Higher cross-ratios and geometric functional equations for polylogarithms

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Danylo Radchenko

aus

Kiew, Ukraine

Bonn 2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Don Zagier

2. Gutachter: Prof. Dr. Catharina Stroppel

Tag der Promotion: 11.07.2016

Erscheinungsjahr: 2016

Summary

In this work we define and study a generalization of the classical cross-ratio. Roughly speaking, a generalized cross-ratio is a function of n points in a projective space that is invariant under the change of coordinates and satisfies an arithmetic condition somewhat similar to the Plücker relation. Our main motivation behind this generalization is an application to the theory of polylogarithms and a potential application to a long-standing problem in algebraic number theory, Zagier's conjecture.

We study S-unit equations of Erdős-Stewart-Tijdeman type in rational function fields. This is the equation $x + y = 1$, with the restriction that x and y belong to some two fixed multiplicative groups of rational functions. We present an algorithm that enumerates solutions of such equations. Then we study the way in which the solutions change if we fix the multiplicative group of one of the summands and vary the multiplicative group of the other. It turns out that there is a maximal set of "interesting" solutions which is finite and depends only on the first multiplicative group, we call such solutions exceptional.

Next, we study the properties of the algebra of SL_d-invariant polynomial functions on n-tuples of points in d-dimensional vector space. The general definition of S-cross-ratios and exceptional S-cross-ratios is then given. After that, we look at the problem of classifying exceptional S-cross-ratios in the special case when S is the set of all $d \times d$ minors. We give such classification for small values of n, d and then, based on these calculations, give a general conjectural description.

Chapter 4 is devoted to the review of the classical and single-valued polylogarithms and the conjecture of Zagier about special values of Dedekind zeta functions. We then outline Goncharov's strategy for proving Zagier's conjecture. It is this strategy that serves as our main motivation for the study of exceptional cross-ratios.

In the last chapter we apply the results about S-unit equations to the theory of polylogarithms by proving a finiteness result for the space of nontrivial functional equations. By making use of exceptional cross-ratios that were computed in Chapter 3, we construct many symmetric functional equations for polylogarithms. We conclude by discussing the prospect of using these functional equations for proving Zagier's conjecture in the next open case.

Contents

CHAPTER 1

Introduction

The classical cross-ratio, which assigns, in a coordinate-free way, a number to any 4 distinct points on a projective line, is one of the most important constructions in mathematics. It is therefore natural to look for interesting coordinate-free ways of assigning numbers to collections of n points in $\mathbb{P}^{m-1}$ for values of (n, m) other than $(4, 2)$. Because of an important potential application to the theory of polylogarithms that we will describe later, the case $n = 2m$ is of particular interest. Specifically, if we can find a higher cross-ratio in this case with a certain magical property, then there is a chance of proving Zagier's conjecture relating the algebraic K-group K_{2m-1} of a number field to the value of the Dedekind zeta function of that field at $s = m$ using the m-th polylogarithm function. This approach was used by Goncharov, who proved the conjecture in the case $m = 3$ by constructing such a cross-ratio. However, the problem of finding an appropriate cross-ratio for $m \geq 4$ has been open for more than 25 years.

Roughly, the "magical property" needed is that the cross-ratio (or cross-ratios: we usually need many of them) is a number r, defined in an invariant fashion on the n-tuple of points, which is very highly factored and for which $1 - r$ also has a non-trivial factorization. For instance, the classical cross-ratio of 4 points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \overline{P}_4 \in \mathbb{P}^1(\mathbb{C})$ is the number

$$r \;=\; \frac{|P_1P_3||P_2P_4|}{|P_1P_4||P_2P_3|},$$

where P_i is the lift of $\overline{P}_i$ to $\mathbb{C}^2$ and $|P_iP_j|$ is the obvious determinant of a 2×2 matrix. Here the magical property is the Plücker identity

$$1 - \frac{|P_1P_3||P_2P_4|}{|P_1P_4||P_2P_3|} \;=\; \frac{|P_1P_2||P_3P_4|}{|P_1P_4||P_2P_3|},$$

where $1-r$ is just as factorizable as r (and is in fact another cross-ratio). Goncharov's construction for $m = 3$, on the other hand, involves the cross-ratio

$$r \;=\; \prod_{i \pmod 3} \frac{|P_iP_{i+1}Q_i|}{|P_iP_{i+1}Q_{i+1}|}$$

of 6 points $\overline{P}_1, \overline{P}_2, \overline{P}_3, \overline{Q}_1, \overline{Q}_2, \overline{Q}_3 \in \mathbb{P}^2$, and the magical property follows from the identity

$$\prod_{i \pmod 3} |P_iP_{i+1}Q_i| - \prod_{i \pmod 3} |P_iP_{i+1}Q_{i+1}| \;=\; |P_1P_2P_3| \cdot |P_1 \times Q_1, P_2 \times Q_2, P_3 \times Q_3|,$$

where $P_i \times Q_i \in \mathbb{C}^3$ is the cross-product. As these examples make clear, the quality of a cross-ratio depends on the set of "prime factors" that we allow. So just as in classical number theory when we replace units by S-units for some finite set of primes S that one decides to accept, one can introduce and study a notion of S-cross-ratios.

The main purpose of this thesis is to develop this notion systematically and to construct a wide class of examples having non-trivial factorization property of the type required. Whether they will suffice to prove further cases of Zagier's conjecture on polylogarithms remains to be seen, but we have already been able to carry out part of Goncharov's program.

This work is organized as follows.

Chapter 2:

In this chapter we study functional S-unit equations of Erdős-Stewart-Tijdeman type. By this we mean equations of the form

$$x + y = 1, \quad x \in \Gamma_1,\ y \in \Gamma_2, \tag{1.1}$$

where Γ_1 and Γ_2 are some multiplicative subgroups (usually finitely generated) of the rational function field $\mathbb{K}$ over the ground field $\mathbf{k}$. Under some simple conditions on Γ_1 and Γ_2 we prove that the set of solutions is finite and give a simple algorithm that enumerates all solutions. This algorithm is given in terms of an oracle that computes the kernel in Γ_1 of the "reduction modulo π", where π is an irreducible polynomial. In Section 2.3 we give an efficient Las Vegas type algorithm for this oracle. We also discuss several methods of enumerating points with small ℓ_1-norm inside a given subgroup of $\mathbb{Z}^k$.

In Section 2.4 we discuss the way in which solutions to (1.1) depend on Γ_2. This turns out to be of great importance in the application to functional equations for polylogarithms. We establish a stability result that, roughly speaking, says that for a fixed finitely generated Γ_1 there exists a maximal group Γ_2 which contains all the "interesting" solutions to (1.1). We call these "interesting" solutions exceptional and, using the results of Maurin [26] and Bombieri, Masser, and Zannier [6] from the theory of unlikely intersections, we prove that there are only finitely many exceptional elements in any finitely generated group Γ_1.

In the last section we describe some known results about more general types of S-unit equations. In particular, we discuss the theorem of Mason, which gives an explicit upper bound on the projective height of $(x_1, \ldots, x_n)$, where x_i are S-units in a function field in one variable that satisfy $x_1 + \cdots + x_n = 0$. We then look at a related problem of finding all linearly dependent (over $\mathbf{k}$) n-tuples $(x_1, \ldots, x_n) \in \Gamma^n$, where Γ is a finitely generated subgroup of $\mathbb{K}^\times$. We show how Wronskian determinants can be used to reduce this problem to that of solving a certain system of polynomial equations.

Chapter 3:

We begin by defining the general notion of a (higher) cross-ratio. In our definition, a cross-ratio is any GL_d-invariant rational function of n-tuples of points in $\mathbb{P}^{d-1}$. It is convenient to think of n-tuples of points in $\mathbb{P}^{d-1}$ as an equivalence class of $d \times n$ matrices under right multiplication by nonsingular diagonal matrices. A cross-ratio is then a rational function defined on $d \times n$ matrices that satisfies two invariance properties: projective invariance and SL_d-invariance.

In the first section, following the book of Sturmfels [32], we define and briefly summarize the main properties of the so-called bracket algebra. This is the subalgebra generated by all $d \times d$ minors inside the polynomial algebra $\mathbb{Q}[x_{11}, \ldots, x_{dn}]$. The $d \times d$ minor formed from the columns $i_1, \ldots, i_d$ is denoted by $\langle i_1 \ldots i_d \rangle$ and is called a bracket. The so-called first fundamental theorem of invariant theory states that any SL_d-invariant polynomial in $\mathbb{Q}[x_{11}, \ldots, x_{dn}]$ can be expressed in terms of brackets, i.e., belongs to the bracket algebra. This allows us to redefine cross-ratios explicitly, as multi-homogeneous rational functions in brackets of multi-degree 0. The second fundamental theorem describes the ideal of polynomial relations between brackets, which turns out to be generated by the Grassmann-Plücker relations. This allows one to define the bracket algebra as a quotient, without any need to embed it into a polynomial algebra, and this point of view is useful for understanding some of its properties.

In section 3.2 we define the general notion of an S-cross-ratio. We use the letter S to denote sets of distinct (modulo $\mathbb{Q}^\times$) irreducible multi-homogeneous polynomials in the bracket algebra. We define S-cross-ratio to be a cross-ratio that can be written as a product of elements (or their inverses) of S times a unit ± 1. Since the set of S-cross-ratios forms a multiplicative group, we can also define and study the set of exceptional (in the sense of Chapter 2) elements in it, and we refer to these elements as exceptional S-cross-ratios. The concept of an exceptional cross-ratio should be seen as the central notion of this thesis.

A very natural choice for the set S in the bracket algebra is the set of all brackets $\langle i_1 \dots i_d \rangle$. Let us call the S-cross-ratios in this case bracket cross-ratios. In other words, bracket cross-ratios are the cross-ratios that can be written as a quotient of two bracket monomials. Section 3.3 is devoted to the problem of finding all exceptional bracket cross-ratios. By applying the algorithms of Chapter 2 we find many examples of exceptional bracket cross-ratios for $d = 2, 3, 4$ and $n \leq 10$. We conjecture that these lists of exceptional cross-ratios are complete. For any given pair (n, d) the set of all such cross-ratios can in principle be computed rigorously, but we expect that such computation would be very expensive. By analyzing these exceptional cross-ratios we then propose a conjectural description of exceptional bracket cross-ratios for all $n > d \geq 2$.

Finally, we also consider the problem of classifying exceptional S-cross-ratios for some other choices of S, which turns out to be necessary for the application to Zagier's conjecture.

Chapter 4:

In Chapter 4 we provide the necessary background on classical and single-valued polylogarithms, the functional equations that they satisfy, and the conjecture of Zagier about special values of Dedekind zeta functions that serves as one of our main motivations for the definition and study of exceptional cross-ratios.

We begin by defining the classical m-th polylogarithm function Li_m. This is the analytic function defined in the unit disc by the Taylor series

$$\mathrm{Li}_m(z) \;=\; \sum_{n=1}^{\infty} \frac{z^n}{n^m}.$$

We then define the single-valued polylogarithm function $\mathcal{L}_m \colon \mathbb{C} \to \mathbb{R}$. This function is related to the classical polylogarithm Li_m in much the same way as the function $\log|z|$ is related to the usual logarithm. These functions satisfy many interesting functional equations, arguably the most famous of them is the five-term identity

$$\mathcal{L}_2(x) + \mathcal{L}_2(y) + \mathcal{L}_2\left(\frac{1-x}{1-xy}\right) + \mathcal{L}_2(1-xy) + \mathcal{L}_2\left(\frac{1-y}{1-xy}\right) \;=\; 0$$

for the dilogarithm. We give several examples of such relations for $\mathcal{L}_2$ and $\mathcal{L}_3$ and then discuss the general algebraic criterion for functional equations for polylogarithms.

In section 4.2 we formulate the weak form of Zagier's conjecture. Roughly, this conjecture states that the Dedekind zeta function $\zeta_F(s) = \sum_{\mathfrak{a}} N\mathfrak{a}^{-s}$ of a number field F evaluated at a positive integer $m \geq 2$ can be expressed in terms of the function $\mathcal{L}_m$ evaluated at some points in the union of all complex embeddings of F.

We describe the higher Bloch groups in section 4.3, where we also state the strong form of Zagier's conjecture and give some examples of calculations in the Bloch groups. In section 4.4 we describe the strategy of Goncharov for proving Zagier's conjecture. The main idea is to construct a morphism between two complexes: the Grassmannian complex, and the so-called Goncharov complex that is constructed from higher Bloch groups. We then show how Goncharov's approach is realized for $m = 2$ and $m = 3$.

Chapter 5:

In this chapter we look for functional equations for polylogarithms that can be constructed from cross-ratios obtained in Chapter 3. We begin by describing some general properties of such functional equations. We define the space $\mathcal{E}_m(Y)$ of functional equations for $\mathcal{L}_m$ with values (arguments) in a set $Y \subset \mathbb{K}^\times$ and the space $\widehat{\mathcal{E}}_m(\Gamma)$ of all nontrivial functional equations for $\mathcal{L}_m$ with values in the group $\Gamma \subseteq \mathbb{K}^\times$ (by nontrivial we mean that it does not follow from the reflection and distribution properties of $\mathcal{L}_m$). We then prove that for any admissible group Γ the space $\widehat{\mathcal{E}}_m(\Gamma)$ is finite-dimensional and it can be computed explicitly if one knows the set of all exceptional elements in Γ. This is one of the main results of this thesis.

We then define the classes of functional equations with cross-ratios of n points in $\mathbb{P}^{d-1}$ as their arguments, we call such functional equations geometric. These geometric functional equations form a vector space with a natural action of $\mathfrak{S}_n$, and in sections 5.3 and 5.4 we investigate the symmetric and skew-symmetric parts of these vectors spaces in the special case of bracket cross-ratios.

We also define a special class of geometric functional equations that we call $\mathcal{L}_m$-cocycles. Our motivation for studying these objects is that the space of $\mathcal{L}_m$-cocycles contains all possible candidates for the morphism that one needs in Goncharov's strategy and thus could be useful in proving Zagier's conjecture. In particular, if Goncharov's "optimistic conjecture" holds, then the space of $\mathcal{L}_m$-cocycles on $2m$ points in $\mathbb{P}^{m-1}$ is nonempty and contains a nontrivial skew-symmetric element. One of our main results is construction of such skew-symmetric elements for $m = 4$. We also give constructions of skew-symmetric $\mathcal{L}_m$-cocycles in several other cases.

We end this work by discussing the prospect of using our methods to complete the proof of Zagier's conjecture (following the strategy outlined by Goncharov) in the case $m = 4$.

Appendix A:

We collect tables of exceptional bracket cross-ratios that were computed in Chapter 3. We also give lists of special irreducible polynomials (i.e., the polynomials that occur in the factorization of $1 - r$ for exceptional cross-ratios.

Appendix B:

In this appendix we collect tables of certain symmetric functional equations for polylogarithms (see Section 5.3 for details).

List of symbols

$\mathbb{N}$ – the natural numbers $\{1, 2, 3, \dots\}$.

$\mathbb{Z}$ – the ring of integers.

$\mathbb{Q}$ – the field of rational numbers.

$\overline{\mathbb{Q}}$ – the algebraic closure of $\mathbb{Q}$.

$\mathbb{R}$ – the field of real numbers.

$\mathbb{C}$ – the field of complex numbers.

$\mathbf{k}$ – a field of characteristic 0.

μ_n – the multiplicative group of n-th roots of unity in $\mathbf{k}$.

μ_∞ – the multiplicative group of all roots of unity in $\mathbf{k}$.

$\mathcal{P}$ – the polynomial algebra $\mathbf{k}[x_1, \dots, x_l]$.

$\mathcal{P}^{(n)}$ – the space of homogeneous polynomials of degree n.

$\deg(p)$ – the degree of a homogeneous polynomial $p \in \mathcal{P}$.

$\mathbb{K}$ – the homogeneous fraction field of $\mathcal{P}$.

S – a set of inequivalent irreducible polynomials in $\mathcal{P}$.

Γ – a subgroup of $\mathbb{K}^\times$.

G_{tor} – the torsion subgroup of a group G.

$\langle Y \rangle$ – the multiplicative subgroup generated by the elements of Y.

U_S – the group of S-units in $\mathbb{K}$, see (2.1).

U_S^μ – the multiplicative subgroup in $\mathbb{K}^\times$ generated by S and μ_∞, see (2.2).

$S'_{max}(\Gamma)$ – the set of all Γ-special elements, see (2.7).

$S'_{max}(S)$ – the set of all Γ-special elements for $\Gamma = U_S^\mu$.

$\mathcal{R}(\Gamma, \Gamma')$ – the set of solutions to an S-unit equation, see (2.4).

$\mathfrak{S}_n$ – the symmetric group of degree n.

Sym_n – the symmetrization operator, see (5.1).

Alt_n – the skew-symmetrization operator, see (5.1).

GL_d – the general linear group of degree d.

SL_d – the special linear group of degree d.

Diag_d – the group of non-degenerate diagonal matrices of degree d.

$\mathbb{M}_{d,n}$ – the affine variety of $d \times n$ matrices.

$\mathcal{P}_{n,d}$ – the algebra of SL_d-invariant polynomials on $\mathbb{M}_{d,n}$, see Section 3.1.

$\langle i_1 \dots i_d \rangle$ – the $d \times d$ minor formed from columns $i_1, \dots, i_d$, viewed as an element of $\mathcal{P}_{n,d}$.

$\langle ab; cd; ef \rangle$ – the polynomial $\langle abc \rangle\langle def \rangle - \langle abd \rangle\langle cef \rangle \in \mathcal{P}_{n,3}$.

$\langle g|ab; cd; ef \rangle$ – the polynomial $\langle gabc \rangle\langle gdef \rangle - \langle gabd \rangle\langle gcef \rangle \in \mathcal{P}_{n,4}$.

$\mathbb{K}_{n,d}$ – the homogeneous field of fractions of $\mathcal{P}_{n,d}$.

(a, b, c, d) – the classical cross-ratio $\frac{\langle ac \rangle\langle bd \rangle}{\langle ad \rangle\langle bc \rangle} \in \mathbb{K}_{n,2}$.

$(v_1, \dots, v_{d-2}|a, b, c, d) \in \mathbb{K}_{n,d}$ – the classical cross-ratio of points a, b, c, d projected along the hyperplane spanned by $v_1, \dots, v_{d-2}$.

Li_m – the classical polylogarithm function of order m, see Section 4.1.

$\mathcal{L}_m$ – the single-valued polylogarithm function of order m.

$\sim_{\mathbb{Q}^\times}$ – the equivalence relation on $\mathbb{R}$ defined by $a \sim_{\mathbb{Q}^\times} b \Leftrightarrow a \in b \cdot \mathbb{Q}^\times$.

$\mathbb{Z}[Y]$ – the free abelian group generated by symbols $\{[y] \mid y \in Y\}$.

$\mathbb{Q}[Y]$ – the free vector space generated by symbols $\{[y] \mid y \in Y\}$.

$\mathcal{B}_n(F)$ – the n-th higher Bloch group of a field F, see Section 4.3.

$B_n(F)$ – the n-th pre-Bloch group of a field F.

$\mathrm{Conf}_n(m)$ – the set of configurations of n points in F^m in general position.

$C_n(m)$ – the free abelian group $\mathbb{Z}[\mathrm{Conf}_n(m)]$.

$\mathcal{E}_m(Y)$ – the space of functional equations for $\mathcal{L}_m$ in $\mathbb{Q}[Y]$, see Definition 5.7.

$\mathcal{E}^0_m(\Gamma)$ – the space of all “trivial” functional equations for $\mathcal{L}_m$ with values in Γ.

$\widehat{\mathcal{E}}_m(\Gamma)$ – the quotient space $\mathcal{E}_m(\Gamma)/\mathcal{E}^0_m(\Gamma)$; the space of nontrivial functional equations for $\mathcal{L}_m$.

$\mathcal{F}_m(\Gamma)$ – the space of $\mathcal{L}_m$-cocycles with values in Γ, see Definition 5.8.

$\mathcal{F}^0_m(\Gamma)$ – the space of all “trivial” $\mathcal{L}_m$-cocycles with values in Γ.

$\widehat{\mathcal{F}}_m(\Gamma)$ – the quotient space $\mathcal{F}_m(\Gamma)/\mathcal{F}^0_m(\Gamma)$.

Acknowledgments

I am deeply grateful to my advisor Don Zagier. From him I have learned a lot of very beautiful and inspiring mathematics, and his strong sense for structure and clarity of thought has immensely influenced my own way of thinking. His numerous comments and remarks have helped to improve this work greatly.

I would like to thank Herbert Gangl for his help and encouragement and also for sharing his views and ideas about the mysterious world of polylogarithms.

I am also thankful to Yuri Manin and Günter Harder for expressing their interest in my work and many helpful and pleasant discussions.

I want to express special thanks to Lars Kühne for helping me find some important relevant results in the literature.

I would also like to thank the Max Planck Society for funding my doctoral research. Without this financial support, this work would not have been possible.

Finally, I would like to express my gratitude to Iuliia for her patience, for moral support, and also for carefully reading this manuscript and pointing out several mistakes.

CHAPTER 2

Functional S-unit equations

In this chapter we develop some general tools for solving the functional S-unit equations that are relevant to functional equations for polylogarithms. The main result of this chapter is Algorithm 1 (together with its supplementary algorithms), which provides a simple but efficient procedure for enumerating solutions of functional S-unit equations of the type similar to the equations in number fields considered by Erdős, Stewart, and Tijdeman in [15].

Some of the results can be adapted to the more general case of function fields in many variables, but we restrict to the case of rational function fields because this is the only case that we will need for application to polylogarithms.

2.1 Notation

Let $\mathbf{k}$ be a field of characteristic 0, and denote by $\mathbf{k}^\times$ the multiplicative group of nonzero elements of $\mathbf{k}$. Denote by $\mu_n \subset \mathbf{k}^\times$ the subgroup of n-th roots of unity, and by $\mu_\infty = \bigcup_{n\geq 1} \mu_n$ the group of all roots of unity in $\mathbf{k}^\times$. Let $\mathcal{P}$ be the algebra of polynomials $\mathbf{k}[x_1, \dots, x_l]$ and let $\mathcal{P}^{(n)}$ be the subspace of all homogeneous polynomials of degree n, so that

$$\mathcal{P} \;=\; \bigoplus_{n\geq 0} \mathcal{P}^{(n)}.$$

We denote the degree of a homogeneous polynomial p by $\deg(p)$. Let $\mathbb{K}$ be the homogeneous fraction field of $\mathcal{P}$, this is the field generated by all elements of the form P/Q, where $P, Q \in \mathcal{P}^{(n)}$ for some $n \geq 0$. The degree of an element $x \in \mathbb{K}$ is defined as the minimal possible degree of P over all representations $x = P/Q$, we also denote this degree by $\deg(x)$. We use the letter Γ to denote multiplicative subgroups of $\mathbb{K}^\times$, and the letter S to denote sets of homogeneous irreducible polynomials in $\mathcal{P} \smallsetminus \mathbf{k}$, which we always assume pairwise inequivalent under the action of $\mathbf{k}^\times$. Recall that equivalence classes of homogeneous irreducible polynomials are in one-to-one correspondence with homogeneous prime ideals of height 1 in $\mathcal{P}$, see [9, p. 502, Th. 1]. For any such set S of irreducible homogeneous polynomials in $\mathcal{P}$ let us denote by U_S the group of S-units in $\mathbb{K}$

$$U_S \;=\; (\mathbf{k}^\times \cdot \langle S \rangle) \cap \mathbb{K}, \tag{2.1}$$

where for any set Y we use the notation

$$\langle Y \rangle \;=\; \Big\{ \prod_{y\in Y} y^{\alpha(y)} \;\Big|\; \alpha \in \mathbb{Z}^Y \Big\}.$$

for the multiplicative subgroup generated by Y. Finally, we denote by

$$U_S^\mu \;=\; (\mu_\infty \cdot \langle S\rangle) \cap \mathbb{K}. \tag{2.2}$$

the multiplicative subgroup of $\mathbb{K}^\times$ generated by S and all roots of unity in $\mathbf{k}$.

2.2 Basic algorithm

Our goal is to find an efficient method for solving S-unit equations of the form

$$x + y = 1, \quad x \in U_S^\mu,\ y \in U_{S'}, \tag{2.3}$$

where $S \subseteq S'$ are some given finite sets of inequivalent irreducible homogeneous polynomials. In general, for any two subgroups $\Gamma, \Gamma' \subseteq \mathbb{K}^\times$ let us define

$$\mathcal{R}(\Gamma, \Gamma') \;=\; \Gamma \cap (1 - \Gamma') \smallsetminus \mathbf{k}^\times \;=\; \{x \in \Gamma \smallsetminus \mathbf{k}^\times \mid 1 - x \in \Gamma'\}\,. \tag{2.4}$$

In particular,

$$\mathcal{R}(S, S') \;:=\; \mathcal{R}(U_S^\mu, U_{S'}) \tag{2.5}$$

is the set of non-constant solutions to the equation (2.3). We will also consider the sets of solutions $\mathcal{R}(\Gamma, U_{S'})$ for multiplicative groups $\Gamma \subset \mathbb{K}^\times$ that satisfy the following finiteness condition.

Definition 2.1. *We call a multiplicative subgroup $\Gamma \subset \mathbb{K}^\times$ **admissible** if the group Γ/Γ_{tor} is finitely generated, where $\Gamma_{tor} = \{x \in \Gamma \mid \exists n > 1,\ x^n = 1\}$ is the torsion subgroup of Γ.*

In particular, $\Gamma = U_S^\mu$ is admissible for any finite S. Similarly, for any $c_1, \dots, c_t \in \mathbf{k}^\times$ the group $\Gamma = U_S^\mu \cdot \langle c_1, \dots, c_t\rangle$ is also admissible. On the other hand, the following lemma, whose simple proof we omit, shows that every admissible group is contained in a group of the form $\Gamma = U_S^\mu \cdot \langle c_1, \dots, c_t\rangle$.

Lemma 2.2. *For any admissible subgroup $\Gamma \subset \mathbb{K}^\times$ there exists a finite set S of inequivalent homogeneous irreducible polynomials in $\mathcal{P} \smallsetminus \mathbf{k}$, and a collection of multiplicatively independent constants $c_1, \dots, c_t \in \mathbf{k}^\times$ such that $\Gamma \;\subseteq\; \langle c_1, \dots, c_t\rangle \cdot U_S^\mu$.*

For any subgroup $\Gamma \subseteq \mathbb{K}^\times$ and any non-constant homogeneous polynomial $p \in \mathcal{P}$ define the subgroup $U_\Gamma(p) \subseteq \Gamma$ by

$$U_\Gamma(p) \;=\; \big\{r \in \Gamma \mid r \equiv 1 \pmod{p}\big\} \tag{2.6}$$

(meaning that p divides the numerator of $1 - r$). We usually omit the subscript Γ. Some simple properties of these abelian groups are collected in the following proposition.

Proposition 2.3. *The subgroups $U(p) \subseteq \Gamma$ for $p \in \mathcal{P} \smallsetminus \mathbf{k}$ have the following properties:*

(1) $U(cp) = U(p)$ for any constant $c \in \mathbf{k}^\times$;

(2) if p divides q, then $U(p) \supseteq U(q)$;

(3) if p and q are relatively prime, then $U(pq) = U(p) \cap U(q)$;

(4) the only constant element of $U(p)$ is 1;

(5) if the group Γ is admissible, then $U(p)$ is a finitely generated free abelian group.

Proof. The properties (1), (2), and (3) follow immediately from the definition.

If $c \in \mathbf{k}$ is an element of $U(p)$, then p divides $c - 1 \in \mathbf{k}$, and so $c = 1$. This proves (4).

Let us prove (5). From (4) we know that $U(p) \cap \mu_\infty = \{1\}$. Since $\Gamma_{tor} = \Gamma \cap \mu_\infty$ this proves that $U(p)$ is torsion-free. Therefore, $U(p)$ is isomorphic to its image in the quotient Γ/Γ_{tor}, which is a finitely generated free abelian group, hence $U(p)$ is also free and finitely generated. □

Remark. Part (5) of the above proposition holds even under the weaker assumption that $\Gamma/(\Gamma \cap \mathbf{k}^\times)$ is finitely generated.

From now on, we will always work with admissible groups Γ. One important property of the subgroups $U(p)$ that makes functional S-unit equations different from S-unit equations in number fields is the following simple lemma.

Lemma 2.4. *Let Γ be an admissible subgroup of $\mathbb{K}^\times$. Then there exists a positive integer $M = M(\Gamma)$ such that $U(p^k) = \{1\}$ for all $k > M$ and all homogeneous $p \in \mathcal{P} \setminus \mathbf{k}$.*

Proof. We will actually prove this result for any subgroup Γ of $\mathbb{K}^\times$ which is finitely generated modulo its intersection with $\mathbf{k}^\times$. Using Lemma 2.2 we can reduce to the case when $\Gamma = U_S$, where $S = \{p_1, \ldots, p_n\}$ and p_i are irreducible and pairwise inequivalent polynomials. If $c \in \mathbf{k}^\times$ and an element $x = cp_1^{\alpha_1} \ldots p_n^{\alpha_n}$ lies in $U(p^k)$, then

$$\prod_i p_i^{\alpha_i} \equiv c^{-1} \pmod{p^k}.$$

Note that if for some i the polynomial p_i divides p, then $\alpha_i = 0$. Therefore, we may assume that no p_i divides p. Let $D\colon \mathcal{P} \to \mathcal{P}$ be any derivation of degree -1. By applying D (extended to the fraction field of $\mathcal{P}$) to the above congruence we get

$$\prod_i p_i^{\alpha_i} \left(\sum_i \alpha_i \frac{Dp_i}{p_i} \right) \equiv 0 \pmod{p^{k-1}}$$

and thus

$$\sum_i \alpha_i \frac{Dp_i}{p_i} \equiv 0 \pmod{p^{k-1}}.$$

From this congruence we see that if $(k-1)\deg p \geq \sum_i \deg p_i$, then we must have that

$$\sum_i \alpha_i \frac{Dp_i}{p_i} = 0.$$

Since the derivation D was arbitrary, we conclude that $\prod_i p_i^{\alpha_i} = c^{-1}$, i.e., $x = 1$. Therefore, we can take $M = \sum_i \deg p_i$. □

Notice that we actually proved the stronger statement with "$(k-1)\deg(p) \geq M$" instead of "$k > M$", but this will not be used.

As a corollary we immediately get that the set of solutions $\mathcal{R}(\Gamma, U_{S'})$ is finite.

Theorem 2.5. *If $\Gamma \subseteq \mathbb{K}^\times$ is an admissible group and S' is a finite set of inequivalent homogeneous polynomials, then the set of solutions $\mathcal{R}(\Gamma, U_{S'})$ is finite. Moreover, for any $x \in \mathcal{R}(\Gamma, U_{S'})$ we have*

$$\deg(x) \leq M(\Gamma) \cdot \sum_{p \in S'} \deg(p).$$

Proof. Let $x \in \mathcal{R}(\Gamma, U_{S'})$ be a non-constant solution of (2.4) and let

$$1 - x \;=\; c \cdot \prod_{p \in S'} p^{\alpha(p)}$$

where $c \in \mathbf{k}^{\times}$ and $\alpha\colon S' \to \mathbb{Z}$. Let us denote $(t)_+ = \max(t, 0)$ for any $t \in \mathbb{Z}$. Then we have

$$\deg(x) = \deg(1-x) = \sum_{p \in S'} (\alpha(p))_+ \deg(p) \leq M(\Gamma) \cdot \sum_{p \in S'} \deg(p),$$

where we have used the result of 2.4 that $(\alpha(p))_+ \leq M(\Gamma)$ for all $p \in S'$. The finiteness follows from the fact that

$$\mathcal{R}(\Gamma, U_{S'}) \;\subseteq\; \bigcup_{p \in S'} U(p)$$

and the fact that there are only finitely many elements of bounded degree in each group $U(p)$. This last fact follows from the properties (4) and (5) in Proposition 2.3. □

Remark. The above proof also works in the case $\Gamma = U_S$.

For $\Gamma = U_S^{\mu}$ a slightly better upper bound can be obtained by a more careful analysis of the proof of Lemma 2.4.

Proposition 2.6. *For any $x \in \mathcal{R}(U_S^{\mu}, U_{S'})$ the following inequality holds:*

$$\deg(x) \;\leq\; \sum_{p \in S} \deg(p) + \sum_{p \in S'} \deg(p).$$

Now we describe an algorithm for enumerating all solutions $x \in \mathcal{R}(\Gamma, U_{S'})$ that, despite its simplicity, works reasonably well in practice and will be our main tool for finding functional equations for polylogarithms.

Algorithm 1. Computation of $\mathcal{R}(\Gamma, U_{S'})$.

Step 1. Initialize $L = \{1\}$ and $\Omega = \emptyset$;

Step 2. For each $p \in L \smallsetminus \mathbf{k}$ generate all the non-constant elements $x \in U(p)$ for which $\deg(x) \leq \deg(p)$ and add them to Ω;

Step 3. Set $L' = \{pq \mid p \in L, q \in S'\}$;

Step 4. Set $L = \{p \in L' \mid U(p) \neq \{1\}\}$ and if $L \neq \emptyset$ then go to Step 3. Otherwise, output the solution set $\mathcal{R}(\Gamma, U_{S'}) = \Omega$ and terminate.

Basically, the algorithm enumerates solutions $x \in \mathcal{R}(\Gamma, S')$ by going through all possible numerators (up to multiplicative constants) of $1 - x$, which by the condition $1 - x \in U_{S'}$ must be products of elements in S'. Lemma 2.4 guarantees that the algorithm terminates.

Remarks.

1. In Algorithm 1 we did not specify the way in which one can compute the subgroups $U(p)$. We will present an algorithm for computing $U(p)$ in the next section.

2. We can replace Step 3 in Algorithm 1 by a slightly more efficient version

Step 3$'$. Set $S'' = \{p \in S' \mid \exists q \in L,\ p \text{ divides } q\}$ and $L' = \{pq \mid p \in L, q \in S''\}$.

3. If G is a finite group of degree-preserving automorphisms of $\mathcal{P}$ and the sets Γ and S' are invariant under the action of G, then so is the set $\mathcal{R}(\Gamma, S')$. One can modify the above algorithm to produce all the representatives of G-orbits in $\mathcal{R}(\Gamma, S')$ without having to compute the whole set of solutions. This is an important observation, because in some computations considered in Chapter 3 we will have $G = \mathfrak{S}_n$ for $n = 4, \ldots, 10$. The difference between enumerating orbits and enumerating individual solutions would increase the computational time (roughly) by a factor of $n!$ and in many cases would make the computation infeasible.

2.3 Supplementary algorithms

2.3.1 Basis computation

To implement the above algorithm, we need a way (preferably efficient) of computing the basis of the subgroup $U(p)$ for any homogeneous $p \in \mathcal{P} = \mathbf{k}[x_1, \dots, x_l]$. In this section we assume for simplicity that $|\mu_\infty| < \infty$ (in fact, we will apply this algorithm only in the case $\mathbf{k} = \mathbb{Q}$). Let $\overline{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$. We will rely on the following two oracles.

Oracle 1. This is a random oracle that generates points from the variety

$$V(p) \;=\; \{\overline{x} \in \overline{\mathbf{k}}^l \mid p(\overline{x}) = 0\}.$$

A simple oracle of this kind can work as follows: pick values of $x_2, \dots, x_l$ at random and find x_1 by solving the 1-variable polynomial equation $p(x_1, \dots, x_l) = 0$. Ideally, we would like to generate points in $\mathbf{k}_1^l$ for some extension $\mathbf{k}_1$ of small degree over $\mathbf{k}$ since the efficiency of the Algorithm 2 below will depend greatly on the field to which x_i belong.

Oracle 2. This is an oracle that implements the following procedure: given nonzero elements $y_1, \dots, y_n \in \overline{\mathbf{k}}$ compute a basis for the lattice $\mathrm{Rel}(y_1, \dots, y_n) \subset \mathbb{Z}^n$ defined as

$$\mathrm{Rel}(y_1, \dots, y_n) \;=\; \Big\{\overline{\alpha} \in \mathbb{Z}^n \;\Big|\; \prod_i y_i^{\alpha_i} = 1\Big\}.$$

In the case when $y_1, \dots, y_n$ lie in a common number field F, this procedure can be realized by computing the factorization of y_i in F.

To compute $U(p)$ we employ the following Las Vegas algorithm. As usual, by a Las Vegas type algorithm we mean a probabilistic algorithm that is “always correct, probably fast”.

Algorithm 2. Computation of the multiplicative basis of $U_\Gamma(p)$.

Step 1. Initialize $\Gamma_1 = \Gamma$.

Step 2. Compute a basis $r_1, \dots, r_N$ of Γ_1 modulo torsion.

Step 3. Check whether all elements $r_i \in \Gamma_1$ lie in $U(p)$. If this is the case, go to step 6.

Step 4. Using Oracle 1, find a finite extension $\mathbf{k}_1 \supseteq \mathbf{k}$ and a point $x_0 \in \mathbf{k}_1^m$ satisfying $p(x_0) = 0$.

Step 5. Using Oracle 2 compute a basis of $\mathrm{Rel}(r_1(x_0), \dots, r_N(x_0))$. Let $\overline{\alpha}_1, \dots, \overline{\alpha}_s$ be this basis. For each $\overline{\alpha}_i$ define the element $r_{\overline{\alpha}} = \prod_i r_i^{\alpha_i}$. Set $N = s$, set $(r_1, \dots, r_N)$ to be the collection $(r_{\overline{\alpha}_1}, \dots, r_{\overline{\alpha}_s})$, and set Γ_1 to be the group generated by $r_{\overline{\alpha_i}}$. Go to step 3.

Step 6. Compute the subgroup of all elements $r \in \Gamma$ such that some power of r belongs to Γ_1 and denote this group by Γ_2. Compute the coset representatives x_j of Γ_1 in Γ_2, i.e., $\Gamma_2/\Gamma_1 = \{x_j\Gamma_1 \,|\, j = 1, \dots, t\}$.

Step 7. Compute all indices j for which $x_j \in U(\pi)$, denote this set by J. Compute the reduced basis $b_1, \dots, b_k$ of the group generated by $r_1, \dots, r_N$ and $\{x_j \,|\, j \in J\}$. Output $\mathcal{B} = \{b_1, \dots, b_k\}$ and terminate.

At each step of the algorithm, the group Γ_1 contains some power $U(p)^M$, where M is the exponent (the lowest common multiple of the orders of all the elements) of the torsion group μ_∞. At each step the rank of Γ_1 is at least that of $U(p)$. If the algorithm reaches step 6, then we also know that $\Gamma_1 \subseteq U(p)$. From $U(p)^M \subseteq \Gamma_1 \subseteq U(p)$ we see that $\Gamma_1 \subseteq U(p) \subseteq \Gamma_2$, where Γ_2 is the set of all elements $r \in \Gamma$ such that $r^M \in \Gamma_1$. Since the ranks of all these groups are equal, there are only finitely many cosets Γ_2/Γ_1. Thus we compute $U(p)$ by finding all cosets that it contains. Note that in the case $\mathbf{k} = \mathbb{Q}$ we have $\mu_\infty = \{\pm 1\}$ so we may take $M = 2$.

Finally, let us remark that the computation of $\mathrm{Rel}(y_1, \dots, y_n)$ through factorization of y_i can be quite expensive, so in practice we do only partial factorization of y_i and find a basis for some group that contains $\mathrm{Rel}(y_1, \dots, y_n)$ (it is easy to see that it does not impact correctness of Algorithm 2, but in principle could make it run indefinitely).

2.3.2 Elements of small degree

In Step 3 of Algorithm 1 we also need a procedure for generating elements in $U(p)$ of small degree. The group $U(p) \cong \mathbb{Z}^r$ embeds into $\mathbb{Z}^S$, let us denote this embedding by φ. The degree of an element $a \in \mathbb{Z}^r$ is simply the ℓ_1-norm of $\varphi(a) \in \mathbb{Z}^S$. Let L_k be the polytope $\{x \in \mathbb{R}^S \mid \|x\|_1 \leq k\}$. Therefore, to find all elements of degree $\leq k$ in $U(p)$ is the same as to find all the points in the intersection of the polytope $\varphi^{-1}(L_k) \subset \mathbb{R}^r$ with the integer lattice $\mathbb{Z}^r$.

Depending on the dimension r and on the complexity (by which we mean the total number of faces of all dimensions) of the polytope $H_k := \varphi^{-1}(L_k)$ we can proceed in a number of ways:

1. The most naïve way is to compute the bounding box $Q = \prod_{j=1}^{r} I_j$ that contains H_k, where I_j are some closed intervals in $\mathbb{R}$. We can then enumerate all integer points in Q by a simple loop, and test for each one whether it belongs to the polytope H_k. Because of the special shape of H_k (it contains the point $x \in \mathbb{R}^r$ if and only if $\|\varphi(x)\|_1 \leq k$) the membership test is very efficient.

2. If the dimension and the complexity is "small", then we can triangulate the polytope and list the points for each simplex. Let us give a sketch of how to list integer points in a simplex. If one of the vertices of the simplex Δ is $v_0 = 0 \in \mathbb{Z}^n$ and the rest, $v_1, \dots, v_r$, are the rows of an upper-triangular matrix, then there is a simple recursive procedure for computing all points in $\Delta \cap \mathbb{Z}^r$ (one simply goes through possible coordinates of such points from left to right, recursively calling the same procedure for each of the corresponding hyperplane sections; we omit further details). If the simplex satisfies only the condition $v_0 = 0$, then one first needs to find an appropriate linear transformation in $\mathrm{GL}_r(\mathbb{Z})$ that puts the simplex into the upper-triangular form. This linear transformation can be found by computing the so-called Hermite normal form of a matrix, see [10, §2.4.2] for further details. With slight modification, this method also works for simplexes, none of whose vertices belongs to $\mathbb{Z}$ (so that we cannot assume $v_0 = 0$).

3. If the rank and/or the complexity is sufficiently large, then it is sometimes easier to list all points using l_2-norm instead. We take some positive definite quadratic form Q and a real number $C > 0$ such that $\varphi^{-1}(L_k) \subseteq \{x \mid Q(x) \leq C\}$. To list all lattice points that satisfy $Q(x) \leq C$ we can apply the Fincke-Pohst algorithm [10, Alg. 2.7.7], which is conveniently implemented in PARI/GP [41] in the function `qfminim`. Then we go through the obtained list and leave only the elements x for which $\|\varphi(x)\|_1 \leq k$.

An efficient implementation would require careful fine-tuning to choose between the three methods above.

2.4 The choice of S'

Let us consider how the set of solutions $\mathcal{R}(\Gamma, U_{S'})$ depends on S'. It is clear from the definition of $\mathcal{R}(\Gamma, U_{S'})$ that if $p \in S'$, then

$$\mathcal{R}(\Gamma, U_{S'}) \ \subset\ \mathcal{R}(\Gamma, U_{S' \smallsetminus \{p\}}) \cup U(p).$$

Therefore, if $U(p) = \{1\}$, then $\mathcal{R}(\Gamma, S') = \mathcal{R}(\Gamma, S' \smallsetminus \{p\})$. Thus, we may assume that for any $p \in S'$ the rank of the group $U(p)$ is at least 1. If the rank of $U(p)$ is exactly 1, then $U(p) = \{r^k \mid k \in \mathbb{Z}\}$ for some $r \in \mathbb{K}^\times$. In application to functional equations for polylogarithms we are usually not interested in arguments of the form r^k. (A rigorous argument for why this is the case will be given in Chapter 5.) This motivates the following definition.

