] =^* y$, and consequently $s^m(e_0) = e_1$ as promised. □

Definition 8.5. A *finite segment* of a sequence $(\alpha_n)_{n\in\omega}$ is a finite sequence $(\beta_0, \ldots, \beta_p)$ such that $(\alpha_{n_0}, \ldots, \alpha_{n_0+p}) = (\beta_0, \ldots, \beta_p)$ for some $n_0 \in \omega$. Each index n_0 with this property is an *occurrence of* $(\beta_0, \ldots, \beta_p)$ *in* $(\alpha_n)_{n\in\omega}$.

The empty sequence, i.e. the sequence of length 0 will not be considered throughout this chapter. If $\eta_0 = (k_0, \ldots, k_p)$ and $\eta_1 = (l_0, \ldots, l_q)$ are finite sequences, we use the notation $\eta_0{}^\frown\eta_1$ to denote the sequence $(k_0, \ldots, k_p, l_0, \ldots, l_q)$. Also, a sequence (k) of length 1 is usually identified with its unique entry k.

Definition 8.6. For $e \in \mathcal{P}(\omega)/\mathtt{fin}$ and a finite sequence η in $\omega \setminus \{0\}$, we define an element $\mathtt{gap}(e;\eta) \in \mathcal{P}(\omega)/\mathtt{fin}$ by induction on the length of η as follows: Given $k \geq 1$ let

$$\mathtt{gap}(e;k) := e \wedge s^{-k}(e) \wedge \bigwedge_{i=1}^{k-1} \neg s^{-i}(e).$$

If $\eta = (k_0, \ldots, k_p)$ and $k_{p+1} \geq 1$, let

$$\mathtt{gap}(e;\eta^\frown k_{p+1}) := \mathtt{gap}(e;\eta) \wedge s^{-(k_0+\cdots+k_p)}(\mathtt{gap}(e;k_{p+1})).$$

Lemma 8.7. *Suppose $e \in \mathcal{P}(\omega)/\mathtt{fin}$ is non-zero, $x \subseteq \omega$ is a representative of e with increasing enumeration $(x_n)_{n\in\omega}$ and η is a finite sequence in $\omega \setminus \{0\}$. Let*

$$y := \{x_n : n \textit{ is an occurrence of } \eta \textit{ in } \mathtt{GS}(x)\}.$$

Then, $[\![y]\!] = \mathtt{gap}(e;\eta)$.

Proof. We proceed by induction on the length of η. If $\eta = (k)$ for some $k \geq 1$, let

$$z := x \cap S^{-k}[x] \cap \bigcap_{i=1}^{k-1} (\omega \setminus S^{-i}[x])$$

so that we have $\mathtt{gap}(e;k) = [\![z]\!]$. If n is an occurrence of k in $\mathtt{GS}(x)$, we have $x_{n+1} = x_n + k$. Thus, $x_n \in x \cap S^{-k}[x]$. Since the chosen enumeration of x is strictly increasing, we know that $x_n + i \notin x$ (that is, $x_n \notin S^{-i}[x]$) for $i = 1, \ldots, k-1$. This shows that $x_n \in z$, and since n is an arbitrary occurrence of k in $\mathtt{GS}(x)$, we have $y \subseteq z$. Similar reasoning shows that if $x_n \in z$, then the smallest element of x greater than x_n, namely x_{n+1}, is precisely $x_n + k$, and therefore n is an occurrence of k in $\mathtt{GS}(x)$. Consequently, $y = z$, which concludes the proof in the case that the length of η is 1.

For the induction step, let $\eta = (k_0, \ldots, k_p)$ and suppose $k_{p+1} \geq 1$. Moreover, let

$$y' := \{x_n : n \text{ is an occurrence of } k_{p+1} \text{ in } \mathtt{GS}(x)\}$$
$$\text{and } y'' := \{x_n : n \text{ is an occurrence of } \eta^\frown k_{p+1} \text{ in } \mathtt{GS}(x)\}.$$

Clearly, $n \in \omega$ is an occurrence of $\eta^\frown k_{p+1}$ in $\mathtt{GS}(x)$ if and only if it is an occurrence of η and $n+p+1$ is an occurrence of k_{p+1}. In other words, $x_n \in y''$ if and only if $x_n \in y$ and $x_{n+p+1} \in y'$. Also, if $x_n \in y$ we have $x_{n+p+1} = x_n + k_0 + \cdots + k_p$. We conclude that

$$y'' = y \cap S^{-(k_0+\cdots+k_p)}[y']$$

and from the induction hypothesis we get

$$\begin{aligned}[\![y'']\!] &= [\![y]\!] \wedge s^{-(k_0+\cdots+k_p)}([\![y']\!]) \\ &= \mathtt{gap}(e;\eta) \wedge s^{-(k_0+\cdots+k_p)}(\mathtt{gap}(e;k_{p+1})) \\ &= \mathtt{gap}(e;\eta^\frown k_{p+1})\end{aligned}$$

as we wanted to show. □

Lemma 8.8. *If $\eta_0 = (k_0,\ldots,k_p)$ and η_1 are finite sequences in $\omega \setminus \{0\}$, then*

$$\mathtt{gap}(e;\eta_0{}^\frown\eta_1) = \mathtt{gap}(e;\eta_0) \wedge s^{-(k_0+\cdots+k_p)}(\mathtt{gap}(e;\eta_1))$$

for every $e \in \mathcal{P}(\omega)/\mathtt{fin}$.

Proof. The proof can be carried out directly from the definitions by induction on the length of η_1, or as an easy corollary of the previous lemma. Either way it is straight-forward. □

Definition 8.9. Given a finite sequence η in $\omega\setminus\{0\}$, we denote by $\mathtt{UNB}_\eta$ the set of all infinite sets $x \subseteq \omega$ such that η is a finite segment of $\mathtt{GS}(x)$ occurring infinitely often. Moreover, we let $\mathtt{Unb}_\eta := \{[\![x]\!] : x \in \mathtt{UNB}_\eta\}$.

The notation UNB stands for "unbounded" (in the sense that the set of occurrences of the given finite segment is unbounded).

Lemma 8.10. *If η is a finite sequence in $\omega \setminus \{0\}$, then*

$$\mathtt{Unb}_\eta = \{e \in \mathcal{P}(\omega)/\mathtt{fin} : \mathtt{gap}(e;\eta) \neq 0\}.$$

Proof. Follows easily from Lemma 8.7 together with the observation that $\mathtt{gap}(0;\eta) = 0$. □

Corollary 8.11. *Let $\mathcal{B}$ be a shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and suppose φ is an automorphism of $\mathcal{B}$ which preserves the shift. Then, for every finite sequence η in $\omega \setminus \{0\}$ we have $\varphi[\mathtt{Unb}_\eta \cap \mathcal{B}] = \mathtt{Unb}_\eta \cap \mathcal{B}$.*

Proof. Since $\mathcal{B}$ is shift-invariant, for every $k \geq 1$ and every $e \in \mathcal{B}$ we have $\mathtt{gap}(e;k) \in \mathcal{B}$. (In fact, the same holds if k is replaced by a finite sequence in $\omega \setminus \{0\}$.) Using this it is easy to prove the equality $\varphi(\mathtt{gap}(e;\eta)) = \mathtt{gap}(\varphi(e);\eta)$ for every $e \in \mathcal{B}$ by induction on the length of η. Combined with this equality the previous lemma completes the proof. □

For the next corollary, given a finite sequence $\eta = (k_0,\ldots,k_p)$, let us denote by η^* the reversed sequence $(k_p,k_{p-1},\ldots,k_0)$.

Corollary 8.12. *Let $\mathcal{B}$ be a shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and suppose φ is an automorphism of $\mathcal{B}$ which inverts the shift. Then, for every finite sequence η in $\omega \setminus \{0\}$ we have $\varphi[\mathtt{Unb}_\eta \cap \mathcal{B}] = \mathtt{Unb}_{\eta^*} \cap \mathcal{B}$.*

Proof. We shall prove that if $\eta = (k_0, \ldots, k_p)$, then for every $e \in \mathcal{B}$ we have $\varphi(\mathtt{gap}(e;\eta)) = s^{k_0+\cdots+k_p}(\mathtt{gap}(\varphi(e);\eta^*))$. The corollary clearly follows from this equality.

If $p = 0$ we have $\eta = \eta^* = k_0$ and

$$\begin{aligned}
\varphi(\mathtt{gap}(e;k_0)) &= \varphi\left(e \wedge s^{-k_0}(e) \wedge \bigwedge_{i=1}^{k_0-1} \neg s^{-i}(e)\right) \\
&= \varphi(e) \wedge s^{k_0}(\varphi(e)) \wedge \bigwedge_{i=1}^{k_0-1} \neg s^{i}(\varphi(e)) \\
&= s^{k_0}(\varphi(e)) \wedge \varphi(e) \wedge \bigwedge_{i=1}^{k_0-1} \neg s^{k_0-i}(\varphi(e)) \\
&= s^{k_0}\left(\varphi(e) \wedge s^{-k_0}(\varphi(e)) \wedge \bigwedge_{i=1}^{k_0-1} \neg s^{-i}(\varphi(e))\right) \\
&= s^{k_0}(\mathtt{gap}(\varphi(e);k_0))
\end{aligned}$$

as promised. By induction, suppose the equality holds for $\eta = (k_0, \ldots, k_p)$ and let $k_{p+1} \geq 1$. Then,

$$\begin{aligned}
\varphi(\mathtt{gap}(e;\eta^\frown k_{p+1})) &= \varphi(\mathtt{gap}(e;\eta) \wedge s^{-(k_0+\cdots+k_p)}(\mathtt{gap}(e;k_{p+1}))) \\
&= \varphi(\mathtt{gap}(e;\eta)) \wedge s^{k_0+\cdots+k_p}(\varphi(\mathtt{gap}(e;k_{p+1}))) \\
&= s^{k_0+\cdots+k_p}(\mathtt{gap}(\varphi(e);\eta^*)) \wedge s^{k_0+\cdots+k_{p+1}}(\mathtt{gap}(\varphi(e);k_{p+1})) \\
&= s^{k_0+\cdots+k_{p+1}}(\mathtt{gap}(\varphi(e);k_{p+1}) \wedge s^{-k_{p+1}}(\mathtt{gap}(\varphi(e);\eta^*))).
\end{aligned}$$

By Lemma 8.8 we have $\mathtt{gap}(\varphi(e);k_{p+1}) \wedge s^{-k_{p+1}}(\mathtt{gap}(\varphi(e);\eta^*)) = \mathtt{gap}(\varphi(e);k_{p+1}{}^\frown(\eta^*))$. Since $k_{p+1}{}^\frown(\eta^*) = (\eta^\frown k_{p+1})^*$, we obtain

$$\varphi(\mathtt{gap}(e;\eta^\frown k_{p+1})) = s^{k_0+\cdots+k_{p+1}}(\mathtt{gap}(\varphi(e);(\eta^\frown k_{p+1})^*))$$

which concludes the induction step. □

Definition 8.13. Let $\mathtt{PER}$ be the set of all infinite $x \subseteq \omega$ such that $\mathtt{GS}(x)$ is eventually periodic, that is, if $(x_n)_{n\in\omega}$ is the increasing enumeration of x, there exists some $n_0 \in \omega$ and some $q \geq 1$ such that $x_{n+1} - x_n = x_{n+q+1} - x_{n+q}$ for every $n \geq n_0$.

Lemma 8.14. *Let* $\mathtt{Per}$ *be the algebra of shift-periodic elements as usual. Then,*

$$\mathtt{Per} = \{[\![x]\!] : x \in \mathtt{PER}\} \cup \{0\}.$$

Proof. Let $e \in \mathtt{Per} \setminus \{0\}$. Then, there is $k \geq 1$ such that $e \in \mathtt{Per}_k$, and there are unique $i_0, \ldots, i_p$ such that $0 \leq i_0 < \cdots < i_p < k$ and $e = s^{i_0}(\mu_k) \vee \cdots \vee s^{i_p}(\mu_k)$. Clearly,

$$x := (k\mathbb{N} + i_0) \cup \cdots \cup (k\mathbb{N} + i_p)$$

is a representative of e. If we let $q := p + 1$, then the increasing enumeration $(x_n)_{n\in\omega}$ of x satisfies $x_{mq+j} = mk + i_j$ for each $m \in \omega$ and $j \in q$. Given $n \in \omega$, let $m \in \omega$ and $j \in q$ be such that $n = mq + j$ and it follows that

$$\begin{aligned}
x_{n+q} &= x_{(m+1)q+j} \\
&= (m+1)k + i_j \\
&= (mk + i_j) + k \\
&= x_{mq+j} + k \\
&= x_n + k
\end{aligned}$$

Therefore, for all $n \in \omega$ we have $x_{n+1} - x_n = (x_{n+1} + k) - (x_n + k) = x_{n+q+1} - x_{n+q}$, proving that $x \in \mathtt{PER}$.

On the other hand, given $x \in \mathtt{PER}$ let $(x_n)_{n \in \omega}$ be its increasing enumeration, and let $q \geq 1$ and $n_0 \in \omega$ be such that $x_{n+1} - x_n = x_{n+q+1} - x_{n+q}$ for all $n \geq n_0$. It holds that $x_{n+q} - x_n = x_{(n+1)+q} - x_{n+1}$ for all $n \geq n_0$. By induction, we obtain the equality $x_{n+q} - x_n = x_{n_0+q} - x_{n_0}$ for all $n \geq n_0$. So let $k := x_{n_0+q} - x_{n_0} \geq 1$ and observe that

$$\begin{aligned} S^k[x] &=^* \{x_n + k : n \geq n_0\} \\ &= \{x_{n+q} : n \geq n_0\} \\ &= \{x_n : n \geq n_0 + q\} \\ &=^* x \end{aligned}$$

This shows that $[\![x]\!] \in \mathtt{Per}_k \subseteq \mathtt{Per}$. □

Definition 8.15. Let $\mathtt{CON}$ be the set of all infinite $x \subseteq \omega$ such that $\mathtt{GS}(x)$ is eventually constant, and let $\mathtt{Con} := \{[\![x]\!] : x \in \mathtt{CON}\}$.

Note that if $x \subseteq \omega$, then $x \in \mathtt{CON}$ if and only if $x \in \mathtt{PER}$ and we can choose $q = 1$ in Definition 8.13. The proof of the following lemma is straight-forward (using Lemma 8.7 for the second part).

Lemma 8.16. *The set* $\mathtt{Con}$ *is precisely the collection of all atoms of the algebras* $\mathtt{Per}_k$ *for* $k \geq 1$*. Alternatively, it is the collection of all* $e \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ *such that* $\mathtt{gap}(e;k) = e$ *for some* $k \geq 1$*.* □

Corollary 8.17. *Let* $\mathcal{B}$ *be a shift-invariant subalgebra of* $\mathcal{P}(\omega)/\mathtt{fin}$ *and suppose* φ *is an automorphism of* $\mathcal{B}$ *which preserves or inverts the shift. Then,* $\varphi[\mathtt{Con} \cap \mathcal{B}] = \mathtt{Con} \cap \mathcal{B}$*.* □

Many similar definitions could be made, and similar lemmata would follow, but these were only included here as examples of how the concept of gap sequences can be applied. The next definition is the one that turned out to be the most useful.

