Analysis of martensitic microstructures in shape-memory-alloys: A low volume-fraction limit

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1 Introduction

Mathematical problems are often motivated by the rigorous modeling and analysis of phenomena arising in physics or material science. One among many of such phenomena is the formation of microstructures at austenite-martensite interfaces in shape-memory alloys, such as Nickel Titanium. In this thesis, we will use variational methods to study these structures in the specific case of a low-volume fraction.

We consider an energy consisting of a bulk term and a surface term, in which the bulk term is geometrically linearized. Two martensitic phases, one of them with a much smaller volume fraction, form microstructures at an austenite-martensite interface. For a transition regime, we derive two energies, one for scalar-valued functions and one for vector-valued functions, given by

$$I^\theta(u) = \int_\Omega |\partial_x u|^2 + \min\{|\partial_y u + 1|^2, |\partial_y u - \frac{1}{\theta}|^2\} \,\mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2 u|(\Omega) \qquad \text{and}$$

$$E^{\theta,p}(u) = \int_\Omega \min\{|(Du + e_1 \otimes e_2)_{sym}|_p^p, |(Du - \frac{1}{\theta} e_1 \otimes e_2)_{sym}|_p^p\} \,\mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2 u|(\Omega).$$

Our goal is to compute the Γ-limit of these functionals, letting the volume fraction θ go to zero, to gain further insight into the original problem. During this process, new subspaces of SBV and SBD together with mathematical tools for these subspaces are developed.

In the following introduction, we shortly explain the physical background of the problem, derive the two energy functionals and introduce the main mathematical tools and concepts that are used in this thesis.

Chapter 2 treats a one-dimensional, scalar-valued toy-model. Many basic concepts and arguments for the later chapters are introduced here and the candidate for the limiting energy for I^θ is motivated. In Chapter 3, the $\liminf$-inequality and a compactness result for I^θ are proven. The space $SBV^2_{e_2}$ is introduced, in which the limiting energy is finite. Chapter 4 treats the recovery sequence for I^θ. A main result of this chapter and the entire work is a density result for $SBV^2_{e_2}$: Each function in $SBV^2_{e_2}$ can be approximated with respect to the energy I^θ and weak-$*$ BV convergence with a sequence of functions whose jump sets are a finite union of segments.

For the energy $E^{\theta,p}$, we have only been able to prove partial results. In Chapter 5, we present a possible candidate for the limiting energy, the limiting function space $SBD^p_{e_2 \odot e_1}$ and prove both a $\liminf$-inequality and a compactness result. In Chapter 6, recovery sequences for functions whose jump sets are a finite union of segments are constructed. Whether or not a density result with such functions holds is still an open problem, Chapter 6 also contains a brief overview of the arising difficulties.

Chapter 7 is dedicated to the proof of a Korn-Poincaré-type inequality for the space $SBD^p_{e_2 \odot e_1}$. Although the result is not sufficient to provide a proof of the density result, it is an important step in that direction and an interesting mathematical statement on its own.

The results of Chapters 3 and 4 have been announced in [27]. Moreover, a shortened version of these chapters has been submitted and a preprint has been published on arXiv by the author together with Conti and Zwicknagl [16].

1.1 The shape memory effect

The mathematical analysis we will perform in this work is motivated by the study of microstructures in so-called shape memory alloys. Shape memory alloys have the interesting and useful property that, after they have been subjected to an elastic deformation at a low temperature, they recover their original shape under heating, see [35]. This effect was first published by Chang and Read in 1951 [14].

The reason for this effect lies in a particular behavior of the atomic lattice, namely a solid-solid phase transition at some specific temperature. If the material is at a high temperature (called austenite phase), it prefers a cubic lattice structure, whereas at a low temperature (in the so-called martensite phase), there are different preferred lattices with fewer symmetries [43].

The first one to observe these different states in a material was the physicist Arne Ölander in 1932 in an Au Cd alloy [41]. However, not before Buehler and coworkers witnessed the same effect in a Nickel Titanium (Ni Ti) alloy in 1963 [11], it became interesting and applicable for first military purposes (see [45], [51]). This was only a starting point and applications in biomedicine and engineering followed soon (see [8], [35], [43] and the references therein).

We want to give a short, and in some parts naïve, explanation of the physical effects that cause the shape memory effect. For a deeper understanding we refer to the textbooks by Bhattacharya [7] or by Otsuka and Waymann [43]. Let us imagine a material whose atoms prefer a cubic lattice at high temperatures and different tetragonal lattices if they are below some critical temperature (see Figure 1).

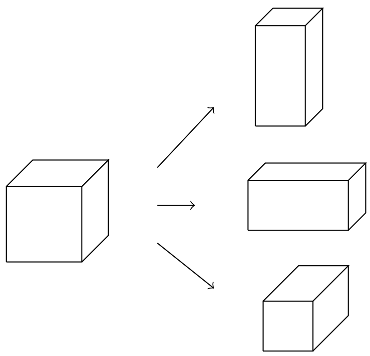

Figure 1: A cubic austenite lattice and three tetragonal martensitic lattice structures.

The material is given in a stable macroscopic shape at a high temperature. When it is cooled down under the critical temperature, the atoms start locally adopting to one of the different tetragonal lattices. In different areas of the material, different lattices are chosen such that the macroscopic shape is essentially preserved, see the third picture of Figure 2. We can also see interfaces between different martensitic states in this sketch, denoted by dotted lines. Since the tetragonal lattice is disturbed along these interfaces, they should bear some interfacial energy.

If the material is macroscopically deformed, it tries to realize this deformation on an atomic level by changing between the different lattices. The second picture in Figure 2 can be imagined as the reaction to a load in vertical direction. The atoms do not change their neighbors during this process. Hence, if the material is heated again above the critical temperature, the atoms are forced back into

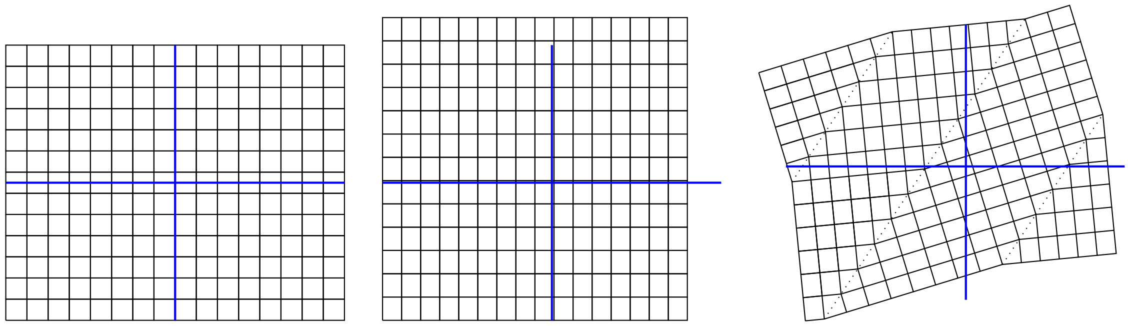

Figure 2: A two-dimensional sketch of 17×14 atoms in a cubic austenite lattice and two different versions of a martensitic, tetragonal lattice. The blue lines have the same length in all three pictures and indicate the change of macroscopic lengths, the dotted lines indicate a concentration of surface energy.

the cubic lattice and it is energetically preferable that the atoms do not change their neighbors. It is, on an atomic level, in the same state as before the cooling process and has therefore the same macroscopic shape.

This is, of course, only an idealized description of the processes. The original material in austenite state might not have a perfect cubic lattice, but bear some defects in the lattice structure, and even more defects may occur on interfaces between different martensitic phases. A repeated cycle of cooling, deforming and heating might lead to an accumulation of these effects and hence to a fatigue in the material (see [35], [38]). There are also other phenomena influencing the fatigue effect, like large loads [37] or specific crystallographic orientations [32].

An important detail is that there is not one critical temperature at which the austenite-martensite phase transition takes place, but a temperature at which martensite starts transforming to austenite under heating and a second, lower temperature at which the martensite starts transforming to austenite under cooling. The difference of these temperatures is called hysteresis, see [7], [44] for more details. Controlling the width of hysteresis is important in many applications. One can easily imagine purposes in which materials of either small or large hysteresis are preferable. In particular, a small hysteresis is also connected to a weak fatigue effect [32], [39]. Notice that during the process of heating (or cooling) both states, martensite and austenite, will be present in the material and hence also interfaces between these phases. These intermediate temperatures for materials with low hysteresis are the situations we want to focus on.

It is important to mention that not only the mathematical analysis is based on and influenced by the work of material scientists. A detailed mathematical description of the problems can conversely help developing materials with specific properties, see the work of Cui et al. [19] and the recent results of Song et al. [47].

1.2 Continuum model for elastic deformations

In this work, we follow a continuous theory of martensitic phases, going back to Bowles and MacKenzie [9] and Wechsler, Lieberman and Read [52]. We are following [6] and [7] in the subsequent introduction.

We identify a crystalline body in its stress-free austenite phase with an open domain $U \subseteq \mathbb{R}^3$. At a fixed temperature, the body is described by a continuous deformation $\varphi : U \to \mathbb{R}^3$, where the gradient encodes the lattice structure. If the material is (locally) in its austenite phase we have $D\varphi = \mathrm{Id}$, and there are matrices $F_i \in \mathbb{R}^{3\times 3}$ that represent the different martensitic phases. Due to frame indifference, we do not distinguish between rotated variants of the same martensitic phase, that is, for every rotation $R \in SO(3)$ we say that RF_i and F_i belong to the same martensitic phase.

An important field of the analysis is the compatibility of different martensitic phases to each other and to the austenite phase. If the material is divided by a C^1-surface Γ into two different areas U_1 and U_2 and the continuous deformation φ has a constant gradient F_i on these sets U_i, then it is a straightforward computation that $F_1 \neq F_2$ if and only if the so-called Hadamard jump condition holds. That is: The matrices F_1 and F_2 are rank-one connected i.e., $F_1 - F_2 = a \otimes n$ for $a, n \in \mathbb{R}^3$ and Γ is a hypersurface with normal n.

Let F be the matrix of a single martensitic phase and let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of F. An interface between the austenite phase and this single martensitic phase is possible if and only if the middle eigenvalue $\lambda_2 = 1$ [6]. We will focus our analysis on the case where $\lambda_2 \neq 1$.

A variant of martensite that has an interface to an other variant of martensite along some hypersurface will then not be compatible to an austenite phase along any other hypersurface. In this case, a fine mixture of the two different martensitic phases occurs at an austenite-martensite interface, see for example the work of Sun et al. for experimental data [49] and Figure 3 for a sketch.

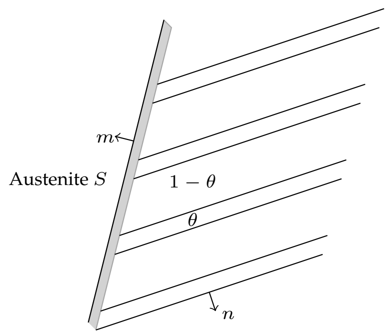

Figure 3: A sketch of an austenite-martensite interface where two different variants of martensite enter with different volume fractions and are compatible to each other in direction n.

The two different martensitic phases will be mixed with different volume fractions θ and $1 - \theta$. These volume fractions are chosen in such a way that there is a rank-one connection between the infinitesimally fine mixture and a rotated version of the identity, representing the austenite. To be formal: There are $b, m \in \mathbb{R}^3$ and $S \in SO(3)$ such that $S - (\theta F_1 + (1-\theta)F_2) = b \otimes m$, see again Figure 3 for a sketch. A necessary and sufficient condition for the solvability of this equation has been developed in the work of Ball and James [6].

It has been conjectured and supported by experimental data by Zhang, James and Müller [53] that there is a connection between the distance of the middle eigenvalue to 1 and the size of the hysteresis, see Figure 4. Moreover, they compute the volume fraction θ of the different martensitic phases and deduce that $|\lambda_2 - 1| \sim 0$ implies $\theta \sim 0$.

The continuous theory assumes that the deformation u minimizes an energy functional I. A possible choice for such a functional is the following sum of a bulk energy and a surface energy:

$$I(\varphi) = \int_U W_T(D\varphi)\, \mathrm{d}\mathcal{L}^3 + \kappa|D^2\varphi|(U).$$

The function W_T has energy wells, or local minima, at the different martensitic states i.e., at $SO(3)F_i$ and at the austenite state i.e., at $SO(3)\,\mathrm{Id}$ and depends on the temperature T. The surface-term penalizes jumps between the different wells and prevents infinitely fine mixtures of different states. It also ensures the existence of minimizers and is given as a total variation norm of the second derivatives. The parameter $\kappa > 0$ might be identified with the measured interfacial energy, this does, however, lead to length scales that are not realistic [7]. We therefore understand it as a mathematically necessary parameter.

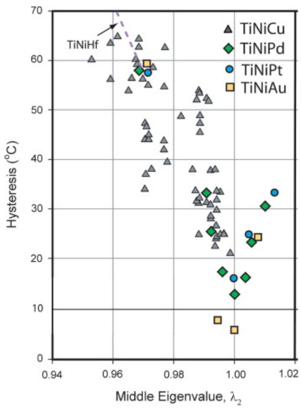

Figure 4: Width of hysteresis vs. middle eigenvalue λ_2[1].

We want to analyze the behavior of energetic minimizers of austenite-martensite interfaces with a mixture of two phases as the volume fraction θ tends to zero. As a first step, we will need to simplify the model we have motivated previously.

