On τ-tilting theory and perpendicular categories

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Zusammenfassung
Lebenslauf

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1 Introduction and main results

1.1 Motivation and summary

Gabriel's Theorem and tilting theory The central problem in the representation theory of finite-dimensional algebras is to describe their category of finite-dimensional modules as completely as possible. Various approaches in trying to solve this problem and partial solutions are variations and applications of tilting theory. The first step in this direction in modern representation theory was Gabriel's Theorem in 1972: A finite-dimensional path algebra $A = KQ$, where Q is a finite connected acyclic quiver, is representation-finite if and only if Q is of Dynkin type A_n, D_n, E_6, E_7, E_8. Moreover, in this case there is a bijection between the isomorphism classes of indecomposable A-modules and the positive roots of the corresponding simple complex Lie algebra. Gabriel thus also established a close connection to Lie Theory.

Shortly after Gabriel's breakthrough Bernstein, Gelfand and Ponomarev proved that all indecomposable modules over a representation-finite path algebra $A = KQ$, can be constructed recursively from the simple modules by using reflection functors. This can be considered as the starting point of tilting theory. It became apparent that the module category does not change too much when changing the orientation of the quiver. This procedure was generalized by Auslander, Platzek and Reiten who showed that the reflection functors are equivalent to functors of the form $\mathrm{Hom}_A(T, -)$, where T is nowadays known as an (APR)-tilting module. Ever since then, tilting theory has appeared in numerous areas of mathematics as a method for constructing functors between categories. In this thesis we focus on two particular theories, whose origins can be traced back to tilting theory.

τ-tilting theory On the one hand there is the rather new concept called τ-tilting theory introduced by Adachi, Iyama and Reiten in 2012. It follows from results by Riedtmann and Schofield and Unger in the early 1990's that any almost complete (support) tilting module over a finite-dimensional algebra can be completed in at least one and at most two ways to a complete (support) tilting module. This was the first approach to a combinatorial study of the set of isomorphism classes of multiplicity-free tilting modules. The (support) τ-tilting modules are a generalization of the classical tilting modules. In this wider class of modules it is possible to model the process of mutation inspired by cluster tilting theory. In other words any basic almost complete support τ-tilting pair over a finite-dimensional algebra is a direct summand of exactly two basic support τ-tilting pairs.

Cluster algebras were introduced by Fomin and Zelevinsky in 2002. Since then cluster (tilting) theory has had a huge impact on the research of representation theory of finite-dimensional algebras. The algebras appearing in connection with cluster theory are Jacobian algebras defined via quivers with potential. These are not finite-dimensional in general. This suggests the need for developing τ-tilting theory for infinite dimensional algebras.

Completed string algebras String algebras are a subclass of the special biserial algebras. The module category of a finite-dimensional string algebra can be described completely in combinatorial terms. Therefore, they are often used to test conjectures. Furthermore, they appear in cluster theory as Jacobian algebras of surfaces. Hence one should consider string algebras as an important class of examples. In this thesis we study the module category of what we call completed string algebras, a generalization of the finite-dimensional string algebras which include infinite dimensional algebras. We extend the combinatorial description of the module category of a finite-dimensional string algebra to the category of finitely generated modules over a completed string algebra. In particular we describe the Auslander-Reiten sequences ending in a finitely generated indecomposable module. This allows us to develop τ-tilting theory for completed string algebras and prove that mutation is possible within the class of finitely generated support τ-tilting pairs.

Perpendicular categories On the other hand we consider the theory of perpendicular categories which goes back to an article by Geigle and Lenzing in 1989 and an article by Schofield in 1991 which focuses on the study of perpendicular categories for path algebras. One of the main results here is the following: Let A be a finite-dimensional algebra and M an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ is a skew-field. Then there exists a finite-dimensional A-module T such that $M \oplus T$ is a basic tilting module and the (right) perpendicular category $M^\perp$ is equivalent to $\mathrm{mod}(A')$, where $A' = \mathrm{End}_A(T)^{\mathrm{op}}$. It follows that the algebra A' has one simple module less than the original algebra A and $\mathrm{gl.dim}(A') \leq \mathrm{gl.dim}(A)$. Thus if A is hereditary, then so is A' and moreover, in this case it is known that any exceptional module M is an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ is a skew-field.

This result opened the possibility for proving statements by induction. This has been used by Crawley-Boevey to prove the existence of a transitive action of the braid group on exceptional sequences for path algebras. Ringel generalized this result to hereditary algebras and in addition developed an inductive procedure for obtaining all exceptional modules from the simple modules.

Algebras associated with symmetrizable Cartan matrices The theory of modulated graphs or species was developed by Dlab and Ringel in a series of papers in the 1970's. It was a first attempt in generalizing path algebras associated to symmetric Cartan matrices to hereditary algebras which can be associated to symmetrizable Cartan matrices. They extended Gabriel's Theorem to include the non-simply laced root systems B_n, C_n, F_4 and G_2. More precisely, they proved that a finite-dimensional hereditary algebra is representation-finite if and only if its corresponding valued graph is of Dynkin type. To ensure the existence of these hereditary algebras one has to make quite strong assumptions on the ground field, and cannot assume it to be algebraically closed in general.

Recently Geiß, Leclerc and Schröer suggested another approach by introducing a new class of algebras which are defined via quivers with relations associated with symmetrizable Cartan matrices. They thus obtain new representation theoretic realizations of all finite root systems without any assumptions on the ground field. These newly defined algebras are in general no longer hereditary but 1-Iwanaga-Gorenstein. An algebra is 1-Iwanaga-Gorenstein if and only if the injective dimension of its regular representation is at most 1. Thus all self-injective and all hereditary algebras are particular examples of 1-Iwanaga-Gorenstein algebras.

Perpendicular categories for 1-Iwanaga-Gorenstein algebras In this thesis we study perpendicular categories for finite-dimensional 1-Iwanaga-Gorenstein algebras. We find that if A is 1-Iwanaga-Gorenstein and $M \in \mathrm{mod}(A)$ an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ is a skew-field, then $M^\perp$ is equivalent to $\mathrm{mod}(A')$, where A' is again 1-Iwanaga-Gorenstein. We then concentrate on the particular class of 1-Iwanaga-Gorenstein algebras defined via quivers with relations associated with symmetrizable Cartan matrices. If H is such an algebra associated with a symmetrizable Cartan matrix C and $M \in \mathrm{mod}(H)$ an indecomposable partial tilting module, the ring $\mathrm{End}_H(M)$ is not a skew-field in general. However, if M is preinjective we still find that $M^\perp$ is a equivalent to $\mathrm{mod}(H')$, where H' is a 1-Iwanaga-Gorenstein algebra associated with a symmetrizable Cartan matrix C', which is of size one smaller than C.

In the following sections we explain our results, which are stated in the Theorems A to E, in more detail.

1.2 Completed string algebras and τ-tilting theory

Finite-dimensional string algebras The representation theory of finite-dimensional string algebras can be considered as being well-understood. The indecomposable modules over a string algebra can be described in completely combinatorial terms usually referred to as strings and bands or more generally words. This classification goes back to Gelfand and Ponomarev [GP] for the algebra $K[x,y]/(xy)$ and Ringel [R1] who adapted their methods to the non-commutative algebra $K\langle x,y\rangle/(x^2,y^2)$. Also the almost split sequences containing string and band modules can be described using the combinatorics of words. Their classification was achieved by Butler and Ringel [BR]. Thus it is possible and usually not difficult to compute the Auslander-Reiten quiver of any finite-dimensional string algebra, which in a sense captures all of the information about the objects and also morphisms in the category of finite-dimensional modules. It is therefore not surprising that string algebras are an important class of examples when testing and presenting new theories.

Infinite dimensional string algebras Recently Crawley-Boevey [CB4] achieved the classification of the finitely generated modules over a possibly infinite dimensional string algebra. Again the indecomposable modules are associated to certain words, where now one also has to consider infinite words. The main tool in his work is the functorial filtration method. This was first used in the original works of Gelfand and Ponomarev and Ringel. It is possible to modify the method to the infinite dimensional case, since the string algebra is a Noetherian algebra over the polynomial ring in one variable.

Unfortunately, this does not help when trying to describe almost split sequences containing finitely generated string modules. It is one of the properties of almost split sequences, that the first and last term in the sequence have local endomorphism ring. However, if A is an infinite dimensional string algebra, there are indecomposable finitely generated A-modules whose endomorphism ring is not local. The easiest example for this is the polynomial ring itself. This shows that, if A is infinite dimensional, there are finitely generated indecomposable modules which do not appear as the first or last term of an almost split sequence in $\mathrm{Mod}(A)$. This problem vanishes when considering completed string algebras.

Completed string algebras A *completed string algebra* is an algebra $\Lambda = \widehat{KQ}/I$, where $I = \overline{(\rho)}$ is the closure of the ideal generated by zero-relations ρ, such that

- At any vertex of Q there are at most two arrows coming in and at most two arrows going out.
- For any arrow y of Q there is at most one arrow x such that $xy \notin \rho$ and at most one arrow z such that $yz \notin \rho$.

Let A be a possibly infinite dimensional string algebra and let $\mathfrak{n}$ be the ideal in A generated by z, where we consider A as a Noetherian algebra over $K[z]$. Then the completed string algebra Λ can also be defined as the completion of A with respect to the ideal $\mathfrak{n}$. It follows that Λ is a Noetherian algebra over the complete local ring $K[[z]]$, the ring of formal power series in one variable.

Classification of finitely generated modules The classification of the finitely generated Λ-modules is achieved in almost the same way as in the non-completed case and in this work is completely analogous to the article by Crawley-Boevey. However, one has to make slight modifications in the definition of the string and band modules. If M is a finitely generated A-string module, then we consider its completion with respect to the $\mathfrak{n}$-adic topology. This turns out to be an indecomposable finitely generated Λ-module. In case M is nilpotent with respect to $\mathfrak{n}$, it follows that M is isomorphic to its completion. If M is finitely generated as an A-module but infinite dimensional, it is not complete, that is it is not isomorphic to its completion. The general slogan here is that one has to replace direct sums in the definition of the string module by direct products. Again, the easiest example is the polynomial ring and its completion, the ring of formal power series.

For the completed string algebra, there do not exist any finite-dimensional modules, that are not nilpotent with respect to the $\mathfrak{n}$-adic topology. Hence it follows that there are no band modules corresponding to non-trivial cycles, or as Crawley-Boevey calls them, primitive simples. This actually makes the classification slightly easier in the completed setting. The following result is Theorem 3.3.15 and is the analogue of [CB4, Theorem 1.2].

Theorem A. *Let Λ be a completed string algebra. Then any finitely generated Λ-module is isomorphic to a direct sum of copies of string modules and finite-dimensional band modules.*

Noetherian algebra over complete local ring Since Λ is a Noetherian algebra over a complete local ring, we can directly deduce various extremely useful properties of the category $\mathrm{Noeth}(\Lambda) = \mathrm{mod}(\Lambda)$ of finitely generated Λ-modules. First of all there exists a duality between $\mathrm{Noeth}(\Lambda)$ and $\mathrm{Art}(\Lambda)$, the category of Artinian Λ-modules and thus the appropriate duals of all of the following statements also hold for $\mathrm{Art}(\Lambda)$. The algebra Λ is a semiperfect ring and thus every finitely generated module has a minimal projective presentation. If M is a finitely generated Λ-module, M is indecomposable if and only if $\mathrm{End}_\Lambda(M)$ is local. Furthermore, the category $\mathrm{mod}(\Lambda)$ has the Krull-Remak-Schmidt property. It was by proven by Auslander [A1, A2] in the more general setting for Noetherian algebras over complete local rings that, if M is an indecomposable finitely generated non-projective module, there exists an almost split sequence in $\mathrm{Mod}(\Lambda)$ that ends in M. Furthermore, one can define the Auslander-Reiten translation τ_Λ as for Artin algebras and the almost split sequence ending in M is of the form

$$0 \to \tau_\Lambda(M) \to E \to M \to 0,$$

where $\tau_\Lambda(M)$ is an indecomposable Artin Λ-module. This is in fact the main reason why we are considering completed string algebras. Here one can actually define what it means for a finitely generated module to be τ_Λ-rigid.

τ-tilting theory As for finite-dimensional algebras in [AIR] we say that $M \in \mathrm{mod}(\Lambda)$ is τ_Λ-*rigid* if $\mathrm{Hom}_\Lambda(M, \tau_\Lambda(M)) = 0$. If in addition we have $|M| = |\Lambda|$, where $|X|$ denotes the number of pairwise non-isomorphic indecomposable summands of any Λ-module X, we say that M is a τ_Λ-*tilting module*. We prove that any finitely generated τ_Λ-rigid module is a direct summand of some finitely generated τ_Λ-tilting module.

A pair (M, P) where $M \in \mathrm{mod}(\Lambda)$ and P is a projective Λ-module is called τ_Λ-*rigid* if M is τ_Λ-rigid and $\mathrm{Hom}_\Lambda(P, M) = 0$. If in addition we have $|M|+|P| = |\Lambda|$ (respectively $|M|+|P| = |\Lambda|-1$), we say that (M, P) is a *support* τ_Λ-*tilting* (respectively *almost complete support* τ_Λ-*tilting*) pair. The following analogue theorem of [AIR, Theorem 2.18] is our main result (see Theorem 3.5.9) concerning τ_Λ-tilting theory.

Theorem B. *Let Λ be a completed string algebra. Then any basic almost complete support τ_Λ-tilting pair for Λ is a direct summand of exactly two basic support τ_Λ-tilting pairs.*

Auslander-Reiten sequences ending in finitely generated string modules For the proof of this result we need the description of almost split sequences containing finitely generated string and band modules in $\mathrm{Mod}(\Lambda)$. As it turns out, the combinatorial description is very similar to the description for finite-dimensional string algebras. Unfortunately, we were not able to simply adopt the proof from Butler and Ringel since one cannot restrict to considering finitely generated modules. It is not quite obvious what happens when one applies the functorial filtration method to arbitrary, not necessarily finitely generated or Artinian Λ-modules. In our proof we use that any finitely generated Λ-string module M is isomorphic to an inverse limit

$$M \cong \varprojlim M_p,$$

where for p large enough M_p is a finitely generated string module over the *p-truncation* A_p of the string algebra A. The p-truncation A_p is a finite-dimensional string algebra and hence in $\mathrm{mod}(A_p)$ we have the description of almost split sequence as in [BR]. We describe appropriate analogues of canonical exact sequences for finite-dimensional string algebras and prove the following result in Theorem 3.4.2.

Theorem C. *The canonical exact sequences are the almost split sequences ending in finitely generated string modules.*

An example arising from cluster algebra theory An interesting example of a completed string algebra, given by the well-studied Markov quiver and relations, arises as the Jacobian algebra of the once-punctured torus and a non-degenerate potential. In this case it is true that any basic finite-dimensional almost complete τ_Λ-tilting module is a direct summand of exactly two basic finite-dimensional τ_Λ-tilting modules. This is implied by results by Derksen, Weyman and Zelevinsky on quivers with potentials and their representations. In fact the support τ_Λ-tilting quiver, as defined in [AIR], in this example consists of two isomorphic components, one containing all the finite-dimensional τ_Λ-tilting modules, and the other containing all finitely generated but infinite dimensional support τ_Λ-tilting pairs. This follows from results in [Ri]. One can deduce (at least for gentle algebras) combinatorial requirements for the quiver and the relations defining a completed string algebra which ensure the same results as in this particular example. However, it is obviously not true in general that mutation of (support) τ_Λ-tilting pairs is possible within the class of finite-dimensional modules.

1.3 Perpendicular categories

Definition of the perpendicular category Perpendicular categories in the context of representation theory of finite-dimensional algebras were first studied by Geigle-Lenzing [GL] and Schofield [Scho]. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ be a system of objects in $\mathcal{A}$. Then the *(right) perpendicular category* $\mathcal{S}^\perp$ is defined as the full subcategory of all objects $M \in \mathcal{A}$ which satisfy both

$$\operatorname{Hom}_{\mathcal{A}}(S, M) = 0$$
$$\operatorname{Ext}^1_{\mathcal{A}}(S, M) = 0$$

for all $S \in \mathcal{S}$. It is easily seen, that if all objects in $\mathcal{S}$ have projective dimension at most 1, the category $\mathcal{S}^\perp$ is abelian again.

Perpendicular categories for path algebras Suppose that $A = KQ$ is the path algebra of a finite acyclic quiver Q. Thus A is a finite-dimensional hereditary algebra, or equivalently an algebra of global dimension at most 1. It follows that any rigid module is a partial tilting module. It was proven by Schofield using Bongartz's exact sequence for partial tilting modules that for an indecomposable rigid A-module M we have an equivalence of categories

$$M^\perp \simeq \operatorname{mod}(KQ'),$$

where Q' is a quiver having one vertex less than Q. In case M is a projective module this process is also referred to as the *deletion of a vertex*. The proof also uses that K is algebraically closed and the fact that any hereditary algebra over an algebraically closed field is isomorphic to the path algebra of some quiver.

Perpendicular categories for finite-dimensional algebras Geigle and Lenzing considered perpendicular categories in the more general context of abelian categories. If M is in object in an abelian category $\mathcal{A}$, such that M is rigid, of projective dimension at most 1 and such that $\operatorname{Hom}_{\mathcal{A}}(M, A)$ and $\operatorname{Ext}^1_{\mathcal{A}}(M, A)$ are of finite length over $\operatorname{End}_{\mathcal{A}}(M)$, then there exists a functor $L\colon \mathcal{A} \to M^\perp$ which is left adjoint to the inclusion functor $M^\perp \to \mathcal{A}$. The proof of this uses again a slight generalization of Bongartz's exact sequence. They apply this to the case where $\mathcal{A}$ is the module category of a ring or in particular of a finite-dimensional algebra to prove the existence of a *homological ring epimorphism* induced by the functor L. The more precise result is the following: Let A be a finite-dimensional algebra and $M \in \operatorname{mod}(A)$ an indecomposable partial tilting module such that $\operatorname{End}_A(M)$ is a skew-field and $\operatorname{Hom}_A(M, A) = 0$. Then there exists a finite-dimensional algebra A' and a homological epimorphism $\varphi\colon A \to A'$ which is also injective, such that

- $M^{\perp} \simeq \operatorname{mod}(\operatorname{End}_A(LA)^{\mathrm{op}}) \simeq \operatorname{mod}(A')$;
- $T = M \oplus \varphi_*(A')$ is a tilting module in $\operatorname{mod}(A)$;
- $\operatorname{gl.dim}(A') \leq \operatorname{gl.dim}(A)$;
- $|_{A'}A'| = |_A A| - 1$.

Iwanaga-Gorenstein algebras Our aim in this work was to generalize Schofield's result to a class of 1-Iwanaga-Gorenstein algebras defined via quivers with relations associated with symmetrizable Cartan matrices. These algebras were recently introduced by Geiß, Leclerc and Schröer in [GLS1] as a generalization of path algebras of quivers associated with symmetric Cartan matrices. Recall that a finite-dimensional algebra A is 1-*Iwanaga-Gorenstein*, if

$$\operatorname{inj.dim}(A) \leq 1 \quad \text{and} \quad \operatorname{proj.dim}(DA) \leq 1$$

where D denotes the standard K-duality. This implies that in fact $\operatorname{inj.dim}(A) = \operatorname{proj.dim}(DA)$ and for any $M \in \operatorname{mod}(A)$ the following are equivalent:

- $\operatorname{proj.dim}(M) \leq 1$;
- $\operatorname{inj.dim}(M) \leq 1$;
- $\operatorname{proj.dim}(M) < \infty$;
- $\operatorname{inj.dim}(M) < \infty$.

Particular examples of 1-Iwanaga-Gorenstein algebras are hereditary and selfinjective algebras.

In general a 1-Iwanaga-Gorenstein algebra A can be of infinite global dimension. Thus the result of Geigle and Lenzing does not tell us much about the new algebra A'. Still the following theorem as in Corollary 4.3.7 is basically a direct consequence of the proofs and results in [GL].

Theorem D. *Let A be a finite-dimensional 1-Iwanaga-Gorenstein algebra and $M \in \operatorname{mod}(A)$ an indecomposable partial tilting H-module such that $\operatorname{End}_A(M)$ is a skew-field and $\operatorname{Hom}_A(M, A) = 0$. Then there is an equivalence of categories*

$$M^{\perp} \simeq \operatorname{mod}(A'),$$

where A' is a 1-Iwanaga-Gorenstein algebra having one simple module less than A.

Algebras associated with symmetrizable Cartan matrices Let C be a symmetrizable generalized Cartan matrix, Ω an acyclic orientation of it and D a diagonal matrix which is a symmetrizer of C. We denote by $H = H(C, D, \Omega)$ the finite dimensional K-algebra, where K is an arbitrary field, as defined in [GLS1]. This algebra is defined via a quiver with relations, where the quiver may have loops but no cycles passing through more than one vertex. In case C is symmetric and D is the identity, then H is the classical path algebra associated with C. In general H is not hereditary but it is 1-Iwanaga-Gorenstein.

For this new class of algebras $H = H(C, D, \Omega)$ Geiß, Leclerc and Schröer prove several generalizations of classical results for hereditary algebras. Let $A = A(C, D, \Omega)$ be the corresponding hereditary algebra defined via species. There appears to be a strong connection between the indecomposable partial H-tilting modules and the indecomposable partial A-tilting modules. The partial H-tilting modules are part of the class of modules, which are called τ-*locally free* modules in [GLS1]. The following analogue of Gabriel's Theorem is one of the main results in [GLS1]: there are only finitely many isomorphism classes of τ-locally free H-modules if and only if C is of Dynkin type. In this case there is a bijection between the isomorphism classes of τ-locally free H-modules and the set of positive roots of the semisimple complex Lie algebra associated with C. Furthermore, if C is of Dynkin type, it is known that the class of indecomposable partial H-tilting modules coincides with the class of indecomposable τ-locally free modules. This leads us to the following conjecture.

Conjecture. *Let M be an indecomposable partial H-tilting module. Then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{mod}(H'),$$

where $H' = H(C', D', \Omega')$ is a 1-Iwanaga-Gorenstein algebra associated with a symmetrizable Cartan matrix C', whose size is one smaller than the size of C. Equivalently, the quiver defining H' has one vertex less than the quiver defining H.

If $H = H(C, D, \Omega)$ and $A = A(C, D, \Omega)$, as before there is a well behaved bijective map between the indecomposable preprojective H-modules and the indecomposable preprojective A-modules. Here a module $M \in \mathrm{mod}(H)$ is indecomposable preprojective if $M \cong \tau_H^{-k}(P)$ for some indecomposable projective H-module P and some non-negative integer k. Indecomposable preinjective H-modules are defined dually. It is easily seen that in this case M is also an indecomposable partial tilting H-module. We try to use this bijection to obtain better results for this special class of 1-Iwanaga-Gorenstein algebras. From now on assume that K is algebraically closed and let $H = H(C, D, \Omega) = KQ/I$. The following theorem is a collection of the results in Theorem 4.4.23 and Theorem 4.4.36.

Theorem E. *Let $M \in \mathrm{mod}(H)$ be an indecomposable partial tilting module. Then the following hold:*

- *If M is preprojective, then*
$$M^{\perp} \simeq \mathrm{mod}(H'),$$
where $H' \cong KQ'/I'$ is a 1-Iwanaga-Gorenstein algebra, Q' has one vertex less than Q and in Q' there are no cycles passing through more than one vertex.
- *If $M \cong \tau_H^{k}(I_i) \in \mathrm{mod}(H)$ is a preinjective module, then*
$$M^{\perp} \simeq \mathrm{mod}(H'),$$
where H' is obtained from H by possibly applying a series of reflections to H and deleting the vertex i from it.

The proof of the part of the theorem concerning the preinjective modules is analogue to the proof in [St] for the hereditary case. The proof uses the description of a projective generator of $(\tau_H M)^{\perp}$ in terms of a projective generator of $M^{\perp}$, where neither M nor $\tau_H M$ are projective H-modules. It follows from this description that if $M^{\perp}$ is equivalent to $\mathrm{mod}(H')$, where $H' = H(C', D', \Omega')$, then $\tau_H M^{\perp}$ is equivalent to $\mathrm{mod}(H'')$, where H'' is obtained from H' by applying a series of reflections to it. The result for the preinjective modules follows then by induction. Unfortunately, we were not able to alter this line of proof to the preprojective modules. However, we strongly conjecture that the corresponding result is true for preprojective modules. If the Cartan matrix C is of Dynkin type, the conjecture follow trivially, since in that case the preprojective modules coincide with the preinjective modules.

Corollary. *If C is of Dynkin type and $M \in \mathrm{mod}(H)$ an indecomposable partial tilting module, then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{mod}(H'),$$

where H' is obtained from H by changing its orientation and deleting a vertex from it.

Finally, we prove the conjecture for preprojective H-modules for $H = H(C, D, \Omega)$, where C is of extended Dynkin type $\tilde{C}_n$ and D is a minimal symmetrizer. We study the algebras of this type more closely for two reasons. Firstly, the Cartan matrix C is not of Dynkin type, and thus there are rigid module, which are not preprojective or preinjective. Secondly, the algebra H is a finite-dimensional string algebra. This allows us to explicitly describe its Auslander-Reiten quiver and to classify the indecomposable rigid modules. We prove that for any indecomposable rigid H-module M, the category $M^{\perp}$ is equivalent to $\mathrm{mod}(H')$, where H' is a 1-Iwanaga-Gorenstein algebra, such that $|H'| = |H| - 1$.

1.4 Structure of this thesis

The thesis is organized as follows: In Chapter 2 we recall some basic definitions and fix notation which will be used throughout the whole thesis.

The results concerning τ-tilting theory for completed string algebras are proven in Chapter 3. In Section 3.1 we recall the classification of finitely generated modules over a possibly infinite dimensional algebra due to Crawley-Boevey and the description of the Auslander-Reiten sequences for finite-dimensional string algebras due to Butler and Ringel. In Section 3.2 we give the definition of a completed string algebra and prove that it is a Noetherian algebra over a complete local ring. Theorem A, that is the classification of finitely generated modules over a completed string algebra is given in Section 3.3, which is organized as the article of Crawley-Boevey. Section 3.4 is devoted to the proof of Theorem C the description of the Auslander-Reiten sequences ending in indecomposable finitely generated modules over a completed string algebra. The main result, Theorem B, i.e. the mutation theorem is proven in Section 3.5.

In chapter 4 we study the theory of perpendicular categories. We begin by recalling the results of Geigle and Lenzing for finite-dimensional algebras in Section 4.1. In Section 4.2 we collect properties of hereditary algebras. This will be needed to deduce properties of the algebras defined via quivers with relations for symmetrizable matrices. Furthermore, we consider the results on perpendicular categories for hereditary algebras. We briefly introduce 1-Iwanaga-Gorenstein algebras and prove Theorem D in Section 4.3. Then, in Section 4.4 we study in more detail the special class of 1-Iwanaga-Gorenstein algebras for symmetrizable Cartan matrices as defined by Geiß, Leclerc, and Schröer and prove Theorem E. Finally, Section 4.5 contains a detailed study of the category $\mathrm{mod}(H)$, where H is a 1-Iwanaga-Gorenstein algebra of extended Dynkin type $\tilde{C}_n$.

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2 Basic definitions and notation

Notation In this thesis K always denotes a field. In Sections 4.4 and 4.5 we will assume that K is algebraically closed. By an *algebra* we mean an associative K-algebra with 1. For an algebra A we denote by $\mathrm{Mod}(A)$ the category of all left A-modules and by $\mathrm{mod}(A)$ the full subcategory of finitely generated A-modules. Furthermore, let $\mathrm{proj}(A)$ and $\mathrm{inj}(A)$ be the full subcategory of $\mathrm{Mod}(A)$ with objects the projective and injective A-modules, respectively.

From now on suppose that A is a finite-dimensional K-algebra. We denote by $D = \mathrm{Hom}_K(-, K)$ the standard K-duality and by $\tau = \tau_A$ the Auslander-Reiten translation of A. For a module $M \in \mathrm{mod}(A)$ we denote by $\mathrm{add}(M)$ the subcategory of modules which are isomorphic to finite direct sums of direct summands of M, and by $\mathrm{Fac}(M)$ the subcategory of modules which are generated by M, where a module X is generated by M if there is an epimorphism $M^n \to X$ for some natural number n. We denote by $|M|$ the number of nonisomorphic indecomposable direct summands of $M \in \mathrm{mod}(A)$. The projective dimension (respectively injective dimension) of M will be denoted by $\mathrm{proj.\,dim}(M)$ (respectively $\mathrm{inj.\,dim}(M)$).

Quivers A *quiver* $Q = (Q_0, Q_1) = (Q_0, Q_1, h, t)$ is a finite directed graph, where Q_0 denotes the finite set of *vertices* and Q_1 the finite set of *arrows*. The maps $h, t\colon Q_1 \to Q_0$ assign a *head* and a *tail* to each arrow. By definition a *path of length k* in Q is a sequence of arrows $p = \alpha_1\alpha_2 \dots \alpha_k$ such that $h(\alpha_{i+1}) = t(\alpha_i)$ for all $1 \leq i \leq k-1$. The head of p is $h(\alpha_1)$ and the tail of p is $t(\alpha_k)$. Note that multiple arrows between two vertices and also loops, where a *loop* is an arrow α with $h(\alpha) = t(\alpha)$, are allowed.

Additionally, there is a path e_v of length 0 for every vertex $v \in Q_0$, and the head and tail of e_v is the vertex v by definition. A *cycle* in Q is a path of positive length with the same head and tail. Thus loops are particular examples of cycles.

Path algebras For every $m \in \mathbb{N}$ we denote by Q_m the set of paths of length m and we define KQ_m to be the K-vector space with basis the elements in Q_m. We do not make any distinction between a path of length m and the corresponding basis vector in KQ_m.

As a vector space we define the *path algebra* of Q by

$$KQ = \bigoplus_{m=0}^{\infty} KQ_m.$$

The *product* $(\alpha_1 \dots \alpha_m)(\alpha_{m+1} \dots \alpha_{m+k})$ of two paths is 0 unless $t(\alpha_m) = h(\alpha_{m+1})$, in which case it is given by concatenation of the paths $(\alpha_1 \dots \alpha_m\alpha_{m+1} \dots \alpha_{m+k})$. The product of arbitrary elements in KQ is given by linear extension of the product on paths. It is well-known that the path algebra is finite-dimensional if and only if there are no cycles in Q.

Representations of quivers A *representation of a quiver* $Q = (Q_0, Q_1, h, t)$ is a tuple $M = (M_v, M_\alpha)_{v \in Q_0, \alpha \in Q_1}$, where M_v is a K-vector space for each vertex $v \in Q_0$ and $M_\alpha\colon M_{t(\alpha)} \to M_{h(\alpha)}$ is a linear map for every arrow $\alpha \in Q_1$. We denote by $\mathrm{Rep}(Q) = \mathrm{Rep}_K(Q)$ the abelian K-category of all representations of Q. Here morphisms between two representations are defined in the obvious way. A representation M is called *finite-dimensional* if M_v is finite-dimensional for all $v \in Q_0$. The full subcategory of finite-dimensional representations is denoted by $\mathrm{rep}(Q) = \mathrm{rep}_K(Q)$.

Denote by $\mathrm{Mod}(KQ)$ the category of left KQ-modules. Then it is well-known that $\mathrm{Rep}_K(Q)$ and $\mathrm{Mod}(KQ)$ are equivalent categories, and we will make no distinction between modules and representations. If KQ is finite-dimensional the category of finitely generated KQ-modules, denoted by $\mathrm{mod}(KQ)$, coincides with the category of finite-dimensional modules and is equivalent to $\mathrm{rep}_K(Q)$.

If $M = (M_v, M_\alpha)_{v,\alpha}$ is a finite-dimensional representation of Q we call

$$\underline{\dim}(M) = (\dim M_v)_{v \in Q_0} \in \mathbb{N}^{Q_0}$$

its *dimension vector*.

For every vertex $v \in Q$ there is a *simple representation*

$$(\mathcal{S}_v)_u = \begin{cases} K & \text{if } v = u, \\ 0 & \text{otherwise.} \end{cases}$$

Note, that if there are cycles in Q, there are many more simple representations than just the $\mathcal{S}_v$.

Quivers with relations A *relation* in Q is an element in KQ of the form

$$r = \sum_{i=1}^{k} \lambda_i p_i,$$

where $\lambda_i \in K^*$ for all i and the p_i are paths of length at least 2 in Q such that $h(p_i) = h(p_j)$ and $t(p_i) = t(p_j)$ for all i, j. If $k = 1$ we call r a *zero-relation* and if $k = 2$ we call it a *commutativity relation*.

If $M = (M_v, M_\alpha)_{v,\alpha}$ is a representation of Q, and $p = \alpha_1 \alpha_2 \ldots \alpha_k$ a path we set

$$M_p = M_{\alpha_1} \circ M_{\alpha_2} \circ \cdots \circ M_{\alpha_k}.$$

For a relation $r = \sum_{i=1}^{k} \lambda_i p_i$ we extend this definition linearly by setting

$$M_r = \sum_{i=1}^{k} \lambda_i M_{p_i}$$

and say that M *satisfies the relation* r if $M_r = 0$.

Let $\rho = \{r_i \mid i \in I\}$ be a set of relations in Q. We denote by $\mathrm{Rep}(Q, \rho)$ (respectively $\mathrm{rep}(Q, \rho)$) the category of all representations (respectively finite-dimensional representations) that satisfy all the relations in ρ. Let (ρ) be the ideal generated by the relations in ρ. Then the category $\mathrm{Mod}(KQ/(\rho))$ is equivalent to the category $\mathrm{Rep}(Q, \rho)$. For more details on path algebras, representations of quivers and quivers with relations we refer to [ARS], [ASS] and [CB2].

Tilting modules Let A be a finite-dimensional K-algebra. We call a module $M \in \mathrm{mod}(A)$ *rigid* if $\mathrm{Ext}^1_A(M, M) = 0$. If in addition M is indecomposable, we call it an *exceptional* module.

We call a module $M \in \mathrm{mod}(A)$ a *partial tilting module* if $\mathrm{proj.dim}(M) \leq 1$ and if it is rigid. A partial tilting module M is called a *tilting module* if there exists a short exact sequence

$$0 \to A \to T' \to T'' \to 0$$

with $T', T'' \in \mathrm{add}(M)$. It is a result of Bongartz [B] that any partial tilting module can be completed to a tilting module, that is if $M \in \mathrm{mod}(A)$ is a partial tilting module, there exists a module $T \in \mathrm{mod}(A)$ such that $M \oplus T$ is a tilting module.

Denote by $|M|$ the number of nonisomorphic indecomposable summands of M. Then it follows from Bongartz's result, that a partial tilting module M is a tilting module if and only if $|M| = |A|$. We say that a partial tilting module M is an *almost complete tilting module* if $|M| = |A| - 1$. Any almost complete tilting module can be completed in 1 or 2 ways to a tilting module. This follows from work by Rietdmann and Schofield [RS] and Unger [U], where tilting modules were considered as combinatorial objects for the first time.

A pair (M, P) with $M \in \mathrm{mod}(A)$ and $P \in \mathrm{proj}(A)$ is called a *support tilting pair* (respectively *almost support tilting pair*) if M is a partial tilting module, $\mathrm{Hom}_A(P, M) = 0$ and $|M| + |P| = |A|$ (respectively $|M| + |P| = |A| - 1$). If A is a finite-dimensional path algebra, any almost complete support tilting pair can be completed in exactly two ways to a support tilting pair. However, even in the more general class of support tilting pairs, this result is not true for finite-dimensional algebras. For this see the example at the end of this chapter. A collection of articles on tilting theory and a lot of references concerning this topic can be found in [AHK].

Auslander-Reiten translation Let A be a finite-dimensional algebra and let

$$D = \mathrm{Hom}_K(-, K)\colon \mathrm{mod}(A) \to \mathrm{mod}(A^{\mathrm{op}}) \quad \text{and} \quad (-)^* = \mathrm{Hom}_A(-, A)\colon \mathrm{proj}(A) \to \mathrm{proj}(A^{\mathrm{op}})$$

be the well-known dualities, where A^{op} denotes opposite algebra of A. Further let

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

be a minimal projective presentation of $M \in \mathrm{mod}(A)$, that is an exact sequence such that $p_0\colon P_0 \to M$ and $p_1\colon P_1 \to \mathrm{Ker}(p_0)$ are projective covers. Applying the functor $(-)*$, yields an exact sequence

$$0 \to M^* \xrightarrow{p_0^*} P_0^* \xrightarrow{p_1^*} P_1^* \to \mathrm{Cok}(p_1^*) \to 0$$

of right A-modules. We define $\mathrm{Tr}(M) = \mathrm{Cok}(p_1^*)$ and call it the *transpose*. We have that $\mathrm{Tr}(M) = 0$ if and only if M is projective.

Denote by $\underline{\mathrm{mod}}(A)$ the *projectively stable category* and by $\overline{\mathrm{mod}}(A)$ the *injectively stable category.* Then the transpose induces a duality

$$\mathrm{Tr}\colon \underline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(A^{\mathrm{op}})$$

called the *Auslander Reiten transpose.* Furthermore, the functors

$$\tau_A = D\,\mathrm{Tr}\colon \underline{\mathrm{mod}}(A) \to \overline{\mathrm{mod}}(A) \qquad \text{and} \qquad \tau_A^{-1} = \mathrm{Tr}\,D\colon \overline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(A)$$

are mutually inverses of categories called the *Auslander Reiten translation* and its inverse.

Recall the Auslander-Reiten formula

$$\mathrm{Ext}^1_A(M, N) \cong D\overline{\mathrm{Hom}}_A(N, \tau_A(M)) \cong D\underline{\mathrm{Hom}}_A(\tau_A^{-1}(N), M)$$

where $\underline{\mathrm{Hom}}_A(M, N)$ respectively $\overline{\mathrm{Hom}}_A(M, N)$ denotes the *stable Homomorphism space*, that is the Homomosphism space modulo morphisms factoring through projectives respectively injectives. The Auslander-Reiten formula simplifies to

$$\mathrm{Ext}^1_A(M, N) \cong D\,\mathrm{Hom}_A(N, \tau_A(M))$$

if $\mathrm{proj.\,dim}(M) \leq 1$ and

$$\mathrm{Ext}^1_A(M, N) \cong D\,\mathrm{Hom}(\tau_A^{-1}(N), M)$$

if $\mathrm{inj.\,dim}(N) \leq 1$.

Auslander-Reiten sequences Let R be an arbitrary ring and $\mathrm{Mod}(R)$ the category of left R-modules.

A morphism $v\colon E \to V$ is called *right almost split* if v is not a split epimorphism, and if $h\colon M \to V$ is not a split epimorphism, there is a morphism $h'\colon M \to E$ such that $h = vh'$.

Dually a morphism $u\colon U \to E$ is called *left almost split* if u is not a split monomorphism, and if $g\colon U \to N$ is not a split monomorphism, there is a morphism $g'\colon E \to N$ such that $g = g'u$.

An exact sequence

$$0 \to U \xrightarrow{u} E \xrightarrow{v} V \to 0$$

is called an *almost split sequence* or *Auslander-Reiten sequence* if it satisfies the following conditions:

- it is not a split sequence;
- the morphism $v\colon E \to V$ is right almost split;
- the morphism $u\colon U \to E$ is left almost split;

The following properties of an almost split sequence $0 \to U \xrightarrow{u} E \xrightarrow{v} V \to 0$ in $\mathrm{Mod}(R)$ are well-known:

- The rings $\mathrm{End}_R(U)$ and $\mathrm{End}_R(V)$ are local, and hence U and V are indecomposable.
- Suppose that $0 \to U' \xrightarrow{u'} E' \xrightarrow{v'} V' \to 0$ is also an almost split sequence. Then the following are equivalent:
  - The sequences $0 \to U \xrightarrow{u} E \xrightarrow{v} V \to 0$ and $0 \to U' \xrightarrow{u'} E' \xrightarrow{v'} V' \to 0$ are isomorphic;
  - $U \cong U'$;
  - $V \cong V'$.

  Let A be a finite-dimensional K-algebra (or more general an Artin algebra). Then we have the following existence theorems for almost split sequences:
  - If M is an indecomposable nonprojective module in $\mathrm{mod}(A)$, there exists an almost split sequence $0 \to \tau_A(M) \to E \to M \to 0$.
  - If N is an indecomposable noninjective module in $\mathrm{mod}(A)$, there exists an almost split sequence $0 \to N \to E \to \tau_A^{-1}(N) \to 0$.

Auslander-Reiten quiver Let A be a finite-dimensional K-algebra. A homomorphism $f\colon M \to N$ in $\mathrm{mod}(A)$ is called *irreducible* if f is neither a split monomorphism nor a split epimorphism and if $f = f_1 f_2$, either f_1 is a split epimorphism or f_2 is a split monomorphism.

Let $M, N \in \mathrm{mod}(A)$ be indecomposable and such that there is an irreducible homomorphism $M \to N$. Further let $g\colon E \to N$ be a right almost split map, and $f\colon M \to E'$ a left almost split map. Then there are positive integers a, b and modules X, Y such that $E \cong M^a \oplus X$, where M is not a direct summand of X and such that $E' \cong N^b \oplus Y$, where N is not a direct summand of Y.

The *Auslander-Reiten quiver* Γ_A is defined as follows. The vertices of Γ_A are given by the isomorphism classes $[M]$ of finite-dimensional indecomposable A-modules. There is an arrow $[M] \to [N]$ if and only if there is an irreducible morphism $M \to N$. The arrow has a label (a, b) if there is a minimal right almost split morphism $M^a \oplus X \to N$, where M is not a summand of X, and a minimal left almost split morphism $M \to N^b \oplus Y$, where N is not a summand of Y. In addition if M is an indecomposable nonprojective A-module we draw an arrow

$$[M] \dashrightarrow [\tau(M)]$$

to indicate the Auslander-Reiten translation. Usually the vertices corresponding to projective vertices are drawn on the left and the vertices corresponding to injective vertices are drawn on the right hand side.

For more details on Auslander-Reiten theory we refer to [ARS], [ASS] and [A1].

τ-tilting modules Let A be a finite-dimensional algebra, $M \in \mathrm{mod}(A)$ and P a projective A-module. Then

- M is *τ-rigid* if $\mathrm{Hom}_A(M, \tau_A M) = 0$,

- M is a *τ-tilting module* if M is τ-rigid and $|M| = |A|$, and
- (M, P) is a *support τ-tilting pair* (respectively *almost complete support τ-tilting pair*) if M is τ-rigid, $\mathrm{Hom}_A(P, M) = 0$ and $|M| + |P| = |A|$ (respectively $|M| + |P| = |A| - 1$).

Hence any rigid module of projective dimension at most 1 is also τ-rigid. Note that any tilting module is a τ-tilting module. Furthermore, it also follows from the Auslander-Reiten formulas that any τ-rigid module is also rigid. For more details on τ-tilting theory we refer to [AIR], where the authors also prove the following theorem.

Mutation Theorem. *Any basic almost complete support τ-tilting pair is a direct summand of exactly two basic support τ-tilting pairs.*

Example Let $A = KQ/I$ be the finite-dimensional algebra defined by the quiver

$$\begin{tikzcd}
 & 2 \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] & & 3 \arrow[ll, "\alpha_3"]
\end{tikzcd}$$

and the ideal I generated by the relations $\alpha_2\alpha_1 = \alpha_3\alpha_2 = \alpha_1\alpha_3 = 0$. Then its Auslander-Reiten quiver, containing representatives of all indecomposable modules indicated by composition factors is given by

$$\begin{tikzcd}
 & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[dr] & \\
2 \arrow[ur] & & 1 \arrow[ll, dashed, "\tau"] \arrow[ur] & & 3 \arrow[ll, dashed, "\tau"] \arrow[ur] & & 2 \arrow[ll, dashed, "\tau"]
\end{tikzcd}$$

where one has to identify the simple module at 2 on the left with the one on the right hand side. Then

$$T = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}$$

is a (support) tilting module, as its 3 nonisomorphic indecomposable summands are the projective indecomposable modules. However, we cannot replace any of its summands to obtain a new (support) tilting module, as the remaining indecomposable modules all have infinite projective dimension. This example shows, that tilting modules are in general not enough objects to model the process of mutation inspired by cluster tilting theory.

It is easy to see that all the simple modules are τ-rigid and starting with the tilting module T we obtain a new τ-tilting module T', by exchanging the summand $\begin{smallmatrix}1\\2\end{smallmatrix}$ with the simple module at vertex 3.

$$T = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \qquad \longleftrightarrow \qquad T' = 3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}$$

Similarly, one can replace any other summand of T to obtain another (support) τ-tilting module.

3 τ-tilting theory for completed string algebras

3.1 String algebras

Finite-dimensional string algebras are a well-studied class of algebras. The module category and in fact the Auslander-Reiten theory of a finite-dimensional string algebra can be described in completely combinatorial terms. The so called string and band modules form a complete list of representatives of isomorphism classes of indecomposable modules. Recently Crawley-Boevey [CB4] achieved the classification of finitely generated (and artinian) modules over a possible infinite dimensional string algebra KQ/I, using the functorial filtration method. In this section we will recall his notation and results. Note that in contrast to Crawley-Boevey, we only consider finite quivers. Thus, his notion of *finitely controlled* (respectively *pointwise artinian*) modules (see [CB4, Introduction]), coincides with finitely generated (respectively artinian) modules and will therefore be omitted. We would like to mention Ringel's work on algebraically compact modules [R5], where infinite strings were also considered.

A *string algebra* is an algebra $A = KQ/I$, where $I = (\rho)$ is an ideal generated by a set of zero-relations ρ, i.e. a set of paths of length ≥ 2 in Q such that

- at any vertex of Q there are at most two arrows coming in and at most two arrows going out,
- for any arrow y of Q there is at most one arrow x such that $xy \notin \rho$, and at most one arrow z such that $yz \notin \rho$.

Example 3.1.1. Let Q be the 1-loop quiver, i.e. the quiver with one vertex and one loop x as below

$$\begin{tikzcd} \bullet \arrow[loop left, "x"] \end{tikzcd}$$

Then $KQ \cong K[x]$ is a string algebra.

Let Q be the the quiver with one vertex and two loops, i.e.

$$\begin{tikzcd} \bullet \arrow[loop left, "x"] \arrow[loop right, "y"] \end{tikzcd}$$

and let $\rho = \{xy, yx\}$. Then $A = KQ/(\rho)$ is a string algebra, and furthermore A is isomorphic to $K[x,y]/(xy)$ the polynomial ring in two commuting variables modulo the ideal generated by (xy).

For the same quiver we can also choose the relations $\rho = \{x^2, y^2\}$, and then $A = KQ/(\rho) \cong K\langle x,y\rangle/(x^2,y^2)$ is a string algebra, where $K\langle x,y\rangle$ is the polynomial ring in two non-commuting variables.

3.1.1 Words

For the combinatorial description of the indecomposable modules over a string algebra we will use words in the direct and inverse letters, satisfying certain conditions. Roughly speaking, a word is a possibly infinite path in the underlying graph of Q, which avoids the zero-relations in ρ. These words are also used in the finite-dimensional case, where they are usually referred to as strings and bands. But in contrast to the finite-dimensional setting we will also have to consider infinite words.

Let $A = KQ/(\rho)$ be a string algebra. For any arrow $x \in Q_1$ we define a formal inverse x^{-1} with $h(x^{-1}) = t(x)$ and $t(x^{-1}) = h(x)$, and we set $(x^{-1})^{-1} = x$. We say, x is a *direct letter*, and x^{-1} is an *inverse letter*.

Let I be one of the sets $\{0, 1, \dots, n\}$ for $n \geq 1$ or $\mathbb{N}$ or $-\mathbb{N} = \{0, -1, -2 \dots\}$ or $\mathbb{Z}$. Then an *I-word* or *word* C is a sequence of letters

$$C = \begin{cases} C_1 C_2 \dots C_n & \text{if } I = \{0, 1, \dots, n\}, \\ C_1 C_2 C_3 \dots & \text{if } I = \mathbb{N}, \\ \dots C_{-2} C_{-1} C_0 & \text{if } I = -\mathbb{N}, \\ \dots C_{-2} C_{-1} C_0 \mid C_1 C_2 \dots & \text{if } I = \mathbb{Z}, \end{cases}$$

satisfying:

- if C_i and C_{i+1} are consecutive letters, then the tail of C_i is the head of C_{i+1} and $C_i^{-1} \neq C_{i+1}$ and
- no zero relation in ρ nor its inverse appears as a sequence of consecutive letters in C.

In addition for any vertex v in Q we define the *trivial* I-words $1_{v,\varepsilon}$ for the set $I = \{0\}$ and for $\varepsilon = \pm 1$. Let C be an I-word. The *inverse* of C is the word denoted by C^{-1} and is given by inverting the letters of C and reversing their order. The inverse of the trivial words is defined by $(1_{v,\varepsilon})^{-1} = 1_{v,-\varepsilon}$. If C is an $\mathbb{N}$-word, then C^{-1} is an $(-\mathbb{N})$-word and vice versa. A word C is called *direct* or *inverse* if every letter of C is direct or inverse respectively.

If C is a $\mathbb{Z}$-word we define the *shift of C by n* for $n \in \mathbb{Z}$, as the word $C[n] = \dots C_n \mid C_{n+1} \dots$. The shift can be extended to any I-word, with $I \neq \mathbb{Z}$, by setting $C[n] = C$. We define an equivalence relation $\sim$ on the set of all words, where two words C and D are equivalent if and only if $D = C[n]$ or $D = C^{-1}[n]$ for some $n \in \mathbb{Z}$. A $\mathbb{Z}$-word C is called *periodic* if $C = C[n]$ for some $n > 0$, and in that case n is called the *period* of C.

3.1.2 Classification of finitely generated modules

String modules For any I-word C we define an A-module $M(C)$ as follows: we choose symbols $b_i\ (i \in I)$ and as a K-vector space we set

$$M(C) = \bigoplus_{i \in I} K b_i,$$

hence any element in $M(C)$ is a finite linear combination in the basis symbols $b_i\ (i \in I)$. Now, we define the action of the trivial paths and arrows on $M(C)$. For any vertex v we define $e_v b_i = b_i$ if the tail of C_i is v, and $e_v b_i = 0$ otherwise. For any arrow $x \in Q_1$ we define

$$x b_i = \begin{cases} b_{i-1} & (\text{if } i-1 \in I \text{ and } C_i = x) \\ b_{i+1} & (\text{if } i+1 \in I \text{ and } C_{i+1} = x^{-1}) \\ 0 & (\text{otherwise}). \end{cases}$$

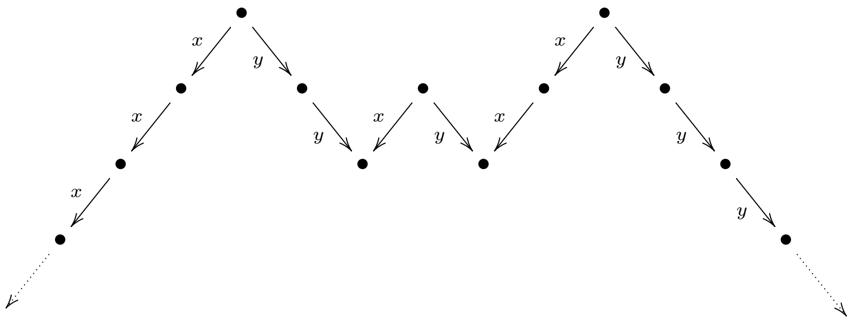
Now it is obvious from the definition of words, that $M(C)$ is a representation of Q, satisfying all relations in ρ, and thus all relations in the ideal (ρ) in KQ. In other words, with the above definition we have a well-defined action of A on the module $M(C)$. If C is a non-periodic word, we call $M(C)$ a *string module*.