Definition 8.18. Let $\mathtt{DIV}$ be the set of all infinite $x \subseteq \omega$ such that $\mathtt{GS}(x)$ diverges to infinity. Let $\mathtt{Div} := \{[\![x]\!] : x \in \mathtt{DIV}\}$.

Lemma 8.19. *The set* $\mathtt{Div}$ *consists of all* $e \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ *such that* $\mathtt{gap}(e;k) = 0$ *for all* $k \geq 1$*. Alternatively, it is precisely the collection of all* $e \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ *such that* $s^m(e) \wedge s^n(e) = 0$ *for all* $m, n \in \mathbb{Z}$ *distinct.*

Proof. It is easy to see that a sequence of natural numbers diverges to infinity if and only if each natural number occurs only finitely many times in it. This being considered, the first part is a corollary of Lemma 8.10.

For the second part, if $e \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ and $s^m(e) \wedge s^n(e) = 0$ for all distinct integers m and n, it is clear (from the definition) that $\mathtt{gap}(e;k) = 0$ for every $k \geq 1$. So the first part shows that $e \in \mathtt{Div}$. On the other hand, if there are distinct integers m and n such that $s^m(e) \wedge s^n(e) \neq 0$, then there is some $k \geq 1$ such that $e \wedge s^{-k}(e) \neq 0$. Assume we have chosen k minimal with this property. Then, for $i = 1, \ldots, k-1$ we have $e \wedge s^{-i}(e) = 0$, and so $e \leq \neg s^{-i}(e)$. It follows that $\mathtt{gap}(e;k) = e \wedge s^{-k}(e) \neq 0$, which implies that $e \notin \mathtt{Div}$. □

Corollary 8.20. *Let* $\mathcal{B}$ *be a shift-invariant subalgebra of* $\mathcal{P}(\omega)/\mathtt{fin}$ *and suppose* φ *is an automorphism of* $\mathcal{B}$ *which preserves or inverts the shift. Then,* $\varphi[\mathtt{Div} \cap \mathcal{B}] = \mathtt{Div} \cap \mathcal{B}$*.* □

Lemma 8.21. $\mathtt{Div}$ *is dense in* $(\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$*.*

Proof. Given $e \in \mathcal{P}(\omega)/\mathtt{fin}$, $e \neq 0$, let $x \subseteq \omega$ be one of its representatives, and let $(x_n)_{n\in\omega}$ be the increasing enumeration of x. For all $m, n \in \omega$ we have $x_{m+n} \geq x_m + n$ (which can be checked by induction on n). Therefore, for every $n \in \omega$ we have $x_{2^{n+1}} - x_{2^n} \geq 2^n$, showing that $y := \{x_{2^n} : n \in \omega\}$ is in $\mathtt{DIV}$. Clearly, $[\![y]\!] < e$. □

It is also worth noticing that $\mathtt{Div}$ is closed downwards in $(\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$, that is, whenever $0 < e \leq e'$ and $e' \in \mathtt{Div}$, we have $e \in \mathtt{Div}$.

Lemma 8.22. *Suppose $e \leq e_1 \vee \cdots \vee e_q$, where $e_1, \ldots, e_q \in \mathtt{Div}$. Then, for every finite sequence η in $\omega \setminus \{0\}$ of length at least q we have $\mathtt{gap}(e; \eta) = 0$.*

Proof. The claim is obvious for $e = 0$, so suppose $e \neq 0$, let $x \subseteq \omega$ be one of its representatives and let $(x_n)_{n\in\omega}$ be the increasing enumeration of x. Furthermore, for each $i = 1, \ldots, q$ let $y_i \subseteq \omega$ be a representative of e_i.

Suppose $\eta = (k_0, \ldots, k_p)$ is a finite sequence in $\omega \setminus \{0\}$ of length $p + 1 \geq q$ and let $k := k_0 + \cdots + k_p$. For each $i \in \{1, \ldots, q\}$, since $\mathtt{GS}(y_i)$ diverges to infinity, it follows that for all $m \in \omega$ large enough $|\{m, m+1, \ldots, m+k\} \cap y_i| \leq 1$. We have $x \subseteq^* y_1 \cup \cdots \cup y_q$, so we can find $M \in \omega$ such that $|\{m, m+1, \ldots, m+k\} \cap x| \leq q$ for all $m \geq M$. If $n \in \omega$ is an occurrence of η in $\mathtt{GS}(x)$, then $\{x_n, x_n + 1, \ldots, x_n + k\} \cap x = \{x_n, x_{n+1}, \ldots, x_{n+p+1}\}$ (which has cardinality $p + 2 > q$), so we have $x_n < M$. By Lemma 8.7, this proves that $\mathtt{gap}(e; \eta) = 0$. □

Corollary 8.23. *No disjunction of elements of* $\mathtt{Div}$ *is equal to* 1.

Proof. Suppose, for a contradiction, that $e_1, \ldots, e_q \in \mathtt{Div}$ are such that $e_1 \vee \cdots \vee e_q = 1$. Let $e := 1$ and let $\eta := (1, \ldots, 1)$ with length q. An application of Lemma 8.7 shows that $\mathtt{gap}(e; \eta) = 1$, contradicting the previous lemma. □

Theorem 8.24. *The algebra $\langle\mathtt{Div}\rangle$ is shift-invariant and atomless, and its cardinality is $2^{\aleph_0}$. Moreover, we have $\langle\mathtt{Div}\rangle \cap \mathtt{Per} = \{0, 1\}$ and $\langle\mathtt{Div} \cup \mathtt{Per}\rangle \neq \mathcal{P}(\omega)/\mathtt{fin}$.*

Proof. Corollary 8.20 applied to $\mathcal{B} := \mathcal{P}(\omega)/\mathtt{fin}$ and $\varphi := s$ shows that $s[\mathtt{Div}] = \mathtt{Div}$. This implies that $\langle\mathtt{Div}\rangle$ is shift-invariant. Lemma 8.21 clearly shows that every subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ containing $\mathtt{Div}$ is atomless. For the cardinality, note that every strictly increasing sequence in $\omega \setminus \{0\}$ is the gap sequence of some element of $\mathtt{DIV}$. Since there are $2^{\aleph_0}$ such sequences and each equivalence class modulo finite is countable, it follows that $\mathtt{Div}$ has cardinality $2^{\aleph_0}$ (and consequently so does $\langle\mathtt{Div}\rangle$).

Let us show that $\langle\mathtt{Div}\rangle \cap \mathtt{Per} = \{0, 1\}$. Let X be the set of all e in $\mathcal{P}(\omega)/\mathtt{fin}$ such that there exist $e_1, \ldots, e_q \in \mathtt{Div}$ satisfying $e \leq e_1 \vee \cdots \vee e_q$. If $e \in \langle\mathtt{Div}\rangle$ and $e \notin X$, we claim that $\neg e \in X$. To see this, we use a disjunctive normal form and write $e = \alpha_1 \vee \cdots \vee \alpha_t$, where for each i we have $\alpha_i = \beta_i \wedge \gamma_i$, such that $\beta_i = b_{i,1} \wedge \cdots \wedge b_{i,u_i}$ and $\gamma_i = \neg c_{i,1} \wedge \cdots \wedge \neg c_{i,v_i}$ for certain $b_{i,1}, \ldots, b_{i,u_i}, c_{i,1}, \ldots, c_{i,v_i}$ in $\mathtt{Div}$. We require that $t \geq 1$ because $e \neq 0$, but for each i we might have $u_i = 0$ (so that $\beta_i = 1$) or $v_i = 0$ (so that $\gamma_i = 1$). Note that it is not possible to have $u_i \neq 0$ for all i, otherwise it would follow that $e \leq b_{1,1} \vee \cdots \vee b_{t,1}$ (contradicting the fact that $e \notin X$). Thus, we can choose $j \in \{1, \ldots, t\}$ such that $u_j = 0$. It follows that $e \geq \gamma_j$ and so $\neg e \leq c_{j,1} \vee \cdots \vee c_{j,v_j}$, proving that $\neg e \in X$.

Take $e \in \langle\mathtt{Div}\rangle \cap \mathtt{Per}$ and suppose $e \neq 0$. For some $k \geq 1$ we have $e \in \mathtt{Per}_k$ and therefore $s^i(\mu_k) \leq e$ for some $i \in k$. Suppose for a contradiction that $e_1, \ldots, e_q \in \mathtt{Div}$ satisfy $e \leq e_1 \vee \cdots \vee e_q$. Then, if $\eta := (k, \ldots, k)$ has length q, Lemma 8.22 implies that $\mathtt{gap}(s^i(\mu_k); \eta) = 0$, which is easily seen to be false. We conclude that $e \notin X$. Thus, we have shown that $\langle\mathtt{Div}\rangle \cap \mathtt{Per} \cap X = \{0\}$. If $e \in \langle\mathtt{Div}\rangle \cap \mathtt{Per}$, then $\neg e \in \langle\mathtt{Div}\rangle \cap \mathtt{Per}$ as well, and by our claim in the previous paragraph we have either $e \in X$ or $\neg e \in X$, so either $e = 0$ or $\neg e = 0$ (and $e = 1$).

Finally, we must show that $\langle \mathtt{Div} \cup \mathtt{Per} \rangle \neq \mathcal{P}(\omega)/\mathtt{fin}$. The fact that $\mathtt{Div}$ is closed downwards in $(\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ also implies that $X \subseteq \langle \mathtt{Div} \rangle$. Each $e \in \langle \mathtt{Div} \cup \mathtt{Per} \rangle$ can be written in the form $e = (a_1 \wedge b_1) \vee \cdots \vee (a_t \wedge b_t)$ where $a_1, \ldots, a_t \in \langle \mathtt{Div} \rangle$ and $b_1, \ldots, b_t \in \mathtt{Per}$. If $e \neq 0$ we have $t \geq 1$ and we may assume that $a_i \wedge b_i \neq 0$ for all i. The two possible cases are that all a_i are in X, or there is some j such that $\neg a_j \in X$. In the first case, note that $e \leq a_1 \vee \cdots \vee a_t$, which clearly shows that $e \in X$. In the second case, let $k \geq 1$ be such that $b_j \in \mathtt{Per}_k$, and since $b_j \neq 0$, let $l \in k$ be such that $s^l(\mu_k) \leq b_j$. We have $e \geq a_j \wedge b_j$ and therefore $\neg e \leq \neg a_j \vee \neg b_j$. It follows that $\neg e \wedge s^l(\mu_k) \leq \neg a_j$, and consequently $\neg e \wedge s^l(\mu_k) \in X$.

Hence, it suffices to find $e \notin X$ such that for all $k \geq 1$ and $l \in k$ we have $\neg e \wedge s^l(\mu_k) \notin X$. For each $q \geq 1$ let η_q be the sequence $(1, \ldots, 1)$ of length q. Suppose $e \in \mathcal{P}(\omega)/\mathtt{fin}$ satisfies

$$\forall q \geq 1 \ (\mathtt{gap}(e; \eta_q) \neq 0 \text{ and } \mathtt{gap}(\neg e; \eta_q) \neq 0) \tag{8.25}$$

Lemma 8.22 shows that $e \notin X$. Given $k \geq 1$ and $l \in k$ we claim that $\neg e \wedge s^l(\mu_k) \notin X$. Again by Lemma 8.22 it suffices to show that for every $q \geq 1$ there is a finite sequence θ_q in $\omega \setminus \{0\}$ of length q such that $\mathtt{gap}(\neg e \wedge s^l(\mu_k); \theta_q) \neq 0$. Given $q \geq 1$, we choose θ_q to be the sequence $(k, \ldots, k)$ with q entries and let $q' := k(q+1) - 1$.

Fix a representative x of $\neg e$. Since $\mathtt{gap}(\neg e; \eta_{q'}) \neq 0$, we know that $\neg e \neq 0$ (so x is infinite) and $\eta_{q'}$ occurs infinitely often in $\mathtt{GS}(x)$. Let $(x_n)_{n \in \omega}$ be the increasing enumeration of x. If $n \in \omega$ is an occurrence of $\eta_{q'}$ in $\mathtt{GS}(x)$, we have $[x_n, x_n + q']_\omega \subseteq x$. Clearly $q' \geq k$, which means that $[x_n, x_n + q']_\omega \cap (k\mathbb{N} + l)$ is non-empty. In particular, this proves that $x \cap (k\mathbb{N} + l)$ is infinite. Let $(x_{n_i})_{i \in \omega}$ be the increasing enumeration of $x \cap (k\mathbb{N} + l)$ (and also a subsequence of $(x_n)_{n \in \omega}$). Again, if $n \in \omega$ is an occurrence of $\eta_{q'}$ in $\mathtt{GS}(x)$, and if x_{n_i} is the minimum of $[x_n, x_n + q']_\omega \cap (k\mathbb{N} + l)$, it is straight-forward to see that

$$[x_n, x_n + q']_\omega \cap (k\mathbb{N} + l) = \{x_{n_i}, x_{n_i} + k, x_{n_i} + 2k, \ldots, x_{n_i} + qk\}.$$

This shows that i is an occurrence of θ_q in $\mathtt{GS}(x \cap (k\mathbb{N} + l))$. Since $n_i \geq n$ and n can be chosen arbitrarily large, it follows that i can be chosen arbitrarily large as well. Thus, θ_q occurs infinitely often in $\mathtt{GS}(x \cap (k\mathbb{N} + l))$ and by Lemma 8.7 we have $\mathtt{gap}(\neg e \wedge s^l(\mu_k); \theta_q) \neq 0$.

It remains to find e satisfying (8.25). For each $m \in \omega$ and each $i \leq m$ let

$$x_{\frac{m(m+1)}{2} + i} := m(m+1) + i$$

It is easy to check that this indeed defines a strictly increasing sequence $(x_n)_{n \in \omega}$. Let $x := \{x_n : n \in \omega\}$ and $e := [\![x]\!]$. Given $q \geq 1$, observe that for every $m \geq q$ we have $[m(m+1), m(m+1) + q]_\omega \subseteq x$ (that is, $n = m(m+1)/2$ is an occurrence of η_q in $\mathtt{GS}(x)$). This shows that $\mathtt{gap}(e; \eta_q) \neq 0$. On the other hand, for every $m \geq q$ we have $[m(m+2)+1, m(m+2)+q+1]_\omega \subseteq \omega \setminus x$. This shows that $\mathtt{gap}(\neg e; \eta_q) \neq 0$ and completes the proof that e satisfies (8.25). □

Definition 8.26. Given $e \in \mathcal{P}(\omega)/\mathtt{fin}$, we denote by $\langle e \rangle_s$ the smallest shift-invariant subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ containing e.