1.3 Reduction of the model to a geometrically linearized energy for low-volume fraction

For the sake of simplicity, we only consider the static problem and hence fix the temperature T such that the austenite phase and martensitic phases are present. We fix two rank-one connected matrices F_1 and F_2 that represent different martensitic states. The most common choice for the potential W_T is $W_T(M) = \mathrm{dist}^2\{M, SO(3)F_1 \cup SO(3)F_2\}$ close to the energy wells [53].
Notice first that in similarity to [53], we can identify a direction in the domain and one in the codomain that is left invariant by the martensitic phases F_1 and F_2. We may therefore assume that $U \subseteq \mathbb{R}^2$ and $\varphi : U \to \mathbb{R}^2$. Secondly, we want to replace the bulk term of the energy with its geometrically linearized version. For this purpose we rewrite the deformation φ as $\varphi(x) = x + \delta u(x)$ where u is the displacement.
We notice that the Hessian of the squared distance to a submanifold in $\mathbb{R}^n$ is given by the projection

[1]Reprinted from Z. Zhang, R.D. James, S. Müller [53], with permission from Elsevier.

on the normal space and hence $\operatorname{dist}^2\{M, SO(3)F_1 \cup SO(3)F_2\} \sim \min_{i=1,2} \|(M - F_i)_{sym}\|^2$ close to the energy wells [4]. After rescaling and for small δ, the energy for the displacement is approximately given by

$$I^\varepsilon(u) = \int_U \min\{\|(Du - F_1 - \mathrm{Id})_{sym}\|^2, \|(Du - F_2 - \mathrm{Id})_{sym}\|^2\} \,\mathrm{d}\mathcal{L}^3 + \varepsilon|D^2u|(U). \tag{1}$$

For our further considerations, we explicitly choose two rank-one connected matrices with volume fractions θ and $1-\theta$, namely

$$F_1 = \mathrm{Id} - \theta e_1 \otimes e_2 = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_2 = \mathrm{Id} + (1-\theta) e_1 \otimes e_2 = \begin{pmatrix} 1 & 1-\theta \\ 0 & 1 \end{pmatrix}.$$

Notice that we expect that our result strongly depends on this choice of the rank-one connection. For other rank-one connections even the energy-scaling might be different, see for example the work of Chan and Conti [13].

Since the symmetrized distance makes the problem quite complicated, we restrict ourselves for the first half of this thesis to a scalar-valued problem as proposed by Kohn and Müller [33], [34]. The energy in (1) then simplifies to

$$\hat{I}^{\varepsilon,\theta}(u) = \int_U |\partial_x u|^2 + \min\{|\partial_y u + \theta|^2, |\partial_y u - 1 + \theta|^2\} \,\mathrm{d}\mathcal{L}^2 + \varepsilon|D^2u|(U)$$

whilst the energy in the vector-valued case is given by

$$\hat{E}^{\varepsilon,\theta}(u) = \int_U \min\{\|(Du + \theta e_1 \otimes e_2)_{sym}\|^2, \|(Du - (1-\theta) e_1 \otimes e_2)_{sym}\|^2\} \,\mathrm{d}\mathcal{L}^2 + \varepsilon|D^2u|(U).$$

Both of these energies do only prefer the two different martensitic states and do not include the austenite phase. The austenite-martensite interface is modeled by boundary data for u at parts of ∂U. For simplicity, we choose the domain to be the unit square throughout the whole thesis and define $\Omega = (0,1)^2$.

The displacement in the austenite phase is given by $u \equiv 0$. We fix zero-boundary data at the left edge $\{0\} \times (0,1)$ for $\hat{I}^{\varepsilon,\theta}$ and on the left and lower edges $\{0\} \times (0,1) \cup (0,1) \times \{0\}$ for $\hat{E}^{\varepsilon,\theta}$. We need to give stronger boundary values for the vector-valued model to avoid trivial minimizers of the energy, see [26].

1.4 Scaling laws and rescaling of the energy

It has been widely discussed that even though it seems difficult to find actual minimizers of energies of type $\hat{I}^{\varepsilon,\theta}$, one can gain insight by proving scaling laws, including explicit constructions for functions of low energy [26], [34], [54]. In the case of $\hat{I}^{\varepsilon,\theta}$, scaling laws for similar energies have been proven in the author's Bachelor's thesis [25] and by Zwicknagl [54]. The following result has been formulated and proven in the appendix of the author's joint work with Conti and Zwicknagl [16].

There exists $c > 0$ such that for all $\theta \in (0, \frac{1}{2}]$ and for all $\varepsilon > 0$ it holds:

$$\frac{1}{c}\theta^2 \min\{1, \varepsilon^{2/3}\theta^{-4/3}\} \le \min \hat{I}^{\varepsilon,\theta} \le c\theta^2 \min\{1, \varepsilon^{2/3}\theta^{-4/3}\}.$$

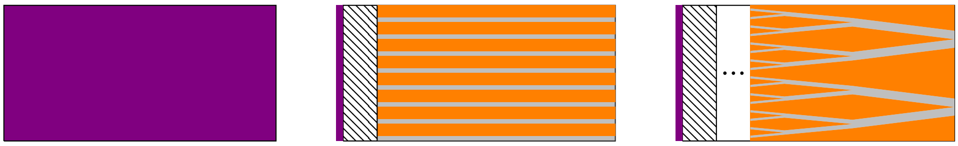

Figure 5: Construction of a deformation: Austenite, laminates or twinning.

The regime where $\hat{I}^{\varepsilon\theta} \sim \theta^2$ is realized by a single phase of austenite whilst the regime where $\hat{I}^{\varepsilon,\theta} \sim \varepsilon^{2/3}\theta^{2/3}$ is achieved by a Kohn-Müller-type twinning construction, see Figure 5. A simple lamination of the domain is not possible due to the hard boundary conditions, a lamination with linear interpolation to the boundary bears more energy then the Kohn-Müller twinning construction.
The aim of this work is to analyze the energy in the transition regime between twinning and a single phase, that is for fixed ratio $\frac{\varepsilon}{\theta^2} \sim \sigma$ for some $\sigma \in \mathbb{R}^+$.
The scaling of the energy in this transition regime is θ^2, hence we rescale it. We also rescale the functions itself since a sequence of functions that has slope θ on a volume fraction of $1-\theta$ would converge against zero. We therefore define the energy

$$\begin{aligned} I^\theta(u) &:= \frac{(1+\theta)^2}{\theta^2}\hat{I}^{\varepsilon(\sigma,\theta),\theta}\left(\frac{\theta}{1+\theta}u - \frac{\theta^2}{1+\theta}y\right) \\ &= \int_\Omega |\partial_x u|^2 + \min\{|\partial_y u + 1|^2, |\partial_y u - \frac{1}{\theta}|^2\}\,\mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2 u|(\Omega) \end{aligned}$$

where we have chosen $\varepsilon = \sigma\frac{\theta^2}{1-\theta}$. The zero-boundary values of u are transformed to $u(0,y) = \frac{\theta^2}{1+\theta}y$. Notice that this rescaling is not unique. In [16], we use a slightly different rescaling that simplifies notation whilst the choice of rescaling in this thesis yields a more descriptive energy.
The energy scaling for the vector-valued energy has been proven in the author's Master's thesis [26] for the case $\theta = \frac{1}{2}$. A joint work with Conti, Melching and Zwicknagl is in preparation, in which we will provide scaling laws for a more general energy [15]. As a special case it will follow that $\min \hat{E}^{\varepsilon,\theta} \sim \min\{\theta^2, \theta^{2/3}\varepsilon^{2/3}\}$. As before we are interested in the transition regime.
A rescaling with $\tilde{u}(x,y) = \left(\frac{\theta}{1+\theta}u_1(x,y) - \frac{\theta^2}{1+\theta}y, \frac{\theta}{1+\theta}u_2(x,y)\right)$ yields the energy

$$E^\theta(u) = \int_\Omega \min\{\|(Du + e_1\otimes e_2)_{sym}\|^2, \|(Du - \frac{1}{\theta}e_1\otimes e_2)_{sym}\|^2\}\,\mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2u|(\Omega).$$

We will derive all results for the more general functional

$$E^{\theta,p}(u) = \int_\Omega \min\{|(Du + e_1\otimes e_2)_{sym}|_p^p, |(Du - \frac{1}{\theta}e_1\otimes e_2)_{sym}|_p^p\}\,\mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2u|(\Omega)$$

where $|.|_p$ denotes the standard p-norm in $\mathbb{R}^n$. This second functional is not derived as an energy-functional for austenite-martensite interface. However, the functional $E^\theta = E^{\theta,2}$ is included as a special case and the former might occur in some different context and its mathematical analysis is interesting by itself. In similarity to this generalization, one can also derive all results for a generalized version $I^{\theta,p}$ of I^θ, see [16] for details.

1.5 Notation and mathematical background

We already used the notation $a \otimes b$ for $a, b \in \mathbb{R}^n$ to denote the rank-one, $n \times n$ matrix with entries $(a \otimes b)_{i,j} = a_i b_j$. Every rank-one matrix can be written in this way. We also denote $a \odot b = (a \otimes b)_{sym}$, that is, $(a \odot b)_{i,j} = \frac{1}{2}(a_i b_j + a_j b_i)$.

We will usually denote the coordinates in $\mathbb{R}^2$ by x and y, the unit square in $\mathbb{R}^2$ will always be denoted by $\Omega = (0,1)^2$.

The derivatives of the functions we consider as arguments of I^θ and $E^{\theta,p}$ will be functions of bounded variation. These functions are defined as follows: For $U \subseteq \mathbb{R}^n$ open a function $f : U \to \mathbb{R}$ is called a *function of bounded variation* if it is integrable and if its distributional derivatives $\partial_{x_i} f$ are finite, signed Radon measures on U for all $i \in \{1, \ldots, n\}$. For details on functions of bounded variation, see the textbooks of Evans and Gariepy [28] or of Ambrosio, Fusco and Pallara [3]. The surface term in the energies should always be interpreted as a total variation measure with respect to the $|.|_2$ norm of these derivatives.

The derivative Df of a function of bounded variation in $\mathbb{R}^n$ can always be divided into three additive parts: An absolute continuous part, denoted by ∇f, which has a density with respect to the n-dimensional Lebesgue measure, a jump part, denoted by $D^J f$, that is concentrated on a set of finite $\mathcal{H}^{n-1}$-measure and a Cantor part, denoted by D^C, that contains the rest of the measure. The limiting energy of the I^θ's will only be finite for so-called special functions of bounded variation. These functions, also called SBV-functions, were introduced by Ambrosio and De Giorgi [2] and are defined as the functions of bounded variation that have vanishing Cantor part.

A generalization of BV-functions are the so-called *functions of bounded deformation*, or BD-functions, as introduced by Suquet [50] and Matthies, Strang and Christiansen [36]. We say that a function $f \in L^1(\Omega, \mathbb{R}^n)$ is of bounded deformation if $Df_{sym} = \frac{1}{2}(Df^T + Df)$, the symmetrized, distributional gradient, is a vector-valued, finite, signed Radon measure. Again, Df_{sym} can be divided into an absolute continuous part, a jump part and a Cantor part, see the work of Ambrosio, Coscia and Dal Maso and the references therein [1]. The subspace of the functions of bounded deformations that have vanishing Cantor part is consequently called SBD, and both spaces, BD and SBD, are objects of ongoing research, see e.g. [17], [23], [30]. In our analysis, a subspace of SBD will be the domain of the limiting energy in the vector-valued case and we will be able to prove a new Poincaré-type inequality for this subspace.

In the previous section we announced that we want to analyze the limit of the energies as $\theta \to 0$. The notion we want to use to interpret this limiting process will be the Γ-limit, as introduced by Dal Maso and Modica [21], which is well-established in variational limits and relaxation problems. For a detailed discussion, see for example the textbook of Dal Maso [20]. We repeat the definition here: Fix a sequence of functionals I^k and a limiting functional I on a function space X together with a notion of convergence $\to_X$ for functions in X. We say that I_k Γ-converges to I with respect to $\to_X$

if the following two assertions hold:

- For each $v \in X$, the $\liminf$-*inequality* holds, that is: For every sequence $v_k \to_X v$ we have that $I(v) \leq \liminf_{k\in\mathbb{N}} I^k(v_k)$.
- For each $v \in X$, there exists a *recovery sequence*, that is: There is a sequence $v_k \to_X v$ such that $I^k(v_k) \to I(v)$.

The functionals are usually bounded from below but are allowed to take the value $+\infty$. If the functionals I or I^k are only defined on some subspace of X, they can be extended by $+\infty$ on the remaining parts of X.
Not only the functionals and the space X are important, the notion of convergence $\to_X$ is also crucial. If the notion is weak i.e., many sequences converge, then it is easier to construct a recovery sequence but more difficult to prove the $\liminf$-inequality and vice versa.
A Γ-converging sequence often comes together with a *compactness result*, that is:

- For each sequence $v_k \in X$ with $I^k(v_k) \leq C$, there exists a subsequence k_l and a function v such that $v_{k_l} \to_X v$.

Assume that a sequence of functionals I^k Γ-converges to some functional I and fulfills a compactness result. Then it is a direct consequence that a sequence of minimizers of I^k converges to a minimizer of I. Depending on the problem, one might be able to gain information about the structure of these minimizers from each other.

