Construction of a Rapoport-Zink space for split $\mathrm{GU}(1,1)$ in the ramified 2-adic case

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CONSTRUCTION OF A RAPOPORT-ZINK SPACE FOR SPLIT $\mathrm{GU}(1,1)$ IN THE RAMIFIED 2-ADIC CASE

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ABSTRACT. Let $F|\mathbb{Q}_2$ be a finite extension. In this paper, we construct an RZ-space $\mathcal{N}_E$ for split $\mathrm{GU}(1,1)$ over a ramified quadratic extension $E|F$. For this, we first introduce the naive moduli problem $\mathcal{N}_E^{\mathrm{naive}}$ and then define $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$ as a canonical closed formal subscheme, using the so-called straightening condition. We establish an isomorphism between $\mathcal{N}_E$ and the Drinfeld moduli problem, proving the 2-adic analogue of a theorem of Kudla and Rapoport. We also give the definition of a local model for $\mathcal{N}_E$ as a flat projective scheme over O_F which, locally for the étale topology, models the singularities of $\mathcal{N}_E$. The formulation of the straightening condition uses the existence of certain polarizations on the points of the moduli space $\mathcal{N}_E^{\mathrm{naive}}$. We show the existence of these polarizations in a more general setting over any quadratic extension $E|F$, where $F|\mathbb{Q}_p$ is a finite extension for any prime p.

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1. Introduction

In this paper, we give a new example of a 2-adic Rapoport-Zink space. Rapoport-Zink spaces are moduli spaces of p-divisible groups that are endowed with certain additional structure. There are two major classes of Rapoport-Zink spaces, namely the (EL) type and the (PEL) type. Here the abbreviations (EL) and (PEL) indicate whether the extra structure comes in form of endomorphisms and level structure (in the (EL) case) or in form of polarizations, endomorphisms and level structure (in the (PEL) case). In [RZ96], Rapoport and Zink define RZ-spaces for the (EL) type, and for the (PEL) type whenever $p \neq 2$. They prove that these moduli spaces are pro-representable by formal schemes.
A general definition for RZ-space of (PEL) type in the case $p = 2$ remains still unknown. In this paper, we construct the 2-adic Rapoport-Zink space $\mathcal{N}_E$ corresponding to the group of unitary similitudes of size 2 relative to any (wildly) ramified quadratic extension $E|F$, where $F|\mathbb{Q}_2$ is a finite extension. Furthermore, we show that there is a natural isomorphism $\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$, where $\mathcal{M}_{Dr}$ is Deligne's formal model of the Drinfeld upper halfplane (cf. [BC91]). This result is in analogy with [KR11], where Kudla and Rapoport construct a corresponding isomorphism for $p \neq 2$ and also for $p = 2$ when $E|F$ is an unramified extension. The formal scheme $\mathcal{M}_{Dr}$ solves a certain moduli problem of p-divisible groups and, in this way, it carries the structure of an RZ-space of (EL) type. In particular, $\mathcal{M}_{Dr}$ is defined even for $p = 2$.

We will now explain the results of this paper in greater detail. Let F be a finite extension of $\mathbb{Q}_2$ and $E|F$ a ramified quadratic extension. Following [Jac62], we consider the following dichotomy for this extension (see section 2):

(R-P) There is a uniformizer $\pi_0 \in F$, such that $E = F[\Pi]$ with $\Pi^2 = \pi_0$. Then the rings of integers O_F and O_E of F and E satisfy $O_E = O_F[\Pi]$.

(R-U) $E|F$ is generated by the square root ϑ of a unit in F. We can choose ϑ such that $\vartheta^2 = 1 + \pi_0^{2k+1}\varepsilon$ for a unit $\varepsilon \in O_F^\times$ and for an integer k with $|2| < |\pi_0^k| \leq |1|$, where $|\cdot|$ is the (normalized) absolute value on F.

An example for an extension of type (R-P) is $\mathbb{Q}_2(\sqrt{2})|\mathbb{Q}_2$, whereas $\mathbb{Q}_2(\sqrt{3})|\mathbb{Q}_2$ is of type (R-U).
The results in the cases (R-P) and (R-U) are similar, but different. We first describe our results in the case (R-P). Let $E|F$ be of type (R-P).

We first define a naive moduli problem $\mathcal{N}_E^{\mathrm{naive}}$, that merely copies the definition from $p \neq 2$ (cp. [KR11]). Let $\breve{F}$ be the completion of the maximal unramified extension of F and $\breve{O}_F$ its ring of integers. Then $\mathcal{N}_E^{\mathrm{naive}}$ is a set-valued functor on $\mathrm{Nilp}_{\breve{O}_F}$, the category of $\breve{O}_F$-schemes where π_0 is locally nilpotent. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, the set $\mathcal{N}_E^{\mathrm{naive}}(S)$ is the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$. Here, X/S is a formal O_F-module of height 4 and dimension 2, equipped with an action $\iota : O_E \to \mathrm{End}(X)$. This action satisfies the Kottwitz condition of signature $(1,1)$, i. e., for any $\alpha \in O_E$, the characteristic polynomial of $\iota(\alpha)$ on $\mathrm{Lie}\, X$ is given by

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \overline{\alpha}).$$

Here, $\alpha \mapsto \overline{\alpha}$ denotes the Galois conjugation of $E|F$. The right hand side of this equation is a polynomial with coefficients in $\mathcal{O}_S$ via the structure map $O_F \hookrightarrow \breve{O}_F \to \mathcal{O}_S$. The third entry λ is a principal polarization $\lambda : X \to X^\vee$ such that the induced Rosati involution satisfies $\iota(\alpha)^* = \iota(\overline{\alpha})$ for all $\alpha \in O_E$. Finally, ϱ is a quasi-isogeny of height 0 (and compatible with all previous data) to a fixed framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\bar{k} = \breve{O}_F/\pi_0$. This framing object is unique up to isogeny under the condition that

$$\{\varphi \in \mathrm{End}^0(\mathbb{X}, \iota_{\mathbb{X}}) \mid \varphi^*(\lambda_{\mathbb{X}}) = \lambda_{\mathbb{X}}\} \simeq \mathrm{U}(C, h),$$

for a split $E|F$-hermitian vector space (C,h) of dimension 2, see Lemma 3.2.
It turns out that the definition of $\mathcal{N}_E^{\mathrm{naive}}$ is not the correct one. In order to illustrate this, let us recall the definition of the Drinfeld moduli problem $\mathcal{M}_{Dr}$. It is the functor on $\mathrm{Nilp}_{\breve{O}_F}$, mapping a scheme S to the set $\mathcal{M}_{Dr}(S)$ of equivalence classes of tuples (X,ι_B,ϱ). Again, X/S is a formal O_F-module of height 4 and dimension 2. Let B be the quaternion division algebra over F and O_B its ring of integers. Then ι_B is an action of O_B on X, satisfying the *special* condition of Drinfeld (see [BC91] or section 3.3 below). The last entry ϱ is an O_B-linear quasi-isogeny of height 0 to a fixed framing object $(\mathbb{X},\iota_{\mathbb{X},B})$ over $\bar{k}$. This framing object is unique up to isogeny (cf. [BC91, II. Prop. 5.2]).
Fix an embedding $O_E \hookrightarrow O_B$ and consider the involution $b \mapsto b^* = \Pi b' \Pi^{-1}$ on B, where $b \mapsto b'$ is the standard involution. By Drinfeld (see Prop. 3.12 below), there exists a principal polarization $\lambda_{\mathbb{X}}$ on the framing object $(\mathbb{X},\iota_{\mathbb{X},B})$ of $\mathcal{M}_{Dr}$, such that the induced Rosati involution satisfies $\iota_{\mathbb{X},B}(b)^* = \iota_{\mathbb{X},B}(b^*)$ for all $b \in O_B$. This polarization is unique up to a scalar in $O_F^\times$. Furthermore, for any $(X,\iota_B,\varrho) \in \mathcal{M}_{Dr}(S)$, the pullback $\lambda = \varrho^*(\lambda_{\mathbb{X}})$ is a principal polarization on X.
We now set

$$\eta(X,\iota_B,\varrho) = (X,\iota_B|_{O_E},\lambda,\varrho).$$

By Lemma 3.13, this defines a closed embedding $\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}$. But η is far from being an isomorphism, as follows from the following proposition:

Proposition 1.1. *The induced map $\eta(\bar{k}) : \mathcal{M}_{Dr}(\bar{k}) \to \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is not surjective.*

Let us sketch the proof here. Using Dieudonné theory, we can write $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ naturally as a union

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union runs over all O_E-lattices Λ in the hermitian vector space (C,h) that are Π^{-1}-modular, i. e., the dual $\Lambda^\sharp$ of Λ with respect to h is given by $\Lambda = \Pi^{-1}\Lambda^\sharp$ (see Lemma 3.6). By Jacobowitz ([Jac62]), there exist different types (i. e. $\mathrm{U}(C,h)$-orbits) of such lattices $\Lambda \subseteq C$ that are parametrized by their norm ideal $\mathrm{Nm}(\Lambda) = \langle \{h(x,x) | x \in \Lambda\} \rangle \subseteq F$. In the case at hand, $\mathrm{Nm}(\Lambda)$ can be any ideal with $2O_F \subseteq \mathrm{Nm}(\Lambda) \subseteq O_F$. If the norm ideal of Λ is minimal, that is, if $\mathrm{Nm}(\Lambda) = 2O_F$, we call Λ *hyperbolic*. Equivalently, the lattice Λ has a basis consisting of isotropic vectors. Now, the image under η of $\mathcal{M}_{Dr}(\bar{k})$ is the union of all lines $\mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ where $\Lambda \subseteq C$ is hyperbolic. This is a consequence of Remark 3.11 and Theorem 3.14 below.
On the framing object $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ of $\mathcal{N}_E^{\mathrm{naive}}$, there exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}$ such that the induced Rosati involution is the identity on O_E. This polarization is unique up to a scalar in $O_E^\times$ (see Thm. 6.2 (1)). On C, the polarization $\widetilde{\lambda}_{\mathbb{X}}$ induces an E-linear alternating form b, such that $\det b$ and $\det h$ differ only by a unit (for a fixed basis of C). After possibly rescaling b by a unit in $O_E^\times$, a Π^{-1}-modular lattice $\Lambda \subseteq C$ is hyperbolic if and only if $b(x,y) + h(x,y) \in 2O_F$ for all $x,y \in \Lambda$. This enables us to describe the "hyperbolic" points of $\mathcal{N}_E^{\mathrm{naive}}$ (i. e., those that lie on a projective line corresponding to a hyperbolic lattice $\Lambda \subseteq C$) in terms of polarizations.
We now formulate the closed condition that characterizes $\mathcal{N}_E$ as a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$. For a suitable choice of $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$, we may assume that $\frac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$. The following definition is a reformulation of Def. 3.10.

Definition 1.2. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. An object $(X,\iota,\lambda,\varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ satisfies the *straightening* condition, if $\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda})$ is a polarization on X. Here, $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$.

We remark that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a polarization on X. This is a consequence of Theorem 6.2, which states the existence of certain polarizations on points of a larger moduli space $\mathcal{M}_E$ containing $\mathcal{N}_E^{\mathrm{naive}}$, see below.

For $S \in \mathrm{Nilp}_{\breve{\mathrm{O}}_{\mathrm{F}}}$, let $\mathcal{N}_E(S) \subseteq \mathcal{N}_E^{\mathrm{naive}}(S)$ be the subset of all tuples $(X, \iota, \lambda, \varrho)$ that satisfy the straightening condition. By [RZ96, Prop. 2.9], this defines a closed formal subscheme $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$. An application of Drinfeld's Proposition (see Prop. 3.12) shows that the image of $\mathcal{M}_{Dr}$ under η lies in $\mathcal{N}_E$. The main theorem in the (R-P) case can now be stated as follows, cf. Theorem 3.14.

Theorem 1.3. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

We next turn our attention towards the construction of a local model $\mathrm{N}_E^{\mathrm{loc}}$ for $\mathcal{N}_E$. We start with an O_E-lattice $\Lambda \subseteq C$ that is selfdual with respect to the hermitian form h and is hyperbolic. After twisting b by a scalar in $O_E^\times$ if necessary, we may assume that $\frac{1}{2}(h + b)$ is integral on Λ. The forms h and b induce O_F-bilinear alternating forms $\langle\, ,\rangle$, $(\, ,)$ and $\langle\, ,\rangle_1$ on Λ, where

$$\langle x, y\rangle_1 = \frac{1}{2}(\langle x, y\rangle + (x, y)),$$

for all $x, y \in \Lambda$. Recall that C has dimension 2 over E, so Λ is an O_F-lattice of rank 4. Let R be an O_F-algebra. We define $\mathrm{N}_E^{\mathrm{loc}}(R)$ as the set of direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ of rank 2 that are O_E-stable and totally isotropic with respect to $\langle\, ,\rangle$, $(\, ,)$ and $\langle\, ,\rangle_1$. Additionally, we impose the Kottwitz condition on the quotient $(\Lambda \otimes_{O_F} R)/\mathcal{F}$, i. e., the characteristic polynomial for the action of any $\alpha \in O_E$ is given by

$$\mathrm{char}((\Lambda \otimes_{O_F} R)/\mathcal{F}, T \mid \alpha) = (T - \alpha)(T - \overline{\alpha}).$$

The functor $\mathrm{N}_E^{\mathrm{loc}}$ is representable by a closed subscheme of a Grassmanian over O_F. In particular, $\mathrm{N}_E^{\mathrm{loc}}$ is projective. Let $F^{(2)}|F$ be the unramified quadratic extension and let $O_F^{(2)}$ be its ring of integers. We denote the local model for the Drinfeld moduli problem by $\mathrm{M}_{Dr}^{\mathrm{loc}}$, cf. [RZ96, Def. 3.27].

Proposition 1.4. *Fix an embedding $O_E \hookrightarrow O_B$. There is an isomorphism*

$$\mu : \mathrm{M}_{Dr}^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \mathrm{N}_E^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)}.$$

In particular, $\mathrm{N}_E^{\mathrm{loc}}$ is flat.

Let $\widehat{\mathrm{N}}_E^{\mathrm{loc}}$ be the π_0-adic completion of $\mathrm{N}_E^{\mathrm{loc}} \otimes_{O_F} \breve{O}_F$. We have a local model diagram for $\mathcal{N}_E$ in the sense of [RZ96], i. e., a diagram

$$\begin{array}{ccc} & \mathcal{M} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathcal{N}_E & & \widehat{\mathrm{N}}_E^{\mathrm{loc}} \end{array} \tag{1.1}$$

of surjective and formally smooth morphisms of formal schemes of identical relative dimension. It follows that the completed local rings at points of $\mathcal{N}_E$ are isomorphic to completed local rings at points of $\mathrm{N}_E^{\mathrm{loc}}$. The diagram (1.1) is compatible with the local model diagram for the Drinfeld case.

Note that there is no good notion of a "naive local model" corresponding to $\mathcal{N}_E^{\mathrm{naive}}$, since there are different types of selfdual lattices $\Lambda \subseteq (C, h)$, cp. Prop. 2.4. Thus there is no such thing as a "standard lattice" and it is not clear how one should define a local model for $\mathcal{N}_E^{\mathrm{naive}}$.

This concludes our discussion of the (R-P) case. From now on, we assume that $E|F$ is of type (R-U).

In the case (R-U), we have to make some adaptions for $\mathcal{N}_E^{\mathrm{naive}}$. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{N}_E^{\mathrm{naive}}(S)$ be the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$ with (X, ι) as in the (R-P) case. But now, the polarization $\lambda : X \to X^\vee$ is supposed to have kernel $\ker \lambda = X[\Pi]$ (in contrast to the (R-P) case, where λ is a principal polarization). As before, the Rosati involution of λ induces the conjugation on O_E. There exists a framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\mathrm{Spec}\, \bar{k}$ for $\mathcal{N}_E^{\mathrm{naive}}$, which is unique under the condition that

$$\{\varphi \in \mathrm{End}^0(\mathbb{X}, \iota_{\mathbb{X}}) \mid \varphi^*(\lambda_{\mathbb{X}}) = \lambda_{\mathbb{X}}\} \simeq \mathrm{U}(C, h),$$

where (C, h) is a split $E|F$-hermitian vector space of dimension 2 (cf. Prop. 4.2). Finally, ϱ is a quasi-isogeny of height 0 from X to $\mathbb{X}$, respecting all structure.
Fix an embedding $E \hookrightarrow B$. Using some subtle choices of elements in B (these are described in Lemma 2.3 (2)) and by Drinfeld's Proposition, we can construct a polarization λ as above for any $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$. This induces a closed embedding

$$\eta : \mathcal{M}_{Dr} \longrightarrow \mathcal{N}_E^{\mathrm{naive}}, (X, \iota_B, \varrho) \longmapsto (X, \iota_B|_{O_E}, \lambda, \varrho).$$

We can write $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ as a union of projective lines,

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union now runs over all selfdual O_E-lattices $\Lambda \subseteq (C, h)$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. As in the (R-P) case, these lattices $\Lambda \subseteq C$ are classified up to isomorphism by their norm ideal $\mathrm{Nm}(\Lambda)$. Since Λ is selfdual with respect to h, the norm ideal can be any ideal satisfying $\frac{2}{\pi_0^k} O_F \subseteq \mathrm{Nm}(\Lambda) \subseteq O_F$. We call Λ *hyperbolic* when the norm ideal is minimal, i. e., $\mathrm{Nm}(\Lambda) = \frac{2}{\pi_0^k} O_F$. Equivalently, the lattice Λ has a basis consisting of isotropic vectors. Recall that here k is an integer, depending on the (R-U) extension $E|F$, with $|2| < |\pi_0^k| \leq |1|$. So we always have $|\frac{2}{\pi_0^k}| \leq |\pi_0|$ and hence there exists at least one type of selfdual lattices $\Lambda \subseteq C$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. In the case (R-U), it may happen that $|2| = |\pi_0^{k+1}|$, in which case all lattices Λ in the description of $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ are hyperbolic.
The image of $\mathcal{M}_{Dr}(\bar{k})$ under η in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is the union of all projective lines corresponding to hyperbolic lattices. Unless $|2| = |\pi_0^{k+1}|$, it follows that $\eta(\bar{k})$ is not surjective and thus η cannot be an isomorphism. For the case $|2| = |\pi_0^{k+1}|$, we will show that η is an isomorphism on reduced loci $(\mathcal{M}_{Dr})_{\mathrm{red}} \xrightarrow{\sim} (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$ (see Remark 4.12), but η is not an isomorphism of formal schemes. This follows from the non-flatness of the "naive local model" $\mathrm{N}_E^{\mathrm{naive}}$ for $\mathcal{N}_E^{\mathrm{naive}}$, see section 5.3.
On the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_E^{\mathrm{naive}}$, there exists a polarization $\widetilde{\lambda}_{\mathbb{X}}$ such that $\ker \widetilde{\lambda}_{\mathbb{X}} = \mathbb{X}[\Pi]$ and such that the Rosati involution induces the identity on O_E. After a suitable choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$, we may assume that $\frac{\pi_0^k}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$. The straightening condition for the (R-U) case is given as follows (cp. Def. 4.11).

Definition 1.5. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. An object $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ satisfies the *straightening* condition, if $\lambda_1 = \frac{\pi_0^k}{2}(\lambda + \widetilde{\lambda})$ is a polarization on X. Here, $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$.

Note that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a polarization on X by Theorem 6.2.
The straightening condition defines a closed formal subscheme $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$ that contains the image of $\mathcal{M}_{Dr}$ under η. The main theorem in the (R-U) case can now be stated as follows, cf. Theorem 4.15.

Theorem 1.6. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

We now define a local model $\mathrm{N}_E^{\mathrm{loc}}$ for $\mathcal{N}_E$. Let $\Lambda \subseteq C$ be a Π-modular lattice with respect to h (i. e., $\Lambda = \Pi\Lambda^\sharp$). The forms h and b induce O_F-linear alternating forms $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ with

$$\langle x, y\rangle_1 = \frac{\pi_0^k}{2}(\langle x, y\rangle + (x, y)),$$

for all $x, y \in \Lambda$.

Let R be an O_F-algebra. Then $\mathrm{N}_E^{\mathrm{loc}}(R)$ is the set of direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ of rank 2 that are O_E-stable and totally isotropic with respect to the forms induced by $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$. Since Λ is Π-modular, the alternating forms $\langle\,,\rangle$ and $(\,,)$ are not perfect on Λ, thus we have to twist by Π^{-1} here, see (5.8). We also impose the Kottwitz condition on $(\Lambda \otimes_{O_F} R)/\mathcal{F}$, see the (R-P) case above. The functor $\mathrm{N}_E^{\mathrm{loc}}$ is representable by a closed subscheme of a Grassmanian over O_F and, in particular, is projective. We have the following proposition, cf. Prop. 5.8.

Proposition 1.7. *Fix an embedding $E \hookrightarrow B$ and let $O_F^{(2)}$ be the unramified quadratic extension of O_F. There is an isomorphism $\mu : \mathrm{M}_{Dr}^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \mathrm{N}_E^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)}$. In particular, $\mathrm{N}_E^{\mathrm{loc}}$ is flat over O_F.*

As in the (R-P) case, there exists a local model diagram (1.1) connecting $\mathcal{N}_E$ and $\mathrm{N}_E^{\mathrm{loc}}$. It is compatible with the local model diagram for the Drinfeld case.

When formulating the straightening condition in the (R-U) and the (R-P) case, we mentioned that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a polarization for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$. This fact is a corollary of Theorem 6.2, that states the existence of this polarization in the following more general setting.

Let $F|\mathbb{Q}_p$ be a finite extension for any prime p and $E|F$ an arbitrary quadratic extension. We consider the following moduli space $\mathcal{M}_E$ of (EL) type. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, the set $\mathcal{M}_E(S)$ consists of equivalence classes of tuples (X, ι_E, ϱ), where X is a formal O_F-module of height 4 and dimension 2 and ι_E is an O_E-action on X satisfying the Kottwitz condition of signature $(1, 1)$, see above. The entry ϱ is an O_E-linear quasi-isogeny of height 0 to a supersingular framing object $(\mathbb{X}, \iota_{\mathbb{X},E})$.

The points of $\mathcal{M}_E$ are equipped with polarizations in the following natural way, cf. Theorem 6.2.

Theorem 1.8. (1) *There exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ such that the Rosati involution induces the identity on O_E, i. e., $\iota(\alpha)^* = \iota(\alpha)$ for all $\alpha \in O_E$. This polarization is unique up to a scalar in $O_E^\times$.*

(2) *Fix $\widetilde{\lambda}_{\mathbb{X}}$ as in part (1). For any $S \in \mathrm{Nilp}_{\breve{O}_F}$ and $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$, there exists a unique principal polarization $\widetilde{\lambda}$ on X such that the Rosati involution induces the identity on O_E and such that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$.*

If $p = 2$ and $E|F$ is ramified of (R-P) or (R-U) type, then there is a canonical closed embedding $\mathcal{N}_E \hookrightarrow \mathcal{M}_E$ that forgets about the polarization λ. In this way, it follows that $\widetilde{\lambda}$ is a polarization for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$.

The statement of Theorem 1.8 can also be expressed in terms of an isomorphism of moduli spaces $\mathcal{M}_{E,\mathrm{pol}} \xrightarrow{\sim} \mathcal{M}_E$. Here $\mathcal{M}_{E,\mathrm{pol}}$ is a moduli space of (PEL) type, defined by mapping $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of tuples $(X, \iota, \widetilde{\lambda}, \varrho)$ where $(X, \iota, \varrho) \in \mathcal{M}_E(S)$ and $\widetilde{\lambda}$ is a polarization as in the theorem.

We now briefly describe the contents of the subsequent sections of this paper. In section 2, we recall some facts about the quadratic extensions of F, the quaternion algebra $B|F$ and hermitian forms. In the next two sections, sections 3 and 4, we define the moduli spaces $\mathcal{N}_E^{\mathrm{naive}}$, introduce the straightening condition describing $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$ and prove our main theorem in both the cases (R-P) and (R-U). Although the techniques are quite similar in both cases, we decided to treat these cases separately, since the results in both cases differ in important details. In section 5, we define the local model $\mathrm{N}_E^{\mathrm{loc}}$, prove that it is isomorphic to $\mathrm{M}_{Dr}^{\mathrm{loc}}$ and construct the local model diagram connecting $\mathrm{N}_E^{\mathrm{loc}}$ and $\mathcal{N}_E$. Again, we explain these results separately for the cases (R-P) and (R-U). The section 5.3 is dedicated to a discussion of the naive local model $\mathrm{N}_E^{\mathrm{naive}}$ for $\mathcal{N}_E^{\mathrm{naive}}$, in the case where $E|F$ is of type (R-U) with $|2| = |\pi_0^{k+1}|$. In particular, we prove the necessity of the straightening condition in that specific case. Finally, in section 6, we prove Theorem 1.8 on the existence of the polarizations $\widetilde{\lambda}$.

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2. Preliminaries on quaternion algebras and hermitian forms

Let $F|\mathbb{Q}_2$ be a finite extension. In this section we will recall some facts about the quadratic extensions of F, the quaternion division algebra $B|F$ and certain hermitian forms. For more information on quaternion algebras, see for example the book by Vigneras [Vig80]. A systematic classification of hermitian forms over local fields has been done by Jacobowitz in [Jac62].
Let $E|F$ be a quadratic field extension and denote by O_F resp. O_E the rings of integers. There are three mutually exclusive possibilities for $E|F$:

- $E|F$ is unramified. Then $E = F[\delta]$ for δ a square root of a unit in F. We can choose δ such that $\delta^2 = 1 + 4\varepsilon_0$ for some $\varepsilon_0 \in O_F^\times$. In this case, $O_E = O_F[\frac{1+\delta}{2}]$. In the following we will write $F^{(2)}$ instead of E and $O_F^{(2)}$ instead of O_E when talking about the unramified extension of F.
- $E|F$ is ramified and E is generated by the square root of a uniformizer in F. That is, $E = F[\Pi]$ and $\Pi^2 = \pi_0$ is a uniformizing element in O_F. We also have $O_E = O_F[\Pi]$. Following Jacobowitz, we will say $E|F$ is of type (R-P) (which stands for "ramified-prime").
- $E|F$ is ramified and E is generated by the square root of a unit in F. Here $E = F[\vartheta]$ with $\vartheta^2 \in O_F^\times$. We can choose ϑ such that ϑ^2 is of the form $\vartheta^2 = 1+\pi_0^{2k+1}\varepsilon$. Here π_0 is a uniformizer of O_F, $\varepsilon \in O_F^\times$ and k is a non-negative integer such that $|2| < |\pi_0|^k \leq |1|$ for the (normalized) absolute value $|\cdot|$ on F. A uniformizer of O_E is given by $\Pi = (1+\vartheta)/\pi_0^k$ and $O_E = O_F[\Pi]$. In this case $E|F$ is said to be of type (R-U) (for "ramified-unit").

Throughout the paper, we use this notation for the (R-U) case.

Remark 2.1. (1) The isomorphism classes of quadratic extension of F correspond to the non-trivial equivalence classes of $F^\times/(F^\times)^2$. We have $F^\times/(F^\times)^2 \simeq \mathrm{H}^1(G_F, \mathbb{Z}/2\mathbb{Z})$ for the absolute Galois group G_F of F and $\dim \mathrm{H}^1(G_F, \mathbb{Z}/2\mathbb{Z}) = 2+d$, where $d = [F : \mathbb{Q}_2]$ is the degree of F over $\mathbb{Q}_2$ (see, for example, [NSW00, Cor. 7.3.9]).

A representative of an equivalence class in $F^\times/F^{\times 2}$ can be chosen to be either a prime or a unit, and exactly half of the classes are represented by prime elements, the others being represented by units. It follows that there are, up to isomorphism, 2^{1+d} different extensions $E|F$ of type (R-P) and $2^{1+d} - 2$ extension of type (R-U). (We have to exclude the trivial element $1 \in F^\times/F^{\times 2}$ and one unit element corresponding to the unramified extension.)

(2) Note that $1+\pi_0^{2k+1}\varepsilon$ is never a square in O_F/π_0^{2k+2} for any unit ε and any k with $|2| < |\pi_0|^k \leq |1|$. It follows that, if two elements $1+\pi_0^{2k+1}\varepsilon$ and $1+\pi_0^{2k'+1}\varepsilon'$ lie in the same equivalence class of $F^\times/F^{\times 2}$ (i.e. they induce the same quadratic extension of type (R-U)), then $k = k'$ and $\varepsilon \equiv \varepsilon' \mod \pi_0$.

Lemma 2.2. *The inverse different of $E|F$ is given by $\mathfrak{D}_{E|F}^{-1} = \frac{1}{2\Pi}O_E$ in the case* (R-P) *and by $\mathfrak{D}_{E|F}^{-1} = \frac{\pi_0^k}{2}O_E$ in the case* (R-U).

Proof. The inverse different is defined as

$$\mathfrak{D}_{E|F}^{-1} = \{\alpha \in E \mid \mathrm{Tr}_{E|F}(\alpha O_E) \subseteq O_F\}.$$

It is enough to check the condition on the trace for the elements 1 and $\Pi \in O_E$. If we write $\alpha = \alpha_1 + \Pi\alpha_2$ with $\alpha_1, \alpha_2 \in F$, we get

$$\begin{aligned}\mathrm{Tr}_{E|F}(\alpha \cdot 1) &= \alpha + \bar{\alpha} = 2\alpha_1 + \alpha_2(\Pi + \overline{\Pi}),\\ \mathrm{Tr}_{E|F}(\alpha \cdot \Pi) &= \alpha\Pi + \overline{\alpha\Pi} = \alpha_1(\Pi + \overline{\Pi}) + \alpha_2(\Pi^2 + \overline{\Pi}^2).\end{aligned}$$

In the case (R-P) we have $\Pi + \overline{\Pi} = 0$ and $\Pi^2 + \overline{\Pi}^2 = 2\pi_0$, while in the case (R-U), $\Pi + \overline{\Pi} = \frac{1+\vartheta}{\pi_0^k} + \frac{1-\vartheta}{\pi_0^k} = \frac{2}{\pi_0^k}$ and $\Pi^2 + \overline{\Pi}^2 = (\frac{1+\vartheta}{\pi_0^k})^2 + (\frac{1-\vartheta}{\pi_0^k})^2 = \frac{4}{\pi_0^{2k}} + 2\pi_0\varepsilon$. It is now easy to deduce that the inverse different is of the claimed form. □

Over F, there exists up to isomorphism exactly one quaternion division algebra B, with unique maximal order O_B. For every quadratic extension $E|F$, there exists an embedding $E \hookrightarrow B$ and this induces an embedding $O_E \hookrightarrow O_B$. If $E|F$ is ramified, a basis for O_E as O_F-module is given by $(1, \Pi)$. We would like to extend this to an O_F-basis of O_B.