Definition 2.7. *A polynomial p is called Γ-**special** if the rank of the abelian group $U(p)$, defined as in (2.6) is greater than 1. If the group Γ is understood from context, we will simply call the polynomial p **special**.*

Let us denote by $S_2'(\Gamma)$ a set of representatives of $\mathbf{k}^\times$-equivalence classes of all Γ-special polynomials, and by $S(\Gamma)$ a set of representatives of all the irreducible polynomials that divide elements of Γ. We then define the *maximal set of special primes* $S'_{max}(\Gamma)$ by

$$S'_{max}(\Gamma) \;=\; S_2'(\Gamma) \cup S(\Gamma). \tag{2.7}$$

This set is defined only up to multiplication of its elements by elements of $\mathbf{k}^\times$, but since this does not affect the group of S-units $U_{S'_{max}(S)}$, we allow this slight abuse of notation. In the case $\Gamma = U_S^\mu$ we shall write $S'_{max}(S)$ instead of $S'_{max}(U_S^\mu)$.

Theorem 2.8. *If the subgroup $\Gamma \subset \mathbb{K}^\times$ is admissible, then the set $S'_{max}(\Gamma)$ is finite.*

The proof relies on the following result from the theory of unlikely intersections.

Proposition 2.9. *Let K be a field of characteristic zero, and for $n \geq 2$ let $\mathcal{C}$ be an irreducible curve in $\mathbb{G}_m^n$ that is defined over the algebraic closure $\overline{K}$ and is not contained in any algebraic subgroup of dimension $< n$. Then the intersection of $\mathcal{C}$ with the union $\mathcal{H}_{n-2}$ of all algebraic subgroups of dimension at most $(n-2)$ is a finite set.*

Here $\mathbb{G}_m$ denotes the affine variety $\mathbb{A}^1 \smallsetminus \{0\}$ endowed with the multiplicative group law. This result was proved by Maurin [26] in the case $K = \overline{\mathbb{Q}}$, and later by Bombieri, Masser, and Zannier [6] in the case $K = \mathbb{C}$. For arbitrary field of characteristic 0 the above result does not seem to be explicitly formulated anywhere in the literature, but it follows from the case $K = \overline{\mathbb{Q}}$ and the specialization arguments from [5]. Alternatively, one can deduce it from the case $K = \mathbb{C}$ by applying the Lefschetz principle.

In [5] Bombieri, Masser, and Zannier have proved that the conclusion of Proposition 2.9 holds under the stronger requirement that $\mathcal{C}$ is not contained in any translate of an algebraic subgroup. This weaker result is important, since it is still sufficient for the proof of Theorem 2.8 in the case $\Gamma = U_S^\mu$, but unlike Proposition 2.9 the proof is effective in the sense that it gives a procedure for computing the intersection $\mathcal{C} \cap \mathcal{H}_{n-2}$.

Before we prove theorem 2.8, let us give two simple examples that illustrate Proposition 2.9. If we take the curve $\mathcal{C} = \{(t-1, t, t+1) \mid t \in \mathbb{C}\}$ in $\mathbb{G}_m^3$, then Proposition 2.9 implies that there are only finitely many values $t \neq -1, 0, 1$ such that there are at least two multiplicative dependencies between numbers $t-1$, t, and $t+1$ (in fact, there are exactly 34 such numbers, as was shown in [12]). Similarly, if we take the curve $\mathcal{C} = \{(t, t-1, t^2-t+1) \mid t \in \mathbb{C}\}$, then it shows that there are only finitely many values of t for which there are two multiplicative relations between numbers t, $t-1$, and t^2-t+1, and it is not hard to show that there are exactly 48 such values. This last example was utilized by Gangl [16] in his construction of functional equations for $\mathcal{L}_7$. Finally, let us remark that Proposition 2.9 is also closely related to the question of reducibility for lacunary polynomials, see [30], [40, Sec. 1.3.5].

Proof of Theorem 2.8. Without loss of generality, we may assume that the field $\mathbf{k}$ is algebraically closed and $\Gamma = \langle c_1, \ldots, c_s \rangle \cdot U_S^\mu$, where $S = \{p_1, \ldots, p_n\}$ and p_i are irreducible and pairwise inequivalent modulo $\mathbf{k}^\times$. Let us first consider the case $\mathcal{P} = \mathbf{k}[x_1, x_2]$. The homogeneous irreducible polynomials are then just linear functions. Let $p_i = x_1 - c_{i+s}x_2$. The claim is then equivalent to the fact that there are only finitely many numbers $z \in \mathbf{k}$ such that

$$\prod_{i=1}^{s} c_i^{\alpha_i} \prod_{i=s+1}^{n} (z - c_i)^{\alpha_i} \;=\; \prod_{i=1}^{s} c_i^{\beta_i} \prod_{i=s+1}^{n} (z - c_i)^{\beta_i} \;=\; 1$$

for two linearly independent vectors $(\alpha_1, \ldots, \alpha_{s+n}), (\beta_1, \ldots, \beta_{s+n}) \in \mathbb{Z}^{s+n}$. This immediately follows from Proposition 2.9 if we take $K = \mathbf{k}$ and

$$\mathcal{C} \;=\; \{(c_1, \ldots, c_s, z - c_{s+1}, \ldots, z - c_{s+n}) \mid z \in \mathbf{k}\}.$$

We prove the theorem in the general case $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_l]$ by induction on l. By making a suitable linear change of variables, we can insure that for every $p \in S$ the coefficient of $x_1^{\deg(p)}$ is nonzero. Let L be the algebraic closure of the field $\mathbf{k}(x_2, \ldots, x_l)$. Since the elements p_i are irreducible and multiplicatively independent, if we factorize each p_i as

$$a_i \cdot \prod_j (x_1 - z_{ij}(x_2, \ldots, x_l)),$$

then $z_{ij} \in L$ are different for all i, j. If we now consider the curve

$$\mathcal{C} \;=\; \{(c_1, \ldots, c_s, a_1(z - z_{1,1}), \ldots, a_n(z - z_{n,k_n})) \mid z \in L\}$$

and apply Proposition 2.9, we get that there are at most finitely many $z \in L$ such that the polynomial $x_1 - z(x_2, \ldots, x_l)$ has S-rank at least 2, hence we get finiteness of the set of all polynomials in $S'_{max}(\Gamma)$ of the from

$$x_1^k + x_1^{k-1} q_1(x_2, \ldots, x_l) + \cdots + q_k(x_2, \ldots, x_l),$$

where for each q_i we have either $q_i = 0$ or $\deg(q_i) = k - i$. The only cases not yet covered are polynomials of the form $\pi = q(x_2, \ldots, x_l)$, and the finiteness in this case follows from inductive assumption by taking a generic linear specialization $x_1 = \phi(x_2, \ldots, x_l)$. $\square$

This result provides us with a canonical choice of S' for any given Γ. Solutions of the equation (2.4) for $S' = S'_{max}(\Gamma)$ will play an important role in Chapter 5.

Definition 2.10. *Let Γ be a subgroup of $\mathbb{K}^\times$. We say that the element $x \in \Gamma$ is **exceptional** if it belongs to the set $\mathcal{R}(\Gamma, U_{S'_{max}(\Gamma)})$. Equivalently, an element $x \in \Gamma$ is **exceptional** if and only if every irreducible polynomial p dividing the numerator of $1 - x$ either divides some element in Γ or satisfies $\mathrm{rk}(U(p)) \geq 2$.*

Combining the results of Theorem 2.5 and Theorem 2.8 we get the following finiteness result for the set of exceptional elements.

Theorem 2.11. *For any admissible group $\Gamma \subset \mathbb{K}^\times$ the set of exceptional elements of Γ is finite.*

Remark. As was noted above, Proposition 2.9 provides a procedure for finding $\mathcal{C} \cap \mathcal{H}_{n-2}$ in the case when $\mathcal{C}$ is not contained in any translate of an algebraic subgroup (see, [40, Th. 1.3]). Thus, in the case $\Gamma = U_S^\mu$, Theorem 2.11 is also effective. The rigorous computation of $S'_{max}(\Gamma)$, however, is computationally expensive, and we could complete it only in few small examples.

2.5 Application of Wronskians to S-unit equations

In this section we describe some known general results for S-unit equations in function fields (in one variable). The main references for this section are the papers [25] and [39].

We assume that $\mathbf{k}$ is an algebraically closed field of characteristic 0 and that $\mathbb{K}/\mathbf{k}$ is a function field in one variable over $\mathbf{k}$ of genus g. In this section exclusively, the letter S will be used to denote sets of places of $\mathbb{K}/\mathbf{k}$. For a collection of n elements $x_1, \ldots, x_n$ define the usual projective height by

$$H(x_1, \ldots, x_n) \;=\; -\sum_v \min(v(x_1), \ldots, v(x_n)),$$

where v runs over all places of $\mathbb{K}/\mathbf{k}$.

In [25] Mason has proved the following result (this formulation is from [39]):

Proposition 2.12. *Let $x_1, \ldots, x_n \in \mathbb{K}$ be such that they span $(n-1)$-dimensional vector space over $\mathbf{k}$ and $x_1 + \cdots + x_n = 0$. Then*

$$H(x_1, \ldots, x_n) \;\leq\; C_n(\#S + 2g - 2),$$

where C_n is some constant that depends only on n, g is the genus of $\mathbb{K}/\mathbf{k}$, and S is the set of places of $\mathbb{K}$ where some x_i is not a unit.

This result also implies Theorem 2.5. In his original paper, Mason gave a value of constant $C_n = 4^{n-2}$. This was later improved in [33] to $C_n = \binom{n-1}{2}$, and in [39] it was shown that if no subsum of x_i vanishes, then the same inequality holds with C_n replaced by $\binom{\mu}{2}$, where μ is the dimension of the $\mathbf{k}$-vector space spanned by $\{x_1, \ldots, x_n\}$.

Note that the set $\Omega_n(\Gamma)$ contains $\binom{n}{k}$ copies of $\Omega_k(\Gamma)$, one for each k-subset of $\{1, 2, \ldots, n\}$. In particular, $\Omega_n(\Gamma)$ contains all the diagonals $\Delta_{ij} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}$. Let us denote by $\widehat{\Omega}_n(\Gamma)$ the set of all "new" solutions, i.e., the solutions that do not come from any $\Omega_k(\Gamma)$ for $k < n$. It follows from the Proposition 2.12 that the set $\widehat{\Omega}_n(\Gamma)$ is effectively bounded. Thus, we can compute the set $\widehat{\Omega}_n(\Gamma)$ by enumerating all the lattice points of norm less than a certain bound. However, even with the best known bound for C_n, this computation becomes impractical for groups Γ of moderately high rank. We will now give an alternative approach to the computation of $\Omega_n(\Gamma)$.

For simplicity, let $\mathbb{K} = \mathbf{k}(x)$ and let Γ be a finitely generated subgroup of $\mathbb{K}^\times$ which satisfies $\Gamma \cap \mathbf{k}^\times = \{1\}$. Let $r_1, \ldots, r_m$ form a multiplicative basis of Γ. We would like to describe solutions of the following equation:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \;=\; 0,\; x_i \in \Gamma, \tag{2.8}$$

where $c_i \in \mathbf{k}$ are some constants which may depend on the solution $(x_1, \ldots, x_n)$. Let

$$x_i \;=\; \prod_{j=1}^m r_j^{\alpha_{ij}}, \tag{2.9}$$

where $\alpha_{ij} \in \mathbb{Z}$. Denote the set of solutions to (2.8) by

$$\Omega_n(\Gamma) \;=\; \left\{(\alpha_{i,j})_{i,j} \mid c_1x_1 + \cdots + c_nx_n = 0 \text{ for some } (c_1, \ldots, c_n) \neq (0, \ldots, 0)\right\}, \tag{2.10}$$

where x_i are given by (2.9).

Recall the following well-known result (see, for example, [4, pp.91–92]).

Proposition 2.13 (Wronskian condition)**.** *The elements $f_1, \ldots, f_n \in \mathbf{k}(x)$ are linearly dependent over $\mathbf{k}$ if and only if the Wronskian determinant*

$$W(f_1, \ldots, f_n) \;=\; \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is zero in $\mathbf{k}(x)$.

If we define the element $A_i \in \mathbf{k}(x)$ to be the logarithmic derivative of x_i, i.e.,

$$A_i \;=\; \frac{x_i'}{x_i} \;=\; \sum_{j=1}^{m} \alpha_{ij} \frac{r_j'}{r_j},$$

then we have

$$x_i^{(n)} \;=\; \Delta_n(A_i)x_i,$$

where Δ_i is a sequence of nonlinear differential operators defined inductively by $\Delta_1(A) = A$ and $\Delta_{i+1}(A) = (\Delta_i(A))' + \Delta_i(A)A$. The first three operators Δ_i are

$$\begin{aligned}
\Delta_1(A) &= A,\\
\Delta_2(A) &= A' + A^2,\\
\Delta_3(A) &= A'' + 3A'A + A^3.
\end{aligned}$$

We can rewrite the Wronskian $W(x_1, \ldots, x_n)$ in the following way

$$\frac{W(x_1, \ldots, x_n)}{x_1 \ldots x_n} \;=\; \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \Delta_1(A_1) & \Delta_1(A_2) & \ldots & \Delta_1(A_n) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1}(A_1) & \Delta_{n-1}(A_2) & \ldots & \Delta_{n-1}(A_n) \end{vmatrix}.$$

We denote the determinant on the right-hand side by $W_{log}(A_1, \ldots, A_n)$. With this notation we get the following equivalent reformulation of Proposition 2.13.

Corollary 2.14. *A collection of nonzero elements $f_1, \ldots, f_n \in \mathbf{k}(x)$ is linearly dependent over $\mathbf{k}$ if and only if*

$$W_{log}(A_1, \ldots, A_n) \;=\; 0,$$

where $A_i = \frac{f_i'}{f_i}$.

We can now prove the following theorem that gives another approach to computing $\Omega_n(\Gamma)$, by presenting this set as the solution set of a polynomial system.

Theorem 2.15. *There is an explicit collection of polynomials $\mathcal{P} = \{P_\nu\}$ in α_{ij}, such that $\Omega_n(\Gamma)$ given by* (2.10) *is the set of common zeros of $\mathcal{P}$ in $\mathbb{Z}^{nm}$.*

Proof. By Proposition 2.13 we know that $(\alpha_{ij}) \in \Omega_n(\Gamma)$ if and only if $W_{log}(A_1, \ldots, A_n) = 0$. We will show that $W_{log}(A_1, \ldots, A_n)$ is a polynomial function in α_{ij} with values in a (fixed) finite-dimensional $\mathbf{k}$-vector space of rational functions. Indeed, it is enough to show that all entries of the matrix $(\Delta_i(A_j))_{i,j}$ lie in some finite-dimensional vector space over $\mathbf{k}$. We can easily do this by induction. By definition $\Delta_1(A_j)$ lies in

$$\mathcal{V}_1 \;=\; \left\langle \frac{r_1'}{r_1}, \frac{r_2'}{r_2}, \ldots, \frac{r_m'}{r_m} \right\rangle_{\mathbf{k}}.$$

Assume that all $\Delta_k(A_j)$ lie in some finite-dimensional vector space $\mathcal{V}_k$. Then, from the relation

$$\Delta_{k+1}(A) \;=\; (\Delta_k(A))' + \Delta_k(A)A$$

we see that $\Delta_{k+1}(A_j) \in \mathcal{V}_k' + \mathcal{V}_k \cdot \mathcal{V}_1 =: \mathcal{V}_{k+1}$, which is again finite-dimensional, and hence the proof is complete. $\square$

Remarks.

1. Theorem 2.13 holds for any collection of meromorphic functions on a domain in $\mathbb{C}$. There is also a higher-dimensional generalization for $f_1, \ldots, f_n \in \mathbf{k}(t_1, \ldots, t_k)$. In this case one needs to consider the vanishing of the Wronskian determinants for all possible chains of derivations $(D_1, D_2D_1, \ldots, D_{n-1} \ldots D_2D_1)$ applied to $(f_1, \ldots, f_n)$.

2. The conclusion of Theorem 2.15 holds for any function field $\mathbb{K}$ in finitely many variables. It is also not necessary to assume that $\mathbf{k}$ is algebraically closed.

CHAPTER 3

Higher cross-ratios

Consider the affine variety $\mathbb{M}_{d,n}$ of $d \times n$ matrices. We will view elements of this variety as n-tuples of d-vectors. The group GL_d acts on $\mathbb{M}_{d,n}$ by left multiplication. Similarly, the group Diag_n of nonsingular diagonal $n \times n$ matrices acts on $\mathbb{M}_{d,n}$ by right multiplication. We will call functions on $\mathbb{M}_{d,n}$ projective invariants if they are invariant under the action of Diag_n and GL_d-invariants if they are invariant under the action of GL_d.

Definition 3.1. *A **higher cross-ratio** (or simply cross-ratio) is a non-constant rational function on $\mathbb{M}_{d,n}$ that is both a projective invariant and a GL_d-invariant.*

The classical cross-ratio can be viewed as a rational function on $\mathbb{M}_{2,4}$, so this definition includes it as a special case. The above definition, however, is too general for our purposes (that is, for application to polylogarithms). Later in this chapter we will specify a class of higher cross-ratios that would be of interest to us with a view towards applications to polylogarithm functional equations. We begin with the description of the general form of GL_d-invariant rational functions on $\mathbb{M}_{d,n}$.

Remark. Note that, since $(\lambda 1_d)A = A(\lambda 1_n)$ for any $A \in \mathbb{M}_{d,n}$ and any scalar λ, it is enough to require SL_d-invariance in the definition above.

3.1 Bracket algebra

The main reference for this section is [32].

3.1.1 Definition

Let $n > d \geq 2$ and let $\mathcal{P}_{n,d}$ be the algebra of SL_d-invariant polynomials with rational coefficients on $\mathbb{M}_{d,n}$. We view $\mathcal{P}_{n,d}$ as a subalgebra in $\mathbb{Q}[x_{ij}]$, where x_{ij} are indeterminate entries of a $d \times n$ matrix. This algebra is nonempty, since any $d \times d$ minor formed from the matrix columns $i_1, \dots, i_d$ is SL_d-invariant. We denote these minors by $\Delta(i_1, \dots, i_d)$, the so-called Plücker coordinates.

Consider the set

$$I(n,d) \;=\; \{\langle i_1 i_2 \dots i_d \rangle \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}.$$

Let $\mathbb{Q}[I(n,d)]$ be the polynomial algebra generated by symbols $\langle i_1 \dots i_d\rangle$. We extend the definition of $\langle\cdot\rangle$ to include

$$\begin{aligned}\langle i_{\pi(1)} \dots i_{\pi(d)}\rangle &= \operatorname{sgn}(\pi)\langle i_1 \dots i_d\rangle, \ \pi \in \mathfrak{S}_d;\\ \langle i_1 \dots i_d\rangle &= 0 \text{ if } i_j = i_k \text{ for some } j \neq k.\end{aligned}$$

Let $\phi_{n,d}\colon \mathbb{Q}[I(n,d)] \to \mathbb{Q}[x_{ij}]$ be the homomorphism

$$\langle i_1 \dots i_d\rangle \ \mapsto \ \Delta(i_1,\dots,i_d).$$

Proposition 3.2. *The map $\phi_{n,d}$ is surjective. Moreover, the ideal of relations $\mathcal{I}_{n,d} := \phi_{n,d}^{-1}(0)$ is generated by all the quadratic polynomials in $\mathbb{Q}[I(n,d)]$ of the form*

$$Q(\bar{i},\bar{j}) \ = \ \sum_{k=1}^{d+1}(-1)^k\langle i_1 \dots \widehat{i_k} \dots i_{d+1}\rangle\langle i_k j_1 \dots j_{d-1}\rangle,$$

where $\bar{i} = (i_1,\dots,i_{d+1})$ and $\bar{j} = (j_1,\dots,j_{d-1})$.

Proof. See [32, Thm. 3.1.7]. □

Thus, the algebra $\mathcal{P}_{n,d}$ is isomorphic to the quotient $\mathbb{Q}[I(n,d)]/\mathcal{I}_{n,d}$. From now on we will identify $\mathcal{P}_{n,d}$ with $\mathbb{Q}[I(n,d)]/\mathcal{I}_{n,d}$ and freely use both points of view for $\mathcal{P}_{n,d}$ (as a subalgebra and as a quotient algebra). The elements $\langle i_1 \dots i_d\rangle \in \mathcal{P}_{n,d}$ are called brackets and the algebra $\mathcal{P}_{n,d}$ is called the bracket algebra.

3.1.2 Main properties

Let us collect some useful properties of the algebras $\mathcal{P}_{n,d}$.

1. $\mathbb{Z}^n_{\geq 0}$-grading. The algebra $\mathcal{P}_{n,d}$ has a natural $\mathbb{Z}^n_{\geq 0}$-grading by degrees in each of the n vectors. Let us denote the graded pieces by $\mathcal{P}^\alpha_{n,d}$, so that

$$\mathcal{P}_{n,d} \ = \ \bigoplus_{\alpha\in\mathbb{Z}^n_{\geq 0}} \mathcal{P}^\alpha_{n,d}.$$

An element $p \in \mathcal{P}^\alpha_{n,d}$ is said to be multi-homogeneous, and α is called its multi-degree. We will usually drop the prefix "multi" and just call p homogeneous, and α its degree. That the grading is well-defined follows from Proposition 3.2 and the fact that all relations $\psi_{\bar{i},\bar{j}}$ are homogeneous. Note that the graded pieces of $R_{n,d}$ that correspond to the generators $\langle i_1 \dots i_d\rangle$ are one-dimensional. In particular, this means that the minimal set of generators $\{\langle i_1 \dots i_d\rangle\}_{i_1<\dots<i_d}$ is uniquely determined by the grading (up to multiplication by nonzero rationals).

2. Unique factorization. Any homogeneous polynomial $p \in \mathcal{P}_{n,d}$ can be uniquely factorized into a product of irreducible homogeneous polynomials in $\mathcal{P}_{n,d}$. This property follows from the fact that $\mathbb{Q}[x_{ij}]$ is a unique factorization domain, and that there are no closed finite index subgroups in SL_d (this latter fact follows from the fact that SL_d is irreducible, and any closed finite index subgroup is also open). As a consequence, we can freely apply to $\mathcal{P}_{n,d}$ the algorithms developed in the previous chapter.

3. $\mathfrak{S}_n$-action. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{P}_{n,d}$ on the left by

$$\sigma\cdot\langle i_1 \dots i_d\rangle \ = \ \langle\sigma(i_1)\dots\sigma(i_d)\rangle.$$

One can see that this action is well-defined either by extending the action of $\mathfrak{S}_n$ to the algebra $\mathbb{Q}[x_{11},\dots,x_{dn}]$ by mapping x_{ij} to $x_{i\sigma(j)}$ or by checking that the ideal $\mathcal{I}_{n,d}$ from Proposition 3.2 is invariant under this action on $\mathbb{Q}[I(n,d)]$.

4. Functoriality. Given n_1, n_2, and a map of sets $\iota\colon \{1,\ldots,n_1\} \to \{1,\ldots,n_2\}$, we define a homomorphism of graded algebras $\varphi_\iota\colon \mathcal{P}_{n_1,d} \to \mathcal{P}_{n_2,d}$ by the formula

$$\langle i_1 \ldots i_d\rangle \;\mapsto\; \langle \iota(i_1)\ldots\iota(i_d)\rangle.$$

These homomorphisms are functorial, by which we mean that if we have two maps $\iota_1\colon \{1,\ldots,n_1\} \to \{1,\ldots,n_2\}$ and $\iota_2\colon \{1,\ldots,n_2\} \to \{1,\ldots,n_3\}$, then

$$\varphi_{\iota_2\circ\iota_1} \;=\; \varphi_{\iota_2}\circ\varphi_{\iota_1}.$$

5. Inclusion. As a special case, if we take $\iota\colon \{1,\ldots,n\} \to \{1,\ldots,n{+}1\}$ to be the inclusion map, then φ_ι is an injective homomorphism from $\mathcal{P}_{n,d}$ to $\mathcal{P}_{n+1,d}$.

6. Folding. Another special case of functoriality. If we take a surjection

$$\iota\colon \{1,\ldots,n+k\} \;\to\; \{1,\ldots,n\},$$

then we will call the corresponding map $\varphi_\iota\colon \mathcal{P}_{n+k,d} \to \mathcal{P}_{n,d}$ a folding morphism.

7. Duality. There is a canonical isomorphism

$$\star\colon \mathcal{P}_{n,d} \;\to\; \mathcal{P}_{n,n-d}$$

which is defined on the level of (nonzero) brackets as

$$\langle i_1 \ldots i_d\rangle \;\mapsto\; \varepsilon\cdot\langle j_1\ldots j_{n-d}\rangle,$$

where $\{i_1,\ldots,i_d\}\cup\{j_1,\ldots,j_{n-d}\} = \{1,\ldots,n\}$, and the sign ε is the sign of the corresponding shuffle permutation of $\{1,\ldots,n\}$. We can prove that this operation is a well-defined group homomorphism by checking that it sends the ideal $\mathcal{I}_{n,d}$ to the ideal $\mathcal{I}_{n,n-d}$, which by Proposition 3.2 is generated by quadratic bracket polynomials $Q(\underline{i};\underline{j})$. Note that for $n = 2d$ the operation $\star$ defines an automorphism of $\mathcal{P}_{2d,d}$. In some literature on invariant theory, this duality operation is called association (see [13, Ch. 3]).

8. Dual inclusion. By combining duality and inclusion maps we get the map

$$\mathcal{P}_{n,d} \xrightarrow{\;\star\;} \mathcal{P}_{n,n-d} \xrightarrow{\;\iota\;} \mathcal{P}_{n+1,n-d} \xrightarrow{\;\star\;} \mathcal{P}_{n+1,d+1},$$

which we call dual inclusion.

9. Projection to $\mathcal{P}_{n,2}$. There is a special morphism $\eta_2\colon \mathcal{P}_{n,d} \to \mathcal{P}_{n,2}$ defined on the level of brackets as

$$\eta_2(\langle i_1\ldots i_d\rangle) \;=\; \prod_{1\le k<l\le d} \langle i_k i_l\rangle.$$

It can be easily checked that this is a well-defined algebra homomorphism and it sends the graded piece $\mathcal{P}^{\alpha}_{n,d}$ to $\mathcal{P}^{(d-1)\alpha}_{n,2}$.

3.2 S-cross-ratios and exceptional S-cross-ratios

Let us denote by $\mathbb{K}_{n,d}$ the homogeneous fraction field of $\mathcal{P}_{n,d}$, i.e., a field generated by all elements of the form p/q, where polynomials $p, q \in \mathcal{P}_{n,d}$ have the same multi-degree. It follows from the definition of $\mathcal{P}_{n,d}$ that higher cross-ratios are simply non-constant elements of $\mathbb{K}_{n,d}$. Because of the inclusion $\mathcal{P}_{n,d} \subset \mathbb{Q}[x_{11},\ldots,x_{dn}]$, we have that the field $\mathbb{K}_{n,d}$ is unirational (over $\mathbb{Q}$), i.e., it is a subfield of a rational function field. The following simple proposition shows that the field $\mathbb{K}_{n,d}$ is in fact rational.

Proposition 3.3. *The field $\mathbb{K}_{n,d}$ is isomorphic to the rational field $\mathbb{Q}(z_{1,1},\dots,z_{d-1,n-d-1})$ in $(d-1)(n-d-1)$ variables.*

Proof. Consider a parametric family of matrices $Z \in \mathbb{M}_{d,n}$ of the form

$$Z \;=\; \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 0 & 1 & z_{1,1} & \dots & z_{1,n-d-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 & z_{d-1,1} & \dots & z_{d-1,n-d-1} \end{pmatrix} \tag{3.1}$$

and define a morphism $\phi\colon \mathbb{K}_{n,d} \to \mathbb{Q}(z_{1,1},\dots,z_{d-1,n-d-1})$ that sends r to $\phi(r) = r(Z)$. To check that this map is well-defined, we need to prove that if a homogeneous polynomial $p \in \mathcal{P}^{\alpha}_{n,d}$ vanishes on the set of all matrices of the form (3.1), then $p = 0$. Since any matrix $A \in \mathbb{M}_{d,n}$ all of whose $d \times d$ minors are nonzero is equivalent under multiplication by an element of GL_d to a matrix of the form (3.1), we get that p must vanish on the complement to the affine subvariety in $\mathbb{M}_{d,n}$ given by the equation

$$\prod_{i_1<\dots<i_d} \langle i_1 \dots i_d \rangle \;=\; 0\,.$$

Since the affine space $\mathbb{M}_{d,n}$ is irreducible, we get that $p = 0$.

To prove the proposition, it is enough to construct the mapping inverse to ϕ. We will do it using the projected cross-ratios. For any set $I = \{i_1,\dots,i_{d-2}\}$ define the projected cross-ratio $(I|j_1,j_2,j_3,j_4) \in \mathbb{K}_{n,d}$ by the formula

$$(I|j_1,j_2,j_3,j_4) \;=\; \frac{\langle Ij_1j_3\rangle\langle Ij_2j_4\rangle}{\langle Ij_1j_4\rangle\langle Ij_2j_3\rangle}.$$

Note that the expression on the right does not change under any involution $i_s \leftrightarrow i_t$, so the projected cross-ratio is well-defined. Define $I_s = \{1,2,\dots,d\} \smallsetminus \{1,s\}$. A simple computation shows that

$$(I_{k+1}|1,k+1,d+l+1,d+1)(Z) \;=\; z_{k,l}\,.$$

Thus we can define the inverse morphism ψ by the formula

$$z_{k,l} \;\mapsto\; (I_{k+1}|1,k+1,d+l+1,d+1)\,.$$

Clearly, $\phi \circ \psi = \mathrm{id}$. To prove the identity $\psi \circ \phi = \mathrm{id}$, we note again that if two cross-ratios coincide on all matrices of the form (3.1), then they must be equal identically. $\square$

As was noted in the introduction, the classical cross-ratio

$$r \;=\; \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle}\,, \tag{3.2}$$

that lies in $\mathbb{K}_{4,2}$, is not only invariant under the actions of GL_2 and Diag_4, but it also satisfies the Plücker identity

$$1 - \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle} \;=\; \frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle}\,. \tag{3.3}$$

We want to define a class of cross-ratios that satisfy identities of similar type. We start by defining S-cross-ratios.

Definition 3.4. *Let S be a finite set of pairwise inequivalent irreducible homogeneous polynomials in $\mathcal{P}_{n,d}$, and let $\Gamma = U_S^\mu$, where U_S^μ is defined in* (2.2)*. Explicitly, the group Γ is defined as*

$$\Gamma \;=\; \Big\{ \pm \prod_{p\in S} p^{\alpha(p)} \in \mathbb{K}_{n,d} \;\Big|\; \alpha \in \mathbb{Z}^S \Big\}.$$

*An S-**cross-ratio** is any non-constant element of Γ.*

Similarly to 2.7 and 2.10 we define the special polynomials and the exceptional cross-ratios. The notion of an exceptional cross-ratio is the main object of study in this work.

Definition 3.5. *Let S and Γ be as in Definition 3.4. A polynomial $p \in \mathcal{P}_{n,d}$ is called S-**special** (or simply **special**) if the rank of the abelian group $U(p)$ (defined in* (2.6)*) is greater than* 1.

Definition 3.6. *Let S be a finite set of pairwise inequivalent irreducible homogeneous polynomials in $\mathcal{P}_{n,d}$ and let $\Gamma = U_S^\mu$ (see* (2.2)*). We call an S-cross-ratio $r \in \Gamma$ **exceptional** if it belongs to the set $\mathcal{R}(\Gamma, U_{S'_{max}(S)})$ (see 2.7).*

In other words, an element $r \in \Gamma$ is exceptional if and only if for every irreducible divisor $\pi \notin S$ of the numerator of $1 - r$ there exists an $r' \in \Gamma$ such that r and r' are multiplicatively independent and $r' \equiv 1 \ (mod\ \pi)$.

Examples. To see that we are still in line with the motivating examples presented in the introduction, let us check that the above definition (for a suitable choice of S) includes both the classical cross-ratio and Goncharov's triple ratio.

For the classical cross-ratio r defined as in (3.2), we choose $S = \{\langle 12\rangle, \ldots, \langle 34\rangle\}$ and the identity (3.3) shows that $r \in \mathcal{R}(U_S^\mu, U_S^\mu)$.

Goncharov's triple ratio r_g, which, using our notation, can be written as

$$r_g \;=\; \frac{\langle 124\rangle\langle 235\rangle\langle 136\rangle}{\langle 125\rangle\langle 236\rangle\langle 134\rangle}$$

satisfies the identity

$$1 - r_g \;=\; \frac{\langle 123\rangle \cdot (\langle 145\rangle\langle 236\rangle - \langle 124\rangle\langle 356\rangle)}{\langle 125\rangle\langle 236\rangle\langle 134\rangle}.$$

If we take $S = \{\langle 123\rangle, \ldots, \langle 456\rangle\}$, then clearly r_g is an S-cross-ratio. To show that r_g is exceptional (with respect to S), we need to check that the polynomial

$$\pi_g \;=\; \langle 145\rangle\langle 236\rangle - \langle 124\rangle\langle 356\rangle \in \mathcal{P}_{6,3}^{(111111)}$$

occurs in the factorization of $1 - r'$ for some S-cross-ratio r' that is multiplicatively independent of r_g. It is easy to see that the cross-ratio

$$r' \;=\; \frac{\langle 145\rangle\langle 236\rangle}{\langle 124\rangle\langle 356\rangle}$$

has this property. In this particular case we have $\operatorname{rk} U(\pi_g) = 4$.

It follows from Theorem 2.11 that for each finite S there are only finitely many exceptional S-cross-ratios. The best possible result that we can hope for is to find all exceptional S-cross-ratios by first computing the set $S' = S'_{max}(S)$ and then computing $\mathcal{R}(U_S^\mu, U_{S'_{max}(S)})$ using Algorithm 1. As was noted in the previous chapter, we do not have a practical algorithm for computing $S'_{max}(S)$. Thus, in each case we have to find a reasonable candidate S' for $S'_{max}(S)$ and then we compute $\mathcal{R}(U^{\mu}_{S}, U_{S'})$ for this candidate.

In what follows we will work only with sets S, S' that are closed under the action of $\mathfrak{S}_n$. For this we will use the following notation for $\mathfrak{S}_n$-closure of a set:

$$\{\pi_1, \dots, \pi_k\}_{\mathfrak{S}_n} \;=\; \{\sigma\pi_i \mid i \in \{1, \dots, k\},\, \sigma \in \mathfrak{S}_n\}.$$

We will also implicitly assume, whenever we use this notation, that the polynomials $\pi_1, \dots, \pi_k$ are inequivalent under the $\mathfrak{S}_n$-action (here we mean that p is equivalent to q under $\mathfrak{S}_n$-action if for some $c \in \mathbb{Q}$ and $\sigma \in \mathfrak{S}_n$ we have $\sigma p = c \cdot q$).

Before we go to explicit computations, we want to briefly discuss stability of the set of exceptional S-cross-ratios of n points in $\mathbb{P}^{d-1}$ as n goes to infinity. Because of the natural inclusions $\mathcal{P}_{n,d} \to \mathcal{P}_{n+1,d}$, we can promote the set $S \subset \mathcal{P}_{n,d}$ to $\mathcal{P}_{n+k,d}$ by taking the closure of S in $\mathcal{P}_{n+k,d}$ under the $\mathfrak{S}_{n+k}$-action. Let us denote the new set by $S^{(n+k)}$ and the group $U^{\mu}_{S^{(n+k)}}$ (that is generated by the elements of $S^{(n+k)}$ and ± 1) by $\Gamma^{(n+k)}$. As is easily verified, the rank function

$$(\pi, k) \;\mapsto\; \operatorname{rk} U_{\Gamma^{(n+k)}}(\pi)$$

is nondecreasing in k for each fixed $\pi \in \mathcal{P}_{n,d}$. Therefore, the set $S'_{max}(S^{(n+k)})$ contains the closure of the set $S'_{max}(S)$ in $\mathcal{P}_{n+k,d}$ under the action of $\mathfrak{S}_{n+k}$. This leads us to the following stability question.

Question 1. *Let $S \subset \mathcal{P}_{n,d} \smallsetminus \mathbb{Q}$ be a set of irreducible homogeneous polynomials that is closed under $\mathfrak{S}_n$-action. Is it true that the sets $S'_{max}(S^{(k)})$ stabilize? By this we mean that there exists some $l \geq n$ such that $S'_{max}(S^{(k)}) = S'_{max}(S^{(l)})_{\mathfrak{S}_k}$ for all $k > l$. If this is true, does the set of exceptional cross-ratios $\mathcal{R}(S^{(k)}, S'_{max}(S^{(k)}))$ also stabilize?*

Our computations below indicate that the answer is positive for $S = \{\langle 1 \dots d \rangle\}_{\mathfrak{S}_n}$ and we believe that in general the answer is positive, but we do not have enough evidence for any other choices of S.

3.3 Classification of exceptional bracket cross-ratios

In this section we consider the problem of classifying exceptional S-cross-ratios in $\mathbb{K}_{n,d}$ for the set $S = S_{n,d} = \{\langle 123 \dots d \rangle\}_{\mathfrak{S}_n}$, we will also call these elements **bracket cross-ratios**. Let us denote by $\Gamma_{n,d}$ the multiplicative group $U^{\mu}_{S_{n,d}}$ (recall from (2.2) that this is simply the group generated by the elements of $S_{n,d}$ and ± 1). We will start by describing our computations for $d = 2, 3$, and 4. Then, for all $d \geq 2$, we will give an explicit finite set of bracket cross-ratios that conjecturally contains all the exceptional ones. (We will explain, although very briefly, the way in which we have computed sets of special polynomials in Section 3.4.)

3.3.1 Cross-ratios on $\mathbb{P}^1$

Theorem 3.7. *The polynomials $\pi^{(1)}_i$, $0 \leq i \leq 7$, are irreducible and $S_{n,2}$-special (in the sense of Definition 3.5).*

i	n	*Rank*	*Degree*	*Polynomial* $\pi^{(1)}_i$
0	≥ 5	$n-3$	11	$\langle 12 \rangle$
1	≥ 4	2	2222	$\langle 13 \rangle^2 \langle 24 \rangle^2 - \langle 12 \rangle \langle 13 \rangle \langle 24 \rangle \langle 34 \rangle + \langle 12 \rangle^2 \langle 34 \rangle^2$

2	≥ 5	2	21111	$\langle 14\rangle\langle 15\rangle\langle 23\rangle - \langle 12\rangle\langle 13\rangle\langle 45\rangle$
3	≥ 5	2	22211	$\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 25\rangle + \langle 12\rangle^2\langle 34\rangle\langle 35\rangle$
4	≥ 6	3	111111	$\langle 16\rangle\langle 25\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle\langle 56\rangle$
5	≥ 6	2	221111	$\langle 15\rangle\langle 16\rangle\langle 23\rangle\langle 24\rangle + \langle 12\rangle\langle 14\rangle\langle 26\rangle\langle 35\rangle$
6	≥ 7	2	2111111	$\langle 13\rangle\langle 17\rangle\langle 26\rangle\langle 45\rangle + \langle 12\rangle\langle 14\rangle\langle 35\rangle\langle 67\rangle$
7	≥ 8	2	11111111	$\langle 13\rangle\langle 27\rangle\langle 46\rangle\langle 58\rangle - \langle 17\rangle\langle 23\rangle\langle 48\rangle\langle 56\rangle$

Table 3.1: $S_{n,2}$-special polynomials

Proof. We check that these polynomials are special by an application of Algorithm 2.

The irreducibility of each polynomial can be easily checked in any computer algebra system, since the element $p \in \mathcal{P}_{n,d}$ is irreducible if and only if the corresponding polynomial in $\mathbb{Q}[x_{ij}]$ is irreducible (this follows from property 2 in Section 3.1.2). ☐

Remark. Notice that the expressions for the above polynomials in terms of brackets are not unique. For instance, the polynomial $\pi_1^{(1)}$ can also be written in the form

$$\langle 12\rangle^2\langle 34\rangle^2 + \langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle,$$

and the other π_i's also have several alternative forms even as binomials in $\langle ij\rangle$.

Note that the above table stops at $n = 8$, i.e., there are no polynomials that involve 9 or more points in $\mathbb{P}^1$. The reason for this is that we do not find any new $S_{n,2}$-special polynomials in $\mathcal{P}_{n,2}$ for $n \geq 9$ except for the ones that belong to $\mathcal{P}_{8,2}$ and are listed in the above table. We conjecture that there are in fact no new $S_{n,2}$-special polynomials for $n \geq 9$. More precisely, set

$$S'_{n,2} \;=\; \{\pi_i^{(1)} \mid 0 \leq i \leq 7,\, \pi_i^{(1)} \in \mathcal{P}_{n,2}\}.$$

We conjecture that this set contains all $S_{n,2}$-special polynomials:

Conjecture 1. *For all $n \geq 4$ the set $S'_{max}(\Gamma_{n,2})$ is equal to $S'_{n,2}$.*

For $n = 4$ this conjecture is easy to prove, and it is equivalent to the following simple fact: the only values of $z \in \mathbb{C}$ for which both z and $1 - z$ are roots of unity are $z = \exp(\pm\pi i/3)$. We have also proved the conjecture in the case $n = 5$, however the proof is rather long and involved. Regardless of whether the conjecture is true in general, we can still compute all the exceptional cross-ratios in $\mathcal{R}(\Gamma_{8,2}, U_{S'_{8,2}})$ (see Definition 3.5).