1.6 Results and outline

The limiting functional for the scalar-valued problem will be given by

$$I(u) = \int_\Omega |\partial_x u|^2 + |\partial_y u + 1|^2 \,\mathrm{d}\mathcal{L}^2 + 2\sigma\mathcal{H}^1(Ju)$$

defined on the space

$$SBV^2_{e_2,0} = \{u \in SBV_{loc}(\Omega) \,|\, D^J u \cdot e_1 = 0,\ D^J u \cdot e_2 \geq 0,\ \nabla u \in L^2(\Omega,\mathbb{R}^2),\ u = 0 \text{ on } \{0\} \times (0,1), \\ \text{and } |Du|((0,1)\times(\delta,1-\delta)) < \infty \text{ for all } \delta > 0\}.$$

It will be a consequence of the results of Chapter 3 and of Chapter 4 that the following theorem of Γ-convergence holds:

Theorem (Compactness and Γ-limit in the scalar-valued case)**.** *Let I and I^θ be defined as above.*

i) Let $\theta_k \searrow 0$ and let $\{u_k \,|\, k \in \mathbb{N}\}$ such that $I^{\theta_k}(u_k) \leq C$. Then there is a subsequence $\{k_l \,|\, l \in \mathbb{N}\}$ and a function $u \in SBV^2_{e_2,0}$ such that $u_{k_l} \overset{*}{\rightharpoonup} u$ in $BV((0,1)\times(\delta,1-\delta))$ for all $\delta > 0$ and $I(u) \leq \liminf_{k\in\mathbb{N}} I^{\theta_k}(u_k)$.

ii) Let $u \in SBV^2_{e_2,0}$ and let $\theta_k \searrow 0$. Then there is $\{u_k \,|\, k \in \mathbb{N}\}$ such that $I^{\theta_k}(u_k) \to I(u)$ and $u_k \overset{*}{\rightharpoonup} u$ in $BV((0,1)\times(\delta,1-\delta))$ for all $\delta > 0$.

A main result that is used in the proof of the second assertion is the following density result:

Theorem (Density for $SBV^2_{e_2,0}$-functions). *Let $u \in SBV^2_{e_2,0}$ and $\delta > 0$.*
Then there is $v \in SBV^2_{e_2,0} \cap C^\infty(\Omega \setminus Jv)$ such that Jv is a finite union of segments and such that $\|u - v\|_{L^1(\Omega)} \le C\delta$, $|Dv|(\Omega) \le C(1 + I(u) + I(u)^{1/2})$, $I(v) \le (1 + C\delta)I(u)$ and $\|v\|_{C^2(\Omega\setminus Jv)} < \infty$.

In the vector-valued problem, the candidate for the limiting energy is given by

$$E^p(u) = \int_\Omega |(Du + e_1 \otimes e_2)_{sym}|_p^p \,\mathrm{d}\mathcal{L}^2 + 2\sigma\mathcal{H}^1(Ju)$$

on the function space

$$\begin{aligned} SBD^p_{e_2\odot e_1,0} = \{u \in SBD_{loc}(\Omega, \mathbb{R}^2) \,|\, e(u) \in L^p(\Omega; \mathbb{R}^{2\times 2}),\ \mathcal{H}^1(Ju) < \infty,\ u_1(0,\cdot) = 0,\ u_2(\cdot, 0) = 0, \\ [u_1]\nu_{Ju} \in [0,\infty)e_2,\ [u_2]\nu_{Ju} \in [0,\infty)e_1\ \mathcal{H}^1\text{-a.e.}\}. \end{aligned}$$

We present the following $\liminf$-inequality in Chapter 5:

Theorem (Compactness and $\liminf$ inequality in the vector-valued case). *Let $p \in (1,\infty)$, $\theta_k \searrow 0$ and let $\{u^k \,|\, k \in \mathbb{N}\} \subseteq W^{1,p}(\Omega)$ such that $\partial_{x_i} u^k \in BV(\Omega)$, $u^k_1(0,y) = \frac{\theta^2}{1+\theta}y$, $u^k_2(\cdot,0) = 0$ and such that $E^{\theta_k,p}(u^k) \le M$.*
Then there is a subsequence $\{k_l \,|\, l \in \mathbb{N}\} \subseteq \mathbb{N}$ and a function $u \in SBD^p_{e_2\odot e_1,0}$ such that $u^{k_l} \overset{*}{\rightharpoonup} u$ in $BD_{loc}(\Omega, \mathbb{R}^2)$ and $u^{k_l} \to u$ in $L^1(\Omega, \mathbb{R}^2)$.
Moreover: $E^p(u) \le \liminf E^{\theta_k,p}(u_k)$.

We have not been able to provide a density result as in the scalar-valued case. Therefore we are only able to construct a recovery sequence for a subspace of $SBD^p_{e_2\odot e_1,0}$ in which the functions and their jump sets have higher regularity. The details are given in Chapter 6.

Theorem (Recovery sequence for regular functions in the vector-valued case). *Let $p \in (1,\infty)$ and let $u \in SBD^p_{e_2\odot e_1,0}$ such that $Ju_1 = \bigcup_{i=0}^I (a_i, b_i) \times \{y_i\}$, $Ju_2 = \bigcup_{j=0}^J \{x_j\} \times (d_j, e_j)$, $u_i \in W^{2,\infty}(\Omega \setminus Ju_i)$ and such that $[u_i] \in C^2(Ju_i)$. Let $\theta_k \searrow 0$. Then there is a sequence of functions $v^k \in W^{1,p}(\Omega)$ such that $\partial_{x_i} v^k \in BV(\Omega)$, $v^k_1(0,y) = \frac{\theta^2}{1+\theta}y$, $v^k_2(\cdot,0) = 0$,*

$$I^{\theta_k}(v_k) \to I(u) \qquad \text{and} \qquad v_k \overset{*}{\rightharpoonup} u \text{ in } BD.$$

The main gap in the proof of the density result is a missing local approximation for $SBD^p_{e_2\odot e_1,0}$ at points at which the jump accumulates that has a small-enough error in energy. As a step in that direction, we have been able to prove a Korn-Poincaré-type estimate. Although this estimate does not suffice to complete the proof of the density result, we present it in Chapter 7 since it deepens our understanding of the space $SBD^p_{e_2\odot e_1}$ and might be useful to understand Poincaré-type inequalities on other subspaces of BD. The result is:

Theorem (Korn-Poincaré-type inequality). *Let $p \in (1,\infty)$. There is a constant $C > 0$ such that for every function $u \in SBD^p_{e_2\odot e_1}((-r,r)^2)$ there exists two finite partitions, $x_1 < \cdots < x_I$ and $y_1 < \cdots < y_J$, values $a_i < a_{i+1}$, $b_j < b_{j+1}$ and a skew-symmetric matrix $\tilde{R}$ such that $I \le \frac{C}{r}\mathcal{H}^1(Ju_2)$, $J \le \frac{C}{r}\mathcal{H}^1(Ju_1)$*

and for $R(x,y) = \tilde{R} \cdot (x,y)^T$ *it holds*

$$\left\| u - R - \sum_{i,j=0}^{I,J} \begin{pmatrix} a_i \\ b_j \end{pmatrix} \chi_{(x_i,x_{i+1})\times(y_j,y_{j+1})} \right\|_{L^p(Q_r)}^p \leq Cr^p \|e(u)\|_{L^p(Q_r)}^p \left(1 + \frac{(\mathcal{H}^1(Ju))^p}{r^p} \right).$$

We also provide an example that this estimate is optimal in some sense.

There are different approaches on which one could focus next. The most natural step is proving the $SBD^p_{e_2 \odot e_1}$ density result. This would complete the proof of the Γ-limit in the vector-valued case which has been a direct goal of the considerations in this work and would probably lead to further insight on SBD functions in general.
A second aim would be the analysis of a different rank-one connection. It has been shown by Chan and Conti that if two matrices connected by $e_2 \otimes e_2$ are considered, a different energy scaling is achieved [13]. In a low-volume fraction case, one could also analyze this model with the technique of Γ-limit.
Finally, one could consider a detailed discussion of the limiting functionals E^2 or I. Can one analytically derive information about minimizers that are not an immediate consequence of known properties of minimizers of $E^{2,\theta}$ or I^θ respectively?
Since we have not only reduced the nonconvex double-well problem to a convex problem, but have in particular obtained a functional of Mumford-Shah-type, different methods and results are applicable that might lead to new results. In particular, there are well-established numerical approaches to Mumford-Shah-type functionals that might be implemented and would lead to a better understanding of the behavior of minimizers and hence of the formation of microstructures.

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2 The scalar-valued problem in one dimension

As a first step, and for a better understanding of the problem, we will consider a one-dimensional, scalar-valued variant of the problem. The most relevant behavior comes from the different preferred values for the y-derivative of the functions. We therefore consider only a vertical slice in the domain and consider the functionals and functions reduced on that slice. The functional I^θ reduces to

$$I^\theta_{e_2}(v) := \begin{cases} \int_{(0,1)} \min\{|v'+1|^2, |v' - \frac{1}{\theta}|^2\} \,\mathrm{d}\mathcal{L}^1 + \sigma\theta|D^2 v|((0,1)) & v \in \mathcal{B}_1 \\ \infty & v \notin \mathcal{B}_1 \end{cases}$$

where the space $\mathcal{B}_\lambda$ is defined by

$$\mathcal{B}_\lambda = \{v \in W^{1,2}((0,\lambda)) \,|\, v' \in BV((0,\lambda))\}.$$

Notice that we will denote derivatives by Dv or D^2v instead of v' or v'' whenever they are only measures to avoid confusion.

Consider a sequence $\{v_k \,|\, k \in \mathbb{N}\}$ of functions that has uniformly bounded energy $I^{\theta_k}_{e_2}$ and is additionally bounded in L^1. Then v'_k is bounded on the area where $v'_k \sim -1$. The part of the domain where $v'_k \sim \frac{1}{\theta_k}$ concentrates due to the L^1-boundedness and creates jumps of the limiting function. An easy example for such a sequence is a sequence of sawtooth functions where each function has the same number of teeth, see Figure 6 for a sketch.

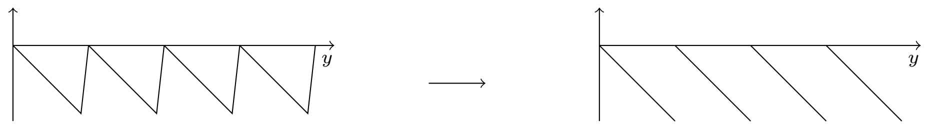

Figure 6: A sequence of sawtooth functions with a constant number of sawteeth and increasing slope has bounded energy. Jumps of the derivatives converge to jumps of positive height of the limiting function.

In the limit, each sawtooth converges to an affine function with slope -1, separated by jumps. The limiting function is no longer continuous but a SBV-function.

At each sawtooth the derivative changes from -1 to $\frac{1}{\theta_k}$ and back to -1. That is: The second derivative has a jump of height approximately $\frac{1}{\theta}$ hence the term $\theta|D^2 v_k|((0,1))$ counts twice the number of jumps in the limit. Notice, that the so-created jumps are always of positive height since they are created by large positive derivatives.

We hence define the space

$$SBV_{+,\lambda} = \{v \in SBV((0,\lambda)) \,|\, v^+(x) > v^-(x) \text{ for } x \in Jv,\ \mathcal{H}^0(Jv) < \infty \text{ and } Dv^{ac} \in L^2((0,\lambda))\}$$

and the candidate for the limiting energy

$$I_{e_2}(v) := \begin{cases} \int_{(0,1)} |Dv^{ac} + 1|^2 \,\mathrm{d}\mathcal{L}^1 + 2\sigma\mathcal{H}^0(Jv) & u \in SBV_{+,1} \\ \infty & u \notin SBV_{+,1}. \end{cases}$$

We will prove the following statement:

Theorem 2.1 (One-dimensional Γ-limit)**.** *Let I_{e_2}, $I_{e_2}^\theta$ be defined on $BV((0,1))$. Then:*

$$I_{e_2}^\theta \xrightarrow[\theta\searrow 0]{\Gamma} I_{e_2} \text{ with respect to weak-}* \ BV_{loc}\text{-convergence.}$$

The proof will be divided into two parts. The $\liminf$-inequality, stated in Lemma 2.2 and the recovery sequence, constructed in Lemma 2.3.

Remark. We prove Γ-convergence with respect to *local* weak-$*$ convergence in BV although we could also prove it with respect to weak-$*$ convergence. In the proof of the two-dimensional result, this theorem will be applied during a slicing argument whilst the recovery sequence is constructed differently. We therefore present the more general $\liminf$-inequality here.

2.1 The $\liminf$-inequality

Lemma 2.2 (One-dimensional $\liminf$-inequality)**.** *Let $I = (0, \lambda)$ and $\theta_k \searrow 0$. Then for all $\{u_k \mid k \in \mathbb{N}\} \subseteq BV_{loc}(I)$, $u \in BV_{loc}(I)$ such that $u_k \overset{*}{\rightharpoonup} u$ in $BV_{loc}(I)$ it holds*

$$I_{e_2}(u) \le \liminf_{k\in\mathbb{N}} I_{e_2}^{\theta_k}(u_k).$$

Moreover it holds that $2\sigma\mathcal{H}^0(Ju) \le \liminf_{k\in\mathbb{N}} \sigma\theta_k|D^2u_k|(I)$ and $u_k'\chi_{\{u_k'\le \frac{1}{\theta_k}-\frac{1}{2}\}} \rightharpoonup Du^{ac}$ in $L^2(I)$.