Lemma 2.3. (1) *If $E|F$ is of type* (R-P)*, there exists an embedding $F^{(2)} \hookrightarrow B$ such that $\delta\Pi = -\Pi\delta$. An O_F-basis of O_B is then given by $(1, \frac{1+\delta}{2}, \Pi, \frac{1+\delta}{2}\cdot\Pi)$.*

(2) *If $E|F$ is of type* (R-U)*, there exists an embedding $E_1 \hookrightarrow B$, where $E_1|F$ is of type* (R-P)*, such that $\vartheta\Pi_1 = -\Pi_1\vartheta$. The tuple $(1, \vartheta, \Pi_1, \vartheta\Pi_1)$ is an F-basis of B. Furthermore, there is also an embedding $\widetilde{E} \hookrightarrow B$ with $\widetilde{E}|F$ of type* (R-U)*, such that $\vartheta\widetilde{\vartheta} = -\widetilde{\vartheta}\vartheta$ and $\widetilde{\vartheta}^2 = 1 + (4/\pi_0^{2k+1})\cdot\widetilde{\varepsilon}$. In terms of this embedding, an O_F-basis of O_B is given by $(1, \Pi, \widetilde{\Pi}, \Pi\cdot\widetilde{\Pi}/\pi_0)$. Also,*

$$\frac{\Pi\cdot\widetilde{\Pi}}{\pi_0} = \frac{1+\delta}{2} \tag{2.1}$$

for some embedding $F^{(2)} \hookrightarrow B$ of the unramified extension and $\delta^2 = 1 + 4\varepsilon_0$ with $\varepsilon_0 = -\varepsilon\widetilde{\varepsilon}$. Hence, $O_B = O_F[\Pi, \frac{1+\delta}{2}]$ as O_F-algebra.

Proof. (1) This is [Vig80, II. Cor. 1.7].

(2) By [Vig80, I. Cor. 2.4], it suffices to find a uniformizer $\Pi_1^2 \in F^\times \setminus \mathrm{Nm}_{E|F}(E^\times)$ in order to prove the first part. But $\mathrm{Nm}_{E|F}(E^\times) \subseteq F^\times$ is a subgroup of order 2 and $F^{\times 2} \subseteq \mathrm{Nm}_{E|F}(E^\times)$. On the other hand, the residue classes of uniformizing elements in $F^\times/F^{\times 2}$ generate the whole group. Thus they cannot all be contained in $\mathrm{Nm}_{E|F}(E^\times)$. For the second part, choose a unit $\delta \in F^{(2)}$ with $\delta^2 = 1 + 4\varepsilon_0 \in F^\times \setminus F^{\times 2}$ for some $\varepsilon_0 \in O_F^\times$. Let $\widetilde{E}|F$ be of type (R-U), generated by $\widetilde{\vartheta}$ with $\widetilde{\vartheta}^2 = 1 + (4/\pi_0^{2k+1})\cdot\widetilde{\varepsilon}$ and $\varepsilon_0 = -\varepsilon\widetilde{\varepsilon}$. We have to show that $\widetilde{\vartheta}^2$ is not contained in $\mathrm{Nm}_{E|F}(E^\times)$.

Assume it is a norm, so $\widetilde{\vartheta}^2 = \mathrm{Nm}_{E|F}(b)$ for a unit $b \in E^\times$. After multiplying with a scalar in $F^\times$, we can write b as $b = 1 + (2/\pi_0^{k+1}) \cdot b_2\Pi$ for some $b_2 \in F$. Now,

$$\begin{aligned}\mathrm{Nm}_{E|F}(b) &= (1 + \frac{2}{\pi_0^{k+1}} \cdot b_2\Pi) \cdot (1 + \frac{2}{\pi_0^{k+1}} \cdot b_2\overline{\Pi}) \\ &= 1 + \frac{4}{\pi_0^{2k+1}}(b_2 - \varepsilon b_2^2),\end{aligned}$$

and it follows from Remark 2.1 (2), that b_2 is a unit with

$$\widetilde{\varepsilon} \equiv b_2 - \varepsilon b_2^2 \mod \pi_0.$$

After multiplying the equation with ε and setting $x = -\varepsilon b_2$, this becomes

$$\varepsilon_0 = x + x^2 \mod \pi_0.$$

But a solution of this equation would lift to O_F by Hensel's Lemma, and then

$$(1 + 2x)^2 = 1 + 4\varepsilon_0,$$

contradicting our assumptions on δ. Hence $\widetilde{\vartheta}^2 \notin \mathrm{Nm}_{E|F}(E^\times)$ and we can choose an embedding $\widetilde{E} \hookrightarrow B$ such that $\vartheta\widetilde{\vartheta} = -\widetilde{\vartheta}\vartheta$.
We have $\Pi = (1 + \vartheta)/\pi_0^k$ and $\widetilde{\Pi} = (1 + \widetilde{\vartheta})/\ (2/\pi_0^{k+1})$, thus

$$\frac{\Pi \cdot \widetilde{\Pi}}{\pi_0} = \frac{(1+\vartheta)\cdot(1+\widetilde{\vartheta})}{2} = \frac{1 + \vartheta + \widetilde{\vartheta} + \vartheta \cdot \widetilde{\vartheta}}{2},$$

and

$$\begin{aligned}(\vartheta + \widetilde{\vartheta} + \vartheta \cdot \widetilde{\vartheta})^2 &= \vartheta^2 + \widetilde{\vartheta}^2 - \vartheta^2 \cdot \widetilde{\vartheta}^2 \\ &= (1 + \pi_0^{2k+1}\varepsilon) + (1 + \frac{4}{\pi_0^{2k+1}}\widetilde{\varepsilon}) - (1 + \pi_0^{2k+1}\varepsilon)(1 + \frac{4}{\pi_0^{2k+1}}\widetilde{\varepsilon}) \\ &= 1 + 4\varepsilon_0.\end{aligned}$$

Hence $\frac{1+\delta}{2} \mapsto \frac{\Pi\cdot\widetilde{\Pi}}{\pi_0}$ induces an embedding $F^{(2)} \hookrightarrow B$.
It remains to prove that the tuple $u = (1, \Pi, \widetilde{\Pi}, \Pi \cdot \widetilde{\Pi}/\pi_0)$ is a basis of O_B as O_F-module. By [Vig80, I. Cor. 4.8], it suffices to check that the discriminant

$$\mathrm{disc}(u) = \det(\mathrm{Trd}(u_i u_j)) \cdot O_F$$

is equal to $\mathrm{disc}(O_B)$. An easy calculation shows $\det(\mathrm{Trd}(u_i u_j)) \cdot O_F = \pi_0 O_F$ and then the assertion follows from [Vig80, V, II. Cor. 1.7]. □

For the remainder of this section, we will consider lattices Λ in a 2-dimensional E-vector space C with a split $E|F$-hermitian form h. Recall from [Jac62] that, up to isomorphism, there are 2 different $E|F$-hermitian vector spaces (C, h) of fixed dimension n, parametrized by the discriminant $\mathrm{disc}(C, h) \in F^\times / \mathrm{Nm}_{E|F}(E^\times)$. A hermitian space (C, h) is called *split* whenever $\mathrm{disc}(C, h) = 1$. In our case, where (C, h) is split of dimension 2, we can find a basis (e_1, e_2) of C with $h(e_i, e_i) = 0$ and $h(e_1, e_2) = 1$.
Denote by $\Lambda^\sharp$ the dual of a lattice $\Lambda \subseteq C$ with respect to h. The lattice Λ is called Π^i-*modular* if $\Lambda = \Pi^i\Lambda^\sharp$ (resp. *unimodular* or *selfdual* when $i = 0$). In contrast to the p-adic case with $p > 2$, there exists more than one type of Π^i-modular lattices in our case (cf. [Jac62]):

Proposition 2.4. *Define the norm ideal* $\mathrm{Nm}(\Lambda)$ *of* Λ *by*

$$\mathrm{Nm}(\Lambda) = \langle\{h(x, x) | x \in \Lambda\}\rangle \subseteq F. \tag{2.2}$$

Any Π^i-modular lattice $\Lambda \subseteq C$ is determined up to the action of $\mathrm{U}(C,h)$ *by the ideal* $\mathrm{Nm}(\Lambda) = \pi_0^\ell O_F \subseteq F$. *For $i = 0$ or 1, the exponent ℓ can be any integer such that*

$$
\begin{aligned}
|2| \le |\pi_0|^\ell \le |1| &\quad \textit{(for } E|F \textit{ (R-P), unimodular } \Lambda),\\
|2\pi_0| \le |\pi_0|^\ell \le |\pi_0| &\quad \textit{(for } E|F \textit{ (R-P), } \Pi\textit{-modular } \Lambda),\\
|2/\pi_0^k| \le |\pi_0|^\ell \le |1| &\quad \textit{(for } E|F \textit{ (R-U), unimodular } \Lambda),\\
|2/\pi_0^k| \le |\pi_0|^\ell \le |\pi_0| &\quad \textit{(for } E|F \textit{ (R-U), } \Pi\textit{-modular } \Lambda),
\end{aligned}
$$

where $|\cdot|$ is the (normalized) absolute value on F. □

For any other i, the possible values of ℓ for a given Π^i-modular lattice Λ are easily obtained by shifting. In fact, we can choose an integer j such that $\Pi^j\Lambda$ is either unimodular or Π-modular. Then $\mathrm{Nm}(\Lambda) = \pi_0^{-j}\,\mathrm{Nm}(\Pi^j\Lambda)$ and we can apply the proposition above.
Since (C,h) is split, any Π^i-modular lattice Λ contains an *isotropic* vector v (i. e., with $h(v,v) = 0$). After rescaling with a suitable power of Π, we can extend v to a basis of Λ. Hence there always exists a basis (e_1,e_2) of Λ such that h is represented by a matrix of the form

$$
H_\Lambda = \begin{pmatrix} x & \Pi^i \\ \overline{\Pi}^i & \end{pmatrix}, \quad x \in F. \tag{2.3}
$$

If $x = 0$ in this representation, then $\mathrm{Nm}(\Lambda) = \pi_0^\ell O_F$ is as small as possible, or in other words, the absolute value of $|\pi_0|^\ell$ is minimal. On the other hand, whenever $|\pi_0|^\ell$ takes the minimal absolute value for a given Π^i-modular lattice Λ, there exists a basis (e_1,e_2) of Λ such that h is represented by H_Λ with $x = 0$. Indeed, this follows because the ideal $\mathrm{Nm}(\Lambda)$ already determines Λ up to isomorphism. In this case (when $x = 0$), we call Λ a *hyperbolic* lattice. By the arguments above, a Π^i-modular lattice is thus hyperbolic if and only if its norm is minimal. In all other cases, where Λ is Π^i-modular but not hyperbolic, we have $\mathrm{Nm}(\Lambda) = xO_F$.
For further reference, we explicitly write down the norm of a hyperbolic lattice for the cases that we need later. For other values of i, the norm can easily be deduced from this by shifting (see also [Jac62, Table 9.1]).

Lemma 2.5. *Let Λ be a hyperbolic Π^i-modular lattice. Then,*

$$
\begin{aligned}
\mathrm{Nm}(\Lambda) &= 2O_F, && \textit{for } E|F \textit{ (R-P), } i = 0 \textit{ or } -1,\\
\mathrm{Nm}(\Lambda) &= 2\pi_0^{-k}O_F, && \textit{for } E|F \textit{ (R-U), } i = 0 \textit{ or } 1.
\end{aligned}
$$

The norm ideal of Λ is minimal among all norm ideals for Π^i-modular lattices in C. □

In the following, we will only consider the cases $i = 0$ or 1. We want to study the following question:

Question 2.6. Fix a selfdual lattice $\Lambda_0 \subseteq C$ (not necessarily hyperbolic). How many Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$ are there and what norms $\mathrm{Nm}(\Lambda_1)$ can appear? Dually, for a fixed Π-modular lattice $\Lambda_1 \subseteq C$, how many unimodular lattices Λ_0 with $\Lambda_1 \subseteq \Lambda_0$ do exist and what are their norms?

Of course, such an inclusion is always of index 1. The inclusions $\Lambda_1 \subseteq \Lambda_0$ of index 1 correspond to lines in $\Lambda_0/\Pi\Lambda_0$. Denote by q the number of elements in the common residue field of O_F and O_E. Then there exist at most $q+1$ such Π-modular lattices Λ_1 for a given Λ_0. The same bound holds in the dual case, i. e., there are at most $q+1$ selfdual lattices containing a given Π-modular lattice Λ_1. The Propositions 2.7 and 2.8 below provide an exhaustive answer to Question 2.6. Since the proofs consist of a lengthy but simple case-by-case analysis, we will leave it to the interested reader.

Proposition 2.7. *Let $E|F$ of type* (R-P)*.*

(1) *Let $\Lambda_0 \subseteq C$ be a hyperbolic selfdual lattice. There are 2 hyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$ and $q-1$ non-hyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$ with $\mathrm{Nm}(\Lambda_1) = 2O_F$.*

(2) *Let $\Lambda_0 \subseteq C$ be selfdual non-hyperbolic with $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$. There exists one Π-modular lattice $\Lambda_1 \subseteq \Lambda_0$ with $\mathrm{Nm}(\Lambda_1) = \pi_0^{\ell+1} O_F$ and, unless $\ell = 0$, there are q non-hyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$ with $\mathrm{Nm}(\Lambda_1) = \pi_0^\ell O_F$.*

(3) *Let $\Lambda_1 \subseteq C$ be a hyperbolic Π-modular lattice. There are $q+1$ hyperbolic unimodular lattices containing Λ_1.*

(4) *Let $\Lambda_1 \subseteq C$ be a non-hyperbolic Π-modular lattice of norm $\mathrm{Nm}(\Lambda_1) = \pi_0^\ell O_F$. Then Λ_1 is contained in q selfdual lattices of norm $\pi_0^{\ell-1} O_F$ and in one selfdual lattice Λ_0 with $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$.*

Note that the total amount of selfdual resp. Π-modular lattices found for $\Lambda = \Lambda_1$ resp. Λ_0 is $q+1$ except in the case of Prop. 2.7 (2) when $\ell = 0$. In that particular case, there is just one Π-modular lattice contained in Λ_0. The same phenomenon also appears in the case (R-U), compare part (2) of the following proposition.

Proposition 2.8. *Let $E|F$ of type* (R-U)*.*

(1) *Let $\Lambda_0 \subseteq C$ be a hyperbolic selfdual lattice. There are $q+1$ hyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$.*

(2) *Let $\Lambda_0 \subseteq C$ be selfdual non-hyperbolic with $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$. There is one Π-modular lattice $\Lambda_1 \subseteq \Lambda_0$ with norm ideal $\mathrm{Nm}(\Lambda_1) = \pi_0^{\ell+1} O_F$ and if $\ell \neq 0$, there are also q non-hyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$ with $\mathrm{Nm}(\Lambda_1) = \pi_0^\ell O_F$.*

(3) *Let $\Lambda_1 \subseteq C$ be a hyperbolic Π-modular lattice. There are 2 selfdual hyperbolic lattices containing Λ_1 and $q-1$ selfdual lattices Λ_0 with $\Lambda_1 \subseteq \Lambda_0$ and $\mathrm{Nm}(\Lambda_0) = (2/\pi_0^{k+1}) \cdot O_F$.*

(4) *Let $\Lambda_1 \subseteq C$ be a non-hyperbolic Π-modular lattice of norm $\mathrm{Nm}(\Lambda_1) = \pi_0^\ell O_F$. The lattice Λ_1 is contained in q selfdual lattices of norm $\pi_0^{\ell-1} O_F$ and in one selfdual lattice Λ_0 with $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$.*

If $E|F$ is a quadratic extension of type (R-U) such that $|\pi_0^{k+1}| = |2|$, there exist only hyperbolic Π-modular lattices in C and hence case (4) of Prop. 2.8 does not appear. (See page 7 for the definition of the parameter k in the (R-U) case.)

3. The moduli problem in the case (R-P)

Throughout this section, $E|F$ is a quadratic extension of type (R-P), i.e. there exists a uniformizing element $\Pi \in E$ such that $\pi_0 = \Pi^2$ is a uniformizer of F. Then $O_E = O_F[\Pi]$ for the rings of integers O_F and O_E of F and E, respectively. Let k be the common residue field with q elements, $\bar{k}$ an algebraic closure, and $\breve{F}$ the completion of the maximal unramified extension of F, with ring of integers $\breve{O}_F = W_{O_F}(\bar{k})$. Let σ be the lift of the Frobenius in $\mathrm{Gal}(\bar{k}|k)$ to $\mathrm{Gal}(\breve{O}_F|O_F)$.

3.1. The definition of the naive moduli problem $\mathcal{N}_E^{\mathrm{naive}}$.

We first construct a functor $\mathcal{N}_E^{\mathrm{naive}}$ on $\mathrm{Nilp}_{\breve{O}_F}$, the category of $\breve{O}_F$-schemes S such that $\pi_0 \mathcal{O}_S$ is locally nilpotent. We consider tuples (X, ι, λ), where

- X is a formal O_F-module over S of dimension 2 and height 4.
- $\iota : O_E \to \mathrm{End}(X)$ is an action of O_E satisfying the *Kottwitz condition*: The characteristic polynomial of $\iota(\alpha)$ on $\mathrm{Lie}\, X$ for any $\alpha \in O_E$ is

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}).$$

Here $\alpha \mapsto \bar{\alpha}$ is the non-trivial Galois automorphism and the right hand side is a polynomial with coefficients in $\mathcal{O}_S$ via the embedding $O_F[T] \hookrightarrow \breve{O}_F[T] \to \mathcal{O}_S[T]$.

- $\lambda : X \to X^\vee$ is a principal polarization on X such that the Rosati involution satisfies $\iota(\alpha)^* = \iota(\overline{\alpha})$ for $\alpha \in O_E$.

Definition 3.1. A *quasi-isogeny* (resp. an *isomorphism*) $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$ of two such tuples (X, ι, λ) and (X', ι', λ') over S is an O_E-linear quasi-isogeny of height 0 (resp. an O_E-linear isomorphism) $\varphi : X \to X'$ such that $\lambda = \varphi^*(\lambda')$.

For $S = \operatorname{Spec} \bar{k}$ we have the following proposition:

Proposition 3.2. *Up to isogeny, there exists only one tuple* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *over* $\operatorname{Spec} \bar{k}$ *such that the group*

$$G_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})} = \left\{ \varphi \in \operatorname{Aut}^0(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \,\middle|\, \det \varphi = 1 \right\} \tag{3.1}$$

is isomorphic to $\mathrm{SU}(C, h)$ *for a* 2*-dimensional* E*-vector space* C *with split* $E|F$*-hermitian form* h*.*

Here $\operatorname{Aut}^0(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is the group of quasi-isogenies $\varphi : (\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \to (\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. Both $G_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})}$ and $\mathrm{SU}(C, h)$ are considered as linear algebraic groups over F.

Remark 3.3. We will show uniqueness of the tuple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ even for the slightly weakened condition that we have just a closed embedding $\mathrm{SU}(C, h) \hookrightarrow G_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})}$ of linear algebraic groups over F (and not necessarily an isomorphism).

Proof. We first show uniqueness. Let $(X, \iota, \lambda)/\operatorname{Spec} \bar{k}$ be such a tuple. Its rational Dieudonné module N_X is a 4-dimensional vector space over $\breve{F}$ with an action of E and an alternating form $\langle\, , \rangle$ such that for all $x, y \in N_X$,

$$\langle x, \Pi y \rangle = -\langle \Pi x, y \rangle. \tag{3.2}$$

The space N_X has the structure of a 2-dimensional vector space over $\breve{E} = E \otimes_F \breve{F}$ and we can define an $\breve{E}|\breve{F}$-hermitian form on it via

$$h(x, y) = \delta(\langle \Pi x, y \rangle + \Pi \langle x, y \rangle), \tag{3.3}$$

where $\delta \in \breve{O}_F$ is a unit generating the unramified quadratic extension of F, chosen in such a way that $\delta^2 = 1 + 4\varepsilon_0$ for some $\varepsilon_0 \in O_F^\times$. The alternating form can be recovered from h by

$$\langle x, y \rangle = \operatorname{Tr}_{\breve{E}|\breve{F}} \left(\frac{1}{2\Pi\delta} \cdot h(x.y) \right). \tag{3.4}$$

Furthermore we have on N_X a σ-linear operator $\mathbf{F}$, the Frobenius, and a σ^{-1}-linear operator $\mathbf{V}$, the Verschiebung, that satisfy $\mathbf{VF} = \mathbf{FV} = \pi_0$. Recall that σ is the lift of the Frobenius on $\breve{O}_F$. Since $\langle\, , \rangle$ comes from a polarization, we have

$$\langle \mathbf{F}x, y \rangle = \langle x, \mathbf{V}y \rangle^\sigma,$$

and together with $\delta^\sigma = -\delta$, this yields

$$h(\mathbf{F}x, y) = -h(x, \mathbf{V}y)^\sigma,$$

for all $x, y \in N_X$. Let us consider the σ-linear operator $\tau = \Pi \mathbf{V}^{-1}$. Its slopes are all zero, since N_X is isotypical of slope $\frac{1}{2}$. (This follows from the condition on $G_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})}$.) We set $C = N_X^\tau$. This is a 2-dimensional vector space over E and $N_X = C \otimes_E \breve{E}$. Now h induces an $E|F$-hermitian form on C since

$$h(\tau x, \tau y) = h(\mathbf{F}\Pi^{-1} x, \Pi \mathbf{V}^{-1} y) = -h(\Pi^{-1} x, \Pi y)^\sigma = h(x, y)^\sigma.$$

A priori, there are up to isomorphism two possibilities for (C, h), either h is split on C or non-split. But automorphisms of (C, h) with determinant 1 correspond to elements of $G_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})}$. The special unitary groups of (C, h) for h split and h non-split are not isomorphic and thus they cannot contain each other as a Zariski-closed subgroup for dimension reasons. Hence the condition on $G_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})}$ implies that h is split.

Assume now we have two different objects (X, ι, λ) and (X', ι', λ') as in the proposition. These give us isomorphic vector spaces (C, h) and (C', h') and an isomorphism between these extends to an isomorphism between N_X and N'_X (respecting all rational structure) which corresponds to a quasi-isogeny between (X, ι, λ) and (X', ι', λ').
The existence of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ now follows from the fact that a 2-dimensional E-vector space (C, h) with split $E|F$-hermitian form contains a selfdual lattice Λ. Indeed, this gives us a lattice $M = \Lambda \otimes_{O_E} \breve{O}_E \subseteq C \otimes_E \breve{E}$. We extend h to $N = C \otimes_E \breve{E}$ and define the $\breve{F}$-linear alternating form $\langle \, , \rangle$ as in (3.4). Now M is selfdual with respect to $\langle \, , \rangle$, because $\frac{1}{2\Pi\delta}\breve{O}_E$ is the inverse different of $\breve{E}|\breve{F}$ (see Lemma 2.2). We choose the operators $\mathbf{F}$ and $\mathbf{V}$ on M such that $\mathbf{FV} = \mathbf{VF} = \pi_0$ and $\Lambda = M^{\tau}$ for $\tau = \Pi\mathbf{V}^{-1}$. This makes M a (relative) Dieudonné module and we define $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as the corresponding formal O_F-module. □

We fix such a framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\operatorname{Spec} \bar{k}$.
For arbitrary $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\bar{S} = S \times_{\operatorname{Spf} \breve{O}_F} \operatorname{Spec} \bar{k}$. We define $\mathcal{N}_E^{\mathrm{naive}}(S)$ as the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$ over S, where (X, ι, λ) as above and

$$\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S}$$

is a quasi-isogeny between the tuple (X, ι, λ) and the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ (after base change to $\bar{S}$). Two objects $(X, \iota, \lambda, \varrho)$ and $(X', \iota', \lambda', \varrho')$ are equivalent if and only if there exists an isomorphism $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$ such that $\varrho = \varrho' \circ (\varphi \times_S \bar{S})$.

Remark 3.4. (1) The morphism ϱ is a quasi-isogeny in the sense of Def. 3.1, i. e., we have $\lambda = \varrho^*(\lambda_{\mathbb{X}})$. Similarly, we have $\lambda = \varphi^*(\lambda')$ for the isomorphism φ. We obtain an equivalent definition of $\mathcal{N}_E^{\mathrm{naive}}$ if we replace strict equality by the condition that, locally on S, λ and $\varrho^*(\lambda_{\mathbb{X}})$ (resp. $\varphi^*(\lambda')$) only differ by a scalar in $O_F^{\times}$. This variant is used in the definition of RZ-spaces of (PEL) type for $p > 2$ in [RZ96]. In this paper we will use the version with strict equality, since it simplifies the formulation of the straightening condition, see Def. 3.10 below.

(2) $\mathcal{N}_E^{\mathrm{naive}}$ is pro-representable by a formal scheme, formally locally of finite type over $\operatorname{Spf} \breve{O}_F$. This follows from [RZ96, Thm. 3.25].

As a next step, we use Dieudonné theory in order to get a better understanding of the special fiber of $\mathcal{N}_E^{\mathrm{naive}}$. Let $N = N_{\mathbb{X}}$ be the rational Dieudonné module of the base point $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_E^{\mathrm{naive}}$. This is a 4-dimensional vector space over $\breve{F}$, equipped with an E-action, an alternating form $\langle \, , \rangle$ and two operators $\mathbf{V}$ and $\mathbf{F}$. As in the proof of Proposition 3.2, the form $\langle \, , \rangle$ satisfies condition (3.2):

$$\langle x, \Pi y \rangle = -\langle \Pi x, y \rangle. \tag{3.2}$$

A point $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ corresponds to an $\breve{O}_F$-lattice $M_X \subseteq N$. It is stable under the actions of the operators $\mathbf{V}$ and $\mathbf{F}$ and of the ring O_E. Furthermore M_X is selfdual under $\langle \, , \rangle$, i.e. $M_X = M_X^{\vee}$, where

$$M_X^{\vee} = \{x \in N \mid \langle x, y \rangle \in \breve{O}_F \text{ for all } y \in M_X\}.$$

We can regard N as a 2-dimensional vector space over $\breve{E}$ with the $\breve{E}|\breve{F}$-hermitian form h defined by

$$h(x, y) = \delta(\langle \Pi x, y \rangle + \Pi \langle x, y \rangle). \tag{3.3}$$

Let $\breve{O}_E = O_E \otimes_{O_F} \breve{O}_F$. Then $M_X \subseteq N$ is an $\breve{O}_E$-lattice and we have

$$M_X = M_X^{\vee} = M_X^{\sharp},$$

where $M_X^\sharp$ is the dual lattice of M_X with respect to h. The latter equality follows from the formula

$$\langle x, y\rangle = \mathrm{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{2\Pi\delta}\cdot h(x.y)\right) \tag{3.4}$$

and the fact that the inverse different of $E|F$ is $\mathfrak{D}_{E|F}^{-1} = \frac{1}{2\Pi}O_E$ (see Lemma 2.2). We can thus write the set $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ as

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \{\breve{O}_E\text{-lattices } M \subseteq N_{\mathbb{X}} \mid M^\sharp = M, \pi_0 M \subseteq \mathbf{V}M \subseteq M\}. \tag{3.5}$$

Let $\tau = \Pi\mathbf{V}^{-1}$. This is a σ-linear operator on N with all slopes zero. The elements invariant under τ form a 2-dimensional E-vector space $C = N^\tau$. The hermitian form h is invariant under τ, hence it induces a split hermitian form on C which we denote again by h. With the same proof as in [KR11, Lemma 3.2], we have:

Lemma 3.5. *Let $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then:*
(1) *$M + \tau(M)$ is τ-stable.*
(2) *Either M is τ-stable and $\Lambda_0 = M^\tau \subseteq C$ is selfdual ($\Lambda_0^\sharp = \Lambda_0$) or M is not τ-stable and then $\Lambda_1 = (M + \tau(M))^\tau \subseteq C$ is Π^{-1}-modular ($\Lambda_1^\sharp = \Pi\Lambda_1$).*

Under the identification $N = C \otimes_E \breve{E}$, we get $M = \Lambda_0 \otimes_{O_E} \breve{O}_E$ for a τ-stable Dieudonné lattice M. If M is not τ-stable, we have $M + \tau M = \Lambda_1 \otimes_{O_E} \breve{O}_E$ and $M \subseteq \Lambda_1 \otimes_{O_E} \breve{O}_E$ is a sublattice of index 1. The next lemma is the analogue of [KR11, Lemma 3.3].

Lemma 3.6. (1) *Fix a Π^{-1}-modular lattice $\Lambda_1 \subseteq C$. There is an injective map*

$$i_{\Lambda_1} : \mathbb{P}(\Lambda_1/\Pi\Lambda_1)(\bar{k}) \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$$

mapping a line $\ell \subseteq (\Lambda_1/\Pi\Lambda_1)\otimes\bar{k}$ to its preimage in $\Lambda_1\otimes\breve{O}_E$. Identify $\mathbb{P}(\Lambda_1/\Pi\Lambda_1)(\bar{k})$ with its image in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then $\mathbb{P}(\Lambda_1/\Pi\Lambda_1)(k) \subseteq \mathbb{P}(\Lambda_1/\Pi\Lambda_1)(\bar{k})$ is the set of τ-invariant Dieudonné lattices $M \subseteq \Lambda_1 \otimes \breve{O}_E$.
(2) *The set $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is a union*

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda_1 \subseteq C} \mathbb{P}(\Lambda_1/\Pi\Lambda_1)(\bar{k}), \tag{3.6}$$

ranging over all Π^{-1}-modular lattices $\Lambda_1 \subseteq C$. The projective lines corresponding to the lattices Λ_1 and Λ_1' intersect in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ if and only if $\Lambda_0 = \Lambda_1 \cap \Lambda_1'$ is selfdual. In this case, their intersection consists of the point $M = \Lambda_0 \otimes \breve{O}_E \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$.

Proof. We only have to prove that the map i_{Λ_1} is well-defined. Denote by M the preimage of $\ell \subseteq (\Lambda_1/\Pi\Lambda_1) \otimes \bar{k}$ in $\Lambda_1 \otimes \breve{O}_E$. We need to show that M is an element in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ under the identification of (3.5). It is clearly a sublattice of index 1 in $\Lambda_1 \otimes \breve{O}_E$, stable under the actions of $\mathbf{F}$, $\mathbf{V}$ and O_E.
Let $e_1 \in \Lambda_1 \otimes \breve{O}_E$ such that $e_1 \otimes \bar{k}$ generates ℓ. We can extend this to a basis (e_1, e_2) of Λ_1 and with respect to this basis, h is represented by a matrix of the form

$$\begin{pmatrix} x & \Pi^{-1} \\ -\Pi^{-1} & y \end{pmatrix},$$

with $x, y \in \Pi^{-1}\breve{O}_E \cap \breve{O}_F = \breve{O}_F$. The lattice $M \subseteq \Lambda_1 \otimes \breve{O}_E$ is generated by e_1 and Πe_2. With respect to this new basis, h is now given by the matrix

$$\begin{pmatrix} x & 1 \\ 1 & -\pi_0 y \end{pmatrix}.$$

Since all entries of the matrix are integral, we have $M \subseteq M^\sharp$. But this already implies $M^\sharp = M$, because they both have index 1 in $\Lambda_1 \otimes \breve{O}_E$. Thus $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ and i_{Λ_1} is well-defined. □

Remark 3.7. (1) Recall from Prop. 2.4 that the isomorphism type of a Π^i-modular lattice $\Lambda \subseteq C$ only depends on its norm ideal $\mathrm{Nm}(\Lambda) = \langle \{h(x,x) | x \in \Lambda\} \rangle = \pi_0^\ell O_F \subseteq F$. In the case that $\Lambda = \Lambda_0$ or Λ_1 is selfdual or Π^{-1}-modular, ℓ can be any integer such that $|1| \geq |\pi_0|^\ell \geq |2|$. In particular, there are always at least two possible values for ℓ. Recall from Lemma 2.5, that Λ is *hyperbolic* if and only if $\mathrm{Nm}(\Lambda) = 2O_F$.

