Essays on Dynamic Sales and Matching Mechanisms

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Contents

Chapter 0

Introduction

For the design of markets and contractual relationships dynamics play an important role. Dynamic aspects are, for example, information that evolves over time, or changing environments such as the arrival of agents and changing preferences. This collection of three essays contributes to the corresponding area of microeconomic theory: Chapters 1 and 2 address dynamic contracting problems and Chapter 3, which is written jointly with Benjamin Schickner, studies a dynamic market design framework.

The three chapters have in common that they study a dynamic design question by applying concepts from the theory of dynamic mechanism design. Formulated in general terms, mechanism design studies which social choice functions a designer can implement when agents have private information about their preferences over outcomes. A social choice function maps agents' private information into outcomes. As the designer does not have access to the agents' information, he needs to provide the right incentives for the agents. Specifically, he needs to design a game in which the agents participate and which has an equilibrium in which for each profile of private information the intended outcome results. A social choice function is said to be implementable if such a game exists. Dynamic mechanism design is the extension of mechanism design to account for dynamic environments.

While all three essays use concepts from dynamic mechanism design, the essays study two substantially different applications. Chapters 1 and 2 study the optimal contract design by a revenue-maximizing monopolist. The designer of the mechanism is, hence, the monopolist. From a theoretical perspective, the problem in this case is to find the social choice function among all implementable functions that maximizes the designer's revenue. The essays are mainly interested in the characterization of this social choice function. The dynamic aspect in both chapters is that the buyer learns his preferences over the goods offered by the monopolist only gradually over time.

Chapter 3 examines the welfare-maximizing design of a dynamic matching market. The designer's problem is to decide when and how to form groups of heterogeneous agents that arrive gradually over time to a matching market. While the welfare-maximizing policy is first found in the absence of informational fictions, the second part relates to dynamic mechanism design. It is shown that the welfare-maximizing social choice rule is implementable when the agents' types are their private information. From the conceptual perspective, the problem differs from Chapters 1 and 2 as a given social choice rule is shown to be implementable. The remainder of the introduction explains the model, its motivation, and the findings of each chapter in more detail.

All three chapters present independent models, however, Chapters 1 and 2 are connected. Both chapters study the optimal contract design of a firm which offers horizontally differentiated goods to consumers. Goods are horizontally differentiated when the consumers do potentially not agree on the ranking of the products according to their preferences. The dimension along which goods are differentiated is for example the departure time flights, or other product characteristics like size and color. Differently from a standard model of horizontal product differentiation, the consumers learn their preferences only gradually over time. This means at the first point in time where contracting is possible, the consumer has some idea about his preferences. However, he learns his exact preferences only when consumption takes place. This information structure applies for example when booking a flight well in advance, when shopping online, when purchasing experience goods, or when buying goods in behalf of other persons. In many environments concerned, the firms react to the consumers' initial uncertainty by offering a menu of contracts which equip the consumer with differently generous exchange and refund policies. Chapters 1 and 2 explore the reasons why firms offer these menus, they study the optimal design of exchange policies, and they identify which characteristics of the model drive the particular design of exchange policies.

Chapter 1 specifies further structure on the information arrival. It studies consumers who are initially uncertain about their favorite variants, but do know how much importance they attach to the feature which differentiates the goods. Only later they learn which variant of the good they favor. For example, when booking a flight well in advance, travelers initially do not know the departure time they will actually prefer. Travelers do, however, know whether their journey is for business or leisure, which indicates their flexibility in terms of time. First, the essay shows that in this framework exchange policies with varying flexibility are employed as a price discrimination device. Second, it provides a novel reason for the occurrence of partially restrictive exchange policies. Third, it studies the optimal design of restrictive exchange policies. Any optimal restrictive exchange policy can be implemented by a Limited Exchange Contract. A deterministic Limited Exchange Contract specifies a price, an initial product choice and a limited range of products within which free exchange is possible. The key property of Limited Exchange Contracts is that prices are completely pinned down at the contracting stage. In particular, the use of exchange fees to govern exchanges is not optimal.

While there is a number of situations in which Limited Exchange Contracts are commonly used to govern exchanges[1], contracts that restrict exchanges via exchange fees are also widespread in real world situations. Therefore, Chapter 2 analyzes when a menu of contracts with exchange fees maximizes the seller's profits. The use of exchange fees is optimal if two key properties are satisfied: When contracting takes place, the buyer is uncertain about his difference in valuations and the seller does not know which magnitude of differences the buyer roughly expects. The chapter provides several foundations when this information structure applies. It furthermore shows that dynamic screening is not beneficial at all when buyers initially only differ in their belief which good they prefer but expect similar magnitudes of valuation differences. The first key property differentiates Chapter 2 from Chapter 1. In Chapter 1 the consumers know how much they suffer from obtaining the "wrong" good already at the contracting stage. A consequence is that the agents' private information at the contracting stage already determines how much flexibility the designer wants to grant each consumer. The most advantageous way to implement this flexibility is via Limited Exchange Contracts. In Chapter 2 the buyer learns only after contracting how much he suffers from ending up with the "wrong" good. This provides an incentive for the designer to adapt the consumer's flexibility to this ex post information. The only way to implement such an allocation is via exchange fees.

Chapter 3 is concerned with dynamic market design. It studies a dynamic version of a simple, cardinal, one-sided matching model. Agents that arrive gradually over time join forces in order to generate output. The agents are heterogeneous and when forming a group their characteristics are complements in the production function. Matches are irrevocable. In a static environment, the complementarity of the agents' characteristics implies that it is beneficial to match agents of similar characteristics. This outcome is called a positive assortative matching. The dynamic arrival of agents combined with impatience, however, poses a challenge to positive assortativeness. If future outcomes are discounted, the desirability of early matches increases both from a social welfare as well as an participating individual's perspective. This chapter studies the resulting trade-off between matching agents early and waiting for a thickening of the market to match assortatively. The model addresses a trade-off underlying a wide range of situations including the formation of teams and task assignment within firms, as well as the establishment of partnerships that constitute organizations themselves. Examples are amongst others consultancy in firms, coauthoring at universities, education in groups,

[1] An example is the contract design by the ferry companies DFDS Seaways and P&O Ferries. Also airlines often offer costless or very cheap same-day exchanges and stand-by options.

or team sports in clubs.

First, the essay addresses the welfare-maximizing matching procedures under complete information. It develops a tool that allows for solving for the welfare-maximizing matching policy in closed form without imposing restrictions on the policy. Second, it studies implementability of the welfare-maximizing matching policy when agents have private information about their types. The essay proves that the policy is implementable in a strong solution concept with contracts that satisfy natural requirements. Furthermore, it identifies situations in which the central authority can abstain from using monetary incentives. Finally, it addresses the case in which the agents can, in addition to their private type, hide their arrival.

Chapter 1

Pricing Heterogeneous Goods under Ex Post Private Information

This paper studies optimal contract design by a firm which faces consumers who know how important the good is to them but only later learn which variant of the good they favor. Exchange policies with varying flexibility are employed as a price discrimination device. I provide a novel reason for the occurrence of partially restrictive exchange policies. Optimal contracts specify a price, an initial product choice, and a limited range of products within which exchange is costless. Crucially, optimal contracts do not use exchange fees to govern exchanges. This contrasts with standard results in the literature on sequential screening.

1.1 Introduction

In many situations, firms contract with consumers who know about their desire to buy a certain kind of good, but only later learn which variant they favor. An example is the sale of tickets for means of transportation such as planes, buses, trains, or ships to customers who do not yet know their favorite departure times. Further instances include online shopping and the sale of experience (e.g. packaged) goods when there are several variants of the good as well as various procurement settings in which the contractor's favorite delivery time of the good or service is uncertain. Firms apparently react to the consumers' uncertainty by designing an elaborate system of exchange policies that are part of the sales contract. For example, in the airline industry, consumers are typically offered comparatively cheap tickets, which, however, entail restrictions regarding refund and exchange. More flexible tickets for the same flight are offered at a higher price.

This paper derives the firm's optimal pricing policy in the situations described above, which amounts to providing a theory of exchange policies. I capture the common structure underlying the examples by considering a firm which sells horizontally differentiated goods to consumers. The consumers are initially uncertain about their favorite variants, but do know how much importance they attach to the feature which differentiates the variants. The paper departs from the classical literature on sequential screening that to date has primarily studied the sale of homogeneous goods to consumers who learn their valuations for the good gradually over time.[1] Extending the model to heterogeneous goods enables me to address product choice.

My first contribution is to give a price-discrimination based explanation for the observation that firms offer menus of contracts with different exchange policies. Second, the paper provides a novel reason for offering contracts with an intermediately restrictive exchange policy. Third, I study the optimal design of restrictive exchange policies. Any optimal restrictive exchange policy can be implemented by a Limited Exchange Contract. A deterministic Limited Exchange Contract specifies a price, an initial product choice and a limited range of products within which free exchange is possible in the second period.[2] The key property of Limited Exchange Contracts is that prices are completely pinned down at the contracting stage. In particular, the use of exchange fees to restrict exchanges is not optimal. The concept of Limited Exchange Contracts is observed in practice: many US airlines offer costless or very cheap same-day exchanges and stand-by options. The use of Limited Exchange Contracts is also widespread among European ferry companies.[3] Finally, I compare the optimal mechanisms of the classical homogeneous goods model with those of my heterogeneous goods model. Both mechanisms have in common that each agent is screened sequentially. However, in the homogeneous goods model, screening necessarily involves type dependent prices in each period, whereas screening in the heterogeneous goods model involves prices only in the initial period.

My results are driven by three main features of the model, which reflect characteristics from the applications in mind. The first feature is the heterogeneity of the goods. I consider a monopolist which may offer horizontally differentiated goods to consumers with unit demand and single-peaked preferences. Applied to the airline example, tickets are differentiated by the departure time of the flight. In the shopping examples, products differ in a characteristic such as size, fit, or color. And in the procurement setting, any order specifies a delivery time for the good or service.

The second main feature of the model is the information arrival. Consumers learn their valuations for the goods in two stages. In the first period, henceforth referred to

[1] The canonical contribution is Courty and Li (2000). More papers are cited in the literature review.
[2] Stochastic Limited Exchange Contracts are an extension to distributions over goods.
[3] An example is the contract design by DFDS Seaways and P&O Ferries. While DFDS Seaways permits customers to take one ferry earlier or later, P&O Ferries specifies time intervals around the booking time in which costless exchange is possible.

as the ex ante stage, consumers are uncertain about their favorite variant, but differ in the privately known valuation of the favorite variant and the relative valuation losses from obtaining non-favorite products.[4] In the second period, the consumers privately learn which variant they prefer most, and consumption takes place. For example, when booking a flight well in advance, travelers initially do not know the departure time they will actually prefer. A traveler's valuation loss associated with flying at a time other than the favorite departure time is a measure of his flexibility in terms of time. This flexibility crucially depends on whether the journey is for business or not, which is known ex ante. Online shopping and the purchase of experience goods have in common that the shopper cannot entirely evaluate the product immediately and hence the favorite variant is uncertain to the consumer. Despite this uncertainty, consumers already know how important the product feature which differentiates the variants is to them. And upon the conclusion of many procurement contracts, the contractor has uncertainty regarding his internal work-flow and hence about his preferred delivery time. He knows, however, how tightly operational procedures are packed in his company which influences the cost of amending delivery times.

The third main feature is a positive relation between the valuation of the most preferred product, which will be referred to as the top valuation, and the relative loss in valuation when obtaining non-favorite products. Compared to leisure travelers, business travelers value the flight at their favorite time more, but are less flexible concerning departure times. Typically, shoppers who attach importance to a product group value its consumption a lot, but are at the same time relatively selective concerning specific product features. And contractors with tighter operational schedules use fewer resources for the same task and hence generate higher revenues, but rearranging processes to hold delivery times is comparatively costly to them.

In this framework, I explain the observation that firms offer menus of contracts with different exchange policies by a price discrimination based motive. The revenue-maximizing menu, which is found using a mechanism design approach without restrictions on contracts, consists of two offers. It contains an expensive contract that allows for costless exchange with any other variant, and a cheaper contract that limits exchanges of products in the second period. Consumers that care a lot about obtaining the favorite variant choose the expensive contract, whereas consumers who attach less importance to whether they consume the favorite variant take the cheaper one. This menu corresponds to common observations in the airline, train, and ship industry.[5] The driving force behind the establishment of the menu of contracts with differing

[4] In other words, the consumers are uncertain about their ordinal preferences over the goods, but have private information about their intensity of preferences.

[5] Many countries forbid the use of restrictive exchange policies in the shopping examples. For these applications, this paper, hence, predicts what changes would occur in case the law was repealed.

exchange policies is a price discrimination motive: the utility consumers derive from a contract which allows them to exchange variants arbitrarily equals their top valuation. If only this contract is offered, the firm has to trade off leaving rent to the consumers with high top valuations against excluding those with low top valuations. The firm can, however, exploit the fact that consumers with high top valuations also have a strong demand for flexibility in product choice in the second stage. This makes it profitable to offer a second contract with little such flexibility in order to extract more rent from consumers with high top valuations.

In particular, the present paper provides a non-standard explanation for the appearance of contracts that *partially* restrict exchanges - an observation that is often made, for example in ticket pricing. This means that the consumer is neither granted free exchange to whatever variant he prefers nor is he restricted to definitively staying with the initially purchased good.[6] In particular, this result shows that the restriction on contracts made by Gale (1993) in his pioneering work on contracting in situations with ex post private information, excludes the optimal solution. The partial flexibility restriction in the cheaper contract is a result of countervailing incentives in the sense of Lewis and Sappington (1989). "Countervailing incentives" means that it depends on the contract whether the consumer's incentive to over- or understate his top valuation is a binding constraint for revenue maximization. The origin of countervailing incentives is that it depends on the specific contract whether it is valued more or less by consumers with higher top valuations.[7] This is the case in the present model: a consumer with a higher top valuation also suffers more from ending up with unfavorable goods. Consequently, contracts with very restrictive exchange policies might be valued less by him. The contract with unrestricted exchange is, however, valued more by consumers with higher top valuations. This paper provides a particularly tractable method for solving problems with countervailing incentives in linear environments. I show that in my model the optimal cheaper contract is designed such that each consumer values it the same. This is the case for the partially distorted contract.

Finally, I answer the question of how the partial restriction in flexibility is optimally designed. The key optimality condition requires that the price is entirely pinned down at the contracting stage. The remaining way to induce the consumer to not always obtain his favorite variant is then to reduce the range of goods the consumer can choose from in the second period.[8] A contract is partially restrictive if this range contains more

[6] The optimality of such contracts is unusual in linear environments like in my model, in which optimal mechanisms typically satisfy the "bang-bang" property.

[7] The connection between these two statements follows from the well known result that binding incentive constraints are the ones where consumers understate the utility they drive from a contract.

[8] In order to provide a clear intuition, in the introduction I describe optimal mechanisms which are deterministic. In the main text, I allow for stochastic allocations, which are probability distributions over goods.

than one variant but not all variants. Contracts with this structure are implementable by Limited Exchange Contracts. Connecting transfers to the consumer's second period choice as a way to induce consumers to not always consume their favorite good is not optimal. This can best be seen when considering the use of exchange fees to limit exchanges. Even though initially consumers do not know their favorite variant, they already have private information about the extent to which they react to the exchange fee: consumers who care a lot about obtaining their favorite variant will often be willing to pay the exchange fee as compared to consumers who don't care much about the variant. As this information is privately known by the consumers already at the contracting stage, this forces the monopolist to leave additional information rents to the agents. The firm can shut down this source of information rents by offering Limited Exchange Contracts: consumers' plans for how to act in the second period then do not depend on their information held at the contracting stage.

The result on the optimality of Limited Exchange Contracts implies an interesting relation to the literature on sequential screening. Similar to my results, the optimal mechanism in Courty and Li (2000) screens the consumers in each period.[9] Because in their model there is just a homogeneous good, the only incentive compatible way to screen agents in the second period is via a price difference, which is the refund for giving back the good. But for the same reason as in my model, this gives rise to information rents to the consumers. The presence of heterogeneous goods in my model equips the monopolist with the ability to screen agents in the second period without setting prices that depend on the second period choice. From this perspective, the new tool is superior for the firm as it gives rise to less information rents left to consumers.

Related Literature. This paper contributes to the growing literature on contracting with agents whose private information evolves dynamically over time, which has its origin in the primary contribution by Baron and Besanko (1984). In particular, I expand the canonical sequential screening model introduced by Courty and Li (2000) to differentiated goods. Courty and Li (2000) set up a theory of intertemporal pricing to explain the prevalence of partial refund contracts. In their model there is one homogeneous type of good and consumers initially have individual uncertainty about their final valuation for it. This uncertainty is then resolved in the second period, which is also when consumption takes place. As both the initial valuation distributions and the final valuations differ between agents and are private information, a revenue-maximizing monopolist sequentially screens the agents. The authors show that for some cases revenue maximization occurs by offering menus of partial refund

[9] In my model, consumers are screened in the second period in the sense that depending on their favorite good, consumers end up with different variants.

contracts.[10] A partial refund contract consists of an initial payment in order to receive the good and a later option to return it and receive a partial refund. Since Courty and Li (2000), the design of revenue-maximizing contracts has been examined in many variants and extensions of their model. Examples are Battaglini (2005), Nocke et al. (2011), Boleslavsky and Said (2013), Akan et al. (2015), and Deb and Said (2015).[11] [12] In contrast to the present paper, all the papers cited above have in common that there is just one homogeneous good involved.

There is a handful of papers in which agents' preferences over differentiated goods are gradually learned over time. Gale (1993) studies intertemporal pricing policies in a setting similar to the two-goods version of my model. In order to obtain a fruitful comparison between monopolistic and oligopolistic pricing behavior, Gale, however, restricts his considerations to two types of contracts: late purchases with a single price for both goods and early purchases at an advance-purchase discount but without any possibility of exchange or refund. As I will show, allowing for the full range of possible contracts further raises the monopolist's revenue. A major contribution of my paper is the characterization of this new type of contract. Furthermore, Gale considers a two-product case with special attention to advance-purchase discounts. However, it turns out that it is specifically the richness of my setting which allows an understanding of the underlying effects and enables a modeling of exchange policies as such. The two papers Gale and Holmes (1992, 1993) start with the same basic framework as Gale (1993) but depart from price discrimination and focus on how intertemporal pricing rules can optimally resolve capacity problems. In contrast, my focus is on pure price discrimination motives without any capacity constraints. Recently, Möller and Watanabe (2016) rediscovered the early model on advance-purchase discounts with differentiated goods as a way to introduce oligopolistic competition. In order to obtain a tractable analysis of strategic interaction, they use a stylized two-goods model and restrict strategies to advance-purchase discounts as well.

My paper points out that in the presence of differentiated goods, the monopolist can eliminate one source of information rents as compared to the standard sequential

[10] In their model, the results better match reality when a business traveler's valuation distribution differs from a leisure traveler's distribution by a spread, rather than by first order stochastic dominance, which they initially considered to be natural. My model provides a foundation for this assumption made by Courty and Li (2000). Paralleling their initial intuition, my model exhibits a higher top valuation for business travelers. However, the relative valuation loss from consuming unfavorable goods is also larger for them. When the variant that can be offered is fixed, as assumed in Courty and Li (2000), distances to the favorite product vary and the steeper loss function for business travelers leads to greater fluctuations in valuation for the offered variant.

[11] For a textbook treatment see Chapter 11 in Borgers et al. (2015).

[12] Additionally, there is an extensive literature restricting considerations to advance-purchase discounts. An advance-purchase discount enables an agent to buy a certain good at an early point in time at a discount, but without any possibility of refund. Examples are DeGraba (1995), Courty (2003b,a), and Möller and Watanabe (2010).

screening setup with one homogeneous good. This relates my work to further contributions to the literature on sequential screening that focus on information rents and the question of whether disclosure of ex post private information to the agents is beneficial for the monopolist. Examples are Eső and Szentes (2007a,b), Krähmer and Strausz (2015a,b), and Li and Shi (2015). While the latter papers study a model with one homogeneous good, Eső and Szentes (2015) study the role of information rents in a more general setting. The design of incentive-compatible mechanisms in dynamic settings in which information gradually evolves over time has been studied in a general environment by Pavan et al. (2014).

Finally, my analysis relates to mechanism design problems with continuous types and type-dependent outside options. The pioneering contribution is Lewis and Sappington (1989). A continuative analysis is done by Maggi and Rodriguez-Clare (1995) and a general exposition is Jullien (2000). While the latter two papers apply results from optimal control theory to obtain a solution, Nöldeke and Samuelson (2007) provide an alternative approach. I add to this literature by providing a particularly tractable method for solving problems with countervailing incentives in linear environments.

The paper is organized as follows. In Section 1.2, I introduce a simple version of my model to present the method I use to solve the maximization problem, to show basic properties of optimal mechanisms and to clarify the relation to Gale (1993). A key feature of optimal contracts is that some consumers are partially restricted in their flexibility to exchange variants in the second period. In Section 1.3, a more general model is introduced in order to study the optimal design of this partial limitation of flexibility and its implementation. Finally, Section 1.4 concludes.

1.2 The two goods model

1.2.1 Model

Consider a firm that sells two differentiated goods to a consumer with unit demand. The consumer learns his valuations for good 1 and good 2 gradually over two periods. In the first period, the consumer learns his valuations for the preferred good and for the alternative good. The consumer is, however, uncertain which good he prefers. In the second period, the consumer's uncertainty is resolved.

The consumer's information in the first period is represented by ex ante types τ and his information in the second period by the ex post type θ.[13] The ex ante type $\tau \in T = [0, \overline{\tau}]$ determines two valuation levels $v^+(\tau)$ and $v^-(\tau)$ with $v^+(\tau) \geq v^-(\tau)$.

[13] The consumer can alternatively be interpreted as representing a continuum of consumers with unit demand and total mass normalized to one.

The valuation of the preferred good, $v^+(\tau)$, is referred to as top valuation and $v^-(\tau)$ is the valuation of the alternative good. The ex post type $\theta \in \Theta = \{\theta_1, \theta_2\}$ determines which valuation level is associated with which good. I denote this information by ex post types θ_1 and θ_2 for preferring good 1 and 2 respectively.[14] Ex post types are equally likely, independent of ex ante types.[15]

The ex ante type determines the valuation premium $v^+(\tau) - v^-(\tau)$, which is the loss of ending up with the wrong good. I assume that both the top valuation and the valuation premium are increasing in ex ante types. Let the lowest ex ante type be indifferent between the goods, implying that the valuation premium is zero.

For the sake of tractability, I assume that valuations are linear in ex ante types. It turns out to be helpful to rescale ex ante types such that valuations are

$$v^+(\tau) = v - \delta\tau + \tau$$
$$\text{and} \qquad v^-(\tau) = v - \delta\tau - \tau$$

with $\delta < 1$.[16] The upper bound on δ ensures that both the top valuation $v^+(\tau)$ and the valuation premium $v^+(\tau) - v^-(\tau) = 2\tau$ are increasing in ex ante types. Note that I allow for $v^-(\tau)$ to be decreasing in ex ante types. Likewise, the common trend in both valuations $-\delta\tau$ may be in- or decreasing in ex ante types depending on δ. Let the basic valuation v be high enough such that $v^+(\tau)$ and $v^-(\tau)$ are positive.

Ex ante types τ are continuously distributed over the type space $T = [0, \overline{\tau}]$ with density function $f(\tau)$ and probability distribution function $F(\tau)$. Let the distribution satisfy the standard assumption of increasing virtual values $\tau - \frac{1-F(\tau)}{f(\tau)}$.

The firm is a revenue-maximizing monopolist that can produce good 1 and good 2 at constant marginal cost which is normalized to zero. The firm has full commitment and can contract the consumer in the first period. At that stage the consumer already privately knows his ex ante type but is still uncertain about his ex post type. At the contracting stage the consumer has an outside option of zero. In the second period, the consumer privately learns his ex post type and afterwards consumption takes place.

1.2.2 First best

In order to convey a basic economic intuition for the firm's problem, I briefly discuss the firm's optimal behavior under complete information. In the absence of private

[14] An abstract way to view the information arrival is the following: The ex post type reveals the preference-order, which is the ordinal dimension of preferences. The ex ante type provides information only about the intensity of preferences, which is the cardinal dimension of preferences.

[15] Given independence, the extension to any distribution is technically without further complications.

[16] The general formulation of the linear valuation levels is $v^+(\tau) = v + \beta\tau$ and $v^-(\tau) = v + \gamma\tau$ with $\beta > 0$ and $\gamma < \beta$. Rescale by multiplying ex ante types with $\frac{\beta-\gamma}{2}$. Then define $\delta = \frac{\beta+\gamma}{\gamma-\beta}$.

information, the firm can extract the entire surplus. Thus the revenue-maximizing firm maximizes welfare. The corresponding first best provision of goods is to always give the consumer his preferred good. This is surplus maximizing, as all valuations by assumption exceed production costs and the provision of all goods is equally costly. The firm achieves first best profits if it implements this allocation rule and then extracts all rents by charging the consumer with ex ante type τ his top valuation $v^+(\tau)$.

Next, consider the case when the consumer has private information. If the consumer's top valuation was independent of the ex ante type τ, which would be the case for $\delta = 1$, the firm could indeed achieve first best profits. A simple way to attain these profits would be to charge the consumer a first period payment equal to his top valuation and then allow him to pick his favorite good in the second period. However, by assumption the top valuation is increasing in τ. To achieve first best profits, the firm would need to induce different types to sign contracts that differ in payments but still guarantee the consumer his top choice. This is not possible. Hence, when only offering contracts that guarantee the consumer to obtain his favorite good, the firm faces the standard monopoly trade-off of leaving rents to high types and excluding low types. As shown in the full analysis of the problem, the firm can, however, do better by offering a contract that does not always guarantee the consumer his favorite good. The reason is that the firm can profitably price discriminate exploiting the fact that with increasing τ also the valuation premium is increasing.

1.2.3 Analysis

As the firm has full commitment power, the revelation principle applies (see Myerson (1986)), which allows me to concentrate on direct and incentive compatible mechanisms. A direct mechanism specifies for any reported pair of types $(\hat{\tau}, \hat{\theta})$ a price p paid by the consumer to the firm and an allocation X. A general allocation is a probability distribution over all possible sets of goods that the consumer can end up with. These are "only good 1", "only good 2", "both goods" and "no good". An allocation is described by $X = (x_1, x_2, x_{1\&2})$ with $x_1, x_2, x_{1\&2} \in [0,1]$ and $x_1+x_2+x_{1\&2} \leq 1$. The three entries denote the probabilities for the first three sets of goods, respectively. Hence, a direct mechanism is the combination of an allocation rule $\{X(\hat{\tau}, \hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$ and a payment rule $\{p(\hat{\tau}, \hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$. For a given report about the ex ante type, $\hat{\tau}$, I call $\{X(\hat{\tau}, \hat{\theta}), p(\hat{\tau}, \hat{\theta}) : \hat{\theta} \in \Theta\}$ a contract. A contract is defined as a mapping from ex post type reports into allocations and prices. The choice of the ex ante report then corresponds to the choice of a contract and the choice of an ex post report determines an option within that contract.

Given a pair of types (τ, θ_i), $i \in \{1, 2\}$, and an allocation determined by a pair of reports $(\hat{\tau}, \hat{\theta})$ the consumer's second period utility is

$$u(\tau,\hat{\tau},\theta_i,\hat{\theta}) = v^+(\tau)\cdot\big(x_i(\hat{\tau},\hat{\theta}) + x_{1\&2}(\hat{\tau},\hat{\theta})\big) + v^-(\tau)\cdot x_j(\hat{\tau},\hat{\theta}) - p(\hat{\tau},\hat{\theta}), \tag{1.1}$$

where $j \neq i$. The consumer values having both goods with $v^+(\tau)$, as he only consumes the favorite good out of the two. The consumer's second period strategy is described by a function $\sigma : \Theta \times T \times T \to \Theta$, where $\sigma(\theta,\tau,\hat{\tau})$ denotes the consumer's ex post report, which may depend on his ex ante and ex post type as well as his ex ante report. The consumer's first period expected utility is then[17]

$$U(\hat{\tau},\tau,\sigma) = \mathbb{E}_\theta[u(\tau,\hat{\tau},\theta,\sigma(\theta,\tau,\hat{\tau}))].$$

Denote $\sigma(\theta,\tau,\hat{\tau}) \equiv \theta$ by the identity id_θ. Define further the first period expected utility from truthful reporting about the ex ante and ex post type by $U(\tau) := U(\tau,\tau,id_\theta)$. The firm‘s maximization problem $(\mathcal{P})$ can then be formulated:

$$\max_{X,p} \int_0^{\overline{\tau}} f(\tau)\mathbb{E}_\theta[p(\tau,\theta)]dr$$

s.t.

$$U(\tau) \geq U(\hat{\tau},\tau,\sigma) \qquad \forall \tau, \hat{\tau} \neq \tau, \sigma, \tag{IC_1}$$

$$U(\tau) \geq 0 \qquad \forall \tau, \tag{IR}$$

$$u(\tau,\tau,\theta,\theta) \geq u(\tau,\tau,\theta,\hat{\theta}) \qquad \forall \tau,\theta,\hat{\theta} \tag{IC_2}$$

$$x_1(\hat{\tau},\hat{\theta}), x_2(\hat{\tau},\hat{\theta}), x_{1\&2}(\hat{\tau},\hat{\theta}) \geq 0, \quad x_1(\hat{\tau},\hat{\theta}) + x_2(\hat{\tau},\hat{\theta}) + x_{1\&2}(\hat{\tau},\hat{\theta}) \leq 1 \quad \forall \hat{\tau},\hat{\theta}. \tag{F}$$

In this application of the dynamic revelation principle, the second period incentive constraints (IC_2) ensure that the consumer, *if he has truthfully reported his ex ante type*, also truthfully reports his ex post type. The first period incentive constraints (IC_1) say that telling the truth in the first and second period must be better than any combination of lying about the ex ante type potentially followed by another lie about the ex post type. This means the first period incentive constraints must ensure against double deviations. Furthermore, the individual rationality constraints (IR) must hold in the first period, and (F) is the feasibility constraint for the allocation.

The following Lemma simplifies the problem. It states that as there is unit demand,

[17] Since θ is uniformly distributed on the binary support $\{\theta_1,\theta_2\}$

$$\mathbb{E}_\theta[u(\tau,\hat{\tau},\theta,\sigma(\theta,\tau,\hat{\tau}))] = \frac{1}{2}u(\tau,\hat{\tau},\theta_1,\sigma(a_1,\tau,\hat{\tau})) + \frac{1}{2}u(\tau,\hat{\tau},\theta_2,\sigma(a_2,\tau,\hat{\tau}))$$

I can restrict attention to allocations that assign at most one good.

Lemma 1. *For any direct mechanism that satisfies the constraints of $\mathcal{P}$, there exists a direct mechanism with the same payments that never assigns both goods and satisfies the constraints of $\mathcal{P}$. Thus without loss of generality, $x_{1\&2} = 0$.*

The proof follows by a replication argument.[18] The assignment of several goods at a time can be replaced by assigning the single good which is claimed to be preferred among those. In equilibrium, consumption and hence on-path utilities are unchanged. Incentive compatibility is preserved as well, because off-path utilities are weakly lowered. As in the modified mechanism payments are unchanged, the mechanisms are equivalent in terms of profit. Therefore it is without loss to only consider mechanisms that assign at most one good. Formally, I from now on set $x_{1\&2}$ to zero for all reports $(\hat{\tau}, \hat{\theta})$ and drop it.

As ex post types are equally likely for each ex ante type, the expected valuation in the first period for any allocation with $x_1 + x_2 = 1$ is $v - \delta\tau$. Whether it increases in or decreases in ex ante types depends on the sign of δ. Since the solution technique and results differ for these two possibilities, the analysis of the problem is split into two cases.

1.2.3.1 Decreasing mean

The analysis begins with the case $\delta \geq 0$. In this case the loss of the increasing valuation premium outweighs the increase in valuation of the preferred alternative such that the expected valuation of a particular good assigned in the first period decreases in τ.

I solve this maximization problem with the help of a technique that is common in the literature on sequential screening:[19] I consider the relaxed problem with publicly observable ex post types and find the set of solutions to it. The relaxed problem equals the original problem except that it does not contain IC_2 constraints as well as all those IC_1 constraints which ensure against first period deviations which are followed by another lie. The profit generated by the solutions to the relaxed problem constitutes an upper bound on the profit that can be achieved in the original maximization problem. Then I show that each solution to the relaxed problem satisfies the constraints which are left out. Hence, these are solutions to the original problem.

[18] The proof of Lemma 1 as well as all subsequent ones are given in the appendix.
[19] See for example Gale and Holmes (1993) and Eső and Szentes (2007b).

The relaxed maximization problem ($\mathcal{P}_o$) is

$$\max_{X,p} \int_0^{\overline{\tau}} f(\tau) \cdot \mathbb{E}_\theta[p(\tau,\theta)]dr$$

s.t.

$$U(\tau) \geq U(\hat{\tau},\tau,id_\theta) \qquad \forall \tau,\hat{\tau},\theta, \qquad (IC_1')$$

$$U(\tau) \geq 0 \qquad \forall \tau, \qquad \text{(IR)}$$

$$x_1(\hat{\tau},\hat{\theta}), x_2(\hat{\tau},\hat{\theta}) \geq 0, \quad x_1(\hat{\tau},\hat{\theta}) + x_2(\hat{\tau},\hat{\theta}) \leq 1 \qquad \forall \hat{\tau},\hat{\theta}. \qquad \text{(F)}$$

In the next step, I exploit the model's symmetry with respect to the two goods. Instead of identifying goods by their name, I distinguish between preferred and undesired goods. Denote $x^+(\hat{\tau}) = \frac{1}{2}x_1(\hat{\tau},\theta_1) + \frac{1}{2}x_2(\hat{\tau},\theta_2)$ and correspondingly $x^-(\hat{\tau}) = \frac{1}{2}x_2(\hat{\tau},\theta_1) + \frac{1}{2}x_1(\hat{\tau},\theta_2)$. The probabilities $x^+(\hat{\tau})$ and $x^-(\hat{\tau})$ are specific to the contract determined by ex ante report $\hat{\tau}$. Formed in the first period, they indicate the probability of the assignment of a preferred and an undesirable good in the second period given truthful revelation of ex post types. With this notation, the expected utility can be rewritten as

$$\begin{aligned} U(\hat{\tau},\tau,id_\theta) &= x^+(\hat{\tau}) \cdot v^+(\tau) + x^-(\hat{\tau}) \cdot v^-(\tau) - \mathbb{E}_\theta[p(\hat{\tau},\theta)] \\ &= v[x^+(\hat{\tau}) + x^-(\hat{\tau})] + r \cdot K(\hat{\tau},\delta) - \mathbb{E}_\theta[p(\hat{\tau},\theta)] \end{aligned} \qquad (1.2)$$

with $K(\hat{\tau},\delta) = x^+(\hat{\tau}) - x^-(\hat{\tau}) - \delta(x^+(\hat{\tau}) + x^-(\hat{\tau}))$.

Lemma 2. *The first period incentive constraints IC_1' are satisfied if and only if*

$$\partial U(\tau)/\partial\tau = K(\tau,\delta) \; a.e. \qquad (ENV)$$

$$\textit{and} \quad K(\tau,\delta) \textit{ is mon. increasing in } \tau. \qquad (MON)$$

Even though Lemma 2 and its proof are familiar from the literature on static mechanism design, it is non-standard in the literature on sequential screening. In the standard sequential screening problem, it is generally not possible to find necessary and sufficient conditions for incentive compatibility using the envelope theorem.[20] Difficulties arise, because the analoga to ENV and MON in Lemma 2 hold only in expectation over the ex post type and are generally not sufficient for incentive compatibility. In my model, this problem is overcome due to the symmetry that stems from the horizontal differentiation of goods: The utility level just depends on whether the obtained good is

[20] For an exposition see Courty and Li (2000) and Eső and Szentes (2007b).

favorite or non-favorite, but the identity of goods does not play any role. This permits to rewrite the expected utility of a contract as a probability distribution over utility levels as done in (1.2).

Lemma 2 implies that maximizing with respect to the constraints (IC_1'), (IR) and (F) is equivalent to taking (ENV), (MON), (IR) and (F) as constraints. If (ENV) and (MON) hold, the ex ante utility is convex in types.

What distinguishes this maximization problem also from a standard static screening problem with continuous types and linear utility is that, depending on the contract, $K(\tau, \delta)$ can take both positive and negative values. Hence, expected utility might monotonically increase or decrease on T, but it might also be the case that expected utility is U-shaped as a function of ex ante types. Consequently, it is not clear which ex ante type will have the lowest expected utility which will then be set to zero in the optimum by the individual rationality constraints.

The economic intuition for this non-standard situation emerges from the horizontal differentiation of goods. While in a usual setting with vertical differentiation the ordering of the consumer's types induced by the valuation of an allocation is independent of the allocation considered, in the case with horizontally differentiated goods there is no such clear ordering. In my model, an ex ante type that values getting the right good more also suffers more from ending up unfavorably. As a consequence, any contract that has a tendency towards assigning the 'wrong' good is valued *less* by higher types. On the contrary, contracts that have a tendency towards assigning the 'right' good, are valued more by higher types. As usual, the binding incentive constraints are those that ensure that the consumer does not understate the valuation he derives from a given contract. In my model it therefore depends on the contract whether understating the utility from a contract amounts to over- or underreporting the ex ante type.

Models in which the type with binding individual rationality constraint is ambiguous have first been considered in screening problems with type-dependent outside options (see Lewis and Sappington (1989), Maggi and Rodriguez-Clare (1995), Jullien (2000) and Samuelson and Nldeke (2007) for an exposition). However, their results cannot directly be applied, because all of the latter papers by assumption exclude my case of pure revenue maximization. Furthermore, in my model a consequence of Lemma 2 is the convexity of the utilities in τ given truthtelling. This allows for a particularly tractable method of solving the maximization problem: Convexity implies that in every solution, there is an ex ante type $z \in [0; \overline{\tau}]$ that has the lowest ex ante utility. Making use of this fact, in a first step I solve the relaxed problem ($\mathcal{P}_o$) with the additional constraint $U(\tau, id_\theta) \geq U(z, id_\theta)$ for all $\tau \in T$ and some arbitrary but fixed ex ante type z. Denote this problem by $\mathcal{P}_o^z$. This results in the description of an optimal allocation dependent on z for all $z \in T$, where z is the exogenously given ex ante type with the lowest expected utility. In a second step I then maximize the profit in z.

Having fixed ex ante type z, Lemma 2 can be employed to reformulate the maximization problem: In the optimum, z's expected utility is zero by the binding individual rationality constraint. By (ENV) and absolute continuity[21] any ex ante type's expected utility can then be written as

$$U(\tau) = U(z) + \int_z^\tau K(y,\delta)dy = \int_z^\tau K(y,\delta)dy. \tag{1.3}$$

By (1.2) and (1.3) prices can be written as a function of the allocation rule:

$$\mathbb{E}_\theta[p(\tau,\theta)] = v[x^+(\tau) + x^-(\tau)] + \tau \cdot K(\tau,\delta) - \int_z^\tau K(y,\delta)dy. \tag{1.4}$$

Plugging (1.4) into the objective reduces problem $\mathcal{P}_o^z$ to:

$$\max_x \int_0^{\overline{\tau}} f(\tau)\Big(v[x^+(\tau) + x^-(\tau)] + \tau \cdot K(\tau,\delta) - \int_z^\tau K(y,\delta)dy\Big)d\tau$$

s.t. (MON), (F) and $U(z) \leq U(\tau) \quad \forall \tau \in T$.

By integration by parts and rearranging terms, the objective can be rewritten as

$$\begin{aligned}\max_x &\int_0^z f(\tau)\left[v[x^+(\tau) + x^-(\tau)] + K(\tau,\delta) \cdot \left(\tau + \frac{F(\tau)}{f(\tau)}\right)\right] d\tau \\ &+ \int_z^{\overline{\tau}} f(\tau)\left[v[x^+(\tau) + x^-(\tau)] + K(\tau,\delta) \cdot \left(\tau - \frac{1-F(\tau)}{f(\tau)}\right)\right] d\tau\end{aligned} \tag{1.5}$$

s.t. (MON), (F) and $U(z) \leq U(\tau) \quad \forall \tau \in T$.

Let $b = sup\{\tau \in T | \tau - \frac{1-F(\tau)}{f(\tau)} \leq 0\}$ be the highest ex ante type with a non-positive virtual value.

[21] For a proof of absolute continuity see for example Theorem 2 in Milgrom and Segal (2002).

Lemma 3. *Any solution to problem $\mathcal{P}_o^z$ has the following properties:*[22]

$$x^+(\tau) = \frac{1+\delta}{2} \quad \text{and} \quad x^-(\tau) = \frac{1-\delta}{2} \quad \text{if } \tau \leq max\{b, z\},$$
$$x^+(\tau) = 1 \quad \text{and} \quad x^-(\tau) = 0 \quad \text{if } \tau > max\{b, z\}.$$

For any $z \in T$ a solution does exist.

Lemma 3 is proven by pointwise maximization of objective (1.5) for every ex ante type τ. Unlike in standard pointwise maximization problems familiar from the literature on mechanism design,[23] the monotonicity constraint is not entirely ignored at that point. Instead, pointwise maximization is done subject to a relaxed version of the constraints. The constraints are weakened in the sense that I only pay attention to the bounds on both $x^+(\tau) + x^-(\tau)$ and $K(\tau, \delta)$ that are implied by (MON), (F) and z to be the type with minimal utility.

As a consequence, the pointwise maximization itself is not trivial. I show that contracts with the properties given in Lemma 3 lead to an upper bound on pointwise profits given the relaxed constraints. Then I show the existence of a feasible allocation rule with these properties. Finally, it can immediately be seen that allocation rules with the derived properties satisfy (MON) and z is the type with lowest expected utility, which completes the proof of the Lemma.

This allows me to turn to the second step now, the maximization with respect to the worst-off type z.

Lemma 4. *z is optimal if and only if $z \leq b$.*

Lemma 4 results from inserting the properties from Lemma 3 into objective (1.5). By the optimality conditions from Lemma 3, both the virtual value for types $\tau < z$ as well as the virtual value for types $z < \tau < b$ are multiplied by zero and hence do not influence profits. Thus for ex ante types $\tau \leq b$ the relative position to z, which determines the virtual value associated with the ex ante type, has no relevance even though for a given τ virtual values are not equal. The virtual value for types $\tau > b$, however, enters the objective strictly positively if and only if $\tau > z$. Therefore, profit maximization requires that $\tau > z$ if $\tau > b$.

[22] W.l.o.g. let b and ex ante types with $\tau - \frac{1-F(\tau)}{f(\tau)} = 0$ get the allocations of low types.
[23] See Myerson (1981)

Lemma 5. *The set of mechanisms which solve problem $\mathcal{P}_o$ is the following: Ex ante types $\tau > b$ always obtain their favorite good and prices satisfy $\mathbb{E}_\theta[p(\tau,\theta)] = v + b(1-\delta)$. Ex ante types $\tau \leq b$ obtain contracts with $\mathbb{E}_\theta[p(\tau,\theta)] = v$. There is a continuum of optimal allocation rules for ex ante types $\tau \leq b$ characterized by $\alpha \in [\delta, 1]$:*

$$x_1(\tau,\theta_1) = \alpha, \qquad x_2(\tau,\theta_1) = 1-\alpha,$$
$$x_1(\tau,\theta_2) = \alpha-\delta, \qquad x_2(\tau,\theta_2) = 1+\delta-\alpha.$$

The set of optimal allocation rules is obtained as the solution to a system of linear equations, which are the feasibility requirements and the optimality conditions given by Lemmas 3 and 4. As the relaxed problem takes into account only first period incentives, only expected prices matter. They are pinned down by (1.4).

A thorough interpretation of the results in Lemma 5 is postponed until the explanation of Proposition 1. Before turning to implementability in the original problem, I introduce a measure for the quality of a contract using a property of the solution to the relaxed problem. Any optimal contract satisfies $x^+(\tau) + x^-(\tau) = 1$. This means that the consumer always obtains a good and the event 'no assignment' does not occur. I call this property 'full market coverage'. It pays off to insert the full market coverage property into expected utility (1.2). This yields

$$U(\hat{\tau},\tau,id_\theta) = v - \tau\delta + \tau[x^+(\hat{\tau}) - x^-(\hat{\tau})] - \mathbb{E}_\theta[p(\tau,\theta)]. \tag{1.6}$$

I denote the term $x^+(\hat{\tau}) - x^-(\hat{\tau})$ as 'responsiveness' of the corresponding contract. The responsiveness is the difference between the ex ante probability to obtain the right and the ex ante probability to obtain the wrong good and is central for the analysis. It is a measure for quality of the contract from an ex ante point of view. Due to feasibility, responsiveness is bounded above by $x^+(\hat{\tau}) - x^-(\hat{\tau}) = 1$, the case in which the consumer always obtains the good he prefers, and bounded below by $x^+(\hat{\tau}) - x^-(\hat{\tau}) = -1$, the case in which the consumer never obtains the preferred good. The responsiveness of a contract that maps any ex post type into the same allocation is zero. These are contracts that fix an allocation in the first period that cannot be influenced by any ex post report. If the responsiveness is positive, the contract is said to positively respond to the consumer's needs. As the first best allocation rule has maximum responsiveness, contracts with lower values of responsiveness are considered as distorted and distortion is measured by the difference of responsiveness to one. Note that any contract with responsiveness unequal to one, minus one, or zero assigns nondegenerate allocations to at least some ex post type. I call such contracts stochastic.

Having found the set of solutions to the relaxed problem with observable ex post types ($\mathcal{P}_o$), I will finally address its implementation in the original problem with private ex post types ($\mathcal{P}$). When ex post types are private, two additional types of incentive constraints have to be satisfied: The second period incentive constraints (IC_2) have to hold, which means if the consumer has truthfully revealed his ex ante type, he does not have an incentive to lie about his ex post type. And in the first period, there may not exist profitable double deviations (IC_1), i.e., a first period lie followed by another lie in the second period. Proposition 1 states that the entire set of optimal allocation rules from Lemma 5 is also implementable in the original problem with private ex post types.

Proposition 1. *Let the mean be decreasing. The set of allocation rules which solve the problem with private ex ante and ex post types is the following:*

- *For $\tau > b$: The consumer always obtains his favorite good.*
- *For $\tau \leq b$: $x_1(\tau, \theta_1) = \alpha, \quad x_1(\tau, \theta_2) = \alpha - \delta$ with arbitrary $\alpha \in [\delta, 1]$ and $x_2(\tau, \theta) = 1 - x_1(\tau, \theta) \quad$ for $\theta \in \{\theta_1, \theta_2\}$.*

Necessary conditions for prices are:

- *For $\tau > b$: $\mathbb{E}_\theta[p(\tau, \theta)] = v + b(1 - \delta)$.*
- *For $\tau \leq b$: $\mathbb{E}_\theta[p(\tau, \theta)] = v$.*

Ex post type independent prices $p(\tau, \theta_1) = p(\tau, \theta_2)$ are always sufficient for incentive compatibility.

In order to prove incentive compatibility, I show that a stronger condition than (IC_2) holds: Each ex post type has an incentive to truthfully reveal his type no matter what his ex ante report was. This is sufficient for incentive compatibility. The (IC_2) constraints are then trivially satisfied. However, (IC_1) is satisfied as well: lying in the second stage and thus complex deviations are never optimal. Unilateral first period deviations are not profitable either, because the solutions satisfy the relaxed problem's (IC_1') constraints. For problem $\mathcal{P}_o$ only expected prices matter and hence by Lemma 5 for each ex ante type only the sum of the prices $p(\tau, \theta_1) + p(\tau, \theta_2)$ is pinned down. When dealing with implementability in the original problem, single ex post prices are relevant due to second period incentives. Setting $p(\tau, \theta_1) = p(\tau, \theta_2)$ for all ex ante types implies that the price is fixed by the reported ex ante type and is independent of the report on the ex post type. Given any report $\hat{\tau}$, the consumer then reports honestly about θ, because the contract provides him with the good which is announced to be the 'good' one with a higher probability. Hence, it is the positive responsiveness that makes the solutions to the relaxed problem implementable in the original problem.