Proof. Assume without loss of generality that $\liminf_{k\in\mathbb{N}} I_{e_2}^{\theta_k}(u_k) < \infty$. Hence $u_k \in \mathcal{B}_\lambda$ eventually. We choose a subsequence such that $\liminf_{k\in\mathbb{N}} I_{e_2}^{\theta_k}(u_k) = \lim_{k\to\infty} I_{e_2}^{\theta_k}(u_k)$. Choosing further subsequences is hence no loss of generality.

A first task is to identify the jump points of u. We expect that these points are approximated by intervals of length $\sim \theta_k$ where $\partial_y u \sim \frac{1}{\theta_k}$. Having much larger intervals would imply an unbounded L^1-norm of the functions whilst shorter intervals can not create jump in the limit. In order to define these approximating intervals we choose $\eta \in (0, \frac{1}{4})$, fix $z_0 = 0$ and define

$$x_n^k = \inf\{x \ge z_{n-1}^k | u_k'(x) \ge \frac{1-\eta}{\theta_k}\} \qquad \text{and}$$
$$z_n^k = \inf\{x \ge x_n^k | u_k'(x) \le \frac{\eta}{\theta_k}\} \qquad \text{recursively.}$$
$$\text{Define moreover} \quad y_n^k = \sup\{x \le x_n^k | u_k'(x) \le \frac{\eta}{\theta_k}\},$$

see Figure 7 for a sketch of this partition. The idea is that in the intervals (y_i^k, z_i^k) the derivative u_k' is not small, where $x_i^k \in (y_i^k, z_i^k)$ ensures that it also gets large enough to possibly create a jump in the limit.

By definition we have that $y_n^k \le x_n^k \le z_n^k \le y_{n+1}^k$. Assume first that we have $y_n^k < x_n^k < z_n^k$. The derivate changes from ~ 0 to $\sim \frac{1}{\theta_k}$ and back to ~ 0 in the interval (y_i^k, z_i^k). This is taken into account

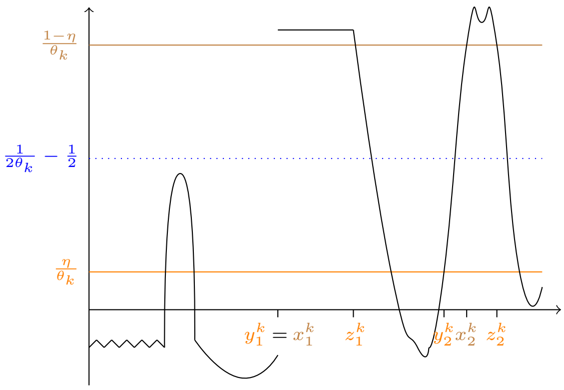

Figure 7: The intervals (y_n^k, z_n^k) will converge to the jump points of the limiting function u. The plotted function is a possible choice for a derivative u_k'.

by the second derivative:

$$\begin{aligned}\sigma\theta_k|D^2u_k|(y_n^k, z_n^k) &\geq \sigma\theta_k(|u_k'(x_n^k) - u_k'(y_n^k)| + |u_k'(z_n^k) - u_k'(y_n^k)| \\ &\geq \sigma\theta_k\left(\frac{1-2\eta}{\theta_k} + \frac{1-2\eta}{\theta_k}\right) \\ &= 2\sigma(1-2\eta).\end{aligned} \tag{2}$$

Hence we derive that for η small enough there are at most $N := \lfloor \frac{C}{2\sigma} \rfloor$ many of these intervals, independent of the choice of k. Assume there were $N+1$ many. We get a contradiction by computing

$$\begin{aligned}C \geq I_{e_2}^{\theta_k}(u_k) &\geq \sum_{n=1}^{N+1} \sigma\theta_k|D^2u_k|(y_n^k, z_n^k) \\ &\geq \sigma\sum_{n=1}^{N+1} 2(1-2\eta) \\ &\geq 2\sigma(N+1)(1-2\eta).\end{aligned}$$

If we only have $y_n^k \leq x_n^k \leq z_n^k$ we can choose by definition $\tilde{y}_n^k < \tilde{x}_n^k < \tilde{z}_n^k < \tilde{y}_{n+1}^k$ such that $u'(\tilde{y}_n^k), u'(\tilde{z}_n^k) \leq \frac{\eta}{\theta_k} + \varepsilon$ and $u'(\tilde{y}_n^k) \geq \frac{1-\eta}{\theta_k} - \varepsilon$ and the same argument holds.

We can therefore without loss of generality assume (going over to at most N many subsequences) that $x_n^k \to x_n \in [0,1]$ as $k \to \infty$. We expect the union of these points to be the jump set of u, but obviously they do not have to be located in the interior of the domain or to be disjoint. This is no real concern since we are interested in an upper bound and a jump that 'flows out of the domain' or two jumps that 'merge to one' will only create additional energy that is not seen in the limit.

If $x_n \in (0,1)$ then there is $\delta > 0$ such that $B_\delta(x_n) \subset\subset I$. Then $x_n^k \in B_{\delta/2}(x_n)$ eventually. Due to the weak-$*$ BV_{loc} convergence there is C_{x_n} such that $\|u_k\|_{W^{1,1}(B_\delta(x_n))} \leq C_{x_n}$.

Then $\mathcal{L}^1(\{u_k' > \frac{\eta}{\theta_k}\} \cap B_\delta(x_n)) \leq C_{x_n}\frac{\theta_k}{\eta}$. If this is the case than we know that y_n^k and z_n^k must lie in $B_\delta(x_n)$ eventually and hence in particular $\mathcal{L}^1((y_n^k, z_n^k)) \leq C_{x_n}\frac{\theta_k}{\eta}$.

We expect that jumps at the boundary disappear in the limit. I.e., if $x_n = 0$ then we expect that

$z_n^k \to 0$. Assume this is not the case. Then there is $\delta > 0$ such that without loss of generality $z_n^k \geq \delta$. It follows that $u_k' \geq \frac{\eta}{\theta_k}$ on $(\frac{\delta}{2}, \delta)$ and $\|u_k'\|_{L^1((\delta/2,\delta))} \geq \frac{\delta}{2}\frac{\eta}{\theta_k}$ leads to a contradiction if we let $k \to \infty$. The same is true if $x_n = \lambda$

We now define

$$A_0^k := \bigcup_{n=1}^N (y_n^k, z_n^k), \quad A_1^k := \left\{ u_k' \leq \frac{1}{\theta_k} - \frac{1}{2} \right\} \setminus A_0^k, \quad A_2^k := I \setminus (A_0^k \cup A_1^k)$$

and

$$f_k = u_k' \chi_{A_1^k}, \quad g_k = u_k' \chi_{A_2^k}, \quad \nu_k = u_k' \chi_{A_0^k} \mathrm{d}\mathcal{L}^1.$$

That is: The functions f_k are the part of the derivative that lives on the 'good' set and will create the absolute continuous part of the derivative of u. The jump part of the derivative of u is created by the intervals (y_n^k, z_n^k) on which $u_k' \sim \frac{1}{\theta_k}$, that is by the functions ν_k. There may remain a part where u_k' is large, but not large enough to create jumps. This will only happen on a small set and therefore this part of the derivative, called g_k, will converge strongly to zero in L^1. We will now make this rigorous. We can estimate

$$\begin{aligned} \|f_k\|_{L^2(I)}^2 = \int_{A_1^k} |u_k'|^2 \, \mathrm{d}\mathcal{L}^1 &\leq \int_{A_1^k} \min\{|u_k' + 1|^2, |u_k' - \frac{1}{\theta_k}|^2\} \, \mathrm{d}\mathcal{L}^1 + \int_{A_1^k} 1 \, \mathrm{d}\mathcal{L}^1. \\ &\leq C + \lambda. \end{aligned}$$

It follows, after choosing a subsequence, that there is $f \in L^2(I)$ such that $f_k \rightharpoonup f$ in $L^2(I)$ and $L^1(I)$. As a next step we want to analyze the limiting behavior of g_k. Notice, that for all $x \in A_2^k$ it holds that:

$$|u_k'(x) - \frac{1}{\theta_k}| \leq |u_k'(x) + 1| \quad \text{and hence} \quad |u_k'(x) - \frac{1}{\theta_k}| \geq \frac{1}{4\theta_k}.$$

Then

$$\begin{aligned} C \geq I_{e_2}^{\theta_k}(u_k) &\geq \int_{A_2^k} \min\{|u_k' + 1|^2, |u_k' - \frac{1}{\theta_k}|^2\} \, \mathrm{d}\mathcal{L}^1 \\ &\geq \mathcal{L}^1(A_2^k) \frac{1}{\theta_k^2} \end{aligned} \tag{3}$$

and we can therefore estimate

$$\begin{aligned} \|g_k\|_{L^1(I)} &\leq \int_{A_2^k} |u_k' - \frac{1}{\theta_k}| \, \mathrm{d}\mathcal{L}^1 + \frac{1}{\theta_k} \mathcal{L}^1(A_2^k) \\ &\leq C \mathcal{L}^1(A_2^k)^{1/2} + \frac{1}{\theta_k} \mathcal{L}^1(A_2^k) \\ &\leq C\theta_k + C\theta_k. \end{aligned}$$

We therefore know that $g_k \to 0$ in $L^1(I)$. Notice that moreover:

$$\|g_k\|_{L^2(I)}^2 \leq C + \frac{1}{\theta_k^2} \mathcal{L}^1(A_2^k) \leq C. \tag{4}$$

Putting this together and using the fact that Du^{ac} is a radon measure we see that on every compact set

$$\nu_k = u_k' \mathrm{d}\mathcal{L}^1 - f_k \mathrm{d}\mathcal{L}^1 - g_k \mathrm{d}\mathcal{L}^1 \overset{*}{\rightharpoonup} Du - f =: \nu,$$

where ν is a radon measure on every compact set. Now choose $K \subseteq I \setminus \bigcup_{n=1}^N \{x_n\}$ compact and arbitrary. Then there is $\varepsilon > 0$ such that

$$\operatorname{dist}\left(B_\varepsilon(K), \bigcup_{n=1}^N \{x_n\} \cup \{0,1\}\right) \geq \varepsilon.$$

Since we have that $(y_n^k, z_n^k) \subseteq B_\varepsilon(\bigcup_{n=1}^N \{x_n\})$ eventually we may conclude

$$|\nu|(K) \leq |\nu|(B_\varepsilon(K)) \leq \liminf_{k\in\mathbb{N}} |\nu_k|(B_\varepsilon(K)) = 0.$$

We hence know that $\nu = \sum_{n=1}^N c_n \delta_{x_n}$ and in particular that $\nu \perp \mathcal{L}^1$. Since $Du = f + \nu$ it follows that $u \in SBV(I)$ and $Du^{ac} = f$. The fact that $f_k \rightharpoonup f$ in L^2 yields

$$\|Du^{ac} - 1\|_{L^2(I)}^2 \leq \liminf_{k\in\mathbb{N}} \|f_k - 1\|_{L^2(I)}^2 \leq \liminf_{k\in\mathbb{N}} \|\min\{|u_k' + 1|, |u_k' - \frac{1}{\theta_k}|\}\|_{L^2(I)}^2. \tag{5}$$

We want to prove that f is even approximated by the sequence $u_k' \chi_{\{u_k' \leq \frac{1}{\theta_k} - \frac{1}{2}\}}$ weakly in L^2. Define $D^k = \{u_k' \in [\frac{\eta}{\theta_k}, \frac{1}{\theta_k} - \frac{1}{2}]\}$. We have that $|f_k - u_k' \chi_{\{u_k' \leq \frac{1}{\theta_k} - \frac{1}{2}\}}|(x) \leq u_k'(x) \chi_{D_k}(x)$. In similarity to (3) we estimate $\mathcal{L}^1(D_k) \leq \theta_k^2$. It follows:

$$\|f_k - u_k' \chi_{\{u_k' \leq \frac{1}{\theta_k} - \frac{1}{2}\}}\|_{L^1(I)} \to 0 \quad \text{and} \quad \|f_k - u_k' \chi_{\{u_k' \leq \frac{1}{\theta_k} - \frac{1}{2}\}}\|_{L^2(I)}^2 \leq C.$$

We have hence proven that $u_k' \chi_{\{u_k' \leq \frac{1}{\theta_k} - \frac{1}{2}\}} \rightharpoonup f = Du^{ac}$ in $L^2(I)$.

As a next step we want to show that jumps of the limiting function are of positive height i.e., $u^+(x_n) > u^-(x_n)$.

Fix $\bar{x} \in Ju$ for this purpose. We have seen that a point is in the jump set if it has been approximated by at least one of the intervals in A_k^1 i.e., $\bar{x} = x_n$ for some $n \in \{1, \dots, N\}$ and $y_n^k, z_n^k \to \bar{x}$ as $k \to \infty$. Let $r > 0$ such that $B_r(\bar{x}) \subseteq I$ and $Ju \cap B_r(\bar{x}) = \{\bar{x}\}$ and let additionally $y_n^k, z_n^k \in B_{r/2}(\bar{x})$.