(2) The intersection behaviour of the projective lines in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ can be deduced from Prop. 2.7. In particular, for a given selfdual lattice $\Lambda_0 \subseteq C$ with $\mathrm{Nm}(\Lambda_0) \subseteq \pi_0 O_F$, there are $q+1$ lines intersecting in $M = \Lambda_0 \otimes \breve{O}_E$. If $\mathrm{Nm}(\Lambda_0) = O_F$, the lattice $M = \Lambda_0 \otimes \breve{O}_E$ is only contained in one projective line. On the other hand, a projective line $\mathbb{P}(\Lambda_1/\Pi\Lambda_1)(\bar{k}) \subseteq \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ contains $q+1$ points corresponding to selfdual lattices in C. By Lemma 3.6 (1), these are exactly the k-rational points of $\mathbb{P}(\Lambda_1/\Pi\Lambda_1)$.

(3) If we restrict the union at the right hand side of (3.6) to hyperbolic Π^{-1}-modular lattices $\Lambda_1 \subseteq C$ (i. e., $\mathrm{Nm}(\Lambda_1) = 2O_F$, see Lemma 2.5), we obtain a canonical subset $\mathcal{N}_E(\bar{k}) \subseteq \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ and there is a description of $\mathcal{N}_E$ as a pro-representable functor on $\mathrm{Nilp}_{\breve{O}_F}$ (see below). We will see later (Theorem 3.14) that $\mathcal{N}_E$ is isomorphic to the Drinfeld moduli space $\mathcal{M}_{Dr}$, described in [BC91, I.3]. In particular, the underlying topological space of $\mathcal{N}_E$ is connected. (The induced topology on the projective lines is the Zariski topology, see Prop. 3.8.) Moreover, each projective line in $\mathcal{N}_E(\bar{k})$ has $q+1$ intersection points and there are 2 projective lines intersecting in each such point (also cp. Prop. 2.7).

We fix such an intersection point $P \in \mathcal{N}_E(\bar{k})$. Now going back to $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$, there are $q-1$ additional lines going through $P \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ that correspond to non-hyperbolic lattices in C (see Prop. 2.7). Each of these additional lines contains P as its only "hyperbolic" intersection point, all other intersection points on this line and the line itself correspond to selfdual resp. Π^{-1}-modular lattices $\Lambda \subseteq C$ of norm $\mathrm{Nm}(\Lambda) = (2/\pi_0)O_F$ (whereas all hyperbolic lattices occuring have the norm ideal $2O_F$, see Lemma 2.5). Assume $\mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}) \subseteq \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is such a line and let $P' \in \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ be an intersection point, where $P \neq P'$. There are again q more lines going through P' (always $q+1$ in total) that correspond to lattices with norm ideal $\mathrm{Nm}(\Lambda) = (2/\pi_0^2)O_F$, and these lines again have more intersection points and so on. This goes on until we reach lines $\mathbb{P}(\Lambda'/\Pi\Lambda')(\bar{k})$ with $\mathrm{Nm}(\Lambda') = O_F$. Each of these lines contains q points that correspond to selfdual lattices $\Lambda_0 \subseteq C$ with $\mathrm{Nm}(\Lambda_0) = O_F$. Such a lattice is only contained in one Π^{-1}-modular lattice (see part 2 of Prop. 2.7). Hence, these points are only contained in one projective line, namely $\mathbb{P}(\Lambda'/\Pi\Lambda')(\bar{k})$.

In other words, each intersection point $P \in \mathcal{N}_E(\bar{k})$ has a "tail", consisting of finitely many projective lines, which is the connected component of P in $(\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) \setminus \mathcal{N}_E(\bar{k})) \cup \{P\}$. Figure 1 shows a drawing of $(\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$ for the cases $F = \mathbb{Q}_2$ (on the left hand side) and $F|\mathbb{Q}_2$ a ramified quadratic extension (on the right hand side). The "tails" are indicated by dashed lines.

Fix a Π^{-1}-modular lattice $\Lambda = \Lambda_1 \subseteq C$. Let X_Λ^+ be the formal O_F-module over $\operatorname{Spec} \bar{k}$ associated to the Dieudonné lattice $M = \Lambda \otimes \breve{O}_E \subseteq N$. It comes with a canonical quasi-isogeny

$$\varrho_\Lambda^+ : \mathbb{X} \longrightarrow X_\Lambda^+$$

of F-height 1. We define a subfunctor $\mathcal{N}_{E,\Lambda} \subseteq \mathcal{N}_E^{\mathrm{naive}}$ by mapping $S \in \mathrm{Nilp}_{\breve{O}_F}$ to

$$\mathcal{N}_{E,\Lambda}(S) = \{(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S) \mid (\varrho_\Lambda^+ \times S) \circ \varrho \text{ is an isogeny}\}. \tag{3.7}$$

Note that the condition of (3.7) is closed, cf. [RZ96, Prop. 2.9]. Hence $\mathcal{N}_{E,\Lambda}$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$. On geometric points, we have a

(A) $e = 1,\ f = 1.$ (B) $e = 2,\ f = 1.$

FIGURE 1. The reduced locus of $\mathcal{N}_E^{\text{naive}}$ for $E|F$ of type (R-P) where $F|\mathbb{Q}_2$ has ramification index e and inertia degree f. Solid lines are given by subschemes $\mathcal{N}_{E,\Lambda}$ for hyperbolic lattices Λ.

bijection

$$\mathcal{N}_{E,\Lambda}(\bar{k}) \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}), \tag{3.8}$$

as a consequence of Lemma 3.6 (1).

Proposition 3.8. *The reduced locus of $\mathcal{N}_E^{\text{naive}}$ is given by*

$$(\mathcal{N}_E^{\text{naive}})_{\text{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda},$$

where Λ runs over all Π^{-1}-modular lattices in C. For each Λ, there is an isomorphism of reduced schemes

$$\mathcal{N}_{E,\Lambda} \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda),$$

inducing the map (3.8) *on $\bar{k}$-valued points.*

Proof. The embedding

$$\bigcup_{\Lambda \subseteq C} (\mathcal{N}_{E,\Lambda})_{\text{red}} \hookrightarrow (\mathcal{N}_E^{\text{naive}})_{\text{red}} \tag{3.9}$$

is closed, because each embedding $\mathcal{N}_{E,\Lambda} \subseteq \mathcal{N}_E^{\text{naive}}$ is closed and, locally on $(\mathcal{N}_E^{\text{naive}})_{\text{red}}$, the left hand side is always only a finite union of $(\mathcal{N}_{E,\Lambda})_{\text{red}}$. It follows already that (3.9) is an isomorphism, since it is a bijection on $\bar{k}$-valued points (see the equations (3.6) and (3.8)) and $(\mathcal{N}_E^{\text{naive}})_{\text{red}}$ is reduced by definition and locally of finite type over $\operatorname{Spec} \bar{k}$ by Remark 3.4 (2).

For the second part of the proposition, we follow the proof presented in [KR11, 4.2]. Fix a Π^{-1}-modular lattice $\Lambda \subseteq C$ and let $M = \Lambda \otimes \breve{O}_E \subseteq N$, as above. Now X_Λ^+ is the formal O_F-module associated to M, but we also get a formal O_F-module X_Λ^- associated to the dual $M^\sharp = \Pi M$ of M. This comes with a natural isogeny

$$\text{nat}_\Lambda : X_\Lambda^- \longrightarrow X_\Lambda^+$$

and a quasi-isogeny $\varrho_\Lambda^- : X_\Lambda^- \to \mathbb{X}$ of F-height 1. For $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$ where $S \in \text{Nilp}_{\breve{O}_F}$, we consider the composition

$$\varrho_{\Lambda,X}^- = \varrho^{-1} \circ (\varrho_\Lambda^- \times S) : (X_\Lambda^- \times S) \longrightarrow X.$$

By [KR11, Lemma 4.2], this composition is an isogeny if and only if $(\varrho_\Lambda^+ \times S) \circ \varrho$ is an isogeny, or, in other words, if and only if $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(S)$. Let $\mathbb{D}_{X_\Lambda^-}(S)$ be the (relative) Grothendieck-Messing crystal of X_Λ^- evaluated at S (cf. [ACZ, Def. 3.24] or [Ahs11, 5.2]). This is a locally free $\mathcal{O}_S$-module of rank 4, isomorphic to $\Lambda/\pi_0\Lambda \otimes_{O_F} \mathcal{O}_S$.

The kernel of $\mathbb{D}(\mathrm{nat}_\Lambda)(S)$ is given by $(\Lambda/\Pi\Lambda)\otimes_{O_F}\mathcal{O}_S$, locally a direct summand of rank 2 of $\mathbb{D}_{X_\Lambda^-}(S)$. For any $(X,\iota,\lambda,\varrho)\in\mathcal{N}_{E,\Lambda}(S)$, the kernel of $\varrho_{\Lambda,X}^-$ is contained in $\ker(\mathrm{nat}_\Lambda)$. It follows from [VW11, Cor. 4.7] (cp. Prop. 4.6 in [KR11]) that $\ker\mathbb{D}(\varrho_{\Lambda,X}^-)(S)$ is locally a direct summand of rank 1 of $(\Lambda/\Pi\Lambda)\otimes_{O_F}\mathcal{O}_S$. This induces a map

$$\mathcal{N}_{E,\Lambda}(S)\longrightarrow\mathbb{P}(\Lambda/\Pi\Lambda)(S),$$

functorial in S, and the arguments of [VW11, 4.7] show that it is an isomorphism. (One easily checks that their results indeed carry over to the relative setting over O_F.) □

3.2. **Construction of the closed formal subscheme $\mathcal{N}_E\subseteq\mathcal{N}_E^{\mathrm{naive}}$.** We now use a result from section 6. By Theorem 6.2 and Remark 6.1 (2), there exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}:\mathbb{X}\to\mathbb{X}^\vee$ on $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$, unique up to a scalar in $O_E^\times$, such that the induced Rosati involution is the identity on O_E. Furthermore, for any $(X,\iota,\lambda,\varrho)\in\mathcal{N}_E^{\mathrm{naive}}(S)$, the pullback $\widetilde{\lambda}=\varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a principal polarization on X.
The next proposition is crucial for the construction of $\mathcal{N}_E$. Recall the notion of a *hyperbolic* lattice from Prop. 2.4 and the subsequent discussion.

Proposition 3.9. *It is possible to choose $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$ such that*

$$\lambda_{\mathbb{X},1}=\frac{1}{2}(\lambda_{\mathbb{X}}+\widetilde{\lambda}_{\mathbb{X}})\in\mathrm{Hom}(\mathbb{X},\mathbb{X}^\vee).$$

Fix such a choice and let $(X,\iota,\lambda,\varrho)\in\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then, $\frac{1}{2}(\lambda+\widetilde{\lambda})\in\mathrm{Hom}(X,X^\vee)$ if and only if $(X,\iota,\lambda,\varrho)\in\mathcal{N}_{E,\Lambda}(\bar{k})$ for some hyperbolic lattice $\Lambda\subseteq C$.

Proof. The polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $\mathbb{X}$ induces an alternating form $(\,,)$ on the rational Dieudonné module $N=M_{\mathbb{X}}\otimes_{\breve{O}_F}\breve{F}$. For all $x,y\in N$, the form $(\,,)$ satisfies the equations

$$(\mathbf{F}x,y)=(x,\mathbf{V}y)^\sigma,$$
$$(\Pi x,y)=(x,\Pi y).$$

It induces an $\breve{E}$-alternating form b on N via

$$b(x,y)=(\Pi x,y)+\Pi(x,y).$$

On the other hand, we can describe $(\,,)$ in terms of b,

$$(x,y)=\mathrm{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{2\Pi}\cdot b(x,y)\right). \tag{3.10}$$

The form b is invariant under $\tau=\Pi\mathbf{V}^{-1}$, since

$$b(\tau x,\tau y)=b(\mathbf{F}\Pi^{-1}x,\Pi\mathbf{V}^{-1}y)=b(\Pi^{-1}x,\Pi y)^\sigma=b(x,y)^\sigma.$$

Hence b defines an E-linear alternating form on $C=N^\tau$, which we again denote by b. Denote by $\langle\,,\rangle$ the alternating form on $M_{\mathbb{X}}$ induced by the polarization $\lambda_{\mathbb{X}}$ and let h be the corresponding hermitian form, see (3.3). On $N_{\mathbb{X}}$, we define the alternating form $\langle\,,\rangle_1$ by

$$\langle x,y\rangle_1=\frac{1}{2}(\langle x,y\rangle+(x,y)).$$

This form is integral on $M_{\mathbb{X}}$ if and only if $\lambda_{\mathbb{X},1}=\frac{1}{2}(\lambda_{\mathbb{X}}+\widetilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$.
We choose $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ such that it corresponds to a selfdual hyperbolic lattice $\Lambda_0\subseteq(C,h)$ under the identifications of (3.5) and Lemma 3.5. There exists a basis (e_1,e_2) of Λ_0 such that

$$h\mathrel{\widehat{=}}\begin{pmatrix}&1\\1&\end{pmatrix},\quad b\mathrel{\widehat{=}}\begin{pmatrix}&u\\-u&\end{pmatrix}, \tag{3.11}$$

for some $u \in E^\times$. Since $\widetilde{\lambda}_{\mathbb{X}}$ is principal, the alternating form b is perfect on Λ_0, thus $u \in O_E^\times$. After rescaling $\widetilde{\lambda}_{\mathbb{X}}$, we may assume that $u = 1$. We now have

$$\frac{1}{2}(h(x,y) + b(x,y)) \in O_E,$$

for all $x, y \in \Lambda_0$. Thus $\frac{1}{2}(h+b)$ is integral on $M_{\mathbb{X}} = \Lambda_0 \otimes_{O_E} \breve{O}_E$. This implies that

$$\begin{aligned}\langle x,y\rangle_1 = \frac{1}{2}(\langle x,y\rangle + (x,y)) &= \frac{1}{2}\operatorname{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{2\Pi\delta}\cdot h(x.y) + \frac{1}{2\Pi}\cdot b(x,y)\right) \\ &= \operatorname{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{2\Pi\delta}\cdot\frac{1}{2}(h(x,y)+b(x,y))\right) + \frac{\delta-1}{2\delta}\cdot\operatorname{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{2\Pi}\cdot b(x,y)\right) \in \breve{O}_F,\end{aligned}$$

for all $x, y \in M_{\mathbb{X}}$. Indeed, in the definition of h (see (3.3)), the unit δ has been chosen such that $\frac{1+\delta}{2} \in \breve{O}_F$, so the second summand is in $\breve{O}_F$. The first summand is integral, since $\frac{1}{2}(h+b)$ is integral. It follows that $\lambda_{\mathbb{X},1} = \frac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$.
Let $(X,\iota,\lambda,\varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ and assume that $\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda}) = \varrho^*(\lambda_{\mathbb{X},1})$ is a polarization on X. Then $\langle\,,\rangle_1$ is integral on the Dieudonné module $M \subseteq N$ of X. By the above calculation, this is equivalent to $\frac{1}{2}(h+b)$ being integral on M. In particular, this implies that

$$h(x,x) = h(x,x) + b(x,x) \in 2\breve{O}_F,$$

for all $x \in M$. Let $\Lambda = M^\tau$ resp. $\Lambda = (M+\tau(M))^\tau$ as in Lemma 3.5. Then $h(x,x) \in 2O_F$ for all $x \in \Lambda$, hence $\mathrm{Nm}(\Lambda) \subseteq 2O_F$. By Lemma 2.5 and because of minimality, we have $\mathrm{Nm}(\Lambda) = 2O_F$ and Λ is a hyperbolic lattice. It follows that $(X,\iota,\lambda,\varrho) \in \mathcal{N}_{E,\Lambda'}(\bar{k})$ for some hyperbolic Π^{-1}-modular lattice $\Lambda' \subseteq C$. Indeed, either Λ is Π^{-1}-modular and $\Lambda' = \Lambda$, or, if Λ is selfdual, it is contained in some Π^{-1}-modular hyperbolic lattice Λ', cp. Prop. 2.7.
Conversely, assume that $(X,\iota,\lambda,\varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ for some hyperbolic lattice $\Lambda \subseteq C$. It suffices to show that $\frac{1}{2}(h+b)$ is integral on Λ. Indeed, it follows that $\frac{1}{2}(h+b)$ is integral on the Dieudonné module M. Thus $\langle\,,\rangle_1$ is integral on M and this is equivalent to $\lambda_1 = \frac{1}{2}(\lambda+\widetilde{\lambda}) \in \mathrm{Hom}(X, X^\vee)$.
Let $\Lambda' \subseteq C$ be the Π^{-1}-modular lattice generated by $\Pi^{-1}e_1$ and e_2, where (e_1,e_2) is the basis of the lattice Λ_0 corresponding to the framing object $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$. By (3.11), h and b have the following form with respect to the basis $(\Pi^{-1}e_1, e_2)$,

$$h \mathrel{\widehat{=}} \begin{pmatrix} & -\Pi^{-1} \\ \Pi^{-1} & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & \Pi^{-1} \\ -\Pi^{-1} & \end{pmatrix}.$$

In particular, Λ' is hyperbolic and $\frac{1}{2}(h+b)$ is integral on Λ'. By Prop. 2.4, there exists an automorphism $g \in \mathrm{SU}(C,h)$ mapping Λ onto Λ'. Since $\det g = 1$, the alternating form b is invariant under g. It follows that $\frac{1}{2}(h+b)$ is also integral on Λ. □

From now on, we assume $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$ chosen in a way such that

$$\lambda_{\mathbb{X},1} = \frac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) \in \mathrm{Hom}(\mathbb{X},\mathbb{X}^\vee).$$

Note that this determines the polarization $\widetilde{\lambda}_{\mathbb{X}}$ up to a scalar in $1+2O_E$. If we replace $\widetilde{\lambda}_{\mathbb{X}}$ by $\widetilde{\lambda}'_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}} \circ \iota_{\mathbb{X}}(1+2u)$ for some $u \in O_E$, then $\lambda'_{\mathbb{X},1} = \lambda_{\mathbb{X},1} + \widetilde{\lambda}_{\mathbb{X}} \circ \iota_{\mathbb{X}}(u)$.
We can now formulate the straightening condition.

Definition 3.10. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. An object $(X,\iota,\lambda,\varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ satisfies the *straightening condition* if

$$\lambda_1 \in \mathrm{Hom}(X, X^\vee), \tag{3.12}$$

where $\lambda_1 = \frac{1}{2}(\lambda+\widetilde{\lambda}) = \varrho^*(\lambda_{\mathbb{X},1})$.

This definition is clearly independent of the choice of the polarization $\widetilde{\lambda}_{\mathbb{X}}$. We define $\mathcal{N}_E$ as the functor that maps $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of all tuples $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ that satisfy the straightening condition. By [RZ96, Prop. 2.9], $\mathcal{N}_E$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$.

Remark 3.11. The reduced locus of $\mathcal{N}_E$ can be written as

$$(\mathcal{N}_E)_{\mathrm{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda} \simeq \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda),$$

where we take the unions over all *hyperbolic* Π^{-1}-modular lattices $\Lambda \subseteq C$. By Prop. 2.7 and Lemma 3.6, each projective line contains $q+1$ points corresponding to selfdual lattices and there are two lines intersecting in each such point. Recall from Remark 3.7 (1) that there exist non-hyperbolic Π^{-1}-modular lattices $\Lambda \subseteq C$, thus we have $\mathcal{N}_E(\bar{k}) \neq \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$, and in particular $(\mathcal{N}_E)_{\mathrm{red}} \neq (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$.

3.3. **The isomorphism to the Drinfeld moduli problem.** We now recall the Drinfeld moduli problem $\mathcal{M}_{Dr}$ on $\mathrm{Nilp}_{\breve{O}_F}$. Let B be the quaternion division algebra over F and O_B its ring of integers. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. Then $\mathcal{M}_{Dr}(S)$ is the set of equivalence classes of objects (X, ι_B, ϱ) where

- X is a formal O_F-module over S of dimension 2 and height 4,
- $\iota_B : O_B \to \mathrm{End}(X)$ is an action of O_B on X satisfying the *special* condition, i.e. $\mathrm{Lie}\, X$ is, locally on S, a free $\mathcal{O}_S \otimes_{O_F} O_F^{(2)}$-module of rank 1, where $O_F^{(2)} \subseteq O_B$ is any embedding of the unramified quadratic extension of O_F into O_B (cf. [BC91]),
- $\varrho : X \times_S \overline{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \overline{S}$ is an O_B-linear quasi-isogeny of height 0 to a fixed framing object $(\mathbb{X}, \iota_{\mathbb{X}}) \in \mathcal{M}_{Dr}(\bar{k})$.

Such a framing object exists and is unique up to isogeny. By a proposition of Drinfeld, cf. [BC91, p. 138], there always exist polarizations on these objects, as follows:

Proposition 3.12 (Drinfeld)**.** *Let $\Pi \in O_B$ a uniformizer with $\Pi^2 \in O_F$ and let $b \mapsto b'$ be the standard involution of B. Then $b \mapsto b^* = \Pi b' \Pi^{-1}$ is another involution on B.*

(1) *There exists a principal polarization $\lambda_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^\vee$ on $\mathbb{X}$ with associated Rosati involution $b \mapsto b^*$. It is unique up to a scalar in $O_F^\times$.*

(2) *Let $\lambda_{\mathbb{X}}$ as in* (1)*. For $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, there exists a unique principal polarization*

$$\lambda : X \longrightarrow X^\vee$$

with Rosati involution $b \mapsto b^*$ such that $\varrho^*(\lambda_{\mathbb{X}}) = \lambda$ on $\overline{S}$.

We now relate $\mathcal{M}_{Dr}$ and $\mathcal{N}_E$. For this, we fix an embedding $E \hookrightarrow B$. Any choice of a uniformizer $\Pi \in O_E$ with $\Pi^2 \in O_F$ induces the same involution $b \mapsto b^* = \Pi b' \Pi^{-1}$ on B. For the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ of $\mathcal{M}_{Dr}$, let $\lambda_{\mathbb{X}}$ be a polarization associated to this involution by Prop. 3.12 (1). Denote by $\iota_{\mathbb{X},E}$ the restriction of $\iota_{\mathbb{X}}$ to $O_E \subseteq O_B$. For any object $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, let λ be the polarization with Rosati involution $b \mapsto b^*$ that satisfies $\varrho^*(\lambda_{\mathbb{X}}) = \lambda$, see Prop. 3.12 (2). Let ι_E be the restriction of ι_B to O_E.

Lemma 3.13. *$(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ is a framing object for $\mathcal{N}_E^{\mathrm{naive}}$. Furthermore, the map*

$$(X, \iota_B, \varrho) \longmapsto (X, \iota_E, \lambda, \varrho)$$

induces a closed embedding of formal schemes

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}.$$

Proof. There are two things to check: that $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})} \simeq \mathrm{SU}(C,h)$, with (C,h) split as in Prop. 3.2, and that ι_E satisfies the Kottwitz condition. Indeed, once these two assertions hold, we can take $(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})$ as a framing object for $\mathcal{N}_E^{\mathrm{naive}}$ and the morphism η is well-defined. For any $S \in \mathrm{Nilp}_{\breve{O}_F}$, the map $\eta(S)$ is injective, because (X,ι_B,ϱ) and $(X',\iota'_B,\varrho') \in \mathcal{M}_{Dr}(S)$ map to the same point in $\mathcal{N}_E^{\mathrm{naive}}(S)$ under η if and only if the quasi-isogeny $\varrho'\circ\varrho$ on $\bar{S}$ lifts to an isomorphism on S, i. e., if and only if (X,ι_B,ϱ) and (X',ι'_B,ϱ') define the same point in $\mathcal{M}_{Dr}(S)$. The functor

$$F : S \longmapsto \{(X,\iota,\lambda,\varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S) \mid \iota \text{ extends to an } O_B\text{-action}\}$$

is pro-representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$ by [RZ96, Prop. 2.9]. Now, the formal subscheme $\eta(\mathcal{M}_{Dr}) \subseteq F$ is given by the special condition. But the special condition is open and closed (see [RZ14, p. 7]), thus η is a closed embedding.
It remains to show the two assertions from the beginning of this proof. We first prove that $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})} \simeq \mathrm{SU}(C,h)$. Let $G_{(\mathbb{X},\iota_{\mathbb{X}})}$ be the group of O_B-linear quasi-isogenies $\varphi : (\mathbb{X},\iota_{\mathbb{X}}) \to (\mathbb{X},\iota_{\mathbb{X}})$ of height 0 and determinant 1. Then we have (non-canonical) isomorphisms $G_{(\mathbb{X},\iota_{\mathbb{X}})} \simeq \mathrm{SL}_{2,F}$ and $\mathrm{SL}_{2,F} \simeq \mathrm{SU}(C,h)$, since h is split (cp. [KR11, p. 3]). The uniqueness of the polarization $\lambda_{\mathbb{X}}$ (up to a scalar in $O_F^\times$) implies that $G_{(\mathbb{X},\iota_{\mathbb{X}})} \subseteq G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})}$. This is a closed embedding of linear algebraic groups over F, since a quasi-isogeny $\varphi \in G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})}$ lies in $G_{(\mathbb{X},\iota_{\mathbb{X}})}$ if and only if it is O_B-linear, and this defines a closed condition on $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})}$. Thus $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})}$ contains a closed subgroup isomorphic to $\mathrm{SU}(C,h)$ and this already implies that $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})} \simeq \mathrm{SU}(C,h)$, see Remark 3.3.
Finally, the special condition implies the Kottwitz condition for any element $b \in O_B$ (see [RZ14, Prop. 5.8]), i. e., the characteristic polynomial for the action of $\iota(b)$ on $\mathrm{Lie}\, X$ is

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(b)) = (T-b)(T-b'),$$

where the right hand side is a polynomial in $\mathcal{O}_S[T]$ via the structure homomorphism $O_F \hookrightarrow \breve{O}_F \to \mathcal{O}_S$. From this, the second assertion follows. □

Let $O_F^{(2)} \subseteq O_B$ be an embedding such that conjugation with Π induces the non-trivial Galois action on $O_F^{(2)}$, cf. Lemma 2.3 (1). Fix a generator $\frac{1+\delta}{2}$ of $O_F^{(2)}$ with $\delta^2 \in O_F^\times$. On $(\mathbb{X},\iota_{\mathbb{X}})$, the principal polarization $\widetilde{\lambda}_{\mathbb{X}}$ given by

$$\widetilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\delta)$$

has a Rosati involution that induces the identity on O_E. For any $(X,\iota_B,\varrho) \in \mathcal{M}_{Dr}(S)$, we set $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}}) = \lambda \circ \iota_B(\delta)$. The tuple $(X,\iota_E,\lambda,\varrho) = \eta(X,\iota_B,\varrho)$ satisfies the straightening condition (3.12), since

$$\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda}) = \lambda \circ \iota_B\left(\frac{1+\delta}{2}\right) \in \mathrm{Hom}(X, X^\vee).$$

In particular, the tuple $(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})$ is a framing object of $\mathcal{N}_E$ and η induces a natural transformation

$$\eta : \mathcal{M}_{Dr} \longhookrightarrow \mathcal{N}_E. \tag{3.13}$$

Note that this map does not depend on the above choices, as $\mathcal{N}_E$ is a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$.

Theorem 3.14. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

We will first prove this on $\bar{k}$-valued points:

Lemma 3.15. *η induces a bijection $\eta(\bar{k}) : \mathcal{M}_{Dr}(\bar{k}) \to \mathcal{N}_E(\bar{k})$.*

Proof. We can identify the $\bar{k}$-valued points of $\mathcal{M}_{Dr}$ with a subset $\mathcal{M}_{Dr}(\bar{k}) \subseteq \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. The rational Dieudonné-module N of $\mathbb{X}$ is equipped with an action of B. Fix an embedding $F^{(2)} \hookrightarrow B$ as in Lemma 2.3 (1). This induces a $\mathbb{Z}/2$-grading $N = N_0 \oplus N_1$ of N, where

$$N_0 = \{x \in N \mid \iota(a)x = ax \text{ for all } a \in F^{(2)}\},$$
$$N_1 = \{x \in N \mid \iota(a)x = \sigma(a)x \text{ for all } a \in F^{(2)}\},$$

for a fixed embedding $F^{(2)} \hookrightarrow \breve{F}$. The operators $\mathbf{V}$ and $\mathbf{F}$ have degree 1 with respect to this decomposition. Recall that λ has Rosati involution $b \mapsto \Pi b' \Pi^{-1}$ on O_B which restricts to the identity on $O_F^{(2)}$. The subspaces N_0 and N_1 are therefore orthogonal with respect to $\langle\, , \rangle$.

Under the identification (3.5), a lattice $M \in \mathcal{M}_{Dr}(\bar{k})$ respects this decomposition, i.e. $M = M_0 \oplus M_1$ with $M_i = M \cap N_i$. Furthermore it satisfies the special condition:

$$\dim M_0/\mathbf{V}M_1 = \dim M_1/\mathbf{V}M_0 = 1.$$

We already know that $\mathcal{M}_{Dr}(\bar{k}) \subseteq \mathcal{N}_E(\bar{k})$, so let us assume $M \in \mathcal{N}_E(\bar{k})$. We want to show that $M \in \mathcal{M}_{Dr}(\bar{k})$, i. e., that the lattice M is stable under the action of O_B on N and satisfies the special condition. It is stable under the O_B-action if and only if $M = M_0 \oplus M_1$ for $M_i = M \cap N_i$. Let $y \in M$ and $y = y_0 + y_1$ with $y_i \in N_i$. For any $x \in M$, we have

$$\langle x, y \rangle = \langle x, y_0 \rangle + \langle x, y_1 \rangle \in \breve{O}_F. \tag{3.14}$$

We can assume that $\lambda_{\mathbb{X},1} = \lambda_{\mathbb{X}} \circ \iota_B\left(\frac{1+\delta}{2}\right)$ with $\frac{1+\delta}{2} \in O_F^{(2)}$ under our fixed embedding $F^{(2)} \hookrightarrow B$. Let $\langle\, , \rangle_1$ be the alternating form on M induced by $\lambda_{\mathbb{X},1}$. Then,

$$\langle x, y \rangle_1 = \frac{1+\delta}{2} \cdot \langle x, y_0 \rangle + \frac{1-\delta}{2} \cdot \langle x, y_1 \rangle \in \breve{O}_F. \tag{3.15}$$

From the equations (3.14) and (3.15), it follows that $\langle x, y_0 \rangle$ and $\langle x, y_1 \rangle$ lie in $\breve{O}_F$. Since $x \in M$ was arbitrary and $M = M^\vee$, this gives $y_0, y_1 \in M$. Hence M respects the decomposition of N and is stable under the action of O_B.