The module $M(C)$ can be pictured, using what we call *the quiver of C*. This quiver has the set $b_i\ (i \in I)$ as vertices. There is an arrow from b_i to b_{i-1} if and only if C_i is a direct letter x and then we label this arrow with x. There is an arrow from b_i to b_{i+1} if and only if C_{i+1} is an inverse letter x^{-1} and then we label this arrow with x. Thus the quiver of an I-word is a quiver of type A_{n+1} if $I = \{0, 1, \dots, n\}$ of type A_∞ if $I = \mathbb{N}$ and of type ${}_\infty A_\infty$ if $I = \mathbb{Z}$. It is convenient to draw the quiver in zickzack shape, such that sources are above sinks.

Example 3.1.2. Let $A = K[x,y]/(xy)$ and consider the $\mathbb{Z}$-word

$$C = \ldots xxxy^{-1}y^{-1}xy^{-1}xxy^{-1}y^{-1}y^{-1}\ldots.$$

Then its quiver is given by the quiver of type ${}_\infty A_\infty$



where in place of the basis vectors b_i we used bullets for the vertices.

Lemma 3.1.3. *If C, D are two words such that $C \sim D$, then the modules $M(C)$ and $M(D)$ are isomorphic.*

Proof. For any I-word C there is an isomorphism $i_C\colon M(C) \to M(C^-)$ given by reversing the basis, and for any $n \in \mathbb{Z}$ there is an isomorphism $t_{C,n}\colon M(C) \to M(C[n])$ given by the identity for $I \neq \mathbb{Z}$, and for a $\mathbb{Z}$-word by $t_{C,n}(b_i) = b_{i-n}$. Now if C and D are any equivalent words, a composition of these isomorphisms yields an isomorphism of $M(C)$ and $M(D)$. $\square$

Since we are mainly interested in the isomorphism classes of modules we sometimes make no distinction between equivalent words. However, there are situations where we have to be more careful and work with the labelling of the letters of a given words.

Band modules If C is a periodic $\mathbb{Z}$-word with period n, the isomorphism $t_{C,n}$ induces an action of $K[T,T^{-1}]$ via

$$b_i \cdot T := t_{C,n}(b_i) = b_{i-n}$$

which is compatible with the action of A, since if $C_i = x$ for an arrow x we have

$$(xb_i)T = t_{C,n}(b_{i-1}) = b_{i-1-n}$$

and since C is n-periodic we have $C_{i-n} = C_i = x$ and thus

$$x(b_iT) = x(b_{i-n}) = b_{i-1-n}$$

and similarly for $C_i = x^{-1}$. Thus $M(C)$ becomes an A-$K[T,T^{-1}]$-bimodule and for any $K[T,T^{-1}]$-module V we can define a new A-module by

$$M(C,V) = M(C) \otimes_{K[T,T^{-1}]} V.$$

Note that, since $M(C)$ is free over $K[T,T^{-1}]$ of rank n, $M(C,V)$ is finite-dimensional if and only if V is finite-dimensional.

If C is periodic and V is an indecomposable $K[T,T^{-1}]$-module, we call $M(C,V)$ a *band module.* In the following we want to recall the main results from [CB4].

Theorem 3.1.4. *String modules and finite-dimensional band modules are indecomposable. The only isomorphisms between such modules are those arising from equivalence relation of words. To be more precise, there are no isomorphisms between string modules and modules of the form*

$M(C,V)$; two string modules $M(C)$ and $M(D)$ are isomorphic if and only if C and D are equivalent words; two modules of the form $M(C,V)$ and $M(D,W)$ are isomorphic if and only if $D = C[m]$ and $W \cong V$ or $D = (C^{-1})[m]$ and $W \cong res_\iota V$ for some m, where ι is the automorphism of $K[T, T^{-1}]$ exchanging T and T^{-1} and res_ι denotes the restriction map via ι.

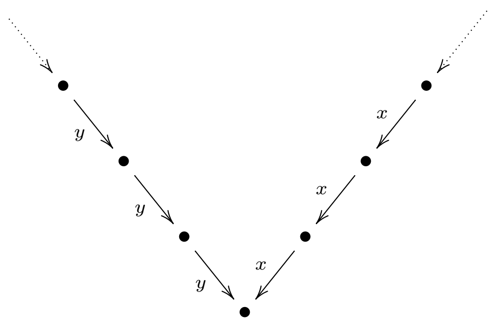
The following is [CB4, Theorem 1.2.] and one of the main results in this article.

Theorem 3.1.5. *Every finitely generated A-module is isomorphic to a direct sum of string and finitely dimensional band modules.*

Example 3.1.6. Let $A = K[x,y]/(xy)$ and consider the $\mathbb{Z}$-word

$$C = \dots y^{-1}y^{-1}y^{-1}xxx \dots .$$

Then its quiver is given by the quiver of type ${}_\infty A_\infty$



where in place of the basis vectors b_i we used bullets for the vertices. Obviously, this represents a string module which is not finitely generated.

Finitely generated string modules We say that an I-word is *eventually inverse* (respectively *direct*) if there are only finitely many $i > 0$ in I such that C_i is a direct letter (respectively inverse). For example, if C is finite or an $-\mathbb{N}$-word, it is eventually inverse and eventually direct. The $\mathbb{Z}$-word in Example 3.1.3. is eventually inverse. The next result is [CB4, Proposition 12.1.] and together with Theorem 3.1.5 yields the classification of finitely generated modules over a string algebra.

Theorem 3.1.7. *A string module $M(C)$ is finitely generated if and only if C and C^- are eventually inverse. A direct sum of string and finite-dimensional band modules is finitely generated if and only if the sum is finite, and the string modules are finitely generated.*

Dually a string module $M(C)$ is Artinian if and only if C and C^{-1} are eventually direct.

Example 3.1.8. Let A be the string algebra $K[x]$ and let $M = K[x] = M(C)$ be the string module of the $\mathbb{N}$-word $xxx\dots$. Then obviously $\mathrm{End}_A(M) \cong K[x]$ is not a local ring. One can see more generally, that if C is an infinite word, such that C or C^- is eventually inverse, then $\mathrm{End}_A(M(C))$ is not local.

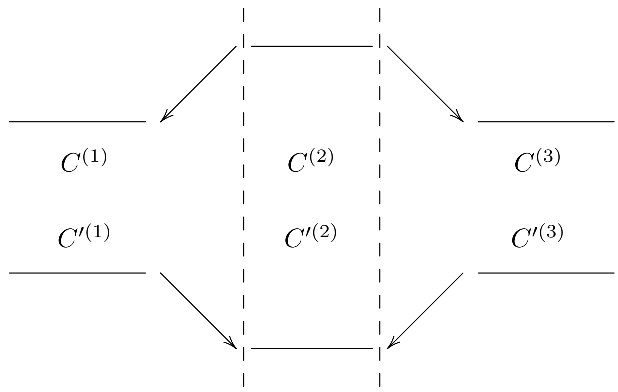
3.1.3 Standard homomorphisms and Auslander-Reiten sequences

In this section we consider a finite-dimensional string algebra A. Hence, all finitely generated modules are also finite-dimensional. So the finitely generated string modules are given by finite words and therefore, in this section we always consider finite words.

Homomorphisms between string modules In the following we will describe a basis of the homomorphism space between two string modules. This combinatorial description of the basis is due to Crawley-Boevey [CB1] and a reformulation of his construction can be found in [Schr]. Let C and C' be two words. We call a pair of triples of words

$$(a, a') = ((C^{(1)}, C^{(2)}, C^{(3)}), (C'^{(1)}, C'^{(2)}, C'^{(3)}))$$

such that $C = C^{(1)}C^{(2)}C^{(3)}$ and $C' = C'^{(1)}C'^{(2)}C'^{(3)}$ *admissible* if $C^{(2)} \sim C'^{(2)}$ and the quivers of C and C' are given by



Here it is possible that $C^{(i)}$ and $C'^{(i)}$ are words of length zero. We also say that $C^{(2)}$ is a *predecessor closed subword* of C, and $C'^{(2)}$ is a *successor closed subword* of C'. We denote the set of all admissible pairs for C and C' by $\mathscr{A}(C, C')$. The following theorem was proven in a more general setup by Crawley-Boevey.

Theorem 3.1.9 ([CB1])**.** *For any admissible pair $(a, a') \in \mathscr{A}(C, C')$ of words C, C' there is a canonical homomorphism $\theta_{(a,a')} \colon M(C) \to M(C')$. Moreover the set*

$$\{\theta_{(a,a')} \mid (a, a') \in \mathscr{A}(C, C')\}$$

is a basis of $\mathrm{Hom}_A(M(C), M(C'))$.

We call the homomorphisms of the form $\theta_{(a,a')} \colon M(C) \to M(C')$ for an admissible pair $(a, a') \in \mathscr{A}(C, C')$ *standard homomorphisms*. This description of homomorphisms can be transferred to band modules and one should keep the same picture in mind.

Remark 3.1.10. Let $M = M(C)$ and $N = M(C')$ be two string modules. Then the following is not hard to see:

(i) If $b_i^M \in M$ and $b_j^N \in N$ are basis vectors, there is at most one standard homomorphism $\theta \colon M \to N$ with $\theta(b_i^M) = \theta(b_j^N)$.

(ii) The composition of two standard homomorphisms is either 0 or again a standard homomorphism.

Auslander-Reiten sequences containing string modules We would like to give a description of Auslander-Reiten sequences containing string and band modules. These descriptions are due to Butler and Ringel [BR]. In order to phrase their results we need some more definitions.

For any arrow x we define $U(x) = M(B) \in \mathrm{mod}(A)$ where B is the longest inverse word such that Bx is a word. Similarly we define $V(x) = M(C) \in \mathrm{mod}(A)$ where C is the longest inverse word such that xC is a word. Finally we define $N(x) = M(D)$, where $D = BxC$. We call the short exact sequence

$$o \to U(x) \xrightarrow{\iota} N(x) \xrightarrow{\pi} V(x) \to 0$$

where ι is the canonical inclusion and π the canonical projection *a canonical exact sequence.*

Let C be a word. We say that

- C *starts on a peak* if there is no arrow x such that $x^{-1}C$ is a word;
- C *ends on a peak* if there is no arrow x such that Cx is a word;
- C *starts in a deep* if there is no arrow y such that yC is a word;
- C *ends in a deep* if there is no arrow y such that Cy^{-1} is a word.

If C is a word not starting on a peak, there is an arrow x such that $x^{-1}C$ is a word and a unique direct word D such that ${}_hC := Dx^{-1}C$ is a word starting in a deep. We call Dx^{-1} a *hook*.

If C is a word not ending on a peak, there is an arrow x such that Cx is a word and a unique direct word D such that $C_h := CxD^{-1}$ is a word ending in a deep. We call xD^{-1} a *hook*.

If C is a word not starting in a deep, there is an arrow y such that yC is a word and a unique direct word D such that ${}_cC := D^{-1}yC$ is a word starting on a peak. We call $D^{-1}y$ a *cohook*.

If C is a word not ending in a deep, there is an arrow y such that Cy^{-1} is a word and a unique direct word D such that $C_c := Cy^{-1}D$ is a word ending on a peak. We call $y^{-1}D$ a *cohook*.

From now on, assume that C is a word such that $M(C)$ is neither isomorphic to an injective module nor to one of the form $U(x)$. Then if C is a word neither starting nor ending on a peak, the words ${}_hC$, C_h and ${}_hC_h$ are defined and we call

$$0 \longrightarrow M(C) \xrightarrow{(\iota,\iota)} M({}_hC) \oplus M(C_h) \xrightarrow{(\iota,-\iota)} M({}_hC_h) \longrightarrow 0$$

where ι is the canonical inclusion (and π will be the canonical projection), *a canonical exact sequence*. If C is a word starting but not ending on a peak the word C_h is defined, C equals ${}_cD$ for some word D not starting on a peak and

$$0 \longrightarrow M(C) \xrightarrow{(\iota,\pi)} M(C_h) \oplus M(D) \xrightarrow{(\pi,-\iota)} M(D_h) \longrightarrow 0$$

is called *a canonical exact sequence*. If C is a word not starting but ending on a peak, the word ${}_hC$ is defined, C equals D_c for some word D not ending on a peak and

$$0 \longrightarrow M(C) \xrightarrow{(\iota,\pi)} M({}_hC) \oplus M(D) \xrightarrow{(\pi,-\iota)} M({}_hD) \longrightarrow 0$$

is *a canonical exact sequence*. If C is a word neither starting nor ending on a peak, C equals ${}_cD_c$ for some word D neither starting nor ending on a peak and

$$0 \longrightarrow M(C) \xrightarrow{(\pi,\pi)} M({}_cD) \oplus M(D_c) \xrightarrow{(\pi,-\pi)} M(D) \longrightarrow 0$$

is *a canonical exact sequence*.

The next proposition is [BR, Prop. on p.172].

Proposition 3.1.11. *The Auslander-Reiten sequences in* $\mathrm{mod}(A)$ *containing string modules are the canonical exact sequences.*

Example 3.1.12. Consider the finite-dimensional string algebra $A = K[x,y]/(xy, x^5, y^5)$ and the word

$$C = y^{-1}y^{-1}y^{-1}y^{-1}xxy^{-1}$$

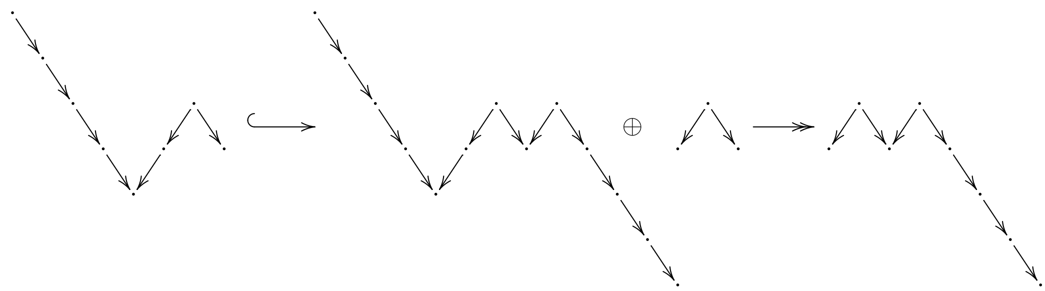
starting but not ending on a peak. We have $C = {}_cD$, where $D = xy^{-1}$ is a finite word. Attaching a hook at the end of C yields the word

$$C_h = y^{-1}y^{-1}y^{-1}y^{-1}xxy^{-1}xy^{-1}y^{-1}y^{-1}y^{-1}$$

starting on a peak and ending in a deep. Hence the canonical exact sequence

$$0 \longrightarrow M(C) \xrightarrow{(\iota,\pi)} M(C_h) \oplus M(D) \xrightarrow{(\pi,-\iota)} M(D_h) \longrightarrow 0$$

can be pictured as follows



where we drew dots corresponding to the basis vectors.

Auslander-Reiten sequences containing band modules Let $V \in \mathrm{mod}(K[T, T^{-1}])$ be an indecomposable finite-dimensional module. Then there exists an Auslander-Reiten sequence in $\mathrm{mod}(K[T, T^{-1}])$ which is of the form

$$0 \to V \to W \to V \to 0$$

for some $W \in \mathrm{mod}(K[T, T^{-1}])$. Furthermore W consists of at most two indecomposable direct summands. Let C be a periodic word for A. Then the above sequence gives rise to a short exact sequence in $\mathrm{mod}(A)$

$$0 \to M(C, V) \to M(C, W) \to M(C, V) \to 0$$

which we refer to as a *canonical exact sequence of band modules.* Note that if $W = W_1 \oplus W_2$ for W_i indecomposable, then $M(C, W) = M(C, W_1) \oplus M(C, W_2)$ and $M(C, W_i)$ are band modules for $i = 1, 2$. The next proposition is also due to Butler and Ringel [BR].

Proposition 3.1.13. *The Auslander-Reiten sequences in* $\mathrm{mod}(A)$ *containing band modules are the canonical exact sequences of band modules.*

We see that the components in the Auslander-Reiten quiver of A containing band modules are all homogeneous tubes. Furthermore, we see that band modules cannot be τ-rigid.

3.2 Completed string algebras

Completed Path algebras Let Q be a finite quiver. Then the *completed path algebra* is defined by

$$\widehat{KQ} := \varprojlim KQ/\mathfrak{m}^n$$

i.e. it is the completion of the path algebra with respect to the maximal ideal $\mathfrak{m}$ in KQ spanned by all arrows, also called *arrow ideal.* More precisely, let KQ_n be the K-span of all paths of length n in Q. One can identify $\widehat{KQ}$ canonically with

$$\prod_{n \geq 0} KQ_n$$

where the multiplication is induced by the multiplication on KQ. We write elements in $\widehat{KQ}$ as infinite sums $\sum_{n \geq 0} a_n$ with $a_n \in KQ_n$ and then the product of two elements is given by

$$(\sum_{i \geq 0} a_i)(\sum_{j \geq 0} b_j) = \sum_{k \geq 0} \sum_{i+j=k} a_i b_j.$$

Thus, the completed path algebra is a topological K-algebra via the $\mathfrak{m}$-adic topology. For a subset X of $\widehat{KQ}$ we denote by $\overline{X}$ the closure of X. For $a \in \widehat{KQ}$ the sets $a+\mathfrak{m}^n$ form a fundamental system of open neighbourhoods. Since the intersection over all $n \in \mathbb{N}$ of $\mathfrak{m}^{\mathrm{n}}$ is $\{0\}$, it follows that $\widehat{KQ}$ is a regular Hausdorff space. In particular, for any subset X of $\widehat{KQ}$ we have

$$\overline{X} = \bigcap_{n\in\mathbb{N}} X + \mathfrak{m}^n.$$

Note, that in general the ideal (ρ) in $\widehat{KQ}$ generated by relations ρ is not the same as its closure $\overline{(\rho)}$. An example for this and a more detailed study of $\widehat{KQ}$ as a topological K-algebra can be found in the master thesis [Ge].

Completed path algebras have appeared more often in the representation theory of finite-dimensional algebra, since the introduction of Jacobian algebras by Derksen, Weyman and Zelevinsky in 2008. In [DWZ], they call a finite-dimensional $\widehat{KQ}$-module M *nilpotent*, if it is annihilated by some power of the arrow ideal, say $\mathfrak{m}^s M$ for $s \gg 0$. Furthermore, using an argument of Crawley-Boevey they prove that any finite-dimensional $\widehat{KQ}$-module is nilpotent. We say that a KQ-module M is *locally nilpotent*, if for any $x \in M$ there exists some $s \gg 0$ such that $\mathfrak{m}^s x = 0$.

Proposition 3.2.1. *Let M be a KQ-module which is locally nilpotent. Then the action of KQ extends naturally to an action of $\widehat{KQ}$. Furthermore, if N is any $\widehat{KQ}$ module, then any KQ-module homomorphism from M to N is also a $\widehat{KQ}$-module homomorphism.*

Proof. Let $a = \sum_{n\geq 0} a_n \in \widehat{KQ}$, where $a_n \in KQ_n$ and let $m \in M$. Then

$$am = \sum_{n\geq 0}(a_n m)$$

is a finite sum since $a_s m = 0$ for all $s \gg 0$, and is hence a well defined action on M.

Let N be any $\widehat{KQ}$-module and $f\colon M \to N$ an KQ-module homomorphism. Then for any $a = \sum_{n\geq 0} a_n \in \widehat{KQ}$ and $m \in M$ we have

$$f(am) = f(\sum_{n\geq 0}(a_n m)) = \sum_{n\geq 0}(a_n f(m)) = af(m)$$

where we used the finiteness of the sum and the KQ-linearity of f. Hence f is a $\widehat{KQ}$-module homomorphism. □

Completed string algebras A *completed string algebra* is an algebra

$$\Lambda = \widehat{KQ}/\overline{(\rho)},$$

where $\overline{(\rho)}$ is the closure of the ideal generated by zero-relations in ρ, such that

- at any vertex of Q there are at most two arrows coming in and at most two arrows going out and
- for any arrow y of Q there is at most one arrow x such that $xy \notin \rho$ and at most one arrow z such that $yz \notin \rho$.

Let $A = KQ/(\rho)$ be a string algebra and $\Lambda = \widehat{KQ}/\overline{(\rho)}$ the completed string algebra, defined by the same quiver and relations. Further for any $n \geq 2$ denote by

$$A_n = KQ/((\rho) + \mathfrak{m}^n) = \widehat{KQ}/(\overline{(\rho)} + \mathfrak{m}^n)$$

the *n-truncation* of A respectively of Λ. Then there is an inverse system

$$\cdots \to A_n \to \cdots \to A_3 \to A_2$$

of algebras and we have

$$\Lambda = \varprojlim(A_n).$$

Later on we will prefer to consider Λ as an inverse system of factor algebras of A. Note that by the third isomorphism theorem for rings we have

$$A_p = \frac{KQ}{(\rho) + \mathfrak{m}^n} \cong \frac{KQ/(\rho)}{((\rho) + \mathfrak{m}^n)/(\rho)} = \frac{A}{((\rho) + \mathfrak{m}^n)/(\rho)}$$

and these isomorphisms are compatible with the transition maps. Thus, if we set

$$\mathfrak{m}_n = ((\rho) + \mathfrak{m}^n)/(\rho) \subseteq A$$

this defines an descending filtration of ideals of A such that

$$\Lambda = \varprojlim A/\mathfrak{m}_n.$$

Example 3.2.2. Let $A = K[x,y]/(xy)$ be the string algebra as in Example 3.1.1. Then the corresponding completed string algebra is given by $\Lambda = K\langle\langle x,y\rangle\rangle/\overline{(xy,yx)}$ and from the above description it is not hard to see that this is isomorphic to $K[[x,y]]/(xy)$. In this case it is easy to see that the closure of the ideal (xy) in $K[[x,y]]$ is the ideal itself, since the algebra is commutative. Thus the two-sided ideal generated by xy, is the same as the left-ideal and it follows from [Ge, Lemma 2.5.5.] that this is a closed subspace.

Let $A = K\langle x,y\rangle/(x^2,y^2)$ be the string algebra from Example 3.1.1 and $\Lambda = K\langle\langle x,y\rangle\rangle/\overline{(x^2,y^2)}$ the corresponding completed string algebra. Then it follows for example from [Ge, Lemma 2.5.5.] that the element

$$\sum_{n\in\mathbb{N}} (xy)^n x^2 (yx)^n$$

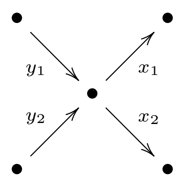
is in $\overline{(x^2,y^2)}$, but with some work one can see, that it is not an element of the ideal (x^2,y^2) in $K\langle\langle x,y\rangle\rangle$.

3.2.1 More on words

The sign for letters For each letter ℓ we choose a *sign* $\varepsilon(\ell) = \pm 1$, such that for two letters ℓ, ℓ' with the same head and same sign we have $\{\ell, \ell'\} = \{x^{-1}, y\}$ where $xy \in \rho$.

Note that if C_i and C_{i+1} are consecutive letters in a word, the head of C_i^{-1} is equal to the head of C_{i+1}, but neither C_iC_{i+1} nor $C_{i+1}^{-1}C_i^{-1}$ is a relation in ρ. Hence, C_i^{-1} and C_{i+1} must have different signs.

Example 3.2.3. Let $A = KQ/(\rho)$ be the string algebra given by the quiver



and relations $\rho = \{x_1y_1, x_2y_2\}$. Then, if we choose the sign $\varepsilon(y_1) = 1$, it follows that $\varepsilon(y_2) = \varepsilon(x_2^{-1}) = -1$ since these letters have the same head as y_1, but neither they nor their inverses form a relation. Further we can choose $\varepsilon(x_1^{-1}) = 1$, since x_1^{-1} has the same head as y_1 and $x_1y_1 \in I$. For the remaining letters we can choose the sign arbitrarily since they all have different heads.

Let $A = K[x,y]/(xy)$ as in Example 3.1.1. Then, if we choose the sign $\varepsilon(x) = 1$, it follows that $\varepsilon(x^{-1}) = -1, \varepsilon(y^{-1}) = 1, \varepsilon(y) = -1$ and these are all letters.

Composition of words The *head* of a finite word or $\mathbb{N}$-word C is the head of C_1 or v for $C = 1_{v,\varepsilon}$. The *sign* of a finite word or $\mathbb{N}$-word C is the sign of C_1 or ε for $C = 1_{v,\varepsilon}$. The *tail* of a finite word of length n is the tail of C_n or v for $C = 1_{v,\varepsilon}$. If C is a $-\mathbb{N}$-word its *tail* is the tail of C_0.

Let C and D be two words. If the tail of C is equal to the head of D, then we define the *composition* CD as the concatenation of the sequences of letters, provided that the result is a word. This implies, that C^- and D have opposite signs. If $C = C_1C_2 \dots C_n$ is a non-trivial word such that all of its powers C^m are words, we write C^∞ and ${}^\infty C^\infty$ for the $\mathbb{N}$-word and $\mathbb{Z}$-word

$$C_1 \dots C_n C_1 \dots C_n C_1 \dots \quad \text{and} \quad \dots C_1 \dots C_n | C_1 \dots C_n C_1 \dots .$$

If C is an $\mathbb{N}$-word of the form D^∞ it is called *repeating*.

A *primitive cycle* in A is a finite direct word P, such that ${}^\infty P^\infty$ is a periodic word, but P itself is not the power of a smaller word. Hence P is given by a cyclic path in Q, such that all its powers are pairwise different non-zero elements in A.

Let C and B be words. We say that B is a *subword* of C, if there exist words C' and C'' such that $C = C'BC''$. If C is an I-word and $i \in I$, we can define new words

$$C_{>i} = C_{i+1}C_{i+2} \dots \quad \text{and} \quad C_{\leq i} = \dots C_{i-1}C_i$$

and these are subwords of C.

An I-word C is called *eventually repeating* if $C_{>i}$ is repeating for some $i \in I$. Note that if C is an eventually inverse or an eventually direct $\mathbb{N}$- or $\mathbb{Z}$-word, then it is also eventually repeating: say C is eventually direct, e.g. $C_{>i}$ is direct. Then there is a unique choice for any C_j for $j \geq i+2$. Since if x is an arrow in Q with tail i there is a unique arrow y with head i and such that xy is not a relation. Since we assume that Q is a finite quiver we must have that $C_{>i} = D^\infty$ for some primitive cycle D.

3.2.2 Noetherian algebras over complete local rings

In [A1] and [A2] Auslander studied Noetherian algebras over a complete local ring in the context of almost split sequences or more general morphisms determined by objects. He discovered that these algebras have nice enough properties such that similar existence theorems - as for Artinian algebras - on almost split sequences still hold. In the following we will show that a completed string algebra is a Noetherian algebra over a complete local ring and hence Auslander's results are applicable.

Let S be a commutative ring. Then an algebra Λ is an *S-algebra* if there exists a ring morphism $f \colon S \to \Lambda$, such that $f(S)$ is contained in the center of Λ. If in addition S is a Noetherian ring and Λ is finitely generated as an S-module, we say that Λ is a *Noetherian S-algebra.* Since $f(S)$ is contained in the center of Λ, it follows that Λ is a Noetherian S-algebra if and only if Λ^{op} is a Noetherian S-algebra. Hence, in this case Λ is a right- and left-Noetherian ring. We call Λ a *Noetherian algebra over complete local ring* if it is a Noetherian S-algebra and S is a complete local ring.

A string algebra is a Noetherian $K[z]$-algebra Let $A = KQ/(\rho)$ be a string algebra. We define for any vertex $v \in Q$ the element $z_v \in e_vAe_v$ as the sum of all primitive cycles with head v. Note that by the definition of a string algebra, there are at most two such cycles. Moreover if P and R are two such distinct primitive cycles with head v we have

$$PR = RP = 0$$

in A which implies for example

$$(P+R)^n = P^n + R^n \qquad \text{and} \qquad (P+R)^n P = P^{n+1}.$$

The following is a consequence of [CB4, Lemma 3.1.].

Lemma 3.2.4. *The element $\sum_{v \in Q_0} z_v$ is in the center of A.*

Proposition 3.2.5. *Any string algebra $A = KQ/(\rho)$ is a Noetherian $K[z]$-algebra, where $K[z]$ denotes the polynomial ring in one variable.*

Proof. By the above lemma $\sum_{v \in Q_0} z_v$ commutes with every other element in A. Thus, by the universal property of the polynomial ring there exists a unique ring homomorphism

$$f \colon K[z] \to A, \qquad z \mapsto \sum_{v \in Q_0} z_v.$$

Now it follows from [CB4, Lemma 3.2.] that A is finitely generated as a $K[z]$-module. Note, that for this we need that Q consists of only finitely many vertices. □

Another descending filtration of ideals of A From now on we will identify z in $K[z]$ with the element $\sum_{v \in Q_0} z_v$ in A. Let

$$\mathfrak{n} = (z)A$$

be the ideal in A generated by z. Note that since z is in the center of A the notions of left-, right and two-sided ideal coincide in this case. The ideal $\mathfrak{n}$ induces an descending filtration

$$A \supset \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \mathfrak{n}_3 \supset \dots$$

where $\mathfrak{n}_i = \mathfrak{n}^i$ for all $i \geq 1$. Let

$$\hat{A} := \varprojlim A/\mathfrak{n}_i$$

be the $\mathfrak{n}$-adic completion of A.

Example 3.2.6. The string algebra A is finite-dimensional if and only if there are no primitive cycles. In that case the element z and thus $\mathfrak{n}$ is 0. Hence we simply have $\hat{A} = A$. Note that in that case we also have that $\mathfrak{m}_i = 0$ for some $i \gg 0$. Hence $\mathfrak{m}_i \subseteq \mathfrak{n}$ and we have $\Lambda = \hat{A}$. In the following lemma we will show, that this example can be generalised.

Proposition 3.2.7. *The descending filtrations*

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \mathfrak{m}_3 \supset \dots$$

and

$$A \supset \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \mathfrak{n}_3 \supset \dots$$

induce the same topologies and therefore there is a natural isomorphism of K-algebras

$$\Lambda = \varprojlim A/\mathfrak{m}_i \cong \varprojlim A/\mathfrak{n}_i = \hat{A}.$$

Proof. We have to show, that for each $\mathfrak{n}_j$ there is some $\mathfrak{m}_i$ such that $\mathfrak{m}_i \subset \mathfrak{n}_j$ and vice versa that for each $\mathfrak{m}_j$ there is some $\mathfrak{n}_i$ such that $\mathfrak{n}_i \subset \mathfrak{m}_j$.

The elements in $\mathfrak{m}_i$ correspond to non-trivial paths in A of length at least i. Let u and v be two vertices in Q and consider all non-trivial paths from u to v which are non-zero in A. By the string algebra condition, all such paths are of the form

$$D, PD, P^2D, \dots$$

where P and D are non-trivial paths (compare the proof of [CB4, Lemma 3.2.]). If there are infinitely many of such non-trivial paths, then P is a primitive cycle and $P^jD = z_v^jD$ in A. If there are only finitely many such paths, their length is bounded and they are not elements in $\mathfrak{m}_i$ for $i \gg 0$. Hence there is some $i \gg 0$ such that all elements in $\mathfrak{m}_i$ are either zero or of the form z^ja for some $a \in A$. Hence we have $\mathfrak{m}_i \subset \mathfrak{n}_j$.

On the other hand, since $\mathfrak{n}$ is generated by sums of classes of paths we have that $\mathfrak{n} \subset \mathfrak{m}_1$ and therefore $\mathfrak{n}_i = \mathfrak{n}^i \subset \mathfrak{m}_i$ for all i. It is well-known that this is enough to ensure the existence of the natural isomorphism of the two completions. See for example [E, Lemma 7.14.] for a proof in the commutative setting. However, the same proof works for noncommutative rings. $\square$

Note that the natural map $A \to \hat{A} = \Lambda$ is injective. Indeed the kernel is given by

$$\bigcap_{i \geq 0} (z)^i A$$

and this intersection is zero. Also note that A is finite-dimensional if and only if Λ is and in that case they are isomorphic.

A completed string algebra is a Noetherian $K[[z]]$-algebra From now on we identify the algebras $\hat{A}$ and Λ and refer to both as the *completed string algebra*.

Theorem 3.2.8. *The completed string algebra $\Lambda = \hat{A}$ is a Noetherian algebra over the complete local ring $K[[z]]$.*

Proof. The unique ring homomorphism $f\colon K[z] \to A$ sending z to $\sum_{v \in Q_0} z_v$ induces unique morphisms

$$K[z] \to A/\mathfrak{n}^i$$

sending z to the class of $\sum_{v \in Q_0} z_v$ which factor through

$$K[[z]]/(z)^i = K[z]/(z)^i \to A/\mathfrak{n}^i.$$

Therefore, we have unique maps

$$K[[z]] \to A/\mathfrak{n}^i$$

sending z to the class of $\sum_{v \in Q_0} z_v$, which induce a unique map

$$K[[z]] \to \hat{A}$$

since $\hat{A}$ is the inverse limit of the $A/\mathfrak{n}^i$. Since A is finitely generated as an $K[z]$-module, $\hat{A}$ is finitely generated as an $K[[z]]$-module. $\square$

The following can be found in [L, Proposition 21.34] in a more general setting.

Corollary 3.2.9. *The algebra $\hat{A}$ is $\hat{\mathfrak{n}} := (z)\hat{A}$ -adically complete.*

On the existence of Auslander-Reiten sequences for completed string algebras Denote by $\mathrm{Noeth}(\Lambda)$ the full subcategory of $\mathrm{Mod}(\Lambda)$ whose objects are noetherian modules and by $\mathrm{Noeth}_P(\Lambda)$ the full subcategory of $\mathrm{Noeth}(\Lambda)$ consisting of those modules with no non-zero projective direct summands. Since Λ is noetherian, we have that $\mathrm{Noeth}(\Lambda) = \mathrm{mod}(\Lambda)$ the full subcategory of finitely generated modules. Further, denote by $\mathrm{Art}(\Lambda)$ the full subcategory of $\mathrm{Mod}(\Lambda)$ whose objects are artinian Λ-modules and by $\mathrm{Art}_I(\Lambda)$ the full subcategory consisting of those modules with no non-zero injective direct summands. The *projectively stable category* is denoted by $\underline{\mathrm{Noeth}}_P(\Lambda)$. It has the same objects as those in $\mathrm{Noeth}(\Lambda)$ and $\underline{\mathrm{Hom}}_\Lambda(M, N)$ in $\mathrm{Noeth}_P(\Lambda)$ is defined to be

$$\underline{\mathrm{Hom}}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)/P(M, N)$$

where $P(M, N)$ is the $K[[z]]$-submodule of $\mathrm{Hom}_\Lambda(M, N)$ consisting of all homomorphisms that factor through a projective Λ-module. Similarly we define the *injectively stable category* $\overline{\mathrm{Art}}_I(\Lambda)$, which has the same objects as $\mathrm{Art}_I(\Lambda)$ and morphisms

$$\overline{\mathrm{Hom}}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)/I(M, N)$$

where $I(M,N)$ is the $K[[z]]$-submodule of $\operatorname{Hom}_\Lambda(M,N)$ consisting of all homomorphisms that factor through an injective Λ-module.

Denote by

$$E = E(K[[z]]/(z))$$

the injective envelope of the simple module $K[[z]]/(z)$ and define the functors

$$\tilde{D}\colon \operatorname{Mod}(\Lambda) \to \operatorname{Mod}(\Lambda^{\mathrm{op}}) \quad \text{and} \quad \tilde{D}\colon \operatorname{Mod}(\Lambda^{\mathrm{op}}) \to \operatorname{Mod}(\Lambda)$$

by $\tilde{D}(-) := \operatorname{Hom}_{K[[z]]}(-,E)$.

We are now able to give a list of some direct very nice consequences of the fact that a completed string algebra Λ is a Noetherian algebra over a complete local ring. Most of these can be found in this more general setting in [A1] and [A2].

- By Matlis duality [M, Paragraph 4] we know, that M is in $\operatorname{Noeth}(\Lambda)$ if and only if $\tilde{D}(M)$ is in $\operatorname{Art}(\Lambda^{\mathrm{op}})$. Thus the functors $\tilde{D}$ induce functors

$$\tilde{D}\colon \operatorname{Noeth}(\Lambda) \to \operatorname{Art}(\Lambda^{\mathrm{op}}) \quad \text{and} \quad \tilde{D}\colon \operatorname{Art}(\Lambda^{\mathrm{op}}) \to \operatorname{Noeth}(\Lambda).$$

  which are inverse dualities.

- Every Noetherian algebra Γ over a complete local ring is *semiperfect*, i.e. every finitely generated Γ-module has a projective cover. Hence the algebra Λ is semiperfect and so is $\operatorname{End}_\Lambda(M)$ for every $M \in \operatorname{Noeth}(\Lambda)$ and by duality for $M \in \operatorname{Art}(\Lambda)$.

- A finitely generated Λ-module M is indecomposable if and only if $\operatorname{End}_\Lambda(M)$ is local. The same holds if M is an artinian Λ-module.

- Every finitely generated Λ-module M has a Krull-Remak-Schmidt decomposition, i.e. $M = M_1 \oplus \cdots \oplus M_r$ where each M_i is an indecomposable Λ-module. Moreover, this decomposition is unique in the sense that for any other such decomposition $M = N_1 \oplus \cdots \oplus N_k$, we have $k = r$, and there is a bijection between the summands such that corresponding summands are isomorphic. By duality the same holds for every $M \in \operatorname{Art}(\Lambda)$.

- Using minimal projective presentations we can define the well-known Auslander-Reiten transpose

$$\operatorname{Tr}\colon \underline{\operatorname{Noeth}}_P(\Lambda) \to \underline{\operatorname{Noeth}}_P(\Lambda^{\mathrm{op}})$$

  which is a duality such that $M \in \underline{\operatorname{Noeth}}_P(\Lambda)$ is indecomposable if and only if $\operatorname{Tr}(M)$ is indecomposable.

- It is not hard to see that the *Auslander-Reiten translation for* Λ defined as

$$\tau_\Lambda = \tilde{D}\operatorname{Tr}\colon \underline{\operatorname{Noeth}}_P(\Lambda) \to \overline{\operatorname{Art}}_I(\Lambda)$$

  is an equivalence of categories with inverse

$$\tau_\Lambda^{-1} = \operatorname{Tr}\tilde{D}\colon \overline{\operatorname{Art}}_I(\Lambda) \to \underline{\operatorname{Noeth}}_P(\Lambda).$$

  This equivalence induces a bijection between the isomorphism classes of indecomposable modules in $\operatorname{Noeth}_P(\Lambda)$ and isomorphism classes of indecomposable modules in $\operatorname{Art}_I(\Lambda)$.

The following is [A1, Theorem 2.3. and 2.4.]

Proposition 3.2.10. *Let $M \in \operatorname{Noeth}_P(\Lambda)$ be an indecomposable module. Then there exists an Auslander-Reiten sequence*

$$0 \to \tau_\Lambda(M) \to U \to M \to 0$$

in $\operatorname{Mod}(\Lambda)$. *If* N *is in* $\operatorname{Art}_I(\Lambda)$ *and indecomposable there exists an Auslander-Reiten sequence*

$$0 \to N \to U \to \tau_\Lambda^{-1}(N) \to 0$$

in $\operatorname{Mod}(\Lambda)$.

The existence of Auslander-Reiten sequences ending in indecomposable finitely generated non-projective modules, is the main reason that we are considering completed string algebras. As we have seen before, if A is an infinite dimensional string algebra there exist indecomposable finitely generated modules, which do not have local endomorphism rings. Hence, there cannot exist an Auslander-Reiten sequence in $\operatorname{Mod}(A)$ ending in such a module.

3.2.3 On the completion of modules

Let A be a string algebra and as before let $\mathfrak{n} = (z)A$ be the ideal in A generated by (z). For any A-module M its $\mathfrak{n}$-*adic completion* is defined as

$$\hat{M} := \varprojlim M/\mathfrak{n}^i M.$$

As a limit of A-modules $\hat{M}$ is again an A-module and also an $\hat{A}$-module. In fact the $\mathfrak{n}$-adic completion is a functor

$$\operatorname{Mod}(A) \to \operatorname{Mod}(\hat{A}), \qquad M \mapsto \hat{M}$$

that restricts to finitely generated modules

$$\operatorname{mod}(A) \to \operatorname{mod}(\hat{A}), \qquad M \mapsto \hat{M},$$

i.e. if M is finitely generated as an A-module then $\hat{M}$ is finitely generated as an $\hat{A}$-module.

One might hope that the completion functor is essentially surjective. This is not the case in general, not even when A is a local commutative noetherian ring. A counterexample for this can be found in [FSW]. Here the authors consider the ring

$$A = \mathbb{C}[X,Y]_{(X,Y)}/(Y^2 - X^3 - X^2)$$

and its completion $\hat{A} \cong \mathbb{C}[[U,V]]/(UV)$, and argue that the finitely generated $\hat{A}$-module $\hat{A}/(U)$ is not an extended A-module, that is it is not isomorphic to $\hat{M}$ for a finitely generated A-module M.

However, when $A = K[z]$ is the polynomial ring and $\hat{A} = K[[z]]$ the ring of formal power series the (z)-adic completion functor $\operatorname{mod}(A) \to \operatorname{mod}(\hat{A})$ is essentially surjective. This follows from the structure theorem of finitely generated modules over principal ideal domains, which implies that any finitely generated $K[[z]]$-module is isomorphic to the finite direct sum of free modules and truncated polynomial rings $K[z]/(z)^{k_i}$ for some $k_i > 0$. Because of the tight relation ship between a (completed) string algebra and $K[z]$ (respectively $K[[z]]$) one might yet hope again that $\mathfrak{n}$-adic completion is essentially surjective on finitely generated modules. Unfortunately, we were not able to prove this directly. However, it will follow from our results a posteriori.

Lemma 3.2.11. *Let* M *be a finitely generated* $\hat{A}$*-module. Then* M *is* $\hat{\mathfrak{n}}$*-adically complete in the sense that the natural map*

$$i_M \colon M \to \varprojlim M/\hat{\mathfrak{n}}^i M$$

is an isomorphism.

Proof. This Lemma is proved in [L, Lemma 21.33] in the commutative setting. We only have to check, that everything that is used, still works in this noncommutative setting.

The kernel of the map i_m is $N = \bigcap_{i=1}^{\infty} \hat{\mathfrak{n}}^i M$. By Krull's Intersection Theorem, see [Sch, Theorem 3] for a noncommutative version that can be applied to our setting, we have $\hat{\mathfrak{n}}N = N$. Since M

is finitely generated and $\hat{A}$ is Noetherian, the module N is finitely generated. Thus we can apply Nakayama's Lemma, see [L, 4.22], to obtain that $N = 0$. Hence i_m is injective.

For the proof of the surjectivity of i_M we refer to [L, Lemma 21.33]. This part does not use the commutativity of the ring, but just that M is finitely generated and that $\hat{A}$ is $\hat{\mathfrak{n}}$-adically complete. □

Corollary 3.2.12. *Let M be a finitely generated $\hat{A} = \Lambda$-module. We can consider M as an (not necessarily finitely generated) A-module and we have that*

$$M \cong \varprojlim M/\mathfrak{n}^i M$$

hence also considered as an A-module M is complete with respect to the $\mathfrak{n}$-adic topology.

Proof. This follows from Lemma 3.2.11 and the fact, that $\mathfrak{n}M = \hat{\mathfrak{n}}M$. □

Remark 3.2.13. If M is a finitely generated A-module which is nilpotent with respect to $\mathfrak{n}$, then the natural map $M \to \hat{M}$ is an isomorphism. Note that since the filtrations $\mathfrak{n}^i$ and $\mathfrak{m}_i$ induce the same topologies, we have for $M \in \operatorname{mod}(A)$ that

$$\hat{M} \cong \varprojlim M/\mathfrak{m}_i M.$$

Hence, if M is a finitely generated nilpotent A-module, then

$$\hat{M} = \varprojlim M/\mathfrak{m}_i M \cong M$$

is complete with respect to the $\mathfrak{m}_i$-adic topology, and is hence a finitely generated indecomposable Λ-module.

3.3 Classification of finitely generated modules

The results and proofs in this chapter are based on Crawley-Boevey's work and are thus very similar to his. In fact Crawley-Boevey mentions in the introduction of his article, that the functorial filtration method should adapt to completed string algebras. We would also like to mention that Burban and Drozd in [BD] study certain algebras given by completions, which they refer to as *nodal algebras*, using matrix reductions. This includes the example $K\langle\langle x, y\rangle\rangle/(x^2, y^2)$.

From now on A will be a string algebra and Λ its completion. Recall that in case one of these algebras is finite-dimensional we have $A \cong \Lambda$.

3.3.1 String and band modules

Crawley-Boevey's definition of string and band modules for infinite dimensional string algebras does not work for completed string algebras: as an example one can consider the ring of polynomials. This is a finitely generated module over itself, but we cannot consider it as a module over its completion, the ring of power series. The slogan here is, that we have to replace direct sums by direct products. Furthermore, band modules for direct bands do not exist. These would be finite-dimensional $\widehat{KQ}$-modules which are not nilpotent.

For any non-periodic I-word C we are going to denote the Λ-string module by $M(C)$ or $M_\Lambda(C)$ if we want to distinguish it from the A-string module $M_A(C)$ explicitly.

String modules

We call a $\mathbb{Z}$-word C *mixed* if either C is eventually inverse but C^{-1} is not ,or C^{-1} is eventually inverse but C is not. Note that C is mixed if and only if C^{-1} is mixed.

Any I-word C satisfies one of the following properties:

- C and C^{-1} are eventually inverse (this includes all finite words);

- neither C nor C^{-1} are eventually inverse (this does not contain finite words, but infinite words which are eventually direct);
- C is a mixed word.

In the following we will define Λ-modules for words C according to these three cases.

The words C and C^{-1} are eventually inverse Let C be an I-word such that both C and C^{-1} are eventually inverse. Then we define a vector space

$$M(C) = \prod_{i \in I} K b_i,$$

and hence any element in $M(C)$ is an infinite linear combination in the symbols b_i $(i \in I)$. Obviously, we have that $M_A(C) \subseteq M(C)$ as vector spaces with equality if and only if C is a finite word.

We define an action of $\widehat{KQ}$ on $M(C)$ as follows: the action of the trivial paths and arrows on $M(C)$ is as in the non-completed case. We have $\varepsilon_v b_i = b_i$ if the tail of C_i is v, and $\varepsilon_v b_i = 0$ otherwise and we set

$$xb_i = \begin{cases} b_{i-1} & (\text{if } i-1 \in I \text{ and } C_i = x) \\ b_{i+1} & (\text{if } i+1 \in I \text{ and } C_{i+1} = x^{-1}) \\ 0 & (\text{otherwise}) \end{cases}$$

for any arrow $x \in Q$. Let $a_n \in KQ_n$ be a finite linear combination of paths of length n in Q. Then for every $i \in I$

$$a_n b_i = \sum_{j \in I} \mu_j^{(n,i)} b_j$$

is given by extending the above defined action to finite linear combinations of finite paths. Note that since C and C^{-1} are eventually inverse, for any $j \in I$ there are only finitely many non-trivial paths p in Q such that $pb_i = b_j$ for some $i \in I$. It follows that $\mu_j^{(n,i)} = 0$ for almost all $(n,i) \in \mathbb{N} \times I$. Therefore, for $a = \sum_{n \geq 0} a_n \in \widehat{KQ}$ and $m = \sum_{i \in I} \lambda_i b_i \in M(C)$ with

$$a_n \lambda_i b_i = \sum_{j \in I} \mu_j^{(n,i)} b_j$$

we can define

$$am = \sum_{n \geq 0} a_n \sum_{i \in I} \lambda_i b_i = \sum_{j \in I} \sum_{(n,i) \in \mathbb{N} \times I} \mu_j^{(n,i)} b_j$$

where $\mu_j^{(n,i)}$ is non-zero for only finitely many $(n,i) \in \mathbb{N} \times I$.

It is still obvious with that definition that $M(C)$ satisfies the defining relations in ρ. However, we are no longer considering just the ideal (ρ) generated by ρ in $\widehat{KQ}$, but the closure of this ideal $\overline{(\rho)}$ with respect to the $\mathfrak{m}$-adic topology. The following lemma will imply that $M(C)$ is annihilated by any element in $\overline{(\rho)}$, and thus is indeed a Λ-module.

Lemma 3.3.1. *Let*

$$a = \sum_{n \geq 0} a_n \in \overline{(\rho)}$$

where $a_n \in KQ_n$. Then we have that $a_n \in (\rho)$ for all $n \geq 0$.

Proof. Let $s \geq 0$ be minimal with $a_s \neq 0$. We will show that $a_s \in (\rho)$, then the lemma follows by induction. We can write $a = a^1 + a^2$ where $a^1 \in (\rho)$ and $a^2 \in \mathfrak{m}^{s+1}$. Write $a^1 = \sum_{n \geq 0} a_n^1$ where we have $a_n^1 \in KQ_n$ for all $n \geq 0$. Since (ρ) is a homogeneous ideal with respect to the length grading, it follows that $a_n^1 \in (\rho)$ for all $n \geq 0$. Furthermore, we must have $a_s = a_s^1$ for degree reasons and thus we see that $a_s \in (\rho)$. $\square$

Corollary 3.3.2. *Let C be and I-word such that both C and C^{-1} are eventually inverse. Then with the above definition*

$$M_\Lambda(C) = M(C)$$

is a Λ-module and from now on will be referred to as a string module.

Truncation of words and modules Let C be an I-word such that both C and C^{-1} are eventually inverse. Thus we can write $C = B\tilde{C}D$, where $\tilde{C}$ is some finite word, B is a word of length zero or an direct $-\mathbb{N}$-word and D is a word of length zero or an inverse $\mathbb{N}$-word. Furthermore, if B is not of length zero, we assume that the first letter of $\tilde{C}D$ is inverse and if D is not of length zero, we assume that the last letter of $B\tilde{C}$ is direct. Then for any $n \in \mathbb{N}_{\geq 2}$ greater than the length of $\tilde{C}$ we define the *n-truncation* of C to be the finite word

$$\pi_n(C) = B_{>-(n+2)}\tilde{C}D_{\leq n-1}.$$

If C is finite we will just have $\pi_n(C) = \tilde{C} = C$. Note that if B is a $-\mathbb{N}$-word, then $B_{>-(n+2)}$ is a finite word of length $n-1$, and if D is a $\mathbb{N}$-word then $D_{\leq n-1}$ is also a finite word of length $n-1$.