Assume for the moment that there are no other intervals (y_l^k, x_l^k) converging to $\bar{x}$. Notice that due to the usual argument and (4) we know that

$$\|u_k'\|_{L^2(B_r(\bar{x}) \setminus (y_n^k, z_n^k))}^2 \leq C.$$

Therefore for all $y < y_n^k$ it holds

$$\begin{aligned} |u_k(y) - u_k(y_n^k)| &\leq \int_y^{y_n^k} |u_k'| \, \mathrm{d}\mathcal{L}^1 \\ &\leq |y - y_n^k|^{1/2} \|u_k'\|_{L^2(B_r(\bar{x}) \setminus (y_n^k, z_n^k))} \\ &\leq C|y - y_n^k|^{1/2} \end{aligned} \tag{6}$$

and by the same argument for all $x > x_n^k$

$$|u_k(x) - u_k(x_n^k)| \leq C|x - x_n^k|^{1/2}.$$

Let $\{a_i | i \in \mathbb{N}\} \subseteq (0, \bar{x})$, $\{b_i | i \in \mathbb{N}\} \subseteq (\bar{x}, 1)$ sequences such that $a_i, b_i \to \bar{x}$, $u(a_i) \to u(\bar{x}^-)$ and $u(b_i) \to u(\bar{x}^+)$. Then there is a subsequence such that $y_n^{k_i} \geq a_i$, $z_n^{k_i} \leq b_i$. Then $u(y_n^{k_i}) \to u(\bar{x}^-)$ and $u(z_n^{k_i}) \to u(\bar{x}^+)$. Since $u(y_n^{k_i}) < u(z_n^{k_i})$ we can conclude that $u(\bar{x}^-) \leq u(\bar{x}^+)$.
If there is more then one interval converging against $\bar{x}$ then we may use equation (6) again to deduce that the space between these intervals does not produce any negative derivative in the limit.
We have hence proven that the limiting function u is indeed an element of $SBV_{+,\lambda}$.
Recalling estimate (2) and the fact that $Ju = \bigcup_{n=1}^N \{x_n\}$ we conclude

$$2\sigma(1-2\eta)\mathcal{H}^0(Ju) \leq \sigma \sum_{n=1}^N 2(1-2\eta) \leq \sigma \liminf_{k\in\mathbb{N}} \sum_{n=1}^N \theta_k |D^2 u_k|(y_n^k, z_n^k) \leq \liminf_{k\in\mathbb{N}} \sigma\theta_k |D^2 u_k|(I).$$

Letting $\eta \to 0$ yields the $\liminf$-inequality for this term, the inequality in the absolute continuous part has already been proven in (5). □

2.2 The recovery sequence

The next step is to construct a recovery sequence. We will first mollify the function away from the jump set and afterwards fill the jumps with affine functions with slope $\frac{1}{\theta}$, see Figure 8 for a sketch of the construction.

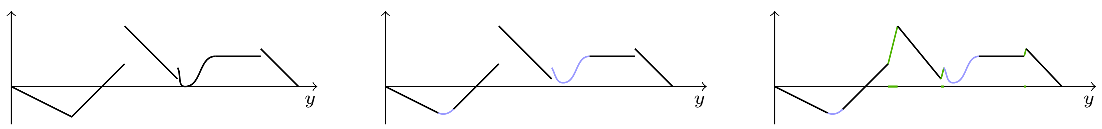

Figure 8: The recovery sequence in one dimension: We first mollify the function, keeping the jump set as it was, and replace the jumps afterwards with a linear interpolation with slope $\frac{1}{\theta}$ on intervals of length $\theta[u]$.

Lemma 2.3 (Existence of a recovery sequence)**.** *Let $I = (0, \lambda)$ and $\theta_k \searrow 0$. Let $u \in SBV_{+,\lambda}$ such that*

$$I_{e_2}(u) = \int_I |Du^{ac} + 1|^2 \, d\mathcal{L}^1 + 2\sigma\mathcal{H}^0(Ju) < \infty.$$

Then there is a sequence $\{u_k \,|\, k \in \mathbb{N}\} \subseteq \mathcal{B}_\lambda$ such that

$$u_k \overset{*}{\rightharpoonup} u \text{ in } BV \quad \text{and} \quad I_{e_2}^{\theta_k}(u_k) \to I_{e_2}(u).$$

Proof. There is $N \in \mathbb{N}$ and points $x_1 < x_2 < \cdots < x_{N-1} < x_N$ such that $Ju = \{x_1, \ldots x_N\}$.
We choose for the sake of simplicity $N = 1$. The case with arbitrary N is a straightforward generalization. Let $u_r = u^+(x_1)$, $u_l = u^-(x_1)$ and $h = u_r - u_l > 0$.
Step 1: Smooth approximation.
We will show that for every $\varepsilon > 0$ there is a function $v \in SBV(I)$ with $v^+(x_1) > v^-(x_1)$ and

$v \in W^{2,1}(I \setminus \{x_1\}) \cap C^\infty(I \setminus \{x_1\})$ such that $\|v-u\|_{L^1(I)} \leq \varepsilon$, $|D^2 v|(I) \leq C(u)$, $\|Dv^{ac}\|_{C^0} \leq C(u)$ and $I(v) \leq I(u) + C\varepsilon$.

Let φ_{ρ_k} a standard-mollifier. We mirror the function at the boundary and at the jump, that is we define

$$u_l(x) = \begin{cases} 2u(0) - u(-x) & x \leq 0 \\ u(x) & 0 \leq x \leq x_1 \\ 2u_l - u(2x_1 - x) & x \geq x_1 \end{cases} \quad \text{and} \quad u_r(x) = \begin{cases} 2u_r - u(2x_1 - x) & x \leq x_1 \\ u(x) & x_1 \leq x \leq \lambda \\ 2u(1) - u(1-x) & x \geq \lambda. \end{cases}$$

We define the mollification of these functions by $v_l^k := u_l * \varphi_{\rho_k}$ and $v_r^k := u_r * \varphi_{\rho_k}$ and the approximating function by $v^k = v_l^k \chi_{(0,x_1)} + v_r^k \chi_{(x_1,\lambda)}$.

Then $v^k \to u$ in $W^{1,2}(I \setminus \{x_1\})$, $Jv^k \subseteq \{x_1\}$ and $v^k \in W^{2,1}(I \setminus \{x_1\}) \cap C^\infty(I \setminus \{x_1\})$.

The only thing that remains to show is that ${v^k}^-(x_1) < {v^k}^+(x_1)$. Due to fundamental theorem of integration theory it holds that

$$\begin{aligned} v_l^k(x_1) &= \int_{x_1-\rho_k}^{x_1+\rho_k} u_l(z)\varphi_k(x_1 - z)\,\mathrm{d}z \\ &= \int_{x_1}^{x_1+\rho_k} \left(\int_{x_1}^{z} u_l'(\tilde{z})\,\mathrm{d}\tilde{z} + u^-(x_1) \right) \varphi_k(x_1 - z)\,\mathrm{d}z \\ &\quad + \int_{x_1-\rho_k}^{x_1} \left(-\int_{z}^{x_1} u_l'(\tilde{z})\,\mathrm{d}\tilde{z} + u^-(x_1) \right) \varphi_k(x_1 - z)\,\mathrm{d}z \\ &= u^-(x_1) - \int_{x_1}^{x_1+\rho_k} \int_{x_1}^{z} u'(2x_1 - \tilde{z})\,\mathrm{d}\tilde{z}\varphi_k(x_1 - z)\,\mathrm{d}z - \int_{x_1-\rho_k}^{x_1} \int_{z}^{x_1} u'(\tilde{z})\,\mathrm{d}\tilde{z}\varphi_k(x_1 - z)\,\mathrm{d}z \end{aligned}$$

Using the same arguments for v_r^k we see that

$$v_r^k(x_1) - v_l^k(x_1) \geq h - 2\|u'\|_{L^1((x_1-\rho_k, x_1+\rho_k))}.$$

It follows that the jump remains of positive height for k large enough.

Step 2: The recovery sequence.

Due to the considerations of Step 1 we may assume without loss of generality that $u \in W^{2,1}(I \setminus \{x_1\})$ and $\|Du^{ac}\|_{C^0}$ is finite. Define $\psi(x) = \frac{\lambda - x_1}{\lambda - x_1 - h\theta_k} x - \frac{h\theta_k}{\lambda - x_1 - h\theta_k}$ the linear interpolation between x_1 and λ on $[x_1 + \theta_k h, \lambda]$ and let

$$u_k(x) = \begin{cases} u(x) & 0 \leq x \leq x_1 \\ u_l + (x - x_1)\frac{1}{\theta_k} & x_1 \leq x \leq x_1 + \theta_k h \\ u(\psi(x)) & x_1 + \theta_k h \leq x \leq \lambda. \end{cases}$$

The function u_{θ_k} is continuous,

$$u_k'(x) = \begin{cases} Du^{ac}(x) & 0 \leq x \leq x_1 \\ \frac{1}{\theta_k} & x_1 \leq x \leq x_1 + \theta_k h \\ Du^{ac}(\psi(x))\psi'(x) & x_1 + \theta_k h \leq x \leq \lambda \end{cases}$$

and

$$\begin{aligned} D^2u_k =& D^2u^{ac}\chi_{[0,x_1]} + \left((D^2u^{ac}\circ\psi)\cdot\psi'^2 + (Du^{ac}\circ\psi)\cdot\psi''\right)\chi_{[x_1+h\theta_k,\lambda]} \\ &+ \left(\frac{1}{\theta_k} - Du^{ac+}(x_1)\right)\delta_{x_1} + \left(Du^{ac-}(x_1)\psi'(x_1+\theta_k h) - \frac{1}{\theta_k}\right)\delta_{x_1+h\theta_k}. \end{aligned}$$

We in particular know that $u_k \in \mathcal{B}_\lambda$. Notice first that since $|\psi'| \leq 2$ and $\psi'' = 0$ we have

$$|D^2u_k|(I) \leq C\|D^2u^{ac}\|_{L^1(I)} + 2(|Du^{ac-}(x_1)| + |Du^{ac+}(x_1)|) + \frac{2}{\theta_k}.$$

Since additionally $\|Du^{ac}\|_{C^0}$ is finite we conclude

$$\lim_{k\to\infty} \theta_k\sigma|D^2u_k|(I) \leq 2\sigma = 2\sigma\mathcal{H}^0(Ju).$$

Moreover we have that

$$\begin{aligned} &\int_I \min\{|u_k'+1|^2, |u_k' - \frac{1}{\theta}|^2\}\,\mathrm{d}\mathcal{L}^1 \\ \leq &\int_{(0,x_1)} |Du^{ac}+1|^2\,\mathrm{d}\mathcal{L}^1 + \int_{(x_1+h\theta_k,\lambda)} |Du^{ac}(\psi_k(x))\psi_k'(x)+1|^2\,\mathrm{d}\mathcal{L}^1. \end{aligned} \tag{7}$$

Since $Du^{ac} \in L^\infty((0,1))$ we can follow, using the dominated convergence theorem, that the right-hand side of (7) converges to $\int_I |Du^{ac}+1|^2\,\mathrm{d}\mathcal{L}^1$. Hence we know that

$$\lim_{k\to\infty} I_{e_2}^{\theta_k}(u_k) \leq I_{e_2}(u).$$

Due to the triangle inequality we see that $\|u_k'\|_{L^1(I\setminus(x_1,x_1+h\theta_k))}$ is uniformly bounded. Additionally we know that $\|u_k'\|_{L^1((x_1,x_1+h\theta_k))} = h$ and hence u_k is bounded in $W^{1,1}(I)$ and converges to u pointwise everywhere. Due to the standard BV compactness we know that there is a subsequence such that $u_k \overset{*}{\rightharpoonup} u$ in $BV(I)$. □

2.3 A compactness result

It is easy to see that this one-dimensional problem can not provide a compactness result since it is stated without any boundary values. Suppose u_k is a sequence of functions of bounded energy that does already converge in L^1 and weakly-$*$ in BV_{loc} and $\{c_k \mid k \in \mathbb{N}\}$ a monotone diverging sequence of constants. Then $u_k + c_k$ has no converging subsequence in L^1 but is of bounded energy.
We will, however, provide a compactness result under the additional assumption of an uniform L^1 bound. This result does not only complete the discussion of the one-dimensional problem but will later be used for the two-dimensional problem. It is stated in a L^p-setting to make it also applicable for the vector-valued problem.

Lemma 2.4. *Let $p \in (1,\infty)$, $\theta_k \searrow 0$, $u_k \in W^{1,p}((0,\lambda))$ and $\delta \in (0,\lambda/4)$.*
Then there is $C > 0$ such that

$$\|u_k'\|_{L^1((\delta,\lambda-\delta))} \leq C\left(\lambda + \lambda^{(p-1)/p}\|\min\{|u_k'+1|^p, |u_k' - \frac{1}{\theta_k}|^p\}\|_{L^p((0,\lambda))} + \frac{\|u_k\|_{L^1((0,\lambda))}}{\delta}\right).$$

Moreover:

$$\begin{aligned}&\mathcal{L}^1\left(\left\{\left|u_k' - \frac{1}{\theta_k}\right| < |u_k'+1|\right\} \cap (\delta, \lambda-\delta)\right)\\ \le & C\theta_k\left(\lambda + \lambda^{(p-1)/p}\|\min\{|u_k'+1|^p, |u_k' - \frac{1}{\theta_k}|^p\}\|_{L^p((0,\lambda))} + \frac{\|u_k\|_{L^1((0,\lambda))}}{\delta}\right)\end{aligned}$$

If additionally u_k is such that $\int_0^\lambda \min\{|u_k'+1|^p, |u_k'-\frac{1}{\theta_k}|^p\} \le M$ and $\|u_k\|_{L^1((0,\lambda))} \le L$ it follows that $\|u_k\|_{BV(\delta,\lambda-\delta)} \le C(\lambda + \lambda^{(p-1)/p}M^{1/2} + \frac{L}{\delta})$ and hence there is $u \in BV_{loc}((0,\lambda))$ such that $u_k \overset{*}{\rightharpoonup} u$ in $BV_{loc}((0,\lambda))$.

