On the relation between K- and L-theory of complex C^*-Algebras

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1 Introduction

In this thesis we study the relationship between two *spectra* valued functors on the category of *separable*[1]*, complex* C^*-algebras, the first one being topological K-theory and the second one being projective symmetric L-theory of the underlying involutive ring:

$$\mathrm{C}^*\mathrm{Alg} \underset{L}{\overset{K}{\rightrightarrows}} \mathrm{Sp}$$

The main motivation of this comparison is to relate the L-theoretic Farrell-Jones conjecture to the Baum-Connes conjecture in topological K-theory.

Motivation: comparing assembly maps

We want to take some time to explain why a natural transformation between these two functors provides the mathematical substance to relate the L-theoretic Farrell-Jones conjecture to the Baum-Connes conjecture.

So let G be a countable discrete group. To any such group we can associate the group ring RG, where (R, τ) is a discrete involutive ring. This group ring again has a canonical involution given by

$$\sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \tau(\lambda_g) \cdot g^{-1}$$

hence it is an involutive ring. The examples that are of particular interest to us are $R = \mathbb{Z}$ with the identity as involution and $R = \mathbb{C}$ with $\tau(\lambda) = \overline{\lambda}$.

To come to the role of L-theory in geometric topology, let us suppose that M is an n-dimensional closed, connected, oriented topological manifold. One can ask how many manifolds exist (up to homeomorphism) that are homotopy equivalent to M? This can be divided into two parts: One can first try to study the set of *manifold structures on* M

$$\mathcal{S}(M) = \{M' \xrightarrow{\simeq} M\}/\sim,$$

the set of equivalence classes of manifolds equipped with a homotopy equivalence to M, and then see that this set admits an action of the group of homotopy automorphisms of M by postcomposition. The quotient of this action is the set of homeomorphism classes of manifolds homotopy equivalent to M.

In order to study $\mathcal{S}(M)$ one defines a modification of $\mathcal{S}(M)$, called the the set of *normal invariants of* M, given by

$$\mathcal{N}(M) = \{M' \xrightarrow{f} M\}/\sim$$

where $f \colon M' \to M$ is a degree one normal map. There is a canonical map

$$\mathcal{S}(M) \to \mathcal{N}(M)$$

using that a homotopy equivalence can be viewed as a degree one normal map.

The *geometric surgery exact sequence* determines the failure of this map being a bijection by virtue of the following long exact sequence

$$\cdots \xrightarrow{\sigma} L^q_{n+1}(\mathbb{Z}\pi_1(M)) \longrightarrow \mathcal{S}(M) \longrightarrow \mathcal{N}(M) \xrightarrow{\sigma} L^q_n(\mathbb{Z}\pi_1(M))$$

[1]This is needed in order to turn C^*-algebras with the Kasparov product into a category.

where the map σ is the *surgery obstruction* of a normal invariant. Here the groups $L^q(\mathbb{Z}\pi_1(M))$ are the *quadratic* L-groups of $\mathbb{Z}\pi_1(M)$ and one needs to assume $\dim(M) \geq 5$ in order for this sequence to be exact in general. Thus the geometric surgery exact sequence exhibits the quadratic L-groups as obstruction groups for improving a normal invariant of M to a homotopy equivalence to M.

Warning. This notation for quadratic L-theory is not standard, usually people write $L^*(R)$ for symmetric L-theory groups and $L_*(R)$ for quadratic L-theory groups and use new symbols like $\mathbb{L}^*(R)$, $\mathbb{L}_*(R)$ or $\mathbf{L}^*(R)$, $\mathbf{L}_*(R)$ for the symmetric and quadratic L-theory *spectra*. But since we want to view the spectrum as the main object of study, we want to use the notation LR for the spectrum. Moreover, the lower and upper star is often mixed up with the variance of the functor, which is misleading: both quadratic and symmetric L-theory are covariant. Thus we choose to denote symmetric L-theory by LR and quadratic L-theory by L^qR.

Hence informations about the set $\mathcal{S}(M)$ imply information about the surgery obstruction map and vice versa. Here is a first instance of this phenomenon:

The Poincaré conjecture in the topological category states that any closed manifold Σ that is homotopy equivalent to S^n is already homeomorphic to S^n. This conjecture is now known for all $n \geq 1$: it is clear in dimensions 1 and 2 since all such manifolds are classified, in dimension 3 it follows from Perelmanns work, in dimension 4 it follows from the classification of simply connected topological 4-manifolds due to Freedman, and in dimensions larger than 4 it follows from the h-cobordism theorem. Since moreover any homotopy automorphism of the sphere is homotopic to a homeomorphism, it follows that

$$\mathcal{S}(S^n) \cong \{\mathrm{id}_{S^n}\}.$$

Thus the geometric surgery exact sequence implies that the surgery obstruction map

$$\mathcal{N}(S^n) \xrightarrow[\cong]{\sigma} L^q_n(\mathbb{Z})$$

for $n \geq 6$ is an isomorphism.

Furthermore it turns out that the set of normal invariants has much more structure than a priori expected: Following an insight of Sullivan it was shown that if M is a manifold (more general a Poincaré complex with reducible Spivak fibration), there is an isomorphism

$$\mathcal{N}(M) \to [M, G/TOP],$$

where G denotes the stable group of homotopy automorphisms of spheres and TOP denotes the stable group of homeomorphisms of spheres.

In particular the previous argument shows that

$$\pi_n(G/TOP) \cong L^q_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0\ (4), \\ \mathbb{Z}/2 & \text{if } n \equiv 2\ (4), \text{ and} \\ 0 & \text{else.} \end{cases}$$

at least for $n \geq 6$. Using the fibration

$$G/TOP \to BTOP \xrightarrow{J} BG \tag{1.1}$$

one can acutally show that the above isomorphism holds for all $n > 0$.

Actually, even more is true: Ranicki produces (quadratic) L-theory spectra, see e.g. [Ran92a, chapter 13],

$$L^{(q)} \colon \mathrm{Ring}^{\mathrm{inv}} \to \mathrm{Sp}$$

and the above argument can be used to show that

$$\Omega^\infty\left((L^q\mathbb{Z})\langle 1\rangle\right) \simeq G/TOP.$$

The fibration (1.1) provides G/TOP with the structure of an infinite loop space and we want to emphasize that the two infinite loop spaces

$$(L^q\mathbb{Z})\langle 1\rangle \text{ and } G/TOP$$

are not even equivalent as H-spaces. We will come back to this later.

Thus one obtains isomorphisms

$$[M, G/TOP] \cong (L^q\mathbb{Z})\langle 1\rangle^0(M) \cong (L^q\mathbb{Z})\langle 1\rangle_n(M)$$

where the second isomorphism is given by Poincaré duality. The fact that topological manifolds satisfy Poincaré duality in $(L^q\mathbb{Z})\langle 1\rangle$ is implied by the existence of a homotopy ring map

$$MSTOP \to L\mathbb{Z}$$

called the Sullivan-Ranicki orientation.

We may thus interpret the surgery obstruction map as a map

$$(L^q\mathbb{Z})\langle 1\rangle_n(M) \xrightarrow{\sigma} L^q_n(\mathbb{Z}\pi_1(M)) .$$

Since the L-group which is the target of the surgery obstruction map only depends on the fundamental group one could ask whether this is already seen in topology, more precisely, does the surgery obstruction map factor through the homology of the 1-type $B\pi_1(M)$ associated to M?

$$\begin{array}{ccc} (L^q\mathbb{Z})\langle 1\rangle_n(M) & \xrightarrow{\sigma} & L^q_n(\mathbb{Z}\pi_1(M)) \\ \downarrow & \nearrow & \\ (L^q\mathbb{Z})\langle 1\rangle_n(B\pi_1(M)) & & \end{array}$$

It is a theorem that the answer to this question is *yes*, and the map making the diagram commutative is an *assembly map* as discussed in [DL98]. We shortly recall their approach.

We denote by Gpd the 1-category of groupoids and consider functors

$$X\colon \mathrm{Gpd} \to \mathrm{Sp}$$

that send equivalences of groupoids to equivalences of spectra. Then given a (discrete) group G and a family $\mathcal{F}$ of subgroups of G one can construct a universal map

$$X^G_*(E_{\mathcal{F}}G) \to \pi_*(X(G))$$

where $X^G_*(-)$ is a G-homology theory, essentially built from X as a coend over the orbit category, and the space $E_{\mathcal{F}}G$ is the universal G-space with isotropy in the family $\mathcal{F}$.

If one uses the family $\mathcal{F} = \{1\}$ consisting of only the trivial subgroup, the assembly map takes the form

$$X_*(BG) \cong X^G_*(EG) \to \pi_*(X(G)).$$

By [DL98] there is a functor $L\colon \mathrm{Gpd} \to \mathrm{Sp}$, preserving equivalences, such that the composite

$$\mathrm{Grp} \longrightarrow \mathrm{Gpd} \xrightarrow{L} \mathrm{Sp}$$

sends a group H to its quadratic L-theory spectrum $L^q(RH)$. Thus we obtain an assembly map

$$(L^qR)^G_*(E_{\mathcal{F}}G) \xrightarrow{\mathrm{FJ}} L^q_*(RG)$$

called the Farrell-Jones assembly map in L-theory. The *Farrell-Jones conjecture* in L-theory states that this map is an isomorphism if we choose the family $\mathcal{F} = Vcyc$ to be the family of virtually cyclic subgroups of G. If one is willing to invert 2 it even suffices to choose the family of finite subgroups and if the group G is torsion free it suffices to use the trivial family $\mathcal{F} = \{1\}$, see [LR05, Proposition 2.18 and Proposition 2.10] in which case the assembly map for the ring $\mathbb{Z}$ reads as

$$L^q\mathbb{Z}_*(BG) \longrightarrow L^q_*(\mathbb{Z}G)$$

and is, for $G = \pi_1(M)$, related to the surgery obstruction map as discussed above.

Let us turn to the analytical point of view now, to K-theory of C^*-algebas. One can complete $\mathbb{C}G$ to a C^*-algebra C^*_rG, the *reduced group* C^*-algebra in the following way. The left multiplication action of G on ℓ^2G extends to an involutive injection

$$\mathbb{C}G \subset \mathcal{B}(\ell^2G)$$

so we can complete $\mathbb{C}G$ in the norm topology of $\mathcal{B}(\ell^2G)$ to obtain C^*_rG. There is also a different completion of $\mathbb{C}G$ to a C^*-algebra, called the *full group* C^*-algebra C^*G which comes with a canonical surjection

$$C^*G \to C^*_rG.$$

This map turns out to be an isomorphism if and only if the group G is amenable, so for instance for abelian or finite groups.

We introduce the full group C^*-algebra mainly to avoid issues in functoriality: The full group C^*-algebra is functorial for all group homomorphisms, whereas the association $G \mapsto C^*_rG$ is in general not functorial. For instance it is a remarkable fact that the algebra $C^*_rF_2$ is a *simple*, where F_2 is the free group on two generators, see [Pow75]. One can use this to show that this contradicts functoriality.

There is also a functor $\mathrm{Gpd} \to \mathrm{Sp}$ which preserves equivalences and has the property that

$$\mathrm{Grp} \longrightarrow \mathrm{Gpd} \xrightarrow{K} \mathrm{Sp}$$

sends a group H to the topological K-theory spectrum of the *full group C^*-algebra* $K(C^*H)$. We obtain such a functor in the appendix, for example. This gives an assembly map in topological K-theory which reads as

$$KU^G_*(E_\mathcal{F}G) \xrightarrow{\mathrm{BC}} K_*(C^*G) \longrightarrow K_*(C^*_rG)$$

and it is the content of the *Baum-Connes conjecture* that this composite is an isomorphism for the family $\mathcal{F} = \mathrm{fin}$ of finite subgroups. There is a version for K-theory of real C^*-algebras which reads as

$$KO^G_*(E_\mathcal{F}G) \xrightarrow{\mathrm{BC}} KO_*(C^*G) \longrightarrow K_*(C^*_rG)$$

where now we insert the corresponding *real* group C^*-algebra by replacing the complex numbers by the reals numbers in the definition. In [Sch04] the author proves that the conjecture is true in the real case if and only if it is true in the complex case. Thus in proofs, usually the complex case is considered, but for the application to e.g. positive scalar curvature questions like the stable Gromov-Lawson-Rosenberg conjecture one needs the real version.

The assembly map in K-theory can be constructed in a purely analytical fashion and has been studied by operator algebraists for a long time, notably in the work of Kasparov [Kas75], [Kas81], [Kas88] on the Novikov conjecture. See also [LR05] for a survery, [Val02] for a gentle introduction and [Lan15] for a construction and comparison of different analytical constructions.

Both the Baum-Connes conjecture and the Farrell-Jones conjecture have been intensively studied and are proven in many but not all cases. We recommend the survery [LR05] for details. There are classes of groups for which Baum-Connes is known but Farrell-Jones is not known (e.g. amenable groups) and vice versa (e.g. $SL_n(\mathbb{Z})$ for $n \geq 3$). Both conjectures imply various other conjectures in various different fields of mathematics: the Borel conjecture is implied by the L-theoretic

Farrell-Jones conjecture (together with a version of this conjecture in algebraic K-theory) and the idempotent conjecture and the stable Gromov-Lawson-Rosenberg conjecture are implied by the Baum-Connes conjecture. The Novikov conjecture about the homotopy invariance of higher signatures of manifolds is implied by the (rational) injectivitiy of both the Baum-Connes map and the L-theoretic Farrell-Jones map for the ring $\mathbb{Z}$. We can thus draw the following schematic picture

$$\begin{array}{ccc} \text{Injectivitiy of } \mathrm{FJ}_{\mathbb{Q}} & \overset{?}{\longleftrightarrow} & \text{Injectivitiy of } \mathrm{BC}_{\mathbb{Q}} \\ & \searrow\!\!\searrow \quad \swarrow\!\!\swarrow & \\ & \text{Novikov conjecture} & \end{array}$$

and ask whether there is any relation between the injectivity of L-theoretic Farrell-Jones map and the injectivity of the Baum-Connes map. The first step in comparing these two conjectures is to notice that there is an obvious intermediate step we can take, namely we can consider the assembly map for the functor that takes a group H to the L-theory spectrum $L(C^*H)$ of the *real* group C^*-algebra. This assembly map sits in a commutative diagram

$$\begin{array}{ccccc} L\mathbb{R}^G_*(\underline{E}G) & \xrightarrow{\mathrm{FJ}} & L_*(\mathbb{R}G) & \longrightarrow & L_*(C^*_rG) \\ \uparrow & & \uparrow & & \\ L^q\mathbb{Z}^G_*(\underline{E}G) & \xrightarrow[\mathrm{FJ}]{} & L^q_*(\mathbb{Z}G) & & \end{array}$$

where $\underline{E}G$ is the space $E_{\mathrm{fin}}G$. The left hand map is an isomorphism after inverting 2 by [Ran92a, Proposition 22.34 (ii)]. In the case of complex and unital C^*-algebras there is the following a priori surprising and important theorem that the K- and L-groups

$$\pi_*(KA) \text{ and } \pi_*(LA)$$

are naturally isomorphic, see [Ros95] and [Mil98] for proofs of this. It turns out that this is also true for real C^*-algebras *after* inverting 2, see again [Ros95]. We can thus consider the following commutative diagram, compare with [LR05, page 56]

$$\begin{array}{ccccc} KO^G_*(\underline{E}G)[\frac{1}{2}] & & \xrightarrow{\mathrm{BC}[\frac{1}{2}]} & & KO_*(C^*_rG)[\frac{1}{2}] \\ \cong \downarrow & & & & \downarrow \cong \\ L\mathbb{R}^G_*(\underline{E}G)[\frac{1}{2}] & \xrightarrow{\mathrm{FJ}[\frac{1}{2}]} & L_*(\mathbb{R}G)[\frac{1}{2}] & \longrightarrow & L_*(C^*_rG)[\frac{1}{2}] \\ \cong \uparrow & & \uparrow & & \\ L^q\mathbb{Z}^G_*(\underline{E}G)[\frac{1}{2}] & \xrightarrow[\mathrm{FJ}[\frac{1}{2}]]{} & L^q_*(\mathbb{Z}G)[\frac{1}{2}] & & \end{array} \tag{1.2}$$

and now a dashed arrow would exist and be an isomorphism (making the diagram commutative) if the two *functors* $K[\frac{1}{2}]$ and $L[\frac{1}{2}]$ were *equivalent* on the category of *real* C^*-algebras. In this thesis we only prove this in the complex case, but the real case will be dealt with in a joint paper with T. Nikolaus which is in preparation.

We call the conjecture that the map

$$L_*(\mathbb{R}G)[\tfrac{1}{2}] \longrightarrow L_*(C^*_rG)[\tfrac{1}{2}] \tag{1.3}$$

is an isomorphism the *completion conjecture* in L-theory for the group G. Just from the existence and commutativity of the top square of diagram (1.2) we obtain that the three conjectures

$$\mathrm{BC}[\tfrac{1}{2}], \mathrm{FJ}[\tfrac{1}{2}], \text{ and conjecture (1.3)}$$

satisfy the two out of three property.

We emphasize the relevance of the completion conjecture to the question whether the Baum-Connes conjecture and the Farrell-Jones conjecture hold for all groups: if the completion conjecture fails it follows that not *both* the Baum-Connes conjecture *and* the Farrell-Jones conjecture hold, so one could potentially disprove one of the two conjectures indirectly. On the other hand if the completion conjecture is true, Baum-Connes and Farrell-Jones are equivalent after inverting 2. Since finding a counter example to both the Baum-Connes conjecture and the Farrell-Jones conjecture is a very hard problem, the completion conjecture in L-theory is in our opinion a reasonable conjecture.

We also want to mention that from diagram (1.2) it also follows that the injectivity of $\mathrm{BC}[\frac{1}{2}]$ implies the injectivity of $\mathrm{FJ}[\frac{1}{2}]$ (even for the ring $\mathbb{Z}$). If the group G is torsion free this result already follows from the equivalence $K[\frac{1}{2}] \simeq L[\frac{1}{2}]$ on *complex* C^*-algebras. For this one writes down the same diagram for the complex group C^*-algebra and notices that the map $L\mathbb{Z}[\frac{1}{2}] \to L\mathbb{C}[\frac{1}{2}]$ is the inclusion of a direct summand.

We want to explain why we have to invert 2 in this diagram. In L-theory there is a *decoration* one has to choose that we have not discussed so far in order to ease the read. In the theorem where K- and L-groups of C^*-algebras are compared one needs to use the *projective* decoration, also denoted by $\langle 0 \rangle$. But in the Farrell-Jones assembly map one is forced to use the decoration $\langle -\infty \rangle$ because of the Shaneson splitting, which says that

$$L^{\langle j \rangle}(R[\mathbb{Z}]) \simeq L^{\langle j \rangle}(R) \oplus \Sigma L^{\langle j-1 \rangle}(R)$$

but the Farrell-Jones assembly map predicts that

$$L^{\langle j \rangle}(R[\mathbb{Z}]) \simeq S^1_+ \otimes L^{\langle j \rangle}(R) \simeq L^{\langle j \rangle}(R) \oplus \Sigma L^{\langle j \rangle}(R)$$

so this is only consistent for $j = -\infty = -\infty - 1$.

But there is a canonical map

$$LR \to L^{\langle -\infty \rangle}(R)$$

whose homotopy fiber is trivial if e.g. the algebraic K-theory KR of R is connective, and always when inverting 2, more precisely the map

$$LR[\tfrac{1}{2}] \to L^{\langle -\infty \rangle}(R)[\tfrac{1}{2}]$$

is an equivalence, see e.g. Corollary 2.2.17 for a more general result. So inverting 2 solves the issue of having to change the decorations in L-theory.

Previous work

The question about the relation between K-theory and L-theory of C^*-algebras has been investigated in the past: we have already said that there exists a natural isomorphism

$$\pi_*(KA) \cong \pi_*(LA)$$

for all unital C^*-algebras A which was proven by [Ros95] and [Mil98] using result of Karoubi and Mishchenko.

Both proofs of this result are obtained by first proving that K- and L-theory are 2-periodic and then showing the claim only for $* \in \{0, 1\}$. Furthermore, this result is specific to *complex* C^*-algebras, the corresponding result is not true for R^*-algebras, i.e. C^*-algebras over the *real* numbers. This is mainly the reason why we do not deal with the real case in this thesis at all.

One could be bold and ask whether this equivalence actually comes from an equivalence of spectra, but this is not the case and was observed in [Ros95, Section 2]. The ingredients were already also known to [TW79]:

The spectra $K\mathbb{C}$ and $L\mathbb{C}$ are not equivalent.

In particular the spectra valued functors K and L are not equivalent. This follows from the following observations. Ranicki showed that the functor

$$L\colon \mathrm{Ring}^{\mathrm{inv}} \to \mathrm{Sp} \to \mathcal{SHC}$$

has a lax symmetric monoidal refinement. Here $\mathcal{SHC}$ stands for the *stable homotopy category*. This implies that for any involutive ring R the spectrum LR is a module spectrum over the ring spectrum $L\mathbb{Z}$. Using the Sullivan-Ranicki orientation

$$MSO \to MSTOP \to L\mathbb{Z}$$

it follows that LR is a module over MSO. Since MSO splits 2-locally, so does LR. In particular $L\mathbb{C}$ splits 2-locally but $K\mathbb{C} = KU$ does *not* split 2-locally. Thus the relation between the spectra KA and LA is more subtle than the relation between their homotopy groups.

In [Ros95, Theorem 2.1] Rosenberg shows that for each unital (real or complex) C^*-algebra A there is an equivalence of spectra $KA[\frac{1}{2}] \simeq LA[\frac{1}{2}]$. He claims that this equivalence is natural. However, his proof does not provide this.

The line of thought is as follows. First one shows that the *homotopy ring spectra* $KO[\frac{1}{2}]$ and $L\mathbb{R}[\frac{1}{2}]$ are equivalent. For instance, in [Lur, lecture 25] Lurie proves that the *formal groups* of $KO[\frac{1}{2}]$ and $L\mathbb{R}[\frac{1}{2}]$ are isomorphic and Landweber exact, which implies that the homotopy ring spectra are equivalent. Rosenbergs argument is different, as we will see later.

Then one uses that $KA[\frac{1}{2}]$ is a module over $KO[\frac{1}{2}]$ and $LA[\frac{1}{2}]$ is a module over $L\mathbb{R}[\frac{1}{2}] \simeq KO[\frac{1}{2}]$. Now it is a theorem due to Bousfield, see [Bou90], that given any two $KO[\frac{1}{2}]$ modules M and M' and an isomorphism

$$\pi_*(M) \to \pi_*(M')$$

of $\pi_*(KO[\frac{1}{2}])$-modules, this isomorphisms lifts (*non-uniquely*) to a module map $M \to M'$ which hence is a weak equivalence. This follows from the spectral sequence calculating the homotopy groups of the mapping spectrum of module maps over a ring spectrum and using that the homotopy ring $KO[\frac{1}{2}]_*$ has global dimension 1. Compare with the remark after Proposition 5.0.10 for a precise argument.

Thus Bousfield's result reduces the claim that $KA[\frac{1}{2}]$ and $LA[\frac{1}{2}]$ are equivalent to the existence of an isomorphism of $\pi_*(KO[\frac{1}{2}])$ modules between $\pi_*(KA[\frac{1}{2}])$ and $\pi_*(LA[\frac{1}{2}])$ which we have already established.

However, since the lift to a module map is not canonical there is no reason why one should be able to choose the equivalence

$$KA[\tfrac{1}{2}] \simeq LA[\tfrac{1}{2}]$$

to be *natural* in A. In summary, the best one can achieve with these methods is that the two functors

$$K[\tfrac{1}{2}], L[\tfrac{1}{2}]\colon \mathrm{C^*Alg} \to \mathrm{Sp} \to \mathcal{SHC}$$

are *pointwise* equivalent.

We want to emphasize that in order to apply such an equivalence to assembly maps as outlined previously, we really need these two functors with values in an honest category of spectra (not the homotopy category) to be equivalent as functors (not only pointwise).

Next we want to explain briefly how Rosenberg argues that the two homotopy ring spectra $KO[\frac{1}{2}]$ and $L\mathbb{R}[\frac{1}{2}]$ are equivalent as ring spectra. We want to do this for two reasons: firstly we think that the arguments given are not sufficient to deduce this equivalence. Secondly in his proof he quotes [MM79, Corollary 4.31], a result that we can improve, as we will describe.

Let us recall that if R is an $\mathbb{E}_\infty$-ring spectrum, there is a *spectrum of units* denoted by $gl_1(R)$. This spectrum is connective and the component of the unit is written as

$$sl_1(R) = gl_1(R)_{(e)}.$$

For instance for the spectrum KO, the H-space underlying the spectrum $sl_1(KO)$ is usually referred to as $BO_{\otimes}$.

Now we notice is that there is a natural map

$$L^q\mathbb{Z} \to L\mathbb{Z} \to L\mathbb{R}$$

which becomes an equivalence after inverting 2. As we have discussed earlier, there is an equivalence of *spaces*

$$\Omega^\infty(L^q\mathbb{Z}) \to G/TOP \times \mathbb{Z}$$

and in [MM79, Theorem 4.28] it is proven that there is an equivalence

$$G/TOP[\tfrac{1}{2}] \to BO[\tfrac{1}{2}],$$

a result attributed to Sullivan. Furthermore in [MM79, Corollary 4.31] the authors prove that a modified version of this map actually provides an equivalence of H-spaces

$$G/TOP[\tfrac{1}{2}] \simeq sl_1(KO[\tfrac{1}{2}]) \tag{1.4}$$

where G/TOP is endowed with the H-space structure coming from the infinite loop space structure provided by the fibration (1.1). Then Rosenberg claims that these observations imply that $KO[\frac{1}{2}]$ and $L\mathbb{R}[\frac{1}{2}]$ are equivalent as homotopy ring spectra but we do not see how this should follow.