Example 3.3.3. If Λ is the completed string algebra $K[[x,y]]/(xy)$ and

$$C = ...xxxy^{-1}y^{-1}y^{-1}xy^{-1}y^{-1}y^{-1}...$$

then $B = ...xxx$, $\tilde{C} = y^{-1}y^{-1}y^{-1}x$ and $D = y^{-1}y^{-1}y^{-1}....$ Then the 5-truncation of C is given by

$$\pi_5(C) = xxxxy^{-1}y^{-1}y^{-1}xy^{-1}y^{-1}y^{-1}y^{-1}.$$

Proposition 3.3.4. *Let C be an I-word such that both C and C^{-1} are eventually inverse and let $M = M_A(C)$ be the finitely generated A-string module. Then there is a natural isomorphism*

$$\hat{M} = \varprojlim M/\mathfrak{m}_n M \cong M_\Lambda(C)$$

of Λ-modules.

Proof. We are going to show, that $M_\Lambda(C)$ satisfies the universal property of the inverse limit. For the finite-dimensional nilpotent string module $M_n = M(\pi_n(C))$ we have

$$M/\mathfrak{m}_n M \cong M_n \cong M_\Lambda(C)/\widehat{\mathfrak{m}_n} M_\Lambda(C)$$

where $\widehat{\mathfrak{m}_n}$ is the ideal in Λ generated by $\mathfrak{m}_n$. Therefore, the natural projections $\pi_n \colon M_\Lambda(C) \to M_n$ are well-defined Λ-module homomorphisms which are compatible with the transition morphisms of the inverse system.

Now let T be any Λ-module with morphisms $f_n \colon T \to M_n$, which are compatible with the transition morphisms. Then the map $f \colon T \to M_\Lambda(C)$ sending

$$t \mapsto f_1(t) + (f_2(t) - f_1(t)) + (f_3(t) - f_2(t)) + \dots$$

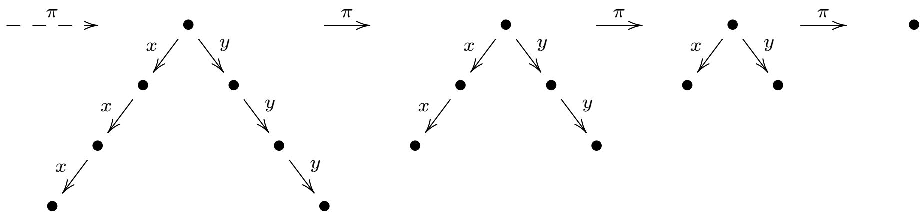
is a well-defined Λ-module homomorphism: each $f_n(t)$ is a finite linear combination in the b_i such that

$$f_n(t) = f_q(t) + (\text{linear combination in } b_i \text{ with } b_i \in \mathfrak{m}_s M, s > \min(n,q)).$$

Therefore, $f_{n+1}(t) - f_n(t)$ is a linear combination in elements b_i with $b_i \in \mathfrak{m}_{n+1}M$ for all i. It is not hard to see, that f is the unique morphism such that $f_n = \pi_n \circ f$ for all $n > 0$. □

Example 3.3.5. Let $\Lambda = K[[x,y]]/(xy)$. If C is the $\mathbb{Z}$-word $...xxxy^{-1}y^{-1}y^{-1}...$ then $\pi_n(C) =$

$x^{(n-1)}(y^{-1})^{(n-1)}$ for all $n \geq 0$. The modules in the inverse system can be pictured as follows



where by π we denote the natural projections. Note that in fact $M_\Lambda(C) \cong \varprojlim M_n$ is the regular representation of Λ.

The words C and C^{-1} are not eventually inverse

Proposition 3.3.6. *Let C be an I-word such that neither C nor C^- are eventually inverse. Then the action defined by A on the string module $M_A(C)$ extends naturally to an action of Λ on $M_A(C)$. We will denote this Λ-module by $M(C)$ or $M_\Lambda(C)$ if we want to distinguish it from $M_A(C)$ explicitly.*

Proof. We will first show, that $M = M_A(C)$ is a module over $\widehat{KQ}$. Since neither C nor C^- are eventually inverse for any $i \in I$, there exists some $s \gg 0$ such that for all paths p of length s in Q we have $pb_i = 0$. Thus, since any $m \in M$ is a finite linear combination in the b_i, the KQ-module M is locally nilpotent and thus by Proposition 3.2.1 a $\widehat{KQ}$-module.

We need to show that M is annihilated by elements in $\overline{(\rho)}$. As M is annihilated by elements in $(\rho) \subseteq KQ$, it is also annihilated by elements in $(\rho) \subseteq \widehat{KQ}$. Let $a \in \overline{(\rho)}$ and $m = \sum_{j=1}^{k} \lambda_{i_j} b_{i_j} \in M$ with $i_j \in I$. Let $s \gg 0$, such that $wm = 0$ for any path of length greater equal s. Since a is in $\overline{(\rho)}$ there exist $a^1 \in (\rho)$ and $a^2 \in \mathfrak{m}^s$ such that $a = a^1 + a^2$ and hence we have

$$am = (a^1 + a^2)m = a^1 m + a^2 m = 0 + 0 = 0$$

and this completes the proof. $\square$

The word C is mixed Finally, let C be mixed and assume that C is eventually inverse and C^{-1} is not. Choose $k \in I$ such that C_i is inverse for all $k < i$. As a vector space we define $M_\Lambda(C)$ to be the subspace of

$$\prod_{i \in \mathbb{Z}} \mathbb{C} b_i$$

consisting of elements $\sum_{i \in \mathbb{Z}} \lambda_i b_i$ with $\lambda_i = 0$ for almost all $i < k$. Then it is not hard to see, that by extending the action of A on $M(C_{<k})$ to Λ and combining with the action of Λ on $M_\Lambda(C_{\geq k})$ yields an action of Λ on $M_\Lambda(C)$. If C is a mixed word such that C^{-1} is eventually inverse we define $M_\Lambda(C) = M_\Lambda(C^{-1})$.

Example 3.3.7. Let Q be the quiver with one vertex and one loop x. Then the completed string algebra $\Lambda = \widehat{KQ}$ is isomorphic to the ring of formal power series $K[[x]]$. The $\mathbb{Z}$-word $C = {}^\infty x^\infty$ is a mixed word and the mixed module $M_\Lambda(C)$ is isomorphic to the ring of formal Laurent series $K((x))$ as a module over $K[[x]]$.

Proposition 3.3.8. *Let C by a mixed word, such that neither C nor C^{-1} are eventually direct and let $M = M_A(C)$ be the A-string module. Then there is a natural isomorphism*

$$\hat{M} = \varprojlim M/\mathfrak{m}_p M \cong M_\Lambda(C)$$

of Λ-modules.

Proof. This is proved similarly to Proposition 3.3.4. ☐

Let C be a non-periodic I-word. Then we call the Λ-module $M_\Lambda(C)$ a *string module*. Note that in general the string module $M_\Lambda(C)$ need not be isomorphic to the completion $\hat{M}$ where $M = M_A(C)$ with respect to the $\mathfrak{n}$-adic or equivalently to the $\mathfrak{m}_i$-adic topology. But in any case $M_\Lambda(C)$ is a module over the completed string algebra Λ.

Example 3.3.9. Let $A = K[x,y]/(xy)$ and $\Lambda = K[[x,y]]/(xy)$. Further consider the $\mathbb{Z}$-word

$$C = \dots y^{-1}y^{-1}y^{-1}xxx\dots$$

and let $M = M_A(C)$. Since $\mathfrak{m}_i M = M$ for all $i \in \mathbb{N}$ we see that

$$\hat{M} = \varprojlim M/\mathfrak{m}_i M = 0.$$

Hence M is not complete with respect to the $\mathfrak{m}_i$-adic topology and since $M_\Lambda(C) = M_A(C)$ as a vector space, $M_\Lambda(C)$ is not isomorphic to $\hat{M} = 0$.

The following Proposition - as in the non-completed case - is obvious.

Proposition 3.3.10. *Let C be a non-periodic I-word. Then*

- $M_\Lambda(C)$ *is in* $\mathrm{Noeth}(\Lambda)$ *if and only if C and C^{-1} are eventually inverse and*
- $M_\Lambda(C)$ *is in* $\mathrm{Art}(\Lambda)$ *if and only if C and C^{-1} are eventually direct.*

Remark 3.3.11. We would like to mention that Ringel studied similar modules corresponding to possibly infinite words in [R5]. Since he considered finite-dimensional algebras in his case infinite words cannot be eventually inverse or eventually direct. However, he makes the distinction between *expanding*, *contracting* and *mixed* words and defines an infinite dimensional module using infinite direct products or infinite direct sums correspondingly and proves that the module is algebraically compact. If the word is infinite and expanding or mixed, his definition differs from ours and one might prefer his definition in certain contexts. But as our main interest lies with the finitely generated modules, it does not make a difference in our case. We have already seen that there is a duality between the Noetherian and Artinian Λ-modules and therefore these modules are all algebraically compact anyway.

String modules are indecomposable In the following we want to show that string modules over a completed string algebras Λ are indecomposable. First, recall that if $A = KQ/(\rho)$ is a finite-dimensional string algebra, the functor

$$D = \mathrm{Hom}_K(-,K)\colon \mathrm{mod}(A) \to \mathrm{mod}(A^{\mathrm{op}})$$

is a duality called the *standard K-duality*. Let $C = (C_i)$ be a finite word over A, and denote by $C^* = (C_i^{-1})$ the *dual word* over A^{op}. Then it is well-known that $D(M(C)) \cong M(C^*)$ in $\mathrm{mod}(A^{\mathrm{op}})$.

We want to show, that a similar statement holds for finitely generated string modules over a completed string algebra Λ and the duality

$$\tilde{D}\colon \mathrm{Noeth}(\Lambda) \to \mathrm{Art}(\Lambda^{\mathrm{op}}).$$

Let $C = (C_i)$ be any I-word over Λ. Define the *the dual word* by $C^* = (C_i^{-1})$ an I-word over Λ^{op}. If C is an I-word, such that C and C^{-1} are eventually inverse, then C^* is an I-word, such that C^* and $(C^*)^{-1}$ are eventually direct.

Proposition 3.3.12. *Let C be an I-word such that C and C^{-1} is eventually inverse, i.e. we have $M_\Lambda(C) \in \mathrm{Noeth}(\Lambda)$. Then the Λ^{op}-module $\tilde{D}M_\Lambda(C) \in \mathrm{Art}(\Lambda^{\mathrm{op}})$ is isomorphic to $M_{\Lambda^{\mathrm{op}}}(C^*)$.*

Proof. We are first going to show the following: if Λ is finite-dimensional, then $D, \tilde{D}\colon \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Lambda)$ are isomorphic functors. By [A2, Prop 4.4.], which we apply to Λ as an Noetherian $K[[z]]$-algebra and as an K-algebra we have

$$\begin{aligned} \tilde{D}(-) = \operatorname{Hom}_{K[[z]]}(-, E) &\cong \operatorname{Hom}_\Lambda(-, \operatorname{Hom}_{K[[z]]}(\Lambda^{\mathrm{op}}, E)) = \operatorname{Hom}_\Lambda(-, \tilde{D}(\Lambda^{\mathrm{op}})), \\ D(-) = \operatorname{Hom}_K(-, K) &\cong \operatorname{Hom}_\Lambda(-, \operatorname{Hom}_K(\Lambda^{\mathrm{op}}, K)) = \operatorname{Hom}_\Lambda(-, D(\Lambda^{\mathrm{op}})) \end{aligned}$$

and thus we need to show, that $D(\Lambda^{\mathrm{op}}) \cong \tilde{D}(\Lambda^{\mathrm{op}})$. But this follows, since they are both injective, of the same dimension and both contain all the simple Λ-modules (it is known that $D(\Lambda^{\mathrm{op}})$ is the injective envelope of the sum of all simples).

Now let Λ be any completed string algebra, and $M = M(C)$ a finite-dimensional Λ-module. Choose $n > \dim(M)$. Hence, M can be considered as a module over the finite-dimensional string algebra A_n. Then we have

$$\tilde{D}_\Lambda(M) = \tilde{D}_{A_n}(M) \cong D_{A_n}(M) = M_{A_n^{\mathrm{op}}}(C^*)$$

as A_n^{op}-modules, but then also as Λ^{op}-modules. Thus for $M(C)$ finite-dimensional we have

$$\tilde{D}_\Lambda(M(C)) \cong M_{\Lambda^{\mathrm{op}}}(C^*).$$

Finally let $M = M_\Lambda(C) \in \operatorname{Noeth}(\Lambda)$ be any finitely generated string module. Then $M \cong \varprojlim M/\mathfrak{m}_n M$ and for $n \gg 0$ we know that $M/\mathfrak{m}_n M \cong M(C_n)$ for finite words $C_n = \pi_n(C)$. We have

$$\tilde{D}(M) = \tilde{D}(\varprojlim M(C_n)) \cong \varinjlim \tilde{D}(M(C_n)) = \varinjlim M((C_n)^*) \cong M(C^*)$$

which proves the proposition. $\square$

Lemma 3.3.13. *Let A be a string algebra and Λ its completion. Let C be a non-periodic I-word, such that as vector spaces we have $M_\Lambda(C) = M_A(C)$. This is for example the case if $M_\Lambda(C)$ is in $\operatorname{Art}(\Lambda)$. Then $M_\Lambda(C)$ is indecomposable.*

Proof. By [CB4, Theorem 1.1.], we know that $M_A(C)$ is indecomposable as an A-module. But since any Λ-module is also an A-module, it follows that $M_\Lambda(C)$ is indecomposable as an Λ-module. $\square$

Theorem 3.3.14. *Let C be an I-word such that $M(C) \in \operatorname{Noeth}(\Lambda) \cup \operatorname{Art}(\Lambda)$. Then $\operatorname{End}_\Lambda(M(C))$ is a local ring.*

Proof. Since Λ is a Noetherian algebra over a complete local ring, and $M(C)$ is by assumption finitely generated or Artinian the ring $\operatorname{End}_\Lambda(M(C))$ is local if and only if $M(C)$ is indecomposable. If $M(C)$ is Artinian it is indecomposable by the above lemma. If $M(C) \in \operatorname{Noeth}(\Lambda)$ then $M(C) \cong \tilde{D}(M(C^*))$ where $M(C^*) \in \operatorname{Art}(\Lambda^{\mathrm{op}})$ is indecomposable again by the above lemma. Therefore, $M(C)$ is also indecomposable. $\square$

Band modules

Let C be a periodic word which is not direct or inverse. Then for any finite-dimensional $K[T, T^{-1}]$-module V the module $M = M_A(C, V)$ is a finite-dimensional nilpotent A-module. It follows that M is complete with respect to the $\mathfrak{m}_\mathfrak{i}$-adic topology and hence

$$M \cong \hat{M} = M_\Lambda(C, V)$$

is also Λ-module. If V is an indecomposable $K[T, T^{-1}]$-module we call $M_\Lambda(C, V)$ a *band module.*

If P is a primitive cycle and V an indecomposable $K[T, T^{-1}]$-module, the finite-dimensional band module $M = M({}^\infty P^\infty, V)$ cannot be a Λ-module. Also note, that in this case the $\mathfrak{n}$-adic completion functor applied to M yields $\hat{M} = 0$, since we have $\mathfrak{n}M = M$. So if C is a periodic word which is direct or inverse we define $M(C, V) = 0$ for any finite-dimensional $K[T, T^{-1}]$-module V.

The rest of this section is dedicated to altering Crawley-Boevey's methods to prove the following theorem.

Theorem 3.3.15. *Let Λ be a completed string algebra. Then any finitely generated Λ-module is isomorphic to a direct sum of copies of string modules and finite-dimensional band modules.*

3.3.2 Linear relations

Since the corresponding section in [CB4, Section 4] on linear relations deals with vector spaces and not with particular modules, we will only recall the necessary definitions and results without giving proofs.

From now on let V and W be possibly infinite dimensional K-vector spaces. A *relation from V to W* is a subspace C of $V \times W$. For instance if $f\colon V \to W$ is a linear map, the graph

$$C_f = \{(v, w) \in V \times W \mid f(v) = w\}$$

is a linear relation.

Let C be a linear relation. Then for $v \in V$ and $H \subseteq V$ we define

$$Cv = \{w \in W \mid (v, w) \in C\} \quad \text{and} \quad CH = \bigcup_{v \in H} Cv$$

and

$$C^{-1} = \{(w, v) \in W \times V \mid (v, w) \in C\}.$$

If D is a relation from U to V we can *compose* the two relations to obtain a new relation from U to W given by

$$CD = \{(u, w) \mid \exists v \in V \text{ such that } w \in Cv \text{ and } v \in Du\}$$

and hence we can define C^n for any $n \in \mathbb{Z}$.

Example 3.3.16. If Λ is a completed string algebra and M a Λ-module, any arrow x with head v and tail u defines a linear map $x\colon e_uM \to e_vM$ given by multiplication with x. Hence the graph C_x is a linear relation from e_uM to e_vM. Thus for any finite word C with head v and tail u we can inductively define a relation from e_uM to e_vM, which will also be denoted by C. We write CM for Ce_vM, when we consider C as a linear relation from M to itself.

If C is a linear relation on a vector space V, i.e. C is a linear relation from V to itself, we define subspaces $C' \subseteq C'' \subseteq V$ by

$$\begin{aligned} C'' &= \{v \in V \mid \exists v_0, v_1, v_2 \ldots \text{ with } v = v_0 \text{ and } v_n \in Cv_{n+1} \text{ for all } n\}, \\ C' &= \bigcup_{n \geq 0} C^n 0. \end{aligned}$$

Furthermore, we define subspaces $C^\flat \subseteq C^\sharp \subseteq V$ by

$$C^\sharp = C'' \cap (C^{-1})'' \quad \text{and} \quad C^\flat = C'' \cap (C^{-1})' + C' \cap (C^{-1})''.$$

The following are [CB4, Lemma 4.2., Lemma 4.4. and Lemma 4.5.].

Lemma 3.3.17. *Let C be a linear relation on V. Then we have*

- $C'' \subseteq \bigcap_{n \geq 0} C^n V$ *and if V is finite-dimensional we have* $C'' = \bigcap_{n \geq 0} C^n V$,
- $C^\sharp \subseteq CC^\sharp$,
- $C^\flat = C^\sharp \cap CC^\flat$,
- $C^\sharp \subseteq C^{-1}C^\sharp$ *and*

- $C^\flat = C^\sharp C^{-1} C^\flat$.

Furthermore, C induces an automorphism θ of $C^\sharp/C^\flat$ with $\theta(C^\flat + v) = C^\flat + w$ if and only if $w \in C^\sharp \cap (C^\flat + Cv)$.

Let C be a relation on a vector space V. We say that C is *split* if there exists a subspace U of V such that $C^\sharp = C^\flat \oplus U$ and C induces an automorphism on U. The following is what Crawley-Boevey calls the *Splitting Lemma* [CB4, Lemma 4.6.].

Lemma 3.3.18. *If $C^\sharp/C^\flat$ is finite-dimensional the relation C is split.*

3.3.3 Torsion submodules

In the classification of finitely generated modules for string algebras, one often recurring means is to consider the torsion submodule instead of the whole module.

Let S be the ring of polynomials $K[z]$ or the ring of formal power series $K[[z]]$, and M an S-module. Then since S is an integral domain the set of all torsion elements

$$T(M) = \{m \in M \mid f(z)m = 0 \text{ for some } f \in S\}$$

is a submodule, which we call *torsion submodule*. It decomposes as a direct sum

$$T(M) = T^0(M) \oplus T^1(M)$$

where

$$\begin{aligned} T^0(M) &= \{m \in M \mid z^n m = 0 \text{ for some } n \geq 0\}, \text{ and} \\ T^1(M) &= \{m \in M \mid f(z)m = 0 \text{ for some } f \in S \text{ with } f(0) = 1\} \end{aligned}$$

are the *nilpotent torsion* and *primitive torsion* submodules of M.

Lemma 3.3.19. *Let M be an S-module.*

- *If $S = K[[z]]$ the primitive torsion submodule $T^1(M)$ of M is zero.*
- *If M is finitely generated over S, then $T^0(M)$ and $T^1(M)$ are finite-dimensional.*

Proof. The first part follows from the fact, that any $f(z) \in K[[z]]$ with $f(0) = 1$ is a unit in $K[[z]]$. More precisely Let $v \in M$ be a torsion element and $f(z) \in K[[z]]$ such that $f(z)v = 0$. Then we can write $f(z) = z^i \tilde{f}(z)$ with $i \geq 0$ and $\tilde{f}(0) \in K \setminus \{0\}$. Then $\tilde{f}(z)$ is a unit in $K[[z]]$ and hence we find $z^i v = 0$ and thus $v \in T^0(M)$.

The second part follows from the structure theorem for finitely generated modules over principal ideal domains. □

Any Λ-module (respectively A-module) can also be considered as a $K[[z]]$-module (respectively $K[z]$-module) and hence we can consider its nilpotent and primitive torsion submodules.

Lemma 3.3.20. *Let M be a Λ-module or an A-module. Then $T^0(M)$ and $T^1(M)$ are Λ-submodules (respectively A-submodules) of M and we have*

$$T(M) = \bigoplus_v T(e_v M) \quad \textit{and} \quad T^i(M) = \bigoplus_v T^i(e_v M).$$

Proof. This follows immediately, since the actions of Λ (respectively A) and $K[[z]]$ (repectively $K[z]$) on M commute. □

If P is a primitive cycle with head v we can also consider $e_v M$ as a $K[[P]]$-module (respectively $K[P]$-module), and hence consider

$$T_P(e_v M) = T_P^0(e_v M) \oplus T_P^1(e_v M)$$

as above, with z replaced by P, and call these the *P-torsion*, *P-nilpotent torsion* and *P-primitive torsion* subspaces of e_vM. They are $K[[z]]$-submodules (respectively $K[z]$-submodules) of e_vM.

Lemma 3.3.21. *Let M be a Λ-module. We have*

$$T^0(e_vM) = \bigcap_P T^0_P(e_vM)$$

where the intersection runs over the (up to two) primitive cycles with head v.

Proof. It is enough to consider the case where there are exactly two primitive cycles P and R with head v. Let $m \in T^0(e_vM)$, that is we have $z^n m = (P+R)^n m = 0$ for some $n \geq 0$. Then we also see

$$0 = P(P+R)^n m = P(P^n + R^n)m = P^{n+1}m.$$

Hence it follows that $m \in T^0_P(M)$ and similarly one shows that $m \in T^0_R(M)$.

On the other hand if $m \in T^0_P(e_vM) \cap T^0_R(e_vM)$, there exist $k, l \geq 0$, such that $P^k m = 0$ and $R^l m = 0$. Then it also follows that

$$z^{\max\{k,l\}} m = (P^{\max\{k,l\}} + R^{\max\{k,l\}})m = 0.$$

□

Lemma 3.3.22. *Let P be a primitive cycle with head v and M a Λ-module. Then P defines a linear relation on e_vM and we have*

- $P' = 0$
- $(P^{-1})' = T^0_P(e_vM)$
- $(P^{-1})'' = e_vM$*, and*
- *if M is finitely generated, then* $P'' = \bigcap_{n\geq 0} P^n M = T^1_P(e_vM) = 0$.

Proof. The first three parts follow directly from the definitions.

By Lemma 3.3.19 and Lemma 3.3.17 we have

$$0 = T^1_P(e_vM) \subseteq P'' \subseteq \bigcap_{n\geq 0} P^n M.$$

Let J be the ideal generated by P in $K[[P]]$ or $K[[P,R]]/(PR)$, if R is another primitive cycle with head v. Define $N = \bigcap_{n\geq 0} J^n M$. Then since M is finitely generated as a Λ-module, it is also finitely generated as a $K[[z]]$-module. Therefore, we can apply Krull's Intersection Theorem to N and the Notherian ring $K[[P]]$ or $K[[P,R]]/(PR)$, to obtain $JN = N$. But then by Nakayama's Lemma it follows that $N = 0$. □

3.3.4 Functorial filtration given by words

The functorial filtration method is based on two ideas: the first idea is to simplify the structure of a given Λ-module by considering it as vector space via the forgetful functor. The second idea is to look at filtrations of the underlying vector space. Graphically speaking we will use words to define different subfunctors of the forgetful functors, cutting the underlying vector spaces into more applicable slices.

Let Λ be a completed string algebra and denote by $\text{Forget}\colon \text{Mod}(\Lambda) \to \text{Mod}(K)$ the forgetful functor from Λ-modules to K-vector spaces. Then for any vertex $v \in Q$ we define a functor

$$\begin{aligned} e_v\colon \text{Mod}(\Lambda) &\longrightarrow \text{Mod}(K) \\ M &\longmapsto e_vM \end{aligned}$$

which is a subfunctor of Forget, i.e. $e_vM \subseteq \mathrm{Forget}(M)$ for all $M \in \mathrm{Mod}(\Lambda)$ and for any homomorphism $f\colon M \to N$ of Λ-modules the linear map e_vf is the restriction of $\mathrm{Forget}(f)$ to e_vM.

For a fixed vertex $v \in Q$ and a sign $\varepsilon = \pm 1$ denote by $\mathcal{W}_{v,\varepsilon}$ the set of finite words and $\mathbb{N}$-words with head v and sign ε. In the following we will define for any word $C \in \mathcal{W}_{v,\varepsilon}$ two subfunctors of Forget which will be denoted by C^- and C^+. In fact we will show, that C^- is a subfunctor of C^+ which in turn is a subfunctor of e_v, such that we have the following chain of inclusions

$$C^-(M) \subseteq C^+(M) \subseteq e_vM$$

for any Λ-module M.

Finite words First consider the case, where C is a finite word. Then we define

$$C^+(M) = \begin{cases} Cx^{-1}0 & \text{if there is an arrow } x \text{ such that } Cx^{-1} \text{ is a word,} \\ CM & \text{otherwise,} \end{cases}$$

and similarly we define

$$C^-(M) = \begin{cases} CyM & \text{if there is an arrow } y \text{ such that } Cy \text{ is a word,} \\ C0 & \text{otherwise.} \end{cases}$$

In this case it is easily seen, that $C^-(M) \subseteq C^+(M)$. Suppose there exist arrows x, y, such that Cx^{-1} and Cy are words. Then it follows that x^{-1} and y have the same head (namely the tail of C) and the same sign (namely the opposite sign of C^{-1}). But then it follows from the conditions on choosing the sign ε that $xy \in \rho$. Therefore, if we consider x and y as linear maps on M, we have $\mathrm{Im}(y) \subseteq \mathrm{Ker}(x)$ and hence

$$C^-(M) = Cy(M) = C(\mathrm{Im}(y)) \subseteq C(\mathrm{Ker}(x)) = Cx^{-1}(0) = C^+(M).$$

The other cases follow similarly.

Example 3.3.23. Let $\Lambda = K[[x,y]]/(xy)$ and let $M = M(D)$ be string module, where $D = xy^{-1}y^{-1}xx$, i.e. we can picture M as

$$\begin{tikzcd}[column sep=small]
& b_1 \arrow[dl, "x"'] \arrow[dr, "y"'] & & & & b_5 \arrow[dl, "x"'] \\
b_0 & & b_2 \arrow[dr, "y"'] & & b_4 \arrow[dl, "x"'] & \\
& & & b_3 & &
\end{tikzcd}$$

For $C = D$ the vector space $C^+M = Cy^{-1}(0) = xy^{-1}y^{-1}xxy^{-1}0$ is spanned by b_0, b_3, b_4 and $C^-M = CxM = xy^{-1}y^{-1}xxxM$ is spanned by b_3, b_4. Assuming that x has sign ε, the vector space $1^+_{v,-\varepsilon}(M) = x^{-1}(0)$ is spanned by b_0, b_2, b_3 and $1^-_{v,-\varepsilon}(M) = y(M)$ is spanned by b_2, b_3.

For $C = y^{-1}y^{-1}xx$ the vector space $C^+M = Cy^{-1}(0) = y^{-1}y^{-1}xxy^{-1}0$ is spanned by all b_i for $i = 1, \ldots, 5$ and $C^-M = CxM = y^{-1}y^{-1}xxxM$ is spanned by b_0, b_2, b_3, b_4, b_5. For $B = x^{-1}$ the vector space $B^+(M) = x^{-1}x^{-1}(0)$ is spanned by b_0, b_1, b_2, b_3, b_4 and $B^-(M) = x^{-1}y(M)$ is spanned by b_0, b_2, b_3, b_4.

Infinite words Now suppose that C is an $\mathbb{N}$-word. Then we define

$$C^+(M) = \{m \in M \mid \exists \text{ a sequence } m_n\ (n \geq 0), m_0 = m, m_{n-1} \in C_n m_n \text{ for all } n \geq 0\}$$

and

$$C^-(M) = \{m \in C^+(M) \mid \exists s \gg 0,\ m_n = 0 \text{ for all } n \geq s\}.$$

Note that

$$C^+(M) \subseteq \bigcap_{n\geq 0} C_{\leq n}M \qquad \text{and} \qquad C^-(M) = \bigcup_{n\geq 0} C_{\leq n}0$$

and that if $C = D^\infty$ is repeating, then

$$C^+(M) = D'' \qquad \text{and} \qquad C^-(M) = D'.$$

Since $\theta(C^\pm(M)) \subseteq C^\pm(N)$ for any homomorphism $\theta\colon M \to N$ of Λ-modules, the functors $C^\pm$ are subfunctors of the forgetful functor from Λ-modules to vector spaces or $K[[z]]$-modules. Also the proof of the following lemma is straightforward.

Lemma 3.3.24. *The functors $C^\pm$ commute with arbitrary direct sums.*

Total order on $\mathcal{W}_{v,\varepsilon}$ We have seen, that for any Λ-module M and any $C \in \mathcal{W}_{v,\varepsilon}$, we have $C^-(M) \subseteq C^+(M)$. Now, we also want a means to compare $C^\pm(M)$ and $C'^\pm(M)$ for two words $C, C' \in \mathcal{W}_{v,\varepsilon}$. To do so, we define a total order on $\mathcal{W}_{v,\varepsilon}$ as follows: for $C, C' \in \mathcal{W}_{v,\varepsilon}$ we say that $C < C'$ if - roughly speaking C has an direct letter where C' does not, or C is shorter than C' - and more explicitly if

- $C = ByD$ and $C' = Bx^{-1}D'$ for a finite word B and arrows x, y and D, D' are words, or
- C' is a finite word and $C = C'yD$ for an arrow y and a word D, or
- C is a finite word and $C' = Cx^{-1}D'$ for an arrow x and a word D'.

Lemma 3.3.25. *For $C, C' \in \mathcal{W}_{v,\varepsilon}$ with $C < C'$ we have $C^+(M) \subseteq C'^-(M)$.*

Proof. First note that if C is a finite word, then $C^+(M) \subseteq C(M)$. Since if there exists an arrow x, such that Cx^{-1} is a word, then we have $C^+(M) = Cx^{-1}0 \subseteq CM$.

- First assume $C = ByD$ and $C' = Bx^{-1}D'$ for a finite word B, arrows x, y and words D, D'. If C is finite, then $C^+(M) = ByDe^{-1}(0)$ if some letter e exists, otherwise $C^+(M) = ByD(M)$. If C is infinite $C^+(M) \subseteq C_{\leq n}M$ for all $n \geq 0$. In any case $C^+(M) \subseteq By(M)$ and
$$By(M) = B^-(M) \subseteq B^+(M) = Bx^{-1}(0) \subseteq (Bx^{-1}D')^-(M) = C'^-(M),$$
where the last inclusion follows since in case C' is finite we have
$$(Bx^{-1}D')^-(M) = Bx^{-1}D'z(M)$$
if such an arrow z exists or $(Bx^{-1}D')^-(M) = Bx^{-1}D'0$ otherwise and in case C' is infinite we have $C'_{\leq n}0 \subseteq C'^-(M)$ for all $n \geq 0$.
- Now assume that C' is a finite word and $C = C'yD$ for an arrow y and a word D. Then we have
$$C^+(M) \subseteq C'y(M) = C'^-M.$$
- If we assume C is a finite word and $C' = Cx^{-1}D'$ for an arrow x and a word D', then we have
$$C^+(M) = Cx^{-1}0 \subseteq C'^-(M).$$

$\square$

3.3.5 Refined functors

In this section we introduce the refined functors given by words from Λ-modules to vector spaces.

Let $B \in \mathcal{W}_{v,\varepsilon}$ and $D \in \mathcal{W}_{v,-\varepsilon}$ be two words and M a Λ-module. Note that $B^{-1}D$ is a word unless it involves a zero-relation or the inverse of a zero-relation. We define the functors

$$\begin{aligned}
F^+_{B,D}(M) &= B^+(M) \cap D^+(M),\\
F^-_{B,D}(M) &= (B^+(M) \cap D^-(M)) + (B^-(M) \cap D^+(M)), \text{ and}\\
F_{B,D}(M) &= F^+_{B,D}(M)/F^-_{B,D}(M).
\end{aligned}$$

If $B^{-1}D$ is a non-periodic word we have thus defined functors $\mathrm{Mod}(\Lambda) \longrightarrow \mathrm{Mod}(K)$, which do not depend on the order of B and D.

If $C = B^{-1}D$ is a periodic word, say $C = {}^\infty E^\infty$ for a finite word E of length n and head v, then E induces a linear relation on e_vM for any Λ-module M. Then we have $F^+_{B,D}(M) = E^\sharp$ and $F^-_{B,D}(M) = E^\flat$, and hence by Lemma 3.3.17 the linear relation E induces an automorphism on $F_{B,D}(M)$. We can thus consider $F_{B,D}$ as a functor from Λ-modules to $K[T,T^{-1}]$-modules, with the action of T given by the induced automorphism.

Example 3.3.26. Let $\Lambda = K[[x,y]]/(xy)$ and $M = M(C)$ the string module, where $C = xy^{-1}y^{-1}xx$. In a previous example we have already computed for $B = 1_{v,-\varepsilon}$ and $D = C$

$$\begin{aligned}
B^+(M) &= \langle b_0, b_2, b_3\rangle_K & B^-(M) &= \langle b_2, b_3\rangle_K\\
D^+(M) &= \langle b_0, b_3, b_4\rangle_K & D^-(M) &= \langle b_3, b_4\rangle_K
\end{aligned}$$

assuming that x has sign ε and thus we have

$$F^+_{B,D}(M) = \langle b_0, b_3\rangle_K, \qquad F^-_{B,D}(M) = \langle b_3\rangle_K, \qquad F_{B,D}(M) = \langle b_0\rangle_K.$$

For $B = x^{-1}$ and $D = y^{-1}y^{-1}xx$ we have

$$\begin{aligned}
B^+(M) &= \langle b_0, b_1, b_2, b_3, b_4\rangle_K & B^-(M) &= \langle b_0, b_2, b_3, b_4\rangle_K\\
D^+(M) &= \langle b_0, b_1, b_2, b_3, b_4, b_5\rangle_K & D^-(M) &= \langle b_0, b_2, b_3, b_4, b_5\rangle_K
\end{aligned}$$

and hence

$$F^+_{B,D}(M) = \langle b_0, b_1, b_2, b_3, b_4\rangle_K, \qquad F^-_{B,D}(M) = \langle b_0, b_2, b_3, b_4\rangle_K, \qquad F_{B,D}(M) = \langle b_1\rangle_K.$$

Lemma 3.3.27. *(i) $F_{B,D}$ commutes with direct sums.*
(ii) If $B^{-1}D$ is not a word, then $F_{B,D} = 0$.
(iii) If $B^{-1}D$ is a periodic word, then $F_{B,D} \cong res_\iota F_{D,B}$.
(iv) For a word C the functors $F_{B,D}$ with $B^{-1}D = C[n]$ are isomorphic for any n.

Proof. (i) This follows from the fact, that the functors $C^\pm$ for any word C commute with direct sums.

(ii) If $B^{-1}D$ is not a word it contains a zero-relation or the inverse of one. In the latter case, we can exchange B and D, since $F_{B,D}$ does not depend on their order and may thus assume without loss of generality, that $B^{-1}D$ involves a zero-relation. Hence we can write $B = x_n^{-1}\dots x_1^{-1}C$ and $D = y_1 \dots y_k E$ for words C, E and such that $x_1 \dots x_n y_1 \dots y_k \in \rho$. Let $m \in F^+_{B,D}(M)$. Then $m = y_1 \dots y_k m'$ for $m' \in E^+(M)$ and as

$$x_1 \dots x_n m = x_1 \dots x_n y_1 \dots y_k m' = 0$$

we have $m \in x_n^{-1} \dots x_1^{-1}(0) \subseteq B^-(M)$. Therefore, we also have $m \in F^-_{B,D}(M)$ proving that the image of m in $F_{B,D}(M)$ is zero.

(iii) This is obvious.

(iv) This is done in the same way, as the lemma on page 25 in [R1].

□

Lemma 3.3.28. *Let $(B, D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ be non-trivial words, such that the first letter of both is direct. Then $B^+(M) \cap D^+(M)$ is finite-dimensional for any finitely generated Λ-module M.*

Proof. Since B and D have opposite signs, their first letters must be two different arrows x and y with head v. But then for any primitive cycle with tail v we have $Px = 0$ or $Py = 0$ in Λ. Hence $B^+(M) \cap D^+(M)$ is contained in $T^0(e_v M)$ which is finite-dimensional since M is finitely generated (see Lemma 3.3.8.). □

Lemma 3.3.29. *Let $C = B^{-1}D$ be a periodic word, say $C = {}^\infty E^\infty$ for some word E of length n and head v. If M is a finitely generated Λ-module, the vector space $F_{B,D}(M)$ is finite-dimensional. In particular, the relation E on $e_v M$ is split.*

Proof. If $C = B^{-1}D$ is not direct or inverse, then by Lemma 3.3.27, we can apply a shift to C and hence we may suppose that we are in the case of Lemma 3.3.28. Otherwise we can assume without loss of generality that C is direct and periodic. Hence $C = {}^\infty P^\infty$ for some primitive cycle P with head v. Then since M is finitely generated $F_{B,D}(M) = P'' = T^1_P(e_v M) = 0$ by Lemma 3.3.22. In either case since $F_{B,D}(M) = E^\sharp / E^\flat$ is finite-dimensional, the relation E on $e_v M$ is split by Lemma 3.3.18. □

3.3.6 Evaluation on string and band modules

In this section we want to investigate what happens, when we apply the refined functors to string and band modules. The proofs and results are mainly based on [CB4, Section 8], where the author in turn gives credit to [R1].

Evaluation on string modules Recall that for any I-word C and for any $i \in I$ we have defined

$$C_{>i} = C_{i+1}C_{i+2}\dots \qquad \text{and} \qquad C_{\leq i} = \dots C_{i-1}C_i.$$

Then $C_{>i}$ has as first letter C_{i+1} and hence has head $v_i(C)$ and the word $C_{\leq i}^{-1}$ has as first letter C_i^{-1}, which has as head the tail of C_i, hence also $v_i(C)$. Since $C_{\leq i}C_{>i}$ is a word, $(C_{\leq i})^{-1}$ and $C_{>i}$ must have opposite signs. From now on let

$$C(i, \varepsilon) = \begin{cases} C_{>i} & \text{if the sign of } C_{>i} \text{ is } \varepsilon, \\ (C_{\leq i})^{-1} & \text{otherwise,} \end{cases}$$

and further let

$$d_i(C, \varepsilon) = \begin{cases} 1 & \text{if } C(i, \varepsilon) = C_{>i}, \\ -1 & \text{if } C(i, \varepsilon) = (C_{\leq i})^{-1}. \end{cases}$$

The result and proof of the next lemma is basically [CB4, Lemma 8.1.]. The only difference is that for string modules over the completed string algebra the set $\{b_i \mid i \in I\}$ is not necessarily a basis. But when looking at the details of the proof, it becomes clear that we only need the uniqueness of the coefficients of possibly infinite linear combinations of the b_i. We do not need any finiteness assumptions on these linear combinations.

Lemma 3.3.30. *Let C be a non-periodic I-word and let $M = M_\Lambda(C)$. If $D \in \mathcal{W}_{v,\varepsilon}$ we have*
(i) $D^+(M) \cap \{b_i \mid i \in I\} = \{b_i \mid v_i(C) = v, C(i, \varepsilon) \leq D\}$ and
(ii) $D^-(M) \cap \{b_i \mid i \in I\} = \{b_i \mid v_i(C) = v, C(i, \varepsilon) < D\}$.

Proof. We are going to prove the following two statements:
(a) We have $b_i \in C(i, \varepsilon)^+(M)$.
(b) If $m = \sum_{j \in I} \lambda_j b_j$ belongs to $C(i, \varepsilon)^-(M)$, then $\lambda_i = 0$.

Let us first point out, how this proves the lemma. Let $i \in I$ such that $v_i(C) = v$ and $C(i, \varepsilon) \leq D$. We want to show, that $b_i \in D^+(M)$. If $D = C(i, \varepsilon)$, then this follows immediately from (a). If $C(i, \varepsilon) < D$ by Lemma 3.3.25 we have

$$C(i, \varepsilon)^+(M) \subseteq D^-(M) \subseteq D^+(M)$$

and hence $b_i \in D^+(M)$ again by (a).

Now let $m \in D^+(M)$ and write $m = \sum_{j \in I} \lambda_j b_j$ a possibly infinite linear combination. Let $i \in I$ such that $v_i(C) = v$ and $C(i, \varepsilon) > D$. Then using again the total order on words we have

$$m \in D^+(M) \subseteq C(i, \varepsilon)^-(M)$$

and hence by assumption (b) the coefficient $\lambda_i = 0$. In particular if $C(i, \varepsilon) > D$ we have $b_i \notin D^+(M)$ and this proves part (i) of the lemma.

For part (ii) we have already shown that

$$\{b_i \mid v_i(C) = v, C(i, \varepsilon) < D\} \subseteq D^-(M) \cap \{b_i \mid i \in I\}.$$

Now let $m \in D^-(M)$ and write again $m = \sum_{j \in I} \lambda_j b_j$. Let $i \in I$ such that $v_i(C) = v$ and $C(i, \varepsilon) \geq D$. If $C(i, \varepsilon) = D$, then $\lambda_i = 0$ by part (b). If $C(i, \varepsilon) > D$, then again by the total order of words we have

$$m \in D^-(M) \subseteq D^+(M) \subseteq C(j, \varepsilon)^-(M)$$

and again $\lambda_i = 0$ by part (b), which proves part (ii).

Let us now prove statement (a). Set $d = d_i(C, \varepsilon)$. First note, that for $n \geq 1$, and not greater than the length of $C(i, \varepsilon)$, we have

$$b_{i+d(n-1)} \in C(i, \varepsilon)_n b_{i+dn}.$$

Indeed suppose that $d = 1$, the case $d = -1$ can be treated similarly. Then $C(i, \varepsilon) = C_{>i}$ and $(C_{>i})_n = C_{i+n}$. If $C_{i+n} = x$ for an arrow x, we have $b_{i+n-1} = x b_{i+n}$ and if $C_{i+n} = x^{-1}$ for an arrow x we have $x b_{i+n-1} = b_{i+n}$ and thus $b_{i+n-1} \in x^{-1} b_{i+n}$. In any case we see that $b_{i+n-1} \in (C_{>i})_n b_{i+n}$ as claimed. Thus we see, if $C(i, \varepsilon)$ is an $\mathbb{N}$-word, we have $b_i \in C(i, \varepsilon)^+(M)$. Now suppose that $C(i, \varepsilon)$ is a finite word of length n and let $1_{u,\eta}$ be the trivial word such that $C(i, \varepsilon) 1_{u,\eta}$ is defined. Then $b_{i+dn} \in 1^+_{u,\eta}(M)$ and hence also in the finite case it follows that $b_i \in C(i, \varepsilon)^+(M)$ proving statement (a).

To prove statement (b) we first show the following statement by induction on n, where n is not greater than the length of $C(i, \varepsilon)$: If $m \in C(i, \varepsilon)_{\leq n} m'$ for some $m' \in M$, and the coefficient of b_i in m is λ, then the coefficient of b_{i+dn} in m' is also λ.

We prove this again only for the case $d = 1$. Let $n = 1$ and $m \in (C_{>i})_1 m' = C_{i+1} m'$. We can write $m = \sum_{j \in I} \lambda_j b_j$ and $m' = \sum_{j \in I} \mu_j b_j$. If $C_{i+1} = x$ for some arrow x we have

$$\sum_{j \in I} \lambda_j b_j = m = x m' = \sum_{j \in I} \mu_j x b_j$$

and $x b_{i+1} = b_i$ and $x b_j \neq b_i$ for any $j \neq i + 1$. Hence by the uniqueness of the coefficients we have $\mu_{i+1} = \lambda_i = \lambda$. The argument is similar if $C_{i+1} = x^{-1}$ for an arrow x.

Now assume the statement has been proven for $n \geq 1$ and let $m \in C(i, \varepsilon)_{\leq n+1} m'$. Then there exists some $m'' \in M$ such that $m \in C(i, \varepsilon)_{\leq n}$ and $m'' \in C(i, \varepsilon)_{i+n+1} m'$. Then by induction the coefficient of b_{i+n} in m'' is λ and similarly as in the case $n = 1$ one shows that hence the coefficient of b_{i+n+1} in m' is also λ.

Thus if $C(i, \varepsilon)$ is an $\mathbb{N}$-word and $m \in C(i, \varepsilon)^-(M)$, then the coefficient λ of b_i in m must be zero. Otherwise this non-zero coefficient appears as a coefficient in any of the members of any sequence m_n with $m_0 = m$ and $m_{n-1} \in C(i, \varepsilon)_n m_n$, which contradicts $m \in C(i, \varepsilon)^-(M)$.

If $C(i, \varepsilon)$ is a finite word of length n, then no element of $1^-_{u,\eta}(M)$ has b_{i+dn} occurring with non-zero coefficient and this concludes the proof of statement (*b*). $\square$

Corollary 3.3.31. *Let C be a non-periodic I-word and let $M = M_\Lambda(C)$. Then for any $i \in I$ we have*

$$F^+_{C(i,1),C(i,-1)}(M) = F^-_{C(i,1),C(i,-1)}(M) \oplus Kb_i$$

and hence for any words B, D with $B^{-1}D = C$ we have $F_{B,D}(M) \cong K$.

Proof. By Lemma 3.3.30 we know that

$$F^+_{C(i,1),C(i,-1)}(M) = F^-_{C(i,1),C(i,-1)}(M) \oplus U$$

where any element in U can be written as a possibly infinite linear combination of the b_j with $C(j,1) = C(i,1)$ and $C(j,-1) = C(i,1)$. But since C is not periodic and no word can be equal to a shift of its inverse, this condition only holds for $j = i$. For words B, D with $B^{-1}D = C$ there exists some $i \in I$ such that $\{B, D\} = \{C(i,1), C(i,-1)\}$. $\square$

Total order on $\mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ For a pair of words $(B, D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ and a Λ-module M, we define

$$G^{\pm}_{B,D}(M) = B^-(M) + D^{\pm}(M) \cap B^+(M) \subseteq e_v M.$$

Note that, since $B^-(M) \subseteq B^+(M)$ we have

$$[B^-(M) + D^{\pm}(M)] \cap B^+(M) = B^-(M) + [D^{\pm}(M) \cap B^+(M)].$$

Furthermore, we define a total order on $\mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ lexicographically, thus

$$(B, D) < (B', D') \iff B < B' \quad \text{or} \quad (B = B' \text{ and } D < D').$$

Lemma 3.3.32. *For $(B, D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ and a Λ-module M, we have*

- *$G^-_{B,D}(M) \subseteq G^+_{B,D}(M)$ and $G_{B,D}(M) = G^+_{B,D}(M)/G^-_{B,D}(M) \cong F_{B,D}(M)$,*
- *if $(B, D) < (B', D')$ we have $G^+_{B,D}(M) \subseteq G^-_{B',D'}(M)$.*

Proof.

- The part $(B, D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ follows immediately from $D^-(M) \subseteq D^+(M)$. Using some well-known isomorphisms and the fact that $D^-(M) \cap B^+(M) \subseteq D^+(M) \cap B^+(M)$, we easily compute

$$\frac{G^+_{B,D}(M)}{G^-_{B,D}(M)} = \frac{[B^- + (D^+ \cap B^+)](M)}{[B^- + (D^- \cap B^+)](M)} \cong \frac{[D^+ \cap B^+](M)}{[(D^- \cap B^+) + (B^- \cap D^+ \cap B^+)](M)}$$

  which proves the first part.

- If $B < B'$ we have

$$[B^- + D^+ \cap B^+](M) \subseteq B^+(M) \subseteq (B')^-(M) \subseteq [(B')^- + (D')^- \cap (B')^+](M)$$

  where the second inclusion follows from Lemma 3.3.25. If $B = B'$ and $D < D'$ the claim follows directly since $D^+(M) \subseteq (D')^-(M)$ again by Lemma 3.3.25.

$\square$

Corollary 3.3.33. *Let C be a non-periodic word and let $M = M_\Lambda(C)$. For any two words B, D such that $B^{-1}D$ is not equivalent to C we have $F_{B,D}(M) = 0$.*

Proof. We can assume without loss of generality that $(B, D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ for some vertex v. If $G_{B,D}(M) = F_{B,D}(M) \neq 0$ then by Lemma 3.3.30 there is some $i \in I$ such that $b_i \in$

$G^+_{B,D}(M) \setminus G^-_{B,D}(M)$ and we know that

$$b_i \in G^+_{C(i,1),C(i,-1)}(M) \setminus G^-_{C(i,1),C(i,-1)}(M).$$

But then by the total ordering on $\mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ we must have $(B,D) = (C(i,1), C(i,-1))$. Indeed, if we assume to the contrary that $(B,D) < (C(i,1), C(i,-1))$ we have

$$b_i \in G^+_{B,D}(M) \subseteq G^-_{C(i,1),C(i,-1)}(M)$$

which is a contradiction and similarly we get a contradiction if $(B,D) > (C(i,1), C(i,-1))$. Thus we have $(B,D) = (C(i,1), C(i,-1))$ and so $B^{-1}D$ is equivalent to C. □

Covering property for string modules The following lemma is what we call the *covering property for string modules.* Later on for any finitely generated Λ-module M we will construct a homomorphism $f\colon N \to M$, where N is a direct sum of string and band modules, such that $F_{B,D}(f)$ is an isomorphism for all refined functors $F_{B,D}$. We will need the covering property for string modules to show that in this situation $f\colon N \to M$ is injective. In [CB4] this is done in Lemma 9.4. The proof in our case is essentially the same. However, it is a bit more involved, since we have to show, that even when considering infinitely many pairs of words $(B,D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ such that $B^{-1}D$ is equivalent to a fixed word C, we can choose a pair which is maximal with respect to the total order on $\mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$.