Our next goal is to study the structures of the form $(\langle e \rangle_s, s)$. Of course, this notation is slightly informal and technically should be written $(\langle e \rangle_s, s \restriction \langle e \rangle_s)$. We will see that we can obtain information about $(\langle e \rangle_s, s)$ using the gap sequence of e.

Lemma 8.27. *Let $\mathcal{B}$ be a Boolean algebra and X an infinite subset of $\mathcal{B} \setminus \{0\}$ whose elements are pairwise disjoint (i.e. have null conjunction). Then, all elements of $\langle X \rangle$ can be written either in the form $\bigvee E$ or in the form $\neg \bigvee E$ for some finite $E \subseteq X$, and this representation is unique.*

Proof. Let $\mathcal{C} := \{\bigvee E : E \subseteq X \text{ and } |E| < \infty\} \cup \{\neg \bigvee E : E \subseteq X \text{ and } |E| < \infty\}$. It is clear that $X \subseteq \mathcal{C} \subseteq \langle X \rangle$. To prove the existence of the proposed representations it suffices to show that $\mathcal{C}$ is a subalgebra of $\mathcal{B}$. Evidently, $0 = \bigvee \emptyset \in \mathcal{C}$ and whenever $e \in \mathcal{C}$ it holds that $\neg e \in \mathcal{C}$. If $E, F \subseteq X$ and both E and F are finite, the fact that the elements of X are pairwise disjoint implies that

$$\Big(\bigvee E\Big) \wedge \Big(\bigvee F\Big) = \bigvee (E \cap F) \tag{8.28}$$

$$\Big(\bigvee E\Big) \wedge \Big(\neg \bigvee F\Big) = \bigvee (E \setminus F) \tag{8.29}$$

$$\Big(\neg \bigvee E\Big) \wedge \Big(\neg \bigvee F\Big) = \neg \bigvee (E \cup F) \tag{8.30}$$

and these equalities clearly show that $\mathcal{C}$ is closed under conjunctions, completing the proof that $\mathcal{C} = \langle X \rangle$.

It remains to prove the uniqueness of these representations. If E is a finite subset of X and $e = \bigvee E$, it is obvious that $E \subseteq \{x \in X : x \leq e\}$. On the other hand, if $x \in X$ and $x \leq e$, then we have $x \wedge y \neq 0$ for some $y \in E$ (recall that $x \neq 0$ because we assumed that $0 \notin X$). Since the elements of X are pairwise disjoint we conclude that $x = y$. This shows that $E = \{x \in X : x \leq e\}$. In particular, if E and F are finite subsets of X and $\bigvee E = \bigvee F$ or (equivalently) $\neg \bigvee E = \neg \bigvee F$, then $E = F$. To complete the proof that the representation is unique, we must show that $\bigvee E \neq \neg \bigvee F$ whenever E and F are finite subsets of X. Indeed, the equality $\bigvee E = \neg \bigvee F$ would imply that $\bigvee (E \cup F) = (\bigvee E) \vee (\bigvee F) = 1$. Then, we would have $E \cup F = \{x \in X : x \leq 1\} = X$, contradicting the fact that X is infinite. □

Corollary 8.31. *In the situation of the lemma above, X is the set of atoms of $\langle X \rangle$.*

Proof. If $x \in X$ we know that $x \neq 0$. Suppose $b \in \langle X \rangle$ and $b \leq x$. Let $E := \{x\}$ (so that $x = \bigvee E$) and let F be a finite subset of X such that either $b = \bigvee F$ or $b = \neg \bigvee F$. The equalities (8.28) and (8.29) show that $x \wedge b$ $(= b)$ is either 0 or x itself. This proves that x is an atom of $\langle X \rangle$.

On the other hand, suppose a is an atom of $\langle X \rangle$. If a is of the form $\bigvee E$ for some finite $E \subseteq X$, its minimality clearly implies that $|E| = 1$, so that $a \in X$. The proof will be complete once we show that a cannot be of the form $\neg \bigvee E$. Indeed, if E is a finite subset of X we can choose $x_0, x_1 \in X \setminus E$ distinct and let $b_0 := \neg \bigvee (E \cup \{x_0\})$ and $b_{01} := \neg \bigvee (E \cup \{x_0, x_1\})$. It follows that $b_{01} < b_0 < \neg \bigvee E$ proving that $\neg \bigvee E$ is not an atom of $\langle X \rangle$. (The inequality $b_{01} < b_0$ is meant to show that $b_0 \neq 0$.) □

Corollary 8.32. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be Boolean algebras and let $X_0 \subseteq \mathcal{B}_0 \setminus \{0\}$ and $X_1 \subseteq \mathcal{B}_1 \setminus \{0\}$. Suppose both X_0 and X_1 are infinite, the elements of X_0 are pairwise disjoint and so are the elements of X_1. Then $\langle X_0 \rangle \simeq \langle X_1 \rangle$ if and only if $|X_0| = |X_1|$. In the affirmative case, every bijection from X_0 onto X_1 extends uniquely to an isomorphism from $\langle X_0 \rangle$ onto $\langle X_1 \rangle$, and every isomorphism from $\langle X_0 \rangle$ onto $\langle X_1 \rangle$ restricts to a bijection from X_0 onto X_1.*

Proof. Suppose $\langle X_0 \rangle \simeq \langle X_1 \rangle$ and let $\varphi : \langle X_0 \rangle \to \langle X_1 \rangle$ be an isomorphism. Since φ maps the atoms of $\langle X_0 \rangle$ precisely onto the atoms of $\langle X_1 \rangle$, Corollary 8.31 implies that $\varphi[X_0] = X_1$. This also proves that $|X_0| = |X_1|$.

For the opposite direction, assume that $|X_0| = |X_1|$ and let $\varphi : X_0 \to X_1$ be a bijection. By Lemma 8.27, each $e \in \langle X_0 \rangle$ has a unique representation either of the form $\bigvee E$ or of the form $\neg \bigvee E$ for some finite set $E \subseteq X_0$. In the first case define $\varphi(e) := \bigvee \varphi[E]$, and in the second case define $\varphi(e) := \neg \bigvee \varphi[E]$. The uniqueness of these representations for elements of $\langle X_0 \rangle$ guarantees that φ is well-defined. The existence of such representations for elements of $\langle X_1 \rangle$ implies that φ is onto $\langle X_1 \rangle$, and the uniqueness on this side implies that φ is injective. Thus, this definition extends φ to a bijection from $\langle X_0 \rangle$ onto $\langle X_1 \rangle$.

It remains to show that φ is a homomorphism. It is clear that $\varphi(0) = 0$ and that $\varphi(\neg e) = \neg\varphi(e)$ for all $e \in \langle X_0 \rangle$. Using the equalities (8.28), (8.29) and (8.30) it is easy to check that $\varphi(e \wedge f) = \varphi(e) \wedge \varphi(f)$ for all $e, f \in \langle X_0 \rangle$, which concludes the proof. □

Corollary 8.33. *If $e_0, e_1 \in \mathtt{Div}$, then there are isomorphisms $\varphi^+ : (\langle e_0 \rangle_s, s) \to (\langle e_1 \rangle_s, s)$ and $\varphi^- : (\langle e_0 \rangle_s, s) \to (\langle e_1 \rangle_s, s^{-1})$ both of which map e_0 to e_1.*

Proof. For each $i \in 2$ let $X_i := s^{\mathbb{Z}}(e_i)$ and note that $\langle e_i \rangle_s = \langle X_i \rangle$. Lemma 8.19 says that for each i the map $m \mapsto s^m(e_i)$ is a bijection between $\mathbb{Z}$ and X_i, and that the elements of X_i are pairwise disjoint. Thus, by the previous corollary, every bijection between X_0 and X_1 extends to an isomorphism between $\langle e_0 \rangle_s$ and $\langle e_1 \rangle_s$. If we let φ^+ be the isomorphism extending the bijection $s^m(e_0) \mapsto s^m(e_1)$ and φ^- be the isomorphism extending the bijection $s^m(e_0) \mapsto s^{-m}(e_1)$, it is not hard to see that φ^+ preserves the shift, while φ^- inverts it. □

When searching for a non-trivial automorphism of $(\mathcal{P}(\omega)/\mathtt{fin}, s)$, one of the first strategies that come to mind is to try to adapt the proof of Theorem 6.14 as follows: Let $\mathbb{P}^+$ be the set of all shift-preserving isomorphisms between countable shift-invariant subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. For each $e \in \mathcal{P}(\omega)/\mathtt{fin}$, let $D_e^+ := \{\varphi \in \mathbb{P}^+ : e \in \mathtt{dom}(\varphi) \cap \mathtt{ran}(\varphi)\}$. Then, prove that $\mathbb{P}^+$ is $\aleph_1$-closed and 2-splitting, and that the sets D_e^+ are dense in $\mathbb{P}^+$. Assume CH and apply the Generalized Rasiowa-Sikorski Theorem. Similarly, to find an isomorphism between $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ one could try the following: Let $\mathbb{P}^-$ be the set of all shift-inverting isomorphisms between countable shift-invariant subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. For each $e \in \mathcal{P}(\omega)/\mathtt{fin}$, let $D_e^- := \{\varphi \in \mathbb{P}^- : e \in \mathtt{dom}(\varphi) \cap \mathtt{ran}(\varphi)\}$. Then, prove that $\mathbb{P}^-$ is $\aleph_1$-closed and that the sets D_e^- are dense in $\mathbb{P}^-$. Assume CH and apply the Generalized Rasiowa-Sikorski Theorem. Neither of these strategies work:

Lemma 8.34. *Let $e := \mu_2 = [\![2\mathbb{N}]\!]$. In the notation described above, D_e^+ is not dense in $\mathbb{P}^+$, and D_e^- is not dense in $\mathbb{P}^-$.*

Proof. Let $e_0, e_1 \in \mathtt{Div}$ be such that $e_0 \leq e$, while $e_1 \not\leq e$ and $e_1 \not\leq s(e)$ (for example, take $e_0 := [\![\{2^n : n \in \omega\}]\!]$ and $e_1 := [\![\{2^n + n : n \in \omega\}]\!]$). By Corollary 8.33, there are isomorphisms $\varphi^+ : (\langle e_0 \rangle_s, s) \to (\langle e_1 \rangle_s, s)$ and $\varphi^- : (\langle e_0 \rangle_s, s) \to (\langle e_1 \rangle_s, s^{-1})$, both of which map e_0 to e_1 (so $\varphi^+ \in \mathbb{P}^+$ and $\varphi^- \in \mathbb{P}^-$). Suppose, for a contradiction, that φ is an extension of φ^+ in D_e^+. On the one hand, since $e_0 \leq e$ we have $e_1 \leq \varphi(e)$. On the other hand, since $\neg e = s(e)$ and φ is shift-preserving, it follows that $\neg\varphi(e) = s(\varphi(e))$ as well. However, e and $s(e)$ are the only solutions to the equality $\neg a = s(a)$, so we conclude that either $\varphi(e) = e$ (contradicting the fact that $e_1 \not\leq e$) or $\varphi(e) = s(e)$ (contradicting the fact that $e_1 \not\leq s(e)$). Similarly, φ^- cannot have an extension in D_e^- because e and $s(e)$ are also the only solutions of the equality $\neg a = s^{-1}(a)$. □

For precise definitions of the elements involved in the next corollary, I recommend Hodges's book on model theory [Hod93]. The vocabulary of Boolean algebras already includes the symbols $\neg$, $\vee$ and $\wedge$ so, to avoid confusion, let us agree that the negation, disjunction and conjunction of formulas will be represented by the monospaced font words `not`, `or` and `and` respectively.

Corollary 8.35 (to the previous proof). *The theory of $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ does not have quantifier elimination.*

Proof. Consider the vocabulary $L := \{0, 1, \vee, \wedge, \neg, \leq, f\}$ with the usual symbols for Boolean algebras plus the unary function symbol f, and consider the first-order L-formula

$$\exists y\ (\neg y = f(y)\ \mathtt{and}\ x \leq y)$$

Let us denote this formula by $\beta(x)$. If e_0, e_1 and φ^+ are as in the previous proof, then $(\mathcal{P}(\omega)/\mathtt{fin}, s) \models \beta[e_0]$, but $(\mathcal{P}(\omega)/\mathtt{fin}, s) \not\models \beta[e_1]$ even though φ^+ is an isomorphism between

$(\langle e_0\rangle_s, s)$ and $(\langle e_1\rangle_s, s)$ and maps e_0 to e_1. This could not happen if β were equivalent to a quantifier-free formula in $(\mathcal{P}(\omega)/\mathtt{fin}, s)$. □

This corollary becomes more interesting paired with the next (known) theorem, which implies that the theory of $\mathcal{P}(\omega)/\mathtt{fin}$ (without the shift) *does* have quantifier elimination. The proof presented here is lengthy because it is elementary. Exercise 13(a) of Section 5 in [Kop89a] provides a different strategy for the proof.

Theorem 8.36. *The theory of atomless Boolean algebras has quantifier elimination.*

Proof. All terms and formulas in this proof are built on the vocabulary $\{0, 1, \vee, \wedge, \neg, \leq\}$ and we agree that *equivalent* formulas are formulas *equivalent under the theory of atomless Boolean algebras*. Following a basic result in model theory it suffices to prove that for every quantifier-free formula $\varphi(x, y_1, \dots, y_n)$:

$(*_\varphi)$ There is $\psi(y_1, \dots, y_n)$ which is quantifier-free and equivalent to $\exists x\varphi$.

Evidently, to prove $(*_\varphi)$, it suffices to prove $(*_{\varphi'})$ for any formula φ' which is equivalent to φ. So our strategy is to replace φ by equivalent formulas several times (and once justifiably by a simpler formula) until we reach a canonical form which is easier to work with.

If $\varphi(x, y_1, \dots, y_n)$ is a quantifier-free formula, it is a Boolean combination of atomic formulas, some of which are equalities and some of which are inequalities. Each equality $t_0 = t_1$ in φ is equivalent to the conjunction of the inequalities $t_0 \leq t_1$ and $t_1 \leq t_0$, so we may assume that all atomic subformulas of φ are inequalities. In Theorem 2.4 we have implicitly shown (or at least presented all the necessary equations to show) that for every term $t(x, y_1, \dots, y_n)$ there are terms $t_0(y_1, \dots, y_n)$ and $t_1(y_1, \dots, y_n)$ such that t is equivalent to $(t_0 \wedge x) \vee (t_1 \wedge \neg x)$ in every Boolean algebra (in the sense that they have precisely the same interpretation). Thus, we may assume that all atomic subformulas of φ are of the form $(t_0 \wedge x) \vee (t_1 \wedge \neg x) \leq (t_2 \wedge x) \vee (t_3 \wedge \neg x)$ where x does not appear in the terms t_0, t_1, t_2, and t_3. On the other hand, such an inequality is easily seen to be equivalent to the conjunction of the inequalities $t_1 \wedge \neg t_3 \leq x$ and $x \leq \neg t_0 \vee t_2$. Therefore, we may assume that φ only has atomic subformulas of two kinds, namely $t(y_1, \dots, y_n) \leq x$ and $x \leq t(y_1, \dots, y_n)$.