Theorem 3.8. *The solution set $\mathcal{R}(\Gamma_{8,2}, S'_{8,2})$ consists of precisely 18 $\mathfrak{S}_8$-orbits, the representatives of which are given in the following table. The last column of the table indicates the factorization of $(1 - r)$ modulo the $\mathfrak{S}_8$-action, where $\pi_i^{(1)}$ are the special polynomials from Table 3.1.*

i	*Degree*	*Numerator of* r_i	*Denominator of* r_i	$j : \pi_j^{(1)} \mid (1 - r_i)$
1	11110000	$\langle 13\rangle\langle 24\rangle$	$\langle 14\rangle\langle 23\rangle$	$0, 0$
2	22220000	$-\langle 12\rangle^2\langle 34\rangle^2$	$\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle$	1
3	21111000	$\langle 12\rangle\langle 13\rangle\langle 45\rangle$	$\langle 14\rangle\langle 15\rangle\langle 23\rangle$	2
4	22211000	$-\langle 12\rangle^2\langle 34\rangle\langle 35\rangle$	$\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 25\rangle$	3
5	22211000	$\langle 12\rangle^2\langle 34\rangle\langle 35\rangle$	$\langle 13\rangle^2\langle 24\rangle\langle 25\rangle$	$0, 2$

6	42222000	$\langle 12\rangle^2\langle 13\rangle^2\langle 45\rangle^2$	$\langle 14\rangle^2\langle 15\rangle^2\langle 23\rangle^2$	$2,2$
7	43322000	$-\langle 12\rangle^3\langle 14\rangle\langle 34\rangle\langle 35\rangle^2$	$\langle 13\rangle^3\langle 15\rangle\langle 24\rangle^2\langle 25\rangle$	$2,3$
8	11111100	$\langle 16\rangle\langle 23\rangle\langle 45\rangle$	$\langle 15\rangle\langle 24\rangle\langle 36\rangle$	4
9	22111100	$-\langle 12\rangle^2\langle 35\rangle\langle 46\rangle$	$\langle 13\rangle\langle 16\rangle\langle 24\rangle\langle 25\rangle$	5
10	22111100	$\langle 12\rangle\langle 15\rangle\langle 23\rangle\langle 46\rangle$	$\langle 13\rangle\langle 14\rangle\langle 25\rangle\langle 26\rangle$	5
11	22111100	$\langle 13\rangle\langle 14\rangle\langle 25\rangle\langle 26\rangle$	$\langle 15\rangle\langle 16\rangle\langle 23\rangle\langle 24\rangle$	$0,4$
12	22222200	$-\langle 13\rangle^2\langle 24\rangle\langle 26\rangle\langle 45\rangle\langle 56\rangle$	$\langle 14\rangle\langle 16\rangle\langle 25\rangle^2\langle 34\rangle\langle 36\rangle$	$4,4$
13	33222200	$\langle 14\rangle\langle 16\rangle^2\langle 23\rangle^2\langle 25\rangle\langle 45\rangle$	$\langle 13\rangle\langle 15\rangle^2\langle 24\rangle^2\langle 26\rangle\langle 36\rangle$	$4,5$
14	44222200	$-\langle 12\rangle^3\langle 15\rangle\langle 23\rangle\langle 35\rangle\langle 46\rangle^2$	$\langle 13\rangle^2\langle 14\rangle\langle 16\rangle\langle 24\rangle\langle 25\rangle^2\langle 26\rangle$	$5,5$
15	21111110	$-\langle 13\rangle\langle 17\rangle\langle 26\rangle\langle 45\rangle$	$\langle 12\rangle\langle 14\rangle\langle 35\rangle\langle 67\rangle$	6
16	42222220	$\langle 13\rangle^2\langle 16\rangle\langle 17\rangle\langle 26\rangle\langle 27\rangle\langle 45\rangle^2$	$\langle 12\rangle^2\langle 14\rangle\langle 15\rangle\langle 34\rangle\langle 35\rangle\langle 67\rangle^2$	$6,6$
17	11111111	$\langle 13\rangle\langle 27\rangle\langle 46\rangle\langle 58\rangle$	$\langle 17\rangle\langle 23\rangle\langle 48\rangle\langle 56\rangle$	7
18	22222222	$\langle 13\rangle\langle 17\rangle\langle 23\rangle\langle 27\rangle\langle 45\rangle^2\langle 68\rangle^2$	$\langle 12\rangle^2\langle 37\rangle^2\langle 46\rangle\langle 48\rangle\langle 56\rangle\langle 58\rangle$	$7,7$

Table 3.2: Exceptional bracket cross-ratios on $\mathbb{P}^1$

Proof. This result is a direct application of Algorithm 1. □

Notice that the last column of this table contains some duplicates. For instance, it indicates that $1-r_{14}$ is divisible by two distinct factors in the orbit of $\pi_5^{(1)}$. Notice also that if Conjecture 1 holds, then the table of Theorem 3.8 gives a complete list of exceptional bracket cross-ratios on $\mathbb{P}^1$.

3.3.2 Cross-ratios on $\mathbb{P}^2$

In this case the list of special polynomials is significantly longer.

Theorem 3.9. *There are at least* 34 $\mathfrak{S}_{10}$*-orbits of* $S_{10,3}$*-special polynomials with representatives* $\{\pi_i^{(2)}\}$ *listed in Table A.3 in the appendix.*

In this case, we also do not find any special polynomials in $\mathcal{P}_{n,3}$ for $n \geq 10$. Once again, let us define $S'_{n,3}$ to be the $\mathfrak{S}_n$-closure of the set $\{\pi_i^{(2)} \mid 0 \leq i \leq 33,\ \pi_i^{(2)} \in \mathcal{P}_{n,3}\}$. In this case we also conjecture that all the special polynomials are contained in the set $S'_{n,3}$.

Conjecture 2. *For all* $n \geq 6$ *the set* $S'_{max}(S_{n,3})$ *is equal to* $S'_{n,3}$.

An application of Algorithm 1 from Chapter 2 allows us to compute the exceptional cross-ratios in the set $\mathcal{R}(\Gamma_{10,3}, U_{S'_{10,3}})$.

Theorem 3.10. *The solution set* $\mathcal{R}(\Gamma_{10,3}, U_{S'_{10,3}})$ *consists of precisely* 91 $\mathfrak{S}_{10}$*-orbits with representatives listed in Table A.4.*

We will not need this full list which is rather long, but let us list the subset of solutions for $\mathcal{P}_{6,3}$ (i.e., for up to 6 points on $\mathbb{P}^2$). In this case the list of special polynomials $S'_{6,3}$ is given in the following table.

i	Rank	Degree	Polynomial $\pi_i^{(2)}$
0	4	111000	$\langle 123\rangle$
1	2	222210	$\langle 123\rangle\langle 124\rangle\langle 345\rangle - \langle 125\rangle\langle 134\rangle\langle 234\rangle$
2	2	332220	$\langle 123\rangle\langle 125\rangle\langle 134\rangle\langle 245\rangle - \langle 124\rangle^2\langle 135\rangle\langle 235\rangle$
3	4	111111	$\langle 123\rangle\langle 456\rangle - \langle 124\rangle\langle 356\rangle$
4	2	321111	$\langle 123\rangle\langle 124\rangle\langle 156\rangle - \langle 125\rangle\langle 126\rangle\langle 134\rangle$
5	2	222111	$\langle 123\rangle\langle 125\rangle\langle 346\rangle + \langle 126\rangle\langle 135\rangle\langle 234\rangle$
6	2	422211	$\langle 123\rangle^2\langle 145\rangle\langle 146\rangle + \langle 124\rangle\langle 126\rangle\langle 134\rangle\langle 135\rangle$
7	2	332211	$\langle 123\rangle\langle 125\rangle\langle 134\rangle\langle 246\rangle + \langle 124\rangle\langle 126\rangle\langle 135\rangle\langle 234\rangle$
8	2	332211	$\langle 123\rangle\langle 124\rangle\langle 145\rangle\langle 236\rangle + \langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 234\rangle$
9	3	322221	$\langle 124\rangle\langle 125\rangle\langle 136\rangle\langle 345\rangle - \langle 126\rangle\langle 134\rangle\langle 135\rangle\langle 245\rangle$
10	2	322221	$\langle 123\rangle\langle 125\rangle\langle 146\rangle\langle 345\rangle - \langle 124\rangle\langle 134\rangle\langle 156\rangle\langle 235\rangle$
11	2	322221	$\langle 124\rangle\langle 135\rangle\langle 156\rangle\langle 234\rangle + \langle 126\rangle\langle 134\rangle\langle 145\rangle\langle 235\rangle$
12	2	222222	$\langle 123\rangle\langle 156\rangle\langle 246\rangle\langle 345\rangle + \langle 125\rangle\langle 146\rangle\langle 234\rangle\langle 356\rangle$
13	2	222222	$\langle 124\rangle\langle 156\rangle\langle 236\rangle\langle 345\rangle - \langle 125\rangle\langle 146\rangle\langle 235\rangle\langle 346\rangle$
14	2	222222	$\langle 124\rangle\langle 156\rangle\langle 235\rangle\langle 346\rangle - \langle 125\rangle\langle 146\rangle\langle 236\rangle\langle 345\rangle$
15	5	222222	$\langle 123\rangle\langle 145\rangle\langle 256\rangle\langle 346\rangle - \langle 125\rangle\langle 134\rangle\langle 236\rangle\langle 456\rangle$

Table 3.3: $S_{6,3}$-special polynomials

The exceptional cross-ratios of 6 points on $\mathbb{P}^2$ are given in the following table.

i	Degree	Numerator of r_i	Denominator of r_i	$j : \pi_i^{(2)} \| 1 - r_i$
1	211110	$\langle 124\rangle\langle 135\rangle$	$\langle 125\rangle\langle 134\rangle$	$0, 0$
2	222210	$\langle 123\rangle\langle 134\rangle\langle 245\rangle$	$\langle 124\rangle\langle 135\rangle\langle 234\rangle$	1
3	332220	$-\langle 123\rangle^2\langle 145\rangle\langle 245\rangle$	$\langle 124\rangle\langle 125\rangle\langle 135\rangle\langle 234\rangle$	2
4	332220	$\langle 123\rangle^2\langle 145\rangle\langle 245\rangle$	$\langle 124\rangle^2\langle 135\rangle\langle 235\rangle$	$0, 1$
5	554430	$-\langle 123\rangle^3\langle 134\rangle\langle 145\rangle\langle 245\rangle^2$	$\langle 124\rangle^3\langle 135\rangle^2\langle 234\rangle\langle 235\rangle$	$1, 2$
6	111111	$\langle 126\rangle\langle 345\rangle$	$\langle 125\rangle\langle 346\rangle$	3
7	321111	$\langle 123\rangle\langle 125\rangle\langle 146\rangle$	$\langle 124\rangle\langle 126\rangle\langle 135\rangle$	4
8	222111	$\langle 124\rangle\langle 136\rangle\langle 235\rangle$	$\langle 125\rangle\langle 134\rangle\langle 236\rangle$	$0, 3$
9	222111	$-\langle 123\rangle\langle 125\rangle\langle 346\rangle$	$\langle 126\rangle\langle 135\rangle\langle 234\rangle$	5
10	422211	$\langle 124\rangle^2\langle 135\rangle\langle 136\rangle$	$\langle 123\rangle^2\langle 145\rangle\langle 146\rangle$	$0, 4$
11	422211	$-\langle 123\rangle^2\langle 145\rangle\langle 146\rangle$	$\langle 124\rangle\langle 126\rangle\langle 134\rangle\langle 135\rangle$	6
12	332211	$-\langle 123\rangle\langle 126\rangle\langle 145\rangle\langle 234\rangle$	$\langle 124\rangle\langle 125\rangle\langle 134\rangle\langle 236\rangle$	7
13	332211	$-\langle 123\rangle^2\langle 145\rangle\langle 246\rangle$	$\langle 124\rangle\langle 126\rangle\langle 135\rangle\langle 234\rangle$	8
14	332211	$-\langle 123\rangle\langle 124\rangle\langle 145\rangle\langle 236\rangle$	$\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 234\rangle$	8
15	332211	$\langle 123\rangle^2\langle 145\rangle\langle 246\rangle$	$\langle 124\rangle^2\langle 135\rangle\langle 236\rangle$	7
16	332211	$\langle 123\rangle^2\langle 146\rangle\langle 245\rangle$	$\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 234\rangle$	$0, 5$

17	322221	$\langle126\rangle\langle134\rangle\langle145\rangle\langle235\rangle$	$\langle125\rangle\langle135\rangle\langle146\rangle\langle234\rangle$	10
18	322221	$\langle126\rangle\langle134\rangle\langle135\rangle\langle245\rangle$	$\langle124\rangle\langle125\rangle\langle136\rangle\langle345\rangle$	9
19	322221	$\langle123\rangle\langle126\rangle\langle145\rangle\langle345\rangle$	$\langle124\rangle\langle135\rangle\langle156\rangle\langle234\rangle$	10
20	322221	$-\langle123\rangle\langle126\rangle\langle145\rangle\langle345\rangle$	$\langle125\rangle\langle135\rangle\langle146\rangle\langle234\rangle$	11
21	222222	$\langle123\rangle\langle145\rangle\langle256\rangle\langle346\rangle$	$\langle125\rangle\langle134\rangle\langle236\rangle\langle456\rangle$	15
22	222222	$\langle126\rangle\langle145\rangle\langle236\rangle\langle345\rangle$	$\langle124\rangle\langle156\rangle\langle234\rangle\langle356\rangle$	14
23	222222	$\langle123\rangle\langle125\rangle\langle346\rangle\langle456\rangle$	$\langle124\rangle\langle126\rangle\langle345\rangle\langle356\rangle$	$3,3$
24	222222	$-\langle126\rangle\langle135\rangle\langle234\rangle\langle456\rangle$	$\langle124\rangle\langle156\rangle\langle235\rangle\langle346\rangle$	12
25	222222	$-\langle126\rangle\langle145\rangle\langle236\rangle\langle345\rangle$	$\langle125\rangle\langle146\rangle\langle234\rangle\langle356\rangle$	13
26	222222	$\langle126\rangle\langle145\rangle\langle235\rangle\langle346\rangle$	$\langle125\rangle\langle146\rangle\langle234\rangle\langle356\rangle$	14
27	333222	$-\langle125\rangle^2\langle134\rangle\langle236\rangle\langle346\rangle$	$\langle126\rangle\langle135\rangle\langle156\rangle\langle234\rangle^2$	$3,5$
28	443322	$\langle126\rangle^2\langle135\rangle\langle145\rangle\langle234\rangle^2$	$\langle125\rangle^2\langle134\rangle^2\langle236\rangle\langle246\rangle$	$3,7$
29	743322	$-\langle124\rangle^3\langle125\rangle\langle135\rangle\langle136\rangle^2$	$\langle123\rangle^3\langle126\rangle\langle145\rangle^2\langle146\rangle$	$4,6$
30	554322	$\langle123\rangle^3\langle145\rangle^2\langle236\rangle\langle246\rangle$	$\langle125\rangle\langle126\rangle^2\langle134\rangle\langle135\rangle\langle234\rangle^2$	$5,8$
31	664422	$-\langle123\rangle^3\langle125\rangle\langle134\rangle\langle145\rangle\langle246\rangle^2$	$\langle124\rangle^3\langle126\rangle\langle135\rangle^2\langle234\rangle\langle236\rangle$	$7,8$
32	433332	$\langle126\rangle\langle134\rangle^2\langle156\rangle\langle235\rangle\langle245\rangle$	$\langle125\rangle^2\langle136\rangle\langle146\rangle\langle234\rangle\langle345\rangle$	$3,9$
33	644442	$\langle124\rangle\langle126\rangle\langle134\rangle^2\langle145\rangle\langle156\rangle\langle235\rangle^2$	$\langle123\rangle\langle125\rangle^2\langle135\rangle\langle146\rangle^2\langle234\rangle\langle345\rangle$	$9,10$
34	644442	$\langle123\rangle\langle126\rangle^2\langle134\rangle\langle145\rangle^2\langle235\rangle\langle345\rangle$	$\langle124\rangle\langle125\rangle\langle135\rangle^2\langle146\rangle\langle156\rangle\langle234\rangle^2$	$10,11$
35	333333	$\langle124\rangle\langle135\rangle\langle146\rangle\langle236\rangle\langle256\rangle\langle345\rangle$	$\langle126\rangle\langle134\rangle\langle145\rangle\langle235\rangle\langle246\rangle\langle356\rangle$	$3,15$
36	333333	$\langle123\rangle\langle126\rangle\langle135\rangle\langle246\rangle\langle345\rangle\langle456\rangle$	$\langle124\rangle\langle125\rangle\langle146\rangle\langle235\rangle\langle346\rangle\langle356\rangle$	$3,12$
37	444444	$\langle125\rangle\langle126\rangle\langle135\rangle\langle146\rangle\langle234\rangle^2\langle356\rangle\langle456\rangle$	$\langle123\rangle\langle124\rangle\langle156\rangle^2\langle235\rangle\langle246\rangle\langle345\rangle\langle346\rangle$	$12,15$
38	444444	$\langle124\rangle\langle126\rangle\langle145\rangle\langle156\rangle\langle235\rangle^2\langle346\rangle^2$	$\langle125\rangle^2\langle146\rangle^2\langle234\rangle\langle236\rangle\langle345\rangle\langle356\rangle$	$14,15$
39	444444	$\langle126\rangle^2\langle145\rangle^2\langle235\rangle\langle236\rangle\langle345\rangle\langle346\rangle$	$\langle124\rangle\langle125\rangle\langle146\rangle\langle156\rangle\langle234\rangle^2\langle356\rangle^2$	$13,14$

Table 3.4: Exceptional bracket cross-ratios of up to 6 points on $\mathbb{P}^2$

3.3.3 Cross-ratios on $\mathbb{P}^3$

We will consider only the case of 8 points in $\mathbb{P}^3$.

Theorem 3.11. *There are at least* 51 $\mathfrak{S}_8$*-orbits* $S_{8,4}$*-special polynomials with representatives* $\{\pi_i^{(3)}\}$ *listed in Table A.5 in the Appendix.*

Once again we conjecture, though with lesser certainty than in the two previous cases, that this list (and hence also the list in the following Theorem 3.12) is complete. Let us define $S'_{8,4}$ to be the set $\{\pi_i^{(3)} \mid 0 \le i \le 51\}_{\mathfrak{S}_8}$.

Theorem 3.12. *The solution set* $\mathcal{R}(\Gamma_{8,4}, U_{S'_{8,4}})$ *consists of precisely* 148 $\mathfrak{S}_8$*-orbits of exceptional cross-ratios with representatives listed in Table A.6.*

3.3.4 Cross-ratios on $\mathbb{P}^{d-1}$ for $d > 4$

We now want to analyze the results of the above computations and put forward a conjectural description of the set of all exceptional bracket cross-ratios. This description agrees with all of our computations for $\mathbb{P}^{d-1}$ for $d = 2, 3, 4$.

First, we observe that in the case of $\mathbb{P}^1$ the ($\mathfrak{S}_8$-orbits of) special primes from Theorem 3.7 can all be obtained from the polynomial

$$\langle13\rangle\langle24\rangle\langle57\rangle\langle68\rangle \;-\; \langle14\rangle\langle23\rangle\langle58\rangle\langle67\rangle \in \mathcal{P}_{8,2}$$

by applying to it a homomorphism of the type φ_ι (see property 4 in Section 3.1.2), where $\iota\colon \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ is an arbitrary map, and collecting all irreducible factors of the resulting polynomial. Similarly, all the exceptional bracket cross-ratios from Table 3.4 can be obtained starting from the two cross-ratios

$$\begin{aligned} r_1 &= \frac{\langle 13\rangle\langle 24\rangle\langle 57\rangle\langle 68\rangle}{\langle 14\rangle\langle 23\rangle\langle 58\rangle\langle 67\rangle}, \\ r_2 &= \frac{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 56\rangle^2\langle 78\rangle^2}{\langle 12\rangle^2\langle 34\rangle^2\langle 67\rangle\langle 68\rangle\langle 57\rangle\langle 58\rangle} \end{aligned}$$

by applying φ_ι to them. In this case, some care needs to be taken, since the homomorphism φ_ι is not always well-defined on the fraction field, so we allow applying φ_ι to a fraction $r = p/q$ only if both $\varphi_\iota(p)$ and $\varphi_\iota(q)$ are nonzero.

If we denote the classical cross-ratios of points $(1, 2, 3, 4)$ and $(5, 6, 7, 8)$ by

$$\begin{aligned} x &= \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle}, \\ y &= \frac{\langle 57\rangle\langle 68\rangle}{\langle 58\rangle\langle 67\rangle}, \end{aligned}$$

then the cross-ratios r_1 and r_2 can be rewritten as

$$\begin{aligned} r_1 &= x \cdot y, \\ r_2 &= \frac{x(1-y)^2}{y(1-x)^2}. \end{aligned} \tag{3.4}$$

In the case of $\mathbb{P}^2$ all the special polynomials from Theorem 3.10 can be obtained from the “generating” polynomial

$$\langle 135\rangle\langle 146\rangle\langle 279\rangle\langle 2,8,10\rangle - \langle 136\rangle\langle 145\rangle\langle 2,7,10\rangle\langle 289\rangle \in \mathcal{P}_{10,3}$$

by the same procedure as above, and a similar statement also holds for exceptional bracket cross-ratios, where the two “generating” cross-ratios have numbers 90 and 91 in the table A.4. In this case we find that exactly the same formulas (3.4) hold for the two “generating” cross-ratios if we define $x = (1|3, 4, 5, 6)$ and $y = (2|7, 8, 9, 10)$, where

$$(a|b, c, d, e) = \frac{\langle abd\rangle\langle ace\rangle}{\langle abe\rangle\langle acd\rangle}$$

is the classical cross-ratio projected from the point a.

This construction is easy to generalize to $\mathbb{P}^{d-1}$. Let $r^{(d-1)}_{1,G}, r^{(d-1)}_{2,G}$ be the cross-ratios defined by (3.4), where

$$\begin{aligned} x &= (I_x|2d-3, 2d-2, 2d-1, 2d), \\ y &= (I_y|2d+1, 2d+2, 2d+3, 2d+4) \end{aligned}$$

are the cross-ratios projected from $d-2$ points, where $I_x = \{1, 2, \ldots, (d-2)\}$, and $I_y = \{d-1, \ldots, 2(d-2)\}$. We also define the “generating” special polynomial $\pi^{(d-1)}_G \in \mathcal{P}_{2d+4,d}$ to be the numerator of $1 - r^{(d-1)}_{1,G}$.

Theorem 3.13. *The polynomial $\pi^{(d-1)}_G$ is S-special, and it occurs in the factorization of the numerator of $1 - r^{(d-1)}_{i,G}$ for $i = 1, 2$.*

Proof. For simplicity, let us only consider the case $d = 2$. The proof in all other cases is almost identical, one just needs to replace all cross-ratios by the corresponding projected cross-ratios (as was done in the definition of the cross-ratios $r_{i,G}^{(d-1)}$).

An application of Algorithm 2 produces the basis for the multiplicative group $U(\pi_G^{(1)})$ showing that it is generated by two cross-ratios

$$\begin{aligned} r_1 &= \frac{\langle 13\rangle\langle 24\rangle\langle 57\rangle\langle 68\rangle}{\langle 14\rangle\langle 23\rangle\langle 58\rangle\langle 67\rangle}, \\ r_2 &= -\frac{\langle 12\rangle\langle 34\rangle\langle 58\rangle\langle 67\rangle}{\langle 13\rangle\langle 24\rangle\langle 56\rangle\langle 78\rangle}. \end{aligned}$$

The identity

$$1 - \frac{x(1-y)^2}{y(1-x)^2} = \frac{(y-x)(1-xy)}{y(1-x)^2}. \tag{3.5}$$

shows that the numerator of $1 - r_{2,G}^{(1)}$ can be factorized as

$$(\langle 13\rangle\langle 24\rangle\langle 57\rangle\langle 68\rangle - \langle 14\rangle\langle 23\rangle\langle 58\rangle\langle 67\rangle)\cdot(\langle 13\rangle\langle 24\rangle\langle 58\rangle\langle 67\rangle - \langle 14\rangle\langle 23\rangle\langle 57\rangle\langle 68\rangle),$$

where the second factor is obtained from the first by permuting $7 \leftrightarrow 8$ and thus also belongs to the $\mathfrak{S}_8$-orbit of $\pi_G^{(1)}$. □

By taking the image of $\pi_G^{(d-1)}$ under φ_ι for various maps ι we can produce a lot of special polynomials. Similarly, we can get new exceptional bracket cross-ratios from $r_{i,G}^{(d-1)}$ by the same procedure. We conjecture that all exceptional bracket cross-ratios can be obtained in this way.

Conjecture 3. *Let $S = \{\langle 123\dots d\rangle\}_{\mathfrak{S}_n}$ and let $\pi_G^{(d-1)}$ and $r_{i,G}^{(d-1)}$ be defined as above. Then*

(i) For every special polynomial $\pi \in S'_{max}(S)$ there exists a map $\iota\colon \{1,2,\dots,2d+4\} \to \{1,\dots,n\}$ such that π divides $\varphi_\iota(\pi_G^{(d-1)})$;

(ii) For every exceptional bracket cross-ratio $r \in \Gamma_{n,d}$ there exists a map $\iota\colon \{1,2,\dots,2d+4\} \to \{1,\dots,n\}$ and an index $i \in \{1,2\}$ such that $r = \varphi_\iota(r_{i,G}^{(d-1)})$.

Remark. The combinatorics of obtaining exceptional cross-ratios and special polynomials from the "generating" ones is far from trivial. For example, the only case in which there is a special prime of total degree 2 is the case $d = 3$, but it is not easy to prove it just using part (i) of Conjecture 3. Moreover, it is not even clear that applying φ_ι to an exceptional cross-ratio always yields another exceptional cross-ratio (for instance, it could happen that for some divisor π of the numerator of $1 - r$ we would have $\operatorname{rk} U(\pi) \geq 2$ but $\operatorname{rk} U(\varphi_\iota(\pi)) = 1$), but in all of our calculations this seems to be the case.

This conjecture has some interesting corollaries. We collect these corollaries in the following proposition, whose trivial proof we omit.

Proposition 3.14. *Let $S = \{\langle 123\dots d\rangle\}_{\mathfrak{S}_n}$. If Conjecture 3 holds, then*

(i) any S-special polynomial can be written as a difference of two bracket monomials;

(ii) any exceptional S-cross-ratio can be written as a product of 2 or 4 projected cross-ratios;

(iii) any exceptional S-cross-ratio involves at most $2d+4$ different points. If it involves exactly $2d+4$ points, then it belongs to the $\mathfrak{S}_{2d+4}$-orbit of one of the two generating cross-ratios;

(iv) the answer to Question 1 (see Section 3.2) is positive for $S = \{\langle 123\dots d\rangle\}_{\mathfrak{S}_n}$.

Note that Conjecture 3 also implies Conjecture 1 and Conjecture 2.

3.4 Other multiplicative systems

For some of the functional equations in Chapter 5 we will need to consider exceptional S-cross-ratios for choices of S other than $S = \{\langle 123 \dots d \rangle\}_{\mathfrak{S}_n}$. In this section we present some of our computations in these cases. The main difficulty in all such computations is to find the set of special polynomials $S'_{max}(S)$ or at least some approximation to this set. We begin by discussing this problem.

3.4.1 General methods for finding special polynomials

There are two practical methods that we have for computing subsets of elements from $S'_{max}(S)$. The first is based on the following very simple observation, which also applies to all S-unit equations of the type (2.3).

Proposition 3.15. *Let $\alpha_1, \dots, \alpha_4 \in \mathbb{Z}^S_{\geq 0}$ and $\varepsilon_1, \dots, \varepsilon_4 \in \{-1, 1\}$ be such that*

$$A_1 + A_2 + A_3 + A_4 \ = \ 0,$$

where $A_i = \varepsilon_i \cdot \prod_{\pi \in S} \pi^{\alpha_i(\pi)}$. Assume that $\gcd(A_1, A_2, A_3, A_4) = 1$, that $A_i \neq \pm A_j$ for all $i \neq j$, and let $k < l$ be such that polynomial $\pi_{kl} = A_k + A_l$ is irreducible. Then $\pi_{kl} \in S'_{max}(S)$.

Proof. Note that we have $\pi_{12} + \pi_{34} = \pi_{13} + \pi_{24} = \pi_{14} + \pi_{23} = 0$. Consider the cross-ratios r_{ij} defined by

$$r_{ij} \ = \ -\frac{A_i}{A_j}.$$

We can assume without loss of generality that $k = 1$ and $l = 2$. Then π_{12} divides both $1 - r_{12}$ and $1 - r_{34}$. Therefore, to prove that $\pi_{12} \in S'_{max}(S)$, it is enough to prove that r_{12} and r_{34} are multiplicatively independent. Since we know that π_{12} is irreducible, the degree of r_{12} and r_{34} is equal to the degree of A_1. Therefore, if $r_{12}^n = r_{34}^m$, then $n = \pm m$. Since both r_{12} and r_{34} lie in $U(\pi_{12})$, which is a free abelian group, we get that if they are multiplicatively dependent, then $r_{12} = r_{34}^{\pm 1}$. By replacing r_{34} by r_{43} if necessary, we can assume that $r_{12} = r_{34}$. Then we get

$$1 - r_{12} \ = \ 1 - r_{34},$$

which implies that

$$(A_1 + A_2)/A_2 \ = \ (A_3 + A_4)/A_4.$$

Using the condition $A_1 + A_2 = -(A_3 + A_4)$ we get that $A_2 + A_4 = 0$, which contradicts our assumption $A_i + A_j \neq 0$. $\square$

This proposition allows us to find elements in $S'_{max}(S)$ by solving equation $A_1 + A_2 + A_3 + A_4 = 0$, where $A_i \in U_S \cap \mathcal{P}^{\alpha}_{n,m}$ for some degree $\alpha \in \mathbb{Z}^n$. Note that, together with special polynomials, we also get a collection of 9 distinct exceptional S-cross-ratios. The divisibility relations between all these objects are represented by edges in the following picture.

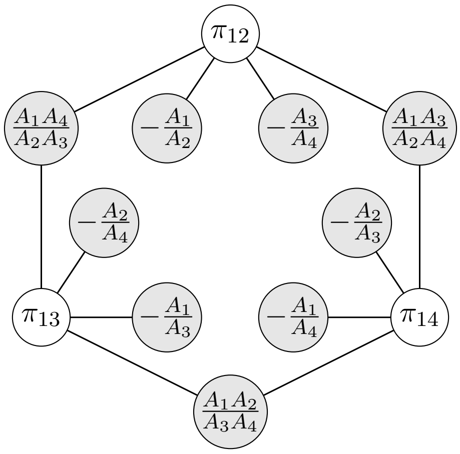

Figure 1. The “nine cross-ratios” construction of special polynomials

Remark. The 9 cross-ratios illustrated above are the same as the arguments that figure in the symmetric formulation of the Kummer-Spence relation (4.5) for $\mathcal{L}_3$ (for details on why these equations are equivalent, see [38, p. 18]).

The other method that we use to generate elements in $S'_{max}(S)$ is based on graph traversal. Let $G_{max}(S)$ be the graph with vertex set $S'_{max}(S)$ in which we connect two polynomials $\pi_1, \pi_2 \in S'_{max}(S)$ by an edge if and only if the intersection $U(\pi_1) \cap U(\pi_2)$ is nonempty. If we have already found some subset $S' \subseteq S'_{max}(S)$, we can search for new elements of $S'_{max}(S)$ by generating cross-ratios r of small degree in $U(\pi)$ for $\pi \in S'$, factorizing the numerator of $(1-r)$ for each such cross-ratio, and then checking whether the new factors belong to $S'_{max}(S)$.

3.4.2 Multiplicative system for 6 points in $\mathbb{P}^2$

Let us denote by $\langle 12;34;56\rangle$ the polynomial $\langle 123\rangle\langle 456\rangle - \langle 124\rangle\langle 356\rangle$ and consider the set $S = \{\langle 123\rangle, \langle 12;34;56\rangle\}_{\mathfrak{S}_6}$. Using the methods described above, we can find a large set of S-special polynomials.

Theorem 3.16. *There are at least* 354 $\mathfrak{S}_6$*-orbits of polynomials in* $S'_{max}(S)$*. The rank distribution of these polynomials is given in the following table.*

Rank	2	3	4	5	6	8	13
# *of orbits*	321	18	10	2	1	1	1

Let us list some of these special polynomials of the highest rank:

Rank	Polynomial
13	$\langle 123\rangle$
8	$\langle 12;34;56\rangle$
6	$\langle 134\rangle\langle 256\rangle\langle 15;23;46\rangle\langle 16;24;35\rangle - \langle 135\rangle\langle 146\rangle\langle 156\rangle\langle 234\rangle\langle 235\rangle\langle 246\rangle$
5	$\langle 134\rangle\langle 156\rangle\langle 235\rangle\langle 246\rangle - \langle 135\rangle\langle 146\rangle\langle 234\rangle\langle 256\rangle$
5	$\langle 123\rangle\langle 125\rangle\langle 346\rangle\langle 456\rangle + \langle 12;36;45\rangle\langle 13;25;46\rangle$

As usual, we compute the set $\mathcal{R}(U_S^{\mu}, U_{S'})$ using Algorithm 1.

Theorem 3.17. *If S' is the set of special polynomials from Theorem 3.16, then there are exactly* 1629 $\mathfrak{S}_6$*-orbits of cross-ratios in* $\mathcal{R}(U_S^\mu, U_{S'})$. *The number of orbits of exceptional cross-ratios of each total degree is given in the following table.*

Degree	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
# of orbits	6	12	179	227	352	122	248	69	143	50	130	22	44	6	19

3.4.3 Multiplicative system for 8 points in $\mathbb{P}^3$

Let us denote by $\langle 1|23;45;67\rangle$ the polynomial $\langle 1234\rangle\langle 1567\rangle - \langle 1235\rangle\langle 1467\rangle$ For cross-ratios defined on 8 points in $\mathbb{P}^3$ we consider the set

$$S = \{\langle 1234\rangle, \langle 1|23;45;67\rangle\}_{\mathfrak{S}_8}.$$

Theorem 3.18. *There are at least* 36814 $\mathfrak{S}_8$*-orbits of polynomials in* $S'_{max}(S)$. *The rank distribution is given in the following table.*

Rank	2	3	4	5	6	7	8	9	10	14	15	19	32	201
# of orbits	35343	1346	87	16	7	4	1	2	2	1	2	1	1	1

Let us list some of these special polynomials of the highest rank:

Rank	Polynomial
201	$\langle 1234\rangle$
32	$\langle 1\|23;45;67\rangle$
19	$\langle 1234\rangle\langle 5678\rangle - \langle 1235\rangle\langle 4678\rangle$
15	$\langle 1234\rangle\langle 1256\rangle\langle 1278\rangle - \langle 1245\rangle\langle 1267\rangle\langle 1238\rangle$
15	$\langle 1234\rangle\langle 1256\rangle\langle 3456\rangle - \langle 1236\rangle\langle 1456\rangle\langle 2345\rangle$
14	$\langle 1235\rangle\langle 1467\rangle\langle 2348\rangle - \langle 1246\rangle\langle 1347\rangle\langle 2358\rangle$

The exceptional S-cross-ratios are given in the following theorem.

Theorem 3.19. *If S' is the set of special polynomials from Theorem 3.18, then there are exactly* 173887 $\mathfrak{S}_8$*-orbits of exceptional cross-ratios in* $\mathcal{R}(U_S^\mu, U_{S'})$. *The number of orbits of exceptional cross-ratios of each total degree is given in the following table.*

Degree	2	3	4	5	6	7	8	9	10
# of orbits	6	178	7601	35795	44528	19155	10141	5691	15104

Degree	11	12	13	14	15	16
# of orbits	11044	13861	3836	5041	961	945

We will use the results of this computation in Chapter 5, where we will construct an interesting skew-symmetric functional equation for $\mathcal{L}_4$ from these cross-ratios.

CHAPTER 4

Polylogarithms and Zagier's conjecture

The main references for this chapter are the papers [36] and [17].

4.1 Classical and single-valued polylogarithms

We start by defining and studying some properties of classical polylogarithms.

4.1.1 Definition and properties

The classical polylogarithm of weight m is the analytic function defined by the power series

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}, \tag{4.1}$$

which converges absolutely in the unit disc $|z| < 1$ (and even in the closed disc $|z| \leq 1$ if $m \geq 2$). For $m = 1$ the function Li_m is the classical logarithm:

$$\mathrm{Li}_1(z) = -\log(1-z).$$

The function Li_2 was first defined by Euler [14], and it was studied by Lobachevsky, Abel, Kummer, and many other mathematicians in the 19th century. For an introduction to this function, its properties, applications, and connections to different areas of mathematics the reader is advised to read the wonderful paper [37]. Numerous identities, functional equations, and general structural results for polylogarithms can be found in the books [23] and [24].

Some of the special values of Li_m include

$$\begin{aligned}
\mathrm{Li}_m(0) &= 0;\\
\mathrm{Li}_m(1) &= \zeta(m);\\
\mathrm{Li}_m(-1) &= \zeta(m)\cdot(2^{1-m}-1);\\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{1}{2}\zeta(2) - \frac{1}{2}\log^2(2);\\
\mathrm{Li}_3\left(\frac{1}{2}\right) &= \frac{7}{8}\zeta(3) - \frac{1}{2}\log(2)\zeta(2) + \frac{1}{6}\log^3(2).
\end{aligned}$$

It is easy to see from (4.1) that these functions satisfy the differential equation

$$\left(z\frac{d}{dz}\right)\mathrm{Li}_m(z) \;=\; \mathrm{Li}_{m-1}(z)$$

and, using the fact that $\mathrm{Li}_0(z) = z/(1-z)$, we get

$$\left(z\frac{d}{dz}\right)^m \mathrm{Li}_m(z) \;=\; \frac{z}{1-z}.$$

This implies that $\mathrm{Li}_m(z)$ can be extended analytically to a multivalued function on $\mathbb{C}\setminus\{0,1\}$. In particular, the principal branch of Li_m can be defined inductively on $\mathbb{C}\setminus[1,\infty)$ by the formula

$$\mathrm{Li}_m(z) \;=\; \int_0^z \mathrm{Li}_{m-1}(z)\frac{dz}{z}.$$

The general solution to the differential equation

$$\left(z\frac{d}{dz}\right)^m F(z) \;=\; \frac{z}{1-z}$$

in the cut plane $\mathbb{C}\setminus(-\infty,0]\cup[1,\infty)$ has form

$$F(z) \;=\; \mathrm{Li}_m(z) + c_0 + c_1\log(z) + \cdots + c_{m-1}\log^{m-1}(z)$$

for some choice of principal branches of Li_m and $\log$ and therefore any branch of the analytic continuation of Li_m can also be written in this form. Explicitly, the monodromy of Li_m has been computed by Ramakrishnan (see [29], [21]).

4.1.2 Functional equations for polylogarithms

One of the most interesting and the most mysterious properties of classical polylogarithms is the functional equations that they satisfy. For example, the classical logarithm satisfies

$$\log(x) + \log(y) \;=\; \log(xy) \pmod{2\pi i\mathbb{Z}}.$$

The Euler's dilogarithm Li_2 satisfies the following simple reflection symmetries

$$\begin{aligned}\mathrm{Li}_2\left(\frac{1}{z}\right) + \mathrm{Li}_2(z) &= -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z),\\ \mathrm{Li}_2(1-z) + \mathrm{Li}_2(z) &= \frac{\pi^2}{6} - \log(z)\log(1-z),\end{aligned}$$

as well as the five-term relation

$$\begin{aligned}&\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right)\\ &= \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right).\end{aligned} \tag{4.2}$$

As one can see from these examples, the "elementary" terms involved in the functional equations for Li_m can get rather complicated. For this reason (another reason to introduce them is for the formulation of Zagier's conjecture), it is much better to work with the following single-valued versions of Li_m

$$\begin{aligned}
\mathcal{L}_1(z) &= \Re(\mathrm{Li}_1(z)) \;=\; -\log|1-z|;\\
\mathcal{L}_2(z) &= \Im(\mathrm{Li}_2(z) - \mathrm{Li}_1(z)\log|z|);\\
\mathcal{L}_3(z) &= \Re(\mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log|z| + \mathrm{Li}_1(z)\log^2|z|/3);\\
&\dots
\end{aligned}$$

The general formula for these functions is (see [36])

$$\mathcal{L}_m(z) \;=\; \Re_m\Big(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!}\mathrm{Li}_{m-k}(z)\log^k|z|\Big),$$

where $\Re_m(z)$ denotes the imaginary part $\Im(z)$ if m is even and the real part $\Re(z)$ if m is odd. The number $B_k \in \mathbb{Q}$ is the k-th Bernoulli number, defined by the generating function

$$\frac{x}{e^x - 1} \;=\; \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

The function $\mathcal{L}_2$ is known as the Bloch-Wigner dilogarithm and was defined in [3]. The functions $\mathcal{L}_m$ are indeed "single-valued" (see [36]):

Proposition 4.1. *For $m \geq 2$ the function $\mathcal{L}_m\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ is continuous and bounded on $\mathbb{P}^1(\mathbb{C})$ and real analytic on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$.*

The immediate benefit of working with $\mathcal{L}_m$ instead of Li_m is that the functional equations become "clean". Let us give some examples to illustrate this.

Dilogarithm. For $\mathcal{L}_2$ the reflection properties become

$$\begin{aligned}
\mathcal{L}_2(z) + \mathcal{L}_2(1/z) &= 0,\\
\mathcal{L}_2(z) + \mathcal{L}_2(1-z) &= 0,
\end{aligned}$$

and the five-term relation (4.2) becomes

$$\mathcal{L}_2(x) + \mathcal{L}_2(y) + \mathcal{L}_2\left(\frac{1-x}{1-xy}\right) + \mathcal{L}_2(1-xy) + \mathcal{L}_2\left(\frac{1-y}{1-xy}\right) \;=\; 0. \tag{4.3}$$

Trilogarithm. For the single-valued trilogarithm $\mathcal{L}_3$ we have one reflection property

$$\mathcal{L}_3(z) - \mathcal{L}_3(1/z) \;=\; 0,$$

and another one-variable functional equation

$$\mathcal{L}_3(z) + \mathcal{L}_3(1-z) + \mathcal{L}_3(1-1/z) \;=\; \mathcal{L}_3(1). \tag{4.4}$$

There is also the two-variable Kummer-Spence relation (for a symmetric version of this identity, see the remark after Proposition 3.15)

$$\begin{aligned}
&2\mathcal{L}_3(x) + 2\mathcal{L}_3(y) + 2\mathcal{L}_3\left(\frac{1-x}{1-y}\right) + 2\mathcal{L}_3\left(\frac{x(1-y)}{y(1-x)}\right) + 2\mathcal{L}_3\left(\frac{-x(1-y)}{1-x}\right)\\
&+2\mathcal{L}_3\left(\frac{-(1-y)}{y(1-x)}\right) - \mathcal{L}_3(xy) - \mathcal{L}_3\left(\frac{x}{y}\right) - \mathcal{L}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) \;=\; 2\mathcal{L}_3(1).
\end{aligned} \tag{4.5}$$

Finally, there is the following 3-variable functional equation that was discovered by Goncharov in his proof of Zagier's conjecture for $m = 3$.

$$\begin{aligned}\mathcal{L}_3(-xyz) + \sum_{cyc}\Big[\mathcal{L}_3(x) + \mathcal{L}_3(xy-y+1) + \mathcal{L}_3\Big(\frac{xy-y+1}{xy}\Big) - \mathcal{L}_3\Big(\frac{xy-y+1}{x}\Big)\\ +\mathcal{L}_3\Big(\frac{x(yz-z+1)}{x-1-xz}\Big) + \mathcal{L}_3\Big(\frac{yz-z+1}{y(xz-x+1)}\Big) - \mathcal{L}_3\Big(\frac{yz-z+1}{yz(xz-x+1)}\Big)\Big] \;=\; 3\mathcal{L}_3(1),\end{aligned} \tag{4.6}$$

where $\sum_{cyc} f(x,y,z) = f(x,y,z) + f(y,z,x) + f(z,x,y)$. In fact, Goncharov found a more general identity, a symmetric form of which will be given later in this chapter.

Higher polylogarithms. In general, the problem of finding functional equations for $\mathcal{L}_m$ is very difficult. There are two functional equations that are known for all $\mathcal{L}_m$. These are the reflection property

$$\mathcal{L}_m(z) + (-1)^m \mathcal{L}_m(1/z) \;=\; 0$$

and the "distribution property"

$$\mathcal{L}_m(z^n) \;=\; n^{m-1} \sum_{\varepsilon^n=1} \mathcal{L}_m(\varepsilon z). \tag{4.7}$$

Aside from these two functional equations and their formal consequences, no nontrivial functional equation of $\mathcal{L}_m$ is known for $m \geq 8$. Examples of nontrivial functional equations of $\mathcal{L}_m$ for m up to 7 were found by Gangl [16].