It remains to show that M satisfies the special condition: The alternating form $\langle\, , \rangle$ is perfect on M, thus the restrictions to M_0 and M_1 are perfect as well. If M is not special, we have $M_i = \mathbf{V}M_{i+1}$ for some $i \in \{0, 1\}$. But then, $\langle\, , \rangle$ cannot be perfect on M_i. In fact, for any $x, y \in M_{i+1}$,

$$\langle \mathbf{V}x, \mathbf{V}y \rangle^\sigma = \langle \mathbf{F}\mathbf{V}x, y \rangle = \pi_0 \cdot \langle x, y \rangle \in \pi_0 \breve{O}_F.$$

Thus M is indeed special, i.e. $M \in \mathcal{M}_{Dr}(\bar{k})$, and this finishes the proof of the lemma. □

Proof (of Theorem 3.14). We already know that η is a closed embedding

$$\eta : \mathcal{M}_{Dr} \longrightarrow \mathcal{N}_E.$$

Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be the framing object of $\mathcal{M}_{Dr}$ and choose an embedding $O_F^{(2)} \subseteq O_B$ and a generator $\frac{1+\delta}{2}$ of $O_F^{(2)}$ as in Lemma 2.3 (1). We take $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ as a framing object for $\mathcal{N}_E$ and set $\widetilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\delta)$.

Let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ and $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$. We have

$$\varrho^{-1} \circ \iota_{\mathbb{X}}\left(\frac{1+\delta}{2}\right) \circ \varrho = \varrho^{-1} \circ \lambda_{\mathbb{X}}^{-1} \circ \lambda_{\mathbb{X},1} \circ \varrho = \lambda^{-1} \circ \lambda_1 \in \mathrm{End}(X),$$

where $\lambda_{\mathbb{X},1} = \frac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$ and $\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda})$. Since $O_B = O_F\left[\Pi, \frac{1+\delta}{2}\right]$, this induces an O_B-action ι_B on X and makes ϱ an O_B-linear quasi-isogeny. We have to check that (X, ι_B, ϱ) satisfies the special condition.

Recall that the special condition is open and closed (see [RZ14, p. 7]), so η is an open and closed embedding. Furthermore, $\eta(\bar{k})$ is bijective and the reduced loci $(\mathcal{M}_{Dr})_{\mathrm{red}}$

and $(\mathcal{N}_E)_{\mathrm{red}}$ are locally of finite type over $\operatorname{Spec}\bar{k}$. Hence η indcues an isomorphism on reduced subschemes. But any open and closed embedding of formal schemes, that is an isomorphism on the reduced subschemes, is already an isomorphism. □

4. The moduli problem in the case (R-U)

Let $E|F$ be a quadratic extension of type (R-U), generated by a unit $\vartheta \in O_E^\times$ with $\vartheta^2 = 1 + \pi_0^{2k+1}\varepsilon$ for some $\varepsilon \in O_F^\times$ and for an integer k such that $|2| < |\pi_0|^k \leq |1|$ for the (normalized) absolute value $|\cdot|$ of F. A uniformizer of E is given by $\Pi = (1+\vartheta)/\pi_0^k$ and for the rings of integers O_F and O_E of F and E, we have $O_E = O_F[\Pi]$. As in the case (R-P), let k be the common residue field, $\bar{k}$ an algebraic closure, $\breve{F}$ the completion of the maximal unramified extension with ring of integers $\breve{O}_F = W_{O_F}(\bar{k})$ and σ the lift of the Frobenius in $\mathrm{Gal}(\bar{k}|k)$ to $\mathrm{Gal}(\breve{O}_F|O_F)$.

4.1. **The naive moduli problem.** Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. Consider tuples (X, ι, λ), where

- X is a formal O_F-module over S of dimension 2 and height 4.
- $\iota : O_E \to \mathrm{End}(X)$ is an action of O_E on X satisfying the *Kottwitz condition*: The characteristic polynomial of $\iota(\alpha)$ for some $\alpha \in O_E$ is given by

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}).$$

Here $\alpha \mapsto \bar{\alpha}$ is the Galois conjugation of $E|F$ and the right hand side is a polynomial in $\mathcal{O}_S[T]$ via the structure morphism $O_F \hookrightarrow \breve{O}_F \to \mathcal{O}_S$.

- $\lambda : X \to X^\vee$ is a polarization on X with kernel $\ker\lambda = X[\Pi]$, where $X[\Pi]$ is the kernel of $\iota(\Pi)$. Further we demand that the Rosati involution of λ satisfies $\iota(\alpha)^* = \iota(\bar{\alpha})$ for all $\alpha \in O_E$.

Definition 4.1. A *quasi-isogeny* (resp. an *isomorphism*) $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$ of two tuples (X, ι, λ) and (X', ι', λ') over $S \in \mathrm{Nilp}_{\breve{O}_F}$ is an O_E-linear quasi-isogeny of height 0 (resp. an O_E-linear isomorphism) $\varphi : X \to X'$ such that $\lambda = \varphi^*(\lambda')$.

Let $\mathrm{Aut}^0(X, \iota, \lambda)$ be the group of quasi-isogenies $\varphi : (X, \iota, \lambda) \to (X, \iota, \lambda)$. We have:

Proposition 4.2. *Up to isogeny, there exists exactly one such tuple* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *over* $S = \operatorname{Spec}\bar{k}$ *under the condition that the group*

$$G_{(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})} = \{\varphi \in \mathrm{Aut}^0(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \mid \det\varphi = 1\} \tag{4.1}$$

is isomorphic to $\mathrm{SU}(C, h)$ *for a* 2*-dimensional* E*-vector space* C *with split* $E|F$*-hermitian form* h*.*

Remark 4.3. As in the case (R-P), we consider $G_{(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})}$ and $\mathrm{SU}(C, h)$ as linear algebraic groups over F. We will prove uniqueness for the slightly weakened condition that $\mathrm{SU}(C, h)$ is only a Zariski-closed subgroup of $G_{(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})}$ (and it will follow implicitly that this is already an isomorphism).

Proof. We first show uniqueness of the object. Let $(X, \iota, \lambda)/\operatorname{Spec}\bar{k}$ be a tuple as in the proposition and consider its rational Dieudonné-module N_X. This is a 4-dimensional vector space over $\breve{F}$ equipped with an action of E and an alternating form $\langle\, ,\rangle$ such that

$$\langle x, \Pi y\rangle = \langle \overline{\Pi} x, y\rangle \tag{4.2}$$

for all $x, y \in N_X$. Let $E^{(2)} = F^{(2)} \otimes_F E$ the unramified quadratic extension of E and choose elements $c_1, c_2 \in E^{(2)}$ as follows: The element $\pi_0 \in O_F$ has even valuation in E, hence we can find an element $c_2 \in E^{(2)}$ with $\mathrm{Nm}_{E^{(2)}|E}(c_2) = \pi_0$. Note that

$$\mathrm{Nm}_{E^{(2)}|E}\left(\frac{\sigma(c_2)}{\overline{c_2}}\right) = \frac{\sigma(c_2)\cdot c_2}{\overline{c_2}\cdot\sigma(\overline{c_2})} = \frac{\pi_0}{\overline{\pi_0}} = 1,$$

where $\alpha \mapsto \overline{\alpha}$ is the conjugation in $\mathrm{Gal}(\breve{E}|\breve{F})$ and σ is the lift of the Frobenius inducing the conjugation in $\mathrm{Gal}(E^{(2)}|E)$. By Hilbert 90, there exists a unit $c_1 \in E^{(2)}$ with

$$\frac{c_1}{\sigma(c_1)} = \frac{\sigma(c_2)}{\overline{c_2}}.$$

Let $\breve{E} = \breve{F} \otimes_F E$. Now N_X is a 2-dimensional vector space over $\breve{E}$ with a hermitian form h given by

$$h(x,y) = c_1(\langle \Pi x, y\rangle - \overline{\Pi}\langle x, y\rangle). \tag{4.3}$$

Let $\mathbf{F}$ and $\mathbf{V}$ be the σ-linear Frobenius and the σ^{-1}-linear Verschiebung on N_X. We have $\mathbf{FV} = \mathbf{VF} = \pi_0$ and, since $\langle\, ,\rangle$ comes from a polarization,

$$\langle \mathbf{F}x, y\rangle = \langle x, \mathbf{V}y\rangle^{\sigma}.$$

Consider the σ-linear operator $\tau = c_2\mathbf{V}^{-1} = \mathbf{F}c_2^{-1}$. The hermitian form h is invariant under τ:

$$\begin{aligned} h(\tau x, \tau y) &= h(\mathbf{F}c_2^{-1}x, c_2\mathbf{V}^{-1}y) = \frac{\overline{c_2}}{\sigma(c_2)} \cdot h(\mathbf{F}x, \mathbf{V}^{-1}y) \\ &= \frac{\overline{c_2}}{\sigma(c_2)} \cdot \frac{c_1}{\sigma(c_1)} \cdot h(x,y)^{\sigma} = h(x,y)^{\sigma}. \end{aligned}$$

From the condition on $G_{(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})}$ it follows that N_X is isotypical of slope $\frac{1}{2}$ and thus the slopes of τ are all zero. Let $C = N_X^{\tau}$. This is a 2-dimensional vector space over E with $N_X = C \otimes_E \breve{E}$ and h induces an $E|F$-hermitian form on C. A priori, there are two possibilities for (C, h), either h is split or non-split. The group $\mathrm{SU}(C, h)$ of automorphisms of determinant 1 is isomorphic to $G_{(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})}$. But the special unitary groups for h split and h non-split are not isomorphic and do not contain each other as a closed subgroup for dimension reasons. Thus the condition on $G_{(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})}$ implies that h is split.

Assume we are given two different objects (X, ι, λ) and (X', ι', λ') as in the proposition. Then there is an isomorphism between the spaces (C, h) and (C', h') extending to an isomorphism of N_X and $N_{X'}$ respecting all structure. This corresponds to a quasi-isogeny $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$.

Now we prove the existence of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. We start with a Π-modular lattice Λ in a 2-dimensional vector space (C, h) over E with split hermitian form. Then $M = \Lambda \otimes_{O_E} \breve{O}_E$ is an $\breve{O}_E$-lattice in $N = C \otimes_E \breve{E}$. The σ-linear operator $\tau = 1 \otimes \sigma$ on N has slopes are all 0. We can extend h to N such that

$$h(\tau x, \tau y) = h(x,y)^{\sigma},$$

for all $x, y \in N$. Furthermore, we choose $c_1, c_2 \in E^{(2)}$ as in the first part of the proof. The operators $\mathbf{F}$ and $\mathbf{V}$ are given by the equations $\tau = c_2\mathbf{V}^{-1} = \mathbf{F}c_2^{-1}$. Finally, the alternating form $\langle\, ,\rangle$ is defined via

$$\langle x, y\rangle = \mathrm{Tr}_{\breve{E}|\breve{F}}\left(\frac{\pi_0^k}{2\vartheta c_1} \cdot h(x,y)\right),$$

for $x, y \in N$. The lattice $M \subseteq N$ is the Dieudonné module of the object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. We leave it to the reader to check that this is indeed an object as considered above. □

We fix such an object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\mathrm{Spec}\,\bar{k}$ from the proposition. We define the functor $\mathcal{N}_E^{\mathrm{naive}}$ on $\mathrm{Nilp}_{\breve{O}_F}$ as follows:

Let $S \in \mathrm{Nilp}_{\breve{O}_F}$ and write $\overline{S} = S \times_{\mathrm{Spf}\,\breve{O}_F} \mathrm{Spec}\,\bar{k}$. Then $\mathcal{N}_E^{\mathrm{naive}}(S)$ is the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$ over S where (X, ι, λ) is a tuple as above and ϱ is a quasi-isogeny

$$\varrho : (X, \iota, \lambda) \times_S \overline{S} \longrightarrow (\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \times_{\mathrm{Spec}\,\bar{k}} \overline{S}$$

in the sense of Def. 4.1. Two objects $(X, \iota, \lambda, \varrho)$ and $(X', \iota', \lambda', \varrho')$ are equivalent if there exists an isomorphism $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$, such that $\varrho = \varrho' \circ (\varphi \times_S \overline{S})$.
We refer to Remark 3.4 (1) for a comparison of the definition of $\mathcal{N}_E^{\mathrm{naive}}$ with [RZ96] (strict equality versus equality up to a unit).

Remark 4.4. $\mathcal{N}_E^{\mathrm{naive}}$ is pro-representable by a formal scheme, formally locally of finite type over $\mathrm{Spf}\, \breve{O}_F$, cf. [RZ96, Thm. 3.25].

We now study the $\bar{k}$-valued points of the space $\mathcal{N}_E^{\mathrm{naive}}$. Let $N = N_{\mathbb{X}}$ be the rational Dieudonné-module of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. This is a 4-dimensional vector space over $\breve{F}$, equipped with an action of E, with two operators $\mathbf{F}$ and $\mathbf{V}$ and an alternating form $\langle\, , \rangle$.
Let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. This corresponds to an $\breve{O}_F$-lattice $M = M_X \subseteq N$ which is stable under the actions of $\mathbf{F}$, $\mathbf{V}$ and O_E. The condition on the kernel of λ implies that $M = \Pi M^\vee$ for

$$M^\vee = \{x \in N \mid \langle x, y \rangle \in \breve{O}_F \text{ for all } y \in M\}.$$

The alternating form $\langle\, , \rangle$ induces an $\breve{E}|\breve{F}$-hermitian form h on N, seen as 2-dimensional vector space over $\breve{E}$:

$$h(x, y) = c_1(\langle \Pi x, y \rangle - \overline{\Pi} \langle x, y \rangle). \tag{4.3}$$

Recall that c_1 is a unit in $\breve{E}$ and that

$$\langle x, y \rangle = \mathrm{Tr}_{\breve{E}|\breve{F}} \left(\frac{\pi_0^k}{2\vartheta c_1} \cdot h(x, y) \right). \tag{4.4}$$

Since the inverse different of $E|F$ is $\mathfrak{D}_{E|F}^{-1} = \frac{\pi_0^k}{2} O_E$ (see Lemma 2.2), this implies that M is Π-modular with respect to h, as $\breve{O}_E$-lattice in N. We denote the dual of M with respect to h by $M^\sharp$. There is a natural bijection

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \{\breve{O}_E\text{-lattices } M \subseteq N \mid M = \Pi M^\sharp, \pi_0 M \subseteq \mathbf{V} M \subseteq M\}. \tag{4.5}$$

Recall that $\tau = c_2 \mathbf{V}^{-1}$ is a σ-linear operator on N with slopes all 0. Further $C = N^\tau$ is a 2-dimensional E-vector space with hermitian form h.

Lemma 4.5. *Let $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then:*
(1) *$M + \tau(M)$ is τ-stable.*
(2) *Either M is τ-stable and $\Lambda_1 = M^\tau \subseteq C$ is Π-modular, or M is not τ-stable and then $\Lambda_0 = (M + \tau(M))^\tau \subseteq C$ is selfdual (with respect to h).*

The proof is the same as that of [KR11, Lemma 3.2]. We identify N with $C \otimes_E \breve{E}$. For any τ-stable lattice $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$, we have $M = \Lambda_1 \otimes_{O_E} \breve{O}_E$. If $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is not τ-stable, there is an inclusion $M \subseteq \Lambda_0 \otimes_{O_E} \breve{O}_E$ of index 1. Recall from Prop. 2.4 that the isomorphism class of a Π-modular or selfdual lattice $\Lambda \subseteq C$ is determined by the norm ideal

$$\mathrm{Nm}(\Lambda) = \langle \{h(x, x) | x \in \Lambda\} \rangle.$$

There are always at least two types of selfdual lattices. However, not all of them appear in the description of $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$.

Lemma 4.6. (1) *Let $\Lambda \subseteq C$ be a selfdual lattice with* $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. *There is an injection*

$$i_\Lambda : \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}) \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}(\bar{k}),$$

that maps a line $\ell \subseteq \Lambda/\Pi\Lambda \otimes_k \bar{k}$ to its inverse image under the canonical projection

$$\Lambda \otimes_{O_E} \breve{O}_E \longrightarrow \Lambda/\Pi\Lambda \otimes_k \bar{k}.$$

The k-valued points $\mathbb{P}(\Lambda/\Pi\Lambda)(k) \subseteq \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ are mapped to τ-invariant Dieudonné modules $M \subseteq \Lambda \otimes_{O_E} \breve{O}_E$ under this embedding.

(2) *Identify* $\mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ *with its image under* i_Λ*. The set* $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ *can be written as*

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union is taken over all lattices $\Lambda \subseteq C$ *as described in part* (1) *of this lemma.*

Proof. Let $\Lambda \subseteq C$ be a selfdual lattice. For any line $\ell \in \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$, denote its preimage in $\Lambda \otimes \breve{O}_E$ by M. The inclusion $M \subseteq \Lambda \otimes \breve{O}_E$ has index 1 and M is an $\breve{O}_E$-lattice with $\Pi(\Lambda \otimes \breve{O}_E) \subseteq M$. Furthermore $\Lambda \otimes \breve{O}_E$ is τ-invariant by construction, hence $\Pi(\Lambda \otimes \breve{O}_E) = \mathbf{V}(\Lambda \otimes \breve{O}_E) = \mathbf{F}(\Lambda \otimes \breve{O}_E)$. It follows that M is stable under the actions of $\mathbf{F}$ and $\mathbf{V}$. Thus $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ if and only if $M = \Pi M^\sharp$. The hermitian form h induces a symmetric form s on $\Lambda/\Pi\Lambda$. Now M is Π-modular if and only if it is the preimage of an isotropic line $\ell \subseteq \Lambda/\Pi\Lambda \otimes \bar{k}$. Note that s is also anti-symmetric since we are in characteristic 2.

We first consider the case $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. We can find a basis of Λ such that h has the form

$$H_\Lambda = \begin{pmatrix} x & 1 \\ 1 & \end{pmatrix}, \quad x \in \pi_0 O_F,$$

cf. (2.3). It follows that the induced form s is even alternating (because $x \equiv 0 \mod \pi_0$). Hence any line in $\Lambda/\Pi\Lambda \otimes \bar{k}$ is isotropic. This implies that i_Λ is well-defined, proving part 1 of the Lemma.

Now assume that $\mathrm{Nm}(\Lambda) = O_F$. There is a basis (e_1, e_2) of Λ such that h is represented by

$$H_\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}.$$

The induced form s is given by the same matrix and $\ell = \bar{k} \cdot e_2$ is the only isotropic line in $\Lambda/\Pi\Lambda$. Since ℓ is already defined over k, the corresponding lattice $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is of the form $M = \Lambda_1 \otimes \breve{O}_E$ for a Π-modular lattice $\Lambda_1 \subseteq \Lambda$. But, by Prop. 2.8, any Π-modular lattice in C is contained in a selfdual lattice Λ' with $\mathrm{Nm}(\Lambda') \subseteq \pi_0 O_F$.

It follows that we can write $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ as a union

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union is taken over all selfdual lattices $\Lambda \subseteq C$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. This shows the second part of the Lemma. ☐

Remark 4.7. We can use Prop. 2.8 to describe the intersection behaviour of the projective lines in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. A τ-invariant point $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ corresponds to the Π-modular lattice $\Lambda_1 = M^\tau \subseteq C$. If $\mathrm{Nm}(\Lambda_1) \subseteq \pi_0^2 O_F$, there are $q+1$ lines going through M. If $\mathrm{Nm}(\Lambda_1) = \pi_0 O_F$, the point M is contained in one or 2 lines, depending on whether Λ_1 is hyperbolic or not, see part (3) and (4) of Prop. 2.8. The former case (i. e., Λ_1 is hyperbolic) appears if and only if $\pi_0 O_F = \mathrm{Nm}(\Lambda_1) = \frac{2}{\pi_0^k} O_F$ (see Lemma 2.5), which is equivalent to $|\pi_0^{k+1}| = |2|$. This is a condition that only depends on the quadratic extension $E|F$, see page 7. We refer to Remark 4.9, Remark 4.12 and Section 5.3 for a further discussion of this special case.

On the other hand, each projective line in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ contains $q+1$ τ-invariant points. Such a τ-invariant point M is an intersection point of 2 or more projective lines if and only if $|\pi_0^{k+1}| = |2|$ or $\Lambda_1 = M^\tau \subseteq C$ has a norm ideal satisfying $\mathrm{Nm}(\Lambda_1) \subseteq \pi_0^2 O_F$.

Let $\Lambda \subseteq C$ as in Lemma 4.6. We denote by X_Λ^+ the formal O_F-module corresponding to the Dieudonné module $M = \Lambda \otimes \breve{O}_E$. There is a canonical quasi-isogeny

$$\varrho_\Lambda^+ : \mathbb{X} \longrightarrow X_\Lambda^+$$

(A) $e = 2,\ f = 1,\ k = 0.$ (B) $e = 2,\ f = 1,\ k = 1.$

FIGURE 2. The reduced locus of $\mathcal{N}_E^{\mathrm{naive}}$ for an (R-U) extension $E|F$ where e and f are the ramification index and the inertia degree of $F|\mathbb{Q}_2$. We always have $0 \leq k \leq e-1$. The solid lines lie in $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$.

of F-height 1. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, we define

$$\mathcal{N}_{E,\Lambda}(S) = \{(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S) \mid (\varrho_\Lambda^+ \times S) \circ \varrho \text{ is an isogeny}\}.$$

By [RZ96, Prop. 2.9], the functor $\mathcal{N}_{E,\Lambda}$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$. On geometric points, we have

$$\mathcal{N}_{E,\Lambda}(\bar{k}) \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}), \tag{4.6}$$

as follows from Lemma 4.6 (1).

Proposition 4.8. *The reduced locus of* $\mathcal{N}_E^{\mathrm{naive}}$ *is a union*

$$(\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda}$$

where Λ *runs over all selfdual lattices in* C *with* $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. *For each* Λ*, there exists an isomorphism*

$$\mathcal{N}_{E,\Lambda} \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda),$$

inducing the bijection (4.6) *on* $\bar{k}$*-valued points.*

The proof is analogous to that of Prop. 3.8.

Remark 4.9. Similar to Remark 3.7 (3), we let $(\mathcal{N}_E)_{\mathrm{red}} \subseteq (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$ be the union of all projective lines $\mathcal{N}_{E,\Lambda}$ corresponding to *hyperbolic* selfdual lattices $\Lambda \subseteq C$. Later, we will define $\mathcal{N}_E$ as a functor on $\mathrm{Nilp}_{\breve{O}_F}$ and show that $\mathcal{N}_E \simeq \mathcal{M}_{Dr}$, where $\mathcal{M}_{Dr}$ is the Drinfeld moduli problem (see Theorem 4.15, a description of the formal scheme $\mathcal{M}_{Dr}$ can be found in [BC91, I.3]). In particular, $(\mathcal{N}_E)_{\mathrm{red}}$ is connected and each projective line in $(\mathcal{N}_E)_{\mathrm{red}}$ has $q+1$ intersection points and there are 2 lines intersecting in each such point.
It might happen that $(\mathcal{N}_E)_{\mathrm{red}} = (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$ (see, for example, Figure 2(B)), if there are no non-hyperbolic selfdual lattices $\Lambda \subseteq C$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. In fact, this is the case if and only if $|\pi_0^{k+1}| = |2|$, cp. Prop. 2.4 and Lemma 2.5. (Note however that we still have $\mathcal{N}_E \neq \mathcal{N}_E^{\mathrm{naive}}$, see Remark 4.12 and Section 5.3.)
Assume $|\pi_0^{k+1}| \neq |2|$ and let $P \in \mathcal{N}_E(\bar{k})$ be an intersection point. Then, as in the case where $E|F$ is of type (R-P) (cp. Remark 3.7 (3)), the connected component of P in $((\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}} \setminus (\mathcal{N}_E)_{\mathrm{red}}) \cup \{P\}$ consists of a finite union of projective lines (corresponding to non-hyperbolic lattices, by definition of $(\mathcal{N}_E)_{\mathrm{red}}$). In Figure 2(A), these components are indicated by dashed lines (they consist of just one projective line in that case).

4.2. **The straightening condition.** As in the case (R-P), cp. section 3.2, we use the results of section 6 to define the straightening condition on $\mathcal{N}_E^{\mathrm{naive}}$. By Theorem 6.2 and Remark 6.1 (2), there exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}^0$ on the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ such that the Rosati involution is the identity on O_E. We set $\widetilde{\lambda}_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}}^0 \circ \iota_{\mathbb{X}}(\Pi)$, which is again a polarization on $\mathbb{X}$ with the Rosati involution inducing the identity on O_E, but with kernel $\ker \widetilde{\lambda}_{\mathbb{X}} = \mathbb{X}[\Pi]$. This polarization is unique up to a scalar in $O_E^{\times}$, i. e., any two polarizations $\widetilde{\lambda}_{\mathbb{X}}$ and $\widetilde{\lambda}'_{\mathbb{X}}$ with these properties satisfy

$$\widetilde{\lambda}'_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}} \circ \iota(\alpha),$$

for some $\alpha \in O_E^{\times}$. For any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$,

$$\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}}) = \varrho^*(\widetilde{\lambda}_{\mathbb{X}}^0) \circ \iota(\Pi)$$

is a polarization on X with kernel $\ker \widetilde{\lambda} = X[\Pi]$, see Theorem 6.2 (2).
Recall that a selfdual or Π-modular lattice $\Lambda \subseteq C$ is called *hyperbolic* if there exists a basis (e_1, e_2) of Λ such that, with respect to this basis, h has the form

$$\begin{pmatrix} & \Pi^i \\ \overline{\Pi}^i & \end{pmatrix},$$

for $i = 0$ resp. 1. By Lemma 2.5, this is the case if and only if $\mathrm{Nm}(\Lambda) = \frac{2}{\pi_0^k} O_F$.

Proposition 4.10. *For a suitable choice of* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *and* $\widetilde{\lambda}_{\mathbb{X}}$, *the quasi-polarization*

$$\lambda_{\mathbb{X},1} = \frac{\pi_0^k}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$$

is a polarization on $\mathbb{X}$. *Let* $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ *and* $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$. *Then* $\lambda_1 = \frac{\pi_0^k}{2}(\lambda + \widetilde{\lambda})$ *is a polarization if and only if* $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ *for a hyperbolic selfdual lattice* $\Lambda \subseteq C$.

Proof. On the rational Dieudonné module $N = M_{\mathbb{X}} \otimes_{\breve{O}_F} \breve{F}$, denote by $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ the alternating forms induced by $\lambda_{\mathbb{X}}$, $\widetilde{\lambda}_{\mathbb{X}}$ and $\lambda_{\mathbb{X},1}$, respectively. The form $\langle\,,\rangle_1$ is integral on $M_{\mathbb{X}}$ if and only if $\lambda_{\mathbb{X},1}$ is a polarization on $\mathbb{X}$. We have

$$\begin{aligned}(\mathbf{F}x, y) &= (x, \mathbf{V}y)^{\sigma}, \\ (\Pi x, y) &= (x, \Pi y), \\ \langle x, y\rangle_1 &= \frac{\pi_0^k}{2}(\langle x, y\rangle + (x, y)),\end{aligned}$$

for all $x, y \in N$. The form $(\,,)$ induces an $\breve{E}$-bilinear alternating form b on N by the formula

$$b(x, y) = c_3((\Pi x, y) - \overline{\Pi}(x, y)). \tag{4.7}$$

Here, $c_3 = \Pi/c_2$ is a unit in $\breve{O}_E$. The dual of M with respect to this form is again $M^{\sharp} = \Pi^{-1}M$, since

$$(x, y) = \mathrm{Tr}_{\breve{E}|\breve{F}}\left(\frac{\pi_0^k}{2\vartheta c_3} \cdot b(x, y)\right),$$

and the inverse different of $E|F$ is given by $\mathfrak{D}_{E|F}^{-1} = \frac{\pi_0^k}{2} O_E$, cf. Lemma 2.2. Now b is invariant under the σ-linear operator $\tau = c_2\mathbf{V}^{-1} = \mathbf{F}c_2^{-1}$, because

$$\begin{aligned} b(\tau x, \tau y) = b(\mathbf{F}c_2^{-1}x, c_2\mathbf{V}^{-1}y) &= \frac{c_3}{\sigma(c_3)} \cdot b(c_2^{-1}x, \sigma^{-1}(c_2)y)^{\sigma} \\ &= \frac{c_3 \cdot c_2}{\sigma(c_3)\cdot\sigma(c_2)} \cdot b(x, y)^{\sigma} = b(x, y)^{\sigma}. \end{aligned}$$

Hence b defines an E-linear alternating form on C.
We choose the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ such that $M_{\mathbb{X}}$ is τ-invariant (see Lemma 4.5) and such that $\Lambda_1 = M_{\mathbb{X}}^{\tau}$ is hyperbolic. We can find a basis (e_1, e_2) of Λ_1 such that

$$h \mathrel{\widehat{=}} \begin{pmatrix} & \Pi \\ \overline{\Pi} & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & u \\ -u & \end{pmatrix},$$

for some $u \in E^{\times}$. Since $\widetilde{\lambda}_{\mathbb{X}}$ has the same kernel as $\lambda_{\mathbb{X}}$, we have $u = \overline{\Pi} u'$ for some unit $u' \in O_E^{\times}$. We can choose $\widetilde{\lambda}_{\mathbb{X}}$ such that $u' = 1$ and $u = \overline{\Pi}$. Now $\frac{\pi_0^k}{2}(h(x,y) + b(x,y))$ is integral for all $x, y \in \Lambda_1$. Hence $\frac{\pi_0^k}{2}(h(x,y) + b(x,y))$ is also integral for all $x, y \in M_{\mathbb{X}}$. For all $x, y \in M_{\mathbb{X}}$, we have

$$\begin{aligned} \langle x, y\rangle_1 = \frac{\pi_0^k}{2}(\langle x, y\rangle + (x,y)) &= \frac{\pi_0^k}{2} \operatorname{Tr}_{\breve{E}|\breve{F}} \left(\frac{\pi_0^k}{2\vartheta c_1} \cdot h(x,y) + \frac{\pi_0^k}{2\vartheta c_3} \cdot b(x,y) \right) \\ &= \operatorname{Tr}_{\breve{E}|\breve{F}} \left(\frac{\pi_0^k}{2\vartheta c_1} \cdot \frac{\pi_0^k}{2}(h(x,y) + b(x,y)) \right) + \operatorname{Tr}_{\breve{E}|\breve{F}} \left(\frac{\pi_0^k}{2\vartheta c_1 c_3} \cdot \frac{\pi_0^k (c_1 - c_3)}{2} b(x,y) \right). \end{aligned}$$

The first summand is integral since $\frac{\pi_0^k}{2}(h(x,y) + b(x,y))$ is integral. Recall that c_1 is a unit in $\breve{O}_E^{\times}$ with $c_1/\sigma(c_1) = \sigma(c_2)/\overline{c_2}$. After multiplying with a suitable scalar in $O_E^{\times}$, we may assume that c_1 is of the form $c_1 = c_3 + \frac{2}{\pi_0^k}\alpha$ for some $\alpha \in O_E$. Indeed, $c_1 c_3^{-1}$ satisfies the equation

$$\frac{c_1 c_3^{-1}}{\sigma(c_1 c_3^{-1})} = \frac{\sigma(c_2) \cdot c_2}{\overline{c_2} \cdot \sigma(c_2)} = \frac{c_2}{\overline{c_2}} = 1 \mod 2/\pi_0^k,$$

thus we can modify c_1 by a scalar in $O_E^{\times}$ such that $c_1 c_3^{-1} = 1 \mod 2/\pi_0^k$. It follows that the second summand above is integral as well. Hence $\langle\,,\rangle_1$ is integral on $M_{\mathbb{X}}$ and this implies that $\lambda_{\mathbb{X},1}$ is a polarization on $\mathbb{X}$.
Now let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ and denote by $M \subseteq N$ its Dieudonné module. Assume that $\lambda_1 = \frac{\pi_0^k}{2}(\lambda + \widetilde{\lambda})$ is a polarization on X. Then $\langle\,,\rangle_1$ is integral on M. But this is equivalent to $\frac{\pi_0^k}{2}(h(x,y) + b(x,y))$ being integral for all $x, y \in M$. For $x = y$, we have

$$h(x,x) = h(x,x) + b(x,x) \in \frac{2}{\pi_0^k} \cdot \breve{O}_F.$$

Let $\Lambda \subseteq C$ be the selfdual or Π-modular lattice given by $\Lambda = M^{\tau}$ resp. $\Lambda = (M + \tau(M))^{\tau}$, see Lemma 4.5. Then $h(x,x) \in (2/\pi_0^k) \cdot O_F$ for all $x \in \Lambda$. Thus $\mathrm{Nm}(\Lambda) \subseteq (2/\pi_0^k) \cdot O_F$ and, by minimality, this implies that $\mathrm{Nm}(\Lambda) = \frac{2}{\pi_0^k} O_F$ and Λ is hyperbolic (cf. Lemma 2.5). Hence, in either case, the point corresponding to $(X, \iota, \lambda, \varrho)$ lies in $\mathcal{N}_{E,\Lambda'}$ for a hyperbolic lattice Λ'.
Conversely, assume that $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ for some hyperbolic lattice $\Lambda \subseteq C$. We want to show that λ_1 is a polarization on X. This follows if $\langle\,,\rangle_1$ is integral on M, or equivalently, if $\frac{\pi_0^k}{2}(h(x,y) + b(x,y))$ is integral on M. For this, it is enough to show that $\frac{\pi_0^k}{2}(h(x,y) + b(x,y))$ is integral on Λ. Let $\Lambda' \subseteq C$ be the selfdual lattice generated by $\overline{\Pi}^{-1} e_1$ and e_2, where (e_1, e_2) is the basis of the Π-modular lattice $\Lambda_1 = M_{\mathbb{X}}$. With respect to the basis $(\overline{\Pi}^{-1} e_1, e_2)$, we have

$$h \mathrel{\widehat{=}} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

In particular, Λ' is a hyperbolic lattice and $\frac{\pi_0^k}{2}(h+b)$ is integral on Λ'. By Prop. 2.4, there exists an element $g \in \mathrm{SU}(C,h)$ with $g\Lambda = \Lambda'$. Since $\det g = 1$, the alternating form b is invariant under g. Thus $\frac{\pi_0^k}{2}(h+b)$ is also integral on Λ. □

From now on, we assume that $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$ are chosen in a way such that

$$\lambda_{\mathbb{X},1} = \frac{\pi_0^k}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) \in \mathrm{Hom}(\mathbb{X}, \mathbb{X}^\vee).$$

Definition 4.11. A tuple $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ satisfies the *straightening condition* if

$$\lambda_1 = \frac{\pi_0^k}{2}(\lambda + \widetilde{\lambda}) \in \mathrm{Hom}(X, X^\vee). \tag{4.8}$$

This condition is independent of the choice of $\widetilde{\lambda}_{\mathbb{X}}$. In fact, we can only change $\widetilde{\lambda}_{\mathbb{X}}$ by a scalar of the form $1 + 2u/\pi_0^k$, $u \in O_E$. But if $\widetilde{\lambda}'_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}} \circ \iota(1 + 2u/\pi_0^k)$, then $\lambda'_{\mathbb{X},1} = \lambda_{\mathbb{X},1} + \widetilde{\lambda}_{\mathbb{X}} \circ \iota(u)$ and $\lambda'_1 = \lambda_1 + \widetilde{\lambda} \circ \iota(u)$. Clearly, λ'_1 is a polarization if and only if λ_1 is one.
For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{N}_E(S)$ be the set of all tuples $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ that satisfy the straightening condition. By [RZ96, Prop. 2.9], the functor $\mathcal{N}_E$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$.