Proof. Let $M_k = \int_0^\lambda \min\{|u_k'+1|^p, |u_k'-\frac{1}{\theta_k}|^p\}$. We distinguish between phases in which the different terms contribute to the elastic energy and define the sets $A^k = \chi_{\{|u_k'+1| \le |u_k' - \frac{1}{\theta_k}|\}}$, $B^k = (0,\lambda)\setminus A^k$, $A^k_\delta = A^k \cap (\delta, \lambda-\delta)$ and $B^k_\delta = B^k \cap (\delta,\lambda-\delta)$. It holds:

$$\begin{aligned}\|u_k'\|_{L^1((\delta,\lambda-\delta))} &\le \|u_k'+1\|_{L^1(A^k_\delta)} + \|1\|_{L^1(A^k_\delta)} + \left\|u_k' - \frac{1}{\theta_k}\right\|_{L^1(B^k_\delta)} + \left\|\frac{1}{\theta_k}\right\|_{L^1(B^k_\delta)}\\ &\le \lambda + \lambda^{(p-1)/p}M_k^{1/p} + \frac{1}{\theta_k}\mathcal{L}^1(B^k_\delta) \qquad (8)\end{aligned}$$

We only need to control the last term to conclude the proof.
The idea is that if the set B^k_δ is large, the function u_k gets large at $\lambda-\delta$ (or negative with large absolute value at δ) and is therefore large in $L^1((0,\lambda)\setminus(\delta,\lambda-\delta))$ and hence in $L^1((0,\lambda))$. Let without loss of generality $u(\frac{\lambda}{2}) \ge 0$. Then it holds that

$$u(\lambda-\delta) \ge \int_{\lambda/2}^{\lambda-\delta} u_k' \, \mathrm{d}\mathcal{L}^1 \ge \int_{B^k_\delta} u_k' \, \mathrm{d}\mathcal{L}^1 - \int_{A^k_\delta} |u_k'| \, \mathrm{d}\mathcal{L}^1 \ge \frac{1}{4\theta_k}\mathcal{L}^1(B^k_\delta) - \lambda^{(p-1)/p}M_k^{1/p} - \lambda.$$

For $x \ge \lambda-\delta$ we compute, using the fact that $\int_{\lambda-\delta}^x u_k' \, \mathrm{d}\mathcal{L}^1 \ge -\lambda^{(p-1)/p}M_k^{1/p} - \delta$,

$$u_k(x) = u_k(\lambda-\delta) + \int_{\lambda-\delta}^x u_k' \, \mathrm{d}\mathcal{L}^1 \ge \frac{1}{4\theta_k}\mathcal{L}^1(B^k_\delta) - \lambda^{(p-1)/p}M_k^{1/p} - \lambda - \lambda^{(p-1)/p}M_k^{1/p} - \delta.$$

This yields, after integrating in x from $\lambda-\delta$ to λ and dividing by δ,

$$\frac{1}{\theta_k}\mathcal{L}^1(B^k_\delta) \le \frac{4}{\delta}\|u_k\|_{L^1(I)} + 8\lambda^{(p-1)/p}M_k^{1/p} + 8\lambda$$

which, together with (8) finishes the proof.

□

3 Compactness and $\liminf$-inequality for the scalar-valued problem

After we understood the behavior of the functional reduced on one-dimensional slices we can now turn to the analysis of the two-dimensional, scalar-valued problem.
We denote the two-dimensional unit square by $\Omega = (0,1)^2$. We remind ourselves that we want to analyze the functional

$$I^\theta(u) = \begin{cases} \int_\Omega |\partial_x u|^2 \,\mathrm{d}\mathcal{L}^2 + \int_\Omega \min\{|\partial_y u + 1|^2, |\partial_y u - \frac{1}{\theta}|^2\} \,\mathrm{d}\mathcal{L}^2 + \sigma\theta |D^2 u|(\Omega) & u \in \mathcal{A} \\ \infty & u \notin \mathcal{A} \end{cases}$$

defined on the space

$$\mathcal{A} = \Big\{ u \in W^{1,2}(\Omega) \,|\, \partial_x u, \partial_y u \in BV(\Omega),\ u(0,y) = \frac{\theta^2}{1+\theta} y \Big\}.$$

Parts of I^θ are given as an integrated version of $I^\theta_{e_2}$. We will see in the proof of the $\liminf$-inequality how the functionals are connected via the method of slicing. It is reasonable to expect the limiting functional and function space also to be given as integrated version of I_{e_2} and SBV_+ respectively. We therefore consider

$$I(u) = \begin{cases} \int_\Omega |\partial_x u|^2 \,\mathrm{d}\mathcal{L}^2 + \int_\Omega |\partial_y u + 1|^2 \,\mathrm{d}\mathcal{L}^2 + 2\sigma \mathcal{H}^1(Ju) & u \in SBV^2_{e_2,0} \\ \infty & u \notin SBV^2_{e_2,0} \end{cases}$$

where the function space $SBV^2_{e_2,0}$ is given by

$$SBV^2_{e_2,0} = \{ u \in SBV_{loc}(\Omega) \,|\, D^J u \cdot e_1 = 0,\ D^J u \cdot e_2 \geq 0,\ u = 0 \text{ on } \{0\} \times (0,1),\ \nabla u \in L^2(\Omega, \mathbb{R}^2)$$
$$\mathcal{H}^1(Ju) < \infty \text{ and } |Du|((0,1) \times (\delta, 1-\delta)) < \infty \text{ for all } \delta > 0 \}.$$

The uppercase 2 denotes that the absolute continuous part of the gradient is in $L^2(\Omega)$, the lowercase e_2 denotes that jumps are only pointing in e_2 direction, the 0 denotes the zero-boundary values at the left edge. Functions in $SBV^2_{e_2,0}$ have jumps of positive height concentrated on horizontal lines, the jump part of the gradient only has to be a finite measure on compact sets of the form $(0,1) \times (\delta, 1-\delta)$. For the convergence we will use weak-$*$-convergence in $BV((0,1) \times (\delta, 1-\delta))$ for all $\delta > 0$ together with $L^1(\Omega)$-convergence.
The fact that the jump set does concentrate in horizontal lines comes from the regularization in x-direction provided from the $\|\partial_x u\|_{L^2}$-term in I^θ_k which has no θ_k-dependence.

3.1 Locality and the choice of convergence

We first want to comment on the choice of convergence for the problem and on the locality in the limiting function space, since they might seem unnecessarily complicated. We will provide two examples in the following that explain that these choices are reasonable. Notice that this choices of locality in the function space and the convergence are made such that the compactness statement holds. The Γ-limit without a compactness statement could also be proven for functions in SBV with respect to weak-$*$ BV-convergence on the entire square.

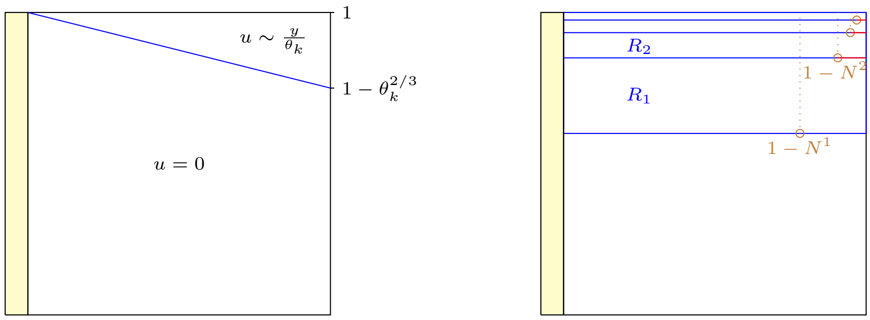

Figure 9: A sequence of bounded energy that does not converge in $BV(\Omega)$ (left) and a function in $SBV^2_{e_2,0} \setminus SBV(\Omega)$ (right). At the left edge, the boundary values are approximated on a small scale.

The following constructions are always performed with zero boundary values at the left edge. Obviously a linear interpolation on a small scale to the boundary values $u(0,y) = \frac{\theta^2}{1+\theta} y$ is still of uniformly bounded energy.

Remark. The problem has no compactness with respect to weak-$*$ BV-convergence. A counterexample is a needle of thickness $x\theta_k^{2/3}$ on which the function equals $\sim y\frac{1}{\theta_k}$, see the left part of Figure 9 for a sketch. Define the needle $T_k = \{(x,y) \in (0,1)^2 \,|\, y \geq 1 - x\theta_k^{2/3}\}$ and the function

$$u_k(x,y) = \begin{cases} \frac{y}{\theta_k} - \frac{1}{\theta_k} + \frac{x}{\theta_k^{1/3}} & (x,y) \in T_k \\ 0 & (x,y) \notin T_k \end{cases}$$

where the x-dependence in T_k ensures the continuity of the function. We easily compute that $\mathcal{L}^2(T_k) \sim \theta_k^{2/3}$, $|D^2 u_k|(\Omega) = \mathcal{H}^1(\partial T_k) \begin{pmatrix} \theta_k^{-1} \\ \theta_k^{-1/3} \end{pmatrix} \cdot \nu_{T_k} \leq 2\frac{1}{\theta_k}$,

$$\|\partial_x u_k\|^2_{L^2(\Omega} = \int_{T_k} \theta_k^{-2/3} \,\mathrm{d}\mathcal{L}^2 \leq 1$$

and

$$\int_\Omega \min\{|\partial_y u_k + 1|^2, |\partial_y u_k - \frac{1}{\theta_k}|^2\} \,\mathrm{d}\mathcal{L}^2 = 0.$$

Hence $I^{\theta_k}(u_k) \leq 3$. But it also holds that

$$\|\partial_y u_k\|_{L^1(\Omega)} = \int_{T_k} \frac{1}{\theta_k} \,\mathrm{d}\mathcal{L}^2 = \frac{1}{2}\theta_k^{-1/3}.$$

That is: Every subsequence is unbounded in $BV(\Omega)$ hence there can be no subsequence that converges weakly-$*$ in $BV(\Omega)$.
But of course $u_k \overset{*}{\rightharpoonup} 0$ in $BV((0,1) \times (\delta, 1-\delta))$ for all $\delta > 0$, so this is no contradiction to the compactness result in Theorem 3.1.

Remark. We have previously seen that it is reasonable to work with weak-$*$-convergence restricted on $(0,1)\times(\delta,1-\delta)$. But since the limit in the foregoing construction was $u\equiv 0$ this does not yet explain the choice of the constraint $u\in SBV((0,1)\times(\delta,1-\delta))$ for the functions in $SBV^2_{e_2,0}$. One might imagine that it suffices to require $u\in SBV(\Omega)$, together with the constraints on the direction of the jump set an the integrability of the absolute continuous part of the gradient.
We want to point out that this is not the case and construct a function $u\in SBV^2_{e_2,0}$ such that $u\notin SBV(\Omega)$ but $I(u)<\infty$. Since we will prove in Chapter 4 that it is possible to construct a recovery sequence for each $u\in SBV^2_{e_2,0}$, we have that the energy of this recovery sequence would be uniformly bounded and each converging subsequence would converge against u. Hence our choice of the space $SBV^2_{e_2,0}$ is reasonable.
Let $L,M,N>1$ and let $R_k=(0,1)\times(1-M^{-k},1-M^{-k-1})$. We define the function u by

$$u|_{R_k}(x,y)=\sum_{i=1}^{k}L^i(x-(1-N^{-i}))\chi_{\{x>1-N^{-i}\}}(x,y)$$

and notice that the absolute continuous part of the y-derivative satisfies $\partial_y u=0$. If $L^2\geq N$ we may estimate

$$\|\partial_x u\|^2_{L^2(\Omega)}\leq\sum_{k\in\mathbb{N}}\sum_{i=1}^{k}(L^2)^iN^{-i}M^{-k}\leq\sum_{k\in\mathbb{N}}k\left(\frac{L^2}{NM}\right)^k$$

and $\mathcal{H}^1(Ju)=\sum_{k\in\mathbb{N}}N^{-k}$. Hence the energy of u is finite if $N>1$ and M is sufficient large in dependence of L, for example $M=L^3$. On the other hand we can estimate

$$\begin{aligned}|Du|(\Omega)\geq|D^Ju|(\Omega)\geq\sum_{k\in\mathbb{N}}\int_{1-N^{-k}}^{1}L^k(x-(1-N^{-k}))\,\mathrm{d}\mathcal{L}^2(x)&=\sum_{k\in\mathbb{N}}L^k\int_0^{N^{-k}}x\,\mathrm{d}\mathcal{L}^2(x)\\&=\frac{1}{2}\sum_{k\in\mathbb{N}}\left(\frac{L}{N^2}\right)^k.\end{aligned}$$

So $u\notin SBV(\Omega)$ if $L\geq N^2$ but $u\in SBV((0,1)\times(\delta,1-\delta))$ for all $\delta>0$.

Remark. The behavior that the functions grows strongly at the horizontal edges of the domain might be unexpected for this problem. There are different possibilities to get rid of this behavior: A uniform L^∞ bound for the functions in $\mathcal{A}$, or periodic boundary conditions would both provide a compactness result in $BV(\Omega)$.