Geometric interpretation of $\mathcal{L}_2$. Let us give a geometric interpretation for $\mathcal{L}_2$ and the five-term relation (4.3). If we define

$$\mathcal{L}_2(z_0, z_1, z_2, z_3) \;=\; \mathcal{L}_2\Big(\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}\Big),$$

then we get (as a consequence of the two reflection properties) that for any permutation $\sigma \in \mathfrak{S}_4$

$$\mathcal{L}_2(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) \;=\; \mathrm{sgn}(\sigma) \cdot \mathcal{L}_2(z_1, z_2, z_3, z_4)$$

and (4.3) becomes equivalent to

$$\sum_{i=0}^{4} (-1)^i \mathcal{L}_2(z_0, \ldots, \widehat{z_i}, \ldots, z_4) \;=\; 0. \tag{4.8}$$

The geometric meaning of $\mathcal{L}_2(z_0, z_1, z_2, z_3)$ is the oriented volume of an ideal hyperbolic tetrahedron in $\mathbb{H}^3$ with vertices at $z_0, z_1, z_2, z_3 \in \mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$, this result goes back to Lobachevsky (see [27]). The functional equation (4.3) can then be interpreted as the so-called Pachner 2-3 move on triangulations.

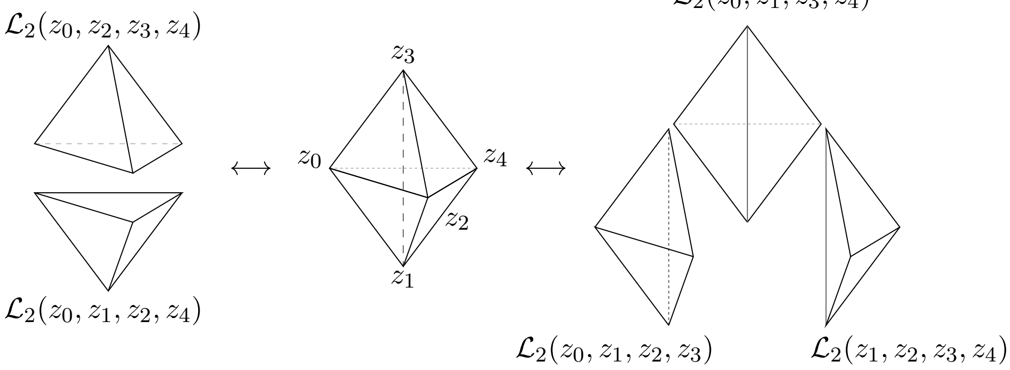

Figure 2. Geometric interpretation of the five-term relation

Similarly, there is a geometric interpretation of $\mathcal{L}_3$ (and the relation (4.6) plays an important role in it), namely, there is a formula due to Goncharov [18] that expresses the volume of a hyperbolic simplex in $\mathbb{H}^5$ in terms of $\mathcal{L}_3$. Unfortunately, for $m \geq 4$ there is no such simple geometric interpretation.

Criterion for functional equations. While finding functional equations is hard, proving a given relation is usually easy. The reason for this is the following result [36, Prop. 1, p. 411].

Proposition 4.2. *The necessary and sufficient condition for a collection of integers n_i and rational functions $f_i \in \mathbb{C}(t)$ to satisfy $\sum_i n_i \mathcal{L}_m(f_i(t)) = \mathrm{const}$ is the identity*

$$\sum_i n_i \cdot (1-f_i) \wedge f_i \otimes \underbrace{f_i \otimes \cdots \otimes f_i}_{m-2} \;=\; \mathrm{const}$$

in $(\Lambda^2(\mathbb{C}(t)^\times) \otimes \mathrm{Sym}^{m-2}(\mathbb{C}(t)^\times)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Note that the restriction to one-variable functional equations is not important, since one can recover the multivariate case by specialization. We refer to the tensor expression

$$\sum_i n_i \cdot (1-f_i) \wedge f_i \otimes \underbrace{f_i \otimes \cdots \otimes f_i}_{m-2}$$

as the **symbol** of the function $\sum_i n_i \mathcal{L}_m(f_i(t_1, \ldots, t_k))$.

Let us look at some examples of computations with the symbol.

Example 1. The reflection property $\mathcal{L}_2(x) + \mathcal{L}_2(1/x) = 0$ follows from the following symbol computation

$$(1-x) \wedge x + (1-1/x) \wedge (1/x) \;=\; \frac{1-x}{1-1/x} \wedge x \;=\; x \wedge x \;=\; 0.$$

Example 2. The reflection property $\mathcal{L}_2(x) + \mathcal{L}_2(1-x) = 0$ follows from the identity

$$(1-x) \wedge x + x \wedge (1-x) \;=\; 0.$$

Example 3. For a more interesting example, let us prove (4.8). From the two reflection properties we get that

$$\mathcal{L}_2(z_1, z_2, z_3, z_4) \;=\; \frac{1}{24} \sum_{\sigma \in \mathfrak{S}_4} \mathrm{sgn}(\sigma) \mathcal{L}_2(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}),$$

and we need to prove that

$$\sum_{\sigma \in \mathfrak{S}_5} \mathrm{sgn}(\sigma) \mathcal{L}_2(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) \;=\; 0.$$

Each of the terms $\mathcal{L}_2(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ has symbol

$$\frac{(z_{\sigma(1)} - z_{\sigma(3)})(z_{\sigma(2)} - z_{\sigma(4)})}{(z_{\sigma(1)} - z_{\sigma(4)})(z_{\sigma(2)} - z_{\sigma(3)})} \wedge \frac{(z_{\sigma(1)} - z_{\sigma(2)})(z_{\sigma(3)} - z_{\sigma(4)})}{(z_{\sigma(1)} - z_{\sigma(4)})(z_{\sigma(2)} - z_{\sigma(3)})}.$$

Expanding this expression we get a collection of terms of the form $(z_i - z_j) \wedge (z_i - z_k)$, and the identity (4.8) follows from

$$\sum_{\sigma \in \mathfrak{S}_5} \mathrm{sgn}(\sigma)(z_1 - z_2) \wedge (z_1 - z_3) \;=\; 0,$$

which holds since the exchange $z_4 \leftrightarrow z_5$ fixes $(z_1 - z_2) \wedge (z_1 - z_3)$ but changes its sign.

4.2 Zagier's conjecture

Let F be a number field, $\mathcal{O}_F$ be its ring of integers, r_1 and r_2 be the numbers of real and pairs of complex conjugate embeddings respectively. Recall that the Dedekind zeta function of F is defined by

$$\zeta_F(s) \;=\; \sum_{\mathfrak{a}} N\mathfrak{a}^{-s} \;=\; \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}\,,$$

where $\mathfrak{a}$ runs over the nonzero ideals of $\mathcal{O}_F$, and $\mathfrak{p}$ – over nonzero prime ideals (both expressions converge absolutely for $\Re(s) > 1$).

Very roughly, Zagier's conjecture states that the value $\zeta_F(m)$ for $m \geq 2$ can be expressed in terms of $\mathcal{L}_m(\sigma(\alpha))$, where $\alpha \in F$ and σ is an embedding of F into $\mathbb{C}$. It is instructive to view it as a generalization of the analytic class number formula.

Proposition 4.3 (Analytic class number formula)**.** *For any number field F the following identity holds*

$$\mathrm{res}_{s=1}\zeta_F(s) \;=\; \frac{2^{r_1}(2\pi)^{r_2} \cdot h_F \cdot \mathrm{Reg}_F}{w_F \cdot \sqrt{|D_F|}}\,,$$

where h_F is the class number, w_F is the root number, Reg_F *is the regulator, and D_F is the discriminant of F.*

Recall that the regulator of F is the absolute value of the determinant of the following $(r_1 + r_2 - 1) \times (r_1 + r_2 - 1)$ matrix

$$\begin{pmatrix} l_1(u_1) & l_1(u_2) & \dots & l_1(u_{r_1+r_2-1}) \\ l_2(u_1) & l_2(u_2) & \dots & l_2(u_{r_1+r_2-1}) \\ \vdots & \vdots & \ddots & \vdots \\ l_{r_1+r_2-1}(u_1) & l_{r_1+r_2-1}(u_2) & \dots & l_{r_1+r_2-1}(u_{r_1+r_2-1}) \end{pmatrix},$$

where $l_i(x) = \log|\sigma_i(x)|$ for $i = 1, \dots, r_1$, $l_i(x) = 2\log|\sigma_i(x)|$ for $i = r_1 + 1, \dots, r_1 + r_2 - 1$, and $u_1, \dots, u_{r_1+r_2-1}$ is the basis of the group of units of $\mathcal{O}_F$.

We will use the notation $a \sim_{\mathbb{Q}^\times} b$ to mean that $a/b \in \mathbb{Q}^\times$. The analytic class number formula has the following simple corollary that does not involve the class number at all.

Corollary 4.4. *For any number field F we have*

$$\mathrm{res}_{s=1}\zeta_F(s) \;\sim_{\mathbb{Q}^\times}\; \pi^{r_2}|D_F|^{-1/2} \cdot \mathrm{Reg}_F\,.$$

In its weak form, Zagier's conjecture seeks to generalize this corollary. To formulate it, we need to set up some notation. For any set Y denote by $\mathbb{Z}[Y]$ the free abelian group generated by the set of symbols $\{[y] \mid y \in Y\}$. We extend $\mathcal{L}_m\colon \mathbb{C} \to \mathbb{R}$ linearly to a homomorphism from $\mathbb{Z}[\mathbb{C}]$ to $\mathbb{R}$. That is, put

$$\mathcal{L}_m\Big(\sum_{\nu} a_\nu [x_\nu]\Big) \;=\; \sum_{\nu} a_\nu \mathcal{L}_m(x_\nu).$$

Similarly, we extend any complex embedding $\sigma\colon F \to \mathbb{C}$ to a homomorphism $\sigma\colon \mathbb{Z}[F] \to \mathbb{Z}[\mathbb{C}]$. Finally, define an integer $d_m = d_m(F)$ by the formula

$$d_m \;=\; \begin{cases} r_2, & m \text{ even;} \\ r_1 + r_2, & m \text{ odd},\ m > 1. \end{cases}$$

Conjecture 4 (Zagier's Conjecture, weak form)**.** *Let $\{\sigma_j\}_{j=1,\ldots,r_1+2r_2}$ be the set of all complex embeddings of F, labeled in such a way that $\sigma_{r_1+r_2+k} = \overline{\sigma_k}$ for $k = 1, \ldots, r_2$. Then there exist elements $y_1, \ldots, y_{d_m} \in \mathbb{Z}[F]$ such that*

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} |D_F|^{-1/2} \cdot \pi^{md_{m+1}} \cdot \det\Big(\mathcal{L}_m(\sigma_i(y_j))\Big)_{1\le i,j\le d_m}. \tag{4.9}$$

Let us look at some examples.

Example 1. For $F = \mathbb{Q}$ we have $\zeta_F(s) = \zeta(s)$ the Riemann zeta function. The number d_m is equal to 1 for m odd and 0 for m even. For even m the determinant on the right-hand side of (4.9) is equal (by definition) to 1, and thus $\zeta(m)$ must be a rational multiple of π^m, the fact that was first established by Euler:

$$\zeta(m) = -\frac{B_m(2\pi i)^m}{2\cdot m!}.$$

For odd m we recover the special value for $\mathcal{L}_m$:

$$\zeta(m) = \mathcal{L}_m(1).$$

Example 2. Let $F = \mathbb{Q}(i)$. Then d_m is equal to 1 for all m. For $m = 6$ we compute $\zeta_F(6)$ numerically (using PARI [41]) and observe that to a high precision

$$\zeta_F(6) = \frac{\pi^6}{945}\cdot\mathcal{L}_6(i).$$

This identity can be proved by noting that there is a factorization

$$\zeta_F(s) = \zeta(s)\cdot L(s,\chi_{-4}),$$

where

$$L(s,\chi_{-4}) = \sum_{n\ge1}\frac{(-1)^n}{(2n-1)^s}$$

is the Dirichlet L-function for the character χ_{-4} and observing that $L(2k,\chi_{-4}) = \mathcal{L}_{2k}(i)$ and that $\zeta(6) = \pi^6/945$.

Example 3. For a more interesting example, consider the cubic field $F = \mathbb{Q}(\alpha)$, where α satisfies the equation $\alpha^3 = 1-\alpha$. Let $\alpha_1 \in \mathbb{R}$ and $\alpha_2, \overline{\alpha_2} \in \mathbb{C}$ be the three solutions of this equation. By computing the value $\zeta_F(2)$ numerically we see that, in accordance with Zagier's conjecture, the following identity holds to a high precision

$$\zeta_F(2) \stackrel{?}{=} \frac{8\pi^4}{3\cdot 31^{3/2}}\cdot\mathcal{L}_2(\alpha_2).$$

Similarly, for $\zeta_F(3)$ we find the following identity (which again holds numerically to a high precision)

$$\zeta_F(3) \stackrel{?}{=} \frac{32\pi^3}{31^{5/2}}\cdot\begin{vmatrix}\mathcal{L}_3(\xi_1) & \mathcal{L}_3(1)\\ \mathcal{L}_3(\xi_2) & \mathcal{L}_3(1)\end{vmatrix},$$

where $\xi_i = 2[\alpha_i] - [-\alpha_i^2] \in \mathbb{Z}[\mathbb{C}]$.

With some more effort we also find that

$$\zeta_F(4) \stackrel{?}{=} \frac{4\pi^8}{135\cdot 31^{7/2}}\cdot\mathcal{L}_4(\xi),$$

where
$$\xi \;=\; 645[\alpha_2^2] - 600[\alpha_2] + 480[-\alpha_2^2] - 40[-\alpha_2^3] - 24[-\alpha_2^5]\,.$$
If we replace each term $[-x]$ above by a linear combination of $[x]$ and $[x^2]$, we see that this is what Lewin called a "ladder", i.e., a polylogarithm relation whose arguments are powers of a fixed algebraic number. The biggest known ladder involves powers of Lehmer's 10th degree algebraic irrationality, see [11], [1].

Note that for some classes of number fields the truth of Zagier's conjecture is known. These include all abelian extensions of $\mathbb{Q}$ and all totally real fields (where the corresponding statement follows from the theorem of Siegel-Klingen, see [31], [22]). In full generality, the conjecture is open for all $m \geq 4$.

For $m = 2$ Conjecture 4 has been proved by Zagier in [35] using the geometric interpretation of $\mathcal{L}_2$ (he proved a slightly weaker statement in which y_j belong to some extension of F of degree ≤ 4). For $m = 3$ it was proved by Goncharov in [17], where he also outlined a general strategy for proving the conjecture for higher m.

Note that Conjecture 4 does not tell anything about how to choose $y_j \in \mathbb{Z}[F]$. Already in Example 3 above it is quite nontrivial to guess the arguments of $\mathcal{L}_m$ without any further information. We shall now proceed to define the higher Bloch groups that will help us specify the y_j's in Zagier's conjecture.

4.3 Higher Bloch groups

In the rest of this chapter, F will denote any field of characteristic zero. Our immediate goal is to define the higher Bloch groups. These groups are of the form $\mathcal{B}_m(F) = \mathcal{A}_m(F)/\mathcal{C}_m(F)$, where $\mathcal{C}_m(F) < \mathcal{A}_m(F)$ are certain subgroups of $\mathbb{Z}[F]$. Roughly speaking, the group $\mathcal{C}_m(F)$ is "the subgroup of $\mathbb{Z}[F]$ generated by all functional equations of $\mathcal{L}_m$", and $\mathcal{A}_m(F)$ is "the subgroup of all admissible elements".

We will define these groups by induction on m.

Base step $m = 1$**.**

Let $\mathcal{A}_1(F) = \mathbb{Z}[F]$ and define the group $\mathcal{C}_1(F)$ as
$$\mathcal{C}_1(F) \;=\; \big\langle [0], [xy] - [x] - [y] \mid x, y \in F^\times \big\rangle \subset \mathbb{Z}[F]\,.$$
As it is easy to see, the group $\mathcal{B}_1(F) = \mathcal{A}_1(F)/\mathcal{C}_1(F)$ is naturally isomorphic to $F^\times$.

Induction step.

Assume that the groups $\mathcal{A}_m(F), \mathcal{B}_m(F)$, and $\mathcal{C}_m(F)$ are already defined for all F. For $m = 1$ define the homomorphism $\delta_2\colon \mathbb{Z}[F] \to \mathcal{B}_1(F) \wedge \mathcal{B}_1(F)$ by $\delta_2(x) = \{1-x\}_1 \wedge \{x\}_1$ and define $\mathcal{A}_{m+1}(F) = \ker \delta_2$. For $m \geq 2$ we define $\mathcal{A}_{m+1}(F)$ as
$$\mathcal{A}_{m+1}(F) \;=\; \{\xi \in \mathbb{Z}[F] \mid \iota_\phi(\xi) \in \mathcal{C}_m(F)\ \forall \phi \in \mathrm{Hom}(F^\times, \mathbb{Z})\},$$
where $\iota_\phi(\xi) = \sum_i n_i \phi(x_i)[x_i]$ for $\xi = \sum_i n_i [x_i] \in \mathbb{Z}[F]$. Finally, define
$$\mathcal{C}_{m+1}(F) \;=\; \langle \xi(0) - \xi(1) \mid \xi \in \mathcal{A}_{m+1}(F(t)) \rangle.$$

Proposition 4.5 ([36])**.** *For all fields F, the inclusion $\mathcal{A}_m(F) \supset \mathcal{C}_m(F)$ holds. For any number field F, any complex embedding $\sigma\colon F \to \mathbb{C}$, and $y \in C_m(F)$, we have*
$$\mathcal{L}_m(\sigma(y)) \;=\; 0.$$

As a corollary, for any complex embedding $\sigma\colon F \to \mathbb{C}$ the function $\mathcal{L}_m \circ \sigma$ is a well-defined mapping from $\mathcal{B}_m(F)$ to $\mathbb{C}$.

The refined form of Zagier's conjecture is then the following.

Conjecture 5. *The group $\mathcal{B}_m(F)$ is finitely generated of rank d_m, and y_i from Conjecture 4 can be taken to be any basis of $\mathcal{B}_m(F) \otimes \mathbb{Q}$.*

The groups $\mathcal{B}_m(F)$ are called *higher Bloch groups*, and the usual Bloch group is $\mathcal{B}(F) = \mathcal{B}_2(F)$. In the construction of the Goncharov complex in the next section, we will also need the so-called *pre-Bloch groups* $B_m(F) = \mathbb{Z}[F]/\mathcal{C}_m(F)$. Similarly to $\mathcal{B}_m(F)$, the m-th polylogarithm is a well-defined function on $B_m(F)$ (for a given embedding $F \hookrightarrow \mathbb{C}$), and for $m = 1$ we have $B_1(F) \cong \mathcal{B}_1(F)$. We denote by η_m the projection from $\mathbb{Z}[F]$ to $B_m(F)$, and by $\{x\}_m$ the projection of the element $[x] \in \mathbb{Z}[F]$.

Let us remark that for $m = 2$, Wojtkowiak [34] proved that the group $\mathcal{C}_2(F)$ can be equivalently defined (as it was initially defined by Bloch in [3]) as

$$\mathcal{C}_2(F) \;=\; \Big\langle [0],\, [1],\, [x] + [y] + \big[\frac{1-x}{1-xy}\big] + [1-xy] + \big[\frac{1-y}{1-xy}\big] \;\big|\; x, y \in F^\times \Big\rangle.$$

For $m \geq 3$ such simple explicit description of $\mathcal{C}_m(F)$ is not known.

Let us give some examples of calculations in Bloch groups.

Example 1. For $F = \mathbb{Q}$ we have $\{1\}_{2k+1} \in \mathcal{B}_{2k+1}(\mathbb{Q})$ and as was noted above

$$\mathcal{L}_{2k+1}(1) \;=\; \zeta(2k+1) \;\neq\; 0,$$

so $\{1\}_{2k+1}$ is in fact a nontrivial element of $\mathcal{B}_{2k+1}(\mathbb{Q})$.

For a more interesting computation, let us show that $\xi = 4\{2/3\}_3 + 4\{3\}_3 + \{4\}_3$ lies in $\mathcal{B}_3(\mathbb{Q})$. By definition, this is equivalent to two inclusions

$$\begin{aligned} -4[2/3] + 4[3] \;&\in\; \mathcal{C}_2(\mathbb{Q}); \\ 4[2/3] + 2[4] \;&\in\; \mathcal{C}_2(\mathbb{Q}). \end{aligned}$$

The first identity easily follows from the functional equation $\mathcal{L}_2(x) = \mathcal{L}_2(1 - 1/x)$, and the second follows from $\mathcal{L}_2(x^2) = 2\mathcal{L}_2(x) + 2\mathcal{L}_2(-x)$ and $\mathcal{L}_2(x) + \mathcal{L}_2(1-x) = 0$. Thus, $\xi \in \mathcal{B}_3(\mathbb{Q})$ and we can check that

$$\mathcal{L}_3(\xi) = \frac{15}{2}\zeta(3),$$

so that $\xi \neq 0$.

Example 2. Let $F = \mathbb{Q}(i)$. Then we find that $\{i\}_{2k} \in \mathcal{B}_{2k}(F)$ (in general, roots of unity always lie in the higher Bloch groups) and the identity $\mathcal{L}_{2k}(i) = L(2k, \chi_{-4}) \neq 0$ shows that $\{i\}_{2k}$ is nonzero.

Example 3. Let $F = \mathbb{Q}(\alpha)$, where $\alpha^3 = 1 - \alpha$ as in Exercise 3 after Conjecture 4. Then one can check (at least numerically) that $\xi_1 = 2\{\alpha\}_3 - \{-\alpha^2\}_3 \in \mathcal{B}_3(F)$ and

$$\xi_2 \;=\; 645\{\alpha^2\}_4 - 600\{\alpha\}_4 + 480\{-\alpha^2\}_4 - 40\{-\alpha^3\}_4 - 24\{-\alpha^5\}_4 \in \mathcal{B}_4(F).$$

This is precisely the way in which we found the identities for $\zeta_F(3)$ and $\zeta_F(4)$.

4.4 Goncharov's strategy

The main idea of Goncharov's strategy is to find a mapping between two complexes constructed from a field F. One of them is defined in purely geometric terms using configurations of points in F^m, and the other is constructed from the pre-Bloch groups. We now proceed to define these two complexes.

The Goncharov complex. For $m > 2$ define the differential

$$\delta_{m+n} \colon B_m(F) \otimes \bigwedge^{n} F^\times \to B_{m-1}(F) \otimes \bigwedge^{n+1} F^\times$$

by the formula

$$\delta_{m+n}\left(\{x\}_m \otimes y_1 \wedge \cdots \wedge y_n\right) \;=\; \{x\}_{m-1} \otimes x \wedge y_1 \wedge \cdots \wedge y_n.$$

For $m = 2$ we put

$$\delta_{2+n}\left(\{x\}_2 \otimes y_1 \wedge \cdots \wedge y_n\right) \;=\; (1-x) \wedge x \wedge y_1 \wedge \cdots \wedge y_n.$$

One can prove that the groups $B_{m-*}(F) \otimes \bigwedge^* F^\times$ together with the map δ_m form a complex modulo 2-torsion, see [17, p. 199]. Moreover, $\ker \delta_m \subset B_m(F)$ is isomorphic to $\mathcal{B}_m(F)$ modulo torsion. In the sequel, for all our statements concerning this complex, it will always be assumed that we work modulo torsion (or we will even replace all the groups with vector spaces obtained by tensoring with $\mathbb{Q}$).

Grassmannian complex. A set of n points in F^m is said to be in general position if any subset of $k \leq \min(n,m)$ points is linearly independent. Let $\widetilde{C}_n(m)$ be the free abelian group generated by n-tuples of vectors in general position in F^m. For $n \geq m$ let us define the differential $d\colon \widetilde{C}_{n+1}(m) \to \widetilde{C}_n(m)$ by the formula

$$(x_0,\ldots,x_n) \;\overset{d}{\mapsto}\; \sum_i (-1)^i (x_0,\ldots,\widehat{x_i},\ldots,x_n).$$

Denote by $C_n(m)$ the group of GL_m-coinvariants of $\widetilde{C}_n(m)$. In other words, $C_n(m)$ is the free abelian group generated by GL_m-orbits of n-tuples of points in F^m. Let $\mathrm{Conf}_n(m)$ be the set of GL_m-equivalence classes of n-tuples of points in F^m, so that $C_n(m) = \mathbb{Z}\left[\mathrm{Conf}_n(m)\right]$. The homogeneous elements of the fraction field of $\mathcal{P}_{n,m}$ of total degree 0 can be viewed as rational functions on $\mathrm{Conf}_n(m)$ with values in F and we will extend them to functions on $C_n(m)$ with values in $\mathbb{Z}\left[F\right]$ by linearity. More generally, we extend this correspondence to linear combinations of such rational functions.

The same formula as above defines a differential $d\colon C_{n+1}(m) \to C_n(m)$. In addition, there is another differential $\partial\colon C_{n+1}(m+1) \to C_n(m)$, defined as follows. For a nonzero vector $x \in F^{m+1}$ let $\pi_x : F^{m+1} \to F^{m+1}/\langle x\rangle$ be the natural projection. Then define the differential ∂ by the formula

$$(x_0,\ldots,x_n) \;\overset{\partial}{\mapsto}\; \sum_i (-1)^i (\pi_{x_i}(x_0),\ldots,\widehat{x_i},\ldots,\pi_{x_i}(x_n)).$$

Another important operation is the duality on configuration spaces (see [17, §7])

$$\star\colon \mathrm{Conf}_{n+m}(m) \to \mathrm{Conf}_{n+m}(n),$$

which can be described as follows. An element $(x_1,\ldots,x_{n+m})$ of $\mathrm{Conf}_{n+m}(m)$ can be represented by the following $m \times (n+m)$-matrix:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & 1 & \ldots & 0 & a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \;=\; (I_m, A).$$

Then we define the dual configuration $\star(x_1,\ldots,x_{n+m})$ by the coordinate matrix $(-A^T, I_n)$. We denote the induced map from $C_{n+m}(m)$ to $C_{n+m}(n)$ by the same symbol. One easily checks that $\star^2 = \mathrm{id}$ and that $\star$ interchanges the two differentials d and ∂, i.e., $d \circ \star = \star \circ \partial$. Note that this operation induces the duality isomorphism $\star\colon \mathcal{P}_{n+m,m} \to \mathcal{P}_{n+m,n}$ that was defined in Chapter 3.

Together with these two differentials, we can form the Grassmannian bicomplex.

$$
\begin{tikzcd}
 & & & & \vdots \arrow[d, "\partial"] \\
 & & & \cdots \arrow[r, "d"] & C_{2m}(2m-1) \arrow[d, "\partial"] \\
 & & \vdots \arrow[d, "\partial"] & \ddots & \vdots \arrow[d, "\partial"] \\
 & \cdots \arrow[r, "d"] \arrow[d, "\partial"] & C_{2m}(m+1) \arrow[r, "d"] \arrow[d, "\partial"] & \cdots \arrow[r, "d"] & C_{m+2}(m+1) \arrow[d, "\partial"] \\
\cdots \arrow[r, "d"] & C_{2m}(m) \arrow[r, "d"] & C_{2m-1}(m) \arrow[r, "d"] & \cdots \arrow[r, "d"] & C_{m+1}(m)
\end{tikzcd}
$$

It is easily verified that the differentials satisfy the relations $d^2 = 0$, $\partial^2 = 0$, and $d\partial + \partial d = 0$. Let $T_*(m)$ denote the total complex of the above bicomplex, that is, $T_n(m) = \bigoplus_{i=0}^{n-m} C_n(m+i)$ with the differential $d + \partial$.

To formulate the main conjecture of Goncharov, we need to talk about morphisms between $C_n(m)$ and $B_k(F) \otimes \Lambda^l F^\times$. The morphisms that we will work with are all of the form

$$(x_1, \ldots, x_n) \;=\; (\underline{x}) \mapsto \sum_{c, r, P_1, \ldots, P_l} c \cdot \{r(\underline{x})\}_k \otimes P_1(\underline{x}) \wedge \cdots \wedge P_l(\underline{x}),$$

where the sum on the right is finite, c runs over rational numbers, r runs over elements in $\mathbb{K}_{n,m}$, and $P_1, \ldots, P_l$ run over multi-homogeneous elements (not necessarily of degree 0) of $\mathcal{P}_{n,m}$, provided that the resulting map is GL_m-invariant. For instance, the map

$$\langle 12\rangle \wedge \langle 13\rangle \;-\; \langle 12\rangle \wedge \langle 23\rangle \;+\; \langle 13\rangle \wedge \langle 23\rangle$$

is a well-defined homomorphism from $C_3(2)$ to $\Lambda^2 F^\times$ even though the individual terms $\langle 12\rangle \wedge \langle 13\rangle$, $\langle 12\rangle \wedge \langle 23\rangle$, and $\langle 13\rangle \wedge \langle 23\rangle$ are not GL_2-invariant.

The following "optimistic" conjecture was stated in [19, Conj. 5.11].

Conjecture 6. *There exist morphisms $\psi_n(m)$ for $n = m+2, \ldots, 2m$ that make the following diagram commute*

$$
\begin{tikzcd}
T_{2m+1}(m) \arrow[r, "d"] \arrow[d] & T_{2m}(m) \arrow[r, "d"] \arrow[d, "\psi_{2m}(m)"] & \cdots \arrow[r, "d"] & T_{m+2}(m) \arrow[r, "d"] \arrow[d, "\psi_{m+2}(m)"] & T_{m+1}(m) \arrow[d, "\psi_{m+1}(m)"] \\
0 \arrow[r] & B_m(F) \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & B_2(F) \otimes \Lambda^{m-2} F^\times \arrow[r, "\delta"] & \Lambda^m F^\times,
\end{tikzcd}
$$

where $\psi_{m+1}(m)$ is given by

$$\psi_{m+1}(m)(x_0, \ldots, x_m) \;=\; \mathrm{Alt}_{m+1} \bigwedge_{i=2}^{m+1} \langle 1, 2, \ldots, \widehat{i}, \ldots, m+1\rangle$$

and $\mathrm{Alt}_n = \sum_\sigma \mathrm{sgn}(\sigma)\sigma \in \mathbb{Z}[\mathfrak{S}_n]$ *is the skew-symmetrization operator.*

The existence of the morphisms $\psi_n(m)$ has a lot of important consequences. For instance, Goncharov proved the following result (see [19, Cor. 5.13]).

Proposition 4.6. *For any $m \geq 2$ the truth of Conjecture 6 implies Zagier's conjecture (Conjecture 4).*

In fact, for this and many other applications, it is enough to construct maps $\psi_n(m)$ between the complexes after tensoring with $\mathbb{Q}$, and in the sequel we will implicitly assume that every group A in the complex has already been replaced by $A \otimes \mathbb{Q}$.

One can restrict these maps to the subcomplex $C_*(m)$ to get the diagram

$$\begin{tikzcd} C_{2m+1}(m) \arrow[r, "d"] \arrow[d] & C_{2m}(m) \arrow[r, "d"] \arrow[d, "\psi_{2m}(m)"] & \cdots \arrow[r, "d"] & C_{m+2}(m) \arrow[r, "d"] \arrow[d, "\psi_{m+2}(m)"] & C_{m+1}(m) \arrow[d, "\psi_{m+1}(m)"] \\ 0 \arrow[r] & B_m(F) \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & B_2(F) \otimes \Lambda^{m-2}F^\times \arrow[r, "\delta"] & \Lambda^m F^\times . \end{tikzcd} \tag{4.10}$$

One can show (see [8]) that commutativity of the diagram (4.10) is still sufficient for the proof of Zagier's conjecture (as formulated in Conjecture 4).

Let us remark, very briefly, on how Goncharov derives Zagier's conjecture from Conjecture 6. The commutativity of the leftmost square is equivalent (in the case $F = \mathbb{C}$) to the statement that the composition $\mathcal{L}_m \circ \psi_{2m}(m) \colon C_{2m}(m) \to \mathbb{R}$ is a locally integrable $(2m-1)$-cocycle. By this we mean that it is a function $F(x_1, \ldots, x_{2m})$ of $2m$-tuples of points $x_1, \ldots, x_{2m} \in \mathbb{C}^m$ that satisfies

$$\sum_{i=1}^{2m+1} (-1)^i F(x_1, \ldots, \widehat{x_i}, \ldots, x_{2m+1}) \;=\; 0$$

for any collection of $2m+1$ points $x_1, \ldots, x_{2m+1} \in \mathbb{C}^m$. Using such function, one can construct a continuous $(2m-1)$-cocycle for GL_m. (One constructs a group cocycle by the formula $\phi(g_1, \ldots, g_{2m}) = F(g_1 x, \ldots, g_{2m} x)$, where $x \in \mathbb{C}^m$ is fixed. To pass from measurable to continuous cocycles, one uses the result of [2]. The details of this construction can be found in [20].) Goncharov has proved that, under the assumption that the diagram (4.10) commutes, this cocycle is homologous to a rational multiple of the so-called Borel class. The weak form of Zagier's conjecture then follows by an application of the beautiful result of Borel [7] that expresses the special value $\zeta_F(m)$ in terms of the Borel regulator on $K_{2m-1}(F)$. A much more detailed discussion can be found in the expository paper [8].

Finally, let us note that the morphism $\psi_{2m}(m)$ can be written as $\eta_m \circ f_m$ for some $f_m \in \mathbb{Z}\left[\mathbb{K}_{2m,m}\right]$. The commutativity of the left-most square in (4.10) is then equivalent to the following maps from $C_{2m+1}(m)$ to $B_m(F)$ composing to 0

$$\begin{tikzcd} C_{2m+1}(m) \arrow[r, "d"] \arrow[drr, bend right, "0"] & C_{2m}(m) \arrow[r, "f_m"] & \mathbb{Z}[\mathbb{P}^1(F)] \arrow[d, "\eta_m"] \\ & & B_m(F). \end{tikzcd}$$

This will serve as motivation for the definition of geometric cocycles for polylogarithms that we will give in the next chapter.

Remarks.

1. Up to rational multiples, $\psi_{m+1}(m)$ can be defined as the unique $\mathfrak{S}_{m+1}$-skew-symmetric map from $T_{m+1}(m) = C_{m+1}(m)$ to $\Lambda^m F^\times$. This follows from the fact that every homogeneous element of $\mathcal{P}_{m+1,m}$ is a bracket monomial.

2. Since $\psi_{m+1}(m)$ is skew-symmetric, we see that, by replacing each morphism $\psi_n(m)$ with an appropriate rational multiple of $\mathrm{Alt}_n \psi_n(m)$, no generality is lost in assuming that all $\psi_n(m)$ are skew-symmetric.

3. It is a simple exercise to prove that the image of an element $(x_1, \ldots, x_{m+1})$ under the morphism $\psi_{m+1}(m)$ is invariant under rescaling $(x_1, \ldots, x_{m+1}) \mapsto (c_1 x_1, \ldots, c_{m+1} x_{m+1})$.

4.5 Proof (sketch) of Zagier's conjecture for $m = 2$ and $m = 3$

Assuming the above results, following [17], let us give proofs of Zagier's conjecture in the cases $m = 2$ and $m = 3$ by constructing the corresponding morphisms $\psi_n(m)$ for the diagram (4.10).

4.5.1 The case $m = 2$

We need to find a morphism $\psi_4(2)$ that fits into the following commutative diagram:

$$\begin{tikzcd} C_5(2) \arrow[r, "d"] \arrow[d] & C_4(2) \arrow[r, "d"] \arrow[d, "\psi_4(2)"] & C_3(2) \arrow[d, "\psi_3(2)"] \\ 0 \arrow[r] & B_2(F) \arrow[r, "\delta_2"] & \Lambda^2 F^\times \end{tikzcd} \tag{4.11}$$

This is done in the following proposition.

Proposition 4.7. *The morphism $\psi_4(2) = 2 \cdot \{r\}_2$, where r is the classical cross-ratio*

$$r = \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle},$$

makes the diagram 4.11 commute.

Proof. We have that $\delta_2 \circ [r]$ is equal to

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 14\rangle\langle 23\rangle} \wedge \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle},$$

and a simple computation shows that it simplifies to

$$\frac{1}{2} \sum_{\sigma \in \mathfrak{S}_4} \operatorname{sgn}(\sigma) \langle \sigma(1)\sigma(2)\rangle \wedge \langle \sigma(1)\sigma(3)\rangle.$$

On the other hand, the composition $\psi_3(2) \circ d$ is the morphism

$$\begin{aligned} &\sum_{k=1}^{4} (-1)^k \sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) \langle \sigma(\iota_k(1))\sigma(\iota_k(2))\rangle \wedge \langle \sigma(\iota_k(1))\sigma(\iota_k(3))\rangle \\ &\qquad = \sum_{\sigma \in \mathfrak{S}_4} \operatorname{sgn}(\sigma) \langle \sigma(1)\sigma(2)\rangle \wedge \langle \sigma(1)\sigma(3)\rangle, \end{aligned}$$

where $\iota_k \colon \{1, 2, 3\} \to \{1, 2, 3, 4\}$ is the strictly monotone map that omits the value k. Therefore, if we choose $\psi_4(2) = 2 \cdot \{r\}_2$, then $\delta_2 \circ \psi_4(2) = \psi_3(2) \circ d$. $\square$

4.5.2 The case $m = 3$

In this case we need to define two morphisms $\psi_6(3)$ and $\psi_5(3)$ that make the following diagram commute.

$$\begin{tikzcd} C_7(3) \arrow[r, "d"] \arrow[d] & C_6(3) \arrow[r, "d"] \arrow[d, "\psi_6(3)"] & C_5(3) \arrow[r, "d"] \arrow[d, "\psi_5(3)"] & C_4(3) \arrow[d, "\psi_4(3)"] \\ 0 \arrow[r] & B_3(F) \arrow[r, "\delta_3"] & B_2(F) \otimes F^\times \arrow[r, "\delta_3"] & \Lambda^3 F^\times. \end{tikzcd} \tag{4.12}$$

Following Goncharov, we define $\psi_5(3)\colon C_5(3) \to B_2(F) \otimes F^\times$ by equation

$$\psi_5(3) \;=\; (1/2)\,\mathrm{Alt}_5\,\{r_1\}_2 \otimes \langle 123\rangle,$$

where

$$r_1 \;=\; (1|2,3,4,5) \;=\; \frac{\langle 124\rangle\langle 135\rangle}{\langle 125\rangle\langle 134\rangle}$$

is the classical cross-ratio projected from the point 1. We define $\psi_6(3)\colon C_6(3) \to B_3(F)$ as $\psi_6(3) = -(1/15)\mathrm{Alt}_6\,\{r_2\}_3$, where r_2 is Goncharov's triple ratio given in the Table 3.4 as

$$r_2 \;=\; \frac{\langle 124\rangle\langle 235\rangle\langle 136\rangle}{\langle 125\rangle\langle 236\rangle\langle 134\rangle}\,.$$

Proposition 4.8 ([20])**.** *With the above choice of $\psi_6(3)$ and $\psi_5(3)$ the diagram 4.12 commutes.*

Sketch of a proof. The commutativity of the leftmost square is equivalent to the following functional equation for $\mathcal{L}_3$, which Goncharov derived from the functional equation (4.6)

$$\sum_{\sigma\in\mathfrak{S}_7} \mathrm{sgn}(\sigma)\cdot\mathcal{L}_3\big(r_2(x_{\sigma(1)},\ldots,x_{\sigma(6)})\big) \;=\; 0. \tag{4.13}$$

Let us prove this relation using the criterion from Proposition 4.2. At the same time we will check that $\psi_5(3)\circ d = \delta\circ\psi_6(3)$.

For any permutation $i_1,\ldots,i_6$ of $1,2,\ldots,6$, define the element $\langle i_1i_2;i_3i_4;i_5i_6\rangle \in \mathcal{P}_{6,3}$ by the formula

$$\langle i_1i_2;i_3i_4;i_5i_6\rangle \;=\; \langle i_1i_2i_3\rangle\langle i_4i_5i_6\rangle - \langle i_1i_2i_4\rangle\langle i_3i_5i_6\rangle.$$

Then we have the following factorization for $1-r_2$:

$$1-r_2 \;=\; \frac{\langle 123\rangle\cdot\langle 14;52;36\rangle}{\langle 125\rangle\langle 236\rangle\langle 134\rangle}.$$

One can show that if $\sigma\in\mathfrak{S}_6$ preserves the set partition $\{\{i_1,i_2\},\{i_3,i_4\},\{i_5,i_6\}\}$, then

$$\sigma\cdot\langle i_1i_2;i_3i_4;i_5i_6\rangle \;=\; \pm\langle i_1i_2;i_3i_4;i_5i_6\rangle.$$

A simple calculation shows that

$$\sum_{\sigma\in G}\mathrm{sgn}(\sigma)(\sigma r_2\otimes\sigma r_2) \;=\; 0,$$

where G is the group of order 8 generated by the transpositions $1\leftrightarrow 4$, $2\leftrightarrow 5$, and $3\leftrightarrow 6$. Therefore, the term

$$\mathrm{Alt}_6\,\big(\langle 14;52;36\rangle\wedge r_2\otimes r_2\big)$$

vanishes and we get that

$$-\frac{1}{15}\mathrm{Alt}_6\,(1-r_2)\wedge r_2\otimes r_2 \;=\; -\frac{1}{15}\mathrm{Alt}_6\,\frac{\langle 123\rangle}{\langle 125\rangle\langle 236\rangle\langle 134\rangle}\wedge r_2\otimes r_2.$$

Another calculation shows that the right-hand side of the above equations simplifies to

$$2\cdot\mathrm{Alt}_6\,\big(\langle 123\rangle\wedge\langle 124\rangle\otimes\langle 135\rangle\big).$$

This proves (4.13), since the above expression vanishes under Alt_7 (because it is fixed under the involution $6\leftrightarrow 7$). To prove $\psi_5(3)\circ d = \delta\circ\psi_6(3)$ we just need to check that

$$\frac{1}{2}\mathrm{Alt}_6\,(1-r_1)\wedge r_1\otimes\langle 123\rangle \;=\; 2\cdot\mathrm{Alt}_6\,\big(\langle 123\rangle\wedge\langle 124\rangle\otimes\langle 135\rangle\big),$$

which is again a simple but tedious calculation.

We leave the proof of an easier identity $\psi_4(3)\circ d = \delta\circ\psi_5(3)$ to the reader. □

CHAPTER 5

Functional equations for polylogarithms

In this chapter we will collect various functional equations that we have found using the exceptional cross-ratios computed in Chapter 3. We do not attempt to be comprehensive, and among many possible relations, our main criterion for selection is the presence of a large symmetry group. For convenience, let us define two special elements in the group ring $\mathbb{Z}[\mathfrak{S}_n]$:

$$\begin{aligned} \mathrm{Sym}_n &= \sum_{\pi\in\mathfrak{S}_n} \pi, \\ \mathrm{Alt}_n &= \sum_{\pi\in\mathfrak{S}_n} \mathrm{sgn}(\pi)\cdot\pi. \end{aligned} \tag{5.1}$$

Whenever we have a linear action of $\mathfrak{S}_n$ on an abelian group, we extend the action to $\mathbb{Z}[\mathfrak{S}_n]$, and then Sym_n and Alt_n represent the symmetrization and skew-symmetrization operators respectively (we have already used the operator Alt_n in the previous chapter).