The next step is to replace φ by a formula in disjunctive normal form with precisely the same atomic subformulas. We borrow/adapt some terminology from propositional logic: A *literal* (here) is either an atomic formula or the negation of one. Thus, we now additionally assume that φ is of the form

$$\varphi_0 \ \mathtt{or} \ \dots \ \mathtt{or} \ \varphi_k$$

where each φ_i is a conjunction of literals. It is not difficult to see that $(*_\varphi)$ follows if we can prove $(*_{\varphi_i})$ for every i. Therefore, we only need to take care of the case $k = 0$, that is, the case that φ itself is a conjunction of literals.

In this case, there are four finite sets A, B, C and D whose elements are terms on the variables $y_1, \dots, y_n$ and such that φ is the conjunction of all the literals $a \leq x$ for $a \in A$, plus all the literals $x \leq b$ for $b \in B$, plus all the literals $c \not\leq x$ for $c \in C$, plus all the literals $x \not\leq d$ for $d \in D$. Of course, if we let $a_0 := \bigvee A$ (with the agreement that $\bigvee \emptyset = 0$ in this context), we can replace the conjunction of all the inequalities $a \leq x$ for $a \in A$ by the single inequality $a_0 \leq x$. Similarly, if we let $b_0 := \bigwedge B$ (with the agreement that $\bigwedge \emptyset = 1$ in this context), we can replace the conjunction of all the inequalities $x \leq b$ for $b \in B$ by the single inequality $x \leq b_0$. With this, we have finally reached the desired canonical form for φ.

Let $\psi(y_1, \dots, y_n)$ be the conjunction of the inequality $a_0 \leq b_0$ with all the literals $c \not\leq a_0$ for $c \in C$, plus all the literals $b_0 \not\leq d$ for $d \in D$. To prove that ψ is equivalent to $\exists x\varphi$, we must show that for every atomless Boolean algebra $\mathcal{B}$ we have $\mathcal{B} \vDash \exists x\varphi \to \psi$ as well as $\mathcal{B} \vDash \psi \to \exists x\varphi$. The first implication is clear. For the second one, suppose $f_1, \dots, f_n \in \mathcal{B}$ are such that $\mathcal{B} \vDash \psi[f_1, \ldots, f_n]$ and let us find $e \in \mathcal{B}$ such that $\mathcal{B} \vDash \varphi[e, f_1, \ldots, f_n]$. To simplify the notation, we use $\bar{f}$ to denote the tuple $(f_1, \ldots, f_n)$.

Write $C = \{c_1, \ldots, c_p\}$ (with the possibility that p might be 0). We now inductively define a finite decreasing sequence $(e_i)_{i \leq p}$ in $\mathcal{B}$ satisfying

- $a_0[\bar{f}] \leq e_i \leq b_0[\bar{f}]$
- $e_i \not\leq d[\bar{f}]$ and
- $c_j[\bar{f}] \not\leq e_i$

for every $i \leq p$, every $d \in D$ and every $j \in \{1, \ldots, i\}$. Once the sequence is defined, we clearly have $\mathcal{B} \vDash \varphi[e_p, f_1, \ldots, f_n]$, which concludes the proof. First, let $e_0 := b_0[\bar{f}]$. For the induction step, assume e_i has already been chosen for some $i < p$. We can obviously choose $e_{i+1} := e_i$ if $c_{i+1}[\bar{f}] \not\leq e_i$, so from now on we assume that $c_{i+1}[\bar{f}] \leq e_i$. In informal terms, making the parallel with algebras of sets, we wish to obtain e_{i+1} by "removing" from e_i some "part" of $c_{i+1}[\bar{f}]$. This "part" must be "out" of $a_0[\bar{f}]$, and we have to be careful not to "remove" too much, to avoid making the resulting element a "subset" of $d[\bar{f}]$ for some $d \in D$. This last issue is where the atomlessness of $\mathcal{B}$ is used.

Let $\mathcal{D} := \langle d[\bar{f}] : d \in D \rangle$. This is a finite subalgebra of $\mathcal{B}$ and therefore the disjunction of its atoms is 1. The fact that $\mathcal{B} \vDash \psi[\bar{f}]$ implies that $c_{i+1}[\bar{f}] \wedge \neg a_0[\bar{f}] \neq 0$ and so there is some atom d_0 of $\mathcal{D}$ such that $c_{i+1}[\bar{f}] \wedge \neg a_0[\bar{f}] \wedge d_0 \neq 0$. Since $\mathcal{B}$ is atomless, we can find $e' \in \mathcal{B}$ such that

$$0 < e' < c_{i+1}[\bar{f}] \wedge \neg a_0[\bar{f}] \wedge d_0.$$

and let $e_{i+1} := e_i \wedge \neg e'$.

As promised, $e_{i+1} \leq e_i$, which already guarantees the conditions $e_{i+1} \leq b_0[\bar{f}]$ and $c_j[\bar{f}] \not\leq e_{i+1}$ for all $j \leq i$. Since $e' \leq \neg a_0[\bar{f}]$, we have $a_0[\bar{f}] \leq \neg e'$ and it follows that $a_0[\bar{f}] \leq e_{i+1}$. Also, note that $c_{i+1}[\bar{f}] \wedge \neg e_{i+1} = c_{i+1}[\bar{f}] \wedge (\neg e_i \vee e') \geq e' > 0$, which shows that $c_{i+1}[\bar{f}] \not\leq e_{i+1}$. It remains to prove that $e_{i+1} \not\leq d[\bar{f}]$ for every $d \in D$. For each element $d \in D$ we have either $d_0 \leq d[\bar{f}]$ or $d_0 \leq \neg d[\bar{f}]$. In the first case we have $e' \leq d[\bar{f}]$, so the claim is an easy consequence of the fact that $e_i \not\leq d[\bar{f}]$. On the other hand, in the second case we have

$$\begin{aligned} 0 &< c_{i+1}[\bar{f}] \wedge d_0 \wedge \neg e' \\ &\leq e_i \wedge \neg d[\bar{f}] \wedge \neg e' \\ &= e_{i+1} \wedge \neg d[\bar{f}] \end{aligned}$$

proving that $e_{i+1} \not\leq d[\bar{f}]$ as desired. □

Corollary 8.33 together with our previous observations about the cardinality of `Div` shows that there is a class of $2^{\aleph_0}$ many pairwise isomorphic structures of the form $(\langle e \rangle_s, s)$ with $e \in \mathcal{P}(\omega)/\mathtt{fin}$. The next theorem is a nice complement to this result.

Theorem 8.37. *There is a class of $2^{\aleph_0}$ many pairwise non-isomorphic structures of the form $(\langle e \rangle_s, s)$ with $e \in \mathcal{P}(\omega)/\mathtt{fin}$.*

The proof will be broken into several lemmata. Let Ξ be the set of all $\xi \in 2^\omega$ with infinite support (i.e. such that $\xi^{-1}(1)$ is infinite) and with $\xi(0) = 1$.

Lemma 8.38. *Given $\xi \in \Xi$, there is $e_\xi \in \mathcal{P}(\omega)/\mathtt{fin}$ such that:*

$$\forall p \in \omega \quad (e_\xi \wedge s^p(e_\xi) = 0 \quad \textit{if and only if} \quad \xi(p) = 0)$$

$$\textit{and} \quad \forall p_0, p_1, p_2 \in \omega \textit{ distinct } \ (s^{p_0}(e_\xi) \wedge s^{p_1}(e_\xi) \wedge s^{p_2}(e_\xi) = 0).$$

Proof. We define a strictly increasing sequence $(x_n)_{n\in\omega}$ in ω by induction on n, combined with a particular representation of each natural number. If I is the set of all triples (k,i,j) with $k\in\omega$, $i\in\{1,\dots,k+1\}$ and $j\in\{1,2\}$, it is straight-forward to verify that the map

$$\begin{aligned} I &\to \omega \\ (k,i,j) &\mapsto k(k+1)+2i-j \end{aligned}$$

is a bijection. Let $x_0 := 0$. For $n>0$, let (k,i,j) be the unique triple in I such that $n=k(k+1)+2i-j$ and define

$$x_n := \begin{cases} x_{n-1}+k+1 & \text{if } j=2 \text{ or } \xi(i)=0 \\ x_{n-1}+i & \text{if } j=1 \text{ and } \xi(i)=1 \end{cases}$$

Let $x:=\{x_n : n\in\omega\}$ and $e_\xi := [\![x]\!]$. To prove the first part of the statement, suppose $p\in\omega$ is given. In the case $p=0$, we know that $\xi(0)=1$ (by the definition of Ξ), and indeed $e_\xi\wedge s^0(e_\xi)=e_\xi\neq 0$ because x is clearly infinite. Now, let us assume $p\geq 1$.

If $n=k(k+1)+2i-j$ where $(k,i,j)\in I$ and $k\geq p$, there are a few cases in which $x_n\notin S^p[x]$ regardless of the value of $\xi(p)$. (Note that $k\geq p\geq 1$ implies that $n\geq 2$.) The first case is when $j=2$ or $\xi(i)=0$: In this case, $x_n=x_{n-1}+k+1>x_{n-1}+p$. (This shows that $x_n\notin S^p[x]$ because the enumeration $(x_m)_{m\in\omega}$ is strictly increasing.) The second case is when $i>p$: Because of the first case, we may assume that $j=1$ and $\xi(i)=1$. Then, we have $x_n=x_{n-1}+i>x_{n-1}+p$ and the conclusion follows as in the previous case. The third and final case is when $i<p$: Again, we assume that $j=1$ and $\xi(i)=1$. This time we have $x_n=x_{n-1}+i<x_{n-1}+p$. On the other hand, we can write $n-1=k(k+1)+2i-2$ and we have $(k,i,2)\in I$ and $n-1>0$, therefore $x_{n-1}=x_{n-2}+k+1>x_{n-2}+p$. This shows that $x_{n-2}+p<x_n<x_{n-1}+p$ and proves that $x_n\notin S^p[x]$. Since $x=^*\{x_{k(k+1)+2i-j} : (k,i,j)\in I \text{ and } k\geq p\}$, we conclude that

$$\begin{aligned} x\cap S^p[x] &\subseteq^* \{x_{k(k+1)+2i-j} : (k,i,j)\in I,\ k\geq p,\ j=1,\ \xi(i)=1,\ i\leq p \text{ and } i\geq p\} \\ &= \begin{cases} \emptyset & \text{if } \xi(p)=0 \\ \{x_{k(k+1)+2p-1} : k\geq p\} & \text{if } \xi(p)=1 \end{cases} \end{aligned}$$

Thus, if $\xi(p)=0$ we have $x\cap S^p[x]=^*\emptyset$, and consequently $e_\xi\wedge s^p(e_\xi)=0$ as we wanted. On the other hand, if $\xi(p)=1$, for every n of the form $k(k+1)+2p-1$ with $k\geq p$ we have $x_n=x_{n-1}+p\in S^p[x]$. It follows that $x\cap S^p[x]=^*\{x_{k(k+1)+2p-1} : k\geq p\}$ and since the set on the right is infinite, it holds that $e_\xi\wedge s^p(e_\xi)\neq 0$.

To prove the second statement of the lemma, it suffices to show that if $0<p_1<p_2$, then $e_\xi\wedge s^{p_1}(e_\xi)\wedge s^{p_2}(e_\xi)=0$. The equality obviously holds if $\xi(p_1)=0$ or $\xi(p_2)=0$. On the other hand, if $\xi(p_1)=\xi(p_2)=1$, then

$$x\cap S^{p_1}[x]\cap S^{p_2}[x]=^*\{x_{k_1(k_1+1)+2p_1-1} : k_1\geq p_1\}\cap\{x_{k_2(k_2+1)+2p_2-1} : k_2\geq p_2\}.$$

The intersection on the right is empty because $p_1\neq p_2$, therefore $e_\xi\wedge s^{p_1}(e_\xi)\wedge s^{p_2}(e_\xi)=0$ in this case as well. □

From the statement of the lemma it is easy to see that for all $\xi\in\Xi$ we have:

$$\forall m_0,m_1\in\mathbb{Z}\ \ (s^{m_0}(e_\xi)\wedge s^{m_1}(e_\xi)=0 \ \text{ if and only if }\ \xi(|m_0-m_1|)=0) \tag{8.39}$$

$$\text{and}\quad \forall m_0,m_1,m_2\in\mathbb{Z}\text{ distinct }\ (s^{m_0}(e_\xi)\wedge s^{m_1}(e_\xi)\wedge s^{m_2}(e_\xi)=0). \tag{8.40}$$

In particular, (8.40) shows that e_ξ is not shift-periodic (since we know that $e_\xi\neq 0$).