As the set of optimal allocation rules from Lemma 5 is identical to those of Proposition 1, the full market coverage property carries over to Proposition 1. The solution is a step function as illustrated in Figure 1.1. High ex ante types above a certain threshold type b always receive the good they prefer $x^+(\tau) - x^-(\tau) = 1$, which implies the classical 'no distortion at the top' result. This is implementable, for example, by selling the goods for a uniform price in the second period or selling a good in the first period but with an option for free exchange. All ex ante types lower than the critical type get a contract from the continuum of contracts with responsiveness δ. Recall, this means that the contracts positively respond to the consumer's needs in the sense that the likelihood to obtain a good is increasing when it is announced to be favorite. Consider for example the contract $\alpha = 1$ from Lemma 5. The contract gives the consumer good 1 with certainty if he announces this good to be the favorite one. However, if the consumer prefers good 2, there is a chance of δ that he obtains good 2. For the firm a possible implementation of this contract is to sell good 1 in the first period, but if the consumer afterwards reports that he would prefer the other good, give him a chance of δ to exchange the good. In practice, the stochastic element can - for example - be implemented by allowing for exchange subject to availability. For the special case $\delta = 0$ and hence $x^+(\tau) - x^-(\tau) = 0$ an optimal contract chosen by low ex ante types fixes an arbitrary allocation that always assigns a good in the first period and does not respond to ex post reports in any way.

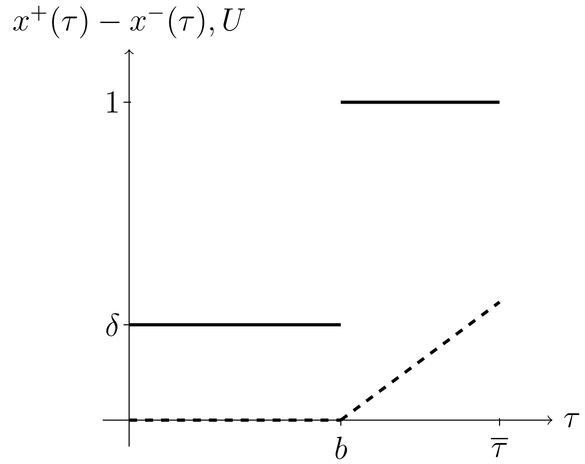

Figure 1.1: Solution to the decreasing mean case. The solid line represents the responsiveness of the contracts, the dashed line expected utility.

The optimal menu of contracts is the result of price discrimination. A higher type has a higher first period expected utility from the contract with full flexibility than lower types because his valuation of the preferred good, which he will get with certainty, is higher. When offering just one contract with full flexibility, the firm would have to trade off leaving rent to high types and excluding low types. However, the monopolist uses responsiveness as a screening device. The firm exploits that flexibility between goods has a higher value to the higher ex ante types. By offering less responsive contracts to lower ex ante types, the loss from the lower types is smaller than the gain from extracting rent from higher types. In technical terms, the firm exploits a single-crossing property with respect to responsiveness: Given full market coverage, the marginal expected utility with respect to $x^+(\tau) - x^-(\tau)$ is increasing in ex ante types. The single crossing condition is obvious in (1.6). From (1.6) it is furthermore apparent that $x^+(\tau) - x^-(\tau)$ enters the expected utility linearly. It is a well-known result in the literature on screening that this linearity leads to step solutions. The problem is linear, as there are no costs for the firm to increase the responsiveness of a contract.

Another common feature of solutions to linear screening problems is the 'bang-bang' property: The consumer either obtains the best allocation of goods or the worst. However, in the present problem it is not optimal to distort contracts for low ex ante types to zero responsiveness. Instead, low ex ante types sign contracts that positively respond to the consumer's needs. The contracts are hence not maximally downward distorted.[24] These contracts for low ex ante types are the main object of study in this paper. I first give economic intuition for this result and then relate it to the literature on mechanism design with type dependent outside options.

In the decreasing-mean case considered in this section, the valuation of the non-favorite good, $v^-(\tau)$, decreases faster in ex ante types than the top valuation $v^+(\tau)$ increases. The expected utility gross of payments derived from a contract with responsiveness zero, $x^+ = x^- = 1/2$, is, hence, decreasing in ex ante types. The optimal distorted contracts are designed such that the expected utility of a given contract gross of payments is the same for all ex ante types: The optimal contract entails a larger probability x^+ than x^- to exactly offset the two 'valuation effects'. The higher δ, the stronger is the low valuation decreasing compared to the increase in the top valuation and, hence, the higher is the responsiveness of contracts for low ex ante types.[25] When implementing this allocation rule, the monopolist can set a price to extract all rents from the ex ante types $\tau \leq b$. Rents left to high types $\tau > b$ are linearly increasing. The firm has no reason to offer more distorted contracts to the lowest ex ante types: Firstly, this reduces the overall surplus as the consumer derives less utility from the allocation rule. Secondly, it induces the firm to leave higher rents to the consumer:

[24] There exist incentive compatible contracts with zero responsiveness; for example any contract whose allocation is independent of ex post reports.

[25] If $\delta = 1$, which means the top valuation is constant in ex ante types, only the first best contract is offered. This is intuitive, because the trade-off between the two 'valuation effects' is balanced by contracts with high responsiveness. If $\delta = 0$, the expected utility of a contract with responsiveness zero is constant among ex ante types.

Expected utility would be U-shaped with an interior type's participation constraint binding; ex ante types at both sides would obtain rents. Hence, in the optimum the expected utility of ex ante types below the type with minimum expected utility is 'ironed' to zero.

The appearance of 'intermediately distorted' contracts is in line with the established literature on mechanism design with type-dependent outside options. The setup can be rewritten as a mechanism design problem with increasing expected utility everywhere and an increasing outside option such that the rents can be potentially U-shaped and correspond to the expected utility in the framework presented.[26] The unusual type of contracts stems from the fact that the binding first period incentive constraints change from upward constraints to downward constraints at the interior ex ante type whose participation constraint is binding.[27] In the standard case, under certain assumptions the optimal incentive compatible contract gives zero rent from participating to an interval of types around this critical type.[28] In my model this interval is $[0, b]$ and providing those types with their outside option is achieved via bunching on that interval.[29]

Another special feature of the optimal menu of contracts is the multiplicity of solutions. Contracts for high ex ante types $\tau > b$ are uniquely determined. For each low ex ante type $\tau \leq b$ there is a continuum of optimal contracts. The multiplicity arises by allowing for stochastic contracts and by the symmetry in the sense that the consumer does not cares about the identity of the goods: In the first period, buying good 1 with a stochastic exchange option for good 2 is valued the same as buying good 2 with a stochastic exchange option for good 1. Contracts for low ex ante types can be arbitrarily combined in the menu. The reason for this is that ex ante types are indifferent among all the contracts for any $\hat{\tau} \leq b$. In particular this means that there exist optimal menus which consist of only two contracts, the first best contract and one distorted contract.

Proposition 1 shows that the restriction on contracts imposed by Gale (1993) is with consequences. Gale looked at a setting which is very close to the one examined here. However, he did not use a general mechanism design approach to derive the revenue-maximizing menu of contracts. Instead, there is a restriction to selling in the

[26] In the presented setup, expected utility is given by (1.6) and the outside option is zero. A possible transformation: Define expected utility as $U(\hat{\tau}, \tau, id_\theta) = v + \tau[1 + x^+(\hat{\tau}) - x^-(\hat{\tau})] - \mathbb{E}_\theta[p(\tau, \theta)]$ and the outside option as $\tau(1 + \delta)$. The monopolist's problem is identical under the original setup and the transformation. By the first order constraint for incentive compatibility, expected utility of the transformation is increasing everywhere.

[27] This is meant by 'countervailing incentives', see Lewis and Sappington (1989) and Maggi and Rodriguez-Clare (1995).

[28] See in particular the exposition by Jullien (2000).

[29] For the same reason bunching occurs in problems with linear type-dependent outside options and linear utility, see Maggi and Rodriguez-Clare (1995).

first period without any later flexibility or selling the good in the second period, which corresponds to a contract with full flexibility. In his pioneering work, this restriction was imposed in order to obtain a fruitful comparison between monopolistic and oligopolistic behavior in a setting with individual demand uncertainty. For the monopolistic case, my analysis shows that by allowing for intermediately distorted contracts, which are stochastic contracts, it is possible to achieve even more. Analytically spoken, Gale allows for the first-best contract, which is shown to actually be optimal for high types. However, as 'discrimination' alternative he only allows for a contract with $x^+(\tau) = x^-(\tau) = \frac{1}{2}$, which differs from the optimal contract for any $\delta > 0$. $\delta = 0$ is the case in which the expected utility of a fixed allocation stays constant over ex ante types. For this case, my model also predicts a contract with zero responsiveness to be optimal for low types. Corollary 1 summarizes the relation to Gale (1993):

Corollary 1. *Whenever the expected utility of a given good is decreasing in ex ante types, the solution to the revenue-maximization problem includes stochastic contracts. They strictly improve upon a menu of contracts without possibility for exchange on the one hand and goods sold in the second period on the other hand.*

1.2.3.2 Increasing mean

The analysis is completed with the examination of the case $\delta < 0$. In this case the increase in top valuation dominates the loss of the increasing valuation premium in the sense that the expected valuation of a particular good assigned in the first period, $v - \tau\delta$, is increasing in ex ante types. More generally, the expected valuation of any contract with zero responsiveness is weakly increasing in ex ante types.

Define $e = sup\{\tau \in T | \tau - \frac{1-F(\tau)}{f(\tau)} \leq \frac{v}{\delta}\}$ whenever the supremum exists and $e = 0$ otherwise. As $\delta < 0$, the constant v/δ is negative and from the increasing virtual value assumption it follows that $e < b$.

Proposition 2. *Let the mean be increasing ($\delta < 0$). The set of allocation rules that solve the problem with private ex ante and ex post types is the following:*

- *For $\tau > b$: The consumer always obtains his favorite good.*
- *For $\tau \in [e, b]$: $x_1(\tau, \theta) = \alpha$, $x_2(\tau, \theta) = 1 - \alpha$ $\forall a$ and $\alpha \in [0, 1]$ arbitrary.*
- *For $\tau < e$: No assignment.*

Necessary conditions for prices are:

- *For* $\tau > b$*:* $\mathbb{E}_\theta[p(\tau,\theta)] = v + b(1-\delta)$.
- *For* $\tau \in [e,b]$*:* $\mathbb{E}_\theta[p(\tau,\theta)] = v - \delta e$.
- *For* $\tau < e$*:* $\mathbb{E}_\theta[p(\tau,\theta)] = 0$.

Ex post type independent prices $p(\tau,\theta_1) = p(\tau,\theta_2)$ *are always sufficient for incentive compatibility.*

For $\delta < 0$, the problem $\mathcal{P}$ is solved again by considering a relaxed problem $\mathcal{P}_*$. $\mathcal{P}_*$ differs from $\mathcal{P}_o$ by the additional constraint

$$x_1(\tau,\theta_1) + x_2(\tau,\theta_2) \geq x_1(\tau,\theta_2) + x_2(\tau,\theta_1) \quad \forall\tau, \tag{*}$$

which is a necessary condition for second period incentive compatibility and states that responsiveness is weakly positive. In $\mathcal{P}_o$ second period incentives are completely ignored. As for positive δ the solution to $\mathcal{P}_o$ is implementable in $\mathcal{P}$, (*) is not strictly binding there.[30] This changes when δ is negative. The solution to $\mathcal{P}_o$, which is stated in Lemma 5, for negative δ violates (*) and hence second period incentive compatibility. The upper bound on profits attained by the solution to $\mathcal{P}_*$ is hence lower than the one derived from $\mathcal{P}_o$.

Due to the similar structure of incentive contraints in the problems $\mathcal{P}_o$ and $\mathcal{P}_*$, Lemma 2 applies. A difference, however, is that from (*) and $\delta < 0$ it follows that the expected valuation of any incentive compatible contract is increasing in ex ante types. Consequently, in the optimum the lowest ex ante type's individual rationality constraint binds and a solution to $\mathcal{P}_*$ is found following standard steps including integration by parts, reformulations and pointwise maximization. By the same way as for the decreasing mean case, the allocation rule is then shown to be implementable in $\mathcal{P}$.

For $e > 0$ the solution is illustrated in Figure 1.2. Ex ante types above b again always obtain their preferred good. Ex ante types between e and b always obtain a good, the contract is however maximally distorted with the limit given by (*). This contract has responsiveness zero and hence does not respond to the announcement of ex post types at all. Types lower than e are excluded, which means the full market coverage property does not hold if $e > 0$. First period expected utility is increasing over all types that obtain a good. This menu of contracts can be implemented using advance-purchase discounts, which are well known from numerous studies.[31]

[30] This can also be seen directly from the properties of optimal contracts as stated in Proposition 1.
[31] See for example Gale and Holmes (1993), Gale (1993), Möller and Watanabe (2010) or Nocke et al. (2011).

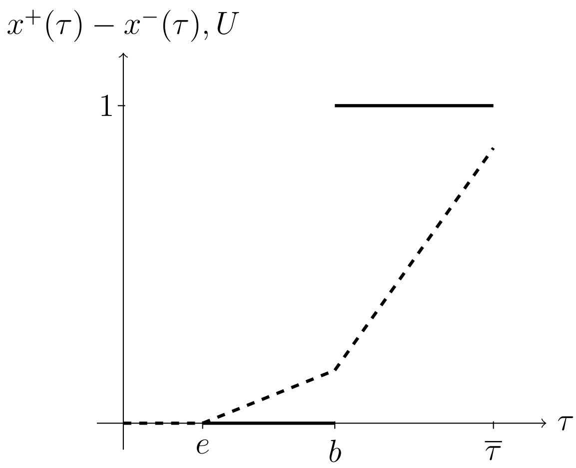

Figure 1.2: Solution to the increasing mean case with interior e. The solid line represents the responsiveness of the chosen contracts, the dashed line expected utility.

The interpretation of the result as one of price discrimination is the logical continuation of the decreasing mean case. Assume for a moment that second period incentive compatibility would be no binding constraint, as is the case when the mean is decreasing in ex ante types. When ignoring second period incentive constraints, the monopolist would like to distort the contracts for ex ante types lower than b such that $x^+(\tau) - x^-(\tau) = \delta < 0$. This is an immediate consequence of Lemma 5. Distortions would be comparatively large, as there is much rent from high ex ante types to be extracted. The upper bound derived from $\mathcal{P}_o$ would be achieved and ex ante utility for these low ex ante types would then be zero. However this contract violates (*), the necessary condition for second period incentive compatibility. By (*) the maximal distortion is $x^+(\tau) - x^-(\tau) = 0$. From the single crossing property it then follows that expected utility is increasing in ex ante types even for the maximally distorted contract. This creates an incentive for the firm to completely exclude very low ex ante types. To put it another way: For the increasing mean case, distortions in the quality of contracts are not sufficient to extract the high types' rents and therefore additional quantity distortions are used.

1.3 The continuous goods model

In the previous section, I solved a model of sequential screening with horizontally differentiated goods without ad-hoc restrictions on contracts. If the ex ante expected valuation of a certain good is decreasing in ex ante types, the optimal menu contains contracts that partially restrict consumers' flexibility. This part further studies the optimal design of the partial restriction of flexibility and thereby characterizes optimal exchange policies. To that end I consider a more comprehensive model with a continuum of goods, which is more specific only in that it focuses on the decreasing mean case exclusively.

1.3.1 Model

A firm can sell a continuum of horizontally differentiated goods $s \in S = [0,1]$ to a consumer with unit demand. The consumer learns his preferences over two periods. In period 1, before contracting takes place, he learns his ex ante type τ, and subsequently after contracting, he learns his ex post type θ. His utility function over S depending on the his ex ante type $\tau \in T = [0;\overline{\tau}]$ and his ex post type $\theta \in \Theta = [0,1]$ is given by

$$v_{\tau,\theta}(s) = v + (1-\delta)\tau - \tau c(\theta, s) \tag{1.7}$$

with $\delta < 1$. The utility loss from getting a non-favorite good is captured by a continuous function $c(\theta, s)$ that is quasi-convex in s for any θ with the minimum at θ where $c(\theta,\theta) = 0$. Hence, the ex post type θ represents the consumer's favorite good $s = \theta$ and consumption of a good $s \neq \theta$ entails a utility loss which is increasing in the distance of the consumed to the favorite good.[32] Assume further that $c(\theta, s)$ is bounded such that $v_{\tau,\theta}(s) \geq 0$ for all τ, θ, and s. The continuum of ex post types $\theta \in \Theta$ is distributed according to G. The distribution G is continuous, independent of ex ante types, and has full support. Thus each good could possibly be the favorite one.[33] The independence assumption means that the ex ante type does not provide any information about the ex post type. The specification (1.7) means further that both the utility derived from the favorite good and the utility loss from getting a non-favorite good are linearly increasing in ex ante types. Let the continuous distribution F of ex ante types τ over type space T satisfy the standard assumption of increasing virtual values $\tau - \frac{1-F(\tau)}{f(\tau)}$. As already explained, I focus on the case where the expected utility from any good is decreasing in ex ante types: $\mathbb{E}_\theta[c(\theta, k)] > 1 - \delta$ for all $k \in S$. Note that like in Section 1.2 the favorite good is valued by $v + \tau - \tau\delta$, and unlike in Section 1.2 $\mathbb{E}_\theta[c(\theta, k)]$ varies in k. The firm can produce arbitrary amounts of each good without cost. Any other assumptions about the firm and timing are as before.

1.3.2 Analysis

I first apply the technique presented in Section 1.2 to solve the principal's relaxed problem with observable second period types. Its solution closely resembles the solutions

[32] Note that I do not restrict the utility loss to be symmetric around θ, and for a given distance $|\theta - s|$ the utility loss may depend on θ.

[33] Note that in this model the valuation for a given good varies gradually in the ex-post type, a key property that helps to reveal more characteristics of exchange-policies.

to the corresponding problem in the two goods model. The key difference to the two goods model is that only a strict subset of the solutions is implementable in the original problem. The main purpose of this section is to characterize this subset.

With the same argument as in Lemma 1, it can be shown that it is without loss of generality to exclude the possibility that multiple goods are assigned to the consumer. An allocation X is, hence, a probability distribution over elements of the set of goods S and 'no assignment'. The payment p is transferred from the consumer to the firm. A direct mechanism consists of an allocation rule $\{X(\hat{\tau},\hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$ and a payment rule $\{p(\hat{\tau},\hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$, where $\hat{\tau}$ and $\hat{\theta}$ are the sequentially reported types.[34] Let $X(\tilde{s}|\hat{\tau},\hat{\theta})$ be the probability that the allocation $X(\hat{\tau},\hat{\theta})$ assigns a good $s \leq \tilde{s}$ to the consumer. Note that $X(s|\hat{\tau},\hat{\theta})$ differs from a cumulative distribution function over S in that $1 - X(1|\hat{\tau},\hat{\theta})$, the probability of no assignment, can be positive. The notion of a contract carries over from Section 1.2. Let $\sigma(\theta,\tau,\hat{\tau})$ be the consumer's strategy for reporting an ex post type upon having reported $\hat{\tau}$, and truthtelling is denoted by id_θ.

The consumer's first period expected utility from reporting $\hat{\tau}$ when being type τ is

$$U(\hat{\tau},\tau,\sigma) = \int_0^1 \left(\int_0^1 v_{\tau,\theta}(s) dX(s|\hat{\tau},\sigma(\theta,\tau,\hat{\tau})) \right) - p(\hat{\tau},\sigma(\theta,\tau,\hat{\tau})) \; dG(\theta).$$

Define further $U(\tau) := U(\tau,\tau,id_\theta)$. Respecting the incentive compatibility, individual rationality, and feasibility constraints, the firm's maximization problem $(\mathcal{P})$ is:

$$\max_{X,p} \int_0^{\overline{\tau}} f(\tau) \int_0^1 p(\tau,\theta) dG(\theta) dr$$

s.t.

$$U(\tau) \geq U(\hat{\tau},\tau,\sigma) \quad \forall \tau, \hat{\tau} \neq \tau, \sigma, \tag{IC_1}$$

$$U(\tau) \geq 0 \quad \forall \tau, \tag{IR}$$

$$\int_0^1 v_{\tau,\theta}(s) dX(s|\tau,\theta) - p(\tau,\theta) \geq \int_0^1 v_{\tau,\theta}(s) dX(s|\tau,\hat{\theta}) - p(\tau,\hat{\theta}) \quad \forall \tau, \hat{\tau}, \hat{\theta}, \tag{IC_2}$$

$$0 \leq X(s|\hat{\tau},\hat{\theta}) \leq X(s'|\hat{\tau},\hat{\theta}) \leq 1 \quad \forall \hat{\tau},\hat{\theta}, s, s' \text{ with } s \leq s'. \tag{F}$$

Similarly to the two goods model, the principal has to account for double deviations, but needs to ensure participation only in the first period. The consumer's expected

[34] The restriction to direct mechanisms is without loss of generality. For an appropriate revelation principle see Myerson (1986).

utility can be rewritten as

$$U(\hat{\tau},\tau,id_\theta) = v\left(\int_0^1 X(1|\hat{\tau},\theta)dG(\theta)\right) + \tau\cdot K(\hat{\tau},\delta) - \int_0^1 p(\hat{\tau},\theta)dG(\theta) \tag{1.8}$$

with $K(\hat{\tau},\delta) = \int_0^1\int_0^1 1-\delta-c(\theta,s)dX(s|\hat{\tau},\theta)dG(\theta)$. As in the two goods model, the expected utility is linear in ex ante types and $K(\hat{\tau},\delta)$ can take both positive or negative values. Hence, I can proceed as in Section 1.2 and first consider the relaxed problem with observable second period types ($\mathcal{P}_o$), which I solve by applying the technique introduced in Section 1.2.

Problem $\mathcal{P}_o$ differs from $\mathcal{P}$ by omitting all IC_2 constraints and relaxing IC_1 to

$$U(\tau) \geq U(\hat{\tau},\tau,id_\theta) \quad \forall\tau,\hat{\tau},a. \tag{IC_1'}$$

To state the solutions to problem $\mathcal{P}_o$ and then to problem $\mathcal{P}$, I introduce the notions of full market coverage and responsiveness corresponding to the definitions for the two goods model.

A mechanism satisfies 'full market coverage' if $\int_0^1 X(1|\tau,\theta)dG(\theta) = 1$ for all τ, which is equivalent to $X(1|\tau,\theta) = 1$ for almost all θ and all τ. This means that generically[35] the consumer ends up with some good, independent of the reported pair of types.

I refer to

$$R(\hat{\tau}) = \int_0^1\int_0^1 1-c(\theta,s)dX(s|\hat{\tau},\theta)dG(\theta) \tag{1.9}$$

as the responsiveness of a contract. If the mechanism satisfies the full market coverage property, $R(\hat{\tau})$ captures the extent to which a contract is distorted by not assigning the favorite good to the consumer.[36] This can be best seen when plugging the full market coverage property into (1.9) in which case

$$R(\hat{\tau}) = 1 - \int_0^1\int_0^1 c(\theta,s)dX(s|\hat{\tau},\theta)dG(\theta).$$

Responsiveness is equal to 1 minus the expected utility loss that accrues when the contract does not always assign the favorite good, where expectations are taken with respect to the ex post type and the possibly stochastic allocations. Hence, $R(\hat{\tau})$ is

[35] Except for a set on Θ of probability measure zero. Note that selling to a mass of consumer-types with probability measure zero on $T\times\Theta$ has no impact on profits.

[36] In the following, a contract is referred to as 'more distorted' than another contract when his responsiveness is lower.

maximal and equal to 1 under the first best contract, where each ex post type θ obtains his favorite good $s = \theta$ with certainty. An immediate consequence of the model assumption $\mathbb{E}_\theta[c(\theta, k)] > 1 - \delta$ for all $k \in S$ is that any contract in which the consumer obtains the same good irrespective of his ex post type has a responsiveness less than δ.

Before stating the set of solutions to problem $\mathcal{P}_o$ in Lemma 7, I finally establish the existence of a contract with $R(\tau) = \delta$ for any $\delta < 1$ which is any possible δ. The existence of such a contract is needed in the proof of Lemma 7 and unlike in the two goods model not trivially obtained. The existence is shown on hand of contract (RD) that is the simplest example from a class of contracts that plays an important role for the upcoming results. Contract (RD) is deterministic; let $x^{RD}(\tau, \theta)$ be the good obtained upon the pair of reports (τ, θ):

$$\begin{aligned} &x^{RD}(\tau,\theta) = I && \text{if} \quad \theta < I, \\ &x^{RD}(\tau,\theta) = \theta && \text{if} \quad \theta \in [I; 1-I], \\ &x^{RD}(\tau,\theta) = 1 - I && \text{if} \quad \theta > I, \\ &\text{and} \quad p(\tau,\theta) = v \quad \forall \theta. \end{aligned} \tag{RD}$$

In the first period, contract (RD) specifies a price to be paid by the consumer and an interval of goods. The price does not depend on the report of the ex post type - the pricing structure is, hence, very simple. In the second period, the consumer obtains his favorite good from that interval. Let the interval be positioned 'in the middle' and define I such that the interval is $[I, 1 - I]$.

Lemma 6. *For any possible $\delta, c(.)$, and G there exists an $I(\delta, c(.), G)$ with $I < 1/2$ such that contract (RD) satisfies $R(\tau) = \delta$.*

Lemma 6 is proved by showing that responsiveness of contract (RD) is continuous in the interval boundary I. As the responsiveness of contract (RD) is 1 for $I = 0$ and less than δ for $I = 1/2$, the lemma then follows from the intermediate value theorem.

Using the technique introduced in Section 1.2, I now obtain the solution to problem $\mathcal{P}_o$. Let $b = sup\{\tau \in T | \tau - \frac{1-F(\tau)}{f(\tau)} \leq 0\}$ be the highest ex ante type with a non-positive virtual value.

Lemma 7. *A mechanism solves problem $\mathcal{P}_o$ if and only if it satisfies (F), the full market coverage property, and has the following properties:*

$$\text{For } \tau \leq b: \qquad R(\tau) = \delta \qquad \text{and} \int_0^1 p(\tau,\theta)dG(\theta) = v;$$

$$\text{For } \tau > b: \qquad R(\tau) = 1 \qquad \text{and} \int_0^1 p(\tau,\theta)dG(\theta) = v + b(1-\delta).$$

Any optimal allocation rule is again a step function in ex ante types with threshold-type b. Types above b always get their favorite good, which implies no distortion at the top and maximum responsiveness. Ex ante types lower than b get a contract with responsiveness δ which gives them utility v gross of payments. On the one hand, this means that types $\tau \leq b$ do not always end up with the favorite good. On the other hand, the contract is less distorted than contracts that assign the same allocation irrespective of the ex post type.

The incentive constraints in problem $\mathcal{P}_o$ pin down only the expected price for each ex ante type. As a consequence, there is a rich class of mechanisms that solve problem $\mathcal{P}_o$. A simple example for a menu of contracts satisfying the conditions of Lemma 7 is the combination of contract (RD) and the first best contract. High ex ante types $\tau > b$ obtain the first best contract with price $p(\tau,\theta) = v + b(1-\delta)$ that is independent of the ex post type, and low ex ante types $\tau \leq b$ obtain contract (RD) with responsiveness δ. I refer to this menu as menu (RD).

In contrast to the two-goods model, the sets of solutions to the relaxed and the original problem do not coincide. In the sequel, I characterize the set of solutions to the original problem. By showing that menu (RD) is incentive compatible in the original problem $\mathcal{P}$, Lemma 8 provides a necessary condition for a mechanism to solve $\mathcal{P}$:

Lemma 8. *Any solution to the original problem satisfies the conditions of Lemma 7.*

Due to the stronger incentive compatibility requirements in $\mathcal{P}$, maximal profits in $\mathcal{P}_o$ are weakly higher than profits in $\mathcal{P}$. If a mechanism that is optimal in $\mathcal{P}_o$ satisfies all constraints of $\mathcal{P}$, it is hence a solution to problem $\mathcal{P}$ and the maximal profit in $\mathcal{P}_o$ can also be attained in $\mathcal{P}$. The set of solutions to problem $\mathcal{P}$ is then a subset of the set solutions to problem $\mathcal{P}_o$. To prove the lemma, it hence suffices to show that the menu (RD) is implementable in the original problem with private ex post types.

As in Section 1.2, I show that the consumer has an incentive to truthfully report his ex post type no matter what his ex ante report was. As already argued, this is sufficient to show incentive compatibility. The intuition for why in the menu (RD) truthtelling in the second period is always optimal is straightforward: Assume some arbitrary type $\tau \in T$ has reported $\hat{\tau} \leq b$. The price he pays is v independent of the ex post type he reports. Furthermore, the contract specifies the interval $[I; 1 - I]$ of potentially assigned goods. Out of this set, the consumer obtains the good which maximizes the utility of his reported ex post type. Therefore, it follows immediately that truthtelling about the ex post type is optimal. Assume some arbitrary type $\tau \in T$ has reported $\hat{\tau} > b$. The price he pays is $v + b(1 - \delta)$ and the consumer simply gets what he claims to prefer. It can immediately be seen that the consumer will report honestly about the ex post type.

Building on Lemma 8, the following lemma provides a second necessary condition for mechanisms to be a solution to the original problem:

Lemma 9. *In any solution to the original problem, prices for low types $\tau \leq b$ generically do not depend on ex post types: $p(\tau, \theta) = p(\tau, \theta')$ for all $\tau \leq b$, for almost all $\theta, \theta' \in \Theta$.*

Even though Lemma 9 is proved by contradiction, the argument gives valuable intuition. Note first that the continuous set of low ex ante types $\tau \in [0, b]$ includes types that care arbitrarily little about which good they get. Assume that there is some low ex ante type $\tau' < b$ who is confronted with two different prices depending on the ex post type he reports. There will always be ex ante types that care sufficiently little about which good they obtain, such that they would irrespectively of their preferences go for the smaller price, if they had the choice. This would imply lying about the ex post type. Any low ex ante type has this choice, when having reported type τ'.

The conditions of Lemma 7 imply that any ex ante type's expected utility from - possibly untruthfully - claiming to be of any type $\tau \leq b$ and then truthtelling about ex post types is zero. Consider the following double deviation for very low ex ante types: First, falsely report to be of ex ante type τ' and then profitably deviate from truthfully reporting about the ex post type. This strategy would yield a positive expected utility for these ex ante types because they profitably deviate from a strategy that gives them zero utility. In the optimum, this cannot occur, because by incentive compatibility, their expected utility from truthful reporting would then have to be strictly positive as well, which has been shown to be not optimal.

Lemma 9 means that in every optimal mechanism the price the consumer of ex ante type $\tau \leq b$ pays is entirely determined by his ex ante report and not dependent on the ex post type report. The ex post type independence of prices significantly reduces the set of optimal contracts as compared to the optimal contracts in $\mathcal{P}_o$: Using (9), from the second period incentive constraints (IC_2) it follows

$$\int_0^1 c(\theta, s) dX(s|\tau, \theta) \leq \int_0^1 c(\theta, s) dX(s|\tau, \theta') \quad \forall \tau \leq b \text{ for almost all } \theta, \theta' \in \Theta. \tag{1.10}$$

Equation (1.10) means that given the consumer has reported a low ex ante type $\tau \leq b$, truthtelling about θ minimizes the expected utility loss from obtaining non-favorite goods. Intuitively, for each ex post type θ the consumer is assigned his favorite allocation from the set $\{X(\tau, \theta) : \theta \in \Theta\}$ which is the set of all allocations he can obtain by varying the ex post type report. I say that a contract has the restricted delegation property if it has an ex post type independent price and (1.10) holds. The reason is that the consumer's choice in the second period is not connected to transfers - a situation that is familiar from the literature on delegation. The menu (RD) is a particularly tractable mechanism in which all contracts satisfy the restricted delegation property.

Finally, Proposition 3 states the set of optimal mechanisms. Therefore, it shows a sufficiency condition: Any menu of contracts that satisfies the conditions of Lemma 7 and in which each contract satisfies the restricted delegation property is a solution to $\mathcal{P}$:

Proposition 3. *The set of allocation rules which solve the original problem with private ex ante and ex post types is the following:*

- *Each type $\tau > b$ obtains his favorite good.*
- *Each type $\tau \leq b$ obtains a contract form the set of all contracts that satisfy (1.10), (F), full market coverage, and responsiveness equals δ.*

Necessary conditions for prices are:

- *For type $\tau > b$: $\int_0^1 p(\tau, \theta) dG(\theta) = v + b(1 - \delta)$.*
- *For type $\tau \leq b$: $p(\tau, \theta) = v$ for almost all θ.*

Ex post type independent prices are sufficient for incentive compatibility.

Lemmas 7 to 9 show that the properties given in the four bullet points are necessary for a solution. In particular, as Lemma 7 pins down expected prices, Lemmas 7 and 9 together completely determine optimal prices for types $\tau \leq b$. It remains to be shown that the necessary conditions joint with ex post type independent prices are sufficient for a mechanism to solve $\mathcal{P}$. The four bullet points of Proposition 3 imply all conditions of Lemma 7. Hence, if the conditions of Lemma 7 and ex post type independent prices are sufficient for incentive compatibility, the proof is completed. The proof of incentive compatibility is a straightforward generalization of the corresponding proof for the menu (RD) as done in Lemma 8. Note that in the optimal mechanism the monopolist can assigns any ex post type his favorite allocation from a set of allocations without knowing the true ex ante type τ. This is crucial for implementability and possible only because the ordinal ranking of goods given one ex post type θ does not depend on the ex ante type.

As the set of solutions is a subset of the set of solutions to the relaxed problem, the properties from Lemma 7 carry over to Proposition 3. As in the two goods model, each solution satisfies the full market coverage property and is a step solution. Ex ante types above the threshold-type always obtain the good they prefer most, which implies the classical 'no distortion at the top' result. Ex ante types below the threshold-type choose intermediately distorted contracts. The corresponding contracts satisfy the restricted delegation property.[37] In each distorted contract, the set of allocations $\{X(\tau, \theta) : \theta \in \Theta\}$ is designed such that the responsiveness is equal to δ.

Again, contracts for low ex ante types $\tau \leq b$ are not uniquely determined. The set of optimal contracts consists of every set of allocations $\{X(\tau, \theta) : \theta \in \Theta\}$ that induces responsiveness δ, and price v. Contracts for low ex ante types can be arbitrarily combined in the menu as ex ante types are indifferent among all the contracts for any $\hat{\tau} \leq b$. In particular this means that there exist optimal menus which consist of only two contracts, the first best contract and one distorted contract. All ex ante types that take the distorted contracts get an expected utility of zero, because the expected utility from the allocation is equal to v, which is completely skimmed by the price.

For low ex ante types, in optimal contracts that are deterministic, the set of allocations takes the form of a subset of goods. This set of goods is chosen such that the responsiveness is equal to δ. As $\delta < 1$, not all goods are contained in this set. Optimal mechanisms that are deterministic do exist; an example is the menu (RD).

In any optimal menu, distorted contracts lead to the consumption of non-favorite

[37] The consumer's choice of contracts then corresponds to the choice amongst pairs that consist of a price and a set of allocations. The consumer always prefers larger sets of allocations to subsets. Preferences over sets of allocations that have this structure are studied in the literature on preferences for flexibility; for a canonical contribution see Kreps (1979).

goods by restricting the set of allocations that can be chosen from in the second period. For the decision which allocation to consume in the second period, prices do not play any role in the sense that the choice does not affect the price.

An alternative way to induce consumer to not always consume the favorite good is to charge prices that depend on the report about the ex post type and hence on the allocation the consumer gets. In contrast to the instrument of restricted delegation, the consumer might voluntarily consume an allocation different from the favorite good if the associated loss in valuation is outweighed by the price difference.

Proposition 3 states that the use of ex post type-dependent prices is not optimal for a revenue-maximizing monopolist. The reason is that differing ex ante types react differently to price differences. Even though the consumer does not know yet how precisely he will act in the second period when choosing a contract, he already has information about his intensity of preferences and hence about the extent to which he reacts to price differences. This enables the consumer to plan ahead deviations in the second period which induces the firm to leave information rents to the consumer at the contracting stage for potential deviations in the second period. As some ex ante types always choose the lowest price, the only way to avoid differing reactions to price differences is to make these differences large enough such that no ex ante type chooses the expensive contract. This is equivalent to restricted delegation.

The result on the optimality of ex post type-independent prices implies an interesting relation to the literature on sequential screening. Since the contribution by Courty and Li (2000), the literature on sequential screening has concentrated on firms that sell homogeneous goods to consumers who learn their valuations for the good gradually over time. In both the homogeneous goods model and the differentiated goods model, a common property of the revenue-maximizing mechanisms is that the consumer is screened sequentially. In the first period, different ex ante types choose different contracts from a menu and these contracts exhibit allocations that depend on the ex post type, which is screening in the second period. In the model with homogeneous goods, the only screening variable in the second period is the probability with which the good is assigned to the consumer. Independently of the second period information, the consumer prefers higher chances to obtain the good to lower chances. Hence, screening ex post types in the second period must involve prices that vary with the report about the ex post type such that some ex ante types prefer having lower chances to obtain the good. These ex post type-dependent prices are undesirable, as they induce the firm to leave information rents to the consumer following the logic described in the previous paragraph. In my model, the presence of differentiated goods allows the firm to set up an allocation rule that exhibits allocations with are ex post type-dependent and incentive compatible without ex post type-dependent prices at the same time. The additional instrument that is available to the firm in the presence of heterogeneous goods is restricted delegation. As shown in Lemma 8, any desired response of the allocation to ex post types, measured as the responsiveness, can be achieved by restricted delegation. From this perspective, the new tool is superior for the firm as it gives rise to less rents left to the consumer.

1.3.3 Implementation

Direct mechanisms are rarely observed in practice. A general concern in the literature on mechanism design is therefore to find simple indirect mechanisms that are equivalent in terms of outcomes to the optimal direct mechanism. Ideally, the indirect mechanisms match observations in the economic context the theory addresses.

In this paper, the simple indirect mechanisms are menus of Limited Exchange Contracts. A Limited Exchange Contract is a triple $\langle \pi, X, \Psi \rangle$, where $\pi \in \mathbb{R}$ is a price, X is an allocation, and Ψ a set of allocations with $X \in \Psi$. In the first period, the consumer pays π and obtains allocation X. In the second period, the consumer can exchange his allocation with any allocation in Ψ without any monetary transfers. There is no possibility at all to exchange with allocations which are not in the set Ψ.

Proposition 4. *Every optimal allocation rule from Proposition 3 can be implemented by a menu of Limited Exchange Contracts.*

The Limited Exchange Contract $\langle v + b(1-\delta), s, S \rangle$, in which s is a good from the set of all goods S, implements the optimal contract for high ex ante types $\tau > b$. For any ex ante type $\tau \leq b$ denote by $\Psi(\tau)$ the set of all allocations the consumer can obtain in the second period in the optimal contract by varying the ex post type report. The corresponding Limited Exchange Contract is $\langle v, y, \Psi(\tau) \rangle$ with y contained in $\Psi(\tau)$. Potentially, these Limited Exchange Contracts differ for each $\tau \leq b$.

Note that the first best contract for high ex ante types can be seen as a special case of a Limited Exchange Contract, where the limitation on the exchange set is not binding in the sense that the consumer never favors an allocation which does not belong to the set. As already argued for direct mechanisms, offering just one optimal restrictive Limited Exchange Contracts joint with the not restrictive Limited Exchange Contract is an optimal menu as well. The menu of Limited Exchange Contracts that implements the optimal menu (RD), which is simple and deterministic, is given in Example 1.

Example 1: Implementation of menu (RD)
Offer the following two Limited Exchange Contracts:
(i) Contract $\langle v + b(1-\delta), s, S\rangle$, in which s is a good from the set of all goods S.
(ii) Contract $\langle v, 1/2, [I, 1-I]\rangle$, in which I is defined as in the optimal menu (RD).

The menu of Example 1 contains only one distorted contract, in which the exchange set is an interval of goods around the initially purchased good. It is easy to verify that for δ close to one, contract (ii) allows for almost free exchange, which means the contract is only slightly distorted. The lower δ, the smaller is the set of goods with which free exchange is possible and for $\delta = 0$ contract (ii) does not give any opportunity for exchange.

Applied to ticket pricing for transportation services, the optimal menu (RD) has an intuitive interpretation. On the one hand, tickets with free exchange to any other departure time are offered . On the other hand, for any departure time tickets are sold that include the option to change departure time for free within a certain time span around the initially purchased departure time. Many airlines have explicitly designed such options by introducing costless same-day exchange possibilities and stand-by options. A same-day exchange option usually is an extra amendment to the terms and conditions of a flight ticket, which allows consumers to change flight within the same day for free or at a symbolic price. Stand-by options are closely related amendments, which - upon availability - enable passengers to take an earlier flight if they arrive early at the airport or to take a later one if they miss their flight. An implicit equivalent to these contracts emerges when airlines create a reputation for being obliging concerning their refund and exchange policy. The use of Limited Exchange Contracts is also common among ferry companies; examples are P&O Ferries and DFDS Seaways. Both companies offer tickets which explicitly specify a time interval around the purchased departure time within which costless exchange is possible. Tickets that provide full flexibility can be obtained at higher prices. Note that the first best contract can also be implemented by offering expensive tickets for each variant at the point in time of consumption.

An alternative concept to abstain the consumer from always purchasing his favorite good is to introduce exchange fees. When the consumer's valuation for the initially acquired good is close to the top valuation, the consumer might prefer to stay with the non-favorite good in order to save the exchange fee.

Proposition 5. *Contracts that limit responsiveness via exchange fees are not optimal.*

The use of exchange fees to limit responsiveness is not optimal, because their use implies ex post type-dependent prices: If the consumer decided to buy a certain allocation for some price p in the first period, it depends on his ex post type whether he prefers to stay with the allocation or to pay an additional exchange fee p_e and get a preferred distribution over goods. This means for some ex post types the price is p and for some ex post types it is $p + p_e$. Ex post type dependent prices are, however, not optimal by Lemma 9.

With the help of Proposition 5, I finally argue that the optimal mechanism cannot simply be constructed as a combination of partial refund contracts. A menu of partial refund contracts implements the revenue-maximizing mechanism in the homogeneous good setting of Courty and Li (2000). As giving one good back and buying a new one is essentially equivalent to exchange, one might think that the optimal mechanism in the heterogeneous goods model is obtained when combining the optimal mechanisms of multiple homogeneous goods models. However, the optimal mechanism in my model can *not* be implemented by a combination of partial refund contracts. To see this, start by noting that the first best contract, which is optimal for high ex ante types, can indeed be implemented by combining full refund contracts for each good. However, the optimal distorted contracts cannot be implemented by giving back a good for a partial refund and buying a new one. The latter procedure entails a cost, which is the money for the returned good which is not being refunded. This is equivalent to an exchange fee whose use is shown to be not optimal.

1.4 Conclusion

In this paper, I have characterized revenue-maximizing contracts for situations in which consumers learn their valuations for horizontally differentiated goods gradually over time. Initially, consumers know about their desire to obtain a certain type of product and differ in terms of their preference intensity and their highest valuation, but are uncertain about their favorite variant. Let higher ex ante types have higher valuations for their favorite good and larger cost from consuming non-favorite goods. The mechanism design approach without ad-hoc restrictions on contracts shows that in the optimum the flexibility to choose among allocations in the second period is used as a price discrimination device. The revenue-maximizing contract is a step solution. Consumers with high ex ante types always receive their favorite good. If the valuation of undesirable goods is sufficiently decreasing in ex ante types, contracts for consumers with low ex ante types partially restrict the flexibility to choose between goods. The optimal device to restrict flexibility is restricted delegation. The key feature of restricted delegation is that prices are completely determined in the first period and cannot be influenced by the consumer in the second period. This contrasts the optimal contracts in the standard sequential screening problem in which a homogeneous good is sold. Restricted delegation can be implemented by Limited Exchange Contracts. A deterministic Limited Exchange Contract consists of an initial good offered in the first period at some price and the option to exchange it to some good out of a fixed subset of goods later on for free. The use of exchange fees as a price discrimination device is not optimal.

There are several versions of and extensions to the model which are worth being examined. This paper studies the benchmark case in which the ex ante type completely the the level of top valuation. Relaxing this assumption about the information structure could have an impact on the model's predictions. A model in which the ex ante type leaves a sufficiently high degree of uncertainty about the top valuation may imply that optimal policies involve both exchanges and refunds. Furthermore, an important task is to check if the optimal contract design changes when moving from the monopoly to an oligopolistic environment; of particular interest is the robustness of the optimality of restricted delegation. Finally, the question of how capacity constraints influence optional exchange policies in the presence of aggregate uncertainty deems interesting as well: The revenue-maximizer then faces an additional trade-off between giving consumers the optimal amount of flexibility and directing them towards available capacity.

1.5 Appendix

Proof of Lemma 1:

Consider any mechanism $\{X(\hat{\tau}, \hat{\theta}), p(\hat{\tau}, \hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$.
Construct an alternative mechanism $\{\tilde{X}(\hat{\tau}, \hat{\theta}), \tilde{p}(\hat{\tau}, \hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$ such that

- $\tilde{p}(\hat{\tau}, \hat{\theta}) = p(\hat{\tau}, \hat{\theta}) \quad \forall \hat{\tau}, \hat{\theta},$
- $\tilde{x}_{1\&2}(\hat{\tau}, \hat{\theta}) = 0 \quad \forall \hat{\tau}, \hat{\theta},$
- $\tilde{x}_i(\hat{\tau}, \theta_i) = x_i(\hat{\tau}, \theta_i) + x_{1\&2}(\hat{\tau}, \theta_i) \quad \forall \hat{\tau}, \forall i \in \{1, 2\},$
- $\tilde{x}_{3-i}(\hat{\tau}, \theta_i) = x_{3-i}(\hat{\tau}, \theta_i) \quad \forall \hat{\tau}, \forall i \in \{1, 2\}.$

From (1.1) can immediately be seen that for any given $\tau, \hat{\tau},$ and a, $u(\tau, \hat{\tau}, \theta, \theta)$ is equal for both mechanisms, whereas for $a \neq \hat{\theta}$ ex post utility $u(\tau, \hat{\tau}, \theta, \hat{\theta})$ is weakly higher under mechanism $\big(X(\hat{\tau}, \hat{\theta}), p(\hat{\tau}, \hat{\theta})\big)$. From this follows that the modified mechanism satisfies (IC_1) and (IC_2) if mechanism $\big(X(\hat{\tau}, \hat{\theta}), p(\hat{\tau}, \hat{\theta})\big)$ does. ■

Proof of Lemma 2:

Recall that $K(\hat{\tau},\delta) = x^+(\hat{\tau}) - x^-(\hat{\tau}) - \delta(x^+(\hat{\tau}) + x^-(\hat{\tau}))$.
By (IC_1'), for any τ, τ' with $\tau > \tau'$ it holds that

$$\begin{aligned} U(\tau) &\geq U(\tau', \tau, id_\theta) \\ &= v[x^+(\tau') + x^-(\tau')] + \tau' \cdot K(\tau',\delta) - \mathbb{E}_\theta[p(\tau',\theta)] \qquad + (\tau - \tau') \cdot K(\tau',\delta) \\ &= U(\tau') \qquad + (\tau - \tau') \cdot K(\tau',\delta). \end{aligned}$$

Analogeously,

$$U(\tau') \geq U(\tau) + (\tau' - \tau) \cdot K(\tau,\delta).$$

Combining the two inequalitities yields

$$(\tau - r') \cdot K(\tau,\delta) \geq U(\tau) - U(\tau') \geq (\tau - \tau') \cdot K(\tau',\delta).$$

Dividing by $(\tau - \tau')$ yields:

$$K(\tau,\delta) \text{ is monotonically increasing in } \tau \qquad (MON).$$

Letting τ' converge to τ yields:

$$\partial U(\tau)/\partial\tau = K(\tau,\delta) \text{ almost everywhere} \qquad (ENV).$$

To proof the invers direction, note that from (ENV) and absolute continuity it follows that for any τ, τ' with $\tau > \tau'$

$$U(\tau) = U(\tau', \tau, id_\theta) + \int_{r'}^{\tau} K(y,\delta)dy.$$

For a proof of absolute continuity see for example Theorem 2 in Milgrom and Segal (2002).
From the monotonicity condition (MON) it follows that

$$\begin{aligned} U(\tau', \tau, id_\theta) + \int_{\tau'}^{\tau} K(y,\delta)dy &\geq U(\tau', \tau, id_\theta) + \int_{\tau'}^{\tau} K(\tau',\delta)dy \\ &= U(\tau') + (\tau - \tau') \cdot K(\tau',\delta) \\ &= U(\tau', \tau, id_\theta). \end{aligned}$$

■

Proof of Lemma 3:

Maximization problem (1.5) is solved by pointwise maximization for every ex ante type τ subject to constraints:
As z by definition has minimal expected utility, from (ENV) and (MON) follows

$$\begin{aligned} & K(\tau,\delta) \leq 0 \quad \text{for } \tau < z \\ \text{and} \quad & K(\tau,\delta) \geq 0 \quad \text{for } \tau > z. \end{aligned} \tag{1.11}$$

Furthermore from the feasibility constraints (F) follows

$$\begin{aligned} & x^+(\tau) + x^-(\tau) \leq 1 \\ \text{and} \quad & K(\tau,\delta) \leq x^+(\tau) + x^-(\tau) - \delta(x^+(\tau) + x^-(\tau)) \leq 1 - \delta. \end{aligned} \tag{1.12}$$

<u>Case 1:</u> $\tau \geq z$

The optimal allocation rule maximizes

$$v[x^+(\tau) + x^-(\tau)] + K(\tau,\delta) \cdot \left(\tau - \frac{1-F(\tau)}{f(\tau)}\right).$$

Case 1.1: $\tau > b$
The virtual value is positive. If there exists a contract such that $x^+(\tau)+x^-(\tau) = 1$ and $K(\tau,\delta) = 1-\delta$, by (1.12) it is optimal at point τ.
Case 1.2: $\tau \leq b$
The virtual value is weakly negative.[38] If there exists a contract such that $x^+(\tau) + x^-(\tau) = 1$ and $K(\tau,\delta) = 0$, by (1.11) and (1.12) it is optimal at point τ.