We will give numerous functional equations for $\mathcal{L}_3$, $\mathcal{L}_4$, and $\mathcal{L}_5$, but we will omit the proofs, save for a select few. The reason for this is that due to the algebraic criterion from Proposition 4.2 such proofs always amount to a simple but tedious calculation with rational functions in many variables. To make sure that each functional equation that we give here and in Appendix B is correct, we have checked them in two different ways. First, we checked each functional equation with a program that does the algebraic calculation needed to prove a functional equation symbolically. This approach gives us a rigorous proof. Second, to make sure that this program does not produce false positives, we also checked each functional equation by computing its value at a randomly selected point numerically to a high precision. In the case of $\mathcal{L}_4$ we check that this number is close to 0; in the cases of $\mathcal{L}_3$ and $\mathcal{L}_5$ we check that the value is close to a rational multiple of $\zeta(3)$ or $\zeta(5)$ respectively.

5.1 General results about functional equations for polylogarithms

In this section, we establish some general results about functional equations for polylogarithms. In particular, these results motivate the definition of exceptional S-cross-ratios, since they form a source of nontrivial functional equations for $\mathcal{L}_m$.

As in Chapter 2, let $\mathcal{P}$ be the ring of polynomials $\overline{\mathbb{Q}}[x_1,\ldots,x_l]$ and let $\mathbb{K}$ be the homogeneous field of fractions of $\mathcal{P}$. Note that we view $\overline{\mathbb{Q}}$ as a subfield in $\mathbb{C}$. For an arbitrary set of rational functions $Y \subset \mathbb{K}$ we define the **space of functional equations for $\mathcal{L}_m$ with values in Y**

$$\mathcal{E}_m(Y) \;=\; \Big\{ \sum_i a_i[r_i] \in \mathbb{Q}\,[Y] \;\Big|\; \sum_i a_i \mathcal{L}_m(r_i) \;\text{ is constant} \Big\}. \tag{5.2}$$

(Recall from Proposition 4.2 that there is a purely algebraic description of this space.) If $\Gamma \subseteq \mathbb{K}^\times$ is a subgroup containing the group μ_∞ of all roots of unity in $\overline{\mathbb{Q}}^\times$, then by the distribution property (4.7) of $\mathcal{L}_m$, the space $\mathcal{E}_m(\Gamma)$ contains the (infinite-dimensional) $\mathbb{Q}$-vector space

$$\mathcal{E}^0_m(\Gamma) \;=\; \Big\langle [r^n] - n^{m-1} \sum_{\lambda^n = 1} [\lambda r],\; [c] \;\Big|\; r \in \Gamma,\; c \in \Gamma \cap \overline{\mathbb{Q}} \Big\rangle_{\mathbb{Q}} \tag{5.3}$$

of "trivial" functional equations. For arbitrary $\Gamma \subseteq \mathbb{K}^\times$, we define $\mathcal{E}^0_m(\Gamma)$ as $\mathcal{E}_m(\Gamma) \cap \mathcal{E}^0_m(\mu_\infty \cdot \Gamma)$. It is natural to interpret the quotient space $\widehat{\mathcal{E}}_m(\Gamma) := \mathcal{E}_m(\Gamma)/\mathcal{E}^0_m(\Gamma)$ as the **space of nontrivial functional equations for $\mathcal{L}_m$ with values in Γ**. We will now use the results of Chapter 2 to prove that it is finite-dimensional for any admissible Γ, as defined in Definition 2.1 (recall that this simply means that $\dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q}) < \infty$).

Theorem 5.1. *The space $\widehat{\mathcal{E}}_m(\Gamma)$ is finite-dimensional for any admissible $\Gamma \subset \mathbb{K}^\times$ and any $m \geq 2$.*

Before we prove this, let us prove a very simple necessary condition for functional equations. For any element $\xi = \sum_i a_i[r_i] \in \mathbb{Q}\,[\mathbb{K}]$ and an irreducible homogeneous polynomial $\pi \in \mathcal{P}$ we define the counting function $N_\pi(\xi)$ as

$$N_\pi(\xi) \;=\; \#\{i \mid a_i \neq 0 \text{ and } r_i \equiv 1 \pmod{\pi}\}. \tag{5.4}$$

In terms of the group $U(\pi)$ (see 2.6) the function $N_\pi(\xi)$ can be equivalently defined as

$$N_\pi(\xi) \;=\; \#\{i \mid a_i \neq 0,\; r_i \in U(\pi)\} \;=\; |\mathrm{supp}(\xi) \cap U(\pi)|.$$

Lemma 5.2. *Let $\xi = \sum_i a_i[r_i] \in \mathbb{Q}\left[\mathbb{K} \smallsetminus \overline{\mathbb{Q}}\right]$ belong to $\mathcal{E}_m(\mathbb{K})$, and let π be an irreducible polynomial that does not occur in the factorization of any r_i. Then $N_\pi(\xi) \neq 1$.*

Proof. From the criterion for functional equations from Proposition 4.2, we see that $\xi \in \mathcal{E}_m(\mathbb{K})$ if and only if

$$\sum_i a_i \cdot (1 - r_i) \wedge r_i \otimes r_i \otimes \cdots \otimes r_i \equiv \text{const}, \tag{5.5}$$

where the identity is in $(\Lambda^2(\mathbb{K}^\times) \otimes \mathrm{Sym}^{m-2}(\mathbb{K}^\times))_{\mathbb{Q}}$. Suppose that $N_\pi(\xi) = 1$ and r_i is the only element such that $r_i \equiv 1 \pmod{\pi}$. If α is the exponent of π that divides $1 - r_i$, i.e., $\pi^\alpha || 1 - r_i$ then it is easy to see that the term

$$\alpha \cdot \pi \wedge r_i \otimes r_i \otimes \cdots \otimes r_i$$

does not cancel with any other term in the sum (5.5) and therefore ξ cannot be a functional equation for $\mathcal{L}_m$. $\square$

Proof of Theorem 5.1. Since the group $\Gamma \cdot \mu_\infty$ is still admissible and $\widehat{\mathcal{E}}_m(\Gamma) \subseteq \widehat{\mathcal{E}}_m(\Gamma \cdot \mu_\infty)$, one can assume without loss of generality that $\Gamma \supset \mu_\infty$ and therefore that $\mathcal{E}^0_m(\Gamma)$ is defined by 5.3. For similar reason, we may assume that Γ is of the form $\langle c_1, \ldots, c_k \rangle \cdot U^\mu_S$.

Define Y to be the set of all elements of Γ that are not perfect powers. That is,

$$Y \;=\; \Gamma \smallsetminus \bigcup_{n \geq 2} (\mathbb{K}^\times)^n.$$

Let Y' be the subset of Y, obtained by throwing out one element from each pair $\{r, 1/r\} \subseteq Y$. Let ξ be any element of $\mathcal{E}_m(\Gamma)$. Then by using linear combinations of the trivial functional equations, we can get an element $\xi' \in \mathcal{E}_m(Y') \subset \mathcal{E}_m(\Gamma)$ such that $\xi - \xi' \in \mathcal{E}_m^0(\Gamma)$ (possibly, one would need to extend Γ by some n-th roots of constant elements in Γ, but it will not affect the argument below since this operation does not change the rank of Γ or any of its subgroups). Let $\xi' = \sum_i a_i[r_i]$ and assume that $a_i \neq 0$ for all i. Let π be an irreducible polynomial that divides $1 - r_i$ for at least one index i. Then, by Lemma 5.2, we know that $N_\pi(\xi') \geq 2$ and therefore there are at least two different elements $r_1, r_2 \in Y' \cap U(\pi)$. By definition of Y' we have $r_1 \neq r_2$ and $r_1 r_2 \neq 1$.

We want to show that $\operatorname{rk} U(\pi) \geq 2$. If this were not the case, then for some integers $s, t \neq 0$ we would have $r_1^s = r_2^t$. If $s \neq \pm t$, then one or both of r_1, r_2 is a power of some element in $\mathbb{K}^\times$, which contradicts the definition of Y'. Therefore, $s = \pm t$, assume that $s = t$ (the other case). Then we have that $r_1/r_2 \in U(\pi)$ is a root of unity, hence $r_1 = r_2$ (here we used part (4) of the Proposition 2.3), which contradicts the choice of r_1 and r_2. Therefore, $\operatorname{rk} U(\pi) \geq 2$.

Since the irreducible polynomial π was arbitrary, we get that all r_i belong to the set $\mathcal{R}(\Gamma, U_{S'_{max}(\Gamma)})$ which is finite by Theorem 2.11. This proves that $\widehat{\mathcal{E}}_m(\Gamma)$ is finite-dimensional. □

Remarks.

1. By the remark after the proof of Theorem 2.8, even the set $\mathcal{R}(\overline{\mathbb{Q}}^\times\Gamma, U_{S'_{max}(\Gamma)})$ is finite, so we can extend Γ (which was necessary for the reduction to ξ') by any $c \in \overline{\mathbb{Q}}$ for which $c^n \in \Gamma$ without affecting the above argument.

2. In all the statements above, we could work over any subfield of $\mathbb{C}$ in place of $\overline{\mathbb{Q}}$.

3. Note that we have actually proved the following slightly stronger statement:

Theorem 5.1′. *Let $Y \subset \mathbb{K} \setminus \mathbf{k}$ be any subset such that $\langle Y \rangle$ is admissible and assume that if for some $r_1, r_2 \in Y$ we have $r_1^s = r_2^t$, then $s = t$. Then the space $\mathcal{E}_m(Y)$ is finite-dimensional.*

4. As a corollary of the proof of Theorem 5.1 we also get the following result.

Proposition 5.3. *If the group $\Gamma \subset \mathbb{K}^\times$ is of rank 1, then $\widehat{\mathcal{E}}_m(\Gamma) = \{0\}$.*

5. The statement of the theorem does not hold if we only require that $\Gamma/(\Gamma \cap \overline{\mathbb{Q}}^\times)$ is finitely generated. For example, if we take $\Gamma = \overline{\mathbb{Q}}^\times \cdot \langle t, t-1 \rangle$ in $\overline{\mathbb{Q}}(t)$, then the space $\widehat{\mathcal{E}}_2(\Gamma)$ has infinite dimension. This can be seen as follows. The five-term relation for $\mathcal{L}_2$ can be rewritten as

$$\mathcal{L}_2(x) + \mathcal{L}_2(y) - \mathcal{L}_2(xy) + \mathcal{L}_2\Big(\frac{x(y-1)}{1-x}\Big) + \mathcal{L}_2\Big(\frac{y(x-1)}{1-y}\Big) \;=\; 0.$$

Then for any value $y \in \overline{\mathbb{Q}}^\times \setminus \{0, 1\}$ the five arguments in the above relation all lie in Γ and it is easy to show that the corresponding elements of $\widehat{\mathcal{E}}_2(\Gamma)$ span an infinite-dimensional subspace.

Let us give a refined version of the necessary condition from Lemma 5.2. For that we will need the following simple result from linear algebra.

Lemma 5.4. *Let V be a vector space and let $v_1, \ldots, v_k \in V$ be elements of V any two of which are linearly independent. Then the elements $v_1^m, \ldots, v_k^m$ of the symmetric power $\operatorname{Sym}^m(V)$ are linearly independent over $\mathbb{Q}$ for all $m \geq k-1$.*

Proof. If $V = \mathbb{Q}^2$, then we can assume that $v_i = (1, x_i)$ and $x_i \neq x_j$. The statement then follows from the non-vanishing of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1} & x_2^{k-1} & \dots & x_k^{k-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j) \neq 0.$$

The case $\dim V > 2$ easily follows from the case $\dim V = 2$. □

Proposition 5.5. *Let $\xi = \sum_i a_i[r_i] \in \mathbb{Q}\left[\mathbb{K} \setminus \overline{\mathbb{Q}}\right]$ be an element of $\mathcal{E}_m(\mathbb{K})$ with the property that if $r_i^s = r_j^t$ for some i, j, s, t, then $s = t$. Let π be an irreducible polynomial that does not occur in the factorization of any r_i. Then the number $N_\pi(\xi)$ defined in* (5.4) *is either equal to 0 or it is greater than m.*

Proof. Without loss of generality, assume that $r_1, \dots, r_s$ are the only elements among r_i that lie in $U(\pi)$ and that $s \geq 2$ (otherwise the result follows from Lemma 5.2). Arguing as in the proof of Theorem 5.1 we see that $\operatorname{rk} U(\pi) \geq 2$. By the algebraic criterion for functional equations there must be some nontrivial linear dependency of the form

$$\sum_{i=1}^{s} \alpha_i \cdot \pi \wedge r_i \otimes r_i \otimes \dots \otimes r_i \;=\; 0.$$

If we denote $\Gamma = \langle r_1, \dots, r_s \rangle$, then the above identity is equivalent to an identity

$$\sum_{i=1}^{s} \alpha_i \cdot r_i^{(m-1)} \;=\; 0$$

in the group $\operatorname{Sym}^{m-1}(\Gamma) \otimes \mathbb{Q}$. By the assumption any two elements r_i and r_j are linearly independent when viewed as vectors in $\Gamma \otimes \mathbb{Q}$. An application of Lemma 5.4 then shows that $s > m$. □

Finally, let us show how one can easily check that a functional equation $f \in \mathcal{E}_m(\mathbb{K})$ is nontrivial, i.e., does not lie in $\mathcal{E}_m^0(\mathbb{K}^\times)$. Let us define an equivalence relation $\sim$ on $\mathbb{K}^\times \setminus \mathbf{k}$:

$$a \sim b \;\Leftrightarrow\; a^n = b^m \text{ for some } (n, m) \neq (0, 0).$$

For any set $Y \subseteq \mathbb{K}$, we denote by θ_Y the projection from $\mathbb{Q}[\mathbb{K}]$ to $\mathbb{Q}[Y]$ that sends $[r]$ to itself if $r \in Y$ and to 0 otherwise. The following proposition, the proof of which trivially follows from Proposition 5.3, allows one to easily check that elements $\xi_1, \dots, \xi_k \in \mathcal{E}_m(\mathbb{K})$ are linearly independent modulo $\mathcal{E}_m^0(\mathbb{K})$.

Proposition 5.6. *The element $\xi \in \mathbb{Q}[\mathbb{K}^\times]$ belongs to $\mathcal{E}_m^0(\mathbb{K}^\times)$ if and only if $\theta_Y(\xi)$ lies in $\mathcal{E}_m(\mathbb{K}^\times)$ for all $\sim$-equivalence classes $Y \subseteq \mathbb{K}^\times$.*

5.2 Geometric functional equations for polylogarithms

Let us describe the class of functional equations that we are interested in. As in Chapter 3, we set $\mathcal{P}_{n,d} = \mathbb{Q}[x_{11}, \dots, x_{dn}]^{\mathrm{SL}_d(\mathbb{Q})}$ (the bracket algebra) and set $\mathbb{K}_{n,d}$ to be the homogeneous fraction field of $\mathcal{P}_{n,d}$ (field of cross-ratios on n points in $\mathbb{P}^{d-1}$). We will use the notations of the previous section.

Definition 5.7. *A* ***geometric functional equation for*** $\mathcal{L}_m$ *is a nonzero element of the space* $\widehat{\mathcal{E}}_m(\mathbb{K}_{n,d}^\times)$ *of nontrivial functional equations with values in* $\mathbb{K}_{n,d}^\times$*, or any representative of such an element.*

We are going to be interested only in geometric functional equations that lie in $\widehat{\mathcal{E}}_m(\Gamma)$ for some admissible subgroups $\Gamma \subset \mathbb{K}_{n,d}$.

We now define an important subclass of geometric functional equations. For this we first define a differential

$$D\colon \mathbb{Q}\,[\mathbb{K}_{n,d}] \to \mathbb{Q}\,[\mathbb{K}_{n+1,d}]\,, \quad [r] \;\mapsto\; \sum_{k=1}^{n+1} (-1)^k [\varphi_{\iota_k}(r)],$$

where ι_k is the unique monotone map from $\{1,\dots,n\}$ to $\{1,\dots,n+1\}$ that omits value k and $\varphi_\iota\colon \mathbb{K}_{n,d} \to \mathbb{K}_{n+1,d}$ is the inclusion homomorphism as defined in Section 3.1.2. A simple computation shows that $D^2 = 0$. In fact, the obvious complex has trivial cohomology, so it is equivalent to speak of cocycles (f with $Df = 0$) and coboundaries (f in the image of D).

Definition 5.8. *A* ***geometric*** $\mathcal{L}_m$***-cocycle*** *(of n points in $\mathbb{P}^{d-1}$) is an element $f \in \mathbb{Q}[\mathbb{K}_{n,d}^\times]$ such that $D(f)$ is an element of $\mathcal{E}_m(\mathbb{K}_{n+1,d}^\times)$. We denote the space of all geometric $\mathcal{L}_m$-cocycles with values in a group $\Gamma \subseteq \mathbb{K}_{n,d}^\times$ by $\mathcal{F}_m(\Gamma)$. We will usually drop the adjective and simply call f an $\mathcal{L}_m$-cocycle. We call an $\mathcal{L}_m$-cocycle f* ***trivial*** *if $D(f) \in \mathcal{E}_m^0(\mathbb{K}_{n,d}^\times)$.*

The reason for the terminology is that if f is an $\mathcal{L}_m$-cocycle, then the function $F = \mathcal{L}_m \circ f$, viewed as a function of n-tuples of vectors in $\mathbb{C}^d$, satisfies a cocycle relation modulo constants:

$$\sum_{i=1}^{n+1} (-1)^i F(v_1,\dots,\widehat{v_i},\dots,v_{n+1}) \;=\; \text{const}.$$

Denote by $\mathcal{F}_m^0(\Gamma)$ the subspace of trivial $\mathcal{L}_m$-cocycles (i.e., f such that $D(f)$ belongs to the space of trivial functional equations for $\mathcal{L}_m$) and by $\widehat{\mathcal{F}}_m(\Gamma)$ the quotient space $\mathcal{F}_m(\Gamma)/\mathcal{F}_m^0(\Gamma)$, which we interpret as the space of nontrivial geometric cocycles.

In the notation of Chapter 4, we can interpret these two types of functional equations as follows. Let $C_n(d)$ be the Grassmannian complex (defined in 4.4). We will identify the elements of $\mathbb{Q}\,[\mathbb{K}_{n,d}]$ with the corresponding morphisms from $C_n(d)_\mathbb{Q}$ to $\mathbb{Q}\,[F]$, where for an abelian group G we detote by $G_\mathbb{Q}$ the vector space $G \otimes \mathbb{Q}$. A geometric functional equation is an element $f \in \mathbb{Q}\,[\mathbb{K}_{n,d}]$ for which $\eta_m \circ f = \text{const}$:

$$\begin{tikzcd} C_n(d)_\mathbb{Q} \arrow[r, "f"] \arrow[dr, "\text{const}"'] & \mathbb{Q}\,[F] \arrow[d, "\eta_m"] \\ & B_m(F)_\mathbb{Q}. \end{tikzcd}$$

Similarly, a geometric $\mathcal{L}_m$-cocycle is an element $f \in \mathbb{Q}\,[\mathbb{K}_{n,d}]$ such that $\eta_m \circ f \circ d_G = \text{const}$ and $\eta_m \circ f \neq \text{const}$, where d_G is the differential of the Grassmannian complex (see 4.4):

$$\begin{tikzcd} C_{n+1}(d)_\mathbb{Q} \arrow[r, "d_G"] \arrow[drr, bend right, "\text{const}"'] & C_n(d)_\mathbb{Q} \arrow[r, "f"] \arrow[dr, "\neq\text{const}"] & \mathbb{Q}\,[F] \arrow[d, "\eta_m"] \\ & & B_m(F)_\mathbb{Q}. \end{tikzcd}$$

Our motivation for studying geometric $\mathcal{L}_m$-cocycles is that the map $\psi_{2m}(m)$ from Conjecture 6, in the diagram

$$\begin{array}{ccccccccc}
C_{2m+1}(m) & \xrightarrow{d} & C_{2m}(m) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C_{m+2}(m) & \xrightarrow{d} & C_{m+1}(m) \\
\big\downarrow & & \big\downarrow{\scriptstyle \psi_{2m}(m)} & & & & \big\downarrow{\scriptstyle \psi_{m+2}(m)} & & \big\downarrow{\scriptstyle \psi_{m+1}(m)} \\
0 & \longrightarrow & B_m(F) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & B_2(F)\otimes\Lambda^{m-2}F^\times & \xrightarrow{\delta} & \Lambda^m F^\times
\end{array}$$

is a nontrivial geometric $\mathcal{L}_m$-cocycle.

Let $\Gamma \subseteq \mathbb{K}_{n,d}^\times$ be an admissible group that is invariant under the $\mathfrak{S}_n$-action. Then it is easy to see that $\mathfrak{S}_n$ acts also on the space of geometric functional equations $\widehat{\mathcal{E}}_m(\Gamma)$. By Theorem 5.1 the space $\widehat{\mathcal{E}}_m(\Gamma)$ is finite-dimensional, so we get a finite-dimensional representation of $\mathfrak{S}_n$. For any vector space V on which $\mathfrak{S}_n$ acts, let us denote by V^+ the subspace on which $\mathfrak{S}_n$ acts trivially and by V^- the subspace of V on which $\sigma \in \mathfrak{S}_n$ acts as multiplication by $\mathrm{sgn}(\sigma)$. While the group $\mathfrak{S}_n$ does not preserve the space $\mathcal{F}_m(\Gamma)$, the skew-symmetrization operator Alt_n does, so we can still define the space $\mathcal{F}_m(\Gamma)^-$ (and also the spaces $\mathcal{F}_m^0(\Gamma)^-$ and $\widehat{\mathcal{F}}_m(\Gamma)^-$) of skew-symmetric f that satisfy $D(f) \in \mathcal{E}_m(\mathbb{K}_{n+1,d})$. The following question is motivated by Gocharov's approach to Zagier's conjecture (more specifically, by the question about existence of a morphism $\psi_{2m}(m)$ fitting into the above diagram).

Question 2. *For which values of (m, n, d) does there exist a nontrivial skew-symmetric geometric $\mathcal{L}_m$-cocycle (nontrivial in the sense of Definition 5.8)?*

We shall call such triples (m, n, d) **good**. Conjecture 6 implies that the triple $(m, 2m, m)$ is good, but for $m \geq 4$ even this weaker statement is not known.

In the next sections we are going to give some information about the functional equation groups $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^\pm$, where $\Gamma_{n,d} \subseteq \mathbb{K}_{n,d}^\times$ is the multiplicative subgroup generated by all brackets. In what follows we restrict to the case $n \geq 2d$, since the duality isomorphism $\star\colon \mathbb{K}_{n,d} \to \mathbb{K}_{n,n-d}$ (see Section 3.1.2) induces a $\mathfrak{S}_n$-equivariant isomorphism of the spaces of functional equations

$$\widehat{\mathcal{E}}_m(\mathbb{K}_{n,d}^\times) \xrightarrow{\sim}_{\mathfrak{S}_n} \widehat{\mathcal{E}}_m(\mathbb{K}_{n,n-d}^\times).$$

5.3 Symmetric geometric functional equations

In this section we establish some results for $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^+$ for $2 \leq d \leq 4$ and $3 \leq m \leq 5$. Before we proceed, let us remark that the map

$$[r] \mapsto \mathrm{Sym}_{n+1}[r],$$

that maps $\mathbb{Q}[\mathbb{K}_{n,d}]$ into $\mathbb{Q}[\mathbb{K}_{n+1,d}]^+$ induces a map from the space $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^+$ into $\widehat{\mathcal{E}}_m(\Gamma_{n+1,d})^+$. Therefore, for each n we will be interested only in the "new" elements of $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^+$.

5.3.1 Functional equations on $\mathbb{P}^1$

The following theorem gives lower bounds, conjecturally sharp in each case, for the space $\widehat{\mathcal{E}}_m(\Gamma_{n,2})^+$.

Theorem 5.9. *Lower bounds for the dimension of $\widehat{\mathcal{E}}_m(\Gamma_{n,2})^+$ are given for $m = 3, 4, 5$ in the following table.*

n	4	5	6	7	≥ 8
$\dim \widehat{\mathcal{E}}_3(\Gamma_{n,2})^+$	≥ 1	≥ 3	≥ 7	≥ 8	≥ 9
$\dim \widehat{\mathcal{E}}_4(\Gamma_{n,2})^+$			≥ 1	≥ 1	≥ 1
$\dim \widehat{\mathcal{E}}_5(\Gamma_{n,2})^+$			≥ 3	≥ 4	≥ 5

The following table gives in each case a list of elements of $\widehat{\mathcal{E}}_m(\Gamma_{n,2})^+$ *that are linearly independent modulo* $\widehat{\mathcal{E}}_m(\Gamma_{n-1,2})^+$.

m	n	*Elements* $\xi \in \mathbb{Q}\,[\Gamma_{n,2}]$ *such that* $\mathrm{Sym}_n\,\xi$ *lies in* $\widehat{\mathcal{E}}_m(\Gamma_{n,2})^+$
3	4	$[r_1]$
3	5	$-2[r_3]+[r_5]$
		$-5[r_3]-3[r_4]+[r_7]$
3	6	$-4[r_9]-2[r_{10}]+[r_{14}]$
		$-5[r_8]+[r_9]-4[r_{10}]+[r_{11}]+[r_{13}]$
		$-4[r_{11}]+[r_{12}]$
		$-4[r_8]+3[r_{11}]$
3	7	$-6[r_{15}]+[r_{16}]$
3	8	$-6[r_{17}]+[r_{18}]$
4	6	$2[r_1]-4[r_9]-2[r_{10}]+[r_{14}]$
5	6	$24[r_1]-32[r_8]-6[r_{11}]+3[r_{12}]$
		$9[r_1]+15[r_8]-9[r_9]-18[r_{10}]-2[r_{12}]+[r_{13}]+[r_{14}]$
		$432[r_1]-288[r_4]-128[r_5]-9[r_6]+32[r_7]$
5	7	$9[r_1]+18[r_3]+18[r_4]+8[r_5]-2[r_7]-54[r_{15}]+3[r_{16}]$
8	5	$18[r_{15}]-[r_{16}]-18[r_{17}]+[r_{18}]$

Table 5.1: Symmetric functional equations on $\mathbb{P}^1$

Sketch of a proof. The identity $\mathrm{Sym}_4[r_1] \in \widehat{\mathcal{E}}_3(\Gamma_{4,2})$ is equivalent to Kummer's functional equation

$$\mathcal{L}_3(z)+\mathcal{L}_3(1-z)+\mathcal{L}_3(1-1/z) \;=\; \mathcal{L}_3(1)$$

for the trilogarithm already mentioned in (4.4).

Let us indicate how one proves, for example, the identity $\mathrm{Sym}_5\left([r_5]-2[r_3]\right) \in \widehat{\mathcal{E}}_3(\Gamma_{5,2})$. We work with the algebraic criterion for functional equations from Proposition 4.2. We have

$$1-r_3 \;=\; \frac{\pi}{\langle 14\rangle\langle 15\rangle\langle 23\rangle},$$

$$1 - r_5 \;=\; \frac{\langle 25\rangle \cdot \pi}{\langle 15\rangle^2 \langle 23\rangle \langle 24\rangle},$$

where $\pi = \langle 14\rangle\langle 15\rangle\langle 23\rangle - \langle 12\rangle\langle 13\rangle\langle 45\rangle$. The group of automorphisms of π (modulo ± 1) is generated by permutations $(3\,4)$ and $(2\,4)(3\,5)$. Using this it is easy to check that

$$\mathrm{Sym}_5\left(\pi \wedge r_5 \otimes r_5 - 2\,\pi \wedge r_3 \otimes r_3\right) \;=\; 0.$$

The rest of the computation involves only tensor products of the brackets $\langle ij\rangle$ and is a simple exercise.

The remaining identities are proved by similar considerations: for $\xi = \sum_i a_i[r_i]$ we consult Table 3.2 to find the factorization of $1 - r_i$, then check that in the symbol of $\mathrm{Sym}_n \xi$ every term that involves some nontrivial factor π_j cancels out, and finally check that the rest of the terms (they will all necessarily be of the form $\langle i_1 j_1\rangle \otimes \cdots \otimes \langle i_m j_m\rangle$) cancel out.

Finally, the linear independence of the given elements modulo $\mathcal{E}^0_m(\Gamma_{n,2})$ follows easily from Proposition 5.6. □

5.3.2 Functional equations on $\mathbb{P}^2$

To obtain the following theorem, we have considered the exceptional bracket cross-ratios from Theorem 3.10 (that are given in Table A.4 in the appendix).

Theorem 5.10. *Lower bounds for the dimension of $\widehat{\mathcal{E}}_m(\Gamma_{n,3})^+$ are given for $m = 3, 4, 5$ in the following table.*

n	6	7	8
$\dim \widehat{\mathcal{E}}_3(\Gamma_{n,3})^+$	≥ 17	≥ 30	≥ 37
$\dim \widehat{\mathcal{E}}_4(\Gamma_{n,3})^+$	≥ 16	≥ 19	≥ 24
$\dim \widehat{\mathcal{E}}_5(\Gamma_{n,3})^+$	≥ 5	≥ 10	≥ 14

A list of linearly independent elements of $\widehat{\mathcal{E}}_m(\Gamma_{n,3})^+$ is given in Table B.2 in the appendix.

5.3.3 Functional equations on $\mathbb{P}^3$

To obtain the following theorem, we have considered the exceptional bracket cross-ratios from Theorem 3.12 (that are given in Table A.6 in the appendix).

Theorem 5.11. *Lower bounds for the dimension of $\widehat{\mathcal{E}}_m(\Gamma_{8,4})^+$ for $m = 3, 4, 5$ are as follows:*

$$\begin{aligned}
\dim \widehat{\mathcal{E}}_3(\Gamma_{8,4})^+ \;&\geq\; 62,\\
\dim \widehat{\mathcal{E}}_4(\Gamma_{8,4})^+ \;&\geq\; 15,\\
\dim \widehat{\mathcal{E}}_5(\Gamma_{8,4})^+ \;&\geq\; 20.
\end{aligned}$$

A list of linearly independent elements of $\widehat{\mathcal{E}}_m(\Gamma_{8,4})^+$ is given in Table B.3 in the appendix.

5.4 Skew-symmetric geometric functional equations

The spaces of skew-symmetric functional equations $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^-$ are much smaller than the corresponding symmetric spaces. We include the computations for all three cases $\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3$ in Theorem 5.12.

Notice that in this case it is no longer true that nonzero elements of $\widehat{\mathcal{E}}_m(\Gamma_{n,d})$ are nonzero when mapped into $\widehat{\mathcal{E}}_m(\Gamma_{n+1,d})$. For this reason, in the following theorem we list complete bases for the spaces of skew-symmetric functional equations and not just the "new" elements.

Theorem 5.12. *Skew-symmetric functional equations with values in $\Gamma_{n,d}$ exist for the following 12 values of (m, n, d):*

(m,d)	$(3,2)$	$(3,3)$	$(3,4)$	$(4,3)$	$(4,4)$	$(5,2)$	$(5,3)$
n	5	6, 7, 8	8	6, 7, 8	8	6	6, 7

More precisely, we have the following lower bounds for the dimension of $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^-$.

(n,d)	$(5,2)$	$(6,2)$	$(6,3)$	$(7,3)$	$(8,3)$	$(8,4)$
$\dim \widehat{\mathcal{E}}_3(\Gamma_{n,d})^-$	≥ 1		≥ 6	≥ 8	≥ 2	≥ 12
$\dim \widehat{\mathcal{E}}_4(\Gamma_{n,d})^-$			≥ 1	≥ 2	≥ 1	≥ 4
$\dim \widehat{\mathcal{E}}_5(\Gamma_{n,d})^-$		≥ 1	≥ 8	≥ 1		

A list of linearly independent elements of $\widehat{\mathcal{E}}_m(\Gamma_{n,2})^-$ is given in the following table. The r_i in each row with index d are taken from the corresponding table in Appendix A (Table A.2 for $d = 2$, Table A.4 for $d = 3$, and Table A.6 for $d = 4$).

d	m	n	*Elements* $\xi \in \mathbb{Q}[\Gamma_{n,d}]$ *such that* $\mathrm{Alt}_n\, \xi$ *lies in* $\widehat{\mathcal{E}}_m(\Gamma_{n,d})^-$
2	3	5	$-3[r_3] + 3[r_4] + [r_7]$
2	5	6	$-15[r_3] + 9[r_4] + [r_7]$
3	3	6	$3[r_2] - 3[r_3] + [r_5]$
			$-3[r_8] - 4[r_9] - [r_{16}] + [r_{27}]$
			$-3[r_7] + 3[r_{11}] + [r_{29}]$
			$-[r_9] - 3[r_{13}] + 2[r_{16}] + [r_{30}]$
			$-3[r_{17}] - 3[r_{20}] + [r_{34}]$
			$-3[r_{25}] - 3[r_{26}] + [r_{39}]$
3	3	7	$[r_8]$
			$-4[r_9] - [r_{16}] + [r_{27}]$
			$-3[r_7] + 3[r_{11}] + [r_{29}]$
			$-4[r_{48}] + 2[r_{49}] - 4[r_{51}] + [r_{64}]$
			$-2[r_{48}] - 2[r_{49}] + [r_{51}] - 3[r_{52}] + [r_{66}]$
			$-3[r_{25}] - 3[r_{26}] + [r_{39}]$
			$-3[r_{17}] - 3[r_{20}] + [r_{34}]$
			$-[r_9] - 3[r_{13}] + 2[r_{16}] + [r_{30}]$

3	3	8	$-4[r_{48}] + 2[r_{49}] - 4[r_{51}] + [r_{64}]$
			$-2[r_{48}] - 2[r_{49}] + [r_{51}] - 3[r_{52}] + [r_{66}]$
3	4	6	$-3[r_9] - 3[r_{12}] - 9[r_{13}] - 3[r_{15}] + 3[r_{16}] + [r_{30}] + [r_{31}]$
3	4	7	$-3[r_9] - 3[r_{12}] - 9[r_{13}] - 3[r_{15}] + 3[r_{16}] + [r_{30}] + [r_{31}]$
			$-12[r_{48}] - 6[r_{49}] - 6[r_{50}] - 6[r_{51}] - 6[r_{52}] + [r_{64}] + 2[r_{66}]$
3	4	8	$-12[r_{48}] - 6[r_{49}] - 6[r_{50}] - 6[r_{51}] - 6[r_{52}] + [r_{64}] + 2[r_{66}]$
3	5	6	$15[r_2] - 9[r_3] + [r_5] - 15[r_7] + 9[r_{11}] + [r_{29}]$
3	5	7	$-15[r_7] + 9[r_{11}] + [r_{29}]$
4	3	8	$-4[r_{17}] + [r_{26}] + [r_{49}]$
			$[r_{17}] + 3[r_{24}] + 2[r_{26}] + [r_{52}]$
			$-3[r_{31}] - 3[r_{32}] + [r_{59}]$
			$-2[r_{34}] - 2[r_{35}] + [r_{36}] - 3[r_{37}] + [r_{60}]$
			$2[r_{34}] - 4[r_{35}] - 4[r_{36}] + [r_{63}]$
			$-3[r_{40}] - 3[r_{44}] + [r_{67}]$
			$-[r_{73}] + [r_{80}] - 4[r_{81}] - 2[r_{82}] + [r_{125}]$
			$-4[r_{84}] - 4[r_{86}] + 2[r_{87}] + [r_{128}]$
			$-2[r_{84}] + 3[r_{85}] + [r_{86}] - 2[r_{87}] + [r_{129}]$
			$-2[r_{88}] - 2[r_{89}] - 2[r_{90}] + [r_{91}] - [r_{92}] + [r_{133}]$
			$[r_{101}] + 2[r_{105}] - 3[r_{107}] + [r_{139}]$
			$-3[r_{100}] - 4[r_{101}] + [r_{105}] + [r_{141}]$
4	4	8	$-3[r_{17}] + 3[r_{22}] + 3[r_{23}] - 9[r_{24}] - 3[r_{26}] - [r_{52}] + [r_{54}]$
			$-6[r_{34}] - 12[r_{35}] + 6[r_{36}] - 6[r_{37}] - 6[r_{38}] + 2[r_{60}] + [r_{63}]$
			$6[r_{83}] - 12[r_{84}] + 6[r_{85}] - 6[r_{86}] + 6[r_{87}] + [r_{128}] + 2[r_{129}]$
			$3[r_{97}] + 3[r_{101}] + 3[r_{103}] + 3[r_{105}] - 9[r_{107}] + [r_{139}] + [r_{140}]$

Table 5.2: Skew-symmetric functional equations

Proof. The proof of this result is a calculation similar to the proof ot Theorem 5.9. □

5.5 Geometric cocycles

For application to the polylogarithm conjecture we are mainly interested in geometric cocycles in the sense of Definition 5.8. For the reason explained in the remarks after Conjecture 6 in Chapter 4 we are actually only interested in skew-symmetric $\mathcal{L}_m$-cocycles. Roughly, the reason is that we can always apply the antisymmetrization operator Alt_n to get a skew-symmetric cocycle. As an example, Goncharov initially found a functional equation (related to his 22-term relation) for the trilogarithm that can be naturally interpreted as a geometric cocycle relation. It was observed by Zagier that the terms of that equation split into two orbits permuted by $\mathfrak{S}_7$ and that one of these orbits trivially vanished under antisymmetrization, so one is left with 840 terms that all fall into a single $\mathfrak{S}_7$-orbit.

5.5.1 Skew-symmetric $\mathcal{L}_m$-cocycles given by bracket cross-ratios

The (symmetric version of the) five term relation for the dilogarithm and Goncharov's functional equation (4.13) show that the following spaces of $\mathcal{L}_m$-cocycles are nontrivial

$$\begin{aligned}\dim \widehat{\mathcal{F}}_2(\Gamma_{4,2})^- &\geq 1,\\ \dim \widehat{\mathcal{F}}_3(\Gamma_{6,3})^- &\geq 1.\end{aligned}$$

In particular, the triples (2,4,2) and (3,6,3) are good. By analyzing the list of identities in Theorem 5.12 we find that furthermore

$$\begin{aligned}\dim \widehat{\mathcal{F}}_5(\Gamma_{5,2})^- &\geq 1,\\ \dim \widehat{\mathcal{F}}_5(\Gamma_{6,3})^- &\geq 1,\end{aligned}$$

thus the triples (5,6,3) and (5,5,2) are also good. More precisely, we get the following result.

Theorem 5.13. *The elements*

$$\begin{aligned}f_1 &= \mathrm{Alt}_5\big(-15[r_3^{(1)}]+9[r_4^{(1)}]+[r_7^{(1)}]\big),\\ f_2 &= \mathrm{Alt}_6\big(-15[r_7^{(2)}]+9[r_{11}^{(2)}]+[r_{29}^{(2)}]\big),\end{aligned} \tag{5.6}$$

where $r_i^{(j)}$ are the cross-ratios from Table A.2 for $j=1$ and Table A.4 for $j=2$, are nontrivial skew-symmetric $\mathcal{L}_5$-cocycles of 5 points in $\mathbb{P}^1$ and 6 points in $\mathbb{P}^2$ respectively (in the sense of Definition 5.8).

Proof. From the identities in Theorem 5.12 it follows that $f_1 \in \mathcal{F}_5(\Gamma_{5,2})^-$ and $f_2 \in \mathcal{F}_5(\Gamma_{6,3})^-$. A simple application of Proposition 5.6 shows that these elements are nontrivial. □

5.5.2 A skew-symmetric $\mathcal{L}_4$-cocycle on 6 points in $\mathbb{P}^2$

We did not succeed in finding geometric $\mathcal{L}_4$-cocycles with arguments that can be written as (quotients of) products of brackets, but we do find such examples if we allow more complicated cross-ratios. Denote by $\langle ab;cd;ef\rangle$ the bracket polynomial

$$\langle abc\rangle\langle def\rangle - \langle abd\rangle\langle cef\rangle \in \mathcal{P}_{6,3}.$$

By making use of the exceptional S-cross-ratios for the set $S=\{\langle 123\rangle,\langle 12;34;56\rangle\}_{\mathfrak{S}_6}$ that were computed in Theorem 3.17 we can prove the following result.

Theorem 5.14. *Let $S=\{\langle 123\rangle,\langle 12;34;56\rangle\}_{\mathfrak{S}_6}$ and let Γ be the multiplicative subgroup of $\mathbb{K}_{6,3}^\times$ generated by S and ± 1. Then the space $\widehat{\mathcal{F}}_4(\Gamma)^-$ of skew-symmetric $\mathcal{L}_4$-cocycles with values in Γ (see Definition 5.8) has dimension ≥ 1, and it has a representative given by*

$$f = \mathrm{Alt}_6 \sum_{i=1}^{27} c_i[r_i]$$

with c_i as given in Table 5.3 below and r_i as in equation (5.7).