Lemma 3.3.34 (Covering property for string modules)**.** *Let C be a non-periodic I-word and let $M = M_\Lambda(C)$. Let v be a vertex and $0 \neq m \in e_v M$. Then there exists a pair $(B,D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ such that $m \in G^+_{B,D}(M)$, but $m \notin G^-_{B,D}(M)$.*

Proof. First assume that as a vector space we have $M_A(C) = M_\Lambda(C)$. Then the element m can be written as a finite linear combination in the basis elements $b_i \in e_v M$. By Corollary 3.3.31 for any b_i there exist $(B_i, D_i) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ with $F^+_{B_i,D_i}(M) = F^-_{B_i,D_i}(M) \oplus K b_i$. Then it follows that $G^+_{B_i,D_i}(M) = G^-_{B_i,D_i}(M) \oplus K b_i$. Since there are only finitely many $i \in I$ for which the coefficient in the linear combination of m is non-zero, we can choose (B,D) maximal among the corresponding pairs (B_i, D_i). Then it follows that m is in $G^+_{B,D}(M)$ but not in $G^-_{B,D}(M)$.

Now suppose that as vector spaces we have $M_A(C) \subsetneq M_\Lambda(C)$. Thus we are in the situation that C is an infinite word such that C or C^{-1} are eventually inverse. The proof is basically the same as in the first case with the difference that m is a possibly infinite linear combination in the elements $b_i \in e_v M$, and hence it is a priori not clear, that we can choose a maximal pair (B,D) among the relevant pairs (B_i, D_i). We will consider the case where C is a $\mathbb{Z}$-word such that C and C^{-1} are eventually inverse. The other cases are similar. Note that if C is a mixed word it might have a part with infinitely many partitions into pairs of words which do not have a maximum. But for that part we only consider finite linear combinations in the corresponding b_i, and hence it is possible to choose a maximum for the finitely many involved pairs of words.

Now since C and C^{-1} are eventually inverse we have

$$C = \dots RRR\tilde{C}P^{-1}P^{-1}P^{-1}\dots$$

for primitive cycles $R = R_1 \dots R_k$ and $P = P_1 \dots P_s$ and a finite word $\tilde{C}$. We assume without loss of generality that $\tilde{C} = C_1 \dots C_n$. Then we have

$$\begin{aligned}
(C_{\leq -1})^{-1} &= (R^{-1})^{\infty} \\
(C_{\leq -2})^{-1} &= R_{k-1}^{-1} R_{k-2}^{-1} \dots R_1^{-1}(R^{-1})^{\infty} \\
(C_{\leq -3})^{-1} &= R_{k-2}^{-1} \dots R_1^{-1}(R^{-1})^{\infty} \\
&\vdots \\
(C_{\leq -k-1})^{-1} &= (R^{-1})^{\infty} \\
(C_{\leq -k-2})^{-1} &= R_{k-1}^{-1} R_{k-2}^{-1} \dots R_1^{-1}(R^{-1})^{\infty}
\end{aligned}$$

and hence we see, that $C_{\leq -j}^{-1} = C_{\leq -tk-j}^{-1}$ for $1 \leq j < k$ and any $t \geq 0$. Furthermore, we have

$$\begin{aligned}
C_{>-1} &= \tilde{C}(P^{-1})^{\infty} \\
C_{>-2} &= R_k \tilde{C}(P^{-1})^{\infty} \\
C_{>-3} &= R_{k-1} R_k \tilde{C}(P^{-1})^{\infty} \\
&\vdots \\
C_{>-k-1} &= R\tilde{C}(P^{-1})^{\infty} \\
C_{>-k-2} &= R_k R\tilde{C}(P^{-1})^{\infty} \\
&\vdots
\end{aligned}$$

and thus we have $C_{>-j} > C_{>-tk-j}$ for $1 \leq j < k$ and for any $t > 0$: indeed note, that if C_1 is direct we must have $C_1 = R_1$ and so on. Thus if $C_s = x^{-1}$ is the first inverse letter to appear in $\tilde{C}(P^{-1})$, we have $C_{>-tk-j} = ByD$ and $C_{>-j} = Bx^{-1}D'$ for some finite (in fact direct) word B, arrows x, y and some words D and D'. Therefore for $1 \leq j < k$ and any $t > 0$ we have

$$((C_{\leq -j})^{-1}, C_{>-j}) > ((C_{\leq -tk-j})^{-1}, C_{>-tk-j})$$

and hence the maximal pair of $((C_{\leq -j})^{-1}, C_{>-j})$ for $1 \leq j < k$, will be maximal for all pairs $((C_{\leq -i})^{-1}, C_{>-i})$ with $i > 0$. Now let us consider "the other side": We have

$$\begin{aligned}
(C_{\leq n})^{-1} &= (\tilde{C})^{-1}(R^{-1})^{\infty} \\
(C_{\leq n+1})^{-1} &= P_s(\tilde{C})^{-1}(R^{-1})^{\infty} \\
(C_{\leq n+2})^{-1} &= P_{s-1} P_s(\tilde{C})^{-1}(R^{-1})^{\infty} \\
&\vdots \\
(C_{\leq n+s})^{-1} &= P(\tilde{C})^{-1}(R^{-1})^{\infty} \\
(C \leq n+1+s)^{-1} &= P_s P(\tilde{C})^{-1}(R^{-1})^{\infty} \\
&\vdots
\end{aligned}$$

and similarly as before we thus see that $(C_{\leq n+j})^{-1} > (C_{\leq n+j+ts})^{-1}$ for $0 \leq j < s$ and for any $t > 0$.

Furthermore, we have

$$\begin{aligned}
C_{>n} &= (P^{-1})^{\infty} \\
C_{>n+1} &= P_{s-1}^{-1}P_{s-2}^{-1}\dots P_1^{-1}(P^{-1})^{\infty} \\
C_{>n+2} &= P_{s-2}^{-1}\dots P_1^{-1}(P^{-1})^{\infty} \\
&\vdots \\
C_{>n+s} &= (P^{-1})^{\infty} \\
C_{>n+1+s} &= P_{s-1}^{-1}P_{s-2}^{-1}\dots P_1^{-1}(P^{-1})^{\infty} \\
&\vdots
\end{aligned}$$

and thus we have $C_{>n+j} = C_{>n+j+ts}$ for $0 \leq j < s$ and for any $t \geq 0$. Therefore, for $0 \leq j < s$ and for any $t > 0$ we have

$$((C_{\leq n+j})^{-1}, C_{>n+j}) > ((C_{\leq n+j+ts})^{-1}, C_{>n+j+ts})$$

and hence the maximal pair of $((C_{\leq n+j})^{-1}, C_{>n+j})$ for $0 \leq j < s$, will be maximal for all pairs $((C_{\leq i})^{-1}, C_{>i})$ with $i \geq n$.

Since there are only finitely many other partitions $C_{>i}$ and $(C_{\leq i})^{-1}$ of C with $1 \leq i \leq n$, we see that we can always choose (B, D) maximal. □

Evaluation on band modules Concerning band modules, our definition does not alter from Crawley-Boevey's, except for when C is direct or inverse repeating. But in that case for M a finitely generated Λ-module we have $F_{B,D}(M) = 0$ anyway. Therefore, we will simply recall the concerning Lemmatas from [CB4] without giving the proofs. For this assume that C is a periodic word which is not direct or inverse and V a finite-dimensional $K[T, T^{-1}]$-module. Then we have

$$M(C, V) = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$$

as $K[T, T^{-1}]$-modules where each $V_i = b_i \otimes V$ is identified with a copy of V.

Lemma 3.3.35. *Let $M = M(C, V)$. Then we have*

(i) $F^{+}_{C(i,1),C(i,-1)}(M) = F^{-}_{C(i,1),C(i,-1)}(M) \oplus V_i$ *for* $0 \leq i < n$,

(ii) $F_{B,D}(M) \cong V$ *for words* B, D *with* $B^{-1}D = C$,

(iii) $F_{B,D}(M) = 0$ *for words* B, D *such that* $B^{-1}D$ *is not equivalent to* C.

Lemma 3.3.36. *Let $M = M(C, V)$, v a vertex and $0 \neq m \in e_v M$. Then there exists a pair $(B, D) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ such that $m \in G^{+}_{B,D}(M)$, but $m \notin G^{-}_{B,D}(M)$.*

Evaluation on direct sums of copies of string and band modules

Corollary 3.3.37. *Let M be a direct sum of copies of string modules and modules of the form $M(C, V)$.*

(i) If $B^{-1}D$ is a non-periodic word, then $\dim F_{B,D}(M)$ is equal to the number of string module summands of strings C with $C \sim B^{-1}D$.

(ii) If $B^{-1}D$ is a periodic word, then $F_{B,D}(M)$ is isomorphic to the direct sum of all the $K[T, T^{-1}]$-modules V such that $M(C, V)$ is a direct summand of M and $C \sim B^{-1}D$.

Proof. This is a direct consequence of Corollary 3.3.31 and Corollary 3.3.33 and Lemma 3.3.35. □

3.3.7 On the existence of homomorphisms from string and band modules

We would like to recall an important fact about the refined functors for string algebras. Since this is in a sense a key property of the refined functors used in the proof of [CB4, Lemma 8.3.], we would like to formulate this as a lemma.

Lemma 3.3.38. *Let A be a string algebra and C a non-periodic I-word. Let M be any A-module and let $B = C(i,1)$ and $D = C(i,-1)$ for some $i \in I$. Then for any $m \in F^+_{B,D}(M)$ such that the class of m in $F_{B,D}(M)$ is not zero, there is an A-module homomorphism*

$$\theta_m \colon M_A(C) \to M$$

sending b_i to m.

Corollary 3.3.39. *Let C be a non-periodic I-word and let $B = C(i,1)$ and $D = C(i,-1)$ for some $i \in I$.*

(i) If $M_\Lambda(C) = M_A(C)$ as vector spaces, then for any Λ-module M there exists a homomorphism of Λ-modules

$$\theta_{B,D,M} \colon M_\Lambda(C) \otimes_K F_{B,D}(M) \to M$$

such that $F_{B,D,M}(\theta_{B,D,M})$ is an isomorphism.

(ii) If M is a finitely generated Λ-module, there exists a homomorphism of Λ-modules

$$\theta_{B,D,M} \colon M_\Lambda(C) \otimes_K F_{B,D}(M) \to M$$

such that $F_{B,D,M}(\theta_{B,D,M})$ is an isomorphism.

Proof. Let M by any Λ-module and let (m'_α) be a basis of $F_{B,D}(M)$. Pick (m_α) in $F^+_{B,D}(M)$ such that the class of m_α in $F_{B,D}(M)$ is m'_α. Since any Λ-module is also an A-module by Lemma 3.3.38 there are A-module homomorphisms

$$\theta_\alpha \colon M_A(C) \to M$$

sending b_i to m_α for all α.

Now first suppose that $M_A(C) = M_\Lambda(C)$ as vector spaces and hence $M_\Lambda(C)$ is locally nilpotent. Therefore, θ_α is also a Λ-module homomorphism by Proposition 3.2.1. Combining all of them yields a Λ-module homomorphism $\theta_{B,D,M} \colon M_\Lambda(C) \otimes_K F_{B,D}(M) \to M$ and by Corollary 3.3.31 it follows that $F_{B,D,M}(\theta_{B,D,M})$ is an isomorphism.

Now assume that $M_A(C) \subsetneq M_\Lambda(C)$ and that M is finitely generated as a Λ-module. If C is a $\mathbb{Z}$-word or an $\mathbb{N}$-word which is eventually direct or a $\mathbb{Z}$-word or $-\mathbb{N}$-word such that C^{-1} is eventually direct we have $F_{B,D}(M) = 0$: assume without loss of generality that C is a $\mathbb{Z}$-word or an $\mathbb{N}$-word which is eventually direct. Since the functors $F_{B,D}$ are isomorphic for all shifts of C, we can assume that $D = P^\infty$, where P is a primitive cycle with head v. But then $D^+(M) = P'' = T^1_P(e_vM) = 0$, where we consider P as a linear relation on e_vM. Hence, in this case the zero-morphism will do the job.

Otherwise we know that $M_\Lambda(C)$ is isomorphic to the completion of $M_A(C)$ with respect to the $\mathfrak{n}$-adic topology and since M is finitely generated as a Λ-module, M is complete with respect to the $\mathfrak{n}$-adic topology. Hence the $\mathfrak{n}$-adic completions of the maps θ_α yield Λ-module homomorphisms

$$\widehat{\theta_\alpha} \colon \widehat{M_A(C)} \cong M_\Lambda(C) \to M \cong \hat{M}$$

sending b_i to m_α for all α. Again combining all of them, yields a Λ-module homomorphism $\theta_{B,D,M} \colon M_\Lambda(C) \otimes_K F_{B,D}(M) \to M$ and by Corollary 3.3.31 it follows that $F_{B,D,M}(\theta_{B,D,M})$ is an isomorphism. $\square$

Let C be a periodic word and V a finite-dimensional $K[T,T^{-1}]$-module. Note that since $M(C,V)$ is a nilpotent module, any A-module homomorphism $\theta \colon M(C,V) \to M$, where M is a Λ-module and hence an A-module, is also a Λ-module homomorphism. Therefore, the following lemma is [CB4, Lemma 8.6.], where we have replaced the assumption that M is a C-split module, by the assumption that M is a finitely generated Λ-module. But this implies that M is C-split by Lemma 3.3.29.

Lemma 3.3.40. *Let C be a periodic word, M a finitely generated Λ-module and $V = F_{B,D}(M)$ for words B, D with $C = B^{-1}D$. Then there is a homomorphism $\theta_{B,D,M} \colon M(C, V) \to M$ such that $F_{B,D}(\theta_{B,D,M})$ is an isomorphism.*

Theorem 3.3.41. *Let M be a finitely generated Λ-module. There is a homomorphism $\theta \colon N \to M$, where N is a direct sum of string and finite-dimensional band modules such that $F_{B,D}(\theta)$ is an isomorphism for all refined functors $F_{B,D}$. Furthermore, θ is injective.*

Proof. For a non-periodic word $C = B^{-1}D$ there exists a map $\theta_{B,D,M}$ from a direct sum of copies of $M(C)$ to M by Corollary 3.3.39. If $C = B^{-1}D$ is a periodic word, then by Lemma 3.3.40 there exists a homomorphism $\theta_{B,D,M}$ from a finite-dimensional module of the form $M(C, V)$ to M, where $V = F_{B,D}(M)$. We can decompose $V = V_1 \oplus \cdots \oplus V_n$ into indecomposable $K[T, T^{-1}]$-modules to obtain a decomposition

$$M(C, V) = M(C, V_1) \oplus \cdots \oplus M(C, V_n)$$

of finite-dimensional band modules. We define N to be the direct sum of all these string and band modules, where (B, D) runs through pairs of words in such a way that $C = B^{-1}D$ runs through the equivalence classes of words, once each. Then taking $\theta \colon N \to M$ as the direct sum of all the maps $\theta_{B,D,M}$ yields a map such that $F_{B,D}(\theta)$ is an isomorphism for all refined functors $F_{B,D}$.

Suppose that θ is not injective and let n be a non-zero element of $e_v N$ such that $\theta(n) = 0$. We can write n as a sum $n = n_1 + \ldots + n_k$, where each component n_i is a non-zero element of $e_v N_i$, where N_i is a string module or a finite-dimensional band module. By the covering property for string modules we know, that for each $1 \leq i \leq k$ there is $(B_i, D_i) \in \mathcal{W}_{v,\varepsilon} \times \mathcal{W}_{v,-\varepsilon}$ with n_i is in $G^+_{B_i,D_i}(N_i)$ but not in $G^-_{B_i,D_i}(N_i)$. Among those finitely many pairs (B_i, D_i) we can choose the maximal pair (B, D). Then n is in $G^+_{B,D}(N)$ but not in $G^-_{B,D}(N)$ and thus induces a non-zero element in $F_{B,D}(N)$. This implies that $\theta(n)$ induces a non-zero element in $F_{B,D}(M)$, which is a contradiction. □

3.3.8 Covering property

In a very simplified way, the covering property for refined functors says the following: if M is a finitely generated Λ-module and $0 \neq m \in M$ is an element satisfying certain conditions, then there exist some words B, D such that m induces a non-zero element in $F_{B,D}(M)$. We need this property to prove the surjectivity of the homomorphism that was constructed in Theorem 3.3.41. The section is based on [CB4, Section 10] and the results and proofs here are essentially the same, where we have to replace $K[z]$ by $K[[z]]$.

Lemma 3.3.42. *Let C be an $\mathbb{N}$-word, which is not (direct and repeating) and let M be a finitely generated Λ-module. Then the descending chain*

$$C_{\leq 1}M \supseteq C_{\leq 2}M \supseteq C_{\leq 3}M \supseteq \ldots \tag{$\star$}$$

stabilizes.

Proof. Since we are only considering finite quivers, any direct $\mathbb{N}$-word is also repeating. Thus we can assume that C is an $\mathbb{N}$-word which is not direct.

First assume that C is eventually inverse, say $C_{>n}$ is an inverse $\mathbb{N}$-word for some $n \geq 0$. Since $f^{-1}(M) = M$ for any map from M to M it follows, that the chain $(\star)$ stabilizes at $C_{\leq n}M$.

Now we can assume that C is not direct and not eventually inverse. Thus, $C = Dx^{-1}yB$ for some words D, B and distinct arrows x, y. So $(\star)$ becomes the chain

$$Dx^{-1}yB_{\leq 1}M \supseteq Dx^{-1}yB_{\leq 2}M \supseteq Dx^{-1}yB_{\leq 3}M \supseteq \ldots \tag{3.1}$$

and we want to prove that it stabilizes. We have $Dx^{-1}yB_{\leq n}M = Dx^{-1}(xM \cap yB_{\leq n}M)$ and by Lemma 3.3.28 the vector space $xM \cap yB_{\leq n}M$ is finite-dimensional. Thus we see, that all terms in the above chain are finite-dimensional and hence it stabilizes. □

Lemma 3.3.43 (Realization lemma)**.** *Let M be a finitely generated Λ-module and C an $\mathbb{N}$-word. Then we have*

$$C^+(M) = \bigcap_{n\geq 0} C_{\leq n}M.$$

Proof. We always have the inclusion $C^+(M) \subseteq \bigcap_{n\geq 0} C_{\leq n}M$. To prove the other inclusion first assume, that C is direct and repeating, thus $C = P^\infty$ for some primitive cycle P of length r. Then

$$\bigcap_{n\geq 0} C_{\leq n}M \subseteq \bigcap_{n\geq 0} C_{\leq n\cdot r}M = \bigcap_{n\geq 0} P^n M = P'' = C^+(M)$$

where the second last equality follows from Lemma 3.3.22 as M is finitely generated by assumption.

Now, it is enough to prove the following: if $C = \ell D$ for some letter ℓ and an $\mathbb{N}$-word D, and if $m \in \bigcap_{n\geq 0} C_{\leq n}M$, there exists some $m' \in \bigcap_{n\geq 0} D_{\leq n}M$ such that $m \in \ell m'$.

If $\ell = x^{-1}$ for some arrow x, then we can take $m' = xm$. Now suppose that $\ell = x$ is a direct letter. We can assume that D is not (direct and repeating). Then by the previous lemma, the chain

$$D_{\leq 1}M \supseteq D_{\leq 2}M \supseteq D_{\leq 3}M \supseteq \dots$$

stabilizes and hence $\bigcap_{n\geq 0} D_{\leq n}M = D_{\leq k}M$ for some $k \geq 0$. Then we have $m = xm'$ for some $m' \in D_{\leq k}M$ and this proves the lemma. ☐

In the following lemma we will state the generalized version of what Ringel calls the *covering property* of the intervals defined by words.

Lemma 3.3.44 (Weak covering property for words)**.** *Let M be a Λ-module, v a vertex, $\varepsilon = \pm 1$ and $S \subseteq e_vM$ a non-empty set with $0 \notin S$. Then there exists a word $C \in \mathcal{W}_{v,\varepsilon}$ such that either*
(1) C is a finite word and $C^+(M) \cap S \neq \emptyset$ but $C^-(M) \cap S = \emptyset$, or
(2) C is an infinite word and $C_{\leq n}(M) \cap S \neq \emptyset$ for all $n \geq 0$ but $C^-(M) \cap S = \emptyset$.

Proof. Suppose that a finite word $C \in \mathcal{W}_{v,\varepsilon}$ satisfying condition (1) does not exist. We inductively construct an infinite word $C \in \mathcal{W}_{v,\varepsilon}$ satisfying condition (2), starting with $1_{v,\varepsilon}$:

Assume that $D = C_{\leq n}$ is constructed such that $C_{\leq n}(M) \cap S \neq \emptyset$ but $C_{\leq n}(0) \cap S = \emptyset$. If there exists an arrow y with Dy a word and such that $Dy(M) \cap S \neq \emptyset$, then we set $C_{n+1} = y$. Note, that in this case $C_{\leq n+1}0 = Dy0 = D0$ and hence $C_{\leq n+1}(0) \cap S = \emptyset$ also holds. If such an arrow y does not exist, we have $D^-(M) \cap S = \emptyset$.

If there exists an arrow x with Dx^{-1} a word and such that $Dx^{-1}(0) \cap S = \emptyset$, then we set $C_{n+1} = x^{-1}$. Note, that in this case $C_{\leq n+1}(M) = Dx^{-1}(M) = D(M)$ and hence $C_{\leq n+1}(M) \cap S \neq \emptyset$ also holds. If such an arrow x does not exist, we have $D^+(M) \cap S \neq \emptyset$.

By our initial assumption, such an arrow y as above or such an arrow x as above must exist. Otherwise the finite word D satisfies condition (1). ☐

Corollary 3.3.45 (Covering property for words)**.** *Let M be a finitely generated Λ-module, v a vertex and $\varepsilon = \pm 1$. Let U be a $K[[z]]$-submodule of e_vM such that $ze_vM \subseteq U$. If H is a subset of e_vM and $m \in H \setminus U$, there exists a word $C \in \mathcal{W}_{v,\varepsilon}$, such that $H \cap (U + m)$ meets $C^+(M)$, but not $C^-(M)$.*

Proof. Since $-m$ is not in U the set $S = H \cap (U + m)$ contains m but not 0. Hence by the weak covering property for words, there exists a word C, such that S does not meet $C^-(M)$. If C is finite, then S meets $C^+(M)$ and we are done.

If C is an $\mathbb{N}$-word which is not direct and repeating, then since S meets $C_{\leq n}(M)$ for all $n \geq 0$, and this chain stabilizes by Lemma 3.3.42, we see that S also meets

$$C^+(M) = \bigcap_{n\geq 0} C_{\leq n}M$$

by the Realization lemma.

If C is direct and repeating we have $C = P^\infty$ for some primitive cycle P. Then since S meets $C_{\leq n}$ for all $n \geq 0$ it follows that $U + m$ meets P^2M. But since $P^2M = zPM$ and this is contained in U by hypothesis, it follows that $m \in U$, which is a contradiction. $\square$

Lemma 3.3.46 (Covering property for refined functors)**.** *Let M be a finitely generated Λ-module and v a vertex. Let U be a $K[[z]]$-submodule of e_vM such that $ze_vM \subseteq U$. For $m \in e_vM \setminus U$, there exists a pair (B, D), such that $U + m$ meets $G^+_{B,D}(M)$, but not $G^-_{B,D}(M)$.*

Proof. By the covering property for words with $H = e_vM$ there exists a word $B \in \mathcal{W}_{v,\varepsilon}$ such that $U + m$ meets $B^+(M)$ but not $B^-(M)$. Then for $m' \in B^+(M) \cap (U + m)$ we have $U + m = U + m'$.

Consider the $K[[z]]$-submodule $U' = U + B^-(M)$ of e_vM. Suppose $m' \in U'$. Then there exists some $u \in U$ and $b \in B^-(M)$ such that $m' = u + b$. But then $b = m' - u$ is in $B^-(M) \cap (U + m)$ which is a contradiction. Hence $m' \notin U'$ and we can apply the covering property for words to $H = B^+(M)$ and $m' \in B^+(M) \setminus U'$ to obtain a word $D \in \mathcal{W}_{v,-\varepsilon}$ such that $B^+(M) \cap (U' + m')$ meets $D^+(M)$ but not $D^-(M)$.

Now let $x \in (U' + m') \cap B^+(M) \cap D^+(M)$. Then since $U' = U + B^-(M)$ we can write $x = u + b + m'$ with $u \in U$ and $b \in B^-(M)$. It follows that

$$x - b = u + m' \in (U + m') \cap (B^-(M) + D^+(M) \cap B^+(M))$$

and hence this intersection is non-empty.

On the other hand suppose there exists some

$$x \in (U + m') \cap (B^-(M) + D^-(M) \cap B^+(M)).$$

Then we can write

$$x = u + m' = b + d$$

with $u \in U$, $b \in B^-(M)$ and $d \in D^-(M)$. But then the element

$$u - b + m' = x - b = d$$

is in

$$(U' + m') \cap B^+(M) \cap D^-(M)$$

which is a contradiction. It follows that $U + m = U + m'$ meets $G^+_{B,D}(M)$ but not $G^-_{B,D}(M)$. $\square$

Lemma 3.3.47. *Let $\theta\colon N \to M$ be a homomorphism such that $F_{B,D}(\theta)$ is an isomorphism for all refined functors $F_{B,D}$. If M is finitely generated, then θ is surjective.*

Proof. We are going to show, that the cokernel of θ is primitive torsion, as in [CB4, Lemma 10.6.]. Since any finitely generated primitive torsion $K[[z]]$-module is zero, this implies the Lemma.

Suppose the cokernel is not primitive torsion. Then we can choose a vertex v such that $e_vM/e_v \operatorname{Im}(\theta)$ is not primitive torsion. Since this is a finitely generated $K[[z]]$-module it has a 1-dimensional quotient, say V, such that $zV = 0$. This implies that there exists a $K[[z]]$-submodule U of e_vM of codimension 1 with $e_v \operatorname{Im}(\theta) \subseteq U$ and $ze_vM \subseteq U$.

Then by the covering property for refined functors, for $m \in e_vM \setminus U$, there exists a pair (B, D) such that $U + m \cap G^+_{B,D}(M) \neq \emptyset$ but $U + m \cap G^-_{B,D}(M) = \emptyset$. Therefore, we can choose $u \in U$ and $b \in B^-(M)$ and $d \in B^+(M) \cap D^+(M)$ such that

$$u + m = b + d.$$

Now since $G_{B,D}(M) \cong F_{B,D}(M)$ and since $F_{B,D}(\theta)$ is an isomorphism, there exists an element $n \in e_vN$ such that $d = \theta(n) + c + c'$ for some $c \in D^-(M) \cap B^+(M)$ and $c' \in D^+(M) \cap B^-(M)$. Since $\theta(n) \in e_v \operatorname{Im}(\theta) \subseteq U$ it follows that

$$u - \theta(n) + m = b + c + c'$$

is contained in $U + m$. But since $b + c' + c$ is also contained in $B^-(M) + D^-(M) \cap B^+(M)$ this contradicts that $U + m$ does not meet $G^-_{B,D}(M)$. This finishes the proof. $\square$

Theorem 3.3.48. *The string modules $M_\Lambda(C)$, where C is a non-periodic I-word such that C and C^{-1} are eventually inverse and the band modules $M_\Lambda(C, V)$, where C is a periodic word which is not direct or inverse and V an indecomposable $K[T, T^{-1}]$-module, form a complete list of indecomposable finitely generated Λ-modules.*

Proof. Let M be a finitely generated Λ-module and $\theta\colon N \to M$ the injective morphism constructed in Theorem 3.3.41. Then θ is also surjective by Lemma 3.3.47. It follows from Proposition 3.3.10 that if $M_\Lambda(C)$ appears as a direct summand of N, the words C and C^{-1} must be eventually inverse. By Theorem 3.3.14 these are indecomposable Λ-modules. $\square$

3.4 Auslander-Reiten sequences

Let A be a string algebra and Λ the corresponding completed string algebra. In this section we want to describe the Auslander-Reiten sequences in $\mathrm{Mod}(\Lambda)$, ending in a finitely generated string module $M = M(C)$. For this we will use that $M \cong \varprojlim M_p$, where $M_p = M/\mathfrak{m}_p M$ is for $p \gg 0$ isomorphic to a finitely generated string module. Furthermore, M_p can be considered as an A_p-module, where A_p is the p-truncation of A. Since A_p is a finite-dimensional string algebra, the Auslander-Reiten sequences in $\mathrm{mod}(A_p)$ containing string or band modules are well-known (see [BR]).

We will consider short exact sequences

$$0 \to U \to W \to V \to 0$$

where $V \in \mathrm{Noeth}_P(\Lambda)$ is a finitely generated non-projective string module and $U \cong \varinjlim \tau_{A_p}(V_p)$ is a module in $\mathrm{Art}_I(\Lambda)$. In order to check if sequences of this form are in fact almost split sequences, the following result as stated in [A1, Prop 3.4.] is very helpful. For a detailed proof of this statement in the language of morphisms determined by objects we refer to [A2, Theorem 10.6.]. For this let Λ be a noetherian algebra over a complete local ring.

Proposition 3.4.1. *Let $0 \to U \xrightarrow{i} W \xrightarrow{f} V \to 0$ be an exact sequence which is not split and suppose that $U \in \mathrm{Art}_I(\Lambda)$ and $V \in \mathrm{Noeth}_P(\Lambda)$ are both indecomposable. Then the following are equivalent:*

- *The sequence $0 \to U \xrightarrow{i} W \xrightarrow{f} V \to 0$ is an almost split sequence.*
- *If $H \in \mathrm{Noeth}(\Lambda)$, then for any morphism $h\colon H \to V$, which is not a split epimorphism, there exists a morphism $g\colon H \to W$ such that $fg = h$.*
- *If $H \in \mathrm{Art}(\Lambda)$, then for any morphism $h\colon U \to H$, which is not a split monomorphism, there exists a morphism $g\colon W \to H$ such that $gi = h$.*

3.4.1 Canonical exact sequences

Canonical exact sequences for arrows For any arrow x we define $U(x) = M(B) \in \mathrm{Art}_I(\Lambda)$ where B is the longest inverse word such that Bx is a word. Then B is a finite or an $-\mathbb{N}$-word. Similarly we define $V(x) = M(C) \in \mathrm{Noeth}_P(\Lambda)$, where C is the longest inverse word such that xC is a word. Note that C is a finite or an $\mathbb{N}$-word. Finally we define $N(x) = M(D)$, where $D = BxC$. We call the short exact sequence

$$0 \to U(x) \xrightarrow{\iota} N(x) \xrightarrow{\pi} V(x) \to 0$$

where ι is the canonical inclusion and π the canonical projection *a canonical exact sequence.*

Hooks and cohooks Let C be an I-word. Similarly as for finite words we say, that

- C *starts on a peak* if either C is a finite or $\mathbb{N}$-word and there is no arrow x such that $x^{-1}C$ is a string or C is a $-\mathbb{N}$-word or $\mathbb{Z}$-word, such that C^{-1} is eventually direct;
- C *ends on a peak* if either C is a finite or $-\mathbb{N}$-word and there is no arrow x such that Cx is a string or C is a $\mathbb{N}$-word or $\mathbb{Z}$-word which is eventually direct;
- C *starts in a deep* if either C is a finite or $\mathbb{N}$-word and there is no arrow y such that yC is a string or C is a $-\mathbb{N}$-word or $\mathbb{Z}$-word, such that C^{-1} is eventually inverse;
- C *ends in a deep* if either C is a finite or $-\mathbb{N}$-word and there is no arrow y such that Cy^{-1} is a string or C is a $\mathbb{N}$-word or $\mathbb{Z}$-word which is eventually inverse.

For any finite or $\mathbb{N}$-word C not starting on a peak, there is an arrow x and a direct word D such that ${}_hC := Dx^{-1}C$ is a word starting in a deep. We call Dx^{-1} a *hook*.

For any finite or $-\mathbb{N}$-word C not ending on a peak, there is an arrow x and a direct word D such that $C_h := CxD^{-1}$ is a word ending in a deep. We call xD^{-1} a *hook*.

For any finite or $\mathbb{N}$-word C not starting in a deep, there is an arrow y and a direct word D such that ${}_cC := D^{-1}yC$ is a word starting on a peak. We call $D^{-1}y$ a *cohook*.

For any finite or $-\mathbb{N}$-word C not ending in a deep, there is an arrow y and a direct word D such that $C_c := Cy^{-1}D$ is a word ending on a peak. We call $y^{-1}D$ a *cohook*.

Canonical exact sequences for finitely generated string modules Consider an I-word C such that $M(C)$ is finitely generated, not isomorphic to a projective module and not isomorphic to $V(x)$ for some arrow x. Hence C and C^{-1} are eventually inverse.

First assume that C starts and ends in a deep. Since $M(C)$ is not projective, C is not of the form $C = C_1C_2$ with C_1 direct and C_2 inverse. Hence $C = Ex^{-1}DyB^{-1}$, where x and y are arrows and E and B are direct words and D is a finite word. Thus $C = {}_hD_h$ and the short exact sequence

$$0 \longrightarrow M(D) \xrightarrow{(\iota,\iota)} M({}_hD) \oplus M(D_h) \xrightarrow{(\iota,-\iota)} M(C) \longrightarrow 0$$

where ι is the natural inclusion (and π will be the canonical projection), is again called *a canonical exact sequence.*

Now assume that C does not start but ends in a deep. Since C^{-1} is eventually inverse, but C does not start in a deep, C is a finite or an $\mathbb{N}$-word and thus ${}_cC$ is well-defined. Furthermore, since C is not isomorphic to $V(x)$, C is not an inverse word. Hence we can write $C = DyB^{-1}$, where D is a finite word, y is an arrow and B a direct word. Thus we have $C = D_h$ and the short exact sequence

$$0 \longrightarrow M({}_cD) \xrightarrow{(\iota,\pi)} M({}_cC) \oplus M(D) \xrightarrow{(\pi,-\iota)} M(C) \longrightarrow 0$$

is again called *a canonical exact sequence.*

If C starts in a deep but does not end in a deep, we find that C_c is well-defined and $C = {}_hD$ for some finite word D and we call the short exact sequence

$$0 \longrightarrow M(D_c) \xrightarrow{(\pi,\iota)} M(D) \oplus M(C_c) \xrightarrow{(\iota,-\pi)} M(C) \longrightarrow 0$$

again *a canonical exact sequence.*

For the last case, if C neither starts nor ends in a deep, C must be a finite word and both ${}_cC$ and C_c and thus also ${}_cC_c$ are well-defined. Then the short exact sequence

$$0 \longrightarrow M({}_cC_c) \xrightarrow{(\pi,\pi)} M({}_cC) \oplus M(C_c) \xrightarrow{(\pi,-\pi)} M(C) \longrightarrow 0$$

is also called *a canonical exact sequence.*

Theorem 3.4.2. *The canonical exact sequences are the Auslander-Reiten sequences ending in finitely generated string modules.*

Example 3.4.3. Consider the completed string algebra $\Lambda = K[[x, y]]/(xy)$ and the eventually inverse $\mathbb{N}$-word

$$C = xy^{-1}x(y^{-1})^{\infty}.$$

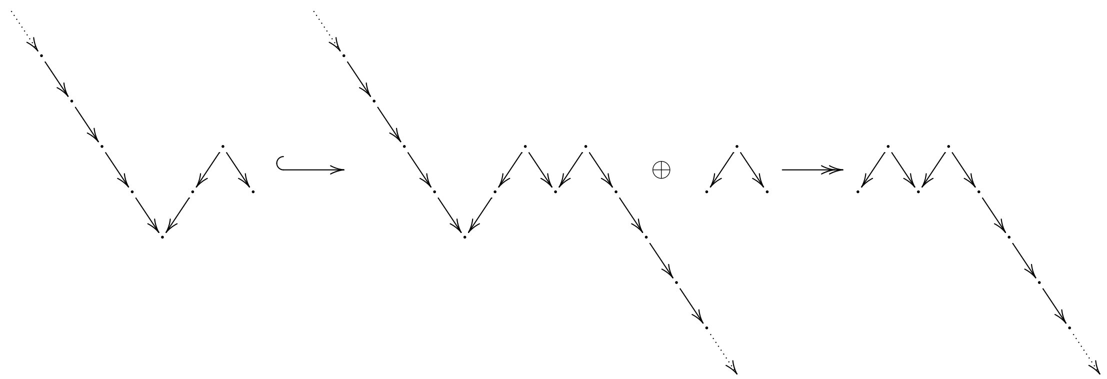
Then C does not start but ends in a deep. We have $C = D_h$, where $D = xy^{-1}$ is a finite word. Attaching a cohook at the start of C yields the mixed $\mathbb{Z}$-word

$$_cC = (y^{-1})^{\infty}xxy^{-1}x(y^{-1})^{\infty}.$$

Hence the canonical exact sequence

$$0 \longrightarrow M(_cD) \xrightarrow{(\iota,\pi)} M(_cC) \oplus M(D) \xrightarrow{(\pi,-\iota)} M(C) \longrightarrow 0$$

can be pictured as follows



where the dotted arrows should indicate, that there are infinitely many more arrows coming from, respectively going in, the pointed direction.

Outline of proof As our proof that these sequences are in fact almost split sequence is rather technical, we would like to explain the general idea first. Let

$$0 \to U \xrightarrow{i} W \xrightarrow{f} V \to 0$$

be a canonical exact sequence. Then by Proposition 3.3.10 we have that $U \in \mathrm{Art}_I(\Lambda)$ and $V \in \mathrm{Noeth}_P(\Lambda)$ and by Theorem 3.3.14 they are indecomposable. Hence by Proposition 3.4.1 it is enough to show, that if $h\colon H \to V$ in $\mathrm{Noeth}(\Lambda)$ is not a split epimorphism, there exists a morphism $g\colon H \to W$, such that $h = fg$.

For any finitely generated Λ-module M and $p > 0$ we will denote by M_p the A_p-module $M/\mathfrak{m}_pM$. Recall, that if M is finitely generated as a Λ-module we have $M = \varprojlim M_p$. We will show, that for any $p \gg 0$ there exist a morphism $h_p\colon H_p \to V_p$ of inverse systems, such that $h = \varprojlim h_p$, and h_p is not a split epimorphism. Further for every $p \gg 0$ we will consider short exact sequences

$$0 \to U_p \xrightarrow{i_p} W_p \xrightarrow{f_p} V_p \to 0$$

which are almost split sequences in $\mathrm{mod}(A_p)$ and such that $V = \varprojlim V_p$. Note that since the modules U and W might not be finitely generated, we have not yet defined U_p and W_p and in particular we cannot assume that $U = \varprojlim U_p$ or $W = \varprojlim W_p$. Anyway, it follows that for all $p \gg 0$ there exist morphisms $s_p\colon H_p \to W_p$ such that $h_p = f_p s_p$.

We will argue, that since H is finitely generated we can assume that there exists some $q \gg 0$ such that for all $p \geq q$ the image of s_p is contained in some finitely generated submodule

$$\tilde{W} = \varprojlim \tilde{W}_p$$

of W. Furthermore, there exists $\tilde{f} = \varprojlim \tilde{f}_p\colon \tilde{W} \to V$, such that $\tilde{f} = f\iota$, where $\iota\colon \tilde{W} \to W$ is the embedding. We will then prove that there actually exists a morphism $\tilde{s}_p\colon H_p \to \tilde{W}_p$ of inverse systems with $h_p = \tilde{f}_p \tilde{s}_p$. Hence for $\tilde{s} = \varprojlim \tilde{s}_p\colon H \to \tilde{W}$ we have $h = \tilde{f}\tilde{s}$. Then it follows that for $g = \iota\tilde{s}\colon H \to W$ we have $h = fg$.

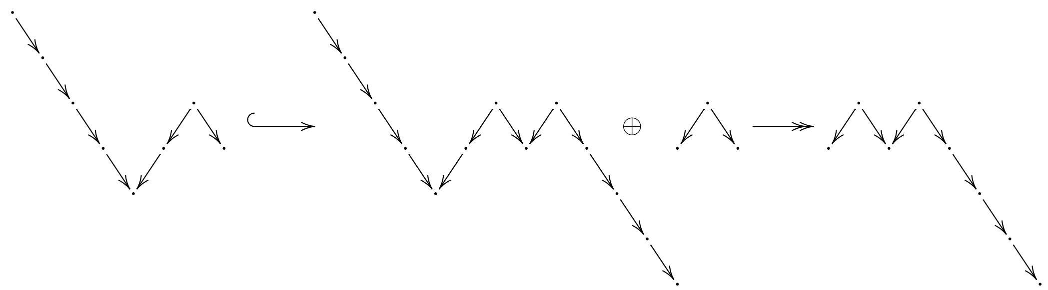
Example 3.4.4. Let Λ and the canonical exact sequence

$$0 \to U \xrightarrow{i} W \xrightarrow{f} V \to 0$$

be given as in Example 3.4.3. Then for $p = 5$ the short exact sequence

$$0 \to U_p \xrightarrow{i_p} W_p \xrightarrow{f_p} V_p \to 0$$

can be pictured as in



Remark 3.4.5. Note that we cannot apply the same methods as in [BR] to show that the maps involved in the canonical sequences are irreducible. The problem is, that even if the map in question is a morphism between string modules, we need to consider factorisations of this map where arbitrary modules in $\mathrm{Mod}(\Lambda)$ can appear. We do not have any knowledge about the Λ-modules that are not artinian or finitely generated. In particular we do not know, whether the refined functors *reflect isomorphisms* in this setting.

3.4.2 Restricting to finite-dimensional string modules

Homomorphisms between finitely generated modules Let $M = M(C), N = M(D)$ be finite-dimensional string modules. Recall that we had given a basis of $\mathrm{Hom}_\Lambda(M, N)$ by certain standard homomorphisms, corresponding to predecessor closed substrings of C, which occur as successor closed substrings of D. Furthermore, if b_i^M is a basis element of M and b_j^N a basis element of N there is at most one standard homomorphism $\theta\colon M \to N$ with $\theta(b_i^M) = b_j^N$.

Let $M = M(C)$ be a finitely generated string module. Recall that for any $p \gg 0$ we had defined the p-truncation as finite words $\pi_p(C)$ such that $M_p = M/\mathfrak{m}_p M \cong M(\pi_p(C))$.

Lemma 3.4.6. *Let $M = M(C)$ and $N = M(D)$ be finitely generated string modules and let $p \gg 0$ such that M_p and N_p are given by the p-truncations of the respective words. Furthermore, let $b_0, b_1, \ldots b_n$ be the basis symbols of N_p and assume that $\theta_p\colon M_p \to N_p$ is a standard homomorphism such that $b_o, b_n \notin \mathrm{Im}(\theta_p)$. Then for all $q \geq p$ there exists a standard homomorphism $\theta_q\colon M_q \to N_q$*

making the diagram

$$\begin{tikzcd} M_q \arrow[r, "\theta_q"] \arrow[d, "\pi"] & N_q \arrow[d, "\pi"] \\ M_p \arrow[r, "\theta_p"] & N_p \end{tikzcd}$$

commutative.

Proof. This is obvious: any predecessor closed substring of $\pi_p(C)$ is also a predecessor closed substring of $\pi_q(C)$ for $q \geq p$. Any successor closed substring of $\pi_p(D)$ not containing b_0 or b_n is also a successor closed substring of $\pi_q(D)$ for $q \geq p$. $\square$

Recall, that for any finitely generated Λ-module M we have $M = \varprojlim M_p$, where $M_p = M/\mathfrak{m}_p M$ is the p-truncation of M and can be considered as an A_p-module.

Lemma 3.4.7. *Let M, N be finitely generated Λ-modules and let $f\colon M \to N$ be a homomorphism. Then there exists a morphism of inverse system $f_p\colon M_p \to N_p$, such that $f = \varprojlim f_p$. In particular for all $q \geq p$ the diagram*

$$\begin{tikzcd} M_q \arrow[r, "f_q"] \arrow[d, "\pi"] & N_q \arrow[d, "\pi"] \\ M_p \arrow[r, "f_p"] & N_p \end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi_p^M"] & N \arrow[d, "\pi_p^N"] \\ M_p \arrow[r, dashed] & N_p \end{tikzcd}$$

where π_p^M are the natural projections. Since $\mathfrak{m}_p M \subseteq \ker(\pi_p^N \circ f)$ there exists a morphism $f_p\colon M_p \to N_p$ making the diagram commutative. Now consider the diagram

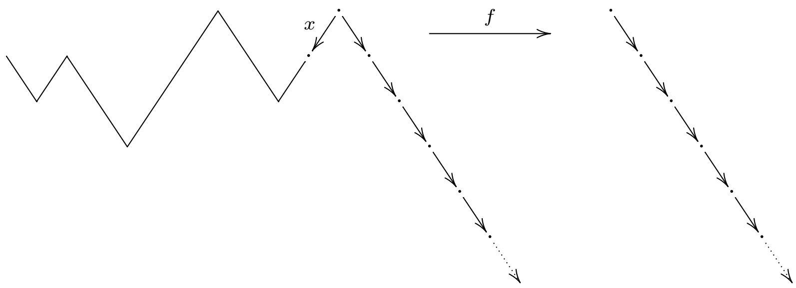
$$\begin{tikzcd} M \arrow[rrr, "f"] \arrow[dr, "\pi_q^M"] \arrow[ddr, "\pi_p^M"'] & & & N \arrow[dl, "\pi_q^N"'] \arrow[ddl, "\pi_p^N"] \\ & M_q \arrow[r, "f_q"] \arrow[d, "\pi_{pq}^M"] & N_q \arrow[d, "\pi_{pq}^N"'] & \\ & M_p \arrow[r, "f_p"] & N_p & \end{tikzcd}$$

and we want to show that the small square in the middle is commutative. But this follows, since any other square and triangle in this diagram is commutative by definition. $\square$

From now on, if $f\colon M \to N$ is a morphism between finitely generated Λ-modules, we will denote by $f_p\colon M_p \to N_p$ the morphisms such that $f = \varprojlim f_p$.

Constructing morphisms of inverse systems In the following assume that $C = DxB$ is an $\mathbb{N}$-word, where D is any finite word, x is an arrow and B an inverse $\mathbb{N}$-word. Set $M = M(C)$,

$N = M(B)$ and let $f\colon M \to N$ be the natural projection. Hence we can picture $f\colon M \to N$ as



where we only indicated the finite word D with the zick-zack-shaped line.

Furthermore, let H be a finitely generated string module and $h\colon H \to N$ a homomorphism. We can choose and fix $p \gg 0$ such that M_p, N_p and H_p are all string modules given by the p-truncation of the respective word. Finally, assume that for all $q \geq p$ there exist morphisms $s_q\colon H_q \to M_q$ such that $h_q = f_q s_q$. Then the following lemma says, that we can assume without loss of generality, that $s_q\colon H_q \to M_q$ is a morphism of inverse systems.

Lemma 3.4.8. *Let $f\colon M \to N$ be the natural projection and $h\colon H \to N$ be a homomorphism as above. Assume that for all $q \geq p$ there exist morphisms $s_q\colon H_q \to M_q$ such that $h_q = f_q s_q$. Then there exists a morphism of inverse systems $g_q\colon H_q \to M_q$ such that $g_p = s_p$ and for all $q \geq p$ we have $h_q = f_q g_q$. Thus, for $g = \varprojlim g_q\colon H \to M$, we have $fg = h$.*

Proof. We will construct $g_q\colon H_q \to M_q$ as described in the Lemma inductively. The reader should keep the following diagram in mind

$$\begin{tikzcd}
H_{p+1} \arrow[rrr, "\pi"] \arrow[dr, "h_{p+1}"] \arrow[dd, "s_{p+1}"'] & & & H_p \arrow[dl, "h_p"'] \arrow[dd, "s_p = g_p"] \\
& N_{p+1} \arrow[r, "\pi"] & N_p & \\
M_{p+1} \arrow[rrr, "\pi"] \arrow[ur, "f_{p+1}"] & & & M_p \arrow[ul, "f_p"']
\end{tikzcd}$$

where π always denotes the canonical projection, and we know that all triangles and squares are commutative, except for possibly the outer one.

First we set $g_p = s_p$ and construct $\tilde{g}_{p+1}\colon H_{p+1} \to M_{p+1}$ with $\pi\tilde{g}_{p+1} = g_p\pi$. For this we can assume without loss of generality that g_p is a standard homomorphism. Now if $f_p g_p = 0$ it follows as in Lemma 3.4.6, that there exists a standard homomorphism $\tilde{g}_{p+1}\colon H_{p+1} \to M_{p+1}$ with $\pi\tilde{g}_{p+1} = g_p\pi$.

So now assume that $f_p g_p \neq 0$. Hence there are basis elements $b_i^H \in H_p$, $b_j^M \in M_p$ and $b_k^N \in N_p$ such that $f_p g_p(b_i^H) = f_p(b_j^M) = b_k^N$ and g_p is uniquely determined by $g_p(b_i^H) = b_j^M$. By abuse of notation denote the basis element $b \in H_{p+1}$ such that $\pi(b) = b_i^H$ also by b_i^H. Then we have

$$b_k^N = f_p g_p(b_i^H) = h_p(b_i^H) = h_p\pi(b_i^H) = \pi h_{p+1}(b_i^H) = \pi f_{p+1} s_{p+1}(b_i^H) = f_p \pi s_{p+1}(b_i^H).$$

Then it follows that $s_{p+1}(b_i^H) = b_j^M + r$, where r is in the kernel of the composition $f_p\pi\colon M_{p+1} \to N_p$. Here again by abuse of notation we denoted the basis element $b \in M_{p+1}$ such that $\pi(b) = b_j^M$ also by b_j^M. This implies that there exists a standard homomorphism $\tilde{g}_{p+1}\colon H_{p+1} \to M_{p+1}$ with $\tilde{g}_{p+1}(b_i^H) = b_j^M$.