Lemma 8.41. *If $c_1,\dots,c_v\in s^{\mathbb{Z}}(e_\xi)$, then $c_1\vee\cdots\vee c_v\neq 1$.*

Proof. Let $b_1 \in s^{\mathbb{Z}}(e_\xi) \setminus \{c_1, \dots, c_v\}$ (which exists because e_ξ is not shift-periodic). Since ξ has infinite support, (8.39) shows that there are infinitely many $b_2 \in s^{\mathbb{Z}}(e_\xi)$ such that $b_1 \wedge b_2 \neq 0$. In particular, we can choose $b_2 \notin \{b_1, c_1, \dots, c_v\}$. By (8.40), it holds that $b_1 \wedge b_2 \wedge c_i = 0$ for every i, and therefore $(b_1 \wedge b_2) \wedge (c_1 \vee \dots \vee c_v) = 0 \neq b_1 \wedge b_2$. □

Corollary 8.42. *Let $c_1, \dots, c_v \in s^{\mathbb{Z}}(e_\xi)$ and $\gamma := \neg c_1 \wedge \dots \wedge \neg c_v$. Then,*

$$\gamma \wedge s(\gamma) \wedge s^2(\gamma) \neq 0.$$

□

Lemma 8.43. *If $b_1, b_2, c_1, \dots, c_v \in s^{\mathbb{Z}}(e_\xi)$ and $b_1 \neq b_2$, then*

$$b_1 \wedge b_2 \wedge \neg c_1 \wedge \dots \wedge \neg c_v = \begin{cases} 0 & \text{if } \{b_1, b_2\} \cap \{c_1, \dots, c_v\} \neq \emptyset \\ b_1 \wedge b_2 & \text{otherwise.} \end{cases}$$

Proof. The case that $\{b_1, b_2\} \cap \{c_1, \dots, c_v\} \neq \emptyset$ is obvious, so suppose instead we have $\{b_1, b_2\} \cap \{c_1, \dots, c_v\} = \emptyset$. Then, $b_1 \wedge b_2 \wedge c_i = 0$ and therefore $b_1 \wedge b_2 \leq \neg c_i$ for every i. It follows that $b_1 \wedge b_2 \leq \neg c_1 \wedge \dots \wedge \neg c_v$, and this implies the desired equality. □

Lemma 8.44. *Let $b_1, b_2, c_{1,1}, \dots, c_{1,v_1}, c_{2,1}, \dots, c_{2,v_2} \in s^{\mathbb{Z}}(e_\xi)$, and suppose that $b_1 \neq b_2$, $b_1 \notin \{c_{1,1}, \dots, c_{1,v_1}\}$, and $b_2 \notin \{c_{2,1}, \dots, c_{2,v_2}\}$. Let $\alpha_1 := b_1 \wedge \neg c_{1,1} \wedge \dots \wedge \neg c_{1,v_1}$ and $\alpha_2 := b_2 \wedge \neg c_{2,1} \wedge \dots \wedge \neg c_{2,v_2}$. Then, there are $m_0, m_1, m_2 \in \mathbb{Z}$ distinct such that*

$$s^{m_0}(\alpha_1 \vee \alpha_2) \wedge s^{m_1}(\alpha_1 \vee \alpha_2) \wedge s^{m_2}(\alpha_1 \vee \alpha_2) \neq 0.$$

Proof. Note that it suffices to find $m_1, m_2 \in \mathbb{Z}$ distinct, both different of 0 and such that $\alpha_1 \wedge s^{m_1}(\alpha_2) \wedge s^{m_2}(\alpha_1) \neq 0$. There is $m_1 \in \mathbb{Z}$ such that $s^{m_1}(b_2) = b_1$ (because b_1 and b_2 are in the same shift-orbit) and we know $m_1 \neq 0$ because $b_1 \neq b_2$. Also, since $b_2 \notin \{c_{2,1}, \dots, c_{2,v_2}\}$, we have $b_1 \notin \{s^{m_1}(c_{2,1}), \dots, s^{m_1}(c_{2,v_2})\}$. By (8.39), there are arbitrarily large $m_2 \in \omega$ such that $b_1 \wedge s^{m_2}(b_1) \neq 0$. By choosing m_2 large enough we can make sure that $m_2 \notin \{0, m_1\}$ and $\{b_1, s^{m_2}(b_1)\} \cap \{c_{1,1}, \dots, c_{1,v_1}, s^{m_1}(c_{2,1}), \dots, s^{m_1}(c_{2,v_2}), s^{m_2}(c_{1,1}), \dots, s^{m_2}(c_{1,v_1})\} = \emptyset$. We have

$$\alpha_1 \wedge s^{m_1}(\alpha_2) \wedge s^{m_2}(\alpha_1) = b_1 \wedge s^{m_2}(b_1) \wedge \neg c_{1,1} \wedge \cdots$$
$$\cdots \wedge \neg c_{1,v_1} \wedge \neg s^{m_1}(c_{2,1}) \wedge \dots \wedge \neg s^{m_1}(c_{2,v_2}) \wedge \neg s^{m_2}(c_{1,1}) \wedge \dots \wedge \neg s^{m_2}(c_{1,v_1})$$

and we conclude from Lemma 8.43 that $\alpha_1 \wedge s^{m_1}(\alpha_2) \wedge s^{m_2}(\alpha_1) = b_1 \wedge s^{m_2}(b_1) \neq 0$. □

Lemma 8.45. *Suppose $a \in \langle e_\xi \rangle_s$ satisfies:*

(1) $a \wedge s^p(a) \neq 0$ for infinitely many $p \in \omega$.

(2) $\forall m_0, m_1, m_2 \in \mathbb{Z}$ distinct $(s^{m_0}(a) \wedge s^{m_1}(a) \wedge s^{m_2}(a) = 0)$.

Then, there is $N_a \in \omega$ such that for all $p \geq N_a$:

$$a \wedge s^p(a) = 0 \quad \text{if and only if} \quad \xi(p) = 0.$$

Proof. Since $a \in \langle e_\xi \rangle_s$ we can write $a = \alpha_1 \vee \dots \vee \alpha_t$, where each α_i is of the form $\beta_i \wedge \gamma_i$ and there are $b_{i,1}, \dots, b_{i,u_i}, c_{i,1}, \dots, c_{i,v_i} \in s^{\mathbb{Z}}(e_\xi)$ such that $\beta_i = b_{i,1} \wedge \dots \wedge b_{i,u_i}$ and $\gamma_i = \neg c_{i,1} \wedge \dots \wedge \neg c_{i,v_i}$. Also, we may assume that for each fixed i the elements $b_{i,1}, \dots, b_{i,u_i}$ are pairwise distinct, and so are the elements $c_{i,1}, \dots, c_{i,v_i}$. Condition (1) implies that $a \neq 0$, so $t \geq 1$ and we may assume that $\alpha_i \neq 0$ for all i. It follows that $\{b_{i,1}, \dots, b_{i,u_i}\} \cap \{c_{i,1}, \dots, c_{i,v_i}\} = \emptyset$ for every i. Condition (2) implies that $a \neq 1$, so we know that $(u_i, v_i) \neq (0, 0)$ for all i.

Let $i \in \{1, \dots, t\}$. By (8.40), we know that $u_i \leq 2$. On the other hand, Corollary 8.42 shows that $\gamma_i \wedge s(\gamma_i) \wedge s^2(\gamma_i) \neq 0$, and it follows easily that $u_i \neq 0$, otherwise condition (2) would not hold. Moreover, if $u_i = 2$, Lemma 8.43 shows that $\alpha_i = b_{i,1} \wedge b_{i,2}$, so we may assume that $v_i = 0$ in this case.

Putting all this information together and rearranging the α_i's we see that there is some $r \leq t$ such that $u_i = 1$ for all $i \leq r$ and $u_j = 2$ and $v_j = 0$ for all $j > r$. We cannot have $r = 0$, otherwise (8.40) would imply that $a \wedge s^p(a) = 0$ for all p large enough, contradicting condition (1). Finally, Lemma 8.44 together with condition (2) implies that $b_{i,1} = b_{1,1}$ for every $i \leq r$. This means that $\alpha_1 \vee \dots \vee \alpha_r \leq b_{1,1}$.

Let $E := \{m \in \mathbb{Z} : s^m(e_\xi) \in \bigcup_{1 \leq i \leq t} \{b_{i,1}, \dots, b_{i,u_i}, c_{i,1}, \dots, c_{i,v_i}\}\}$ and let

$$N_a := \mathtt{max}(E) - \mathtt{min}(E) + 1$$

(so that for all $p \geq N_a$ and all $m \in E$ we have $p + m \notin E$). Using (8.40) it is easy to see that for all $p \geq N_a$

$$a \wedge s^p(a) = (\alpha_1 \vee \dots \vee \alpha_r) \wedge s^p(\alpha_1 \vee \dots \vee \alpha_r)$$

(notice the maximum index r). Therefore, $a \wedge s^p(a) \leq b_{1,1} \wedge s^p(b_{1,1})$. On the other hand, we obviously have $\alpha_1 \wedge s^p(\alpha_1) \leq a \wedge s^p(a)$ and Lemma 8.43 shows that $\alpha_1 \wedge s^p(\alpha_1) = b_{1,1} \wedge s^p(b_{1,1})$. Thus, $a \wedge s^p(a) = b_{1,1} \wedge s^p(b_{1,1})$. By (8.39), it holds that $b_{1,1} \wedge s^p(b_{1,1}) = 0$ if and only if $\xi(p) = 0$ so the proof is complete. □

Proof of Theorem 8.37. Let $\xi, \xi' \in \Xi$ and suppose $\varphi : (\langle e_\xi \rangle_s, s) \to (\langle e_{\xi'} \rangle_s, s)$ is an isomorphism. Clearly, conditions (1) and (2) of Lemma 8.45 hold for $a = e_\xi$. It follows that the same conditions hold for $a = \varphi(e_\xi)$. By Lemma 8.45, there is $N_{\varphi(e_\xi)} \in \omega$ such that for all $p \geq N_{\varphi(e_\xi)}$ we have $\varphi(e_\xi) \wedge s^p(\varphi(e_\xi)) = 0$ if and only if $\xi'(p) = 0$. On the other hand, for all $p \in \omega$ we know that $\varphi(e_\xi) \wedge s^p(\varphi(e_\xi)) = \varphi(e_\xi \wedge s^p(e_\xi)) = 0$ if and only if $e_\xi \wedge s^p(e_\xi) = 0$, and the latter holds precisely when $\xi(p) = 0$. We conclude that $\xi(p) = \xi'(p)$ for every $p \geq N_{\varphi(e_\xi)}$, and so $\xi =^* \xi'$.

This shows that the number of isomorphism types of the structures $(\langle e_\xi \rangle_s, s)$ is at least $|\,\Xi \,/ =^*|$. Since $|\Xi| = 2^{\aleph_0}$ and the equivalence classes modulo $=^*$ are countable, we conclude that $|\,\Xi \,/ =^*| = 2^{\aleph_0}$. □

The next theorem gives another example of how gap sequences can be used. We turn back our attention to how automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ which preserve or invert the shift permute the set of shift-orbits.

Theorem 8.46. *If $\varphi \in \mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ preserves the shift and induces the identity on $\mathtt{Orb}$, then $\varphi = s^m$ for some $m \in \mathbb{Z}$.*

For the proof we need the following lemma (which is quite interesting on its own):

Lemma 8.47. *Let $e \in \mathcal{P}(\omega)/\mathtt{fin}$ and $m \in \mathbb{Z}$ and suppose $e \leq s^m(e)$. Then, $e = s^m(e)$.*

Proof. The claim is trivial for $m = 0$. Note that it suffices to take care of the case $m > 0$: Indeed, if the lemma holds in the positive case, suppose $e \leq s^m(e)$ for some $e \in \mathcal{P}(\omega)/\mathtt{fin}$ and $m < 0$. This implies that $s^{-m}(e) \leq e$ and consequently $\neg e \leq s^{-m}(\neg e)$. Since $-m > 0$, it follows that $\neg e = s^{-m}(\neg e)$, which in turn implies that $e = s^m(e)$.

Given $e \in \mathcal{P}(\omega)/\mathtt{fin}$ and $m > 0$ such that $e \leq s^m(e)$, let $x \subseteq \omega$ be a representative of e and let $N \geq m$ be such that $x \setminus N \subseteq S^m[x]$. Then, $S^{-m}[x \setminus N] \subseteq x$. This implies that $S^{-m}[x \setminus (N + m)] \subseteq x \setminus N$ and therefore $S^{-2m}[x \setminus (N + m)] \subseteq x$. By induction (using an analogous argument) we obtain that $S^{-km}[x \setminus (N + (k-1)m)] \subseteq x$ for all $k \in \omega$.

Suppose, for a contradiction, that $S^m[x] \setminus x$ is infinite. Then, we can find elements $y_0, \dots, y_m \in S^m[x] \setminus x$ satisfying $N - m \leq y_0 < y_1 < \dots < y_m$. We can then find i and

j such that $0 \le i < j \le m$ and $y_i \equiv y_j \,(\texttt{mod}\ m)$. Since $y_i < y_j$, there is some $k \in \omega$ such that $y_j = y_i + (k+1)m$. The fact that $y_j \in S^m[x]$ implies that $y_i + km = y_j - m \in x$. On the other hand, since $y_i \ge N - m$, it follows that $y_i + km \ge N + (k-1)m$. Thus, we have $y_i \in S^{-km}[x \setminus (N + (k-1)m)] \subseteq x$ contradicting our assumption that $y_i \in S^m[x] \setminus x$.

This contradiction shows that $S^m[x] \setminus x$ is finite, (that is, $S^m[x] \subseteq^* x$) and hence $s^m(e) \le e$ as we wanted to prove. □

Proof of Theorem 8.46. If φ is an automorphism of $\mathcal{P}(\omega)/\texttt{fin}$ which preserves the shift and maps each shift-orbit onto itself, then for every $e \in \mathcal{P}(\omega)/\texttt{fin}$ there is $m(e) \in \mathbb{Z}$ such that $\varphi(e) = s^{m(e)}(e)$.

Let $e_0, e_1 \in \texttt{Div}$ and suppose $e_0 \le e_1$. Then, $s^{m(e_0)}(e_0) \le s^{m(e_0)}(e_1)$ and also $s^{m(e_0)}(e_0) = \varphi(e_0) \le \varphi(e_1) = s^{m(e_1)}(e_1)$. Thus, $s^{m(e_0)}(e_1) \wedge s^{m(e_1)}(e_1) \ne 0$, and since $e_1 \in \texttt{Div}$ we get $m(e_0) = m(e_1)$.

Again, let $e_0, e_1 \in \texttt{Div}$. If $s^p(e_0) \wedge e_1 \ne 0$ for some $p \in \mathbb{Z}$, then $s^p(e_0) \wedge e_1 \in \texttt{Div}$ (because $\texttt{Div}$ is closed downwards in $(\mathcal{P}(\omega)/\texttt{fin}) \setminus \{0\}$) and so the argument in the previous paragraph implies that $m(s^p(e_0)) = m(s^p(e_0) \wedge e_1) = m(e_1)$. Of course,

$$s^{m(s^p(e_0))}(s^p(e_0)) = \varphi(s^p(e_0)) = s^p(\varphi(e_0)) = s^p(s^{m(e_0)}(e_0)) = s^{m(e_0)}(s^p(e_0))$$

and since $s^p(e_0) \in \texttt{Div}$, it follows that $m(s^p(e_0)) = m(e_0)$. Therefore, $m(e_0) = m(e_1)$ in this case. On the other hand, if $s^p(e_0) \wedge e_1 = 0$ for all $p \in \mathbb{Z}$, it is not difficult to see that $s^p(e_0 \vee e_1) \wedge s^q(e_0 \vee e_1) = 0$ for all distinct $p, q \in \mathbb{Z}$. Hence, $e_0 \vee e_1 \in \texttt{Div}$ and it follows that $m(e_0) = m(e_0 \vee e_1) = m(e_1)$. This shows that $m(e_0) = m(e_1)$ whenever e_0 and e_1 are elements of $\texttt{Div}$, and we conclude that there is $m \in \mathbb{Z}$ such that $\varphi \restriction \texttt{Div} = s^m \restriction \texttt{Div}$.