<u>Case 2:</u> $\tau < z$

The optimal allocation rule maximizes

$$v[x^+(\tau) + x^-(\tau)] + K(\tau,\delta) \cdot \left(\tau + \frac{F(\tau)}{f(\tau)}\right).$$

The virtual value $\left(\tau + \frac{F(\tau)}{f(\tau)}\right)$ is positive for all $\tau \in R$. If there exists a contract such that $x^+(\tau) + x^-(\tau) = 1$ and $K(\tau,\delta) = 0$, by (1.11) and (1.12) it is optimal

[38] w.l.o.g. let b and types with $\tau - \frac{1-F(\tau)}{f(\tau)} = 0$ get the contract that is optimal for types with $\tau - \frac{1-F(\tau)}{f(\tau)} < 0$

at point τ.

The two pairs of optimality conditions can be solved for $x^+(\tau)$ and $x^-(\tau)$:

$$\begin{aligned}
&x^+(\tau)+x^-(\tau)=1, K(\tau,\delta)=0 && \Leftrightarrow x^+(\tau)=\frac{1+\delta}{2}, && x^-(\tau)=\frac{1-\delta}{2}\\
&x^+(\tau)+x^-(\tau)=1, K(\tau,\delta)=1-\delta && \Leftrightarrow x^+(\tau)=1, && x^-(\tau)=0
\end{aligned}$$

The properties suggested above are stated in the lemma. It is left to be proven that there exists an allocation rule with the properties of Lemma 3 that satisfies (F), and any allocation rule satisfying the properties of Lemma 3 satisfies (MON) and $U(z) \geq U(\tau) \quad \forall \tau \in T$. The conditions (MON) and $U(z) \geq U(\tau) \quad \forall \tau \in T$ are immediately implied by the properties of Lemma 3. Existence of a feasible allocation rule with the desired properties is shown by construction of an example:

$$\begin{aligned}
&\text{For } \tau > max\{b,z\}: && x_i(\tau,\theta_i)=1, \quad x_i(\tau,\theta_{3-i})=0 && \forall i \in \{1,2\}\\
&\text{For } \tau \leq max\{b,z\}: && x_i(\tau,\theta_i)=\frac{1+\delta}{2}, \quad x_i(\tau,\theta_{3-i})=\frac{1-\delta}{2} && \forall i \in \{1,2\}
\end{aligned}$$

■

Proof of Lemma 4:

Insert the optimality conditions from Lemma 3 into objective (1.5):

$$\begin{aligned}
&\int_0^z f(\tau)\left[v[x^+(\tau)+x^-(\tau)]+K(\tau,\delta)\cdot\left(\tau+\frac{F(\tau)}{f(\tau)}\right)\right]dr\\
&+\int_z^{\overline{\tau}} f(\tau)\left[v[x^+(\tau)+x^-(\tau)]+K(\tau,\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)\right]dr\\
&=\int_0^z f(\tau)\left[v+0\cdot\left(\tau+\frac{F(\tau)}{f(\tau)}\right)\right]dr+\int_z^{\overline{\tau}} f(\tau)v dr\\
&+\int_{max\{b,z\}}^{\overline{\tau}} f(\tau)(1-\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)dr\\
&=v+\int_{max\{b,z\}}^{\overline{\tau}} f(\tau)(1-\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)dr
\end{aligned}$$

By definition, $\tau-\frac{1-F(\tau)}{f(\tau)}>0 \quad \forall r>b$. Hence, z is optimal if and only if $z \leq b$. ■

Proof of Lemma 5:

Lemma 4 implies that $max\{b, z\} = b$. The conditions from Lemma 3 are then

$$\begin{aligned} x^+(\tau) &= \frac{1+\delta}{2} && \text{and } x^-(\tau) = \frac{1-\delta}{2} && \text{if } \tau \le b, \\ x^+(\tau) &= 1 && \text{and } x^-(\tau) = 0 && \text{if } \tau > b. \end{aligned}$$

Via (1.4) these characteristics determine the sum of prices $p(\tau, \theta_1) + p(\tau, \theta_2)$ for each ex ante type τ.
Finally, it is left to state the nonempty set of feasible allocation rules that satisfy the optimality conditions. For all types $\tau > b$ from the optimality condition $x^+(\tau) = 1$ and $x^-(\tau) = 0$ together with feasibility (F), it follows $x_1(\tau, \theta_1) = x_2(\tau, \theta_2) = 1$ and $x_2(\tau, \theta_1) = x_1(\tau, \theta_2) = 0$. For $\tau \le b$ feasibility and optimality can be described by the following system of equations:

$$\begin{aligned} x_1(\tau, \theta_1) + x_2(\tau, \theta_2) &= 1 + \delta, \\ x_2(\tau, \theta_1) + x_1(\tau, \theta_2) &= 1 - \delta, \\ x_1(\tau, \theta) + x_2(\tau, \theta) &\le 1 \quad \forall \theta, \\ x_i(\tau, \theta) &\in [0, 1] \quad \forall \theta, i. \end{aligned}$$

There is one degree of freedom and the non-empty set of solutions to this system is the following:

$$\begin{aligned} x_1(\tau, \theta_1) &= \alpha \in [\delta, 1], \\ x_2(\tau, \theta_1) &= 1 - \alpha, \\ x_1(\tau, \theta_2) &= \alpha - \delta, \\ x_2(\tau, \theta_2) &= 1 + \delta - \alpha. \end{aligned}$$

■

Proof of Proposition 1:

It is left to prove that for $p(\tau, \theta_1) = p(\tau, \theta_2) \quad \forall \tau$ any contract satisfies (IC_1) and (IC_2). Proposition 1 follows then from Lemma 5.
Define the following strengthening of condition (IC_2):

$$u(\tau, \hat{\tau}, \theta, \theta) \ge u(\tau, \hat{\tau}, \theta, \hat{\theta}) \quad \forall \tau, \hat{\tau}, \theta, \hat{\theta}. \qquad (IC_2^s)$$

(IC_2^s) states that in the second period truthtelling is optimal for any ex ante report.

<u>Claim 1:</u> *(IC_2^s) and (IC_1') imply (IC_2) and (IC_1).*

(IC_2) trivially follows from (IC_2^s). (IC_1) is implied as well: Consider some ex ante type and an arbitrary reporting strategy. By (IC_2^s), the consumer can always weakly improve by reporting truthfully about his ex post type. Given truthful reporting about the second period type, by (IC_1') the consumer can then weakly improve by reporting truthfully about the ex ante type.

<u>Claim 2:</u> *Any element from the set of allocation rules from Proposition 1 satisfies (IC_2^s) if $p(\tau, \theta_1) = p(\tau, \theta_2) \quad \forall \tau$.*

The claim is shown by plugging an arbitrary element of the set and corresponding prices into (IC_2^s) using (1.1):

Case 1: $\hat{\tau} \leq b$

(IC_2^s) is satisfied, as

$$\alpha(v-\delta\tau+\tau)+(1-\alpha)(v-\delta\tau-\tau)-v \geq (\alpha-\delta)(v-\delta\tau+\tau)+(1-\alpha+\delta)(v-\delta\tau-\tau)-v \quad \forall \tau, \alpha$$

and

$$(1-\alpha+\delta)(v-\delta\tau+\tau)+(\alpha-\delta)(v-\delta\tau-\tau)-v \geq (1-\alpha)(v-\delta\tau+\tau)+\alpha(v-\delta\tau-\tau)-v \quad \forall \tau, \alpha$$

hold if and only if $\delta \geq 0$.

Case 2: $\hat{\tau} > b$

(IC_2^s) is satisfied, as

$$1\cdot(v-\delta\tau+\tau)+0\cdot(v-\delta\tau-\tau)-v-b(1-\delta) \geq 0\cdot(v-\delta\tau+\tau)+1\cdot(v-\delta\tau-\tau)-v-b(1-\delta) \quad \forall \tau, \alpha.$$

■

Proof of Proposition 2:

To prove the proposition, I first solve a relaxed problem, which gives an upper bound on profits, and then show that any solution to the relaxed problem is implementable in $\mathcal{P}$.

Define $\mathcal{P}_*$ as $\mathcal{P}_o$ with the additional constraint

$$x_1(\tau, \theta_1) + x_2(\tau, \theta_2) \geq x_1(\tau, \theta_2) + x_2(\tau, \theta_1) \quad \forall \tau. \tag{*}$$

<u>Claim 1:</u> *$\mathcal{P}_*$ is a relaxed problem of $\mathcal{P}$*

It is sufficient to show that (*) follows from IC_2. IC_2 states that $\forall \tau$ hold

$$\begin{aligned} &u(\tau, \tau, \theta_1, \theta_1) \geq u(\tau, \hat{\tau}, \theta_1, \theta_2) \quad \text{and} \\ &u(\tau, \tau, \theta_2, \theta_2) \geq u(\tau, \hat{\tau}, \theta_2, \theta_1). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & v^+(\tau) \cdot (x_1(\tau, \theta_2) - x_1(\tau, \theta_1)) + v^-(\tau) \cdot (x_2(\tau, \theta_2) - x_2(\tau, \theta_1)) \\ \leq \; & p(\tau, \theta_2) - p(\tau, \theta_1) \\ \leq \; & v^+(\tau) \cdot (x_2(\tau, \theta_2) - x_2(\tau, \theta_1)) + v^-(\tau) \cdot (x_1(\tau, \theta_2) - x_1(\tau, \theta_1)) \quad \forall \tau. \end{aligned}$$

An immediate consequence is

$$\begin{aligned} & v^+(\tau) \cdot [(x_2(\tau, \theta_2) - x_2(\tau, \theta_1)) - (x_1(\tau, \theta_2) - x_1(\tau, \theta_1))] \\ \geq \; & v^-(\tau) \cdot [(x_2(\tau, \theta_2) - x_2(\tau, \theta_1)) - (x_1(\tau, \theta_2) - x_1(\tau, \theta_1))] \quad \forall \tau, \end{aligned}$$

which is equivalent to

$$x_1(\tau, \theta_1) + x_2(\tau, \theta_2) \geq x_1(\tau, \theta_2) + x_2(\tau, \theta_1) \quad \forall \tau. \tag{*}$$

Define $e = sup\{\tau \in R | r - \frac{1-F(\tau)}{f(\tau)} \leq \frac{v}{\delta}\})$ whenever the supremum exists and $e = 0$ otherwise.

<u>Claim 2:</u> *The solution to $\mathcal{P}_*$ is the following:*

$$\begin{aligned} \textit{For } r > b: \quad & x_1(\tau, \theta_1) = x_2(\tau, \theta_2) = 1, \quad x_1(\tau, \theta_2) = x_2(\tau, \theta_1) = 0 \\ \textit{and} \quad & \mathbb{E}_\theta[p(\tau', \theta)] = v + b(1 - \delta). \\ \textit{For } r \in [e, b]: \quad & x_1(\tau, \theta_1) = x_1(\tau, \theta_2) = \alpha, \quad x_2(\tau, \theta_2) = x_2(\tau, \theta_1) = 1 - \alpha \quad \alpha \in [0, 1] \\ \textit{and} \quad & \mathbb{E}_\theta[p(\tau', \theta)] = v - \delta e. \\ \textit{For } r < e: \quad & x_i(\tau, \theta_j) = 0 \quad \forall i, j \in 1, 2 \\ \textit{and} \quad & \mathbb{E}_\theta[p(\tau', \theta)] = 0. \end{aligned}$$

As the first period incentive constraints are identical in $\mathcal{P}_o$ and $\mathcal{P}_*$, Lemma 2 applies.

Lemma 2: The first period incentive constraints IC_1' are satisfied if and only if

$$\partial U(\tau)/\partial\tau = K(\tau,\delta) \ a.e. \qquad (ENV)$$

$$and \quad K(\tau,\delta) \ is \ mon. \ increasing \ in \ r. \qquad (MON)$$

(*) is equivalent to $x^+(\tau) \geq x^-(\tau)$ and from (*) and $\delta < 0$ follows $K(\tau,\delta) \geq 0$. Hence in the optimum expected utility is increasing in ex ante types everywhere and the lowest ex ante type's participation constraint is binding. Following the standard approach, the problem can be restated as

$$\max_x \int_0^{\overline{\tau}} f(\tau)\Big[v[x^+(\tau)+x^-(\tau)] + K(\tau,\delta)\Big(\tau - \frac{1-F(\tau)}{f(\tau)}\Big)\Big] dr$$

s.t. MON, (F), (*).

A solution to this problem is found by pointwise maximization of the relaxed version without the monotonicity constraint.

Case 1: $\tau > b$

Virtual value is positive. Pointwise maximization gives $x^+(\tau) = 1$ and $x^-(\tau) = 0$. For a formal derivation see Lemma 3. The contract trivially satisfies (*).

Case 2: $\tau \leq b$

Virtual value is negative. Maximization is done in two steps:
First, for any fixed $x^+(\tau) + x^-(\tau) = m$, under the restriction (*) virtual surplus is maximized for $x^+(\tau) = x^-(\tau) = m/2$.
Second, m is chosen to satisfy (F) and maximize

$$v \cdot m - \delta \cdot m\Big(1 - \frac{1-F(\tau)}{f(\tau)}\Big).$$

The solution to this linear problem is

$$m = \begin{cases} 1, & \text{if } \tau - [1-F(\tau)]/f(\tau) \geq v/\delta \\ 0, & \text{if } \tau - [1-F(\tau)]/f(\tau) < v/\delta \end{cases}$$

Cases 1 and 2 give the allocation rules of Claim 2, which satisfy the monotonicity constraint. Expected prices are fixed by equation (1.4).

<u>Claim 3:</u> *Any solution to $\mathcal{P}_*$ is implementable in $\mathcal{P}$ if prices are ex post type indepen-*

dent.

It is left to prove that for $p(\tau,\theta_1) = p(\tau,\theta_2) \quad \forall r$ any solution to $\mathcal{P}_*$ satisfies (IC_1) and (IC_2).
According to the proof of Proposition 1 it suffices to show that for $p(\tau,\theta_1) = p(\tau,\theta_2)$ any solution to $\mathcal{P}_*$ satisfies IC_2^s.
As shown in the proof of Proposition 1, the contract for types $\tau > b$ satisfies IC_2^s. The contract for types $\tau \leq b$ trivially satisfies IC_2^s, as the report about the ex post type has no influence on the allocation.

Claims one, two and three imply the proposition. ■

Proof of Lemma 6:

First, note that contract (RD) satisfies

$$\int_0^1 X(1|\tau,\theta)dG(\theta) = 1 \quad \forall I \in [0;1/2]. \tag{1.13}$$

Second, plugging contract (RD) and (1.13) into (1.9) yields

$$R(\tau) = 1 - \int_0^I c(\theta,I)dG(\theta) - \int_{1-I}^1 c(\theta,1-I)dG(\theta) =: R(\tau,I). \tag{1.14}$$

Note that $R(\tau,0) = 1$, and from the model assumption $\mathbb{E}_\theta[c(\theta,k)] > 1-\delta$ it follows that $R(\tau,1/2) < \delta$. If $R(\tau,I)$ is continuous in I, there is an I' such that $R(\tau,I') = 0$ by the intermediate value theorem. I show that the second summand in (1.14) is continuous. The argument for the third summand is analogous.
To begin with, note that

$$\int_0^{I+\gamma} c(\theta,I+\gamma)dG(\theta) - \int_0^I c(\theta,I)dG(\theta) = \int_0^I c(\theta,I+\gamma) - c(\theta,I)dG(\theta) + \int_I^{I+\gamma} c(\theta,1-I)dG(\theta).$$

$c(.)$ is bounded above by v and hence

$$\int_I^{I+\gamma'} c(\theta,1-I)dG(\theta) \leq \gamma' v.$$

As $c(.)$ is a continuous function that maps from a compact subset of the $\mathbb{R}^2$ into the the real numbers, it is uniformly continuous by the Heine-Cantor theorem. Hence, for any ϵ' there is a γ'' such that

$$\int_0^I c(\theta, I+\gamma'') - c(\theta, I)dG(\theta) \leq \int_0^I \epsilon' dG(\theta) \leq \epsilon'.$$

For an arbitrary ϵ, choose $\gamma' = \epsilon/2v$, γ'' such that $\epsilon' = \epsilon/2$, and $\gamma = \min\{\gamma', \gamma''\}$. Then

$$\int_0^{I+\gamma} c(\theta, I+\gamma)dG(\theta) - \int_0^I c(\theta, I)dG(\theta) \leq \epsilon.$$

■

Proof of Lemma 7:

Recall that $K(\hat{\tau}, \delta) = \int_0^1 \int_0^1 1 - \delta - c(\theta, s)dX(s|\hat{\tau}, \theta)dG(\theta)$.
An immediate consequence from the similar structure of the expected utilities (1.8) and (1.2) is the following characterization of incentive compatibility:

The first period incentive constraints (IC_1') are satisfied if and only if

$$\partial U(\tau)/\partial\tau = K(\tau, \delta) \text{ a.e.} \qquad (ENV)$$

$$\text{and} \quad K(\tau, \delta) \text{ is mon. increasing in } \tau. \qquad (MON)$$

Therefore maximizing with respect to the constraints (IC_1'), (IR) and (F) is equivalent to taking (ENV), (MON), (IR) and (F) as constraints. Depending on τ, the term $K(\tau, \delta)$ can take both negative and positive values.
The solution concept introduced for the two-goods model is applied here as well: In every solution, there exists an ex ante type $z \in [0, \overline{\tau}]$ such that $U(z) \leq U(\tau) \quad \forall \tau \in T$. First, I arbitrarily fix z and solve problem $\mathcal{P}_o^z$, which is problem $\mathcal{P}_o$ with the additional constraint $U(\tau) \geq U(z) \quad \forall \tau \in T$. Second, I maximize profit in z.

Preliminary Step: Reformulation of $\mathcal{P}_o^z$

By (ENV) and individual rationality, in the optimum, $U(z) = 0$ and any ex ante type's expected utility can then be written as

$$U(\tau) = U(z) + \int_z^\tau K(y, \delta)dy = \int_z^\tau K(y, \delta)dy. \qquad (1.15)$$

By (1.8) and (1.15) prices can be written as a function of the allocation:

$$\int_0^1 p(\tau,\theta)dG(\theta) = v\left(\int_0^1 X(1|\tau,\theta)dG(\theta)\right) + \tau\cdot K(\tau,\delta) - \int_z^\tau K(y,\delta)dy. \tag{1.16}$$

Plugging (1.16) into the objective reduces problem $\mathcal{P}_o^z$ to

$$\max_x \int_0^{\overline{\tau}} f(\tau)\Big(v\int_0^1 X(1|\tau,\theta)dG(\theta) + \tau\cdot K(\tau,\delta) - \int_z^\tau K(y,\delta)dy\Big)d\tau$$

s.t. (MON), (F) and $U(z)\geq U(\tau) \quad \forall\tau\in T$.

By integration by parts and rearranging terms, the problem can be rewritten as

$$\begin{aligned}\max_x &\int_0^z f(\tau)\left[v\int_0^1 X(1|\tau,\theta)dG(\theta) + K(\tau,\delta)\cdot\left(\tau+\frac{F(\tau)}{f(\tau)}\right)\right]d\tau \\ +&\int_z^{\overline{\tau}} f(\tau)\left[v\int_0^1 X(1|\tau,\theta)dG(\theta) + K(\tau,\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)\right]d\tau\end{aligned} \tag{1.17}$$

s.t. (MON), (F) and $U(z)\geq U(\tau) \quad \forall\tau\in T$.

Claim 1: *A feasible allocation rule which satisfies condition (RO) is a solution to $\mathcal{P}_o^z$. If there exists a feasible allocation rule that satisfies (RO) then this condition is also necessary for an allocation rule to be a solution to $\mathcal{P}_o^z$.*

$$\begin{aligned}\int_0^1 X(1|\tau,\theta)dG(\theta) =&1 \quad \text{and } K(\tau,\delta)=0 \quad \text{if } \tau\leq max\{b,z\} \\ \int_0^1 X(1|\tau,\theta)dG(\theta) =&1 \quad \text{and } K(\tau,\delta)=1-\delta \quad \text{if } \tau > max\{b,z\}\end{aligned} \tag{RO}$$

Maximization problem (1.17) is solved by pointwise maximization for every ex ante type τ. Recall $b = sup\{\tau\in R|(\tau-\frac{1-F(\tau)}{f(\tau)}\leq 0\}$.
As z by definition has minimal expected utility, from (ENV) and (MON) it follows

$$\begin{aligned}&K(\tau,\delta)\leq 0 \quad \text{for } \tau<z \\ \text{and} \quad &K(\tau,\delta)\geq 0 \quad \text{for } \tau>z.\end{aligned} \tag{1.18}$$

Furthermore from the feasibility constraints (F) it follows

$$\int_0^1 X(1|\tau,\theta)dG(\theta)\leq 1 \quad \text{and} \tag{1.19}$$

$$
\begin{aligned}
K(\tau,\delta) &= \int_0^1 \int_0^1 (1-\delta-c(\theta,s))dX(s|\tau,\theta)dG(\theta) \\
&\leq \int_0^1 \int_0^1 (1-\delta)dX(s|\tau,\theta)dG(\theta) \\
&\leq 1-\delta.
\end{aligned} \tag{1.20}
$$

Case 1: $\tau > z$

The optimal allocation rule maximizes

$$
v\int_0^1 X(1|\tau,\theta)dG(\theta) + K(\tau,\delta)\cdot\left(\tau - \frac{1-F(\tau)}{f(\tau)}\right).
$$

Case 1.1: $\tau > b$
The virtual value is positive. If there exists a contract such that $\int_0^1 X(1|\tau,\theta)dG(\theta) = 1$ and $K(\tau,\delta) = 1-\delta$, by (1.19) and (1.20) it is optimal at point τ.

Case 1.2: $\tau \leq b$
The virtual value is weakly negative.[39] If there exists a contract such that $\int_0^1 X(1|\tau,\theta)dG(\theta) = 1$ and $K(\tau,\delta) = 0$, by (1.18), (1.19) and (1.20) it is optimal at point τ.

Case 2: $\tau < z$

The optimal allocation rule maximizes

$$
v\int_0^1 X(1|\tau,\theta)dG(\theta) + K(\tau,\delta)\cdot\left(\tau + \frac{F(\tau)}{f(\tau)}\right).
$$

The virtual value $\left(\tau + \frac{F(\tau)}{f(\tau)}\right)$ is positive for all $\tau \in T$. If there exists a contract such that $\int_0^1 X(1|\tau,\theta)dG(\theta) = 1$ and $K(\tau,\delta) = 0$, by (1.18), (1.19) and (1.20) it is optimal at point τ.

Provided the existence of a feasible allocation rule with the determined characteristics, any solution to the relaxed problem has the properties (RO). Monotonicity is satisfied.

Claim 2: *Given feasible allocation rules that satisfy (RO) exist, z is optimal if and only if $z \leq b$.*

[39] w.l.o.g. let b and types with $\tau - \frac{1-F(\tau)}{f(\tau)} = 0$ get the allocation of low types

Insert the optimality conditions (RO) into objective (1.17):

$$\int_0^z f(\tau)\left[v\int_0^1 X(1|\tau,\theta)dG(\theta) + K(\tau,\delta)\cdot\left(\tau+\frac{F(\tau)}{f(\tau)}\right)\right]d\tau$$
$$+\int_z^{\overline{\tau}} f(\tau)\left[v\int_0^1 X(1|\tau,\theta)dG(\theta) + K(\tau,\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)\right]d\tau$$
$$=\int_0^z f(\tau)\left[v+0\cdot\left(\tau+\frac{F(\tau)}{f(\tau)}\right)\right]d\tau + \int_z^{\overline{\tau}} f(\tau)v d\tau + \int_{max\{b,z\}}^{\overline{\tau}} f(\tau)(1-\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)d\tau$$
$$=v+\int_{max\{b,z\}}^{\overline{\tau}} f(\tau)(1-\delta)\cdot\left(\tau-\frac{1-F(\tau)}{f(\tau)}\right)d\tau$$

By definition, $\tau-\frac{1-F(\tau)}{f(\tau)}>0 \quad \forall\tau>b$. Hence, z is optimal if and only if $z\leq b$.

<u>Claim 3:</u> *A feasible allocation rules that satisfies (RO) does exist.*

I construct a mechanism whose deterministic allocation rule is feasible and satisfies (RO). I refer to the mechanism as menu (RD): Ex ante types $\tau>b$ obtain the first best contract for price $p(\tau,\theta)=v+b(1-\delta) \quad \forall\theta$, and low ex ante types $\tau\leq b$ obtain contract (RD) with responsiveness δ. Contract (RD) with responsiveness δ does exist by Lemma 6.

By construction, the allocation rule is feasible. The final step is to show that menu (RD) satisfies (RO): First, note that obviously the mechanism satisfies the full market coverage property

$$\int_0^1 X(1|\tau,\theta)dG(\theta)=1 \quad \forall\tau. \tag{1.21}$$

Second, plugging the full market coverage property into $K(\tau,\delta)$ yields

$$K(\tau,\delta)=R(\tau)-\delta. \tag{1.22}$$

When plugging menu (RD) into (1.22), I obtain $K(\tau,\delta)=0$ if $\tau\leq b$, and $K(\tau,\delta)=1-\delta$ if $\tau>b$.

The lemma follows from claims 1 to 3. ■

Proof of Lemma 8:

It suffices to show that menu (RD) is implementable in the original problem with

private ex post types. Lemma 8 then follows from Lemma 5.
In line with Section 1.2 I show that menu (RD) satisfies the following strengthened version of IC_2:

$$\int_0^1 v_{\tau,\theta}(s)dX(s|\hat{\tau},\theta) - p(\hat{\tau},\theta) \geq \int_0^1 v_{\tau,\theta}(s)dX(s|\hat{\tau},\hat{\theta}) - p(\hat{\tau},\hat{\theta}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}. \qquad (IC_2^s)$$

(IC_2^s) states that the consumer has an incentive to truthfully report his ex post type independent of his ex ante report. As argued in the proof of Proposition 1, this is sufficient to show incentive compatibility.

As menu (RD) satisfies $p(\hat{\tau},\hat{\theta}) = p(\hat{\tau},\hat{\theta}') \ \forall \tau \leq b \ \forall \hat{\theta},\hat{\theta}' \in \Theta$, menu (RD) satisfies ($IC_2^s$) if and only if:

$$\int_0^1 v_{\tau,\theta}(s)dX(s|\hat{\tau},\theta) \geq \int_0^1 v_{\tau,\theta}(s)dX(s|\hat{\tau},\hat{\theta}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}.$$

As menu (RD) is deterministic, denote the allocation by $x^{RD}_{\hat{\tau},\hat{\theta}} \in S$. Then menu (RD) satisfies (IC_2^s) if and only if:

$$\begin{aligned} v_{\tau,\theta}(x^{RD}_{\hat{\tau},\theta}) &\geq v_{\tau,\theta}(x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta} \\ \Leftrightarrow v + (1-\delta)\tau - \tau c(\theta, x^{RD}_{\hat{\tau},\theta}) &\geq v + (1-\delta)\tau - \tau c(\theta, x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}. \end{aligned} \qquad (1.23)$$

A sufficient condition for (1.23) is:

$$\Leftrightarrow c(\theta, x^{RD}_{\hat{\tau},\theta}) \leq c(\theta, x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}. \qquad (1.24)$$

<u>Case 1</u>: $\hat{\tau} \leq b$

Case 1.1: $\theta \in [I; 1-I]$

By definition, $x^{RD}_{\hat{\tau},\hat{\theta}} = \hat{\theta} \quad \forall \hat{\theta} \in [I; 1-I]$.
(1.24) is satisfied, as $c(\theta; x^{RD}_{\hat{\tau},\theta}) = 0 \leq c(\theta, x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau,\theta,\hat{\theta}$.

Case 1.2: $\theta < I$

By definition, $x^{RD}_{\hat{\tau},\hat{\theta}} \geq I \quad \forall \hat{\theta}$.
(1.24) is satisfied, as $c(\theta, x^{RD}_{\hat{\tau},\theta}) = c(\theta, I) \leq c(\theta, x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau,\theta,\hat{\theta}$.

Case 1.3: $\theta > 1-I$

By definition, $x^{RD}_{\hat{\tau},\hat{\theta}} \leq I \quad \forall \hat{\theta}$.
(1.24) is satisfied, as $c(\theta, x^{RD}_{\hat{\tau},\theta}) = c(\theta, 1-I) \leq c(\theta, x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau, \theta, \hat{\theta}$.

Case 2: $\hat{\tau} > b$

By definition, $x^{RD}_{\hat{\tau},\hat{\theta}} = \hat{\theta} \quad \forall \hat{\theta}$.
(1.24) is satisfied, as $c(\theta, x^{RD}_{\hat{\tau},\theta}) = 0 \leq c(\theta, x^{RD}_{\hat{\tau},\hat{\theta}}) \quad \forall \tau, \theta, \hat{\theta}$. ■

Proof of Lemma 9:

Assume there is a solution such that $\exists \tau' < b$ and $\Theta', \Theta'' \subseteq \Theta$ with positive probability measure such that $p(\tau', \theta') > p(\tau', \theta'') \quad \forall \theta' \in \Theta', \theta'' \in \Theta''$.

Claim 1: *For all* $(\theta', \theta'') \in \Theta' \times \Theta'' \quad \exists \tau_{\theta',\theta''} > 0$ *such that* $\forall \tau \leq \tau_{\theta',\theta''}$ *truthtelling about the ex post type is not optimal for at least one* $\theta \in \{\theta', \theta''\}$.

Take any pair θ', θ'' with $\theta' \in \Theta'$ and $\theta'' \in \Theta''$. An arbitrary ex ante type τ that has reported τ' will report honestly about his ex post type only if the following inequalities hold:

$$\int_0^1 v_{\tau,\theta'}(s) dX(s|\tau', \theta') - p(\tau', \theta') \geq \int_0^1 v_{\tau,\theta'}(s) dX(s|\tau', \theta'') - p(\tau', \theta''), \tag{1.25}$$

$$\int_0^1 v_{\tau,\theta''}(s) dX(s|\tau', \theta'') - p(\tau', \theta'') \geq \int_0^1 v_{\tau,\theta''}(s) dX(s|\tau', \theta') - p(\tau', \theta'). \tag{1.26}$$

Using the full market coverage property, (1.25) and (1.26) are equivalent to

$$\begin{aligned} &\left(\int_0^1 c(\theta', s) dX(s|\tau', \theta') - \int_0^1 c(\theta', s) dX(s|\tau', \theta'') \right) * \tau \\ &\leq p(\tau', \theta'') - p(\tau', \theta') \\ &\leq \left(\int_0^1 c(\theta'', s) dX(s|\tau', \theta') - \int_0^1 c(\theta'', s) dX(s|\tau', \theta'') \right) * \tau. \end{aligned} \tag{1.27}$$

Since $p(\tau', \theta'') - p(\tau', \theta') \neq 0$, $\exists \tau_{\theta',\theta''} > 0$ such that $\forall \tau \leq \tau_{\theta',\theta''}$ (1.27) does not hold. Hence, by (1.25) and (1.26) any type $\tau \leq \tau_{\theta',\theta''}$ that has claimed to be of type τ' has a strict incentive to lie about his ex post type when being either θ' or θ''.

Claim 2: *Define* $\tau_{\Theta',\Theta''} =$ *inf* $\{\tau_{\theta',\theta''} | \theta' \in \Theta', \theta'' \in \Theta''\}$. *Any type* $\tau \leq \tau_{\Theta',\Theta''}$ *that has* *claimed to be of type τ' has a strict incentive to lie on a set of ex post types that has positive probability measure.*

By construction, any type $\tau \leq \tau_{\Theta',\Theta''}$ has a strict incentive to lie when being either θ' or θ'' for any pair $(\theta', \theta'') \in (\Theta', \Theta'')$. Assume first $\exists \theta' \in \Theta'$ and $\Theta''_s \subseteq \Theta''$ with positive probability measure such that types (τ, θ'), $\tau \leq \tau_{\Theta',\Theta''}$ have *no* strict incentive to deviate to any $\theta \in \Theta''_s$. But then by Claim 1 the types $\tau \leq \tau_{\Theta',\Theta''}$ have an incentive to deviate on Θ''_s, which has positive measure. Second, assume that for any θ' there is no such subset Θ''_s. But then by Claim 1 the types $\tau \leq \tau_{\Theta',\Theta''}$ have a strict incentive to deviate on the entire set Θ', which has positive measure.

Claim 3: *If an ex ante type τ reports $\tau'' \leq b$ and then truthfully reveals his ex post type θ, his first period expected utility is zero ($U(\tau'', \tau, id_\theta) = 0$).*

By assumption, the allocation rule is optimal and therefore by Lemma 8 satisfies the properties of Lemma 7. Inserting the optimality properties from Lemma 7 into utility (1.8) reveals that

$$U(\tau'', \tau, id_\theta) = U(\tau'''', \tau''', id_\theta) \quad \forall \tau, \tau''' \in T, \ \forall \tau'', \tau'''' < b.$$

It follows that

$$U(\tau'', \tau, id_\theta) = U(\tau'') = U(z) = 0 \quad \forall \tau \in T, \ \forall \tau'' < b.$$

Final Step: By Claim 3 $U(\tau', \tau, id_\theta) = 0$. If an ex ante type τ with $\tau \leq \min\{\tau_{\Theta',\Theta''}, b\}$ reports τ', by Claim 2 he has a strict incentive to deviate from truthfully revealing his ex post type θ on a set of ex post types with positive probability measure. From this follows $U(\tau', \tau, \sigma^*) > 0 \quad \forall r \leq \tau_{\Theta',\Theta''}$, where $\sigma^*(\theta, \tau, r')$ is the consumer's optimal strategy about reporting ex post types as a function of his true θ, when being of type τ and having reported τ'. From IC_1 follows then $U(\tau) \geq U(\tau', \tau, \sigma^*) > 0$. This contradicts optimality condition $U(\tau) = v - v = 0$ which follows from Lemma 7. ■

Proof of Proposition 3:

Lemmas 7 to 9 show that the properties given in the four bullet points are necessary for a solution. The conditions of Proposition 3 imply all conditions of Lemma 7. Hence, if the properties and ex post type independent prices are sufficient for incentive compatibility, the proof is completed.

Again, it is sufficient to show that (IC_2^s) is satisfied. Using ex post type independence of prices $p(\tau,\theta) = p(\tau,\theta')\ \forall \tau \in R\ \forall \theta,\theta' \in \Theta$, (IC_2^s) can be reformulated:

$$\int_0^1 v_{\tau,\theta}(s)dX(s|\hat{\tau},\theta) - p(\hat{\tau},\theta) \geq \int_0^1 v_{\tau,\theta}(s)dX(s|\hat{\tau},\hat{\theta}) - p(\hat{\tau},\hat{\theta}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}$$

$$\Leftrightarrow \int_0^1 v + (1-\delta)\tau - \tau c(\theta,s)dX(s|\hat{\tau},\theta) \geq \int_0^1 v + (1-\delta)\tau - \tau c(\theta,s)dX(s|\hat{\tau},\hat{\theta}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}$$

$$\Leftrightarrow \int_0^1 c(\theta,s)dX(s|\hat{\tau},\theta) \leq \int_0^1 c(\theta,s)dX(s|\hat{\tau},\hat{\theta}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}$$

Case 1: $\hat{\tau} \leq b$

(IC_2^s) is independent of the true ex ante type τ and relabeling $\hat{\tau}$ as τ gives (1.10). By the second property of Proposition 3 (1.10) is satisfied.

Case 2: $\hat{\tau} > b$

By the first property,

$$\int_0^1 c(\theta,s)dX(s|\hat{\tau},\theta) = 0 \leq \int_0^1 c(\theta,s)dX(s|\hat{\tau},\hat{\theta}) \quad \forall \tau,\hat{\tau},\theta,\hat{\theta}.$$

■

Proof of Proposition 4:

The proof follows from the main text.

■

Proof of Proposition 5:

The proof follows from the main text.

■

Chapter 2

Exchange Fees as a Price Discrimination Device

We consider a monopolist who sells horizontally differentiated goods to a buyer who learns his valuations for the goods gradually over time. We analyze when a menu of contracts with exchange fees maximizes the seller's profits. The use of exchange fees is optimal if two key properties are satisfied: When contracting takes place, the buyer is uncertain about his difference in valuations and the seller does not know which magnitude of differences the buyer roughly expects. We provide several foundations when this is the case. The contracts in the optimal menu consist of a price payed upfront for an initially obtained good and an exchange fee. Contracts with higher upfront prices have lower exchange fees. Exchange fees are not beneficial when buyers initially only differ in their belief which good they prefer but expect similar magnitudes of valuation differences.

2.1 Introduction

In a substantial part of seller-buyer relationships, the buyer is granted the possibility to exchange his good or service after contracting has taken place. Often the buyer's choice to exchange his good is connected to some extra cost. This cost may explicitly take the form of exchange fees, it may consist of a partial refund for the returned good combined with the necessity to buy the new good, or it takes the form of service charges and other kinds of fees. Examples are the purchase of tickets for public transport, the hotel industry, car rental services, buying experience goods, but it also fits the pricing policy of online shopping platforms and mail order companies. More or less explicitly, firms often offer a menu of contracts which differ in the height of the exchange fee and the base price. An economically important example is the airline industry. Typically, the consumer can buy a ticket for the same seating class in the same flight for different prices which depend on how costly exchange options are.

The underlying prerequisite for these exchange options to create value to the buyer is that the buyer's preferences or his information is changing over time. Therefore, this paper considers a monopolist who sells horizontally differentiated goods to a buyer who learns his valuations for the goods gradually over time. We identify conditions under which it is optimal for the firm to offer a menu of contracts that leave the buyer different degrees of flexibility in product choice in order to price-discriminate. In particular, we study when the seller optimally uses exchange fees to govern exchanges. We identify two key properties under which the use of exchange fees is optimal. First, the buyer is uncertain about his difference in valuations for the goods when contracting takes place. Second, the seller does not know which magnitude of differences the buyer roughly expects. In particular, the second property is not satisfied when buyers initially only differ in their belief which good they prefer but expect similar magnitudes of valuation differences. We show that in this case the use of exchange fees is not beneficial. While we show our results by means of a reduced-form model, we provide three foundations which help to identify which kind of situations the reduced-form model represents.

The model consists of a seller who can sell two horizontally differentiated goods to a buyer with unit demand. The variable along which the goods are differentiated might be the departure time of a transportation ticket or any other product feature. At the point in time when contracting takes place, the buyer is, however, uncertain about his own valuations for the two goods. Reasons for this uncertainty are that his preferences might change when there is time between contracting and consumption, he cannot perfectly assess certain product features because he buys the good online, the good is an experience good, or he buys the good for another person. Still, the buyer does already have some private information about whether he tends to have a large or small difference in valuations. This information about the difference is not perfect in that any difference may realize.[1] Before consumption takes place, the buyer learns his actual valuations.

We show that in our reduced-form model the seller sequentially screens the buyer. At the time of contracting, the seller optimally price-discriminates according to how large the buyer's valuation differences tend to be by offering contracts with different degrees of flexibility to change between the goods. At a later point in time, the seller screens according to the realized difference in valuations. The revenue-maximizing menu of contracts can be implemented by a menu of contracts that allow for exchanges subject to an exchange fee: A contract specifies a good, a price, and an exchange fee. At the contracting stage, the buyer obtains the goods against a price. Later, the buyer has the option to exchange goods when additionally paying the exchange fee. The menu

[1] The three foundations for the model are foundations for when this information structure applies.

satisfies the classical no-distortion-at-the-top condition: The buyer type that expects the largest valuation differences obtains the first best contract, which is a contract with a costless exchange option. The menu contains a continuum of contracts with positive exchange fees. The menu is designed such that there is a negative relation between the height of the exchange fee and the initial price for the good.

In order to better assess when our results apply, we provide several foundations for the model. First, we can transform our model to one in which the seller can sell the goods at both ends of a Hotelling line. The buyer's transportation cost is known, but when contracting takes place, his position on the line is uncertain. The buyer's initial information is about the spread of the distribution from which his position on the line is drawn. The interpretation enabled by this foundation is that the consumers simply differ in their uncertainty about which good they favor. This uncertainty might for example depend on how far in advance contracting takes place or whether the consumer buys the good for himself or as a gift. Very uncertain consumers then choose expensive contracts which give them flexibility through low exchange fees.

Another interpretation of this model as a Hotelling model is that all buyer types share the same belief about the distribution their position is drawn from, but they differ in their transportation cost functions. The interpretation of this foundation is that the buyer knows how choosy he is in general, which is his private information. More choosy consumers know their desire to obtain a good which perfectly matches their taste and are hence willing to pay higher prices initially in order to avoid high charges when exchanging the good.

An important further foundation is not based on preferences but purely on the information structure: Let all buyer types share the same belief about their final distribution of valuations. Instead, the buyer types differ in how much they expect to know about their valuations when they have to make the final decision which good to consume. When the buyer is not perfectly sure about his final valuations, he decides based on the expected valuations. Note that expectations are less dispersed than the actual valuations. If the buyer expects to obtain less information until the last point in time when he must take his decision, his distribution of expected valuations is less dispersed. The buyer typically has less information when he has to fix the terms of trade early. We will argue that a business trip is often subject to less rigid circumstances than a holiday trip, meaning that leisure travelers have to fix the terms of trade earlier and with less information than business travelers. Interpreting the optimal menu that way means that business travelers choose contracts with low exchange fees because they simply know more about their preferences and hence have a stronger opinion which good is best at the latest point when there is the possibility to exchange goods.

Next, we give here a short intuition why price discrimination is profitable in our setup. We refer to buyer types which expect larger valuation differences as choosy and to the other buyer types as unconcerned. The first prerequisite for price discrimination to be profitable is that choosy buyer types have a higher expected valuation for the first best contract. This follows as the valuation for the favorite good, which is the valuation for the first best contract, is on average higher when the buyer expects large differences in the valuations. When only offering the first best contract, the seller would have to trade-off excluding unconcerned buyer types and leaving rents to choosy buyer types. The seller can, however, profitably price-discriminate by offering an additional contract that is cheaper but has a positive exchange fee. The exchange fee is chosen such that price and exchange fee together exceed the price for the first best contract. As choosy buyer types expect to pay the exchange fee often, this alternative contract is unattractive for them compared to unconcerned buyer types. When offering the alternative contract to the unconcerned buyer types, the seller can extract more rent from the choosy buyer types.

The paper furthermore shows that price discrimination is only profitable when the buyer initially has private information about the size of valuation differences he expects. The seller does not screen the buyer's information about which good he tends to prefer. To make that point, we consider the extreme case in which the buyer initially has only information about which good he is likely to prefer but the expected valuation difference is kept constant across buyer types. We show that a sequential screening structure is not profitable in this environment. Furthermore, we argue intuitively why dynamic screening is also not possible in this setup even if it would be desirable.

Our paper adds to the literature on sequential screening, which studies the optimal design of contracts when buyers privately learn their final valuations only after contracting but before consumption takes place. With a few exceptions, the literature has focused on the sale of homogeneous goods to buyers who learn their valuation for this good over time.[2] Although we study a model with heterogeneous goods, our model shares many features with this literature. In particular, the buyer learns the payoff relevant variable only after contracting. We follow most of the literature on sequential screening by solving our problem via a first order approach. However, because we are analyzing a model with horizontally differentiated goods, several steps require careful consideration. The technical difficulty is that at both the contracting stage and the consumption stage it is a priori unclear which buyer type has the lowest utility.[3] While

[2] Important contributions to the literature on sequential screening are Baron and Besanko (1984), Courty and Li (2000), Battaglini (2005), Eső and Szentes (2007a,b, 2015), Inderst and Peitz (2012), Boleslavsky and Said (2013), Pavan et al. (2014), Deb and Said (2015), Li and Shi (2015), Battaglini and Lamba (2015), and Krähmer and Strausz (2015a,b).

[3] This kind of problems is studied in the literature on mechanism design with type-dependent outside options. For an exposition see Lewis and Sappington (1989), Maggi and Rodriguez-Clare (1995), Jullien (2000), and Nöldeke and Samuelson (2007).

it turns out that this peculiarity does not have any impact in the consumption-stage, we identify conditions under which the expected utility is monotone in the buyer's "choosiness" in the contracting stage.

There is only a small number of papers which explicitly treat a dynamic screening model in which the buyer learns about his preferences over heterogeneous goods. The most related paper is Herbst (2016). In both papers, exchange policies with varying flexibility are employed as a price discrimination device. However, optimal contracts specify a price, an initial product choice, and a limited range of products within which exchange is costless. These optimal contracts contrast with the results presented in this paper: While in our paper the seller uses transfers to sometimes prevent the buyer from exchanging his good, in the other paper flexibility is restricted by an appropriately designed interdiction. The key difference in the setup in Herbst (2016) is that consumers know already at the point in time of contracting how large valuation differences are for them but only later learn which good they favor. As one main objective of our paper is to advance the understanding of which model features drive the optimal design of exchange policies, the comparison of the two papers is discussed in Section 2.6.1.

Furthermore, Gale (1993) studies intertemporal pricing policies in a setting which is similar to the two-goods version of Herbst (2016). In order to obtain a fruitful comparison between monopolistic and oligopolistic pricing behavior, the paper, however, restricts to setting a uniform price in each period. Instead, the major contribution of our paper consists of the characterization of optimal contracts. Furthermore, we generalize the model in Gale (1993) in that the buyer initially does not perfectly know his difference in valuations, but still every difference can realize. The two papers Gale and Holmes (1992, 1993) start with the same basic framework as Gale (1993) but depart from price discrimination and focus on how intertemporal pricing rules can optimally resolve capacity problems. In contrast, we focus on pure price discrimination motives without any capacity constraints. Recently, Möller and Watanabe (2016) rediscovered the early model on advance-purchase discounts with differentiated goods as a way to introduce oligopolistic competition. In order to obtain a tractable analysis of strategic interaction, they as well restrict strategies to uniform prices per period.