Now we know that there exists $\tilde{g}_p\colon H_{p+1} \to M_{p+1}$ with $\pi\tilde{g}_{p+1} = g_p\pi$ and furthermore, we have

$$\pi f_{p+1}\tilde{g}_{p+1} = f_p\pi\tilde{g}_{p+1} = f_p g_p \pi = h_p\pi = \pi h_{p+1}.$$

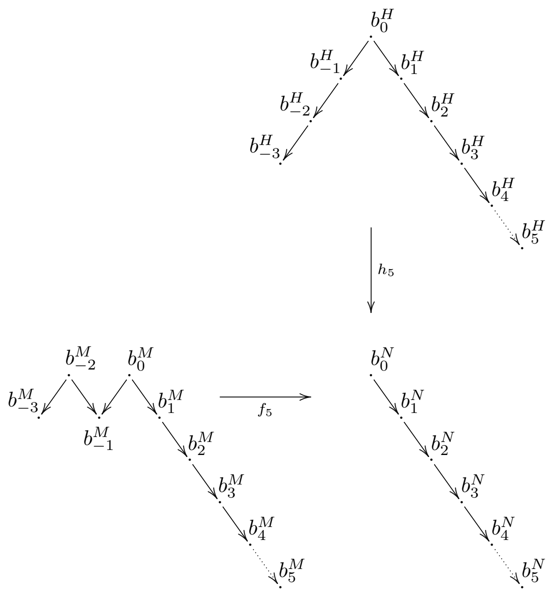


Figure 3.1: The p-truncation of the modules in Example 3.4.9 for $p = 5$ and $p = 6$.

Thus, if $b_{p+1}^N \in N_{p+1}$ denotes the basis symbol such that $\pi(b_{p+1}^N) = 0$, we have for any basis symbol $b_s^H \in H_{p+1}$ that

$$h_{p+1}(b_s^H) = f_{p+1}s_{p+1}(b_s^H) = f_{p+1}\tilde{g}_{p+1}(b_s^H) + \mu b_{p+1}^N$$

for some $\mu \in K$. If $\mu \neq 0$ there exists a standard homomorphism $\theta_{p+1}\colon H_{p+1} \to M_{p+1}$ with $\theta_{p+1}(b_s^H) = b_{p+1}^M$ where $f_{p+1}(b_{p+1}^M) = b_{p+1}^N$ and such that $\operatorname{Im}(\theta_{p+1}) = Kb_{p+1}^M$. Then we add $\mu\theta_{p+1}$ to $\tilde{g}_{p+1}$. Since we only have finitely many basis vectors of H_{p+1} to consider in this way, we eventually obtain $g_{p+1}\colon H_{p+1} \to M_{p+1}$ with

$$f_{p+1}g_{p+1} = h_{p+1} \qquad \text{and} \qquad \pi g_{p+1} = g_p\pi$$

and the lemma follows by induction.

□

Example 3.4.9. Let $A = K[x,y](xy)$ and $\Lambda = K[[x,y]]/(xy)$. We consider the strings

$$C = xy^{-1}x(y^{-1})^{\infty}, \qquad B = (y^{-1})^{\infty} \qquad \text{and} \qquad D = xxx(y^{-1})^{\infty}$$

and the Λ-modules $M = M_\Lambda(C)$, $N = M_\Lambda(B)$ and $H = M_\Lambda(D)$. Denote by

$$b_{-3}^M, b_{-2}^M, b_{-1}^M, b_0^M, b_1^M, \dots, \qquad b_0^N, b_1^N, b_2^N, \dots \qquad \text{and} \qquad b_{-3}^H, b_{-2}^H, b_{-1}^H, b_0^H, b_1^H, \dots$$

the basis symbols of the A-modules $M_A(C)$, $M_A(B)$ and $M_A(D)$ respectively. Let $h\colon H \to N$ be the homomorphism which sends

$$b_0^H \mapsto \sum_{i\geq 0} b_i^N.$$

In this example we can choose $p = 5$ and we assume that $s_5\colon H_5 \to M_5$ is defined by

$$b_0^h \mapsto b_{-2}^M + \sum_{i=0}^{4} b_i^M$$

and $s_6 \colon H_6 \to M_6$ is defined by

$$b_0^h \mapsto \sum_{i=0}^{5} b_i^M.$$

Then we have $h_5 = f_5 s_5$ and $h_6 = f_6 s_6$ but, $s_5 \pi \neq \pi s_6$. We set $g_5 = s_5$ and then the procedure described in the proof of Lemma 3.4.8 yields $g_6 \colon H_6 \to M_6$ defined by

$$b_0^h \mapsto b_{-2}^M + \sum_{i=0}^{5} b_i^M.$$

It follows that $h_5 = f_5 g_5$ and $h_6 = f_6 g_6$ and $g_5 \pi = \pi g_6$. Figure 3.1 shows the truncated modules in this example.

Restricting to finitely generated submodules of the middle term In the following we want to prove a Lemma which allows us to consider a finitely generated submodule of the possibly infinitely generated middle term of a canonical exact sequence. For this let $D = BxC$ be a word, where B is an inverse $-\mathbb{N}$-word, x an arrow and C an eventually inverse finite or $\mathbb{N}$-word. Let $N = M(D)$, $M = M(C)$ and let $\pi \colon N \to M$ be the canonical projection. Then we can assume that for all $p \gg 0$ we have $M_p = M(\pi_p(C))$ and that $\pi_p(xC) = x\pi_p(C)$. For any $q > 1$ we define the finite dimensional module

$${}_qN_p = M(B_{>-(q-1)} x \pi_p(C))$$

such that ${}_pN_p$ can be considered as a module over A_p. Note that for any $1 < q \leq p$ we have natural inclusions ${}_qN_p \hookrightarrow {}_pN_p$ and natural projections

$$\tilde{\pi}_p \colon {}_qN_p \to M_p \qquad \text{and} \qquad \pi_p \colon {}_pN_p \to M_p.$$

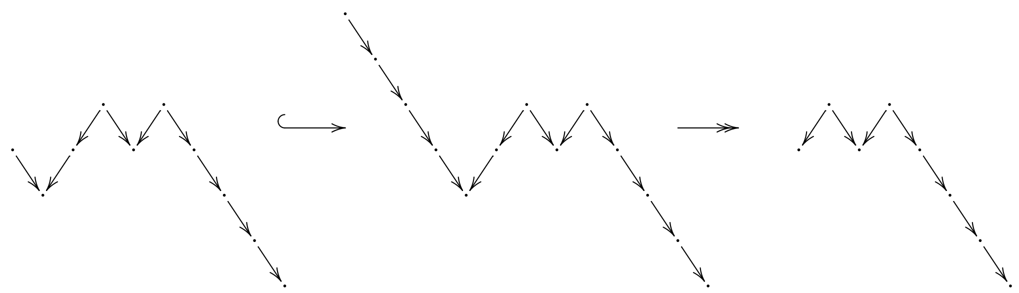
Example 3.4.10. Consider the completed string algebra $\Lambda = K[[x,y]]/(xy)$ and the $\mathbb{Z}$-word

$$D = \dots y^{-1}y^{-1}y^{-1}xxy^{-1}x(y^{-1})^{\infty}$$

and thus $B = \dots y^{-1}y^{-1}y^{-1}$ is an inverse $-\mathbb{N}$-word and $C = xy^{-1}x(y^{-1})^{\infty}$ an eventually inverse $\mathbb{N}$-word. Then for $N = M(D)$ we have

$$\begin{aligned} {}_2N_5 &= M(B_{>-1} x \pi_5(C)) = M(y^{-1}xxy^{-1}xy^{-1}y^{-1}y^{-1}y^{-1}), \\ {}_5N_5 &= M(B_{>-4} x \pi_5(C)) = M(y^{-1}y^{-1}y^{-1}y^{-1}xxy^{-1}xy^{-1}y^{-1}y^{-1}y^{-1}) \end{aligned}$$

and ${}_2N_5 \hookrightarrow {}_5N_5 \twoheadrightarrow M_5$, where $M = M(C)$ can be pictured as



Lemma 3.4.11. *Let $H \in \operatorname{Noeth}(\Lambda)$ be indecomposable and suppose that for all $p \gg 0$ there exist morphisms $s_p \colon H_p \to {}_pN_p$. Then there exists some $n > 0$ such that there exist morphisms $\tilde{s}_p \colon H_p \to {}_nN_p$ for all $p \geq n$ with $\tilde{\pi}_p \circ \tilde{s}_p = \pi_p \circ s_p$.*

Proof. Suppose that
$$B_{>-(p-1)} = y_1^{-1}y_2^{-1}\dots y_{p-1}^{-1}$$
where y_i are arrows for $1 \le i \le p-1$. Let b be the basis vector of ${}_pN_p$, corresponding to the head of $B_{>-(p-1)}x\pi_p(C)$. If $m \in {}_pN_p$ is an element such such that b appears with non-zero coefficient in the linear combination of m, then the elements

$$m,\ y_1m,\ y_2y_1m,\ \dots,\ y_{p-1}\dots y_2y_1m$$

are p linearly independent elements in ${}_pN_p$. Hence, if U is a submodule of ${}_pN_p$ of dimension $q \le p$ it can also be considered as a submodule in ${}_qN_p$.

Now first suppose that H is finite-dimensional and let $n > \dim(H)$. Then since for $p \ge n$ any submodule of ${}_pN_p$ of dimension smaller than n is also a submodule of ${}_nN_p$ we can define $\tilde{s}_p = s_p \colon H_p = H \to \mathrm{Im}(s_p) \subseteq {}_nN_p$.

Now suppose that H is a possibly infinite dimensional but finitely generated string module. Let $n \gg 0$ such that for all $p \ge n$ the module H_p is a string module given by the p-truncation of the word defining H. If $\theta\colon H_p \to {}_pN_p$ is a standard homomorphism which does not have image in ${}_nN_p$ it must correspond to a successor closed substring of $B_{>-(p-1)}$ of length greater than n. But for those we have $\pi_p \circ \theta = 0$. Hence we can define $\tilde{s}_p$ as the sum of standard homomorphisms defining s_p, where we just leave out those θ with $\pi_p \circ \theta = 0$. □

Proof of Theorem 3.4.2

Remark 3.4.12. Let $f\colon M \to N$ be a morphism in $\mathrm{Noeth}(\Lambda)$, which is not a split epimorphism. Then since M is a finite sum of string and band modules, i.e. $M = \oplus_{i=1}^n M_i$, we have $f = (f_i)_i \colon \oplus_{i=1}^n M_i \to N$ and none of the f_i is a split epimorphism. Let $f\colon M \to N$ be a morphism between finitely generated string modules or band modules, and let $f_p\colon M_p \to N_p$ such that $f = \varprojlim f_p$. If f is not a split epimorphism, there exists some $n \gg 0$, such that $f_p\colon M_p \to N_p$ is also not a split epimorphism for all $p \ge n$. Otherwise, since M_p is indecomposable, f_p would be an isomorphism. But if f_p was an isomorphism for infinitely many p, then f would also be an isomorphism.

Proposition 3.4.13. *The short exact sequence*

$$o \to U(x) \xrightarrow{\iota} N(x) \xrightarrow{\pi} V(x) \to 0$$

is an Auslander-Reiten sequence.

Proof. We know that $U = U(x) \in \mathrm{Art}_I(\Lambda)$, $V = V(x) \in \mathrm{Noeth}_P(\Lambda)$ are indecomposable. Hence, by Proposition 3.4.1 it is enough to show that for any morphism $h\colon H \to V = V(x)$ in $\mathrm{Noeth}_P(\Lambda)$ which is not a splittable epimorphism, there is a morphism $g\colon H \to N$ such that $\pi g = h$. By Remark 3.4.12 we can assume that H is indecomposable and therefore either a finitely generated string module or finite-dimensional band module. Again by Remark 3.4.12 we can assume, that $h = \varprojlim h_p \colon H \to V$, where $h_p\colon H_p \to V_p$ is not a split epimorphism for all $p \gg 0$.

Let B and C be the words such that $U = M(B)$, $V = M(C)$ and $N = M(BxC)$. Now we consider all $p \gg 0$ such that H_p and V_p are given by the p-truncation of the corresponding words and as for Lemma 3.4.11 for any $q > 1$ we consider the modules

$${}_qN_p = M(B_{>-(q-1)}x\pi_p(C)) \qquad \text{and} \qquad {}_qU = M(B_{>-(q-1)})$$

where in case that B is a finite word of length n we chose to label it as $B = B_{-(n-1)}\dots B_{-1}B_0$. Then it follows that
$$0 \to {}_pU \to {}_pN_p \xrightarrow{\pi_p} V_p \to 0$$
is an Auslander-Reiten sequence in $\mathrm{mod}(A_p)$. Hence for the morphisms $h_p\colon H_p \to V_p$ there exist morphisms $s_p\colon H_p \to {}_pN_p$, such that $\pi_p s_p = h_p$.

By Lemma 3.4.11 we know that there exists some $n > 0$ such that there exist morphisms $\tilde{s}_p\colon H_p \to {}_nN_p$ for all $p \geq n$ with $\tilde{\pi}_p\tilde{s}_p = \pi_p s_p = h_p$, where $\tilde{\pi}_p\colon {}_nN_p \to V_p$ and $\pi\colon {}_pN_p \to V_p$ are the corresponding projections.

If V is finite-dimensional, then there exists some $p \gg 0$ such that $C = \pi_p(C)$ and hence $V = V_p$. Then we can choose $g\colon H \to N$ as the composition

$$H \twoheadrightarrow H_p \xrightarrow{\tilde{s}_p} {}_nN_p \hookrightarrow N$$

where the maps which are not labelled are canonical projections or inclusions.

If $V(x)$ is infinite-dimensional we are precisely in the situation of Lemma 3.4.8 and there exists a morphism $\tilde{g}_q\colon H_q \to {}_nN_q$ of inverse systems such that $h_q = \tilde{\pi}_q\tilde{g}_q$. Then we define

$$\tilde{g} = \varprojlim \tilde{g}_q\colon H \to M(B_{>-(q-1)}xC) = {}_nN_\infty$$

and $g\colon H \to N$ as the composition

$$H \xrightarrow{\tilde{g}} {}_nN_\infty \hookrightarrow N$$

and this satisfies $h = \pi g$. □

Proposition 3.4.14. *Let C be a word that starts and ends in a deep, such that $C = {}_hD_h$ where D is a finite word. Then*

$$0 \longrightarrow M(D) \xrightarrow{(\iota,\iota)} M({}_hD) \oplus M(D_h) \xrightarrow{(\iota,-\iota)} M(C) \longrightarrow 0$$

where ι is the natural inclusion, is an almost split sequence.

Proof. We know that $M(D) \in \mathrm{Art}_I(\Lambda)$, $M(C) \in \mathrm{Noeth}_P(\Lambda)$ are indecomposable. Hence, it is enough show that for any morphism $h\colon H \to M(C)$ in $\mathrm{Noeth}_P(\Lambda)$ which is not a splittable epimorphism, there is a morphism $g\colon H \to M({}_hD) \oplus M(D_h)$ such that $fg = h$ where $f = (\iota, -\iota)$. From now on set, $M^1 = M({}_hD)$, $M^2 = M(D_h)$ and $N = M(C)$.

Now we consider all $p \gg 0$ such that H_p, N_p, M^1_p, M^2_p and $M(D)_p$ are given by the p-truncations of the corresponding words. Since D is a finite word we have $M(D) = M(D)_p$. Again we can assume that $H \in \mathrm{Noeth}(\Lambda)$ is a finitely generated string or band module and by Lemma 3.4.7 that $h = \varprojlim h_p$ where $h_p\colon H_p \to N_p$ is not a split epimorphism. Furthermore, we know that the sequence

$$0 \longrightarrow M(D) \longrightarrow M^1_p \oplus M^2_p \xrightarrow{f_p} N_p \longrightarrow 0$$

where $f_p = (\iota_p, -\iota_p)$, is an almost split sequence in $\mathrm{mod}(A_p)$ for all $p \gg 0$. Therefore, there exist morphisms $s_p\colon H_p \to M^1_p \oplus M^2_p$ such that $f_p s_p = h_p$.

Assume first that H is finite-dimensional. If in addition N and thus also E are finite-dimensional, then we have $H = H_p$, $N = N_p$, $M^1 = M^1_p$ and $M^2 = M^1_p$ for some $p \gg 0$. Hence we have $h = h_p$ and $f = f_p$ and thus we can choose $g = s_p$. If N is infinite dimensional, then one or both of its hooks are infinite. Assume for simplicity that both hooks are infinite. If we consider $M(D)$ as a submodule of N, then it follows that any finite-dimensional submodule of $M(C)$ is also a submodule of $M(D)$. Since H is finite-dimensional the image of h is a finite-dimensional submodule of N. This implies, that the image of s_p can be considered as a submodule in $M(D) \oplus M(D)$. We can choose $g = \iota s_p$, where $\iota\colon \mathrm{Im}(s_p) \to M^1 \oplus M^2$ is the canonical inclusion.

From now on we can assume that H is a possibly infinite dimensional but finitely generated string module. We can write $s_p = (s^1_p, s^2_p)$ with $s^i_p\colon H_p \to M^i_p$ for $i = 1, 2$.

If M^1 is finite-dimensional, we have $M^1 = M^1_{p_1}$ for some $p_1 \gg 0$. In that case define $g^1\colon H \to M^1$ as the composition

$$H \to H_{p_1} \xrightarrow{s_{p_1}} M^1.$$

Now assume that M^1 is not finite-dimensional. Thus

$$M^1 = M({}_hD) \cong M(({}_hD)^{-1}) = M(D^{-1}xB),$$

where x is an arrow and B an inverse $\mathbb{N}$-word. Let $\tilde{N} = M(B)$ and $\pi\colon N \to \tilde{N}$ the natural projection. Then π can be considered as the projection onto the hook of M^1. Let $\tilde{f} = \pi\iota$ and $\tilde{h} = \pi h$. Then for all $p \gg 0$ we have

$$\tilde{h}_p = \pi_p h_p = \pi_p f_p s_p^1 = \tilde{f}_p s_p^1.$$

By Lemma 3.4.8 for some fixed $p_1 \gg 0$ there exist morphisms $g_q^1\colon H_q \to M_q^1$ for all $q \geq p_1$, such that $g_{p_1}^1 = s_{p_1}^1$ and

$$\pi_q h_q = \tilde{h}_q = \tilde{f}_q g_q^1 = \pi_q f_q g_q^1$$

for all $q \geq p_1$ and such that (g_q^1) is a morphism of inverse systems. Then for $g^1 = \varprojlim g_q\colon H \to M^1$ we have $\tilde{h} = \tilde{f} g^1$.

Similarly we can define $g^2\colon H \to M^2$. Then for $g = (g^1, g^2)\colon H \to M^1 \oplus M^2$ there is some $p \gg 0$ such that $g_p = s_p$. Hence for this p we have $f_p g_p = h_p$. We claim that $f_q g_q = h_q$ for all $q \geq p$. Let b_i^H be a basis symbol in H_q. Then we can write $h_q(b_i^H) = n_1 + x + n_2$, where n_i is a linear combination of the basis symbols in the hook corresponding to M^i and x is a linear combination of the remaining basis symbols that do not occur in the hooks. Similarly we write $f_q g_q(b_i^H) = \tilde{n}_1 + \tilde{x} + \tilde{n}_2$. Further denote by π_q^i the natural projections to the hooks for $i = 1, 2$. Then by the choice of g we have

$$n_i = \pi_q^i(n_1 + r + n_2) = \pi_q^i h_q(b_i^H) = \pi_q^i f_q g_q(b_i^H) = \pi_q^i(\tilde{n}_1 + \tilde{r} + \tilde{n}_2) = \tilde{n}_i.$$

Now let $\pi_{qp}\colon N_q \to N_p$ be the projection. Then we have $\pi_{pq}(n_1 + x + n_2) = x + r$ and $\pi_{pq}(n_1 + \tilde{x} + n_2) = \tilde{x} + r$, where r is a linear combination in the basis symbols on the hooks of N_p. Then we have

$$x + r = \pi_{pq} h_q(b_i^H) = h_p \pi_{pq}(b_i^H) = f_p g_p \pi_{pq}(b_i^H) = f_p \pi_{pq} g_q(b_i^H) = \pi_{pq} f_q g_q(b_i^H) = \tilde{x} + r$$

and thus in total $f_q g_q(b_i^H) = h_q(b_i^H)$. Since this holds for arbitrary basis vectors of H_q we have $f_q g_q = h_q$ for all $q \geq p$, and hence $fg = h$.

□

Proposition 3.4.15. *If C does not start but ends in a deep, then $C = D_h$ where D is a finite word, ${}_cD$ and ${}_cC$ are defined and*

$$0 \longrightarrow M({}_cD) \xrightarrow{(\iota,\pi)} M({}_cC) \oplus M(D) \xrightarrow{(\pi,-\iota)} M(C) \longrightarrow 0$$

is an almost split sequence.

Proof. The proof is similar to the proof of the last proposition. We have $C = D_h = DxB$, where x is an arrow and B is a finite or $\mathbb{N}$-word which is inverse and ${}_cC = EyDxB$, where E is a finite or $-\mathbb{N}$-word which is inverse and y is an arrow. W set $U = M({}_cD)$, $M^1 = M({}_cC)$, an $M^2 = M(D)$ and $N = M(C)$. Again let $h\colon H \to M(C)$ be a morphism in $\mathrm{Noeth}(\Lambda)$ which is not a split epimorphism. We can assume, that H is a finitely generated string module or band module. We can assume that for all $p \gg 0$ the modules H_p, N_p and $M_p^2 = M^2$ are given by the p-truncations of the respective words. Further for each $q > 1$ let ${}_qM_p^1$ and ${}_qU_p$ be as in Lemma 3.4.11. Then we know that

$$0 \longrightarrow {}_pU_p \xrightarrow{(\iota_p,\pi_p)} {}_pM_p^1 \oplus M^2 \xrightarrow{(\pi_p,-\iota_p)} N_p \longrightarrow 0$$

is an Auslander-Reiten sequence in $\mathrm{mod}(A_p)$ for all $p \gg 0$. Hence there exist morphisms $s_p\colon H_p \to {}_pM_p^1 \oplus M^2$ such that $h_p = (\pi_p, -\iota_p)s_p$.

Now if E is in fact a finite word, the poof is exactly the same as in the last proposition. If E is an $-\mathbb{N}$-word, then by Lemma 3.4.11 we can assume that there exists some $n \gg 0$, such that for all $p \geq n$ we have

$$\operatorname{Im}(s_p) \subseteq {}_nM_p^1 \oplus M^2$$

where we consider ${}_nM_p^1$ as a submodule of ${}_pM_p^1$. In this way, we can again restrict to the case, where E is a finite word. □

The proof that the remaining canonical exact sequences are Auslander-Reiten sequences is similar.

Corollary 3.4.16. *Let $M \in \operatorname{mod}(\Lambda)$ be a finitely generated string module. Then we can write $M = \varprojlim M_q$ where M_q is a finite-dimensional string module in $\operatorname{mod}(A_q)$ for all $q \gg 0$. The modules $\tau_{A_q}(M_q)$ form a direct system of finite-dimensional string modules and we have*

$$\tau_\Lambda(M) \cong \varinjlim \tau_{A_q}(M_q).$$

3.4.3 Auslander-Reiten sequences containing band modules

Let $M = M(C,V) \in \operatorname{mod}(\Lambda)$ be a band module of dimension n. Then M can be considered as a module in $\operatorname{mod}(A_p)$ for all $p > n$. Then there exists some $N \cong M(C,W) \in \operatorname{mod}(A_{n+1})$ such that

$$0 \to M \to N \xrightarrow{f} M \to 0$$

is an Auslander-Reiten sequence. Here we have $W \in \operatorname{mod}(K[T,T^{-1}])$ such that

$$0 \to V \to W \to V \to 0$$

is an Auslander-Reiten sequence in $\operatorname{mod}(K[T,T^{-1}])$.

Proposition 3.4.17. *The short exact sequence*

$$0 \to M \to N \xrightarrow{f} M \to 0$$

is an Auslander-Reiten sequence in $\operatorname{mod}(\Lambda)$.

Proof. Again we use Proposition 3.4.1. So let $H \in \operatorname{mod}(\Lambda)$ be an indecomposable module and $h\colon H \to M$ a homomorphism which is not a split epimorphism. If H is finite-dimensional, we can consider H as a module in $\operatorname{mod}(A_p)$ for some $p \gg 0$. Then, since

$$0 \to M \to N \xrightarrow{f} M \to 0$$

is an Auslander-Reiten sequence in $\operatorname{mod}(A_p)$, there exists a morphsim $g\colon H \to N$ of A_p-modules such that $fg = h$. Since g is also a morphism of Λ-module we are done in that case. If H is infinite-dimensional, then since M is finite-dimensional there exists some $p \gg 0$ such that there exists a morphism $h_p\colon H_p \to M$ and such that h is equal to the composition

$$H \xrightarrow{\pi} H_p \xrightarrow{h_p} H_p \to M.$$

Then there exists some morphism $s_p\colon H_p \to N$, such that $fs_p = h_p$. Then for $g = s_p\pi\colon H \to N$ we have $fg = h$. □

3.5 On τ-tilting theory

In this section we will prove the Mutation Theorem for finitely generated modules over completed string algebras. The crucial observation needed in the proof is the following: if $M(C)$ is a τ-rigid finitely generated string module, the word C cannot have very long direct or inverse subwords, except for possibly in the hooks.

Let

$$\Lambda = \widehat{KQ}/\overline{(\rho)}$$

be a completed string algebra and let $n = |Q_0|$ the number of vertices in Q. Then Q has at most $2n$ arrows and thus if there exists a primitive cycle it is of length at most $2n$. If B is a finite direct word with head v and tail u of length at least $2n+1$ it is of the form

$$B = P^k \tilde{B} = \tilde{B} R^k$$

for some non-trivial words $\tilde{B}, P$ and R and $k > 0$ and we assume that $\tilde{B}$ is chosen minimal with respect to the length. Here P has head and tail v and R has head and tail u and $\tilde{B}$ has head v and tail u. Then we see that there is a standard homomorphism $\theta_{\tilde{B}} \colon M(B) \to M(B)$ corresponding to the first occurrence of $\tilde{B}$ as a predecessor closed substring in B and to the last occurrence of $\tilde{B}$ as a successor closed substring.

If B is a direct $\mathbb{N}$-word, we have $B = P^\infty$ where P is a primitive cycle. In both cases, we see that there are non-trivial homomorphisms from $M(B)$ to $M(B)$ which are not the identity.

Let C be an I-word such that $M(C)$ is a finitely generated non-projective Λ-module. We denote by $\tau_\Lambda(C)$ the word such that $\tau_\Lambda(M(C)) = M(\tau_\Lambda(C))$. Furthermore, we define

$$C \cap \tau_\Lambda(C)$$

the intersection of C *and* $\tau_\Lambda(C)$ to be the finite word D, such that

$$C = \begin{cases} {}_hD_h & \text{if } C \text{ starts and ends in a deep} \\ D_h & \text{if } C \text{ does not start but ends in a deep} \\ {}_hD & \text{if } C \text{ starts but does not end in a deep} \\ D & \text{if } C \text{ neither starts nor ends in a deep.} \end{cases}$$

We say that a finite word B is a *subword* of an I-word D, if $D = D_{\leq i} B D_{>j}$ for some $i \leq j \in I$.

Lemma 3.5.1. *Let* $M(C) \in \operatorname{mod}(\Lambda)$ *be a* τ_Λ*-rigid string module. Then the finite word* $D = C \cap \tau_\Lambda(C)$ *does not contain a direct or inverse subword of length greater than* $2n$.

Proof. Suppose otherwise, that $D = D_{\leq i} B D_{>j}$ where B is a direct word of length greater than $2n$ and that the letters adjacent to B (if there are any) are inverse. Note that if $D = D_{\leq i} B$ it might happen that $C = C_{\leq i} B y C_{>j+1}$, where then $C_{>j+1}$ is an inverse word. In that case we consider the direct subword By of C. In any case since D is the intersection of C and $\tau_\Lambda(C)$ we have a non-trivial standard homomorphism $\theta_{\tilde{B}} \colon M(C) \to M(\tau_\Lambda(C))$ similarly as described above. Hence M is not τ_Λ-rigid. Dual arguments or considering C^{-1} shows that D does not contain an inverse subword of length greater than $2n$. □

Proposition 3.5.2. *Let* $M(C) \in \operatorname{mod}(\Lambda)$ *be a finitely generated* τ_Λ*-rigid string module. Then* $M_p = M(\pi_p(C))$ *is a* τ_{A_p}*-rigid string module in* $\operatorname{mod}(A_p)$ *for any* $p \geq 2n+1$.

Proof. It follows from Lemma 3.5.1 that M_p is isomorphic to the string module $M(\pi_p(C))$. In particular we have

$$\pi_p(C) \cap \tau_{A_p}(\pi_p(C)) = C \cap \tau_\Lambda C$$

and that means that C only differs from $\pi_p(C)$ by possibly having longer hooks and $\tau_\Lambda C$ differs from $\tau_{A_p}(\pi_p(C))$ by possibly having longer cohooks. If $f \colon M(\pi_p(C)) \to M(\tau_{A_p}(\pi_p(C)))$ is a non-trivial homomorphism then the composition

$$M(C) \twoheadrightarrow M(\pi_p(C)) \xrightarrow{f} M(\tau_{A_p}(\pi_p(C))) \hookrightarrow M(\tau_\Lambda C)$$

is a non-trivial homomorphism from $M(C)$ to $\tau_\Lambda M(C)$, contradicting that $M(C)$ is τ_Λ-rigid. □

Proposition 3.5.3. *Let $p \geq 2n+1$ and let $M \in \operatorname{mod}(A_p)$ be a finitely generated string module which is τ_{A_p}-rigid. Then there exists a finitely generated string module $M(C) \in \operatorname{mod}(\Lambda)$ which is τ_Λ-rigid, such that $M = M(\pi_p(C))$.*

Proof. We will inductively construct a word C as in the Lemma such that $M(\pi_q(C))$ is τ_{A_q}-rigid for all $q \geq p$. Let B be the word for A_p such that $M = M(B)$ and set $\pi_p(C) = B$.

If there exists an arrow x such that xB is a word for A_{p+1} but not for A_p (in other words B starts in a deep considered as a word for A_p), we set $B' = xB$. Otherwise set $B' = B$. Similarly if there exists an arrow y such that $B'y^{-1}$ is a word for A_{p+1} but not for A_p, we set $B'' = B'y^{-1}$. Otherwise set $B'' = B'$. Note that by Lemma 3.5.1 applied to A_p we know that the finite word $B \cap \tau_{A_p} B$ does not contain a direct or inverse subword of length greater than $2n$ and it follows that

$$B \cap \tau_{A_p} B = B'' \cap \tau_{A_{p+1}}(B'').$$

We define $\pi_{p+1}(C) = B''$ and we claim that $N = M(\pi_{p+1}(C))$ is $\tau_{A_{p+1}}$-rigid. There is natural projection and a natural inclusion

$$N \xrightarrow{\pi} M \qquad \text{and} \qquad \tau_{A_p}(M) \xrightarrow{\iota} \tau_{A_{p+1}}(N)$$

which might be identities. Obviously any homomorphism from $N \to \tau_{A_p}(M)$ factors through $N \xrightarrow{\pi} M$. Hence if $f\colon N \to \tau_{A_{p+1}}(N)$ is a non-zero homomorphism with image in $\tau_{A_p}(M)$, there exists a non-zero homomorphism $M \to \tau_{A_p}(M)$. This is a contradiction to M being τ_{A_p}-rigid.

If $\theta\colon N \to \tau_{A_{p+1}}(N)$ is a standard homomorphism, which does not have image in $\tau_{A_p}(M)$, it follows that $\tau_{A_{p+1}}(B'')$ has a cohook of length $p \geq 2n+1$, and that the basis vector b at the top of this cohook is in the image of θ. Then θ can only correspond to direct or inverse equivalent subwords of B'' and $\tau_{A_{p+1}}(B'')$ of length $p \geq 2n+1$. Then similarly as before, we see that there also exists a standard homomorphism $\theta_{\tilde{B}}\colon N \to \tau_{A_{p+1}}(N)$ which does not hit the basis vector b and thus has image in $\tau_{A_p}(M)$. We have already seen that this is a contradiction. Hence N is $\tau_{A_{p+1}}$-rigid. □

Corollary 3.5.4. *Let $p \geq 2n+1$. Then there is a bijection between indecomposable τ_Λ-rigid modules in $\operatorname{mod}(\Lambda)$ and indecomposable τ_{A_p}-rigid modules in $\operatorname{mod}(A_p)$.*

Proof. This is a direct consequence of Proposition 3.5.2 and Proposition 3.5.3. □

Corollary 3.5.5. *Let $M = M(C) \in \operatorname{mod}(\Lambda)$ be a finitely generated string module. Then M is τ_Λ-rigid if and only if $M(\pi_p(C))$ is τ_{A_p}-rigid for every $p \geq 2n+1$.*

Proposition 3.5.6. *Let $M(C), M(D) \in \operatorname{mod}(\Lambda)$ be τ_Λ-rigid string modules. Then for any $p \geq 2n+1$ we have*

$$\dim \operatorname{Hom}_\Lambda(M(C), \tau_\Lambda M(D)) = 0$$

if and only if

$$\dim \operatorname{Hom}_{A_p}(M(\pi_p(C)), \tau_{A_p} M(\pi_p(D)) = 0.$$

Proof. The proof is very similar to the proofs of Proposition 3.5.2 and Proposition 3.5.3. □

Corollary 3.5.7. *Let $p \geq 2n+1$. Then there is a bijection between basic τ_Λ-tilting modules in* $\operatorname{mod}(\Lambda)$ *and basic τ_{A_p}-tilting modules in* $\operatorname{mod}(A_p)$.

Proof. This follows from Corollary 3.5.4 and Proposition 3.5.6. □

Corollary 3.5.8. *Any τ_Λ-rigid module in* $\operatorname{mod}(\Lambda)$ *is a direct summand of some τ_Λ-tilting Λ-module.*

Let u be a vertex in Q and denote by P_u the indecomposable projective module isomorphic to Ae_u. Then P_u is an indecomposable finitely generated Λ-module and hence isomorphic to a string module. In fact it is not hard to see, that $P_u = M(CD)$ where C is a direct word with tail u and

D is an inverse word with head u, such that CD both starts and ends in a deep. Note that for $M \in \operatorname{mod}(\Lambda)$ with $\dim(M_u)$ finite we have

$$\dim \operatorname{Hom}_\Lambda(P_u, M) = \dim(M_u)$$

and if M_u is infinite dimensional so is $\operatorname{Hom}_\Lambda(P_u, M)$.

As for finite-dimensional algebras we say that a pair (M, P) with $M \in \operatorname{mod}(\Lambda)$ and P a finitely generated projective Λ-module is *τ_Λ-rigid* if M is τ_Λ-rigid and $\operatorname{Hom}_\Lambda(P, M) = 0$. If in addition we have that $|M| + |P| = |\Lambda|$ (respectively $|M| + |P| = |\Lambda| - 1$) we say that (M, P) is a *support τ_Λ-tilting* (respectively *almost support τ_Λ-tilting*) pair.

Theorem 3.5.9. *Let Λ be a completed string algebra. Then any basic almost complete support τ-tilting pair for Λ is a direct summand of exactly two basic support τ-tilting pairs.*

Proof. Let (U, Q) be an almost complete support τ_Λ-tilting pair for Λ. Then for $p \geq 2n+1$ there are $(U_p, Q_p) \in \operatorname{mod}(A_p)$ almost complete support τ_{A_p}-tilting pairs for A_p. By [AIR, Theorem 2.18] we know that $(U_p, Q_p) \in \operatorname{mod}(A_p)$ is a direct summand of exactly two basic support τ_{A_p}-tilting pairs (T_p, P_p) and (T'_p, P'_p) for A_p. If the added summand M_p is not projective, then we know by Proposition 3.5.3 that there exists a corresponding τ_Λ-rigid summand $M \in \operatorname{mod}(\Lambda)$. By Proposition 3.5.6 we know that then $U \oplus M$ is also τ_Λ-rigid. Furthermore, it is not hard to see, that $\operatorname{Hom}_{A_p}(Q_p, M_p) = 0$ implies that $\operatorname{Hom}_\Lambda(Q, M) = 0$. If the added summand is projective, we can add the corresponding indecomposable projective Λ-module. In this way, we see that there are at least two ways to complete (U, Q) to a basic support τ_Λ-tilting module.

Suppose there exists a third basic support τ_Λ-tilting pair, which has (U, Q) as a direct summand. Then there would also exist a third basic support τ_{A_p}-tilting pair, which had (U_p, Q_p) as a direct summand. This contradicts [AIR, Theorem 2.18]. □

4 On perpendicular categories

In this chapter we are going to recall results from [GL] on perpendicular categories and see how they can be applied or adapted to module categories of 1-Iwanaga-Gorenstein algebras. Here we pay particular attention to a new class of algebras, recently introduced by Geiß, Leclerc and Schröer in [GLS1]. These algebras are defined via quivers with relations associated with symmetrizable Cartan matrices. Let $H = H(C)$ be such an algebra associated with an $n \times n$ symmetrizable Cartan matrix C. We find that for an indecomposable partial tilting module $M \in \operatorname{rep}(H)$, such that $\operatorname{End}_H(M)$ is a skew-field, the orthogonal category $M^\perp$ is equivalent to the module category of a 1-Iwanaga-Gorenstein-algebra. In case that M is preinjective we see even more, namely that $M^\perp$ is equivalent to $\operatorname{rep}(H')$, where $H' = H(C')$ and C' is a symmetrizable Cartan matrix of size $n-1$.

4.1 Perpendicular categories for finite-dimensional algebras after Geigle and Lenzing

In this section we will recall results from [GL] and specialise them to the case of finite-dimensional algebras. Most of the proofs will be omitted, except where we need to go into more details in order to see how the proofs can be adapted to 1-Iwanaga-Gorenstein algebras. Note that our notation may change from the literature, since we are considering left-modules, whereas in the above reference the authors consider right-modules.

4.1.1 Definition and existence of left adjoints

Abelian categories and Morita theory Recall that *an abelian category* is an additive category $\mathcal{A}$ such that each morphism $f\colon M \to N$ admits a *kernel* $\operatorname{Ker}(f)$ and a *cokernel* $\operatorname{Cok}(f)$, and such that the induced morphism f' from the *coimage* $\operatorname{Coim}(f)$ to the *image* $\operatorname{Im}(f)$ is an isomorphism. It is well-known that for any ring R the category $\operatorname{Mod}(R)$ of all left R-modules is abelian. If R is left-Noetherian, as in the case of finite-dimensional algebras, then the category $\operatorname{mod}(R)$ of all finitely generated left R-modules is also abelian.

Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ is *cocomplete* if for each set-indexed family $(M_i)_{i\in I}$ of objects in $\mathcal{A}$ there exists the direct sum or *coproduct* $\oplus_{i\in I} M_i$ in $\mathcal{A}$. An object $P \in \mathcal{A}$ is called a *generator* if for each $M \in \mathcal{A}$, there is an epimorphism $\oplus_I P \to M$ for some set I. If in addition the functor $\operatorname{Hom}_{\mathcal{A}}(P,-)\colon \mathcal{A} \to \operatorname{Mod}(\mathbb{Z})$ is exact, the generator P is called *projective* and if the functor commutes with arbitrary set-indexed coproducts it is called *compact*. In this case we also refer to P as a *progenerator* or *projective generator*.

By definition two rings R and S are *Morita equivalent* if there is an equivalence $\operatorname{Mod}(R) \to \operatorname{Mod}(S)$. This notion of equivalence was first introduced by Morita [Mo]. The following result and its consequences for module categories are the corner stone of *Morita theory*.

Theorem 4.1.1 ([Fr], [Ga])**.** *Let P be an object in an abelian category $\mathcal{A}$ and let $R = \operatorname{End}_{\mathcal{A}}(P)^{\mathrm{op}}$. Then $\operatorname{Hom}_{\mathcal{A}}(P,-)\colon \mathcal{A} \to \operatorname{Mod}(R)$ is an equivalence of categories with inverse functor $P \otimes_R -$ if and only if $\mathcal{A}$ is cocomplete and P is a compact projective generator in $\mathcal{A}$.*

Characterization of perpendicular categories which are abelian categories From now on let A be a finite-dimensional algebra and $M \in \operatorname{mod}(A)$. The *right perpendicular category* $M^\perp$ is the full subcategory of $\operatorname{mod}(A)$ whose objects are all $X \in \operatorname{mod}(A)$ such that

$$\operatorname{Hom}_A(M,X) = 0 \quad \text{and} \quad \operatorname{Ext}^1_A(M,X) = 0$$

and the *left perpendicular category* ${}^{\perp}M$ is the full subcategory of $\mathrm{mod}(A)$ whose objects are all $X \in \mathrm{mod}(A)$ such that

$$\mathrm{Hom}_A(X, M) = 0 \quad \text{and} \quad \mathrm{Ext}^1_A(X, M) = 0.$$

We will concentrate on right perpendicular categories, but the dual statements hold for left perpendicular categories. The next lemma is [GL, Proposition 1.1.].

Lemma 4.1.2. *Let* $M \in \mathrm{mod}(A)$ *with* $\mathrm{proj.dim}(M) \leq 1$. *Then* $M^{\perp}$ *is an abelian category and the inclusion* $M^{\perp} \to \mathrm{mod}(A)$ *is exact.*

It is a result by Mitchell [Mi], often referred to as *full embedding theorem*, that any abelian category can be embedded into a module category over a suitable ring. However, we want to know, when the perpendicular category $M^{\perp}$, where M is a finite-dimensional module over a finite-dimensional algebra, is again equivalent to $\mathrm{mod}(A')$, where A' is a finite-dimensional algebra. In the case of finite-dimensional algebras we have the following consequence of Theorem 4.1.1. For more details on Morita theory for finite-dimensional algebras we refer to the book [SY].

Corollary 4.1.3. *Let* A *be a finite-dimensional algebra and* $\mathcal{A}$ *an abelian subcategory of* $\mathrm{mod}(A)$. *If* $T \in \mathrm{mod}(A)$ *is a projective generator of* $\mathcal{A}$, *there is an equivalence of categories*

$$\mathrm{Hom}_A(T, -)\colon \mathcal{A} \to \mathrm{mod}(\mathrm{End}_A(T)^{\mathrm{op}})$$

with inverse functor $T \otimes_A -$.

Example 4.1.4. We would like to give an example that the condition $\mathrm{proj.dim}(M) \leq 1$ is necessary for $M^{\perp}$ to be abelian. Let $A = KQ/(\rho)$ be the algebra given by the quiver

$$1 \xrightarrow{b} 2 \xrightarrow{a} 3$$

and relation $\rho = ab$. Then the simple module at vertex 1, denoted by $\mathcal{S}_1$ has projective resolution

$$0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0$$

and thus $\mathrm{proj.dim}(\mathcal{S}_1) = 2$. The modules $P_1 = I_2 = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $P_2 = I_3 = \begin{smallmatrix}2\\3\end{smallmatrix}$ are in $\mathcal{S}_1^{\perp}$, as there are no non-zero homomorphisms from $\mathcal{S}_1$ to either of them, and they are both injective. Furthermore, we have $\mathrm{Ext}^1_A(\mathcal{S}_1, \mathcal{S}_2) \neq 0$, since there is the non-splitting short exact sequence

$$0 \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0.$$

Another way to see that $\dim \mathrm{Ext}^1_A(\mathcal{S}_1, \mathcal{S}_2) = 1$ is that there is one arrow from 1 to 2 in Q and the number of arrows between two vertices i and j in the quiver of a bounded path algebra equals the dimension of $\mathrm{Ext}^1_A(\mathcal{S}_i, \mathcal{S}_j)$. However, there is a homomorphism

$$f\colon \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$$

in $\mathcal{S}_1^{\perp}$ with $\mathrm{Im}(f) = \mathcal{S}_2 \notin \mathcal{S}_1^{\perp}$.

Remark 4.1.5. At this point we would like to reference an article of Jasso [J], where he defines and studies an analogue of the perpendicular category. If U is a τ_A-rigid A-module then the *τ-perpendicular category associated to* U denoted by $\mathcal{U}$ is the full subcategory of $\mathrm{mod}(A)$ whose objects are all $X \in \mathrm{mod}(A)$ such that

$$\mathrm{Hom}_A(U, X) = 0 \quad \text{and} \quad \mathrm{Hom}_A(X, \tau_A U) = 0.$$

In case U is a partial tilting module, that is if in addition $\mathrm{proj.dim}(U) \leq 1$, then we have

$$\mathcal{U} = U^{\perp}.$$

Let T_U be the Bongartz completion (see [J, Proposition 2.13.]) of U, and set $C = \mathrm{End}_A(T_U)^{\mathrm{op}}/\langle e_U \rangle$, where e_U is the idempotent corresponding to the $\mathrm{End}_A(T_U)^{\mathrm{op}}$-projective module $\mathrm{Hom}_A(T_U, U)$. Then Jasso's main result is that there is an equivalence of categories

$$\mathcal{U} \simeq \mathrm{mod}(C).$$

Note that in the above example, the simple module $\mathcal{S}_1$ is τ_A-rigid and thus it follows that its τ-perpendicular category is abelian and even equivalent to the module category of a finite-dimensional algebra.

Adjoint functors In the following we will establish conditions for an A-module, which ensure that $M^{\perp}$ is not only an abelian category, but even equivalent to a module category over a finite-dimensional algebra. In order to do so, we will construct a left adjoint functor of the embedding $M^{\perp} \to \mathrm{mod}(A)$. Therefore, we would like to recall the following definition. Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

be a pair of additive covariant functors between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then we say, that F is *left adjoint* to G and G is *right adjoint* to F if for all $X \in \mathcal{A}$ and all $Y \in \mathcal{B}$ there exists an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, G(Y))$$

which is functorial in X and Y. Note that F is left adjoint to G if and only if G is right adjoint to F.

The following theorem (see for example [M, IV Satz 2]) gives a very useful equivalent definition for a given functor to be a right adjoint of an adjoint pair of functors.

Theorem 4.1.6. *A functor $G\colon \mathcal{B} \to \mathcal{A}$ is a right adjoint functor if for each object $X \in \mathcal{A}$ there exists an (initial) object $F_0(X) \in \mathcal{B}$ and a universal morphism $\eta_X\colon X \to GF_0(X)$ from X to G. Then the left adjoint functor F is given by F_0 on objects, and on morphisms $f\colon X \to X'$ it is defined by the equation*

$$GF(f) \circ \eta_X = \eta_{X'} \circ f.$$

Generalisation of Bongartz's short exact sequence Let $M \in \mathrm{mod}(A)$ be a rigid module of projective dimension at most 1, or in other words let M be a partial tilting module. Recall that in [B] Bongartz proved that a partial tilting module can always be completed to a tilting module by constructing a short exact sequence

$$0 \to A \to B \to M^k \to 0$$

where $k = \dim \mathrm{Ext}^1_A(M, A)$. This sequence is referred to as *Bongartz's short exact sequence.* It is not difficult to see that $\mathrm{proj.dim}(B) \leq 1$, the number of pairwise non-isomorphic direct summands of $M \oplus B$ equals the number of simple A-modules and $\mathrm{Ext}^1_A(M \oplus B, M \oplus B) = 0$. Hence, $M \oplus B$ is a tilting module, called the *Bongartz's completion*, and B is called the *Bongartz's complement* of M. The following proposition compare [GL, Lemma 3.1.] can be considered as a generalisation of Bongartz's exact sequence.

Proposition 4.1.7. *Let $M \in \mathrm{mod}(A)$ be a rigid module. Then for all $N \in \mathrm{mod}(A)$ there exists some $k \in \mathbb{N}$ and an exact sequence*

$$0 \to N \to N' \to M^k \to 0$$

such that $\mathrm{Ext}^1_A(M, N') = 0$. If in addition $\mathrm{End}_A(M)$ is a skew-field, then

$$\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(M, N').$$

Construction of a left adjoint functor of $M^\perp \to \operatorname{mod}(A)$ Let $M \in \operatorname{mod}(A)$ be a partial tilting module. In the following we will recall the construction of a functor $L_M\colon \operatorname{mod}(A) \to M^\perp$ which is left adjoint to the embedding functor $i_M\colon M^\perp \to \operatorname{mod}(A)$ as was given in [GL, Proposition 3.2.].

Let $N \in \operatorname{mod}(A)$ be arbitrary. By Proposition 4.1.7 there is a short exact sequence

$$0 \to N \to N' \to M^k \to 0$$

such that $\operatorname{Ext}^1_A(M, N') = 0$. Since $\operatorname{Hom}_A(M, N')$ is finite-dimensional, it is in particular finitely generated over $\operatorname{End}_A(M)$. Hence we can choose a finite generating set $\{f_1, \ldots, f_m\}$ of $\operatorname{Hom}_A(M, N')$ as an $\operatorname{End}_A(M)$-module. Define U as the image of the map $(f_1, \ldots, f_m)\colon M^m \to N'$. First of all notice, that $\operatorname{Ext}^1_A(M, U) = 0$. Indeed, there is a short exact sequence

$$0 \to \operatorname{Ker}(f_1, \ldots, f_m) \to M^m \to U \to 0$$

induced by $(f_1, \ldots, f_m)\colon M^m \to N'$. Applying the left-exact covariant functor $\operatorname{Hom}_A(M, -)$ yields the exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_A(M, \operatorname{Ker}(f_1, \ldots, f_m)) \to \operatorname{Hom}_A(M, M^m) \to \operatorname{Hom}_A(M, U) \\ &\to \operatorname{Ext}^1_A(M, \operatorname{Ker}(f_1, \ldots, f_m)) \to \operatorname{Ext}^1_A(M, M^m) \to \operatorname{Ext}^1_A(M, U) \to 0. \end{aligned}$$

Since M is rigid, it follows that $\operatorname{Ext}^1_A(M, U) = 0$.

We claim that the quotient $L_0(N) = N'/U$ is in $M^\perp$. We consider the short exact sequence

$$0 \to U \to N' \to L_0(N) \to 0$$

and apply again the functor $\operatorname{Hom}_A(M, -)$ to obtain the long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_A(M, U) \xrightarrow{\varphi} \operatorname{Hom}_A(M, N') \to \operatorname{Hom}_A(M, L_0(N)) \\ &\to \operatorname{Ext}^1_A(M, U) \to \operatorname{Ext}^1_A(M, N') \to \operatorname{Ext}^1_A(M, L_0(N)) \to 0. \end{aligned}$$

By the choice of U, the morphism φ is an epimorphism and thus $\operatorname{Hom}_A(M, L_0(N)) = 0$. Furthermore, since $\operatorname{Ext}^1_A(M, N') = 0$, it follows that also $\operatorname{Ext}^1_A(M, L_0(N)) = 0$. We now define $L_M(N) = L_0(N) \in M^\perp$ on objects and the composition $N \to N' \to L_0(N)$ is the universal morphism $\eta_N\colon N \to L_0(N)$.