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
c_i	1	−4	−4	1	4	4	−1	−2	−1	−1	−2	−1	−4	22

i	15	16	17	18	19	20	21	22	23	24	25	26	27
c_i	4	-16	4	-16	4	-4	-10	2	6	4	4	-4	-4

Table 5.3: Coefficients of a $\mathcal{L}_4$-cocycle

$$
\begin{array}{rclrcl}
r_1 &=& -\dfrac{\langle 123\rangle^2\langle 124\rangle\langle 134\rangle\langle 245\rangle\langle 456\rangle^2}{\langle 12;36;45\rangle\langle 126\rangle\langle 145\rangle^2\langle 234\rangle^2} & r_{14} &=& \dfrac{\langle 123\rangle\langle 124\rangle\langle 156\rangle\langle 345\rangle}{\langle 16;23;45\rangle\langle 125\rangle\langle 134\rangle} \\
r_2 &=& \dfrac{\langle 12;34;56\rangle\langle 123\rangle\langle 145\rangle}{\langle 124\rangle\langle 135\rangle\langle 156\rangle\langle 234\rangle} & r_{15} &=& \dfrac{\langle 123\rangle\langle 124\rangle\langle 135\rangle\langle 256\rangle\langle 456\rangle}{\langle 12;34;56\rangle\langle 13;24;56\rangle\langle 125\rangle} \\
r_3 &=& \dfrac{\langle 12;34;56\rangle\langle 123\rangle\langle 145\rangle}{\langle 124\rangle\langle 126\rangle\langle 135\rangle\langle 345\rangle} & r_{16} &=& -\dfrac{\langle 123\rangle\langle 124\rangle\langle 156\rangle\langle 345\rangle}{\langle 125\rangle\langle 135\rangle\langle 146\rangle\langle 234\rangle} \\
r_4 &=& \dfrac{\langle 123\rangle^2\langle 124\rangle\langle 125\rangle\langle 146\rangle\langle 456\rangle^2}{\langle 14;23;56\rangle\langle 126\rangle^2\langle 145\rangle^2\langle 234\rangle} & r_{17} &=& -\dfrac{\langle 123\rangle\langle 124\rangle\langle 156\rangle\langle 356\rangle}{\langle 125\rangle\langle 136\rangle\langle 146\rangle\langle 235\rangle} \\
r_5 &=& \dfrac{\langle 123\rangle\langle 124\rangle\langle 135\rangle\langle 456\rangle}{\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 345\rangle} & r_{18} &=& -\dfrac{\langle 123\rangle\langle 124\rangle\langle 156\rangle\langle 356\rangle}{\langle 14;23;56\rangle\langle 125\rangle\langle 136\rangle} \\
r_6 &=& -\dfrac{\langle 12;34;56\rangle\langle 123\rangle\langle 145\rangle}{\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 345\rangle} & r_{19} &=& -\dfrac{\langle 123\rangle\langle 124\rangle\langle 135\rangle\langle 456\rangle}{\langle 125\rangle\langle 134\rangle\langle 146\rangle\langle 235\rangle} \\
r_7 &=& -\dfrac{\langle 12;35;46\rangle\langle 12;34;56\rangle^2\langle 135\rangle\langle 136\rangle}{\langle 123\rangle^2\langle 125\rangle\langle 146\rangle^2\langle 345\rangle\langle 356\rangle^2} & r_{20} &=& -\dfrac{\langle 123\rangle^2\langle 126\rangle\langle 145\rangle^2\langle 236\rangle\langle 346\rangle}{\langle 15;23;46\rangle\langle 124\rangle\langle 125\rangle\langle 136\rangle^2\langle 234\rangle} \\
r_8 &=& -\dfrac{\langle 123\rangle^2\langle 145\rangle\langle 246\rangle\langle 456\rangle}{\langle 15;23;46\rangle\langle 12;35;46\rangle\langle 124\rangle} & r_{21} &=& -\dfrac{\langle 12;34;56\rangle\langle 123\rangle}{\langle 125\rangle\langle 134\rangle\langle 236\rangle} \\
r_9 &=& -\dfrac{\langle 13;24;56\rangle\langle 123\rangle^2\langle 124\rangle\langle 145\rangle\langle 456\rangle}{\langle 125\rangle\langle 126\rangle\langle 134\rangle^2\langle 136\rangle\langle 245\rangle^2} & r_{22} &=& \dfrac{\langle 123\rangle^2\langle 145\rangle\langle 146\rangle\langle 256\rangle\langle 356\rangle}{\langle 14;25;36\rangle^2\langle 126\rangle\langle 135\rangle} \\
r_{10} &=& -\dfrac{\langle 123\rangle^2\langle 126\rangle\langle 145\rangle^2\langle 236\rangle\langle 245\rangle\langle 456\rangle}{\langle 16;23;45\rangle^2\langle 14;23;56\rangle\langle 124\rangle\langle 125\rangle} & r_{23} &=& \dfrac{\langle 123\rangle\langle 124\rangle\langle 156\rangle\langle 256\rangle}{\langle 125\rangle^2\langle 136\rangle\langle 246\rangle} \\
r_{11} &=& \dfrac{\langle 123\rangle^2\langle 145\rangle\langle 246\rangle\langle 456\rangle}{\langle 13;26;45\rangle\langle 13;24;56\rangle\langle 124\rangle} & r_{24} &=& -\dfrac{\langle 12;34;56\rangle\langle 123\rangle\langle 145\rangle}{\langle 125\rangle\langle 134\rangle\langle 156\rangle\langle 234\rangle} \\
r_{12} &=& -\dfrac{\langle 12;35;46\rangle\langle 123\rangle^2\langle 124\rangle\langle 145\rangle\langle 456\rangle}{\langle 125\rangle^2\langle 134\rangle\langle 146\rangle^2\langle 234\rangle\langle 235\rangle} & r_{25} &=& \dfrac{\langle 123\rangle\langle 124\rangle\langle 135\rangle\langle 256\rangle\langle 456\rangle}{\langle 12;34;56\rangle\langle 13;24;56\rangle\langle 125\rangle} \\
r_{13} &=& -\dfrac{\langle 12;34;56\rangle\langle 123\rangle\langle 145\rangle}{\langle 124\rangle\langle 134\rangle\langle 156\rangle\langle 235\rangle} & r_{26} &=& \dfrac{\langle 123\rangle^2\langle 145\rangle^2\langle 234\rangle\langle 236\rangle\langle 256\rangle}{\langle 13;26;45\rangle\langle 124\rangle\langle 126\rangle\langle 134\rangle\langle 235\rangle^2} \\
 && & r_{27} &=& -\dfrac{\langle 12;34;56\rangle\langle 123\rangle\langle 145\rangle}{\langle 124\rangle\langle 125\rangle\langle 136\rangle\langle 345\rangle}
\end{array}
\tag{5.7}
$$

Proof. Since this functional equation is rather complicated, let us give some additional information that helps with the proof. There are 4 $\mathfrak{S}_6$-orbits of special polynomials that occur in the factorization of $1-r_i$ (aside from $\langle 123\rangle$ and $\langle 12;34;56\rangle$):

$$
\begin{aligned}
\pi_1 &= \langle 123\rangle\langle 125\rangle\langle 346\rangle - \langle 124\rangle\langle 135\rangle\langle 236\rangle;\\
\pi_2 &= \langle 123\rangle\langle 126\rangle\langle 145\rangle\langle 345\rangle - \langle 124\rangle\langle 125\rangle\langle 134\rangle\langle 356\rangle;\\
\pi_3 &= \langle 123\rangle\langle 124\rangle\langle 156\rangle\langle 345\rangle + \langle 126\rangle\langle 134\rangle\langle 135\rangle\langle 245\rangle;\\
\pi_4 &= \langle 123\rangle\langle 145\rangle\langle 156\rangle\langle 234\rangle - \langle 124\rangle\langle 125\rangle\langle 134\rangle\langle 356\rangle.
\end{aligned}
$$

The polynomials π_2 and π_4 have no nontrivial automorphisms, the polynomial π_1 is fixed by $(1\,3)(5\,6)$, and π_3 is fixed by permutation $(3\,5)$. The divisibility relations between (orbits of) r_i and π_j are represented in the picture below. The edge between r_i and π_j means that $r_i \equiv 1 \pmod{\pi'_j}$ for some π'_j in the $\mathfrak{S}_6$-orbit of π_j (the double edge means that there are two polynomials in the same orbit that divide $1-r_i$).

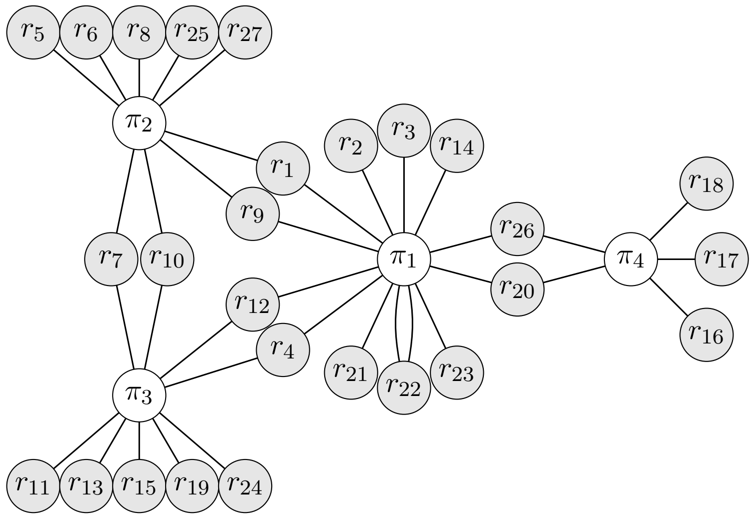

Figure 3. A relation graph for arguments of a $\mathcal{L}_4$-cocycle

The rest of the proof is a calculation with tensor symbols. □

Notice that this result shows that the triple (4,6,3) is good.

5.5.3 A skew-symmetric $\mathcal{L}_4$-cocycle on 8 points in $\mathbb{P}^3$

Denote by $\langle a|bc;de;fg\rangle$ the polynomial

$$\langle abcd\rangle\langle aefg\rangle - \langle abce\rangle\langle adfg\rangle \in \mathcal{P}_{8,4}. \tag{5.8}$$

Let $S = \{\langle 1234\rangle, \langle 1|23;45;67\rangle\}_{\mathfrak{S}_8}$ and let Γ be the multiplicative group generated by S. By making use of the exceptional cross-ratios that were computed in Theorem 3.19 and carefully analyzing that list, we obtain the following result, whose proof is similar to (but simpler than) the proof of Theorem 5.14.

Theorem 5.15. *Let $S = \{\langle 1234\rangle, \langle 1|23;45;67\rangle\}_{\mathfrak{S}_8}$ and let Γ be the multiplicative subgroup of $\mathbb{K}^\times_{8,4}$ generated by S. Then the space $\widehat{\mathcal{F}}_4(\Gamma)^-$ of skew-symmetric $\mathcal{L}_4$-cocycles with values in Γ has dimension ≥ 1 and contains a nontrivial element*

$$f \;=\; \mathrm{Alt}_8\big([r_1] - [r_2] + 9[r_3] + 9[r_4]\big) \in \mathbb{Q}\,[\Gamma]\,,$$

where r_i are defined as

$$\begin{aligned}
r_1 &= -\frac{\langle 1|23;45;67\rangle^2\langle 2348\rangle\langle 2368\rangle\langle 2478\rangle\langle 2568\rangle\langle 4568\rangle\langle 4678\rangle}{\langle 1|25;36;47\rangle\langle 1|27;34;56\rangle\langle 8|23;47;56\rangle^2\langle 2468\rangle^2},\\
r_2 &= -\frac{\langle 1|23;45;67\rangle^2\langle 8|23;47;56\rangle\langle 2348\rangle\langle 2468\rangle\langle 2568\rangle\langle 4678\rangle}{\langle 1|25;36;47\rangle\langle 1|27;34;56\rangle\langle 2368\rangle^2\langle 2478\rangle^2\langle 4568\rangle^2},\\
r_3 &= -\frac{\langle 8|24;36;57\rangle\langle 1234\rangle\langle 1256\rangle\langle 1357\rangle}{\langle 1|24;37;56\rangle\langle 8|26;37;45\rangle\langle 1235\rangle},\\
r_4 &= \frac{\langle 8|24;36;57\rangle\langle 1234\rangle\langle 1256\rangle\langle 1357\rangle}{\langle 1|24;37;56\rangle\langle 8|27;34;56\rangle\langle 1235\rangle}.
\end{aligned}$$

Proof. In this case there are just two $\mathfrak{S}_n$-orbits of polynomials that divide $1 - r_i$:

$$\begin{aligned} \pi_1 &= \text{numerator}(1 - r_3), \\ \pi_2 &= \text{numerator}(1 - r_4). \end{aligned}$$

The divisibility relation graph is given in the picture below.

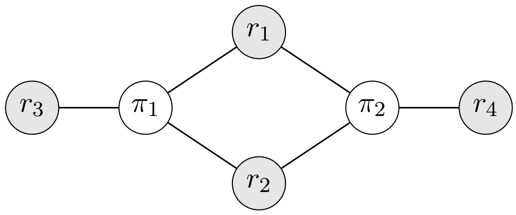

Figure 4. A relation graph for arguments of a $\mathcal{L}_4$-cocycle

The rest of the proof is a calculation with tensor symbols. □

Notice that this result shows that the triple (4,8,4) is good.

5.6 The prospect of proving Zagier's conjecture for $m = 4$

Let us discuss the question to what extent constructions like the ones in this thesis could be used to prove Zagier's conjecture in the case $m = 4$. First, we need to talk about the several levels at which an $\mathcal{L}_m$-cocycle $f \in \mathbb{Q}[\mathbb{K}_{n,d}]$ can be considered non-trivial:

N1. There does not exist an element $g \in \mathbb{Q}[\mathbb{K}_{n-1,d}]$ such that $f = D(g)$.

N2. There does not exist an element $g \in \mathbb{Q}[\mathbb{K}_{n-1,d}]$ such that $f - D(g) \in \mathcal{E}^0_m(\mathbb{K}_{n,d})$, i.e., $f = D(g)$ modulo trivial functional equations for $\mathcal{L}_m$.

N3. There does not exist an element $g \in \mathbb{Q}[\mathbb{K}_{n-1,d}]$ such that $f - D(g) \in \mathcal{E}_m(\mathbb{K}_{n,d})$.

N4. The measurable function $F\colon (\mathbb{C}^d)^n \to \mathbb{R}$ defined by $F(v_1, \ldots, v_n) = \mathcal{L}_m \circ f$ is a nontrivial locally integrable cocycle. In other words, F is locally L_1-integrable and for any function $G(v_1, \ldots, v_{n-1})$ that is locally L_1-integrable we have

$$F(v_1, \ldots, v_n) \neq \sum_{i=1}^{n} (-1)^i G(v_1, \ldots, \widehat{v_i}, \ldots, v_n),$$

where the non-equality is true on a set of positive Lebesgue measure.

It is trivial to prove that N1 $\Leftarrow$ N2 $\Leftarrow$ N3 $\Leftarrow$ N4. It is also easy to see that N1 $\not\Rightarrow$ N2 (one can take, for example, $f = [r] + (-1)^{m-1}[1/r]$, where r is any cross-ratio in $\mathbb{K}_{n,d}$ that depends non-trivially on all n points). Let us also remark that nontriviality in the sense of Definition 5.8 is equivalent to the property N2. Proposition 5.17 below shows that N2 $\not\Rightarrow$ N3 and the following conjecture, based on a rather long analysis that we omit, would imply that N3 $\not\Rightarrow$ N4.

Conjecture 7. *The element $\xi = f_1 \in \mathcal{F}_5(\Gamma_{5,2})$, given in equation* (5.6), *satisfies the property N3 but does not satisfy N4.*

To prove Conjecture 6 (see Chapter 4) for a given m, and thus get a positive answer to Zagier's conjecture for $\zeta_F(m)$, one needs to find a skew-symmetric $\mathcal{L}_m$-cocycle $\xi \in \mathbb{Q}[\mathbb{K}_{2m,m}]$ that satisfies N4. (Goncharov proved that any $\mathcal{L}_m$-cocycle corresponding to the morphism $\psi_{2m}(m)$ in Conjecture 6 has the property N4.)

For $m = 2$ and $m = 3$ the $\mathcal{L}_m$-cocycles that correspond to the classical cross-ratio and Goncharov's triple ratio respectively (that also correspond to the morphisms $\psi_4(2)$ and $\psi_6(3)$ constructed in Section 4.5) satisfy N4. Moreover, in these two cases one finds a cocycle with property N4 as soon as one finds one with N2.

For $m = 4$, unfortunately, the situation is much more complicated. First of all, even finding skew-symmetric $\mathcal{L}_4$-cocycles of 8 points in $\mathbb{P}^3$ that satisfy N2 is much more difficult, as the following negative result shows.

Proposition 5.16. *Assuming Conjecture 3 for $d = 4$, there are no skew-symmetric $\mathcal{L}_m$-cocycles with values in $\Gamma_{8,4}$ that satisfy N2.*

Proof. Conjecture 3 implies that Table A.6 in the appendix lists all exceptional bracket cross-ratios of 8 points in $\mathbb{P}^3$, and there are 148 $\mathfrak{S}_8$-orbits of them in total. The result then follows from a linear algebra computation involving the skew-symmetrizations of these 148 orbits of cross-ratios. □

In particular, this result suggests that to find a $\mathcal{L}_4$-cocycle of 8 points in $\mathbb{P}^3$ which satisfies even the weaker property N2, one must look at multiplicative subgroups of $\mathbb{K}_{8,4}^\times$ larger than the group generated by all 4×4-minors. We have managed to find one such $\mathcal{L}_4$-cocycle in Theorem 5.15, but it took us a considerable amount of effort to decide whether this $\mathcal{L}_4$-cocycle satisfies the stronger property N4, and the following result shows that, alas, it does not.

Proposition 5.17. *The element f constructed in Theorem 5.15 satisfies the property N2 but does not satisfy N3.*

Proof. The fact that f satisfies N2 was part of the statement of Theorem 5.15. To prove that f fails to satisfy N3 we use the following identity (the proof of which, as usual, reduces to a computation with tensor symbols)

$$f + 6 \cdot \mathrm{Alt}_8[1 - r] \in \mathcal{E}_4(\mathbb{K}_{8,4}),$$

where

$$r = \frac{\langle 1237\rangle\langle 1246\rangle\langle 1345\rangle}{\langle 1236\rangle\langle 1245\rangle\langle 1347\rangle}.$$

Thus, if we take $g = 6 \cdot \mathrm{Alt}_7[1 - r]$, then $f - D(g) \in \mathcal{E}_4(\mathbb{K}_{8,4})$. □

This proposition shows that in the case $m = 4$, unlike the previous cases $m = 2$ and $m = 3$, even if we manage to find a cocycle that satisfies N2, it can easily fail to satisfy the property N3 and hence also the more important property N4. This situation is, in fact, even more striking. We have found a huge number of exceptional cross-ratios in the multiplicative group Γ generated by $\{\langle 1234\rangle, \langle 1|23;45;67\rangle\}_{\mathfrak{S}_8}$ (as described in Theorem 3.19), and using them we have constructed a 13-dimensional space of nontrivial $\mathcal{L}_4$-cocycles (i.e., cocycles having the property N2), the simplest element of this space being the element f from Theorem 5.15. It is a priori not unreasonable to expect that at least one of these cocycles should satisfy the needed property N4. However, this is not the case, as we found that every cocycle in this 13-dimensional space fails to satisfy N3.

Thus, it seems that the main problem with extending Goncharov's strategy for proving Zagier's conjecture from the case $m = 3$ to the case $m = 4$ is to find a suitable generalization of the multiplicative group generated by 4×4-minors in the case of 8 points in $\mathbb{P}^3$, and such generalization should presumably capture some additional geometric properties of point configurations in projective spaces.

To summarize:

- it is almost certainly not sufficient to use only cross-ratios defined by products of 4×4-minors, as in the cases $m = 2$ and $m = 3$;
- we have succeeded in finding one such larger class of cross-ratios (that includes products of polynomials $\langle a|bc; de; fg\rangle$ defined in (5.8)) which in any case gives many functional equations for the 4-logarithm;
- it is not clear whether there are geometric $\mathcal{L}_4$-cocycles within even this larger class that satisfy the required non-triviality condition N4, but we at least know very effective methods to test whether any given candidate or type of candidate works.

In short, there is still no reason not to believe that Goncharov's vision of higher cross-ratios can be realized, but there is still work to be done...

Appendices

APPENDIX A

Tables of cross-ratios

In this appendix we collect the results of the computation of exceptional bracket cross-ratios that was described in Chapter 3. In each of the cases $\mathbb{P}^1$, $\mathbb{P}^2$, and $\mathbb{P}^3$ we give two tables: a table of special polynomials (see Definition 3.5), and a table of exceptional bracket cross-ratios (see Definition 3.6).

As in Chapter 3, $\mathcal{P}_{n,d}$ is the bracket algebra, and $S_{n,d}$ denotes the set of all brackets in $\mathcal{P}_{n,d}$.

A.1 Cross-ratios on $\mathbb{P}^1$

The following table lists special polynomials for $S_{8,2}$ (see Theorem 3.7 in Chapter 3).

i	Rank	Degree	Polynomial $\pi_i^{(1)}$
0	5	11000000	$\langle 12\rangle$
1	2	22220000	$\langle 13\rangle^2\langle 24\rangle^2 - \langle 12\rangle\langle 13\rangle\langle 24\rangle\langle 34\rangle + \langle 12\rangle^2\langle 34\rangle^2$
2	2	21111000	$\langle 14\rangle\langle 15\rangle\langle 23\rangle - \langle 12\rangle\langle 13\rangle\langle 45\rangle$
3	2	22211000	$\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 25\rangle + \langle 12\rangle^2\langle 34\rangle\langle 35\rangle$
4	3	11111100	$\langle 16\rangle\langle 25\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle\langle 56\rangle$
5	2	22111100	$\langle 15\rangle\langle 16\rangle\langle 23\rangle\langle 24\rangle + \langle 12\rangle\langle 14\rangle\langle 26\rangle\langle 35\rangle$
6	2	21111110	$\langle 13\rangle\langle 17\rangle\langle 26\rangle\langle 45\rangle + \langle 12\rangle\langle 14\rangle\langle 35\rangle\langle 67\rangle$
7	2	11111111	$\langle 13\rangle\langle 27\rangle\langle 46\rangle\langle 58\rangle - \langle 17\rangle\langle 23\rangle\langle 48\rangle\langle 56\rangle$

Table A.1: $S_{8,2}$-special polynomials in $\mathcal{P}_{8,2}$

The following table lists exceptional cross-ratios for $S_{8,2}$ (see Theorem 3.8 in Chapter 3).

i	Degree	Numerator of r_i	Denominator of r_i	$j : \pi_j^{(1)} \| (1-r_i)$
1	11110000	$\langle 13\rangle\langle 24\rangle$	$\langle 14\rangle\langle 23\rangle$	$0, 0$

2	22220000	$-\langle 12\rangle^2\langle 34\rangle^2$	$\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle$	1
3	21111000	$\langle 12\rangle\langle 13\rangle\langle 45\rangle$	$\langle 14\rangle\langle 15\rangle\langle 23\rangle$	2
4	22211000	$-\langle 12\rangle^2\langle 34\rangle\langle 35\rangle$	$\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 25\rangle$	3
5	22211000	$\langle 12\rangle^2\langle 34\rangle\langle 35\rangle$	$\langle 13\rangle^2\langle 24\rangle\langle 25\rangle$	$0, 2$
6	42222000	$\langle 12\rangle^2\langle 13\rangle^2\langle 45\rangle^2$	$\langle 14\rangle^2\langle 15\rangle^2\langle 23\rangle^2$	$2, 2$
7	43322000	$-\langle 12\rangle^3\langle 14\rangle\langle 34\rangle\langle 35\rangle^2$	$\langle 13\rangle^3\langle 15\rangle\langle 24\rangle^2\langle 25\rangle$	$2, 3$
8	11111100	$\langle 16\rangle\langle 23\rangle\langle 45\rangle$	$\langle 15\rangle\langle 24\rangle\langle 36\rangle$	4
9	22111100	$-\langle 12\rangle^2\langle 35\rangle\langle 46\rangle$	$\langle 13\rangle\langle 16\rangle\langle 24\rangle\langle 25\rangle$	5
10	22111100	$\langle 12\rangle\langle 15\rangle\langle 23\rangle\langle 46\rangle$	$\langle 13\rangle\langle 14\rangle\langle 25\rangle\langle 26\rangle$	5
11	22111100	$\langle 13\rangle\langle 14\rangle\langle 25\rangle\langle 26\rangle$	$\langle 15\rangle\langle 16\rangle\langle 23\rangle\langle 24\rangle$	$0, 4$
12	22222200	$-\langle 13\rangle^2\langle 24\rangle\langle 26\rangle\langle 45\rangle\langle 56\rangle$	$\langle 14\rangle\langle 16\rangle\langle 25\rangle^2\langle 34\rangle\langle 36\rangle$	$4, 4$
13	33222200	$\langle 14\rangle\langle 16\rangle^2\langle 23\rangle^2\langle 25\rangle\langle 45\rangle$	$\langle 13\rangle\langle 15\rangle^2\langle 24\rangle^2\langle 26\rangle\langle 36\rangle$	$4, 5$
14	44222200	$-\langle 12\rangle^3\langle 15\rangle\langle 23\rangle\langle 35\rangle\langle 46\rangle^2$	$\langle 13\rangle^2\langle 14\rangle\langle 16\rangle\langle 24\rangle\langle 25\rangle^2\langle 26\rangle$	$5, 5$
15	21111110	$-\langle 13\rangle\langle 17\rangle\langle 26\rangle\langle 45\rangle$	$\langle 12\rangle\langle 14\rangle\langle 35\rangle\langle 67\rangle$	6
16	42222220	$\langle 13\rangle^2\langle 16\rangle\langle 17\rangle\langle 26\rangle\langle 27\rangle\langle 45\rangle^2$	$\langle 12\rangle^2\langle 14\rangle\langle 15\rangle\langle 34\rangle\langle 35\rangle\langle 67\rangle^2$	$6, 6$
17	11111111	$-\langle 13\rangle\langle 27\rangle\langle 45\rangle\langle 68\rangle$	$\langle 12\rangle\langle 37\rangle\langle 48\rangle\langle 56\rangle$	7
18	22222222	$\langle 13\rangle\langle 17\rangle\langle 23\rangle\langle 27\rangle\langle 45\rangle^2\langle 68\rangle^2$	$\langle 12\rangle^2\langle 37\rangle^2\langle 46\rangle\langle 48\rangle\langle 56\rangle\langle 58\rangle$	$7, 7$

Table A.2: Exceptional bracket cross-ratios on $\mathbb{P}^1$

A.2 Cross-ratios on $\mathbb{P}^2$

The following table lists special polynomials for $S_{10,3}$ (see Theorem 3.9 in Chapter 3).

i	Rank	Degree	Polynomial $\pi_i^{(2)}$
0	12	1110000000	$\langle 123\rangle$
1	2	2222100000	$\langle 125\rangle\langle 134\rangle\langle 234\rangle - \langle 123\rangle\langle 124\rangle\langle 345\rangle$
2	2	3322200000	$\langle 124\rangle^2\langle 135\rangle\langle 235\rangle - \langle 123\rangle\langle 125\rangle\langle 134\rangle\langle 245\rangle$
3	4	1111110000	$\langle 124\rangle\langle 356\rangle - \langle 123\rangle\langle 456\rangle$
4	2	3211110000	$\langle 125\rangle\langle 126\rangle\langle 134\rangle - \langle 123\rangle\langle 124\rangle\langle 156\rangle$
5	2	2221110000	$\langle 126\rangle\langle 135\rangle\langle 234\rangle + \langle 123\rangle\langle 125\rangle\langle 346\rangle$
6	2	4222110000	$\langle 124\rangle\langle 126\rangle\langle 134\rangle\langle 135\rangle + \langle 123\rangle^2\langle 145\rangle\langle 146\rangle$
7	2	3322110000	$\langle 124\rangle\langle 126\rangle\langle 135\rangle\langle 234\rangle + \langle 123\rangle\langle 125\rangle\langle 134\rangle\langle 246\rangle$
8	2	3322110000	$\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 234\rangle + \langle 123\rangle\langle 124\rangle\langle 145\rangle\langle 236\rangle$
9	3	3222210000	$\langle 126\rangle\langle 134\rangle\langle 135\rangle\langle 245\rangle - \langle 124\rangle\langle 125\rangle\langle 136\rangle\langle 345\rangle$
10	2	3222210000	$\langle 124\rangle\langle 134\rangle\langle 156\rangle\langle 235\rangle - \langle 123\rangle\langle 125\rangle\langle 146\rangle\langle 345\rangle$
11	2	3222210000	$\langle 124\rangle\langle 135\rangle\langle 156\rangle\langle 234\rangle + \langle 126\rangle\langle 134\rangle\langle 145\rangle\langle 235\rangle$
12	2	2222220000	$\langle 123\rangle\langle 156\rangle\langle 246\rangle\langle 345\rangle + \langle 125\rangle\langle 146\rangle\langle 234\rangle\langle 356\rangle$
13	2	2222220000	$\langle 124\rangle\langle 156\rangle\langle 236\rangle\langle 345\rangle - \langle 125\rangle\langle 146\rangle\langle 235\rangle\langle 346\rangle$
14	2	2222220000	$\langle 125\rangle\langle 146\rangle\langle 236\rangle\langle 345\rangle - \langle 124\rangle\langle 156\rangle\langle 235\rangle\langle 346\rangle$
15	5	2222220000	$\langle 134\rangle\langle 156\rangle\langle 235\rangle\langle 246\rangle - \langle 135\rangle\langle 146\rangle\langle 234\rangle\langle 256\rangle$
16	3	3111111000	$\langle 126\rangle\langle 137\rangle\langle 145\rangle - \langle 127\rangle\langle 135\rangle\langle 146\rangle$
17	2	2211111000	$\langle 127\rangle\langle 156\rangle\langle 234\rangle - \langle 126\rangle\langle 135\rangle\langle 247\rangle$
18	2	4221111000	$\langle 125\rangle\langle 127\rangle\langle 134\rangle\langle 136\rangle + \langle 123\rangle\langle 126\rangle\langle 135\rangle\langle 147\rangle$
19	2	3321111000	$\langle 123\rangle\langle 127\rangle\langle 145\rangle\langle 236\rangle - \langle 124\rangle\langle 126\rangle\langle 135\rangle\langle 237\rangle$
20	2	3222111000	$\langle 127\rangle\langle 136\rangle\langle 145\rangle\langle 234\rangle - \langle 124\rangle\langle 135\rangle\langle 146\rangle\langle 237\rangle$
21	2	3222111000	$\langle 124\rangle\langle 135\rangle\langle 146\rangle\langle 237\rangle - \langle 123\rangle\langle 136\rangle\langle 145\rangle\langle 247\rangle$
22	2	2222211000	$\langle 124\rangle\langle 135\rangle\langle 257\rangle\langle 346\rangle - \langle 125\rangle\langle 134\rangle\langle 246\rangle\langle 357\rangle$
23	2	2222211000	$\langle 125\rangle\langle 146\rangle\langle 237\rangle\langle 345\rangle - \langle 124\rangle\langle 156\rangle\langle 235\rangle\langle 347\rangle$
24	2	2222211000	$\langle 126\rangle\langle 145\rangle\langle 237\rangle\langle 345\rangle + \langle 125\rangle\langle 146\rangle\langle 234\rangle\langle 357\rangle$
25	2	4211111100	$\langle 123\rangle\langle 126\rangle\langle 147\rangle\langle 158\rangle - \langle 124\rangle\langle 125\rangle\langle 137\rangle\langle 168\rangle$
26	2	3311111100	$\langle 125\rangle\langle 128\rangle\langle 137\rangle\langle 246\rangle - \langle 126\rangle\langle 127\rangle\langle 135\rangle\langle 248\rangle$

27	2	3221111100	$\langle 126\rangle\langle 134\rangle\langle 157\rangle\langle 238\rangle - \langle 123\rangle\langle 125\rangle\langle 167\rangle\langle 348\rangle$
28	2	2222111100	$\langle 124\rangle\langle 156\rangle\langle 237\rangle\langle 348\rangle + \langle 126\rangle\langle 145\rangle\langle 234\rangle\langle 378\rangle$
29	2	2222111100	$\langle 128\rangle\langle 137\rangle\langle 246\rangle\langle 345\rangle - \langle 124\rangle\langle 134\rangle\langle 268\rangle\langle 357\rangle$
30	2	4111111110	$\langle 125\rangle\langle 139\rangle\langle 147\rangle\langle 168\rangle + \langle 126\rangle\langle 134\rangle\langle 158\rangle\langle 179\rangle$
31	2	3211111110	$\langle 128\rangle\langle 134\rangle\langle 159\rangle\langle 267\rangle + \langle 126\rangle\langle 135\rangle\langle 149\rangle\langle 278\rangle$
32	2	2221111110	$\langle 126\rangle\langle 149\rangle\langle 237\rangle\langle 358\rangle - \langle 129\rangle\langle 146\rangle\langle 235\rangle\langle 378\rangle$
33	2	2211111111	$\langle 139\rangle\langle 145\rangle\langle 2,6,10\rangle\langle 278\rangle + \langle 134\rangle\langle 159\rangle\langle 268\rangle\langle 2,7,10\rangle$

Table A.3: $S_{10,3}$-special polynomials in $\mathcal{P}_{10,3}$

The following table lists exceptional cross-ratios for $S_{10,3}$ (see Theorem 3.10 in Chapter 3).

i	Degree	r_i	$j : \pi_j^{(2)} \mid 1 - r_i$
1	2111100000	$\frac{\langle 124\rangle\langle 135\rangle}{\langle 125\rangle\langle 134\rangle}$	$0, 0$
2	2222100000	$\frac{\langle 123\rangle\langle 134\rangle\langle 245\rangle}{\langle 124\rangle\langle 135\rangle\langle 234\rangle}$	1
3	3322200000	$\frac{-\langle 123\rangle^2\langle 145\rangle\langle 245\rangle}{\langle 124\rangle\langle 125\rangle\langle 135\rangle\langle 234\rangle}$	2
4	3322200000	$\frac{\langle 123\rangle^2\langle 145\rangle\langle 245\rangle}{\langle 124\rangle^2\langle 135\rangle\langle 235\rangle}$	$0, 1$
5	5544300000	$\frac{-\langle 123\rangle^3\langle 134\rangle\langle 145\rangle\langle 245\rangle^2}{\langle 124\rangle^3\langle 135\rangle^2\langle 234\rangle\langle 235\rangle}$	$1, 2$
6	1111110000	$\frac{\langle 126\rangle\langle 345\rangle}{\langle 125\rangle\langle 346\rangle}$	3
7	3211110000	$\frac{\langle 123\rangle\langle 125\rangle\langle 146\rangle}{\langle 124\rangle\langle 126\rangle\langle 135\rangle}$	4
8	2221110000	$\frac{\langle 124\rangle\langle 136\rangle\langle 235\rangle}{\langle 125\rangle\langle 134\rangle\langle 236\rangle}$	$0, 3$
9	2221110000	$\frac{-\langle 123\rangle\langle 125\rangle\langle 346\rangle}{\langle 126\rangle\langle 135\rangle\langle 234\rangle}$	5
10	4222110000	$\frac{\langle 124\rangle^2\langle 135\rangle\langle 136\rangle}{\langle 123\rangle^2\langle 145\rangle\langle 146\rangle}$	$0, 4$
11	4222110000	$\frac{-\langle 123\rangle^2\langle 145\rangle\langle 146\rangle}{\langle 124\rangle\langle 126\rangle\langle 134\rangle\langle 135\rangle}$	6
12	3322110000	$\frac{-\langle 123\rangle\langle 126\rangle\langle 145\rangle\langle 234\rangle}{\langle 124\rangle\langle 125\rangle\langle 134\rangle\langle 236\rangle}$	7
13	3322110000	$\frac{-\langle 123\rangle^2\langle 145\rangle\langle 246\rangle}{\langle 124\rangle\langle 126\rangle\langle 135\rangle\langle 234\rangle}$	8
14	3322110000	$\frac{-\langle 123\rangle\langle 124\rangle\langle 145\rangle\langle 236\rangle}{\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 234\rangle}$	8
15	3322110000	$\frac{\langle 123\rangle^2\langle 145\rangle\langle 246\rangle}{\langle 124\rangle^2\langle 135\rangle\langle 236\rangle}$	7
16	3322110000	$\frac{\langle 123\rangle^2\langle 146\rangle\langle 245\rangle}{\langle 125\rangle\langle 126\rangle\langle 134\rangle\langle 234\rangle}$	$0, 5$
17	3222210000	$\frac{\langle 126\rangle\langle 134\rangle\langle 145\rangle\langle 235\rangle}{\langle 125\rangle\langle 135\rangle\langle 146\rangle\langle 234\rangle}$	10
18	3222210000	$\frac{\langle 126\rangle\langle 134\rangle\langle 135\rangle\langle 245\rangle}{\langle 124\rangle\langle 125\rangle\langle 136\rangle\langle 345\rangle}$	9
19	3222210000	$\frac{\langle 123\rangle\langle 126\rangle\langle 145\rangle\langle 345\rangle}{\langle 124\rangle\langle 135\rangle\langle 156\rangle\langle 234\rangle}$	10

20	3222210000	$\frac{-\langle123\rangle\langle126\rangle\langle145\rangle\langle345\rangle}{\langle125\rangle\langle135\rangle\langle146\rangle\langle234\rangle}$	11
21	2222220000	$\frac{\langle123\rangle\langle145\rangle\langle256\rangle\langle346\rangle}{\langle125\rangle\langle134\rangle\langle236\rangle\langle456\rangle}$	15
22	2222220000	$\frac{\langle126\rangle\langle145\rangle\langle236\rangle\langle345\rangle}{\langle124\rangle\langle156\rangle\langle234\rangle\langle356\rangle}$	14
23	2222220000	$\frac{\langle123\rangle\langle125\rangle\langle346\rangle\langle456\rangle}{\langle124\rangle\langle126\rangle\langle345\rangle\langle356\rangle}$	$3,3$
24	2222220000	$\frac{-\langle126\rangle\langle135\rangle\langle234\rangle\langle456\rangle}{\langle124\rangle\langle156\rangle\langle235\rangle\langle346\rangle}$	12
25	2222220000	$\frac{-\langle126\rangle\langle145\rangle\langle236\rangle\langle345\rangle}{\langle125\rangle\langle146\rangle\langle234\rangle\langle356\rangle}$	13
26	2222220000	$\frac{\langle126\rangle\langle145\rangle\langle235\rangle\langle346\rangle}{\langle125\rangle\langle146\rangle\langle234\rangle\langle356\rangle}$	14
27	3332220000	$\frac{-\langle125\rangle^2\langle134\rangle\langle236\rangle\langle346\rangle}{\langle126\rangle\langle135\rangle\langle156\rangle\langle234\rangle^2}$	$3,5$
28	4433220000	$\frac{\langle126\rangle^2\langle135\rangle\langle145\rangle\langle234\rangle^2}{\langle125\rangle^2\langle134\rangle^2\langle236\rangle\langle246\rangle}$	$3,7$
29	7433220000	$\frac{-\langle124\rangle^3\langle125\rangle\langle135\rangle\langle136\rangle^2}{\langle123\rangle^3\langle126\rangle\langle145\rangle^2\langle146\rangle}$	$4,6$
30	5543220000	$\frac{\langle123\rangle^3\langle145\rangle^2\langle236\rangle\langle246\rangle}{\langle125\rangle\langle126\rangle^2\langle134\rangle\langle135\rangle\langle234\rangle^2}$	$5,8$
31	6644220000	$\frac{-\langle123\rangle^3\langle125\rangle\langle134\rangle\langle145\rangle\langle246\rangle^2}{\langle124\rangle^3\langle126\rangle\langle135\rangle^2\langle234\rangle\langle236\rangle}$	$7,8$
32	4333320000	$\frac{\langle126\rangle\langle134\rangle^2\langle156\rangle\langle235\rangle\langle245\rangle}{\langle125\rangle^2\langle136\rangle\langle146\rangle\langle234\rangle\langle345\rangle}$	$3,9$
33	6444420000	$\frac{\langle124\rangle\langle126\rangle\langle134\rangle^2\langle145\rangle\langle156\rangle\langle235\rangle^2}{\langle123\rangle\langle125\rangle^2\langle135\rangle\langle146\rangle^2\langle234\rangle\langle345\rangle}$	$9,10$
34	6444420000	$\frac{\langle123\rangle\langle126\rangle^2\langle134\rangle\langle145\rangle^2\langle235\rangle\langle345\rangle}{\langle124\rangle\langle125\rangle\langle135\rangle^2\langle146\rangle\langle156\rangle\langle234\rangle^2}$	$10,11$
35	3333330000	$\frac{\langle124\rangle\langle135\rangle\langle146\rangle\langle236\rangle\langle256\rangle\langle345\rangle}{\langle126\rangle\langle134\rangle\langle145\rangle\langle235\rangle\langle246\rangle\langle356\rangle}$	$3,15$
36	3333330000	$\frac{\langle123\rangle\langle126\rangle\langle135\rangle\langle246\rangle\langle345\rangle\langle456\rangle}{\langle124\rangle\langle125\rangle\langle146\rangle\langle235\rangle\langle346\rangle\langle356\rangle}$	$3,12$
37	4444440000	$\frac{\langle125\rangle\langle126\rangle\langle135\rangle\langle146\rangle\langle234\rangle^2\langle356\rangle\langle456\rangle}{\langle123\rangle\langle124\rangle\langle156\rangle^2\langle235\rangle\langle246\rangle\langle345\rangle\langle346\rangle}$	$12,15$
38	4444440000	$\frac{\langle124\rangle\langle126\rangle\langle145\rangle\langle156\rangle\langle235\rangle^2\langle346\rangle^2}{\langle125\rangle^2\langle146\rangle^2\langle234\rangle\langle236\rangle\langle345\rangle\langle356\rangle}$	$14,15$
39	4444440000	$\frac{\langle126\rangle^2\langle145\rangle^2\langle235\rangle\langle236\rangle\langle345\rangle\langle346\rangle}{\langle124\rangle\langle125\rangle\langle146\rangle\langle156\rangle\langle234\rangle^2\langle356\rangle^2}$	$13,14$
40	3111111000	$\frac{\langle123\rangle\langle146\rangle\langle157\rangle}{\langle124\rangle\langle137\rangle\langle156\rangle}$	16
41	2211111000	$\frac{\langle125\rangle\langle136\rangle\langle247\rangle}{\langle124\rangle\langle156\rangle\langle237\rangle}$	17
42	4221111000	$\frac{-\langle123\rangle\langle126\rangle\langle135\rangle\langle147\rangle}{\langle125\rangle\langle127\rangle\langle134\rangle\langle136\rangle}$	18
43	4221111000	$\frac{\langle124\rangle\langle126\rangle\langle135\rangle\langle137\rangle}{\langle125\rangle\langle127\rangle\langle134\rangle\langle136\rangle}$	$0,16$
44	4221111000	$\frac{\langle123\rangle^2\langle147\rangle\langle156\rangle}{\langle124\rangle\langle125\rangle\langle136\rangle\langle137\rangle}$	18
45	3321111000	$\frac{\langle123\rangle\langle127\rangle\langle145\rangle\langle236\rangle}{\langle124\rangle\langle126\rangle\langle135\rangle\langle237\rangle}$	19
46	3321111000	$\frac{\langle125\rangle\langle127\rangle\langle136\rangle\langle234\rangle}{\langle124\rangle\langle126\rangle\langle135\rangle\langle237\rangle}$	$0,17$
47	3321111000	$\frac{-\langle123\rangle^2\langle145\rangle\langle267\rangle}{\langle125\rangle\langle126\rangle\langle134\rangle\langle237\rangle}$	19
48	3222111000	$\frac{\langle127\rangle\langle136\rangle\langle145\rangle\langle234\rangle}{\langle124\rangle\langle135\rangle\langle146\rangle\langle237\rangle}$	20
49	3222111000	$\frac{-\langle123\rangle\langle136\rangle\langle145\rangle\langle247\rangle}{\langle124\rangle\langle134\rangle\langle156\rangle\langle237\rangle}$	20