In a joint project in preparation with F. Hebestreit and G. Laures we improve the equivalence (1.4) to an equivalence of infinite loop spaces: Our results imply that there is an equivalence of connective spectra

$$G/TOP[\tfrac{1}{2}] \simeq sl_1(L\mathbb{Z}[\tfrac{1}{2}]). \tag{1.5}$$

Lurie's result that the homotopy ring spectra $KO[\frac{1}{2}]$ and $L\mathbb{Z}[\frac{1}{2}]$ are equivalent implies that there is an equivalence

$$gl_1(L\mathbb{Z}[\tfrac{1}{2}]) \simeq gl_1(KO[\tfrac{1}{2}])$$

of underlying H-spaces, which together with (1.5) recovers the equivalence (1.4). In order for this equivalence to be one of infinite loop spaces, i.e. of grouplike $\mathbb{E}_\infty$-spaces, we would need to have that the $\mathbb{E}_\infty$-ring spectra $KO[\frac{1}{2}]$ and $L\mathbb{Z}[\frac{1}{2}]$ are equivalent as $\mathbb{E}_\infty$-rings.

In joint work with T. Nikolaus, which is also in preparation, we prove among other things that these two spectra are indeed equivalent as $\mathbb{E}_\infty$-ring spectra. Hence we obtain an equivalence of connective spectra

$$G/TOP[\tfrac{1}{2}] \simeq sl_1(L\mathbb{Z}[\tfrac{1}{2}]) \simeq sl_1(KO[\tfrac{1}{2}]) \simeq BO_{\otimes}[\tfrac{1}{2}]$$

as claimed.

We want to summarize that in [TW79, Theorem A] it is shown that the two spectra $KO[\frac{1}{2}]$ and $L\mathbb{Z}[\frac{1}{2}]$ are equivalent (disregarding the multplicative structure) and that [MM79, Corollary 4.31] says that the H-spaces underlying the spectra $gl_1(KO[\frac{1}{2}])$ and $gl_1(L\mathbb{Z}[\frac{1}{2}])$ are equivalent. In general this is certainly not enough to deduce that the homotopy ring spectra $KO[\frac{1}{2}]$ and $L\mathbb{Z}[\frac{1}{2}]$ are equivalent.

Our results

We hoped for quite some time that we could be able to construct a transformation between K- and L-theory that becomes an equivalence after inverting 2. After failing for a while to construct such a transformation we realized it was in fact impossible.

Theorem A *We have that*

$$[L\mathbb{C}, KU] = [KU, L\mathbb{C}] = [\ell\mathbb{C}, KU] = [\ell\mathbb{C}, ku] = 0.$$

In other words, every such map is null homotopic, see Theorem 5.0.3.

In particular there cannot be any natural transformation between the functors K and L that becomes an equivalence *after* inverting 2. Thus the only way one can hope to find a transformation that induces an equivalence after inverting two is studying transformations between the connective theories

$$\tau\colon k \to \ell.$$

Here we find the following

Theorem B *There exists a natural transformation*

$$\tau \in \mathrm{Map}_{\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg},\mathrm{Sp})}(k,\ell)$$

satisfying the following properties.

(1) For all $A \in \mathrm{C}^*\mathrm{Alg}$ we have that the map

$$\tau_A\colon \pi_n(kA) \to \pi_n(\ell A)$$

is an isomorphism for $n = 0, 1$, see Corollary 4.2.4 and Proposition 4.2.5.

(2) In general this map is not an isomorphism in higher homotopy groups, for instance it follows from Corollary 4.2.6 that there is an exact sequence

$$0 \longrightarrow \pi_2(ku) \xrightarrow{\tau_{\mathbb{C}}} \pi_2(\ell\mathbb{C}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

(3) The space of all natural transformations can be calculated as

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg},\mathrm{Sp})}(k,\ell) \simeq \Omega^\infty(\ell\mathbb{C}),$$

see Corollary 4.2.2. In particular, up to homotopy any transformation η corresponds to a number

$$\alpha(\eta) \in \mathbb{Z} \cong \pi_0(\Omega^\infty \ell\mathbb{C}).$$

(4) The number $\alpha(\eta)$ is determined by the effect of the map $\eta_{\mathbb{C}}$ on π_0.

(5) We have that $\alpha(\tau) = 1$, see Lemma 4.1.2.

(6) There is an equivalence $\hat{\tau}\colon K[\frac{1}{2}] \xrightarrow{\simeq} L[\frac{1}{2}]$ making the diagram

$$\begin{array}{ccc} K[\frac{1}{2}] & \xrightarrow{\hat{\tau}} & L[\frac{1}{2}] \\ \uparrow & & \uparrow \\ k & \xrightarrow[\tau]{} & \ell \end{array}$$

commutative, see Theorem 4.2.7.

As explained earlier this implies the following

Corollary C *Suppose G is a torsion free group. If $\mathrm{BC}[\frac{1}{2}]$ is injective, then so is $\mathrm{FJ}[\frac{1}{2}]$, compare to diagram (1.2).*

Remark. One can ask whether the transformation τ is lax symmetric monoidal with respect to the lax symmetric monoidal structures on K- and L-theory. We will not deal with this question in this thesis. It will be part of a joint paper with T. Nikolaus that this is indeed the case.

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2 K- and L-theory

2.1 C^*-Algebras and K-theory

In this section we want to collect the basic definitions and properties of C^*-algebras that are relevant for our purposes. The mathematics we describe is well known, though some formulations might not be in the literature the way we state them. We will not give detailed proofs, rather introduce notation and properties we need later on. There are many good textbooks about C^*-algebras and their K-theory, e.g. [Tak02], [Bla06], [RLL00], [WO93], [HR00] and [Bla98] just to mention a few.

Definition 2.1.1. A C^*-algebra is a Banach algebra A over $\mathbb{C}$ together with a (complex antilinear) involution $x \mapsto x^*$ on A that satisfies the C^*-*identity*:

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

A morphism $f\colon A \to B$ is a bounded operator which is multiplicative and commutes with the involutions on A and B. A C^*-algebra is called *separable* if it is separable as topological space. We denote the category of separable C^*-algebras by $\mathrm{C^*Alg}$. The subcategory of *unital* algebras with morphisms that preserve the unit will be denoted by $\mathrm{C^*Alg}^{\mathrm{unit}}$.

Examples. (1) Let X be a compact Hausdorff space, then $C(X;\mathbb{C})$ is a C^*-algebra. The norm is the supremum norm and $f^*(x) = \overline{f(x)}$. The algebra $C(X;\mathbb{C})$ is separable if and only if X is second countable.

(2) If X is a compact Hausdorff space and A is any C^*-algebra, then $C(X;A)$ is again a C^*-algebra by forming the involution pointwise and using the supremum norm.

(3) Let $\mathcal{H}$ be a Hilbert space. Then the bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ form a C^*-algebra. The norm is the operator norm, and for a bounded operator T the operator T^* is the *adjoint operator* which is uniquely characterized by the formula

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x, y \in \mathcal{H}$$

and whose existence is guaranteed by the Riesz representation theorem.

(4) Given any C^*-algebra B then any norm-closed and $*$-closed subalgebra $A \subset B$ is also a C^*-algebra. In particular every such $A \subset \mathcal{B}(\mathcal{H})$ is a C^*-algebra.

There is an evident forgetful functor

$$U\colon \mathrm{C^*Alg}^{\mathrm{unit}} \to \mathrm{Ring}^{\mathrm{inv}}$$

that forgets the topology on A.

Proposition 2.1.2. *(1) The functor $U\colon \mathrm{C^*Alg}^{\mathrm{unit}} \to \mathrm{Ring}^{\mathrm{inv}}$ is fully-faithful and any morphism is automatically norm-decreasing.*

(2) Any injective morphism is norm increasing, thus is an isometry.

(3) Any morphism $f\colon A \to B$ has closed image.

(4) Given a C^*-algebra A one can reconstruct the norm on A from the involutive ring UA.

Proof. This is in [Tak02]. □

There is an obvious inclusion functor $\mathrm{C^*Alg}^{\mathrm{unit}} \to \mathrm{C^*Alg}$.

Lemma 2.1.3. *This inclusion admits a left adjoint, called the* unitalization

$$\begin{array}{ccc} \mathrm{C^*Alg} & \longrightarrow & \mathrm{C^*Alg}^{\mathrm{unit}} \\ A & \longmapsto & A^+ \end{array}$$

which comes with a natural split short exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \xrightarrow{\pi_A} \mathbb{C} \longrightarrow 0$$

in the sense that if $f \in \mathrm{Hom}_{\mathrm{C^*Alg}}(A, B)$ then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A^+ & \xrightarrow{\pi_A} & \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^+} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & B^+ & \xrightarrow[\pi_B]{} & \mathbb{C} & \longrightarrow & 0 \end{array}$$

commutes.

Furthermore if A happens to be unital, then $A^+ \cong A \times \mathbb{C}$.

Proof. We only sketch the argument. The furthermore part is clear from the universal property. If A does not have a unit one considers the embedding

$$A \subset \mathcal{B}(A)$$

by left-multiplication. It is injective, hence isometric and the image does not contain the unit (by the assumption that A does not have a unit element). The smallest subalgebra containing both A and the identity of $\mathcal{B}(A)$ is the C^*-algebra A^+. □

Theorem 2.1.4. *This theorem has two parts called the GNS-construction (for Gelfand-Naimark-Segal) and the theorem of Gelfand-Naimark.*

(1) Every C^*-algebra A has a faithful representation on a Hilbert space, i.e. is a subalgebra of $\mathcal{B}(\mathcal{H})$ for a suitably constructed $\mathcal{H}$.

(2) Every commutative unital C^*-algebra A is isomorphic to $C(X)$ for some compact Hausdorff space X which is uniquely determined by A.

Remark. The second part has the following strengthening: We denote the category of commutative and unital C^*-algebras with unital morphisms by $\mathrm{ComC^*Alg}^{\mathrm{unit}}$ and the category of compact Hausdorff spaces by CH. Then the functor

$$\begin{array}{ccc} (\mathrm{CH})^{\mathrm{op}} & \longrightarrow & \mathrm{ComC^*Alg}^{\mathrm{unit}} \\ X & \longmapsto & C(X) \end{array}$$

is an equivalence of categories.

This theorem is useful for various applications. We give a few here:

Corollary 2.1.5. *(1) If $A \in \mathrm{C^*Alg}$, then $M_n(A)$ is also a C^*-algebra for any $n \geq 1$.*

(2) If $A, B \in \mathrm{C^*Alg}$ then there are C^*-algebras $A \otimes_{\min} B$ and $A \otimes_{\max} B$, the minimal *and* maximal *tensor products.*

Proof. We construct the minimal tensor product as follows. Suppose A is faithfully represented on $\mathcal{H}_A$ and B is faithfully represented on $\mathcal{H}_B$. Then there is an evident representation of the *algebraic* tensor product $A \otimes_{\text{alg}} B$ on $\mathcal{H}_A \otimes \mathcal{H}_B$. One can see that this is faithful, so we can define $A \otimes_{\min} B$ to be the norm closure of the algebraic tensor product in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Similarly we obtain a faithful representation of $M_n(A)$ on $\bigoplus_{i=1}^{n} \mathcal{H}_A$ which explains the first part. □

Remark. We want to emphasize that even the first part of this corollary is non-trivial since it follows from Proposition 2.1.2 part (4) that the only norm on $M_n(\mathbb{C})$ that is a C^*-norm is the operator norm, which we cannot define purely in terms of norms of the entries of the matrix.

Remark. The names *minimal* and *maximal* tensor products are justified by the following property. If $A, B \in \mathrm{C^*Alg}$ and $A \otimes_\sigma B$ is a C^*-algebraic tensor product of A and B (i.e. σ is a C^*-norm on $A \otimes_{\text{alg}} B$, then there is a canonical surjective morphism

$$A \otimes_{\max} B \xrightarrow{p_\sigma} A \otimes_\sigma B.$$

In particular this is true for the minimal tensor product and the map $p_{\min}$ factors through p_σ, i.e. the diagram

$$\begin{array}{ccc} A \otimes_{\max} B & \xrightarrow{p_\sigma} & A \otimes_\sigma B \\ & \searrow^{p_{\min}} & \downarrow \\ & & A \otimes_{\min} B \end{array}$$

exists and is commutative.

Definition 2.1.6. A C^*-algebra A is called *nucelar* if for every C^*-algebra B the algebraic tensor product $A \otimes B$ admits exactly one C^*-norm.

Remark. This is equivalent to the condition that the canonical map

$$A \otimes_{\max} B \xrightarrow{\cong} A \otimes_{\min} B$$

is an isomorphism.

Examples. (1) Finite dimensional algebras are nuclear, see [WO93, Remark T.6.18].

(2) Commutative algebras are nuclear, see [WO93, Theorem T.6.20].

Remark. We will usually use the maximal tensor product, and thus write $A \otimes B$ for $A \otimes_{\max} B$.

Examples. (1) $M_n(A) \cong M_n(\mathbb{C}) \otimes A$. Here the minimal and maximal tensor product coincide as $M_n(\mathbb{C})$ is finite dimensional and thus nuclear.

(2) If X is a compact Hausdorff space and A is any C^*-algebra, we have $C(X) \otimes A \cong C(X; A)$. This is again consistent since $C(X)$ is commutative and thus nuclear.

Proposition 2.1.7. *The maximal tensor product is an* exact *functor. More precisely, given a short exact sequence of C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

and a C^*-algebra D, then also the sequence

$$0 \longrightarrow J \otimes D \longrightarrow A \otimes D \longrightarrow B \otimes D \longrightarrow 0$$

is exact.

Proof. This is [Bla06, II.9.6.6]. □

We want to continue by studying functors out of the category $\mathrm{C^*Alg}$.

Definition 2.1.8. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be the C^*-algebra of compact operators on $\mathcal{H}$. Any rank one projection in $\mathcal{K}$ determines a map $A \to A \otimes \mathcal{K}$. If C is any category we say that a functor $F \in \mathrm{Fun}(\mathrm{C^*Alg}, C)$ is *stable* if the induced map $F(A) \to F(A \otimes \mathcal{K})$ is an isomorphism.

Definition 2.1.9. For a category T with the notion of an exact sequence (e.g. an abelian or triangulated category) we say that a functor $F \in \mathrm{Fun}(\mathrm{C^*Alg}, T)$ is *split-exact* if for any split exact sequence

$$0 \longrightarrow J \xrightarrow{j} A \underset{p}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

the induced sequence

$$0 \longrightarrow FJ \xrightarrow{Fj} FA \underset{Fp}{\overset{Fs}{\leftrightarrows}} FB \longrightarrow 0$$

is also split exact.

Next we want to discuss a specific functor from $\mathrm{C^*Alg}$ to abelian groups called K-theory.

Definition 2.1.10. Let $A \in \mathrm{C^*Alg}^{\mathrm{unit}}$. We define the zeroth topological K-theory group by

$$K_0(A) = K_0^{\mathrm{alg}}(A)$$

i.e. $K_0(A)$ is the Grothendieck group of finitely generated projective A-modules under direct sum.

To define higher K-groups we consider the topological group

$$GL(A) = \operatorname{colim} GL_n(A)$$

where $GL_n(A) \subset M_n(A)$ is the group of invertible elements in the unital C^*-algebra $M_n(A)$ with the subspace topology.

Definition 2.1.11. For all $n \geq 1$ one defines

$$K_n(A) = \pi_n(BGL(A))$$

where $BGL(A)$ is the classifying space of the topological group $GL(A)$.

Lemma 2.1.12. *K-theory commutes with finite products, i.e. the canonical map*

$$K_n(A \times B) \to K_n(A) \oplus K_n(B)$$

is an isomorphism for all $n \geq 0$.

Proof. This is classical algebra for the case $n = 0$. For the other cases we observe that $GL(A \times B) \cong GL(A) \times GL(B)$. Since the classifying space functor commutes with finite products the lemma follows from the fact the homotopy groups commute with finite products. □

Using this we can extend this definition to non-unital algebras via the unitalization functor.

Definition 2.1.13. Let $A \in \mathrm{C^*Alg}$. Then we define for all $n \geq 0$

$$K_n(A) = \ker\left(K_n(A^+) \to K_n(\mathbb{C})\right).$$

Remark. Lemma 2.1.12 guarantees that we have not changed the definition of K-theory for unital C^*-algebras up to canonical isomorphism.

Lemma 2.1.14. *Let $\mathcal{K}$ be the algebra of compact operators on a Hilbert space. Then for any rank one projection in $\mathcal{K}$ and any $n \geq 0$, the induced map*

$$K_n(A) \to K_n(A \otimes \mathcal{K})$$

is an isomorphism, i.e. K-theory is a stable functor.

Proposition 2.1.15. *K-theory has a lax symmetric monoidal structure, i.e. for all $A, B \in \mathrm{C^*Alg}$ there is a multiplication map*

$$K_n(A) \otimes K_m(B) \to K_{n+m}(A \otimes B).$$

Theorem 2.1.16. *There exists a specific element $\beta \in K_2(\mathbb{C})$ called the* Bott element. *It has the property that*

$$\begin{aligned} K_n(A) &\longrightarrow K_{n+2}(A) \\ x &\longmapsto \beta \cdot x \end{aligned}$$

is an isomorphism.

Remark. This extends topological K-theory to negatively graded groups just by using the periodicity.

Remark. We think it is worthwhile to think about Bott periodicity in C^*-algebras rather than in spaces. The main reason is that we can describe an explicit inverse of the Bott map: We consider the Toeplitz algebra $\mathcal{T}$ which is the C^*-algebra generated by the unilateral shift on $\ell^2(\mathbb{N})$. It contains the compact operators as an ideal. Thus we can form the extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{K} \longrightarrow 0$$

It is easy to calculate that the quotient algebra $\mathcal{T}/\mathcal{K}$ is commutative, and hence of the form $C(X)$, recall Theorem 2.1.4 part (2). More precisely if $C^*(a)$ is the sub-C^*-algebra generated by a normal element $a \in A$ then $C^*(a)$ is isomorphic to $C(\mathrm{spec}(a))$. In our case the element a is the image of the unilateral shift in the quotient by the compact operators. This element is unitary and hence has spectrum contained in S^1. A K-theory argument (the Toeplitz index theorem) now shows that the spectrum has to be all of S^1. We therefore obtain a sequence (the Toeplitz sequence)

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

and hence also the reduced Toeplitz sequence induced by $S\mathbb{C} \subset C(S^1)$

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_0 \longrightarrow S\mathbb{C} \longrightarrow 0$$

By tensoring with any $A \in \mathrm{C^*Alg}$ we obtain a sequence

$$0 \longrightarrow \mathcal{K} \otimes A \longrightarrow \mathcal{T}_0 \otimes A \longrightarrow S\mathbb{C} \otimes A \longrightarrow 0$$

recall Proposition 2.1.7. Using Corollary 2.1.22 and the fact that $S\mathbb{C} \otimes A \cong SA$ the boundary map in K-groups of this sequence can be identified with a map

$$K_{n+2}(A) \cong K_{n+1}(SA) \xrightarrow{\delta} K_n(\mathcal{K} \otimes A) \cong K_n(A)$$

which turns out to be an inverse to the Bott map.

One can give a more homotopy theoretical construction of K-theory.

Theorem 2.1.17. *There is a functor $K\colon \mathrm{C^*Alg^{unit}} \to \mathrm{Sp}$ such that the diagram of functors*

$$\begin{array}{ccc} \mathrm{C^*Alg^{unit}} & \xrightarrow{K} & \mathrm{Sp} \\ & \searrow^{K} & \downarrow{\scriptstyle \pi_*} \\ & & \mathrm{Ab}_{\mathbb{Z}} \end{array}$$

commutes.

Proof. There are various equivalent definitions of K-theory spectra. See e.g. [DEKM11], [Joa04], [Joa03], and [Sch16]. We will give a different but equivalent description of K-theory spectra later, see Theorem 3.2.3. □

Remark. Similar to Definition 2.1.13 we extend the K-theory functor to non-unital algebras by the formula

$$KA = \mathrm{hofib}\left(K(A^+) \to K\mathbb{C}\right).$$

We want to collect some further properties of K-theory.

Theorem 2.1.18. *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then the associated sequence

$$KJ \longrightarrow KA \longrightarrow KB$$

is a fibration sequence of spectra.

Remark. The associated long exact sequence in homotopy groups is the usual long exact sequence of K-theory groups associated to a short exact sequence of C^*-algebras.

Corollary 2.1.19. *The K-theory functor is* split exact. *In particular the extension to non-unital algebras via spectra is compatible with Definition 2.1.13.*

Proof. It follows directly from Theorem 2.1.18 that if

$$0 \longrightarrow J \xrightarrow{\;j\;} A \underset{p}{\overset{s}{\rightleftarrows}} B \longrightarrow 0$$

is a split short exact sequence then

$$KA \simeq KB \oplus KJ$$

and in particular that

$$K(A^+) \simeq KA \oplus K\mathbb{C}.$$

□

Definition 2.1.20. We define $SA = \{f : [0,1] \to A \mid f(0) = 0 = f(1)\}$ and $CA = \{f\colon [0,1] \to A \mid f(0) = 0\}$ with the obvious structures of C^*-algebras coming from $C(S^1; A)$. These algebras are called the C^*-algebraic *suspension* and *cone* of A.

Lemma 2.1.21. *These algebras sit in a short exact sequence*

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{\mathrm{ev}_1} A \longrightarrow 0\,.$$

Moreover the algebra CA is contractible *and hence has trivial K-theory.*

Corollary 2.1.22. *From the induced fibration sequence of K-theory spectra we see that the canonical map*

$$\Sigma K(SA) \to KA$$

is an equivalence.

For later purposes we need to talk about KK-theory. It is sufficient for us to introduce it in the following way.

Theorem 2.1.23. *There is a category KK with the following properties, see e.g. [Bla98]:*

(1) $\mathrm{Ob}(KK) = \mathrm{Ob}(\mathrm{C^*Alg})$ *and there is a functor* $\mathrm{C^*Alg} \to KK$*, which we denote by* $f \mapsto [f]$ *on morphisms,*

(2) This functor is a homotopy functor, i.e. if f and g are homotopic, then $[f] = [g]$.

(3) The category KK is triangulated, exact sequences are short exact sequences of C^*-algebras, and the loop functor is the C^*-algebraic suspension functor.

(4) The groups $KK(A, B)$ can be described as equivalence classes of triples $(\mathcal{E}, \pi, F)$, where $\mathcal{E}$ is a Hilbert-B-module, $\pi\colon A \to \mathcal{L}(\mathcal{E})$ is a representation and $F \in \mathcal{L}(\mathcal{E})$ satisfiying compactness conditions.

Definition 2.1.24. Let $f \in \mathrm{Hom}_{\mathrm{C^*Alg}}(A, B)$. We call f a KK-equivalence if $[f] \in \mathrm{Hom}_{KK}(A, B)$ is invertible, i.e. if there exists an element $\Phi \in KK(B, A)$ such that $[f] \circ \Phi = \mathrm{id}_B$ and $\Phi \circ [f] = \mathrm{id}_A$.

Theorem 2.1.25. *Let* $F \in \mathrm{Fun}(\mathrm{C^*Alg}, \mathrm{Ab})$*. If F is stable and split exact, then it is KK-invariant, i.e. F sends KK-equivalences to isomorphisms.*

Proof. This is done in two steps. Fist, in [Hig87] Higson shows that if F is in addition *homotopy invariant* it sends KK-equivalences to isomorphisms. In [Hig88] Higson then proved that stable and split exact functors actually are already homotopy invariant. □

We will make use of the following

Proposition 2.1.26. *The category* $\mathrm{C^*Alg}$ *admits the structure of a fibration category, with fibrations the* Schochet fibrations *and weak equivalences the KK-equivalences. Its homotopy category is* KK*, the category whose objects are separable C^*-algebras and whose morphism sets are Kasparov's KK-groups.*

Proof. This is [Uuy13, Theorem 2.29]. See also the article [BJM15]. □

Corollary 2.1.27. *The functor* $\mathrm{C^*Alg} \to KK$ *is a localization along the KK-equivalences, i.e. for every category C the induced functor*

$$\mathrm{Fun}(KK, C) \to \mathrm{Fun}(\mathrm{C^*Alg}, C)$$

is fully faithful and has image precisely those functors that send KK-equivalences to isomorphisms. These are characterized by Theorem 2.1.25 as those functors that are stable and split exact.

Remark. It is much more familiar to C^*-algebraists to say that the functor

$$\mathrm{Fun}^{\Pi}(KK, \mathrm{Ab}) \to \mathrm{Fun}^{\Pi}(\mathrm{C^*Alg}, \mathrm{Ab})$$

is a fully-faithful and the image consists of those functors that send KK-equivalences to isomorphisms. Here $\mathrm{Fun}^{\Pi}(-,-)$ refers to functors that preserve finite products. We notice that the functor $\mathrm{C^*Alg} \to KK$ preserves finite products which makes the above functor well defined. Again we have the more general statement that whenever C is a category with finite limits, then the functor

$$\mathrm{Fun}^{\Pi}(KK, C) \to \mathrm{Fun}^{\Pi}(\mathrm{C^*Alg}, C)$$

is fully faithful. For this we consider the diagram

$$\begin{array}{ccc}
\mathrm{Fun}^{\Pi}(KK, C) & \longrightarrow & \mathrm{Fun}^{\Pi}(\mathrm{C^*Alg}, C) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(KK, C) & \longrightarrow & \mathrm{Fun}(\mathrm{C^*Alg}, C)
\end{array}$$

Since both vertical maps in this diagram are (by definition) fully faithful, it follows from the fact that the lower horizontal map is fully faithful that also the top horizontal map is fully faithful. Now we can apply this to $C = \mathrm{Ab}$ or even to $C = \mathrm{Set}$ by using that

$$\mathrm{Fun}^{\Pi}(KK, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Set})) \xrightarrow{\sim} \mathrm{Fun}^{\Pi}(KK, \mathrm{Set})$$

and

$$\mathrm{Fun}^{\Pi}(\mathrm{C^*Alg}, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Set})) \xrightarrow{\sim} \mathrm{Fun}^{\Pi}(\mathrm{C^*Alg}, \mathrm{Set})$$

are equivalences since both KK and $\mathrm{C^*Alg}$ are additive categories and the observation that $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Set}) = \mathrm{Ab}$, see [GGN15]. We will deal in more detail with the object $\mathrm{Grp}_{\mathbb{E}_\infty}(C)$ in the context of ∞-categories later and obtain very similar results, see Chapter 3. Since ordinary categories embedd in ∞-categories it makes sense to look at $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Set})$.