Given $e \in \mathcal{P}(\omega)/\texttt{fin}$, let us show that $\varphi(e) = s^m(e)$. Suppose, for a contradiction, that $s^m(e) \not\le \varphi(e)$. Then, $s^m(e) \wedge \neg\varphi(e) \ne 0$, so Lemma 8.21 implies that there is some $e_0 \in \texttt{Div}$ such that $e_0 \le s^m(e) \wedge \neg\varphi(e)$. However, $e_0 \le s^m(e)$ implies that $s^{-m}(e_0) \le e$, and so $e_0 = \varphi(s^{-m}(e_0)) \le \varphi(e)$. This means that $e_0 \le \varphi(e) \wedge \neg\varphi(e) = 0$, contradicting the fact that $e_0 \in \texttt{Div}$. We conclude that $s^m(e) \le \varphi(e) = s^{m(e)}(e)$. A simple application of the previous lemma shows that $s^m(e) = s^{m(e)}(e)$. □

Lemma 7.11 and Corollary 7.12 show that Theorem 8.46 cannot be generalized to automorphisms of subalgebras of $\mathcal{P}(\omega)/\texttt{fin}$.

Corollary 8.48. *Suppose $\varphi_0, \varphi_1 \in \texttt{Aut}(\mathcal{P}(\omega)/\texttt{fin})$ both preserve or both invert the shift. If they induce the same permutation on $\texttt{Orb}$, then $\varphi_0 = \varphi_1 s^m$ for some $m \in \mathbb{Z}$.*

Proof. Simply note that $\varphi_1^{-1}\varphi_0$ preserves the shift and induces the identity on $\texttt{Orb}$. □

Corollary 8.49. *If $\varphi \in \texttt{Aut}(\mathcal{P}(\omega)/\texttt{fin})$ preserves the shift and induces a permutation on $\texttt{Orb}$ whose cycles are all finite and of bounded length, then there are $n \ge 1$ and $m \in \mathbb{Z}$ such that $\varphi^n = s^m$.*

Proof. Let L be the set of all lengths of cycles of the permutation of $\texttt{Orb}$ induced by φ. The hypothesis implies that L is finite. If $n \ge 1$ is any common multiple of all elements of L, then φ^n induces the identity on $\texttt{Orb}$. Since φ^n preserves the shift, the result follows from Theorem 8.46. □

Lemma 8.50. *There is no automorphism of $\mathcal{P}(\omega)/\texttt{fin}$ that inverts the shift and induces the identity on $\texttt{Orb}$.*

Proof. Suppose, for a contradiction, that $\varphi \in \texttt{Aut}(\mathcal{P}(\omega)/\texttt{fin})$ inverts the shift and for every $e \in \mathcal{P}(\omega)/\texttt{fin}$ there is some $m(e) \in \mathbb{Z}$ such that $\varphi(e) = s^{m(e)}(e)$. With the same argumentation as in the proof of Theorem 8.46 we see that if $e_0, e_1 \in \texttt{Div}$ and $e_0 \le e_1$, then $m(e_0) = m(e_1)$.

Let $x \in \mathtt{DIV}$ and partition x into two infinite parts y_0 and y_1. Let $e := [\![x]\!]$, $e_0 := [\![y_0]\!]$ and $e_1 := [\![y_1]\!]$. Since $e, e_0, e_1 \in \mathtt{Div}$ and $e_0, e_1 \leq e$, we have $m(e_0) = m(e) = m(e_1)$. Finally, let $e' := s(e_0) \vee e_1$. On the one hand,

$$\varphi(e') = s^{m(e')}(e') = s^{m(e')}(s(e_0)) \vee s^{m(e')}(e_1) = s^{m(e')+1}(e_0) \vee s^{m(e')}(e_1)$$

On the other hand,

$$\varphi(e') = \varphi(s(e_0)) \vee \varphi(e_1) = s^{-1}(\varphi(e_0)) \vee \varphi(e_1) = s^{m(e)-1}(e_0) \vee s^{m(e)}(e_1)$$

Therefore, $s^{m(e')+1}(e_0) \vee s^{m(e')}(e_1) = s^{m(e)-1}(e_0) \vee s^{m(e)}(e_1)$. The fact that $e \in \mathtt{Div}$ and $e_0, e_1 \leq e$ implies that $s^p(e_0) \wedge s^q(e_1) = 0$ for all distinct $p, q \in \mathbb{Z}$. Moreover, we have chosen e_0 and e_1 disjoint, so we have $s^p(e_0) \wedge s^q(e_1) = 0$ when $p = q$ as well. It follows that $s^{m(e')+1}(e_0) = s^{m(e)-1}(e_0)$ and $s^{m(e')}(e_1) = s^{m(e)}(e_1)$. Since $e_0 \in \mathtt{Div}$ we conclude that $m(e') + 1 = m(e) - 1$, and since $e_1 \in \mathtt{Div}$ we also conclude that $m(e') = m(e)$, which is clearly a contradiction. □

We finish this chapter with a remark about the conclusion of Corollary 8.49. If φ is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ which preserves the shift and we have $\varphi^n = s^m$ for some $n \geq 1$ and $m \in \mathbb{Z}$, then $n \mid m$: Indeed, we know that there is $i \in n$ such that $\varphi \restriction \mathtt{Per}_n = s^i \restriction \mathtt{Per}_n$, therefore $s^m(\mu_n) = \varphi^n(\mu_n) = s^{in}(\mu_n)$. This shows that $m \equiv in \equiv 0\,(\mathtt{mod}\ n)$. In particular:

Lemma 8.51. *If $\varphi \in \mathtt{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ and $n \in \mathbb{Z}$ are such that $\varphi^n = s$, then $n = \pm 1$.*

Proof. Note that $\varphi s = \varphi\varphi^n = \varphi^n\varphi = s\varphi$, that is, φ preserves the shift. Obviously, $n \neq 0$. If $n \geq 1$, the remark above implies that $n \mid 1$, and so $n = 1$. On the other hand, if $n \leq -1$ we have $\varphi^{-n} = (\varphi^n)^{-1} = s^{-1}$. In this case we have $-n \mid -1$, and clearly $n = -1$. □

9 Permuting and flipping intervals

It was mentioned in the introduction that the shift on $\mathbb{Z}$ can be inverted by an automorphism which "flips the integers around 0". This was meant as follows: If $\sigma : \mathbb{Z} \to \mathbb{Z}$ is given by $\sigma(m) := -m$, and the map $m \mapsto m + 1$ defined on $\mathbb{Z}$ is also denoted by S, then $\sigma S = S^{-1}\sigma$. This method cannot be used in ω (for obvious reasons), but it can be adapted to invert the shift on certain subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. In very vague terms, the idea is to partition ω into finite intervals and flip each interval separately, which has the effect of inverting the shift as long as we avoid the areas where two intervals meet. This method was discovered by Geschke in 2010 [Ges10]. Recall that the notation $[a, b]_\omega$ is used to represent the interval $\{k \in \omega : a \leq k \leq b\}$.

Definition 9.1. An *interval decomposition of* ω is a partition P of ω whose elements are non-empty finite intervals. The *increasing enumeration of* P is its unique enumeration $(I_n)_{n\in\omega}$ such that $(\mathtt{max}(I_n))_{n\in\omega}$ is an increasing sequence in ω. (There is indeed a partial ordering on the set of intervals of ω which makes $(I_n)_{n\in\omega}$ the unique increasing enumeration of P in the usual sense.) For each $n \in \omega$ let $I_n = [a_n, b_n]_\omega$. Throughout this chapter α_P denotes the permutation of ω that maps each $k \in I_n$ to

$$\alpha_P(k) := a_n + b_n - k.$$

(This permutation is the "flipping of intervals" mentioned above.)

If $k \in I_n$ above, it is clear that $\alpha_P(k) \in I_n$ as well. Using this, we see that $\alpha_P^2 = \mathtt{id}_\omega$.

Lemma 9.2. *Let P be an interval decomposition of ω and let $D := \{\mathtt{max}(I) : I \in P\}$. If $x \subseteq \omega$ satisfies either $D \cap x = \emptyset$ or $D \subseteq x$, then $\alpha_P S[x] =^* S^{-1}\alpha_P[x]$.*

Proof. Let $(I_n)_{n\in\omega}$ be the increasing enumeration of P and write $I_n = [a_n, b_n]_\omega$ for each $n \in \omega$ (so that $D = \{b_n : n \in \omega\}$). If $k \in \omega \setminus D$, there is a unique $n \in \omega$ such that $a_n \leq k < b_n$. Thus, $S(k) \in I_n$ as well and it follows that

$$\begin{aligned}\alpha_P S(k) &= a_n + b_n - S(k)\\ &= a_n + b_n - k - 1\\ &= S^{-1}\alpha_P(k)\end{aligned}$$

Therefore, if $D \cap x = \emptyset$, we have $\alpha_P S[x] = S^{-1}\alpha_P[x]$. On the other hand, for each $n \in \omega$ it holds that

$$\begin{aligned}\alpha_P S(b_n) &= \alpha_P(a_{n+1})\\ &= b_{n+1}\\ &= S^{-1}\alpha_P(b_{n+2})\end{aligned}$$

So, if $D \subseteq x$, we have

$$\begin{aligned}\alpha_P S[x] &= \alpha_P S[D] \cup \alpha_P S[x \setminus D]\\ &= S^{-1}\alpha_P[D \setminus \{b_0, b_1\}] \cup S^{-1}\alpha_P[x \setminus D]\\ &= S^{-1}\alpha_P[x \setminus \{b_0, b_1\}]\\ &=^* S^{-1}\alpha_P[x]\end{aligned}$$

□

Definition 9.3. Given an infinite set $D \subseteq \omega$, there is a unique interval decomposition P of ω such that $\{\mathtt{max}(I) : I \in P\} = D$. This decomposition is denoted by $P(D)$.

Lemma 9.4. *Let $D, D' \subseteq \omega$ be infinite. Then, $D =^* D'$ if and only if $\alpha_{P(D)} =^* \alpha_{P(D')}$.*

Proof. The map $\alpha_{P(D)}$ can be described without explicitly mentioning the increasing enumeration of $P(D)$ as follows:

$$\alpha_{P(D)}(k) = \mathtt{max}\{a \in S[D] \cup \{0\} : a \leq k\} + \mathtt{min}\{b \in D : b \geq k\} - k$$

If $D =^* D'$, let $n_0 \in \omega$ be such that $D \setminus n_0 = D' \setminus n_0$. Since D and D' are infinite, there is $n_1 \geq n_0$ such that $n_1 \in D \cap D'$. The formula for $\alpha_{P(D)}(k)$ above and its version with D replaced by D' show that $\alpha_{P(D)}(k) = \alpha_{P(D')}(k)$ for every $k > n_1$, and hence $\alpha_{P(D)} =^* \alpha_{P(D')}$.

For the opposite direction, suppose $\alpha_{P(D)} =^* \alpha_{P(D')}$. If $k \in \omega \setminus D$, we have seen that $\alpha_{P(D)}S(k) = S^{-1}\alpha_{P(D)}(k)$. If $b_0 := \alpha^{-1}_{P(D)}(0)$, then $b_0 \in D \setminus \mathtt{dom}(S^{-1}\alpha_{P(D)})$ and finally if $k \in D \setminus \{b_0\}$, we have seen that $\alpha_{P(D)}S(k) \neq S^{-1}\alpha_{P(D)}(k)$. Therefore,

$$D = \{b_0\} \cup \{k \in \omega \setminus \{b_0\} : \alpha_{P(D)}S(k) \neq S^{-1}\alpha_{P(D)}(k)\}$$

This description and the analogous equality for D' show that $D =^* D'$. □

Definition 9.5. Given $d \in \mathcal{P}(\omega)/\mathtt{fin}$, $d \neq 0$, let $D \subseteq \omega$ be one of its representatives. The very trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by $\alpha_{P(D)}$ is denoted ν_d.

Lemma 9.4 not only implies that ν_d is independent of the choice of D, but also that the map $d \mapsto \nu_d$ is injective.

Definition 9.6. Let $d \in \mathcal{P}(\omega)/\mathtt{fin}$ and $X \subseteq \mathcal{P}(\omega)/\mathtt{fin}$. We say that d *diagonalizes* X (or that d is a *diagonalizer of* X) if

$$\forall e \in X \ (d \wedge e = 0 \text{ or } d \leq e)$$

The set $\{e \in \mathcal{P}(\omega)/\mathtt{fin} : d \wedge e = 0 \text{ or } d \leq e\}$ (which is a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$) is denoted by $\Delta(d)$. Moreover, the set $\{e \in \mathcal{P}(\omega)/\mathtt{fin} : s^{\mathbb{Z}}(e) \subseteq \Delta(d)\}$ (which is a shift-invariant subalgebra of $\Delta(d)$) is denoted by $\Delta_s(d)$.

A few facts follow almost immediately from these definitions. First, if $d \in \mathcal{P}(\omega)/\mathtt{fin}$ and $X \subseteq \mathcal{P}(\omega)/\mathtt{fin}$, then d diagonalizes X if and only if $X \subseteq \Delta(d)$. In particular, in the affirmative case (since $\Delta(d)$ is a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$) d also diagonalizes the algebra generated by X. It also follows that $\Delta(d)$ is the largest subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ that is diagonalized by d, and $\Delta_s(d)$ is the largest shift-invariant one.

Lemma 9.7. *If $d \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$, then $\nu_d \restriction \Delta_s(d)$ is an automorphism of $\Delta_s(d)$ which inverts the shift.*

Proof. Lemma 9.2 implies that $\nu_d s(e) = s^{-1}\nu_d(e)$ for every $e \in \Delta(d)$. Since $\Delta_s(d) \subseteq \Delta(d)$, it suffices to show that $\Delta_s(d)$ is invariant under ν_d. Since $\nu_d^2 = \mathtt{id}$, we only need to prove that $\nu_d[\Delta_s(d)] \subseteq \Delta_s(d)$. Let us first prove that for all $e \in \Delta_s(d)$ and $m \in \mathbb{Z}$ we have $\nu_d s^m(e) = s^{-m}\nu_d(e)$. For $m \geq 0$ we use induction (the case $m = 0$ being obvious): if we assume that $\nu_d s^m(e) = s^{-m}\nu_d(e)$ for some $m \geq 0$, since $s^m(e) \in \Delta(d)$ it follows that

$$\begin{aligned} \nu_d s^{m+1}(e) &= \nu_d s(s^m(e)) \\ &= s^{-1}\nu_d(s^m(e)) \\ &= s^{-1}s^{-m}\nu_d(e) \\ &= s^{-(m+1)}\nu_d(e) \end{aligned}$$

Now suppose $m \leq 0$. Since $s^m(e) \in \Delta_s(d)$ and $-m \geq 0$, the induction above has shown that $\nu_d s^{-m}(s^m(e)) = s^{-(-m)}\nu_d(s^m(e))$, that is, $\nu_d(e) = s^m \nu_d s^m(e)$. Applying s^{-m} to both sides we obtain $s^{-m}\nu_d(e) = \nu_d s^m(e)$ as desired.