Proposition 4.1.8. *Let $M \in \operatorname{mod}(A)$ be a partial tilting module. Then $M^\perp$ is an abelian subcategory of $\operatorname{mod}(A)$ and there exists a functor $L_M\colon \operatorname{mod}(A) \to M^\perp$ which is left adjoint to the inclusion functor $i_M\colon M^\perp \to \operatorname{mod}(A)$.*

Remark 4.1.9. Note the following: If in the above construction $\operatorname{Hom}_A(M, N') = 0$, we have $L_M(N) = N'$ and hence there is a short exact sequence

$$0 \to N \to L_M(N) \to M^k \to 0.$$

Thus if in addition N is of projective dimension at most 1, then $L_M N$ is of projective dimension at most 1. We are particularly interested in the object that is obtained when applying the functor L_M to the regular representation of A. In that case, the short exact sequence

$$0 \to A \to A' \to M^k \to 0$$

is exactly Bongartz's short exact sequence. If $\operatorname{Hom}_A(M, A') = 0$, then we know that $L_M A = A' \in M^\perp$ is of projective dimension at most 1. This holds for example if $\operatorname{Hom}_A(M, A) = 0$ and $\operatorname{End}_A(M)$ is a skew-field.

Characterization of perpendicular categories which are module categories The next theorem is a specialization of [GL, Proposition 3.8. and Corollary 3.9.]. Note that since A is a finite-dimensional algebra, it is in particular a Noetherian ring, and therefore we are in the situation of [GL, Corollary 3.9.] which considers finitely generated modules.

Recall that a morphism of rings $f\colon R \to S$ is an *epimorphism of rings* if given any morphisms $g_1, g_2\colon S \to Q$ such that $g_1 f = g_2 f$, we have $g_1 = g_2$. Note, that epimorphisms of rings need not be surjective in general. For example the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism of rings.

Theorem 4.1.10. *Let $M \in \mathrm{mod}(A)$ be a partial tilting module, further let $i\colon M^\perp \to \mathrm{mod}(A)$ be the inclusion and $L = L_M\colon \mathrm{mod}(A) \to M^\perp$ its left adjoint functor. Then we have the following:*

- *The module LA is a progenerator of $M^\perp$, and hence we have an equivalence of categories*

$$\begin{tikzcd} M^\perp \arrow[r, shift left, "\mathrm{Hom}_A(LA,-)"] & \mathrm{mod}(A') \arrow[l, shift left, "LA\otimes_{A'}-"] \end{tikzcd}$$

  where $A' = \mathrm{End}_A(LA)^{\mathrm{op}}$ is a finite-dimensional algebra.

- *There exists an epimorphism of rings $\varphi\colon A \to A'$ such that the natural functor $\varphi_*\colon \mathrm{mod}(A') \to \mathrm{mod}(A)$ and its left adjoint $\varphi^* = - \otimes_A A'$ make the diagrams*

$$\begin{tikzcd} \mathrm{mod}(A) & \\ M^\perp \arrow[u, "i"] & \mathrm{mod}(A') \arrow[l, "LA\otimes_{A'}-"] \arrow[ul, "\varphi_*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{mod}(A) \arrow[d, "L"'] \arrow[dr, "\varphi^*"] & \\ M^\perp \arrow[r, "\mathrm{Hom}_A(LA,-)"'] & \mathrm{mod}(A') \end{tikzcd}$$

  commutative up to isomorphism. In particular there is an isomorphism

$$LA \cong A'$$

  of left A-modules.

We would like to recall some parts of the proof of the theorem. We will first show that LA is a projective generator of $M^\perp$. Since the functor L is the left adjoint of the additive functor i, it is right exact and also additive (see for example [M, IV.1. Satz 3] and [ASS, A.2. Lemma 4.]). Also, since i is full and faithful, we have $LN \cong N$ for all $N \in M^\perp$ (see for example [M, IV.3. Satz 1]). Then, since N is finitely generated as an A-module, there exists $n \in \mathbb{N}$ and an epimorphism $A^n \to N$. Applying the functor L yields an epimorphism $(LA)^n \to N$. Thus LA is a generator of $M^\perp$.

Furthermore, by adjunction we have $\mathrm{Hom}_A(LA, N) \cong \mathrm{Hom}_A(A, i(N)) = \mathrm{Hom}_A(A, N)$ for all objects $N \in M^\perp$ and therefore it follows that the functor $\mathrm{Hom}_A(LA, -)\colon M^\perp \to \mathrm{Mod}(\mathbb{Z})$ is exact, which is equivalent to LA being projective in $M^\perp$. Since LA is finitely generated, it is compact and to conclude a progenerator.

We will now recall the construction of the morphism $\varphi\colon A \to A'$. There is an isomorphism of rings

$$A \to \mathrm{Hom}_A({}_AA, {}_AA)^{\mathrm{op}}$$

given by

$$h \mapsto f_a, \qquad f_a(b) = ba.$$

Now we define $\varphi\colon A \to A'$ to be the map

$$A \cong \mathrm{Hom}_A({}_AA, {}_AA)^{\mathrm{op}} \to \mathrm{Hom}_A({}_ALA, {}_ALA)^{\mathrm{op}} = A', \qquad f \mapsto L(f)$$

induced by L. Since L is a functor, this is in fact a morphism of rings.

Next, we want to show that

$$\varphi_*(A') \cong i(LA \otimes_{A'} {}_{A'}A') = {}_ALA$$

as left A-modules. We have isomorphisms of left A-modules

$$LA \cong \mathrm{Hom}_A(A, LA) \cong \mathrm{Hom}_A(LA, LA) = A' = \varphi_*(A')$$

where the second isomorphism follows from adjunction, and the last equalities only hold, since we consider the objects as left A-modules, not as rings.

Further we have isomorphisms

$$\mathrm{Hom}_A(LA, LA)^{\mathrm{op}} \cong A' \cong A \otimes_A A' = \varphi^*(A)$$

as left A-modules.

We still need to show that φ is an epimorphism of rings. Since i is a full embedding, so is φ_*. This implies that $\varphi^*(N) \cong N$ for all $N \in \mathrm{mod}(A')$ as before by [M, IV.3. Satz 1]. In particular we have

$$A' \cong \varphi^*(A') \cong A' \otimes_A A'$$

implying that $\varphi\colon A' \to A$ is an epimorphism of rings by [S, Proposition 1.1.]. This completes the proof.

4.1.2 Homological epimorphisms

Let A and A' be finite-dimensional algebras and $\varphi\colon A \to A'$ a homomorphism. Denote by $\varphi_*\colon \mathrm{mod}(A) \to \mathrm{mod}(A')$ the functor induced by φ. For $M, N \in \mathrm{mod}(A')$, the natural map $\mathrm{Hom}_{A'}(M, N) \to \mathrm{Hom}_A(M, N)$ induces natural homomorphisms $\mathrm{Ext}^i_{A'}(M, N) \to \mathrm{Ext}^i_A(M, N)$ of groups for all $i \in \mathbb{N}$. The following is a consequence of [GL, Theorem 4.4. and Proposition 4.9.].

Proposition 4.1.11. *Let $\varphi\colon A \to A'$ be a homomorphism of finite-dimensional algebras, such that A' is of finite projective dimension as a left A-module. Then the following conditions are equivalent:*

- *The natural homomorphism $\mathrm{Hom}_A({}_AA', {}_AA') \to A'$ is an isomorphism and $\mathrm{Ext}^i_A({}_AA', {}_AA') = 0$ for all $i \geq 1$.*
- *For all left A'-modules M the natural homomorphism $\mathrm{Hom}_A({}_AA', {}_AM) \to {}_AM$ is an isomorphism and $\mathrm{Ext}^i_A({}_AA', {}_AM) = 0$ for all $i \geq 1$.*
- *For all right A'-modules M the natural homomorphism $\mathrm{Hom}_A(A'_A, M_A) \to M_A$ is an isomorphism and $\mathrm{Ext}^i_A(A'_A, M_A) = 0$ for all $i \geq 1$.*
- *For all left A'-modules M and N the natural homomorphism*

  $$\mathrm{Ext}^i_{A'}({}_{A'}M, {}_{A'}N) \to \mathrm{Ext}^i_A({}_AM, {}_AN)$$

  is an isomorphism for all $i \geq 0$.
- *For all right A'-modules M and N the natural homomorphism*

  $$\mathrm{Ext}^i_{A'}(M_{A'}, N_{A'}) \to \mathrm{Ext}^i_A(M_A, M_A)$$

  is an isomorphism for all $i \geq 0$.
- *The induced functor $D^b(\varphi_*)\colon D^b(\mathrm{mod}(A')) \to D^b(\mathrm{mod}(A))$ is a full embedding.*

If φ satisfies these equivalent conditions we call it a homological epimorphism .

The next corollary is [GL, Corollary 4.8. and 4.10.] and is easily verified using Theorem 4.1.10 and the second condition in the above proposition.

Corollary 4.1.12. *Let A be a finite-dimensional algebra and $M \in \operatorname{mod}(A)$ a partial tilting module. Let $i\colon M^\perp \to \operatorname{mod}(A)$ be the inclusion functor and $L = L_M\colon \operatorname{mod}(A) \to M^\perp$ its left adjoint.*

If $\operatorname{proj.dim}(LA) \leq 1$, *then the morphism $\varphi\colon A \to A' = \operatorname{End}_A(LA)^{\mathrm{op}}$ induced by L is a homological epimorphism.*

The next is just a special case of [GL, Theorem 4.16.].

Proposition 4.1.13. *Let A be a finite-dimensional algebra, and $M \in \operatorname{mod}(A)$ be an indecomposable partial tilting module, such that $\operatorname{End}_A(M)$ is a skew-field and $\operatorname{Hom}_A(M, A) = 0$. Then there exists a finite-dimensional algebra A' and a homological epimorphism $\varphi\colon A \to A'$ which is also injective, such that*

1. $M^\perp \simeq \operatorname{mod}(A')$;
2. $T = M \oplus \varphi_*(A')$ *is a tilting module in* $\operatorname{mod}(A)$;
3. $\operatorname{gl.dim}(A') \leq \operatorname{gl.dim}(A)$;
4. $|A'| = |A| - 1$.

4.1.3 Perpendicular categories of projective modules

In the following we recall a process called *deletion of vertices* from [R3]. Let $A = KQ/I$ be a path algebra modulo some ideal I, where I is given by relations ρ. Let i be any vertex in Q and denote by Q' the quiver obtained from Q after deleting the vertex i and any arrow incident to i. Further let I' be the ideal generated ρ', where the relation ρ' is obtained from the relation ρ, after deleting any summands which are multiples of paths through the vertex i. Then $A' = KQ'/I'$ is again a path algebra and any A'-module is an A-module with $\underline{\dim}(M)_i = 0$. Hence we have a full exact embedding

$$\operatorname{mod}(A') \to \operatorname{mod}(A)$$

where we can consider $\operatorname{mod}(A')$ as an extension closed full subcategory of $\operatorname{mod}(A)$. The following is a direct consequence of [GL, Proposition 5.1.]

Proposition 4.1.14. *Let $A = KQ/I$ be a finite-dimensional basic algebra, P_i the indecomposable projective module with $\operatorname{top}(P_i) = S_i$. Further let A' be the algebra obtained from A by deleting the vertex i. Then*

$$P_i^\perp \simeq \operatorname{mod}(A') \simeq \operatorname{mod}(A/\operatorname{Tr}(P_i))$$

where $\operatorname{Tr}(P_i)$ denotes the trace ideal of P_i in A. If moreover, $\operatorname{Tr}(P_i)$ is a projective A-module, the morphism

$$A \to A/\operatorname{Tr}(P_i)$$

is a homological epimorphism of algebras.

4.2 Hereditary algebras

4.2.1 Cartan matrices and Coxeter transformations

Symmetrizable Cartan matrices In representation theory Cartan matrices first appeared in Lie Theory. Using the combinatorics of the corresponding root system, they were used to classify simple and affine complex Lie algebras. It was discovered later that Cartan matrices can also be used in the classification of the representation type of finite-dimensional hereditary algebras. More precisely, a finite-dimensional hereditary algebra is of finite representation type, if and only if it is associated with a Cartan matrix of Dynkin type. In this case there is a one-to-one correspondence between the positive roots of the Cartan matrix and the dimension vectors of the indecomposable modules of the algebra. This was first proven by Gabriel for path algebras and then generalised to the setup of modulated graphs by Dlab and Ringel.

We call a matrix $C \in M_n(\mathbb{Z})$ a *generalized Cartan matrix* if we have

- $c_{ii} = 2$ for all $1 \leq i \leq n$,
- $c_{ij} \leq 0$ for all $i \neq j$,
- $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$.

If in addition there exists a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n) \in M_n(\mathbb{Z})$ with $d_i \geq 1$ for all i, such that DC is symmetric, then C is called a *symmetrizable generalized Cartan matrix* or from now on for short *Cartan matrix*. The matrix D is the referred to as a *symmetrizer* of C. Note that D is not unique. In fact, if D is a symmetrizer of C, then so is aD for any positive integer $a \geq 1$. We call D a *minimal symmetrizer* if $d_1 + \cdots + d_n$ is minimal.

Graphs associated with Cartan matrices Let $C \in M_n(\mathbb{Z})$ be a Cartan matrix. We define a connected *valued graph* of C, denoted $\Gamma(C)$ as follows: the set of vertices is denoted by $\{1, \ldots, n\}$ and there is a labelled edge

$$i \xrightarrow{(|c_{ji}|,|c_{ij}|)} j$$

if and only if $c_{ij} \neq 0$. If $c_{ij} = c_{ji} = 1$ we will neglect to write a label on the edge. We say that C is a *connected* Cartan matrix if $\Gamma(C)$ is connected.

Example 4.2.1. Figure 4.1 shows a list of valued graphs, called *Dynkin graphs*, corresponding to Cartan matrices of *Dynkin type*.

In order to define a quiver corresponding to a Cartan matrix, we need to define an orientation. An *orientation of a Cartan matrix* C is a set $\Omega \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that the following hold:

- $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} \neq 0$ and
- Ω is acyclic, i.e. for each sequence $((i_1, i_2), (i_2, i_3), \ldots (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$ we have $i_1 \neq i_t$.

Given a Cartan matrix $C \in M_n(\mathbb{Z})$ with orientation Ω, we associate an *oriented valued graph* $\Gamma(C, \Omega)$ with underlying valued graph $\Gamma(C)$ and if $(i, j) \in \Omega$ we replace the edge between i and j with an arrow

$$i \xrightarrow{(|c_{ji}|,|c_{ij}|)} j$$

Reflections and admissible sequences Let Q be any finite quiver with vertices $Q_0 = \{1, \ldots, n\}$. We define the *reflection at vertex i of Q* by reversing the direction of all arrows starting or ending in i and denote the new quiver by $s_i(Q)$. We say that a sequence $(i_1, \ldots, i_n)$ is an $(+)$-*admissible sequence* for Q, if

- $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$,
- the vertex i_1 is a sink in Q and
- i_k is a sink in $s_{i_{k-1}}, \ldots s_{i_1}(Q)$ for $2 \leq k \leq n$.

A $(-)$-*admissible sequence* for Q is defined similarly, where we replace sinks by sources. If $(i_1, \ldots, i_n)$ is an $(+)$-admissible sequence, then $(i_n, \ldots, i_1)$ is an $(-)$-admissible sequence. It is an easy inductive argument, that a $(+)$-admissible sequence exists if and only if Q has no oriented cycles.

We also want to define the reflection of the orientation Ω of a Cartan matrix C. For $1 \leq i \leq n$ we define the reflection of Ω as the new orientation

$$s_i(\Omega) := \{(r, s) \in \Omega \mid i \notin \{r, s\}\} \cup \{(s, r) \in \Omega^\star \mid i \in \{r, s\}\}$$

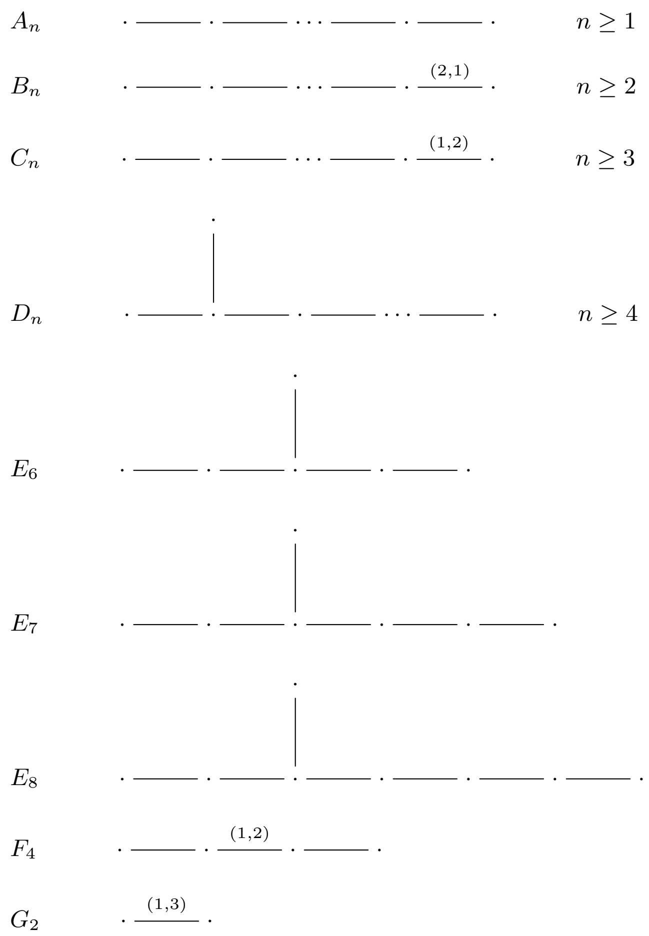


Figure 4.1: Dynkin graphs.

where $\Omega^\star := \{(j,i) \mid (i,j) \in \Omega\}$ is the *opposite orientation* of Ω. Then we have $s_i(\Gamma(C,\Omega)) = \Gamma(C, s_i(\Omega))$.

Let $C \in M_n(\mathbb{Z})$ be a Cartan matrix with orientation Ω. We say that a sequence $(i_1, \ldots, i_n)$ is an (+)-*admissible sequence* for (C,Ω), if it is a (+)-admissible sequence for $\Gamma(C,\Omega)$. Since Ω and thus $\Gamma(C,\Omega)$ are acyclic by definition, there always exists a (+)-admissible sequence for (C,Ω).

Coxeter transformations Denote by $\alpha_1, \ldots, \alpha_n$ the standard basis vectors of $\mathbb{Z}^n$. Then for all $1 \leq i, j \leq n$ we define

$$s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

and obtain thus *simple reflections* $s_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ for $1 \leq i \leq n$. Let $(i_1, \ldots, i_n)$ be a (+)-admissible sequence of (C,Ω). Then define

$$\beta_k := \begin{cases} \alpha_{i_1} & \text{if } k = 1, \\ s_{i_1} s_{i_2} \ldots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } 2 \leq k \leq n, \end{cases}$$

where s_i are the simple reflections. Similarly we define the vectors

$$\gamma_k := \begin{cases} \alpha_{i_n} & \text{if } k = n, \\ s_{i_n} s_{i_{n-1}} \ldots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } 1 \leq k \leq n-1. \end{cases}$$

For an (+)-admissible sequence $(i_1, \ldots, i_n)$ of (C, Ω) we define the *Coxeter transformations* as the linear transformations

$$c^+ := s_{i_n} s_{i_{n-1}} \cdots s_{i_1} : \mathbb{Z}^n \to \mathbb{Z}^n \qquad \text{and} \qquad c^- := s_{i_1} s_{i_2} \cdots s_{i_n} : \mathbb{Z}^n \to \mathbb{Z}^n.$$

The Coxeter transformations do not depend on the chosen admissible order, since s_i and s_j commute for any pair of neighbours i, j which are not neighbours in $\Gamma(C, \Omega)$.

4.2.2 Hereditary algebras and modulations of graphs

Definition of hereditary algebra In this section we recall definitions and classical results concerning hereditary algebras. Since these results are well-described in the literature we will not give proofs in this section, but refer to the sources accordingly.

The *(left) global dimension* of an algebra A is defined as

$$\mathrm{gl.dim}(A) = \max\{\mathrm{proj.dim}(M) \mid M \in \mathrm{mod}(A)\}.$$

An algebra A is called *(left) hereditary* if its (left) global dimension is at most 1. As we are dealing with finite-dimensional algebras, the left- and right- global dimension and thus the notions of left- and right-hereditary coincide. Furthermore, we have

$$\mathrm{gl.dim}(A) = \max\{\mathrm{inj.dim}(N) \mid N \in \mathrm{mod}(A)\}.$$

For a proof of the following different characterizations of hereditary algebras see for example [ASS, Theorem VII.1.4.]

Theorem 4.2.2. *Let A be an algebra. The following are equivalent:*

- *A is hereditary.*
- *Any left ideal of A is projective as an A-module.*
- *Every submodule of a projective A-module is projective.*

Example 4.2.3. Let Q be a finite quiver without oriented cycles and K be a field. Then the path algebra $A = KQ$ is an hereditary algebra. In fact, any basic, connected, finite-dimensional, hereditary algebra A over an algebraically closed field K is of the form $A \cong KQ$, that is, a path algebra of a finite, connected and acyclic quiver.

Hereditary algebras and Cartan matrices Let A be a finite-dimensional connected hereditary algebra, with n simple modules $\mathcal{S}_1, \ldots, \mathcal{S}_n$. To any such algebra A one can attach a Cartan matrix $C = C_A$ as follows: For two simple modules $\mathcal{S}_i, \mathcal{S}_j$ we have $\mathrm{Ext}^1_A(\mathcal{S}_i, \mathcal{S}_j) = 0$ or $\mathrm{Ext}^1_A(\mathcal{S}_j, \mathcal{S}_i) = 0$. Assume that $i \neq j$ and $\mathrm{Ext}^1_A(\mathcal{S}_j, \mathcal{S}_i) = 0$. Set $c_{ii} = 2$ and define

$$c_{ij} = -\dim_{\mathrm{End}_A(S_j)} \mathrm{Ext}^1_A(\mathcal{S}_i, \mathcal{S}_j)$$
$$c_{ji} = -\dim_{\mathrm{End}_A(S_i)^{\mathrm{op}}} \mathrm{Ext}^1_A(\mathcal{S}_i, \mathcal{S}_j)$$

and then for $d_i = \dim_K(\mathrm{End}_A(\mathcal{S}_i))$, we have $d_i c_{ij} = d_j c_{ji}$, showing that C is a symmetrizable Cartan matrix with symmetrizer $D = \mathrm{diag}(d_1, \ldots, d_n)$. Define the orientation Ω_A of C_A by $(j, i) \in \Omega_A$ if $\mathrm{Ext}^1_A(\mathcal{S}_i, \mathcal{S}_j) \neq 0$.

Conversely, using modulations of an oriented valued graph $\Gamma(C, \Omega)$ one can define an artin hereditary ring $A(C, D, \Omega)$ to any Cartan matric C with symmetrizer D and orientation Ω: let F be field and for $1 \leq i \leq n$ let F_i be F-skew-fields with $\dim_F(F_i) = d_i$. Further for $(i, j) \in \Omega$ let ${}_iF_j$ be an F_i-F_j-bimodule such that F acts centrally on ${}_iF_j$, and we have ${}_iF_j \cong F_i^{|c_{ij}|}$ as left-F_i-modules and ${}_iF_j \cong F_j^{|c_{ji}|}$ as right F_j-modules. Then $(F_i, {}_iF_j)$ is called a *modulation* or *realization* of (C, D, Ω). Such a modulation always exists for certain fields F. For example we can choose F

as the field with p elements for some prime number p. Then we can choose F_i as the field with p^{d_i} elements and ${}_iF_j$ the field with $p^{|c_{ij}|d_j}$ elements. However, we can not in general assume, that there exists a modulation over an algebraically closed field.

Let
$$A^{(0)} := \prod_{1\leq i\leq n} F_i \qquad \text{and} \qquad A^{(1)} := \bigoplus_{(i,j)\in\Omega} {}_iF_j$$

and hence $A^{(1)}$ is an $A^{(0)}$-$A^{(0)}$-bimodule. Then the tensor algebra $A := A(C, D, \Omega) := T_{A^{(0)}}(A^{(1)})$ is a finite-dimensional hereditary F-algebra. Furthermore, the category of representations of the modulation $(F_i, {}_iF_j)$ is equivalent to the category $\mathrm{mod}(A)$ of finite-dimensional A-modules. The theory of modulations was developed by Dlab and Ringel ([DR1, DR2, R2]). Note that they used the dual notions.

4.2.3 Auslander-Reiten theory for hereditary algebras

Dimension vectors and Coxeter functors In the following let $A = A(C, D, \Omega)$ be the hereditary algebra of some modulation associated with (C, D, Ω). For any vertex i of $\Gamma(C, \Omega)$ denote by
$$S_i^{\pm}\colon \mathrm{mod}(A) \to \mathrm{mod}(A(C, D, s_i\Omega))$$
the *generalized reflection functors* as defined in [DR2]. For a $(+)$-admissible sequence $(i_1, \dots, i_n)$ of (C, Ω) let
$$C^+ = S_{i_n}^+ \circ \ldots \circ S_{i_1}^+$$
be the *Coxeter functor*. Dually one defines $C^- = S_{j_n}^- \circ \ldots \circ S_{j_1}^-$ for a $(-)$-admissible sequence $(j_1, \dots, j_n)$ of (C, Ω).

Let $(i_1, \dots, i_n)$ be a $(+)$-admissible sequence of (C, Ω) and assume without loss of generality that $i_k = k$ for $1 \leq k \leq n$. Recall that we defined vectors β_k and γ_k for $1 \leq k \leq n$. It is well-known that these vectors are positive roots of the Cartan matrix C. Furthermore, we have the following lemma.

Proposition 4.2.4. *We have*
$$\underline{\dim}(P_k^A) = \beta_k \qquad \text{and} \qquad \underline{\dim}(I_k^A) = \gamma_k$$
where P_k^A and I_k^A are the indecomposable projective and injective A-modules.

Proof. By [DR2, Proposition 2.4.] the indecomposable projective module P_k^A is isomorphic to
$$S_1^- S_2^- \ldots S_{k-1}^- \mathcal{S}_k$$
where $\mathcal{S}_k$ is the simple representation at vertex k of $A(C, s_k s_{k+1} \ldots s_n \Omega)$. Thus, by [DR2, Proposition 2.1.(ii)] we have
$$\underline{\dim}(P_k^A) = \begin{cases} \alpha_k & \text{if } k = 1, \\ s_1 s_2 \ldots s_{k-1}(\alpha_k) & \text{if } 2 \leq k \leq n \end{cases}$$
and the claim follows for the indecomposable projective modules. The claim for the indecomposable injective modules follows dually. $\square$

Proposition 4.2.5. *Let M be a non-projective and N a non-injective indecomposable A-module. Then we have*
$$\underline{\dim}(\tau_A M) = c^+(\underline{\dim}(M)) \qquad \text{and} \qquad \underline{\dim}(\tau_A^{-1} N) = c^-(\underline{\dim}(N)).$$

Proof. Let M be a non-projective indecomposable A-module. Then by [DR2, Proposition 2.5.] we know, that
$$\underline{\dim} C^+ X = c^+(\underline{\dim}(X))$$

where $C^+ = S^+_{i_n} \dots S^+_{i_1}$ is the Coxeter functor for an (+)-admissible ordering $(i_1, \dots, i_n)$. It was proven by Brenner and Butler [BB] that C^+ and C^- coincide with τ_A and τ_A^{-1} respectively. □

Preprojective and preinjective modules Let A be a finite-dimensional hereditary algebra. An indecomposable module $M \in \mathrm{mod}(A)$ is called *preprojective* (respectively *preinjective*) if there exists some indecomposable projective module P (resp. indecomposable injective module I) and some positive integer $k \in \mathbb{Z}$ such that

$$P \cong \tau_A^k(M) \qquad (\text{resp. } \tau_A^k(I) \cong M).$$

We call the corresponding vertex $[M]$ in the Auslander-Reiten quiver of A also preprojective (respectively preinjective). An arbitrary, not necessarily indecomposable, A-module, is *preprojective* (respectively *preinjective*) if it is isomorphic to a sum of indecomposable preprojective (respectively preinjective) modules. Note, that M is preprojective or preinjective if and only if there exists some $k \in \mathbb{Z}$ such that

$$\tau_A^k(M) = 0.$$

An indecomposable A-module is called *regular* if it is neither preprojective nor preinjective.

The following is a direct consequence of [ARS, Lemma VIII.1.8, Corollary VIII.1.10].

Proposition 4.2.6. *Let A be a hereditary algebra and Γ a connected component of its Auslander-Reiten quiver. If Γ contains some projective vertices, than all its vertices are preprojective. In this case Γ is called a preprojective component.*

If Γ contains some injective vertices, than all its vertices are preinjective. In this case Γ is called a preinjective component.

Furthermore, we have the following: the algebra A is indecomposable as an algebra if and only if the Auslander-Reiten quiver of A has only one preprojective component.

We call a tuple $(M_1, \dots, M_n)$ of indecomposable A-modules a *path* in $\mathrm{mod}(A)$ if there exists a chain of non-zero non-isomorphisms $M_1 \to M_2 \to \cdots \to M_n$. We say that a path $(M_1, \dots, M_n)$ is a *cycle* if $M_1 \cong M_n$. The next proposition follows for example from [ASS, Corollary VIII.2.6].

Proposition 4.2.7. *Let A be a hereditary algebra and Γ a preprojective component of its Auslander-Reiten quiver. Let M, N be two non-isomorphic indecomposable modules in Γ. Then M lies on no cycle in Γ, and if $\mathrm{Hom}_A(M, N) \neq 0$, then there exists a path from M to N in Γ.*

Lemma 4.2.8. *Let A be a finite-dimensional hereditary algebra and*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

a short exact sequence of A-modules. Then if X and Z are preprojective, so is Y.

Proof. Suppose there is an indecomposable direct summand Y' of Y which is not preprojective (hence it is preinjective or regular). Then since Z is preprojective, there cannot be any non-zero homomorphisms from Y' to Z . This follows for example from [ARS, Corollary VIII.1.4.]. Thus we have $Y' \subseteq \mathrm{Ker}(g) = \mathrm{Im}(f) \cong X$. But this implies that Y' is isomorphic to a direct summand of X and is thus preprojective, which is a contradiction. Hence the claim follows. □

4.2.4 Perpendicular categories for hereditary algebras

The next result was proven independently by Schofield [Scho] for path algebras and by Geigle and Lenzing [GL] for arbitrary hereditary algebras.

Corollary 4.2.9. *Let A be an artin hereditary algebra with n simple modules. Let $M \in \mathrm{mod}(A)$ be an indecomposable rigid module. Then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{mod}(A'),$$

where A' is a hereditary algebra with $n-1$ simple modules.

The following is [St, Proposition 5.3.].

Proposition 4.2.10. *Let A be an artin hereditary algebra. If $M \cong \tau_A^{-k} P_i$ or $M \cong \tau_A^k I_i$ for some $k \geq 0$, then*

$$M^{\perp} \simeq \mathrm{mod}(A'),$$

where the valued graph of A' is obtained by removing the vertex i from A.

Let A be a hereditary artin algebra. Recall that an A-module M is called *exceptional* if it is indecomposable and $\mathrm{Ext}^1_A(M, M) = 0$. Thus, M is an indecomposable partial tilting module and $\mathrm{End}_A(M)$ is a skew-field. A pair (M, N) of exceptional A-modules is called an *exceptional pair*, if $\mathrm{Hom}_A(N, M) = 0$ and $\mathrm{Ext}^1_A(N, M) = 0$. A sequence $(M_1, \dots, M_n)$ of A-modules is called *complete exceptional sequence* if n is the number of simple A-modules and if (M_i, M_j) is an exceptional pair for any $i < j$.

The next proposition was first proven by Crawley-Boevey [CB3] for hereditary algebras over algebraically closed fields and then generalized by Ringel [R4]. It is one particularly nice application of the theory of perpendicular categories.

Proposition 4.2.11. *Let A be an artin hereditary algebra with n simple modules. Then the braid group on $n-1$ generators acts naturally on the set of complete exceptional sequences of A-modules. Furthermore, this action is transitive.*

4.3 Iwanaga-Gorenstein algebras

4.3.1 Definition and basic properties

Definition of m-Iwanaga-Gorenstein algebras Let A be a finite-dimensional algebra. Then A is called *m-Iwanaga-Gorenstein* if

$$\mathrm{inj.dim}({}_A A) \leq m \qquad \text{and} \qquad \mathrm{inj.dim}(A_A) \leq m.$$

Note that

$$\mathrm{inj.dim}({}_A A) \leq m \Leftrightarrow \mathrm{proj.dim}(D({}_A A)) \leq m$$

and

$$\mathrm{inj.dim}(A_A) \leq m \Leftrightarrow \mathrm{proj.dim}(D(A_A)) \leq m.$$

In this case we have the following generalization of "projectivity=injectivity" for modules over a selfinjective algbra, due to Iwanaga [I].

Proposition 4.3.1. *Let A be an m-Iwanaga-Gorenstein algebra. Then we have*

$$\mathrm{inj.dim}({}_A A) = \mathrm{inj.dim}(A_A)$$

and the following are equivalent for an A-module M:

- $\mathrm{proj.dim}(M) < \infty$,
- $\mathrm{proj.dim}(M) \leq m$,
- $\mathrm{inj.dim}(M) < \infty$,
- $\mathrm{inj.dim}(M) \leq m$.

Example 4.3.2. Recall that a finite-dimensional algebra A is *selfinjective* if it is injective as well as projective as an A-module. An equivalent definition is that an A-module is projective if and only if it is injective. Also, an algebra is self-injective if and only if it is 0-Iwanaga-Gorenstein.

Hereditary and selfinjective algebras are special examples of 1-Iwanaga-Gorenstein algebras. As can be seen from the example of selfinjective algebras, a given algebra can be m-Iwanga-Gorenstein for different values of m.

Remark 4.3.3. The property of being m-Iwanaga-Gorenstein is Morita invariant.

1-**Iwanaga-Gorenstein algebras** In the special case of 1-Iwanaga-Gorenstein algebras we have the following, which was observed by Happel [H] and also Auslander and Reiten [AR, Page 121].

Proposition 4.3.4. *For a finite-dimensional algebra A the following are equivalent:*

- $\operatorname{inj.dim}({}_AA) \leq 1$,
- $\operatorname{inj.dim}(A_A) \leq 1$.

Proof. Assume that $\operatorname{inj.dim}({}_AA) \leq 1$. Then since $\operatorname{proj.dim}(D({}_AA)) \leq 1$ and

$$\operatorname{Ext}^1_A(D({}_AA), D({}_AA)) = 0,$$

it follows that $D({}_AA)$ is a right A-tilting module. Therefore, it follows form Bongartz [B, Tilting Theorem d)] that $D({}_AA)$ is a left B-tilting module, where $B = \operatorname{End}_A(D({}_AA))$. In particular, $D({}_AA)$ is a left B-module of projective dimension ≤ 1. Now, we have the following chain of ismorphisms of rings

$$B = \operatorname{End}_A(D({}_AA)) \cong \operatorname{End}_A({}_AA)^{\mathrm{op}} \cong A$$

and therefore it follows, that $D({}_AA)$ is a left A-module of projective dimension ≤ 1, which is equivalent to $\operatorname{inj.dim}(A_A) \leq 1$. The other direction follows dually. □

The following conjecture has been considered by several authors, see for example [AR, Chapter 6].

Conjecture 4.3.5. *Let A be a finite-dimensional algebra. Then* $\operatorname{inj.dim}({}_AA) \leq m$ *if and only if* $\operatorname{inj.dim}(A_A) \leq m$ *for any* $m \in \mathbb{N}$.

4.3.2 Perpendicular categories for Iwanaga-Gorenstein algebras

Let A be a finite-dimensional 1-Iwanaga-Gorenstein algebra. Then it does not necessarily have finite global dimension and therefore Proposition 4.1.13 (3) is not particularly helpful. However, we can replace it by the following:

Theorem 4.3.6. *Let $M \in \operatorname{mod}(A)$ be an indecomposable partial tilting module, and let*

$$L\colon \operatorname{mod}(A) \to M^{\perp}$$

be the left adjoint to the inclusion functor. If $\operatorname{proj.dim}(LA) \leq 1$ *as a left A-module, then the finite-dimensional algebra $A' = \operatorname{End}_A(LA)^{\mathrm{op}}$ is a 1-Iwanaga-Gorenstein algebra.*

Proof. We want to show that $\operatorname{inj.dim}({}_{A'}A') \leq 1$. Then by Proposition 4.3.4 A' is 1-Iwanaga-Gorenstein.

Since the epimorphism $\varphi\colon A \to A'$ induced by L is a homological epimorphsim, we know that for all left A'-modules N the natural map

$$\operatorname{Ext}^i_{A'}({}_{A'}N, {}_{A'}A') \to \operatorname{Ext}^i_A({}_AN, {}_AA')$$

is an isomorphism for all $i \geq 0$. Here the right hand side is zero for all $i \geq 2$: By assumption, A is 1-Iwananga-Gorenstein and hence $\operatorname{proj.dim}(LA) \leq 1$ is equivalent to $\operatorname{inj.dim}(LA) \leq 1$. By Theorem 4.1.10 there is an isomorphism

$${}_AA' \cong LA$$

of left A-modules. This completes the proof. □

Corollary 4.3.7. *Let $M \in \mathrm{mod}(A)$ be an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ a skew-field and $\mathrm{Hom}_A(M, A) = 0$. Then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{mod}(A'),$$

where A' is again a finite-dimensional 1-Iwanaga-Gorenstein algebra.

Proof. This is a direct consequence of Theorem 4.1.10 and Theorem 4.3.6. □

Proposition 4.3.8. *If Conjecture 4.3.5 is true, the above theorem and corollary hold more general for m-Iwanaga-Gorenstein algebras.*

4.4 On quivers with relations for symmetrizable Cartan matrices

4.4.1 Definition of the algebra $H(C, D, \Omega)$ and basic properties

Definition of the algebra In the following we recall the definition of the 1-Iwanaga-Gorenstein algebras associated with symmetrizable Cartan matrices as given in [GLS1]. This generalizes the notion of a path algebra associated with a symmetric Cartan matrix. This kind of generalization was attempted and achieved before in the setup of modulated graphs by Gabriel and Dlab and Ringel. However, results in this setup depend on quite strong assumptions on the ground field.

Let C be a Cartan matrix of size $n \times n$ with symmetrizer D and orientation Ω. For all $c_{ij} < 0$ we define numbers

$$g_{ij} := |\gcd(c_{ij}, c_{ji})| \qquad \text{and} \qquad f_{ij} := |c_{ij}|/g_{ij}.$$

Furthermore, we define a quiver $Q := Q(C, \Omega) = (Q_0, Q_1)$ as follows: the set of vertices is $Q_0 = \{1, 2, \dots, n\}$ and the set of arrows is

$$Q_1 := \{\alpha_{ij}^{g} \colon j \to i \mid (i, j) \in \Omega, 1 \le g \le g_{ij}\} \cup \{\varepsilon_i \colon i \to i \mid 1 \le i \le n\}.$$

Finally, we define the *algebra associated to a Cartan matrix C with symmetrizer D and orientation Ω* as

$$H := H(C, D, \Omega) = KQ/I$$

where K is a field, KQ the path algebra of $Q = Q(C, \Omega)$ and I the ideal generated by the relations

$$\varepsilon_i^{d_i} = 0$$

for all $1 \le i \le n$ and

$$\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_i^{f_{ij}}$$

for each $(i, j) \in \Omega$ and $1 \le g \le g_{ij}$.

Remark 4.4.1. Let Q° be the quiver obtained from $Q = Q(C, \Omega)$ by deleting all loops. Then Q° is acyclic, because the orientation Ω is acyclic. Thus, it is easy to see, that the algebras $H = H(C, D, \Omega)$ are finite-dimensional. Also note, that the definition of H does depend on the chosen symmetrizer D. If C is symmetric and D is minimal, then H is isomorphic to the path algebra KQ°, showing that it can be understood as a generalization of the classical path algebras indeed.

Locally free modules Denote by $e_1, e_2, \dots, e_n$ the idempotents in H corresponding to the vertices. Then $H_i := e_i H e_i$ is isomorphic to the truncated polynomial ring $K[\varepsilon_i]/(\varepsilon_i^{d_i})$. For any H-module M let $M_i := e_i M$. Then M_i has an H_i-module structure and we call M *locally free* if M_i is free as an H_i-module for $1 \le i \le n$. We denote by $\mathrm{rep}_{\mathrm{l.f.}}(H)$ the full subcategory of locally free modules of $\mathrm{rep}(H)$. One of the main results of [GLS1] is the following

Theorem 4.4.2. *The algebra H is 1-Iwanaga-Gorenstein. Furthermore, for $M \in \operatorname{rep}(H)$ the following are equivalent*

- $\operatorname{proj.dim}(M) \leq 1$;
- $\operatorname{inj.dim}(M) \leq 1$;
- $\operatorname{proj.dim}(M) < \infty$;
- $\operatorname{inj.dim}(M) < \infty$;
- $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$.

τ-locally free modules Denote by τ_H the Auslander-Reiten translate of the algebra H. Let $M \in \operatorname{rep}(H)$ be indecomposable. Then M is called *τ-locally free* if $\tau_H^k(M)$ is locally free for all $k \in \mathbb{Z}$. It follows from [GLS1, Proposition 11.4.] that any indecomposable partial tilting H-module M is τ-locally free and that $\tau_H^k(M)$ is again a partial tilting module for all $k \in \mathbb{Z}$.

An indecomposable τ-locally free H-module M is called

- *preprojective* if there exists some $k \geq 0$ such that $M \cong \tau_H^{-k}(P)$ for some indecomposable projective H-module P;
- *preinjective* if there exists some $k \geq 0$ such that $M \cong \tau_H^k(I)$ for some indecomposable injective H-module I;
- *regular* if $\tau_H^k(M) \neq 0$ or equivalently if it is neither preprojective nor preinjective.

An arbitrary, not necessarily indecomposable, τ-locally free H-module M, is *preprojective* (respectively *preinjective*) if it is isomorphic to a sum of indecomposable preprojective (respectively preinjective) modules. Note, that M is preprojective or preinjective if and only if there exists some $k \in \mathbb{Z}$ such that

$$\tau_H^k(M) = 0.$$

Furthermore, if M is indecomposable and preprojective or preinjective, it follows from the Auslander-Reiten formulas (see also [GLS1, Proposition 11.6.]) that $\operatorname{Ext}_H^1(M, M) = 0$. Hence, we see that in this case M is a partial tilting module.

For $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$ we define the *rank*-vector as follows: for each $i \in Q_0$ let r_i be the rank of M_i as a free H_i-module. Then we have $\dim_K(M_i) = r_i c_i$ and we put

$$\underline{\operatorname{rank}}(M) = (r_1, \ldots, r_n).$$

The following analogue result of Gabriel's Theorem indicates a close connection between the module categories of hereditary algebras and the τ-locally free H-modules. This is again one of the main results of [GLS1].

Theorem 4.4.3. *There are only finitely many isomorphism classes of τ-locally free H-modules if and only if C is of Dynkin type. Furthermore, in this case we have the following:*

- *There is a bijection between the set of isomorphism classes of τ-locally free H-modules and the set $\Delta^+(C)$ of positive roots of the semisimple complex Lie algebra associated with C given by the map $M \mapsto \underline{\operatorname{rank}}(M)$.*
- *If M is an indecomposable H-module, the following are equivalent:*
  - *M is preprojective;*
  - *M is preinjective;*
  - *M is τ-locally free;*
  - *M is locally free and rigid.*

Example 4.4.4. We would like to refer to [GLS1] for a list of examples of algebras $H(C, D, \Omega)$ and their Auslander-Reiten quiver and also to Section 4.5 of this thesis. However, we would like to include at least one example at this point.

Let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

be of Dynkin type C_3 with minimal symmetrizer $D = \operatorname{diag}(1, 1, 2)$ and $\Omega = \{(1, 2), (3, 2)\}$. In [GLS1] the reader can find the same example with orientation $\Omega = \{(1, 2), (2, 3)\}$. The graph $\Gamma(C)$ looks as follows:

$$1 \;\text{———}\; 2 \overset{(1,2)}{\text{———}} 3$$

We have $f_{12} = f_{21} = 1$, $f_{23} = 2$ and $f_{32} = 1$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{12}"'] \arrow[r, "\alpha_{23}"'] \arrow[loop above, "\varepsilon_2"] & 3\varepsilon \arrow[loop above, "\varepsilon_3"] \end{tikzcd}$$

with relations $\varepsilon_1 = \varepsilon_2 = 0$ and $\varepsilon_3^2 = 0$. The Auslander-Reiten quiver of H is shown in Figure 4.2. The numbers in the figure correspond to composition factors and basis vectors. Furthermore, we use the same notation as in [GLS1] that is as follows: the τ-locally free H-modules are marked with a double frame, the locally free H-modules, which are not τ-locally free, are marked with a single solid frame. (In the last three rows the three modules on the left have to be identified with the corresponding three modules on the right.)

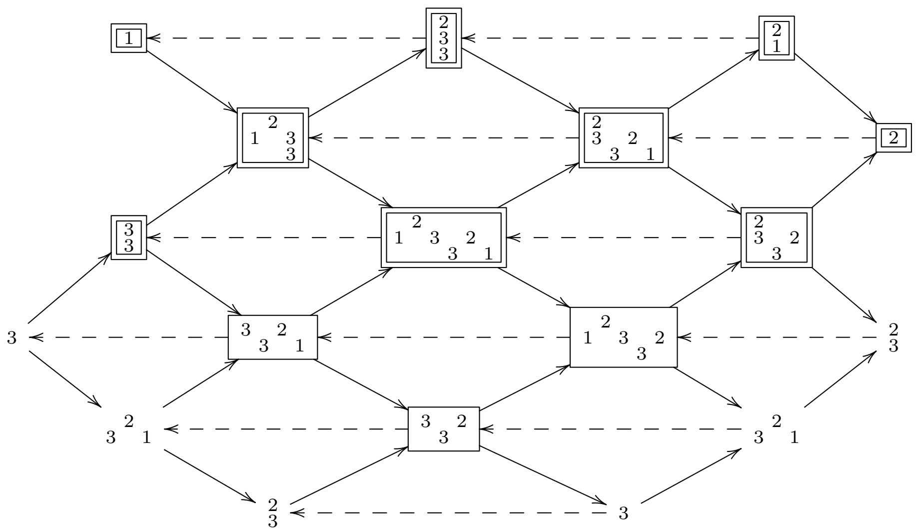


Figure 4.2: The Auslander-Reiten quiver of $H(C, D, \Omega)$ of type C_3 with D minimal.

4.4.2 Reflection functors

Generalized simple H-modules Let (C, D, Ω) and $H = H(C, D, \Omega)$ as defined above. We denote by $E_1, \ldots, E_n$ the indecomposable locally free modules with $\underline{\operatorname{rank}}(E_i) = \alpha_i$, where $\alpha_1, \ldots, \alpha_n$ is the standard basis of $\mathbb{Z}^n$. Thus E_i corresponds to the regular representation of H_i and we refer to it as *generalized simple* H-module. Note, that if i is a sink in $Q^\circ(C, \Omega)$, then $E_i = P_i$ is the indecomposable projective at vertex i.

Reflection functors Recall that for an orientation Ω of C and some $1 \leq i \leq n$ the change of orientation at the vertex i was defined as

$$s_i(\Omega) := \{(r,s) \in \Omega \mid i \notin \{r,s\}\} \cup \{(s,r) \in \Omega^* \mid i \in \{r,s\}\}$$

and we also define

$$s_i(H) := s_i(H(C,D,\Omega) := H(C,D,s_i(\Omega)).$$

We also say that a sequence $(i_1,\ldots,i_n)$ is a $(+)$-*admissible sequence* for H, if it is a $(+)$-admissible sequence for (C,Ω).

Associated to a sink k in $Q^\circ(C,\Omega)$ there is a *reflection functor*

$$F_k^+ \colon \operatorname{rep}(H) \to \operatorname{rep}(s_k(H))$$

and dually if k is a source, there is a reflection functor

$$F_k^- \colon \operatorname{rep}(H) \to \operatorname{rep}(s_k(H)).$$

For details on the definition we refer to [GLS1].

For any vertex k we consider the full subcategories

$$\mathcal{T}_k = \{M \in \operatorname{rep}(H) \mid \operatorname{top}_k(M) = 0\}$$

and

$$\mathcal{S}_k = \{M \in \operatorname{rep}(H) \mid \operatorname{soc}_k(M) = 0\}.$$

Then the following is [GLS1, Corollary 9.2. and Corollary 9.3.]:

Lemma 4.4.5. *Let k be a sink in $Q^\circ(C,\Omega)$. Then the reflection functor*

$$F_k^+ \colon \operatorname{rep}(H) \to \operatorname{rep}(s_k(H))$$

induces an equivalence of the subcategories of $\operatorname{rep}(H)$ *defined by $F_k^+ : \mathcal{T}_k \to \mathcal{S}_k$ with quasi-inverse $F_k^- : \mathcal{S}_k \to \mathcal{T}_k$. Furthermore, if $M, N \in \mathcal{T}_k$ then F_k^+ induces an isomorphism*

$$\operatorname{Ext}^1_H(M,N) \cong \operatorname{Ext}^1_{s_kH}(F_k^+(M), F_k^+(M)),$$

and similarly for $M, N \in \mathcal{S}_k \subseteq \operatorname{rep}(s_k(H))$ the functor F_k^- induces an isomorphism

$$\operatorname{Ext}^1_{s_kH}(M,N) \cong \operatorname{Ext}^1_H(F_k^-(M), F_k^-(M)).$$

Coxeter functors Similarly as for hereditary algebras one defines the Coxeter functors. For a $(+)$-admissible sequence $(i_1,\ldots,i_n)$ of (C,Ω), let

$$C^+ = F_{i_n}^+ \circ \ldots \circ F_{i_1}^+$$

be the *Coxeter functor*. Dually one defines $C^- = F_{j_n}^- \circ \ldots \circ F_{j_1}^-$ for a $(-)$-admissible sequence $(j_1,\ldots,j_n)$ of (C,Ω). Note that the Coxeter functors do not depend on the choice of the admissible sequence. Furthermore, we denote by

$$\mathcal{R}(H) := \{M \in \operatorname{rep}(H) \mid M \text{ is regular}\}$$

the full subcategory of regular modules.

Proposition 4.4.6. *Let k be a sink in $Q^\circ(C,\Omega)$. Then the reflection functor*

$$F_k^+ \colon \operatorname{rep}(H) \to \operatorname{rep}(s_k(H))$$

induces an equivalence of the subcategories

$$F_k^+ : \mathcal{R}(H) \to \mathcal{R}(s_i(H))$$

with quasi-inverse $F_k^- : \mathcal{R}(s_i(H)) \to \mathcal{R}(H)$.