Remark. Since the functor $\mathrm{C^*Alg} \to KK$ is homotopy invariant it follows that any split exact and stable functor $F \in \mathrm{Fun}(\mathrm{C^*Alg}, \mathrm{Ab})$ is also homotopy invariant. This is a crucial theorem of Higson and used heavily in the study of *algebraic K-theory* of C^*-algebras and the Karoubi conjecture (now a theorem) which states that for *stable* C^*-algebras the canonical map $K^{\mathrm{alg}}(A) \to KA$ is an equivalence.

Corollary 2.1.28. *The K-theory functor factors through the KK-category. More precisely, there is a functor $K \in \mathrm{Fun}(KK, \mathrm{Ab})$ such that the diagram*

$$\begin{array}{ccc} \mathrm{C^*Alg} & \xrightarrow{K} & \mathrm{Ab} \\ \downarrow & \nearrow_{K} & \\ KK & & \end{array}$$

commutes.

Proof. Corollary 2.1.19 says that K-theory (as abelian group valued functor) is split exact. Furthermore by Lemma 2.1.14 K-theory is stable and hence factors over KK by Theorem 2.1.25 and Corollary 2.1.27. ☐

Much more is true, see e.g. [Bla98], this is the *Fredholm picture* of KK-theory.

Proposition 2.1.29. *The K-theory functor, when viewed on KK becomes corepresentable by the (tensor unit) object $\mathbb{C}$. In other words there is an isomorphism*

$$\left(KK(\mathbb{C}, -) \xrightarrow{\cong} K\right) : KK \to \mathrm{Ab}_{\mathbb{Z}}.$$

The main objective of chapter 3 is to prove an analogue of this statement for the functor

$$K\colon \mathrm{C^*Alg} \to \mathrm{Sp}_\infty$$

where Sp_∞ denotes the ∞-category of spectra, see Theorem 3.2.3.

2.2 L-theory

In this section we want to recall basic definitions from algebraic L-theory. L-theory has its origins in surgery theory, where L-groups appear as obstruction groups to deciding wether a given degree 1 normal map between manifolds is bordant to a homotopy equivalence, see for instance [Wal99], [Ran79], and [CLM16]. Moreover there are connections to the algebraic theory of forms, relating L-groups to Witt groups of forms (in favourable cases). Algebraic L-theory has been developed by Ranicki in the series of papers [Ran73a], [Ran73b], [Ran73c], [Ran74] and the two books [Ran81] and [Ran92a].

Unless otherwise stated, by a chain complex over a ring R we always mean a *perfect* chain complex, i.e. one which is quasi isomorphic to a finite complex which is levelwise finitely generated projective.

2.2.1 Basic constructions

We define *symmetric* L-groups as follows.

Definition 2.2.1. For any involutive ring (R,τ) we let $L^n(R,\tau)$ be the group of bordism classes of n-dimensional symmetric algebraic Poincaré complexes (C,φ) over the ring R.

We have to explain the words that appear in this definition. First we define what a symmetric structure on a (perfect) complex is.

We need to make some conventions. If C is a chain complex we let $\Sigma^n(C)$ be the chain complex with $\Sigma^n(C)_k = C_{k-n}$. With this convention we have that for the internal hom complex we get

$$\mathrm{Hom}(C,D)_n = \mathrm{Hom}(\Sigma^n C, D)$$

So let C be a chain complex over R. Using the involution τ we can consider the chain complex

$$C \otimes_R C \simeq \mathrm{Hom}_R(C^{-*}, C) \in \mathrm{Perf}(\mathbb{Z})$$

where C^{-*} refers to the internal Hom complex $\mathrm{Hom}(C,R)$. Notice that

$$C_k^{-*} = \mathrm{Hom}(C,R)_k = \mathrm{Hom}(C_{-k},R).$$

The chain complex $C \otimes_R C$ has an evident action of the symmetric group on two letters Σ_2 by switching factors (respectively by dualizing a morphism). We can form the homotopy fixpoints which come with a forgetful map

$$(C \otimes_R C)^{h\Sigma_2} \xrightarrow{\mathrm{ev}} C \otimes_R C \simeq \mathrm{Hom}_R(C^{-*}, C)$$

Remark. A model for the homotopy fixpoints of a complex $X \in \mathrm{Perf}(\mathbb{Z}[\Sigma_2])$ is given by the following. We let

$$W = C^{\mathrm{cell}}_{\Sigma_2}(S^\infty)$$

be cellular chain complex of S^∞ with its canonical Σ_2-equivariant CW-structure with one equivariant cell in each dimension. Then we have that

$$\mathrm{Hom}_{\mathbb{Z}[\Sigma_2]}(W,X) \simeq X^{h\Sigma_2}$$

is a model for the fixpoints.

Definition 2.2.2. An *n-dimensional symmetric structure* on C is an element

$$\varphi \in H_n\left((C \otimes_R C)^{h\Sigma_2}\right).$$

Via the above forgetful map, such a symmetric structure induces an element

$$\mathrm{ev}(\varphi) \in H_n(\mathrm{Hom}_R(C^{-*}, C))$$

which gives a homotopy class of a chain map

$$\Sigma^n C^{-*} \xrightarrow{\varphi_0} C$$

Definition 2.2.3. A symmetric chain complex (C,φ) is called *Poincaré* if φ_0 is an equivalence.

Next we need to define the notion of a bordism. For this we first consider a morphism of chain complexes

$$f\colon C \to D.$$

We now consider the following diagram of horizontal cofiber sequences

$$\begin{array}{ccccccc} C \otimes_R C & \xrightarrow{f\otimes f} & D \otimes_R D & \longrightarrow & C(f \otimes f) & \longrightarrow & \Sigma(C \otimes_R C) \\ \downarrow & & \| & & \downarrow & & \\ D \otimes_R C & \longrightarrow & D \otimes_R D & \longrightarrow & D \otimes_R C(f) & & \end{array}$$

where the third vertical arrow is induced by the canonical null homotopy of the lower composite. Since the upper horizontal sequence is equivariant with respect to the Σ_2-action we obtain morphisms

$$D \otimes_R C(f) \longleftarrow C(f \otimes f)^{h\Sigma_2} \longrightarrow \Sigma(C \otimes_R C)^{h\Sigma_2}$$

Definition 2.2.4. An n-dimensional *symmetric pair* is a morphism $C \xrightarrow{f} D$ together with an element

$$\Phi \in H_n(C(f \otimes f)^{h\Sigma_2}).$$

It is called *Poincaré* if the induced morphism

$$\Sigma^n D^* \xrightarrow{\phi} C(f)$$

is an equivalence.

Remark. An n-dimensional symmetric pair $(C \xrightarrow{f} D), \Phi)$ gives rise to an $(n-1)$-dimensional symmetric complex (C, φ) where φ is obtained from Φ along the map

$$H_n(C(f \otimes f)^{h\Sigma_2}) \to H_{n-1}((C \otimes_R C)^{h\Sigma_2})$$

Definition 2.2.5. Let (C, φ) and (C', φ') be $(n-1)$-dimensional symmetric Poincaré complexes. Then a *bordism* between (C, φ) and (C', φ') is an n-dimensional Poincaré pair

$$((C \oplus C' \xrightarrow{f} D), \Phi)$$

such that the induced structure on $C \oplus C'$ is given by $\varphi \oplus -\varphi'$.

We next give two very different types of examples of symmetric Poincaré complexes, one coming from topology and the other coming from algebra.

Example. Let X be a topological space, which is homotopy equivalent to a finite CW-complex. Then the singular chain complex $C_*(X)$ is a perfect complex over $\mathbb{Z}$. A choice of diagonal approximation gives a chain map

$$C_*(X) \to C_*(X) \otimes_{\mathbb{Z}} C_*(X)$$

But then the map

$$C_*(X) \to C_*(X) \otimes_{\mathbb{Z}} C_*(X) \xrightarrow{\cong} C_*(X) \otimes_{\mathbb{Z}} C_*(X)$$

where the last map is the symmetry isomorphism, is also a diagonal approximation, and hence homotopic to the previous one. Iterating this principle produces a Σ_2-equivariant morphism

$$W \otimes C_*(X) \to C_*(X) \otimes_{\mathbb{Z}} C_*(X)$$

which is adjoint to a map

$$C_*(X) \to (C_*(X) \otimes_{\mathbb{Z}} C_*(X))^{h\Sigma_2},$$

where we use the model of homotopy fixpoints as discussed earlier.

Being more careful with fundamental groups one can promote this to a map

$$C_*(X) \to \left(C_*(\widetilde{X}) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})\right)^{h\Sigma_2}$$

where $\pi = \pi_1(X)$ denotes the fundamental group of X.

Thus if $x \in H_n(X)$ is any element in homology, then we obtain a symmetric complex $(C_*(X), \varphi(x))$ or $(C_*(\widetilde{X}), \varphi(x))$ in the equivariant setting.

This construction is called the *(equivariant) symmetric construction.*

Example. Let P be a finitely generated projective (left) module over an involutive ring R. A *hermitian form* on P is an R-linear map

$$P \xrightarrow{\varphi} P^*$$

where $P^* = \mathrm{Hom}_R(P, R)$ is turned into a left module via the involution on R such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P^* \\ & \overset{\mathrm{ev}}{\searrow} & \uparrow{\scriptstyle \varphi^*} \\ & & P^{**} \end{array}$$

is commutative. The form is called non-degenerate if this map is an isomorphism, in which case we may also consider the associated *coform*

$$P^* \xrightarrow{\varphi^{-1}} P.$$

We now consider the chain complex $C(P)$ which consists of P in degree zero and trivial modules otherwise. Via the isomorphism

$$P \otimes_R P \cong \mathrm{Hom}_R(P^*, P)$$

the coform gives an element of the complex $P \otimes_R P$ which is a *strict* fixpoint of the Σ_2-action on P due to the condition that φ is hermitian. In particular it gives rise to a homotopy fixpoint and thus provides a 0-dimensional symmetric structure on the chain complex $C(P)$ which is Poincaré since φ^{-1} is an isomorphism.

Thus any non-degenerate hermitian form over a finitely generated projective module gives rise to a symmetric Poincaré complex over R and hence an element in $L^0(R)$.

Next we want to explain that if $2 \in R^\times$ that these small examples coming from hermitian forms are (up to bordism) all examples. This is a big theorem of Ranicki and uses the process of *algebraic surgery.*

Theorem 2.2.6. *Let $R \in \mathrm{Ring}^{\mathrm{inv}}$ such that $2 \in R^\times$. Then every $2n$-dimensional symmetric complex (C, φ) is bordant to a chain complex that is concentrated in degree n and every $2n+1$-dimensional symmetric complex (C, φ) is bordant to one that is concentrated in degrees n and $n+1$.*

Moreover the L-groups defined as in Definition 2.2.1 are isomorphic to the classical *L-groups defined via forms and formations.*

Remark. This uses heavily that under the assumption that $2 \in R^\times$ the natural map from *quadratic* to symmetric L-theory is an isomorphism. The theorem should then be read as saying that algebraic surgery works in quadratic L-theory and identifies quadratic L-groups in terms of chain complexes with the classical L-groups studied in geometric surgery theory.

Example. The symmetric L-groups of $\mathbb{Z}$ are given as follows

$$L^n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 4k, \\ \mathbb{Z}/2 & \text{if } n = 4k+1 \\ 0 & \text{else }. \end{cases}$$

The integer is given by the signature of the non-degenerate form that represents the class in the L-group, and the $\mathbb{Z}/2$-invariant is called the deRham invariant.

We now turn to some properties of L-theory.

Proposition 2.2.7. *L-theory satisfies the following properties.*

(1) Algebraic L-theory is naturally 4-periodic, i.e. $L^{n+4}(R) \cong L^n(R)$ for all $n \in \mathbb{Z}$ and all involutive rings R.

(2) If -1 has a square root α in (R,τ) which satisfies $\tau(\alpha) = -\alpha$, then L-theory becomes 2-periodic. As an example $L^*(\mathbb{C}, x \mapsto \overline{x})$ is 2-periodic, but $L^*(\mathbb{C}, \mathrm{id})$ is not 2-periodic.

(3) L-theory commutes with finite products of involutive rings (recall we use perfect complexes throughout).

(4) The L-groups of a ring are indeed the homotopy groups of an L-theory spectrum *LR, i.e. the L-functor factors through the category of spectra:*

$$\begin{array}{ccc} & \overset{L}{\nearrow} & \mathrm{Sp} \\ & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Ring}^{\mathrm{inv}} & \xrightarrow[L]{} & \mathrm{Ab}_{\mathbb{Z}} \end{array}$$

(5) L-theory admits external products $LS \otimes LT \to L(S \otimes T)$, more precisely the functor $L\colon \mathrm{Ring}^{\mathrm{inv}} \to \mathcal{SHC}$ admits a lax symmetric monoidal refinement. In particular for every commutative ring S the spectrum LS is a ring spectrum, and for every S-algebra T, the spectrum LT is a module spectrum over LS. In particular every spectrum LR is a module over $L\mathbb{Z}$ and for all complex C^*-algebras A, the spectrum LA is a module over $L\mathbb{C}$.

(6) Using the notion of ad-theories, in [LM14] and [LM13] the authors discuss multiplicativity and commutativity questions of L-theory which can be used to show that the associated ∞-functor

$$L\colon \mathrm{Ring}^{\mathrm{inv}} \to \mathrm{Sp}_\infty$$

admits a lax symmetric monoidal refinement. In particular it follows that the above monoidal properties not only hold in the homotopy category of spectra, but indeed hold highly structured in the ∞-category of spectra (or a closed monoidal model category of spectra like symmetric spectra). We will not need this fact in this thesis though.

Remark. The symmetric L-theory spectrum $L\mathbb{Z}$ admits a ring map, the Sullivan-Ranicki orientation

$$MSO \xrightarrow{\sigma} L\mathbb{Z}$$

that lifts the Hirzebruch genus, i.e. the induced map on homotopy sends a $4k$-dimensional closed oriented manifold M to its symmetric construction, which is by the previous example an integer. This integer is the signature $\sigma(M)$. In dimensions $4k+1$ there are various ways to interpret the deRham invariant of the symmetric construction of M, and one way is to take the characteristic number $w_{4k-1}w_2$.

In [LM13] the authors show that in the version for topological bordism there is a map of $\mathbb{E}_\infty$-ring spectra

$$MSTOP \to L\mathbb{Z}$$

such that the composite

$$MSO \to MSTOP \to L\mathbb{Z}$$

refines the Sullivan-Ranicki orientation to an $\mathbb{E}_\infty$-map.

We want to remark that also the equivariant symmetric construction can be understood from this point of view. Given an n-dimensional closed oriented topological manifold M with fundamental group $\pi = \pi_1(M)$ we want to produce an element in $L^n(\mathbb{Z}\pi)$. This can be done via the following composite

$$MSTOP_n(M) \xrightarrow{\sigma} L\mathbb{Z}_n(M) \xrightarrow{c_*} L\mathbb{Z}_n(B\pi) \xrightarrow{\mathcal{A}} L^n(\mathbb{Z}\pi)$$

by noticing that $[M \xrightarrow{\text{id}} M]$ represents an element in $MSTOP_n(M)$. Here $c\colon M \to B\pi$ is a map that induces the identity on fundamental groups and

$$\mathcal{A}\colon L\mathbb{Z}_*(B\pi) \to L^n(\mathbb{Z}\pi)$$

denotes the *assembly map* in symmetric L-theory (which is not conjectured to be an isomorphism, but exists).

That this composite gives the equivariant symmetric construction is not clear and requires proof, it is done by Ranicki in [Ran92a] and similar in spirit to showing that the assembly map in *quadratic* L-theory is related to the surgery obstruction map of a degree 1 normal map as we have outlined in the introduction.

2.2.2 Changing control in K-theory

In this section we want to deal with the question wether or not L-theory is excisive. By this we mean the following. Recall that Ring^{inv} is the category of involutive rings.

Definition 2.2.8. We call a diagram

$$\begin{array}{ccc} R & \longrightarrow & S_1 \\ \downarrow & & \downarrow p \\ S_2 & \longrightarrow & T \end{array}$$

a homotopy pullback diagram, or homotopy cartesian if the diagram is a pullback in rings and the map $p\colon S_1 \to T$ is surjective.

We call a functor $F\colon \text{Ring}^{\text{inv}} \to \text{Sp}$ *excisive* if it sends homotopy cartesian squares to homotopy cartesian squares.

Remark. Usually functors $F\colon \mathcal{S} \to \text{Sp}$ are called excisive if they send homotopy pushouts to homotopy pullbacks. But since the passage from spaces to rings is contravariant and homotopy pushouts are sent to homotopy pullbacks the above definition is compatible with the usual definition.

In order to deal with the excisiveness of L-theory we need to introduce decorations on L-theory. We start out with the observation that in both symmetric and quadratic L-theory one can consider only those chain complexes (C, φ) whose Euler characteristic $\chi(C)$ lies in a certain Σ_2-invariant subgroup $X \subset K_0(R)$ (recall that the involution on R induces a Σ_2 action on its algebraic K-theory groups). Our goal now is to explain how to use this idea to produce a map

$$L^{\langle -\infty \rangle}(R) \xrightarrow{\Xi} KR^{t\Sigma_2}$$

which we will use to define L-theory with *decorations*. Here KR refers to the non-connective algebraic K-theory spectrum of R. This was first done in [WW98] and [WW89].

The setup we consider is the following. We let $\mathcal{C}$ be a Waldhausen category with duality and will define a map

$$L\mathcal{C} \xrightarrow{\Xi} K\mathcal{C}^{t\Sigma_2}.$$

where here $K\mathcal{C}$ denotes the connective K-theory spectrum of $\mathcal{C}$. We apply this to the $S^\bullet$-construction of the category of (perfect) chain complexes over an involutive ring R. Doing this construction for the Laurent polynomial rings in several variables over R we obtain

$$L(R[\mathbb{Z}^n]) \to K(R[\mathbb{Z}^n])^{t\Sigma_2}$$

compatible with the inclusions $\mathbb{Z}^{n-1} \subset \mathbb{Z}^n$. We use that this provides a non-connective delooping on the level of algebraic K-theory due to Bass and that it lowers the decoration on the level of L-theory due to the Shaneson splitting, see [Ran92b]. Thus in the colimit we obtain the map as claimed and it hence suffices to produce the desired map for each Waldhausen category $\mathcal{C}$ with duality.

Definition 2.2.9. By $\mathcal{C}(m)$ we refer to the category of functors from the face-poset of the m-simplex Δ^m to $\mathcal{C}$ with appropriate duality as in [WW98, Definition 1.5].

Lemma 2.2.10. *The association $m \mapsto K\mathcal{C}(m)$ is a (semi)-simplicial spectrum whose geometric realization $|K\mathcal{C}(\bullet)| = |m \mapsto K\mathcal{C}(m)|$ is contractible.*

Proof. In [WW98, Lemma 9.3] the authors show that the simplicial space

$$m \mapsto \Omega^\infty(K\mathcal{C}(m))$$

has contractible geometric realization. Now we observe that the functor

$$\Omega^\infty \colon \mathrm{Sp}^{\geq 0} \to \mathcal{S}$$

commutes with filtered colimits (since its left adjoint sends the compact generators to compact generators), and also sifted colimits, [Lur14, Proposition 1.4.3.9], hence with geometric realizations. More precisely we have that

$$|\Omega^\infty(K\mathcal{C}(\bullet))| \simeq \Omega^\infty|K\mathcal{C}(\bullet)|.$$

Furthermore we observe that since $K\mathcal{C}(m)$ is a connective spectrum for all $m \in \Delta$ it follows that the geometric realization $|K\mathcal{C}(\bullet)|$ is connective, e.g. because the inclusion $\mathrm{Sp}^{\geq 0} \to \mathrm{Sp}$ is a left-adjoint and thus commutes with realizations. We thus obtain that $|K\mathcal{C}(\bullet)|$ is a connective spectrum whose infinite loop space is contractible. Thus $|K\mathcal{C}(\bullet)|$ is contractible as claimed. □

Remark. One can replace the category $\mathcal{C}(m)$ by a Waldhausen category $\mathcal{C}[m]$ such that $\mathcal{C}[\bullet]$ is a simplicial Waldhausen category and the canonical map $K\mathcal{C}(m) \to K(\mathcal{C}[m])$ is an equivalence for all m, see [WW98, Definition 9.5]. In other words, one can get rid of the technial issue that $m \mapsto K\mathcal{C}(m)$ is only a semi-simplicial space. We will thus ignore this issue in our notation for the rest of this thesis.

The inclusion of the zeroth vertex $\Delta^0 \subset \Delta^m$ induces a functor $\mathcal{C}(m) \to \mathcal{C}(0)$. This extends to a functor of simplicial Waldhausen categories $\mathcal{C}(\bullet) \to \mathcal{C}(0) = \mathcal{C}$ where the target is viewed as constant simplicial category.

Lemma 2.2.11. *The homotopy fiber F of the induced map on algebraic K-theory*

$$F(m) \longrightarrow K\mathcal{C}(m) \longrightarrow cK\mathcal{C}$$

is induced as spectrum with Σ_2-action. Here $cK\mathcal{C}$ denotes the constant simplicial spectrum with value $K\mathcal{C}$.

Proof. This is [WW98, Lemma 9.4]. □

Corollary 2.2.12. *The canonical map induced by $K\mathcal{C}(\bullet) \to cK\mathcal{C}$*

$$|K\mathcal{C}(\bullet)^{t\Sigma_2}| \longrightarrow |cK\mathcal{C}^{t\Sigma_2}| \simeq K\mathcal{C}^{t\Sigma_2}$$

is an equivalence of spectra.

Proof. We consider the map of simplicial spectra

$$K\mathcal{C}(\bullet) \to cK\mathcal{C}$$

whose homotopy fiber is the simplicial spectrum $m \mapsto F(m)$. In other words we have a cofiber sequence of simplicial spectra

$$F(\bullet) \longrightarrow K\mathcal{C}(\bullet) \longrightarrow cK\mathcal{C}$$

Since the Tate construction is exact we obtain again a cofiber sequence of simplicial spectra

$$F(\bullet)^{t\Sigma_2} \longrightarrow K\mathcal{C}(\bullet)^{t\Sigma_2} \longrightarrow cK\mathcal{C}^{t\Sigma_2}$$

Applying the totalization functor gives a cofiber sequence of spectra

$$|F(\bullet)^{t\Sigma_2}| \longrightarrow |K\mathcal{C}(\bullet)^{t\Sigma_2}| \longrightarrow |K\mathcal{C}^{t\Sigma_2}|$$

By Lemma 2.2.11 the spectrum $F(m)^{t\Sigma_2}$ is contractible, and thus so is the geometric realization $|F(\bullet)^{t\Sigma_2}|$. Thus we obtain the claimed equivalence since

$$|K\mathcal{C}(\bullet)^{t\Sigma_2}| \simeq |cK\mathcal{C}^{t\Sigma_2}| \simeq K\mathcal{C}^{t\Sigma_2}$$

since the latter is a constant simplicial spectrum. □

Proposition 2.2.13. *The canonical map*

$$|K\mathcal{C}(\bullet)^{h\Sigma_2}| \longrightarrow |K\mathcal{C}(\bullet)^{t\Sigma_2}|$$

is an equivalence.

Proof. By the very definition of the Tate contruction we have that there is a cofiber sequence of simpicial spectra

$$K\mathcal{C}(\bullet)_{h\Sigma_2} \longrightarrow K\mathcal{C}(\bullet)^{h\Sigma_2} \longrightarrow K\mathcal{C}(\bullet)^{t\Sigma_2}$$

Since geometric realization commutes with cofiber sequences we thus obtain a cofiber sequence

$$|K\mathcal{C}(\bullet)_{h\Sigma_2}| \longrightarrow |K\mathcal{C}(\bullet)^{h\Sigma_2}| \longrightarrow |K\mathcal{C}(\bullet)^{t\Sigma_2}|$$

Since homotopy orbits commutes with geometric realization we obtain that

$$|K\mathcal{C}(\bullet)_{h\Sigma_2}| \simeq |K\mathcal{C}(\bullet)|_{h\Sigma_2}.$$

By Lemma 2.2.10 the spectrum $|K\mathcal{C}(\bullet)|$ is contractible and thus also its homotopy orbits spectrum is contractible. Thus the proposition follows from the above cofiber sequence. □

Corollary 2.2.14. *We have an equivalence of spectra*

$$K\mathcal{C}^{t\Sigma_2} \simeq |K\mathcal{C}(\bullet)^{h\Sigma_2}|.$$

Using this we can describe the map Ξ as follows. Recall that

$$L\mathcal{C} = |sp_0(\mathcal{C}(\bullet))|$$

where $sp_0(\mathcal{C}(m))$ denotes the set of symmetric Poincaré objects over $\mathcal{C}(m)$. Then we observe that there is a canonical simplicial map

$$sp_0(\mathcal{C}(\bullet)) \to K\mathcal{C}(\bullet)^{h\Sigma_2}$$

by taking a symmetric Poincaré complex (C, φ) to the underlying K-theory element C and use φ to obtain the structure of a homotopy fixed point.