We are ready to show that $\nu_d[\Delta_s(d)] \subseteq \Delta_s(d)$. Given $e \in \Delta_s(d)$ and $m \in \mathbb{Z}$, we must prove that $s^m(\nu_d(e)) \in \Delta(d)$. We know that $s^{-(m+1)}(e) \in \Delta(d)$, so either $d \wedge s^{-(m+1)}(e) = 0$ or $d \leq s^{-(m+1)}(e)$. Applying ν_d to all terms and using the equality proven above we obtain either $\nu_d(d) \wedge s^{m+1}\nu_d(e) = 0$ or $\nu_d(d) \leq s^{m+1}\nu_d(e)$. With a quick look at the definitions involved we see that $\nu_d(d) = s(d)$, so it follows that either $d \wedge s^m\nu_d(e) = 0$ or $d \leq s^m\nu_d(e)$, proving that $s^m(\nu_d(e)) \in \Delta(d)$ as we wanted. □

Lemma 9.8. *If $d \in \mathtt{Div}$, then $|\Delta_s(d)| = 2^{\aleph_0}$.*

Proof. Let D be a representative of d and let $(x_n)_{n\in\omega}$ be the increasing enumeration of D. For each $k \in \omega$ there is $p(k) \in \omega$ such that $x_{n+1} - x_n > 2k$ for all $n \geq p(k)$. We may assume that the sequence $(p(k))_{k\in\omega}$ is strictly increasing. For each map $f : \omega \to 2$ let $Y(f) := \{x_{p(k)} + k + f(k) : k \in \omega\}$ and $e(f) := [\![Y(f)]\!]$. It is easy to see that $\{x_{p(k)} + k + f(k)\} = Y(f) \cap \{x_{p(k)} + k, x_{p(k)} + k + 1\}$ for every k, which implies that the map $f \mapsto Y(f)$ is injective. Thus, there are $2^{\aleph_0}$ many distinct sets of the type $Y(f)$ with $f \in 2^\omega$, and since the equivalence classes modulo finite are countable, we conclude that there are $2^{\aleph_0}$ many distinct elements $e(f)$ (even though the map $f \mapsto e(f)$ is not injective). It remains to show that $\{e(f) : f \in 2^\omega\} \subseteq \Delta_s(d)$.

Given $f \in 2^\omega$ and $m \in \mathbb{Z}$, we claim that $d \wedge s^m(e(f)) = 0$ (and hence $s^m(e(f)) \in \Delta(d)$). Indeed, if $k \in \omega$ and $k > |m|$, it follows that

$$x_{p(k)} < x_{p(k)} + k + f(k) + m < x_{p(k)+1}$$

This proves that $S^m(x_{p(k)} + k + f(k)) \notin D$ because the enumeration $(x_n)_{n\in\omega}$ was chosen strictly increasing. In particular, $D \cap S^m[Y(f)]$ is finite, which completes the proof. □

Lemma 9.9. *Every countable subset of $\mathcal{P}(\omega)/\mathtt{fin}$ has a non-zero diagonalizer.*

Proof. Let X be a countable subset of $\mathcal{P}(\omega)/\mathtt{fin}$ and let $(e_n)_{n<|X|}$ be an enumeration of X. By induction, we choose a decreasing sequence $(d_n)_{n\leq|X|}$ of non-zero elements of $\mathcal{P}(\omega)/\mathtt{fin}$ such that d_n diagonalizes $\{e_i : i < n\}$ for every $n \leq |X|$. To begin the induction we define $d_0 := 1$. For the induction step, suppose $n < |X|$ and $d_n \neq 0$ diagonalizes $\{e_i : i < n\}$. If $d_n \wedge e_n = 0$, let $d_{n+1} := d_n$. Otherwise, let $d_{n+1} := d_n \wedge e_n$. If X is finite, the induction ends with the definition of $d_{|X|}$, which diagonalizes X, so we are done. If X is infinite, the induction defines d_n for every $n \in \omega$, but not $d_\omega = d_{|X|}$. By Lemma 3.4, we can choose a lower bound $d_\omega \neq 0$ for $(d_n)_{n\in\omega}$, and it is easy to see that d_ω diagonalizes X. □

Theorem 9.10 (Geschke)**.** *If X is a countable subset of $\mathcal{P}(\omega)/\mathtt{fin}$, then there is a countable shift-invariant subalgebra $\mathcal{B}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $X \subseteq \mathcal{B}$ and there is an automorphism of $\mathcal{B}$ which inverts the shift.*

Proof. Let $X_s := \{s^m(e) : e \in X \text{ and } m \in \mathbb{Z}\}$ (which is countable) and choose a diagonalizer $d \neq 0$ of X_s. Let $Y := \nu_d[X_s]$ and finally $\mathcal{B} := \langle X_s \cup Y\rangle$ (which is, again, countable). The choice of d implies that $X_s \subseteq \Delta(d)$, which implies that $X_s \subseteq \Delta_s(d)$ because X_s is shift-invariant. From Lemma 9.7 we know that $\mathcal{B} \subseteq \Delta_s(d)$ as well, and that ν_d inverts the shift on $\mathcal{B}$. The fact that X_s is shift-invariant (and ν_d inverts the shift on it) shows that Y is also shift-invariant, and therefore so is $\mathcal{B}$. It remains to show that $\nu_d[\mathcal{B}] = \mathcal{B}$. For this, simply note that $\nu_d[X_s \cup Y] = \nu_d[X_s] \cup \nu_d[\nu_d[X_s]] = Y \cup X_s$ because $\nu_d^2 = \mathtt{id}$. □

Next, we develop a method to create automorphisms which preserve the shift on specific subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. The idea is to use interval decompositions of ω again, but instead of flipping the intervals onto themselves as before, we permute the intervals with each other while preserving the ordering of the elements of each individual interval. The mental image should consist of intervals being moved around as rigid blocks.

Definition 9.11. Let P be an interval decomposition of ω and let $(J_n)_{n\in\omega}$ be an enumeration of P (not necessarily the increasing enumeration). For each $n \in \omega$ write $J_n = [a_n, b_n]_\omega$. Recall that the notation $(J_n)_{n\in\omega}$ is simply an abbreviation of the fact that J is a map defined on ω and $J(n)$ is denoted as J_n for each $n \in \omega$. Throughout this chapter β_J denotes the permutation of ω that maps each $k \in J_n$ to

$$\beta_J(k) := k - b_n - 1 + \sum_{i\leq n} |J_i|.$$

Checking that β_J is a permutation of ω is straight-forward, but the following description makes the task even easier, while helping to understand the action of β_J on the natural numbers: For each $n \in \omega$ let $c_n := \sum_{i<n} |J_i|$ (so that $(c_n)_{n\in\omega}$ is a strictly increasing sequence in ω) and define $K_n := [c_n, c_{n+1} - 1]_\omega$. Then, $\{K_n : n \in \omega\}$ is an interval decomposition of ω, $(K_n)_{n\in\omega}$ is its increasing enumeration, and for each $n \in \omega$ the map $\beta_J \restriction J_n$ is the unique order-preserving bijection from J_n onto K_n.

Lemma 9.12. *Let P be an interval decomposition of ω and let $D := \{\mathtt{max}(I) : I \in P\}$. If $x \subseteq \omega$ satisfies either $D \cap x = \emptyset$ or $D \subseteq x$, then for every enumeration J of P we have $\beta_J S[x] =^* S\beta_J[x]$.*

Proof. Let J be an enumeration of P and write $J_n = [a_n, b_n]_\omega$ for each $n \in \omega$ (so that $D = \{b_n : n \in \omega\}$). If $k \in \omega \setminus D$, let $n \in \omega$ be such that $k \in J_n$ and observe that $k + 1 \in J_n$ as well. This clearly implies that $\beta_J S(k) = S\beta_J(k)$. In particular, if $D \cap x = \emptyset$ we have $\beta_J S[x] = S\beta_J[x]$.

Note that $S[D] = \{a_n : n \in \omega\} \setminus \{0\}$, so $\beta_J S[D] =^* \{a_n - b_n - 1 + \sum_{i \leq n} |J_i| : n \in \omega\}$. For each $n \in \omega$ we have $|J_n| = b_n + 1 - a_n$, so we obtain

$$\beta_J S[D] =^* \left\{ \sum_{i<n} |J_i| \,:\, n \in \omega \right\}$$

On the other hand, for each $n \in \omega$ we have $\beta_J(b_n) = -1 + \sum_{i \leq n} |J_i|$, so

$$\begin{aligned} S\beta_J[D] &= \left\{ \sum_{i \leq n} |J_i| \,:\, n \in \omega \right\} \\ &= \left\{ \sum_{i<n} |J_i| \,:\, n \in \omega \setminus \{0\} \right\} \end{aligned}$$

It follows that $\beta_J S[D] =^* S\beta_J[D]$. Finally, if $D \subseteq x$ we have

$$\begin{aligned} \beta_J S[x] &= \beta_J S[D] \cup \beta_J S[x \setminus D] \\ &=^* S\beta_J[D] \cup S\beta_J[x \setminus D] \\ &= S\beta_J[x] \end{aligned}$$

and the proof is complete. □

Definition 9.13. Let D be an infinite subset of ω and $\sigma \in \mathtt{Sym}(\omega)$. Let $(p_n)_{n\in\omega}$ be the increasing enumeration of $\sigma[D]$ and for each $n \in \omega$ let $b_n := \sigma^{-1}(p_n)$. There is a unique enumeration J of $P(D)$ such that $b_n \in J_n$ for all $n \in \omega$. This enumeration is denoted by $J(D, \sigma)$.

For the next few lemmata we adopt some more notation to facilitate our calculations. Given an infinite set $D \subseteq \omega$ and an element $b \in D$, we let $l_b := |I|$, where I is the unique element of $P(D)$ containing b. Moreover, for each $k \in \omega$ we denote by c_k the minimum of the set $D_{\geq k}$. If D is replaced by a set D', we use l'_b and c'_k instead of l_b and c_k respectively.

The proof of the next lemma is fairly straight-forward.

Lemma 9.14. *Suppose $D \subseteq \omega$ is infinite and $\sigma \in \mathtt{Sym}(\omega)$. Then, for all $k \in \omega$ we have*

$$\beta_{J(D,\sigma)}(k) = k - c_k - 1 + \sum \{l_b \,:\, b \in D \text{ and } \sigma(b) \leq \sigma(c_k)\}.$$

□

Lemma 9.15. *Suppose $D \subseteq \omega$ is infinite. Then, for all $c \in D$ we have*

$$\sum \{l_b \,:\, b \in D \text{ and } b \leq c\} = c + 1.$$

Proof. Let $(b_n)_{n\in\omega}$ be the increasing enumeration of D. We prove the statement for $c = b_n$ by induction. For $n = 0$ we have,

$$\sum \{l_b \,:\, b \in D \text{ and } b \leq b_0\} = l_{b_0} = |[0, b_0]_\omega| = b_0 + 1$$

If we assume that the equality holds for $c = b_n$ for some $n \in \omega$, then

$$\begin{aligned} \sum \{l_b \,:\, b \in D \text{ and } b \leq b_{n+1}\} &= (b_n + 1) + l_{b_{n+1}} \\ &= b_n + 1 + |[b_n + 1, b_{n+1}]_\omega| \\ &= b_n + 1 + (b_{n+1} - b_n) \\ &= b_{n+1} + 1 \end{aligned}$$

□

Lemma 9.16. *Suppose $D \subseteq \omega$ is infinite and $\sigma \in \mathtt{Sym}(\omega)$. Then, $\beta^{-1}_{J(D,\sigma)}$ is also of the form $\beta_{J(D',\sigma')}$. For example, D' may be chosen as $\beta_{J(D,\sigma)}[D]$ and σ' may be chosen as $\beta^{-1}_{J(D,\sigma)}$.*

Proof. Chose D' and σ' be as suggested in the statement. To avoid a few subscripts, let $\beta := \beta_{J(D,\sigma)}$ and $\beta' := \beta_{J(D',\sigma')}$. Since we already know that β is invertible, it suffices to show that β' is a left-inverse of β, i.e. that $\beta'\beta = \mathtt{id}_\omega$.

It is evident that $c_b = b$ for all $b \in D$. Far less evident, but still straight-forward using Lemma 9.14, is that $l'_{\beta(b)} = l_b$ for all $b \in D$, and that $c'_{\beta(k)} = \beta(c_k)$ for all $k \in \omega$. Thus, if $k \in \omega$ we have

$$\begin{aligned}\beta'(\beta(k)) &= \beta(k) - c'_{\beta(k)} - 1 + \sum\{l'_{b'} : b' \in D' \text{ and } \sigma'(b') \leq \sigma'(c'_{\beta(k)})\}\\ &= \beta(k) - \beta(c_k) - 1 + \sum\{l'_{\beta(b)} : b \in D \text{ and } \sigma'(\beta(b)) \leq \sigma'(\beta(c_k))\}\end{aligned}$$

Using Lemma 9.14 we see that $\beta(k) - \beta(c_k) = k - c_k$. Replacing $l'_{\beta(b)}$ with l_b and σ' with β^{-1} above, and applying Lemma 9.15 we get

$$\begin{aligned}\beta'(\beta(k)) &= k - c_k - 1 + \sum\{l_b : b \in D \text{ and } b \leq c_k\}\\ &= k - c_k - 1 + (c_k + 1)\\ &= k\end{aligned}$$

as we wanted to show. □

Lemma 9.17. *Let $D, D' \subseteq \omega$ be infinite and $\sigma, \sigma' \in \mathtt{Sym}(\omega)$. If $D =^* D'$ and $\sigma =^* \sigma'$, then $\beta_{J(D,\sigma)} =^* \beta_{J(D',\sigma')}$.*

Proof. Let $N_0 \in \omega$ be large enough so that $D \setminus N_0 = D' \setminus N_0$ and $\sigma \restriction (\omega \setminus N_0) = \sigma' \restriction (\omega \setminus N_0)$. Since D and D' are infinite, we may assume that $N_0 \in D \cap D'$. If $k > N_0$, then $c_k = c'_k$. Also, if $b \in D \cap D'$ and $b > N_0$, then $l_b = l'_b$.

Choose an upper bound $N_1 \in \omega$ for the set $\{\sigma(n) : n \leq N_0\} \cup \{\sigma'(n) : n \leq N_0\}$. Then, let $N_2 \geq N_0$ be an upper bound for the set $\{m \in \omega : \sigma(m) \leq N_1 \text{ or } \sigma'(m) \leq N_1\}$. This implies that if $m > N_2$ and $n \leq N_0$, then $\sigma(n) < \sigma(m)$ and $\sigma'(n) < \sigma'(m)$.