The paper is organized as follows. In Section 2.2, we introduce the model in reduced form. In Section 2.3, we discuss three foundations for the reduced-form model, which help to identify situations in which the use of our model is appropriate. In Section 2.4 the model is solved. The indirect implementation through contracts that govern exchanges via exchange fees is presented in Section 2.5. Section 2.6 relates our model to Herbst (2016) and discusses the case of first order stochastic dominance. Finally, Section 2.7 concludes.

2.2 Model

Consider a seller (she) that sells two differentiated goods to a buyer (he) with unit demand. The buyer's valuations v_1 for good 1 and v_2 for good 2 depend on his *ex post type* θ in the following way:

$$v_1(\theta) = v + \theta$$
$$\text{and} \qquad v_2(\theta) = v - \theta.$$

The ex post type $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$ with $0 \in (\underline{\theta}, \overline{\theta})$ determines both the valuation premium $|v_1(\theta) - v_2(\theta)|$ and which good is preferred. Let the basic valuation v be large enough such that valuations are always positive, $v > \max\{\underline{\theta}, \overline{\theta}\}$.

The seller's constant marginal cost of production is c with $c \in (0, v)$. The terms of trade are described by an allocation $X = (x_1, x_2)$, where x_i is the probability to trade good i such that $x_1 + x_2 \leq 1$, and a payment t from the buyer to the seller.[4] Given the terms of trade (x_1, x_2, t), the buyer of ex post type θ has *ex post utility*

$$\tilde{u}(\theta, X, t) = v_1(\theta) \cdot x_1 + v_2(\theta) \cdot x_2 - t. \tag{2.1}$$

When the seller and buyer contract, the buyer's true ex post type θ is unknown to both parties. The buyer is, however, better informed about θ than the seller. The buyer's informational advantage at the contracting stage is expressed through a privately known *ex ante type* $\tau \in T = [\underline{\tau}, \overline{\tau}]$. The ex ante type is drawn from the distribution $F_\tau(\tau)$ with density $f_\tau(\tau)$ that has an increasing hazard rate $f_\tau(\tau)/(1 - F_\tau(\tau))$. Each ex ante type τ specifies a distribution $F(\cdot|\tau)$ over Θ from which his ex post type is privately drawn at a later point in time. For any τ, $F(\cdot|\tau)$ permits a density function $f(\cdot|\tau)$, which is bounded from above and bounded away from 0 on Θ. The derivatives $\partial F(\theta|\tau)/\partial\tau$ and $\partial F_\tau(\tau)/\partial\tau$ also exist and are bounded.

The family of distributions $\{F(\cdot|\tau)|\tau \in T\}$ is ordered by the rotation order with constant means, which is a form of mean preserving spread.[5]

Definition 1. *Rotation Order. The family of distributions $\{F(\cdot|\tau)|\tau \in T\}$ is rotation-ordered if there exists a $\theta^\dagger \in \Theta$ such that for all $\tau, \tau' \in T$ with $\tau > \tau'$ holds $F(\theta|\tau) \geq F(\theta|\tau')$ if $\theta < \theta^\dagger$ and $F(\theta|\tau) \leq F(\theta|\tau')$ if $\theta > \theta^\dagger$.*

The rotation order has been introduced by Johnson and Myatt (2006). The definition implies that higher ex ante types have more spread distributions of θ implying

[4] Excluding the possibility to trade both goods at the same time is without loss of generality: As the buyer has unit demand, it is never optimal for the seller to assign both goods to the buyer at the same time. For a formal argument see Herbst (2016).

[5] Theorem 3.A.44 in Shaked and Shanthikumar (2007) shows that joint with a constant mean the rotation order implies second order stochastic dominance and hence a mean preserving spread.

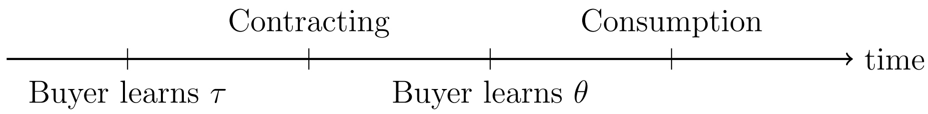

Figure 2.1: timeline

larger differences in valuations on average. Section 2.3 provides several formal foundations for this assumption.

The seller has full commitment and can contract with the buyer after he has privately learned his ex ante type τ but before he learns his ex post type θ. At the contracting stage, the buyer has an outside option of zero. After contracting, the consumer privately learns his ex post type and then consumption takes place. The timing is illustrated in Figure 2.1.

2.3 Foundations

2.3.1 Differing Degrees of Uncertainty Resolution

The model presented can be interpreted in several ways other than the buyer having some prior information about his choosiness. Instead of being a statement about preferences, the model also results from the buyer expecting differing degrees of uncertainty resolution.

Assume there is just one distribution F from which the buyer draws his ex post type θ, independent of his ex ante type. In this case the buyer does not posses private information about his preferences in the first period. However, it might happen that the buyer is forced to fix the terms of trade before his actual ex post type is fully revealed. There are many reasons why this may occur in practice, one being that the buyer must make other investments which are connected to the purchase but which cannot be adapted ex post. Think for example of a leisure traveler planning holiday: He must request holiday, he has to coordinate the timing with his companions or the persons he visits, he has to book a hotel, etc. In that case the buyer has to choose an option within his contract based on his latest *expectation* about his ex post type.

The buyer types differ ex ante in how precise information they expect to obtain until they have to make a final decision on the good they finally consume. Typically, the reason for this difference is that buyer types differ in how far in advance of consumption they have to fix the terms of trade. For example, a business trip is often subject to less rigid circumstances than a holiday trip. Compared to a business traveler, a leisure traveler hence has to decide which option to take far in advance of the flight, when he has obtained less information.

If the buyer chooses the final terms of trade when he is still uncertain about his type, he acts as if he was of the ex post type that equals the expected ex post type given all the information he has obtained up to that point.[6] The important point to note is that when the buyer has less information, his expected values are more concentrated around the mean. Since for each ex post distribution over θ the buyer evaluates the terms of trade as if he was of the expected ex post type, we can equivalently replace ex post distributions by the corresponding expected ex post types.[7] Those buyer types which expect to obtain more information are, hence, the more dispersed ones. This might for example be business travelers.

We illustrate the verbal execution by means of a simple formal demonstration.[8] Assume that the buyer's ex post type θ is drawn from the uniform distribution H on $[-1, 1]$. The buyer's ex ante type τ learned in the first period does not provide information about the true θ such that $H(\theta|\tau) = H(\theta)$ for all τ. Instead, the ex ante type provides information about the informativeness of the signal s drawn in the second period. Signal s can take values in $[-1, 1]$. Its distribution is informative about θ and has a truth-or-noise structure: With probability r signal s equals θ, with probability $1 - r$ signal s is drawn from the uniform distribution H. We refer to $r \in (0, 1]$ as the informativeness of the signal.[9] The signal does always contain some information and can be perfectly informative. In the second period, the buyer learns the signal s and its informativeness r. This means that he updates and has a posterior distribution about θ. The ex ante type learned in the first period specifies a distribution $G(\cdot|\tau)$ from which r is drawn. For any τ the distribution $G(\cdot|\tau)$ has full support $(0, 1]$ and the family of distributions $\{G(\cdot|\tau)|\tau \in T\}$ is ordered by first order stochastic dominance such that $G(\cdot|\overline{\tau})$ first order stochastically dominates all other distributions.[10] This means that higher ex ante types expect more informative signals which results in less dispersed posterior distributions.

Given the buyer observes signal s with informativeness r, his posterior expected ex

[6] To state this point formally, denote the information by a signal s the buyer obtains ex post, which induces ex post distribution $F_s(\theta)$. Due to the linear utility

$$\mathbb{E}(\tilde{u}(\theta, X, t)|s) = \mathbb{E}(v(x_1 + x_2) + \theta(x_1 - x_2) - t|s) = v(x_1 + x_2) + \mathbb{E}(\theta|s)(x_1 - x_2) - t = \tilde{u}(\mathbb{E}(\theta|s), X, t).$$

[7] We are not the first to do this transformation, see e.g. Eső and Szentes (2007b).

[8] Our foundation further develops an idea from Johnson and Myatt (2006). Dai et al. (2006) also captures the idea that screening can be based on the buyer's information structure.

[9] The assumption $r \neq 0$ is for notational convenience only.

[10] For convenience we assume that G is continuously differentiable with respect to θ and τ.

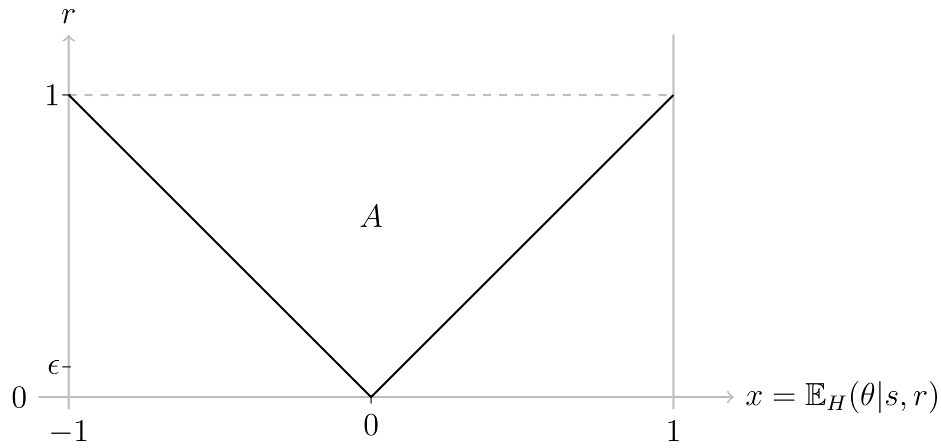

Figure 2.2: Support of the joint distribution of r and $\mathbb{E}_H(\theta|s,r)$.

post type $\mathbb{E}_H(\theta|s,r)$ is

$$\mathbb{E}_H(\theta|s,r) := r \cdot s + (1-r) \cdot \mathbb{E}_H(\theta) = r \cdot s + (1-r) \cdot 0. \tag{2.2}$$

For a fixed r, $\mathbb{E}_H(\theta|s,r)$ is a random variable which is distributed uniformly on $[-r,r]$, because it is a function of s which is drawn from $H[-1,1]$.[11] Denote the corresponding cumulative distribution function by F_r with density $f_r(x) = 1/2r$.

Each ex ante type τ induces a distribution over r. Joint with the posterior expected ex post type we can say each ex ante type τ induces a two-dimensional distribution over $(r, \mathbb{E}_H(\theta|s,r))$ with density $g(r|\tau) f_r(\mathbb{E}_H(\theta|s,r)))$.

The marginal distribution over $\mathbb{E}_H(\theta|s,r)$ indicates the probability with which ex ante type τ ends up with a particular expected ex post type. We denote this distribution $F(\cdot|\tau)$ with density $f(\cdot|\tau)$, which is[12]

$$f(x|\tau) = \int_0^1 f_r(x) g(r|\tau) dr = \int_{|x|}^1 \frac{g(r|\tau)}{2r} dr. \tag{2.3}$$

Finally, we find that the family of functions $\{F(\cdot|\tau)|\tau \in T\}$ satisfies the conditions specified for our model:

Proposition 1. *The family of functions $\{F(\cdot|\tau)|\tau \in T\}$ is rotation-ordered.*

To prove the proposition, we calculate $F(\cdot|\tau)$ explicitly to show that $\partial F(x|\tau)/\partial\tau > 0$ for $x < 0$ and $\partial F(x|\tau)/\partial\tau < 0$ for $x > 0$.

The intuition for Proposition 1 can be conveyed by means of Figure 2.2. Figure 2.2 displays the support A of the two-dimensional distribution induced by ex ante

[11] We ease notation by not distinguishing formally between random variables and realizations.

[12] At $x = 0$ we have $f(0|\tau) = \lim_{\epsilon \to 0} \int_\epsilon^1 f_r(x) g(r|\tau) dr$.

type τ over $(r, \mathbb{E}_H(\theta|s, r))$. The support A is independent of τ. Each ex ante type τ induces a two-dimensional distribution with density $g(r|\tau)f_r(x)$ on support A. The second multiplier $f_r(x)$ is decreasing in r, independent of x, and independent of τ. The density $f(\cdot|\tau)$ is the marginal distribution of the two-dimensional distribution over x as stated in (2.3).

The crucial step is to consider $f(x|\tau)$ as a convex combination of the densities $f_r(x) = 1/(2r)$ with weights $g(r|\tau)$ for $r \geq |x|$, and 0 with weight $G(|x||\tau)$. Note that the weights depend on the ex ante type. The first order stochastic dominance relation on $G(\cdot|\tau)$ implies that for a higher ex ante type $\tau' > \tau$ these weights are more concentrated on lower values of r. The effect of this shift on $f(x|\tau)$ can be decomposed into two components. The first component can be seen best when considering $x = 0$ such that $G(0|\tau) = 0$. As the density $1/(2r)$ decreases in r, we conclude $f(0|\tau') < f(0|\tau)$. However, for values $x \neq 0$ there is an opposing second effect on $f(x|\tau)$: Weights of mass $G(|x||\tau) - G(|x||\tau')$ are shifted from 0 to positive densities. This second aspect is effective in favor of $f(x|\tau)$ to increase in τ. When $|x|$ is large, the second effect dominates, and for small $|x|$ the first effect dominates.

2.3.2 Differing Uncertainty about the Position on the Hotelling Line

This section interprets the model as a Hotelling model where consumers have uncertainty about their position on the Hotelling line.

More specifically, the seller can sell the two goods at both ends of the Hotelling line. The buyer's transportation cost is linear and common knowledge. In the first period, the buyer is, however, uncertain about his position on the Hotelling line. The ex ante types determines the buyer's distribution over positions on the Hotelling line. Larger ex ante types indicate more uncertainty about the position, leading to extremer positions on average and hence larger valuation differences. The situation is illustrated in Figure 2.3.

Formally, we can reformulate the preferences given in the model section as typical Hotelling preferences: First, define $x' = -\theta$. We obtain $v_1 = v - x'$ and $v_2 = v + x'$. Second, define $x = x' + 1$ such that $v_1 = v + 1 - x$ and $v_2 = v - 1 + x$. Third, define $v' = v + 1$ and obtain the valuations for a Hotelling model with linear transportation cost and length 2:

$$v_1 = v' - x \tag{2.4}$$

$$\text{and} \quad v_2 = v' - (2 - x). \tag{2.5}$$

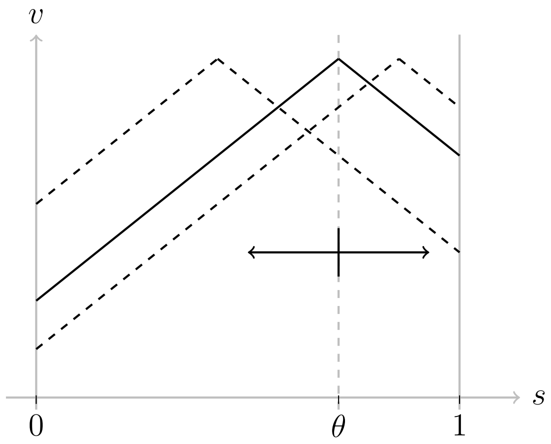

Figure 2.3: Denote by s the Hotelling line. An "extreme" position on the line means that the peak of the valuation function is close to 0 or 1. In that case the difference between the valuations of good 0 and good 1 is large.

This interpretation nicely contrasts the setup in Herbst (2016) in which the ex ante type defines the transportation cost, but the distributions over the postition on the line is the same for all ex ante types. Instead, in our paper the ex ante type defines the distribution from which the position on the Hotelling line, but the transportation cost is the same for any ex ante type.

2.3.3 Differing Transportation Cost

Finally, we can interpret the model as a Hotelling model in which the buyer is initially uncertain about his position, but always faces the same distribution over positions on the Hotelling line. However, ex ante types specify differing transportation cost functions. The transportation cost functions at one side of the favorite good are rotation ordered as illustrated in Figure 2.4. The solid line represents a low ex ante type, the dotted line a high ex ante type and the dashed line an intermediate type.

This interpretation of the model also relates to the setup in Herbst (2016). The key difference is that the seller can only sell the goods at the ends of the Hotelling line, whereas she can sell any good in the other paper.

2.4 Analysis

As the seller has full commitment power, the revelation principle applies (see Myerson (1986)), which allows us to restrict to direct and incentive compatible mechanisms. A direct mechanism specifies for any reported pair of types $(\hat{\tau}, \hat{\theta})$ an allocation X and a price t. Hence, a direct mechanism is the combination of an allocation rule $\{X(\hat{\tau}, \hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$ and a payment rule $\{t(\hat{\tau}, \hat{\theta}) : \hat{\tau} \in T, \hat{\theta} \in \Theta\}$. For a given

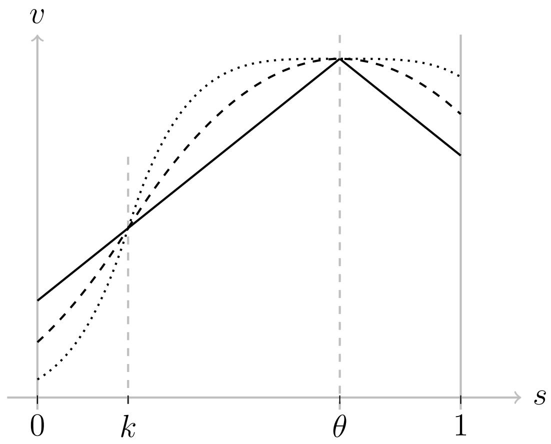

Figure 2.4: Denote by s the Hotelling line. In this example, the transportation cost functions are rotation ordered around point k on the Hotelling line.

report about the ex ante type, $\hat{\tau}$, we call $\{X(\hat{\tau},\hat{\theta}), p(\hat{\tau},\hat{\theta}) : \hat{\theta} \in \Theta\}$ a contract. A contract is defined as a mapping from ex post type reports into allocations and prices. The choice of the ex ante report then corresponds to the choice of a contract and the choice of an ex post report determines an option within that contract.

Given a pair of types (τ, θ) and an allocation determined by a pair of reports $(\hat{\tau}, \hat{\theta})$ we write $u(\hat{\tau}, \hat{\theta}, \theta) := \tilde{u}(\theta, X(\hat{\tau}, \hat{\theta}), t(\hat{\tau}, \hat{\theta}))$ and define $u(\tau, \theta) := u(\tau, \theta, \theta)$. Given the agent always truthfully reports the ex post type, the *ex ante utility* is

$$U(\tau, \tau') = \int_{\Theta} u(\hat{\tau}, \theta) dF(\theta|\tau). \tag{2.6}$$

To simplify notation, we define $U(\tau) := U(\tau, \tau)$. A direct mechanism is incentive compatible, if the following two conditions hold:

$$U(\tau) \geq U(\tau, \tau') \qquad \forall \tau, \tau', \tag{IC_1}$$

$$u(\tau, \theta) \geq u(\tau, \theta, \hat{\theta}) \qquad \forall \theta, \hat{\theta}, \tau. \tag{IC_2}$$

Applying of the dynamic revelation principle, the first-period incentive constraints (IC_1) state that telling the truth in the first period is optimal given the second period type is always revealed truthfully. This requirement is less restrictive than the general requirement of incentive compatibility, which is that truthtelling in both periods must be better than any combination of lying about the ex ante type potentially followed by another lie about the ex post type. Conditions (IC_1) and (IC_2) are sufficient for incentive compatibility because the model satisfies two properties: First, the true ex ante type is not payoff relevant, meaning that the true ex ante type does not appear in (2.1). Second, the model satisfies a "non-shifting support property", which means that the support of ex post types does not depend on the ex ante type. These two assumptions and the resulting simplified characterization of incentive compatibility are standard in the literature on sequential screening since the seminal contribution Courty and Li (2000).[13]

At the point in time of contracting, the buyer has an outside option of 0. The seller hence faces the individual rationality constraints

$$U(\tau) \geq 0 \quad \forall \tau. \tag{IR}$$

Note that there is no outside option for the buyer after he learned his ex post type.[14]

Clearly, the seller is also restricted in that he may offer only feasible allocations:

$$x_1(\tau,\theta), x_2(\tau,\theta) \geq 0, \quad x_1(\tau,\theta) + x_2(\tau,\theta)) \leq 1 \quad \forall \tau, \theta. \tag{F}$$

Now, we are ready to write down the seller's maximization problem ($\mathcal{P}$):

$$\max_{X(\tau,\theta),t(\tau,\theta)} \int_{\underline{\tau}}^{\overline{\tau}} \int_{\underline{\theta}}^{\overline{\theta}} t(\tau,\theta) dF(\theta|\tau) dF_\tau(\tau)$$
$$\text{s.t.} \qquad (IC_1), (IC_2), (IR), \text{ and } (F).$$

In order to solve problem ($\mathcal{P}$), we first exploit the incentive compatibility conditions to rewrite expected transfers as a function of the allocation rule. Then we maximize a relaxed version of ($\mathcal{P}$) and obtain an upper bound on the profits which are achievable. Finally, we identify regularity conditions which ensure that the upper bound is achievable.

To begin with, we provide necessary and sufficient conditions for the second period incentive constraints IC_2 to be satisfied.

Lemma 1. *IC_2 is satisfied if and only if $u(\tau,\theta)$ is absolutely continuous in θ and the following conditions are satisfied:*

$$\partial u(\tau,\theta)/\partial\theta = x_1(\theta,\tau) - x_2(\theta,\tau) \tag{FOC_2}$$
$$\textit{and} \quad x_1(\theta,\tau) - x_2(\theta,\tau) \textit{ is non-decreasing in } \theta. \tag{MON_2}$$

To understand Lemma 1 note that the ex post utility can be rewritten as

$$u(\theta,\tau) = v \cdot [x_1(\theta,\tau) + x_2(\theta,\tau)] + \theta \cdot [x_1(\theta,\tau) - x_2(\theta,\tau)] - t(\theta,\tau). \tag{2.7}$$

[13] For an in-depth discussion of this aspect, see Krähmer and Strausz (2008). A textbook treatment is Borgers et al. (2015).

[14] Krähmer and Strausz (2015b) show that with an outside option after learning the ex post type dynamic screening is not profitable.

The true ex ante type enters (2.7) linearly in the second summand and the first summand is independent of the true type but only depends on reported types. The remainder of the proof is then standard in the literature on mechanism design.

An immediate consequence of Lemma 1 is that the ex post utility is convex as a function of the ex post type. However, the sign of $x_1(\theta,\tau)-x_2(\theta,\tau)$ is unclear a priori. This implies that the ex post utility might increase, decrease, or be U-shaped in the ex post type. Which ex post types have the lowest ex post utility thus depends on the allocation rule. As there is no ex post outside option, this will not complicate the analysis, while it would have strong effects otherwise.[15]

In contrast to the characterization of second period incentive compatibility, we cannot provide necessary and sufficient conditions for first-period incentive compatibility. This difficulty is standard in the literature on sequential screening.[16] The following lemma states necessary conditions for first-period incentive compatibility, which we take into account when maximizing the objective. The solution is then checked for incentive compatibility.

Lemma 2. *IC_1 is satisfied only if the following conditions are satisfied:*

$$\partial U(\tau)/\partial\tau = \int_{\underline{\theta}}^{\bar{\theta}} (x_1(\tau,\theta)-x_2(\tau,\theta)) \cdot -\frac{\partial F(\theta|\tau)}{\partial\tau} d\theta, \qquad (FOC_1)$$

$$\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial[x_1(\tau,\theta)-x_2(\tau,\theta)]}{\partial\tau} \cdot -\frac{\partial F(\theta|\tau)}{\partial\tau} d\theta \geq 0. \qquad (SOC_1)$$

The proof of Lemma 2 closely resembles the part of the proof in Lemma 1 which shows necessity. In particular, Lemma 2 shows that for any incentive compatible mechanism the ex ante utility is pinned down by the allocation rule up to a constant. Thus, also the expected payment is pinned down by the allocation rule implying a "revenue-equivalence" result in the sense of Myerson (1981).

In contrast to the standard framework as introduced in Courty and Li (2000), in our model it is not obvious that the ex ante utility is increasing in the buyer's ex ante type. As the following lemma shows, this is, however, still the case.

Lemma 3. *For any incentive compatible mechanism holds $\partial U(\tau)/\partial\tau \geq 0$.*

Lemma 3 is not trivial in our setup. Indeed it only follows when combining not only first-period and second-period incentive compatibility but also the rotation ordering and the constant mean.

[15] For static mechanism design problems this issue is addressed in the literature on "countervailing incentives". For an exposition see Jullien (2000) and Maggi and Rodriguez-Clare (1995).

[16] See for example Courty and Li (2000); Pavan et al. (2014) and Eső and Szentes (2007a).

An immediate consequence of the rotation order is $\partial F(\theta|\tau)/\partial\tau \geq 0$ for all $\theta \leq \theta^\dagger$ and $\partial F(\theta|\tau)/\partial\tau \leq 0$ for all $\theta \geq \theta^\dagger$. From Lemma 1 we furthermore know that $x_1(\tau,\theta) - x_2(\tau,\theta)$ increases in θ. Combining these two observations with (FOC_1) which follows from first-period incentive compatibility, we obtain

$$\begin{aligned} \partial U(\tau)/\partial\tau &= \int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta) - x_2(\tau,\theta)) \cdot -\frac{\partial F(\theta|\tau)}{\partial\tau} d\theta \\ &\geq \int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta^\dagger) - x_2(\tau,\theta^\dagger)) \cdot -\frac{\partial F(\theta|\tau)}{\partial\tau} d\theta. \end{aligned} \tag{2.8}$$

As all distributions $F(\theta|\tau)$ have the same mean, $\partial\mathbb{E}_\theta[\theta|\tau]/\partial\tau = 0$. Integration by parts reveals that this is equivalent to $\int_{\underline{\theta}}^{\overline{\theta}} \partial F(\theta|\tau)/\partial\tau d\theta = 0$, which in turn implies

$$\int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta^\dagger) - x_2(\tau,\theta^\dagger)) \cdot -\frac{\partial F(\theta|\tau)}{\partial\tau} d\theta = 0. \tag{2.9}$$

The proof follows from (2.8) and (2.9).

The lemma states that the buyer's ex ante utility is weakly higher if he expects to have larger valuation differences. As we now know that the lowest ex ante type $\underline{\tau}$ has the lowest ex ante utility, we can proceed as is standard in the literature on mechanism design. We rewrite the payments as the difference between surplus and ex ante utility, use (FOC_1) to express the ex ante utility as a function of the allocation and $U(\underline{\tau})$, and do integration by parts. We solve the resulting optimization problem $\mathcal{P}'$.

$$\begin{aligned} &\max_{X(\tau,\theta),U(\underline{\tau})} \int_{\underline{\tau}}^{\overline{\tau}} \int_{\underline{\theta}}^{\overline{\theta}} (v-c)\cdot(x_1+x_2) + (x_1-x_2)\psi(\theta,\tau)dF(\theta|\tau)dF_\tau(\tau) - U(\underline{\tau}) \\ &\text{s.t.} \qquad (MON_2), (IR), \text{ and } (F), \end{aligned} \tag{$\mathcal{P}'$}$$

where $\psi(\theta,\tau)$ is the virtual valuation

$$\psi(\theta,\tau) = \theta - \frac{1-F_\tau(\tau)}{f_\tau(\tau)} \frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)}. \tag{2.10}$$

Problem $\mathcal{P}'$ is not equivalent to problem $\mathcal{P}$ as the constraints in $\mathcal{P}'$ are not sufficient to guarantee incentive compatibility. In particular, $\mathcal{P}'$ does neither contain SOC_1 nor are FOC_1 and SOC_1 sufficient for first-period incentive compatibility. Mechanisms that solve problem $\mathcal{P}'$ are only solutions to the original problem $\mathcal{P}$ if they are incentive compatible.

The virtual value (2.10) is well known from the literature on sequential screening.[17] As in our setting the virtual value is associated with $x_1 - x_2$, it consists of the additional value for obtaining the preferred good, θ, and a virtual cost. The virtual cost consists of the inverse hazard rate $\frac{1-F_\tau(\tau)}{f_\tau(\tau)}$ reflecting the fact that when improving type (θ, τ)'s allocation all higher ex ante types have to be left additional rent, which is multiplied by an informativeness measure. The informativeness measure accounts for the fact that larger ex ante types only profit through having advantageous ex post types more often.

However, whether the advantageous ex post types are those which are larger than θ or those which are smaller than θ depends on which ex post type has the lowest utility in our setting. Notably, the functional form of ψ does not depend on whether the second period incentive constraints are binding upwards or downwards, which is is different from the standard static mechanism design framework. This fact leads to a significant facilitation when solving problem $\mathcal{P}'$, as it implies that we do not need any prior information about which incentive constraints are binding in the second period.[18]

In order to state the first property of solutions to $\mathcal{P}'$, we need one further definition.

Definition 2. *Full market coverage. An allocation rule satisfies the full market coverage property, if* $x_1(\theta, \tau) + x_2(\theta, \tau) = 1 \quad \forall \theta, \tau$.

We say a mechanism satisfies the full market property if its allocation rule does. Full market coverage means that the buyer always obtains some good. Note that this does not mean the buyer always obtains his favorite good. Now, we can state Lemma 4.

Lemma 4. *Each mechanism that solves problem $\mathcal{P}'$ satisfies the full market coverage property.*

To prove the first part of Lemma 4, note that by the feasibility conditions (F) we know $x_1 + x_2 \in [0, 1]$ and $x_1 - x_2 \in [-1, 1]$. The key observation that leads to the full market coverage property is that any feasible level of $x_1 - x_2$ can also be obtained when fixing $x_1 + x_2 = 1$. This means that the two summands in the argument of the integral in the objective of $\mathcal{P}'$ can essentially be maximized separately. When maximizing the first of these summands, the full market coverage obtains.

In the following paragraph we identify conditions on the virtual valuation under which the solution to $\mathcal{P}'$ is incentive compatible. To this end, we present and interpret

[17] See for example Baron and Besanko (1984); Courty and Li (2000); Pavan et al. (2014) and Eső and Szentes (2007a).

[18] The combination of two facts explains why ψ does not depend on which incentive constraints are binding: First, the constant in the information measure does not matter, as $\frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)} = \frac{\partial(-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)}$. Second, when doing pointwise maximization, ex post types which prefer good 2 have a valuation of $-\theta$ and obtain their favorite good when ψ is *smaller* than zero.

assumptions on the distributions $F_\tau(\tau)$ and $F(\theta|\tau)$, and relate them to assumptions which are frequently used in standard models of sequential screening.

Assumption 1. *Regularity Conditions*

1.1) $\left|\frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)}\right|$ *decreases in* τ *for all* θ.

1.2) $\psi(\theta,\tau)$ *increases strictly in* θ *for all* τ.

We call our model *regular* if it satisfies Assumption 1. Assumption 1.1 implies that for higher ex ante types τ a change in the ex ante type has less impact on the distribution of the ex post type. We can hence say that the informativeness of the ex ante type is decreasing in ex ante types. As the first summand in ψ is increasing in θ with slope one, Assumption 1.2 basically means that the distortion due to information rents is not excessively high.

Assumption 1 is the natural extension of the corresponding assumptions introduced for the first order stochastic dominance ordering in the literature on sequential screening to the rotation order.[19] When considering the the family of distribution functions $\{F_\tau(\tau)|\tau \in T\}$ only on $[\theta^\dagger, \overline{\theta}]$, the ordering resembles the first order stochastic dominance ordering. On $[\theta^\dagger, \overline{\theta}]$, Assumption 1 equals the standard assumptions in the literature on sequential screening. On $[\underline{\theta}, \theta^\dagger]$ the distributions' ordering also resembles a first order stochastic dominance, but the ordering is inverted. Assumption 1 is chosen to account for this fact.

An illustration of two examples for ψ that satisfy the regularity condition is given in Figure 2.5. Note that $\frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)} = 0$ at $\underline{\theta}$, $\overline{\theta}$ and at the rotation point $\theta^\dagger$. For $\theta > \theta^\dagger$ the virtual value is distorted downwards and for $\theta < \theta^\dagger$ the virtual value is distorted upwards. A consequence of our model assumptions is that the virtual value is continuous in θ.

In the following, we solve the model taking Assumption 1 as given. Using the regularity condition, we can characterize the mechanisms that solve problem $\mathcal{P}$ as stated in Proposition 2.

Proposition 2. *If the regularity condition is satisfied, each mechanism that solves problem $\mathcal{P}'$ is incentive compatible, and hence solves problem $\mathcal{P}$. Every optimal allocation has the following properties:*

(i) *For each* $\tau \in T$ *there exists a cutoff* $m(\tau)$ *such that* $x_2(\theta,\tau) = 1$ *if* $\theta < m(\tau)$ *and* $x_1(\theta,\tau) = 1$ *if* $\theta > m(\tau)$,

(ii) $|m(\tau)|$ *decreases in* τ *and* $m(\overline{\tau}) = 0$,

(iii) $m(\tau) \geq 0$ *if* $\theta^\dagger < 0$ *and* $m(\tau) \leq 0$ *if* $\theta^\dagger > 0$.

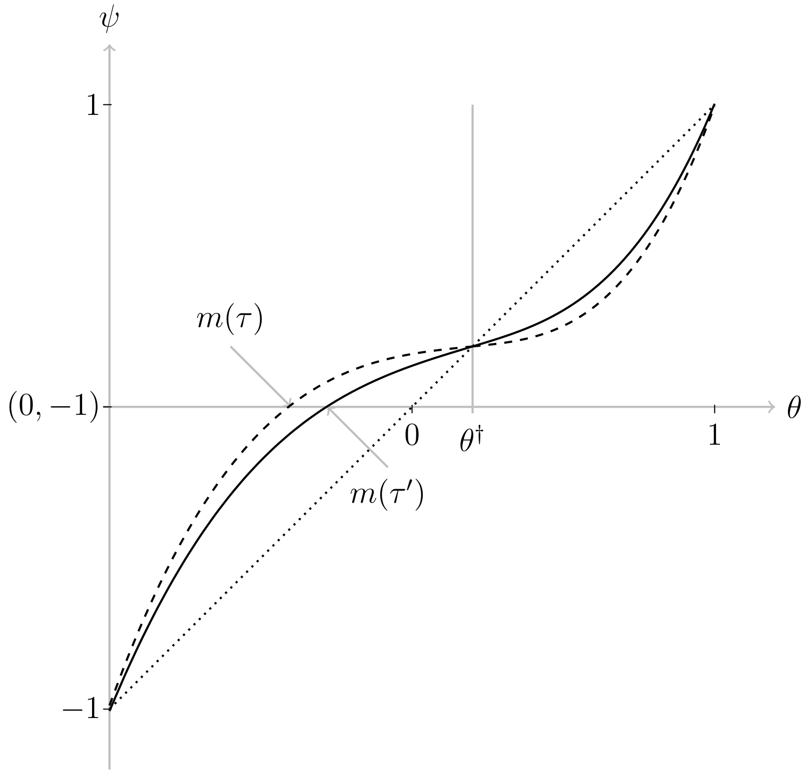

Figure 2.5: The graph shows the virtual values for two ex ante types τ (dashed line) and τ' (solid line) such that $\tau' > \tau$. For illustrative purposes we chose $[\underline{\theta}, \overline{\theta}] = [-1, 1]$.

The optimal allocation rule maximizes the objective in $\mathcal{P}'$ pointwise. Because of the second part of the regularity assumption, the pointwise maximization of the objective in $\mathcal{P}'$ results in an allocation rule that satisfies (MON_2) and hence solves problem $\mathcal{P}'$. The subtle part of proof is to show that the solution to $\mathcal{P}'$ is incentive compatible.

For a better understanding of the optimal allocation we first describe the welfare-maximizing allocation rule. As the production of both goods is equally costly to the seller and the buyer's basic valuation v exceeds this cost, the first best allocation rule satisfies the full market coverage property and the buyer always obtains the preferred good. In terms of cutoffs, the first best cutoff $m_e(\tau)$ is 0 for all τ. We thus call a contract more distorted than another, if the absolute value of the cutoff $|m(\tau)|$ is larger. Obviously, a more distorted contract generates lower welfare, because inefficient consumption decisions are more likely.

The optimal allocation is a cutoff solution: There is a cutoff ex post type. If the buyer has stronger preferences for good one than the cutoff type, he obtains this good and vice versa. Hence, the contract is deterministic in the sense that the optimal allocation is no non-degenerate lottery over goods. In particular, agents that have strong preferences ex post about which good they prefer always end up with this good.

Furthermore, the cutoff has the same sign for each ex ante type. This means that

[19] See for example Eső and Szentes (2007b).

independently of the ex ante type there is overconsumption of the same good. The good which is more likely to be consumed is the one which is preferred by the ex post type that equals the rotation point. Apart from extreme cases this equals the good which is preferred in expectation.

Our optimal allocation yields the classical no distortion at the top-result: The highest ex ante type always obtains his favorite good. Contracts for lower ex ante types are increasingly distorted. This means that if the buyer is more uncertain about his ex post type, he obtains contracts which are less distorted. In particular, this implies that depending on the ex ante type the buyer can obtain different allocations even when being of the same ex post type. The next section provides an in-depth discussion of the optimal mechanism from a price discrimination based perspective.

2.5 Implementation

Direct mechanisms are rarely observed in practice. A general concern in the literature on mechanism design is therefore to find simple indirect mechanisms that are equivalent in terms of outcomes to the optimal direct mechanism. Ideally, the indirect mechanisms match observations in the economic context the theory addresses. In this paper, the simple indirect mechanisms are menus of contracts which allow for exchanges subject to exchange fees. A contract consists of a price p for a specific good z and an exchange fee w. If the buyer chooses this contract, he pays p and obtains good z in the first period. In the second period, he may either consume his good or exchange it with the other good, which costs him the exchange fee w in addition.

Proposition 3. *If the regularity condition is satisfied, every mechanism that solves problem $\mathcal{P}$ is implementable by a menu of contracts which allow for exchanges subject to exchange fees, $\{z, p(\tau), w(\tau) | \tau \in T\}$. The menu has the following properties:*

(i) $p(\tau)$ increases in τ,

(ii) $w(\tau)$ decreases in τ, and $w(\overline{\tau}) = 0$

Note that in our model the good which is sold in the first period is always the same, which is the good that is preferred by the ex post type that equals the rotation point. Of course, the buyer stays with his initially purchased good when he learns in the second period that he prefers it to the alternative good. But also when his valuation for the alternative good slightly exceeds the valuation of the good he possesses, he stays with his good in order to save the exchange fee. Only when the difference in valuations exceeds the exchange fee w, the buyer is willing to incur the cost w of exchanging goods. The exchange fee is chosen such that the cutoff ex post type equals $m(\tau)$.

The price p and the exchange fee w both depend on the ex ante type τ. If the buyer is of the highest ex ante type, he chooses the contract with the highest price which however guarantees him free exchange. The contracts for lower ex ante types have cheaper prices, however, they feature higher exchange fees.

The seller offers the menu of contracts in order to price-discriminate following a logic which has similarities to the intuition in Courty and Li (2000). The first step to understand why price discrimination is profitable in our environment is to note that different ex ante types have different expected valuations for the first best allocation. Higher ex ante types have on average more extreme values of θ, which implies that on average the valuation for the favorite good is larger for them. As the first best contract always guarantees the buyer his favorite variant, the expected valuation of this allocation increases in the ex ante type. If the seller would offer only the first best allocation, she would hence have to trade off leaving rent to high ex ante types and excluding low ex ante types. The second step is to understand how price discrimination is possible in this framework. Therefore, we compare the first best contract to the contract for τ' with $p(\tau') < p(\overline{\tau})$ and $w(\tau') > w(\overline{\tau}) = 0$. For illustration, assume $z = 2$. We know from first-period incentive compatibility that $p(\tau') + w(\tau') > p(\overline{\tau})$. This means that conditional on consuming good 1, the buyer is better off when having chosen the first best contract. Conditional on consuming good 2, the buyer is better off when having chosen the first best contract for τ'. The crucial observation is that a higher ex ante type expects to pay the exchange fee more often than a lower ex ante type, which is the case because he draws θ from a more dispersed distribution. This makes the contract for τ' less attractive for $\overline{\tau}$ than for τ' in the following sense: When choosing $p(\tau'), w(\tau')$ and $p(\overline{\tau})$ such that in the first period ex ante type τ' is indifferent between the contracts, ex ante type $\overline{\tau}$ strictly prefers the first best contract. Then we can increase profits by increasing $p(\overline{\tau})$.

The use of exchange fees to govern exchanges is widespread in important industries. Most prominently, menus of tickets that differ in the height of their exchange fees are common in the transportation industry, the hotel business, and the car rental industry. In particular for those industries we believe that firms discriminate between buyers that expect differing degrees of resolution of uncertainty before they have to make a final decision. This means that business travelers know in advance that they may change their travel plans until shortly before the trip. Consequently they are better informed than leisure travelers at the point in time they eventually have to fix the terms of trade. Therefore they expect to have stronger preferences over the time of the flight, car rental, and hotel stay resulting in a more frequent desire to change the plans they set up initially. Hence, business travelers choose contracts with higher prices and lower exchange fees compared to leisure travelers.

In other situations it seems more plausible that firms discriminate between buyers that are just differently choosy. Examples are online shopping platforms and mail order companies. Note that another indirect mechanism that is equivalent to the optimal direct mechanism is to offer a partial refund for giving back the initially purchased good and then give the buyer the opportunity to purchase the other good. Besides explicitly declared fees for refunding goods, e.g. tickets for events, they take the form of service fees as frequently observed for example for online shopping platforms and mail order companies.

2.6 Discussion

2.6.1 Exchange Fees vs Limited Exchange Contacts

In the optimal direct mechanism, the transfer from the buyer to the seller depends on the reported ex post type. This is obvious when considering the implementation via a menu of contracts with exchange fees: The final price depends on whether the buyer exchanges the good in the second period or not. The difference is equal to the exchange fee. This means that in principle the agent is able to consume any good in the second period, but through an appropriate design of transfers he is incentivized to not always consume his favorite good.

Herbst (2016) studies an alternative setup in which the seller also offers a menu of contracts that leave the buyer with differing degrees of flexibility to exchange products. However, in that paper the seller optimally uses an alternative concept to govern exchanges, namely Limited Exchange Contracts. In the two goods model of our paper, a Limited Exchange Contract gives the buyer one of the goods in the first period. In the second period, the buyer may either stay with his good or take the option for a costless exchange with the other good. Exchange is, however, possible only with a pre-specified probability. More flexible contracts feature a higher probability with which exchange is possible. While in our paper exchange is limited by incentivizing the buyer in the second period to not exchange, in Herbst (2016) exchange is impossible to a certain probability.

In this paragraph we relate the differences in the setups to the difference in how the restriction of flexibility is optimally designed. First, we compare our model with the two goods case in Herbst (2016). In Herbst (2016) the ex ante type fixes two valuations and the only information contained in the ex post type is which good is valued with which of the two valuations. Hence the buyer perfectly knows the difference in valuations, which is his cost of ending up with the wrong good, in the first period.[20] When the seller learns the buyer's ex ante type in the first period, she already knows how much flexibility she wants to grant the buyer in the second period. When governing this flexibility via restricted but costless exchange, restricting the flexibility is not connected to additional information rents in the second period. In our paper, the buyer does not know his difference in valuations in the first period. Instead, the buyer has noisy information about the extent of the difference in valuations, but he learns the exact difference only in the second period. This makes it desirable for the seller to adapt the flexibility between the goods to the realized valuation difference ex post. The only way to prevent the buyer from misrepresenting his ex post type then to incentivize him via payments which depend on the ex post type.

A natural question concerning the comparison of the two papers is when which design of exchange policies fits best. Therefore we first consider the foundation for our model in which the consumer is uncertain about his position on the hotelling line. Also Herbst (2016) can be generalized to a Hotelling model. The crucial difference is that while in our model the seller can only trade the two goods at the end of the hotelling line, she can trade any good on the line in the other paper.[21] This comparison suggests that we observe Limited Exchange Contracts primarily in environments in which many varieties of the good can be offered. Applied to the transportation industry this means that Limited Exchange Contracts should appear for connections which are served in small intervals. Indeed we can observe this phenomenon in the ferry industry: While P&O Ferries has established Limited Exchange Contracts for the highly frequented connection from Dover to Calais, they charge exchange fees for less frequently served connections.[22]

So far we have discussed how the restriction of flexibility is optimally designed in both models and why this is the case. One further aspect which differentiates the models is the motive for offering contracts which partially restrict exchanges. In Herbst (2016) the seller's motivation stems from the fact that higher ex ante types have higher transportation cost and higher maximal valuations but on average valuations are lower due to the high cost. When adding this feature to the model presented in this paper, it seems likely that optimal contracts will combine Limited Exchange Contracts with exchange fees: Already without paying an exchange fee the buyer is

[20] This setup is interpreted as the buyer obtaining the cardinal dimension of his preferences in the first period and the ordinal information in the second period.

[21] Herbst (2016) also differs in that all ex ante types expect the same distribution for the position on the Hotelling line but differ in the transportation cost function instead. The foundation which is close with that respect is the one with uncertainty about the transportation cost function. Herbst (2016) differs also from this foundation in that the seller may offer any good.

[22] Note that with multiple goods a Limited Exchange Contract can for example consist of a time interval around the initially booked departure time in with changes are free.

given some flexibility to exchange goods. This flexibility is granted as in Limited Exchange Contracts and will be the same for any contract of the menu. When paying an extra exchange fee, the buyer can, however, change to any good he prefers.[23] Menus of contracts of this shape are common in the US airline industry. Many US airlines offer costless same day exchanges and stand-by options for any flight ticket. Only if the departure time needs to be changed a different day, exchange fees apply.

2.6.2 The FOSD Ordering

In this paper we study a model which can be interpreted as one in which agents gradually receive information about their position on the Hotelling line. Ex ante types differ in that the respective distributions of the ex post type θ are ordered by a mean preserving spread. This means the ex ante type only provides information about how extreme valuation differences are and how large the valuation for the favorite good is in expectation, but the ex ante type does not provide information about which good is preferred. In this section, we consider the other extreme in which the ex ante type only provides information about which good is preferred but the expected valuation for the favorite good is kept constant across ex ante types.

Our first result in this section, stated in Proposition 4, is that price discrimination is not desirable. After providing the intuition, we argue why we think price discrimination is also not possible in this modified model even if it would be desirable from the seller's perspective.

The model we consider differs from the main model of this paper only in the assumptions about the distributions $F_\tau(\tau)$ and $F(\theta|\tau)$. In addition, we assume $\underline{\theta} = -\overline{\theta}$. First, we assume that $F_\tau(\tau)$ is symmetric around $(\underline{\tau}+\overline{\tau})/2$, which means $f_\tau(\underline{\tau}+\tau) = f_\tau(\underline{\tau}-\tau)$. Second, we put structure on $F(\theta|\tau)$, which amounts to a special case of first order stochastic dominance in τ: We assume that $F(\theta|\tau)$ is linear in the ex ante type, in particular that there is a "most left" distribution $F_{\underline{\tau}}(\theta)$ and a "most right" distribution $F_{\overline{\tau}}(\theta)$ which are symmetric to each other in that $f_{\underline{\tau}}(\theta) = f_{\underline{\tau}}(-\theta)$ for all ex post types. $F_{\underline{\tau}}(\theta)$ is first order stochastically dominated by $F_{\overline{\tau}}(\theta)$. For any intermediate τ, we define

$$f_\tau(\theta) = \frac{\overline{\tau}-\tau}{\overline{\tau}-\underline{\tau}} f_{\underline{\tau}}(\theta) + \frac{\tau-\underline{\tau}}{\overline{\tau}-\underline{\tau}} f_{\overline{\tau}}(\theta). \tag{2.11}$$

[23] While the result seems rather trivial given the results of Herbst (2016) and this paper, deriving the result is analytically challenging because in this dynamic mechanism design problem the support of the ex post type depends on the ex ante type.