On realization this gives a map

$$\Xi\colon L\mathcal{C} = |sp_0(\mathcal{C}(\bullet)| \to |K\mathcal{C}(\bullet)^{h\Sigma_2}| \simeq K\mathcal{C}^{t\Sigma_2}$$

as claimed.

Remark. As discussed this is only a map of spaces, not of spectra (or infinite loop spaces). In [WW98] it is explained how one produces out of this a map of infinite loop spaces using variants of the duality.

We will use this construction to define variants of L-theory with *control in algebraic K-theory*. First we need the following preliminary definition in algebraic K-theory.

Definition 2.2.15. Let $j \in \mathbb{Z}$ and $X \subset K_j(R)$ be a Σ_2-invariant subgroup. The spectrum $K^X(R)$ is the universal j-connective spectrum over $K(R)$ such that the induced map

$$\pi_i(K^X(R)) \to \pi_i(K(R))$$

is bijective for $i > j$ and the inclusion $X \to K_0(R)$ for $i = j$. By j-connective we mean that $\pi_i(K^X(R)) = 0$ for $k < j$. Furthermore we let

$$K_X(R) = \operatorname{hocofib}\left(K^X(R) \to K(R)\right).$$

Definition 2.2.16. We define a spectrum $L^X(R)$ as the homotopy pullback of the diagram

$$\begin{array}{ccc} L^X(R) & \longrightarrow & L^{\langle -\infty \rangle}(R) \\ \downarrow & & \downarrow \\ K^X(R)^{t\Sigma_2} & \longrightarrow & K(R)^{t\Sigma_2} \end{array}$$

Remark. If $X = K_0(R)$ this is known as *projective* L-theory and for

$$X = \overline{K}_0(R) \overset{\text{Def}}{=} \ker\Big(K_0(R) \to \widetilde{K}_0(R)\Big)$$

this yields *free* L-theory.

Since the Tate construction is exact we obtain the following

Corollary 2.2.17. *We have a cofiber sequence*

$$L^X(R) \longrightarrow L^{\langle -\infty \rangle}(R) \longrightarrow K_X(R)^{t\Sigma_2}$$

In particular the induced morphism

$$L^X(R)[\tfrac{1}{2}] \longrightarrow L^{\langle -\infty \rangle}(R)[\tfrac{1}{2}]$$

is an equivalence.

Proof. The first part follows directly from Definition 2.2.16 because the Tate construction is an exact functor. For the second statement we recall that given a spectrum E with Σ_2 action on it, there is a spectral sequence converging to the homotopy groups of the Tate construction whose E_2-term looks like

$$E_2^{*,*} = \hat{H}^*(\Sigma_2; \pi_*(E)) \Longrightarrow \pi_*(E^{t\Sigma_2})$$

In particular the spectral sequence implies that $E^{t\Sigma_2}[\frac{1}{2}]$ is contractible because Tate cohomology of Σ_2 is always 2-torsion. This in turn follows from the fact that for every finite group G the composition

$$H^*(G;\mathbb{Z}) \xrightarrow{\operatorname{res}^G_{\{e\}}} H^*(\{e\};\mathbb{Z}) \xrightarrow{\operatorname{tr}^G_{\{e\}}} H^*(G;\mathbb{Z})$$

is given by multiplication with $|G|$ and that $H^*(G; M)$ is a module over $H^*(G;\mathbb{Z})$ for any $\mathbb{Z}G$-module M.

□

Recalling Definition 2.2.16 we can consider a further Σ_2-invariant subgroup $Y \subset X \subset K_0(R)$ and consider the diagram

$$\begin{array}{ccccc} L^Y(R) & \longrightarrow & L^X(R) & \longrightarrow & L^{\langle -\infty \rangle}(R) \\ \downarrow & & \downarrow & & \downarrow \\ K^Y(R)^{t\Sigma_2} & \longrightarrow & K^X(R)^{t\Sigma_2} & \longrightarrow & K(R)^{t\Sigma_2} \end{array}$$

where both the right and the large square are homotopy cartesian. It thus follows that also the left square is homotopy cartesian.

Now it is immediate form the definitions that

$$\operatorname{hocofib}\left(K^Y(R) \to K^X(R)\right)$$

is an Eilenberg-MacLane spectrum concentrated in degree 0 and with

$$\pi_0\left(\operatorname{hocofib}\left(K^Y(R) \to K^X(R)\right)\right) \cong X/Y$$

Since the Tate construction is an exact functor we therefore obtain that there is a cofiber sequence

$$L^Y(R) \longrightarrow L^X(R) \longrightarrow H(X/Y)^{t\Sigma_2}$$

Again we recall the spectral sequence converging to the homotopy groups of the Tate construction whose E_2-term looks like

$$E_2^{*,*} = \hat{H}^*(\Sigma_2; \pi_*(E)) \Longrightarrow \pi_*(E^{t\Sigma_2})$$

In our case, where $E = H(X/Y)$ is an Eilenberg-MacLane spectrum, the spectral sequence collapes at the E_2-term since it is concentrated in one line and gives an isomorphism

$$\pi_*(H(X/Y)^{t\Sigma_2}) \cong \hat{H}^*(\Sigma_2; X/Y).$$

We have thus proved the following

Proposition 2.2.18. *Associated to the cofiber sequence*

$$L^Y(R) \longrightarrow L^X(R) \longrightarrow H(X/Y)^{t\Sigma_2}$$

there is a long exact Rothenberg sequence*:*

$$\cdots \longrightarrow \hat{H}^{n+1}(\Sigma_2; X/Y) \longrightarrow L_n^Y(R) \longrightarrow L_n^X(R) \longrightarrow \hat{H}^n(\Sigma_2; X/Y) \longrightarrow \cdots$$

We want to recall what is known about excision properties of L-theory, see [Ran81, §6].

Definition 2.2.19. Consider a pullback diagram of (unital) involutive rings

$$\begin{array}{ccc} R & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ S_2 & \longrightarrow & T \end{array}$$

in which the morphism $S_1 \to T$ is surjective. We call a Σ_2-invariant subdiagram

$$\begin{array}{ccc} X & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Z \end{array} \quad \subset \quad \begin{array}{ccc} K_0(R) & \longrightarrow & K_0(S_1) \\ \downarrow & & \downarrow \\ K_0(S_2) & \longrightarrow & K_0(T) \end{array}$$

K-admissible if for

$$I_m = \ker\left(K_0(R) \to K_0(S_1) \oplus K_0(S_2)\right)$$

we have the following properties:

(1) $I_m \subset X$, and

(2) the sequence

$$0 \longrightarrow X/I_m \longrightarrow Y_1 \oplus Y_2 \longrightarrow Z \longrightarrow 0$$

is exact.

Remark. The condition of being K-admissible is called *cartesian* in [Ran81, §6]. But notice that the square

$$\begin{array}{ccc} I_m & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

is K-admissible but not in general a cartesian square in abelian groups in the ordinary sense. That is why we chose to give this a different name in order to avoid confusion. The name K-admissible is justified by the following

Proposition 2.2.20. *Given a K-admissible diagram as in Definition 2.2.19 the diagram*

$$\begin{array}{ccc} K_X(R) & \longrightarrow & K_{Y_1}(S_1) \\ \downarrow & & \downarrow \\ K_{Y_2}(S_2) & \longrightarrow & K_Z(T) \end{array}$$

is homotopy cartesian.

Proof. We need to show that the associated Mayer-Vietoris sequence is exact. For this we first calculate the homotopy groups of $K_X(R)$ using the cofiber sequence from Definition 2.2.15. We get

$$\pi_*(K_X(R)) \cong \begin{cases} 0 & \text{if } * > 0, \\ K_0(R)/X & \text{if } * = 0, \text{ and} \\ K_*(R) & \text{if } * < 0 \end{cases}$$

By [Mil71] algebraic K-theory admits a Mayer-Vietoris sequence in low dimensional homotopy groups. More precisely the sequence

$$K_1(T) \longrightarrow K_0(R) \longrightarrow K_0(S_1) \oplus K_0(S_2) \longrightarrow K_0(T) \longrightarrow K_{-1}(R) \longrightarrow \dots \tag{2.1}$$

is exact. Strictly speaking, in [Mil71, Theorem 3.3] it is proven that the sequence

$$K_1(R) \longrightarrow K_1(S_1) \oplus K_1(S_2) \longrightarrow K_1(T) \longrightarrow K_0(R) \longrightarrow K_0(S_1) \oplus K_0(S_2) \longrightarrow K_0(T)$$

is exact. Using the definition of negative K-groups due to Bass it follows immediately that one can extend the sequence to the right by the negative groups, see e.g. [Wei13, III Theorem 4.3]. Now given a K-admissible diagram

$$\begin{array}{ccc} X & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & Z \end{array}$$

we need to calculate that this implies that also the sequence

$$0 \longrightarrow \frac{K_0(R)}{X} \longrightarrow \frac{K_0(S_1)}{Y_1} \oplus \frac{K_0(S_2)}{Y_2} \longrightarrow \frac{K_0(T)}{Z} \longrightarrow K_{-1}(R) \longrightarrow \dots \tag{2.2}$$

is also exact.

So we consider the following diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X/I_m & \longrightarrow & Y_1 \oplus Y_2 & \longrightarrow & Z & \longrightarrow & 0 & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0(R)/I_m & \longrightarrow & K_0(S_1) \oplus K_0(S_2) & \longrightarrow & K_0(T) & \longrightarrow & K_{-1}(R) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K_0(R)}{X} & \longrightarrow & \frac{K_0(S_1)}{Y_1} \oplus \frac{K_0(S_2)}{Y_2} & \longrightarrow & \frac{K_0(T)}{Z} & \longrightarrow & K_{-1}(R) & \longrightarrow & \dots
\end{array}$$

This is obviously a short exact sequence of chain complexes. K-admissibility implies that the top chain complex is exact. The definition of I_m together with exactness of (2.1) says that the middle chain complex is also exact.

The long exact sequence in homology now implies that also the quotient chain complex is exact which shows the proposition. □

Theorem 2.2.21. *In the above situation we have that the diagram*

$$\begin{array}{ccc}
L^X(R) & \longrightarrow & L^{Y_1}(S_1) \\
\downarrow & & \downarrow \\
L^{Y_2}(S_2) & \longrightarrow & L^Z(T)
\end{array}$$

is a homotopy pullback diagram of spectra provided that 2 is invertible in all rings.

Proof. In [Ran81, 6.3.1] using [Ran81, 6.1.3] it is proven that the canonical map between the relative L-groups is an isomorphism. The relative L-groups are also the homotopy groups of a spectrum, which is equivalent to the homotopy fiber of the induced map on L-spectra. □

Remark. The condition that 2 is invertible implies that quadratic and symmetric L-spectra coincide. Ranicki proves this theorem in the quadratic context, in the symmetric case there is in general only a portion of a long exact Mayer-Vietoris sequence.

Corollary 2.2.22. *The diagram*

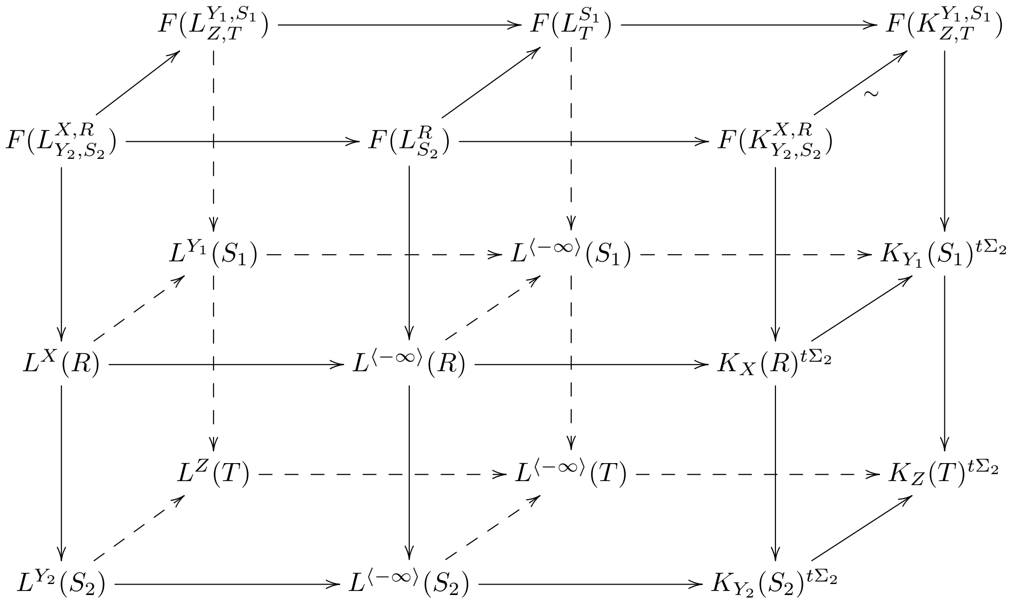
$$\begin{array}{ccc}
L^{\langle -\infty \rangle}(R) & \longrightarrow & L^{\langle -\infty \rangle}(S_1) \\
\downarrow & & \downarrow \\
L^{\langle -\infty \rangle}(S_2) & \longrightarrow & L^{\langle -\infty \rangle}(T)
\end{array}$$

is homotopy cartesian.

Proof. Recall from Corollary 2.2.17 that there is a cofiber sequence

$$L^X(R) \longrightarrow L^{\langle -\infty \rangle}(R) \longrightarrow K_X(R)^{t\Sigma_2} .$$

We consider the following diagram in which all horizontal sequences are cofiber sequences.



Since by Proposition 2.2.20 the top right map is an equivalence, it follows that the top left flat square of homotopy fibers is cartesian. Thus the left most map is an equivalence if and only the middle map is an equivalence. This implies that the square of L-spectra with control in subgroups is cartesian if and only if the square of L-spectra with decoration $\langle -\infty \rangle$ is cartesian. Thus we are done by Theorem 2.2.21. $\square$

Remark. We think that Theorem 2.2.21 should be thought of from this perspective. The functor $R \mapsto L^{\langle -\infty \rangle}(R)$ is excisive and one should be able to prove that without referring to Theorem 2.2.21. The fact that Proposition 2.2.20 is true then implies Theorem 2.2.21.

Remark. This should imply that in a suitable sense (depending on the homotopy theory we intend to study on rings) the map $L \to L^{\langle -\infty \rangle}$ is the *right-excisive* approximation in the sense of Goodwillie.

We will use this Theorem 2.2.21 several times. One immediate consequence of the theorem is the following

Proposition 2.2.23. *Suppose k is a unital ring and R is a (possibly) non-unital k-algebra. We denote by $R^+_{\mathbb{Z}}$ the unitalization in involutive rings and by R^+_k the unitalization in involutive k-algebras. Then the natural map*

$$\mathrm{hofib}\left(L(R^+_{\mathbb{Z}}) \to L\mathbb{Z}\right) \longrightarrow \mathrm{hofib}\left(L(R^+_k) \to Lk\right)$$

is an equivalence.

Proof. We consider the diagram

$$\begin{array}{ccc} R^+_{\mathbb{Z}} & \longrightarrow & R^+_k \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & k \end{array}$$

which is a pullback diagram in which the right vertical map is surjective. In [Mil71] it is shown that for such diagrams there is a Mayer-Vietoris sequence in algebraic K-groups in degrees ≤ 1. In particular, and using that the map $R^+_k \to k$ is split surjective, we have an exact sequence

$$0 \longrightarrow K_0(R^+_{\mathbb{Z}}) \longrightarrow K_0(R^+_k) \oplus K_0(\mathbb{Z}) \longrightarrow K_0(k) \longrightarrow 0$$

This means that the square

$$\begin{array}{ccc} K_0(R^+_{\mathbb{Z}}) & \longrightarrow & K_0(R^+_k) \\ \downarrow & & \downarrow \\ K_0(\mathbb{Z}) & \longrightarrow & K_0(k) \end{array}$$

is K-admissible (with $I_m = 0$ in this case) and hence we obtain a pullback diagram of projective L-theory spectra

$$\begin{array}{ccc} L(R^+_{\mathbb{Z}}) & \longrightarrow & L(R^+_k) \\ \downarrow & & \downarrow \\ L\mathbb{Z} & \longrightarrow & Lk \end{array} .$$

Thus the vertical homotopy fibers of this diagram are equivalent which proves the proposition. □

Remark. The diagram

$$\begin{array}{ccc} \overline{K}_0(R^+_{\mathbb{Z}}) & \longrightarrow & \overline{K}_0(R^+_k) \\ \downarrow & & \downarrow \\ \overline{K}_0(\mathbb{Z}) & \longrightarrow & \overline{K}_0(k) \end{array}$$

is also K-admissible. Thus also free L-theory satisfies the conclusion of Proposition 2.2.23.

More generally we obtain the following

Corollary 2.2.24. *The functor L-theory is split exact, i.e. given a split short exact sequence of involutive rings*

$$0 \longrightarrow J \xrightarrow{j} S \underset{p}{\overset{s}{\rightleftarrows}} T \longrightarrow 0$$

then we have

$$LS \simeq LJ \oplus LT.$$

Proof. We show that the sequence

$$LJ \longrightarrow LS \longrightarrow LT$$

is a fibration sequence in spectra. The existence of the multiplicative split then implies the claim. So we consider the diagram

$$\begin{array}{ccc} J^+ & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & T \end{array}$$

Again using the Mayer-Vietoris sequence in low algebraic K-groups from [Mil71] we obtain that the sequence

$$0 \longrightarrow K_0(J^+) \longrightarrow K_0(S) \oplus K_0(\mathbb{Z}) \longrightarrow K_0(T) \longrightarrow 0$$

is (split) exact. In particular in this case $I_m = 0$ and thus the square

$$\begin{array}{ccc} K_0(J^+) & \longrightarrow & K_0(S) \\ \downarrow & & \downarrow \\ K_0(\mathbb{Z}) & \longrightarrow & K_0(T) \end{array}$$

is again K-admissible. Thus the claim follows from Theorem 2.2.21 which says that the square

$$\begin{array}{ccc} L(J^+) & \longrightarrow & LS \\ \downarrow & & \downarrow \\ L\mathbb{Z} & \longrightarrow & LT \end{array}$$

is cartesian. □

Remark. It follows trivially that projective L-theory commutes with finite products.

2.2.3 Applications to C^*-algebras

Algebraic L-theory for C^*-algebras is defined by the composite

$$\mathrm{C^*Alg}^{\mathrm{unit}} \longrightarrow \mathrm{Ring}^{\mathrm{inv}} \xrightarrow{L} \mathrm{Ab}_{\mathbb{Z}} .$$

This does not define algebraic L-theory for *non-unital* C^*-algebras, which is what we will do next. Recall that the forgetful functor

$$\mathrm{C^*Alg}^{\mathrm{unit}} \rightleftarrows \mathrm{C^*Alg}$$

has a left adjoint, the unitalization $A \mapsto A^+$.

Definition 2.2.25. For every $A \in \mathrm{C^*Alg}$ and $n \in \mathbb{Z}$ we define

$$L^n(A) = \ker\left(L^n(A^+) \to L^n(\mathbb{C})\right).$$

Remark. Since the unitalization is a functor the above definition extends to morphisms and makes L-theory a functor on $\mathrm{C^*Alg}$.

Remark. If A is unital then $A^+ \cong A \times \mathbb{C}$. Since L-theory commutes with products, we have not changed the definition of L-groups on $\mathrm{C^*Alg}^{\mathrm{unit}}$ up to canonical isomorphism.

Remark. It follows that also for non-unital C^*-algebras J, the symmetrization map $L_*(J) \to L^*(J)$ is an isomorphism.

We can define L-theory spectra similary:

Definition 2.2.26. Let $A \in \mathrm{C^*Alg}$. We define its L-theory *spectrum* by the formula

$$LA = \mathrm{hofib}\left(L(A^+) \to L\mathbb{C}\right).$$

Proposition 2.2.27. *This is compatible with Definition 2.2.25 in the sense that $\pi_n(LA) \cong L^n(A)$ for all $n \in \mathbb{Z}$. Moreover we have that $L(A^+) \simeq LA \vee L\mathbb{C}$. Furthermore if A was unital then we have not changed the definition of the L-spectrum up to canonical equivalence.*

Proof. This follows immediately from the long exact sequence of homotopy groups and the existence of the split $\mathbb{C} \to A^+$. □

Remark. We want to notice that we have made the choice to unitalize the algebra A within the category of C^*-algebras. The intention of this is to make comparisons to K-theory where the unitalization has to be made within C^*-algebras. Nevertheless since L-theory only depends on the underlying involutive ring a natural way to define L-theory of non-unital such rings would be to unitalize within the category of involutive rings. Recall from Proposition 2.2.23 that this does not make a difference.

Remark. Proposition 2.2.23 also implies that L-theory is split exact as a functor on C^*-algebras, because the unitalization in C^*-algebras is (as an involutive ring) the same as the unitalization in $\mathbb{C}$-algebras.

Next we want to describe how to calculate $L_0(A)$ for $A \in \mathrm{C^*Alg}^{\mathrm{unit}}$. It will be crucial for later purposes to also calculate free L-theory and understand the Rothenberg sequence for the change from projective to free L-theory.

We begin with the following observation. Let A be a unital C^*-algebra and let P be a finitely generated projective module over A. It is a result of Karoubi that any such module has a (up to homotopy unique) non-degenerate positive definite sesquilinear form σ over it (it comes from the fact that P becomes a Hilbert-A-module through any embedding $P \subset A^n$), see [Kar80, Lemme 2.9].

Since K-theory is the group completion of the monoid of isomorphism classes of finitely generated projective module the construction of above induces a natural map

$$\begin{aligned} K_0(A) &\longrightarrow L^0(A) \\ [P] &\longmapsto [P,\sigma] \end{aligned}$$

Proposition 2.2.28. *The natural map $K_0(A) \to L^0(A)$ is an isomorphism for all unital C^*-algebras A.*

Proof. We use the standard form σ to identify $P^* \cong P$. Making use of the spectral theorem we see that if $\varphi\colon P \to P^*$ is a non-degenerate hermitian form it follows that the composite $P \xrightarrow{\varphi} P^* \xrightarrow{\sigma} P$ is a self-adjoint invertible in the C^*-algebra $\mathcal{L}(P)$ of adjointable operators of P. In particular its spectrum is contained in $\mathbb{R} \setminus \{0\}$. Taking the spectral projections decomposes P into a direct sum $P = P^+ \oplus P^-$ such that φ becomes positive definite on P^+ and negative definite on P^-. Using again [Kar80, Lemme 2.9] it follows that this defines an inverse map to the map of the lemma. □

Remark. It follows that K_0 and L^0 are naturally isomorphic for all (possibly non-unital) C^*-algebras.

Next we want to argue how to use this method to also give a description of $L_0^{\langle h\rangle}(A)$ for $A \in \mathrm{C^*Alg}^{\mathrm{unit}}$. For this we consider the following group. We let

$$M(A) = \ker\Big(K_0(A) \times K_0(A) \xrightarrow{p\circ\oplus} \widetilde{K}_0(A)\Big).$$

There is an evident diagonal map

$$\overline{K}_0(A) \xrightarrow{\Delta} M(A)$$

and so we define a group

$$Q(A) = \mathrm{coker}\left(\overline{K}_0(A) \to M(A)\right)$$

This fits in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K}_0(A) & \longrightarrow & K_0(A) & \xrightarrow{p} & \widetilde{K}_0(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \cdot 2} & & \\
0 & \longrightarrow & M(A) & \longrightarrow & K_0(A) \times K_0(A) & \xrightarrow[p\circ\oplus]{} & \widetilde{K}_0(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \ominus} & & & & \\
 & & Q(A) & \xrightarrow{\vartheta} & K_0(A) & & & &
\end{array}$$

where $\ominus\colon K_0(A) \times K_0(A) \to K_0(A)$ is given by $(\alpha, \beta) \mapsto \alpha - \beta$. We describe a map

$$Q(A) \to L_0^{\langle h\rangle}(A)$$

as follows. Let P and Q be projective modules over A such that $P \oplus Q \in \overline{K}_0(A)$, i.e. such that $([P],[Q]) \in M(A)$. Then the association

$$([P],[Q]) \mapsto (P,\sigma_P) \oplus (Q,-\sigma_Q) \in L_0^{\langle h \rangle}(A).$$

induces a well-defined map

$$Q(A) \to L_0^{\langle h \rangle}(A).$$

Proposition 2.2.29. *This map is an isomorphism and makes the diagram*

$$\begin{array}{ccc} Q(A) & \xrightarrow{\vartheta} & K_0(A) \\ \cong\downarrow & & \downarrow\cong \\ L_0^{\langle h \rangle}(A) & \longrightarrow & L_0(A) \end{array}$$

commute.

Proof. The commutativity of the square is clear from the description of all involved maps. Thus it remains to prove that $\vartheta\colon Q(A) \to L_0^{\langle h \rangle}(A)$ is a bijection. For this we observe that given an element in $L_0^{\langle h \rangle}(A)$ represented by a unimodular hermitian form (F,φ) where F is a finitely generated free module, then the same argument as in Proposition 2.2.28 shows that we can decompose F into a sum $(F,\varphi) \cong (P,\sigma_P) \oplus (Q,-\sigma_Q)$. The map

$$\begin{array}{ccc} L_0^{\langle h \rangle}(A) & \longrightarrow & Q(A) \\ (F,\varphi) & \longmapsto & \big[[P],[Q]\big] \end{array}$$

is well-defined and is an inverse to ϑ. □

Next we want to calculate the difference between free and projective L-theory for complex C^*-algebras.