Suppose $k > N_2$. By Lemma 9.14, if we define

$$\begin{aligned}C_0 &:= \sum\{l_b : b \in D,\ b > N_0, \text{ and } \sigma(b) \leq \sigma(c_k)\}\\ C_1 &:= \sum\{l_b : b \in D,\ b \leq N_0, \text{ and } \sigma(b) \leq \sigma(c_k)\}\\ C'_0 &:= \sum\{l'_b : b \in D',\ b > N_0, \text{ and } \sigma'(b) \leq \sigma'(c'_k)\}\\ \text{and } C'_1 &:= \sum\{l'_b : b \in D',\ b \leq N_0, \text{ and } \sigma'(b) \leq \sigma'(c'_k)\}\end{aligned}$$

then it holds that $\beta_{J(D,\sigma)}(k) = k - c_k - 1 + C_0 + C_1$ and $\beta_{J(D',\sigma')}(k) = k - c'_k - 1 + C'_0 + C'_1$. We have already noted that $c_k = c'_k$. Since $c_k \geq k > N_0$, we also have $\sigma(c_k) = \sigma'(c_k) = \sigma'(c'_k)$. It follows from these and our previous remarks that $C_0 = C'_0$. Observe that for all $b \leq N_0$ we have $\sigma(b) < \sigma(c_k)$ and $\sigma'(b) < \sigma'(c'_k)$ (because $c_k = c'_k > N_2$). Consequently,

$$\begin{aligned}C_1 &= \sum\{l_b : b \in D \text{ and } b \leq N_0\}\\ \text{and } C'_1 &= \sum\{l'_b : b \in D' \text{ and } b \leq N_0\}\end{aligned}$$

Since $N_0 \in D \cap D'$, Lemma 9.15 implies that $C_1 = C'_1 = N_0 + 1$. It follows that, $\beta_{J(D,\sigma)}(k) = \beta_{J(D',\sigma')}(k)$ for all $k > N_2$, which concludes the proof. □

Definition 9.18. Given $d \in (\mathcal{P}(\omega)/\mathtt{fin})\setminus\{0\}$ and $\varphi \in \mathtt{VTriv}$, let $D \subseteq \omega$ be a representative of d and let $\sigma \in \mathtt{Sym}(\omega)$ be such that $\varphi = \varphi_\sigma$. Then, the very trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by $\beta_{J(D,\sigma)}$ is denoted by $\xi_{(d,\varphi)}$.

As before, Lemma 9.17 shows that $\xi_{(d,\varphi)}$ is independent of the choices of D and σ, but in this case the map $(d,\varphi) \mapsto \xi_{(d,\varphi)}$ is not injective. For example, $\xi_{(d,\mathtt{id})} = \mathtt{id}$ for all $d \neq 0$.

Lemma 9.19. *If $d \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ and $\varphi \in \mathtt{VTriv}$, then $\xi_{(d,\varphi)} \restriction \Delta_s(d)$ is an isomorphism onto $\Delta_s(\xi_{(d,\varphi)}(d))$ which preserves the shift.*

Proof. To simplify the notation, let $d' := \xi_{(d,\varphi)}(d)$. By Lemma 9.12, we have $\xi_{(d,\varphi)}s(e) = s\xi_{(d,\varphi)}(e)$ for all $e \in \Delta(d)$, so it suffices to show that $\xi_{(d,\varphi)}[\Delta_s(d)] = \Delta_s(d')$. We need the equality $\xi_{(d,\varphi)}s^m(e) = s^m\xi_{(d,\varphi)}(e)$ for all $e \in \Delta_s(d)$ and $m \in \mathbb{Z}$, but the proof is skipped here because it is similar to what was done in Lemma 9.7.

Given $e \in \Delta_s(d)$ and $m \in \mathbb{Z}$, we have either $d \wedge s^m(e) = 0$ or $d \leq s^m(e)$. Applying $\xi_{(d,\varphi)}$ we get $d' \wedge s^m(\xi_{(d,\varphi)}(e)) = 0$ or $d' \leq s^m(\xi_{(d,\varphi)}(e))$, which shows that $s^m(\xi_{(d,\varphi)}(e)) \in \Delta(d')$. We conclude that $\xi_{(d,\varphi)}(e) \in \Delta_s(d')$ and, by the generality of $e \in \Delta_s(d)$, it follows that $\xi_{(d,\varphi)}[\Delta_s(d)] \subseteq \Delta_s(d')$.

By Lemma 9.16, if we let $\varphi' := \xi^{-1}_{(d,\varphi)}$, then $\xi_{(d',\varphi')} = \xi^{-1}_{(d,\varphi)}$. The argument above shows that $\xi_{(d',\varphi')}[\Delta_s(d')] \subseteq \Delta_s(\xi_{(d',\varphi')}(d')) = \Delta_s(d)$. Applying $\xi_{(d,\varphi)}$ to both sides we obtain $\Delta_s(d') \subseteq \xi_{(d,\varphi)}[\Delta_s(d)]$, which completes the proof. □

Corollary 9.20. *If $d \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus \{0\}$ and $\varphi \in \mathtt{VTriv}$, then $\xi_{(d,\varphi)} \circ \nu_d \restriction \Delta_s(d)$ is an isomorphism onto $\Delta_s(\xi_{(d,\varphi)}(d))$ which inverts the shift.* □

There are still many important unanswered questions about this method of building shift-preserving and shift-inverting isomorphisms. For example, in most cases there are larger shift-invariant algebras $\mathcal{B} \supsetneq \Delta_s(d)$ on which $\xi_{(d,\varphi)}$ preserves the shift, but we do not know what the largest of these algebras is. It is also not well understood how the automorphisms ν_d and $\xi_{(d,\varphi)}$ "change" when we choose different d's and φ's. In particular, if a certain automorphism ν_d inverts the shift on some countable algebra $\mathcal{B}$, it would be interesting to know for which d' it holds that $\nu_d \restriction \mathcal{B} = \nu_{d'} \restriction \mathcal{B}$. The analogous question regarding $\xi_{(d,\varphi)}$ is just as relevant. So far, besides Theorem 9.10, the usefulness of this method has been restricted to the construction of examples to test different conjectures. The next theorem is one such application. The sets $\mathbb{P}^+$, $\mathbb{P}^-$, D^+_e and D^-_e in the statement are those defined in Chapter 8 directly above Lemma 8.34.

After proving Lemma 8.34, I tried to come up with more examples using the same recipe: Find elements $e_0, e_1, e \in \mathcal{P}(\omega)/\mathtt{fin}$, isomorphisms $\varphi^+ \in \mathbb{P}^+$ and $\varphi^- \in \mathbb{P}^-$ both of which map e_0 to e_1 and whose domains do not contain e, and quantifier-free formulas $\alpha(x)$ and $\beta(x,y)$ in the language of Boolean algebras with an additional unary function symbol such that

(1) Both $\alpha[e]$ and $\beta[e,e_0]$ hold in $(\mathcal{P}(\omega)/\mathtt{fin}, s)$

(2) Whenever $\alpha[e']$ holds in $(\mathcal{P}(\omega)/\mathtt{fin}, s)$, the formula $\beta[e', e_1]$ does not.

(3) Whenever $\alpha[e']$ holds in $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$, the formula $\beta[e', e_1]$ does not.

Then, conclude that if ψ were an extension of φ^+ in $\mathbb{P}^+$ or an extension of φ^- in $\mathbb{P}^-$ and e were in its domain, then $\alpha[\psi(e)]$ and $\beta[\psi(e), e_1]$ would necessarily be true in $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ or $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ respectively, and this would contradict (2) or (3) respectively.

However, due to the complexity of this construction, I was only able to find examples in which α only has finitely many solutions in $(\mathcal{P}(\omega)/\mathtt{fin}, s)$. Since the shift is an automorphism of this structure, it follows that the set of solutions of α is shift-invariant. Thus, if α only has finitely many solutions, they must all be shift-periodic and, in particular, e

is shift-periodic. So, this method only provided examples in which there were still shift-periodic elements to be added to the domains of φ^+ and φ^-. The following theorem uses the tools developed in the current chapter to fix this problem.

Theorem 9.21. *There are $\varphi^+ \in \mathbb{P}^+$, $\varphi^- \in \mathbb{P}^-$ and $e \in \mathcal{P}(\omega)/\mathtt{fin}$ such that*

$$\mathtt{Per} \subseteq \mathtt{dom}(\varphi^+) \cap \mathtt{dom}(\varphi^-)$$

and neither φ^+ has an extension in D_e^+ nor φ^- has an extension in D_e^-.

Proof. We define a sequence $(x_n)_{n\in\omega}$ by letting $x_0 := 0$ and then inductively defining $x_{n+1} := x_n + 8(n!)$ for each $n \in \omega$. Next, we let

$$D := \bigcup_{n\in\omega} \{x_n, x_n + 2(n!), x_n + 4(n!), x_n + 6(n!)\}, \quad d := [\![D]\!],$$
$$A := \bigcup_{n\in\omega} \{x_n + n!, x_n + 5(n!) + 1\}, \qquad a := [\![A]\!],$$
$$B := \{x_n + 3(n!) \,:\, n \in \omega\}, \qquad b := [\![B]\!],$$
$$\text{and} \;\; \mathcal{B} := \langle \mathtt{Per} \cup s^{\mathbb{Z}}(a) \cup s^{\mathbb{Z}}(b)\rangle$$

The algebra $\mathcal{B}$ is the domain of both isomorphisms φ^+ and φ^-.

The first step is to prove that d diagonalizes $\mathcal{B}$. It is easy to see that both $D \cap S^m[A]$ and $D \cap S^m[B]$ are finite for all $m \in \mathbb{Z}$, so d diagonalizes $s^{\mathbb{Z}}(a) \cup s^{\mathbb{Z}}(b)$. For each $k \geq 1$, note that all elements of $D \setminus x_k$ are in the same congruence class modulo k. This means that for some $i \in k$ we have $d \leq s^i(\mu_k)$, and (obviously) for every $j \in k \setminus \{i\}$ we have $d \wedge s^j(\mu_k) = 0$. In particular, d diagonalizes the set of atoms of $\mathtt{Per}_k$, so it follows that it diagonalizes all of $\mathtt{Per}_k$. Putting all of this together we conclude that d diagonalizes $\mathtt{Per} \cup s^{\mathbb{Z}}(a) \cup s^{\mathbb{Z}}(b)$ and therefore also $\mathcal{B}$. Since $\mathcal{B}$ is clearly shift-invariant, we have $\mathcal{B} \subseteq \Delta_s(d)$.

This information already suffices to define φ^- as $\nu_d \restriction \mathcal{B}$ and conclude from Lemma 9.7 that $\varphi^- \in \mathbb{P}^-$. To define φ^+ we need an auxiliary automorphism $\varphi \in \mathtt{VTriv}$. Let $\sigma \in \mathtt{Sym}(\omega)$ be the permutation that maps $x_n + 4(n!)$ to x_{n+1} and vice versa for every $n \in \omega$ and which leaves every other element fixed (in symbols: $\sigma = \prod_{n\in\omega}(x_n + 4(n!) \quad x_{n+1})$). Then, let φ be the automorphism induced by σ and define $\varphi^+ := \xi_{(d,\varphi)} \restriction \mathcal{B}$. Lemma 9.19 implies that $\varphi^+ \in \mathbb{P}^+$.

The next step is to define e. It shall be the unique solution to the system

$$\begin{cases} e \wedge s(e) = 0 \\ e \vee s(e) = \neg a \end{cases}$$

Let us prove the uniqueness part, and the process will show how to prove that such a solution exists. Assuming e satisfies both equalities, let $E \subseteq \omega$ be one of its representatives. Then, there is $N \in \omega$ such that $E \cap S[E] \subseteq N$ and $(E \cup S[E]) \setminus N = (\omega \setminus A) \setminus N$. The first condition means that for all $k \geq N$ we have $\{k-1, k\} \not\subseteq E$. The second condition means that for $k \geq N$ we have $E \cap \{k-1, k\} \neq \emptyset$ if and only if $k \notin A$. Therefore, if I is an interval in $\omega \setminus N$ and is disjoint from A, then E contains either precisely the even elements of I, or precisely the odd ones. After checking a few details it is easy to see that for n large (which here means that $n!$ is even, i.e. $n \geq 2$, and $x_n \geq N$) we have:

(1) If $k \in [x_n, x_n + n! - 1]_\omega$, then $k \in E$ if and only if k is even.

(2) If $k \in [x_n + n!, x_n + 5(n!)]_\omega$, then $k \in E$ if and only if k is odd.

(3) If $k \in [x_n + 5(n!) + 1, x_{n+1} - 1]_\omega$, then $k \in E$ if and only if k is even.

These necessary conditions show the uniqueness of e. To prove existence, we define the set E using conditions (1), (2) and (3) (for $n \geq 2$) and check that $E \cap S[E] \subseteq (x_2 + 1)$ and $(E \cup S[E]) \setminus (x_2 + 1) = (\omega \setminus A) \setminus (x_2 + 1)$, which is straight-forward.

Suppose, for a contradiction, that ψ is an extension of φ^+ in D_e^+. The remaining of the proof requires several lengthy but easy computations, which are skipped here. The first example is proving that $\varphi^+(a) = a$. It follows that

$$\begin{cases} \psi(e) \wedge s(\psi(e)) = 0 \\ \psi(e) \vee s(\psi(e)) = \neg a \end{cases}$$

and so (by the uniqueness proved in the previous paragraph) we have $\psi(e) = e$. Let $B' := \{x_n + 7(n!) : n \in \omega\}$ and $b' := [\![B']\!]$. Another lengthy calculation shows that $\varphi^+(b) = b'$. Condition (2) above implies that $e \wedge b = 0$ and therefore $e \wedge b' = \psi(e) \wedge \psi(b) = 0$. However, condition (3) implies that $b' \leq e$, which is clearly a contradiction.

Finally, for another contradiction, suppose that ψ is an extension of φ^- in D_e^-. Let $A' := \bigcup_{n \in \omega}\{x_n + n! + 1, x_n + 5(n!)\}$ and $a' := [\![A']\!]$. One more tedious computation shows that $\varphi^-(a) = a'$. It follows that

$$\begin{cases} \psi(e) \wedge s^{-1}(\psi(e)) = 0 \\ \psi(e) \vee s^{-1}(\psi(e)) = \neg a' \end{cases}$$

If we take a representative E' of $\psi(e)$ and proceed as we did with E above, we obtain that the following conditions hold whenever n is large enough:

(1') If $k \in [x_n, x_n + n! + 1]_\omega$, then $k \in E'$ if and only if k is even.

(2') If $k \in [x_n + n! + 2, x_n + 5(n!)]_\omega$, then $k \in E'$ if and only if k is odd.

(3') If $k \in [x_n + 5(n!) + 1, x_{n+1} - 1]_\omega$, then $k \in E'$ if and only if k is even.

The last tedious computation shows that $\varphi^-(b) = s(b)$. Again, since $e \wedge b = 0$, we have $\psi(e) \wedge s(b) = 0$. However, condition (2') implies that $s(b) \leq \psi(e)$, which is clearly another contradiction and completes the proof. $\square$

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