Remark 4.12. The reduced locus of $\mathcal{N}_E$ is given by

$$(\mathcal{N}_E)_{\mathrm{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda} \simeq \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda),$$

where the union goes over all *hyperbolic* selfdual lattices $\Lambda \subseteq C$. Note that, depending on the form of the (R-U) extension $E|F$, it may happen that all selfdual lattices are hyperbolic (when $|\pi_0^{k+1}| = |2|$) and in that case, we have $(\mathcal{N}_E)_{\mathrm{red}} = (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$. However, the equality does not extend to an isomorphism between $\mathcal{N}_E$ and $\mathcal{N}_E^{\mathrm{naive}}$. This will be discussed in section 5.3.

4.3. The main theorem for the case (R-U).

As in the case (R-P), we want to establish a connection to the Drinfeld moduli problem. Therefore, fix an embedding of E into the quaternion division algebra B. Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be the framing object of the Drinfeld problem. We want to construct a polarization $\lambda_{\mathbb{X}}$ on $\mathbb{X}$ with $\ker \lambda_{\mathbb{X}} = \mathbb{X}[\Pi]$ and Rosati involution given by $b \mapsto \vartheta b' \vartheta^{-1}$ on B. Here $b \mapsto b'$ denotes the standard involution on B.
By Lemma 2.3 (2), there exists an embedding $E_1 \hookrightarrow B$ of a ramified quadratic extension $E_1|F$ of type (R-P), such that $\Pi_1\vartheta = -\vartheta\Pi_1$ for a prime element $\Pi_1 \in E_1$. From Proposition 3.12 (1) we get a principal polarization $\lambda^0_{\mathbb{X}}$ on $\mathbb{X}$ with associated Rosati involution $b \mapsto \Pi_1 b' \Pi_1^{-1}$. If we assume fixed choices of E_1 and Π_1, this is unique up to a scalar in $O_F^\times$. We define

$$\lambda_{\mathbb{X}} = \lambda^0_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\Pi_1\vartheta).$$

Obviously $\ker \lambda_{\mathbb{X}} = \mathbb{X}[\Pi]$ and the Rosati involution of $\lambda_{\mathbb{X}}$ is indeed $b \mapsto \vartheta b' \vartheta^{-1}$. On the other hand, any polarization on $\mathbb{X}$ satisfying these two conditions can be constructed in this way (using the same choices for E_1 and Π_1). Hence:

Lemma 4.13. (1) *There exists a polarization $\lambda_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^\vee$, unique up to a scalar in $O_F^\times$, with $\ker \lambda_{\mathbb{X}} = \mathbb{X}[\Pi]$ and associated Rosati involution $b \mapsto \vartheta b' \vartheta^{-1}$.*

(2) *Fix $\lambda_{\mathbb{X}}$ as in (1) and let $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$. There exists a unique polarization λ on X with $\ker \lambda = X[\Pi]$ and Rosati involution $b \mapsto \vartheta b' \vartheta^{-1}$ such that $\varrho^*(\lambda_{\mathbb{X}}) = \lambda$ on $\overline{S} = S \times_{\mathrm{Spf}\, \breve{O}_F} \bar{k}$.*

Note also that the involution $b \mapsto \vartheta b' \vartheta^{-1}$ does not depend on the choice of $\vartheta \in E$. We write $\iota_{\mathbb{X},E}$ for the restriction of $\iota_{\mathbb{X}}$ to $E \subseteq B$ and, in the same manner, we write ι_E for the restriction of ι_B to E for any $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$. Fix a polarization $\lambda_{\mathbb{X}}$ of $\mathbb{X}$ as in Lemma 4.13 (1). Accordingly for a tuple $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, let λ be the polarization given by Lemma 4.13 (2).

Lemma 4.14. *The tuple $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ is a framing object of $\mathcal{N}_E^{\mathrm{naive}}$. Moreover, the map*

$$(X, \iota_B, \varrho) \longmapsto (X, \iota_E, \lambda, \varrho)$$

induces a closed embedding of formal schemes

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}.$$

Proof. We follow the same argument as in the proof of Lemma 3.13. Again it is enough to check that $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})} \simeq \mathrm{SU}(C,h)$ and that ι_E satisfies the Kottwitz condition.
By [RZ14, Prop. 5.8], the special condition on ι_B implies the Kottwitz condition for ι_E. It remains to show that $G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})} \simeq \mathrm{SU}(C,h)$. But the group $G_{(\mathbb{X},\iota_{\mathbb{X}})}$ of automorphisms of determinant 1 of $(\mathbb{X}, \iota_{\mathbb{X}})$ is isomorphic to $\mathrm{SL}_{2,F}$ and $G_{(\mathbb{X},\iota_{\mathbb{X}})} \subseteq G_{(\mathbb{X},\iota_{\mathbb{X},E},\lambda_{\mathbb{X}})}$ is a Zariski-closed subgroup by the same argument as in Lemma 3.13. Hence the statement follows from the exceptional isomorphism $\mathrm{SL}_{2,F} \simeq \mathrm{SU}(C,h)$ and Remark 4.3. □

As a next step, we want to show that this already induces a closed embedding

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E. \tag{4.9}$$

Let $\widetilde{E} \hookrightarrow B$ an embedding of a ramified quadratic extension $\widetilde{E}|F$ of type (R-U) as in Lemma 2.3 (2). On the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ of $\mathcal{M}_{Dr}$, we define a polarization $\widetilde{\lambda}_{\mathbb{X}}$ via

$$\widetilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\widetilde{\vartheta}),$$

where $\widetilde{\vartheta}$ is a unit in $\widetilde{E}$ of the form $\widetilde{\vartheta}^2 = 1 + (4/\pi_0^{2k+1}) \cdot \widetilde{\varepsilon}$, see Lemma 2.3 (2). The Rosati involution of $\widetilde{\lambda}_{\mathbb{X}}$ induces the identity on O_E and we have

$$\begin{aligned} \lambda_{\mathbb{X},1} = \frac{\pi_0^k}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) = \frac{\pi_0^k}{2} \cdot \lambda_{\mathbb{X}} \circ \iota_B(1 + \widetilde{\vartheta}) = \lambda_{\mathbb{X}} \circ \iota_B(\widetilde{\Pi}/\pi_0) \\ = \lambda_{\mathbb{X}} \circ \iota_B(\Pi^{-1} \cdot \frac{1+\delta}{2}) \in \mathrm{Hom}(\mathbb{X}, \mathbb{X}^\vee), \end{aligned}$$

using the notation of Lemma 2.3 (2). For $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, we set $\widetilde{\lambda} = \lambda \circ \iota_B(\widetilde{\vartheta})$. By the same calculation, we have $\lambda_1 = \frac{\pi_0^k}{2}(\lambda + \widetilde{\lambda}) \in \mathrm{Hom}(X, X^\vee)$. Thus the tuple $(X, \iota_E, \lambda, \varrho) = \eta(X, \iota_B, \varrho)$ satisfies the straightening condition. Hence we get a closed embedding of formal schemes $\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ which is independent of the choice of $\widetilde{E}$.

Theorem 4.15. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

We first check this for $\bar{k}$-valued points:

Lemma 4.16. *η induces a bijection $\eta(\bar{k}) : \mathcal{M}_{Dr}(\bar{k}) \to \mathcal{N}_E(\bar{k})$.*

Proof. We only have to show surjectivity and we will use for this the Dieudonné theory description of $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$, see (4.5). The rational Dieudonné-module $N = N_{\mathbb{X}}$ of $\mathbb{X}$ now carries additionally an action of B. The embedding $F^{(2)} \hookrightarrow B$ given by

$$\frac{1+\delta}{2} \longmapsto \frac{\Pi \cdot \widetilde{\Pi}}{\pi_0}, \tag{2.1}$$

cf. Lemma 2.3 (2), induces a $\mathbb{Z}/2$-grading $N = N_0 \oplus N_1$. Here,

$$\begin{aligned} N_0 &= \{x \in N \mid \iota(a)x = ax \text{ for all } a \in F^{(2)}\}, \\ N_1 &= \{x \in N \mid \iota(a)x = \sigma(a)x \text{ for all } a \in F^{(2)}\}, \end{aligned}$$

for a fixed embedding $F^{(2)} \hookrightarrow \breve{F}$. The operators $\mathbf{F}$ and $\mathbf{V}$ have degree 1 with respect to this grading. The principal polarization

$$\lambda_{\mathbb{X},1} = \frac{\pi_0^k}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\Pi^{-1} \cdot \frac{1+\delta}{2})$$

induces an alternating form $\langle\,,\rangle_1$ on N that satisfies

$$\langle x, y \rangle_1 = \langle x, \iota(\Pi^{-1} \cdot \frac{1+\delta}{2}) \cdot y \rangle,$$

for all $x, y \in N$. Let $M \in \mathcal{N}_E(\bar{k}) \subseteq \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ be an $\breve{O}_F$-lattice in N. We claim that $M \in \mathcal{M}_{Dr}(\bar{k})$. For this, it is necessary that M is stable under the action of $O_F^{(2)}$ (since $O_B = O_F[\Pi, \frac{1+\delta}{2}] = O_F^{(2)}[\Pi]$, cf. Lemma 2.3 (2)) or equivalently, that M respects the grading of N, i.e. $M = M_0 \oplus M_1$ for $M_i = M \cap N_i$. Furthermore M has to satisfy the *special* condition:

$$\dim M_0/\mathbf{V}M_1 = \dim M_1/\mathbf{V}M_0 = 1.$$

We first show that $M = M_0 \oplus M_1$. Let $y = y_0 + y_1 \in M$ with $y_i \in N_i$. Since $M = \Pi M^\vee$, we have

$$\langle x, \iota(\Pi)^{-1} y \rangle = \langle x, \iota(\Pi)^{-1} y_0 \rangle + \langle x, \iota(\Pi)^{-1} y_1 \rangle \in \breve{O}_F,$$

for all $x \in M$. Together with

$$\begin{aligned}\langle x, y \rangle_1 = \langle x, y_0 \rangle_1 + \langle x, y_1 \rangle_1 &= \langle x, \iota(\widetilde{\Pi}/\pi_0) y_0 \rangle + \langle x, \iota(\widetilde{\Pi}/\pi_0) y_1 \rangle \\ &= \frac{1+\delta}{2} \cdot \langle x, \iota(\Pi^{-1}) y_0 \rangle + \frac{1-\delta}{2} \cdot \langle x, \iota(\Pi^{-1}) y_1 \rangle \in \breve{O}_F,\end{aligned}$$

this implies that $\langle x, \iota(\Pi^{-1}) y_0 \rangle$ and $\langle x, \iota(\Pi^{-1}) y_1 \rangle$ lie in $\breve{O}_F$ for all $x \in M$. Hence, $y_0, y_1 \in M$ and this means that M respects the grading. It follows that M is stable under the action of O_B.

In order to show that M is special, note that

$$\langle \mathbf{V}x, \mathbf{V}y \rangle_1^\sigma = \langle \mathbf{F}\mathbf{V}x, y \rangle_1 = \pi_0 \cdot \langle x, y \rangle_1 \in \pi_0 \breve{O}_F,$$

for all $x, y \in M$. The form $\langle\,,\rangle_1$ comes from a principal polarization, so it induces a perfect form on M. Now it is enough to show that also the restrictions of $\langle\,,\rangle_1$ to M_0 and M_1 are perfect. Indeed, if M was not special, we would have $M_i = \mathbf{V}M_{i+1}$ for some i and this would contradict $\langle\,,\rangle_1$ being perfect on M_i. We prove that $\langle\,,\rangle_1$ is perfect on M_i by showing $\langle M_0, M_1 \rangle_1 \subseteq \pi_0 \breve{O}_F$.

Let $x \in M_0$ and $y \in M_1$. Then,

$$\begin{aligned}\langle x, y \rangle_1 &= \frac{1-\delta}{2} \langle x, \iota(\Pi)^{-1} y \rangle, \\ \langle x, y \rangle_1 &= -\langle y, x \rangle_1 = -\frac{1+\delta}{2} \langle y, \iota(\Pi)^{-1} x \rangle = \frac{1+\delta}{2} \langle x, \iota(\overline{\Pi})^{-1} y \rangle.\end{aligned}$$

We take the difference of these two equations. From $\Pi \equiv \overline{\Pi} \mod \pi_0$, it follows that $\langle x, \iota(\Pi)^{-1} y \rangle \equiv 0 \mod \pi_0$ and thus also $\langle x, y \rangle_1 \equiv 0 \mod \pi_0$. The form $\langle\,,\rangle_1$ is hence perfect on M_0 and M_1 and the special condition follows. This finishes the proof of Lemma 4.16. □

Proof (of Theorem 4.15). Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be a framing object for $\mathcal{M}_{Dr}$ and let further

$$\eta(\mathbb{X}, \iota_{\mathbb{X}}) = (\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$$

be the corresponding framing object for $\mathcal{N}_E$. We fix an embedding $F^{(2)} \hookrightarrow B$ as in Lemma 2.3 (2). For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ and $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$. We have

$$\begin{aligned} \varrho^{-1} \circ \iota_{\mathbb{X}} \left(\frac{1+\delta}{2}\right) \circ \varrho &= \varrho^{-1} \circ \iota_{\mathbb{X}}(\Pi) \circ \lambda_{\mathbb{X}}^{-1} \circ \lambda_{\mathbb{X},1} \circ \varrho \\ &= \iota(\Pi) \circ \lambda^{-1} \circ \lambda_1 \in \mathrm{End}(X), \end{aligned}$$

for $\lambda_1 = \frac{\pi_0^k}{2}(\lambda + \widetilde{\lambda})$, since $\ker \lambda = X[\Pi]$. But $O_B = O_F[\Pi, \frac{1+\delta}{2}]$ (see Lemma 2.3 (2)), so this already induces an O_B-action ι_B on X. It remains to show that (X, ι_B, ϱ) satisfies the *special* condition (see the discussion before Prop. 3.12 for a definition).
The special condition is open and closed (see [RZ14, p. 7]) and η is bijective on $\bar{k}$-points. Hence η induces an isomorphism on reduced subschemes

$$(\eta)_{\mathrm{red}} : (\mathcal{M}_{Dr})_{\mathrm{red}} \xrightarrow{\sim} (\mathcal{N}_E)_{\mathrm{red}},$$

because $(\mathcal{M}_{Dr})_{\mathrm{red}}$ and $(\mathcal{N}_E)_{\mathrm{red}}$ are locally of finite type over $\mathrm{Spec}\, \bar{k}$. It follows that $\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism. ☐

5. The local model of $\mathcal{N}_E$

In this section we will construct a local model of the moduli problem $\mathcal{N}_E$ in both the cases (R-P) and (R-U). As before, we will treat these two cases separately, although the methods used are similar. See [PRS13] for an introduction to the theory of local models. Note that there is no good notion of a “naive local model” for the functor $\mathcal{N}_E^{\mathrm{naive}}$ (in the sense of [RZ96]). In fact, let $E|F$ be of type (R-P), or of type (R-U) with $|\pi_0^{k+1}| > |2|$, and let (C, h) be a 2-dimensional E-vector space with split hermitian form h. Then, by Prop. 2.4, there are at least two isomorphism classes of selfdual resp. Π-modular lattices $\Lambda \subseteq C$. This means that there is no such thing as a “standard lattice” and hence it is not clear how one should define a local model of $\mathcal{N}_E^{\mathrm{naive}}$. In the remaining case, $E|F$ of type (R-U) such that $|\pi_0^{k+1}| = |2|$, all Π-modular lattices $\Lambda \subseteq C$ are hyperbolic and we can write down a naive local model functor $\mathrm{N}_E^{\mathrm{naive}}$. However, this naive local model is not flat over $\mathrm{Spec}\, O_F$, as we will show in section 5.3.

5.1. **The case (R-P).** In this paragraph, $E|F$ will be a ramified quadratic extension of type (R-P). Let C be a 2-dimensional vector space over E and let h be a split $E|F$-hermitian form on C. We can (and will) choose a basis (e_1, e_2) of C such that h has the form

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

and, with respect to this basis, we define an E-linear alternating form b given by the matrix

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

We denote by Λ the O_E-lattice generated by e_1 and e_2. It is hyperbolic with respect to h and it satisfies the straightening condition in the sense that

$$\frac{1}{2}(h(x,y) + b(x,y)) \in O_E,$$

for all $x, y \in \Lambda$. This setting is unique in the following sense.

Lemma 5.1. *Let Λ be a free O_E-module of rank 2 with a perfect split $E|F$-hermitian form h and a perfect alternating form b, such that $\frac{1}{2}(h(x,y) + b(x,y)) \in O_E$ for all $x, y \in \Lambda$. Then there exists an isomorphism $\Lambda \otimes_{O_E} E = C$, a basis (e_1, e_2) of Λ and a unit $u \in 1 + 2O_E$ such that h and $u \cdot b$ are as given above.*

Proof. We have an isomorphism $\Lambda \otimes_{O_E} E \xrightarrow{\sim} C$, since there exists only one split hermitian vector space of dimension 2. If we identify $\Lambda \otimes_{O_E} E$ with C, the perfectness of h implies that $\Lambda \subseteq C$ is a selfdual lattice. From the condition that $\frac{1}{2}(h(x,y)+b(x,y)) \in O_E$, we get that $h(x,x) \in 2O_F$ for all $x \in \Lambda$. By Lemma 2.5, it follows that Λ is hyperbolic, i. e., there exists a basis (e_1, e_2) of Λ such that h is as given above. With respect to this basis, the alternating form b has the form

$$\begin{pmatrix} & u \\ -u & \end{pmatrix},$$

for some unit $u \in O_E^\times$. But we have $u \in 1+2O_E$, since $\frac{1}{2}(h(e_1,e_2)+b(e_1,e_2)) \in O_E$. □

We can also interpret C as a 4-dimensional F-vector space. The forms h and b induce F-linear symplectic forms $\langle\, ,\rangle$ and $(\, ,)$ via

$$\langle x, y\rangle = \mathrm{Tr}_{E|F}\left(\frac{1}{2\Pi}\cdot h(x,y)\right),$$
$$(x, y) = \mathrm{Tr}_{E|F}\left(\frac{1}{2\Pi}\cdot b(x,y)\right),$$

for $x, y \in C$. Then Λ is selfdual with respect to these forms and the equation

$$\langle x, y\rangle_1 = \frac{1}{2}(\langle x, y\rangle + (x,y))$$

defines another alternating form $\langle\, ,\rangle_1$ that is integral on Λ. In terms of the O_F-basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ of Λ, we have

$$\langle\, ,\rangle \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right), \quad (\, ,) \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right), \tag{5.1}$$
$$\langle\, ,\rangle_1 \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & & & 1 \\ & & 0 & \\ \hline & 0 & & \\ -1 & & & \end{array}\right).$$

Remark 5.2. Note that the forms $\langle\, ,\rangle$ and $\langle\, ,\rangle_1$ are defined not quite in the same way as on the rational Dieudonné module $N = N_{\mathbb{X}}$ in section 3. On N, there is in fact a twist by δ in the formulae relating h and $\langle\, ,\rangle$, see (3.3) and (3.4). But since we want to define the local model $\mathrm{N}_E^{\mathrm{loc}}$ over O_F (and $\delta \notin O_F$), we cannot make this twist here. However, if we change the base to an unramified extension O_F' of O_F and twist h or b by a unit in $(O_F' \otimes_{O_F} O_E)^\times$ such that the conditions of Lemma 5.1 are still fulfilled, then the description of $\mathrm{N}_E^{\mathrm{loc}}$ itself will remain the same, as condition (2) is invariant under such a twist.

We now define the local model $\mathrm{N}_E^{\mathrm{loc}}$ as a functor on schemes over $\mathrm{Spec}\, O_F$. For an O_F-scheme S, we let $\mathrm{N}_E^{\mathrm{loc}}(S)$ be the set of all locally free direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S$ of rank 2 over $\mathcal{O}_S$ that satisfy the following conditions:

(1) $\mathcal{F}$ is O_E-linear, i. e., it is an $O_E \otimes_{O_F} \mathcal{O}_S$-submodule of $\Lambda \otimes_{O_F} \mathcal{O}_S$.

(2) Via base change, $\langle\, ,\rangle$, $(\, ,)$ and $\langle\, ,\rangle_1$ induce alternating forms of the type

$$(\Lambda \otimes \mathcal{O}_S) \times (\Lambda \otimes \mathcal{O}_S) \longrightarrow \mathcal{O}_S.$$

The direct summand $\mathcal{F}$ is totally isotropic with respect to all these forms.

(3) The last condition is the *Kottwitz condition*. For any $\alpha \in O_E$, the characteristic polynomial for the action of $\alpha \otimes 1$ on the quotient $(\Lambda \otimes_{O_F} \mathcal{O}_S)/\mathcal{F}$ is given by

$$\mathrm{char}((\Lambda \otimes_{O_F} \mathcal{O}_S)/\mathcal{F}, T \mid \alpha \otimes 1) = (T-\alpha)(T-\bar{\alpha}).$$

Here the polynomial on the right hand side has a priori coefficients in O_F and it becomes a polynomial in $\mathcal{O}_S[T]$ via the structure morphism $O_F \to \mathcal{O}_S$.

The functor $\mathrm{N}_E^{\mathrm{loc}}$ is representable by a closed subscheme of $\mathrm{Gr}(2,\Lambda)_{O_F}$, the Grassmanian of rank 2 direct summands of Λ. In particular, $\mathrm{N}_E^{\mathrm{loc}}$ is projective over $\operatorname{Spec} O_F$.
Recall from [RZ96] the definition of the local model $\mathrm{M}_{Dr}^{\mathrm{loc}}$ for the Drinfeld moduli problem. Let Λ' be a free O_F-module of rank 4 with an O_B-action. Let S be a scheme over $\operatorname{Spec} O_F$. Then $\mathrm{M}_{Dr}^{\mathrm{loc}}(S)$ is the set of locally free direct summands $\mathcal{F} \subseteq \Lambda' \otimes_{O_F} \mathcal{O}_S$ of rank 2 over $\mathcal{O}_S$ that are O_B-stable.
The functor $\mathrm{M}_{Dr}^{\mathrm{loc}}$ is representable by a flat closed subscheme of $\mathrm{Gr}(2,\Lambda)_{O_F}$, cf. [PRS13, Example 2.4]. Clearly, $\mathrm{M}_{Dr}^{\mathrm{loc}}(F) = \emptyset$, hence $\mathrm{N}_E^{\mathrm{loc}}$ and $\mathrm{M}_{Dr}^{\mathrm{loc}}$ cannot be isomorphic over O_F. However, they are isomorphic after base change to the unramified quadratic extension $O_F^{(2)}$ of O_F.

Proposition 5.3. *Fix an embedding $O_E \hookrightarrow O_B$. Then $\Lambda' \otimes_{O_F} O_F^{(2)}$ and $\Lambda \otimes_{O_F} O_F^{(2)}$ are isomorphic as free $O_E \otimes_{O_F} O_F^{(2)}$-modules. For a fixed isomorphism*

$$\varphi : \Lambda' \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \Lambda \otimes_{O_F} O_F^{(2)},$$

there is a canonical isomorphism

$$\mu : \mathrm{M}_{Dr}^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \mathrm{N}_E^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)}.$$

In particular, $\mathrm{N}_E^{\mathrm{loc}}$ is flat.

Proof. Once the isomorphism is established, the flatness of $\mathrm{N}_E^{\mathrm{loc}}$ follows from the flatness of $\mathrm{M}_{Dr}^{\mathrm{loc}}$.
In order to construct the isomorphism, we want to identify $\Lambda' \otimes_{O_F} O_F^{(2)}$ with $\Lambda \otimes_{O_F} O_F^{(2)}$ and then we show that the conditions on the direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S$ for $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(S)$ and $\mathcal{F} \in \mathrm{M}_{Dr}^{\mathrm{loc}}(S)$ are equivalent for any scheme S over $O_F^{(2)}$.
Therefore, we define alternating forms $\langle\, , \rangle$, $(\, ,)$ and $\langle\, , \rangle_1$ on Λ'. First of all, if we choose an isomorphism $\Lambda' \simeq O_B$ as O_B-modules, there is a symmetric form s on Λ' given by

$$s(x,y) = \mathrm{Trd}(xy'),$$

where $x, y \in O_B$ and $y \mapsto y'$ is the standard involution of B. For any $b \in O_B$, we have $s(bx,y) = s(x,b'y)$. If we change the isomorphism $\Lambda' \simeq O_B$ by a scalar in $b \in O_B^\times$, the form s is replaced by s' with

$$s'(x,y) = s(bx,by) = bb' \cdot s(x,y).$$

Thus s is canonical up to scalar in $O_F^\times$. Note that s is not a perfect bilinear form. In fact, $(\det s) \cdot O_F = \operatorname{disc} O_B = \pi_0 O_F$ by definition of the discriminant. We now fix the embedding $O_E \hookrightarrow O_B$ and take an embedding of $O_F^{(2)}$ as in Lemma 2.3 (1). We set

$$\begin{aligned} \langle x,y\rangle &= s(x,\Pi^{-1}y), \\ (x,y) &= s(x,\Pi^{-1}\delta y), \\ \langle x,y\rangle_1 &= s(x,\Pi^{-1}\cdot \frac{1+\delta}{2} y). \end{aligned}$$

Then these forms satisfy all conditions of Lemma 5.1 with the one exception that $\langle\, , \rangle$ induces a *non-split* hermitian form. After base change to $O_F^{(2)}$, the latter becomes equivalent to a split hermitian form and we can identify $\Lambda' \otimes_{O_F} O_F^{(2)}$ with $\Lambda \otimes_{O_F} O_F^{(2)}$.