Proposition 2.2.30. *For a unital C^*-algebra A we have an exact sequence*

$$0 \longrightarrow \hat{H}^{2k+1}(\Sigma_2;\widetilde{K}_0(A)) \longrightarrow \pi_{2k}(L^{\langle h \rangle}(A)) \longrightarrow \pi_{2k}(LA) \longrightarrow \hat{H}^{2k}(\Sigma_2;\widetilde{K}_0(A)) \longrightarrow 0$$

Furthermore we have an isomorphism

$$\pi_{2k+1}(L^{\langle h \rangle}(A)) \xrightarrow{\cong} \pi_{2k+1}(LA).$$

Proof. We follow the argument of [Ros95]. Recall from Definition 2.2.16 that $L^{\langle h \rangle}(A)$ was denoted $L^{\overline{K}_0(A)}(A)$. Clearly we have that

$$K_0(A)/\overline{K}_0(A) \cong \widetilde{K}_0(A).$$

So from the fibration sequence (2.2.18)

$$L^{\langle h \rangle}(A) \longrightarrow LA \longrightarrow H(\widetilde{K}_0(A))^{t\Sigma_2}$$

we obtain a corresponding long exact sequence which reads as

$$\cdots \longrightarrow \hat{H}^{n+1}(\Sigma_2;\widetilde{K}_0(A)) \longrightarrow L_n^{\langle h \rangle}(A) \longrightarrow L_n(A) \longrightarrow \hat{H}^{n}(\Sigma_2;\widetilde{K}_0(A)) \longrightarrow \cdots$$

We need to argue that this sequence behaves the way we describe in the statement of the proposition. For this we first observe that since L-theory is 2-periodic the Rothenberg sequence is

6-periodic. Moreover, since every element in $K_0(A)$ can be represented by *self-adjoint* projections it follows that the action of Σ_2 in $\widetilde{K}_0(A)$ is trivial and thus we have identifications of the Tate cohomology groups

$$\hat{H}^{2k+1}(\Sigma_2; \widetilde{K}_0(A)) \cong \{x \in \widetilde{K}_0(A) \mid 2x = 0\}$$

and

$$\hat{H}^{2k}(\Sigma_2; \widetilde{K}_0(A)) \cong \frac{\widetilde{K}_0(A)}{2\widetilde{K}_0(A)}.$$

Next we investigate the map

$$L_0^{\langle h \rangle}(A) \longrightarrow L_0(A).$$

We consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & \hat{H}^1(\Sigma_2; \widetilde{K}_0(A)) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K}_0(A) & \longrightarrow & K_0(A) & \xrightarrow{p} & \widetilde{K}_0(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \cdot 2} & & \\
0 & \longrightarrow & M(A) & \longrightarrow & K_0(A) \times K_0(A) & \xrightarrow[p \circ \oplus]{} & \widetilde{K}_0(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Q(A) & \longrightarrow & K_0(A) & \longrightarrow & \hat{H}^0(\Sigma_2; \widetilde{K}_0(A) & \longrightarrow & 0
\end{array}$$

In Proposition 2.2.29 we will argue that the diagram

$$\begin{array}{ccc}
Q(A) & \longrightarrow & K_0(A) \\
\cong \downarrow & & \downarrow \cong \\
L_0^{\langle h \rangle}(A) & \longrightarrow & L_0(A)
\end{array}$$

commutes.

From the snake lemma for the above big diagram we thus obtain an exact sequence

$$0 \longrightarrow \hat{H}^1(\Sigma_2; \widetilde{K}_0(A)) \longrightarrow L_0^{\langle h \rangle}(A) \longrightarrow L_0(A) \longrightarrow \hat{H}^0(\Sigma_2; \widetilde{K}_0(A)) \longrightarrow 0$$

Since this is part of a the long exact Rothenberg sequence of Proposition 2.2.18 it automatically follows that we have the described isomorphism in odd degrees. □

The following theorem was the starting point for this thesis.

Theorem 2.2.31. *There is a natural isomorphism of functors*

$$\mathrm{C^*Alg} \underset{L}{\overset{K}{\rightrightarrows}} \mathrm{Ab}_{\mathbb{Z}} \quad (\tau \Downarrow)$$

from topological K-theory to projective algebraic L-theory on the category of separable C^*-algebras.

Proof. We need to prove that for any $A \in \mathrm{C^*Alg}^{\mathrm{unit}}$ and $n \in \mathbb{Z}$ there is a natural isomorphism

$$K_n(A) \cong L_n(A).$$

Recall that Proposition 2.2.28 provides such an isomorphism for $n = 0$. By Proposition 2.2.7 part (2) we know that both functors are naturally 2-periodic. Thus if it is true that $K_1(A)$ and $L^1(A)$ are naturally isomorphic we deduce the theorem by using the periodicities.

In [Ros95] and [Mil98] it is proven that indeed $K_1(A)$ and $L_1(A)$ are naturally isomorphic. This uses the fact that for a complex C^*-algebra A the comparison map $L_1^{\langle h \rangle}(A) \to L_1(A)$ is an isomorphism, see Proposition 2.2.30. □

Corollary 2.2.32. *L-theory is KK-invariant, i.e. we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{C^*Alg} & \xrightarrow{\ L\ } & \mathrm{Ab}_{\mathbb{Z}} \\ \downarrow & \nearrow_{L} & \\ \mathrm{KK} & & \end{array} .$$

Proof. This is obvious from Theorem 2.2.31 because K-theory factors over KK. □

Remark. Recall from Theorem 2.1.25 that any C^*-stable and split exact functor $F\colon \mathrm{C^*Alg} \to \mathrm{Ab}$ factors over KK. In Corollary 2.2.24 we established that L-theory is split exact. So in order to deduce Corollary 2.2.32 it would suffice to find any argument that L-theory is C^*-stable. We notice that L-theory is matrix-stable, or Morita-invariant in the sense that for all $n \geq 1$ there is an isomorphism

$$L_*(R) \cong L_*(M_n(R))$$

and that L-theory commutes with filtered colimits. Furthermore we have that

$$\operatorname{colim} M_n(A) \subset A \otimes \mathcal{K}$$

is dense and so the question of C^*-stability can be reduced to a question about *continuity* of L-theory, in the sense that if we have a C^*-algebra A and a sequence $A_i \subset A$ of sub-C^*-algebras with inclusions $A_{i-1} \subset A_i$ such that the union of the A_i is dense in A, then we can ask whether the natural map

$$\operatorname{colim} L_*(A_i) \to L_*(A)$$

is an isomorphism. Notice the similarity to the completion conjecture in the introduction. Of course the fact that L-theory is isomorphic to K-theory implies that this is true. But we are interested in this for the following reason.

In all what comes next, the essential point is that L-theory factors over KK. For example, in Corollary 2.2.37 using only the fact that L-theory factors over KK we give a proof that there is a natural isomorphism $L_1(A) \cong K_1(A)$.

Another application would be the question about L-theory of R^*-algebras, by which we mean C^*-algebras over the real numbers. It is well known that it is not true anymore that $L_*(A) \cong KO_*(A)$ for R^*-algebras A, the easiest example being

$$KO_1(\mathbb{R}) \cong \mathbb{Z}/2 \text{ but } L_1(\mathbb{R}) = 0.$$

In particular at the moment we cannot deduce that L-theory viewed as a functor $\mathrm{R^*Alg} \to \mathrm{Ab}$ factors over the real KK-category. But maybe an independent (or L-theoretic) proof of the fact that L-theory of C^*-algebras is C^*-stable could carry over to R^*-algebras, providing the desired factorization.

It will turn out to be crucial to understand in which sense the spectrum LA depends on the way we embedd A into a unital algebra (as an ideal). In particular this will be important for the algebra SA, the C^*-algebraic suspension of A. We will need the following preliminary

Lemma 2.2.33. *For any C^*-algebra A we have that the unit inclusion map induces a weak equivalence*

$$L\mathbb{C} \to L(CA^+).$$

This is true for both the projective and the free version of L-theory.

Proof. First we observe that the unitalization functor preserves homotopy equivalences. In particular it follows that the unit inclusion map $\mathbb{C} \to CA^+$ is a KK-equivalence. From Corollary 2.2.32 we see that the claim of the lemma holds true for the projective L-theory. Using the Rothenberg sequence and the 5-lemma it now follows easily that also the induced map on free L-theory is an equivalence.

Notice that since $\mathbb{C}$ is a field, there is no difference between the free and the projective version. □

Theorem 2.2.34. *Let A be a C^*-algebra. Then the following diagram is a homotopy pullback diagram in spectra.*

$$\begin{array}{ccc} L(SA^+) & \longrightarrow & L\mathbb{C} \\ \downarrow & & \downarrow \\ L\mathbb{C} & \longrightarrow & L^{\langle h \rangle}(A^+) \end{array}$$

The maps $L\mathbb{C} \to L^{\langle h \rangle}(A^+)$ are both induced by the unit inclusion $\mathbb{C} \to A^+$. In particular there is a homotopy fibration sequence

$$L\mathbb{C} \longrightarrow L^{\langle h \rangle}(A^+) \longrightarrow \Sigma L(SA).$$

Proof. We apply the previous theorem to the following situation We consider the short exact sequence of C^*-algebras

$$0 \longrightarrow SA \longrightarrow CA^+ \longrightarrow A^+ \longrightarrow 0\ .$$

This fits into a pullback diagram

$$\begin{array}{ccc} SA^+ & \longrightarrow & CA^+ \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & A^+ \end{array}$$

Using that the unit inclusion induces a weak equivalence $L\mathbb{C} \to L(CA^+)$ it follows that the composite

$$L\mathbb{C} \to L(CA^+) \to L^{\langle h \rangle}(A^+)$$

is also given by the unit inclusion.

The square

$$\begin{array}{ccc} K_0(SA^+) & \longrightarrow & K_0(\mathbb{C}) \\ \downarrow & & \downarrow \\ K_0(\mathbb{C}) & \longrightarrow & \overline{K}_0(A^+) \end{array}$$

is K-admissible. Thus we obtain the pullback diagram as claimed. □

Corollary 2.2.35. *We have that*

$$L^{\langle h \rangle}(A^+) \simeq L\mathbb{C} \oplus \Sigma L(SA).$$

Proof. We consider the homotopy fiber sequence

$$L\mathbb{C} \longrightarrow L^{\langle h \rangle}(A^+) \longrightarrow \Sigma L(SA)$$

and observe that the induced map of the canonical projection map $A^+ \to \mathbb{C}$ splits the fibration. □

Consider the commutative diagram

$$\begin{array}{ccc} SA & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A \end{array} .$$

If we apply L-theory to this diagram we obtain a commutative diagram and hence obtain a map (from the universal property of pullbacks which are pushouts)

$$\Theta_A \colon \Sigma L(SA) \to LA.$$

Theorem 2.2.36. *The following diagram is a homotopy pullback diagram of spectra*

$$\begin{array}{ccc} L^{\langle h \rangle}(A^+) & \longrightarrow & \Sigma L(SA) \\ \downarrow & & \downarrow \Theta_A \\ L(A^+) & \longrightarrow & LA \end{array}$$

Proof. We first consider the commutative diagram

$$\begin{array}{ccc} L\mathbb{C} & \longrightarrow & L^{\langle h \rangle}(A^+) \\ \| & & \downarrow \\ L\mathbb{C} & \longrightarrow & L(A^+) \end{array}$$

that comes from mapping free to projective L-theory. Inserting the horizontal homotopy cofibers gives the diagram

$$\begin{array}{ccccc} L\mathbb{C} & \longrightarrow & L^{\langle h \rangle}(A^+) & \longrightarrow & \Sigma L(SA) \\ \| & & \downarrow & & \downarrow \Theta_A \\ L\mathbb{C} & \longrightarrow & L(A^+) & \longrightarrow & LA \end{array}$$

where the upper horizontal homotopy cofiber is computed by Theorem 2.2.34 and the lower homotopy cofiber again comes from the fact that the fibration

$$LA \to LA^+ \to L\mathbb{C}$$

is split by the unit inclusion $\mathbb{C} \to A^+$, see Proposition 2.2.27. To compute that the map induced on cofibers is given by Θ_A one observes that one can also consider the diagram

$$\begin{array}{ccc} SA^+ & \longrightarrow & CA^+ \simeq \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & A^+ \end{array} .$$

which is the old diagram unitalized. Applying L-theory to it gives a diagram

$$\begin{array}{ccc} L(SA) \oplus L\mathbb{C} & \longrightarrow & L\mathbb{C} \\ \downarrow & & \downarrow \\ L\mathbb{C} & \longrightarrow & LA \oplus L\mathbb{C} \end{array}$$

Since pullbacks commute with products (any two limits commute) the universal property provides a map

$$\Sigma L(SA) \oplus \Sigma L\mathbb{C} \xrightarrow{\Theta_A \oplus \mathrm{id}_{L\mathbb{C}}} LA \oplus L\mathbb{C}$$

We thus have proven that in the claimed diagram the horizontal homotopy fibers are equivalent and thus the square is cartesian. □

Corollary 2.2.37. *For every complex C^*-algebra A the map induced by Θ_A on π_1*

$$L_0(SA) \xrightarrow{\Theta_A} L_1(A)$$

is an isomorphism. In particular we have a chain of natural isomorphisms

$$L_1(A) \cong L_0(SA) \cong K_0(SA) \cong K_1(A)$$

between L-theory and K-theory in degree 1.

Proof. Theorem 2.2.36 says that the horizontal homotopy fibers in the diagram in Theorem 2.2.36 are equivalent. Using that the forgetful map from free to projective L-theory gives the Rothenberg sequence of Proposition 2.2.30 we obtain a commutative diagram

$$\begin{array}{ccccccc}
\hat{H}^2(\Sigma_2; K_0(A)) & \xrightarrow{0} & L_1^{\langle h \rangle}(A^+) & \xrightarrow{\cong} & L_1(A^+) & \xrightarrow{0} & \hat{H}^1(\Sigma_2; K_0(A)) \\
\| & & \downarrow & & \downarrow & & \| \\
\hat{H}^2(\Sigma_2; K_0(A)) & \longrightarrow & L_0(SA) & \longrightarrow & L_1(A) & \longrightarrow & \hat{H}^1(\Sigma_2; K_0(A))
\end{array}$$

Again from Proposition 2.2.30 we see that the top middle map is an isomorphism. In particular the outer top maps are the zero morphisms. It follows that the lower left map is zero as well. Using that the map $L_1(A^+) \to L_1(A)$ is surjective (because $L(A^+) \simeq LA \oplus L\mathbb{C}$) we see that also the lower right map is zero. Thus the Corollary follows. □

Next we want to investigate the boundary map also in the next degree:

Theorem 2.2.38. *There is an exact sequence*

$$0 \longrightarrow \hat{H}^1(C_2; K_0(A)) \longrightarrow L_1(SA) \longrightarrow L_2(A) \longrightarrow \hat{H}^0(C_2; K_0(A)) \longrightarrow 0$$

In particular for the algebra $A = \mathbb{C}$ we have a short exact sequence

$$0 \longrightarrow L_1(S\mathbb{C}) \longrightarrow L_2(\mathbb{C}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

Proof. Again from Theorem 2.2.36 and the Rothenberg sequence Proposition 2.2.30 we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
L_3(A^+) & \xrightarrow{0} & \hat{H}^3(\Sigma_2; K_0(A)) & \longrightarrow & L_2^{\langle h \rangle}(A^+) & \longrightarrow & L_2(A^+) & \longrightarrow & \hat{H}^2(\Sigma_2; K_0(A)) & \xrightarrow{0} & \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
L_3(A) & \longrightarrow & \hat{H}^3(\Sigma_2; K_0(A)) & \longrightarrow & L_1(SA) & \longrightarrow & L_2(A) & \longrightarrow & \hat{H}^2(\Sigma_2; K_0(A)) & \xrightarrow{0} &
\end{array}$$

Again it follows from the surjectivity of $L_3(A^+) \to L_3(A)$ that the lower left most map is zero. Thus the claim follows. □

Remark. It is worthwile to notice that for a C^*-algebra A, all elements in $K_0(A)$ can be represented by (formal differences) of *self-adjoint* idempotents. This implies that the action of C_2 on $K_0(A)$ is trivial. Thus we obtain that

$$\hat{H}^0(C_2; K_0(A)) \cong \frac{K_0(A)}{2K_0(A)}$$

and

$$\hat{H}^1(C_2; K_0(A)) \cong \{x \in K_0(A) \mid 2x = 0\} .$$

In particular under the isomorphism $L_1(SA) \cong L_0(A)$ we obtain that the map

$$L_0(A) \cong L_1(SA) \to L_2(A) \cong L_0(A)$$

is given by multiplication by ± 2.

3 ∞-categories

3.1 Preliminaries

We work in the setup of ∞-categories as developed by Joyal and Lurie, see [Lur09] and [Lur14]. We mainly introduce notation.

Definition 3.1.1. An ∞-category is a simplicial set $\mathcal{C}$ satisfying the left lifting property with respect to all *inner horn inclusions* i.e. for all $1 < i < n$ and any given diagram of solid arrows

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

a dotted arrow exists making the diagram commutative.

Definition 3.1.2. If $\mathcal{C}$ is an ∞-category, we denote its homotopy category by $h(\mathcal{C})$, see [Lur09, section 1.2.3].

Definition 3.1.3. Given an ∞-category $\mathcal{C}$ we define the largest Kan complex $\mathcal{C}^{\sim}$ inside $\mathcal{C}$ by the pullback

$$\begin{array}{ccc} \mathcal{C}^{\sim} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ h(\mathcal{C})^{\sim} & \longrightarrow & h(\mathcal{C}) \end{array}$$

where $h(\mathcal{C})^{\sim}$ is the subcategory of isomorphisms of $h(\mathcal{C})$. The ∞-category $\mathcal{C}^{\sim}$ is also called the *groupoid core* of $\mathcal{C}$.

Examples. (1) Let C be an ordinary category. Then the *nerve* NC of C is an ∞-category.

(2) More generally, if C is a simplicially enriched category then the *coherent nerve* is an ∞-category which we denote by $cN(C)$, see [Lur09, Definition 1.1.5.5]. This construction preserves the homotopy type of the mapping spaces if the simplicial mapping spaces are Kan complexes.

(3) The ∞-category $\mathcal{S}$ of spaces is the homotopy coherent nerve of the simplicial category of Kan-complexes.

(4) If $\mathcal{C}$ and $\mathcal{C}'$ are ∞-categories, then the internal mapping simplicial set is again an ∞-category and will be denoted by $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$, see [Lur09, Proposition 1.2.7.3].

(5) The ∞-category Cat_∞ is the homotopy coherent nerve of the simplicial category of ∞-categories, where we take as mapping simplicial sets the groupoid core of the usual mapping simplicial sets which are ∞-categories by (4).

Next we consider a construction within ∞-categories called a *Dwyer-Kan* localization.

Proposition 3.1.4. *Suppose $\mathcal{C}$ is an ∞-category and $\mathcal{W}$ is a subcategory closed under the 2 out of 3 property such that $\mathcal{W}$ contains all equivalences of $\mathcal{C}$. For our purposes it is approriate to call* *the morphisms of* $\mathcal{W}$ local equivalences. *Then there is an* ∞*-category* $\mathcal{C}[\mathcal{W}^{-1}]$ *and a functor of* ∞*-categories*

$$i\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

that is an ∞*-categorical* Dwyer-Kan localization *at the local equivalences, i.e. for every* ∞*-category* $\mathcal{D}$ *the functor*

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}],\mathcal{D}) \xrightarrow{i^*} \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

is fully-faithful and the image consists of those functors that send local equivalences in $\mathcal{C}$ *to equivalences in* $\mathcal{E}$*.*

Proof. This essentially goes back to [DK80]. An argument in the language of ∞-categories is given for instance in [Lur14, Def. 1.3.4.1 and Rmk 1.3.4.2]. One can define the localization to be a fibrant replacement of the object $(\mathcal{C},\mathcal{W})$ in the cartesian model structure on marked simplicial sets Set_Δ^+. This universal property characterizes the category $\mathcal{C}[\mathcal{W}^{-1}]$ up to equivalence. □

Definition 3.1.5. Suppose $(\mathcal{C}, W)$ is a relative category i.e. a category $\mathcal{C}$ with a collection of morphisms W containing the isomorphisms. Then we define the associated ∞-category denoted by M_∞ to be $N\mathcal{C}[W^{-1}]$. This applies for instance for fibration categories or model categories.

We are now in the situation that given a simplicial category with a class of weak equivalences there are a priori different ways of obtaining a localization. The following proposition states that all these construction agree if the simplicial category comes from a simplicial model category.

Proposition 3.1.6. *Let* M *be a simplicial model category and let* W *be the class of weak equivalences. We denote by* M_{cf} *the full subcategory of cofibrant and fibrant objects. Then we have equivalences*

$$cN(M)[W^{-1}] \simeq M_\infty \simeq cN(M_{cf})$$

in other words, also the functors $NM \to cN(M)[W^{-1}]$ *and* $NM \to cN(M_{cf})$ *exhibit the target category as Dwyer-Kan localizations along the weak equivalences of* M*.*

Proof. This is [Lur14, Example 1.3.4.8] and [Lur14, Theorem 1.3.4.20]. □

Proposition 3.1.7. *Suppose that* M *is a combinatorial simplicial model category. Then for any category* $\mathcal{C}$ *the category of functors* $\mathrm{Fun}(\mathcal{C}, M)$ *is again combinatorial and simplicial and its associated* ∞*-category is equivalent to the* ∞*-category of functors* $\mathrm{Fun}(N\mathcal{C}, M_\infty)$*, more precisely we have*

$$\mathrm{Fun}(\mathcal{C}, M)_\infty \simeq \mathrm{Fun}(N\mathcal{C}, M_\infty).$$

Proof. This is a special case of [Lur09, Proposition 4.2.4.4]. □

Since the category of spectra is a simplicial and combinatorial model category, combining Proposition 3.1.6 and Proposition 3.1.7 we obtain the following

Corollary 3.1.8. *We have an equivalence*

$$\mathrm{Fun}(N\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty) \simeq cN\left(\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp})_{cf}\right).$$

We will need the following properties of ∞-categories.

Definition 3.1.9. Let $\mathcal{C}$ be an ∞-category.

(1) $\mathcal{C}$ is called *pointed* if there exists an object $0 \in \mathcal{C}$ which is both terminal and initial.

(2) $\mathcal{C}$ is called *preadditive* if it is pointed and $\mathcal{C}$ admits both finite coproducts and finite products and for all objects $X, Y \in \mathcal{C}$ the canonical morphism $X \cup Y \to X \times Y$ is an equivalence. In that case we will denote any such object as $X \oplus Y$.

(3) $\mathcal{C}$ is called *additive* if it is preadditive and its homotopy category $h(\mathcal{C})$ is additive.

(4) $\mathcal{C}$ is called *stable* if it is pointed, every morphism has a fiber and a cofiber, and triangles are fiber sequences if and only if they are cofiber sequences.

Remark. For any ∞-category $\mathcal{C}$ with finite products one can construct universally a preadditive ∞-category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ of $\mathbb{E}_\infty$-monoids in $\mathcal{C}$. It comes with a natural evaluation map $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$. In [GGN15] it is shown that $\mathcal{C}$ is preadditive if and only if the evaluation map $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In particular if $\mathcal{C}$ is preadditive then every object $X \in \mathcal{C}$ gets the structure of an $\mathbb{E}_\infty$-monoid in $\mathcal{C}$ which induces a fold map $\nabla\colon X \oplus X \to X$. Using this we obtain the shear map

$$X \oplus X \xrightarrow{(\mathrm{pr}_1, \nabla)} X \oplus X$$

and it is shown in [GGN15] that $\mathcal{C}$ is additive if and only if this shear map is an equivalence for all objects $X \in \mathcal{C}$.

Proposition 3.1.10. *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if $\mathcal{C}$ admits all finite limits and the loop functor $\Omega\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.*

Proof. This is [Lur14, Corollary 1.4.2.27]. □

Lemma 3.1.11. *Let (C, W) be a fibration category. Then the associated ∞-category $\mathcal{C}$ admits all finite limits.*

Proof. Since (C, W) is a fibration category it follows directly that the associated ∞-category has a terminal object and admits all pullbacks, thus admits all finite limits, see [Cis10]. Also consult [Szu14] for a discussion of the relation between (co)fibration categories and ∞-categories. □

Lemma 3.1.12. *Let $\mathcal{C}$ be an ∞-category.*

(1) If $\mathcal{C}$ admits all finite limits then $\mathcal{C}$ is pointed if and only if its homotopy category $h\mathcal{C}$ is.

(2) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a limit-preserving functor between pointed ∞-categories that admit all finite limits. Then F is an equivalence if and only if $hF\colon h\mathcal{C} \to h\mathcal{D}$ is.

Proof. To show (1) let $* \in \mathcal{C}$ be a terminal object. We need to show that $\mathrm{Map}_{\mathcal{C}}(*, Z)$ is contractible for all objects $Z \in \mathcal{C}$. The condition that $h\mathcal{C}$ is pointed implies that $\pi_0(\mathrm{Map}_{\mathcal{C}}(*, Z)) = \{*\}$. The diagram

$$\begin{array}{ccc} \Omega Z & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

is a pullback in the ∞-category $\mathcal{C}$. Thus we obtain that

$$\mathrm{Map}_{\mathcal{C}}(*, \Omega Z) \simeq \Omega \mathrm{Map}_{\mathcal{C}}(*, Z)$$

and hence for all objects $Z \in \mathcal{C}$ we obtain

$$\pi_n \mathrm{Map}_{\mathcal{C}}(*, Z) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(*, \Omega^n Z) = \{*\}$$

so part (1) follows.