Proposition 4. *The revenue-maximizing mechanism assigns the first best contract to all ex ante types.*

As payoffs are unchanged compared to the main model, the first best contract coincides with the one from Section 2.4. The key step to understand this result is to note that by construction every ex ante type has the same expected utility from the first best contract. The valuation of the first best contract is determined by the expected valuation of the favorite good, which is proportional to the expected minimum of $\overline{\theta} - \theta$ and $\theta - \underline{\theta}$. As the distributions $F_{\overline{\tau}}(\theta)$ and $F_{\underline{\tau}}(\theta)$ are symmetric, their expected minimum and hence their expected valuation for the first best contract coincides. As any other distribution $F_{\tau}(\theta)$ is constructed as a linear combination of the two distributions, also their expected valuation is a linear combination of the expected valuations of the two extreme distributions, and is hence constant over ex ante types. This implies that the seller can extract the first best surplus by offering the first best allocation-rule and skimming all utility, which is clearly revenue-maximizing. This means that there is no need for the seller to price discriminate in order to extract more rent from the buyer.

We finish this section by explaining why we think price discrimination is generally not possible in this modified model. Following the intuition of this paper or Courty and Li (2000), the scheme to do price discrimination would be the following: The lowest ex ante type obtains the first best contract. Increasing ex ante types sign contracts consisting of a decreasing upfront price for $x_2 = 1$, but an increasing exchange fee for obtaining good one. This means the difference in prices that is payed ex post for the left good minus the right good is increasing. As the lower ex ante type expects to consume the left good more often, the contracts with larger price differences are unattractive for him.

However, in contrast to the intuition provided Section 2.5, we would like to do the same type of price discrimination starting from the highest ex ante type by making the price for good 1 larger than the price for good 2 ex post. The problem with this intuition is that both discrimination schemes jointly violate global incentive compatibility: For example, the highest ex ante type prefers any contract in which good 2 is cheaper than good 1 to the first best contract which is designed for him. This ex ante type would hence misreport a low ex ante type. This is an essential problem the seller faces when she intends to do price discrimination according to the position on the Hotelling line, when ex ante types are ordered by first order stochastic dominance.[24]

[24] While this impossibility-result is intuitive, it is hard to prove in environments which are more general than the particular order defined in this section. The reason is that when ex ante types are ordered by first order stochastic dominance, the on-path expected utility is not necessarily increasing in the ex ante type. As on-path the ex ante utility is furthermore a non-trivial function of the ex ante type, it is difficult to apply results from optimal control theory as done for example in Jullien (2000)

2.7 Conclusion

In this paper we studied environments in which a monopolist sells two horizontally differentiated goods to a buyer who only gradually learns his valuations for the goods. The seller optimally price discriminates by offering a menu of contracts that allow for exchange subject to an exchange fee. The optimal menu consists of contracts with high base prices for which the exchange fees are comparatively low and contracts with smaller base prices but higher exchange fees. There are two key ingredients to the model that lead to the optimality of restricting the buyer's flexibility between the goods via exchange fees. First, the buyer is initially uncertain about the difference in valuations ex post. This means it is possible that he ends up having very strong preferences for one of the goods but there is also a chance he is indifferent. Second, at the point of contracting the buyer has private information about the extent of valuation differences he expects on average.

The paper provides several foundations for the reduced form model. One foundation is that buyers have different amounts of uncertainty about their position on the Hotelling line, another foundation is that buyers have the same uncertainty about their position on the line but differ in their transportation cost. Finally, there is s foundation which is not based on preferences but on information structure instead: Buyers have different expectations about how much information they obtain until they eventually have to decide among their options.

When buyers differ in that they have tendencies to prefer one of the goods, the characterization of the revenue-maximizing mechanism is technically challenging. For a special case we show that price discrimination is not desirable from the seller's perspective. We provide an intuition for why price discrimination is difficult in this setup even when it would be desirable. Solving a general model where ex ante types are ranked by first order stochastic dominance is hence an interesting avenue for future research. An additional benefit from solving such a model is that it enables a comparison to the competitive setting in which each good is offered by one seller respectively.[25]

We also compared our model to Herbst (2016) which is related in that it also studies optimal exchange policies. The optimal design of the partial restriction of exchange is implementable via Limited Exchange Contracts which is a fundamentally different concept than using exchange fees to govern exchanges. The difference originates in two differences in the setup: Buyers know already ex ante their precise difference in valuations and the expected valuation for a given good varies in the ex ante type. A formal approach to combining the two models is a further aspect to be studied. Studying this combinations seems worthwhile as we regularly observe combinations of

for canonical economic settings.

[25] Conceptually, this is a generalization of Gale (1993) and Möller and Watanabe (2016).

the two concepts in applications.

2.8 Appendix

Proof of Proposition 1:

To proof Proposition 1, we first calculate $F_r(x)$ for all $x \neq 0$. Then we show that $\partial F_r(x|\tau)/\partial\tau > 0$ for $x < 0$ and $\partial F_r(x|\tau)/\partial\tau < 0$ for $x > 0$. The last step is to proof continuity of F_r at 0.

For $x \neq 0$ the cumulative distribution $F_r(x)$ is defined as

$$F_r(x) = \int_{-1}^{x} f_r(l)dl = \int_{-1}^{x}\int_{|l|}^{1} \frac{g(r|\tau)}{2r} drdl. \tag{2.12}$$

We first consider the case $x < 0$. By changing the order of integration we obtain

$$\begin{aligned}\int_{-1}^{x}\int_{|l|}^{1} \frac{g(r|\tau)}{2r} drdl &= \int_{-1}^{x}\int_{l}^{1} \frac{g(r|\tau)}{2r} drdl \\ &= \int_{-x}^{1}\int_{-r}^{x} \frac{g(r|\tau)}{2r} dldr \\ &= \int_{-x}^{1} (x+r)\frac{g(r|\tau)}{2r} dr.\end{aligned}$$

Through integration by parts we obtain

$$F_r(x) = \frac{1}{2} + \frac{x}{2}\Big(1 + \int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr\Big). \tag{2.13}$$

Analogously we obtain for $x > 0$

$$F_r(x) = \frac{1}{2} + \frac{x}{2}\Big(1 + \int_{x}^{1} \frac{G(r|\tau)}{r^2} dr\Big). \tag{2.14}$$

The next step is to show that $\partial F_r(x|\tau)/\partial\tau > 0$ for $x < 0$ and $\partial F_r(x|\tau)/\partial\tau < 0$ for $x > 0$. This follows from directly taking the derivative and from $\partial G(r|\tau)/\partial\tau < 0$ which holds by the assumption that $G(r|\tau)$ is ordered by first order stochastic dominance:

$$\frac{\partial F_r(x)}{\partial\tau} = \begin{cases} \frac{x}{2}\Big(\int_{-x}^{1} \frac{\partial G(r|\tau)/\partial\tau}{r^2} dr\Big) > 0 & \text{for } x < 0, \\ \frac{x}{2}\Big(\int_{x}^{1} \frac{\partial G(r|\tau)/\partial\tau}{r^2} dr\Big) < 0 & \text{for } x > 0. \end{cases} \tag{2.15}$$

The inequalities (2.15) only imply that the family $\{F_r | r \in (0,1]\}$ is rotation-ordered if F_r is continuous at 0. This is proven in the final step by showing $\lim_{x \nearrow 0} F_r(x|\tau) = 1/2 = \lim_{x \searrow 0} F_r(x|\tau)$, which is sufficient for continuity as F_r is increasing by definition. We proof $\lim_{x \nearrow 0} F_r(x|\tau) = 1/2$. Showing $\lim_{x \searrow 0} F_r(x|\tau) = 1/2$ is analogous.

We start by rewriting the expression as

$$\lim_{x \nearrow 0} F_r(x|\tau) = \lim_{x \nearrow 0} \frac{1}{2} + \frac{x}{2}\left(1 + \int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr\right) = \frac{1}{2} + \lim_{x \nearrow 0} \frac{\int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr}{\frac{2}{x}}.$$

As we do not have information about $\lim_{x \nearrow 0} \int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr$, we make a case distinction.

Case 1: $\lim_{x \nearrow 0} \int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr = m$ with $|m| < \infty$

It follows immediately $\lim_{x \nearrow 0} F_r(x|\tau) = 1/2$.

Case 2: $\left|\lim_{x \nearrow 0} \int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr\right| = \infty$

We can apply l'Hospital's rule and obtain

$$\lim_{x \nearrow 0} F_r(x|\tau) = \frac{1}{2} + \lim_{x \nearrow 0} \frac{G(x|\tau)/x^2}{-2/x^2} = \frac{1}{2} + \lim_{x \nearrow 0} -\frac{G(x|\tau)}{2} = \frac{1}{2}.$$

As $\int_{-x}^{1} \frac{G(r|\tau)}{r^2} dr$ is monotone in x for $x \in (\infty, 0)$, one of the two cases applies.

As $\lim_{x \nearrow 0} F_r(x|\tau) = 1/2 = \lim_{x \searrow 0} F_r(x|\tau)$ and $F_r(x|\tau)$ is increasing, we know that $F_r(0|\tau) = 1/2$ and hence 0 is the rotation point.

■

Proof of Lemma 1:

The proof is standard in the literature on mechanism design and hence omitted. ■

Proof of Lemma 2:

We first derive condition (FOC_1). By the envelope-theorem we know

$$\frac{\partial U(\tau)}{\partial \tau} = \left.\frac{\partial U(\tau, \hat{\tau})}{\partial \tau}\right|_{\tau=\hat{\tau}} = \int_{\underline{\theta}}^{\overline{\theta}} u(\tau, \theta) \frac{\partial f(\theta|\tau)}{\partial \tau} d\theta.$$

By integration by parts and using Lemma IC_1 we obtain

$$\frac{\partial U(\tau)}{\partial \tau} = \int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau, \theta) - x_2(\tau, \theta)) \cdot \left(-\frac{\partial F(\theta|\tau)}{\partial \tau}\right) d\theta. \qquad (FOC_1)$$

Second, we derive (SOC_1). By (2.6) we know

$$\begin{aligned} U(\tau) &\geq U(\tau,\hat{\tau}) \\ &= \int_\Theta u(\hat{\tau},\theta)dF(\theta|\tau) \\ &= U(\hat{\tau}) + \int_\Theta u(\hat{\tau},\theta)d[F(\theta|\tau) - F(\theta|\hat{\tau})] \end{aligned}$$

which is equivalent to

$$U(\tau) - U(\hat{\tau}) \geq \int_\Theta u(\hat{\tau},\theta)d[F(\theta|\tau) - F(\theta|\hat{\tau})]. \tag{2.16}$$

Analogously we obtain

$$U(\hat{\tau}) - U(\tau) \geq \int_\Theta u(\tau,\theta)d[F(\theta|\hat{\tau}) - F(\theta|\tau)]. \tag{2.17}$$

Combining (2.16) and (2.17) we obtain

$$\int_\Theta [u(\tau,\theta) - u(\hat{\tau},\theta)]d[F(\theta|\hat{\tau}) - F(\theta|\tau)] \geq 0.$$

Dividing by $(\tau - \hat{\tau})^2$ and taking limits we obtain

$$\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial x_1(\tau,\theta) - x_2(\tau,\theta)}{\partial \tau} \cdot \left(-\frac{\partial F(\theta|\tau)}{\partial \tau}\right) d\theta \geq 0. \tag{SOC_1}$$

■

Proof of Lemma 3:

From (FOC_1) we obtain

$$\partial U(\tau)/\partial \tau = \int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta) - x_2(\tau,\theta)) \cdot \left(-\frac{\partial F(\theta|\tau)}{\partial \tau}\right) d\theta. \tag{2.18}$$

As by assumption $F(\theta|\tau)$ is differentiable in both arguments, the rotation order implies $\partial F(\theta|\tau)/\partial\tau \geq 0$ for all $\theta \leq \theta^\dagger$ and $\partial F(\theta|\tau)/\partial\tau \leq 0$ for all $\theta \geq \theta^\dagger$. From Lemma 1 we furthermore know that $x_1(\tau,\theta) - x_2(\tau,\theta)$ increases in θ for any τ. Consequently, for any τ we know that $x_1(\tau,\theta) - x_2(\tau,\theta) \leq x_1(\tau,\theta^\dagger) - x_2(\tau,\theta^\dagger)$ for all $\theta < \theta^\dagger$ and $x_1(\tau,\theta) - x_2(\tau,\theta) \geq x_1(\tau,\theta^\dagger) - x_2(\tau,\theta^\dagger)$ for all $\theta > \theta^\dagger$. By a pointwise comparison we can conclude

$$\begin{aligned} &\int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta) - x_2(\tau,\theta)) \cdot \left(-\frac{\partial F(\theta|\tau)}{\partial \tau} \right) d\theta \\ \geq &\int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta^\dagger) - x_2(\tau,\theta^\dagger)) \cdot \left(-\frac{\partial F(\theta|\tau)}{\partial \tau} \right) d\theta. \end{aligned} \tag{2.19}$$

By assumption all distributions $F(\theta|\tau)$ have the same mean. A consequence is that $\partial \mathbb{E}_\theta[\theta|\tau]/\partial\tau = 0$ for all τ. Through integration by parts we obtain

$$\begin{aligned} 0 = \partial \mathbb{E}_\theta[\theta|\tau]/\partial\tau =& \partial \int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta|\tau) d\theta \Big/ \partial\tau \\ =& \partial \Big(\overline{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} F(\theta|\tau) d\theta \Big) \Big/ \partial\tau. \\ =& \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial F(\theta|\tau)}{\partial\tau} d\theta. \end{aligned} \tag{2.20}$$

This in turn implies

$$\int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau,\theta^\dagger) - x_2(\tau,\theta^\dagger)) \cdot -\frac{\partial F(\theta|\tau)}{\partial\tau} d\theta = 0. \tag{2.21}$$

The proof follows from (2.18), (2.19), and (2.21). ■

Proof of Lemma 4:

The proof is done in the main text.

Proof of Proposition 2:

The first step is to solve Problem $\mathcal{P}'$. The second step is to show that the solution is incentive compatible.

The seller minimizes $U(\underline{\tau})$ while maintaining individual rationality. When setting $U(\underline{\tau}) = 0$, (IR) is satisfied by Lemma 3 and Problem $\mathcal{P}'$ simplifies to

$$\begin{aligned} \max_{X(\tau,\theta)} \quad & (v - c) + \int_{\underline{\tau}}^{\overline{\tau}} \int_{\underline{\theta}}^{\overline{\theta}} \big[x_1(\tau,\theta) - x_2(\tau,\theta) \big] \psi(\theta,\tau) dF(\theta|\tau) dF_\tau(\tau) \\ \text{s.t.} \quad & (MON_2) \text{ and } (F), \end{aligned} \tag{2.22}$$

where $\psi(\theta,\tau)$ is the virtual valuation

$$\psi(\theta,\tau)=\theta-\frac{1-F_\tau(\tau)}{f_\tau(\tau)}\frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)}.$$

Maximizing objective (2.22) subject to feasibility (F) gives

$$x_1(\tau,\theta)-x_2(\tau,\theta)=\begin{cases}1 & \text{if } \psi(\theta,\tau)>0,\\ -1 & \text{if } \psi(\theta,\tau)<0.\end{cases} \tag{2.23}$$

By Assumption 1.2 the allocation defined by (2.23) and the full market coverage property satisfies (MON_2) and hence solves Problem $\mathcal{P}'$. For each ex ante type denote the ex post type defined via $\psi(\theta,\tau)=0$ by $m(\tau)$. $m(\tau)$ is the cutoff-ex post type at which the virtual valuation $\psi(\theta,\tau)$ gets positive. The cutoff exists by the intermediate-value theorem, as ψ is continuous, strictly increasing, $\psi(\underline{\theta},\tau)=\underline{\theta}<0$, and $\psi(\overline{\theta},\tau)=\overline{\theta}>0$. From Assumption 1.1 joint with the continuity of $\psi(\theta,\tau)$ follows furthermore that $m(\tau)$ is continuous and $|m(\tau)|$ decreases in τ. For the highest ex ante type $\overline{\tau}$ there are no distortions:

$$\begin{aligned}\psi(\theta,\overline{\tau})=&\theta-\frac{1-F_\tau(\overline{\tau})}{f_\tau(\overline{\tau})}\frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)}\\ =&\theta-\frac{1-1}{f_\tau(\overline{\tau})}\frac{\partial(1-F(\theta|\tau))/\partial\tau}{f(\theta|\tau)}\\ =&\theta.\end{aligned}$$

As the solution is obtained by pointwise maximization, $\psi(\theta,\overline{\tau})=0$ immediately implies $m(\overline{\tau})=0$. In the main text we furthermore show that $m(\tau)<0$ if $\theta^\dagger>0$ and $m(\tau)>0$ if $\theta^\dagger<0$.

The final step is to show that the solution to Problem $\mathcal{P}'$ satisfies IC_1. Assume without loss of generality that $\tau>\tau'$. By construction the solution to Problem $\mathcal{P}'$ satisfies FOC_1. Hence, we can write

$$U(\tau)=U(\tau')+\int_{\tau'}^{\tau}\int_{\underline{\theta}}^{\overline{\theta}}(x_1(y,\theta)-x_2(y,\theta))\cdot-\frac{\partial F(\theta|y)}{\partial y}d\theta dy.$$

The crucial step is then to note that

$$\begin{aligned}&\int_{\tau'}^{\tau}\int_{\underline{\theta}}^{\overline{\theta}}(x_1(y,\theta)-x_2(y,\theta))\cdot-\frac{\partial F(\theta|y)}{\partial y}d\theta dy\\ \geq&\int_{\tau'}^{\tau}\int_{\underline{\theta}}^{\overline{\theta}}(x_1(\tau',\theta)-x_2(\tau',\theta))\cdot-\frac{\partial F(\theta|y)}{\partial y}d\theta dy.\end{aligned} \tag{2.24}$$

To explain inequality (2.24), we do a case distinction. Let us first consider $m(\tau) < 0$. We know $x_1(\tau',\theta) - x_2(\tau',\theta) > x_1(y,\theta) - x_2(y,\theta)$ for all $y \in [\underline{\theta}, \overline{\theta}]$. We also know that $x_1(\tau',\theta) - x_2(\tau',\theta) = x_1(y,\theta) - x_2(y,\theta)$ for all $y \in [\theta^\dagger, \overline{\theta}]$, where $\theta^\dagger$ is the rotation point. A consequence of the rotation order is $\big(-\frac{\partial F(\theta|y)}{\partial y}\big) < 0$ for all $\theta < \theta^\dagger$. Combined, this implies (2.24) pointwise.

Second, consider $m(\tau) > 0$. In that case $x_1(\tau',\theta) - x_2(\tau',\theta) < x_1(y,\theta) - x_2(y,\theta)$ for all $y \in [\underline{\theta}, \overline{\theta}]$. However, in that case $x_1(\tau',\theta) - x_2(\tau',\theta) = x_1(y,\theta) - x_2(y,\theta)$ for all $y \in [\underline{\theta}, \theta^\dagger]$. A consequence of the rotation order is $\big(-\frac{\partial F(\theta|y)}{\partial y}\big) > 0$ for all $\theta > \theta^\dagger$. Combined, this implies (2.24) pointwise.

Using (2.24) and interchanging the order of intervals we obtain

$$U(\tau) \geq U(\tau') - \int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau',\theta) - x_2(\tau',\theta)) \cdot (F(\theta|\tau) - F(\theta|\tau'))d\theta. \tag{2.25}$$

Employing Lemma 1 and using integration by parts we finally obtain

$$\begin{aligned} U(\tau) \geq & U(\tau') - \int_{\underline{\theta}}^{\overline{\theta}} (x_1(\tau',\theta) - x_2(\tau',\theta)) \cdot (F(\theta|\tau) - F(\theta|\tau'))d\theta \\ = & U(\tau') + \int_{\underline{\theta}}^{\overline{\theta}} u(\tau',\theta)(f(\theta|\tau) - f(\theta|\tau'))d\theta \\ = & U(\tau,\tau'). \end{aligned}$$

■

Proof of Proposition 3:

We show that for any optimal mechanism as described in Proposition 2 we can construct a menu $\{z, p(\tau), w(\tau)|\tau \in T\}$ that is equivalent in terms of allocation rule and payment rule.

We start by writing down the payment rule of the optimal direct mechanism. Using the definition of the ex post utility (2.7) and Lemma 1, the payment structure of the direct mechanism is

$$t(\theta,\tau) = v + \theta(x_1(\theta,\tau) - x_2(\theta,\tau)) - u(m(\tau),\tau) - \int_{m(\tau)}^{\theta} x_1(\theta,\tau) - x_2(\theta,\tau))dy. \tag{2.26}$$

Using the optimality properties from Lemma 4 and Proposition 2 we can rewrite the payments (2.26) as

$$t(\theta,\tau)=\begin{cases} v+\theta-u(m(\tau),\tau)-(\theta-m(\tau)) & \text{if } \theta>m(\tau),\\ v-\theta-u(m(\tau),\tau)-(m(\tau)-\theta) & \text{if } \theta<m(\tau),\end{cases}$$
$$=\begin{cases} v-u(m(\tau),\tau)+m(\tau) & \text{if } \theta>m(\tau),\\ v-u(m(\tau),\tau)-m(\tau) & \text{if } \theta<m(\tau).\end{cases} \tag{2.27}$$

Now we can construct the indirect mechanism: First, if $m(\tau)>0$ set $z=1$ and if $m(\tau)<0$ set $z=2$. Second, let the exchange fee be $w(\tau)=2|(m(\tau)|$. And third, $p(\tau)=v-u(m(\tau),\tau)$ is the upfront price. The indirect mechanism is obviously equivalent to the optimal indirect in terms of payments and the allocation.

Finally, we need to show that $p(\tau)$ increases in τ. The proof is done by contradiction. Assume there exist two ex ante types τ and τ' such that $p(\tau)>p(\tau')$. Then we know that $p(\tau)>p(\tau')$, $w(\tau)>w(\tau')$, and $z(\tau)>z(\tau')$. It follows immediately that for ex ante type τ it pays off to misreport his ex ante type and report τ' instead. This is a contradiction to incentive compatibility. ■

Proof of Proposition 4:

We need to show that any ex ante type has the same utility from the first best allocation. The rest of the proof is then obvious.

To circumvent notational complications, we do an indirect proof. First, note that

$$\frac{\partial F_\tau(\theta)}{\partial\tau}=\frac{-1}{\overline{\tau}-\underline{\tau}}F_{\underline{\tau}}(\theta)+\frac{1}{\overline{\tau}-\underline{\tau}}F_{\overline{\tau}}(\theta) \tag{2.28}$$

which is independent of τ. Second, using (2.28) we see that from symmetry of $F_{\overline{\tau}}$ and $F_{\underline{\tau}}$ follows that $\frac{\partial F_\tau(\theta)}{\partial\tau}$ is symmetric in θ. Third, note that FOC_1 as a necessary condition for incentive compatibility is unchanged. When plugging the allocation of the first best contract into FOC_1 and using the symmetry of $\frac{\partial F_\tau(\theta)}{\partial\tau}$ in θ, we obtain $\partial U(\tau)/\partial\tau=0$ for all τ. This means that in any incentive compatible mechanism in which each ex ante type obtains the first best allocation, the ex ante utility is the same for each ex ante type. Since each ex ante type obtains the same allocation, the only incentive compatible payment rule is to charge each ex ante type the same expected payment. From this follows that the expected utility derived from the first best allocation is constant across ex ante types. ■

Chapter 3

Dynamic Formation of Teams: When Does Waiting for Good Matches Pay Off?

This paper studies the trade-off between realizing match values early and waiting for good matches that arises in a dynamic matching model with discounting. We consider heterogeneous agents that arrive stochastically over time to a centralized matching market. First, we derive the welfare-maximizing assignment rule, which displays the subtle trade-off between matching agents early and accumulating agents to form assortative matches. Second, we show that the welfare-maximizing policy is implementable when agents have private information about their types. The corresponding mechanism satisfies natural requirements. Furthermore, we identify situations in which the designer can abstain from using monetary incentives.

3.1 Introduction

We study a canonical situation in which agents that arrive gradually over time join forces in order to generate output. Agents are heterogeneous and when forming a group their characteristics are complements in the production function. In a static world, when all agents are present from the beginning, positive assortative matchings are both stable and efficient with this kind of production function.[1] The dynamic arrival of agents combined with impatience, however, poses a challenge to positive assortativeness. If future outcomes are discounted, the desirability of early matches increases both from a social welfare as well as an participating individual's perspective.

This paper analyses the emerging trade-off between realizing match values early and

[1] This is a well established result in the matching literature. For a study of necessary and sufficient conditions for positive assortative matchings see Legros and Newman (2002).

waiting for good matches. For this purpose it tackles the question of assortativeness in a centralized dynamic matching market. We address both the welfare-maximizing matching procedures under complete information, and socially optimal mechanisms when agents have private information. We develop a tool that allows us to solve for the welfare-maximizing matching policy in closed form without imposing any restriction on the policy. This provides clear insights into the effects involved. Then we prove implementability of the welfare-maximizing matching policy when agents have private information about their types. Furthermore, we identify situations in which the market organizer can abstain from using monetary incentives. Finally, we address the case in which the agents can, in addition to their private type, hide their arrival.

Applications comprise a wide range of situations including the formation of teams and task assignment within firms, as well as the establishment of partnerships that constitute organizations themselves. Nowadays, output within organizations is mostly created by teams: Examples are consultancy in firms, coauthoring at universities, or team sports in clubs. Complementarities of experts' skill in production processes is clearly illustrated in the well-known O-Ring Theory in Kremer (1993). Employees arrive over time when having finished previous projects or being hired newly. There are furthermore various industries, in which entrepreneurs team up to found companies; an example are doctors who found group practices to share the burden of the large investment in medical equipment. Complementarity arises when patients are risk-averse concerning the quality of treatment and the dynamic friction is that doctors arrive over time in a local market. Our framework equally well fits situations of education in groups, e.g. language courses, as group member's skills typically are complements and participants arrive over time. Finally, the model applies to the wide-spread practice of group-lending in the market for microcredits: Borrowers who cannot offer collateral obtain loans for individual projects only if they pool the default risk with peers conducting independent projects. A borrower's type is the individual default risk. Following the standard assumption that the default risks of the group members are independent leads to the complementarity of the individual default risks.[2]

Our paper aims at closing the gap between static matching models and the literature on search and matching. The growing literature on search and matching studies matching patterns using search models. Each agent from a continuous population meets random fellows one by one and then decides whether to match with that partner or to continue search. Major contributions are Shimer and Smith (2000), Smith (2006), and Atakan (2006).[3] While search and matching models modify the static matching

[2] Explicit models of group-lending with complementarities in individual default risks are for example given in the publications on microcredit group-lending Ghatak (1999, 2000) and Ghatak and Guinnane (1999).

[3] See also the early contributions by Sattinger (1995), Lu and McAfee (1996), and Burdett and Coles

model by introducing a time and a search friction jointly, we isolate the effect of the time friction. To that end, we study the centralized organization of matching markets using a mechanism design approach.[4]

We consider a population of heterogeneous agents that differ in a binary characteristic. Matched agents jointly produce socially valuable output according to a production function which is supermodular in the agents' characteristics. Once matches are made, they are irrevocable. Following most of the literature, we assume that matchings are pairwise.[5] Agents arrive according to a Poisson process and types are drawn independent of past arrivals. This model is flexible with respect to four key features: The degree of complementarity of the partners' characteristics in the output function, the relative size of absolute values of output generated by two possible matchings of similar agents, the probability distribution of arriving agents' types, and the patience represented by discounting.[6]

The irrevocability of matches may originate from the matched group's need to initially make sunk investments, which make any later split economically unprofitable. Investments may be capital investments for example in medical equipment or advertisement for a newly founded company, or social investments like trust and social arrangements within a group of workers. An alternative view is that pairs simply leave the market and do not return even in case of a split.

For the sake of tractability, we assume characteristics to be binary. This enables us to find a closed-form solution to an otherwise still unsolved optimization problem. Apart from the insights provided by the solution to this problem, both the knowledge of the exact shape of this solution and the solution technique developed in this paper may help to tackle more comprehensive problems. We refer to the type that generates the higher output when being paired with itself as 'productive'.

In the first part of the paper, we derive the welfare-optimal matching policy when the designer can observe both arrivals and the arriving agents' types. As opposed to the literature on search and matching, we do not impose any restrictions on the technology the central organizer may use.[7] This allows us to analyze the role of assortativeness

(1997) on two-sided matching. For a literature survey on search and matching models see Smith (2011).

[4] The environment can be interpreted as small in the sense that the arrival process is discrete, representing single agents arriving. In small matching markets central organization is naturally more appropriate than decentralized search models.

[5] For an exception see Ahlin (2015).

[6] More precisely, the combination of the discount rate and the frequency with which arrivals are expected. While in the introduction little discounting between two arrivals is interpreted as patience, a higher frequency of arrivals, which can be interpreted as a larger market, has the same effect.

[7] In the literature on search and matching there is little work on social optimality. Shimer and Smith

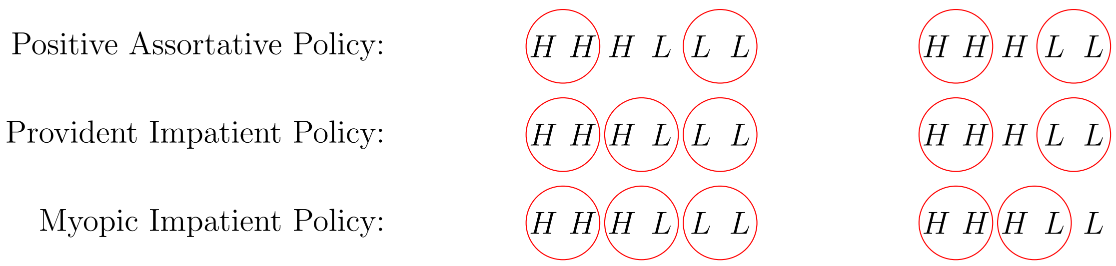

Figure 3.1: H stands for a productive agent, L represents an unproductive agent, and circles indicate matches. The policies are illustrated by means of two freely chosen sets of agents. The first set of agents consists of three productive agents and three unproductive agents. The second set consists of three productive agents and two unproductive agents.

from an efficiency perspective. We obtain the welfare-optimal policy in closed form depending on the four key characteristics. This allows for a clear-cut analysis of the dynamic friction on efficiency.

First, assume that the outputs produced by two productive agents and two unproductive agents do not differ too drastically. Depending on the remaining three key features, always one of three matching policies is optimal. The *Positive Assortative Policy* matches agents with equal types whenever possible and lets every agent wait otherwise. The *Provident Impatient Policy* matches two similar agents with priority whenever possible, but it also matches two unequal types if only those are left. Finally, the *Myopic Impatient Policy* always matches pairs of productive agents. If the number of productive agents is uneven, the remaining one is then matched with priority to an 'unproductive' agent. Only then remaining unproductive agents are matched. The differences between the matching policies are illustrated in Figure 3.1.

The relation between patience and the optimal policies is monotone in the sense that there are two cut-off levels: When discounting is weak, the Positive Assortative Policy is optimal, for intermediate levels of patience the Provident Impatient Policy maximizes welfare, and in an impatient environment it is best to apply the Myopic Impatient Policy. The intuition for this result is perspicuous: The stronger future payoffs are discounted, the less willing is the designer to give up immediate output for the option of realizing gains from positive assortativeness in the future. Whereas the Positive Assortative Policy always respects these options, the Myopic Impatient Policy only maximizes current payoff.

An immediate insight is that positive assortative matchings are not always welfare-maximizing. This means in particular that a failure of positive assortativeness in search models triggered by agents that are 'too' impatient to wait for good matches

(2001b) and Shimer and Smith (2001a) study socially optimal policies, however, the search friction is taken as given.

may indeed be welfare-enhancing.[8]

Considered from the opposite point of view, the result tells us that for small rates of discounting the efficient matching pattern resembles the standard pattern from frictionless matching. The dynamic model approaches the static frictionless version when discounting gets negligible. This implies that the result that positive assortative matchings are efficient in static environments is robust to small dynamic frictions.

The role of the degree of complementarity of the matched agents' characteristics is closely related to the degree of patience. The stronger the complementarity, the greater are the gains from positive assortativeness. Consequently, *ceteris paribus*, for little complementarities the Myopic Impatient Policy is optimal, for intermediate levels the Provident Impatient Policy, and for strong complementarities the Positive Assortative Policy maximizes welfare.

Surprisingly, the relation between the distribution of arriving agents' types and optimal policies may be non-monotone. There are situations in which *ceteris paribus* the Positive Assortative Policy is optimal for intermediate probabilities of arriving agents to be productive but for both small and very high probabilities the Provident Impatient Policy is optimal. For illustrative purposes, consider the state in which there is one agent of either type in the market. When applying the Positive Assortative Policy, both agents are matched with the next arriving peer. When the probability of arriving agents' types takes extreme values, abstaining from the creation of mixed matches implies that one type has large expected waiting cost. The designer might be more willing to enforce the Positive Assortative Policy if he knows that arrivals of both types happen such that all agents get matched in near future.

Next, we consider the situation in which the output produced by two unproductive agents is drastically smaller than the one generated by two productive agents. If in addition complementarities are weak such that the Myopic Impatient Policy is strongly preferred to the Positive Assortative Policy, different matching policies may be welfare-maximizing. It may become optimal to store unproductive types on the market only for the purpose of matching productive agents with them immediately upon arrival and thereby avoiding losses from letting productive agents wait.[9] The reason for the optimality of these policies is the strongly heterogeneous waiting cost induced by the big differences in productivity joint with discounting.

Besides the insights gained from the design of the optimal policies, there is a technical contribution in the paper. We solve for the welfare-maximizing matching policy

[8] Examples from the search and matching literature in which supermodular output functions are not sufficient for positive assortativeness are Shimer and Smith (2000) and Smith (2006).

[9] This aspect relates to the basic thought of optimal inventory. See Arrow et al. (1951) for the fundamental thought and Whitin (1954) and Veinott Jr (1966) for early literature surveys.

using methods from dynamic programming. As the problem has discrete but infinitely many states, a guess and verify approach implies the need to check all possible deviations on infinitely many states. We develop a tool to do an involved form of induction, which we call a 'State Space Reduction'. This tool enables us to solve the problem via a guess and verify approach by only checking deviations on a small set of states. This tool might well be applicable to a broader range of problems, in particular, it might be used to solve the problem with richer types spaces.

In the second part of the paper, we treat implementability of the welfare-maximizing matching policies by an intermediary which faces agents that have private information and care only about their own matches' output. Match values are split equally. We follow Bergemann and Välimäki (2010) and consider mechanisms that satisfy 'efficient exit' and are interim incentive compatible. We show that the welfare-maximizing policy is always implementable if agents have private information about their type but the designer can observe their arrival. This holds even under the most disadvantageous information structure for implementation: Reports are public such that agents in equilibrium have all information about the set of agents in the market when arriving.[10] Note that in our setup no general implementation theorem applies, as agents' values are interdependent and types are uncorrelated in this dynamic setting. While with observable arrivals the implementation of the Provident Impatient Policy and the Myopic Impatient Policy turns out to be generally unproblematic, the possibility result is surprising concerning the Positive Assortative Policy. In static environments, positive assortativeness can be implemented using a single-crossing property with respect to each other agent's type. By this we mean that the gain from matching with a productive agent instead of an unproductive one is higher for agents which are productive themselves. However, the time friction is not only a friction to efficiency but also to the incentive constraints. When the expected time until getting matched depends on the reported type, it might be more attractive for unproductive agents to report being productive than it does for productive agents. Whenever this is the case, the Positive Assortative Policy is not implementable. However, it turns out that whenever this happens, the Positive Assortative Policy is not welfare-maximizing either.

In addition, we show that if the complementarity of the match value function is sufficiently strong or equivalently the environment is sufficiently patient, the welfare-maximizing policy can be implemented with transfers that only depend on the agent's reported type. This simple structure of payments is reminiscent of the one that implements the positive assortative matching in the static model. Thus, besides the optimal

[10] In our model this is equivalent to the notion of periodic ex post equilibrium as defined in Bergemann and Välimäki (2010).

policy also the implementation in the static matching model is robust to small dynamic frictions. We prove further that whenever this is possible, there exists a splitting rule for the match value of mixed matches such that the optimal policy can be implemented without transfers.

Finally, we address implementation when agents' arrivals are unobservable to the principal. In this case the agents' private information is two-dimensional: It consists of the type and the arrival time. Deviations from truthful reporting may, hence, consist of misreports about the type combined with strategic delays of the report about the arrival. We proof that even in this environment the optimal policy is always implementable. The contract that implements the Positive Assortative Policy when arrivals are observable is also incentive compatible with unobservable arrivals. Concerning the Provident Impatient Policy and the Myopic Impatient Policy, we crucially exploit that for their implementation the authority does not always need to elicit information about arriving agents' types.

Besides the literature on search and matching, our paper relates to further contributions that study the efficiency of positive assortativeness in dynamic matching models. Shi (2005) considers two-sided matching with a supermodular production function where matches get split after random durations. Shi, however, endogenizes one side's quality choice which turns out to make the problem of our paper uninteresting. The focus of his paper is on an additionally introduced coordination friction that cannot be overcome by the central authority. Anderson and Smith (2010) analyzes the trade-off between creating a payoff-maximizing positive assortative matching in the current period and having an advantageous distribution over characteristics in the next period. Agents can be rematched each period, but the trade-off arises as the evolution of agents characteristics depends on their current match.

Similar to our model, Baccara et al. (2015) considers a matching market with dynamically arriving agents that is organized by a benevolent central planner. There are three key differences in their setup: The market is two-sided, waiting cost are homogeneous, and each period two agents arrive. A consequence of these differences is that the analytical problem and hence both the solution technique and the optimal matching policies differ remarkably from ours. Furthermore, the authors do not consider implementation with private information. Instead, they are interested in a welfare-comparison between the outcome of a decentralized organization of the market and the socially optimal outcome.

Dynamic matching markets that are organized by a central authority are further studied in the growing literature on dynamic kidney exchange. Respective papers are Ünver (2010), Ashlagi et al. (2013), Anderson et al. (2015), Akbarpour et al. (2016), and

Ashlagi et al. (2016). The objective in these papers is to minimize waiting times and therefore maximize the number of matches respecting restrictions on feasible matches that are exogenously given on medical grounds. Opposed to this literature, we focus on maximizing total match value in an environment in which any match is feasible.

Fershtman and Pavan (2015) studies a centralized two-sided matching market in which the agent's private valuations for partners change over time. The profit-maximizing intermediary faces restrictions on re-matching agents. Besides the invariant set of agents in the market, an important difference to our study is that agents have idiosyncratic valuations for partners. First, this means that their paper is not about assortativeness, and second, values are not interdependent.

In a broader perspective, our paper adds to the literature on dynamic assignment problems. One strand of this literature considers the assignment of dynamically arriving agents to goods which are present from the beginning. Examples are Gallien (2006), Gershkov and Moldovanu (2009, 2010), Mierendorff (2015), Board and Skrzypacz (2015), Dizdar et al. (2011), and Pai and Vohra (2013). Another strand treats the assignment of dynamically arriving goods to agents that are queuing for these goods. Examples are Leshno (2015) and Bloch and Cantala (2016). The housing literature combines these two strands: Agents arrive over time and are matched with houses that get back to the market when the assigned agents have moved out. Examples are Kurino (2014), Bloch and Houy (2012), and Bloch and Cantala (2013). The housing literature and our paper share the property that both matching partners arrive over time. There are, however, two substantial differences. First, whereas in the housing literature the arriving stream of houses is determined by the allocation, in our paper it is entirely exogenous. Second, in our environment both matching partners have preferences over partners and both have private information.

Finally, we relate to a small literature that treats the implementability of welfare-maximizing social choice functions in general dynamic environments. Bergemann and Välimäki (2010) presents a dynamic VCG mechanism that implements socially optimal social choice functions in dynamic environments with private values. Liu (2014) and Noda (2016) develop dynamic versions of payment schemes found in Cremer and McLean (1985, 1988) to implement welfare-optimal social choice functions in environments with interdependent values and correlated types. Nath et al. (2015) extents the idea of Mezzetti (2004) to dynamic settings. Their payment scheme allows to implement the optimal social choice in general environments if payments are made in periods subsequent to the agent's report. As in our environment values are interdependent, types are uncorrelated and payments have to be made immediately upon reports, none of the latter results can be applied.

The rest of the paper is organized as follows: Section 3.2 presents the setup, Section 3.3 derives the welfare-maximizing matching policies, Section 3.4 addresses implementability of the policies under private information and Section 3.5 concludes.

3.2 Model

We consider agents that arrive over time to a matching market. Time is continuous, and the time horizon is infinite, $t \in [0, \infty)$. Having arrived to the market, agents remain in the market until they are matched, i.e., agents are long-lived. Agents are characterized by the tuple (θ, a), where θ is the agent's type and $a \in [0, \infty)$ is his arrival time. An agent's type reflects his productivity; he is either productive H or unproductive L, $H > L > 0$.

Arrivals are described by a Poisson process $(N_t)_{t \geq 0}$ with arrival rate λ. A Poisson process is a counting processes and thus describes discrete arrivals. The random variable N_t describes the number of arrivals up to time t. Let t_n be the time of the n-th arrival. Arriving agents' types are drawn from a Bernoulli distribution that is independent of the process $(N_t)_{t \geq 0}$ and i.i.d. across time; we denote the probability of the productive type by $p \in (0, 1)$. We refer to the process induced by $(N_t)_{t \geq 0}$ joint with the Bernoulli distributions as arrival process.

There exists a central authority, the designer, which organizes the market. Once an agent arrives, the designer may assign him to another agent that is in the market. After being assigned a partner, an agent cannot be reassigned. This could be, for example, because agents leave the market after forming a group and are thus no longer available for the designer or because they make sunk investments that are too costly to forfeit. Together agents produce a match value depending on the pair's types. Formally,

$$\begin{aligned} m : \mathbb{R}_{>0} \times \mathbb{R}_{>0} &\longrightarrow \mathbb{R}_{\geq 0}, \\ \theta_1 \times \theta_2 &\longmapsto m(\theta_1, \theta_2). \end{aligned}$$

In accordance with the literature, the match value function m is assumed to be symmetric and strictly increasing in both arguments. Given the binary type space, m can be described by three match values m_{LL}, m_{HL}, and m_{HH}, where $m_{\theta_1\theta_2} := m(\theta_1, \theta_2)$.[11] We refer to pairs where both agents have the same type as homogeneous matches; pairs of agents with different types are termed mixed matches. We assume that the match value function is supermodular, which in our setup boils down to $2m_{HL} \leq m_{HH} + m_{LL}$. Supermodularity implies that in a static model, where all agents are present simulta-

[11] As will become clear from the analysis below, one could normalize only one of these three values.

neously, positive assortative matching maximizes the sum of match values.[12] Alternatively, if we regard the two match partners as contributing to the match value, the types, interpreted as input factors, may vary from negligible degrees of complementarity to perfect complementarity. In addition, we require that the match value of the unproductive pair is not too small compared to the match value of the productive pair, $3m_{LL} \geq m_{HH}$.[13]

In the absence of additional payments, an agent's utility from a match, his premuneration value cf. Mailath et al. (2013, 2015), equals his share of the match value. In the first part of the paper, the precise share and the way it is determined, endogenously or exogenously, may be arbitrary. All agents discount future payoffs with a common discount rate $r \in (0, \infty)$.

The designer seeks to maximize the expected discounted sum of match values, i.e., the expected sum of discounted utilities. If we assume that all agents are present from $t = 0$ but only enter the market at their arrival time a, our objective corresponds to maximizing the expected sum of utilities. In yet another, less benevolent, interpretation the designer maximizes output. For a formal description of the designer's objective denote by $a^t = (a_1, a_2, \ldots, a_{N_t})$ the history of arrival times up to time t and by $\theta^t = (\theta_1, \theta_2, \ldots, \theta_{N_t})$ the corresponding history of types. Let $\varphi_s = (\varphi_s^{HH}, \varphi_s^{HL}, \varphi_s^{LL}) \in \mathbb{Z}^3_{\geq 0}$ be the action taken by the designer at time s, where φ_s^{HH} is the number of homogeneous pairs of productive agents, φ_s^{HL} is the number of mixed pairs, and φ_s^{LL} is the number of homogeneous pairs of unproductive agents formed at time s. Denote by φ^t the history of the designer's actions up to time t, $\varphi^t = \{\varphi_s\}_{0 \leq s < t}$. Altogether, a history up to time t, h^t, is given by $h^t = (t, a^t, \theta^t, \varphi^t)$. Let $\mathcal{H}^t$ be the set of all histories up to time t. A matching policy ρ is a family of functions $\rho = (\rho_t)_{t \in \mathbb{R}_{\geq 0}}$ where ρ_t is defined as

$$\begin{aligned} \rho_t : \mathcal{H}^t &\longrightarrow \mathbb{Z}^3_{\geq 0} \\ \rho_t(h^t) &\longmapsto \varphi_t. \end{aligned}$$

We write $\rho_t = (\rho_t^{HH}, \rho_t^{HL}, \rho_t^{LL})$, where $\rho_t^{\theta_1\theta_2}$ maps the history h^t into the number of $\theta_1\theta_2$-pairs created at time t. The value generated by this policy at time t after history h^t is

$$v_t^\rho(h^t) = \rho_t^{HH}(h^t) m_{HH} + \rho_t^{HL}(h^t) m_{HL} + \rho_t^{LL}(h^t) m_{LL}.$$

[12] A positive assortative matching is a pairing of all agents in the market, in which productive types pair with productive types and unproductive types pair with unproductive types.

[13] In Section 3.3.2 we analyze the case $3m_{LL} < m_{HH}$.

Denote by ϑ_t the corresponding realized value at time t. Let $x(h^t)$ and $y(h^t)$ be the number of productive and unproductive types that are still available in the market given history h^t. Formally, $x(h^t)$ and $y(h^t)$ are given by[14]

$$x(h^t) = \#\{i \mid \theta_i \in \theta^t,\ \theta_i = H\} - \sum_{\substack{s:\vartheta_s>0 \\ s<t}} (2\varphi_s^{HH} + \varphi_s^{HL}),$$

and

$$y(h^t) = \#\{i \mid \theta_i \in \theta^t,\ \theta_i = L\} - \sum_{\substack{s:\vartheta_s>0 \\ s<t}} (2\varphi_s^{LL} + \varphi_s^{HL}).$$

We call a matching policy ρ feasible if it never matches more agents than available in the market, i.e., for all t and h^t,

$$\begin{aligned} 2\rho_t^{HH}(h^t) + \rho_t^{HH}(h^t) &\le x(h^t) \\ 2\rho_t^{LL}(h^t) + \rho_t^{HL}(h^t) &\le y(h^t). \end{aligned}$$

The designer's expected payoff from ρ can be written as

$$\mathbb{E}[\int_{s=0}^{\infty} e^{-rs} v_s^{\rho}(h^s)\ \mathrm{d}s], \tag{3.1}$$

where the expectation is taken over histories with respect to the probability measure induced by ρ and the arrival process. We refer to a feasible matching policy ρ that maximizes (3.1) as welfare-maximizing. Define the expected payoff at time t after history h^t, i.e., the continuation value, from ρ

$$V_\rho(h^t) = \mathbb{E}[\int_{s=t}^{\infty} e^{-r(s-t)} v_s^{\rho}(h^s)\ \mathrm{d}s | h^t], \tag{3.2}$$

where the expectation is taken with respect to the probability measure induced by the arrival process and $(\rho_s)_{s\ge t}$. We call a feasible matching policy ρ optimal if it maximizes (3.2) for any time t and any history h^t. Therefore, an optimal matching policy is a refinement of a welfare-maximizing matching policy, requiring maximization of the designer's payoff not only ex ante but after any path of play. In particular, every optimal matching policy is welfare-maximizing.

Recursive Formulation. As a first step, we derive a recursive formulation. Start by noting that at every point in time, after any history, x and y are a sufficient statistic to summarize the maximization problem: Firstly, feasibility depends only on x and y. Secondly, the arrival process is independent of the history. This is a consequence of

[14] Here we slightly abuse notation. By $\theta_i \in \theta^t$ we refer to the components of the vector θ^t.

the memorylessness of the Poisson process, the independence of interarrival times, and the independence of the Bernoulli distributions from the arrival times. Thus, the state of our problem is (x, y) with state space $S := \{(x, y) \in \mathbb{N}^2_{\geq 0} |\ x \geq 0, y \geq 0, x + y > 0\}$.[15]

In particular, time is not part of the state. Thus, there exists an optimal policy that matches only upon agents' arrival: Whenever an action is optimal on a given state, the policy that takes this action on this state is optimal and only matches upon arrival.