Now $\Lambda^{(2)} = \Lambda \otimes_{O_F} O_F^{(2)}$ splits into a direct sum $\Lambda^{(2)} = \Lambda_0^{(2)} \oplus \Lambda_1^{(2)}$ with

$$\Lambda_0^{(2)} = \{x \in \Lambda^{(2)} \mid \iota(\alpha)x = \alpha x \text{ for all } \alpha \in O_F^{(2)}\},$$
$$\Lambda_1^{(2)} = \{x \in \Lambda^{(2)} \mid \iota(\alpha)x = \sigma(\alpha)x \text{ for all } \alpha \in O_F^{(2)}\}.$$

For clarity, we denote the O_B-action on $\Lambda^{(2)}$ by ι here, and σ is the lift of the Frobenius, as usual. Let $x \in \Lambda_0^{(2)}$ and $y \in \Lambda_1^{(2)}$. Then,

$$\begin{aligned}\frac{1+\delta}{2} \cdot \langle x, y\rangle = \langle x, \iota(\frac{1-\delta}{2})y\rangle = \langle \iota(\Pi \cdot \frac{1+\delta}{2} \cdot \Pi^{-1})x, y\rangle = \\ = \langle \iota(\frac{1-\delta}{2})x, y\rangle = \frac{1-\delta}{2} \cdot \langle x, y\rangle. \end{aligned} \tag{5.2}$$

Hence $\langle x, y\rangle = 0$ for all $x \in \Lambda_0^{(2)}$, $y \in \Lambda_1^{(2)}$. By the same argument, we also get $(x, y) = 0$ and $\langle x, y\rangle_1 = 0$.
Let R be an $O_F^{(2)}$-algebra and assume $\mathcal{F} \in \mathrm{M}_{Dr}^{\mathrm{loc}}(R)$. Then $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ is an O_B-stable direct summand of rank 2. We want to show that $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$.
Clearly, $\mathcal{F}$ is stable under the action of O_E, and the Kottwitz condition follows from the fact that $(\Lambda \otimes_{O_F} R)/\mathcal{F}$ is O_B-stable. It remains to prove that $\mathcal{F}$ is totally isotropic with respect to $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$. Since $\mathcal{F}$ is stable under O_B, it splits into a direct sum $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ where $\mathcal{F}_i \subseteq \Lambda_i^{(2)} \otimes_{O_F^{(2)}} R$ is a direct summand of rank 1. But now $\langle \mathcal{F}_0, \mathcal{F}_1\rangle = 0$ by the calculation of (5.2), i. e., $\mathcal{F}$ is totally isotropic with respect to $\langle\,,\rangle$. Analogously, this also follows for the alternating forms $(\,,)$ and $\langle\,,\rangle_1$. Hence $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$.
Conversely, assume that $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$. We claim that $\mathcal{F} \in \mathrm{M}_{Dr}^{\mathrm{loc}}(R)$, i. e., $\mathcal{F}$ is stable under the O_B-action.
First we show that $\mathcal{F}$ is stable under the action of $O_F^{(2)} \subseteq O_B$. Assume it is not, i. e., there is a $x \in \mathcal{F}$ such that $\iota(\frac{1+\delta}{2})x \notin \mathcal{F}$. Since $\mathcal{F}$ is totally isotropic with respect to $\langle\,,\rangle$ and $\langle\,,\rangle_1$, we get $\langle y, \iota(\frac{1+\delta}{2})x\rangle = 0$ for all $y \in \mathcal{F}$. But $\langle\,,\rangle$ is a perfect form, hence a maximal totally isotropic direct summand has rank 2, contradicting our assumption. It follows that $\mathcal{F}$ is O_B-stable, since $O_B = O_F^{(2)}[\Pi]$ by Lemma 2.3 (1). Thus $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$ if and only if $\mathcal{F} \in \mathrm{M}_{Dr}^{\mathrm{loc}}(R)$, q.e.d. □

Remark 5.4. We can also compute explicit equations for $\mathrm{N}_E^{\mathrm{loc}}$ by restricting standard affine charts of $\mathrm{Gr}(2, \Lambda)_{O_F}$ and prove the flatness of $\mathrm{N}_E^{\mathrm{loc}}$ directly this way.
Let R be an O_F-algebra and consider a direct summand $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ such that $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$. We choose a basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ of $\Lambda \otimes_{O_F} R$ such that the alternating forms $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ are given by the matrices in (5.1). Since $\mathcal{F}$ is O_E-linear, we can choose a R-basis (v_1, v_2) of $\mathcal{F}$ such that either $v_2 = \Pi v_1$ or $\mathcal{F}$ lies in the chart around $(\Pi e_1, \Pi e_2)$. Thus we only need to consider three of the six standard affine charts.
(1) The chart around $(\Pi e_1, \Pi e_2)$. We assume that the vectors v_1 and v_2 have the form

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ 1 & \\ & 1 \end{pmatrix},$$

for some $x_{ij} \in R$. The conditions (1) - (3) on $\mathcal{F}$ can be reduced to the following set of equations:

$$\begin{aligned} x_{11} + x_{22} &= 0, \\ x_{11}x_{22} - x_{12}x_{21} &= -\pi_0, \\ x_{11} &= 0. \end{aligned}$$

Hence, the restriction of $\mathrm{N}_E^{\mathrm{loc}}$ to this affine chart is given by

$$\mathcal{U}_{(\Pi e_1, \Pi e_2)} = \operatorname{Spec} O_F[x_{12}, x_{21}]/(x_{12}x_{21} - \pi_0).$$

(2) The chart around $(e_1, \Pi e_1)$. We now assume that $v_2 = \Pi v_1$ and that they are of the form

$$[v_1\, v_2] = \begin{pmatrix} 1 & \\ x_{12} & \pi_0 x_{14} \\ & 1 \\ x_{14} & x_{12} \end{pmatrix},$$

for $x_{ij} \in R$. The conditions on $\mathcal{F}$ are equivalent to $x_{12} = 0$. Thus this affine chart of $\mathrm{N}_E^{\mathrm{loc}}$ is isomorphic to $\mathcal{U}_{(e_1, \Pi e_1)} = \operatorname{Spec} O_F[x_{14}]$.
(3) The chart around $(e_2, \Pi e_2)$. Equations for this chart are obtained by exchanging e_1 and e_2 in the calculation for the second chart. It follows that on this chart $\mathrm{N}_E^{\mathrm{loc}}$ is again an affine line over $\operatorname{Spec} O_F$. The flatness of $\mathrm{N}_E^{\mathrm{loc}}$ is now evident.

We want to relate $\mathcal{N}_E$ and $\mathrm{N}_E^{\mathrm{loc}}$ via a local model diagram, cf. [RZ96, Chapter 3]. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{M}(S)$ be the set of tuples $(X, \iota, \lambda, \varrho; \gamma)$, where $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ and γ is an isomorphism of $O_E \otimes_{O_F} \mathcal{O}_S$-modules

$$\gamma : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S. \tag{5.3}$$

Here, $\mathbb{D}_X(S)$ denotes the Grothendieck-Messing crystal of X evaluated at S (in other words, it is the Lie algebra of the universal O_F-vector extension of X, cf. [Ahs11, 5.2]). The polarizations λ, $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ and $\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda})$ induce alternating forms $\langle\, , \rangle^X$, $(\, ,)^X$ and $\langle\, , \rangle_1^X$ on $\mathbb{D}_X(S)$. We demand that the isomorphism γ induces the following equalities for all $x, y \in \mathbb{D}_X(S)$:

$$\begin{aligned} \gamma^*\langle x, y\rangle &= \delta \cdot \langle x, y\rangle^X, \\ \gamma^*(x, y) &= (x, y)^X, \\ \gamma^*\langle x, y\rangle_1 &= \langle x, y\rangle_1^X + \frac{\delta - 1}{2} \cdot \langle x, y\rangle^X. \end{aligned}$$

Recall that $\delta \in \breve{O}_F$ is a unit generating the unramified quadratic extension of F with $\frac{1+\delta}{2} \in \breve{O}_F$. These become elements in $\mathcal{O}_S$ via the structure map $\breve{O}_F \hookrightarrow \mathcal{O}_S$. For an explanation why we need these twists, see Remark 5.2.
This defines the functor $\mathcal{M}$ on $\mathrm{Nilp}_{\breve{O}_F}$. There is a forgetful morphism

$$f : \mathcal{M} \longrightarrow \mathcal{N}_E. \tag{5.4}$$

Let $\operatorname{Aut}(\Lambda)$ over $\operatorname{Spec} O_F$ be the affine group scheme of automorphisms of Λ respecting all structure, i. e., for any O_F-scheme S, let $\operatorname{Aut}(\Lambda)(S)$ be the group of automorphisms of $O_E \otimes_{O_F} \mathcal{O}_S$-modules

$$\varphi : \Lambda \otimes_{O_F} \mathcal{O}_S \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S,$$

that leave all alternating forms invariant. Let $\mathcal{P} = \operatorname{Aut}(\Lambda) \times_{\operatorname{Spec} O_F} \operatorname{Spf} \breve{O}_F$. Then $\mathcal{M}$ is an $\mathcal{P}$-torsor over $\mathcal{N}_E$, hence it is pro-representable by a formal scheme which is of finite type over $\mathcal{N}_E$.
We denote by $\widehat{\mathrm{N}}_E^{\mathrm{loc}}$ the π_0-adic completion of $\mathrm{N}_E^{\mathrm{loc}} \otimes_{O_F} \breve{O}_F$. We have a second morphism

$$g : \mathcal{M} \longrightarrow \widehat{\mathrm{N}}_E^{\mathrm{loc}} \tag{5.5}$$

that maps a point $(X, \iota, \lambda, \varrho; \gamma) \in \mathcal{M}(S)$ to

$$\mathcal{F} = \ker(\Lambda \otimes_{O_F} \mathcal{O}_S \xrightarrow{\gamma^{-1}} \mathbb{D}_X(S) \longrightarrow \operatorname{Lie} X) \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S.$$

This morphism is formally smooth by Grothendieck-Messing theory (see [Mes72, V.1.6] for $O_F = \mathbb{Z}_p$, the other cases follow from the definition of $\mathbb{D}_X$ in [Ahs11, Chap. 5.2]). The local model diagram now looks as follows:

$$\begin{array}{ccc} & \mathcal{M} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathcal{N}_E & & \widehat{\mathrm{N}}_E^{\mathrm{loc}} \end{array} \tag{5.6}$$

In order to be able to use the results from [RZ96, Chapter 3], we need the following result:

Proposition 5.5. *f is smooth and surjective.*

Remark 5.6. Here, f is smooth in the sense that for any scheme S and any morphism $S \to \mathcal{N}_E$, the morphism of schemes $S \times_{\mathcal{N}_E} \mathcal{M} \to S$ is smooth. Thus f is smooth if and only if $\mathrm{Aut}(\Lambda)$ is smooth over O_F.
By saying f is surjective, we mean that f is surjective as a map of étale sheaves. However, if f is smooth, this is equivalent to f being surjective on geometric points.

Proof. For an O_F-algebra R, let $g \in \mathrm{Aut}(\Lambda)(R)$. Then g leaves the form $\langle\,,\rangle_1$ invariant on $\Lambda \otimes R$, hence it also leaves invariant $\ker \langle\,,\rangle_1 = \{x \in \Lambda \otimes R \mid \langle x, y\rangle_1 = 0 \,\forall y\}$, which is a direct summand of rank 2 of $\Lambda \otimes R$. Furthermore $g\Pi = \Pi g$, i. e., with respect to the basis $(e_1, e_2, \Pi e_1, \Pi e_2)$, the element g is given by a 4×4 matrix of the form

$$g \mathrel{\widehat{=}} \begin{pmatrix} a & & & \pi_0 b \\ & d & \pi_0 c & \\ & b & a & \\ c & & & d \end{pmatrix},$$

with coefficients in R. From $\langle g e_1, g\Pi e_2\rangle_1 = \langle e_1, \Pi e_2\rangle_1$, we deduce the condition

$$ad - \pi_0 bc = 1.$$

Conversely, it is easily checked that any $g \in \mathrm{GL}_R(\Lambda \otimes R)$ of this form is an element of $\mathrm{Aut}(\Lambda)(R)$. Therefore we have an isomorphism of schemes,

$$\mathrm{Aut}(\Lambda) \simeq \mathrm{Spec}\, O_F[a, b, c, d]/(ad - \pi_0 bc - 1).$$

The smoothness of $\mathrm{Aut}(\Lambda)$ follows.
By the previous remark, this shows the smoothness of f. It now suffices to check the surjectivity of f on geometric points. Thus we have to show that for any tuple $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(\bar{k})$, there exists an isomorphism

$$\gamma : \mathbb{D}_X(\bar{k}) \xrightarrow{\sim} \Lambda \otimes_{O_F} \bar{k}$$

as in (5.3). We can write $\mathbb{D}_X(\bar{k}) = M_X/\pi_0 M_X$, where M_X is the Dieudonné module of X. Hence, we only need to construct an isomorphism

$$\gamma : M_X \xrightarrow{\sim} \Lambda \otimes_{O_F} \breve{O}_F.$$

This isomorphism has to be O_E-equivariant and compatible with the alternating forms $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ on both sides. Such an isomorphism is equivalent to an isomorphism of $\breve{O}_E$-lattices

$$\gamma : M_X \xrightarrow{\sim} \Lambda \otimes_{O_E} \breve{O}_E,$$

such that the induced forms h and b on Λ and on M_X coincide. But it is easy to check that, up to $\breve{O}_E$-linear isomorphism of M_X, there is only one possible choice for the forms h and b such that $\frac{1}{2}(h + b)$ is integral. Hence, there exists such an isomorphism γ. $\square$

Thus (5.6) is indeed a local model diagram in the sense of [RZ96]. It is compatible with the local model diagram for $\mathcal{M}_{Dr}$ in the following sense. Let $\mathcal{M}'$ be the functor on $\mathrm{Nilp}_{\breve{O}_F}$ that maps S to the set of tuples $(X, \iota_B, \varrho; \gamma')$, where $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$ and γ' is isomorphism of $O_B \otimes_{O_F} \mathcal{O}_S$-modules

$$\gamma' : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda' \otimes_{O_F} \mathcal{O}_S.$$

By [RZ96, Chapter 3], $\mathcal{M}'$ is pro-representable by a formal scheme of finite type. Furthermore, we have a smooth projection $p : \mathcal{M}' \to \mathcal{M}_{Dr}$ and a formally smooth morphism $q : \mathcal{M}' \to \mathrm{M}_{Dr}^{\mathrm{loc}}$ that maps $(X, \iota_B, \varrho; \gamma')$ to

$$\mathcal{F} = \ker(\Lambda' \otimes_{O_F} \mathcal{O}_S \xrightarrow{\gamma^{-1}} \mathbb{D}_X(S) \longrightarrow \mathrm{Lie}\, X) \subseteq \Lambda' \otimes_{O_F} \mathcal{O}_S.$$

It factors over $\widehat{\mathrm{M}}_{Dr}^{\mathrm{loc}}$, the π_0-adic completion of $\mathrm{M}_{Dr}^{\mathrm{loc}} \otimes_{O_F} \breve{O}_F$. Now the following diagram is commutative.

$$\begin{array}{ccccc} \mathcal{M}_{Dr} & \xleftarrow{p} & \mathcal{M}' & \xrightarrow{q} & \widehat{\mathrm{M}}_{Dr}^{\mathrm{loc}} \\ \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{N}_E & \xleftarrow{f} & \mathcal{M} & \xrightarrow{g} & \widehat{\mathrm{N}}_E^{\mathrm{loc}} \end{array} \tag{5.7}$$

In fact, let

$$\varphi : \Lambda' \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \Lambda \otimes_{O_F} O_F^{(2)}$$

be the isomorphism inducing μ (cf. Prop. 5.3). The morphism in the middle then maps $(X, \iota_B, \varrho; \gamma')$ to $(\eta(X, \iota_B, \varrho); (\varphi \otimes_{O_F^{(2)}} \mathcal{O}_S) \circ \gamma')$.

5.2. **The case (R-U).** Let $E|F$ be a quadratic extension of type (R-U) and let (C, h) be a 2-dimensional vector space over E with split hermitian form h. We choose a basis (e_1, e_2) of C such that h has the form

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

and we define an E-linear alternating form b given by the matrix

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Let Λ be the O_E-lattice generated by $(e_1, \Pi e_2)$. The lattice Λ is Π-modular hyperbolic with respect to h and satisfies the straightening condition, i. e.,

$$\frac{\pi_0^k}{2} \cdot (h(x, y) + b(x, y)) \in O_E,$$

for all $x, y \in \Lambda$. The datum (C, h, b, Λ) with the above conditions is unique up to E-linear isomorphism of C and up to multiplying b by a scalar $u \in 1 + (2/\pi_0^k) \cdot O_E$, cp. Lemma 5.1. We now interpret C as an F-vector space of dimension 4 and define F-linear symplectic forms $\langle\, ,\rangle$ and $(\, ,)$ on C via

$$\langle x, y \rangle = \mathrm{Tr}_{E|F}\left(\frac{\pi_0^k}{2\vartheta} \cdot h(x, y)\right),$$

$$(x, y) = \mathrm{Tr}_{E|F}\left(\frac{\pi_0^k}{2\vartheta} \cdot b(x, y)\right).$$

The dual of Λ with respect to each of these forms is $\Lambda^\sharp = \Pi^{-1}\Lambda$. Furthermore the alternating form $\langle\, ,\rangle_1$ defined by

$$\langle x, y \rangle_1 = \frac{\pi_0^k}{2}(\langle x, y \rangle + (x, y)),$$

for all $x, y \in C$, is integral on Λ. In terms of the basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ of C, the forms $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ are represented by the following matrices:

$$\langle\,,\rangle \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right), \quad (\,,) \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & 1 & & \frac{2}{\pi_0^k} \\ -1 & & -\frac{2}{\pi_0^k} & \end{array}\right),$$

$$\langle\,,\rangle_1 \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & & & \pi_0^k \\ & & 0 & \\ \hline & 0 & & 1 \\ -\pi_0^k & & -1 & \end{array}\right).$$

Remark 5.7. As in the (R-P) case, there will be some twists by units in $\breve{O}_F^\times$ involved if one compares the forms $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ to those coming from a point $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$. However, this does not affect the construction of $\mathrm{N}_E^{\mathrm{loc}}$, see Remark 5.2.

We now give the definition of the local model functor $\mathrm{N}_E^{\mathrm{loc}}$. For an O_F-scheme S, we let $\mathrm{N}_E^{\mathrm{loc}}(S)$ be the set of locally free direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S$ of rank 2 over $\mathcal{O}_S$ such that

(1) $\mathcal{F}$ is an $O_E \otimes_{O_F} \mathcal{O}_S$-submodule of $\Lambda \otimes_{O_F} \mathcal{O}_S$.

(2) The forms $\langle\,,\rangle$ and $(\,,)$ induce perfect bilinear forms on $\Lambda \otimes_{O_F} \mathcal{O}_S$ via base change of the compositions

$$\Lambda \times \Lambda \xrightarrow{(1,\Pi^{-1})} \Lambda \times \Lambda^\sharp \longrightarrow O_F. \tag{5.8}$$

We require that $\mathcal{F}$ is totally isotropic with respect to these induced forms.

(3) $\mathcal{F}$ is totally isotropic with respect to the alternating form induced by $\langle\,,\rangle_1$ on $\Lambda \otimes \mathcal{O}_S$.

(4) The *Kottwitz condition* holds. For any element $\alpha \in O_E$, the action of $\alpha \otimes 1$ on the quotient $(\Lambda \otimes_{O_F} \mathcal{O}_S)/\mathcal{F}$ has the characteristic polynomial

$$\mathrm{char}((\Lambda \otimes_{O_F} \mathcal{O}_S)/\mathcal{F}, T \mid \alpha \otimes 1) = (T - \alpha)(T - \overline{\alpha}).$$

The functor $\mathrm{N}_E^{\mathrm{loc}}$ is representable by a closed subscheme of the Grassmanian $\mathrm{Gr}(2, \Lambda)_{O_F}$. In particular, it is projective over $\mathrm{Spec}\, O_F$. For the following proposition, recall from section 5.1 that $\mathrm{M}_{Dr}^{\mathrm{loc}}$ is a flat closed subscheme of $\mathrm{Gr}(2, \Lambda)_{O_F}$.

Proposition 5.8. *Fix an embedding $E \hookrightarrow B$ and let $O_F^{(2)}$ be the unramified quadratic extension of O_F. Then, $\Lambda' \otimes_{O_F} O_F^{(2)}$ and $\Lambda \otimes_{O_F} O_F^{(2)}$ are isomorphic as free $O_E \otimes_{O_F} O_F^{(2)}$-modules. For a fixed isomorphism $\varphi : \Lambda' \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \Lambda \otimes_{O_F} O_F^{(2)}$, there is a canonical isomorphism $\mu : \mathrm{M}_{Dr}^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)} \xrightarrow{\sim} \mathrm{N}_E^{\mathrm{loc}} \otimes_{O_F} O_F^{(2)}$. In particular, $\mathrm{N}_E^{\mathrm{loc}}$ is flat over O_F.*

Proof. Let Λ' be a free O_F-module of rank 4 with an O_B-action. For an O_F-algebra R, the set $\mathrm{M}_{Dr}^{\mathrm{loc}}(R)$ is the set of direct summands $\mathcal{F} \subseteq \Lambda' \otimes_{O_F} R$ of rank 2 that are stable under the O_B-action. Assume now that R is even an $O_F^{(2)}$-algebra. In order to construct the claimed isomorphism, we want to identify $\Lambda' \otimes_{O_F} O_F^{(2)}$ with $\Lambda \otimes_{O_F} O_F^{(2)}$. Then we only have to show that a direct summand $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ lies in $\mathrm{M}_{Dr}^{\mathrm{loc}}(R)$ if and only if it lies in $\mathrm{N}_E^{\mathrm{loc}}(R)$.

Recall from section 5.1 that there is a canonical symmetric form s on Λ' that is defined up to a scalar in $O_F^\times$. We now fix the embedding $E \hookrightarrow B$ and choose embeddings $\widetilde{E} \hookrightarrow B$ and $F^{(2)} \hookrightarrow B$ as in Lemma 2.3 (2). In order to identify $\Lambda' \otimes_{O_F} O_F^{(2)}$ with $\Lambda \otimes_{O_F} O_F^{(2)}$, we have to define alternating forms $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ on $\Lambda' \otimes_{O_F} O_F^{(2)}$. These forms are given by the following equations:

$$\begin{aligned}\langle x,y\rangle &= s(x,\vartheta y),\\ (x,y) &= s(x,\vartheta\widetilde{\vartheta} y),\\ \langle x,y\rangle_1 &= s(x,\vartheta\cdot\frac{1+\widetilde{\vartheta}}{2/\pi_0^k}y) = s(x,\vartheta\cdot\Pi^{-1}\frac{1+\delta}{2}y).\end{aligned}$$

One easily checks that, after twisting these forms by some units in $(O_F^{(2)} \otimes_{O_F} O_E)^\times$ if necessary, we indeed have $\Lambda' \otimes_{O_F} O_F^{(2)} \simeq \Lambda \otimes_{O_F} O_F^{(2)}$. In the following, we denote the O_B-action by ι for clarity.

Let $\mathcal{F} \in \mathrm{M}_{Dr}^{\mathrm{loc}}(R)$ for some $O_F^{(2)}$-algebra R. Then $\mathcal{F}$ is clearly O_E-linear and the Kottwitz condition follows from the fact that $(\Lambda\otimes_{O_F} R)/\mathcal{F}$ is O_B-stable. Furthermore, $\mathcal{F}$ is totally isotropic with respect to all forms of the type $(x,y) \mapsto s(x,\iota(b)y)$ for some $b \in \Pi^{-1}O_B$. Indeed, it suffices to show this for one $b \in \Pi^{-1}O_B^\times$ and this has already been done in the proof of Prop. 5.3. Thus $\mathcal{F}$ also satisfies the conditions (2) and (3) above, hence $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$.

Conversely, let $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$. Then,

$$\begin{aligned}\langle x,\iota(\Pi^{-1})y\rangle &= 0,\\ \langle x,y\rangle_1 = \langle x,\iota(\Pi^{-1}\cdot\frac{1+\delta}{2})y\rangle &= 0,\end{aligned}$$

for all $x,y \in \mathcal{F}$. It follows that $\iota(\frac{1+\delta}{2})\cdot\mathcal{F} \subseteq \mathcal{F}$, since $(x,y) \mapsto \langle x,\iota(\Pi^{-1})y\rangle$ is a perfect bilinear form on $\Lambda \otimes_{O_F} R$ and $\mathcal{F}$ is a maximal totally isotropic direct summand. But O_E and $\frac{1+\delta}{2}$ generate O_B, so $\mathcal{F}$ is O_B-stable. Hence we have shown that $\mathcal{F} \in \mathrm{M}_{Dr}^{\mathrm{loc}}(R)$ if and only if $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$. □

Remark 5.9. We can also calculate local equations for $\mathrm{N}_E^{\mathrm{loc}}$ by using standard affine charts of $\mathrm{Gr}(2,\Lambda)_{O_F}$. As in the case (R-P), it suffices to consider only three of the six affine charts. Indeed, let R be an O_F-algebra and let $\mathcal{F} \subseteq \Lambda\otimes_{O_F} R$ be a direct summand with $\mathcal{F} \in \mathrm{N}_E^{\mathrm{loc}}(R)$. Since $\mathcal{F}$ is O_E-linear, we may assume that it is generated by an R-basis (v_1,v_2) such that either $\Pi v_1 = \Pi v_2 = 0$ or $v_2 = \Pi v_1$. We fix $(e_1, e_2', \Pi e_1, (\pi_0/\Pi)e_2')$ as an R-basis for $\Lambda \otimes_{O_F} R$ where $e_2' = \Pi e_2$ and thus $(\pi_0/\Pi)e_2' = \pi_0 e_2$. Then the charts one by one are:

(1) The chart around $(\Pi e_1, (\pi_0/\Pi)e_2')$. In terms of our fixed basis, we assume that the vectors (v_1,v_2) have the form

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21}\\ x_{12} & x_{22}\\ 1 & \\ & 1\end{pmatrix},$$

for some $x_{ij} \in R$. The conditions on $\mathcal{F}$ are equivalent to

$$\begin{aligned}x_{11} + \varepsilon x_{22} &= 0,\\ (x_{11} + \frac{2}{\pi_0^k})\cdot x_{22} - x_{12}x_{21} &= -\pi_0,\\ x_{22} &= -\pi_0^{k+1}.\end{aligned}$$

Recall that $E = F(\vartheta)$ for $\vartheta \in O_E^\times$ with $\vartheta^2 = 1 + \pi_0^{2k+1}\varepsilon$ for some unit $\varepsilon \in O_F^\times$. The restriction of $\mathrm{N}_E^{\mathrm{loc}}$ to this affine chart is thus given by

$$\mathcal{U}_{(\Pi e_1,(\pi_0/\Pi)e_2')} \simeq \operatorname{Spec} O_F[x_{12},x_{21}]/(x_{12}x_{21} + \vartheta^2\pi_0).$$

(2) The chart around $(e_1, \Pi e_1)$. Here $v_2 = \Pi v_1$ and the basis (v_1, v_2) has the form

$$[v_1\, v_2] = \begin{pmatrix} 1 & \\ x_{12} & x_{22} \\ & 1 \\ x_{14} & x_{24} \end{pmatrix},$$

for $x_{ij} \in R$. We have

$$\begin{aligned} x_{24} &= \varepsilon x_{12}, \\ x_{22} &= \pi_0 x_{14} + \frac{2}{\pi_0^k} x_{12}, \\ x_{12} &= -\pi_0^{k+1} x_{14}. \end{aligned}$$

Hence this affine chart intersected with $\mathrm{N}_E^{\mathrm{loc}}$ is an affine line $\mathcal{U}_{(e_1, \Pi e_1)} \simeq \operatorname{Spec} O_F[x_{14}]$.
(3) The chart around $(e_2', \Pi e_2')$. We assume that $v_2 = \Pi v_1$ and

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{22} \\ 1 & \frac{2}{\pi_0^k} \\ x_{13} & x_{23} \\ & \varepsilon \end{pmatrix},$$

for some $x_{ij} \in R$. The affine chart $\mathcal{U}_{(e_2', \Pi e_2')}$ is then given by the equations

$$\begin{aligned} x_{21} &= \varepsilon \pi_0 x_{13}, \\ x_{23} &= x_{11} + \frac{2}{\pi_0^k} x_{13}, \\ x_{11} &= \pi_0^k x_{21}. \end{aligned}$$

Hence we have $\mathcal{U}_{(e_2', \Pi e_2')} \simeq \operatorname{Spec} O_F[x_{13}]$.

We can now relate the moduli problem $\mathcal{N}_E$ and the local model $\mathrm{N}_E^{\mathrm{loc}}$ via a local model diagram as in (5.6). Here, $\mathcal{M}(S)$ for some $S \in \mathrm{Nilp}_{\breve{O}_F}$ is the set of tuples $(X, \iota, \lambda, \varrho; \gamma)$ where $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ and γ is an isomorphism of $O_E \otimes_{O_F} \mathcal{O}_S$-modules

$$\gamma : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S,$$

satisfying the conditions below. The polarizations λ, $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ and λ_1 on X induce alternating forms $\langle\, , \rangle^X$, $(\, ,)^X$ and $\langle\, , \rangle_1^X$ on $\mathbb{D}_X(S)$. Under the isomorphism γ, we can compare these forms to the alternating forms $\langle\, , \rangle$, $(\, ,)$ and $\langle\, , \rangle_1$ on $\Lambda \otimes \mathcal{O}_S$. We demand that

$$\begin{aligned} \gamma^* \langle x, y \rangle \;\; &= \langle x, c_1 y \rangle^X, \\ \gamma^* (x, y) \;\; &= (x, c_3 y)^X, \\ \gamma^* \langle x, y \rangle_1 &= \langle x, c_3 y \rangle_1^X + \langle x, \frac{\pi_0^k (c_1 - c_3)}{2} y \rangle, \end{aligned}$$

for any $x, y \in \mathbb{D}_X(S)$. Here, c_1 and c_3 are units in $\breve{O}_E$, acting via the structure map $\breve{O}_E = \breve{O}_F \otimes_{O_F} O_E \to \mathcal{O}_S \otimes_{O_F} O_E$. Note that these are exactly the twists in the equations (4.3) and (4.7) defining the forms h and b on the Dieudonné module M_X for $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(\bar{k})$.
With the same reasoning as in the case (R-P), the functor $\mathcal{M}$ is pro-representable by a formal scheme of finite type over $\mathcal{N}_E$. We have to show that the forgetful morphism $f : \mathcal{M} \to \mathcal{N}_E$ (as in (5.4)) is smooth and surjective. This is a consequence of the two propositions below.

Proposition 5.10. *Let* $\mathrm{Aut}(\Lambda)$ *be the affine group scheme over* $\operatorname{Spec} O_F$ *whose* S*-valued points are automorphisms*

$$\varphi : \Lambda \otimes_{O_F} \mathcal{O}_S \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S,$$

respecting all structure. Then $\mathrm{Aut}(\Lambda)$ *is smooth over* $\operatorname{Spec} O_F$*.*

Proof. Let R be an O_F-algebra and let $g \in \mathrm{Aut}(\Lambda)(R)$. Now, g is an automorphism of $\Lambda \otimes_{O_F} R$ that commutes with the action of Π and satisfies

$$\langle x, y\rangle_1 = \langle gx, gy\rangle_1, \tag{5.9}$$

for all $x, y \in \Lambda \otimes R$. In particular, it leaves $\ker\langle\, ,\rangle_1$ invariant which is a direct summand of rank 2 generated by $(e_1 - \pi_0^k \cdot \Pi e_1, (\pi_0/\Pi)e_2')$. We set $e_1' = e_1 - \pi_0^k \cdot \Pi e_1$. With respect to the basis $(e_1', e_2', \Pi e_1', (\pi_0/\Pi)e_2')$, the automorphism g is given by a matrix in $\mathrm{GL}_R(\Lambda \otimes R)$ of the form

$$g \mathrel{\widehat{=}} \begin{pmatrix} a & & & \pi_0 b \\ & d & \pi_0 c & \\ & b & a & \\ c & & & d \end{pmatrix},$$

for some $a, b, c, d \in R$. From (5.9), we get that $ad - \pi_0 bc = 1$. Conversely, any g of this form lies in $\mathrm{Aut}(\Lambda)(R)$. Hence,

$$\mathrm{Aut}(\Lambda) \simeq \operatorname{Spec} O_F[a, b, c, d]/(ad - \pi_0 bc - 1),$$

which is obviously smooth. □

Proposition 5.11. *Let* $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ *for some* $S \in \mathrm{Nilp}_{\breve{O}_F}$*. Locally on* S *for the étale topology, there exists an isomorphism* $\gamma : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S$*, such that* $(X, \iota, \lambda, \varrho; \gamma) \in \mathcal{M}(S)$*.*

Proof. We can prove this in the same way as in the (R-P) case. In particular, it suffices to check this on geometric points, see Remark 5.6. Hence, for $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(\bar{k})$, it is enough to find an $\breve{O}_F$-linear isomorphism

$$\gamma : M_X \xrightarrow{\sim} \Lambda \otimes_{O_F} \breve{O}_F,$$

compatible with the O_E-action and all alternating forms. This is equivalent to an $\breve{O}_E$-linear isomorphism γ such that the induced forms h and b on both sides coincide. It is now easy to check that, up to $\breve{O}_E$-linear isomorphism, there exists only one possible choice for the forms h and b on M_X. It follows that such an isomorphism γ exists. □

Thus we have local model diagram connecting $\mathcal{N}_E$ and $\mathrm{N}_E^{\mathrm{loc}}$ as in the (R-P) case, see (5.6). This is compatible with the local model diagram of the Drinfeld case, in the sense that the diagram (5.7) is commutative.