50	3222111000	$\frac{\langle 123\rangle\langle 136\rangle\langle 145\rangle\langle 247\rangle}{\langle 124\rangle\langle 135\rangle\langle 146\rangle\langle 237\rangle}$	21
51	3222111000	$\frac{-\langle 127\rangle\langle 134\rangle\langle 156\rangle\langle 234\rangle}{\langle 123\rangle\langle 135\rangle\langle 146\rangle\langle 247\rangle}$	20
52	3222111000	$\frac{-\langle 127\rangle\langle 136\rangle\langle 145\rangle\langle 234\rangle}{\langle 124\rangle\langle 134\rangle\langle 156\rangle\langle 237\rangle}$	21
53	2222211000	$\frac{\langle 126\rangle\langle 145\rangle\langle 235\rangle\langle 347\rangle}{\langle 125\rangle\langle 146\rangle\langle 234\rangle\langle 357\rangle}$	23
54	2222211000	$\frac{\langle 126\rangle\langle 145\rangle\langle 237\rangle\langle 345\rangle}{\langle 124\rangle\langle 156\rangle\langle 234\rangle\langle 357\rangle}$	23
55	2222211000	$\frac{-\langle 126\rangle\langle 145\rangle\langle 237\rangle\langle 345\rangle}{\langle 125\rangle\langle 146\rangle\langle 234\rangle\langle 357\rangle}$	24
56	2222211000	$\frac{\langle 134\rangle\langle 157\rangle\langle 235\rangle\langle 246\rangle}{\langle 135\rangle\langle 146\rangle\langle 234\rangle\langle 257\rangle}$	22
57	6222222000	$\frac{\langle 123\rangle\langle 126\rangle\langle 137\rangle\langle 145\rangle^2\langle 167\rangle}{\langle 127\rangle^2\langle 134\rangle\langle 135\rangle\langle 146\rangle\langle 156\rangle}$	$16, 16$
58	4422222000	$\frac{\langle 125\rangle\langle 126\rangle\langle 135\rangle\langle 136\rangle\langle 247\rangle^2}{\langle 124\rangle\langle 127\rangle\langle 156\rangle^2\langle 234\rangle\langle 237\rangle}$	$17, 17$
59	7332222000	$\frac{-\langle 124\rangle^2\langle 126\rangle\langle 135\rangle^2\langle 137\rangle\langle 167\rangle}{\langle 125\rangle\langle 127\rangle^2\langle 134\rangle\langle 136\rangle^2\langle 145\rangle}$	$16, 18$
60	5532222000	$\frac{-\langle 125\rangle\langle 127\rangle^2\langle 134\rangle\langle 145\rangle\langle 236\rangle^2}{\langle 124\rangle^2\langle 126\rangle\langle 135\rangle^2\langle 237\rangle\langle 267\rangle}$	$17, 19$
61	8442222000	$\frac{-\langle 123\rangle^3\langle 126\rangle\langle 135\rangle\langle 147\rangle^2\langle 156\rangle}{\langle 124\rangle\langle 125\rangle^2\langle 127\rangle\langle 134\rangle\langle 136\rangle^2\langle 137\rangle}$	$18, 18$
62	6642222000	$\frac{-\langle 123\rangle^3\langle 127\rangle\langle 145\rangle^2\langle 236\rangle\langle 267\rangle}{\langle 124\rangle\langle 125\rangle\langle 126\rangle^2\langle 134\rangle\langle 135\rangle\langle 237\rangle^2}$	$19, 19$
63	6444222000	$\frac{\langle 123\rangle^2\langle 135\rangle\langle 136\rangle\langle 145\rangle\langle 146\rangle\langle 247\rangle^2}{\langle 124\rangle\langle 127\rangle\langle 134\rangle^2\langle 156\rangle^2\langle 234\rangle\langle 237\rangle}$	$20, 20$
64	6444222000	$\frac{-\langle 127\rangle^2\langle 134\rangle\langle 136\rangle\langle 145\rangle\langle 156\rangle\langle 234\rangle^2}{\langle 123\rangle\langle 124\rangle\langle 135\rangle^2\langle 146\rangle^2\langle 237\rangle\langle 247\rangle}$	$20, 20$
65	6444222000	$\frac{\langle 127\rangle^2\langle 135\rangle\langle 136\rangle\langle 145\rangle\langle 146\rangle\langle 234\rangle^2}{\langle 123\rangle\langle 124\rangle\langle 134\rangle^2\langle 156\rangle^2\langle 237\rangle\langle 247\rangle}$	$21, 21$
66	6444222000	$\frac{-\langle 123\rangle\langle 127\rangle\langle 136\rangle^2\langle 145\rangle^2\langle 234\rangle\langle 247\rangle}{\langle 124\rangle^2\langle 134\rangle\langle 135\rangle\langle 146\rangle\langle 156\rangle\langle 237\rangle^2}$	$20, 21$
67	4444422000	$\frac{\langle 126\rangle^2\langle 145\rangle^2\langle 235\rangle\langle 237\rangle\langle 345\rangle\langle 347\rangle}{\langle 124\rangle\langle 125\rangle\langle 146\rangle\langle 156\rangle\langle 234\rangle^2\langle 357\rangle^2}$	$23, 24$
68	4444422000	$\frac{\langle 124\rangle\langle 126\rangle\langle 145\rangle\langle 156\rangle\langle 235\rangle^2\langle 347\rangle^2}{\langle 125\rangle^2\langle 146\rangle^2\langle 234\rangle\langle 237\rangle\langle 345\rangle\langle 357\rangle}$	$22, 23$
69	4211111100	$\frac{-\langle 124\rangle\langle 128\rangle\langle 137\rangle\langle 156\rangle}{\langle 123\rangle\langle 125\rangle\langle 146\rangle\langle 178\rangle}$	25
70	3311111100	$\frac{\langle 123\rangle\langle 128\rangle\langle 157\rangle\langle 246\rangle}{\langle 124\rangle\langle 127\rangle\langle 135\rangle\langle 268\rangle}$	26
71	3221111100	$\frac{-\langle 125\rangle\langle 138\rangle\langle 167\rangle\langle 234\rangle}{\langle 127\rangle\langle 134\rangle\langle 156\rangle\langle 238\rangle}$	27
72	3221111100	$\frac{-\langle 126\rangle\langle 138\rangle\langle 157\rangle\langle 234\rangle}{\langle 123\rangle\langle 127\rangle\langle 156\rangle\langle 348\rangle}$	27
73	2222111100	$\frac{-\langle 126\rangle\langle 145\rangle\langle 234\rangle\langle 378\rangle}{\langle 124\rangle\langle 156\rangle\langle 237\rangle\langle 348\rangle}$	28
74	2222111100	$\frac{\langle 126\rangle\langle 145\rangle\langle 238\rangle\langle 347\rangle}{\langle 125\rangle\langle 146\rangle\langle 237\rangle\langle 348\rangle}$	28
75	2222111100	$\frac{\langle 128\rangle\langle 137\rangle\langle 246\rangle\langle 345\rangle}{\langle 124\rangle\langle 134\rangle\langle 268\rangle\langle 357\rangle}$	29
76	2222111100	$\frac{\langle 126\rangle\langle 137\rangle\langle 248\rangle\langle 345\rangle}{\langle 124\rangle\langle 135\rangle\langle 268\rangle\langle 347\rangle}$	29
77	8422222200	$\frac{\langle 124\rangle\langle 125\rangle^2\langle 127\rangle\langle 134\rangle\langle 137\rangle\langle 168\rangle^2}{\langle 123\rangle^2\langle 126\rangle\langle 128\rangle\langle 147\rangle^2\langle 156\rangle\langle 158\rangle}$	$25, 25$
78	6622222200	$\frac{\langle 123\rangle^2\langle 126\rangle\langle 128\rangle\langle 157\rangle^2\langle 246\rangle\langle 248\rangle}{\langle 124\rangle^2\langle 125\rangle\langle 127\rangle\langle 135\rangle\langle 137\rangle\langle 268\rangle^2}$	$26, 26$
79	6442222200	$\frac{-\langle 126\rangle^2\langle 134\rangle\langle 138\rangle\langle 157\rangle^2\langle 234\rangle\langle 238\rangle}{\langle 123\rangle^2\langle 125\rangle\langle 127\rangle\langle 156\rangle\langle 167\rangle\langle 348\rangle^2}$	$27, 27$

80	6442222200	$\frac{\langle 125\rangle\langle 126\rangle\langle 138\rangle^2\langle 157\rangle\langle 167\rangle\langle 234\rangle^2}{\langle 123\rangle\langle 127\rangle^2\langle 134\rangle\langle 156\rangle^2\langle 238\rangle\langle 348\rangle}$	$27, 27$
81	4444222200	$\frac{\langle 125\rangle\langle 126\rangle\langle 145\rangle\langle 146\rangle\langle 234\rangle^2\langle 378\rangle^2}{\langle 124\rangle^2\langle 156\rangle^2\langle 237\rangle\langle 238\rangle\langle 347\rangle\langle 348\rangle}$	$28, 28$
82	4444222200	$\frac{\langle 126\rangle\langle 128\rangle\langle 137\rangle^2\langle 246\rangle\langle 248\rangle\langle 345\rangle^2}{\langle 124\rangle^2\langle 134\rangle\langle 135\rangle\langle 268\rangle^2\langle 347\rangle\langle 357\rangle}$	$29, 29$
83	4444222200	$\frac{\langle 126\rangle^2\langle 134\rangle\langle 137\rangle\langle 248\rangle^2\langle 345\rangle\langle 357\rangle}{\langle 124\rangle\langle 128\rangle\langle 135\rangle^2\langle 246\rangle\langle 268\rangle\langle 347\rangle^2}$	$28, 29$
84	4111111110	$\frac{-\langle 125\rangle\langle 137\rangle\langle 149\rangle\langle 168\rangle}{\langle 128\rangle\langle 134\rangle\langle 156\rangle\langle 179\rangle}$	30
85	3211111110	$\frac{\langle 128\rangle\langle 139\rangle\langle 145\rangle\langle 267\rangle}{\langle 127\rangle\langle 135\rangle\langle 149\rangle\langle 268\rangle}$	31
86	2221111110	$\frac{\langle 129\rangle\langle 146\rangle\langle 235\rangle\langle 378\rangle}{\langle 126\rangle\langle 149\rangle\langle 237\rangle\langle 358\rangle}$	32
87	8222222220	$\frac{\langle 125\rangle^2\langle 137\rangle\langle 139\rangle\langle 147\rangle\langle 149\rangle\langle 168\rangle^2}{\langle 126\rangle\langle 128\rangle\langle 134\rangle^2\langle 156\rangle\langle 158\rangle\langle 179\rangle^2}$	$30, 30$
88	6422222220	$\frac{-\langle 126\rangle\langle 128\rangle\langle 139\rangle^2\langle 145\rangle^2\langle 267\rangle\langle 278\rangle}{\langle 127\rangle^2\langle 134\rangle\langle 135\rangle\langle 149\rangle\langle 159\rangle\langle 268\rangle^2}$	$31, 31$
89	4442222220	$\frac{-\langle 124\rangle\langle 129\rangle\langle 146\rangle\langle 169\rangle\langle 235\rangle^2\langle 378\rangle^2}{\langle 126\rangle^2\langle 149\rangle^2\langle 237\rangle\langle 238\rangle\langle 357\rangle\langle 358\rangle}$	$32, 32$
90	2211111111	$\frac{-\langle 139\rangle\langle 145\rangle\langle 2{,}6{,}10\rangle\langle 278\rangle}{\langle 134\rangle\langle 159\rangle\langle 268\rangle\langle 2{,}7{,}10\rangle}$	33
91	4422222222	$\frac{-\langle 139\rangle^2\langle 145\rangle^2\langle 267\rangle\langle 2{,}6{,}10\rangle\langle 278\rangle\langle 2{,}8{,}10\rangle}{\langle 134\rangle\langle 135\rangle\langle 149\rangle\langle 159\rangle\langle 268\rangle^2\langle 2{,}7{,}10\rangle^2}$	$33, 33$

Table A.4: Exceptional bracket cross-ratios on $\mathbb{P}^2$

A.3 Cross-ratios on $\mathbb{P}^3$

The following table lists special polynomials for $S_{8,4}$ (see Theorem 3.11 in Chapter 3).

i	Rank	Degree	Polynomial $\pi_i^{(3)}$
0	9	11110000	$\langle 1234\rangle$
1	2	32222100	$\langle 1236\rangle\langle 1245\rangle\langle 1345\rangle + \langle 1234\rangle\langle 1235\rangle\langle 1456\rangle$
2	3	22222200	$\langle 1235\rangle\langle 1346\rangle\langle 2456\rangle - \langle 1234\rangle\langle 1256\rangle\langle 3456\rangle$
3	2	44222200	$\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle - \langle 1234\rangle^2\langle 1256\rangle^2$
4	2	44222200	$\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle + \langle 1234\rangle^2\langle 1256\rangle^2$
5	2	43322200	$\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1346\rangle + \langle 1234\rangle^2\langle 1256\rangle\langle 1356\rangle$
6	2	33332200	$\langle 1235\rangle\langle 1246\rangle\langle 1345\rangle\langle 2346\rangle - \langle 1234\rangle^2\langle 1256\rangle\langle 3456\rangle$
7	4	21111110	$\langle 1246\rangle\langle 1357\rangle - \langle 1235\rangle\langle 1467\rangle$
8	2	33211110	$\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle + \langle 1234\rangle\langle 1235\rangle\langle 1267\rangle$
9	2	32221110	$\langle 1236\rangle\langle 1247\rangle\langle 1345\rangle + \langle 1234\rangle\langle 1237\rangle\langle 1456\rangle$
10	2	22222110	$\langle 1234\rangle\langle 1357\rangle\langle 2456\rangle + \langle 1237\rangle\langle 1245\rangle\langle 3456\rangle$
11	2	44222110	$\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1247\rangle + \langle 1234\rangle^2\langle 1256\rangle\langle 1257\rangle$
12	2	43322110	$\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1347\rangle + \langle 1234\rangle^2\langle 1256\rangle\langle 1357\rangle$
13	2	43322110	$\langle 1235\rangle^2\langle 1246\rangle\langle 1347\rangle - \langle 1234\rangle^2\langle 1256\rangle\langle 1357\rangle$
14	2	33332110	$\langle 1235\rangle\langle 1246\rangle\langle 1345\rangle\langle 2347\rangle - \langle 1234\rangle^2\langle 1256\rangle\langle 3457\rangle$
15	3	43222210	$\langle 1237\rangle\langle 1246\rangle\langle 1256\rangle\langle 1345\rangle - \langle 1234\rangle\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle$
16	2	43222210	$\langle 1237\rangle\langle 1245\rangle\langle 1256\rangle\langle 1346\rangle + \langle 1234\rangle\langle 1246\rangle\langle 1257\rangle\langle 1356\rangle$
17	2	43222210	$\langle 1236\rangle\langle 1247\rangle\langle 1256\rangle\langle 1345\rangle + \langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1456\rangle$
18	2	33322210	$\langle 1235\rangle\langle 1245\rangle\langle 1367\rangle\langle 2346\rangle - \langle 1234\rangle\langle 1236\rangle\langle 1357\rangle\langle 2456\rangle$
19	2	33322210	$\langle 1237\rangle\langle 1245\rangle\langle 1356\rangle\langle 2346\rangle - \langle 1234\rangle\langle 1236\rangle\langle 1357\rangle\langle 2456\rangle$
20	5	42222220	$\langle 1234\rangle\langle 1256\rangle\langle 1357\rangle\langle 1467\rangle - \langle 1235\rangle\langle 1246\rangle\langle 1347\rangle\langle 1567\rangle$
21	2	42222220	$\langle 1246\rangle\langle 1257\rangle\langle 1347\rangle\langle 1356\rangle + \langle 1234\rangle\langle 1235\rangle\langle 1467\rangle\langle 1567\rangle$
22	2	42222220	$\langle 1234\rangle\langle 1257\rangle\langle 1367\rangle\langle 1456\rangle + \langle 1237\rangle\langle 1246\rangle\langle 1345\rangle\langle 1567\rangle$
23	2	42222220	$\langle 1234\rangle\langle 1257\rangle\langle 1367\rangle\langle 1456\rangle + \langle 1236\rangle\langle 1245\rangle\langle 1357\rangle\langle 1467\rangle$
24	2	33222220	$\langle 1237\rangle\langle 1247\rangle\langle 1456\rangle\langle 2356\rangle - \langle 1236\rangle\langle 1246\rangle\langle 1457\rangle\langle 2357\rangle$
25	2	33222220	$\langle 1237\rangle\langle 1245\rangle\langle 1567\rangle\langle 2346\rangle - \langle 1234\rangle\langle 1257\rangle\langle 1367\rangle\langle 2456\rangle$
26	2	33222220	$\langle 1246\rangle\langle 1257\rangle\langle 1347\rangle\langle 2356\rangle + \langle 1234\rangle\langle 1235\rangle\langle 1467\rangle\langle 2567\rangle$
27	3	33111111	$\langle 1235\rangle\langle 1247\rangle\langle 1268\rangle - \langle 1234\rangle\langle 1256\rangle\langle 1278\rangle$
28	2	32211111	$\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle - \langle 1235\rangle\langle 1268\rangle\langle 1347\rangle$
29	2	22221111	$\langle 1235\rangle\langle 1467\rangle\langle 2348\rangle - \langle 1246\rangle\langle 1347\rangle\langle 2358\rangle$
30	2	44221111	$\langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1248\rangle - \langle 1234\rangle^2\langle 1256\rangle\langle 1278\rangle$

31	2	43321111	$\langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1348\rangle - \langle 1234\rangle^2\langle 1256\rangle\langle 1378\rangle$
32	2	33331111	$\langle 1235\rangle\langle 1246\rangle\langle 1347\rangle\langle 2348\rangle - \langle 1234\rangle^2\langle 1256\rangle\langle 3478\rangle$
33	2	43222111	$\langle 1238\rangle\langle 1246\rangle\langle 1257\rangle\langle 1345\rangle - \langle 1234\rangle\langle 1235\rangle\langle 1268\rangle\langle 1457\rangle$
34	2	43222111	$\langle 1234\rangle\langle 1257\rangle\langle 1268\rangle\langle 1345\rangle - \langle 1235\rangle\langle 1238\rangle\langle 1246\rangle\langle 1457\rangle$
35	2	33322111	$\langle 1237\rangle\langle 1258\rangle\langle 1345\rangle\langle 2346\rangle - \langle 1234\rangle\langle 1235\rangle\langle 1278\rangle\langle 3456\rangle$
36	2	33322111	$\langle 1234\rangle\langle 1268\rangle\langle 1345\rangle\langle 2357\rangle - \langle 1235\rangle\langle 1236\rangle\langle 1248\rangle\langle 3457\rangle$
37	2	42222211	$\langle 1235\rangle\langle 1246\rangle\langle 1367\rangle\langle 1458\rangle + \langle 1236\rangle\langle 1258\rangle\langle 1345\rangle\langle 1467\rangle$
38	2	42222211	$\langle 1246\rangle\langle 1258\rangle\langle 1347\rangle\langle 1356\rangle - \langle 1247\rangle\langle 1256\rangle\langle 1346\rangle\langle 1358\rangle$
39	2	42222211	$\langle 1246\rangle\langle 1258\rangle\langle 1347\rangle\langle 1356\rangle + \langle 1234\rangle\langle 1235\rangle\langle 1467\rangle\langle 1568\rangle$
40	2	33222211	$\langle 1237\rangle\langle 1268\rangle\langle 1456\rangle\langle 2345\rangle - \langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle$
41	2	33222211	$\langle 1236\rangle\langle 1248\rangle\langle 1567\rangle\langle 2345\rangle - \langle 1234\rangle\langle 1267\rangle\langle 1356\rangle\langle 2458\rangle$
42	2	33222211	$\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle + \langle 1236\rangle\langle 1245\rangle\langle 1278\rangle\langle 3456\rangle$
43	2	33222211	$\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle + \langle 1234\rangle\langle 1256\rangle\langle 1358\rangle\langle 2467\rangle$
44	2	33222211	$\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle - \langle 1234\rangle\langle 1246\rangle\langle 1358\rangle\langle 2567\rangle$
45	2	33222211	$\langle 1236\rangle\langle 1247\rangle\langle 1358\rangle\langle 2456\rangle + \langle 1238\rangle\langle 1245\rangle\langle 1356\rangle\langle 2467\rangle$
46	2	32222221	$\langle 1237\rangle\langle 1467\rangle\langle 1568\rangle\langle 2345\rangle - \langle 1235\rangle\langle 1456\rangle\langle 1678\rangle\langle 2347\rangle$
47	2	32222221	$\langle 1235\rangle\langle 1456\rangle\langle 1678\rangle\langle 2347\rangle + \langle 1234\rangle\langle 1468\rangle\langle 1567\rangle\langle 2357\rangle$
48	2	32222221	$\langle 1235\rangle\langle 1456\rangle\langle 1678\rangle\langle 2347\rangle - \langle 1234\rangle\langle 1467\rangle\langle 1568\rangle\langle 2357\rangle$
49	2	22222222	$\langle 1237\rangle\langle 1346\rangle\langle 2578\rangle\langle 4568\rangle + \langle 1236\rangle\langle 1347\rangle\langle 2458\rangle\langle 5678\rangle$
50	2	22222222	$\langle 1234\rangle\langle 1367\rangle\langle 2568\rangle\langle 4578\rangle - \langle 1236\rangle\langle 1347\rangle\langle 2458\rangle\langle 5678\rangle$
51	2	22222222	$\langle 1245\rangle\langle 1378\rangle\langle 2468\rangle\langle 3567\rangle + \langle 1246\rangle\langle 1357\rangle\langle 2458\rangle\langle 3678\rangle$

Table A.5: $S_{8,4}$-special polynomials in $\mathcal{P}_{8,4}$

The following table lists exceptional cross-ratios for $S_{8,4}$ (see Theorem 3.12 in Chapter 3).

i	Degree	r_i	$j : \pi_j^{(3)} \mid 1 - r_i$
1	22111100	$\frac{\langle 1235\rangle\langle 1246\rangle}{\langle 1236\rangle\langle 1245\rangle}$	$0, 0$
2	32222100	$\frac{-\langle 1236\rangle\langle 1245\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1456\rangle}$	1
3	22222200	$\frac{\langle 1256\rangle\langle 1345\rangle\langle 2346\rangle}{\langle 1245\rangle\langle 1346\rangle\langle 2356\rangle}$	2
4	44222200	$\frac{\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle}{\langle 1234\rangle^2\langle 1256\rangle^2}$	3
5	44222200	$\frac{-\langle 1236\rangle^2\langle 1245\rangle^2}{\langle 1234\rangle\langle 1235\rangle\langle 1246\rangle\langle 1256\rangle}$	4
6	43322200	$\frac{\langle 1235\rangle^2\langle 1246\rangle\langle 1346\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 1356\rangle}$	$0, 1$
7	43322200	$\frac{\langle 1234\rangle\langle 1236\rangle\langle 1256\rangle\langle 1345\rangle}{\langle 1235\rangle^2\langle 1246\rangle\langle 1346\rangle}$	5
8	33332200	$\frac{\langle 1235\rangle\langle 1246\rangle\langle 1345\rangle\langle 2346\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 3456\rangle}$	6

9	33332200	$\frac{-\langle 1236\rangle\langle 1245\rangle\langle 1345\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 1256\rangle\langle 1346\rangle\langle 2345\rangle}$	6
10	33332200	$\frac{\langle 1236\rangle\langle 1245\rangle\langle 1346\rangle\langle 2345\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 1345\rangle\langle 2346\rangle}$	$0,2$
11	75544300	$\frac{-\langle 1235\rangle^3\langle 1246\rangle^2\langle 1345\rangle\langle 1346\rangle}{\langle 1234\rangle^3\langle 1245\rangle\langle 1256\rangle\langle 1356\rangle^2}$	$1,5$
12	44444400	$\frac{\langle 1234\rangle\langle 1256\rangle^2\langle 1345\rangle\langle 2346\rangle\langle 3456\rangle}{\langle 1235\rangle\langle 1245\rangle\langle 1346\rangle^2\langle 2356\rangle\langle 2456\rangle}$	$2,2$
13	55554400	$\frac{-\langle 1236\rangle^2\langle 1245\rangle^2\langle 1346\rangle\langle 2345\rangle\langle 3456\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 1256\rangle\langle 1345\rangle^2\langle 2346\rangle^2}$	$2,6$
14	66664400	$\frac{-\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle\langle 1345\rangle^2\langle 2346\rangle^2}{\langle 1234\rangle^3\langle 1256\rangle^2\langle 1346\rangle\langle 2345\rangle\langle 3456\rangle}$	$6,6$
15	21111110	$\frac{-\langle 1267\rangle\langle 1345\rangle}{\langle 1247\rangle\langle 1356\rangle}$	7
16	33211110	$\frac{-\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1267\rangle}$	8
17	32221110	$\frac{-\langle 1236\rangle\langle 1247\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1237\rangle\langle 1456\rangle}$	9
18	32221110	$\frac{\langle 1236\rangle\langle 1245\rangle\langle 1347\rangle}{\langle 1237\rangle\langle 1246\rangle\langle 1345\rangle}$	$0,7$
19	22222110	$\frac{-\langle 1237\rangle\langle 1245\rangle\langle 3456\rangle}{\langle 1234\rangle\langle 1357\rangle\langle 2456\rangle}$	10
20	44222110	$\frac{\langle 1235\rangle^2\langle 1246\rangle\langle 1247\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 1257\rangle}$	$0,8$
21	44222110	$\frac{-\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1247\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 1257\rangle}$	11
22	43322110	$\frac{-\langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1236\rangle\langle 1245\rangle\langle 1357\rangle}$	13
23	43322110	$\frac{\langle 1235\rangle^2\langle 1246\rangle\langle 1347\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 1357\rangle}$	13
24	43322110	$\frac{\langle 1234\rangle\langle 1237\rangle\langle 1256\rangle\langle 1345\rangle}{\langle 1235\rangle^2\langle 1246\rangle\langle 1347\rangle}$	12
25	43322110	$\frac{-\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1246\rangle\langle 1357\rangle}$	12
26	43322110	$\frac{\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle\langle 1345\rangle}{\langle 1234\rangle^2\langle 1257\rangle\langle 1356\rangle}$	$0,9$
27	33332110	$\frac{\langle 1235\rangle\langle 1246\rangle\langle 1345\rangle\langle 2347\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 3457\rangle}$	14
28	33332110	$\frac{\langle 1237\rangle\langle 1245\rangle\langle 1345\rangle\langle 2346\rangle}{\langle 1235\rangle\langle 1247\rangle\langle 1346\rangle\langle 2345\rangle}$	$0,10$
29	33332110	$\frac{-\langle 1236\rangle\langle 1245\rangle\langle 1345\rangle\langle 2347\rangle}{\langle 1234\rangle\langle 1256\rangle\langle 1347\rangle\langle 2345\rangle}$	14
30	43222210	$\frac{\langle 1237\rangle\langle 1246\rangle\langle 1256\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle}$	15
31	43222210	$\frac{-\langle 1236\rangle\langle 1247\rangle\langle 1256\rangle\langle 1345\rangle}{\langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1456\rangle}$	17
32	43222210	$\frac{-\langle 1237\rangle\langle 1245\rangle\langle 1256\rangle\langle 1346\rangle}{\langle 1234\rangle\langle 1246\rangle\langle 1257\rangle\langle 1356\rangle}$	16
33	43222210	$\frac{\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle\langle 1456\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 1247\rangle\langle 1356\rangle}$	16
34	33322210	$\frac{\langle 1235\rangle\langle 1245\rangle\langle 1367\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 1236\rangle\langle 1357\rangle\langle 2456\rangle}$	18
35	33322210	$\frac{\langle 1237\rangle\langle 1245\rangle\langle 1356\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 1246\rangle\langle 1357\rangle\langle 2345\rangle}$	18
36	33322210	$\frac{\langle 1235\rangle\langle 1246\rangle\langle 1367\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 1237\rangle\langle 1356\rangle\langle 2456\rangle}$	18
37	33322210	$\frac{\langle 1237\rangle\langle 1245\rangle\langle 1356\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 1236\rangle\langle 1357\rangle\langle 2456\rangle}$	19
38	33322210	$\frac{\langle 1235\rangle\langle 1245\rangle\langle 1367\rangle\langle 2346\rangle}{\langle 1236\rangle\langle 1246\rangle\langle 1357\rangle\langle 2345\rangle}$	19

39	42222220	$\frac{\langle 1237\rangle\langle 1267\rangle\langle 1345\rangle\langle 1456\rangle}{\langle 1247\rangle\langle 1257\rangle\langle 1346\rangle\langle 1356\rangle}$	$7,7$
40	42222220	$\frac{-\langle 1234\rangle\langle 1235\rangle\langle 1467\rangle\langle 1567\rangle}{\langle 1246\rangle\langle 1257\rangle\langle 1347\rangle\langle 1356\rangle}$	21
41	42222220	$\frac{\langle 1246\rangle\langle 1267\rangle\langle 1345\rangle\langle 1357\rangle}{\langle 1245\rangle\langle 1257\rangle\langle 1346\rangle\langle 1367\rangle}$	22
42	42222220	$\frac{\langle 1246\rangle\langle 1257\rangle\langle 1347\rangle\langle 1356\rangle}{\langle 1247\rangle\langle 1256\rangle\langle 1346\rangle\langle 1357\rangle}$	20
43	42222220	$\frac{-\langle 1236\rangle\langle 1245\rangle\langle 1357\rangle\langle 1467\rangle}{\langle 1234\rangle\langle 1257\rangle\langle 1367\rangle\langle 1456\rangle}$	23
44	42222220	$\frac{-\langle 1237\rangle\langle 1246\rangle\langle 1345\rangle\langle 1567\rangle}{\langle 1234\rangle\langle 1257\rangle\langle 1367\rangle\langle 1456\rangle}$	22
45	33222220	$\frac{\langle 1236\rangle\langle 1246\rangle\langle 1457\rangle\langle 2357\rangle}{\langle 1237\rangle\langle 1247\rangle\langle 1456\rangle\langle 2356\rangle}$	24
46	33222220	$\frac{-\langle 1236\rangle\langle 1245\rangle\langle 1467\rangle\langle 2357\rangle}{\langle 1235\rangle\langle 1247\rangle\langle 1456\rangle\langle 2367\rangle}$	24
47	33222220	$\frac{-\langle 1234\rangle\langle 1256\rangle\langle 1467\rangle\langle 2357\rangle}{\langle 1247\rangle\langle 1257\rangle\langle 1346\rangle\langle 2356\rangle}$	26
48	33222220	$\frac{\langle 1237\rangle\langle 1245\rangle\langle 1567\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 1257\rangle\langle 1367\rangle\langle 2456\rangle}$	25
49	53332220	$\frac{\langle 1236\rangle\langle 1247\rangle\langle 1267\rangle\langle 1345\rangle^2}{\langle 1237\rangle^2\langle 1245\rangle\langle 1346\rangle\langle 1456\rangle}$	$7,9$
50	64433220	$\frac{\langle 1237\rangle^2\langle 1246\rangle\langle 1256\rangle\langle 1345\rangle^2}{\langle 1236\rangle^2\langle 1245\rangle^2\langle 1347\rangle\langle 1357\rangle}$	$7,13$
51	77433220	$\frac{-\langle 1235\rangle^3\langle 1236\rangle\langle 1246\rangle\langle 1247\rangle^2}{\langle 1234\rangle^3\langle 1237\rangle\langle 1256\rangle^2\langle 1257\rangle}$	$8,11$
52	75543220	$\frac{\langle 1236\rangle^2\langle 1237\rangle\langle 1245\rangle^2\langle 1345\rangle\langle 1347\rangle}{\langle 1234\rangle^3\langle 1246\rangle\langle 1256\rangle\langle 1357\rangle^2}$	$9,12$
53	44444220	$\frac{-\langle 1237\rangle^2\langle 1245\rangle\langle 1456\rangle\langle 2345\rangle\langle 3456\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1347\rangle\langle 1357\rangle\langle 2456\rangle^2}$	$10,10$
54	86644220	$\frac{-\langle 1234\rangle^3\langle 1237\rangle\langle 1256\rangle^2\langle 1345\rangle\langle 1357\rangle}{\langle 1235\rangle^3\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle\langle 1347\rangle^2}$	$12,13$
55	55554220	$\frac{\langle 1237\rangle\langle 1245\rangle\langle 1257\rangle\langle 1345\rangle^2\langle 2346\rangle^2}{\langle 1235\rangle^2\langle 1247\rangle^2\langle 1346\rangle\langle 2345\rangle\langle 3456\rangle}$	$10,14$
56	66664220	$\frac{-\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle\langle 1345\rangle^2\langle 2347\rangle^2}{\langle 1234\rangle^3\langle 1256\rangle^2\langle 1347\rangle\langle 2345\rangle\langle 3457\rangle}$	$14,14$
57	64333320	$\frac{\langle 1237\rangle\langle 1247\rangle\langle 1256\rangle^2\langle 1345\rangle\langle 1346\rangle}{\langle 1234\rangle^2\langle 1257\rangle\langle 1267\rangle\langle 1356\rangle\langle 1456\rangle}$	$7,15$
58	86444420	$\frac{\langle 1235\rangle\langle 1237\rangle\langle 1245\rangle\langle 1247\rangle\langle 1256\rangle^2\langle 1346\rangle^2}{\langle 1234\rangle^2\langle 1236\rangle\langle 1246\rangle\langle 1257\rangle^2\langle 1356\rangle\langle 1456\rangle}$	$15,16$
59	86444420	$\frac{\langle 1234\rangle\langle 1236\rangle\langle 1247\rangle\langle 1256\rangle^2\langle 1267\rangle\langle 1345\rangle^2}{\langle 1235\rangle\langle 1237\rangle^2\langle 1245\rangle\langle 1246\rangle^2\langle 1356\rangle\langle 1456\rangle}$	$16,17$
60	66644420	$\frac{\langle 1235\rangle\langle 1237\rangle\langle 1245\rangle^2\langle 1356\rangle\langle 1367\rangle\langle 2346\rangle^2}{\langle 1234\rangle\langle 1236\rangle^2\langle 1246\rangle\langle 1357\rangle^2\langle 2345\rangle\langle 2456\rangle}$	$18,19$
61	66644420	$\frac{\langle 1235\rangle^2\langle 1245\rangle\langle 1246\rangle\langle 1367\rangle^2\langle 2345\rangle\langle 2346\rangle}{\langle 1234\rangle^2\langle 1236\rangle\langle 1237\rangle\langle 1356\rangle\langle 1357\rangle\langle 2456\rangle^2}$	$18,18$
62	66644420	$\frac{\langle 1237\rangle^2\langle 1245\rangle\langle 1246\rangle\langle 1356\rangle^2\langle 2345\rangle\langle 2346\rangle}{\langle 1234\rangle^2\langle 1235\rangle\langle 1236\rangle\langle 1357\rangle\langle 1367\rangle\langle 2456\rangle^2}$	$19,19$
63	66644420	$\frac{\langle 1234\rangle\langle 1237\rangle^2\langle 1245\rangle\langle 1356\rangle^2\langle 2346\rangle\langle 2456\rangle}{\langle 1235\rangle\langle 1236\rangle\langle 1246\rangle^2\langle 1357\rangle\langle 1367\rangle\langle 2345\rangle^2}$	$18,18$
64	63333330	$\frac{\langle 1237\rangle\langle 1246\rangle\langle 1257\rangle\langle 1345\rangle\langle 1356\rangle\langle 1467\rangle}{\langle 1235\rangle\langle 1247\rangle\langle 1267\rangle\langle 1346\rangle\langle 1357\rangle\langle 1456\rangle}$	$7,20$
65	63333330	$\frac{\langle 1235\rangle\langle 1236\rangle\langle 1246\rangle\langle 1357\rangle\langle 1457\rangle\langle 1467\rangle}{\langle 1234\rangle\langle 1237\rangle\langle 1257\rangle\langle 1346\rangle\langle 1456\rangle\langle 1567\rangle}$	$7,23$
66	84444440	$\frac{\langle 1237\rangle^2\langle 1245\rangle\langle 1246\rangle\langle 1345\rangle\langle 1346\rangle\langle 1567\rangle^2}{\langle 1234\rangle^2\langle 1257\rangle\langle 1267\rangle\langle 1357\rangle\langle 1367\rangle\langle 1456\rangle^2}$	$20,22$
67	84444440	$\frac{-\langle 1237\rangle\langle 1246\rangle^2\langle 1267\rangle\langle 1345\rangle^2\langle 1357\rangle\langle 1567\rangle}{\langle 1234\rangle\langle 1245\rangle\langle 1257\rangle^2\langle 1346\rangle\langle 1367\rangle^2\langle 1456\rangle}$	$21,22$
68	84444440	$\frac{\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle^2\langle 1346\rangle\langle 1357\rangle\langle 1467\rangle\langle 1567\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1246\rangle\langle 1257\rangle\langle 1367\rangle^2\langle 1456\rangle\langle 1457\rangle}$	$20,23$

69	66444440	$\frac{\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle^2\langle 1467\rangle^2\langle 2356\rangle\langle 2357\rangle}{\langle 1235\rangle^2\langle 1246\rangle\langle 1247\rangle\langle 1456\rangle\langle 1457\rangle\langle 2367\rangle^2}$	$24, 25$
70	66444440	$\frac{-\langle 1236\rangle^2\langle 1245\rangle\langle 1246\rangle\langle 1457\rangle\langle 1467\rangle\langle 2357\rangle^2}{\langle 1235\rangle\langle 1237\rangle\langle 1247\rangle^2\langle 1456\rangle^2\langle 2356\rangle\langle 2367\rangle}$	$24, 26$
71	33111111	$\frac{-\langle 1237\rangle\langle 1245\rangle\langle 1268\rangle}{\langle 1234\rangle\langle 1258\rangle\langle 1267\rangle}$	27
72	32211111	$\frac{\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle}{\langle 1235\rangle\langle 1268\rangle\langle 1347\rangle}$	28
73	22221111	$\frac{\langle 1235\rangle\langle 1467\rangle\langle 2348\rangle}{\langle 1246\rangle\langle 1347\rangle\langle 2358\rangle}$	29
74	44221111	$\frac{-\langle 1234\rangle\langle 1236\rangle\langle 1245\rangle\langle 1278\rangle}{\langle 1235\rangle\langle 1238\rangle\langle 1246\rangle\langle 1247\rangle}$	30
75	44221111	$\frac{\langle 1234\rangle^2\langle 1256\rangle\langle 1278\rangle}{\langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1248\rangle}$	30
76	44221111	$\frac{\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle\langle 1248\rangle}{\langle 1235\rangle\langle 1238\rangle\langle 1246\rangle\langle 1247\rangle}$	$0, 27$
77	43321111	$\frac{\langle 1236\rangle\langle 1237\rangle\langle 1248\rangle\langle 1345\rangle}{\langle 1235\rangle\langle 1238\rangle\langle 1247\rangle\langle 1346\rangle}$	$0, 28$
78	43321111	$\frac{\langle 1235\rangle\langle 1237\rangle\langle 1246\rangle\langle 1348\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 1378\rangle}$	31
79	43321111	$\frac{-\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle\langle 1348\rangle}{\langle 1234\rangle\langle 1238\rangle\langle 1256\rangle\langle 1347\rangle}$	31
80	33331111	$\frac{\langle 1235\rangle\langle 1246\rangle\langle 1347\rangle\langle 2348\rangle}{\langle 1234\rangle^2\langle 1256\rangle\langle 3478\rangle}$	32
81	33331111	$\frac{-\langle 1236\rangle\langle 1245\rangle\langle 1347\rangle\langle 2348\rangle}{\langle 1234\rangle\langle 1256\rangle\langle 1348\rangle\langle 2347\rangle}$	32
82	33331111	$\frac{\langle 1235\rangle\langle 1247\rangle\langle 1346\rangle\langle 2348\rangle}{\langle 1238\rangle\langle 1246\rangle\langle 1347\rangle\langle 2345\rangle}$	$0, 29$
83	43222111	$\frac{\langle 1238\rangle\langle 1245\rangle\langle 1246\rangle\langle 1357\rangle}{\langle 1235\rangle\langle 1236\rangle\langle 1248\rangle\langle 1457\rangle}$	33
84	43222111	$\frac{\langle 1236\rangle\langle 1248\rangle\langle 1257\rangle\langle 1345\rangle}{\langle 1238\rangle\langle 1245\rangle\langle 1246\rangle\langle 1357\rangle}$	34
85	43222111	$\frac{\langle 1238\rangle\langle 1246\rangle\langle 1257\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1268\rangle\langle 1457\rangle}$	33
86	43222111	$\frac{\langle 1234\rangle\langle 1257\rangle\langle 1268\rangle\langle 1345\rangle}{\langle 1235\rangle\langle 1238\rangle\langle 1246\rangle\langle 1457\rangle}$	34
87	43222111	$\frac{\langle 1234\rangle\langle 1245\rangle\langle 1268\rangle\langle 1357\rangle}{\langle 1235\rangle\langle 1236\rangle\langle 1248\rangle\langle 1457\rangle}$	34
88	33322111	$\frac{-\langle 1238\rangle\langle 1257\rangle\langle 1345\rangle\langle 2346\rangle}{\langle 1235\rangle\langle 1278\rangle\langle 1346\rangle\langle 2345\rangle}$	35
89	33322111	$\frac{\langle 1238\rangle\langle 1246\rangle\langle 1345\rangle\langle 2357\rangle}{\langle 1236\rangle\langle 1248\rangle\langle 1357\rangle\langle 2345\rangle}$	36
90	33322111	$\frac{-\langle 1237\rangle\langle 1258\rangle\langle 1346\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 1238\rangle\langle 1257\rangle\langle 3456\rangle}$	35
91	33322111	$\frac{\langle 1237\rangle\langle 1258\rangle\langle 1345\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1278\rangle\langle 3456\rangle}$	35
92	33322111	$\frac{\langle 1234\rangle\langle 1268\rangle\langle 1345\rangle\langle 2357\rangle}{\langle 1235\rangle\langle 1236\rangle\langle 1248\rangle\langle 3457\rangle}$	36
93	42222211	$\frac{\langle 1246\rangle\langle 1258\rangle\langle 1347\rangle\langle 1356\rangle}{\langle 1247\rangle\langle 1256\rangle\langle 1346\rangle\langle 1358\rangle}$	38
94	42222211	$\frac{\langle 1258\rangle\langle 1267\rangle\langle 1345\rangle\langle 1346\rangle}{\langle 1245\rangle\langle 1246\rangle\langle 1358\rangle\langle 1367\rangle}$	37
95	42222211	$\frac{-\langle 1247\rangle\langle 1258\rangle\langle 1346\rangle\langle 1356\rangle}{\langle 1234\rangle\langle 1256\rangle\langle 1358\rangle\langle 1467\rangle}$	39
96	42222211	$\frac{-\langle 1236\rangle\langle 1258\rangle\langle 1345\rangle\langle 1467\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 1367\rangle\langle 1458\rangle}$	37
97	33222211	$\frac{\langle 1245\rangle\langle 1267\rangle\langle 1356\rangle\langle 2348\rangle}{\langle 1248\rangle\langle 1256\rangle\langle 1367\rangle\langle 2345\rangle}$	41
98	33222211	$\frac{-\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle}{\langle 1236\rangle\langle 1245\rangle\langle 1278\rangle\langle 3456\rangle}$	42