A policy that conditions the action, for some state (x, y), on variables different from x and y, e.g. time, is only optimal if all actions that the policy might take in that state are optimal. Vice versa, if there is a unique optimal policy among those that match only upon arrival, then at every state exactly one action is optimal which implies that this policy is the unique optimal policy in the set of all policies.[16]

In the following, we restrict attention to feasible matching policies that condition solely on (x, y). As we will prove generic uniqueness of the optimal policy, the restriction will turn out to be without loss, and we will obtain a sharp characterization of all optimal matching policies. Accordingly, we suppress the dependence on time and history in the notation for continuation values and policies and write them merely as a function of x and y.

It is convenient to define the expected discount factor until the arrival of the next agent

$$\delta := \mathbb{E}[e^{-rt_1}] = \frac{\lambda}{\lambda + r}. \tag{3.3}$$

Building upon the preceding insights, we are thus led to study the Bellman equation

$$\begin{aligned} V(x, y) = \max_{\varphi^{HH},\, \varphi^{HL},\, \varphi^{LL}} \ & \{\varphi^{HH} m_{HH} + \varphi^{HL} m_{HL} + \varphi^{LL} m_{LL} \\ & + \delta \left[pV(x - 2\varphi^{HH} - \varphi^{HL} + 1, y - 2\varphi^{LL} - \varphi^{HL}) \right. \\ & + \left. (1 - p)V(x - 2\varphi^{HH} - \varphi^{HL}, y - 2\varphi^{LL} - \varphi^{HL} + 1) \right] \} , \\ \text{subject to:} & \\ & 2\varphi^{HH} + \varphi^{HL} \leq x, \quad 2\varphi^{LL} + \varphi^{HL} \leq y, \quad \forall x, y \in S. \end{aligned} \tag{3.4}$$

Given the recursive formulation of the designer's problem (3.4), the difference between optimal matching policies and welfare-maximizing matching policies has an intuitive interpretation. Every matching policy defines, together with the arrival process,

[15] Note, that this formulation implies that a state represents the set of agents in the market including the new arrival.

[16] Whenever the optimal policy is unique, there does not exist a stochastic policy which yields a higher payoff.

a Markov chain over states in S. As matching policies are deterministic, there are two possible successors for each state with transition probabilities p and $1-p$. Depending on the policy, this Markov chain might have a recurrent set. If the Markov chain has a recurrent set and the market is initially, i.e., at the point in time at which the policy is enforced, in a state within the recurrent set, then no state outside the recurrent set is reached with positive probability. A policy is then welfare-maximizing if it maximizes the designer's expected payoff at all states in the recurrent set. The criterion welfare maximization does not restrict policies on states outside of the recurrent set. An optimal matching policy maximizes the designer's expected payoff at every state in S. In the following, we solve for optimal policies.[17] Firstly, this allows us to solve the designer's problem for any initial state of the market. Secondly, an optimal policy maximizes the designer's payoff even if in the past several agents arrived at the same point in time. Also, as we will see below, focusing on optimal policies elucidates the economic trade-offs connected to welfare maximization and their resolution.

3.3 Optimal Policy

We start by considering the welfare maximization problem under complete information. This means at the point in time of an agent's arrival, the authority observes both the agent's arrival and his type. Incentive constraints arising from the agents' informational advantage are added in Section 3.4.

3.3.1 The Regular Case

By means of the assumption $m_{HH} \leq 3m_{LL}$ we have guaranteed that the match value of a pair of productive agents is not extremely high compared to the output generated by two unproductive agents. In this case there are three important policies to be considered, which are portrayed in the following.

Definition 1. ***Positive Assortative Policy***
The Positive Assortative Policy creates in each state (x, y) the maximal number of homogeneous pairs of both kinds of agents. Mixed matches are never created.

The matching pattern produced by this policy is positive assortative. In the absence of discounting, this policy maximizes the overall match value. Whenever there is exactly one agent of either kind left in the market, the policy lets both agents wait despite waiting costs.

[17] As will become clear from the analysis below, solving for optimal policies is indispensable even if we are only interested in welfare-maximizing policies. See the outline of the proof of Theorem 1 for an explanation.

Definition 2. ***Provident Impatient Policy***
The Provident Impatient Policy creates in each state (x, y) the maximal number of homogeneous pairs of both kinds of agents. If both x and y are uneven, one mixed match is created in addition.

The matching pattern produced by this policy is not positive assortative but contains a relatively large number of homogeneous matches. Only when there is exactly one mixed pair left in the market after all homogeneous pairs have been created, the mixed match is created as well. In that sense homogeneous matches are given priority over mixed matches. However, the policy never lets two agents wait.

Definition 3. ***Myopic Impatient Policy***
The Myopic Impatient Policy creates in each state (x, y) the maximal number of pairs of productive agents. If x is uneven and $y \geq 1$, one mixed match is created. The maximal number of pairs from the pool of remaining unproductive agents is formed.

The matching pattern produced by this policy is not positive assortative and contains a relatively little number of homogeneous matches. Again, the policy first creates the maximal number of productive matches. If afterwards there is a productive agent left, it is matched to an unproductive agent with priority. Only then pairs of unproductive agents are formed. In that sense productive agents are given priority over homogeneous matches. The policy acts entirely myopic in the sense that it maximizes the sum of immediate match values. The three matching policies are illustrated in Figure 3.1 in the introduction.

In order to state the main theorem, we define two functions that partition the space of parameter constellations $(p, \delta, m_{HH}, m_{LL}, m_{HL})$. Denote

$$m_{HL}^1 := m_{HH} \frac{\delta p}{1 - \delta(1 - 2p)} + m_{LL} \frac{\delta(1 - p)}{1 + \delta(1 - 2p)} \tag{3.5}$$

and

$$m_{HL}^2 := m_{HH} \frac{\delta p}{1 - \delta(1 - 2p)} + m_{LL} \frac{1 - \delta(1 - p)}{1 - \delta(1 - 2p)}. \tag{3.6}$$

Note that for any $m_{HH}, m_{LL}, m_{HL}, \delta$ and p it holds that $m_{HL}^1 < m_{HL}^2$.

Theorem 1. *For any given parameter constellation $(p, \delta, m_{HH}, m_{LL}, m_{HL})$ one of three matching policies is optimal:*

If $m_{HL} \leq m_{HL}^1$, the Positive Assortative Policy is optimal.

If $m_{HL} \in [m_{HL}^1, m_{HL}^2]$, the Provident Impatient Policy is optimal.

If $m_{HL} \geq m_{HL}^2$, the Myopic Impatient Policy is optimal.

The optimal policy is unique if $m_{HL} \notin \{m_{HL}^1, m_{HL}^2\}$.

In the following we first elaborate on the statement of Theorem 1. Then we provide an economic intuition for the result and the underlying trade-offs. The treatment of the theorem is completed with an outline of the proof.

Immediate implications. First, Theorem 1 is surprisingly simple in the sense that only three policies are generically optimal. In particular, it is generically not optimal to let two unproductive agents wait when there is nobody else in the market. As we show in the extension to this section, this crucially hinges on the regularity assumption $3m_{LL} \geq m_{HH}$.

The second comment addresses the difference between optimal and welfare-maximizing policies. The Provident Impatient Policy and the Myopic Impatient Policy do not differ on states which are reached once one of the two policies is established. Once they are installed, the total number of agents in the market never exceeds two and both policies form a pair whenever possible, which is each second arrival. We call the set of policies which always form a pair as soon as two agents are present the set of *Impatient Policies*. For a discussion why to focus on optimal policies we refer the reader to Section 3.2.

Intuition. There are two major driving forces. We refer to the first one as the *gain of assortative matching*. The value of matching two productive agents and two unproductive agents is higher than the value of two mixed matches. In order to achieve positive assortative matchings, it might be necessary to accumulate agents. This is the case whenever pairing agents in the order of their arrivals induces creating mixed pairs. Hence, the gain of assortative matching is effective in favor of waiting with agents in the market. The second force is the *loss from deferring matches*. Having agents wait in the market is costly because of discounting. This force is effective in favor of creating match values early; this may include creating mixed pairs.

We first comment on the influence of complementarity of the match value function on optimal policies. Given all other parameters, the match value of a mixed pair characterizes the degree of supermodularity of the match value function in the agents' types. The larger m_{HL}, the more are the partners' types substitutes, and perfect substitutability is achieved at the upper bound $m_{HL} = 1/2(m_{HH} + m_{LL})$. The theorem states that the relation between the match value of a mixed pair and the optimal policies is monotone in the sense that there are two cut-off levels: When complementarity is strong, the Positive Assortative Policy is optimal, for intermediate levels the Provident Impatient Policy is optimal, and for high degrees of substitutability it is optimal to apply the Myopic Impatient Policy. The intuition is perspicuous: The higher the value of mixed matches, the smaller is the gain from assortative matching. As all other parameters are kept fixed, the loss from deferring matches is unchanged. Hence, with increasing match values for mixed pairs the losses from deferring matches more and more outweigh the gains from positive assortativeness. Whereas the Positive Assortative Policy fully realizes gains from positive assortativeness, the Myopic Impatient Policy solely prevents losses from deferring matches.

Second, for given match values we discuss how the choice of optimal policies depends on the impatience, represented by the discount factor δ. Recall that δ is a compound expression of the arrival rate and the discounting rate. The discount factor increases when the discount rate decreases, or when the arrival rate increases. The arrival rate of agents can be interpreted as the size of the market. We obtain the following consequence of Theorem 1:

Corollary 1. *There exist two functions δ^1 and δ^2 that map any parameter constellation $(p, m_{HH}, m_{LL}, m_{HL})$ into $[0,1]$ with $\delta^1 > \delta^2$ such that:*
If $\delta \geq \delta^1$, the Positive Assortative Policy is optimal.
If $\delta \in [\delta^2, \delta^1]$, the Provident Impatient Policy is optimal.
If $\delta \leq \delta^2$, the Myopic Impatient Policy is optimal.

We call δ^1 and δ^2 cut-off levels. The existence of two cut-off levels follows from showing that $\partial m^1_{HL}/\partial\delta \geq 0$ and $\partial m^2_{HL}/\partial\delta \geq 0$ independent of the specific choice of parameters. This means that when fixing all parameters but δ, the monotonicity in m_{HL} carries over to monotonicity in δ. $1 \geq \delta^1 \geq \delta^2 \geq 0$ implies that given any parameter constellation $(p, m_{HH}, m_{LL}, m_{HL})$, for each of the three policies there exists a δ such that the policy is optimal.

Implication 1: The relation between patience and the optimal policies is also monotone. The stronger discounting, the greater are the losses from deferring matches. Hence, the stronger future payoffs are discounted, the less willing is the designer to give up immediate output for the option of realizing gains from positive assortativeness in the future.

Implication 2: The result tells us that for little rates of discounting the efficient matching pattern resembles the standard pattern from frictionless matching. More precisely, note that for $\delta = 1$ holds $m^1_{HL} = 1/2(m_{HH} + m_{LL})$, which is the maximum value for m_{HL}, and whenever m^1_{HL} is interior, then δ^1 is interior as well. m^1_{HL} and thus δ^1 is interior for all $p, \delta \neq \{0,1\}$. From an economic perspective, the dynamic model approaches the static frictionless model when discounting gets negligible. Combining the above statements, this implies that the efficiency result of positive assortative matchings in static environments is robust to small dynamic frictions.

Third, we examine the role of the distribution of arriving agents' types represented by p. Surprisingly, there are situations in which *ceteris paribus* the Provident Impatient Policy is optimal for small and high probabilities but for intermediate values of p the Positive Assortative Policy is optimal. This non-monotonicity arises from the presence of two different effects. For an illustration, consider the state in which there is one agent of either type in the market. First, a higher probability of productive arrivals decreases the expected time until the next productive arrival. For the productive agent in the market this means that the likelihood that he can be matched with a productive peer in the near future increases. This raises the expected value of letting the productive agent wait instead of creating a mixed match. This effect implies that the attractiveness of the Positive Assortative Policy is increasing in p, implying that the cut-off δ^1 should decrease in p. However, there is an opposite second effect originating from the unproductive agent. For high levels of p abstaining from the creation of mixed matches implies a large expected waiting time for the unproductive agent. This implies that the value the unproductive agent contributes to is strongly discounted. In isolation, δ^1 should increase in p . In particular, if the values m_{LL} and m_{HH} do not differ much, the second effect is strong enough such that as p approaches one, δ^1 increases in p. Then δ^1 has its minimum at some interior value, decreases for small p, and increases for high values of p. In this case the designer is more willing to enforce the Positive Assortative Policy because he knows that arrivals of both types happen regularly such that all agents get matched in near future.

The dependence of the optimal matching policy on p and δ is graphically illustrated for match values that represent three canonical match value functions: The case of perfect complements (Figure 3.2), the multiplicative case (Figure 3.3), and the case of (almost) perfect substitutes (Figure 3.4). The red line depicts δ_1, the boundary between the parameter regions in which the Positive Assortative Policy and the Provident Impatient Policy are optimal. The blue line depicts δ^2, the boundary between the Provident Impatient Policy and the Myopic Impatient Policy. As mentioned, for large values of δ the Positive Assortative Policy is optimal, for intermediate values the Provident Impatient Policy, and for small values the Myopic Impatient Policy.

In the case of perfect complements, Figure 3.2, the match value of a mixed match equals the match value of a homogeneous match of unproductive agents. A consequence is that the Provident Impatient Policy dominates the Myopic Impatient Policy in the sense that the expected welfare from the Provident Impatient Policy is weakly higher on each possible state. The reason is that, starting on a given state, both policies generate the same sum of match values in the first period using the same number of agents. However, the Myopic Impatient policy uses weakly more productive agents for

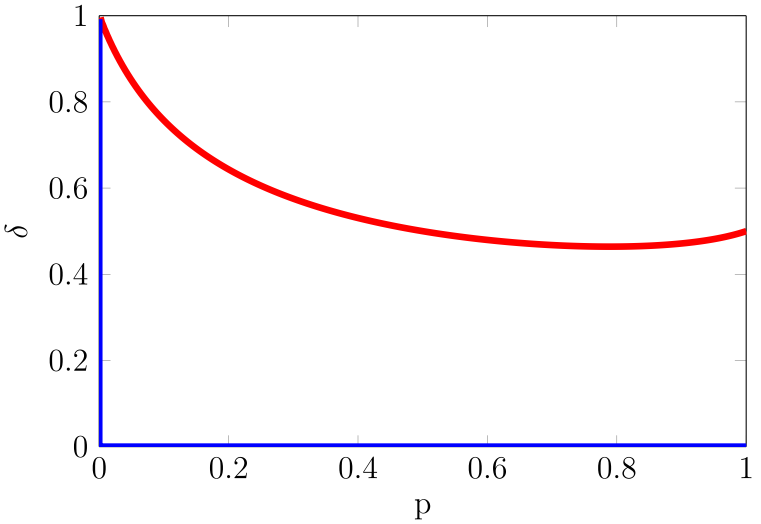

Figure 3.2: Perfect complements: $m(H,L) = \min\{H,L\}$; Here $H = 3$ and $L = 1$. The red line depicts δ^1; the blue line depicts δ^2.

it than the Provident Impatient Policy.[18] Furthermore, as the match value function is strongly supermodular, the gain of assortative matching is high, and the parameter region in which the Positive Assortative Policy is optimal is large.

The case of multiplication, illustrated in Figure 3.3, is regularly used to model complementarities in the match value function. All three policies from Theorem 1 are optimal on a parameter region with positive measure. Note that for p close to one, the Positive Assortative Policy is never optimal - the designer always wants to avoid leaving unproductive agents unmatched.

In the case of perfect substitutes, approximated by Figure 3.4, there are no complementarities in the match value function. As there are no gains from assortative matching, the parameter regions, for which the Positive Assortative Policy and the Provident Impatient Policy are optimal, vanish.[19]

Outline of the proof. The problem is solved using a Guess & Verify method. We guess three candidate matching policies, which turn out to be optimal on some subset of the parameter space. As a side product of the verification, we obtain both the precise parameter region where the verified policy is optimal and its (generic) uniqueness on that parameter region. It turns out that the respective parameter regions of the three matching policies constitute a partitioning of the parameter space. The challenging

[18] Consider, for example, the state $(1,2)$. The Provident Impatient Policy matches the two unproductive agents, whereas the Myopic Impatient Policy creates a mixed match. Both match values are equal, but the Provident Impatient Policy leaves a productive agent in the market whereas the Myopic Impatient Policy leaves an unproductive agent in the market.

[19] From $\partial m^1_{HL}/\partial\delta \geq 0$ and $\partial m^2_{HL}/\partial\delta \geq 0$ joint with $m^1_{HL} = m^2_{HL} = 1/2(m(H,H) + m(L,L))$ at $\delta = 1$ follows that when $m(H,L)$ approaches $1/2(m(H,H) + m(L,L))$, $m(H,L) > m^1_{HL}, m^2_{HL}$ for almost all δ.

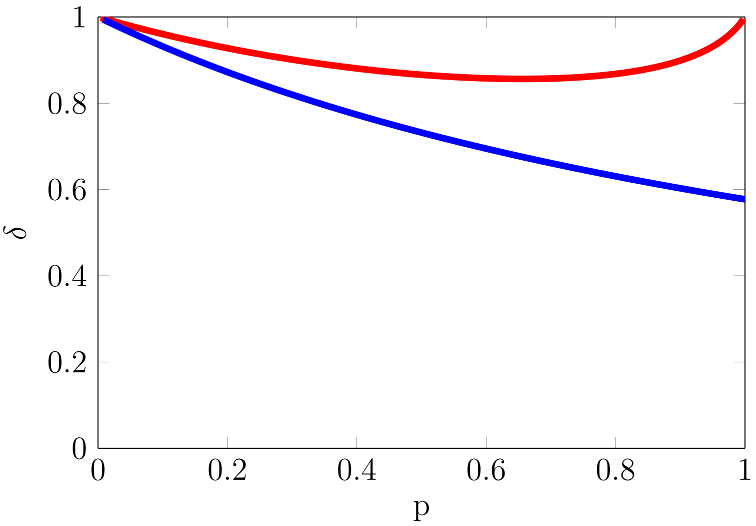

Figure 3.3: Product: $m(H, L) = H \cdot L$; Here $H = \sqrt{3}$ and $L = 1$.

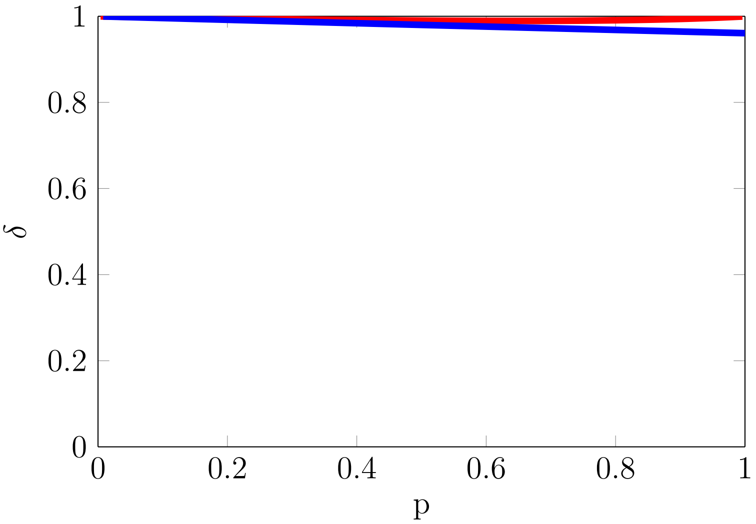

Figure 3.4: Almost perfect substitutes: Almost $m(H, L) = H + L$; Here $H = 1.5$, $L = 0.5$ and $m(H, L) = 1,98 < H + L$.

step is the verification, as it involves checking deviations on a discrete but infinite state space. We cope with the situation by developing a procedure we call 'State Space Reduction'.

Guess. We guess the Positive Assortative Policy, the Provident Impatient Policy, and the Myopic Impatient Policy as candidate matching policies. Remember that joint with the arrival process a matching policy defines a Markov chain over future states S. For all three candidates, the induced Markov chains jumps to a finite recurrent set after the first period. This means that (apart from the initial state) only a finite number of states realize. For the Provident Impatient Policy and the Myopic Impatient Policy only the five states $(1,0),(0,1),(0,2),(2,0)$, and $(1,1)$ can occur. For the Positive Assortative Policy the recurrent set is $\{(1,0),(0,1),(1,1),(0,2),(2,0),(2,1),(1,2)\}$.[20]

As these recurrent sets are finite, the value function of each candidate at every state in the respective recurrent set can be computed as the solution to a finite system of equations. The reason is that the value function at an element in the recurrent set only depends on payoffs generated from states in that set and the transition probabilities. For illustrative reasons, we state here the system of equations for the Myopic Impatient Policy. Denote the corresponding values by V_{MIP}:

$$\begin{aligned}
V_{MIP}(0,1) =&\delta\big[pV_{MIP}(1,1) + (1-p)V_{MIP}(0,2)\big]\\
V_{MIP}(1,0) =&\delta\big[pV_{MIP}(2,0) + (1-p)V_{MIP}(1,1)\big]\\
V_{MIP}(0,2) =&m_{HH} + \delta\big[pV_{MIP}(1,0) + (1-p)V_{MIP}(0,1)\big]\\
V_{MIP}(2,0) =&m_{LL} + \delta\big[pV_{MIP}(1,0) + (1-p)V_{MIP}(0,1)\big]\\
V_{MIP}(1,1) =&m_{HL} + \delta\big[pV_{MIP}(1,0) + (1-p)V_{MIP}(0,1)\big].
\end{aligned} \tag{3.7}$$

The value functions on states outside the respective recurrent sets can then easily be computed as they only differ from the ones on the recurrent set by the payoff generated in the starting period. We illustrate this by a short example.

[20] Note that the state represents the set of agents in the market including the new arrival.

Example 1. *The example shows how the value function of the Myopic Impatient Policy at state* $(6,5)$ *can be written as a function of the immediate payoff and the value at a state in the recurrent set:*

$$\begin{aligned} V_{MIP}(6,5) =& 3m_{HH} + 2m_{LL} + \delta\big[pV_{MIP}(1,1) + (1-p)V_{MIP}(0,2)\big] \\ =& 3m_{HH} + 2m_{LL} + \delta V_{MIP}(0,1). \end{aligned}$$

Policies that differ only on states outside the recurrent set have the same value function at states inside the recurrent set and the recurrent sets coincide. In particular, the values of the Provident Impatient Policy (V_{PIP}) equal those of the Myopic Impatient Policy on the recurrent set. If the initial state lies in the recurrent set, a policy is welfare-maximizing if it maximizes the value on each state in the recurrent set. Any policy that differs outside of the recurrent set is welfare-maximizing as well.[21]

Verification. Verifying the optimality of the three candidates can potentially be cumbersome. We describe the procedure that we apply to all three candidate policies. Fix one candidate. In the previous step, we determined the value on each state. The verification consists of showing for any state that following the policy is better than deviating on the particular state and subsequently following the candidate solution. By the principle of optimality, if all conditions are checked, we have shown both the matching policy, defined as a course of actions, and the associated value function to be optimal. When following the candidate is strictly better than any deviation on any state, the optimal policy is unique.

The difficulty in the verification is that there is a large number of potential deviations to be checked: First, the number of states on which deviations have to be checked is (countably) infinite; second, on states with many agents in the market there is a large number of deviations possible.

This problem can not be avoided by restricting attention to welfare-maximizing policies that start with the very first arrival. Even though in the latter case welfare maximization demands optimality only on a finite set of states, the verification still demands guessing the optimal policy on each possible state. The reasoning is as follows: On each state in the recurrent set, applying the candidate policy must be better than any alternative action. These alternative actions might, however, lead to states outside the recurrent set as illustrated in Example 2.

[21] For the difference of optimality and welfare maximization see the model section.

Example 2. *The example illustrates how checking deviations from the Myopic Impatient Policy at state* $(1,1)$*, a state in the respective recurrent state, involves the value on states* $(2,1)$ *and* $(2,1)$*, which are not in the recurrent set for that candidate. The (only possible) deviation considered here is to not create the mixed pair:*

$$\begin{aligned} V_{MIP}(1,1) =& m_{HL} + \delta\big[pV_{MIP}(1,0) + (1-p)V_{MIP}(0,1)\big] \\ >& \delta\big[pV_{MIP}(2,1) + (1-p)V_{MIP}(1,2)\big]. \end{aligned}$$

In order to evaluate whether the value associated with a deviation is strictly lower than the candidate course of action, we must make statements about the upper bound of the value when being in this 'off-path' state. As we want to determine for which parameter configurations precisely the candidate policy is optimal, we need to know exactly the maximal value of the 'off-path' state. Therefore, we need to know what the optimal policy does on that state. In order to find the optimal policy on that 'off-path' state we guess it and again check deviations, which may lead to more 'off-path' states. As there is always the possibility of not matching anything, any state in the state space is reached by the procedure. Hence, it is necessary to set up a candidate that is optimal and verify it.

We tackle the problem of checking the large number of deviations by a proof strategy that involves what we call 'State Space Reduction'. The State Space Reduction is an elaborate induction argument and is the crucial step of the proof. We believe that the concept of the 'State Space Reduction' can be used to address comprehensive maximization problems, in particular the related problem with more than two types of agents. The State Space Reduction works as follows: Instead of checking deviations on each state, we set up a number of general statements. On an arbitrary state, these statements identify deviations, which are not optimal, given the candidate policy will be continued in the following states and given the candidate policy is optimal on smaller states.[22] We call these statements 'principles' that specify which kind of deviations do not have to be considered. Then we identify states on which these principles capture every possible deviation. We show that there is only a finite number of states including the smallest ones, on which the principles do not capture every possible deviation. When explicitly showing that on this finite set there are no profitable deviations, we have shown that there are no profitable deviations at all. Note, that this set of states does not equal the recurrent set and also differs between the Provident Impatient Policy and the Myopic Impatient Policy. To state the principles, we use one further definition.

[22] State (x,y) is smaller than state (x',y') if and only if $x \leq x'$ and $y \leq y'$ with at least one inequality being strict.

Definition 4. *A policy ρ is consistent iff for any state $(x, y) \in S$ and any $(\theta_1, \theta_2) \in \Theta^2$ holds: $\rho_{\theta_1\theta_2}(x, y) > 0 \Rightarrow \rho_{\theta_1\theta_2}(x - \mathbb{1}_{\{\theta_1=L\}} - \mathbb{1}_{\{\theta_2=L\}}, y - \mathbb{1}_{\{\theta_1=H\}} - \mathbb{1}_{\{\theta_2=H\}}) = \rho_{\theta_1\theta_2}(x, y) - 1$ and $\rho_{\theta_3\theta_4}(x - \mathbb{1}_{\{\theta_1=L\}} - \mathbb{1}_{\{\theta_2=L\}}, y - \mathbb{1}_{\{\theta_1=H\}} - \mathbb{1}_{\{\theta_2=H\}}) = \rho_{\theta_3\theta_4}(x, y)$ for all $(\theta_3, \theta_4) \notin \{(\theta_1, \theta_2), (\theta_2, \theta_1)\}$.*

The meaning of consistency is best illustrated by an example: Suppose on a given state the policy creates matches with at least one pair of productive agents amongst them, $\theta_1 = \theta_2 = H$. Then, on the state with two productive agents less, the policy creates the same matches except for one pair of productive agents. This definition is used in order to formulate the first principle. Each of our candidates is consistent.

Lemma 1. ***(Principle 1)*** *Assume that the candidate policy is consistent. Then in every state, deviations that form a pair that is also formed under the candidate policy, do not have to be checked.*

For example, consider a consistent candidate policy and a state on which it forms a homogeneous pair of productive types. On that state no deviations have to be considered that also match two productive agents.

The reason is based on two observations. First, note that the match value of the pair that is created under the candidate and the deviation can be canceled out from the inequality that corresponds to checking the deviation. Second, due to consistency, the candidate policy creates the same matches except for one pair of productive agents on the state with two productive agents less. Combining the two observations, the deviation is not profitable given there was no profitable deviation on the state with two productive agents less. The same holds for homogeneous pairs of unproductive agents and mixed pairs. The following example illustrates this point:

Example 3. *The example shows how a deviation from the Positive Assortative Policy as in Principle 1 on state $(6, 5)$ can be traced back to a deviation on a smaller state. Denote V_{PAP} the value under the Positive Assortative Policy. The deviation considered matches exactly one pair of productive agents:*

$$\begin{aligned}
V_{PAP}(6,5) =& 3m_{HH} + 2m_{LL} + \delta\big[pV_{PAP}(1,1) + (1-p)V_{PAP}(0,2)\big] \\
>& m_{HH} + \delta\big[pV_{PAP}(5,5) + (1-p)V_{PAP}(4,6)\big] \\
\Leftrightarrow \qquad & 2m_{HH} + 2m_{LL} + \delta\big[pV_{PAP}(1,1) + (1-p)V_{PAP}(0,2)\big] \\
>& \delta\big[pV_{PAP}(5,5) + (1-p)V_{PAP}(4,6)\big] \\
\Leftrightarrow \qquad & V_{PAP}(4,5) > \delta\big[pV_{PAP}(5,5) + (1-p)V_{PAP}(4,6)\big].
\end{aligned}$$

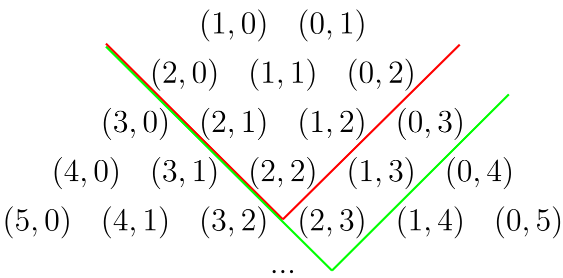

Figure 3.5: This graph represents the state space. It visualizes Lemmas 4, 5 and 6. Deviations on states below the lines do not have to be verified. The red line corresponds to Lemma 4 and 5, the green line to Lemma 6.

Lemma 2. ***(Principle 2)*** *Assume that the candidate policy is consistent. Consider a state and a deviation on it that leaves two agents with types θ_1 and θ_2 unmatched. If after the following arrival the candidate policy creates a pair (θ_1, θ_2) independent of the arriving agent's type, this deviation does not have to be checked.*

For example, consider a deviation which lets two productive types in the market. The deviation does not have to be checked, if in the next period (upon the next arrival) a match of two productive agents is created independently of the type of arrival. The reason is that a better deviation exists. The more profitable deviation equals the excluded deviation except that the match, which would be made later for sure, is created immediately. Because of discounting, this deviation has a higher value. It suffices to check against the best deviation instead of all deviations.

Lemma 3. ***(Principle 3)*** *In every state, deviations that create more than one mixed pair do not have to be considered.*

The reason is similar to the one of Principle 2. There exists another deviation which is more profitable. The more profitable deviation exploits the supermodularity of the match value function and creates one homogeneous pair of either type instead of two mixed pairs.

In the next step, we apply the principles to all three candidate matching policies. For each candidate we identify the set of states for which all deviations can be excluded.

Lemma 4. *When the Positive Assortative Policy is the candidate, there is no need to verify deviations on all states that contain more than two agents of the same type.*

The proof consists of applying Principles 1 to 3 to the Positive Assortative Policy and then combining them. By Principle 1, deviations that create a homogeneous pair do not have to be checked. By Principle 2, deviations that leave more than one agent of the same type in the market do not have to be checked. The application of Principle 3 to the Positive Assortative Policy is straightforward.

Finally, we combine the applications of the principles: Consider a state with three productive types. Principle 1 implies that there is no need to consider deviations that match two productive types. Principle 2 states that we do not need to treat deviations that leave two or more productive types unmatched. The only deviations left to consider match two or more productive types with unproductive ones. For those deviations, Principle 2 applies. The proof for unproductive agents is analogous.

Lemma 5. *When the Provident Impatient Policy is the candidate, there is no need to verify deviations on all states that contain more than two agents of the same type.*

The application of the three principles to the Provident Impatient Policy follows similar thoughts, even though there are differences. There are states in which the candidates do not create homogeneous matches even though this is possible and in addition there are states in which mixed matches are created.

Lemma 6. *When the Myopic Impatient Policy is the candidate, there is no need to verify deviations on all states that contain more than two productive agents or more than three unproductive agents.*

When the Myopic Impatient Policy is the candidate solution, the sets of remaining states that are to be verified one by one is slightly larger than for the other two candidates. The reason is that if two unproductive agents stay in the market, it might happen that one of them is matched with a productive arrival in the next period. Hence, the statement that two unproductive agents get matched in the next period anyways if they are not matched, does not hold.

The deviations on the remaining states are verified one by one. Some deviations are unprofitable only under certain conditions on parameters. These conditions define the region of the parameter space in which a candidate is optimal. It turns out that these regions constitute a partitioning of the entire parameter space. The condition $m_{HL} \leq m_{HL}^1$ ensures that creating the mixed match in $(1, 1)$ is unprofitable if the candidate is the Positive Assortative Policy. $m_{HL} \geq m_{HL}^1$ ensures that creating the mixed match in $(1, 1)$ is profitable if the candidate is the Provident Impatient Policy. m_{HL}^2 takes the corresponding role for the question whether to create the homogeneous match or the mixed match in $(1, 2)$: If the candidate is the Provident Impatient Policy, the condition $m_{HL} \leq m_{HL}^2$ ensures that the homogeneous match is optimal, and if the candidate is the Myopic Impatient Policy, the condition $m_{HL} \geq m_{HL}^2$ ensures that the mixed match is value-maximizing.

3.3.2 Extension: Non-regular Case

In this extension we drop the assumption $m_{HH} \leq 3m_{LL}$. This means we allow for extreme differences in the productivity of agents. When the value of unproductive agents is very low compared to the value of productive agents, new matching policies can be optimal. Optimality sometimes requires two unproductive agents to wait in the market, when nobody else is in the market. Therefore we need to define a new class of matching policies.

Definition 5. ***Matching Policy*** $\mathcal{P}_k$
The Matching Policy $\mathcal{P}_k$ *creates in each state* (x, y) *the maximal number of pairs of productive agents. If* x *is uneven and* $y \geq 1$*, one mixed matches is created. If a mixed match is created, the maximal number of pairs from the pool of* $\max\{0, y-1-k\}$ *unproductive agents is formed. If no mixed match is created, the maximal number of pairs from the pool of* $\max\{0, y-k\}$ *unproductive agents is formed.*

Policy $\mathcal{P}_k$ has similarities to the Myopic Impatient Policy. The difference is that less than the maximum number of homogeneous matches of unproductive agents is created. When there are only unproductive agents in the market, the policy always keeps at least k of them in the market. For example, Policy $\mathcal{P}_1$ creates no match in state $(0, 2)$.

Proposition 1. *For any given* m_{LL}, m_{HH} *such that* $m_{HH} > 3m_{LL}$*, there exist parameter constellations* (p, δ, m_{HL}) *for which matching policy* $\mathcal{P}_1$ *is optimal.*

In order to proof Proposition 1 we apply the same strategy as for proving Theorem 1. An application of the three principles reduces the verification to a finite set of states on which deviations are verified by hand.

At first glance it might be surprising that it can be optimal to abstain from creating the homogeneous match but to let two unproductive agents wait in the market. To gain intuition for this result, we reconsider the two basic effects in the maximization problem. Consider a situation in which the gain of assortative matching is very small such that the sorting into matches is not decisive. The effect of the loss from deferring matches is then to match agents early with priority given to productive agents. The implication, however, requires careful consideration: Strong discounting does not only make it attractive to match agents which are currently present in the market. It also provides incentives to lay the foundation to quickly match productive agents which arrive in the future. The latter can be achieved by storing an unproductive agent in the market such that a productive agent that arrives in the future can be matched immediately upon arrival. If the value of this availability exceeds the cost of deferring the match of two unproductive agents, it is optimal to abstain from creating a match in order to keep an unproductive agent as stock in the market. Proposition 1 states that there are parameter constellations for which this does happen.

Policy $\mathcal{P}_1$ is optimal when complementarities are low, the differences in the productivity of agents are large, arrivals of productive agents are likely, and discounting is intermediate. A prerequisite for Policy $\mathcal{P}_1$ to be optimal is that there is a strong preference to match a single productive agent with an unproductive agent even if there are two unproductive agents in the market. Therefore, the gain of assortative matching and hence complementarities must be small. Under Policy $\mathcal{P}_1$, two unproductive types incur waiting costs in order to save the waiting cost of a potentially arriving productive type in the future. This is optimal if the productive agent's loss from waiting significantly exceeds the unproductive agents' loss, which is the case only for strong differences in productivity. Furthermore, letting two unproductive agents wait in the market is only optimal if the probability that it pays off, namely that a productive agent arrives in the next period, is sufficiently high. Finally, Policy $\mathcal{P}_1$ is optimal for intermediate values of δ. When discounting is very little, there is no desire to create mixed matches anyways and clearly Policy $\mathcal{P}_1$ is not optimal. The smaller δ, the higher is the value of having unproductive agents available when an productive agent arrives. This dominates the increased cost of accumulating unproductive agents. However, when discounting is very strong, Policy $\mathcal{P}_1$ is not optimal either: The designer does not care about the option to match productive agents earlier, because the option only increases payoffs in the strongly discounted future.

In Figure 3.6 we fix match values such that for each possible (p, δ) one of the policies listed in Theorem 1 or the Policy $\mathcal{P}_1$ is optimal. The figure illustrates the respective parameter regions (p, δ). The red and the blue line are δ^1 and δ^2 as before. The black line depicts the boundary between the parameter regions on which the Myopic Impatient Policy and Matching Policy $\mathcal{P}_1$ are optimal. Matching Policy $\mathcal{P}_1$ has no boundary to the Provident Impatient Policy. Policy $\mathcal{P}_1$ is optimal for large values of p.

Policy $\mathcal{P}_1$ is optimal for low levels of complementarity. Corollary 2 shows that the monotonicity of optimal policies with respect to m_{HL} extends to Matching Policy $\mathcal{P}_1$.

Corollary 2. *If $m_{HH} > 3m_{LL}$, there exist two continuous functions m^3_{HL} and m^4_{HL} that map any parameter constellation $(p, \delta, m_{HH}, m_{LL})$ into $\mathbb{R}_{\geq 0}$ satisfying $m^3_{HL} < 1/2(m_{LL} + m_{HH})$ and $m^2_{HL} \leq m^3_{HL} \leq m^4_{HL}$ such that:*

If $m_{HL} \leq m^1_{HL}$, the Positive Assortative Policy is optimal.

If $m_{HL} \in [m^1_{HL}, m^2_{HL}]$, the Provident Impatient Policy is optimal.

If $m_{HL} \in [m^2_{HL}, m^3_{HL}]$, the Myopic Impatient Policy is optimal.

If $m_{HL} \in [m^3_{HL}, min\{m^4_{HL}, 1/2(m_{LL} + m_{HH})\}]$, the Matching Policy $\mathcal{P}_1$ is optimal.

A consequence of Corollary 2 is that the condition $m_{HH} \leq 3m_{LL}$ is not only

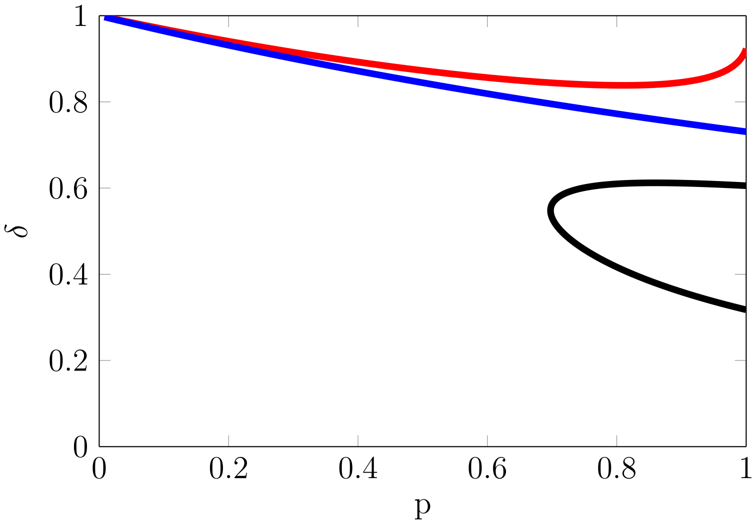

Figure 3.6: Parameter choice: $m(H,H) = 10, m(L,L) = 1, m(H,L) = 4.8$. The black line depicts the boundary between the parameter regions on which the Myopic Impatient Policy and Matching Policy $\mathcal{P}_1$ are optimal. Matching Policy $\mathcal{P}_1$ has no boundary to the Provident Impatient Policy.

sufficient for the statement of Theorem 1 to hold but also necessary. Whenever $m_{HH} > 3m_{LL}$, there exist parameters (p, δ) such that none of the three matching policies from Theorem 1 is optimal if $m_{HL} = 1/2(m_{LL} + m_{HH})$.

Note, that for some parameter constellations $m^4_{HL} < 1/2(m_{LL} + m_{HH})$. This may happen for some values of (p, δ) if the ratio m_{HH}/m_{LL} is extremely high and m_{HL} is close to its upper limit. In that case none of the four matching policies is optimal. Following the logic presented in this extension, our guess is that in these special cases it would be optimal to hold even more than one unproductive agent on stock to prepare for the case that several productive agents arrive to the market in row. Proposition 2 describes an extreme case in which this stock is even infinite.

Proposition 2. *Policy $\mathcal{P}_\infty$ is optimal only if $m_{LL} = 0$.*

Policy $\mathcal{P}_\infty$ never matches two unproductive agents. Potentially, it accumulates an unbounded stock of unproductive agents. When two unproductive agents generate no value, there is no loss of deferring homogeneous matches of unproductive agents. If in addition the creation of mixed pairs is strictly profitable when a single productive agent is in the market, Policy $\mathcal{P}_\infty$ is uniquely optimal.

However, we show that apart from the extreme case $m_{LL} = 0$, Policy $\mathcal{P}_\infty$ is never optimal: This means that generically Policy $\mathcal{P}_\infty$ is not optimal. Hence, if a policy that keeps unproductive agents on stock is optimal and there is a positive cost of waiting with the agents, there is a maximum number of unproductive agents above which two of them get matched. The reason for keeping k unproductive agents on stock is to prepare for the event that k productive agents arrive in row. The probability for this event is exponentially decreasing in k; the cost of holding an additional agent on stock is, however, not decreasing in k. Therefore, at some number of agents the additional cost from accumulating a larger stock exceeds the additional expected profit.

The previous propositions have identified parameter constellations on which none of the three initially introduced policies is optimal. Recall that we interpret the three match values as describing the possible outcomes of a function that maps tuples of types (θ_1, θ_2) into match values $m(\theta_1, \theta_2)$. Despite the results of this extension, there is a large number of natural match value functions for which on each parameter constellation either the Positive Assortative Policy, the Provident Impatient Policy, or the Myopic Impatient Policy is optimal. An important functional form, which is regularly used to represent complementarities in matching, is the product case.

Proposition 3. *Assume $m(\theta_1, \theta_2) = \theta_1 \cdot \theta_2$. For any values of H, L, p and δ one of the following matching policies is optimal: The Positive Assortative Policy, the Provident Impatient Policy, or the Myopic Impatient Policy.*

Note, that this result holds irrespective of the ratio H/L. This means that even when the ration is large such that $m_{HH} > 3m_{LL}$ unproductive agents are never accumulated.

3.4 Incomplete Information and Implementation

In our model, agents are characterized by their productivity θ and their arrival time a. In Section 3.3, we assumed that the designer can observe agents' characteristics. We now consider situations in which the designer cannot observe agents' entry into the market or their productivity, which are thus private information to the agents. Therefore, the designer needs to elicit private information from agents. As the designer's and agents' interests are not aligned, e.g. either type of agent wants to be assigned a productive partner, the presence of private information gives rise to an incentive problem. In this section we analyze ways of implementing the welfare-maximizing policies under various information structures.

Henceforth, we assume that the match value is divided equally among the two partners. This splitting-rule can be justified as the Nash Bargaining Solution: In our model, the designer assigns two agents to a pair. Once the match is formed, both agents leave the market and can not return. This implies that once they are matched, both partners have an outside option of zero. If both agents have equal bargaining power, they share the surplus of their cooperation, the match value, equally.[23] We allow the designer to use monetary transfers and assume that agents have quasilinear utility. Thus, they maximize (half of the) match value minus payments.[24] We study the market beginning with the first arrival. Therefore, initially the market is in the recurrent set of all optimal policies, and we may focus on the implementation of welfare-maximizing policies. In particular, we only distinguish between the Positive Assortative Policy and the Impatient Policy.

In the following, we will prove a possibility result for the implementation of the optimal policies. To strengthen this result, we consider a setting which impedes implementation. Firstly, we will impose strong requirements on the mechanism. Secondly, agents will draw on a rich information structure, allowing for many deviations. Finally, regarding the designer's information, we assume that he does not observe arriving agents' types. We consider both observable and unobservable arrivals.

3.4.1 Observable Arrivals

First, we analyze the case in which the designer observes arrivals. We consider direct mechanisms in which agents report their type. Upon arrival, an agent observes the past reports of all other agents in the market and reports his type. With a slight abuse of notation we denote by $\mathcal{S}$ the set of agents that are already in the market and by Θ_S the vector containing their types. We adopt the convention to denote reported types with hats. We call the vector $\hat{\Theta}_S$ market report. Given policy ρ, market report $\hat{\Theta}_S$, and an agent entering the market and reporting type $\hat{\theta}$, we denote the (random) variable describing the type of that agent's partner by $\tilde{\theta}^{\rho(\hat{\Theta}_S,\hat{\theta})}$ and the random variable describing the time when the agent will be matched by $t_{\rho(\hat{\Theta}_S,\hat{\theta})}$. A mechanism maps the market report into the allocation given by the welfare-maximizing policy and a payment.

We begin by stating the properties of the mechanism. We concentrate on mechanisms that support *efficient exit* meaning that an agent who stops to affect current and future matches also stops to receive and pay transfers.[25] In particular, payments do not condition on realized match values and agents cannot reveal their partner's type to the designer after being matched. This is in accordance with our interpretation that agents leave the market after forming a group. Thus, we focus on payments $\tau^{\hat{\Theta}_S}(\hat{\theta})$

[23] For a strong justification of uniform sharing rules in static settings see also Dizdar and Moldovanu (2013).

[24] Recall that agents discount the future.

[25] See also Bergemann and Välimäki (2010). In the dynamic assignment literature an analogous condition is the requirement that mechanisms are *online*, cf. Gershkov and Moldovanu (2010).

that are charged upon arrival and depend on the market report and the agent's report.

We study direct mechanisms that have a truthful equilibrium in which welfare is maximized. Agents arrive to the market, observe past reports of all agents in the market, form Bayesian expectations with respect to the future, and maximize their utility given that all other agents report truthfully. Observe that this notion of incentive compatibility coincides with interim incentive compatibility in Bergemann and Välimäki (2010). Formally, the *incentive compatibility* constraints are given by:

$$\frac{1}{2}\mathbb{E}[e^{-rt_{\rho(\Theta_S,\theta)}}m(\tilde{\theta}^{\rho(\Theta_S,\theta)},\theta)] - \tau^{\Theta_S}(\theta) \geq \frac{1}{2}\mathbb{E}[e^{-rt_{\rho(\Theta_S,\hat{\theta})}}m(\tilde{\theta}^{\rho(\Theta_S,\hat{\theta})},\theta)] - \tau^{\Theta_S}(\hat{\theta}),\ \forall\theta,\hat{\theta},\Theta_S, \tag{3.8}$$

where the expectation is taken with respect to the partner's type and the matching time.[26] There are other perceivable specifications of the agents' information structure in which agents observe only the number of reports, i.e., the number of agents in the market, or do not observe reports at all. If the designer can implement the welfare-maximizing policies under this information structure, he can implement the welfare-maximizing policies under any information structure in which agents have less information.[27]

As agents participate in the mechanism voluntarily, the following *individual rationality* constraints have to be satisfied

$$\frac{1}{2}\mathbb{E}[e^{-rt_{\rho(\Theta_S,\theta)}}m(\tilde{\theta}^{\rho(\Theta_S,\theta)},\theta)] - \tau^{\Theta_S}(\theta) \geq 0,\ \forall\theta,\Theta_S. \tag{3.9}$$

Observe that (3.9) entails a strong notion of individual rationality because, in addition to observing his type, an agent also observes the market report before he decides whether to participate.[28]

As last condition, we impose that the mechanism requires no external injection of money:

$$\tau^{\Theta_S}(\theta) \geq 0,\ \forall\theta,\Theta_S \tag{3.10}$$

i.e., all payments are positive, which implies that the mechanism runs *no deficit* at

[26] As types do not change over time, (3.8) implies that the mechanism is periodic ex post incentive compatible in the sense of Bergemann and Välimäki (2010). Thus, the mechanism exhibits the no-regret property with respect to past agents' types. Also, as noted by Dizdar and Moldovanu (2013), ex post implementation is intuitively closer to the complete information environment of traditional matching models.