5.3. **The naive local model in the case of minimal different.** Let $E|F$ be of type (R-U) such that $|\pi^{k+1}| = |2|$ (see page 7 for the definition of the parameter k). As in the previous paragraphs, let (C, h) be a 2-dimensional E-vector space with $E|F$-hermitian form h. With respect to a suitable chosen basis (e_1, e_2), the hermitian form is given by

$$h \mathrel{\widehat{=}} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Now the O_E-lattice Λ generated by e_1 and $e_2' = \Pi e_2$ is Π-modular and hence automatically hyperbolic by Prop. 2.8. The datum (C, h, Λ) as above is unique up to isomorphism. The form h induces an F-linear alternating form $\langle\, ,\rangle$ on C via the formula

$$\langle x, y\rangle = \mathrm{Tr}_{E|F}\left(\frac{\pi_0^k}{2\vartheta} \cdot h(x, y)\right).$$

We will now define an O_F-scheme $\mathrm{N}_E^{\mathrm{naive}}$, called the *naive local model*, which mimics the construction of the scheme $\mathrm{N}_E^{\mathrm{loc}}$, but "forgets" about the forms $(\,,)$ and $\langle\,,\rangle_1$ (cp. the (PEL)-case of [RZ96, Def. 3.27]). In other words, $\mathrm{N}_E^{\mathrm{naive}}(S)$ for an O_F-scheme S is the set of direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S$ of rank 2 over $\mathcal{O}_S$ such that

(1) $\mathcal{F}$ is an $O_E \otimes_{O_F} \mathcal{O}_S$-submodule of $\Lambda \otimes_{O_F} \mathcal{O}_S$,

(2) $\mathcal{F}$ is totally isotropic with respect to the bilinear form induced by $\langle\,,\rangle$ on $\Lambda \otimes_{O_F} \mathcal{O}_S$, cf. (5.8), and

(3) the Kottwitz condition holds, see condition (4) on page 39.

This functor is representable by a closed subscheme of the Grassmanian $\mathrm{Gr}(2,\Lambda)_{O_F}$, in particular it is a projective scheme over $\operatorname{Spec} O_F$.

Proposition 5.12. $\mathrm{N}_E^{\mathrm{naive}}$ *is not flat over* $\operatorname{Spec} O_F$.

Proof. Let $(e_1, e_2', \Pi e_1, (\pi_0/\Pi)e_2')$ be an O_F-basis of Λ, where $e_2' = \Pi e_2$. It suffices to show that $\mathrm{N}_E^{\mathrm{naive}}$ is not flat when restricted to one of the standard affine charts of $\mathrm{Gr}(2,\Lambda)_{O_F}$. We consider the chart around $(\Pi e_1, (\pi_0/\Pi)e_2')$. Let R be an O_F-algebra. A direct summand $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ in $\mathrm{N}_E^{\mathrm{naive}}(R)$ is given by a basis (v_1, v_2) of the form

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ 1 & \\ & 1 \end{pmatrix}.$$

The elements $x_{ij} \in R$ satisfy the following equations:

$$\begin{gathered} x_{11} + \varepsilon x_{22} = 0, \\ (x_{11} + \frac{2}{\pi_0^k}) \cdot x_{22} - x_{12}x_{21} = -\pi_0, \\ \pi_0(x_{22} + \pi_0^{k+1}) = 0, \\ x_{12}(x_{22} + \pi_0^{k+1}) = x_{21}(x_{22} + \pi_0^{k+1}) = x_{22}(x_{22} + \pi_0^{k+1}) = 0. \end{gathered}$$

The restriction of $\mathrm{N}_E^{\mathrm{naive}}$ to this chart is isomorphic to $\operatorname{Spec} O_F[x_{12}, x_{21}, x_{22}]$ (modulo the equations above). This is indeed not flat over O_F, since $O_F[x_{12}, x_{21}, x_{22}]$, seen as O_F-module, decomposes into a direct sum

$$O_F[x_{12}, x_{21}, x_{22}] = O_F[x_{12}, x_{21}, x_{22}]/(x_{22} + \pi_0^{k+1}) \oplus x_{22} \cdot O_F/\pi_0 O_F,$$

and the second summand is obviously torsion. Geometrically, the special fiber of this chart is a union of two affine lines intersecting transversally, but with an infinitesimally thickened intersection point (see Figure 3). $\square$

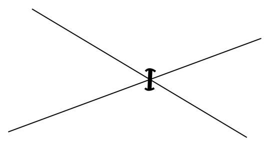

FIGURE 3. The special fiber of $\mathrm{N}_E^{\mathrm{naive}}$, with thickened intersection point.

Let $\mathcal{M}^{\mathrm{naive}}$ be the functor mapping $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of tuples $(X, \iota, \lambda, \varrho; \gamma)$ with $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ and γ an isomorphism of $O_E \otimes_{O_F} \mathcal{O}_S$-modules

$$\gamma : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S,$$

such that $\gamma^*\langle x, y\rangle = \langle x, c_1 y\rangle^X$ for the bilinear form $\langle\, ,\rangle^X$ induced by the polarization λ. We can write down a naive local model diagram,

$$\begin{array}{ccc} & \mathcal{M}^{\mathrm{naive}} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathcal{N}_E^{\mathrm{naive}} & & \widehat{\mathrm{N}}_E^{\mathrm{naive}} \end{array}$$

where $\widehat{\mathrm{N}}_E^{\mathrm{naive}}$ is the π_0-adic completion of the base change $\mathrm{N}_E^{\mathrm{naive}} \otimes_{O_F} \breve{O}_F$. This is a local model diagram in the sense of [RZ96], in particular we have the following result.

Proposition 5.13. *f is smooth and surjective.*

We will omit the proof of this proposition. It uses exactly the same methods as in the cases (R-P) and (R-U) (in the non-naive setting). Instead, we will end this chapter with the following proposition.

Proposition 5.14. *$\mathcal{N}_E^{\mathrm{naive}}$ is not flat over $\operatorname{Spf} \breve{O}_F$.*

Proof. Let $x \in \mathcal{N}_E^{\mathrm{naive}}$ be the intersection point of two projective lines in the reduced locus (cf. Remark 4.7 and Prop. 4.8). Consider its first infinitesimal neighborhood $\operatorname{Spec} \mathcal{O}_{\mathcal{N}_E^{\mathrm{naive}},x}/\mathfrak{m}_x^2$ in $\mathcal{N}_E^{\mathrm{naive}}$. For $\mathcal{N}_E^{\mathrm{naive}}$ to be flat over $\operatorname{Spf} \breve{O}_F$, this neighborhood would have to be flat over $\operatorname{Spec} \breve{O}_F/\pi_0^2\breve{O}_F$. But by Grothendieck-Messing theory, it is isomorphic to the first infinitesimal neighborhood of the intersection point in $\widehat{\mathrm{N}}_E^{\mathrm{naive}}$, which is not flat (see the proof of the Prop. 5.12 and Figure 3). □

In particular, $\mathcal{N}_E^{\mathrm{naive}}$ is not isomorphic to $\mathcal{N}_E$ and this shows the necessity of the straightening condition, cf. Remark 4.12.

6. A theorem on the existence of polarizations

In this section, we will prove the existence of the polarization $\widetilde{\lambda}$ for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ as claimed in the sections 3.2 and 4.2 in both the cases (R-P) and (R-U). In fact, we will show more generally that $\widetilde{\lambda}$ exists even for the points of a larger moduli space $\mathcal{M}_E$ where we forget about the polarization λ.

We start with the definition of the moduli space $\mathcal{M}_E$. Let $F|\mathbb{Q}_p$ be a finite extension (not necessarily $p = 2$) and let $E|F$ be a quadratic extension (not necessarily ramified). We denote by O_F and O_E the rings of integers, by k the residue field of O_F and by $\bar{k}$ the algebraic closure of k. Furthermore, $\breve{F}$ is the completion of the maximal unramified extension of F and $\breve{O}_F$ its ring of integers. Let B be the quaternion division algebra over F and O_B the ring of integers.

If $E|F$ is unramified, we fix a common uniformizer $\pi_0 \in O_F \subseteq O_E$. If $E|F$ is ramified and $p > 2$, we choose a uniformizer $\Pi \in O_E$ such that $\pi_0 = \Pi^2 \in O_F$. If $E|F$ is ramified and $p = 2$, we use the notations of section 2 for the cases (R-P) and (R-U).

For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{M}_E(S)$ be the set of isomorphism classes of tuples (X, ι_E, ϱ) over S. Here, X is a formal O_F-module of dimension 2 and height 4 and ι_E is an action of O_E on X satisfying the Kottwitz condition for the signature $(1, 1)$, i. e., the characteristic polynomial for the action of $\iota_E(\alpha)$ on $\mathrm{Lie}(X)$ is

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \overline{\alpha}), \tag{6.1}$$

for any $\alpha \in O_E$, cp. the definition of $\mathcal{N}_E^{\mathrm{naive}}$ in the sections 3 and 4. The last entry ϱ is an O_E-linear quasi-isogeny

$$\varrho : X \times_S \overline{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \overline{S},$$

of height 0 to the framing object $(\mathbb{X}, \iota_{\mathbb{X},E})$ defined over $\operatorname{Spec} \bar{k}$. The framing object for $\mathcal{M}_E$ is the Drinfeld framing object $(\mathbb{X}, \iota_{\mathbb{X},B})$ where we restrict the O_B-action to O_E for an arbitrary embedding $O_E \hookrightarrow O_B$. The special condition on $(\mathbb{X}, \iota_{\mathbb{X},B})$ implies the Kottwitz condition for any $\alpha \in O_E$ by [RZ14, Prop. 5.8].

Remark 6.1. (1) Up to isogeny, there is more than one pair (X, ι_E) over $\operatorname{Spec} \bar{k}$ satisfying the conditions above. Indeed, let N_X be the rational Dieudonné module of (X, ι_E). This is a 4-dimensional $\breve{F}$-vector space with an action of O_E. The Frobenius $\mathbf{F}$ on N_X commutes with the action of O_E. For a suitable choice of a basis of N_X, it may be of either of the following two forms,

$$\mathbf{F} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ \pi_0 & & & \\ & \pi_0 & & \end{pmatrix} \sigma \quad \text{or} \quad \mathbf{F} = \begin{pmatrix} \pi_0 & & & \\ & \pi_0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \sigma.$$

This follows from the classification of isocrystals, see for example [RZ96, p. 3]. In the left case, $\mathbf{F}$ is isoclinic of slope 1/2 (the supersingular case), and in the right case, the slopes are 0 and 1. Our choice of the framing object above assures that we are in the supersingular case, since the framing object for the Drinfeld moduli problem can be written as a product of two formal O_F-modules of dimension 1 and height 2 (cf. [BC91, p. 136-137]).

(2) Let $p = 2$ and $E|F$ ramified of type (R-P) or (R-U). We can identify the framing objects $(\mathbb{X}, \iota_{\mathbb{X},E})$ for $\mathcal{N}_E^{\mathrm{naive}}$, $\mathcal{M}_{Dr}$ and $\mathcal{M}_E$ by Lemma 3.12 and Lemma 4.14. In this way, we obtain a forgetful morphism $\mathcal{N}_E^{\mathrm{naive}} \to \mathcal{M}_E$. This is a closed embedding, since the existence of a polarization λ for $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$ is a closed condition by [RZ96, Prop. 2.9].

By [RZ96, Thm. 3.25], $\mathcal{M}_E$ is pro-representable by a formal scheme, formally locally of finite type over $\operatorname{Spf} \breve{O}_F$. We will prove the following theorem in this section.

Theorem 6.2. (1) *There exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ such that the Rosati involution induces the identity on O_E, i. e., $\iota(\alpha)^* = \iota(\alpha)$ for all $\alpha \in O_E$. This polarization is unique up to a scalar in $O_E^\times$, that is, for any two polarizations $\widetilde{\lambda}_{\mathbb{X}}$ and $\widetilde{\lambda}'_{\mathbb{X}}$ of this form, there exists an element $\alpha \in O_E^\times$ such that $\widetilde{\lambda}'_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}} \circ \iota_{\mathbb{X},E}(\alpha)$.*

(2) *Fix $\widetilde{\lambda}_{\mathbb{X}}$ as in part* (1)*. For any $S \in \mathrm{Nilp}_{\breve{O}_F}$ and $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$, there exists a unique principal polarization $\widetilde{\lambda}$ on X such that the Rosati involution induces the identity on O_E and such that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$.*

We will split the proof of this theorem into several lemmata. As a first step, we use Dieudonné theory to prove the statement for all geometric points.

Lemma 6.3. *Part* (1) *of theorem holds. Furthermore, for a fixed polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ and for any $(X, \iota_E, \varrho) \in \mathcal{M}_E(\bar{k})$, the pullback $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a polarization on X.*

Proof. We consider 4 different cases for the quadratic extension $E|F$. The cases are

(1) $E|F$ is unramified,

(2) $E|F$ is ramified and $p > 2$,

(3) $p = 2$ and $E|F$ is ramified of (R-P) type,

(4) $p = 2$ and $E|F$ is ramified of (R-U) type.

(1) We start with the case where $E|F$ is unramified. Let $N = M_{\mathbb{X}} \otimes_{\breve{O}_F} \breve{F}$ be the rational Dieudonné module of $(\mathbb{X}, \iota_{\mathbb{X},E})$. This is a 4-dimensional vector space over $\breve{F}$

with two operators $\mathbf{F}$ and $\mathbf{V}$, the Frobenius and the Verschiebung. The action $\iota = \iota_{\mathbb{X},E}$ of $O_E \subseteq \breve{O}_F$ induces a direct sum decomposition $N = N_0 \oplus N_1$ where

$$N_0 = \{x \in N \mid \iota(a)x = ax \text{ for all } a \in O_E\},$$
$$N_1 = \{x \in N \mid \iota(a)x = \sigma(a)x \text{ for all } a \in O_E\}.$$

Here σ is the Frobenius of $\breve{F}|F$. The operators $\mathbf{F}$ and $\mathbf{V}$ have degree 1 with respect to this decomposition, since they are σ-linear resp. σ^{-1}-linear and commute with ι.
A point $(X, \iota_E, \varrho) \in \mathcal{M}_E(\bar{k})$ corresponds to a lattice $M_X \subseteq N$ that respects the above decomposition, i. e., $M_X = M_0 \oplus M_1$ for $M_i = M \cap N_i$. Moreover, $\pi_0 M_X \subseteq V M_X \subseteq M_X$. The relative index of M_X and $M_{\mathbb{X}}$ is 0, in other words,

$$[M_X : M_X \cap M_{\mathbb{X}}] = [M_{\mathbb{X}} : M_X \cap M_{\mathbb{X}}].$$

Write $M_{\mathbb{X}} = M_{\mathbb{X},0} \oplus M_{\mathbb{X},1}$. The Kottwitz condition implies that the inclusions $VM_0 \subseteq M_1$ resp. $VM_{\mathbb{X},0} \subseteq M_{\mathbb{X},1}$ are of index 1. It follows that

$$\begin{aligned} [M_0 : M_0 \cap M_{\mathbb{X},0}] &= [M_{\mathbb{X},0} : M_0 \cap M_{\mathbb{X},0}], \\ [M_1 : M_1 \cap M_{\mathbb{X},1}] &= [M_{\mathbb{X},1} : M_1 \cap M_{\mathbb{X},1}]. \end{aligned} \tag{6.2}$$

Let $\tau = \mathbf{F}\mathbf{V}^{-1}$. This is a σ^2-linear operator on N of degree 0 and with all slopes zero. Thus $C = N_0^{\tau}$ is a vector space of dimension 2 over E. We may assume that $\mathbf{F}M_{\mathbb{X},0} = \mathbf{V}M_{\mathbb{X},0}$ for $M_{\mathbb{X}} = M_{\mathbb{X},0} \oplus M_{\mathbb{X},1}$, cf. [BC91, II.4]. Then $M_{\mathbb{X},0}$ is τ-invariant and $\Lambda = M_{\mathbb{X},0}^{\tau} \subseteq C$ is an O_E-lattice.
The datum of a polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $\mathbb{X}$ as in the Theorem corresponds to an alternating form $(\, , \,)$ on N such that $M_{\mathbb{X}}$ is selfdual with respect to this form and such that

$$\begin{aligned} (\mathbf{F}x, y) &= (x, \mathbf{V}y)^{\sigma}, \\ (\iota(\alpha)x, y) &= (x, \iota(\alpha)y), \end{aligned} \tag{6.3}$$

for all $x, y \in N$ and $\alpha \in O_E$. The second equation of (6.3) implies that $(x, y) = 0$ for $x \in N_0$ and $y \in N_1$. By the first equation, the form $(\, , \,)$ is already determined by its values on N_0. For all $x, y \in N$, the alternating form $(\, , \,)$ satisfies

$$(\tau x, \tau y) = (\mathbf{F}\mathbf{V}^{-1}x, \mathbf{F}\mathbf{V}^{-1}y) = (\mathbf{V}^{-1}x, \mathbf{F}y)^{\sigma} = (x, y)^{\sigma^2}.$$

Hence $(\, , \,)$ corresponds to an alternating form b on C such that $\Lambda \subseteq C$ is selfdual with respect to b. Such an alternating form exists and is unique up to a unit in $O_E^{\times}$. Thus $\widetilde{\lambda}_{\mathbb{X}}$ exists and is unique up to a unit in $O_E^{\times}$, which proves part (1) of the theorem in this case.
In order to show that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a polarization on X, we have to see that $(\, , \,)$ is integral on M_X. But $(\, , \,)$ respects the decomposition $N = N_0 \oplus N_1$, so $(\, , \,)$ induces an alternating form on the 2-dimensional $\breve{F}$-vector space N_i for $i = 0, 1$, and it is invariant under the action of $\mathrm{Sp}_2(\breve{F}) = \mathrm{SL}_2(\breve{F})$. The lattices M_0 and $M_{\mathbb{X},0}$ resp. M_1 and $M_{\mathbb{X},1}$ have relative index 0, cp. the equations in (6.2). Thus there exists a $g_i \in SL_2(\breve{F})$ such that $M_i = g_i M_{\mathbb{X},i}$, and since $(\, , \,)$ is integral on $M_{\mathbb{X}}$, it is also integral on M_X.

(2) We now assume that $E|F$ is ramified and $p > 2$. Again, let $N = M_{\mathbb{X}} \otimes_{\breve{O}_F} \breve{F}$ be the rational Dieudonné module of $(\mathbb{X}, \iota_{\mathbb{X},E})$. This is a 4-dimensional vector space over $\breve{F}$ with two operators, the Frobenius $\mathbf{F}$ and the Verschiebung $\mathbf{V}$, and an O_E-action. Let $\breve{E} = \breve{F} \otimes_F E$. Then N has the structure of a 2-dimensional $\breve{E}$-vector space.
A point $(X, \iota_E, \varrho) \in \mathcal{M}_E(\bar{k})$ corresponds to an $\breve{O}_E$-lattice $M_X \subseteq N$ such that

$$\pi_0 M_X \subseteq \mathbf{V} M_X \subseteq M_X.$$

The $\breve{O}_E$-lattices M_X and $M_{\mathbb{X}}$ have relative index 0, i. e.,

$$[M_X : M_X \cap M_{\mathbb{X}}] = [M_{\mathbb{X}} : M_X \cap M_{\mathbb{X}}].$$

Let $\tau = \Pi\mathbf{V}^{-1} = \mathbf{F}\Pi^{-1}$. This is a σ-linear operator with slopes 0. The τ-invariant points $C = N^{\tau}$ form a 2-dimensional vector space over E. By [BC91, II.4], we may assume that $M_{\mathbb{X}}$ is invariant under τ. Then $\Lambda = M_{\mathbb{X}}^{\tau} \subseteq C$ is an O_E-lattice.
A polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $\mathbb{X}$ as in the Theorem corresponds to an alternating form $(\, ,)$ on N such that $M_{\mathbb{X}}$ is selfdual with respect to this form and such that

$$\begin{aligned}(\mathbf{F}x, y) &= (x, \mathbf{V}y)^{\sigma}, \\ (\alpha x, y) &= (x, \alpha y),\end{aligned} \tag{6.4}$$

for all $x, y \in N$ and $\alpha \in O_E$. Such a form $(\, ,)$ induces an $\breve{E}$-bilinear alternating form b on N by setting

$$b(x, y) = (\Pi x, y) + \Pi(x, y),$$

We can recover $(\, ,)$ from b via the formula

$$(x, y) = \mathrm{Tr}_{\breve{E}|\breve{F}} \left(\frac{1}{2\Pi} \cdot b(x, y) \right).$$

The form b is then invariant under $\tau = \Pi\mathbf{V}^{-1}$, since

$$b(\tau x, \tau y) = b(\mathbf{F}\Pi^{-1}x, \Pi\mathbf{V}^{-1}y) = b(\Pi^{-1}x, \Pi y)^{\sigma} = b(x, y)^{\sigma}.$$

Hence b defines an E-bilinear alternating form on C, again denoted by b. The lattice $\Lambda = M_{\mathbb{X}}^{\tau} \subseteq C$ is selfdual with respect to b. Thus b is unique up to a unit in $O_E^{\times}$. It follows that $\widetilde{\lambda}_{\mathbb{X}}$ is unique up to a unit in $O_E^{\times}$. On the other hand, such an alternating form b exists and it induces an alternating form $(\, ,)$ on N satisfying the conditions of (6.4). This implies the existence of $\widetilde{\lambda}_{\mathbb{X}}$.
We have to show that the alternating form b is integral on $M_X \subseteq N$, where M_X is the Dieudonné module for a point $(X, \iota_E, \varrho) \in \mathcal{M}_E(\bar{k})$. But the $\breve{O}_E$-lattices M_X and $M_{\mathbb{X}}$ have relative index 0, so there exists an element $g \in \mathrm{SL}_2(\breve{E}) = \mathrm{Sp}_2(\breve{E})$ such that $M_X = gM_{\mathbb{X}}$. Since b is invariant under the action of $\mathrm{Sp}_2(\breve{E})$ and integral on $M_{\mathbb{X}}$, it is also integral on M_X.

(3) Let $p = 2$ and assume $E|F$ is ramified of type (R-P). The proof here is verbatim the same as in case (2) where $E|F$ ramified and $p > 2$.

(4) Finally, consider the case where $p = 2$ and $E|F$ is ramified of type (R-U). We follow the argumentation of the case (2) (where $E|F$ ramified, $p > 2$) and make the following adaptions.
The operator τ on N is now given by $\tau = c_2\mathbf{V}^{-1} = \mathbf{F}c_2^{-1}$ where $c_2 \in \breve{E}$ is a uniformizer satisfying $c_2 \cdot \sigma(c_2) = \pi_0$. Then τ is σ-linear and has slopes 0 on N.
An alternating form $(\, ,)$ on N satisfying the conditions in (6.4) induces an $\breve{E}$-linear alternating form b by

$$b(x, y) = \Pi / c_2 \cdot ((\Pi x, y) - \overline{\Pi}(x, y)).$$

Accordingly,

$$(x, y) = \mathrm{Tr}_{\breve{E}|\breve{F}} \left(\frac{\pi_0^k c_2}{2\vartheta\Pi} \cdot b(x, y) \right).$$

The form b is invariant under τ, since

$$b(\tau x, \tau y) = b(\mathbf{F}c_2^{-1}x, c_2\mathbf{V}^{-1}y) = \frac{\sigma(c_2)}{c_2} \cdot b(c_2^{-1}x, \sigma^{-1}(c_2)y)^{\sigma} = b(x, y)^{\sigma},$$

for all $x, y \in N$. The rest of the proof is analogous to the case (2). □

In the following, we fix a polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ as in Theorem 6.2 (1). Let $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$ for $S \in \mathrm{Nilp}_{\breve{O}_F}$ and consider the pullback $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$. In general, this is only a quasi-polarization. It suffices to show that $\widetilde{\lambda}$ is a polarization on X. Indeed,

since ϱ is O_E-linear and of height 0, this is then automatically a principal polarization on X such that the Rosati involution is the identity on O_E.
Define a subfunctor $\mathcal{M}_{E,\mathrm{pol}} \subseteq \mathcal{M}_E$ by

$$\mathcal{M}_{E,\mathrm{pol}}(S) = \{(X, \iota_E, \varrho) \in \mathcal{M}_E(S) \mid \widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}}) \text{ is a polarization on } X\}.$$

This is a closed formal subscheme by [RZ96, Prop. 2.9]. Moreover, Lemma 6.3 shows that $\mathcal{M}_{E,\mathrm{pol}}(\bar{k}) = \mathcal{M}_E(\bar{k})$.

Remark 6.4. Equivalently, we can describe $\mathcal{M}_{E,\mathrm{pol}}$ as follows. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, we define $\mathcal{M}_{E,\mathrm{pol}}(S)$ to be the set of equivalence classes of tuples $(X, \iota_E, \widetilde{\lambda}, \varrho)$ where

- X is a formal O_F-module over S of height 4 and dimension 2,
- ι_E is an action of O_E on X that satisfies the Kottwitz condition in (6.1) and
- $\widetilde{\lambda}$ is a principal polarization on X such that the Rosati involution induces the identity on O_E.
- Furthermore, we fix a framing object $(\mathbb{X}, \iota_{\mathbb{X},E}, \widetilde{\lambda}_{\mathbb{X}})$ over $\operatorname{Spec} \bar{k}$, where $(\mathbb{X}, \iota_{\mathbb{X},E})$ is the framing object for $\mathcal{M}_E$ and $\widetilde{\lambda}_{\mathbb{X}}$ is a polarization as in Theorem 6.2 (1). Then ϱ is an O_E-linear quasi-isogeny

$$\varrho : X \times_S \overline{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \overline{S},$$

of height 0 such that, locally on $\overline{S}$, the (quasi-)polarizations $\varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ and $\widetilde{\lambda}$ on X only differ by a scalar in $O_E^\times$, i. e., there exists an element $\alpha \in O_E^\times$ such that $\varrho^*(\widetilde{\lambda}_{\mathbb{X}}) = \widetilde{\lambda} \circ \iota_E(\alpha)$.
Two tuples $(X, \iota_E, \widetilde{\lambda}, \varrho)$ and $(X', \iota_E', \widetilde{\lambda}', \varrho')$ are equivalent if there exists an O_E-linear isomorphism $\varphi : X \xrightarrow{\sim} X'$ such that $\varphi^*(\widetilde{\lambda}')$ and $\widetilde{\lambda}$ only differ by a scalar in $O_E^\times$.
In this way, we gave a definition for $\mathcal{M}_{E,\mathrm{pol}}$ by introducing extra data on points of the moduli space $\mathcal{M}_E$, instead of extra conditions. It is now clear, that $\mathcal{M}_{E,\mathrm{pol}}$ describes a moduli problem for p-divisible groups of (PEL) type. It is easily checked that the two descriptions of $\mathcal{M}_{E,\mathrm{pol}}$ give rise to the same moduli space.

Theorem 6.2 now holds if and only if $\mathcal{M}_{E,\mathrm{pol}} = \mathcal{M}_E$. This equality is a consequence of the following statement.

Lemma 6.5. *For any point $x = (X, \iota_E, \varrho) \in \mathcal{M}_{E,\mathrm{pol}}(\bar{k})$, there is an identity of completed local rings $\widehat{\mathcal{O}}_{\mathcal{M}_{E,\mathrm{pol}},x} = \widehat{\mathcal{O}}_{\mathcal{M}_E,x}$.*

For the proof of this Lemma, we use the theory of local models, cf. [RZ96, Chap. 3]. We postpone the proof of this lemma to the end of this section and we first introduce the local models $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ for $\mathcal{M}_E$ and $\mathcal{M}_{E,\mathrm{pol}}$.

Remark 6.6. Note here that $\mathcal{M}_E$ is an RZ-space of (EL) type. The local model $\mathrm{M}_E^{\mathrm{loc}}$ has been defined in [RZ96, Chap. 3] for any prime p. Furthermore, loc. cit. establishes a local model diagram connecting $\mathcal{M}_E$ and $\mathrm{M}_E^{\mathrm{loc}}$. In particular, it follows that the completed local rings at geometric points of $\mathcal{M}_E$ and $\mathrm{M}_E^{\mathrm{loc}}$ are isomorphic.
On the other hand, $\mathcal{M}_{E,\mathrm{pol}}$ is an RZ-space of (PEL) type. In loc. cit., the authors always make the assumption that $p > 2$ for the (PEL) case. Hence, for $p > 2$, the definition of the local model $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ is known and there also is a local model diagram. In this case, Lemma 6.5 will already follow from the fact that $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}} = \mathrm{M}_E^{\mathrm{loc}}$, cf. Lemma 6.7. For the case $p = 2$, we give a definition of the local model $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ and establish a local model diagram, see Prop. 6.9 below. Lemma 6.5 then again follows from the equality $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}} = \mathrm{M}_E^{\mathrm{loc}}$ in Lemma 6.7.

We now give the definition of the local models $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ for $\mathcal{M}_E$ and $\mathcal{M}_{E,\mathrm{pol}}$. Let C be a 4-dimensional F-vector space with an action of E and let $\Lambda \subseteq C$ be an

O_F-lattice that is stable under the action of O_E. Furthermore, let $(\,,)$ be an F-bilinear alternating form on C with

$$(\alpha x, y) = (x, \alpha y), \tag{6.5}$$

for all $\alpha \in E$ and $x, y \in C$ and such that Λ is selfdual with respect to $(\,,)$. It is easily checked that $(\,,)$ is unique up to an isomorphism of C that commutes with the E-action and that maps Λ to itself.

For an O_F-algebra R, let $\mathrm{M}_E^{\mathrm{loc}}(R)$ be the set of all direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ of rank 2 that are O_E-linear and satisfy the *Kottwitz condition*. That means, for all $\alpha \in O_E$, the action of α on the quotient $(\Lambda \otimes_{O_F} R)/\mathcal{F}$ has the characteristic polynomial

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \alpha) = (T - \alpha)(T - \bar{\alpha}).$$

The subset $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}(R) \subseteq \mathrm{M}_E^{\mathrm{loc}}(R)$ consists of all direct summands $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ that are in addition totally isotropic with respect to $(\,,)$ on $\Lambda \otimes_{O_F} R$.

The functor $\mathrm{M}_E^{\mathrm{loc}}$ is representable by a closed subscheme of $\mathrm{Gr}(2, \Lambda)_{O_F}$, the Grassmanian of rank 2 direct summands of Λ, and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ is representable by a closed subscheme of $\mathrm{M}_E^{\mathrm{loc}}$. In particular, both $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ are projective schemes over $\mathrm{Spec}\, O_F$.