Proof. First note, that if k is a sink in $Q^\circ(C,\Omega)$, then $\mathcal{R}(H) \subseteq \mathcal{T}_k$: the module E_k is projective and we have $\operatorname{top}_k(M) = 0$ for any other indecomposable locally free module. Now, by [GLS1, Proposition 11.8.] if M is τ-locally free, then so is $F_k^+(M)$. Now what is left to show is, that if M is regular, then so is $F_k^+(M)$. This follows from the fact that

$$\tau^m(F_k^+(M)) \cong F_k^+(\tau^m(M))$$

for all $m \in \mathbb{Z}$: since k is a sink there exists a $(+)$-admissible sequence $(i_1, \dots, i_n)$ of (Q,Ω) with $i_1 = k$. Then $(i_2, \dots, i_n, k)$ is a $(+)$-admissible sequence of $s_k(H)$. By [**?**][Theorem 1.3.]GLS we have $\tau(M) \cong TC(M)$ for any $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$, where T is the twist functor and

$$C \cong F_{i_n}^+ \cdots F_{i_2}^+ F_{i_1}^+ \cong F_{i_n}^+ \cdots F_{i_2}^+ F_k^+$$

is the Coxeter functor. This implies

$$F_k^+(\tau(M)) \cong F_k^+(TF_{i_n}^+ \cdots F_{i_1}^+)(M) \cong T(F_k^+ F_{i_n}^+ \cdots F_{i_2}^+) F_k^+(M) \cong \tau(F_k^+(M)).$$

Now it follows by induction that $\tau^m(F_k^+(M)) \cong F_k^+(\tau^m(M))$ for all $m \geq 0$. For $m < 0$ a similar argument with a $(-)$-admissible sequence proves the claim. For more details see also the proof of [GLS1, Proposition 11.8.]. □

Reflection functors and APR-tilting Let i be a sink in $Q^\circ(C,\Omega)$ and assume that $Q(C,\Omega)$ is connected. It follows that i cannot be a source in $Q^\circ(C,\Omega)$ and thus the generalized simple projective module E_i cannot be injective. By [GLS1, Theorem 9.7] we know that the H-module

$$T = {}_H H/E_i \oplus \tau_H^-(E_i) = T_1 \oplus \cdots \oplus T_n$$

where

$$T_j = \begin{cases} P_j & \text{if } j \neq i \\ \tau_H^-(E_i) & \text{if } j = i \end{cases}$$

is a tilting H-module. Furthermore, for $B = \operatorname{End}_H(T)^{\mathrm{op}}$ there is an equivalence of categories

$$S\colon \operatorname{rep}(s_i(H)) \to \operatorname{rep}(B)$$

such that we have an isomorphism of functors $S \circ F_i^+ \cong \operatorname{Hom}_H(T, -)$. From this it follows that $F_i^+(T_j) \cong s_i(H)e_j$ for all $1 \leq j \leq n$ and there is an isomorphism of algebras $s_i(H) \cong B$ (see [GLS1] remark after Lemma 9.14). By choosing a $(+)$-admissible sequence for H and applying the above inductively, we get the following result.

Corollary 4.4.7. *Assume without loss of generality that* $(1, 2, \dots, n)$ *is a* $(+)$*-admissible sequence in* H. *Then there is an isomorphism of algebras*

$$\operatorname{End}_H(\tau_H^-(P_1 \oplus \cdots \oplus P_i) \oplus P_{i+1} \oplus \cdots \oplus P_n)^{\mathrm{op}} \cong s_i s_{i-1} \dots s_1 H$$

for every $1 \leq i \leq n$.

4.4.3 On the connection with hereditary algebras

Rank vectors and the Coxeter transformation From now on, let C be a generalized symmetrizable Cartan matrix and let $H = H(C, D, \Omega)$ be the associated 1-Iwanaga-Gorenstein algebra and denote by $A = A(C, D, \Omega)$ the corresponding hereditary artin algebra. In the following we will line out an analogy between τ-locally free H-modules and A-modules.

Lemma 4.4.8. *Let $M \in \operatorname{rep}(H)$ be τ-locally free and non-projective. Then we have*

$$\underline{\operatorname{rank}}(\tau_H(M)) = c^+(\underline{\operatorname{rank}}(M))$$

where c^+ denotes the Coxeter transformation corresponding to a $(+)$-admissible sequence of (C, Ω).

Proof. By [GLS1, Proposition 11.5.] we have

$$\underline{\dim}(\tau_H(M)) = \Phi_H(\underline{\dim}(M))$$

where Φ_H denotes the Coxeter matrix. Now, if we express Φ_H in the basis given by the vectors $\underline{\dim}(E_k)$, we can identify it with the Coxeter transformation c^+. □

Lemma 4.4.9. *For $i = 1, \ldots, n$ we have*

$$\underline{\dim}(P_i^A) = \underline{\operatorname{rank}}(P_i^H) \text{ and}$$
$$\underline{\dim}(I_i^A) = \underline{\operatorname{rank}}(I_i^H).$$

Proof. Assume without loss of generality that $(1, \ldots, n)$ is a $(+)$-admissible sequence of Ω. Then by [GLS1, Lemma 3.2.] we have

$$\underline{\operatorname{rank}}(P_k^H) = \beta_k = \begin{cases} \alpha_k & \text{if } k = 1, \\ s_1 s_2 \ldots s_{k-1}(\alpha_k) & \text{if } 2 \leq k \leq n. \end{cases}$$

and hence by Proposition 4.2.4 we have

$$\underline{\operatorname{rank}}(P_k^H) = \beta_k = \underline{\dim}(P_k^A).$$

□

Correspondence of preprojective H-modules and preprojective A-modules In the following we establish a close connection between preprojective H-modules and preprojective A-modules. This can also be found in [GLS2, Section 5].

Proposition 4.4.10. *Let $\alpha \in \mathbb{N}^n$. Then there exists an indecomposable preprojective (or preinjective) H-module M with $\underline{\operatorname{rank}}(M) = \alpha$ if and only if there exists an indecomposable preprojective (or preinjective) A-module N with $\underline{\dim}(N) = \alpha$. In this case M and N are uniquely determined by α up to isomorphism.*

Proof. Let $M \in \operatorname{rep}(H)$ be a preprojective module with $\underline{\operatorname{rank}}(M) = \alpha$. Then $M \cong \tau_H^{-k}(P_i^H)$ for some vertex i and $k \geq 0$. Then for $N = \tau_A^{-k}(P_i^A)$ we have

$$\underline{\dim}(N) = \underline{\dim}(\tau_A^{-k}(P_i^A)) = c^{-k}\underline{\dim}(P_i^A) = c^{-k}\underline{\operatorname{rank}}(P_i^H) = \underline{\operatorname{rank}}(\tau_H^{-k}(P_i^H)) = \alpha$$

where we used the previous Lemmas and their corresponding versions for hereditary algebras. The dual statements hold for preinjective modules. By [GLS1, Proposition 11.6.] preprojective and preinjective H-modules are uniquely determined up to isomorphism by their dimension-vectors. □

Proposition 4.4.11. *Let $M(\alpha), M(\beta) \in \operatorname{mod}(H)$ and $N(\alpha), N(\beta) \in \operatorname{mod}(A)$ be indecomposable preprojective or preinjective modules with rank- and dimension-vectors α and β respectively such* *that $\alpha \neq \beta$. Then we have*

$$\dim \operatorname{Hom}_H(M(\alpha), M(\beta)) = \dim \operatorname{Hom}_A(N(\alpha), N(\beta)),$$
$$\dim \operatorname{Ext}^1_H(M(\alpha), M(\beta)) = \dim \operatorname{Ext}^1_A(N(\alpha), N(\beta)).$$

Proof. First assume that $M(\alpha) \cong \tau_H^{-s}(P_i)$. If $M(\beta) \cong \tau_H^{-k}(P_j)$ such that $s \leq k$ or if $M(\beta)$ is preinjective (but not preprojective) by the Auslander-Reiten-formula for modules of projective and injective dimension at most 1 we have

$$\begin{aligned}\dim \operatorname{Hom}_H(M(\alpha), M(\beta)) &= \dim \operatorname{Hom}_H(P_i, \tau_H^s M(\beta)) \\ &= \underline{\dim}(\tau_H^s M(\beta))_i \\ &= d_i(\underline{\operatorname{rank}}(\tau_H^s M(\beta))_i \\ &= d_i c^s(\beta)_i\end{aligned}$$

where c is the Coxeter transformation and d_i the entry of the symmetrizer D of C. We also have

$$\begin{aligned}\dim \operatorname{Hom}_A(N(\alpha), N(\beta)) &= \dim \operatorname{Hom}_H A(P_i^A, \tau_A^s N(\beta)) \\ &= \dim \operatorname{End}_A(S_i^A)[\tau_A^s N(\beta) : S_i^A] \\ &= d_i \underline{\dim}(\tau_A^s N(\beta))_i \\ &= d_i c^s(\beta)_i\end{aligned}$$

where again c is the Coxeter transformation and $d_i = \dim \operatorname{End}_A(S_i^A)$.

If $M(\beta) \cong \tau_H^{-k}(P_j)$ such that $s > k$ we have

$$\dim \operatorname{Hom}_H(M(\alpha), M(\beta)) = \dim \operatorname{Hom}_H(\tau_H^k(M(\alpha)), P_j^H) = 0$$

since $\tau_H^k(M(\alpha))$ is non-projective and by [GLS1, Corollary 11.3.]. We also have

$$\dim \operatorname{Hom}_A(N(\alpha), N(\beta)) = \dim \operatorname{Hom}_A(\tau_A^k(N(\alpha)), P_j^A) = 0$$

since again $\tau_A^k(N(\alpha))$ is not projective.

The case, where both $M(\alpha)$ and $M(\beta)$ are preinjective is treated with dual arguments. If $M(\alpha)$ is preinjective (but not preprojective, hence if C is not of Dynkin type) and $M(\beta)$ is preprojective, then we have

$$\dim \operatorname{Hom}_H(M(\alpha), M(\beta)) = 0$$

by [GLS1, Lemma 11.7.] and

$$\dim \operatorname{Hom}_A(N(\alpha), N(\beta)) = 0$$

by the corresponding statement for hereditary algebras. For this see for example [ASS, Corollary VIII.2.13.].

The formula for Ext^1 now follows by applying the Auslander-Reiten formula appropriately to the different cases and then using the result for the dimension of the Hom-spaces. One proves this case, using the homological bilinear forms of H and A and using [GLS1, Corollary 4.3.]. □

Homological properties of preprojective and preinjective H-modules The last proposition establishes a nice bridge between preprojective or preinjective H- and A-modules and has several useful consequences.

Corollary 4.4.12. *There is no cycle in* $\operatorname{mod}(H)$ *consisting of preprojective or preinjective indecomposable modules.*

Proof. This is a direct consequence of Proposition 4.4.11 and the equivalent argument for hereditary algebras as in Proposition 4.2.7. □

As for hereditary algebras we say that a module $M \in \mathrm{rep}_{\mathrm{l.f.}}(H)$ is called *exceptional* if it is indecomposable and $\mathrm{Ext}^1_H(M,M) = 0$. Thus, M is an indecomposable partial tilting module. However, in general $\mathrm{End}_H(M)$ is not a skew-field. A pair (M,N) of exceptional H-modules is called an *exceptional pair*, if $\mathrm{Hom}_H(N,M) = 0$ and $\mathrm{Ext}^1_H(N,M) = 0$. A sequence $(M_1,\dots,M_n)$ of H-modules is called *complete exceptional sequence* if n is the number of vertices of $Q(C,\Omega)$ and if (M_i,M_j) is an exceptional pair for any $i<j$.

Corollary 4.4.13. *Let $\alpha,\beta \in \mathbb{N}^n$. Then the following are equivalent:*

- *There exist preprojective/preinjective modules $M(\alpha), M(\beta) \in \mathrm{rep}(H)$ such that $(M(\beta),M(\alpha))$ is an exceptional pair.*
- *There exist preprojective/preinjective modules $N(\alpha), N(\beta) \in \mathrm{rep}(A)$ such that $(N(\beta),N(\alpha))$ is an exceptional pair.*

Corollary 4.4.14. *If C is a Cartan matrix of Dynkin type, then there is a bijection between the exceptional sequences in $\mathrm{rep}(A)$ and the exceptional sequences in $\mathrm{rep}_{\mathrm{l.f.}}(H)$. In particular the braid group acts transitively on the exceptional sequences in $\mathrm{rep}_{\mathrm{l.f.}}(H)$.*

Lemma 4.4.15. *Let $H = H(C,D,\Omega)$ and let i,j be two vertices in Q, such that there is an arrow from i to j, but no commutativity relation between i and j. Then there is an irreducible morphism from P_j to P_i.*

Proof. It follows from the description of the indecomposable projective modules in [GLS1] that in this case P_j is a direct summand of $\mathrm{rad}(P_i)$. Thus the claim follows. □

Corollary 4.4.16. *Let $H = H(C,D,\Omega)$, such that C is not of Dynkin type, and such that $k_{ij} = 1$ for all $1 \le i < j \le n$. Further let $\Delta = (Q^\circ(C,D,\Omega))^{\mathrm{op}}$, be the opposite quiver. Then the preprojective vertices form a connected subquiver (not component) of the Auslander-Reiten quiver of H of the form $\mathbb{N}\Delta$.*

Proof. By the last lemma we can identify the isoclasses of the projective modules $\{P_1,\dots,P_n\}$, hence the projective vertices with the vertices of Δ. Then the properties of the Auslander-Reiten sequences yield, that we can identify the isoclass of $\tau_H^{-k}(P_i)$ with the vertex (k,i) for $k \ge 0$ and $i \in \Delta_0$. □

4.4.4 Perpendicular categories for $H(C,D,\Omega)$

From now on assume that the ground field K is algebraically closed.

Generalization of Bongartz's short exact sequence for preprojective H-modules From now on let C be a generalized Cartan matrix, D a minimal symmetrizer, Ω an orientation of C and $H = H(C,D,\Omega)$ and $A = A(C,D,\Omega)$. The following result and its proof can be found in [GLS2, Section 5.4.]

Proposition 4.4.17. *Let $M,N \in \mathrm{rep}_{\mathrm{l.f.}}(H)$ be rigid modules with $\mathrm{Ext}^1_H(M,N) = 0$. Further let $E \in \mathrm{rep}_{\mathrm{l.f.}}(H)$ be a rigid module with*

$$\underline{\mathrm{rank}}(E) = \underline{\mathrm{rank}}(M) + \underline{\mathrm{rank}}(N).$$

Then there exists a short exact sequence

$$0 \to M \to E \to N \to 0.$$

Lemma 4.4.18. *Let $M,N \in \mathrm{mod}(H)$ be preprojective modules with $\mathrm{Ext}^1_H(N,M) = 0$ and M be indecomposable. There exists a short exact sequence*

$$0 \to N \to N' \to M^k \to 0$$

with $k \in \mathbb{N}$ and such that $\operatorname{Ext}^1_H(M, N') = 0$ and $\operatorname{Hom}_H(M, N) = \operatorname{Hom}_H(M, N')$. Hence if $\operatorname{Hom}_H(M, N) = 0$, it follows that $L(N) = N'$ where $L\colon \operatorname{mod}(H) \to M^\perp$ is the left adjoint of the inclusion functor.

Proof. Let X, Y be the preprojective A-modules with $\underline{\dim}(X) = \underline{\operatorname{rank}}(M)$ and $\underline{\dim}(Y) = \underline{\operatorname{rank}}(N)$. Then by Lemma 4.4.11 the indecomposable module X is rigid. Thus it follows, for example from [HR, Lemma 4.1.], that the algebra $\operatorname{End}_A(X)$ is a skew field. Therefore, by Proposition 4.1.7 there exists a short exact sequence of A-modules

$$0 \to Y \to Y' \to X^k \to 0$$

with $\operatorname{Ext}^1_A(X, Y') = 0$ and $\operatorname{Hom}_A(X, Y) = \operatorname{Hom}_A(X, Y')$. By Lemma 4.2.8 the module Y' is also preprojective. Hence, there exists a preprojective H-module N' with

$$\underline{\operatorname{rank}}(N') = \underline{\dim}(Y') = \underline{\operatorname{rank}}(N) + \underline{\operatorname{rank}}(M^k).$$

Then by Proposition 4.4.17 there is a short exact sequence

$$0 \to N \to N' \to M^k \to 0$$

of H-modules and by Lemma 4.4.11 we have

$$\dim \operatorname{Hom}_H(M, N) = \dim \operatorname{Hom}_A(X, Y) = \dim \operatorname{Hom}_A(X, Y') = \dim \operatorname{Hom}_H(M, N').$$

Thus, we see that in the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_H(M, N) &\xrightarrow{g} \operatorname{Hom}_H(M, N') \to \operatorname{Hom}_H(M, M^k) \\ &\to \operatorname{Ext}^1_H(M, N) \to \operatorname{Ext}^1_H(M, N') \to 0 \end{aligned}$$

the monomorphism g is in fact an isomorphism. □

The left-adjoint functor $L\colon \operatorname{mod}(H) \to M^\perp$ for preprojective and preinjective H-modules

Proposition 4.4.19. *Let $M \in \operatorname{mod}(H)$ be indecomposable and preprojective or preinjective, and non-projective. Let $i\colon M^\perp \to \operatorname{mod}(H)$ be the inclusion functor and $L\colon \operatorname{mod}(H) \to M^\perp$ its left adjoint. Then there is some $k \in \mathbb{N}$ and a short exact sequence*

$$0 \to H \to LH \to M^k \to 0$$

such that $M \oplus LH$ is a tilting module, and in particular $\operatorname{proj.dim}(LH) \leq 1$.

Proof. Let $M \in \operatorname{mod}(H)$ be an indecomposable partial tilting module, which is not projective. Let

$$0 \to H \xrightarrow{g} H' \xrightarrow{f} M^k \to 0$$

be the short exact sequence which is used in the construction of the left-adjoint functor

$$L\colon \operatorname{mod}(H) \to M^\perp,$$

in this case also known as Bongartz's short exact sequence. In particular, $M \oplus H'$ is a tilting H-module. It follows that all indecomposable summands of $M \oplus H'$ are τ-locally free and rigid. We will prove that $\operatorname{Hom}_H(M, H') = 0$ and then it follows from Remark 4.1.9 that $LH = H' \in M^\perp$ is of projective dimension at most 1.

Let T be an indecomposable summand of H'. If $\operatorname{Hom}_H(T, M) = 0$, it follows in particular that

$$T \subseteq \operatorname{Ker}(f) \cong \operatorname{Im}(g) \cong H$$

and thus T is isomorphic to a direct summand of H and therefore projective. Since M is assumed to be not projective in this case we have

$$\mathrm{Hom}_H(M,T)=0$$

and hence $T\in M^{\perp}$.

From now on assume that $\mathrm{Hom}_H(T,M)\neq 0$ and that T is not isomorphic to M. Note that since M and T are τ-locally free and indecomposable, they are preprojective, preinjective or regular.

First assume that M is preprojective. Then since $\mathrm{Hom}_H(T,M)\neq 0$, it follows from [GLS1, Lemma 11.7.] that T is also preprojective. Hence, again since $\mathrm{Hom}_H(T,M)\neq 0$ by Corollary 4.4.12 we have $\mathrm{Hom}_H(M,T)=0$ and thus $T\in M^{\perp}$.

Secondly assume that M is preinjective. Now, if T is preprojective or regular it follows again from [GLS1, Lemma 11.7.] that we have $\mathrm{Hom}_H(M,T)=0$ and thus $T\in M^{\perp}$. If T is preinjective it follows from the dual version of Corollary 4.4.12 and $\mathrm{Hom}_H(T,M)\neq 0$ that $\mathrm{Hom}_H(M,T)=0$ and thus $T\in M^{\perp}$. □

Remark 4.4.20. Assume that in the above situation M is regular. Again by [GLS1, Lemma 11.7.] and $\mathrm{Hom}_H(T,M)\neq 0$ it follows that T is preprojective or regular. If T is preprojective then again by [GLS1, Lemma 11.7.] we have $\mathrm{Hom}_H(M,T)=0$.

Unfortunately, the case where both M and T are regular remains open.

We also have another useful consequence from the last proof.

Corollary 4.4.21. *Let $M\in\mathrm{mod}(H)$ be a preprojective module. Then the indecomposable summands of its Bongartz complement LH are also preprojective.*

Lemma 4.4.22. *Let $M\in\mathrm{mod}(H)$ be an indecomposable preprojective or preinjective module. If $T\in\mathrm{mod}(H)$ such that $M\oplus T$ is a tilting module, $T\in M^{\perp}$ and if $\mathrm{Ext}^1_H(T,X)=0$ for all $X\in M^{\perp}$ which are τ-locally free, then T is a projective generator of $M^{\perp}$.*

Proof. Since $T\oplus M$ is a tilting module, we have

$$\mathrm{Fac}(T\oplus M)=\{X\in\mathrm{mod}(H)\mid \mathrm{Ext}^1_H(T\oplus M,X)=0\}.$$

As M is preprojective or preinjective, we know that there is a projective generator P of $M^{\perp}$ which is τ-locally free. Since $\mathrm{Ext}^1_H(T\oplus M,P)=0$ it follows that $P\in\mathrm{Fac}(T\oplus M)$ and since $\mathrm{Hom}_H(M,P)=0$ we have in fact $P\in\mathrm{Fac}(T)$. But this implies that

$$M^{\perp}\subseteq\mathrm{Fac}(P)\subseteq\mathrm{Fac}(T)\subseteq\{X\in\mathrm{mod}(H)\mid \mathrm{Ext}^1_H(T,X)=0\}$$

where for the last inclusion we used that T is a partial tiling module. Thus, the lemma follows. □

Theorem 4.4.23. *Let $M\in\mathrm{mod}(H)$ be an indecomposable partial tilting module.*

- *If $\mathrm{End}_H(M)$ is a skew-field and $\mathrm{Hom}_A(M,A)=0$, then there is an equivalence of categories*

  $$M^{\perp}\simeq\mathrm{mod}(H')$$

  where $H'\cong KQ'/I'$ is a 1-Iwanaga-Gorenstein algebra and $|Q'_0|=|Q_0|-1$.

- *If M is preprojective or preinjective, then there is an equivalence of categories*

  $$M^{\perp}\simeq\mathrm{mod}(H')$$

  where $H'\cong KQ'/I'$ is a 1-Iwanaga-Gorenstein algebra and $|Q'_0|=|Q_0|-1$. In the case where M is preprojective in Q' there are no cycles passing through more than one vertex.

Proof.
- By Corollary 4.3.7 we have $M^\perp \simeq \operatorname{rep}(H')$, where H' is 1-Iwanaga-Gorenstein. Since being Iwanaga-Gorenstein is a Morita-invariant, we can assume that H' is a basic algebra. Now, since K is algebraically closed, we know that $H' \cong KQ'/I$ for some quiver Q' and admissible ideal I'. It follows from Proposition 4.1.13 that $Q'_0 = Q_0 - 1$.

- By Proposition 4.4.19, we have that $LH \in M^\perp$ is the Bongartz complement of M and has projective dimension at most 1 as an H-module. Therefore, by Theorem 4.3.6 we have an equivalence of categories $M^\perp \simeq \operatorname{rep}(H')$, where H' is 1-Iwanaga-Gorenstein. By the same arguments as in the first part, we can assume that $H' \cong KQ'/I'$ for some quiver Q' with $Q'_0 = Q_0 - 1$ and admissible ideal I'.

  Suppose that M is preprojective and let $T_1, \dots, T_{n-1}$ be the pairwise non-isomorphic indecomposable summands of LH, where we suppose that $n = |Q_0|$. Then by Corollary 4.4.21 the summands T_i are also preprojective. Then the quiver of $\operatorname{End}_H(T)^{\mathrm{op}}$ for $T = T_1 \oplus \cdots \oplus T_{n-1}$ does not contain a cycle by Corollary 4.4.12.

□

Example 4.4.24. We know that the perpendicular category $M^\perp$ for $M \in \operatorname{mod}(H)$ an indecomposable locally free module is abelian. In the above theorem we always assume, that M is also rigid. The following simple example shows, that this condition is necessary if we want the number of simple modules to reduce by exactly one.

The example is taken from [GLS1, Section 13]. Let C be a Cartan matrix of Dynkin type B_2, more precisely let

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and choose the minimal symmetrizer $D = \operatorname{diag}(2,1)$ and orientation $\Omega = \{(1,2)\}$. We have $f_{12} = 1$ and $f_{21} = 2$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{12}"] \arrow[loop above, "\varepsilon_2"] \end{tikzcd}$$

with relations $\varepsilon_1^2 = 0$ and $\varepsilon_2 = 0$. The Auslander-Reiten quiver of H is shown in Figure 4.3. The numbers in the figure correspond to composition factors and basis vectors. (In the last two rows the two modules on the left have to be identified with the corresponding two modules on the right.)

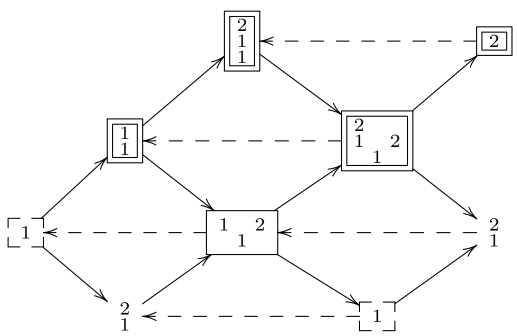


Figure 4.3: The Auslander-Reiten quiver of $H(C, D, \Omega)$ of type B_2 with D minimal.

Let $M \in \operatorname{mod}(H)$ be the indecomposable locally free module corresponding to the vertex

$$\boxed{\begin{smallmatrix} 1 & & 2 \\ & 1 & \end{smallmatrix}}$$

in the Auslander-Reiten quiver of H. Then, since we have $\mathrm{Hom}_H(M, X) \neq 0$ for all $X \in \mathrm{mod}(H)$, it follows that $M^\perp$ consists only of the zero object. In particular it does not contain a non-trivial simple object.

4.4.5 The perpendicular category of the Auslander-Reiten translate

The results in this section are based on [St, section 4], where the author considers hereditary algebras and proves that the perpendicular category does not change too much after applying the Auslander-Reiten translate.

Lemma 4.4.25. *Let $M \in \mathrm{mod}(H)$ be an indecomposable, non-projective module, such that both M and $\tau_H(M)$ are locally free. Then there is a bijection between the isomorphism classes of indecomposable τ-locally free modules in $M^\perp$ and the isomorphism classes of indecomposable τ-locally free modules in $(\tau_H M)^\perp$ given by*

$$N \mapsto \begin{cases} \tau_H N & \text{if } N \text{ is non-projective,} \\ I_i & \text{if } N = P_i \text{ is projective.} \end{cases}$$

Proof. If N is non-projective, we have

$$\mathrm{Hom}_H(M, N) \cong \mathrm{Hom}_H(\tau_H M, \tau_H M)$$

since $\mathrm{proj.dim}(\tau_H M) \leq 1$ and $\mathrm{inj.dim}(N) \leq 1$, and

$$\mathrm{Ext}^1_H(M, N) \cong D\,\mathrm{Hom}_H(N, \tau_H M) \cong \mathrm{Ext}^1_H(\tau_H M, \tau_H N)$$

since $\mathrm{proj.dim}(\tau_H M) \leq 1$ and $\mathrm{inj.dim}(\tau_H N) \leq 1$.

If $N = P_i$ is projective, we know $\mathrm{Hom}_H(M, P_i) = 0$, since M is non-projective. We always have $\mathrm{Ext}^1_H(\tau_H(M), I_i) = 0$. Further we have

$$\dim \mathrm{Ext}^1_H(M, P_i) = \dim \mathrm{Hom}_H(P_i, \tau_H M) = \underline{\dim}(\tau_H M)_i = \dim \mathrm{Hom}_H(\tau_H M, I_i)$$

which proves the Lemma. □

Recall that M is *sincere* if $\mathrm{Hom}_H(P_i, M) \neq 0$ for all indecomposable projective modules P_i.

Corollary 4.4.26. *Let $M \in \mathrm{mod}(H)$ be an indecomposable, non-injective preprojective module. If M is sincere, there is an equivalence of categories*

$$M^\perp \simeq (\tau_H^- M)^\perp.$$

Proof. Let I_i be the indecomposable injective module at vertex i. Then $I_i \in M^\perp$ if and only if $\mathrm{Hom}_H(M, I_i) = 0$ if and only if $\underline{\dim}(M)_i = 0$. Hence, since by assumption M is sincere it follows that there is no injective module in $M^\perp$. Thus the dual version of Lemma 4.4.25 implies that if $T \in M^\perp$ is a projective generator, then $\tau_H^-(T)$ is a projective generator of $(\tau_H^- M)^\perp$ and we have $\mathrm{End}_H(T) \cong \mathrm{End}_H(\tau_H^-(T))$. □

Remark 4.4.27. Let $H = H(C, D, \Omega)$ and suppose there exists an indecomposable projective-injective H-module. Thus by [GLS1, Lemma 11.7.] the Cartan matrix C must be of Dynkin type. Let $A = A(C, D, \Omega)$ be the corresponding representation-finite hereditary algebra. Using the correspondence between the preprojective H-modules and the preprojective A-modules we see, that there exists an indecomposable projective-injective A-module. It follows that A is a Nakayama algebra, hence A is of Dynkin type A_n with linear orientation, and the indecomposable projective-injective A-module is uniquely determined. Thus, we see that there exists an indecomposable projective-injective H-module if and only if A is a Nakayama algebra, and if and only if C is of Dynkin type A_n with linear orientation Ω.

Proposition 4.4.28. *Let $M \in \operatorname{mod}(H)$ be an indecomposable preprojective or preinjective module such that neither M nor $\tau_H(M)$ is projective. Further let*

$$L\colon \operatorname{mod}(H) \to M^{\perp} \simeq \operatorname{mod}(H'(M))$$
$$L'\colon \operatorname{mod}(H) \to (\tau_H M)^{\perp} \simeq \operatorname{mod}(H'(\tau_H M))$$

be the left adjoints of the respective inclusion functors. Here $H'(M)$ (respectively $H'(\tau_H M)$) denotes the 1-Iwanaga-Gorenstein algebra as described in Theorem 4.4.23.

- *If $P \oplus T$ is a projective generator of $M^{\perp}$, where P is H-projective and T has no H-projective direct summands, then $L'(P) \oplus \tau_H(T)$ is a projective generator of $(\tau_H M)^{\perp}$. Moreover, L' preserves indecomposability of direct summands of P.*

- *If H is not (of Dynkin type A_n and linearly oriented), then $H'(\tau_H M)$ is Morita-equivalent to $\operatorname{End}_{H'}(\tau_{H'}^{-} P \oplus T)$, where $H' = H'(M)$.*

- *If H is of Dynkin type A_n and linearly oriented, then $H'(\tau_H M)$ is Morita-equivalent to $H' = H'(M)$.*

Proof.

- Let P' be an indecomposable summand of P. Since $\tau_H M$ is τ-locally free, not projective, we have $\operatorname{Hom}_H(\tau_H M, P') = 0$ and thus by Proposition 4.4.19 there is a short exact sequence
$$0 \to P' \to L'P' \to (\tau_H M)^k \to 0$$
for some $k \in \mathbb{N}$. We have $\operatorname{Hom}_H(P', \tau_H M) \cong D\operatorname{Ext}^1(M, P') = 0$ and therefore the short exact sequence
$$0 \to \operatorname{Hom}_H(P', P') \to \operatorname{Hom}_H(P', L'P') \to \operatorname{Hom}_H(P', (\tau_H M)^k) \to 0$$
shows that
$$\operatorname{Hom}_H(P', P') \cong \operatorname{Hom}_H(P', L'P') \cong \operatorname{Hom}_H(L'P', L'P')$$
where the last isomorphism follows since L' is the left adjoint of the inclusion functor. Hence, $\operatorname{End}_H(L'P', L'P')$ is a local ring and $L'P'$ is indecomposable. Similarly, we see that for another indecomposable summand P'' of P we have $\operatorname{Hom}_H(P', P'') \cong \operatorname{Hom}_H(L'P', L'P'')$.

  It is easily seen from Lemma 4.4.25 that $\tau_H T$ is Ext-projective for all τ-locally free modules in $(\tau_H M)^{\perp}$. By applying the contravariant functor $\operatorname{Hom}_H(-, \tau_H T)$ to the above short exact sequence for $P' \in \operatorname{add}(P)$, we obtain
$$\begin{aligned} 0 &\to \operatorname{Hom}_H(\tau_H M^k, \tau_H T) \to \operatorname{Hom}_H(L'P', \tau_H T) \to \operatorname{Hom}_H(P', \tau_H T) \\ &\to \operatorname{Ext}^1_H(\tau_H M^k, \tau_H T) \to \operatorname{Ext}^1_H(L'P', \tau_H T) \to 0. \end{aligned}$$
which shows that $\operatorname{Hom}_H(L'P', \tau_H T) \cong \operatorname{Hom}_H(P', \tau_H T)$ since $\tau_H T \in (\tau_H M)^{\perp}$. Therefore, we have that $\operatorname{Hom}_H(L'P, \tau_H T) \cong \operatorname{Hom}_H(P, \tau_H T) \cong D\operatorname{Ext}^1_H(T, P) = 0$ and hence $L'P$ and $\tau_H T$ have no isomorphic direct summands. It follows that $L'P \oplus \tau_H T \in (\tau_H M)^{\perp}$ is a partial tilting module consisting of $n-1$ indecomposable pairwise non-isomorphic direct summands. Thus $(\tau_H M)^{\perp} \subseteq \operatorname{Fac}(L'(P) \oplus \tau_H(T))$ by Lemma 4.4.22 and the claim follows.

- Since C is not of Dynkin type A_n linearly oriented, there is no indecomposable projective-injective module. We want to prove that $\operatorname{End}_H(L'P \oplus \tau_H T) \cong \operatorname{End}_{H'}(\tau_{H'}^{-} P \oplus T)$. We consider the short exact sequence
$$0 \to P \to L'P \to (\tau_H M)^k \to 0$$
for some $k \in \mathbb{N}$. Now let X be any H-module. Then applying the functor $\operatorname{Hom}_H(X, -)$ to

the short exact sequence yields an exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_H(X,P) \to \mathrm{Hom}_H(X,L'P) \to \mathrm{Hom}_H(X,(\tau_H M)^k)\\ &\to \mathrm{Ext}^1_H(X,P) \to \mathrm{Ext}^1_H(X,L'P) \to \mathrm{Ext}^1_H(X,(\tau_H M)^k) \to 0.\end{aligned}$$

Hence we see, that if $L'P$ is injective so is $\tau_H M$, which is impossible. Thus $L'P$ is non-injective and we see that $\tau_H^-(L'P) \in M^\perp$.

Since $LH \in M^\perp$ it follows that $\mathrm{Hom}_H(LH, \tau_H M) \cong D\,\mathrm{Ext}^1_H(M, LH) = 0$, and thus we see, by setting $X = LH$ in the above exact sequence that

$$\mathrm{Hom}_H(LH, L'P) \cong \mathrm{Hom}_H(LH, P) \cong {}_{H'}P$$

where the last isomorphism follows since H' is Morita-equivalent to $\mathrm{End}_H(LH)$ and $P \in \mathrm{add}(LH)$. Therefore, we see that

$$\begin{aligned}\tau_{H'}(\tau_H^- L'P) &\cong \mathrm{Hom}_{H'}(H', \tau_{H'}(\tau_H^- L'P)) \cong \mathrm{Hom}_{H'}(LH, \tau_{H'}(\tau_H^- L'P))\\ &\cong D\,\mathrm{Ext}^1_{H'}(\tau_H^- L'P, LH) \cong D\,\mathrm{Ext}^1_H(\tau_H^- L'P, LH) \cong \mathrm{Hom}_H(LH, L'P) \cong {}_{H'}P,\end{aligned}$$

where we used that there is a homological epimorphism $H \to H'$ and $\mathrm{proj.dim}_{H'}(\tau_H^- L'P) \leq 1$. This shows that $\tau_H^- L'P \cong \tau_{H'}^- P$ as H'-modules and thus

$$\mathrm{End}_{H'}(\tau_{H'}^- P \oplus T) \cong \mathrm{End}_H(\tau_H^- L'P \oplus T) \cong \mathrm{End}_H(L'P \oplus \tau_H T).$$

- Let $H = H(C, D, \Omega)$ be such that $A = A(C, D, \Omega)$ is a Nakayama algebra. Hence C is of Dynkin type A_n and Ω is of linear orientation. Using the correspondence between preprojective H-modules and preprojective A-modules and the proof of [St, Proposition 4.2.(c)] we have the following:
  - the module $L'(P)$ is projective;
  - the module $\tau_H(T)$ has no projective direct summand;
  - $\mathrm{Hom}_H(P, T) = 0$.

  This directly implies

  $$\mathrm{End}_H(P \oplus T) \cong \mathrm{End}_H(P) \times \mathrm{End}_H(T) \cong \mathrm{End}_H(L'P) \times \mathrm{End}_H(\tau_H T) \cong \mathrm{End}_H(L'P \oplus \tau_H T),$$

  which finishes the proof of the Proposition.

□

Corollary 4.4.29. *Let $M \in \mathrm{mod}(H)$ be an indecomposable preprojective or preinjective module, such that neither M nor $\tau_H(M)$ are projective. Assume there is an equivalence of categories $M^\perp \simeq \mathrm{mod}(H')$, where $H' = H(C', D', \Omega')$ for a Cartan matrix C'. Then there is an equivalence of categories*

$$(\tau_H M)^\perp \simeq \mathrm{mod}(H'')$$

where H'' is obtained from H' by possibly applying some reflections.

Proof. Let $P \oplus T \in M^\perp \simeq \mathrm{rep}(H')$ be the projective generator as in Proposition 4.4.28. Then we know that $H'' \cong \mathrm{End}_{H'}(\tau_{H'}^-(P) \oplus T)$. It follows from

$$\mathrm{Hom}_{H'}(T, P) \cong \mathrm{Hom}_H(T, P) = 0$$

that there are no arrows in H' from the vertices corresponding to the H'-projective modules in $\mathrm{add}(P)$ to the vertices corresponding to the H'-projective modules in $\mathrm{add}(T)$. Thus it follows from Corollary 4.4.7 that

$$\mathrm{End}_{H'}(\tau_{H'}^-(P) \oplus T) \cong s_{i_j} \dots s_{i_1}(H'),$$

where $(i_1, i_2, \dots i_{n-1})$ is a $(+)$-admissible sequence for H' and j is the number of indecomposable summands of P. $\square$

4.4.6 Perpendicular categories of preprojective and preinjective modules

Corollary 4.4.30. *Let $P_i \in \operatorname{rep}(H)$ be the indecomposable projective module at vertex i. Then*

$$P_i^{\perp} \simeq \operatorname{mod}(H')$$

where H' is obtained by deleting the vertex i from H'.

Lemma 4.4.31. *Let $i \in Q_0$ be any vertex such that P_i is not injective and set $T_i = \tau_H^-(P_i)$. The module*

$$T_k = \begin{cases} P_k & \text{if there is no path from } i \text{ to } k \text{ in } Q, \\ \tau_H^-(P_k) & \text{there is a path from } i \text{ to } k \text{ in } Q. \end{cases}$$

is Ext*-projective in $T_i^{\perp}$ for all $k \neq i$*

Proof. We first show that for $k \neq i$ we have $T_k \in T_i^{\perp}$. In case there is no path from i to k we have $T_k = P_k$ and it follows that $\operatorname{Hom}_H(P_k, P_i) = 0$. Hence we see that

$$\operatorname{Ext}^1_H(\tau_H^-(P_i), P_k) \cong D\operatorname{Hom}_H(P_k, P_i) = 0.$$

Further we have $\operatorname{Hom}_H(\tau_H^-(P_i), P_k) = 0$ by [GLS1, Corollary 11.2.].

First note, that in case there is a path from i to k, the projective module P_k cannot be injective and thus $T_k = \tau_H^-(P_k)$ is not zero. It also follows that there is no path from k to i, and therefore we have $\operatorname{Hom}_H(P_i, P_k) = 0$. Hence we see

$$\operatorname{Hom}_H(\tau_H^-(P_i), \tau_H^-(P_k)) \cong \operatorname{Hom}_H(P_i, P_k) = 0$$

and

$$\operatorname{Ext}^1_H(\tau_H^-(P_i), \tau_H^-(P_k)) \cong D\operatorname{Hom}_H(\tau_H^-(P_k), P_i) = 0$$

and thus in both cases we see that $T_k \in T_i^{\perp}$.

It remains to show that $T_k \in T_i^{\perp}$ is Ext-projective. This obviously holds if T_k is projective. So assume that $T_k = \tau_H^-(P_k)$ and hence there is a path from i to k and a monomorphism $P_k \hookrightarrow P_i$. For $X \in T_i^{\perp}$ we have

$$\operatorname{Ext}^1_H(T_k, X) \cong D\operatorname{Hom}_H(X, P_k)$$

by the Auslander-Reiten formula. If there exists a non-zero morphism $f\colon X \to P_k$, the composition $X \to P_k \hookrightarrow P_i$ is a non-zero homomorphism from X to P_i showing that

$$\operatorname{Ext}^1_H(T_i, X) \cong D\operatorname{Hom}_H(X, P_i)$$

is not zero, which is a contradiction. Here we used the Auslander-Reiten-formula and that $\operatorname{proj.dim}(T_i) \leq 1$. This finishes the proof of the Lemma. $\square$

Corollary 4.4.32. *Let $T_i = \tau_H^-(P_i)$ as in the above lemma. Then there is an equivalence of categories*

$$T_i^{\perp} \simeq \operatorname{mod}(H')$$

where H' is obtained from H by possibly applying reflections and deleting the vertex i from it.

Proof. Let $T_k \in T_i^{\perp}$ as in the last lemma and set $T = \oplus_{k \neq i} T_k$. Then $T \oplus T_i$ is a tilting H-module and $T \in T_i^{\perp}$ is Ext-projective. Thus it follows by Lemma 4.4.22 that T is a projective generator in $T_i^{\perp}$. Therefore, we have

$$T_i^{\perp} \simeq \operatorname{mod}(\operatorname{End}_H(T)^{\mathrm{op}}).$$

We can choose a $(+)$-admissible sequence $(1, 2, \ldots, i, \ldots, n)$ of H such that $k \leq i$ if and only if there is a path from i to k. Then it follows that

$$\mathrm{End}_H(T \oplus T_i)^{\mathrm{op}} = \mathrm{End}_H(\tau_H^-(P_1 \oplus \cdots \oplus P_i) \oplus P_{i+1} \oplus \cdots \oplus P_n) \cong s_i s_{i-1} \ldots s_1 H$$

by Corollary 4.4.7 and therefore $\mathrm{End}_H(T)^{\mathrm{op}} \cong s_i s_{i-1} \ldots s_1 H'$, where $s_i s_{i-1} \ldots s_1 H'$ is obtained from $s_i s_{i-1} \ldots s_1 H$ by deleting the vertex i from it. $\square$

Lemma 4.4.33. *Let $M \in \mathrm{mod}(H)$ be a partial indecomposable tilting module. Further let $P \in M^\perp$ be a projective generator, $H' = \mathrm{End}_H(P)^{\mathrm{op}}$ and $Q \in M^\perp$ an injective cogenerator and $H'' = \mathrm{End}_H(Q)^{\mathrm{op}}$. Then there are equivalences of categories*

$$\mathrm{mod}(H') \simeq M^\perp \simeq \mathrm{mod}(H'').$$

In particular, $\mathrm{End}_H(P)^{\mathrm{op}}$ and $\mathrm{End}_H(Q)^{\mathrm{op}}$ are Morita-equivalent.

Proof. We already know that there is an equivalence of categories $\mathrm{mod}(H') \simeq M^\perp$. Now ${}_HQ \in \mathrm{mod}(H')$ is an injective cogenerator if and only if $D({}_HQ) \in \mathrm{mod}(H'^{\mathrm{op}})$ is a projective generator. Hence there is an equivalence of categories

$$\mathrm{mod}(H'^{\mathrm{op}}) \simeq \mathrm{mod}(\mathrm{End}_{H'^{\mathrm{op}}}(D({}_HQ))^{\mathrm{op}})$$

which is equivalent to the existence of an equivalence of categories

$$\mathrm{mod}(H') \simeq \mathrm{mod}(\mathrm{End}_{H'^{\mathrm{op}}}(D({}_HQ))).$$

Now the duality D induces an isomorphism of algebras $\mathrm{End}_{H'}(Q) \cong \mathrm{End}_{H'^{\mathrm{op}}}(D({}_HQ))^{\mathrm{op}}$ or equivalently an isomorphism of algebras $\mathrm{End}_{H'}(Q)^{\mathrm{op}} \cong \mathrm{End}_{H'^{\mathrm{op}}}(D({}_HQ))$, which proves the lemma. $\square$

Lemma 4.4.34. *The module*

$$T_k = \begin{cases} I_k & \text{if there is no path from } k \text{ to } i, \\ \tau_H(I_k) & \text{if there is a path from } k \text{ to } i, \end{cases}$$

is Ext*-injective in $I_i^\perp$ for all $k \neq i$.*

Proof. We first show, that $T_k \in I_i^\perp$ for all $k \neq i$. For the injective module I_k we have

$$\begin{aligned} I_k \in I_i^\perp &\Leftrightarrow \mathrm{Hom}_H(I_i, I_k) = 0 \\ &\Leftrightarrow \underline{\dim}(I_i)_k = 0 \\ &\Leftrightarrow \text{ there is no path from } k \text{ to } i. \end{aligned}$$

Note that if there is a path from k to i the injective module I_k cannot be projective and hence $\tau_H(I_k)$ is a non-zero τ-locally free module and thus $\mathrm{Hom}_H(I_i, \tau_H(I_k)) = 0$ by [GLS1, Corollary 11.2.]. Therefore, in this case we have

$$\begin{aligned} \tau_H(I_k) \in I_i^\perp &\Leftrightarrow \mathrm{Ext}^1_H(I_i, \tau_H I_k) = 0 \\ &\Leftrightarrow D\,\mathrm{Hom}_H(I_k, I_i) = 0 \\ &\Leftrightarrow \text{ there is no path from } i \text{ to } k. \end{aligned}$$

Since we are in the case that there is a path from k to i and Q° is acyclic, the claim follows.

Now if $T_k = I_k$ this is obviously Ext-injective in $I_i^\perp$. So assume that $T_k = \tau_H I_k$ and that there is a path from k to i. Then there is an epimorphism $I_i \to I_k$. For $X \in I_i^\perp$ we have $\mathrm{Hom}_H(I_i, X) = 0$ and thus we must also have

$$\mathrm{Ext}^1_H(X, \tau_H I_k) \cong D\,\mathrm{Hom}_H(I_k, X) = 0$$

which finishes the proof of the lemma. □

Corollary 4.4.35. *For every vertex i there is an equivalence of categories*

$$I_i^{\perp} \simeq \mathrm{mod}(H')$$

where H' is obtained from H after possibly applying reflections and deleting the vertex i from it.

Proof. Let $T_k \in I_i^{\perp}$ as in the last lemma and $T = \oplus_{k \neq i} T_k$. Then $I_i \oplus T$ is a tilting H-module and thus it follows that T is an injective cogenerator in $I_i^{\perp}$. We can choose an $(-)$-admissible sequence $(1, 2, \ldots, i, \ldots, n)$ of H such that $k \leq i$ if and only if there exists a path from k to i. Then it follows that

$$I_i^{\perp} \simeq \mathrm{mod}(\mathrm{End}_H(\tau_H(I_1 \oplus \cdots \oplus I_{i-1}) \oplus I_{i+1} \oplus \cdots \oplus I_n)^{\mathrm{op}})$$

and we have

$$\begin{aligned}
&\mathrm{End}_H(\tau_H(I_1 \oplus \cdots \oplus I_{i-1}) \oplus I_i \oplus I_{i+1} \oplus \cdots \oplus I_n) \\
=&\,\mathrm{End}_H(D\tau_H^{-}(e_1 H \oplus \cdots \oplus e_{i-1} H) \oplus e_i H \oplus e_{i+1} H \oplus \cdots \oplus e_n H) \\
\cong&\,\mathrm{End}_{H^{\mathrm{op}}}(\tau_H^{-}(e_1 H \oplus \cdots \oplus e_{i-1} H) \oplus e_i H \oplus e_{i+1} H \oplus \cdots \oplus e_n H)^{\mathrm{op}} \\
\cong&\, s_{i-1} s_{i-2} \ldots s_1 H^{\mathrm{op}}
\end{aligned}$$

where for the last isomorphism we used that $(1, 2, \ldots, i, \ldots, n)$ is a $(+)$-admissible sequence in H^{op} and that $e_j H$ is the projective indecomposable H^{op}-module at vertex j. Thus we see that

$$I_i^{\perp} \simeq \mathrm{mod}(s_{i-1} s_{i-2} \ldots s_1 H')$$

where $s_{i-1} s_{i-2} \ldots s_1 H'$ is obtained from $s_{i-1} s_{i-2} \ldots s_1 H$ by deleting the vertex i. □

Theorem 4.4.36. *Let $M \cong \tau_H^k(I_i) \in \mathrm{rep}(H)$ be an indecomposable preinjective module. Then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{rep}(H'),$$

where H' is obtained from H by possibly applying a series of reflections to H and deleting the vertex i from it.

Proof. The theorem is true for $M = I_i$ by Corollary 4.4.35. The conclusion follows for $M \cong \tau_H^k(I_i)$ by inductively applying Corollary 4.4.29. □

Corollary 4.4.37. *If C is of Dynkin type and $M \in \mathrm{mod}(H)$ an indecomposable preprojective module then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{mod}(H')$$

where H' is obtained from H by changing its orientation and deleting a vertex from it.

Corollary 4.4.38. *Let $M \cong \tau_H^{-k}(P_i) \in \mathrm{mod}(H)$ be an indecomposable preprojective module. Then there is an equivalence of categories*

$$^{\perp}M \simeq \mathrm{mod}(H')$$

where H' is obtained from H by possibly applying a series of reflections to H and deleting the vertex i from it.

The above corollary follows directly from the theorem for the corresponding preinjective module and duality. Note that this only yields the result for the left-perpendicular category of a preprojective module M. However, we strongly conjecture that the same holds true for the right-perpendicular category of M. In fact we will prove this for the symmetrizable Cartan matrix of type $\tilde{C}_n$ in the next chapter.

4.5 The example $\tilde{C}_n$

4.5.1 The algebra $H(C, D, \Omega)$ for C of type $\tilde{C}_n$

In this section we study the perpendicular categories of indecomposable partial tilting modules over the algebra $H = H(C, D, \Omega)$, where C is a Cartan matrix of extended Dynkin type $\tilde{C}_n$ for $n \geq 3$ and D a minimal symmetrizer. We study the algebras of this type more closely for two reasons. Firstly, the Cartan matrix C is not of Dynkin type, and thus there are rigid modules, which are not preprojective or preinjective. Secondly, if we choose the minimal symmetrizer, the algebra H is a finite-dimensional string algebra.