To see the second part we recall that a functor is an equivalence if it is essentially surjective and fully-faithful (meaning the induced map on mapping *spaces* is an equivalence). Essential surjectivity follows from the fact that hF is an equivalence. Now we consider the induced map

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(FX, FY).$$

which is a bijection on π_0 for all objects $X, Y \in \mathcal{C}$ by the assumption that hF is an equivalence. Using that F preserves limits shows that it also induces a bijection on π_n for all objects $x, y \in \mathcal{C}$ and all $n \geq 1$ by considering the following diagram.

$$\begin{array}{ccc}
\pi_n \mathrm{Map}_{\mathcal{C}}(X,Y) & \longrightarrow & \pi_n \mathrm{Map}_{\mathcal{D}}(FX,FY) \\
\downarrow & & \downarrow \\
\pi_0 \Omega^n \mathrm{Map}_{\mathcal{C}}(X,Y) & \longrightarrow & \pi_0 \Omega^n \mathrm{Map}_{\mathcal{D}}(FX,FY) \\
\downarrow & & \downarrow \\
\pi_0 \mathrm{Map}_{\mathcal{C}}(X,\Omega^n Y) & \longrightarrow & \pi_0 \mathrm{Map}_{\mathcal{D}}(FX,\Omega^n FY) \\
\downarrow & & \downarrow \\
\pi_0 \mathrm{Map}_{\mathcal{C}}(X,\Omega^n Y) & \longrightarrow & \pi_0 \mathrm{Map}_{\mathcal{D}}(FX,F\Omega^n Y)
\end{array}$$

□

The previous two lemmas allow to show that the ∞-category associated to a fibration category has certain properties provided its homotopy category has these properties (e.g. being additive or stable which is implied by being additive or being triangulated on the level of homotopy categories). We will use this in the next section when we consider an ∞-category of separable C^*-algebras.

Recall that we denote by $\mathcal{S}$ the ∞-category of spaces and let Cat_∞^Π be the ∞-category of ∞-categories with finite products and product preserving functors. In [GGN15] a *colocalization* is constructed

$$\mathrm{Grp}_{\mathbb{E}_\infty} : \mathrm{Cat}_\infty^\Pi \to \mathrm{Cat}_\infty^\Pi$$

where the colocal objects are precisely the *additive* ∞-categories. For two ∞-categories $\mathcal{C}, \mathcal{C}'$ that admit finite products we denote the functor category of *product preserving* functors by $\mathrm{Fun}^\Pi(\mathcal{C}, \mathcal{C}')$, and if $\mathcal{C}, \mathcal{C}'$ admit all finite limits we denote the functor category of limit-preserving functors by $\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{C}')$.

Lemma 3.1.13. *Let $\mathcal{C}$ be an ∞-category. Then the Yoneda embedding*

$$\begin{array}{ccc}
\mathcal{C}^{op} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \\
x & \longmapsto & \mathrm{Map}_{\mathcal{C}}(x, -)
\end{array}$$

is fully-faithful. Moreover, for each functor $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ and each object $x \in \mathcal{C}$ the evaluation is an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathrm{Map}_{\mathcal{C}}(x,-), F) \xrightarrow{\simeq} F(x)$$

of spaces.

Proof. Fully-faithfulness of the Yoneda embedding is proven in [Lur09, Proposition 5.1.3.1]. The moreover part is precisely [Lur09, Lemma 5.5.2.1]. □

Remark. If $\mathcal{C}$ has all (small) limits, then the Yoneda embedding factors over the limit preserving and additive functors

$$\begin{array}{ccc}
 & \nearrow & \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{S}) \\
 & & \downarrow \\
 & \nearrow & \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S}) \\
 & & \downarrow \\
\mathcal{C}^{op} & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{S})
\end{array}$$

Lemma 3.1.14. *Let $\mathcal{C}$ be an* additive *∞-category. Then the forgetful functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathcal{S}$ induces an equivalence of functor categories*

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})) \xrightarrow{\simeq} \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S}) .$$

Proof. This is a special case of [GGN15, Corollary 2.10, (iii)]. □

Corollary 3.1.15. *In particular for every object $x \in \mathcal{C}$ there is an essentially unique functor*

$$\begin{aligned} \mathcal{C}^{op} &\longrightarrow \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq \mathrm{Sp}_\infty^{\geq 0} \\ y &\longmapsto \mathrm{map}_{\mathcal{C}}(x, y) \end{aligned}$$

such that

(1) *the composite*

$$x \mapsto \big(\Omega^\infty(\mathrm{map}_{\mathcal{C}}(x, -)) \simeq \mathrm{Map}_{\mathcal{C}}(x, -)\big)$$

is the usual Yoneda embedding, and

(2) *the functor $\mathrm{map}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{Sp}_\infty^{\geq 0}$ preserves finite products.*

The corresponding statement for stable ∞-categories holds as well, more precisely we have the following lemma.

Lemma 3.1.16. *Let $\mathcal{C}$ be a* stable *∞-category. Then the functor $\Omega^\infty \colon \mathrm{Sp}_\infty \to \mathcal{S}$ induces an equivalence of functor categories*

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathrm{Sp}_\infty) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{Lex}}(\mathcal{C}, \mathcal{S}).$$

Corollary 3.1.17. *In particular for every object $x \in \mathcal{C}$ there is a unique (up to contractible choice) functor*

$$\begin{aligned} \mathcal{C}^{op} &\xrightarrow{\mathrm{map}_{\mathcal{C}}(x,-)} \mathrm{Sp}_\infty \\ y &\longmapsto \mathrm{map}_{\mathcal{C}}(x, y) \end{aligned}$$

such that

(1) *the composite*

$$x \mapsto \big(\Omega^\infty(\mathrm{map}_{\mathcal{C}}(x, -)) \simeq \mathrm{Map}_{\mathcal{C}}(x, -)\big)$$

is the usual Yoneda embedding, and

(2) *the functor $\mathrm{map}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathrm{Sp}_\infty$ preserves finite limits.*

Remark. In a way this encodes that a stable ∞-category is *enriched* in the stable ∞-category of spectra and an additive category is enriched in grouplike $\mathbb{E}_\infty$-spaces which are equivalent to connective spectra.

3.2 The stable ∞-category KK_∞

Now we want to apply the general machinery of ∞-categories to our situation.

Definition 3.2.1. We define the ∞-category KK_∞ to be the ∞-category associated to the fibration category $\mathrm{C}^*\mathrm{Alg}$, see Definition 3.1.5 and Proposition 2.1.26.

Remark. The ∞-category KK_∞ is a full subcategory of the ∞-category associated to the (combinatorial) model category of pro-C^*-algebras, see [BJM15]. It is shown there that $\mathrm{C^*Alg}$ admits the structure of a simplicial fibration category. One can verify the assumptions of [Lur14, Proposition 1.3.4.7], so it follows that there is an equivalence

$$\mathrm{KK}_\infty \simeq cN(\mathrm{C^*Alg})[W^{-1}].$$

Lemma 3.2.2. *The ∞-category KK_∞ is a* stable *∞-category.*

Proof. By [Lur14, Corollary 1.4.2.27] it will suffice to show that KK_∞ admits all finite limits, is pointed and that the loop functor $\Omega\colon \mathrm{KK}_\infty \to \mathrm{KK}_\infty$ is an equivalence. Lemma 3.1.11 says that KK_∞ admits all finite limits, the first part of Lemma 3.1.12 implies that KK_∞ is pointed, because in KK the trivial algebra $\{0\}$ is a zero object. Furthermore by the second part of Lemma 3.1.12 it suffices to see that the loop functor is an equivalence on the ordinary KK-category. The loop functor in our sense is the C^*-algebraic suspension functor which is an equivalence by Bott periodicity. Thus the lemma follows. □

Since we know that both K and L-theory can be viewed as functors $\mathrm{C^*Alg} \to \mathrm{Sp}$ we want to argue that we can view them as functors on KK_∞.

Theorem 3.2.3. *We have two parts*

(1) *The corepresented functor* $\mathrm{map}_{\mathrm{KK}_\infty}(\mathbb{C},-)\colon \mathrm{KK}_\infty \to \mathrm{Sp}_\infty$ *is equivalent to K-theory,*

(2) *The functor* $L\colon \mathrm{C^*Alg} \to \mathrm{Sp} \to \mathrm{Sp}_\infty$ *factors over* KK_∞.

Proof. We prove the first statement first. Obviously the functor

$$K\colon \mathrm{C^*Alg} \to \mathrm{Sp}_\infty$$

sends KK-equivalences to equivalences. Thus by the universal property of the localization functor $N\mathrm{C^*Alg} \to \mathrm{KK}_\infty$ it follows that K-theory factors over KK_∞. Using Theorem 2.1.18 K-theory can be viewed as an object of

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty) \simeq \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{KK}_\infty, \mathcal{S}).$$

where the equivalence comes from the fact that KK_∞ is stable, recall Lemma 3.2.2, and Corollary 3.1.17. The Yoneda lemma, recall Lemma 3.1.13, thus tells us that

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty,\mathcal{S})}(\mathrm{map}_{\mathrm{KK}_\infty}(\mathbb{C},-), K) \simeq \Omega^\infty(K\mathbb{C})$$

We may thus consider the commutative diagram

$$\begin{array}{ccc} \pi_0\left(\mathrm{Map}_{\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{KK}_\infty,\mathcal{S})}(\mathrm{map}_{\mathrm{KK}_\infty}(\mathbb{C},-), K)\right) & \longrightarrow & \pi_0\left(\Omega^\infty(K\mathbb{C})\right) \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{Hom}_{\mathrm{Fun}(\mathrm{KK}_\infty,\mathcal{S})}(KK(\mathbb{C},-), K) & \longrightarrow & K_0(\mathbb{C}) \end{array}$$

where both horizontal arrows are isomorphisms by the Yoneda lemma. We have already argued that the element $1 \in K_0(\mathbb{C})$ comes from some isomorphism between the corepresented functor and the K-theory functor. By the fact that the right vertical arrow and top horizontal arrow are isomorphisms it follows that there exists an exact transformation (i.e. a transformation in the category of exact functors) $\eta\colon \mathrm{KK}_\infty(\mathbb{C},-) \to K$ that induces an isomorphism

$$\eta_A\colon \pi_0(\mathrm{map}_{\mathrm{KK}_\infty}(\mathbb{C},A)) \xrightarrow{\cong} \pi_0(KA)$$

for all C^*-algebras A. Since both functors and η are exact it thus follows that the diagram

$$\begin{array}{ccc} \pi_n(\mathrm{map}_{\mathrm{KK}_\infty}(\mathbb{C}, A)) & \longrightarrow & \pi_n(KA) \\ \downarrow & & \downarrow \\ \pi_0(\mathrm{map}(\mathbb{C}, S^n A)) & \longrightarrow & \pi_0(K(S^n A)) \end{array}$$

commutes. The lower horizontal map is an isomorphism by the previous argument, and both vertical maps are isomorphisms since the functors are exact. Thus η is an equivalence as claimed.

In order to prove (2) we need to see that L-theory sends KK-equivalences to weak equivalences of spectra. A weak equivalence of spectra is a map inducing isomorphisms on homotopy groups. Thus the we are reduced to argue that the group-valued L-theory functor sends KK-equivalences to isomorphisms which we have already seen. Hence the theorem follows. $\square$

4 The natural transformation

4.1 Reinterpreting the abelian group valued case

The idea of this section is to give a proof of the existence of a natural transformation between K and L-theory as group valued functor that will have a direct analogue in spectra. It will turn out that it is enough to understand the natural transformation $\tau\colon K_0 \to L^0$ of Proposition 2.2.28 in terms of a universal property.

We recall that we have the following

Lemma 4.1.1. *The canonical map*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{KK},\mathrm{Ab}_{\mathbb{Z}})}(K_0, L^0) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Fun}(\mathrm{C^*Alg},\mathrm{Ab}_{\mathbb{Z}})}(K, L)$$

is a bijection.

Proof. This is Corollary 2.1.27 on homomorphism sets. □

Remark. Notice that this makes sense since we have already argued that we may view L-theory as a functor on KK.

We recall from Proposition 2.1.29 that K-theory is corepresentable when viewed as a functor on KK.

Lemma 4.1.2. *From the Yoneda lemma we see that*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{KK},\mathrm{Ab}_{\mathbb{Z}})}(K_0, L^0) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathrm{KK},\mathrm{Ab}_{\mathbb{Z}})}(KK(\mathbb{C}, -), L^0) \cong L^0(\mathbb{C}) \cong \mathbb{Z}.$$

In particular the transformation $\tau\colon K_0 \to L^0$ as given in Proposition 2.2.28 corresponds to an element of $\mathbb{Z}$. Under this isomorphism τ is sent to $1 \in \mathbb{Z}$.

Proof. We only need to check the image of τ in $\mathbb{Z}$ under the above chain of isomorphisms. By definition the isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{KK},\mathrm{Ab})}(K_0, L^0) \cong L^0(\mathbb{C})$$

maps τ to $\tau_{\mathbb{C}}(\mathrm{id})$ where

$$\tau_{\mathbb{C}}\colon \mathrm{KK}(\mathbb{C},\mathbb{C}) \cong K_0(\mathbb{C}) \to L^0(\mathbb{C}).$$

Under the isomorphism $\mathrm{KK}(\mathbb{C},\mathbb{C}) \cong K_0(\mathbb{C})$ the element [id] is mapped to the projective module $\mathbb{C}$. The isomorphism of Proposition 2.2.28 takes this module to $\mathbb{C}$ equipped with the standard hermitian form over it. Furthermore the isomorphism $L^0(\mathbb{C}) \cong \mathbb{Z}$ takes the signature of this hermitian form which is obviously 1. □

4.2 Homotopical enhancement

We have argued how we can view the transformation from K to L-theory using the universal property of KK-theory. The main idea now is to mimic the universal properties we used.

Thus to obtain a natural transformation between K- and L-theory (viewed as spectra-valued functors) we need the following

Proposition 4.2.1. *The functor $L\colon \mathrm{KK}_\infty \to \mathrm{Sp}_\infty$ commutes with finite products, i.e. $L \in \mathrm{Fun}^{\Pi}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty)$.*

Proof. Recall form Proposition 3.1.4 that

$$\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty) \to \mathrm{Fun}(N(\mathrm{C}^*\mathrm{Alg}), \mathrm{Sp}_\infty)$$

is fully faithful. Since the map $N(\mathrm{C}^*\mathrm{Alg}) \to \mathrm{KK}_\infty$ preserves products also $\mathrm{Fun}^\Pi(\mathrm{KK}_\infty, \mathrm{Sp}_\infty)$ is a full subcategory of $\mathrm{Fun}^\Pi(N(\mathrm{C}^*\mathrm{Alg}), \mathrm{Sp}_\infty)$. Thus it suffices to see that $L\colon \mathrm{C}^*\mathrm{Alg} \to \mathrm{Sp}_\infty$ preserves finite products.

It is clear that $L(\{0\}) \simeq *$ since it is by definition the homotopy fiber of the map $L\mathbb{C} \to L\mathbb{C}$ coming from the fact that $\{0\}^+ = \mathbb{C}$. Furthermore it is known that the functor

$$L\colon \mathrm{Ring}^{\mathrm{inv}} \to \mathrm{Sp}$$

where Sp are some 1-category of spectra (e.g. sequential spectra) commutes with finite products: It is know that for two involutive rings R and S, the canonical map

$$L^n(R \times S) \to L^n(R) \times L^n(S)$$

is an isomorphism for all $n \in \mathbb{Z}$ (recall that we always take *symmetric and projective* L-theory. This directly implies that the map of spectra

$$L(R \times S) \to LR \times LS$$

is a π_*-isomorphism and thus a weak equivalence. Hence the functor $L\colon \mathrm{C}^*\mathrm{Alg} \to \mathrm{Sp}_\infty$ preserves products and thus also as needed. □

This implies that there is the following

Corollary 4.2.2. *We have a chain of equivalences*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty^{\geq 0})}(k, \ell) \xleftarrow{\simeq} \mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty^{\geq 0})}(k, \ell) \xrightarrow{\simeq} \Omega^\infty(\ell\mathbb{C}).$$

Proof. The first equivalence follows from Proposition 3.1.4 and the second equivalence is precisely Lemma 3.1.13. Notice that the last map has both target and source a grouplike $\mathbb{E}_\infty$-space but the map is apriori only a map of spaces. Using the techniques of above one can indeed show that it has a unique (up to contractible choice) refinement to a map of grouplike $\mathbb{E}_\infty$-spaces, i.e. of connective spectra. □

In particular we have

$$\pi_0\big(\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty^{\geq 0})}(k, \ell)\big) \xrightarrow{\cong} \pi_0(\ell\mathbb{C}) \cong \mathbb{Z}$$

Furthermore the diagram

$$\begin{array}{ccccc} \pi_0\big(\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty^{\geq 0})}(k, \ell)\big) & \longrightarrow & \pi_0(\ell\mathbb{C}) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Fun}(\mathrm{KK}, \mathrm{Ab}_\mathbb{Z})}(k, \ell) & \longrightarrow & \ell^0\mathbb{C} & \longrightarrow & \mathbb{Z} \end{array}$$

commutes.

Definition 4.2.3. This implies that there exists a transformation

$$\tau \in \mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty^{\geq 0})}(k, \ell)$$

that is sent under the above construction to the transformation of Proposition 2.2.28 and Lemma 4.1.2 when applying π_0.

Remark. Since connective spectra are a full subcategory of spectra, it follows that the functor

$$\mathrm{Fun}(N\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty^{\geq 0}) \to \mathrm{Fun}(N\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty)$$

is also fully faithful. We thus obtain a transformation

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty^{\geq 0})}(k,\ell) & \xrightarrow{\sim} & \mathrm{Map}_{\mathrm{Fun}(N\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty^{\geq 0})}(k,\ell) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(N\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty)}(k,\ell) \\ \tau & & \longmapsto & & \overline{\tau} \end{array}$$

By Corollary 3.1.8 the transformation $\overline{\tau}$ can be strictified to a zig-zag of transformations in the 1-category of functors $\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp})$.

We thus obtain the following Corollary, compare Proposition 2.2.28.

Corollary 4.2.4. *This transformation satisfies that the map*

$$\tau_*\colon \pi_0(kA) \xrightarrow{\cong} \pi_0(\ell A)$$

is an isomorphism for all $A \in \mathrm{KK}_\infty$.

Proposition 4.2.5. *The natural transformation $\tau\colon k \to \ell$ satisfies that the map*

$$\tau_*\colon \pi_1(kA) \xrightarrow{\cong} \pi_1(\ell A)$$

is an isomorphism for all $A \in \mathrm{KK}_\infty$.

Proof. We consider the diagram

$$\begin{array}{ccc} \pi_1(kA) & \longrightarrow & \pi_1(\ell A) \\ \cong\downarrow & & \downarrow\cong \\ \pi_0(\Omega kA) & \longrightarrow & \pi_0(\Omega \ell A) \\ \cong\uparrow & & \uparrow\cong \\ \pi_0(k(SA)) & \longrightarrow & \pi_0(\ell(SA)) \end{array}$$

where the last vertical maps come from the canonical maps of spectra

$$k(SA) \to \Omega kA \text{ and } \ell(SA) \to \Omega \ell A$$

We have argued that for K-theory this map is an equivalence, and it is the content of Corollary 2.2.37 that it also an isomorphism on π_0 for L-theory

□

Recall that

$$\pi_*(L\mathbb{C}) = \mathbb{Z}[b^{\pm 1}]$$

with $|b| = 2$. We then have the following

Corollary 4.2.6. *The map $\tau_{\mathbb{C}}\colon ku \to \ell\mathbb{C}$ satisfies*

$$\begin{array}{ccc} \pi_2(ku) & \longrightarrow & \pi_2(\ell\mathbb{C}) \\ \beta & \longmapsto & \pm 2b \end{array}$$

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \pi_2(ku) & \xrightarrow{\tau_{\mathbb{C}}} & \pi_2(L\mathbb{C}) \\ \cong\downarrow & & \downarrow\cong \\ \pi_1(\Omega ku) & \xrightarrow{\Omega\tau_{\mathbb{C}}} & \pi_1(\Omega(L\mathbb{C})) \\ \cong\uparrow & & \uparrow\cdot 2 \\ \pi_1(k(S\mathbb{C})) & \xrightarrow[\tau_{S\mathbb{C}}]{} & \pi_1(L(S\mathbb{C})) \end{array}$$

and recall that the lower right vertical map is given by multiplication by 2, see Theorem 2.2.38. □

We end this section with the following

Theorem 4.2.7. *There is a canonical transformation*

$$\hat{\tau} \in \mathrm{map}_{\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg},\mathrm{Sp}_\infty)}(K, L[\tfrac{1}{2}])$$

which has the property that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\hat{\tau}} & L[\tfrac{1}{2}] \\ \uparrow & & \uparrow \\ k & \xrightarrow[\tau]{} & \ell \end{array}$$

is commutative. Furthermore for each $A \in \mathrm{C}^*\mathrm{Alg}$ the induced map

$$\hat{\tau}_A \colon KA[\tfrac{1}{2}] \to LA[\tfrac{1}{2}]$$

is a weak equivalence.

Proof. We have from Corollary 3.1.17 that

$$\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{KK}_\infty, \mathcal{S})$$

is an equivalence. Moreover from the fibration sequence

$$L(SA) \longrightarrow L\mathbb{C} \longrightarrow L^{\langle h \rangle}(A^+)$$

it follows that $L[\frac{1}{2}] \in \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty)$. Moreover as before using the classical Yoneda Lemma we thus see that

$$\pi_0\left(\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty,\mathrm{Sp}_\infty)}(K, L[\tfrac{1}{2}])\right) \cong \pi_0\left(\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty,\mathcal{S})}(K, L[\tfrac{1}{2}])\right) \cong \pi_0(L\mathbb{C}[\tfrac{1}{2}])$$

We choose $\hat{\tau}$ such that it maps (under the above identifications) to the element $1 \in \mathbb{Z}[\frac{1}{2}] \cong \pi_0(L\mathbb{C}[\frac{1}{2}])$. It follows from the naturality of the construction that comparison diagram with τ commutes. It hence remains to prove the last part saying that $\hat{\tau}_A$ is a weak equivalence. By comparing with τ one sees that it induces an isomorphism on π_0 for all $A \in \mathrm{KK}_\infty$. Shifting the algebra around (using that we are not in connective spectra anymore) proves that it induces an isomorphism on π_k for all $k \in \mathbb{Z}$, compare to the argument in the proof of Theorem 3.2.3 part (1). □

5 On the relation between KU and $L\mathbb{C}$

In this section we want to investigate spaces of maps between KU and $L\mathbb{C}$. A first observation is the following general

Proposition 5.0.1. *Let R be a ring spectrum and M be an R-module spectrum. If R is admits the structure of an* $\mathrm{H}\mathbb{Z}$*-module then so does M.*

Proof. We first notice that it is equivalent for a spectrum X to admit the structure of an $\mathrm{H}\mathbb{Z}$-module and to be equivalent to the generalized Eilenberg-MacLane spectrum on its homotopy groups. The module multiplication map and the unit of the ring spectrum give factorization of the identity of M as follows

$$M \longrightarrow R \otimes M \longrightarrow M$$

which shows that M is a retract of $R \otimes M$. Since R is an $\mathrm{H}\mathbb{Z}$-module, so is $R \otimes M$. By the above this implies that $R \otimes M$ is a generalized Eilenberg-MacLane spectrum. So the proposition follows if we show that the category of generalized Eilenberg-MacLane spectra is closed under retracts. It follows from functoriality of homotopy groups that if Y is a retract of X then $\pi_*(Y)$ is a retract of $\pi_*(X)$ and it is easy to see that the map $Y \to X \to \mathrm{H}\pi_*(X) \to \mathrm{H}\pi_*(Y)$ is an equivalence. □

Corollary 5.0.1. *For all $A \in \mathrm{C}^*\mathrm{Alg}$, the spectrum $LA_{(2)}$ admits the structure of an* $\mathrm{H}\mathbb{Z}$*-module.*

Proof. First we need to recall that $L\mathbb{Z}$ is an algebra over MSO due to the Sullivan-Ranicki orientation $\mathrm{MSO} \xrightarrow{\sigma} L\mathbb{Z}$. In particular for any $A \in \mathrm{C}^*\mathrm{Alg}$ the spectrum LA is a module over MSO und thus $LA_{(2)}$ is a module over $\mathrm{MSO}_{(2)}$ which is an $\mathrm{H}\mathbb{Z}$-module, see e.g. [TW79, Theorem A]. □

Remark. The $\mathrm{H}\mathbb{Z}$-module structure on $LA_{(2)}$ is not canonical, but for our purposes it suffices to choose some $\mathrm{H}\mathbb{Z}$-module structure for each algebra A.

Corollary 5.0.2. *The spectra KU and $L\mathbb{C}$ are* not *equivalent, although their homotopy groups are naturally isomorphic.*

Proof. It is well known that KU does not split 2-locally because else also ku would split 2-locally. But it is known that $\mathrm{H}\mathbb{F}_2^*(ku_{(2)}) \cong \mathrm{H}\mathbb{F}_2^*(ku)$ does not split as a module over the Steenrod algebra. □

This implies that there will not be any natural transformation between K-theory and L-theory viewed as spectra valued functors that induces an isomorphism on *all* homotopy groups, i.e. the functors K and L are not equivalent as spectra valued functors.

Remark. The proof of Corollary 5.0.2 shows that at the prime 2 the spectra are not equivalent. It turns out that this is the only prime at which they differ, indeed in joint work with T. Nikolaus in preparation we show that the ($\mathbb{E}_\infty$-ring) spectra $KU[\frac{1}{2}]$ and $L\mathbb{C}[\frac{1}{2}]$ are equivalent (as $\mathbb{E}_\infty$-ring spectra). It is also possible to deduce directly that the underlying homotopy ring spectra are equivalent by comparing their formal groups (both are even spectra, and hence complex *orientable*). See the lecture notes by Lurie, [Lur], for a similar argument in the case of $KO[\frac{1}{2}]$ and $L\mathbb{Z}[\frac{1}{2}]$.