[27] See also Myerson (1986). For example, the designer can reveal any information that is missing to agents and use the original mechanism.

[28] As will become clear from the analysis below the outside option could be any sufficiently small positive value.

any point in time.

Implementation in our setting is not straightforward. Consider the Positive Assortative Policy: An agent that reports a productive type will be assigned a productive partner; an agent that reports an unproductive type will be assigned an unproductive partner. Consider an agent that arrives to a market with one productive agent and one unproductive agent. This situation resembles the static model: Independently of his report, the third agent will be matched immediately with a partner whose report coincides with his report. Either type of arriving agent would like to form a group with the productive agent. Supermodularity of the match value is equivalent to increasing differences, $m_{HH} - m_{HL} \geq m_{HL} - m_{LL}$.[29] Intuitively, increasing differences implies that a match with a productive partner instead of a match with an unproductive partner is valued more by a productive agent than by an unproductive agent. This gap allows the designer to construct payments that make truthful revelation incentive compatible.[30]

Next, consider an agent that arrives to a market with one unproductive agent. Reporting the productive type is less attractive because of the waiting costs incurred until the arrival of the next productive agent. If he reports the unproductive type, he is matched immediately. Therefore, in this case, supermodularity does not necessarily imply increasing differences. The theorem below establishes that, despite these time constraints, the optimal policy is implementable whenever it is welfare-maximizing.

As mentioned in the introduction, there is no general implementation result for our environment as we have a dynamic setting with interdependent values, independent types, and payments that satisfy efficient exit.[31] Our positive implementation result is also surprising in light of the impossibility result for implementing efficient allocations in static settings with interdependent values, cf. Jehiel and Moldovanu (2001).

Theorem 2. *There exist payments such that the implementation of the welfare-maximizing policies is incentive compatible, individual rational, runs no deficit, and supports efficient exit.*

Impatient Policy. If the designer observes arrivals, the implementation of the Impatient Policy is straightforward. As the policy does not condition on the type but matches every two consecutive agents, irrespective of their types, there is no need to elicit agents' private information. Hence, it is possible to set all payments equal to

[29] This is the discrete analogon of the single-crossing property from mechanism design with continuous types.

[30] If the unproductive agent would have a larger incentive to report the productive type than the productive agent, separation would be possible but would induce both types of agents to lie.

[31] For settings with: (i) private values cf. Bergemann and Välimäki (2010), (ii) correlated types cf. Liu (2014), and (iii) without efficient exit cf. Nath et al. (2015).

zero. Furthermore, this is individual rational, runs no deficit, and satisfies efficient exit.

Positive Assortative Policy. In case of the Positive Assortative Policy, the situation is more intricate as the agents' report affects their match. It is useful to aggregate the market report Θ_S into a tuple. Under the Positive Assortative Policy there can be either no agent (0,0), an agent with a productive report (1,0), an agent with an unproductive report (0,1), or two agents with one productive and one unproductive report (1,1) in the market. Denote by $\Delta^\theta(p,\delta)$ the expected discount factor until the next arrival of type θ. In accordance with our intuition, $\Delta^\theta(p,\delta)$ is increasing in δ, $\Delta^L(p,\delta)$ is decreasing in p, and $\Delta^H(p,\delta)$ is increasing in p.

The incentive constraint for the productive type given market report $(0,1)$ can be written as

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} - \tau^{(0,1)}(H) \geq \frac{1}{2}m_{HL} - \tau^{(0,1)}(L). \tag{3.11}$$

Analogously, the incentive constraint for the unproductive type in $(0,1)$ is

$$\frac{1}{2}\Delta^H(p,\delta)m_{HL} - \tau^{(0,1)}(H) \leq \frac{1}{2}m_{LL} - \tau^{(0,1)}(L). \tag{3.12}$$

Rearranging yields the following condition on the payment difference

$$\frac{1}{2}\left(\Delta^H(p,\delta)m_{HH} - m_{HL}\right) \geq \tau^{(0,1)}(H) - \tau^{(0,1)}(L) \geq \frac{1}{2}\left(\Delta^H(p,\delta)m_{HL} - m_{LL}\right). \tag{3.13}$$

Therefore,

$$\Delta^H(p,\delta)m_{HH} - m_{HL} \geq \Delta^H(p,\delta)m_{HL} - m_{LL} \tag{3.14}$$

is a necessary and sufficient condition for the existence of an incentive compatible payment pair in $(0,1)$.

Observe that the left side of (3.14) decreases quicker than the right side as the excepted discount factor decreases. Especially for low discount factors δ or low values of p, (3.14) might be violated. Similarly, we can derive conditions for the existence of incentive compatible payments for all other market reports, $(0,0)$, $(1,0)$, $(1,1)$. It turns out, if (3.14) holds, the conditions for the other market reports are also satisfied. Comparing (3.14) to the boundary of the Positive Assortative Policy (3.5), shows that for all parameters for which the Positive Assortative Policy is welfare-maximizing (3.14) holds. Thus there exist incentive compatible payments that satisfy efficient exit, for all market reports.

We proceed by charging the unproductive type the maximum individual rational payment for all market reports,

$$\tau^{(0,0)}(L) = \tau^{(1,0)}(L) = \frac{1}{2}\Delta^L(p,\delta)m_{LL}, \tag{3.15}$$

$$\tau^{(0,1)}(L) = \tau^{(1,1)}(L) = \frac{1}{2}m_{LL}. \tag{3.16}$$

Given the unproductive type's payment, we choose, for every market report, the maximal payment for the productive type such that the payment pair is incentive compatible, cf. e.g. (3.13). Individual rationality of the unproductive type and incentive compatibility of the productive type yield individual rationality for the productive type. The proof concludes by verifying that all payments are positive.

The proof of Theorem 2 reveals that if positive assortative matching fails to be incentive compatible, it also fails to be welfare-maximizing. Intuitively, if (3.14) is violated, the incentive for an unproductive agent to report the productive type is stronger than the incentive for a productive agent. This means that an unproductive agent's gain of being matched with a productive agent instead of being matched with an unproductive agent outweighs the respective loss for a productive agent. Then, it is plausible that it is welfare-maximizing to create mixed matches.

Note that the mechanism constructed in the proof of Theorem 2 generates revenues: Firstly, we set the unproductive type's expected utility to zero for all market reports by charging the highest payment that is individual rational. Secondly, we choose the maximal incentive compatible payment for the productive type. By reducing payments, the mechanism could account for more lucrative outside options.

The payments of the unproductive type that implement the Positive Assortative Policy depend on the market report only through the presence or absence of an agent in the market whose type coincides with the agent's reported type. That is, payments are equal for market reports $(0,0)$, $(1,0)$ and for market reports $(0,1)$, $(1,1)$. This is a consequence of the agent's report fixing his (future) partner's type and his expected waiting costs under the Positive Assortative Policy.

3.4.2 Extension: Simple Payments

The payments that implement the Positive Assortative Policy in Theorem 2 depend on the market report. In applications it is often desirable to use simple mechanisms that condition on as few parameters as possible. In this section, we examine under which conditions the Positive Assortative Policy is implementable with transfers that depend solely on the reported type but not on the market report. We refer to these payments as simple payments. In addition, this elucidates the relation of implementation of positive

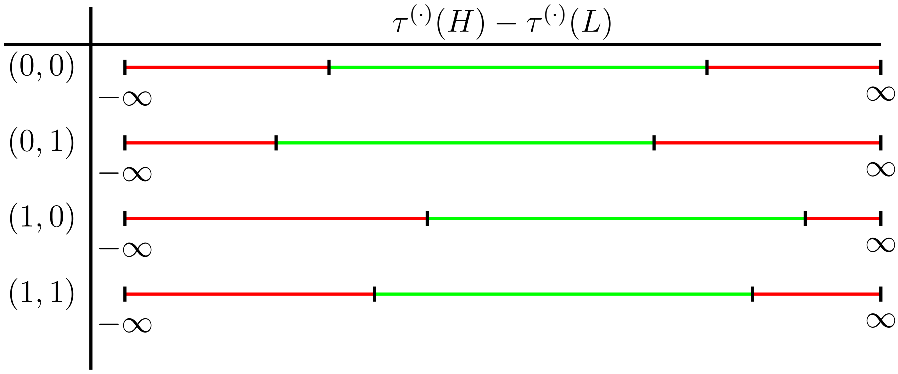

Figure 3.7: Incentive compatible payment differences

assortative matching in our dynamic model and in the static analogon. In the static model payments condition only on the agent's own report. Therefore, our analysis investigates when positive assortative matching can be implemented in the dynamic model with 'static' payments.

Proposition 4. *The Positive Assortative Policy is implementable with simple payments if*

$$m_{HL} \leq \Delta^H(p,\delta)\frac{m_{HH}}{2} + \Delta^L(p,\delta)\frac{m_{LL}}{2}. \tag{3.17}$$

This parameter region is a strict subset of the parameter region where the Positive Assortative Policy is optimal.

In Theorem 2 we proved that the conditions for the existence of incentive compatible payment differences hold. However, the conditions differ across market reports. Therefore, the main issue is to find a single payment pair $(\tau(H), \tau(L))$ that is incentive compatible for all possible market reports. To this end, it is instructive to consider Figure 3.7. Rows correspond to market reports. The green part of each line marks the region where payment differences are incentive compatible, whereas payment differences that lie within the red region are not incentive compatible. We are looking for a payment difference $\tau(H) - \tau(L)$ which lies in the green interval across all market reports. As the boundaries vary significantly with p and δ, existence of such a payment difference is not guaranteed. As illustrated in Figure 3.7, the left boundary of market report $(1, 0)$ and the right boundary of market report $(0, 1)$ are most restrictive. Intuitively, $(1, 0)$ is the most attractive market report for reporting the productive type, whereas $(0, 1)$ is the most attractive market report for reporting the unproductive type. Combining these two conditions yields (3.17).

Observe that (4) holds as δ approaches one, that is, as the time friction vanishes. This means that as the time constraints fades, we can draw on simple payments, which reflect the payments used in the static model to implement the Positive Assortative Policy. Similarly, an increase in complementarities, i.e., a decrease in m_{HL}, strengthens the increasing differences property and thus allows for an implementation with simple payments.

3.4.3 Extension: Asymmetric Match Value Splits

Hitherto, we assumed that partners share their match value equally. While this seems intuitive if partners are homogeneous, i.e., have the same type, one can imagine other sharing rules in case of mixed pairs.[32] An appropriate sharing rule might alleviate the incentive problem.[33] This section investigates under which conditions there exists a sharing rule which induces truthful revelation of types without further intervention, that is to say, without incentivizing agents with payments. In the following, denote by α the productive type's share of the match value when he forms a group with an unproductive type.

Proposition 5. *There exists a share α such that the welfare-maximizing policies are implementable without payments if and only if the welfare-maximizing policies are implementable with simple payments.*

Recall that the implementation of the Impatient Policy in Theorem 2 is straightforward. Hence, we may concentrate on the Positive Assortative Policy. Consider the incentive constraints for market report $(0, 1)$. The productive agent reports truthfully if

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} \geq \alpha m_{HL}. \tag{3.18}$$

Analogously, the unproductive agent reports truthfully if

$$\frac{1}{2}m_{LL} \geq (1-\alpha)\Delta^H(p,\delta)m_{HL}. \tag{3.19}$$

Observe that the incentive constraint of the productive agent gives an upper bound on α, whereas the incentive constraint of the unproductive agent gives a lower bound on α. We proceed similar for the remaining market reports. As in the previous section, the most restrictive conditions arise in $(1, 0)$ and $(0, 1)$. In particular, (3.18) yields the lowest upper bound on α, whereas the incentive constraint of the unproductive type for market report $(1, 0)$

[32] The precise way of how these 'premuneration values' are determined may depend on the specific legal or institutional environment and may lie beyond the designer's control, see Mailath et al. (2015) for an exhaustive discussion.

[33] A different interpretation is that agents share the match value equally but the designer can prescribe internal transfers within matched pairs.

$$\frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq (1-\alpha)m_{HL} \tag{3.20}$$

yields the highest lower bound on α. Thus, an incentive compatible sharing rule exists if

$$\frac{1}{2}\frac{\Delta^H(p,\delta)m_{HH}}{m_{HL}} \geq \frac{m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL}}{m_{HL}}. \tag{3.21}$$

Reformulating (3.21) shows that it coincides with (3.17), which concludes the proof.

To get some intuition for Proposition 5, we reformulate the crucial incentive constraints (3.18) and (3.20). The difference between the equal split and the α split can be interpreted as a substitute for payments:

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} \geq \frac{1}{2}m_{HL} + \left(\alpha m_{HL} - \frac{1}{2}m_{HL}\right), \tag{3.22}$$

$$\frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq \frac{1}{2}m_{HL} - \left(\alpha m_{HL} - \frac{1}{2}m_{HL}\right). \tag{3.23}$$

Recall that the boundaries on the difference of incentive compatible, simple payments in the proof of Proposition 4 are determined by exactly the same incentive constraints:

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} \geq \frac{1}{2}m_{HL} + (\tau(H) - \tau(L)), \tag{3.24}$$

$$\frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq \frac{1}{2}m_{HL} - (\tau(H) - \tau(L)). \tag{3.25}$$

Because α is contained in $[0,1]$, (3.22) and (3.23) provide less flexibility than (3.24) and (3.25). The value that can be redistributed through a sharing rule is bounded by the total match value that is generated in the mixed match, whereas there is no bound on the payment difference. Hence, if there exists an incentive compatible sharing rule, we can also find an incentive compatible, simple payment difference. Proposition 5 states, however, that the converse is true as well. A conclusion is that for any incentive compatible pair of simple payments, the payment difference never exceeds the total match value. Put differently, incentive compatibility does not require extreme transfer differences.

Inequality (3.20) implies that for any incentive compatible sharing rule α, it holds $\alpha > \frac{1}{2}$, i.e., the productive type receives a larger share of the match value when he forms a group with an unproductive type. In the Positive Assortative Policy, the mixed group never occurs on path. Thus, changes in the sharing rule only affect the attractiveness of deviations. When the market consists of one productive agent, an unproductive agent's misreport increases the match value that he creates with his partner from m_{LL} to m_{HL} and reduces the time until he is matched. Consequently, in the absence of payments, truth-telling is incentive compatible only if the unproductive agent's share of the output is lower when being matched with an productive agent. Under the condition identified in Proposition 5, also the productive agent's incentive constraints are satisfied, even though he receives a larger share of the match value when misreporting. The reason is that for the productive agent the total value that is shared is smaller if he misreports.

3.4.4 Unobservable Arrivals

Depending on the organizational details of the market, the designer might not observe agents' arrivals to the market. Instead, agents report their arrival to the designer. Given the welfare-maximizing policies, agents may want to exploit this additional source of private information by strategically delaying their arrival report. We maintain the assumption of private types. This renders implementation of the welfare-maximizing policy a multidimensional screening problem. The current section examines conditions under which the designer can overcome this additional challenge and implement the welfare-maximizing policies.

As in Section 3.4.1, we give arriving agents the informational advantage of past reports being public. We focus on incentive compatible, individual rational, direct mechanisms that run no deficit, satisfy efficient exit, and implement the welfare-maximizing policies. We modify the market report to contain the reported types of all agents in the market that have reported their type and arrival. We construct payments that depend on the reported type, the market report, and the reported arrival time. Agents can report their arrival only after actually arriving to the market, and only agents who have reported their arrival may report their type.

Because of substantially different issues, we discuss the Positive Assortative Policy and the Impatient Policy separately.

Positive Assortative Policy. When type spaces have more than one dimension, incentive constraints pose a severe challenge to the design of incentive compatible mechanisms as one has to account for double deviations, i.e., deviations in several dimensions at the same time. Surprisingly, the mechanism constructed in the proof of Theorem 2 also implements the Positive Assortative Policy with unobservable arrivals.

Proposition 6. *There exists an incentive compatible, individual rational mechanism that runs no deficit, supports efficient exit, and implements the Positive Assortative Policy when both, arrivals and types, are private information to the agents.*

Proof. To prove Proposition 6, we show that agents report their type truthfully and that also the timing of the report remains unchanged, which means that agents reveal their arrival immediately, i.e., truthfully.

We naturally adjust the payments constructed in the proof of Theorem 2 to account for the two-dimensional type space: If an agent reveals his arrival and his type at the same time, payments are as in the proof of Theorem 2. If an agent reveals his arrival strictly before his type, he is punished by a payment of m_{HH}.

To tackle the issue of double deviations in the framework of our model, we divide the problem of showing incentive compatibility into two steps:

(i) First, we show that whenever agents report their type, they report truthfully.

(ii) Second, we argue that given agents report their type truthfully, agents report their arrival time truthfully.

By the memorylessness of the Poisson process, the incentive problem faced by an agent at an arbitrary point in time is the same as the incentive problem at the time of the last arrival. This latter problem, however, resembles the incentive problem with observable arrivals. As the payments solve the incentive problem with observable arrivals, we deduce that (i) holds.

Now, we prove (ii). Given our specification of payments, agents report arrival time and type simultaneously. By the first step, agents report their type truthfully. It remains to be shown that agents want to report their arrival as early as possible. Under the Positive Assortative Policy, an agent's report fixes his match partner's type. The agent's partner is, depending on the market report, either an agent with the same type that is already in the market, or the next agent of his type that arrives to the market. By memorylessness of the Poisson process, delaying an arrival may only be profitable for an agent if the market report changes compared to the market report at the arrival time. By our choice of payments, the unproductive type receives zero expected utility upon arrival for every market report. Therefore, it is an optimal strategy for the unproductive agent to report his arrival time truthfully. Given our payments, the expected utility of a productive type at the point of his arrival is $\Delta^L(p, \delta)\,(m_{HL} - m_{LL})$ for market reports $(0, 0)$ and $(1, 0)$, and $m_{HL} - m_{LL}$ for market reports $(0, 1)$ and $(1, 1)$.

We see that the productive agent's expected utility is highest if an unproductive agent is already in the market. Thus, if the productive agent arrives in $(1, 1)$ or $(0, 1)$, he reports his arrival immediately. On the other hand, if the productive agent arrives in $(0, 0)$ or $(1, 0)$, he might consider waiting for the arrival of an unproductive agent before he reports his arrival to get a higher level of expected utility. Yet, the waiting time until the next arrival of an unproductive agent discounts future payoffs with an expected discount factor of at least $\Delta^L(p, \delta)$ thereby mitigating the advantage of waiting. Hence, also in $(0, 0)$ and $(1, 0)$ it is unprofitable for the productive type to delay his arrival report.[34] This concludes the proof of Step (ii).

Jointly, (i) and (ii) imply that the Positive Assortative Policy together with our payments is incentive compatible even when arrivals are unobservable. Individual rationality, no deficit, and efficient exit remain satisfied, completing the construction of the mechanism. ■

Impatient Policy. Implementing the Impatient Policy in a market where the designer can observe arrivals turned out to be straightforward. As the designer may ignore agents' private information to implement the Impatient Policy, he can abstain from using payments. Yet, if the designer cannot observe agents' arrivals, information relevant for implementing the welfare-maximizing policies, implementation of the Impatient Policy becomes more difficult.

In contrast to the Positive Assortative Policy, the agent's reported type does not fix his match partner's type in the Impatient Policy. The agent's partner is, depending on the market report, either the only agent that is present in the market or the next agent that arrives to the market, irrespective of his type. If the designer asks an agent for his type, future agents may condition their reporting strategy on that report. Consider, for example, a productive agent that arrives to a market with one agent that has reported an unproductive type. If the productive agent reveals his arrival immediately, he will form a group with the unproductive agent. If the productive agent delays his arrival report until after the next arrival, he has the opportunity to be matched with a productive agent. Therefore, depending on the parameter constellation, it might be profitable for the productive agent to delay his arrival report.

[34] To avoid issues with large states that occur because several agents report their arrival simultaneously, we punish agents reporting the same arrival time with a sufficiently high payment, say, m_{HH}. In equilibrium this entails no welfare loss. The deviations checked are, thus, an upper bound for the most profitable deviation.

The designer can circumvent this problem by separating the agents' arrival report from their type report. To implement the Impatient Policy, the designer only needs agents' arrival times but not their types. If the designer asks agents only for their arrival time, future agents only observe arrival reports. Given that agents only observe arrival reports, it is optimal for the agents to report their arrival as early as possible, i.e., truthfully. Therefore, anticipating the agents' informational advantage from reported types, the designer strategically chooses not to ask the agents for their type in order to implement the Impatient Policy.

Combining the insights of the last two sections, we find that even if the designer does not observe arrivals to the market, the welfare-maximizing policies are implementable.

3.4.5 Concluding Remarks

Remark 1. When implementing the Impatient Policy with unobservable arrivals, we demonstrated that it can be beneficial for the designer to strategically not ask agents for their type. Transferring this thought, we can construct another mechanism which implements the Positive Assortative Policy with observable arrivals: As opposed to the Impatient Policy, the Positive Assortative Policy exploits information about agents' types. There exists exactly one situation in which the designer does not need this information upon an agent's arrival: When an agent arrives to an empty market. In this case, the designer needs the agent's information only upon arrival of the next agent, as there is no decision to be taken before. Thus, the designer could set up a mechanism in which an agent that arrives to an empty market does not report his type immediately but only upon arrival of the next agent. The difference to the mechanism studied in Section 3.4.1 is that the second agent arriving to the market does not know the first agent's type. Recall, the most critical situation when implementing the Positive Assortative Policy in Section 3.4.1 arose when the market consisted of one unproductive agent and for small values of p. Under the new mechanism, the subtle difference is that in this situation the agent does not know that the first agent is unproductive but attaches a high probability to this event.

Remark 2. Observe that throughout Section 3.4, we did not use the assumption of Section 3.3 that the value of the productive pair is not too large compared to the value of the unproductive pair. Hence, our implementation results carry over to the case $m_{HH} > 3m_{LL}$ whenever the Positive Assortative Policy and the Impatient Policy are welfare-maximizing.

3.5 Conclusion

This paper studies a dynamic matching market organized by a central authority. Agents of different types that arrive to the market according to a discrete process are matched by a social planner. The model is flexible with respect to four key features: The degree of complementarity of the partners' characteristics in the match value function, the relative size of absolute values of output generated by the two possible homogeneous matchings, the probability distribution of arriving agents' types and the patience represented by discounting. We first address the optimal matching policies under complete information. We develop a tool that helps us to solve for the optimal matching policy in closed form without imposing any restriction on the policy. Whenever the agents' productivities do not differ too much, one of three policies is optimal: The Positive Assortative Policy, the Provident Impatient Policy, or the Myopic Impatient Policy. The social planner is more willing to abstain from creating mixed matches in order to wait for positive assortative matchings when discounting is little or complementarities are strong. This has two immediate implications: a) The optimality of positive assortative matchings in static matching is robust to small discounting frictions. b) When because of impatience mixed matches are created in models of search and matching, this might be welfare-enhancing. The role of the distribution of arriving agents' types is more sophisticated: The designer might abstain from mixed matches only for intermediate probabilities of productive arrivals. When the match value of two productive agents exceeds the match value of the unproductive counterpart by far, it is sometimes optimal to stock unproductive agents in the market in order to ensure that arriving productive agents can get paired immediately. In the second part, we consider implementability of the optimal policy in the presence of private information. We prove implementability of the optimal matching policy when agents have private information about their types and can hide their arrival to the market. We show that if the complementarity of the match value function is sufficiently strong or the environment is sufficiently patient, the welfare-maximizing policy can be implemented with payments that are reminiscent of those that implement the welfare-maximizing policy in the static model. Finally, we identify situations in which the market organizer can abstain from using monetary incentives.

The simple structure of our model helps to expose the trade-off between accumulating agents to achieve positive assortative matchings and matching agents early in order to avoid waiting costs. We conjecture that the State Space Reduction developed in this paper can also be employed to find optimal policies in the extended model with an arbitrary but finite number of types. While we focus on a supermodular match value, we conjecture that in the submodular case it is optimal to form exclusively mixed pairs as the time friction vanishes. Therefore, policies which store both homogeneous groups of productive and unproductive agents might be optimal. Analyzing a model with a continuum of types, but discrete arrivals, would allow for a more detailed comparison between the centralized matching market and decentralized search and matching models. This is an interesting avenue for future research.

3.6 Appendix

The proof of Theorem 1 uses Lemma 1 to 6, which are, hence, proven first.

Preliminaries for Lemma 1 to 3. We denote the candidate policy by ρ, fix an arbitrary state $(x, y) \in S$, and denote by $d = (d_{HH}, d_{HL}, d_{LL})$ a one-period deviation that matches on (x, y) d_{HH} homogeneous pairs of productive agents, d_{HL} mixed pairs, and d_{LL} homogeneous pairs of unproductive agents. The value of ρ on (x, y) can be written as

$$\begin{aligned} V_\rho(x,y) = \rho^{HH}(x,y)m_{HH} + &\rho^{HL}(x,y)m_{HL} + \rho^{LL}(x,y)m_{LL} \\ &+\delta[pV_\rho(x'+1,y') + (1-p)V_\rho(x',y'+1)] \end{aligned} \tag{3.26}$$

with

$$x' = x - 2\rho^{HH}(x,y) - \rho^{HL}(x,y), \quad y' = y - 2\rho^{LL}(x,y) - \rho^{HL}(x,y).$$

Similarly, the value of deviation d from ρ on (x, y) can be expressed as

$$\begin{aligned} V_\rho^d(x,y) = d_{HH}m_{HH} + &d_{HL}m_{HL} + d_{LL}m_{LL} \\ &+\delta[pV_\rho(x''+1,y'') + (1-p)V_\rho(x'',y''+1)] \end{aligned} \tag{3.27}$$

with

$$x'' = x - 2d_{HH}(x,y) - d_{HL}(x,y), \quad y'' = y - 2d_{LL}(x,y) - d_{HL}(x,y).$$

In each of the lemmas we will argue that the value of ρ exceeds the value of a certain class of deviations.

Proof of Lemma 1. Assume that $\rho^{HH}(x, y) > 0$ and $d_{HH} > 0$. Subtracting m_{HH} from $V_\rho(x, y)$ and $V^d_\rho(x, y)$, we observe that the deviation is unprofitable if and only if

$$\begin{aligned} &(\rho^{HH}(x,y)-1)m_{HH} + \rho^{HL}(x,y)m_{HL} + \rho^{LL}(x,y)m_{LL} + V_\rho(x',y') \\ &\geq (d_{HH}-1)m_{HH} + d_{HL}m_{HL} + d_{LL}m_{LL} + \delta[pV_\rho(x''+1,y'') + (1-p)V_\rho(x'',y''+1)]. \end{aligned} \tag{3.28}$$

By consistency the left hand side of the above inequality coincides with $V_\rho(x - 2, y)$.

As the deviation $d_{HH}-1$, d_{HL}, and d_{LL} is feasible on $(x-2,y)$, (3.28) describes a deviation from ρ on $(x-2,y)$. Given that no deviation is profitable on smaller states, the inequality holds. Thus, deviation (d_{HH},d_{HL},d_{LL}) is not profitable on (x,y) either. The proof is analogous for the cases $\rho^{HL}(x,y)$, $d_{HL}>0$ and $\rho^{LL}(x,y)$, $d_{LL}>0$. ■

Proof of Lemma 2. Consider a deviation (d_{HH},d_{HL},d_{LL}) such that $\rho^{HH}(x''+1,y'')>0$ and $\rho^{HH}(x'',y''+1)>0$. The proof constructs an auxiliary deviation $d'=(d'_{HH},d'_{HL},d'_{LL})$ on (x,y) with $V_\rho^{d'}(x,y)\geq V_\rho^{d}(x,y)$. Hence, if $(d'_{HH},d'_{HL},d'_{LL})$ is not profitable, then (d_{HH},d_{HL},d_{LL}) is not profitable either.
Set $(d'_{HH},d'_{HL},d'_{LL})=(d_{HH}+1,d_{HL},d_{LL})$. By our choice of (d_{HH},d_{HL},d_{LL}), $(d'_{HH},d'_{HL},d'_{LL})$ is feasible. From consistency of ρ, $\rho^{HH}(x''+1,y'')>0$, and $\rho^{HH}(x'',y''+1)>0$ follows $V_\rho(x''+1,y'')=m_{HH}+V_\rho(x''-1,y'')$ and $V_\rho(x'',y''+1)=m_{HH}+V_\rho(x''-2,y''+1)$. Together with $\delta<1$ this implies for $V_\rho^d(x,y)$:

$$\begin{aligned}
&d_{HH}m_{HH}+d_{HL}m_{HL}+d_{LL}m_{LL}+\delta m_{HH}\\
&\qquad+\delta[pV_\rho(x''-1,y'')+(1-p)V_\rho(x''-2,y''+1)]\\
<\ &(d_{HH}+1)m_{HH}+d_{HL}m_{HL}+d_{LL}m_{LL}\\
&\qquad+\delta[pV_\rho(x''-1,y'')+(1-p)V_\rho(x''-2,y''+1)].
\end{aligned}$$

We conclude by observing that the latter term is $V_\rho^{d'}(x,y)$. The two remaining cases $\rho^{HL}(x''+1,y'')$, $\rho^{HL}(x'',y''+1)>0$ and $\rho^{LL}(x''+1,y'')$, $\rho^{LL}(x'',y''+1)>0$ follow from an analogous argument. ■

Proof of Lemma 3. Consider a deviation (d_{HH},d_{HL},d_{LL}) with $d_{HL}\geq 2$. As in Lemma 2, we construct an auxiliary deviation $d'=(d'_{HH},d'_{HL},d'_{LL})$ with higher value. Set $(d'_{HH},d'_{HL},d'_{LL})=(d_{HH}+1,d_{HL}-2,d_{LL}+1)$. $(d'_{HH},d'_{HL},d'_{LL})$ is feasible because (d_{HH},d_{HL},d_{LL}) is feasible. As next period states are identical under both deviations, $V_\rho^{d'}(x,y)-V_\rho^{d}(x,y)=m_{HH}+m_{LL}-2m_{HL}\geq 0$, where the inequality follows from the supermodularity of the match value function. Thus, $V_\rho^{d'}(x,y)\geq V_\rho^{d}(x,y)$. ■

Proof of Lemma 4. By construction, the Positive Assortative Policy ρ_{PAP} is consistent. Fix a state (x,y) with $x\geq 3$ and consider a deviation (d_{HH},d_{HL},d_{LL}) on (x,y). As $\rho_{PAP}^{HH}(x,y)>0$, only deviations with $d_{HH}=0$ have to be verified by Lemma 1. Subsequent to following (d_{HH},d_{HL},d_{LL}) there are at least $x-2d_{HH}-d_{HL}$ productive agents in the market after the next arrival. As $\rho_{PAP}^{HH}(x',y')>0$, $\forall x'\geq 2$, only deviations with $x-2d_{HH}-d_{HL}<2$ have to be checked by Lemma 2. By Lemma 3, only deviations with $d_{HL} < 2$ have to be checked. As $x \geq 3$, the set of deviations satisfying the latter three conditions is empty, i.e., no deviation on (x, y) with $x \geq 3$ has to be checked. Similarly, no deviation on (x, y) with $y \geq 3$ has to be checked. ■

Proof of Lemma 5. As all arguments used for the Positive Assortative Policy ρ_{PAP} also apply to the Provident Impatient Policy ρ_{PIP}, the proof is exactly the same as the proof of Lemma 4. ■

Proof of Lemma 6. By construction, the Myopic Impatient Policy ρ_{MIP} is consistent. The proof for states (x, y) with $x \geq 3$ parallels the proof of Lemma 4. For states with many unproductive agents, however, we need to alter the argument slightly. When applying Lemma 2, we can only exclude deviations with $y - 2d_{LL} - d_{HL} \geq 3$ because $\rho^{LL}_{MIP}(x', y') > 0$ holds only $\forall y' \geq 3$. We can apply Lemma 1 and Lemma 3 as before. Thus, no deviation has to checked on states (x, y) with $y \geq 4$. ■

Proof of Theorem 1. For each candidate policy, Lemmas 4 to 6 identify the set of states on which every possible deviation has to be verified for its unprofitability by hand. As is shown in the following, for all parameter constellations $(p, \delta, m_{HH}, m_{LL}, m_{HL})$ such that $m_{HL} \notin \{m^1_{HL}, m^2_{HL}\}$, deviations from the respective candidate policy give a strictly lower payoff. This implies uniqueness of the optimal policy.

Claim 1: The Positive Assortative Policy is optimal for all parameter constellations $(p, \delta, m_{HH}, m_{LL}, m_{HL})$ such that $m_{HL} \leq m^1_{HL}$.

The value function V_{PAP} at states in the recurrent set is determined by the following equations:

$$V_{PAP}(1,0) = \delta[pV_{PAP}(2,0) + (1-p)V_{PAP}(1,1)], \tag{3.29}$$
$$V_{PAP}(0,1) = \delta[pV_{PAP}(1,1) + (1-p)V_{PAP}(0,2)], \tag{3.30}$$
$$V_{PAP}(1,1) = \delta[pV_{PAP}(2,1) + (1-p)V_{PAP}(1,2)], \tag{3.31}$$
$$V_{PAP}(2,0) = m_{HH} + \delta[pV_{PAP}(1,0) + (1-p)V_{PAP}(0,1)], \tag{3.32}$$
$$V_{PAP}(0,2) = m_{LL} + \delta[pV_{PAP}(1,0) + (1-p)V_{PAP}(0,1)], \tag{3.33}$$
$$V_{PAP}(2,1) = m_{HH} + \delta[pV_{PAP}(1,1) + (1-p)V_{PAP}(0,2)], \tag{3.34}$$
$$V_{PAP}(1,2) = m_{LL} + \delta[pV_{PAP}(2,0) + (1-p)V_{PAP}(1,1)]. \tag{3.35}$$

Define $V_{PAP}(0,0) := \delta[pV_{PAP}(1,0) + (1-p)V_{PAP}(0,1)]$. The value at all remaining states (x, y) is given by

$$\begin{aligned} V_{PAP}(x,y) =& \rho^{HH}_{PAP}(x,y) + \rho^{LL}_{PAP}(x,y) + \rho^{HL}_{PAP}(x,y) + V_{PAP}(x',y') \\ \text{with} \quad & x' = x - 2\rho^{HH}_{PAP}(x,y) - \rho^{HL}_{PAP}(x,y) < 2 \\ \text{and} \quad & y' = y - 2\rho^{LL}_{PAP}(x,y) - \rho^{HL}_{PAP}(x,y) < 2. \end{aligned}$$

Observe that this determines $V_{PAP}(x, y)$ uniquely on the entire state space. By Lemma 4, the set of states on which the unprofitability of deviations has to be verified is $\{(x, y)|x \leq 2, y \leq 2, x + y \geq 2)\}$.[35]

On states $(2, 0)$ and $(0, 2)$ the only possible deviation is $(d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)$. In both states this deviation does not have to be considered by Lemma 2.
On state $(1, 2)$ there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 1, 0) is not profitable if

$$V_{PAP}(1, 2) = m_{LL} + V_{PAP}(1, 0) \geq m_{HL} + V_{PAP}(0, 1). \tag{3.36}$$

On state $(2, 1)$ there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 1, 0) is not profitable if

$$V_{PAP}(2, 1) = m_{HH} + V_{PAP}(0, 1) \geq m_{HL} + V_{PAP}(1, 0). \tag{3.37}$$

On state $(2, 2)$ there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0), (1, 0, 0) and (0, 0, 1) do not have to be considered by Lemma 2. Deviation (0, 2, 0) does not have to be considered by Lemma 3. Deviation (0, 1, 0) is not profitable if

$$V_{PAP}(2, 2) = m_{HH} + m_{LL} + V_{PAP}(0, 0) \geq m_{HL} + V_{PAP}(1, 1). \tag{3.38}$$

On state $(1, 1)$ there is one possible deviation, which is (0, 1, 0). The condition for deviation (0, 1, 0) to be unprofitable is

$$V_{PAP}(1, 1) \geq m_{HL} + V_{PAP}(0, 0). \tag{3.39}$$

The final step is to observe that inequalities (3.36) to (3.39) hold if and only if $m_{HL} \leq m^1_{HL}$, in particular, (3.39) corresponds exactly to $m_{HL} \leq m^1_{HL}$. For com-

[35] Note that on states $(1, 0)$ and $(0, 1)$ there is no possible deviation and hence no profitable deviation.

putational details see Appendix 3.7.

Claim 2: The Provident Impatient Policy is optimal for all parameter constellations $(p, \delta, m_{HH}, m_{LL}, m_{HL})$ *such that* $m^1_{HL} \leq m_{HL} \leq m^2_{HL}$.

The value function V_{PIP} at states in the recurrent set is given by the following equations:

$$V_{PIP}(1,0) = \delta[pV_{PIP}(2,0) + (1-p)V_{PIP}(1,1)], \tag{3.40}$$

$$V_{PIP}(0,1) = \delta[pV_{PIP}(1,1) + (1-p)V_{PIP}(0,2)], \tag{3.41}$$

$$V_{PIP}(1,1) = m_{HL} + \delta[pV_{PIP}(1,0) + (1-p)V_{PIP}(0,1)], \tag{3.42}$$

$$V_{PIP}(2,0) = m_{HH} + \delta[pV_{PIP}(1,0) + (1-p)V_{PIP}(0,1)], \tag{3.43}$$

$$V_{PIP}(0,2) = m_{LL} + \delta[pV_{PIP}(1,0) + (1-p)V_{PIP}(0,1)]. \tag{3.44}$$

Define $V_{PIP}(0,0) := \delta[pV_{PIP}(1,0)+(1-p)V_{PIP}(0,1)]$. The value at all remaining states (x, y) is determined as follows:

$$\begin{aligned} V_{PIP}(x,y) =& \rho^{HH}_{PIP}(x,y) + \rho^{LL}_{PIP}(x,y) + \rho^{HL}_{PIP}(x,y) + V_{PIP}(x',y') \\ \text{with} \quad & x' = x - 2\rho^{HH}_{PIP}(x,y) - \rho^{HL}_{PIP}(x,y) < 2 \\ \text{and} \quad & y' = y - 2\rho^{LL}_{PIP}(x,y) - \rho^{HL}_{PIP}(x,y) < 2. \end{aligned}$$

Solving the system gives

$$V_{PIP}(0,0) = \frac{\delta^2}{1-\delta^2}[(p^2 + (1-p)^2)m_{HL} + (1-p)p(m_{HH} + m_{LL})], \tag{3.45}$$

$$V_{PIP}(1,0) = \delta V_{PIP}(0,0) + \delta(pm_{HH} + (1-p)m_{HL}), \tag{3.46}$$

$$V_{PIP}(0,1) = \delta V_{PIP}(0,0) + \delta(pm_{HL} + (1-p)m_{LL}), \tag{3.47}$$

$$V_{PIP}(1,1) = m_{HL} + \delta V_{PIP}(0,0), \tag{3.48}$$

$$V_{PIP}(2,0) = m_{HH} + \delta V_{PIP}(0,0), \tag{3.49}$$

$$V_{PIP}(0,2) = m_{LL} + \delta V_{PIP}(0,0). \tag{3.50}$$

By Lemma 5, the set of states on which the unprofitability of deviations has to be verified is $\{(x, y)|x \leq 2, y \leq 2, x + y \geq 2)\}$.

On states $(2, 0)$ and $(0, 2)$ the only possible deviation is $(d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)$. In both states this deviation does not have to be considered by Lemma 2.
On state $(2, 1)$ there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. The inequality corresponding to

deviation (0, 1, 0) is

$$V_{PIP}(2,1) = m_{HH} + V_{PIP}(0,1) \geq m_{HL} + V_{PIP}(1,0). \tag{3.51}$$

On state $(2,2)$ there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0), (1, 0, 0) and (0, 0, 1) do not have to be considered by Lemma 2. Deviation (0, 2, 0) does not have to be considered by Lemma 3. Deviation (0, 1, 0) is not profitable if

$$m_{HH} + m_{LL} + V_{PIP}(0,0) \geq m_{HL} + \delta[pV_{PIP}(2,1) + (1-p)V_{PIP}(1,2)]. \tag{3.52}$$

On state $(1,1)$ there is one possible deviation, which is (0, 0, 0). The condition for deviation (0, 0, 0) to be not profitable is

$$V_{PIP}(1,1) = m_{HL} + V_{PIP}(0,0) \geq \delta[pV_{PIP}(2,1) + (1-p)V_{PIP}(1,2)]. \tag{3.53}$$

On state $(1,2)$ there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 1, 0) is not profitable if

$$V_{PIP}(1,2) = m_{LL} + V_{PIP}(1,0) \geq m_{HL} + V_{PIP}(0,1). \tag{3.54}$$

The final step is to show that inequalities (3.51) to (3.54) hold if and only if $m^1_{HL} \leq m_{HL} \leq m^2_{HL}$. Inserting the explicit solution for V_{PIP} into (3.53) and solving for m_{HL} shows that (3.53) corresponds to $m_{HL} \geq m^1_{HL}$. Similarly, (3.54) yields $m_{HL} \leq m^2_{HL}$. See Appendix 3.7 for calculatory details.

Claim 3: The Myopic Impatient Policy is optimal for all parameter constellations $(p, \delta, m_{HH}, m_{LL}, m_{HL})$ such that $m_{HL} \geq m^2_{HL}$.

The Myopic Impatient Policy has the same recurrent set as the Provident Impatient Policy, $R := \{(x,y)|1 \leq x + y \leq 2\}$. Define $V_{MIP}(0,0) := \delta[pV_{MIP}(1,0) + (1-p)V_{MIP}(0,1)]$. By construction, $\rho_{MIP}(x,y) = \rho_{PIP}(x,y)$, $\forall(x,y) \in R$, hence, $V_{MIP}(x,y) = V_{PIP}(x,y)$, $\forall(x,y) \in R$ with explicit solution (3.45) to (3.50). At all remaining states (x,y) the value is given by

$$\begin{aligned} V_{MIP}(x,y) =& \rho^{HH}_{MIP}(x,y) + \rho^{LL}_{MIP}(x,y) + \rho^{HL}_{MIP}(x,y) + V_{MIP}(x',y') \\ \text{with} \quad & x' = x - 2\rho^{HH}_{MIP}(x,y) - \rho^{HL}_{MIP}(x,y) < 2 \\ \text{and} \quad & y' = y - 2\rho^{LL}_{MIP}(x,y) - \rho^{HL}_{MIP}(x,y) < 2. \end{aligned}$$

By Lemma 6, the set of states on which the unprofitability of deviations has to be verified is $\{(x, y)|x \leq 2, y \leq 3, x + y \geq 2)\}$.

On state $(2, 0)$ the only possible deviation is $(d_{HH}, d_{HL}, d_{LL}) = (0, 0, 0)$. This deviation does not have to be considered by Lemma 2.
On state $(2, 1)$ there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 1, 0) is not profitable either: The corresponding inequality is

$$V_{MIP}(2, 1) = m_{HH} + V_{MIP}(0, 1) \geq m_{HL} + V_{MIP}(1, 0). \tag{3.55}$$

As $V_{MIP}(0, 1) = V_{PIP}(0, 1)$ and $V_{MIP}(1, 0) = V_{PIP}(1, 0)$, (3.55) equals (3.51) which holds.
On state $(1, 1)$ the only possible deviation is (0, 0, 0) which is unprofitable if

$$V_{MIP}(1, 1) = m_{HL} + V_{MIP}(0, 0) \geq \delta[pV_{MIP}(2, 1) + (1 - p)V_{MIP}(1, 2)]. \tag{3.56}$$

On state $(2, 2)$ there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0), and (1, 0, 0) do not have to be considered by Lemma 2. Deviation (0, 2, 0) does not have to be considered by Lemma 3. Deviation (0, 0, 1) does not have to be considered by Lemma 1. Deviation (0, 1, 0) is not profitable if

$$m_{HH} + m_{LL} + V_{MIP}(0, 0) \geq m_{HL} + \delta[pV_{MIP}(2, 1) + (1 - p)V_{MIP}(1, 2)]. \tag{3.57}$$

On state $(0, 3)$ the only possible deviation is (0, 0, 0) which does not have to be considered by Lemma 2.
On state $(1, 3)$ there are three possible deviations: (0, 0, 0), (0, 0, 1) and (0, 1, 0). Deviations (0, 0, 1) and (0, 1, 0) do not have to be considered by Lemma 1. Deviation (0, 0, 0) does not have to be considered by Lemma 2.
On state $(2, 3)$ there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0), and (1, 0, 0) do not have to be considered by Lemma 2. Deviation (0, 2, 0) does not have to be considered by Lemma 3. Deviation (0, 0, 1) does not have to be considered by Lemma 1. Deviation (0, 1, 0) is not profitable if

$$m_{HH} + m_{LL} + V_{MIP}(0, 1) \geq m_{HL} + \delta[pV_{MIP}(2, 2) + (1 - p)V_{MIP}(1, 3)]. \tag{3.58}$$

On state $(0, 2)$ the only possible deviation is (0, 0, 0) which is unprofitable if

$$V_{MIP}(0, 2) = m_{LL} + V_{MIP}(0, 0) \geq \delta[pV_{MIP}(1, 2) + (1 - p)V_{MIP}(0, 3)]. \tag{3.59}$$

On state $(1,2)$ there are two possible deviations: (0, 0, 0) and (0, 0, 1). Deviation (0, 0, 1) is not profitable if

$$m_{HL} + V_{MIP}(0,1) \geq m_{LL} + V_{MIP}(1,0), \tag{3.60}$$

and deviation (0, 0, 0) is not profitable if

$$m_{HL} + V_{MIP}(0,1) \geq \delta[pV_{MIP}(2,2) + (1-p)V_{MIP}(1,3)]. \tag{3.61}$$

The final step is to show that inequalities (3.56) to (3.61) hold if and only if $m_{HL} \geq m^2_{HL}$. Using the explicit solution for V_{MIP} and solving for m_{HL}, we find that (3.60) is equivalent to $m_{HL} \geq m^2_{HL}$ and that (3.59) corresponds to $m_{HL} \leq m^3_{HL}$. Supporting calculations can be found in Appendix 3.7.

To conclude the proof, observe that $m_{HL} \leq m^3_{HL}$ for all $p, \delta, m_{HH}, m_{LL}, m_{HL}$ such that $m_{HH} \leq 3m_{LL}$. Thus, the parameter regions on which the three candidates are optimal span the entire parameter space. ■

Proof of Corollary 1. The existence of two cut-off levels follows from showing that $\frac{\partial m^1_{HL}}{\partial \delta} \geq 0$ and $\frac{\partial m^2_{HL}}{\partial \delta} \geq 0$, independent of the specific choice of parameters. Using the definitions of m^1_{HL} and m^2_{HL} from (3.5) and (3.6), we obtain

$$\frac{\partial m^1_{HL}}{\partial \delta} = m_{HH}\frac{p}{[1-\delta(1-2p)]^2} + m_{LL}\frac{1-p}{[1+\delta(1-2p)]^2} > 0$$

and

$$\begin{aligned}\frac{\partial m^2_{HL}}{\partial \delta} =& m_{HH}\frac{p}{[1-\delta(1-2p)]^2} + m_{LL}\frac{-p}{[1-\delta(1-2p)]^2}\\ >& m_{HH}\frac{p}{[1-\delta(1-2p)]^2} + m_{HH}\frac{-p}{[1-\delta(1-2p)]^2} = 0.\end{aligned}$$

Furthermore, $m^1_{HL} < m^2_{HL}$ implies $\delta^1 > \delta^2$. Finally, $m^1_{HL} = m^2_{HL} = 0$ for $\delta = 0$ and $m^1_{HL} = m^2_{HL} = 1/2(m_{HH} + m_{LL})$ for $\delta = 1$ and imply $\delta^1, \delta^2 \in [0,1]$. ■

Proof of Proposition 1. We follow the same steps as for the other candidate policies above. Denote Matching Policy $\mathcal{P}_1$ by ρ_{P1}.