99	33222211	$\frac{\langle 1236\rangle\langle 1278\rangle\langle 1345\rangle\langle 2456\rangle}{\langle 1237\rangle\langle 1245\rangle\langle 1268\rangle\langle 3456\rangle}$	40
100	33222211	$\frac{-\langle 1238\rangle\langle 1245\rangle\langle 1356\rangle\langle 2467\rangle}{\langle 1236\rangle\langle 1247\rangle\langle 1358\rangle\langle 2456\rangle}$	45
101	33222211	$\frac{\langle 1238\rangle\langle 1246\rangle\langle 1345\rangle\langle 2567\rangle}{\langle 1235\rangle\langle 1256\rangle\langle 1348\rangle\langle 2467\rangle}$	43
102	33222211	$\frac{\langle 1235\rangle\langle 1245\rangle\langle 1368\rangle\langle 2467\rangle}{\langle 1236\rangle\langle 1246\rangle\langle 1358\rangle\langle 2457\rangle}$	45
103	33222211	$\frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle\langle 2348\rangle}{\langle 1234\rangle\langle 1256\rangle\langle 1367\rangle\langle 2458\rangle}$	41
104	33222211	$\frac{-\langle 1237\rangle\langle 1245\rangle\langle 1268\rangle\langle 3456\rangle}{\langle 1238\rangle\langle 1267\rangle\langle 1456\rangle\langle 2345\rangle}$	42
105	33222211	$\frac{-\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle}{\langle 1234\rangle\langle 1256\rangle\langle 1358\rangle\langle 2467\rangle}$	43
106	33222211	$\frac{\langle 1237\rangle\langle 1268\rangle\langle 1345\rangle\langle 2456\rangle}{\langle 1236\rangle\langle 1278\rangle\langle 1456\rangle\langle 2345\rangle}$	42
107	33222211	$\frac{\langle 1238\rangle\langle 1256\rangle\langle 1345\rangle\langle 2467\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 1348\rangle\langle 2567\rangle}$	44
108	33222211	$\frac{\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle}{\langle 1237\rangle\langle 1268\rangle\langle 1456\rangle\langle 2345\rangle}$	40
109	33222211	$\frac{\langle 1238\rangle\langle 1267\rangle\langle 1345\rangle\langle 2456\rangle}{\langle 1234\rangle\langle 1246\rangle\langle 1358\rangle\langle 2567\rangle}$	44
110	32222221	$\frac{\langle 1234\rangle\langle 1468\rangle\langle 1567\rangle\langle 2357\rangle}{\langle 1237\rangle\langle 1456\rangle\langle 1678\rangle\langle 2345\rangle}$	48
111	32222221	$\frac{\langle 1234\rangle\langle 1467\rangle\langle 1568\rangle\langle 2357\rangle}{\langle 1235\rangle\langle 1456\rangle\langle 1678\rangle\langle 2347\rangle}$	48
112	32222221	$\frac{\langle 1234\rangle\langle 1467\rangle\langle 1568\rangle\langle 2357\rangle}{\langle 1235\rangle\langle 1468\rangle\langle 1567\rangle\langle 2347\rangle}$	46
113	32222221	$\frac{-\langle 1234\rangle\langle 1468\rangle\langle 1567\rangle\langle 2357\rangle}{\langle 1235\rangle\langle 1456\rangle\langle 1678\rangle\langle 2347\rangle}$	47
114	22222222	$\frac{\langle 1246\rangle\langle 1367\rangle\langle 2458\rangle\langle 3578\rangle}{\langle 1248\rangle\langle 1378\rangle\langle 2456\rangle\langle 3567\rangle}$	51
115	22222222	$\frac{-\langle 1237\rangle\langle 1346\rangle\langle 2578\rangle\langle 4568\rangle}{\langle 1236\rangle\langle 1347\rangle\langle 2458\rangle\langle 5678\rangle}$	49
116	22222222	$\frac{-\langle 1245\rangle\langle 1367\rangle\langle 2468\rangle\langle 3578\rangle}{\langle 1248\rangle\langle 1357\rangle\langle 2456\rangle\langle 3678\rangle}$	51
117	22222222	$\frac{\langle 1234\rangle\langle 1367\rangle\langle 2578\rangle\langle 4568\rangle}{\langle 1237\rangle\langle 1346\rangle\langle 2458\rangle\langle 5678\rangle}$	50
118	66222222	$\frac{-\langle 1234\rangle^2\langle 1256\rangle\langle 1258\rangle\langle 1267\rangle\langle 1278\rangle}{\langle 1235\rangle\langle 1237\rangle\langle 1245\rangle\langle 1247\rangle\langle 1268\rangle^2}$	27, 27
119	64422222	$\frac{-\langle 1236\rangle\langle 1238\rangle\langle 1267\rangle\langle 1278\rangle\langle 1345\rangle^2}{\langle 1234\rangle\langle 1235\rangle\langle 1268\rangle^2\langle 1347\rangle\langle 1357\rangle}$	28, 28
120	77332222	$\frac{\langle 1235\rangle^2\langle 1238\rangle\langle 1246\rangle^2\langle 1247\rangle\langle 1278\rangle}{\langle 1236\rangle\langle 1237\rangle^2\langle 1245\rangle\langle 1248\rangle^2\langle 1256\rangle}$	27, 30
121	75532222	$\frac{\langle 1235\rangle\langle 1238\rangle^2\langle 1247\rangle^2\langle 1346\rangle\langle 1356\rangle}{\langle 1236\rangle^2\langle 1237\rangle\langle 1248\rangle\langle 1278\rangle\langle 1345\rangle^2}$	28, 31
122	44442222	$\frac{\langle 1235\rangle\langle 1238\rangle\langle 1467\rangle^2\langle 2345\rangle\langle 2348\rangle}{\langle 1246\rangle\langle 1247\rangle\langle 1346\rangle\langle 1347\rangle\langle 2358\rangle^2}$	29, 29
123	88442222	$\frac{-\langle 1234\rangle^3\langle 1236\rangle\langle 1245\rangle\langle 1256\rangle\langle 1278\rangle^2}{\langle 1235\rangle^2\langle 1237\rangle\langle 1238\rangle\langle 1246\rangle^2\langle 1247\rangle\langle 1248\rangle}$	30, 30
124	86642222	$\frac{-\langle 1235\rangle\langle 1236\rangle\langle 1237\rangle^2\langle 1245\rangle\langle 1246\rangle\langle 1348\rangle^2}{\langle 1234\rangle^3\langle 1238\rangle\langle 1256\rangle^2\langle 1347\rangle\langle 1378\rangle}$	31, 31
125	55552222	$\frac{\langle 1236\rangle^2\langle 1245\rangle^2\langle 1347\rangle\langle 2348\rangle\langle 3478\rangle}{\langle 1235\rangle\langle 1246\rangle\langle 1256\rangle\langle 1348\rangle^2\langle 2347\rangle^2}$	29, 32
126	66662222	$\frac{-\langle 1235\rangle\langle 1236\rangle\langle 1245\rangle\langle 1246\rangle\langle 1347\rangle^2\langle 2348\rangle^2}{\langle 1234\rangle^3\langle 1256\rangle^2\langle 1348\rangle\langle 2347\rangle\langle 3478\rangle}$	32, 32
127	86444222	$\frac{\langle 1236\rangle\langle 1238\rangle\langle 1246\rangle\langle 1248\rangle\langle 1257\rangle^2\langle 1345\rangle^2}{\langle 1234\rangle^2\langle 1235\rangle\langle 1245\rangle\langle 1268\rangle^2\langle 1357\rangle\langle 1457\rangle}$	33, 33
128	86444222	$\frac{\langle 1234\rangle\langle 1236\rangle\langle 1248\rangle\langle 1257\rangle^2\langle 1268\rangle\langle 1345\rangle^2}{\langle 1235\rangle\langle 1238\rangle^2\langle 1245\rangle\langle 1246\rangle^2\langle 1357\rangle\langle 1457\rangle}$	34, 34

129	86444222	$\frac{\langle 1235\rangle\langle 1236\rangle^2\langle 1248\rangle^2\langle 1257\rangle\langle 1345\rangle\langle 1457\rangle}{\langle 1234\rangle\langle 1238\rangle\langle 1245\rangle^2\langle 1246\rangle\langle 1268\rangle\langle 1357\rangle^2}$	$33,34$
130	86444222	$\frac{\langle 1234\rangle^2\langle 1245\rangle\langle 1257\rangle\langle 1268\rangle^2\langle 1345\rangle\langle 1357\rangle}{\langle 1235\rangle^2\langle 1236\rangle\langle 1238\rangle\langle 1246\rangle\langle 1248\rangle\langle 1457\rangle^2}$	$34,34$
131	66644222	$\frac{-\langle 1237\rangle^2\langle 1258\rangle^2\langle 1345\rangle\langle 1346\rangle\langle 2345\rangle\langle 2346\rangle}{\langle 1234\rangle^2\langle 1235\rangle\langle 1238\rangle\langle 1257\rangle\langle 1278\rangle\langle 3456\rangle^2}$	$35,35$
132	66644222	$\frac{\langle 1234\rangle^2\langle 1268\rangle^2\langle 1345\rangle\langle 1357\rangle\langle 2345\rangle\langle 2357\rangle}{\langle 1235\rangle^2\langle 1236\rangle\langle 1238\rangle\langle 1246\rangle\langle 1248\rangle\langle 3457\rangle^2}$	$36,36$
133	66644222	$\frac{\langle 1234\rangle\langle 1238\rangle\langle 1246\rangle\langle 1268\rangle\langle 1345\rangle^2\langle 2357\rangle^2}{\langle 1235\rangle\langle 1236\rangle^2\langle 1248\rangle^2\langle 1357\rangle\langle 2345\rangle\langle 3457\rangle}$	$35,36$
134	66644222	$\frac{-\langle 1237\rangle\langle 1238\rangle\langle 1257\rangle\langle 1258\rangle\langle 1345\rangle^2\langle 2346\rangle^2}{\langle 1234\rangle\langle 1235\rangle^2\langle 1278\rangle^2\langle 1346\rangle\langle 2345\rangle\langle 3456\rangle}$	$35,35$
135	84444422	$\frac{\langle 1234\rangle\langle 1246\rangle\langle 1258\rangle^2\langle 1347\rangle\langle 1356\rangle^2\langle 1467\rangle}{\langle 1235\rangle\langle 1247\rangle^2\langle 1256\rangle\langle 1346\rangle^2\langle 1358\rangle\langle 1568\rangle}$	$37,38$
136	84444422	$\frac{\langle 1246\rangle\langle 1247\rangle\langle 1258\rangle^2\langle 1346\rangle\langle 1347\rangle\langle 1356\rangle^2}{\langle 1234\rangle^2\langle 1235\rangle\langle 1256\rangle\langle 1358\rangle\langle 1467\rangle^2\langle 1568\rangle}$	$37,39$
137	66444422	$\frac{-\langle 1237\rangle^2\langle 1245\rangle\langle 1268\rangle^2\langle 1345\rangle\langle 2456\rangle\langle 3456\rangle}{\langle 1236\rangle\langle 1238\rangle\langle 1267\rangle\langle 1278\rangle\langle 1456\rangle^2\langle 2345\rangle^2}$	$40,42$
138	66444422	$\frac{-\langle 1237\rangle\langle 1238\rangle\langle 1267\rangle\langle 1268\rangle\langle 1345\rangle^2\langle 2456\rangle^2}{\langle 1236\rangle^2\langle 1245\rangle\langle 1278\rangle^2\langle 1456\rangle\langle 2345\rangle\langle 3456\rangle}$	$42,42$
139	66444422	$\frac{\langle 1238\rangle^2\langle 1256\rangle\langle 1267\rangle\langle 1345\rangle^2\langle 2456\rangle\langle 2467\rangle}{\langle 1234\rangle\langle 1235\rangle\langle 1246\rangle^2\langle 1348\rangle\langle 1358\rangle\langle 2567\rangle^2}$	$43,44$
140	66444422	$\frac{\langle 1234\rangle\langle 1238\rangle\langle 1256\rangle^2\langle 1345\rangle\langle 1358\rangle\langle 2467\rangle^2}{\langle 1235\rangle^2\langle 1246\rangle\langle 1267\rangle\langle 1348\rangle^2\langle 2456\rangle\langle 2567\rangle}$	$41,44$
141	66444422	$\frac{-\langle 1235\rangle\langle 1238\rangle\langle 1245\rangle^2\langle 1356\rangle\langle 1368\rangle\langle 2467\rangle^2}{\langle 1236\rangle^2\langle 1246\rangle\langle 1247\rangle\langle 1358\rangle^2\langle 2456\rangle\langle 2457\rangle}$	$43,45$
142	66444422	$\frac{\langle 1238\rangle^2\langle 1245\rangle\langle 1246\rangle\langle 1356\rangle^2\langle 2457\rangle\langle 2467\rangle}{\langle 1235\rangle\langle 1236\rangle\langle 1247\rangle^2\langle 1358\rangle\langle 1368\rangle\langle 2456\rangle^2}$	$41,45$
143	66444422	$\frac{\langle 1238\rangle^2\langle 1267\rangle^2\langle 1345\rangle\langle 1456\rangle\langle 2345\rangle\langle 2456\rangle}{\langle 1236\rangle\langle 1237\rangle\langle 1245\rangle^2\langle 1268\rangle\langle 1278\rangle\langle 3456\rangle^2}$	$42,42$
144	66444422	$\frac{\langle 1236\rangle^2\langle 1278\rangle^2\langle 1345\rangle\langle 1456\rangle\langle 2345\rangle\langle 2456\rangle}{\langle 1237\rangle\langle 1238\rangle\langle 1245\rangle^2\langle 1267\rangle\langle 1268\rangle\langle 3456\rangle^2}$	$40,40$
145	64444442	$\frac{\langle 1234\rangle^2\langle 1467\rangle\langle 1468\rangle\langle 1567\rangle\langle 1568\rangle\langle 2357\rangle^2}{\langle 1235\rangle\langle 1237\rangle\langle 1456\rangle^2\langle 1678\rangle^2\langle 2345\rangle\langle 2347\rangle}$	$47,48$
146	64444442	$\frac{\langle 1234\rangle\langle 1237\rangle\langle 1467\rangle^2\langle 1568\rangle^2\langle 2345\rangle\langle 2357\rangle}{\langle 1235\rangle^2\langle 1456\rangle\langle 1468\rangle\langle 1567\rangle\langle 1678\rangle\langle 2347\rangle^2}$	$46,48$
147	44444444	$\frac{-\langle 1245\rangle\langle 1246\rangle\langle 1367\rangle^2\langle 2458\rangle\langle 2468\rangle\langle 3578\rangle^2}{\langle 1248\rangle^2\langle 1357\rangle\langle 1378\rangle\langle 2456\rangle^2\langle 3567\rangle\langle 3678\rangle}$	$49,51$
148	44444444	$\frac{\langle 1234\rangle\langle 1236\rangle\langle 1347\rangle\langle 1367\rangle\langle 2578\rangle^2\langle 4568\rangle^2}{\langle 1237\rangle^2\langle 1346\rangle^2\langle 2458\rangle\langle 2568\rangle\langle 4578\rangle\langle 5678\rangle}$	$50,51$

Table A.6: Exceptional bracket cross-ratios on $\mathbb{P}^3$

APPENDIX B

Symmetric functional equations for $\mathcal{L}_m$ for $m = 3, 4, 5$

In this appendix we collect some long lists of functional equations of symmetric type, see Section 5.3 for the notation.

B.1 Symmetric functional equations on $\mathbb{P}^1$

The cross-ratios r_i in the following table are taken from Table A.2 in Appendix A.

m	n	Elements $\xi \in \mathbb{Q}[\Gamma_{n,2}]$ such that $\mathrm{Sym}_n\,\xi$ lies in $\widehat{\mathcal{E}}_m(\Gamma_{n,2})^+$
3	4	$[r_1]$
3	5	$-2[r_3] + [r_5]$
		$-5[r_3] - 3[r_4] + [r_7]$
3	6	$-4[r_9] - 2[r_{10}] + [r_{14}]$
		$-5[r_8] + [r_9] - 4[r_{10}] + [r_{11}] + [r_{13}]$
		$-4[r_{11}] + [r_{12}]$
		$-4[r_8] + 3[r_{11}]$
3	7	$-6[r_{15}] + [r_{16}]$
3	8	$-6[r_{17}] + [r_{18}]$
4	6	$2[r_1] - 4[r_9] - 2[r_{10}] + [r_{14}]$
5	6	$24[r_1] - 32[r_8] - 6[r_{11}] + 3[r_{12}]$
		$9[r_1] + 15[r_8] - 9[r_9] - 18[r_{10}] - 2[r_{12}] + [r_{13}] + [r_{14}]$
		$432[r_1] - 288[r_4] - 128[r_5] - 9[r_6] + 32[r_7]$
5	7	$9[r_1] + 18[r_3] + 18[r_4] + 8[r_5] - 2[r_7] - 54[r_{15}] + 3[r_{16}]$
5	8	$18[r_{15}] - [r_{16}] - 18[r_{17}] + [r_{18}]$

Table B.1: Symmetric functional equations on $\mathbb{P}^1$

B.2 Symmetric functional equations on $\mathbb{P}^2$

The cross-ratios r_i in the following table are taken from Table A.4 in Appendix A.

m	n	Elements $\xi \in \mathbb{Q}[\Gamma_{n,3}]$ such that $\mathrm{Sym}_n \xi$ lies in $\widehat{\mathcal{E}}_m(\Gamma_{n,3})^+$
3	6	$[r_1]$
		$-2[r_2] + [r_4]$
		$-5[r_2] - 3[r_3] + [r_5]$
		$-2[r_7] + [r_{10}]$
		$-4[r_6] + [r_{23}]$
		$-2[r_6] - [r_8] - 4[r_9] + [r_{16}] + [r_{27}]$
		$[r_6] - 4[r_8] - 4[r_{12}] + [r_{15}] + [r_{28}]$
		$-[r_7] - 2[r_{10}] - 3[r_{11}] + [r_{29}]$
		$-[r_9] - [r_{13}] - 2[r_{14}] - 2[r_{16}] + [r_{30}]$
		$-[r_{12}] - 4[r_{13}] + [r_{14}] - 2[r_{15}] + [r_{31}]$
		$2[r_6] - 4[r_8] - 2[r_{18}] + [r_{32}]$
		$-4[r_{17}] - 3[r_{18}] + [r_{19}] + [r_{33}]$
		$-[r_{17}] - 2[r_{19}] - 3[r_{20}] + [r_{34}]$
		$2[r_6] + 4[r_{18}] - 3[r_{21}] - 2[r_{32}] + 2[r_{35}]$
		$-2[r_{18}] - [r_{21}] - [r_{23}] - 4[r_{24}] + [r_{32}] - [r_{35}] + [r_{36}] + [r_{37}]$
		$-3[r_{21}] + [r_{22}] - 4[r_{26}] + [r_{38}]$
		$-2[r_{22}] - 3[r_{25}] - [r_{26}] + [r_{39}]$
3	7	$-4[r_{40}] + 3[r_{43}]$
		$-4[r_{43}] + [r_{57}]$
		$-8[r_{41}] + 2[r_{46}] + [r_{58}]$
		$-[r_{40}] - 4[r_{42}] - 2[r_{43}] + [r_{44}] + [r_{59}]$
		$-[r_{41}] - 4[r_{45}] - 2[r_{46}] + [r_{47}] + [r_{60}]$
		$-2[r_{42}] - 4[r_{44}] + [r_{61}]$
		$-2[r_{45}] - 4[r_{47}] + [r_{62}]$
		$2[r_{48}] - 4[r_{49}] - 4[r_{51}] + [r_{63}]$
		$-4[r_{48}] + 2[r_{49}] - 4[r_{51}] + [r_{64}]$
		$2[r_{50}] - 8[r_{52}] + [r_{65}]$
		$-2[r_{48}] - 2[r_{49}] - 2[r_{50}] + [r_{51}] - [r_{52}] + [r_{66}]$
		$-[r_{53}] - 2[r_{54}] - 3[r_{55}] + [r_{67}]$
		$-4[r_{53}] + [r_{54}] - 3[r_{56}] + [r_{68}]$
3	8	$-6[r_{69}] + [r_{77}]$
		$-6[r_{70}] + [r_{78}]$
		$2[r_{71}] - 8[r_{72}] + [r_{79}]$
		$-4[r_{71}] - 2[r_{72}] + [r_{80}]$
		$-8[r_{73}] + 2[r_{74}] + [r_{81}]$
		$-4[r_{75}] - 2[r_{76}] + [r_{82}]$
		$-[r_{73}] - 2[r_{74}] + [r_{75}] - 4[r_{76}] + [r_{83}]$

4	6	$-3[r_9] - 3[r_{13}] - 3[r_{14}] - 3[r_{16}] + [r_{30}]$
4	7	$-6[r_{42}] - 6[r_{44}] + [r_{61}]$
		$-6[r_{45}] - 6[r_{47}] + [r_{62}]$
		$-6[r_{48}] + 6[r_{49}] - 12[r_{51}] - [r_{63}] + [r_{64}]$
		$-6[r_{48}] - 12[r_{49}] + 6[r_{51}] - 18[r_{52}] + [r_{63}] + [r_{65}] + 2[r_{66}]$
4	8	$-18[r_{72}] + [r_{79}] + [r_{80}]$
		$-6[r_{75}] - 6[r_{76}] + [r_{82}]$
5	6	$27[r_1] - 18[r_2] - 18[r_3] - 8[r_4] + 2[r_5]$
		$9[r_2] + 9[r_3] + 4[r_4] - [r_5] - 9[r_7] - 4[r_{10}] - 9[r_{11}] + [r_{29}]$
		$27[r_1] + 54[r_2] + 54[r_3] + 24[r_4] - 6[r_5] - 9[r_6] - 18[r_8] - 36[r_9] - 18[r_{12}] - 36[r_{13}] - 18[r_{14}]$ $-9[r_{15}] - 18[r_{16}] + 2[r_{27}] + [r_{28}] + 4[r_{30}] + 2[r_{31}]$
		$9[r_1] + 18[r_2] + 18[r_3] + 8[r_4] - 2[r_5] - 18[r_{17}] - 9[r_{18}] - 9[r_{19}] - 18[r_{20}] + [r_{33}] + 2[r_{34}]$
		$9[r_1] + 18[r_7] + 8[r_{10}] + 18[r_{11}] - 9[r_{21}] - 9[r_{22}] - 18[r_{25}] - 18[r_{26}] - 2[r_{29}] + [r_{38}] + 2[r_{39}]$
5	7	$-81[r_1] + 297[r_{40}] - 126[r_{42}] + 36[r_{43}] - 63[r_{44}] - 32[r_{57}] + 7[r_{59}] + 7[r_{61}]$
		$9[r_1] + 18[r_2] + 18[r_3] + 8[r_4] - 2[r_5] - 18[r_{53}] - 9[r_{54}] - 18[r_{55}] - 9[r_{56}] + 2[r_{67}] + [r_{68}]$
		$45[r_1] + 144[r_2] + 144[r_3] + 64[r_4] - 16[r_5] + 3[r_{40}] - 234[r_{42}] - 36[r_{43}] - 117[r_{44}]$ $-8[r_{57}] + 13[r_{59}] + 13[r_{61}]$
		$27[r_1] + 54[r_2] + 54[r_3] + 24[r_4] - 6[r_5] - 36[r_{41}] - 72[r_{45}] - 18[r_{46}] - 36[r_{47}] + [r_{58}]$ $+4[r_{60}] + 4[r_{62}]$
		$27[r_1] + 54[r_2] + 54[r_3] + 24[r_4] - 6[r_5] - 36[r_{48}] - 36[r_{49}] - 18[r_{50}] - 36[r_{51}]$ $-36[r_{52}] + 2[r_{63}] + 2[r_{64}] + [r_{65}] + 4[r_{66}]$
5	8	$9[r_1] + 18[r_2] + 18[r_3] + 8[r_4] - 2[r_5] - 54[r_{69}] + 3[r_{77}]$
		$54[r_{70}] - 18[r_{71}] - 36[r_{72}] - 3[r_{78}] + [r_{79}] + 2[r_{80}]$
		$9[r_1] + 18[r_2] + 18[r_3] + 8[r_4] - 2[r_5] - 54[r_{70}] + 3[r_{78}]$
		$27[r_1] + 54[r_2] + 54[r_3] + 24[r_4] - 6[r_5] - 36[r_{73}] - 18[r_{74}] - 36[r_{75}] - 72[r_{76}]$ $+[r_{81}] + 4[r_{82}] + 4[r_{83}]$

Table B.2: Symmetric functional equations on $\mathbb{P}^2$

B.3 Symmetric functional equations on $\mathbb{P}^3$

The cross-ratios r_i in the following table are taken from Table A.6 in Appendix A.

m	Elements $\xi \in \mathbb{Q}[\Gamma_{8,4}]$ such that $\mathrm{Sym}_8\,\xi$ lies in $\widehat{\mathcal{E}}_m(\Gamma_{8,4})^+$
3	$[r_1]$
	$-2[r_2] + [r_6]$
	$-4[r_3] + 3[r_{10}]$
	$-[r_2] - 2[r_6] - 3[r_7] + [r_{11}]$
	$-4[r_{10}] + [r_{12}]$
	$-[r_3] + [r_8] - 4[r_9] - 2[r_{10}] + [r_{13}]$
	$-4[r_8] - 2[r_9] + [r_{14}]$

$-2[r_{16}] + [r_{20}]$
$-4[r_{15}] + [r_{39}]$
$-2[r_{15}] - 4[r_{17}] - [r_{18}] + [r_{26}] + [r_{49}]$
$[r_{15}] - 4[r_{18}] - 4[r_{22}] + [r_{23}] + [r_{50}]$
$-[r_{16}] - 2[r_{20}] - 3[r_{21}] + [r_{51}]$
$-[r_{17}] - [r_{24}] - 2[r_{25}] - 2[r_{26}] + [r_{52}]$
$-8[r_{19}] + 2[r_{28}] + [r_{53}]$
$-[r_{22}] - 2[r_{23}] - 4[r_{24}] + [r_{25}] + [r_{54}]$
$-[r_{19}] + [r_{27}] - 2[r_{28}] - 4[r_{29}] + [r_{55}]$
$-4[r_{27}] - 2[r_{29}] + [r_{56}]$
$2[r_{15}] - 4[r_{18}] - 2[r_{30}] + [r_{57}]$
$-3[r_{30}] - 4[r_{32}] + [r_{33}] + [r_{58}]$
$-3[r_{31}] - [r_{32}] - 2[r_{33}] + [r_{59}]$
$-2[r_{34}] - 2[r_{35}] + [r_{36}] - [r_{37}] - 2[r_{38}] + [r_{60}]$
$-4[r_{34}] + 2[r_{35}] - 4[r_{36}] + [r_{61}]$
$-8[r_{37}] + 2[r_{38}] + [r_{62}]$
$2[r_{34}] - 4[r_{35}] - 4[r_{36}] + [r_{63}]$
$2[r_{15}] + 4[r_{30}] - 3[r_{42}] - 2[r_{57}] + 2[r_{64}]$
$[r_{41}] - 3[r_{42}] - 4[r_{44}] + [r_{66}]$
$-3[r_{40}] - 2[r_{41}] - [r_{44}] + [r_{67}]$
$-2[r_{30}] - [r_{39}] - [r_{42}] - 4[r_{43}] + [r_{57}] - [r_{64}] + [r_{65}] + [r_{68}]$
$[r_{45}] - 4[r_{46}] - 3[r_{48}] + [r_{69}]$
$-2[r_{45}] - [r_{46}] - 3[r_{47}] + [r_{70}]$
$-4[r_{71}] + 3[r_{76}]$
$-4[r_{76}] + [r_{118}]$
$-8[r_{72}] + 2[r_{77}] + [r_{119}]$
$-[r_{71}] - 4[r_{74}] + [r_{75}] - 2[r_{76}] + [r_{120}]$
$-[r_{72}] - 2[r_{77}] + [r_{78}] - 4[r_{79}] + [r_{121}]$
$-8[r_{73}] + 2[r_{82}] + [r_{122}]$
$-2[r_{74}] - 4[r_{75}] + [r_{123}]$
$-4[r_{78}] - 2[r_{79}] + [r_{124}]$
$-[r_{73}] + [r_{80}] - 4[r_{81}] - 2[r_{82}] + [r_{125}]$
$-4[r_{80}] - 2[r_{81}] + [r_{126}]$
$2[r_{83}] - 8[r_{85}] + [r_{127}]$
$-4[r_{84}] - 4[r_{86}] + 2[r_{87}] + [r_{128}]$
$-2[r_{83}] - 2[r_{84}] - [r_{85}] + [r_{86}] - 2[r_{87}] + [r_{129}]$
$2[r_{84}] - 4[r_{86}] - 4[r_{87}] + [r_{130}]$
$2[r_{88}] - 4[r_{90}] - 4[r_{91}] + [r_{131}]$
$2[r_{89}] - 8[r_{92}] + [r_{132}]$
$-2[r_{88}] - 2[r_{89}] - 2[r_{90}] + [r_{91}] - [r_{92}] + [r_{133}]$
$-4[r_{88}] + 2[r_{90}] - 4[r_{91}] + [r_{134}]$

	$-3[r_{93}] + [r_{94}] - 4[r_{96}] + [r_{135}]$
	$-2[r_{94}] - 3[r_{95}] - [r_{96}] + [r_{136}]$
	$[r_{98}] - [r_{99}] - 2[r_{104}] - 2[r_{106}] - 2[r_{108}] + [r_{137}]$
	$-4[r_{98}] + 2[r_{104}] - 4[r_{106}] + [r_{138}]$
	$-[r_{101}] - 2[r_{105}] - [r_{107}] - 2[r_{109}] + [r_{139}]$
	$-[r_{97}] - 2[r_{103}] - 4[r_{107}] + [r_{109}] + [r_{140}]$
	$-[r_{100}] - 4[r_{101}] - 2[r_{102}] + [r_{105}] + [r_{141}]$
	$-4[r_{97}] - 4[r_{100}] + [r_{102}] + [r_{103}] + [r_{142}]$
	$-4[r_{98}] - 4[r_{104}] + 2[r_{106}] + [r_{143}]$
	$-8[r_{99}] + 2[r_{108}] + [r_{144}]$
	$-2[r_{110}] - [r_{111}] - 3[r_{113}] + [r_{145}]$
	$[r_{110}] - 4[r_{111}] - 3[r_{112}] + [r_{146}]$
	$-2[r_{114}] - 3[r_{115}] - [r_{116}] + [r_{147}]$
	$[r_{114}] - 4[r_{116}] - 3[r_{117}] + [r_{148}]$
4	$-6[r_8] - 6[r_9] + [r_{14}]$
	$-3[r_{17}] - 3[r_{24}] - 3[r_{25}] - 3[r_{26}] + [r_{52}]$
	$-6[r_{27}] - 6[r_{29}] + [r_{56}]$
	$6[r_{34}] - 6[r_{35}] + 12[r_{36}] - [r_{61}] + [r_{63}]$
	$-6[r_{34}] - 12[r_{35}] + 6[r_{36}] - 18[r_{37}] + 2[r_{60}] + [r_{62}] + [r_{63}]$
	$-6[r_{74}] - 6[r_{75}] + [r_{123}]$
	$-6[r_{78}] - 6[r_{79}] + [r_{124}]$
	$-6[r_{80}] - 6[r_{81}] + [r_{126}]$
	$-6[r_{84}] - 12[r_{86}] - 6[r_{87}] + [r_{128}] + [r_{130}]$
	$6[r_{84}] + 18[r_{85}] - 6[r_{86}] - 12[r_{87}] - [r_{127}] - 2[r_{129}] + [r_{130}]$
	$6[r_{88}] - 12[r_{90}] - 6[r_{91}] - 18[r_{92}] + [r_{131}] + [r_{132}] + 2[r_{133}]$
	$-6[r_{88}] - 6[r_{90}] - 12[r_{91}] + [r_{131}] + [r_{134}]$
	$-3[r_{101}] - 3[r_{105}] - 3[r_{107}] - 3[r_{109}] + [r_{139}]$
	$-12[r_{98}] + 6[r_{104}] - 6[r_{106}] + [r_{138}] + [r_{143}]$
	$-6[r_{98}] - 18[r_{99}] + 12[r_{104}] + 6[r_{106}] - 2[r_{137}] + [r_{143}] + [r_{144}]$
5	$9[r_2] + 4[r_6] + 9[r_7] - [r_{11}] - 9[r_{16}] - 4[r_{20}] - 9[r_{21}] + [r_{51}]$
	$9[r_1] + 18[r_2] + 8[r_6] + 18[r_7] - 2[r_{11}] - 9[r_{30}] - 18[r_{31}] - 18[r_{32}] - 9[r_{33}] + [r_{58}] + 2[r_{59}]$
	$9[r_1] + 18[r_2] + 8[r_6] + 18[r_7] - 2[r_{11}] - 9[r_{93}] - 9[r_{94}] - 18[r_{95}] - 18[r_{96}] + [r_{135}] + 2[r_{136}]$
	$9[r_1] + 18[r_{16}] + 8[r_{20}] + 18[r_{21}] - 2[r_{51}] - 9[r_{110}] - 18[r_{111}] - 9[r_{112}] - 18[r_{113}] + 2[r_{145}] + [r_{146}]$
	$9[r_1] + 18[r_{16}] + 8[r_{20}] + 18[r_{21}] - 18[r_{40}] - 9[r_{41}] - 9[r_{42}] - 18[r_{44}] - 2[r_{51}] + [r_{66}] + 2[r_{67}]$
	$9[r_1] + 18[r_{16}] + 8[r_{20}] + 18[r_{21}] - 2[r_{51}] - 9[r_{114}] - 18[r_{115}] - 18[r_{116}] - 9[r_{117}] + 2[r_{147}] + [r_{148}]$
	$9[r_1] + 18[r_{16}] + 8[r_{20}] + 18[r_{21}] - 9[r_{45}] - 18[r_{46}] - 18[r_{47}] - 9[r_{48}] - 2[r_{51}] + [r_{69}] + 2[r_{70}]$
	$-15[r_3] + 9[r_8] + 18[r_9] + 2[r_{12}] - [r_{13}] - [r_{14}] + 15[r_{71}] - 18[r_{74}] - 9[r_{75}] - 2[r_{118}] + [r_{120}] + [r_{123}]$
	$32[r_3] + 6[r_{10}] - 3[r_{12}] - 32[r_{71}] - 6[r_{76}] + 3[r_{118}]$
	$27[r_1] - 9[r_{15}] + 54[r_{16}] - 36[r_{17}] - 18[r_{18}] + 24[r_{20}] + 54[r_{21}] - 18[r_{22}] - 9[r_{23}] - 36[r_{24}] - 18[r_{25}]$ $-18[r_{26}] + 2[r_{49}] + [r_{50}] - 6[r_{51}] + 4[r_{52}] + 2[r_{54}]$

$27[r_1] - 36[r_3] - 36[r_8] - 72[r_9] - 18[r_{10}] + [r_{12}] + 4[r_{13}] + 4[r_{14}] + 54[r_{16}] + 24[r_{20}] + 54[r_{21}] - 6[r_{51}]$

$27[r_1] + 54[r_{16}] + 24[r_{20}] + 54[r_{21}] - 6[r_{51}] - 18[r_{97}] - 18[r_{100}] - 36[r_{101}] - 9[r_{102}] - 9[r_{103}]$
$\quad -18[r_{105}] - 36[r_{107}] - 18[r_{109}] + 4[r_{139}] + 2[r_{140}] + 2[r_{141}] + [r_{142}]$

$27[r_1] + 54[r_{16}] + 24[r_{20}] + 54[r_{21}] - 6[r_{51}] - 36[r_{98}] - 36[r_{99}] - 36[r_{104}] - 36[r_{106}] - 18[r_{108}]$
$\quad +4[r_{137}] + 2[r_{138}] + 2[r_{143}] + [r_{144}]$

$27[r_1] + 54[r_{16}] + 24[r_{20}] + 54[r_{21}] - 6[r_{51}] - 18[r_{83}] - 36[r_{84}] - 36[r_{85}] - 36[r_{86}] - 36[r_{87}]$
$\quad +[r_{127}] + 2[r_{128}] + 4[r_{129}] + 2[r_{130}]$

$27[r_1] + 54[r_2] + 24[r_6] + 54[r_7] - 6[r_{11}] - 36[r_{34}] - 36[r_{35}] - 36[r_{36}] - 36[r_{37}]$
$\quad -18[r_{38}] + 4[r_{60}] + 2[r_{61}] + [r_{62}] + 2[r_{63}]$

$-81[r_1] + 297[r_{71}] - 126[r_{74}] - 63[r_{75}] + 36[r_{76}] - 32[r_{118}] + 7[r_{120}] + 7[r_{123}]$

$-27[r_1] - 54[r_{16}] - 24[r_{20}] - 54[r_{21}] + 6[r_{51}] + 36[r_{88}] + 18[r_{89}] + 36[r_{90}] + 36[r_{91}] + 36[r_{92}]$
$\quad -2[r_{131}] - [r_{132}] - 4[r_{133}] - 2[r_{134}]$

$27[r_1] + 54[r_2] + 24[r_6] + 54[r_7] - 6[r_{11}] - 36[r_{72}] - 18[r_{77}] - 36[r_{78}] - 72[r_{79}]$
$\quad +[r_{119}] + 4[r_{121}] + 4[r_{124}]$

$27[r_1] + 54[r_{16}] + 24[r_{20}] + 54[r_{21}] - 6[r_{51}] - 36[r_{73}] - 36[r_{80}] - 72[r_{81}] - 18[r_{82}]$
$\quad +[r_{122}] + 4[r_{125}] + 4[r_{126}]$

$27[r_1] + 54[r_{16}] - 36[r_{19}] + 24[r_{20}] + 54[r_{21}] - 36[r_{27}] - 18[r_{28}] - 72[r_{29}] - 6[r_{51}] + [r_{53}]$
$\quad +4[r_{55}] + 4[r_{56}]$

Table B.3: Symmetric functional equations on $\mathbb{P}^3$

Bibliography

[1] D. Bailey, D. Broadhurst, *A seventeenth-order polylogarithm ladder*, arxiv:math.CA/9906134.

[2] P. Blanc, *Sur la cohomologie continue des groupes localement compacts*, Annales scientifiques de l'É.N.S. 4e série, tome 12, no 2 (1979), pp. 137–168.

[3] S. Bloch, *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry*, Proc. of the International Symp. on Alg. Geometry, Kinokuniya, Tokyo (1978).

[4] M. Bôcher, *The theory of linear dependence*, Ann. of Math. (2) **2** (1900/01), pp. 81–96.

[5] E. Bombieri, D. Masser, U. Zannier, *Finiteness results for multiplicatively dependent points on complex curves*, Michigan Math. J. 51 (2003), pp. 451–466.

[6] E. Bombieri, D. Masser, U. Zannier, *On unlikely intersections of complex varieties with tori*, Acta Arithmetica 133 (2008), pp. 309–323.

[7] A. Borel, *Cohomologie de SL_n et valeurs de fonctions zêta aux points entiers*, Ann. Sc. Norm. Sup. Pisa 4 (1977), pp. 613–636.

[8] A. Borel, *Values of zeta-functions at integers, cohomology and polylogarithms*, Current Trends in Mathematics and Physics, a Tribute to Harish-Chandra (1995), pp. 1–44.

[9] N. Bourbaki, *Commutative algebra*, Addison-Wesley (1972).

[10] H. Cohen, *A Course in Computational Algebraic Number Theory*, Springer-Verlag (1993).

[11] H. Cohen, L. Lewin, D. Zagier, *A sixteenth-order polylogarithm ladder*, Experimental Math. 1 (1992), pp. 25–34.

[12] P.B. Cohen, U. Zannier, *Multiplicative dependence and bounded height, an example*, Algebraic Number Theory and Diophantine Analysis (Graz, 1998), de Gruyter, Berlin (2000).

[13] I. Dolgachev, D. Ortland, *Point sets in projective spaces and theta functions*, Astérisque No. 165 (1988).

[14] L. Euler, *Meditationes circa singulare serierum genus*, Novi Comm. Acad. Sci. Petropol. 20 (1775), pp. 140–186.

[15] P. Erdős, C.L. Stewart, R. Tijdeman, *Some Diophantine equations with many solutions*, Compositio Math. 66 (1988), pp. 37–56.

[16] H. Gangl, *Funktionalgleichungen von Polylogarithmen*, Bonner Math. Schriften 278 (1995).

[17] A.B. Goncharov, *Geometry of Configurations, Polylogarithms, and Motivic Cohomology*, Adv. in Math. 114 (1995), pp. 197–318.

[18] A.B. Goncharov, *Volumes of hyperbolic manifolds and mixed Tate motives*, J. Amer. Math. Soc. 12, 2 (1999), pp. 569–618.

[19] A.B. Goncharov, *Polylogarithms and motivic Galois groups*, Motives, Proc. Sympos. Pure Math., vol. 55, part 2, (1994) pp. 43–96.

[20] A.B. Goncharov, *Explicit Construction of Characteristic Classes, I*, M. Gelfand Seminar, Adv. Soviet Math., vol. 16, part 1, (1993), pp. 169–210.

[21] R. Hain, *Classical Polylogarithms*, Proceedings of AMS Research Summer Conference "Motives", vol. 2 (1994), pp. 1–42.

[22] H. Klingen, *Über die Werte der Dedekindschen Zetafunktion*, Math. Ann. 145 (1961/1962), pp. 265–272.

[23] L. Lewin, *Polylogarithms and associated functions*. North Holland, (1981).

[24] L. Lewin, *Structural properties of polylogarithms*, Mathematical Surveys and Monographs vol. 37, AMS (1991).

[25] R.C. Mason, *Norm form equations I*, J. Number Theory 22 (1986), pp. 190–207.

[26] G. Maurin, *Courbes algébriques et équations multiplicatives*, Math. Ann. 341 no. 4 (2008), pp. 789–824.

[27] J. Milnor, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. 6 (1982), pp. 9–24.

[28] J. Milnor, J. Moore, *On the structure of Hopf algebras*, Annals of Math. 81 (1965), pp. 211–264.

[29] D. Ramakrishnan, *On the monodromy of higher logarithms*, Proc. Amer. Math. Soc. 85 (1982), pp. 596–599.

[30] A. Schinzel, *Polynomials with special regard to reducibility*, Encyclopedia of Mathematics and its Applications, Cambridge Univ. Press (2000).

[31] C.L. Siegel, *Über die analytische Theorie der quadratischen Formen. III*, Ann. of Math. (2) 38, no. 1 (1937), pp. 212–291.

[32] B. Sturmfels, *Algorithms in Invariant Theory*, Texts and Monographs in Symbolic Computation. Springer-Verlag (1993).

[33] J.F. Voloch, *Diagonal equations over function fields*, Bol. Soc. Brasil. Mat. 16 (1985), pp. 29–39.

[34] Z. Wojtkowiak, *Functional equations of iterated integrals with regular singularities*, Nagoya Math. J. 142 (1996), pp. 145–159.

[35] D. Zagier, *Hyperbolic manifolds and special values of Dedekind zeta-functions*, Invent. Math. 83 (1986), pp. 285–301.

[36] D. Zagier, *Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields*, Progr. Math. 89 (1991), pp. 391–430.

[37] D. Zagier, *The Dilogarithm Function*, Frontiers in Number Theory, Physics, and Geometry II (P. Cartier, P. Moussa, B. Julia and P. Vanhove, eds.) (2007), pp. 3–65.

[38] D. Zagier, *Special values and functional equations of polylogarithms*, in "The Structural Properties of Polylogarithms", ed. L. Lewin, Mathematical Surveys and Monographs 37, AMS, Providence (1991), pp. 377–400.

[39] U. Zannier, *Some remarks on the S-unit equation in function fields*, Acta Arith. 64, no. 1, (1993), pp. 87–98.

[40] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies 181 (2012). With appendixes by David Masser.

[41] The PARI Group, PARI/GP version `2.7.0`, Bordeaux, 2014, `http://pari.math.u-bordeaux.fr/`.