So one might ask whether there is a natural transformation between K- and L-theory that induces an equivalence after inverting 2. We are able to calculate π_0 of all relevant mapping spaces.

Theorem 5.0.3. *We have that*

$$[L\mathbb{C}, KU] = [KU, L\mathbb{C}] = [\ell\mathbb{C}, KU] = [\ell\mathbb{C}, ku] = 0.$$

The proof of this theorem will proceed in several steps. We need a couple of lemmas to get started.

Lemma 5.0.4. *If E and F are even and complex oriented ring spectra and E is Landweber exact, then $E \otimes F$ is even, i.e. has no odd homotopy. In particular $KU \otimes L\mathbb{C}$ and $KU \otimes \ell\mathbb{C}$ are even.*

Proof. By assumption we have

$$E_*F \simeq MU_*F \otimes_{MU_*} E_*$$

since MU_* and E_* are even it thus suffices to prove that MU_*F is even. For this we see that

$$MU_*F = F_*MU \cong F_*(BU)$$

by the Thom-isomorphism for F. But since F is even and BU has even homology the Atiyah Hirzebruch spectral sequence implies that $F_*(BU)$ is even. □

Lemma 5.0.5. *Suppose R is a commutative ring such that the additive and the multiplicative formal group law are isomorphic. Then R is a $\mathbb{Q}$-algebra.*

Proof. We can formally write down the logarithm of the multiplicative formal group law and see that this forces all primes to act invertibly on R. □

Corollary 5.0.6. *The spectrum $KU \otimes \mathrm{H}\mathbb{Z}$ is rational.*

Proof. This is a classical fact. A nice proof using formal groups goes as follows. The spectrum $KU \otimes \mathrm{H}\mathbb{Z}$ has two complex orientations, one coming from KU and one coming from $\mathrm{H}\mathbb{Z}$. Thus on $\pi_*(KU \otimes \mathrm{H}\mathbb{Z})$ the additive and the multiplicative formal group law are isomorphic. Since by Lemma 5.0.4 the spectrum $KU \otimes \mathrm{H}\mathbb{Z}$ is an even periodic one can shift the coefficients of the formal group law to degree 0. We then obtain that $\pi_0(KU \otimes \mathrm{H}\mathbb{Z})$ is a ring on which the additive and the multiplicative formal group law are isomorphic hence is a $\mathbb{Q}$-algebra. Since $\pi_*(KU \otimes \mathrm{H}\mathbb{Z})$ is a module over $\pi_0(KU \otimes \mathrm{H}\mathbb{Z})$ the corollary follows. □

Lemma 5.0.7. *Let $\mathbb{S}$ be the sphere spectrum and p be a prime. Then the diagram*

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \mathbb{S}[\frac{1}{p}] \\ \downarrow & & \downarrow \\ \mathbb{S}_{(p)} & \longrightarrow & \mathbb{S}_{\mathbb{Q}} \end{array}$$

is a pullback diagram of spectra.

Proof. The lemma follows if we show that the spectrum $M(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})$ is p-local because then we consider the map of cofiber sequences coming from p-localizing vertically

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{S}[\frac{1}{p}] & \longrightarrow & M(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{S}_{(p)} & \longrightarrow & \mathbb{S}[\frac{1}{p}]_{(p)} & \longrightarrow & M(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})_{(p)} \end{array}$$

and use that $\mathbb{S}[\frac{1}{p}]_{(p)} \simeq \mathbb{S}_{\mathbb{Q}}$.

The group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is isomorphic to the Prüfer group $\mathbb{Z}(p^\infty)$ which is p-local. Thus the lemma follows. □

We apply this observation as follows.

Lemma 5.0.8. *The canonical map*

$$L\mathbb{C} \otimes KU \longrightarrow L\mathbb{C} \otimes KU \otimes \mathbb{S}[\tfrac{1}{2}] = (L\mathbb{C} \otimes KU)[\tfrac{1}{2}]$$

is an equivalence of $L\mathbb{C} \otimes KU$-modules.

Proof. The map is clearly a module map. So it suffices to argue that it is an equivalence of spectra. For this we consider the pullback diagram

$$\begin{array}{ccc} L\mathbb{C} \otimes KU & \longrightarrow & (L\mathbb{C} \otimes KU)[\tfrac{1}{2}] \\ \downarrow & & \downarrow \\ (L\mathbb{C} \otimes KU)_{(2)} & \longrightarrow & (L\mathbb{C} \otimes KU)_{\mathbb{Q}} \end{array}$$

wich is obtained by smashing the pullback diagram of Lemma 5.0.7 with the spectrum $KU \wedge \mathrm{H}\mathbb{Z}$. Since pullbacks are pushouts smashing a pullback diagram with a spectrum gives again a pullback diagram. Now we observe that $(L\mathbb{C} \otimes KU)_{(2)}$ is an $\mathrm{H}\mathbb{Z} \otimes KU$ module (since $L\mathbb{C}_{(2)}$ is an $\mathrm{H}\mathbb{Z}$-module). By Corollary 5.0.6 $\mathrm{H}\mathbb{Z} \otimes KU$ is rational. Hence also all modules over this spectrum are rational. But this implies that in the above pullback diagram the lower horizontal arrow is an equivalence, thus also the upper horizontal one is. □

Remark. The same is true if we replace $L\mathbb{C}$ by $\ell\mathbb{C}$ (with the same proof).

Corollary 5.0.9. *There is an equivalence of mapping spaces*

(1) $\mathrm{Map}(L\mathbb{C}, KU) \simeq \mathrm{Map}(L\mathbb{C}[\tfrac{1}{2}], KU)$*, and*

(2) $\mathrm{Map}(KU, L\mathbb{C}) \simeq \mathrm{Map}(KU[\tfrac{1}{2}], L\mathbb{C})$*, and*

(3) $\mathrm{Map}(\ell\mathbb{C}, ku) \simeq \mathrm{Map}(\ell\mathbb{C}, KU) \simeq \mathrm{Map}(\ell\mathbb{C}[\tfrac{1}{2}], KU)$*.*

Proof. Lemma 5.0.8 implies that $L\mathbb{C} \otimes KU \simeq L\mathbb{C}[\tfrac{1}{2}] \otimes KU$ as KU-modules and $L\mathbb{C} \otimes KU \simeq L\mathbb{C} \otimes KU[\tfrac{1}{2}]$ as $L\mathbb{C}$-modules. Thus we obtain

$$\begin{aligned} \mathrm{Map}(L\mathbb{C}, KU) &\simeq \mathrm{Map}_{KU}(L\mathbb{C} \otimes KU, KU) \\ &\simeq \mathrm{Map}_{KU}(L\mathbb{C}[\tfrac{1}{2}] \otimes KU, KU) \\ &\simeq \mathrm{Map}(L\mathbb{C}[\tfrac{1}{2}], KU) \end{aligned}$$

Statement (2) follows similarly and statement (3) from the fact that Lemma 5.0.8 is true for $\ell\mathbb{C}$ instead of $L\mathbb{C}$ and the universal property of connective covers. □

Proposition 5.0.10. *There are short exact sequences*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(KU_{-1}(L\mathbb{C}[\tfrac{1}{2}]), \mathbb{Z}) \longrightarrow KU^0(L\mathbb{C}[\tfrac{1}{2}]) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(KU_0(L\mathbb{C}[\tfrac{1}{2}]), \mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(KU_{-1}(\ell\mathbb{C}[\tfrac{1}{2}]), \mathbb{Z}) \longrightarrow KU^0(\ell\mathbb{C}[\tfrac{1}{2}]) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(KU_0(\ell\mathbb{C}[\tfrac{1}{2}]), \mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(L\mathbb{C}_{-1}(KU[\tfrac{1}{2}]), \mathbb{Z}) \longrightarrow L\mathbb{C}^0(KU[\tfrac{1}{2}]) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L\mathbb{C}_0(KU[\tfrac{1}{2}]), \mathbb{Z}) \longrightarrow 0$$

Proof. The exact sequences follow from the general UCT sequence relating a spectrum and its Anderson dual, see [And70], using that both KU and $L\mathbb{C}$ are Anderson self-dual (see [SH14, below Prop. 2.2]). □

Remark. The existence of the short exact sequences of above can also be proven without appealing to Anderson duality. To explain this we consider the following situation. Suppose R is a ring spectrum and M and N are R-modules. Then there is a spectral sequence converging to R-module maps $\pi_*(\mathrm{Map}_R(M,N))$ with E_2-term given by

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\pi_*(R)}(\pi_*(M), \pi_*(N))$$

here the Ext-group is calculated in the category of *graded modules* over the *graded ring* $\pi_*(R)$. The number p refers to the homological degree and q to the internal degree of the graded object.

The following is an immediate consequence of the existence of the spectral sequence. Suppose R is such that the graded ring $\pi_*(R)$ has global dimension one. This is the case for instance if $\pi_*(R) = \mathbb{Z}[u^{\pm 1}]$ with $u \in \pi_2(R)$ as e.g. in the case $R = KU$ or $R = L\mathbb{C}$. In this case the global dimension is one because the category of graded modules over $\mathbb{Z}[u^{\pm 1}]$ is equivalent to the category $\mathrm{Ab} \times \mathrm{Ab}$ by taking a graded module $\{M_k\}_{k \in \mathbb{Z}}$ to the pair of $\mathbb{Z}$-modules (M_0, M_1).

In the case where $\pi_*(R)$ has global dimension one the spectral sequence collapses on the E_2-page and we obtain a short exact sequence

$$0 \to \mathrm{Ext}^{1,1}_{\pi_*(R)}(\pi_*(M), \pi_*(N)) \to \pi_0(\mathrm{Map}_R(M,N)) \to \mathrm{Hom}_{\pi_*(R)}(\pi_*(M), \pi_*(N)) \to 0$$

Applying this for instance in the case $R = KU$, $M = KU \otimes X$ for some spectrum X and $N = KU$ and using that

$$KU^0(X) \cong \pi_0(\mathrm{Map}(X, KU)) \cong \pi_0(\mathrm{Map}_{KU}(KU \otimes X, KU))$$

gives a short exact sequence

$$0 \to \mathrm{Ext}^{1,1}_{\pi_*(KU)}(KU_*X, KU_*) \to KU^0(X) \to \mathrm{Hom}_{\pi_*(KU)}(KU_*X, KU_*) \to 0$$

Now using the above equivalence of module categories and using the fact that $\pi_1(KU) = 0$ we obtain an isomorphic sequence which reads as

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(KU_{-1}(X), \mathbb{Z}) \longrightarrow KU^0(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(KU_0(X), \mathbb{Z}) \longrightarrow 0$$

for any spectrum X.

Proof of Theorem 5.0.3. The Ext-terms vanish due to Lemma 5.0.4, and certainly

$$KU_0(L\mathbb{C}[\tfrac{1}{2}]) \cong L\mathbb{C}_0(KU[\tfrac{1}{2}]) \text{ and } KU_0(\ell\mathbb{C}[\tfrac{1}{2}])$$

are $\mathbb{Z}[\frac{1}{2}]$-modules. For all $\mathbb{Z}[\frac{1}{2}]$-modules M we have that $\mathrm{Hom}(M, \mathbb{Z}) = 0$. Thus we obtain that

$$\begin{aligned} KU^0(L\mathbb{C}) &= 0 \\ L\mathbb{C}^0(KU) &= 0 \\ ku^0(\ell\mathbb{C}) &= KU^0(\ell\mathbb{C}) = 0. \end{aligned}$$

□

Remark. Indeed it is not true that $ku \wedge \mathrm{H}\mathbb{Z}$ is rational, for instance $\pi_0(ku \wedge H\mathbb{Z}) \cong \mathbb{Z}$. So we cannot make any conclusions about $\mathrm{map}(ku, \ell\mathbb{C})$ just yet. Indeed we have seen that there is a map $ku \to \ell\mathbb{C}$ which induces an isomorphism on π_0 and multiplication by 2 on π_2.

At last we want to say something about the last remaining mapping space we have not investigated yet, namely $\mathrm{map}(L\mathbb{C}, ku)$. Here we have the following

Theorem 5.0.11. *We have that*

(1) $\mathrm{map}(L\mathbb{C}, ku) \simeq \prod_{k\in\mathbb{Z}} \mathrm{map}(\Sigma^{2k}H\mathbb{Z}_{(2)}, ku)$,

(2) For each $k \in \mathbb{Z}$ *there is an exact sequence*

$$\mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \longrightarrow \pi_0(\mathrm{map}(\Sigma^{2k+3}H\mathbb{Z}_{(2)}, I_{\mathbb{Z}}ku)) \longrightarrow \pi_0(\mathrm{map}(\Sigma^{2k}H\mathbb{Z}_{(2)}, ku)) \longrightarrow 0$$

(3) There is an isomorphism

$$\pi_0(\mathrm{map}(\Sigma^{2k+3}H\mathbb{Z}_{(2)}, I_{\mathbb{Z}}ku)) \cong \mathrm{Ext}(ku_{2k+2}H\mathbb{Z}_{(2)}, \mathbb{Z}).$$

(4) Thus we obtain, substituting $n = k+1$, *that*

$$\pi_0(\mathrm{map}(L\mathbb{C}, ku)) \cong \prod_{n\in\mathbb{Z}} \Big(\mathrm{coker}\left(\mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \to \mathrm{Ext}(ku_{2n}H\mathbb{Z}_{(2)}, \mathbb{Z})\right)\Big)$$

where the map of which we form the cokernel is induced by the map

$$\pi_2(ku \otimes H\mathbb{Z}_{(2)}) \to \pi_{2n}(KU \otimes H\mathbb{Z}_{(2)}) \cong \mathbb{Q}$$

Remark. We want to point out that this mapping space is not relevant for our purposes: We wanted to argue that there is a unique choice of direction for a natural transformation between K- and L-theory (or their connective counterparts) which at least induces an isomorphism on π_0 after inverting 2. But we observe that any map $L\mathbb{C} \to ku$ gives a map

$$\ell\mathbb{C} \to L\mathbb{C} \to ku$$

which on the one hand detects the behaviour on homotopy of the map we started with and on the other hand is null homotopic by Theorem 5.0.3. Thus any map $L\mathbb{C} \to ku$ will induce the trivial map on all homotopy groups.

Proof. To prove part (1) we again begin by mapping the cartesian square

$$\begin{array}{ccc} L\mathbb{C} & \longrightarrow & L\mathbb{C}[\frac{1}{2}] \\ \downarrow & & \downarrow \\ L\mathbb{C}_{(2)} & \longrightarrow & L\mathbb{C}_{\mathbb{Q}} \end{array}$$

to ku. We thus obtain a cartesian square

$$\begin{array}{ccc} \mathrm{map}(L\mathbb{C}_{\mathbb{Q}}, ku) & \longrightarrow & \mathrm{map}(L\mathbb{C}[\frac{1}{2}], ku) \\ \downarrow & & \downarrow \\ \mathrm{map}(L\mathbb{C}_{(2)}, ku) & \longrightarrow & \mathrm{map}(L\mathbb{C}, ku) \end{array}$$

We now claim that the upper horizontal map is an equivalence. To see this we consider the Postnikov tower of ku which has the following slices

$$\Sigma^{2i}H\mathbb{Z} \longrightarrow P_{2i}(ku) \longrightarrow P_{2i-2}(ku)$$

for $i \geq 1$. Then we notice that for any spectrum E there is an equivalence

$$\mathrm{map}(E, ku) \simeq \mathrm{holim}_{i\geq 0}\, \mathrm{map}(E, P_{2i}(ku))$$

It will thus suffice the the map $L\mathbb{C}[\frac{1}{2}] \to L\mathbb{C}_{\mathbb{Q}}$ induces an equivalence on mapping spectra

$$\mathrm{map}(L\mathbb{C}_{\mathbb{Q}}, P_{2i}(ku)) \xrightarrow{\simeq} \mathrm{map}(L\mathbb{C}[\tfrac{1}{2}], P_{2i}(ku))$$

for all $i \geq 0$. This we do by induction. For $i = 0$ we have $P_0(ku) \simeq H\mathbb{Z}$. There we have

$$\begin{aligned}\mathrm{map}(L\mathbb{C}[\tfrac{1}{2}], H\mathbb{Z}) &\simeq \mathrm{map}_{H\mathbb{Z}}(L\mathbb{C}[\tfrac{1}{2}] \otimes H\mathbb{Z}, H\mathbb{Z}) \\ &\simeq \mathrm{map}_{H\mathbb{Z}}(L\mathbb{C}_{\mathbb{Q}} \otimes H\mathbb{Z}, H\mathbb{Z}) \\ &\simeq \mathrm{map}(L\mathbb{C}_{\mathbb{Q}}, H\mathbb{Z})\end{aligned}$$

where we use that $L\mathbb{C}[\frac{1}{2}] \simeq KU[\frac{1}{2}]$ and hence that $L\mathbb{C}[\frac{1}{2}] \otimes H\mathbb{Z}$ is rational.

Then we proceed by induction and consider the diagram of fiber sequences

$$\begin{array}{ccccc}\mathrm{map}(L\mathbb{C}_{\mathbb{Q}}, \Sigma^{2i} H\mathbb{Z}) & \longrightarrow & \mathrm{map}(L\mathbb{C}_{\mathbb{Q}}, P_{2i}(ku)) & \longrightarrow & \mathrm{map}(L\mathbb{C}_{\mathbb{Q}}, P_{2i-2}(ku)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{map}(L\mathbb{C}[\frac{1}{2}], \Sigma^{2i} H\mathbb{Z}) & \longrightarrow & \mathrm{map}(L\mathbb{C}[\frac{1}{2}], P_{2i}(ku)) & \longrightarrow & \mathrm{map}(L\mathbb{C}[\frac{1}{2}], P_{2i-2}(ku))\end{array}$$

By the induction start the left vertical map is an equivalence, and by induction hypothesis the right vertical map is an equivalence. Thus the 5-lemma implies that also the middle vertical map is an equivalence.

We can thus conlude that the map

$$\mathrm{map}(L\mathbb{C}_{(2)}, ku) \longrightarrow \mathrm{map}(L\mathbb{C}, ku)$$

is an equivalence. Since $L\mathbb{C}_{(2)} \simeq \bigoplus_{k \in \mathbb{Z}} \Sigma^{2k} H\mathbb{Z}_{(2)}$ the theorem follows.

To prove part (2) we recall that there exists a fiber sequence

$$\Sigma^{-1} KU \longrightarrow \Sigma^{-3} I_{\mathbb{Z}} ku \longrightarrow ku \longrightarrow KU$$

and we can map the spectrum $\Sigma^{2k} H\mathbb{Z}_{(2)}$ into that fiber sequence to obtain a the fiber sequence

$$\mathrm{map}(\Sigma^{2k+3} H\mathbb{Z}_{(2)}, I_{\mathbb{Z}} ku) \longrightarrow \mathrm{map}(\Sigma^{2k} H\mathbb{Z}_{(2)}, ku) \longrightarrow \mathrm{map}(\Sigma^{2k} H\mathbb{Z}_{(2)}, KU)$$

We are interested in the induced sequence on homotopy groups in degree 0. All we need to observe is that

$$\mathrm{map}(\Sigma^{2k} H\mathbb{Z}_{(2)}, KU) \simeq \mathrm{map}(H\mathbb{Z}_{(2)}, KU)$$

and that

$$\pi_i(\mathrm{map}(H\mathbb{Z}_{(2)}, KU)) \cong KU^i(H\mathbb{Z}_{(2)}) \cong \begin{cases} \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

where the last isomorphism follows from Anderson duality and the fact that

$$KU \otimes H\mathbb{Z}_{(2)} \simeq KU_{\mathbb{Q}}.$$

Thus part (2) follows.

The calculation of part (3) of the theorem is again an application of the Anderson exact sequence for

$$\pi_0(\mathrm{map}(\Sigma^{2k+3} H\mathbb{Z}_{(2)}, I_{\mathbb{Z}} ku)) \cong I_{\mathbb{Z}} ku^0(\Sigma^{2k+3} H\mathbb{Z}_{(2)})$$

which reads as

$$0 \longrightarrow \mathrm{Ext}(ku_{2k+2} H\mathbb{Z}_{(2)}, \mathbb{Z}) \longrightarrow I_{\mathbb{Z}} ku^0(\Sigma^{2k+3} H\mathbb{Z}_{(2)}) \longrightarrow \mathrm{Hom}(ku_{2k+3} H\mathbb{Z}_{(2)}, \mathbb{Z}) \longrightarrow 0.$$

Since $ku_{2k+3} H\mathbb{Z}_{(2)}$ is a module over $\mathbb{Z}_{(2)}$ it follows that

$$\mathrm{Hom}(ku_{2k+3} H\mathbb{Z}_{(2)}, \mathbb{Z}) = 0$$

and thus also part (3) of the theorem follows.

Part (4) follows immediately now. □

Corollary 5.0.12.

$$\pi_0(\mathrm{map}(L\mathbb{C}, ku)) \neq 0$$

Proof. By Theorem 5.0.11 we see that it suffices to show that there exists torsion elements $ku_{2n}(H\mathbb{Z}_{(2)})$ for some $n \in \mathbb{Z}$. This is because certainly this homology is finitely generated as a $\mathbb{Z}_{(2)}$ module, so that we can decompose

$$ku_{2n}(H\mathbb{Z}_{(2)}) \cong T(n) \oplus \mathbb{Z}_{(2)}$$

where $T(n)$ is a torsion module. Under the rationalization this torsion module is mapped to zero, and thus we obtain an isomorphism

$$\mathrm{coker}\big(\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Ext}(T \oplus \mathbb{Z}_{(2)}, \mathbb{Z})\big) \cong \mathrm{Ext}(T(n), \mathbb{Z}) \neq 0.$$

We consider the Bockstein sequence

$$\begin{array}{ccccc} ku_{2n+1}(H\mathbb{F}_2) & \xrightarrow{\beta_{\mathbb{Z}}} & ku_{2n}(H\mathbb{Z}) & \xrightarrow{\cdot 2} & ku_{2n}(H\mathbb{Z}) \\ & \searrow^{\beta} & \downarrow & & \\ & & ku_{2n}(H\mathbb{F}_2) & & \end{array}$$

Thus to find elements of order two it suffices to prove that the map $\beta_{\mathbb{Z}}$ is not trivial, which is implied by the map β being non trivial. This is the case if and only if its dual map is non trivial, and the dual of the homological Bockstein is the map

$$\mathrm{Sq}^1 \colon H\mathbb{F}_2^{2n}(ku) \to H\mathbb{F}_2^{2n+1}(ku).$$

Now it is well known that as a module over the Steenrod algebra we have

$$H\mathbb{F}_2^*(ku) \cong \mathcal{A}/\mathrm{Sq}^1, \mathrm{Sq}^3$$

where the quotient is by the submodule generated by Sq^1 and Sq^3. It is easy to see that for instance $\mathrm{Sq}^7 \neq 0$ in this quotient. But we have

$$\mathrm{Sq}^1\mathrm{Sq}^6 = \mathrm{Sq}^7$$

so we obtain the desired result. □

6 Appendix

So far in this thesis we have discussed the relationship between two functors from the (one)-category $\mathrm{C}^*\mathrm{Alg}$ to the ∞-category Sp of spectra. In this chapter we want to explain how this fits into the framework of [DL98] where assembly maps are studied. For this we want to argue about two things.

(1) The natural transformation τ can be lifted to a zig-zag of natural transformations between the two 1-categorical functors $k, \ell \colon \mathrm{C}^*\mathrm{Alg} \to \mathrm{Sp}$.

(2) In order to compare assembly maps for these functors we need a transformation of functors from groupoids to spectra rather than from C^*-algebras to spectra. We explain how to resolve this issue.

The first part is easy and we have already established all tools to do this. Recall that Proposition 3.1.7 states that the functor category

$$\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty)$$

is the ∞-category associated to the model category of functors from $\mathrm{C}^*\mathrm{Alg}$ to the model category of spectra. Thus we obtain that the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp}_\infty)}(k, \ell)$$

is equivalent to the mapping space in the model category $\mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathrm{Sp})$ between cofibrant and fibrant replacements of k and ℓ respectively. Thus we obtain a zig-zag of transformations as claimed.

6.0.1 Some abstract homotopy theory and applications to assembly

Recall that in the approach to assembly taken in [DL98] one starts out with the category

$$\mathrm{Fun}^W(\mathrm{Gpd}, \mathrm{Sp}),$$

in other words with functors from the 1-category of groupoids to the 1-category of spectra that send equivalences of groupoids to equivalences of spectra. In Proposition 3.1.4 we have seen that the canonical functor

$$\mathrm{Fun}(N\mathrm{Gpd}[W^{-1}], \mathrm{Sp}_\infty) \to \mathrm{Fun}^W(N\mathrm{Gpd}, \mathrm{Sp}_\infty)$$

of ∞-functor categories is an equivalence. Moreover since Sp_∞ is the ∞-category associated to the simplicial and combinatorial model category of spectra, we see by the previous argument that we do not lose information by going to ∞-functors.

Thus we want to calculate the category $N\mathrm{Gpd}[W^{-1}]$ which is what we aim for next. In order to do this some notation is needed.

Definition 6.0.1. We say that $\mathcal{C}$ is a strict $(2,1)$-category if there is a class of objects and for two such objects x and y of $\mathcal{C}$ there is a *groupoid* of morphisms $\mathrm{Hom}(x, y)$ such that composition is a functor. The fact that this category is a groupoid is reflected in the notation $(2,1)$-category as opposed to a 2-category.

Definition 6.0.2. Let $\mathcal{C}$ be a strict $(2,1)$-category. We define its associated *simplicial category* $\mathcal{C}_\Delta$ by taking the same class of objects and for objects $x, y \in \mathcal{C}$ we define the simplicial mapping space by

$$\mathrm{Map}(x,y) = N(\mathrm{Hom}(x,y)).$$

Definition 6.0.3. We denote again by $\mathcal{C}$ the associated ∞-category, i.e. we write

$$\mathcal{C} = cN(\mathcal{C}_\Delta)$$

and via this view $(2,1)$ categories as ∞-categories.