We show, that in this case the perpendicular category of a preprojective indecomposable module is equivalent to $\mathrm{mod}(H')$, where H' is obtained from H by applying a series of reflections and deleting one vertex from it. Furthermore, we classify all indecomposable rigid modules and show that they are τ-locally free. It follows from this classification that for any indecomposable rigid H-module M, the category $M^{\perp}$ is equivalent to $\mathrm{mod}(H')$, where H' is a 1-Iwanaga-Gorenstein algebra with $n-1$ simple modules.

Recall that the extended Cartan matrix of type $\tilde{C}_n$ for $n \geq 3$ is given by the $n \times n$-matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and its minimal symmetrizer D is given by $\mathrm{diag}(2, 1, \ldots, 1, 2)$. The corresponding graph is given by

$$\tilde{C}_n \qquad \cdot \overset{(2,1)}{\text{———}} \cdot \text{———} \cdots \text{———} \cdot \overset{(1,2)}{\text{———}} \cdot$$

and if we choose the linear orientation $\Omega = \{(1,2), (2,3), \ldots, (n-1, n)\}$, then $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l] \arrow[loop above, "\varepsilon_2"] & \cdots \arrow[l] & n \arrow[l] \arrow[loop above, "\varepsilon_n"] \end{tikzcd}$$

with relations $\varepsilon_2 = \cdots = \varepsilon_{n-1} = 0$ and $\varepsilon_1^2 = \varepsilon_n^2 = 0$.

From now on let $H = H(C, D, \Omega)$ be of type $\tilde{C}_n$, with minimal symmetrizer and fixed but arbitrary orientation. Choosing the symmetrizer minimal, ensures that H is in fact a string algebra, and thus all indecomposable modules are string or band modules.

4.5.2 The component containing the preprojective and preinjective modules

Recall that for any arrow α from i to j there is an Auslander-Reiten sequence

$$0 \to U(\alpha) \to N(\alpha) \to V(\alpha) \to 0$$

where $U(\alpha)$ is given by the unique maximal path w_α starting at vertex i but not with the arrow α and $V(\alpha)$ is given by the unique maximal path ending in the vertex j but not with the arrow α.

Lemma 4.5.1. *The preprojective and the prepinjective vertices lie in the same connected component of the Auslander-Reiten quiver of H. More precisely, let*

$$0 \to U(\varepsilon_i) \to N(\varepsilon_i) \to V(\varepsilon_i) \to 0$$

be the Auslander-Reiten sequence with indecomposable middleterm, corresponding to the arrow ε_i for $i = 1$ or $i = n$ respectively. Then there are irreducible monomorphisms $C_i \to P_i$ and irreducible epimorphisms $I_i \to A_i$ for $i = 1$ or $i = n$, respectively.

Proof. We will just consider the case $i = 1$, the case $i = n$ is similar. The module $U(\varepsilon_1)$ is the string module of the longest directed string ending in the vertex 1 and not with ε_1, and the module $V(\varepsilon_1)$ is the string module of the longest directed string starting in the vertex 1 and not with ε_1. Note that one of them is the simple module S_1 depending on the orientation of the edge $\{1, 2\}$. Now consider the Auslander-Reiten sequence ending in $U(\varepsilon_1)$. One of its two indecomposable middleterms is obtained by attaching a cohook to the socle 1, starting with ε_1. But this yields exactly the indecomposable injective module with socle S_1. Dually the Auslander-Reiten sequence starting in $V(\varepsilon_1)$ has one middleterm which is obtained from $V(\varepsilon_1)$ by attaching a hook which starts with ε_1. This yields a string module with simple top S_1 and the longest directed paths starting in 1, thus exactly the indecomposable projective module P_1. □

Example 4.5.2. Let $H = H(C, D, \Omega)$ be of type $\tilde{C}_3$ with minimal symmterizer D and orientation $\Omega = \{(2, 1), (3, 2)\}$. In Figure 4.4 we have depicted part of the Auslander-Reiten quiver of H containing the indecomposable projective and injective modules and also the Auslander-Reiten sequences with indecomposable middleterm corresponding to the arrow ε_i for $i = 1, 2$. Note that this is merely a part of the component. In the following we will describe the whole component more precisely.

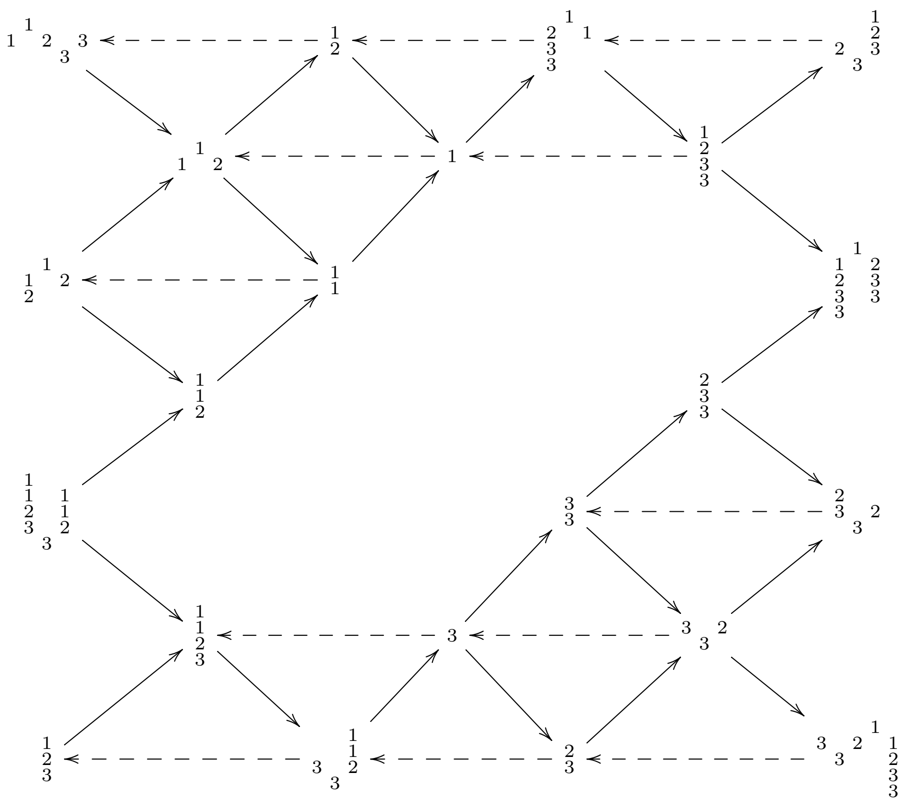


Figure 4.4: Part of the Auslander-Reiten quiver of $H(C, D, \Omega)$ of type $\tilde{C}_3$ with D minimal.

Lemma 4.5.3. *All irreducible morphisms starting in preprojective modules are monomorphisms. All irreducible morphisms ending in preinjective modules are epimorphisms.*

Proof. It is always possible to attach hooks (respectively cohooks): if a longest directed string ends at a vertex $1 < i < n$, then the next arrow adjacent to i points in the other direction and thus can be the beginning of the new hook. If a longest directed string of a preprojective module ends in the vertex 1, it ends with the arrow ε_1. Then the arrow from 2 to 1 is the start of the new hook. A similar argument holds for the vertex n. □

Let M be a rigid indecomposable module and denote the corresponding vertex in the Auslander-Reiten quiver also by M. Since H is a string algebra we know by [BR] that there are at most two arrows coming in and at most two arrows going out of M. We call the longest linear paths starting or ending in M *rays starting or ending in* M. Furthermore, we call the vertices corresponding to the modules $P_1, \tau_H^{-1}(P_1), \tau_H^{-2}(P_2), \ldots$ and $P_n, \tau_H^{-1}(P_n), \tau_H^{-2}(P_n), \ldots$ the *preprojective border* of the component containing the preprojective modules. Note, that these modules do not necessarily lie on the border of the component: they have neighbours, that are not τ-locally free.

Let Q° be the quiver $Q(C, D, \Omega)$ without loops and $(Q^\circ)^{\text{op}}$ be its opposite quiver. We denote by $\mathbb{Z}A_\infty^\infty - (Q^\circ)^{\text{op}}$ the infinite quiver obtained from $\mathbb{Z}A_\infty^\infty$ after deleting one copy of $(Q^\circ)^{\text{op}}$ and all its adjacent arrows in it.

Proposition 4.5.4. *The component $\mathcal{C}$ containing the preprojective and preinjective vertices of the Auslander-Reiten quiver of H is of the form $\mathbb{Z}A_\infty^\infty - (Q^\circ)^{\text{op}}$.*

Proof. We already know, that the preinjective and preprojective modules sit in the same component and that they are pairwise non-isomorphic. We also know, that all modules, except of the ones connecting the preprojective and preinjective component, have Auslander-Reiten sequences with exactly two middle terms. Thus, what is left to show is, that this component does not form a tube. Hence we have to show, that modules sitting above the preprojectives/preinjectives are not isomorphic to modules sitting below them.

Assume that 1 is a sink in Q° such that $P_1 = E_1$. We consider the strings $C^{(i)}$ of the modules M_i in the ray, that starts in $M_0 := P_1$ and goes upwards. The string $C^{(i+1)}$ is obtained by attaching a hook to the right side of $C^{(i)}$, which is always possible. Also note that $\tau_H(M_1) \cong S_1$. We also consider the strings $D^{(i)}$ of the modules N_i in the ray, that starts in $N_1 := S_1$ and goes upwards. Thus we see, that the module M_i always has the simple module S_1 in the top and $N_i \cong \tau_H(M_i)$ always has the simple module S_1 in the socle. Thus M_i cannot be τ-rigid for any $i \geq 1$. Now assume that M_i was isomorphic to one of the modules sitting below the preprojective modules. Then the ray starting in P_1 and going upwards, at some point goes through the preprojective zone. This is a contradiction, since none of the M_i are τ-rigid. Note, that it could not sit below the preinjective zone, because there the irreducible morphisms are epimorphisms.

If 1 is not a sink a similar argument can be used. Then it is the longest directed string starting with the arrow from 1 to 2, which shows that the modules M_i are not τ-rigid. □

4.5.3 Perpendicular categories of preprojective modules

Lemma 4.5.5. *Let $T_1, T_2 \in \text{mod}(H)$ be indecomposable preprojective modules. If*

$$\dim \text{Hom}_H(T_1, T_2) \neq 0,$$

then there exists a monomorphism from T_1 to T_2.

Proof. Let A be the corresponding hereditary algebra and $N_1, N_2 \in \text{mod}(A)$ the preprojective A-modules with

$$\underline{\dim}(N_i) = \underline{\text{rank}}(T_i).$$

Then since $\dim \text{Hom}_A(N_1, N_2) \neq 0$, there is a path from N_1 to N_2 in the Auslander-Reiten quiver of A. Since we can embed the unvalued quiver corresponding to the preprojective component of A into the preprojective component of H, we see that there is a path from T_1 to T_2 only passing through preprojective modules. It follows from Lemma 4.5.3, that the composition of the irreducible morphisms corresponding to the arrows along the path, yields a monomorphism from T_1 to T_2. □

Lemma 4.5.6. *Let T_i be an indecomposable preprojective module in the τ_H^--orbit of P_i, but not P_i itself. Further suppose that there exist indecomposable modules T_k in the τ_H^--orbit of P_k, such that $T_k \in T_i^\perp$ is* Ext*-projective for $k \neq i$ and $T = \oplus T_k$ is a tilting module. Also assume that if* $\mathrm{Hom}_H(T_k, T_i) = 0$*, then $T_k = P_k$. Then we have that*

$$T'_k = \begin{cases} T_k & \text{if } \mathrm{Hom}_H(T_k, T_i) = 0 \\ \tau_H^-(T_k) & \text{if } \mathrm{Hom}_H(T_k, T_i) \neq 0 \end{cases}$$

is Ext*-projective in $\tau_H^-(T_i)^\perp$ for $k \neq i$ and $T' = \oplus_k T'_k$ is a tilting module. Furthermore, if* $\mathrm{Hom}_H(T'_k, T'_i) = 0$ *then $T'_k = P_k$.*

Proof. We first show that $T'_k \in (T'_i)^\perp$. This follows from the dual of Lemma 4.4.25 if $T'_k = \tau_H^-(T_k)$. If $\mathrm{Hom}_H(T_k, T_i) = 0$ and thus $T'_k = T_k = P_k$ is projective, then we have $\mathrm{Hom}_H(\tau_H^-(T_i), P_k) = 0$ and

$$\mathrm{Ext}_H^1(\tau_H^-(T_i), T_k) \cong D\,\mathrm{Hom}_H(T_k, T_i) = 0$$

by assumption.

Assume that $T'_k = \tau_H^-(T_k)$. We have to show that $\mathrm{Ext}_H^1(\tau_H^-(T_k), X) = 0$ for all $X \in \tau_H^-(T_i)^\perp$ which are τ-locally free. If X is non-projective this follows from the dual of Lemma 4.4.25. Suppose that $X = P_j$ is projective. Then we have

$$D\,\mathrm{Hom}_H(P_j, T_i) \cong \mathrm{Ext}_H(\tau_H^-(T_i), P_j) = 0$$

which is equivalent to $\underline{\dim}(T_i)_j = 0$. Since all irreducible morphisms starting in preprojective modules are monomorphisms this implies that $\underline{\dim}(\tau_H^-(T_i))_j = 0$ and it follows that $P_j \in T_i^\perp$. Since by assumption there is a monomorphism $T_k \to T_i$ and since $\mathrm{Hom}_H(P_j, T_i) = 0$, it follows that

$$\mathrm{Ext}_H^1(\tau_H^- T_k, P_j) \cong D\,\mathrm{Hom}_H(P_j, T_k) = 0.$$

□

Proposition 4.5.7. *Let $T_i \cong \tau^{-s}(P_i) \in \mathrm{mod}(H)$ be an indecomposable preprojective module. Then there is an equivalence of categories*

$$T_i^\perp \simeq \mathrm{mod}(H')$$

where H' is obtained from H by applying a series of reflections and deleting the vertex i from it.

Proof. We know that this is true for $T_i = P_i$ and $T_i = \tau_H^-(P_i)$ and in the latter case by Lemma 4.4.31 there exist modules $T_k \in T_i^\perp$ as in Lemma 4.5.6. Now we can proceed by induction, using Lemma 4.5.6 and Corollary 4.4.7. □

Corollary 4.5.8. *Let M be a preprojective module, say $M \cong \tau^{-s}(P_i)$, for some $1 \leq i \leq n$ and $s \in \mathbb{N}$. Then there is an equivalence of categories*

$$M^\perp \simeq \mathrm{rep}(H_{i-1}) \times \mathrm{rep}(H_{n-i})$$

where $H_0 = \emptyset$ and $H_j = H(C, D, \Omega)$ is of Dynkin type C_j with minimal symmetrizer. If the rays ending in M intersect the preprojective border, then H_j is of linear orientation.

Proof. It only remains to be proven, that if the rays ending in M intersect the preprojective border, then H_j is of linear orientation. We claim that the preprojective modules T_k, corresponding to the vertices lying on the rays ending in M, are a progenerator of $M^\perp$.

We first show that in fact $T_k \in M^\perp$ for all k. There is a path from T_k to M in the Auslander-Reiten quiver of H and thus in particular $\mathrm{Hom}_H(T_k, M) \neq 0$ for all k, implying that

$$\mathrm{Hom}_H(M, T_k) = 0.$$

Note, that there are no paths from T_k to $\tau(M)$ in the Auslander-Reiten quiver of H. Therefore, we have $\mathrm{Ext}^1(M, T_k) \cong D\,\mathrm{Hom}(T_k, \tau(M)) = 0$ for all k. Hence we have $T_k \in M^\perp$ for all k.

We will now prove that T_k is Ext-projective in $M^\perp$ for all k. The monomorphism $T_k \to M$ yields a short exact sequence

$$0 \to T_k \to M \to K_k \to 0$$

for all k, where K_k is a locally free module. Applying the functor $\mathrm{Hom}(-, N)$ for any $N \in M^\perp$ yields the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_H(K_k, N) &\to \mathrm{Hom}_H(M, N) \to \mathrm{Hom}_H(T_k, N) \\ \xrightarrow{\delta} \mathrm{Ext}^1_H(K_k, N) &\to \mathrm{Ext}^1_H(M, N) \to \mathrm{Ext}^1_H(T_k, N) \to 0 \end{aligned}$$

where we use, that K_k, M, T_k are all of projective dimension at most 1. Since by assumption $\mathrm{Ext}^1_H(M, N) = 0$, it follows that $\mathrm{Ext}^1_H(T_k, N) = 0$ for all k.

In order to see, that $\bigoplus_k T_k$ generates $M^\perp$, it is now enough to note that

$$T = M \oplus \bigoplus_k T_k$$

is a tilting module. All summands have projective dimension at most 1 and also $\mathrm{Ext}^1(T, T) = 0$ can be easily seen. $\square$

Example 4.5.9. Let $H = H(C, D, \Omega)$ be of type $\tilde{C}_7$, given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 5 \arrow[l] \arrow[r] & 6 \arrow[r] & 7 \arrow[loop above, "\varepsilon_7"] \end{tikzcd}$$

with relations $\varepsilon_1^2 = \varepsilon_7^2 = 0$. Figure 4.5 shows part of the component of the Auslander-Reiten quiver containing the projective and injective modules. The black vertices correspond to preprojective and preinjective modules. The white vertices correspond to modules which are not locally free. Furthermore, we have indicated the preprojective modules that are in $M^\perp$ for $M = \tau_H^{-3}(P_5)$. We have also indicated the vertices lying on the rays ending in M, whose corresponding preprojective modules form a projective generator of $M^\perp$. Note, that there are regular modules in $M^\perp$, that cannot be seen in this picture.

4.5.4 The regular modules

Lemma 4.5.10. *There is a unique non-homogeneous tube of rank $n-1$ of regular modules. The modules at the mouth of this tube are given by the endterms of the Auslander-Reiten-sequences with indecomposable middleterm, corresponding to the $n-1$ arrows which are not loops.*

Proof. We number the $n-1$ arrows in Q which are not loops in the following way: first we number the arrows going from left to right increasingly starting on the left side, and then we order the arrows going from right to left increasingly, starting on the right side. It is then easy to see that the modules $U(\alpha_i)$ are pairwise non-isomorphic and

$$U(\alpha_i) \cong V(\alpha_{i+1})$$

for $i = 1, \ldots, n-2$ and

$$U(\alpha_{n-1}) \cong V(\alpha_1)$$

showing that $\tau^{n-1}(V(\alpha_i)) \cong V(\alpha_i)$ for all $i = 1, \ldots, n-1$.

Now the modules at the mouth of this tube are obviously locally free, since we just take longest direct paths in the quiver. Thus, if we reach one of the ends, we also walk the loop. Since the

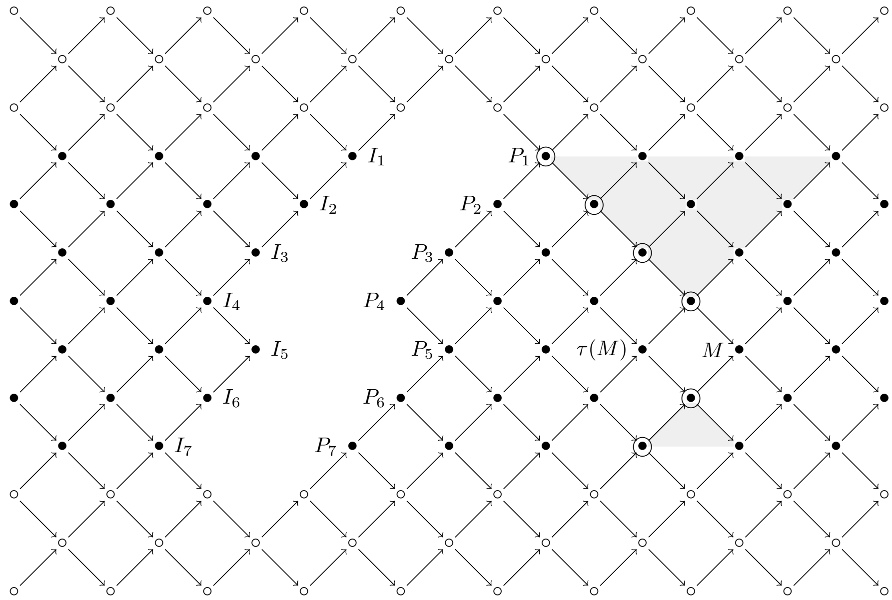
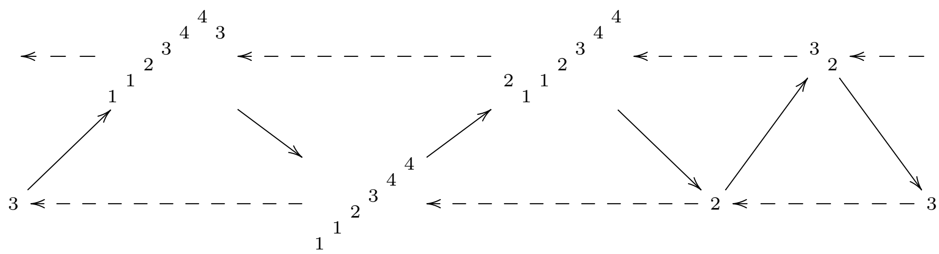


Figure 4.5: Here we see part of the component containing the projective and injective vertices.

locally free modules are extension closed, it then follows inductively, that all the modules in the tube are locally free.

The uniqueness of the tube follows from the description of the Auslander-Reiten quiver of string algebras in [BR]. $\square$

Example 4.5.11. Let H be of type $\tilde{C}_4$ and linearly ordered, say $\Omega = \{(1,2),(2,3),(3,4)\}$. In the picture below we have depicted the first two rows of the tube of rank 3.



Corollary 4.5.12. *The full subcategory consisting of the tube of rank $n-1$ does not depend on the orientation Ω.*

Proof. This follows directly from Proposition 4.4.6 and the uniqueness of the tube of rank $n-1$. $\square$

Lemma 4.5.13. *Let M be an indecomposable H-module corresponding to one of the vertices in the first $n-2$ rows of the tube of rank $n-1$. Then M is an indecomposable partial tilting module such that $\operatorname{End}_H(M)$ is a skew-field.*

Proof. By the last Corollary it is enough to consider the linear orientation. Since the modules are locally free, it is enough to consider one module from each τ-orbit. Choosing the orientation Ω linearly, it is easy to compute, that we have a ray of modules in the tube with rank vectors

$$(0,1,0,\ldots,0),(0,1,1,0,\ldots,0),\ldots,(0,1,1,\ldots,1,0)$$

and obviously the corresponding modules cannot have non-trivial endomorphisms. Furthermore it is not hard to see, that all of these rank vectors r satisfy

$$q_H(r) = 1 = \dim \operatorname{End}_H(M(r)),$$

where q_H denotes the quadratic form as defined in [GLS1]. Therefore, all these modules are rigid. □

Conjecture 4.5.14. *Let M be a regular module in the ith row of the tube of rank $n-1$ for some $1 \leq i \leq n-2$. Then*

$$M^{\perp} \simeq \operatorname{rep}(H_{i-1}) \times \operatorname{rep}(\tilde{H}_{n-i})$$

where $H_0 = \tilde{H}_0 = \emptyset$ and $H_j = H(C, D, \Omega)$ is of type Dynkin type A_j and $\tilde{H}_j = H(C, D, \Omega)$ is of type extended Dynkin type $\tilde{C}_j$, with minimal symmetrizer and some orientation Ω.

Assume that $j > 1$ and consider a maximal chain of irreducible monomorphisms

$$T_1 \to T_2 \to \cdots \to T_i = M,$$

where T_j is in the jth row of the tube. It is rather easy to see, that $T_j \in M^{\perp}$ for $1 \leq j < i$ and is Ext-projective in $M^{\perp}$. We believe that there exist $n-i$ preprojective modules $T_{i+1}, \ldots, T_n$, such that

$$T = \bigoplus_{j=1}^{i-1} T_j \oplus M \oplus \bigoplus_{j=i+1}^{n} T_j$$

is a tilting module and

$$\bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i+1}^{n} T_j$$

is a projective generator in $M^{\perp}$. The endomorphism algebra of $\bigoplus_{j=1}^{i-1} T_j$ is of type A_n with linear orientation. We know, that there are no homomorphisms from regular to preprojective modules. Furthermore, there can be no homomorphisms from any preprojective module $N \in M^{\perp}$ to any T_j for $1 \leq j < i$. This follows inductively: The module T_1 is the endterm of an Auslander-Reiten sequence with indecomposable middleterm, which can be embedded into $\tau(M)$. Thus, if $\operatorname{Hom}(N, T_1) \neq 0$, it follows that

$$\operatorname{Ext}^1(M, N) \cong D \operatorname{Hom}(N, \tau(M)) \neq 0$$

contradicting the assumption that $N \in M^{\perp}$.

Example 4.5.15. Let $H = H(C, D, \Omega)$ be of type $\tilde{C}_7$, given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 5 \arrow[l] \arrow[r] & 6 \arrow[r] & 7 \arrow[loop above, "\varepsilon_7"] \end{tikzcd}$$

with relations $\varepsilon_1^2 = \varepsilon_7^2 = 0$. Let $M = \tau_H^{-3}(P_5)$. Figure 4.6 shows the unique tube in the Auslander-Reiten quiver of H. The vertices indicated by black cycles correspond to rigid regular modules. The locally free modules in $M^{\perp}$ are in the gray shaded area.

4.5.5 Classification of rigid modules

Lemma 4.5.16. *Let M be an indecomposable rigid H-module. Then M is locally free and thus τ-locally free.*

Proof. First note, that band modules cannot be rigid: Band modules are contained in homogeneous tubes, that is a band module is isomorphic to its Auslander-Reiten translate. Therefore, we can

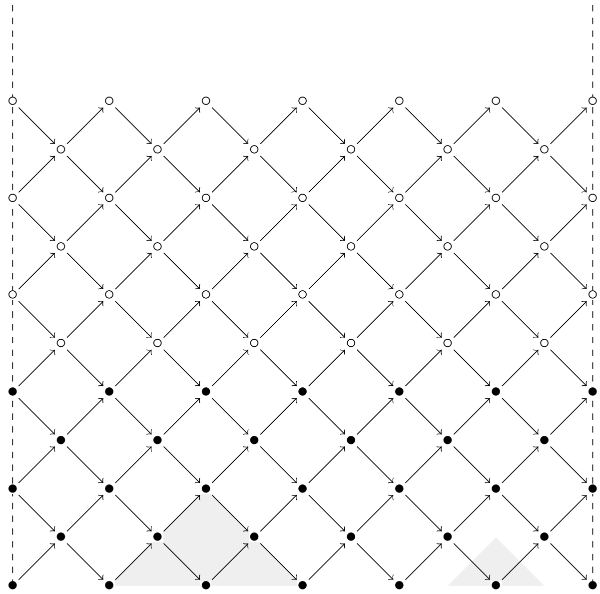
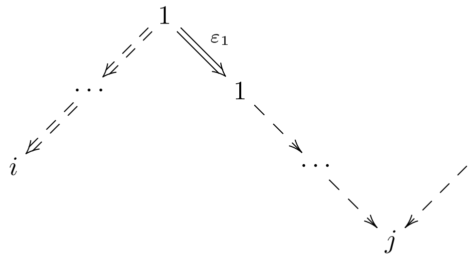


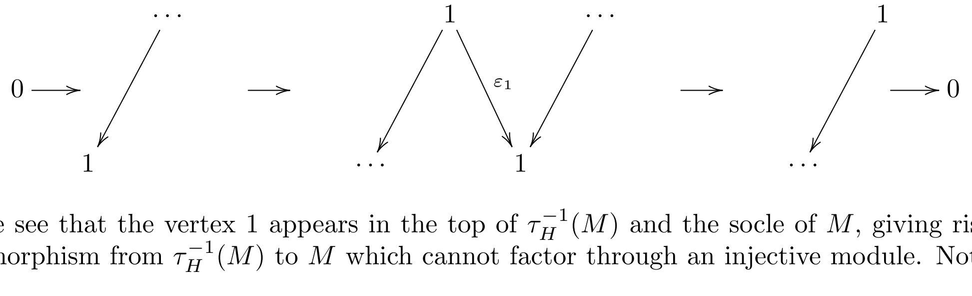
Figure 4.6: The tube where the locally free modules in $M^{\perp}$ are in the gray shaded area.

assume that M is a string module. If M is not locally free, by symmetry we can assume without loss of generality that M_1 is not a free H_1-module. This implies that the string starts or ends with the vertex one, which is not followed or preceded by the letters ε_1 or ε_1^{-1}. Now first assume that the Auslander-Reiten sequence starting in M has two middle terms. Thus in the construction of the string corresponding to $\tau_H^{-1}M$ one adds a hook to the string having the vertex 1 in the top as depicted in the following picture:



here the double arrows picture the added hook and $i \geq j$. Now it is easy to see, that there is a morphism from $\tau_H^{-1}M$ to M. It is left to check, that this does not factor through an injective module. The only possible candidate would be I_i, and obviously this is impossible.

If the Aulander-Reiten sequence starting in M has only one middle term, it can be pictured as follows:



and we see that the vertex 1 appears in the top of $\tau_H^{-1}(M)$ and the socle of M, giving rise to a homomorphism from $\tau_H^{-1}(M)$ to M which cannot factor through an injective module. Note that in the places of the dots, in the picture there does not appear the vertex one, but if one of the strings continues, it will be with the vertex 2.

After working out the easy combinatorics in this case, we noticed, that it could be treated with the more general work in [Z]. With the notation of that paper, it is easy to see that the string corresponding to M has self-extensions. More precisely, we are in the case of [Z, Corollary 2.6] part 1 and 2, where the arrow in our case is ε_1. $\square$

A combinatorial model to picture string modules In order to help visualize the strings, we would like to introduce the following combinatorial model: we consider infinite copies of $Q^{\circ}(C,\Omega)$ glued together with alternating direction, thus yielding an A_{∞}^{∞} quiver. We draw an edge, not an arrow, where we glue the quivers at its start- or endpoints. This should indicate that once we reach the vertex 1 or the vertex n, it can be followed by the corresponding loop or its inverse. Thus, an *interval* that is a finite connected subquiver in this A_{∞}^{∞} quiver, can correspond to different strings and string modules, but they all have the same dimension or rank vector. Also note, that in a string corresponding to a locally free module, once the vertices 1 or n are reached, it is followed by the corresponding loop, or its inverse.

Example 4.5.17. Let us consider the example where $Q^{\circ}(C,\Omega)$ is the quiver

$$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4$$

which in the model is glued to

$$\cdots\, 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow\!\!\!\!- \; 4 \longrightarrow 3 \longrightarrow 2 \longleftarrow 1 \;-\!\!\!- \; 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \;-\!\!\!-\; 4 \,\cdots$$

and we can consider for example the following intervals: The interval

$$\cdots\, 1 -[\!\!\to 2 \longleftarrow 3 \longleftarrow 4 \;-\!\!\!-\; 4 \longrightarrow 3 \longrightarrow 2 \longleftarrow 1 \;-\!\!\!-\; 1 \longrightarrow 2 \leftarrow\!\!]- 3 \longleftarrow 4 \;-\!\!\!-\; 4 \,\cdots$$

corresponds to four different string modules all with dimension vector $(2,3,2,2)$ and rank vector $(1,3,2,1)$ if the modules are locally free. The interval

$$\cdots\, 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 -[- 4 \longrightarrow 3 \longrightarrow 2 \longleftarrow 1 \;-\!\!\!-\; 1 \longrightarrow 2 \leftarrow\!\!]- 3 \longleftarrow 4 \;-\!\!\!-\; 4 \,\cdots$$

corresponds to two different string modules, which cannot be locally free, thus in particular in this case are not rigid and have dimension vector $(2,2,1,1)$. The interval

$$\cdots\, 1 -[\!\!\to 2 \longleftarrow 3 \leftarrow\!\!]- 4 \;-\!\!\!-\; 4 \longrightarrow 3 \longrightarrow 2 \longleftarrow 1 \;-\!\!\!-\; 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \;-\!\!\!-\; 4 \,\cdots$$

corresponds to one string module, which is locally free and has the same dimension as rank vector $(0,1,1,0)$.

Proposition 4.5.18. *The preprojective, preinjective and the regular modules in the first $n-2$ rows of the rank $n-1$ tube are a complete list of indecomposable rigid modules.*

Proof. It follows from the geometry of extension varieties that if there exists a rigid indecomposable module with a given rank vector, it is unique up to isomorphism. Thus, it is enough to check that for any rank vector, for which there might exist a rigid string module, there is a module in our list, with given rank vector. Since we have already shown that any rigid module is locally free, it is enough to consider intervals whose bordering arrows, are in fact arrows and not edges.

We distinguish between the following three types of intervals:

$$-\!\!\!-[\!\!\to \cdots \leftarrow\!\!]-\!\!\!-$$

$$\longleftarrow\!\![- \cdots -]\!\!\longrightarrow$$

$$-\!\!\!-[\!\!\to \cdots -]\!\!\longrightarrow \qquad \longleftarrow\!\![- \cdots \leftarrow\!\!]-\!\!\!-$$

and claim, that for the first type, there exists a preprojective module with corresponding rank vector, for the second there exists a preinjective and for the third a regular module in the tube of rank $n-1$.

Let us consider an interval of the first type. If it is possible to move both borders of the interval towards each other, without them crossing, such that we obtain a shorter interval of the same type within

$$-[\rightarrow \cdot \longleftarrow \cdot \leftarrow\cdots\cdots \cdot \longleftarrow \cdot -[\rightarrow \qquad \cdots \qquad \leftarrow]- \cdot \longrightarrow \cdot \cdots\cdots\rightarrow \cdot \longrightarrow \leftarrow]-$$

then this process, corresponds to applying the Auslander-Reiten translate to a string module by removing hooks. Thus, by induction the shorter interval corresponds to a preprojective module. Note that in this picture an arrow might correspond to a loop or its inverse, chosen such that it "fits" into the hook. However, the arrows lying on the border are always arrows connecting two different vertices.

If it is not possible to move borders as described above, then the only possibilities for this are intervals as

$$-[\rightarrow \cdot \leftarrow]-$$

$$-[\rightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \cdots\cdots\rightarrow \cdot \longrightarrow \cdot \leftarrow]-$$

$$-[\rightarrow \cdot \longleftarrow \cdot \longleftarrow \cdot \leftarrow\cdots\cdots \cdot \longleftarrow \cdot \leftarrow]-$$

$$-[\rightarrow \cdot \longleftarrow \cdot \leftarrow\cdots\cdots \cdot \longleftarrow \cdot \longrightarrow \cdot \cdots\cdots\rightarrow \cdot \longrightarrow \cdot \leftarrow]-$$

each of which corresponds to projective modules.

A dual argument proves that for intervals of the second type, there exists a preinjective module with corresponding rank vector.

Let us consider an interval of the third type. For symmetry reasons it is enough to consider an interval such as

$$-[\rightarrow \cdots -]\rightarrow$$

and show that if there exists a rigid module with the corresponding rank vector, it is a regular module in the tube of rank $n-1$. We already know that if this rigid module exists, it must be regular, since the preprojective and preinjective modules correspond to the intervals of first and second type. Computing τ^{-1} of a string module corresponding to that interval, means attaching a hook on the left side and deleting a cohook on the right side. Thus, we reach the corresponding interval by pushing the borders of the original one to the left, until we reach the next interval of this third type. Note again, that arrows lying on the borders always have to be arrows connecting two different vertices. After repeating this process $n-1$ times, we reach an equivalent interval in different copies of our quiver, showing that

$$\underline{\dim}(\tau^{n-1}(M)) = \underline{\dim}(M)$$

for any string module M corresponding to that interval. Now, if M is rigid, then it is unique with that dimension vector and we have

$$\tau^{n-1}(M) \cong M.$$

Obviously the modules $M, \tau(M), \ldots, \tau^{n-2}(M)$ are pairwise non-isomorphic, showing that M lies in the unique tube of rank $n-1$.

Note that only the regular modules in the first $n-2$ rows of the tube are regular. For the modules above the first $n-2$ rows it is easy to see, that there is a homomorphism from M to $\tau(M)$: it is enough to see, that the arrows going downwards in the tube correspond to surjective morphisms, and the ones going upwards to injective morphisms.

Also note, that the "shortest" (in the sense, that we cannot move one border closer to the other to obtain an interval of the same type) intervals of third type are of the form

$$-[\rightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \cdots\cdots\rightarrow \cdot \longrightarrow \cdot -]\rightarrow$$

or of the form

$$-[\rightarrow \cdot \longleftarrow \cdot \longleftarrow \cdot \leftarrow \cdots \cdot \longleftarrow \cdot -] \rightarrow$$

and the corresponding modules appear as starting or endterm in the Auslander-Reiten sequences with indecomposable middle term, corresponding to the arrows on the border of the intervals. $\square$

Corollary 4.5.19. *Let M be a rigid indecomposable H-module. Then there is an equivalence of categories*

$$M^{\perp} \simeq \mathrm{rep}(H')$$

where $H' \cong KQ'/I'$ is a 1-Iwanaga-Gorenstein algebra and $|Q_0'| = n-1$.

Proof. By Proposition 4.5.18 the module M is preprojective, preinjective or a regular module in the first $n-2$ rows of the unique tube of rank $n-1$. If it is regular, then by Lemma 4.5.13 we have $\mathrm{End}_H(M)$ is a skew-field and thus in any case the claim follows from Theorem 4.4.23. $\square$

Bibliography

[AIR] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory.* Compos. Math. 150 (2014), no. 3, 415–452.

[AHK] *Handbook of tilting theory.* Edited by Lidia Angeleri Hügel, Dieter Happel and Henning Krause. London Mathematical Society Lecture Note Series, 332. Cambridge University Press, Cambridge, 2007. viii+472 pp.

[A1] M. Auslander, *Existence theorems for almost split sequences.* Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), pp. 1–44. Lecture Notes in Pure and Appl. Math., Vol. 26, Dekker, New York, 1977.

[A2] M. Auslander, *Functors and morphisms determined by objects.* Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), pp. 1–244. Lecture Notes in Pure Appl. Math., Vol. 37, Dekker, New York, 1978.

[AR] M. Auslander, I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. 86 (1991), no. 1, 111–152.

[ARS] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*, Corrected reprint of the 1995 original. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997. xiv+425 pp.

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory.* London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006. xiv+458 pp.

[B] K. Bongartz, *Tilted algebras.* Representations of algebras (Puebla, 1980), pp. 26–38, Lecture Notes in Math., 903, Springer, Berlin - New York, 1981.

[BB] S. Brenner, M. C. R. Butler, *The equivalence of certain functors occurring in the representation theory of Artin algebras and species.* J. London Math. Soc. (2) 14 (1976), no. 1, 183–187.

[BD] I. Burban, Y. Drozd, *Derived categories of nodal algebras.* J. Algebra 272 (2004), no. 1, 46–94.

[BR] M. C. R. Butler, C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras.* Comm. Algebra 15 (1987), no. 1–2, 145–179.

[CB1] W. W. Crawley-Boevey, *Maps between representations of zero-relation algebras.* J. Algebra 126 (1989), no. 2, 259–263.

[CB2] W. Crawley-Boevey, *Lectures on representations of quivers.* A graduate course given in 1992 at Oxford University.

[CB3] W. Crawley-Boevey, *Exceptional sequences of representations of quivers.* Proceedings of ICRA VI, Carleton-Ottawa Math. LNS. 14 (1992).

[CB4] W. Crawley-Boevey, *Classification of modules for infinite-dimensional string algebras.* Preprint (2016), 25pp, arXiv:1308.6410v2.

[DR1] V. Dlab, C. M. Ringel, *Representations of graphs and algebras.* Carleton Mathematical Lecture Notes, No. 8. Department of Mathematics, Carleton University, Ottawa, Ont., 1974. iii+86 pp.

[DR2] V. Dlab, C. M. Ringel, *Indecomposable representations of graphs and algebras.* Mem. Amer. Math. Soc. 6 (1976), no. 173, v+57 pp.

[DWZ] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations. I. Mutations.* Selecta Math. (N.S.) 14 (2008), no. 1, 59–119.

[E] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785 pp.

[Fr] P. Freyd, *Abelian Categories.* Harper, 1966.

[FSW] A. J. Frankild, S. Sather-Wagstaff, R. Wiegand, *Ascent of module structures, vanishing of Ext, and extended modules.* Special volume in honor of Melvin Hochster. Michigan Math. J. 57 (2008), 321–337.

[Ga] P. Gabriel, *Des catégories abéliennes.* Bull. Soc. Math. France, 90 (1962), 323–448.

[Ge] J. Geuenich, *Quivers with potentials on three vertices.* Master Thesis, University of Bonn (2013).

[GL] W. Geigle, H. Lenzing, *Perpendicular categories with applications to representations and sheaves.* J. Algebra 144 (1991), no. 2, 273 – 343.

[GLS1] C. Geiß, B. Leclerc, J. Schröer, *Quivers with relations for symmetrizable Cartan matrices I: Foundations.* Preprint (2016), 67pp, arXiv:1410.1403v4.

[GLS2] C. Geiß, B. Leclerc, J. Schröer, *Quivers with relations for symmetrizable Cartan matrices III: Convolution algebras.* Preprint (2016), 37pp, arXiv:1511.06216v2.

[GP] I. M. Gel'fand, V. A. Ponomarev, *Indecomposable representations of the Lorentz group.* (Russian) Uspehi Mat. Nauk 23 1968 no. 2 (140), 3–60.

[H] D. Happel, *On Gorenstein algebras.* Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 389–404, Progr. Math., 95, Birkhäuser, Basel, 1991.

[HR] D. Happel, C. M. Ringel, *Tilted algebras.* Trans. Amer. Math. Soc. 274 (1982), no. 2, 399–443.

[I] Y. Iwanaga, *On rings with self-injective dimension* ≤ 1. Osaka J. Math. 15 (1978), no. 1, 33–46.

[J] G. Jasso, *Reduction of* τ*-tilting modules and torsion pairs.* Int. Math. Res. Not. IMRN (2015), no. 16, 7190–7237.

[L] T. Y. Lam, *A first course in noncommutative rings.* Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 1991. xvi+397 pp.

[M] S. MacLane, *Kategorien.* Begriffssprache und mathematische Theorie. Hochschultext. Springer-Verlag, Berlin-New York, 1972. vii+295 pp.

[Mi] B. Mitchell, *The full imbedding theorem.* Am. J. Math. 86 (1964), 619–637.

[Mo] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition.* Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A 6 (1958), 83–142.

[R1] C. M. Ringel, *The indecomposable representations of the dihedral* 2*-groups.* Math. Ann. 214 (1975), 19–34.

[R2] C. M. Ringel, *Representations of* K*-species and bimodules.* J. Algebra 41 (1976), no. 2, 269–302.

[R3] C. M. Ringel, *On algorithms for solving vector space problems. II.* Tame algebras. Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), pp. 137–287, Lecture Notes in Math., 831, Springer, Berlin, 1980.

[R4] C. M. Ringel, *The braid group action on the set of exceptional sequences of a hereditary Artin algebra.* Abelian group theory and related topics (Oberwolfach, 1993), 339–352, Contemp. Math., 171, Amer. Math. Soc., Providence, RI, 1994.

[R5] C. M. Ringel, *Some algebraically compact modules. I.* Abelian groups and modules (Padova, 1994), 419–439, Math. Appl., 343, Kluwer Acad. Publ., Dordrecht, 1995.

[Ri] C. Ricke, *On Jacobian algebras associated with the once-punctured torus.* J. Pure Appl. Algebra 219 (2015), no. 11, 4998–5039.

[RS] C. Riedtmann, A. Schofield, *On a simplicial complex associated with tilting modules.* Comment. Math. Helv. 66 (1991), no. 1, 70–78.

[Sch] W. Schelter, *Intersection theorems for some noncommutative noetherian rings.* J. Algebra 38 (1976), no. 1, 244–250.

[Scho] A. Schofield, *Semi-invariants of quivers.* J. London Math. Soc. (2) 43 (1991), no. 3, 385–395.

[Schr] J. Schröer, *Modules without self-extensions over gentle algebras.* J. Algebra 216 (1999), no. 1, 178–189.

[S] L. Silver, *Noncommutative localizations and applications.* J. Algebra 7 (1967), 44–67.

[St] H. Strauss, *On the perpendicular category of a tilting module.* J. Algebra 144 (1991), no. 1, 43–66.

[SY] A. Skowroński, K. Yamagata *Frobenius algebras. I. Basic representation theory.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. xii+650 pp.

[U] L. Unger, *Schur modules over wild, finite-dimensional path algebras with three simple modules.* J. Pure Appl. Algebra 64 (1990), no. 2, 205–222.

[Z] J. Zhang, *On indecomposable exceptional modules over gentle algebras.* Comm. Algebra 42 (2014), no. 7, 3096–3119.

Summary

The central problem in the representation theory of finite-dimensional algebras is to describe their category of finite-dimensional modules as completely as possible. Various approaches in trying to solve this problem and partial solutions are variations and applications of tilting theory. The first step in this direction in modern representation theory was Gabriel's Theorem in 1972: A finite-dimensional path algebra $A = KQ$, where Q is a finite connected acyclic quiver, is representation-finite if and only if Q is of Dynkin type A_n, D_n, E_6, E_7, E_8. Moreover, in this case there is a bijection between the isomorphism classes of indecomposable A-modules and the positive roots of the corresponding simple complex Lie algebra. Gabriel thus also established a close connection to Lie Theory.

Shortly after Gabriel's breakthrough Bernstein, Gelfand and Ponomarev proved that all indecomposable modules over a representation-finite path algebra $A = KQ$, can be constructed recursively from the simple modules by using reflection functors. This can be considered as the starting point of tilting theory. It became apparent that the module category does not change too much when changing the orientation of the quiver. This procedure was generalized by Auslander, Platzek and Reiten who showed that the reflection functors are equivalent to functors of the form $\mathrm{Hom}_A(T, -)$, where T is nowadays known as an (APR)-tilting module. Ever since then, tilting theory has appeared in numerous areas of mathematics as a method for constructing functors between categories. In this thesis we focus on two particular theories, whose origins can be traced back to tilting theory.

On the one hand there is the rather new concept called τ-tilting theory introduced by Adachi, Iyama and Reiten in 2012. It follows from results by Riedtmann and Schofield and Unger in the early 1990's that any almost complete (support) tilting module over a finite-dimensional algebra can be completed in at least one and at most two ways to a complete (support) tilting module. This was the first approach to a combinatorial study of the set of isomorphism classes of multiplicity-free tilting modules. The (support) τ-tilting modules are a generalization of the classical tilting modules. In this wider class of modules it is possible to model the process of mutation inspired by cluster tilting theory. In other words any basic almost complete support τ-tilting pair over a finite-dimensional algebra is a direct summand of exactly two basic support τ-tilting pairs.

Cluster algebras were introduced by Fomin and Zelevinsky in 2002. Since then cluster (tilting) theory has had a huge impact on the research of representation theory of finite-dimensional algebras. The algebras appearing in connection with cluster theory are Jacobian algebras defined via quivers with potential. These are not finite-dimensional in general. This suggests the need for developing τ-tilting theory for infinite dimensional algebras.

String algebras are a subclass of the special biserial algebras. The module category of a finite-dimensional string algebra can be described completely in combinatorial terms. Therefore, they are often used to test conjectures. Furthermore, they appear in cluster theory as Jacobian algebras of surfaces. Hence one should consider string algebras as an important class of examples. In this thesis we study the module category of what we call completed string algebras, a generalization of the finite-dimensional string algebras which include infinite dimensional algebras. We extend the combinatorial description of the module category of a finite-dimensional string algebra to the category of finitely generated modules over a completed string algebra. In particular we describe the Auslander-Reiten sequences ending in a finitely generated indecomposable module. This allows us to develop τ-tilting theory for completed string algebras and prove that mutation is possible within the class of finitely generated support τ-tilting pairs.

On the other hand we consider the theory of perpendicular categories which goes back to an article by Geigle and Lenzing in 1989 and an article by Schofield in 1991 which focuses on the study of perpendicular categories for path algebras. One of the main results here is the following: Let A be a finite-dimensional algebra and M an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ is a skew-field. Then there exists a finite-dimensional A-module T such that $M \oplus T$ is a basic tilting module and the (right) perpendicular category $M^{\perp}$ is equivalent to $\mathrm{mod}(A')$, where $A' = \mathrm{End}_A(T)^{\mathrm{op}}$. It follows that the algebra A' has one simple module less than the original algebra A and $\mathrm{gl.dim}(A') \leq \mathrm{gl.dim}(A)$. Thus if A is hereditary, then so is A' and moreover, in

this case it is known that any exceptional module M is an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ is a skew-field.

This result opened the possibility for proving statements by induction. This has been used by Crawley-Boevey to prove the existence of a transitive action of the braid group on exceptional sequences for path algebras. Ringel generalized this result to hereditary algebras and in addition developed an inductive procedure for obtaining all exceptional modules from the simple modules.

The theory of modulated graphs or species was developed by Dlab and Ringel in a series of papers in the 1970's. It was a first attempt in generalizing path algebras associated to symmetric Cartan matrices to hereditary algebras which can be associated to symmetrizable Cartan matrices. They extended Gabriel's Theorem to include the non-simply laced root systems B_n, C_n, F_4 and G_2. More precisely, they proved that a finite-dimensional hereditary algebra is representation-finite if and only if its corresponding valued graph is of Dynkin type. To ensure the existence of these hereditary algebras one has to make quite strong assumptions on the ground field, and cannot assume it to be algebraically closed in general.

Recently Geiß, Leclerc and Schröer suggested another approach by introducing a new class of algebras which are defined via quivers with relations associated with symmetrizable Cartan matrices. They thus obtain new representation theoretic realizations of all finite root systems without any assumptions on the ground field. These newly defined algebras are in general no longer hereditary but 1-Iwanaga-Gorenstein. An algebra is 1-Iwanaga-Gorenstein if and only if the injective dimension of its regular representation is at most 1. Thus all self-injective and all hereditary algebras are particular examples of 1-Iwanaga-Gorenstein algebras.

In this thesis we study perpendicular categories for finite-dimensional 1-Iwanaga-Gorenstein algebras. We find that if A is 1-Iwanaga-Gorenstein and $M \in \mathrm{mod}(A)$ an indecomposable partial tilting module such that $\mathrm{End}_A(M)$ is a skew-field, then $M^\perp$ is equivalent to $\mathrm{mod}(A')$, where A' is again 1-Iwanaga-Gorenstein. We then concentrate on the particular class of 1-Iwanaga-Gorenstein algebras defined via quivers with relations associated with symmetrizable Cartan matrices. If H is such an algebra associated with a symmetrizable Cartan matrix C and $M \in \mathrm{mod}(H)$ an indecomposable partial tilting module, the ring $\mathrm{End}_H(M)$ is not a skew-field in general. However, if M is preinjective we still find that $M^\perp$ is a equivalent to $\mathrm{mod}(H')$, where H' is a 1-Iwanaga-Gorenstein algebra associated with a symmetrizable Cartan matrix C', which is of size one smaller than C.