Claim 1: To verify candidate policy ρ_{P1} it is sufficient to verify deviations on $\{(x,y)|x \leq 2, y \leq 4, x+y \geq 2)\}$.

By construction, ρ_{P1} is consistent. For states (x,y) with $x \geq 3$ the argument is the same as in Lemma 4. Analogously to Lemma 6, we need to adjust the proof slightly for states with many unproductive agents when applying Lemma 2. In this case, we can only exclude deviations on states (x,y) with $y \geq 5$.

Claim 2: There exists a parameter region on which there is no profitable deviation from ρ_{P1} *on* $\{(x,y)|x \leq 2, y \leq 4, x+y \geq 2)\}$.

Define $V_{P1}(0,0) := \delta[pV_{P1}(1,0) + (1-p)V_{P1}(0,1)]$. The value function V_{P1} at states in the recurrent set is determined by

$$\begin{aligned}
V_{P1}(1,0) =&\delta[pV_{P1}(2,0) + (1-p)V_{P1}(1,1)], && (3.62)\\
V_{P1}(0,1) =&\delta[pV_{P1}(1,1) + (1-p)V_{P1}(0,2)] && (3.63)\\
V_{P1}(1,1) =&m_{HL} + \delta[pV_{P1}(1,0) + (1-p)V_{P1}(0,1)], && (3.64)\\
V_{P1}(2,0) =&m_{HH} + \delta[pV_{P1}(1,0) + (1-p)V_{P1}(0,1)], && (3.65)\\
V_{P1}(0,2) =&\delta[pV_{P1}(1,2) + (1-p)V_{P1}(0,3)], && (3.66)\\
V_{P1}(0,3) =&m_{LL} + \delta[pV_{P1}(1,1) + (1-p)V_{P1}(0,2)], && (3.67)\\
V_{P1}(1,2) =&m_{HL} + \delta[pV_{P1}(1,2) + (1-p)V_{P1}(0,3)]. && (3.68)
\end{aligned}$$

The value at all remaining states (x,y) is

$$\begin{aligned}
V_{P1}(x,y) =&\rho_{P1}^{HH}(x,y) + \rho_{P1}^{LL}(x,y) + \rho_{P1}^{HL}(x,y) + V_{P1}(x',y')\\
\text{with} \quad & x' = x - 2\rho_{P1}^{HH}(x,y) - \rho_{P1}^{HL}(x,y) < 2\\
\text{and} \quad & y' = y - 2\rho_{P1}^{LL}(x,y) - \rho_{P1}^{HL}(x,y) < 3.
\end{aligned}$$

On state $(2,0)$ the only possible deviation is $(d_{HH}, d_{HL}, d_{LL}) = (0,0,0)$ which does not have to be considered by Lemma 2.
On state $(2,1)$ there are two possible deviations: (0, 0, 0) and (0, 1, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 1, 0) is not profitable if

$$V_{P1}(2,1) = m_{HH} + V_{P1}(0,1) \geq m_{HL} + V_{P1}(1,0). \tag{3.69}$$

On state $(1,1)$ the only possible deviation is (0, 0, 0) which is unprofitable if

$$V_{P1}(1,1) = m_{HL} + V_{P1}(0,0) \geq \delta[pV_{P1}(2,1) + (1-p)V_{P1}(1,2)]. \tag{3.70}$$

On state $(2,2)$ there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 1), (0, 0, 1) and (0, 2, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 2, 0) does not have to be considered by Lemma 3. Deviations (0, 0, 1) and (1, 0, 1) do not have to be considered by Lemma 1. Deviation (0, 1, 0) is not profitable if

$$m_{HH} + m_{LL} + V_{P1}(0,0) \geq m_{HL} + \delta[pV_{P1}(2,1) + (1-p)V_{P1}(1,2)]. \tag{3.71}$$

On state $(0,3)$ the only possible deviation is (0, 0, 0) which is unprofitable if

$$V_{P1}(0,3) = m_{LL} + V_{P1}(0,1) \geq \delta[pV_{P1}(1,3) + (1-p)V_{P1}(0,4)]. \tag{3.72}$$

On state $(1,3)$ there are three possible deviations: (0, 0, 0), (0, 0, 1) and (0, 1, 1). Deviation (0, 1, 1) does not have to be considered by Lemma 1. Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 0, 1) is not profitable if

$$m_{HL} + V_{P1}(0,2) \geq m_{LL} + \delta[pV_{P1}(2,1) + (1-p)V_{P1}(1,2)]. \tag{3.73}$$

On state $(0,2)$ the only possible deviation is (0, 0, 1) which is unprofitable if

$$V_{P1}(0,2) = \delta[pV_{P1}(1,2) + (1-p)V_{P1}(0,3)] \geq m_{LL} + V_{P1}(0,0). \tag{3.74}$$

On state $(1,2)$ there are two possible deviations: (0, 0, 0) and (0, 0, 1). Deviation (0, 0, 1) is not profitable if

$$m_{HL} + V_{P1}(1,0) \geq m_{LL} + V_{P1}(1,0), \tag{3.75}$$

and deviation (0, 0, 0) is not profitable if

$$m_{HL} + V_{P1}(0,1) \geq \delta[pV_{P1}(2,2) + (1-p)V_{P1}(1,3)]. \tag{3.76}$$

On state $(2,3)$ there are five possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1) and (0, 2, 0). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 2, 0) does not have to be considered by Lemma 3. Deviations (1, 0, 0) and (0, 0, 1) do not have to be considered by Lemma 1. Deviation (0, 1, 0) is not profitable if

$$m_{HH} + m_{LL} + V_{P1}(0,1) \geq m_{HL} + \delta[pV_{P1}(2,2) + (1-p)V_{P1}(1,3)]. \tag{3.77}$$

On state $(0,4)$ there are two possible deviations: (0, 0, 0) and (0, 0, 2). Deviation (0, 0, 0) does not have to be considered by Lemma 2. Deviation (0, 0, 2) does not have to

be considered by Lemma 1.
On state $(1, 4)$ there are four possible deviations: (0, 0, 0), (0, 1, 0), (0, 0, 1) and (0, 0, 2). Deviations (0, 0, 0) and (0, 1, 0) do not have to be considered by Lemma 2. Deviations (0, 0, 1) and (0, 0, 2) do not have to be considered by Lemma 1.
On state $(2, 4)$ there are nine possible deviations: (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (0, 0, 2), (0, 2, 1), (1, 0, 2), (0, 0, 1) and (0, 2, 0). Deviations (0, 0, 0) and (0, 1, 0) do not have to be considered by Lemma 2. Deviations (0, 2, 0) and (0, 2, 1) do not have to be considered by Lemma 3. Deviations (1, 0, 0), (1, 0, 2), (0, 0, 2), (0, 1, 1), (0, 2, 1) and (0, 0, 1) do not have to be considered by Lemma 1.

(3.69) to (3.77) hold if and only if $m^3_{HL} \leq m_{HL} \leq m^4_{HL}$, where

$$m^4_{HL} = \frac{1}{1-\delta(1-2p)}\left[m_{HH}\delta p + m_{LL}\left(2 - \delta(2-p) + \frac{1-\delta+\delta p(1-\delta)^2}{\delta^2 p^2}\right)\right]. \quad (3.78)$$

(3.72) coincides with $m_{HL} \leq m^4_{HL}$, and (3.74) is equivalent to $m_{HL} \geq m^3_{HL}$. All other inequalites are then implied. See Appendix 3.7 for supporting calculations.

We conclude the proof by showing that P_1 actually arises. First, we argue that $m^3_{HL} \leq m^4_{HL}$. Note that in the definition of m^4_{HL} and m^3_{HL} the factors in front of m_{HH} coincide. Therefore, $m^3_{HL} \leq m^4_{HL}$ is equivalent to

$$2(1-\delta) + \delta p + \frac{1-\delta}{\delta^2 p^2} + \frac{(1-\delta)^2}{\delta p} \geq 1 - \delta + \delta p + \frac{1-\delta}{\delta p} \quad (3.79)$$

which holds as $\delta, p \in (0, 1)$. Second, as argued in the last part of the proof to Theorem 1, if $m_{HH} > 3m_{LL}$, then there exist p, δ, m_{HL} such that $m^3_{HL} < \frac{1}{2}m_{HH} + \frac{1}{2}m_{LL}$. ■

Proof of Corollary 2. The corollary follows from Theorem 1 and the proof of Proposition 1. ■

Proof of Proposition 2. Assume that there exists an optimal policy ρ that never matches two unproductive agents and denote its value function on (x, y) by $V_\rho(x, y)$. We derive a lower bound $\underline{a}$ and an upper bound $\overline{a}$ for $V_\rho(x, y)$ such that $\underline{a} > \overline{a}$, which yields a contradiction.

Lower bound. Observe that by optimality

$$V_\rho(0, k) \geq \left\lfloor \frac{k}{2} \right\rfloor m_{LL} + V_\rho(0, k - 2\left\lfloor \frac{k}{2} \right\rfloor) \geq \left\lfloor \frac{k}{2} \right\rfloor m_{LL} = \underline{a}, \quad (3.80)$$

where the second inequality holds because $V_\rho(0,0), V_\rho(0,1) \geq 0$.

Upper bound. As ρ never matches two unproductive agents, we can derive an upper bound on the number of matches created in each period when starting in state $(0,k)$ and following policy ρ. When being in state $(0,k)$ in period t, the maximal number of matches in period $t+s$ is bounded from above by s, for any $s \in \mathbb{N}$. Hence, $V_\rho(0,k)$ is bounded from above by the value generated from creating the highest match value as often as possible and as early as possible, i.e., matching two productive agents in every subsequent period which yields

$$V_\rho(0,k) \leq m_{HH}\frac{1}{1-\delta} = \overline{a}.$$

For every parameter constellation $(p, \delta, m_{HH}, m_{LL}, m_{HL})$ there exists a k such that $\underline{a} > \overline{a}$, which is a contradiction. ■

Proof of Proposition 3. It is sufficient to proof that if $m(\theta_1, \theta_2) = \theta_1 \cdot \theta_2$ then

$$m_{HL} \leq m_{HL}^3 = m_{HH}\frac{\delta p}{1-\delta(1-2p)} + m_{LL}\frac{1-\delta(1-p)+\frac{1-\delta}{\delta p}}{1-\delta(1-2p)}, \quad \forall H, L, p, \delta$$

with $H > L > 0$. Inserting $m(\theta_1, \theta_2) = \theta_1 \cdot \theta_2$, dividing by $(L)^2$, and rearranging terms yields

$$0 \leq \left(\frac{H}{L}\right)^2 \delta p - \frac{H}{L}\left(1-\delta(1-2p)\right) + \left(1-\delta(1-p)+\frac{1-\delta}{\delta p}\right). \tag{3.81}$$

The parabola in $\frac{H}{L}$ on the right side of (3.81) is minimized at

$$\frac{H}{L} = 1 + \frac{1-\delta}{2p\delta}. \tag{3.82}$$

Plugging (3.82) into (3.81) gives

$$0 \leq 3 - 2\delta - \delta^2,$$

which holds true as $\delta < 1$. This completes the proof. ■

Proof of Theorem 2. The Impatient Policy is implementable by setting $\tau^{\Theta_S}(\theta) = 0$, for all θ and Θ_S. The implementability of the Positive Assortative Policy requires a proof. We define

$$\Delta^H(p,\delta) = \frac{\delta p}{1-\delta(1-p)}, \qquad \Delta^L(p,\delta) = \frac{\delta(1-p)}{1-\delta p}. \tag{3.83}$$

The incentive constraint for the productive and the unproductive type are, for market report (0,0),

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} - \tau^{(0,0)}(H) \geq \frac{1}{2}\Delta^L(p,\delta)m_{HL} - \tau^{(0,0)}(L),$$
$$\frac{1}{2}\Delta^H(p,\delta)m_{HL} - \tau^{(0,0)}(H) \leq \frac{1}{2}\Delta^L(p,\delta)m_{LL} - \tau^{(0,0)}(L),$$

for market report (1,1),

$$\frac{1}{2}m_{HH} - \tau^{(1,1)}(H) \geq \frac{1}{2}m_{HL} - \tau^{(1,1)}(L),$$
$$\frac{1}{2}m_{HL} - \tau^{(1,1)}(H) \leq \frac{1}{2}m_{LL} - \tau^{(1,1)}(L),$$

for market report (1,0),

$$\frac{1}{2}m_{HH} - \tau^{(1,0)}(H) \geq \frac{1}{2}\Delta^L(p,\delta)m_{HL} - \tau^{(1,0)}(L),$$
$$\frac{1}{2}m_{HL} - \tau^{(1,0)}(H) \leq \frac{1}{2}\Delta^L(p,\delta)m_{LL} - \tau^{(1,0)}(L),$$

and for market report (0,1),

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} - \tau^{(0,1)}(H) \geq \frac{1}{2}m_{HL} - \tau^{(0,1)}(L),$$
$$\frac{1}{2}\Delta^H(p,\delta)m_{HL} - \tau^{(0,1)}(H) \leq \frac{1}{2}m_{LL} - \tau^{(0,1)}(L).$$

Combining the incentive constraint of the productive type with the incentive constraint of the unproductive type yields the following conditions on the payment differences:

$$\Delta^H(p,\delta)\frac{m_{HH}}{2} - \Delta^L(p,\delta)\frac{m_{HL}}{2} \geq \tau^{(0,0)}(H) - \tau^{(0,0)}(L) \geq \Delta^H(p,\delta)\frac{m_{HL}}{2} - \Delta^L(p,\delta)\frac{m_{LL}}{2}, \tag{3.84}$$

$$\frac{1}{2}\left(m_{HH} - m_{HL}\right) \geq \tau^{(1,1)}(H) - \tau^{(1,1)}(L) \geq \frac{1}{2}\left(m_{HL} - m_{LL}\right), \tag{3.85}$$

$$\frac{m_{HH}}{2} - \Delta^L(p,\delta)\frac{m_{HL}}{2} \geq \tau^{(1,0)}(H) - \tau^{(1,0)}(L) \geq \frac{m_{HL}}{2} - \Delta^L(p,\delta)\frac{m_{LL}}{2}, \tag{3.86}$$

$$\Delta^H(p,\delta)\frac{m_{HH}}{2} - \frac{m_{HL}}{2} \geq \tau^{(0,1)}(H) - \tau^{(0,1)}(L) \geq \Delta^H(p,\delta)\frac{m_{HL}}{2} - \frac{m_{LL}}{2}. \tag{3.87}$$

Thus, an incentive compatible payment difference exists if and only if the following conditions are satisfied:

$$\Delta^H(p,\delta)m_{HH} - \Delta^L(p,\delta)m_{HL} \geq \Delta^H(p,\delta)m_{HL} - \Delta^L(p,\delta)m_{LL}, \tag{3.88}$$

$$m_{HH} - m_{HL} \geq m_{HL} - m_{LL}, \tag{3.89}$$

$$m_{HH} - \Delta^L(p,\delta)m_{HL} \geq m_{HL} - \Delta^L(p,\delta)m_{LL}, \tag{3.90}$$

$$\Delta^H(p,\delta)m_{HH} - m_{HL} \geq \Delta^H(p,\delta)m_{HL} - m_{LL}. \tag{3.91}$$

Observe that $\Delta^\theta(p,\delta) \leq 1$, for all θ. Hence, (3.91) implies (3.88) to (3.90). To prove that (3.91) holds whenever the Positive Assortative Policy is optimal, we show that (3.91) holds if $m_{HL} \leq m^1_{HL}$. Reformulating (3.91) gives

$$m_{HL} \leq \frac{\Delta^H(p,\delta)}{1+\Delta^H(p,\delta)} m_{HH} + \frac{1}{1+\Delta^H(p,\delta)} m_{LL}. \tag{3.92}$$

We argue that the right-hand side of (3.92) is larger than m^1_{HL}. To this end, we will show that the multipliers of m_{HH} and m_{LL} in (3.92) are (weakly) larger than the corresponding factors in m^1_{HL}. First, consider the factor attached to m_{HH}. By (3.83),

$$\frac{\Delta^H(p,\delta)}{1+\Delta^H(p,\delta)} = \frac{\delta p}{1-\delta+2\delta p}$$

which coincides with the multiplier of m_{HH} in m^1_{HL}. Second, for the factor attached to m_{LL} we obtain

$$\frac{1}{1+\Delta^H(p,\delta)} = \frac{1-\delta+\delta p}{1-\delta+2\delta p}. \tag{3.93}$$

(3.93) is larger than the multiplier of m_{LL} in m^1_{HL} if and only if

$$\frac{1-\delta+\delta p}{1-\delta+2\delta p} \geq \frac{\delta(1-p)}{1+\delta-2\delta p} \quad \Leftrightarrow \quad \delta \leq 1.$$

Thus, whenever the Positive Assortative Policy is optimal, we can find an incentive compatible payment pair, for every market report Θ_s.

We construct payments which are positive and individual rational: Set

$$\tau^{(0,0)}(L) = \tau^{(1,0)}(L) = \frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq 0, \tag{3.94}$$

$$\tau^{(0,1)}(L) = \tau^{(1,1)}(L) = \frac{1}{2}m_{LL} \geq 0. \tag{3.95}$$

By construction, payments (3.94) and (3.95) set the unproductive agent's expected utility to zero and are therefore individual rational. For every market report, choose, given the unproductive type's payment, the maximal payment for the productive type that is consistent with (3.84) - (3.87), i.e., such that the payment pair is incentive compatible:

$$\tau^{(0,0)}(H) = \frac{1}{2}\left(\Delta^L(p,\delta)m_{LL} + \Delta^H(p,\delta)m_{HH} - \Delta^L(p,\delta)m_{HL}\right),$$
$$\tau^{(1,1)}(H) = \frac{1}{2}\left(m_{LL} + m_{HH} - m_{HL}\right),$$
$$\tau^{(1,0)}(H) = \frac{1}{2}\left(\Delta^L(p,\delta)m_{LL} + m_{HH} - \Delta^L(p,\delta)m_{HL}\right),$$
$$\tau^{(0,1)}(H) = \frac{1}{2}\left(m_{LL} + \Delta^H(p,\delta)m_{HH} - m_{HL}\right).$$

Individual rationality of the payments for the unproductive type and incentive compatibility yield individual rationality for the productive type.
Given that (3.88) - (3.91) are satisfied whenever the Positive Assortative Policy is optimal, we can deduce that

$$\tau^{(0,0)}(H) \geq \frac{1}{2}\left(\Delta^L(p,\delta)m_{LL} + \Delta^H(p,\delta)m_{HL} - \Delta^L(p,\delta)m_{LL}\right) = \frac{1}{2}\Delta^H(p,\delta)m_{HL} \geq 0,$$
$$\tau^{(1,1)}(H) \geq \frac{1}{2}\left(m_{LL} + m_{HL} - m_{LL}\right) = \frac{1}{2}m_{HL} \geq 0,$$
$$\tau^{(1,0)}(H) \geq \frac{1}{2}\left(\Delta^L(p,\delta)m_{LL} + m_{HL} - \Delta^L(p,\delta)m_{LL}\right) = \frac{1}{2}m_{HL} \geq 0,$$
$$\tau^{(0,1)}(H) \geq \frac{1}{2}\left(m_{LL} + \Delta^H(p,\delta)m_{HL} - m_{LL}\right) = \frac{1}{2}\Delta^H(p,\delta)m_{HL} \geq 0.$$

As all payments are positive, the mechanism runs no deficit. Furthermore, payments support efficient exit because they are charged upon arrival. ■

Proof of Proposition 4. For incentive compatibility, we need to find a single payment pair $(\tau(H), \tau(L))$ such that the difference $\tau(H) - \tau(L)$ satisfies conditions (3.84) to (3.87). Observe that (3.87) yields the lowest upper bound, whereas (3.86) yields the highest lower bound on the payment difference. Hence, $(\tau(H), \tau(L))$ is incentive compatible if and only if

$$\frac{1}{2}\left(\Delta^H(p,\delta)m_{HH} - m_{HL}\right) \geq \tau(H) - \tau(L) \geq \frac{1}{2}\left(m_{HL} - \Delta^L(p,\delta)m_{LL}\right). \tag{3.96}$$

Rearranging terms, note that incentive compatible payments exist iff

$$m_{HL} \leq \Delta^H(p,\delta)\frac{m_{HH}}{2} + \Delta^L(p,\delta)\frac{m_{LL}}{2}. \tag{3.97}$$

To see that (3.97) describes a strict subset of the parameter region in which the Positive Assortative Policy is optimal, we compare it to the boundary of the Positive Assortative Policy m^1_{HL}. We show that (3.97) is more restrictive than $m_{HL} \leq m^1_{HL}$ by separately comparing the factors in front of m_{HH} and m_{LL}. For the factor attached to m_{HH} we note that

$$\frac{1}{2}\frac{\delta p}{(1-\delta(1-p))} < \frac{\delta p}{1-\delta(1-2p)} \quad \Leftrightarrow \quad \delta < 1. \tag{3.98}$$

Similarly, for the factor in front of m_{LL} observe that

$$\frac{1}{2}\frac{\delta(1-p)}{1-\delta p} < \frac{\delta(1-p)}{1+\delta(1-2p)} \quad \Leftrightarrow \quad \delta < 1. \tag{3.99}$$

Set $\tau(L) = \frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq 0$. For market reports (1,1) and (0,1), an arriving unproductive type's expected utility from reporting truthfully is

$$\frac{1}{2}m_{LL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq 0.$$

For market reports (1,0) and (0,0), an arriving unproductive type's expected utility is

$$\frac{1}{2}\Delta^L(p,\delta)m_{LL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL} = 0.$$

Given $\tau(L)$, we choose

$$\tau(H) = \frac{1}{2}\left(\Delta^L(p,\delta)m_{LL} + \Delta^H(p,\delta)m_{HH} - m_{HL}\right)$$

which is consistent with incentive compatibility by (3.96). Given $\tau(H)$, the productive type's expected utility from truthtelling is

$$\frac{1}{2}(1-\Delta^H(p,\delta))m_{HH} + \frac{1}{2}m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq 0,$$

for market reports (1,1) and (1,0), and

$$\frac{1}{2}m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL} \geq 0$$

for market reports (0,0) and (0,1). Thus, the pair $(\tau(H), \tau(L))$ is individual rational. Furthermore, for the parameter region characterized by (3.97) it holds that

$$\tau(H) \geq \frac{1}{2}\left(\Delta^L(p,\delta)m_{LL} + m_{HL} - \Delta^L(p,\delta)m_{LL}\right) = \frac{1}{2}m_{HL} \geq 0,$$

therefore, the mechanism runs no deficit. Payments are charged only upon arrival and hence support efficient exit. ■

Proof of Proposition 5. Fix the share α of the productive agent in the mixed pair.

The incentive constraint for the productive and the unproductive type are, for market report (0,0),

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} \geq \Delta^L(p,\delta)\alpha m_{HL}, \quad \Delta^H(p,\delta)(1-\alpha)m_{HL} \leq \frac{1}{2}\Delta^L(p,\delta)m_{LL},$$

for market report (1,1),

$$\frac{1}{2}m_{HH} \geq \alpha m_{HL}, \quad (1-\alpha)m_{HL} \leq \frac{1}{2}m_{LL},$$

for market report (1,0),

$$\frac{1}{2}m_{HH} \geq \Delta^L(p,\delta)\alpha m_{HL}, \quad (1-\alpha)m_{HL} \leq \frac{1}{2}\Delta^L(p,\delta)m_{LL},$$

and for market report (0,1),

$$\frac{1}{2}\Delta^H(p,\delta)m_{HH} \geq \alpha m_{HL}, \quad \Delta^H(p,\delta)(1-\alpha)m_{HL} \leq \frac{1}{2}m_{LL}.$$

Observe that, for every market report, the incentive constraint of the productive agent provides an upper bound on α, whereas the incentive constraint of the unproductive agent gives a lower bound on α:

$$\frac{1}{2}\frac{\Delta^H(p,\delta)m_{HH}}{\Delta^L(p,\delta)m_{HL}} \geq \alpha \geq \frac{\Delta^H(p,\delta)m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL}}{\Delta^H(p,\delta)m_{HL}}, \tag{3.100}$$

$$\frac{1}{2}\frac{m_{HH}}{m_{HL}} \geq \alpha \geq \frac{m_{HL} - \frac{1}{2}m_{LL}}{m_{HL}}, \tag{3.101}$$

$$\frac{1}{2}\frac{m_{HH}}{\Delta^L(p,\delta)m_{HL}} \geq \alpha \geq \frac{m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL}}{m_{HL}}, \tag{3.102}$$

$$\frac{1}{2}\frac{\Delta^H(p,\delta)m_{HH}}{m_{HL}} \geq \alpha \geq \frac{\Delta^H(p,\delta)m_{HL} - \frac{1}{2}m_{LL}}{\Delta^H(p,\delta)m_{HL}}. \tag{3.103}$$

The incentive constraint of the productive agent given market report (0,1) yields the lowest upper bound, cf. (3.103), and the incentive constraint of the unproductive agent for market report (1,0) provides the highest lower bound, cf. (3.102). Thus, any incentive compatible match value split has to satisfy

$$\frac{1}{2}\frac{\Delta^H(p,\delta)m_{HH}}{m_{HL}} \geq \alpha \geq \frac{m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL}}{m_{HL}}. \tag{3.104}$$

Note that

$$\frac{1}{2}\frac{\Delta^H(p,\delta)m_{HH}}{m_{HL}} \geq 0 \quad \text{and} \quad 1 \geq \frac{m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL}}{m_{HL}} \geq \frac{1}{2}.$$

(3.104) reveals that an incentive compatible match value split exists iff

$$\frac{1}{2}\frac{\Delta^H(p,\delta)m_{HH}}{m_{HL}} \geq \frac{m_{HL} - \frac{1}{2}\Delta^L(p,\delta)m_{LL}}{m_{HL}}.$$

Rearranging terms yields

$$m_{HL} \leq \Delta^H(p,\delta)\frac{m_{HH}}{2} + \Delta^L(p,\delta)\frac{m_{LL}}{2} \tag{3.105}$$

which coincides with (3.97).

The Positive Assortative Policy without payments supports efficient exit and provides all agents with (expected) utility of at least zero. Thus, individual rationality is satisfied which concludes the proof. ■

3.7 Appendix: Supporting Calculations

Calculations for the proof of Theorem 1.

Preliminaries for equations (3.36)-(3.39). We derive a couple of useful relationships. Using the definition of $V_{PAP}(0,0)$ and inserting $V_{PAP}(2,0)$, $V_{PAP}(0,2)$, $V_{PAP}(2,1)$, $V_{PAP}(1,2)$, the system (3.29) - (3.35) can be reformulated to

$$V_{PAP}(0,0) = \delta[pV_{PAP}(1,0) + (1-p)V_{PAP}(0,1)], \tag{3.106}$$
$$V_{PAP}(1,0) = \delta[p(m_{HH} + V_{PAP}(0,0)) + (1-p)V_{PAP}(1,1)], \tag{3.107}$$
$$V_{PAP}(0,1) = \delta[pV_{PAP}(1,1) + (1-p)(m_{LL} + V_{PAP}(0,0))], \tag{3.108}$$
$$V_{PAP}(1,1) = \delta[p(m_{HH} + V_{PAP}(0,1)) + (1-p)(m_{LL} + V_{PAP}(1,0))]. \tag{3.109}$$

Firstly, consider $V_{PAP}(1,0) - V_{PAP}(0,1)$. By (3.107) and (3.108), we obtain

$$\begin{aligned} &V_{PAP}(1,0) - V_{PAP}(0,1) = \\ &= \delta[p(m_{HH} + V_{PAP}(0,0) - V_{PAP}(1,1)) + (1-p)(-m_{LL} + V_{PAP}(1,1) - V_{PAP}(0,0))], \end{aligned}$$

which yields, inserting (3.106) and (3.109), the equation

$$\begin{aligned} V_{PAP}(1,0) - V_{PAP}(0,1) =& \delta p\,[m_{HH} + \delta p(V_{PAP}(1,0) - m_{HH} - V_{PAP}(0,1)) \\ &+\delta(1-p)(V_{PAP}(0,1) - m_{LL} + V_{PAP}(1,0))] \\ &+\delta(1-p)\,[-m_{LL} + \delta p(m_{HH} + V_{PAP}(0,1) - V_{PAP}(1,0)) \\ &+\delta(1-p)(m_{LL} + V_{PAP}(1,0) - V_{PAP}(0,1))]\,. \end{aligned}$$

Solving for $V_{PAP}(1,0) - V_{PAP}(0,1)$ gives

$$V_{PAP}(1,0) - V_{PAP}(0,1) = \frac{\delta[pm_{HH}(1-2p\delta+\delta) - (1-p)m_{LL}(1-\delta+2p\delta)]}{1-\delta^2(1-2p)^2}. \quad (3.110)$$

Similarly, by (3.109) and (3.106), we obtain

$$V_{PAP}(1,1) - V_{PAP}(0,0) =$$
$$\delta[p(m_{HH} + V_{PAP}(0,1) - V_{PAP}(1,0)) + (1-p)(m_{LL} + V_{PAP}(1,0) - V_{PAP}(0,1))],$$

which gives, inserting (3.107) and (3.108), the equation

$$\begin{aligned} V_{PAP}(1,1) - V_{PAP}(0,0) =& \delta p \left[m_{HH} + \delta p (V_{PAP}(1,1) - m_{HH} - V_{PAP}(0,0)) \right. \\ & \left. + \delta(1-p)(m_{LL} + V_{PAP}(0,0) - V_{PAP}(1,1)) \right] \\ & + \delta(1-p) \left[m_{LL} + \delta p (m_{HH} + V_{PAP}(0,0) - V_{PAP}(1,1)) \right. \\ & \left. + \delta(1-p)(V_{PAP}(1,1) - m_{LL} - V_{PAP}(0,0)) \right]. \end{aligned}$$

Solving for $V_{PAP}(1,1) - V_{PAP}(0,0)$ yields

$$V_{PAP}(1,1) - V_{PAP}(0,0) = \frac{\delta[pm_{HH}(1-2p\delta+\delta) + (1-p)m_{LL}(1-\delta+2p\delta)]}{1-\delta^2(1-2p)^2}. \quad (3.111)$$

Comparing (3.111) to (3.110), observe that

$$V_{PAP}(1,1) - V_{PAP}(0,0) = V_{PAP}(1,0) - V_{PAP}(0,1) + m_{LL} \cdot \underbrace{\frac{2\delta(1-p)}{1+\delta-2p\delta}}_{:=A} \quad (3.112)$$

with

$$0 < A < 1. \quad (3.113)$$

Exploiting (3.112), inequalities (3.36) to (3.39) can be reformulated to

$$V_{PAP}(1,0) - V_{PAP}(0,1) \geq m_{HL} - m_{LL}, \quad (3.114)$$

$$m_{HH} - m_{HL} \geq V_{PAP}(1,0) - V_{PAP}(0,1), \quad (3.115)$$

$$m_{HH} + m_{LL} - m_{HL} \geq V_{PAP}(1,0) - V_{PAP}(0,1) + m_{LL}A, \quad (3.116)$$

$$V_{PAP}(1,0) - V_{PAP}(0,1) \geq m_{HL} - m_{LL}A. \quad (3.117)$$

Deriving equation (3.36). It is immediate that (3.117) implies (3.114), i.e., that (3.39) implies (3.36).

Deriving equation (3.38). Similarly (3.115) implies (3.116), i.e., (3.37) implies (3.38).

Deriving equation (3.39). Inserting (3.110), (3.117) can be written as

$$m_{HL} \leq m_{HH} \frac{\delta p}{1 - \delta + 2p\delta} + m_{LL} \frac{\delta(1-p)}{1 + \delta - 2p\delta} \tag{3.118}$$

which corresponds exactly to $m_{HL} \leq m^1_{HL}$.

Deriving equation (3.37). We argue that (3.39) implies (3.37), i.e., (3.118) implies (3.115).
Inserting (3.110), (3.115) can be written as

$$m_{HL} \leq m_{HH} \left(1 - \frac{\delta p}{1 - \delta + 2p\delta}\right) + m_{LL} \frac{\delta(1-p)}{1 + \delta - 2p\delta}. \tag{3.119}$$

Therefore (3.118) implies (3.115) if

$$\frac{2\delta p}{1 - \delta + 2p\delta} \leq 1 \quad \Leftrightarrow \quad \delta \leq 1, \tag{3.120}$$

which holds.

Deriving equation (3.51). Rearranging terms to isolate $V_{PIP}(1,0) - V_{PIP}(0,1)$ and inserting the expression for $V_{PIP}(1,0) - V_{PIP}(0,1)$, (3.51) can be rewritten as

$$m_{HH} - m_{HL} > \delta[p m_{HH} + (1-2p) m_{HL} - (1-p) m_{LL}]. \tag{3.121}$$

If (3.121) holds for $\delta = 1$, it holds for any δ. Setting $\delta = 1$ and rearranging terms yields $m_{HH} + m_{LL} > 2m_{HL}$ which is satisfied by assumption.

Deriving equation (3.61). We argue that (3.60) implies (3.61):

$$\begin{aligned}
\delta[pV_{MIP}(2,2) + (1-p)V_{MIP}(1,3)] & \\
&= \delta[p(m_{HH} + m_{LL} + V_{MIP}(0,0)) + (1-p)(m_{HL} + m_{LL} + V_{MIP}(0,0))] \\
&= \delta m_{LL} + V_{MIP}(1,0) \\
&< m_{LL} + V_{MIP}(1,0) \\
&\leq m_{HL} + V_{MIP}(0,1),
\end{aligned}$$

where the last inequality follows from (3.60).

Deriving equation (3.58). We show (3.58) exploiting supermodularity and optimality of (0, 1, 0) on state $(1,2)$:

$$\begin{aligned} m_{HL} + \delta[pV_{MIP}(2,2)) + (1-p)V_{MIP}(1,3)] &\leq m_{HL} + V_{MIP}(1,2) \\ &= 2m_{HL} + V_{MIP}(0,1) \\ &\leq m_{HH} + m_{LL} + V_{MIP}(0,1). \end{aligned}$$

Deriving equation (3.57). We show (3.57) exploiting supermodularity and optimality of (0, 1, 0) on state $(1,1)$:

$$\begin{aligned} m_{HL} + \delta[pV_{MIP}(2,1)) + (1-p)V_{MIP}(1,2)] &\leq m_{HL} + V_{MIP}(1,1) \\ &= 2m_{HL} + V_{MIP}(0,0) \\ &\leq m_{HH} + m_{LL} + V_{MIP}(0,0). \end{aligned}$$

Deriving equation (3.56). (3.59) and (3.60) imply (3.56): (3.56) is equivalent to

$$\delta V_{MIP}(0,1) - V_{MIP}(0,0) \leq m_{HL} - \delta p m_{HH} - \delta(1-p)m_{HL},$$

similarly, (3.59) is equivalent to

$$\delta V_{MIP}(0,1) - V_{MIP}(0,0) \leq m_{LL} - \delta p m_{HL} - \delta(1-p)m_{LL}.$$

Therefore, a sufficient condition for (3.56) to hold is

$$m_{LL} - \delta p m_{HL} - \delta(1-p)m_{LL} \leq m_{HL} - \delta p m_{HH} - \delta(1-p)m_{HL},$$

which is equivalent to $m_{HL} \geq m_{HL}^2$, i.e. (3.60).

Deriving $m_{HL} \leq m_{HL}^3$. First, note that whenever there exists a m_{HL} such that $m_{HL} > m_{HL}^3$, then $\frac{1}{2}m_{HH} + \frac{1}{2}m_{LL} > m_{HL}^3$. Reformulating $m_{HL} \leq m_{HL}^3$, $\forall p, \delta, m_{HH}, m_{LL}$, gives

$$m_{HL} - m_{LL} \leq \frac{1-\delta}{\delta p} m_{LL} + \delta[p(m_{HH} - m_{HL}) + (1-p)(m_{HL} - m_{LL})], \quad \forall p, \delta, m_{HH}, m_{LL}.$$

Inserting $m_{HL} = \frac{1}{2}m_{HH} + \frac{1}{2}m_{LL}$ and rearranging terms yields

$$\frac{1}{2}(m_{HH} - m_{LL}) \leq \frac{m_{LL}}{\delta p}, \quad \forall p, \delta, m_{HH}, m_{LL}.$$

This holds if and if

$$\frac{1}{2}(m_{HH} - m_{LL}) \leq m_{LL}, \quad \forall m_{HH}, m_{LL},$$

i.e., $m_{HH} \leq 3m_{LL}, \forall m_{HH}, m_{LL}$.

Calculations for the proof of Proposition 1.

Preliminaries for equations (3.69) - (3.77). It is instructive to rewrite (3.62) - (3.68):

$$V_{P1}(0,0) =\delta[pV_{P1}(1,0) + (1-p)V_{P1}(0,1)], \tag{3.122}$$
$$V_{P1}(1,0) =\delta[p(m_{HH} + V_{P1}(0,0)) + (1-p)(m_{HL} + V_{P1}(0,0))], \tag{3.123}$$
$$V_{P1}(0,1) =\delta[p(m_{HL} + V_{P1}(0,0)) + (1-p)V_{P1}(0,2)], \tag{3.124}$$
$$V_{P1}(0,2) =\delta[p(m_{HL} + V_{P1}(0,1)) + (1-p)(m_{LL} + V_{P1}(0,1))]. \tag{3.125}$$

Plugging (3.123) into (3.122) gives

$$V_{P1}(0,0) = \delta p\left(\delta V_{P1}(0,0) + \delta(pm_{HH} + (1-p)m_{LL})\right) + \delta(1-p)V_{P1}(0,1). \tag{3.126}$$

For $V_{P1}(0,1) - \delta V_{P1}(0,2)$ we obtain, inserting (3.124) and (3.125),

$$\begin{aligned} V_{P1}(0,1) - \delta V_{P1}(0,2) =& \delta p(m_{HL} + V_{P1}(0,0) - \delta m_{HL} - \delta V_{P1}(0,1)) \\ &+ \delta(1-p)(V_{P1}(0,2) - \delta m_{LL} - \delta V_{P1}(0,1)). \end{aligned} \tag{3.127}$$

Rearranging (3.125) we see that

$$V_{P1}(0,2) - \delta V_{P1}(0,1) = \delta(pm_{HL} + (1-p)m_{LL}). \tag{3.128}$$

Inserting (3.124), (3.126), and (3.128) into (3.127), we can solve for $V_{P1}(0,1) - \delta V_{P1}(0,2)$ which is explicitly given by

$$V_{P1}(0,1) - \delta V_{P1}(0,2) = \frac{\delta p\left[m_{HL} - \delta m_{LL} + m_{HH}p^2\delta^2 - m_{HL}p^2\delta^2 + m_{LL}p\delta - m_{HL}p\delta\right]}{1 + p^2\delta^2 - p\delta^2}. \tag{3.129}$$

Next, we derive a closed-form expression for $V_{P1}(0,0) - \delta V_{P1}(0,1)$. To this end, note that

$$\begin{aligned} V_{P1}(0,0) - \delta V_{P1}(0,1) =& \delta p(V_{P1}(1,0) - \delta m_{HL} - \delta V_{P1}(0,0)) \\ &+ \delta(1-p)(V_{P1}(0,1) - \delta V_{P1}(0,2)). \end{aligned} \tag{3.130}$$

Furthermore by (3.123)

$$V_{P1}(1,0) - \delta V_{P1}(0,0) = \delta(pm_{HH} + (1-p)m_{HL}). \tag{3.131}$$

Inserting (3.131) and (3.129) into (3.130) gives

$$V_{P1}(0,0)-\delta V_{P1}(0,1)=\frac{p\delta^2\left[m_{HH}p+m_{HL}(1-2p-p\delta+p^2\delta)+m_{LL}(-\delta+2p\delta-p^2\delta)\right]}{1-p\delta^2+p^2\delta^2}. \tag{3.132}$$

Deriving equation (3.72). Rearranging terms in (3.72) gives

$$V_{P1}(0,1)-\delta V_{P1}(0,2)\geq\delta\left[pm_{HL}+(1-p)m_{LL}\right]-m_{LL}. \tag{3.133}$$

Plugging (3.129) into (3.133) and rewriting (3.133) as a condition on m_{HL}, we obtain

$$m_{HL}\leq\frac{1}{1-\delta(1-2p)}\left[m_{HH}\delta p+m_{LL}\left(2-\delta(2-p)+\frac{1-\delta+\delta p(1-\delta)^2}{\delta^2p^2}\right)\right]. \tag{3.134}$$

The term on the right side of (3.134) is m^4_{HL}.

Deriving equation (3.74). We argue that (3.74) holds if and only if $m_{HL}\geq m^3_{HL}$. Inserting $V_{P1}(1,2)$ and $V_{P1}(0,3)$ in (3.74) gives

$$\delta\left[p(m_{HL}+V_{P1}(0,1))+(1-p)(m_{LL}+V_{P1}(0,1))\right]\geq m_{LL}+V_{P1}(0,0).$$

Rearranging terms yields

$$\delta\left[pm_{HL}+(1-p)m_{LL}\right]-m_{LL}\geq V_{P1}(0,0)-\delta V_{P1}(0,1). \tag{3.135}$$

Plugging (3.132) into (3.135) and some algebra yields

$$m_{HL}\geq m_{HH}\frac{\delta p}{1-\delta(1-2p)}+m_{LL}\frac{1-\delta(1-p)+\frac{1-\delta}{\delta p}}{1-\delta(1-2p)}. \tag{3.136}$$

Observe that the right side of (3.136) coincides with m^3_{HL}.

Deriving equation (3.76). We show that (3.72) and (3.74) imply (3.76). Reformulating (3.72) gives

$$m_{LL}-\delta pm_{HL}-\delta(1-p)m_{LL}\geq\delta V_{P1}(0,2)-V_{P1}(0,1), \tag{3.137}$$

and (3.76) gives

$$m_{HL}-\delta pm_{HH}-\delta(1-p)m_{HL}\geq\delta V_{P1}(0,2)-V_{P1}(0,1). \tag{3.138}$$

(3.137) implies (3.138) if

$$m_{HL} - m_{LL} \geq \delta p(m_{HH} - m_{HL}) + \delta(1-p)(m_{HL} - m_{LL}). \tag{3.139}$$

Note that (3.139) coincides with $m_{HL} \geq m_{HL}^2$. As (3.74) requires $m_{HL} \geq m_{HL}^3$ and $m_{HL}^3 \geq m_{HL}^2$, (3.139) is implied by (3.74).

Deriving equation (3.70). We argue that (3.74) implies (3.70). Note that we can rewrite (3.74) as

$$V_{P1}(0,0) - \delta V_{P1}(0,1) \leq \delta \left(p m_{HL} + (1-p) m_{LL}\right) - m_{LL}, \tag{3.140}$$

and (3.70) as

$$V_{P1}(0,0) - \delta V_{P1}(0,1) \geq \delta \left(p m_{HH} + (1-p) m_{HL}\right) - m_{HL}. \tag{3.141}$$

Plugging (3.132) into (3.140) and (3.141) yields, after some algebra, for (3.140)

$$\delta^2 p^2 (m_{HH} - m_{HL}) + m_{LL}(1-\delta) \leq \delta^2 p^2 (m_{HL} - m_{LL}) + (1-\delta) p \delta (m_{HL} - m_{LL}), \tag{3.142}$$

and for (3.141)

$$\begin{aligned} \delta p(m_{HH} - m_{HL})(1 - \delta p - \delta^2 p(1-p)) + (1-p)p^2\delta^3(m_{HL} - m_{LL}) \leq \\ m_{HL}(1-\delta) + (1-p)p\delta^3(m_{HL} - m_{LL}). \end{aligned} \tag{3.143}$$

It is sufficient for (3.140) to imply (3.141), i.e. (3.142) to imply (3.143), if it holds that

$$\begin{aligned} (\delta p(m_{HL} - m_{LL}) + (1-\delta)(m_{HL} - m_{LL}))(1 - \delta p - \delta^2 p(1-p)) \leq \\ m_{HL}(1-\delta) + (1-p)^2 p\delta^3(m_{HL} - m_{LL}). \end{aligned} \tag{3.144}$$

For (3.144) in turn it is sufficient if

$$(\delta p + (1-\delta))(1 - \delta p - \delta^2 p(1-p)) \leq (1-\delta) + (1-p)^2 p \delta^3. \tag{3.145}$$

Simplifying (3.145) shows that the term on the left side coincides with the term on the right side.

Deriving equation (3.69). Supermodularity and (3.74) imply (3.69). Inserting (3.123) and (3.124) into (3.69) and rearranging terms yields

$$m_{HH} - m_{HL} \geq \delta \left[p(m_{HH} - m_{HL}) + (1-p)(m_{HL} + V_{P1}(0,0) - V_{P1}(0,2))\right]. \tag{3.146}$$

By (3.74) it is sufficient for (3.146) to hold that

$$m_{HH} - m_{HL} \geq \delta \left[p m_{HH} + (1-2p) m_{HL} - (1-p) m_{LL} \right]. \tag{3.147}$$

Reformulating yields

$$\frac{m_{HH} - m_{HL}}{m_{HL} - m_{LL}} \geq \frac{\delta(1-p)}{1-\delta p}. \tag{3.148}$$

By supermodularity the left side of (3.148) is larger than one, whereas the right side of (3.148) is smaller than one. Therefore, (3.74) implies (3.69).

Deriving equation (3.77). (3.77) is implied by (3.75), (3.76), and supermodularity: Given (3.75) and (3.76), from optimality on state $(1,2)$ we know

$$m_{HL} + \delta[pV_{P1}(2,2) + (1-p)V_{P1}(1,3)] \leq 2m_{HL} + V_{P1}(0,1), \tag{3.149}$$

and from supermodularity follows

$$2m_{HL} + V_{P1}(0,1) \leq m_{HH} + m_{LL} + V_{P1}(0,1). \tag{3.150}$$

Combining (3.149) and (3.150) gives (3.77).

Deriving equation (3.71). (3.71) is implied by (3.70) and supermodularity: Given (3.70), from optimality on state $(1,1)$ we know

$$m_{HL} + \delta[pV_{P1}(2,1) + (1-p)V_{P1}(1,2)] \leq 2m_{HL} + V_{P1}(0,0), \tag{3.151}$$

and from supermodularity follows

$$2m_{HL} + V_{P1}(0,0) \leq m_{HH} + m_{LL} + V_{P1}(0,0). \tag{3.152}$$

Combining (3.151) and (3.152) gives (3.71).

Deriving equation (3.73). (3.73) is implied by (3.74), (3.70), and supermodularity: Given (3.74), from optimality on state $(0,2)$ we know

$$m_{HL} + V_{P1}(0,2) \geq m_{HL} + m_{LL} + V_{P1}(0,0), \tag{3.153}$$

and given (3.70), from optimality on state $(1,1)$ follows

$$m_{HH} + m_{LL} + V_{P1}(0,0) \leq m_{LL} + \delta[pV_{P1}(2,1) + (1-p)V_{P1}(1,2)]. \tag{3.154}$$

Combining (3.153) and (3.154) gives (3.73).

Deriving equation (3.75). We argue that (3.74) implies (3.75) by showing that 'not (3.75)' implies 'not (3.74)'. Applying 'not (3.75)' and (3.74) leads to a contradiction:

$$\begin{aligned} m_{LL} + V_{P1}(0,0) &\leq \delta[p(m_{HL} + V_{P1}(0,1)) + (1-p)(m_{LL} + V_{P1}(0,1))] \\ &\leq \delta[p(m_{LL} + V_{P1}(1,0)) + (1-p)(m_{LL} + V_{P1}(0,1))] \\ &= \delta m_{LL} + V_{P1}(0,0). \end{aligned}$$

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