Remark. The condition of starting with a $(2,1)$-category implies that the simplicial mapping spaces of the associated simplicial category are Kan complexes, hence the homotopy coherent nerve preserves the homotopy type of these mapping spaces.

This is relevant for us for the following reason.

Proposition 6.0.4. *We have that*

$$(N\mathrm{Gpd})[W^{-1}] \simeq cN(\mathrm{Gpd}_\Delta) = \mathrm{Gpd}_2$$

where Gpd_2 *is the* $(2,1)$*-category of groupoids with natural transformations as* 2*-morphisms.*

Proof. In [CGT06] it is shown that the category Gpd admits a simplicial model structure with

(1) equivalences are the equivalences of groupoids,

(2) fibrations are the functors F such that the map NF of simplicial sets is a Kan fibration,

(3) cofibrations are the functors that are injective on the set of objects, and

(4) the simplicial mapping spaces are given by

$$\mathrm{Map}(\mathcal{G}, \mathcal{G}') = N\mathrm{Fun}(\mathcal{G}, \mathcal{G}')$$

where the functor category has as objects functors and as morphisms natural transformations.

Thus from Proposition 3.1.6 we obtain the proposition as every groupoid is cofibrant and fibrant in this model structure. □

In particular, if $\mathcal{C}$ is an ∞-category we obtain an equivalence

$$\mathrm{Fun}(\mathrm{Gpd}_2, \mathcal{C}) \longrightarrow \mathrm{Fun}^W(N\mathrm{Gpd}, \mathcal{C}) \ .$$

We observe that the $(2,1)$-category Grp_2 of groups, where the 2-morphisms are conjugations, is equivalent to the full subcategory of Gpd_2 on connected groupoids. In particular, if $\mathcal{C}$ has coproducts we also obtain an equivalence

$$\mathrm{Fun}(\mathrm{Grp}_2, \mathcal{C}) \longrightarrow \mathrm{Fun}^{W,\oplus}(N\mathrm{Gpd}, \mathcal{C})$$

where the superscript $\oplus$ refers to functors that in addition respect coproducts.

We notice that homotopy left Kan extension along the inclusion $\mathrm{Grp}_2 \to \mathrm{Gpd}_2$ provides a functor

$$\mathrm{Fun}(\mathrm{Grp}_2, \mathrm{Sp}_\infty) \to \mathrm{Fun}^W(N\mathrm{Gpd}, \mathrm{Sp}_\infty).$$

Thus functors $X\colon \mathrm{Gpd} \to \mathrm{Sp}$ that send equivalences to equivalences and coproducts to sums as in [DL98] may be interpreted as

$$X \in \mathrm{Fun}^{W,\oplus}(N\mathrm{Gpd}, \mathrm{Sp}_\infty) \simeq \mathrm{Fun}(\mathrm{Grp}_2, \mathrm{Sp}_\infty).$$

Since the functors we compare in this thesis (and the natural transformation) have as domain KK_∞ it will suffice to prove the following

Theorem 6.0.5. *There exists a functor*

$$\mathrm{Grp}_2 \xrightarrow{C^*} \mathrm{KK}_\infty$$

which on objects takes a group G and sends it to the full group C^*-algebra C^*G.

An obvious strategy for a proof is to see that there is an evident functor

$$\mathrm{Grp}_2 \to \mathrm{C^*Alg}_2$$

where by $\mathrm{C^*Alg}_2$ we denote the $(2,1)$-category of C^*-algebras where 2-morphisms are conjugations by unitary elements in the target. Then we would like to argue that the inclusion

$$\mathrm{C^*Alg} \to \mathrm{C^*Alg}_2$$

induces an equivalence after localizing along the KK-equivalences. A good reason to believe that this should be true is that given two morphisms $f, g\colon A \to B$ between C^*-algebras that are conjugated by a unitary of B, i.e. there exists a $u \in U(B)$ such that $g = c_u(f)$, then in the usual KK-*group* we have a canonical equality

$$[f] = [g] \in KK(A, B).$$

This is given as follows. Recall that

$$[f] = [B, f, 0] \text{ and } [g] = [B, g, 0]$$

in terms of Kasparov triples. It is easily seen that the morphism

$$\begin{aligned} B &\longrightarrow B \\ x &\longmapsto ux \end{aligned}$$

is an automorphism of the Hilbert-B-module B that intertwines the two represenations f and g. We hoped that this canonical equivalence would imply that one can explicitely write down an ∞-functor $\mathrm{C^*Alg}_2 \to \mathrm{KK}_\infty$ but since we obtained the localization KK_∞ in such abstract way we struggled to do this.

It turns out that one can construct such a functor, but one is naturally led to consider C^*-*categories*. We will come back to this point of view a little later.

For the moment we want to continue to prove the existence of a functor $\mathrm{Grp}_2 \to \mathrm{KK}_\infty$ as in the theorem. As we have argued earlier this will follow if we are able to construct a functor

$$\mathrm{Gpd} \to \mathrm{KK}_\infty$$

that sends equivalences to equivalences. For this we will construct a functor

$$\mathrm{Gpd} \to \mathrm{C^*Alg}$$

that sends equivalences of groupoids to KK-equivalences of C^*-algebras.

We construct a C^*-algebra out of a groupoid $\mathcal{G}$ as follows. We let $\mathbb{C}\mathcal{G}$ be the $\mathbb{C}$-linearization of the set of morphisms of $\mathcal{G}$. This is a ring by linearization of the multiplication on morphisms given by

$$f \cdot g = \begin{cases} f \circ g & \text{if } f \text{ and } g \text{ are composable} \\ 0 & \text{else.} \end{cases}$$

Then we complete this $\mathbb{C}$-algebra in a universal way (like for the full group C^*-algebra) to obtain a C^*-algebra $C^*\mathcal{G}$. In other words this is the C^*-algebra associated to the maximal groupoid C^*-*category* as in [Del12, section 3.3] and using [Joa03, section 3].

Remark. If $\mathcal{G}$ has a countable set of objects, then $C^*\mathcal{G}$ is separable.

Lemma 6.0.6. *This is functorial with respect to cofibrations of groupoids, i.e. functors that are injective on the set of objects.*

Proof. If $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ is such a functor then it induces a functorial morphism of $\mathbb{C}$-algebras

$$F_*\colon \mathbb{C}\mathcal{G}_1 \to \mathbb{C}\mathcal{G}_2.$$

The completion is made in a way that any such map extends to a morphism

$$F_*\colon C^*\mathcal{G}_1 \to C^*\mathcal{G}_2.$$

In other words, the construction of [Joa04, section 3] that takes a C^*-category $\mathcal{C}$ to the C^*-algebra $A_\mathcal{C}$ is functorial when we restrict to functors that are injective on the set of objects. Only to obtain a full functor one is led to do the more fancy construction $\mathcal{C} \mapsto A^f_\mathcal{C}$ of [Joa03, section 3]. □

Proposition 6.0.7. *If $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ is an acyclic cofibration, then $F_*\colon C^*\mathcal{G}_1 \to C^*\mathcal{G}_2$ is a KK-equivalence.*

Proof. It suffices to show that the acyclic cofibration given by including the full subcategory on a single object induces a KK-equivalence on C^*-algebras. In fact they are even Morita equivalent, compare [DL98, Remark 2.3]. □

Now we use the following result from abstract homotopy theory. Here is the setup. Let $(\mathcal{C}, W)$ be the relative category underlying a category of fibrant objects. Assume that $(\mathcal{C}, W)$ admits functorial factorization and that every object is fibrant. Let $(\mathcal{C}', W')$ be the relative category given as follows. The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$ and the morphisms of $\mathcal{C}'$ are the fibrations of $\mathcal{C}$. The collection W' is the collection of acyclic fibrations.

Theorem 6.0.8. *The obvious inclusion*

$$(\mathcal{C}', W') \to (\mathcal{C}, W)$$

of relative categories induces an equivalence on the associated ∞-categories.

Remark. The dual statement for categories of cofibrant objects holds as well.

Remark. We want to thank Karol Szumiło for providing a proof of a lemma which we use to prove this theorem. We postpone the proof to the end of this appendix and will then explain precisely what he contributed to this theorem.

We recall that the category of groupoids is a category of cofibrant objects in which the weak equivalences are the equivalences of groupoids and the cofibrations are the functors that are injective on the set of objects.

The same is true if we take the category of groupoids with at most countable many objects. Thus if we denote by Gpd^{cof} the relative category associated to the cofibrations in Gpd, we obtain the following

Corollary 6.0.9. *We have equivalences of functor categories*

$$\mathrm{Fun}^W(\mathrm{Gpd}, \mathrm{KK}_\infty) \simeq \mathrm{Fun}(\mathrm{Gpd}_2, \mathrm{KK}_\infty) \simeq \mathrm{Fun}^W(\mathrm{Gpd}^{cof}, \mathrm{KK}_\infty).$$

where the superscript W refers to functors that send equivalences to equivalences.

Thus in order to construct an ∞-functor

$$\mathrm{Gpd}_2 \to \mathrm{KK}_\infty$$

it suffices to construct a functor of 1-categories

$$\mathrm{Gpd}^{cof} \to \mathrm{C}^*\mathrm{Alg}$$

which has the *property* that it sends equivalences of groupoids to KK-equivalences. We have just constructed such a functor in Lemma 6.0.6. This finishes the proof of Theorem 6.0.5.

To come back to the question whether this comes from a functor

$$\mathrm{C}^*\mathrm{Alg}_2 \to \mathrm{KK}_\infty$$

one can indeed do similar arguments.

For this we first consider the category of C^*-categories which we denote by $\mathrm{C}^*\mathrm{CAT}$. There is a model structure on $\mathrm{C}^*\mathrm{CAT}$ called the unitary model structure, see [Del12]. Its weak equivalences are what are called unitary equivalences and similarly to the case of groupoids one can see that the full subcategory of

$$\mathrm{C}^*\mathrm{CAT}[W^{-1}]$$

on connected C^*-categories is equivalent to $\mathrm{C}^*\mathrm{Alg}_2$, the $(2,1)$-category of C^*-algebras as decribed above.

The cofibrations in this model structure are functors that are injective on objects, thus the construction $\mathcal{C} \mapsto A_{\mathcal{C}}$ of [Joa03, section 3] is a functor

$$\mathrm{C}^*\mathrm{CAT}^{cof} \to \mathrm{C}^*\mathrm{Alg}$$

sending equivalences to KK-equivalences. Again by using Theorem 6.0.8 we see that this gives a functor

$$\mathrm{C}^*\mathrm{Alg}_2 \to \mathrm{KK}_\infty$$

as claimed.

6.0.2 A proof of Theorem 6.0.8

We recall the setup. Let $(\mathcal{C}, w, f)$ be a fibration category in which all objects are fibrant. We aim to compare the ∞-categories associated to the relative categories $(\mathcal{C}, w)$ and $(f\mathcal{C}, wf)$. For this we first recall that the homotopy theory of $(\infty,1)$-categories is modelled by bisimplicial sets (which we will refer to as simplicial spaces) with the Rezk model structure, see [Rez01]. Furthermore we will use the following

Lemma 6.0.10. *Let $f\colon X \to Y$ be a map of simplicial spaces that is a level weak equivalence. Then it is a weak equivalence in the Rezk model structure.*

The strategy now is to first explain the simplicial space associated to a relative category $(\mathcal{D}, w)$ called the *classification diagram* of Rezk which is given by

$$[n] \mapsto N(w(\mathcal{C}^{[n]})),$$

compare [Rez01, section 3.3], where the weak equivalences in $\mathcal{C}^{[n]}$ are levelwise weak equivalences.

Next we need the following properties.

Lemma 6.0.11. *Suppose $(\mathcal{C}, w, f)$ is a fibration category with functorial factorization. We let $\mathcal{C}_f$ denote the full subcategory of fibrant objects of $\mathcal{C}$. Then the canonical map*

$$N(w(\mathcal{C}_f)) \to N(w(\mathcal{C}))$$

is a weak equivalence.

Proof. We suppress the inclusion $\mathcal{C}_f \to \mathcal{C}$ of the full subcategory from the notation. By assumption we have a functorial fibrant replacement functor

$$R\colon \mathcal{C} \to \mathcal{C}_f$$

which comes with a natural weak equivalence $\mathrm{id} \to R$. Namely given a morphism $x \to y$ in $\mathcal{C}$ we consider the diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

and functorially replace the vertical arrows by a weak equivalence followed by a fibration to obain a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \sim\downarrow & & \downarrow\sim \\ Rx & \xrightarrow{Rf} & Ry \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

Moreover it follows from the commutativity of this diagram that the functor R preserves weak equivalences. We thus obtain the lemma. □

Lemma 6.0.12. *Let $(\mathcal{C}, w, f)$ be a fibration category with functorial factorization. Then the canonical map*

$$N(wf(\mathcal{C})) \to N(w(\mathcal{C}))$$

is a weak equivalence.

Proof. The proof of this lemma was provided to us by Karol Szumiło whom we wish to thank for his time and effort to do this.

We consider the bisimplicial set $W\mathcal{C}$ whose (m, n) simplices are given diagrams of shape $[m] \times [n]$ in $\mathcal{C}$ whose arrows in the n-direction are weak equivalences and whose arrows in m-direction are acyclic fibrations. We thus have that

$$(W\mathcal{C})_{m,\bullet} = N(w(\mathcal{C}^{\widehat{[m]}}_{Rf})) \text{ and } (W\mathcal{C})_{\bullet,n} = N(wf(\mathcal{C}^{\widehat{[n]}}))$$

where the subscript Rf refers to the Reedy fibrant objects in $\mathcal{C}^{[m]}$ and the hat refers to homotopical functors with respect to all morphisms being weak equivalences in the category $[m]$ or $[n]$. Now we claim that the bisimiplicial set $W\mathcal{C}$ is homotopically constant along both the m and the n-direction. Then it follows from a lemma of Quillen, see [Qui73, page 94], that there is a weak equivalence

$$N(w(\mathcal{C})) = (W\mathcal{C})_{0,\bullet} \simeq \mathrm{diag}(W\mathcal{C}) \simeq (W\mathcal{C})_{\bullet,0} = N(wf(\mathcal{C}))$$

which proves the lemma.
We thus continue to show that the bisimplicial set $W\mathcal{C}$ is homotopically constant in both directions. In [Szu14, Lemma 1.17] it is shown that any simplicial operator $[k] \to [l]$ (for instance $[0] \to [m]$) induces an equivalence of fibration categories

$$\mathcal{C}^{\widehat{[l]}} \xrightarrow{\sim} \mathcal{C}^{\widehat{[k]}} \text{ and } \mathcal{C}^{\widehat{[l]}}_{Rf} \xrightarrow{\sim} \mathcal{C}^{\widehat{[k]}}_{Rf}.$$

In the second case this induces a weak equivalence on nerves of weak equivalences

$$N(w(\mathcal{C}^{\widehat{[l]}}_{Rf})) \xrightarrow{\sim} N(w(\mathcal{C}^{\widehat{[k]}}_{Rf}))$$

which follows from [KS16, Prop. 2.1, Cor. 3.8 and Lemmas 5.2 and 5.8].

To finish the argument that the bisimplicial set $W\mathcal{C}$ is homotopocially constant it thus suffices to prove that

$$N(wf(\mathcal{C}^{\widehat{[0]}})) \simeq N(wf(\mathcal{C}^{\widehat{[n]}}))$$

for all $n \geq 0$. We have already stated that the two fibration categories $\mathcal{C}^{\widehat{[0]}}$ and $\mathcal{C}^{\widehat{[n]}}$ are equivalent but this is not enough to deduce an equivalence of the nerve of the category of *acyclic fibrations*. But we can argue as follows. Suppose more generally that $\mathcal{D}$ and $\mathcal{D}'$ are equivalent fibration categories, that is there are exact functors

$$F \colon \mathcal{D} \rightleftarrows \mathcal{D}' \colon G$$

and zig-zags of natural weak equivalences connecting FG to $\mathrm{id}_{\mathcal{D}'}$ and GF to $\mathrm{id}_{\mathcal{D}}$. If the natural weak equivalences are acyclic fibrations then the induced map

$$N(wf(\mathcal{D})) \xrightarrow{\sim} N(wf(\mathcal{D}'))$$

is an equivalence.

For the case in question we consider the two functors

$$\mathrm{ev}_0 \colon \mathcal{C}^{\widehat{[n]}} \rightleftarrows \mathcal{C}^{\widehat{[0]}} \colon \mathrm{const}$$

Obviously the compostion

$$\mathcal{C}^{\widehat{[0]}} \xrightarrow{\mathrm{const}} \mathcal{C}^{\widehat{[n]}} \xrightarrow{\mathrm{ev}_0} \mathcal{C}^{\widehat{[0]}}$$

is the identity. So we only need to consider the functor

$$\mathcal{C}^{\widehat{[n]}} \xrightarrow{\mathrm{const} \circ \mathrm{ev}_0} \mathcal{C}^{\widehat{[n]}}$$

which is given by taking the top row in the following diagram to the bottom row.

$$\begin{array}{ccccccc}
x_0 & \xrightarrow{\sim} & x_1 & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & x_n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
x_0 & \longrightarrow & x_0 & \longrightarrow & \cdots & \longrightarrow & x_0
\end{array}$$

The obvious vertical arrows (the compositions of the top horizontal arrows) provide a natural equivalence

$$\mathrm{const} \circ \mathrm{ev}_0 \to \mathrm{id}_{\mathcal{C}^{\widehat{[n]}}}.$$

This natural map is not a fibration but we can replace it by a fibration as follows. For all $i \in \{0, \ldots, n\}$ we consider the diagram

$$\begin{array}{ccc}
x_0 & \longrightarrow & x_0 \times x_i \\
{\scriptstyle\sim}\downarrow & \nearrow\!\!\!\nearrow & \\
\overline{x}_i & &
\end{array} \tag{6.1}$$

Since we have functorial factorization it follows that there is a commutative diagram

$$\begin{array}{ccccccc}
x_0 & \longrightarrow & x_1 & \longrightarrow & \ldots & \longrightarrow & x_n \\
\uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
\overline{x}_0 & \longrightarrow & \overline{x}_1 & \longrightarrow & \ldots & \longrightarrow & \overline{x}_n \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
x_0 & \longrightarrow & x_0 & \longrightarrow & \ldots & \longrightarrow & x_0
\end{array}$$

in which the vertical arrows are fibrations by construction. Moreover since both components of the map

$$x_0 \longrightarrow x_0 \times x_i$$

are weak equivalences it follows from the 2 out of 3 property for weak equivalences in the diagram (6.1) that the vertical maps are acyclic fibrations. Hence we have constructed a natural zig-zag of acyclic fibrations connecting the two functors

$$\text{const} \circ \text{ev}_0 \text{ and } \text{id}_{\widehat{\mathcal{C}^{[n]}}}.$$

We are thus in the position to conclude that

$$N(wf(\mathcal{C}^{\widehat{[n]}})) \simeq N(wf(\mathcal{C}^{\widehat{[0]}}))$$

for all n and thus the bisimplicial set $W\mathcal{C}$ is homotopically constant. This finishes the proof of the lemma. □

Having these lemmas we are now able to prove Theorem 6.0.8. We have the following chain of equivalences.

$$N(w(\mathcal{C}^{[n]})) \xleftarrow{\sim} N(w(\mathcal{C}^{[n]}_{Rf})) \xleftarrow{\sim} N(wf(\mathcal{C}^{[n]}_{Rf}))$$

where the left equivalence is given by Lemma 6.0.11 and the right equivalence is given by Lemma 6.0.12. Here, the subscript Rf again refers to the *Reedy fibrant objects* of the category $\mathcal{C}^{[n]}$ (as opposed to the level fibrant objects).

In order to prove Theorem 6.0.8 it thus suffices to verify that

$$N(wf(\mathcal{C}^{[n]}_{Rf}))$$

is the classification diagram associated to the relative category $(f\mathcal{C}, wf)$. This follows easily as soon as we notice that Reedy fibrant objects in $\mathcal{C}^{[n]}$ are precisely diagrams $x_\bullet$ of the form

$$x_1 \twoheadrightarrow x_2 \twoheadrightarrow \cdots \twoheadrightarrow x_n \ .$$

For this we recall first that in this situation the matching object satisfies

$$(Mx)_k = (Mx_\bullet)_k = x_{k+1}$$

So the object $x_\bullet$ is Reedy fibrant if and only if all x_i are fibrant (recall we assumed that every object in $\mathcal{C}$ is fibrant) and the canonical map

$$x_k \to (Mx)_k = x_{k+1}$$

is a fibration as claimed.

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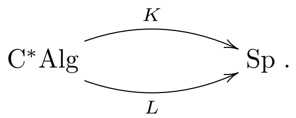
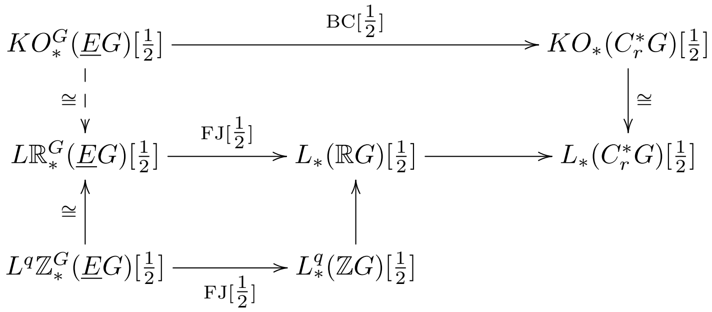
Summary

In this thesis we study the relationship between two *spectra* valued functors on the category of *separable, complex* C^*-algebras, the first one being topological K-theory and the second one being projective symmetric L-theory of the underlying involutive ring:



The main motivation of this comparison is to relate the L-theoretic Farrell-Jones conjecture to the Baum-Connes conjecture in topological K-theory. This is done by considering the following diagram. Let G be a countable discrete group.



The dashed arrow would exist and be an isomorphism (making the diagram commutative) if the two functors $K[\frac{1}{2}]$ and $L[\frac{1}{2}]$ were equivalent on the category of real C^*-algebras.

In this thesis we only discuss the case of complex C^*-algebras. As input we use that there exists a natural transformation

$$\tau\colon \pi_*(K) \to \pi_*(L)$$

on the level of homotopy groups, see Theorem 2.2.31.

In Section 2.1 we introduce the main properties of the topological K-theory functor $K\colon \mathrm{C^*Alg} \to \mathrm{Ab}_{\mathbb{Z}}$, the most important property being that if factors over the KK-category and becomes corepresentable by the tensor unit $\mathbb{C}$, see Proposition 2.1.29. We use the transformation τ to deduce that L-theory also factors over the KK-category and that we can thus reinterpret the transformation via the Yoneda lemma. The main idea is to translate the universal property of K-theory (being corepresentable on the KK-category) to a statement about the *spectrum valued functor* $K\colon \mathrm{C^*Alg} \to \mathrm{Sp}$.

For this we prove the existence of a stable ∞-category KK_∞ which is a Dwyer-Kan localization of $\mathrm{C^*Alg}$ along the KK-equivalences, see Definition 3.2.1 and Lemma 3.2.2. We show that the spectrum valued K-functor factors over KK_∞ and again becomes corepresentable and that furthermore L-theory factors over KK_∞, see Theorem 3.2.3.

We recall that L-theory commutes with finite products, see Proposition 4.2.1 but is not an exact functor due to problems with the control in algebraic K-theory, the decorations, see Theorem 2.2.21. We remark that L-theory becomes exact if we invert 2, compare Corollary 2.2.17.

We then use the Yoneda lemma in ∞-categories to calculate the space of transformations

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty, \mathrm{Sp}_\infty)}(k, \ell) \simeq \Omega^\infty(\ell(\mathbb{C})),$$

see Corollary 4.2.2 and find a transformation $\tau\colon k \to \ell$ that induces an isomorphism

$$\pi_i(\tau_A)\colon \pi_i(kA) \to \pi_i(\ell A)$$

for $i = 0$ and 1 but not on higher homotopy groups, see Corollary 4.2.4, Proposition 4.2.5, and Corollary 4.2.6. This uses all calculations established in Section 2.2.

Furthermore we calculate

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{KK}_\infty,\mathrm{Sp}_\infty)}(K, L[\tfrac{1}{2}]) \simeq \Omega^\infty(L\mathbb{C}[\tfrac{1}{2}])$$

in which we find a transformation that induces an equivalence

$$\hat{\tau}\colon K[\tfrac{1}{2}] \xrightarrow{\simeq} L[\tfrac{1}{2}],$$

see Theorem 4.2.7.

In Chapter 5 we prove that one cannot improve the above transformations in the following sense. If we look for integral maps (maps existing prior to any form of localization) that induce an equivalence *after* inverting 2 we see that the only possible way for this to happen is to consider the mapping space

$$\mathrm{Map}(ku, \ell\mathbb{C}),$$

compare Theorem 5.0.3.

At last, in the appendix we sketch how to translate between the language of ∞-categories and the language of ordinary category theory. We give a proof of the well known theorem in higher categories that the ∞-category obtain from the 1-category of groupoids by universally inverting the equivalences of groupoids is represented by the $(2,1)$-category of groupoids, see Proposition 6.0.4. We use this to relate functors $\mathrm{Gpd} \to \mathrm{Sp}$ that send equivalences to equivalences (as discussed in the usual approach to assembly as in [DL98]) to ∞-functors $\mathrm{Grp}_2 \to \mathrm{Sp}_\infty$. At last we prove that there is an ∞-functor

$$\mathrm{Grp}_2 \to \mathrm{KK}_\infty$$

which on objects sends a group G to the group C^*-algebra C^*G, see Theorem 6.0.5.