Lemma 6.7. $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}} = \mathrm{M}_E^{\mathrm{loc}}$. *In other words, for an O_F-algebra R, any direct summand $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ is totally isotropic with respect to $(\,,)$.*

Proof. As in the proof of Lemma 6.3, we will split this proof into several cases, depending on whether $p > 2$ or $p = 2$ and depending on whether $E|F$ is unramified or ramified. The main idea here is always the same, namely fix a basis for Λ and then check the assertion for each chart of the Grassmanian $\mathrm{Gr}(2, \Lambda)_{O_F}$. However, the calculations in each case differ at certain points.

We first consider the case where $E|F$ is unramified and $p > 2$. Here $O_E = O_F[\delta]$, where δ is the square root of a unit in O_F. We choose a basis $(e_1, e_2, \delta e_1, \delta e_2)$ of Λ such that the alternating form $(\,,)$ is given by the matrix

$$\begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}. \tag{6.6}$$

Let $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ for an O_F-algebra R. Then $\mathcal{F}$ is direct summand $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ of rank 2 over R. Let v_1, v_2 be an R-basis for $\mathcal{F}$. We have to prove that $\mathcal{F}$ is totally isotropic with respect to $(\,,)$. Since $(\,,)$ is alternating, it suffices to see that

$$(v_1, v_2) = 0.$$

We can check this hypothesis on each of the affine charts of the Grassmanian $\mathrm{Gr}(2, \Lambda)_{O_F}$. We claim that, on each chart, the conditions describing $\mathrm{M}_E^{\mathrm{loc}} \subseteq \mathrm{Gr}(2, \Lambda)_{O_F}$ as a closed subscheme already imply that $(v_1, v_2) = 0$. The charts one by one are:

(1) The chart around $(e_1, \delta e_1)$. In terms of the basis $(e_1, e_2, \delta e_1, \delta e_2)$, we have

$$[v_1\, v_2] = \begin{pmatrix} 1 & \\ x_{12} & x_{22} \\ & 1 \\ x_{14} & x_{24} \end{pmatrix},$$

for some $x_{ij} \in R$. The O_E-linearity of $\mathcal{F}$ implies that

$$x_{24} = x_{12},$$
$$x_{22} = \delta^2 x_{14}.$$

It follows that $(v_1, v_2) = x_{24} - x_{12} = 0$. The calculation for the chart $(e_2, \delta e_2)$ is analogous to this one.

(2) The chart around $(\delta e_1, \delta e_2)$. We write

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ 1 & \\ & 1 \end{pmatrix}.$$

From the Kottwitz condition, we get

$$\begin{aligned} x_{11} + x_{22} &= 0, \\ x_{11}x_{22} - x_{21}x_{12} &= -\delta^2. \end{aligned}$$

Thus we have $(v_1, v_2) = x_{11} + x_{22} = 0$.

Since $\delta \in O_E$ is a unit and $\mathcal{F}$ is O_E-linear, we can replace v_i by δv_i without loss of generality. Hence from the calculation in (2), the claim also follows for the charts around (e_1, e_2), $(e_1, \delta e_2)$ and $(\delta e_1, e_2)$. Thus $(v_1, v_2) = 0$ for all $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$. In other words, $\mathcal{F}$ is totally isotropic with respect to $(\,,)$, which implies that $\mathcal{F} \in \mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}(R)$. Thus $\mathrm{M}_E^{\mathrm{loc}}(R) = \mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}(R)$ and, since R is an arbitrary O_F-algebra, we have $\mathrm{M}_E^{\mathrm{loc}} = \mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$. This finishes the proof for the case where $E|F$ is unramified and $p > 2$.

Next we treat the case where $E|F$ is unramified and $p = 2$. In this case, we have $O_E = O_F[\frac{1+\delta}{2}]$, where δ is a unit in O_E such that $\delta^2 = 1 + 4\varepsilon_0 \in O_F$ (cp. section 2). We can choose a basis $(e_1, e_2, \frac{1+\delta}{2}e_1, \frac{1+\delta}{2}e_2)$ for Λ, such that, with respect to this basis, the alternating form $(\,,)$ is given by the matrix

$$\begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & 1 \\ -1 & & -1 & \end{pmatrix}. \tag{6.7}$$

Let $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ and let v_1, v_2 be an R-basis for $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$. We want to show that $(v_1, v_2) = 0$.

(1) The chart around $(e_1, \frac{1+\delta}{2}e_1)$. In terms of the basis $(e_1, e_2, \frac{1+\delta}{2}e_1, \frac{1+\delta}{2}e_2)$, we have

$$[v_1\, v_2] = \begin{pmatrix} 1 & \\ x_{12} & x_{22} \\ & 1 \\ x_{14} & x_{24} \end{pmatrix}.$$

From the O_E-linearity of $\mathcal{F}$, we get that

$$\begin{aligned} x_{24} &= x_{12} + x_{14}, \\ x_{22} &= \varepsilon_0 x_{14}. \end{aligned}$$

Hence, $(v_1, v_2) = x_{24} - x_{12} - x_{14} = 0$. For symmetry reasons, this also follows for the chart $(e_2, \frac{1+\delta}{2}e_2)$.

(2) The chart around $(\frac{1+\delta}{2}e_1, \frac{1+\delta}{2}e_2)$. Here, we write

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ 1 & \\ & 1 \end{pmatrix}.$$

From the Kottwitz condition, we deduce that

$$\begin{aligned} x_{11} + x_{22} &= -1, \\ x_{11}x_{22} - x_{21}x_{12} &= \varepsilon_0. \end{aligned}$$

Thus, we have $(v_1, v_2) = x_{11} + x_{22} + 1 = 0$.

Note that $\frac{1+\delta}{2} \in O_E$ is a unit, hence we can replace v_i by $\frac{1+\delta}{2} v_i$ without loss of generality. The calculations of (2) then also prove our claim for the charts around (e_1, e_2), $(e_1, \frac{1+\delta}{2} e_2)$ and $(\frac{1+\delta}{2} e_1, e_2)$. Hence, we have $(v_1, v_2) = 0$ for all $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ for any O_F-algebra R.

Assume now that $E|F$ is ramified and $p > 2$. Then $O_E = O_F[\Pi]$ for some uniformizer $\Pi \in O_E$ with $\Pi^2 = \pi_0 \in O_F$. We choose a basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ for Λ such that $(\,,)$ is represented by the matrix in (6.6) with respect to this basis.

Let $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ and let v_1, v_2 be an R-basis for $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$. It suffices to show that $(v_1, v_2) = 0$. As in the unramified case, we check this on the charts of the Grassmanian $\mathrm{Gr}(2, \Lambda)_{O_F}$.

(1) The chart around $(e_1, \Pi e_1)$. In terms of the basis $(e_1, e_2, \Pi e_1, \Pi e_2)$, we write

$$[v_1\, v_2] = \begin{pmatrix} 1 & \\ x_{12} & x_{22} \\ & 1 \\ x_{14} & x_{24} \end{pmatrix},$$

for some $x_{ij} \in R$. Since $\mathcal{F}$ is O_E-linear, we have

$$\begin{aligned} x_{24} &= x_{12}, \\ x_{22} &= \pi_0 x_{14}. \end{aligned}$$

Now, $(v_1, v_2) = x_{24} - x_{12} = 0$. The calculation for the chart $(e_2, \Pi e_2)$ is analogous to this one.

(2) The chart around $(\Pi e_1, \Pi e_2)$. Let

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ 1 & \\ & 1 \end{pmatrix}.$$

The Kottwitz condition gives us the following equations,

$$\begin{aligned} x_{11} + x_{22} &= 0, \\ x_{11}x_{22} - x_{21}x_{12} &= -\pi_0. \end{aligned}$$

Thus, we have $(v_1, v_2) = x_{11} + x_{22} = 0$.

It is easily checked that the 3 remaining charts of $\mathrm{Gr}(2, \Lambda)_{O_F}$ contain no additional points of $\mathrm{M}_E^{\mathrm{loc}}$, so we do not need to consider these. It follows that $(v_1, v_2) = 0$ for all $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ and any O_F-algebra R. Thus $\mathrm{M}_E^{\mathrm{loc}} = \mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ also in this case.

The proof in the case where $p = 2$ and $E|F$ is ramified of type (R-P) is exactly the same as in the case where $p > 2$ and $E|F$ is ramified.

Finally, let $p = 2$ and $E|F$ be ramified of type (R-U). We use the notation of section 2 for this case. In particular, $O_E = O_F[\Pi]$ for some uniformizer $\Pi \in O_E$ of the form $\Pi = \frac{1+\vartheta}{\pi_0^k}$ and ϑ is the square root of a unit in O_F, satisfying $\vartheta^2 = 1 + \varepsilon \pi_0^{2k+1}$ for some unit $\varepsilon \in O_F$.

We fix a basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ of Λ such that

$$(\,,) \mathrel{\widehat{=}} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \frac{2}{\pi_0^k} \\ -1 & & -\frac{2}{\pi_0^k} & \end{pmatrix}$$

with respect to this basis.

As usual, let v_1, v_2 be a basis of $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$. We want to show that $(v_1, v_2) = 0$. We check this on each chart of $\mathrm{Gr}(2, \Lambda)_{O_F}$.

(1) First we consider the chart around $(e_1, \Pi e_1)$. We have

$$[v_1\, v_2] = \begin{pmatrix} 1 & \\ x_{12} & x_{22} \\ & 1 \\ x_{14} & x_{24} \end{pmatrix},$$

for some $x_{ij} \in R$. The O_E-linearity of $\mathcal{F}$ implies that

$$x_{24} = x_{12} + \frac{2}{\pi_0^k} x_{14},$$
$$x_{22} = \varepsilon \pi_0 x_{14}.$$

Now, $(v_1, v_2) = x_{24} - x_{12} - \frac{2}{\pi_0^k} x_{14} = 0$. The same argument also works for the chart around $(e_2, \Pi e_2)$.
(2) The chart around $(\Pi e_1, \Pi e_2)$. We write

$$[v_1\, v_2] = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ 1 & \\ & 1 \end{pmatrix}.$$

Then the Kottwitz condition implies that

$$x_{11} + x_{22} = -\frac{2}{\pi_0^k},$$
$$x_{11}x_{22} - x_{21}x_{12} = \varepsilon \pi_0.$$

It follows that $(v_1, v_2) = x_{11} + x_{22} + \frac{2}{\pi_0^k} = 0$.
As in the case $E|F$ ramified and $p > 2$, it suffices to consider these 3 charts of $\mathrm{Gr}(2, \Lambda)_{O_F}$, because the 3 remaining charts contain no additional points of $\mathrm{M}_E^{\mathrm{loc}}$. Thus we have shown that $(v_1, v_2) = 0$ for all $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$. Hence $\mathrm{M}_E^{\mathrm{loc}}(R) = \mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}(R)$ for any O_F-algebra R. It follows that $\mathrm{M}_E^{\mathrm{loc}} = \mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$. □

The moduli spaces $\mathcal{M}_E$ and $\mathcal{M}_{E,\mathrm{pol}}$ are related to the local models $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ via local model diagrams, cf. [RZ96, Chap. 3]. Let $\mathcal{M}_E^{\mathrm{large}}$ be the functor that maps a scheme $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of isomorphism classes of tuples $(X, \iota_E, \varrho; \gamma)$. Here,

$$(X, \iota_E, \varrho) \in \mathcal{M}_E(S),$$

and γ is an O_E-linear isomorphism

$$\gamma : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S.$$

On the left hand side, $\mathbb{D}_X(S)$ denotes the (relative) Grothendieck-Messing crystal of X evaluated at S, cf. [Ahs11, 5.2].
Let $\widehat{\mathrm{M}}_E^{\mathrm{loc}}$ be the π_0-adic completion of $\mathrm{M}_E^{\mathrm{loc}} \otimes_{O_F} \breve{O}_F$. Then there is a local model diagram:

$$\begin{array}{ccccc} & & \mathcal{M}_E^{\mathrm{large}} & & \\ & \swarrow^{f} & & \searrow^{g} & \\ \mathcal{M}_E & & & & \widehat{\mathrm{M}}_E^{\mathrm{loc}} \end{array}$$

The morphism f on the left hand side is the projection $(X, \iota_E, \varrho; \gamma) \mapsto (X, \iota_E, \varrho)$. The morphism g on the right hand side maps $(X, \iota_E, \varrho; \gamma) \in \mathcal{M}_E^{\mathrm{large}}(S)$ to

$$\mathcal{F} = \ker(\Lambda \otimes_{O_F} \mathcal{O}_S \xrightarrow{\gamma^{-1}} \mathbb{D}_X(S) \longrightarrow \mathrm{Lie}\, X) \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S.$$

By [RZ96, Thm. 3.11], the morphism f is smooth and surjective. The morphism g is formally smooth by Grothendieck-Messing theory, see [Mes72, V.1.6], resp. [Ahs11, Chap. 5.2] for the relative setting (i. e., when $O_F \neq \mathbb{Z}_p$).

Remark 6.8. Recall from [RZ96, 3.29] that $\mathcal{M}_E^{\mathrm{large}}$ is a torsor over $\mathcal{M}_E$ via f for (the π_0-adic completion of) the smooth affine group scheme representing the functor

$$S \longmapsto \{O_E\text{-linear automorphisms of } \Lambda \otimes_{O_F} \mathcal{O}_S\}.$$

In particular, $\mathcal{M}_E^{\mathrm{large}}$ is representable by a formal scheme and f is relatively representable in the category of schemes. Now, f is smooth in the sense that for any scheme S and any morphism $S \to \mathcal{M}_E$, the morphism of schemes $S \times_{\mathcal{M}_E} \mathcal{M}_E^{\mathrm{large}} \to S$ is smooth.
When we say that the morphism f is surjective, we mean here that f is surjective as a map of étale sheaves. However, since f is smooth, this is equivalent to saying that f is surjective on geometric points or, again equivalently, that f induces a surjective map on underlying topological spaces.

We also have a local model diagram for the space $\mathcal{M}_{E,\mathrm{pol}}$. We define $\mathcal{M}_{E,\mathrm{pol}}^{\mathrm{large}}$ as the subfunctor of $\mathcal{M}_E^{\mathrm{large}}$ that maps $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of tuples $(X, \iota_E, \varrho; \gamma) \in \mathcal{M}_E^{\mathrm{large}}(S)$ where $(X, \iota_E, \varrho) \in \mathcal{M}_{E,\mathrm{pol}}(S)$ and where γ satisfies the following compatibility condition. The polarization $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ on X induces an alternating form $(\,,)^X$ on $\mathbb{D}_X(S)$. We demand in addition that $(\,,)^X$ is the pullback of the alternating form $(\,,)$ on $\Lambda \otimes_{O_F} \mathcal{O}_S$ under the isomorphism γ.
The local model diagram for $\mathcal{M}_{E,\mathrm{pol}}$ now looks as follows.

$$\begin{array}{ccc} & \mathcal{M}_{E,\mathrm{pol}}^{\mathrm{large}} & \\ {}^{f_{\mathrm{pol}}}\swarrow & & \searrow^{g_{\mathrm{pol}}} \\ \mathcal{M}_{E,\mathrm{pol}} & & \widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}} \end{array} \tag{6.8}$$

Here, $\widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}}$ is the π_0-adic completion of $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}} \otimes_{O_F} \breve{O}_F$ and f_{pol} and g_{pol} are the restrictions of the morphisms f and g above. Again, g_{pol} is formally smooth by Grothendieck-Messing theory. For $p > 2$, the morphism f_{pol} is smooth and surjective by [RZ96, Thm. 3.16]. For $p = 2$, we prove the analogous result in Prop. 6.9 below.
We can now finish the proof of Lemma 6.5.

Proof (of Lemma 6.5). We have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{M}_{E,\mathrm{pol}} & \xleftarrow{f_{\mathrm{pol}}} & \mathcal{M}_{E,\mathrm{pol}}^{\mathrm{large}} & \xrightarrow{g_{\mathrm{pol}}} & \widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}} \\ \big\downarrow & & \big\downarrow & & \big\| \\ \mathcal{M}_E & \xleftarrow{f} & \mathcal{M}_E^{\mathrm{large}} & \xrightarrow{g} & \widehat{\mathrm{M}}_E^{\mathrm{loc}} \end{array} \tag{6.9}$$

The equality on the right hand side follows from Lemma 6.7. The other vertical arrows are closed embeddings.
Let $x \in \mathcal{M}_{E,\mathrm{pol}}(\bar{k})$. Since f_{pol} is surjective, we can choose $y \in \mathcal{M}_{E,\mathrm{pol}}^{\mathrm{large}}(\bar{k})$ such that $f_{\mathrm{pol}}(y) = x$. Moreover, we set $x' = g_{\mathrm{pol}}(y) \in \widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}}$. The theory of local model diagrams (cf. [RZ96, Chap. 3]) induces an isomorphism of completed local rings,

$$\widehat{\mathcal{O}}_{\mathcal{M}_{E,\mathrm{pol}},x} \simeq \widehat{\mathcal{O}}_{\widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}},x'}.$$

We have an analogous isomorphism for the bottom row,

$$\widehat{\mathcal{O}}_{\mathcal{M}_E,x} \simeq \widehat{\mathcal{O}}_{\widehat{\mathrm{M}}_E^{\mathrm{loc}},x'}.$$

The equality $\widehat{\mathrm{M}}^{\mathrm{loc}}_{E,\mathrm{pol}} = \widehat{\mathrm{M}}^{\mathrm{loc}}_{E}$ on the right hand side implies that $\widehat{\mathcal{O}}_{\widehat{\mathrm{M}}^{\mathrm{loc}}_{E,\mathrm{pol}},x'} = \widehat{\mathcal{O}}_{\widehat{\mathrm{M}}^{\mathrm{loc}}_{E},x'}$, and this in turn implies that $\widehat{\mathcal{O}}_{\mathcal{M}_{E,\mathrm{pol}},x} = \widehat{\mathcal{O}}_{\mathcal{M}_{E},x}$. □

It remains to prove that the diagram (6.8) is a local model diagram in the sense of [RZ96] even in the case where $p = 2$. This is a consequence of the following proposition.

Proposition 6.9. *Let $p = 2$. The morphism f_{pol} in the diagram* (6.8) *is smooth and surjective.*

Recall that, for $p > 2$, the morphism f_{pol} is smooth and surjective by [RZ96, Thm. 3.16].

Proof. Let $\mathrm{Aut}(\Lambda)$ be the affine group scheme over $\mathrm{Spec}\, O_F$ representing the following set-valued functor. We map an O_F-algebra R to the set of all O_E-linear automorphisms

$$\varphi : \Lambda \otimes_{O_F} R \xrightarrow{\sim} \Lambda \otimes_{O_F} R,$$

such that the alternating form $(\,,)$ on $\Lambda \otimes_{O_F} R$ is invariant under pullback of φ, i. e., $(x, y) = (\varphi x, \varphi y)$ for all $x, y \in \Lambda \otimes_{O_F} R$. Via the morphism f_{pol}, the formal scheme $\mathcal{M}^{\mathrm{large}}_{E,\mathrm{pol}}$ is a $\mathrm{Aut}(\Lambda)$-torsor over $\mathcal{M}_{E,\mathrm{pol}}$, thus the formal smoothness of f_{pol} follows if $\mathrm{Aut}(\Lambda)$ is smooth over $\mathrm{Spec}\, O_F$, cf. [RZ96, 3.29].
We now prove the smoothness of $\mathrm{Aut}(\Lambda)$ via explicit calculation. We consider three different cases, where $E|F$ is unramified, $E|F$ is of type (R-P) and $E|F$ is of type (R-U), respectively. (Note that we are already assuming that $p = 2$.)
Let $E|F$ be unramified. Then $O_E = O_F[\frac{1+\delta}{2}]$, where $\delta \in O_E$ is a unit such that $\delta^2 = 1 + 4\varepsilon_0 \in O_F$. We fix a basis $(e_1, e_2, \frac{1+\delta}{2}e_1, \frac{1+\delta}{2}e_2)$ of Λ such that $(\,,)$ is given by the matrix

$$B_\Lambda = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & 1 \\ -1 & & -1 & \end{pmatrix}.$$

Let R be an O_F-algebra. We can now express an element $g \in \mathrm{Aut}(\Lambda)(R)$ with respect to the chosen basis of Λ by a 4×4-matrix A with values in R, such that $\frac{1+\delta}{2} \cdot A = A \cdot \frac{1+\delta}{2}$ and $A^t B_\Lambda A = B_\Lambda$. After some calculations, we get

$$A = \begin{pmatrix} a & b & \varepsilon_0 e & \varepsilon_0 f \\ c & d & \varepsilon_0 g & \varepsilon_0 h \\ e & f & a+e & b+f \\ g & h & c+g & d+h \end{pmatrix},$$

for some elements $a, \dots, h \in R$ satisfying the equations

$$\begin{aligned} ah - bg - cf + de + eh - fg &= 0, \\ ad - bc + \varepsilon_0 eh - \varepsilon_0 fg &= 1. \end{aligned}$$

Hence $\mathrm{Aut}(\Lambda) = \mathrm{Spec}\, O_F[a, \dots, h]$ modulo these equations. Using the Jacobi criterion, one now easily checks that $\mathrm{Aut}(\Lambda)$ is smooth.
Let $E|F$ be ramified of type (R-P). We write $O_E = O_F[\Pi]$ where Π is a uniformizer of O_E such that $\Pi^2 = \pi_0 \in O_F$. We fix a basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ of Λ such that $(\,,)$ is given by

$$B_\Lambda = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

Let R be an O_F-algebra. In terms of the basis above, we can write $g \in \mathrm{Aut}(\Lambda)(R)$ as a 4×4-matrix A such that $\Pi A = A\Pi$ and $A^t B_\Lambda A = B_\Lambda$. It follows that A is of the form

$$A = \begin{pmatrix} a & b & \pi_0 e & \pi_0 f \\ c & d & \pi_0 g & \pi_0 h \\ e & f & a & b \\ g & h & c & d \end{pmatrix},$$

where $a, \dots, h \in R$ satisfy the following equations.

$$ah - bg - cf + de = 0,$$
$$ad - bc + \pi_0 eh - \pi_0 fg = 1.$$

Now, $\mathrm{Aut}(\Lambda) = \mathrm{Spec}\, O_F[a, \dots, h]$ modulo these equations. By the Jacobi criterion, we have that $\mathrm{Aut}(\Lambda)$ is smooth over $\mathrm{Spec}\, O_F$.

Let $E|F$ be ramified of type (R-U). We have $O_E = O_F[\Pi]$ for some uniformizer $\Pi \in O_E$ of the form $\Pi = (1+\vartheta)/\pi_0^k$, where ϑ is a unit with $\vartheta^2 = 1 + \varepsilon\pi_0^{2k+1} \in O_F$. We fix a basis $(e_1, e_2, \Pi e_1, \Pi e_2)$ of Λ such that $(\,,)$ is given by

$$B_\Lambda = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \frac{2}{\pi_0^k} \\ -1 & & -\frac{2}{\pi_0^k} & \end{pmatrix}.$$

Let R be an O_F-algebra and let $g \in \mathrm{Aut}(\Lambda)(R)$. The 4×4-matrix A representing g satisfies the equations $\Pi A = A\Pi$ and $A^t B_\Lambda A = B_\Lambda$. This time, it has the following form:

$$A = \begin{pmatrix} a & b & \varepsilon\pi_0 e & \varepsilon\pi_0 f \\ c & d & \varepsilon\pi_0 g & \varepsilon\pi_0 h \\ e & f & a + \frac{2}{\pi_0^k} e & b + \frac{2}{\pi_0^k} f \\ g & h & c + \frac{2}{\pi_0^k} g & d + \frac{2}{\pi_0^k} h \end{pmatrix},$$

with $a, \dots, h \in R$ satisfying

$$ah - bg - cf + de + \frac{2}{\pi_0^k} eh - \frac{2}{\pi_0^k} fg = 0,$$
$$ad - bc + \varepsilon\pi_0 eh - \varepsilon\pi_0 fg = 1.$$

We have $\mathrm{Aut}(\Lambda) = \mathrm{Spec}\, O_F[a, \dots, h]$ modulo these equations. It is smooth by the Jacobi criterion.

We have shown that $\mathrm{Aut}(\Lambda)$ is smooth in all cases, hence f_{pol} is smooth. We want to show the surjectivity of f_{pol}. It suffices to check this for geometric points, see Remark 6.8.

Let $(X, \iota, \widetilde{\lambda}, \varrho) \in \mathcal{M}_{E,\mathrm{pol}}(\bar{k})$. Then the Grothendieck-Messing crystal $\mathbb{D}_X$ and the Dieudonné module M_X are related via the equation $\mathbb{D}_X(\bar{k}) = M_X/\pi_0 M_X$. We want to show that there is an O_E-linear isomorphism

$$\gamma : \mathbb{D}_X(\bar{k}) \xrightarrow{\sim} \Lambda \otimes_{O_F} \bar{k},$$

such that the pullback of the alternating form $(\,,)$ on Λ under γ coincides with the alternating form induces by $\widetilde{\lambda}$. It is enough to show that there exists an isomorphism

$$\gamma : M_X \xrightarrow{\sim} \Lambda \otimes_{O_F} \breve{O}_F.$$

But both sides are free modules of rank 2 over $O_E \otimes_{O_F} \breve{O}_F$ and, up to automorphism of M_X, there exists only one perfect alternating form $(\,,)$ on M_X such that

$$(\alpha x, y) = (x, \alpha y),$$

for all $x, y \in M_X$ and $\alpha \in O_E$. Hence such an isomorphism γ exists. This proves the surjectivity of f_{pol}. □

Remark 6.10. With the Theorem 6.2 established, one can now give an easier proof of the isomorphism $\mathcal{N}_E \xrightarrow{\sim} \mathcal{M}_{Dr}$ for the cases where $E|F$ is unramified or $E|F$ is ramified and $p > 2$, which is the main theorem of [KR11]. Indeed, the main part of the proof in loc. cit. consists of the Propositions 2.1 and 3.1, which claim the existence of a certain principal polarization λ_X^0 for any point $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$. But there is a canonical closed embedding $\mathcal{N}_E \hookrightarrow \mathcal{M}_E$ and under this embedding, λ_X^0 is just the polarization $\widetilde{\lambda}$ of Theorem 6.2, for a suitable choice of $\widetilde{\lambda}_{\mathbb{X}}$ on the framing object. More explicitly, using the notation on page 2 of loc. cit., we take $\widetilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}^{-1}(\Pi) = \lambda_{\mathbb{X}}^0 \circ \iota_{\mathbb{X}}(-\delta)$ in the unramified case and $\widetilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\zeta^{-1})$ in the ramified case.

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Zusammenfassung

In der vorliegenden Arbeit befasse ich mich mit der Konstruktion von 2-adischen Rapoport-Zink-Räumen (kurz RZ-Räumen) von PEL-Typ, mit anderen Worten, Modulräumen von p-dividierbaren Gruppen mit Zusatzdaten in Form von Polarisierungen, Endomorphismen und Levelstrukturen. Derartige Räume wurden erstmals von Rapoport und Zink in ihrem Buch von 1996 definiert, allerdings wird dort neben Modulräumen von EL-Typ der PEL-Typ im p-adischen Fall für beliebiges p, $p > 2$ betrachtet.

In meiner Untersuchung beschränke ich mich auf den RZ-Raum $\mathcal{N}_E$ zugehörig zur spaltenden unitären Gruppe $\mathrm{GU}(1,1)$ über einer (wild) verzweigten quadratischen Erweiterung $E|F$ von einer endlichen Erweiterung $F|\mathbb{Q}_2$. Hierbei unterscheide ich zwei Fälle, (R-P) und (R-U), je nachdem ob die quadratische Erweiterung erzeugt wird von der Quadratwurzel eines uniformisierenden Elements oder der Quadratwurzel einer Einheit. (Als Beispiele seien hier $\mathbb{Q}_2(\sqrt{2})|\mathbb{Q}_2$ und $\mathbb{Q}_2(\sqrt{3})|\mathbb{Q}_2$ erwähnt.) In beiden Fällen definiere ich zuerst das naive Modulproblem $\mathcal{N}_E^{\mathrm{naive}}$, das der bereits bekannten Definition für $p > 2$ nachempfunden ist. Im weiteren Verlauf kann ich jedoch nachweisen, dass $\mathcal{N}_E^{\mathrm{naive}}$ in keinem der betrachteten Fälle die Mindestanforderung von Flachheit über O_F, dem Ganzheitsring von F, erfüllt und diese naive Definition somit nicht ausreichend ist. Vereinfacht gesagt kann diese Aussage zurückgeführt werden auf die Existenz mehrerer Isomorphieklassen von selbstdualen Gittern in einem zweidimensionalen Vektorraum mit spaltender hermitescher Form bezüglich der quadratischen Erweiterung $E|F$. Es ergibt sich daraus die Notwendigkeit einer zusätzlichen Bedingung auf den Punkten des Modulproblems $\mathcal{N}_E^{\mathrm{naive}}$, die $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$ als abgeschlossenes formales Unterschema beschreibt. Diese „ausrichtende Bedingung" (engl. *straightening condition*) formuliere ich mithilfe einer zusätzlichen Polarisierung auf den formalen O_F-Moduln (mit Zusatzdaten), deren Isomorphieklassen den Punkten von $\mathcal{N}_E^{\mathrm{naive}}$ entsprechen. Ich beweise, dass der so erhaltene Modulraum $\mathcal{N}_E$ flach ist über O_F und überdies isomorph zum Drinfeld-Modulproblem $\mathcal{M}_{Dr}$, einem RZ-Raum von EL-Typ, der dargestellt wird durch Delignes formales Modell der Drinfeldschen oberen Halbebene. Dieses Resultat steht in Analogie zu einem Satz von Kudla und Rapoport, der einen Isomorphismus zwischen den entsprechenden RZ-Räumen für $p > 2$ beschreibt.

Für den Modulraum $\mathcal{N}_E$ definiere ich ein lokales Modell $\mathrm{N}_E^{\mathrm{loc}}$, für das ein lokales-Modell-Diagramm im Sinne von Rapoport und Zink existiert. Insbesondere kann man damit für das formale Schema $\mathcal{N}_E$, lokal für die étale Topologie, explizite Gleichungen angeben. Das lokale Modell ist, wie erwartet, flach und, nach Übergang zur unverzweigten quadratischen Erweiterung von F, kanonisch isomorph zum lokalen Modell des Drinfeld-Modulproblems $\mathcal{M}_{Dr}$. Ich berechne außerdem Gleichungen, die auf affinen Karten das lokale Modell $\mathrm{N}_E^{\mathrm{loc}}$ als abgeschlossenes Unterschema einer Grassmannschen beschreiben. Abschließend zeige ich die Existenz und Eindeutigkeit (bis auf Multiplikation mit einer Einheit) der in der ausrichtenden Bedingung verwendeten Polarisierungen auf einem größeren Modulraum $\mathcal{M}_E$, der $\mathcal{N}_E^{\mathrm{naive}}$ als formales abgeschlossenes Unterschema enthält.

Dabei betrachte ich nicht nur 2-adische verzweigte quadratische Erweiterungen $E|F$, sondern allgemeiner beliebige quadratische Erweiterungen $E|F$ über endlichen Erweiterungen $F|\mathbb{Q}_p$ für eine beliebige Primzahl p. Unter Verwendung dieses Satzes kann ich unter anderem auch einen vereinfachten Beweis für den Satz von Kudla und Rapoport geben.