Remark. In general no trace theorem holds for $BV(\Omega)$ equipped with weak-$*$ convergence, since jumps can converge to the boundary and disappear in the limit. Take for example $u_\delta=\chi_{(\delta,1)\times(0,1)}$ which weakly-$*$ converges to $u=1$. However, as we will see in the proof of the compactness result, this is of no concern for our analysis since the jump set lies perpendicular to the part of the boundary on which the boundary values are located. The boundedness of the x-derivatives in L^2 yields a trace theorem for the left edge.

3.2 Compactness

We first prove the compactness result using slicing results and integration on slices. The main difficulty is to prove that the limiting function is indeed element of the function space $SBV^2_{e_2,0}$. The $\liminf$-inequality will afterwards be an easy consequence.

Theorem 3.1 (Compactness). *Let $\{u_k \mid k \in \mathbb{N}\} \subseteq \mathcal{A}$, $\{\theta_k \mid k \in \mathbb{N}\} \subseteq \mathbb{R}^+$ such that $\theta_k \searrow 0$, $I^{\theta_k}(u_k) \leq M$. Then there is a subsequence $\{k_l \mid l \in \mathbb{N}\} \subseteq \mathbb{N}$ and a function $u \in BV_{loc}(\Omega)$ such that*

$$u_{k_l} \overset{*}{\rightharpoonup} u \quad \text{in } BV((0,1) \times (\delta, 1-\delta)) \text{ for all } \delta < 0.$$

Moreover: $u \in SBV^2_{e_2,0}$ and for each $\delta > 0$ it holds that $\|u_k\|_{W^{1,1}((0,1)\times(\delta,1-\delta))} \leq C(1 + M + \frac{1+M^{1/2}}{\delta})$.

Proof. Step 1: Convergence
Let $M_k := I^{\theta_k}(u_k)$ and define the reduction of the energies on the horizontal slice $\{x\} \times (0,1)$ by $M_k^x = I_{e_2}^{\theta_k}(u(x,\cdot))$. Notice that for all $k \in \mathbb{N}$ and for almost every $x \in (0,1)$ we have that M_k^x is finite. We fix $\delta > 0$.
Due to fundamental theorem of integration theory and the boundary values we have

$$\|u_k(x,\cdot)\|_{L^1((0,1))} \leq C + \int_0^1 \int_0^x |\partial_x u_k(z,y)| \, \mathrm{d}z \, \mathrm{d}y \leq C + M_k^{1/2}.$$

Applying the one-dimensional compactness result (see Lemma 2.4) we can conclude

$$\|\partial_y u_k(x,\cdot)\|_{L^1((\delta,1-\delta))} \leq C\left(1 + (M_k^x)^{1/2} + \frac{1+M_k^{1/2}}{\delta}\right). \tag{9}$$

Notice, that either $M_k^x \leq 1$ or $(M_k^x)^{1/2} \leq M_k^x$ and therefore $\int_0^1 (M_k^x)^{1/2} \, \mathrm{d}x \leq 1 + M_k$. Integrating (9) in x-direction yields

$$\|u_k\|_{W^{1,1}((0,1)\times(\delta,1-\delta))} \leq C\left(1 + M_k + \frac{1+M_k^{1/2}}{\delta}\right) \leq C\left(1 + M + \frac{1+M^{1/2}}{\delta}\right).$$

So due to BV-compactness there exists a subsequence $\{k_l \mid l \in \mathbb{N}\} \subseteq \mathbb{N}$, selected with a diagonal argument, and a function $u \in BV_{loc}(\Omega)$ such that $u_{k_l} \overset{*}{\rightharpoonup} u$ in $BV((0,1) \times (\delta, 1-\delta))$ for all $\delta > 0$.
Step 2: $u \in SBV^2_{e_2,0}$
Since $\partial_x u_k$ is bounded in L^2 we get that, up to a subsequence, $\partial_x u_k \rightharpoonup v$ for some $v \in L^2(\Omega)$ and $v = Du \cdot e_1$. Hence $D^J u \cdot e_1 = D^C u \cdot e_1 = 0$. We do also know that $\partial_y u_k \overset{*}{\rightharpoonup} Du \cdot e_2$ locally as measures. We divide $\partial_y u_k$ into two additive terms. Let

$$f_k = \chi_{\{|\partial_y u_k + 1| \leq |\partial_y u_k - \frac{1}{\theta}{}_k|\}} \partial_y u_k \qquad \text{and} \qquad g_k = \partial_y u_k - f_k$$

Then $\|f_k\|_{L^2(\Omega)} \leq C$ and, after choosing a subsequence, $f_k \rightharpoonup f$ in $L^2(\Omega)$ for some $f \in L^2(\Omega)$. Moreover $g_k \geq 0$ and $g_k \overset{*}{\rightharpoonup} D^s u \cdot e_2 + \partial_y u - f$ locally as measures. We see that the locally finite, signed measure $D^s u \cdot e_2 + \partial_y u - f$ is actually positive and conclude, since $D^s u \cdot e_2 \perp \partial_y u - f$, that $D^s u \cdot e_2 \geq 0$.

We now want to prove that the Cantor part vanishes. We have $\int_{(0,1)} M_k^x \, \mathrm{d}\mathcal{L}^1(x) \leq I^{\theta_k}(u_k) \leq M$. Using Fatou's lemma we see that

$$\int_{(0,1)} \liminf_{k\in\mathbb{N}} M_k^x \, \mathrm{d}\mathcal{L}^1(x) \leq M$$

and hence that for almost every $x \in (0,1)$ we get that $\liminf M_k^x$ is finite. For almost every of these x we find a subsequence such that $M_{k_l}^x$ is uniformly bounded and use Lemma 2.4 to conclude that $u_{k_l}(x,\cdot) \overset{*}{\rightharpoonup} u(x,\cdot)$ in $BV((\delta,1-\delta))$ for all $\delta > 0$. The one dimensional $\liminf$-inequality tells us that $u(x,\cdot) \in SBV((\delta,1-\delta))$. This is independent of the subsequence so due to a slicing argument (see [3], Chapter 3.11) we know that this implies $D^c u \cdot e_2|_{((0,1)\times(\delta,1-\delta))} = 0$ for all $\delta > 0$.
As a last step we need make sure that u satisfies the zero boundary values. Define the extension $\hat{u}_k : (-1,1) \times (0,1) \to 0$ by

$$\hat{u}_k(x,y) = \begin{cases} u_k(x,y) & x \geq 0 \\ \frac{\theta^2}{1+\theta} y & x < 0. \end{cases}$$

Since u_k is bounded in $BV((0,1) \times (\delta, 1-\delta))$ for every $\delta > 0$ we immediately follow that $\hat{u}_k$ is bounded in $BV((-1+\delta,1-\delta) \times (\delta,1-\delta))$ for every $\delta > 0$. We therefore have that there exists a subsequence and a function $\hat{u} \in BV_{loc}((-1,1) \times (0,1))$ such that $\hat{u}_k \overset{*}{\rightharpoonup} \hat{u}$ in $BV_{loc}((-1,1 \times (0,1))$. Moreover we have that $\hat{u}_k \to u$ in $BV((-1,0) \times (0,1))$.
Hence $\hat{u}(x,y) = 0$ almost every where on $(-1,0) \times (0,1)$ $x < 0$ and $\hat{u} = u$ for $x > 0$, $D\hat{u}_k \cdot e_1 = \partial_x \hat{u}_k$ and $\partial_x \hat{u}_k \rightharpoonup \partial_x \hat{u}$ in L^2. We therefore know that $\hat{u}(\cdot, y)$ is continuous for almost every $y \in (0,1)$ and it follows that $u(0,y) = \hat{u}(0,y) = 0$ for $\mathcal{H}^1$-almost every $y \in (0,1)$.
Hence all properties are proven and we know that $u \in SBV^2_{e_2,0}$. □

3.3 The $\liminf$-inequality

The $\liminf$-inequality is now an straightforward application of the one-dimensional result together with the slicing method and the previous compactness result.

Theorem 3.2 (The $\liminf$-inequality). *Let $\{u_k | k \in \mathbb{N}\} \subseteq BV(\Omega)$, $u \in BV(\Omega)$ such that*

$$u_k \overset{*}{\rightharpoonup} u \quad \textit{in } BV_{loc}((0,1) \times (\delta, 1-\delta)) \textit{ for all } \delta > 0$$

Let $\theta_k \searrow 0$. Then

$$I(u) \leq \liminf_{k\in\mathbb{N}} I^{\theta_k}(u_k).$$

Proof. Let us without loss of generality assume that $\liminf_{k\in\mathbb{N}} I^{\theta_k}(u_k) < \infty$ and choose a subsequence that realizes the $\liminf$.
The compactness result tells us immediately that $u \in SBV^2_{e_2,0}$ and we know that $\partial_x u_k \rightharpoonup \partial_x u$ in L^2. As in the compactness result it is a consequence of Fatou's lemma that for almost every $x \in (0,1)$ it holds that $\liminf_{k\in\mathbb{N}} I^{\theta_k}_{e_2}(u_k(x,\cdot))$ and $\liminf_{k\in\mathbb{N}} \|u_k(x,\cdot)\|_{L^1((0,1))}$ are finite. Fix one of these $x \in (0,1)$. We go to a subsequence that realizes the $\liminf$ for this choice of x, fix $\delta > 0$ and use the one-dimensional compactness result (see Lemma 2.4) to see that for a further subsequence $\{k_l \,|\, l \in \mathbb{N}\}$ it holds that $u_{k_l}(x,\cdot) \overset{*}{\rightharpoonup} u(x,\cdot)$ in $BV((\delta,1-\delta))$.

Application of the one-dimensional lim inf-inequality on the interval $(\delta, 1-\delta)$ tells us

$$\int_{(\delta,1-\delta)} |\partial_y u(x,\cdot) - 1|^2 \, \mathrm{d}\mathcal{L}^1 + |\partial_y \partial_y u(x,\cdot)|((\delta, 1-\delta)) \leq \liminf_{k\in\mathbb{N}} I_{e_2}^{\theta_k}(u_k(x,\cdot)).$$

This is independent of the chosen subsequence and holds for all $\delta > 0$ hence for almost every $x \in (0,1)$ it holds

$$\int_{(0,1)} |\partial_y u(x,\cdot) - 1|^2 \, \mathrm{d}\mathcal{L}^1 + |\partial_y \partial_y u(x,\cdot)|((0,1)) \leq \liminf_{k\in\mathbb{N}} I_{e_2}^{\theta_k}(u_k(x,\cdot)).$$

Putting this together with Fatou's lemma we can conclude

$$\begin{aligned}
&\liminf_{k\in\mathbb{N}} I^{\theta_k}(u_k) \\
&\geq \liminf_{k\in\mathbb{N}} \left(\int_\Omega |\partial_x u_k|^2 \, \mathrm{d}\mathcal{L}^2 + \int_\Omega \min\{|\partial_y u_k + 1|^2, |\partial_y u_k - \frac{1}{\theta}|^2\} \, \mathrm{d}\mathcal{L}^2 + \sigma\theta_k |\partial_y \partial_y u_k|(\Omega) \right) \\
&\geq \liminf_{k\in\mathbb{N}} \int_\Omega |\partial_x u_k|^2 \, \mathrm{d}\mathcal{L}^2 + \liminf_{k\in\mathbb{N}} \int_{(0,1)} I_{e_2}^{\theta_k}(u_k(x,\cdot)) \, \mathrm{d}\mathcal{L}^1(x) \\
&\geq \int_\Omega |\partial_x u|^2 \, \mathrm{d}\mathcal{L}^2 + \int_{(0,1)} \liminf_{k\in\mathbb{N}} I_{e_2}^{\theta_k}(u_k(x,\cdot)) \, \mathrm{d}\mathcal{L}^1(x) \\
&\geq \int_\Omega |\partial_x u|^2 \, \mathrm{d}\mathcal{L}^2 + \int_{(0,1)} I_{e_2}(u(x,\cdot) \, \mathrm{d}\mathcal{L}^1(x) \\
&= \int_\Omega |\partial_x u|^2 \, \mathrm{d}\mathcal{L}^2 + \int_\Omega |\partial_y u + 1|^2 \, \mathrm{d}\mathcal{L}^2 + \sigma \int_{(0,1)} \mathcal{H}^0(J(u(x,\cdot)) \, \mathrm{d}\mathcal{L}^1(x).
\end{aligned}$$

We use a standard result about slicing (see [3], Chapter 3.11) to prove that

$$\int_{(0,1)} \mathcal{H}^0(J(u(x,\cdot)) \, \mathrm{d}\mathcal{L}^1(x) = \int_{Ju} |v_{Ju}(z) \cdot e_2| \, \mathrm{d}\mathcal{H}^1(z) = \mathcal{H}^1(Ju),$$

which concludes the proof of the proof of the lim inf-inequality. □

4 The recovery sequence for the scalar-valued problem

The following chapter is one of the most central to this work. The aim is to provide a recovery sequence for an arbitrary function $u \in SBV^2_{y,0}$. As often in related problems we will do this in two steps: First provide a density result that tells us that only functions with some additional structure and regularity properties need to be recovered. Afterwards construct a recovery sequence for any function in this smaller space.

Notice that the limiting functional I differs from the well-known Mumford-Shah functional (see [40]) used in image segmentation only by an affine translation and the underlying function space. A detailed analysis of this well-understood functional can be found in the literature (beginning with [5] and [22]), and also density results are provided (see [18] and [24]). However, these results can not be used explicitly in our case since the function space in this setting does not include the constraints $D^J u \cdot e^1 = 0$ and $D^J u \cdot e_2 \geq 0$. We need to provide different constructions such that these constraints are satisfied. This includes modified covering arguments and involved local constructions.

We first want to provide the idea for the construction of a recovery sequence for a regular function u with a single segment as jump set. It originates from the one-dimensional construction and will be made rigorous at the end of this chapter. We want to include an affine function with slope $\frac{1}{\theta_k}$ in y-direction on a set of vertical height $\theta[u]|_{Ju}$ (see Figure 10). Along the boundary of this set the y-derivative jumps from $\frac{1}{\theta}$ to something small and hence the term $\theta|D^2 u|$ counts approximately the diameter of this set, which converges uniformly to $2\mathcal{H}^1(Ju)$ as $\theta \to 0$ if $[u] \in C^1$.

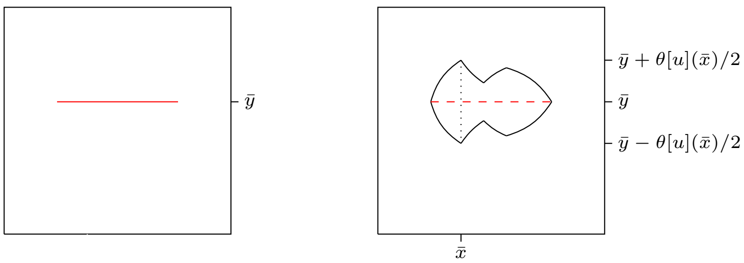

Figure 10: A sketch of the idea for the recovery sequence. The vertical height of the area in which we insert an affine function with slope $\frac{1}{\theta}$ in y-direction is given by $\theta[u]$.

However, this is not the general situation in which we want to construct a recovery sequence since the jump set of a SBV-function in two dimensions can be quite irregular. Imagine as a jump set the union of horizontal segments of length 2^{-n} centered at $\{q_n \mid n \in \mathbb{N}\} = (\mathbb{Q} \times \mathbb{Q}) \cap \Omega$. This is not only a dense set of finite $\mathcal{H}^1$-measure but also jump set of a function of finite energy I and hence we must be able to provide a recovery sequence for this function. Obviously, a direct application of the idea indicated in Figure 10 seems difficult, since the different areas in which we want to insert the affine function would overlap.

We will prove two different density results for the space $SBV^2_{e_2,0}$ and the energy I such that finally we will only have to recover functions with a jump set that is finite union of horizontal segments. Proving these density results will be a lengthy and sometimes technical task which will fill large parts of the following chapter. The first density result will approximate arbitrary functions with functions whose jump set is a compact set, whilst in the second result functions with compact jump set will be approximated by functions whose jump set is a finite union of segments. The general strategy of this chapter follows the approach of other density results as in the work of Braides and Chiadò Piat (see [10]), we will comment on the similarities and differences at a later point.

4.1 Preliminaries

We again denote the unit square in two dimension by $\Omega = (0,1)^2$.
We explained in Chapter 3 that the compactness result implies that there are $SBV_{loc} \setminus SBV$ functions of finite energy. We will see in the following lemma that it is sufficient to construct recovery sequences for functions in $SBV^2_{e_2,0} \cap SBV$. We are moreover able to construct the recovery sequence with respect to the stronger $BV(\Omega)$-convergence.

Lemma 4.1. *Let $\theta_k \searrow 0$. Assume for every function $v \in SBV(\Omega)$ with $D^J v \cdot e_2 \geq 0$, $D^J v \cdot e_1 = 0$ and $v(0,\cdot) = 0$ there exits a recovery sequence $v_k \in \mathcal{A}$ such that $v_k \overset{*}{\rightharpoonup} v$ in $BV(\Omega)$ and $I^{\theta_k}(v_k) \to I(v)$.*
Then, for every function $u \in SBV^2_{e_2,0}$ there exits a recovery sequence $u_k \in \mathcal{A}$ such that $u_k \overset{*}{\rightharpoonup} u$ in $BV((0,1) \times (\delta, 1-\delta))$ for all $\delta > 0$ and $I^{\theta_k}(u_k) \to I(u)$.

Proof. The essence of the proof is a simple diagonal argument. Let $\eta_k \searrow 0$ such that $\eta_k \leq \frac{1}{2}$. Define $v_k(x,y) = u(x,(1-2\delta)y + \delta)$. Then $v_k \in SBV(\Omega)$, $\limsup_{k\in\mathbb{N}} I(v_k) \leq I(v_k)$ and for each $\delta > 2\eta_k$ we know that $|Dv_k|((0,1) \times (\frac{\delta}{2}, 1-\frac{\delta}{2})) \leq (1+2\eta_k)|Du|((0,1) \times (\delta, 1-\delta))$ and $v_k \to v$ in $L^1((0,1) \times (\delta, 1-\delta))$.
Due to the assumption there is a recovery sequence v_k^l for each of these v_k's and so a diagonal sequence of the v_k^l's provides the recovery sequence for u. □

There is also a notational modification we want to introduce for this chapter. We will work with the Mumford-Shah functional J instead of the functional I. Define

$$J(u) = \int_\Omega \|\nabla u\|^2 \, \mathrm{d}\mathcal{L}^2 + \sigma \mathcal{H}^1(Ju)$$

which will now operate on the function space

$$SBV^2_{e_2,a} = \{u \in SBV(\Omega) \,|\, \nabla u \in L^2(\Omega, \mathbb{R}^2),\ D^J u \cdot e_1 = 0,\ D^J u \cdot e_2 \geq 0,\ \mathcal{H}^1(Ju) < \infty,\ u(0,y) = y\}.$$

This function space does not only have different boundary values, we also dropped the assumption that u is only in locally in SBV.
Let us also introduce a localized version of $SBV^2_{e_2}$: For a open set $A \subset\subset \Omega$ define

$$SBV^2_{e_2}(A) = \{u \in SBV(A) \,|\, \nabla u \in L^2(A, \mathbb{R}^2),\ D^J u \cdot e_1 = 0,\ D^J u \cdot e_2 \geq 0,\ \mathcal{H}^1(Ju) < \infty\}.$$

Notice that in our publication [16] the rescaling of the original problem is done differently such that also compactness and $\liminf$ inequality are proven for the space $SBV^2_{e_2,a}$. In that work the notation differs. The space $SBV^2_{e_2,a}$ is there denoted by $\overline{SBV}^2_y$.

Remark 4.2. Adding the affine transformation $g(x,y) = y$ to a function $u \in SBV^2_{e_2,0}$ yields a function $v \in SBV^2_{e_2,a}$, up to the fact that functions are only in SBV_{loc} and it holds $I(u) = J(v)$. A

recovery sequence for v then transforms to a recovery sequence for u under the subtraction of g and the application of Lemma 4.1.
It therefore suffices to prove the density results for the Mumford-Shah functional J on the space $SBV^2_{e_2,a}$.

Remark. The structure theorem for BV-functions tells us that Ju is a rectifiable set i.e., there are sets $K_0, K_1, \ldots \subseteq \Omega$ such that $Ju = \bigcup_{i\in\mathbb{N}} K_i$, $\mathcal{H}^1(K_0) = 0$ and for $i \geq 1$: $K_i \subseteq S_i$ for some C^1-curve S_i. The normal to a curve coincides with the normal to the jump set for $\mathcal{H}^1$-almost every point in K_i, which is e_2 $\mathcal{H}^1$-almost everywhere. We can therefore choose the S_k's without loss of generality to be C^1-graphs. It will in general not be true, that Ju lies in countably many vertical slices of the domain.
However, it is a consequence of Sard's theorem, used on each of the countably many C^1-curves S_k, that $\mathcal{L}^1(\{y \in (0,1) \,|\, (0,1) \times \{y\} \cap Ju \neq \emptyset\}) = 0$.

4.2 Local constructions

As noted above, the function u that we want to recover might have a jump set that is not regular. We will approximate u with a sequence of functions for which we can guarantee a higher regularity of the jump set.
We therefore first introduce different local constructions on small squares. These constructions will differ, in dependence on the amount and regularity of the jump set of u on these squares. We will distinguish between three different types of squares:

- Type-I squares, on which the amount of jump set of u is small. On these squares, all jump will be erased.
- Type-II squares on which the amount of jump set of u might be large and has no regularity. In this case, many additional jumps will be introduced and the error in energy will be relatively large. Hence this construction should not be applied too often.
- Type-III squares on which the amount of jump set of u is close to the sidelength of the square and on which the jump set does not expand much in vertical direction. On these squares we will construct a function whose jump set is the union of two segments each of length approximately half of the sidelength of the square.

We denote by $Q_R(x) = ((x_1 - R, x_1 + R) \times (x_2 - R, x_2 + R))$ the square of sidelength $2R$ and center x. The square of sidelength $2R$ centered at zero is denoted by $Q_R = (-R, R)^2$.

4.2.1 Construction on type-I squares

At first we want to construct the approximation on the so-called type-I squares, in which there is not much jump set i.e., $\mathcal{H}^1(Ju \cap Q_R(x)) \leq \eta R$ for some fixed η. The idea is to approximate the squares via a convolution with a standard mollifier. A main tool will be two Poincaré-type estimates that use the fact that the amount of jump in the squares is small and concentrated in horizontal slices.

Depending on the context we shall denote the components of the absolute continuous part of the derivative during this chapter by $\partial_x u$ or $\partial_1 u$.

Lemma 4.3 (Poincaré-type estimate)**.** *Let $r > 0$, $\eta \in (0, \frac{1}{8})$ and $\rho \in (2\eta r, \frac{1}{2}r)$. Let $u \in SBV^2_{e_2}(Q_r)$ such that $\mathcal{H}^1(Ju) \leq \eta r$. Then the following holds:*

i) There is $C > 0$ such that for all $\bar{x} \in Q_{r-\rho}$ there is $\tilde{u} \in \mathbb{R}$ such that for all $y \in Q_\rho(\bar{x})$ it holds

$$\|u - \tilde{u}\|^2_{L^2(I_y)} \leq C\rho \|\nabla u\|^2_{L^2(Q_\rho(\bar{x}))}.$$

The set I_y denotes the vertical segment through y in $Q_\rho(\bar{x})$ i.e., $I_y = y + (\{0\} \times \mathbb{R}) \cap Q_\rho(\bar{x})$.

ii) There is $C > 0$ such that for each rectangle $R = [-w, w] \times [-h, h]$ that is chosen such that the enlarged rectangle $Q_\rho(R) = [-w - \rho, w + \rho] \times [-h - \rho, h + \rho]$ is still contained in Q_r it holds

$$\|u - u * \varphi_\rho\|^2_{L^2(R)} \leq C\rho \min\{\rho, w\} \|\nabla u\|^2_{L^2(Q_\rho(R))}.$$

The function φ_ρ is a standard mollifier.

Notice that ρ is chosen in both cases such that it is still larger then the overall length of Ju. At first it seems surprising that the scaling of the convolution estimate on the rectangle should only depend on the width but not on the height of the rectangle. However, since the jumps are concentrated in horizontal direction the two directions are completely different. In applications we will use the second result on rectangles that have sidelengths at least ρ

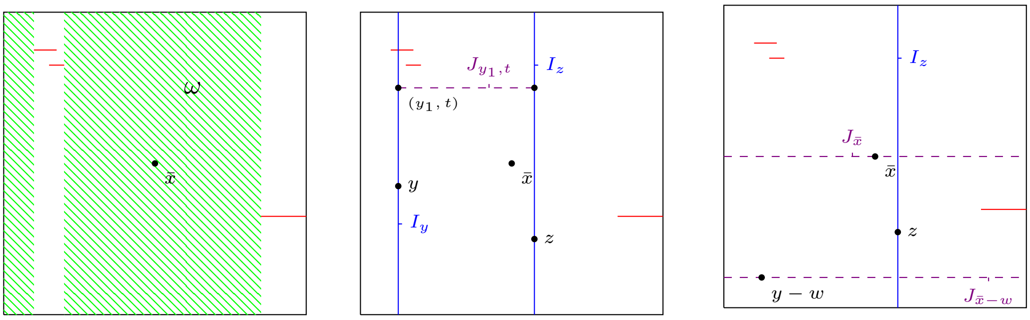

Figure 11: Type-I squares: Avoiding the jumps to prove two different Poincaré-type inequalities.

Proof. For both of the estimates we will need to find pointwise estimates for the integrand.
We fix $\bar{x} \in Q_r$ and define for each $y \in Q_\rho(\bar{x})$ the horizontal and vertical segment through y in $Q_\rho(x)$ by $I_y = y + (\{0\} \times \mathbb{R}) \cap Q_\rho(\bar{x})$ and $J_y = y + (\mathbb{R} \times \{0\}) \cap Q_\rho(\bar{x})$. Let $\omega = \{y \in Q_\rho(\bar{x}) \mid I_y \cap Ju = \emptyset\}$ be the set of points $y \in Q_\rho(\bar{x})$ for which I_y does not intersect the jump set Ju, see Figure 11.
Since $\rho \geq 2\eta r \geq 2\mathcal{H}^1(Ju)$ there is at least one point $z \in Q_\rho(\bar{x})$ depending only on $\bar{x}$ such that $z \in \omega$ and such that

$$\int_{I_z} |\partial_2 u|^2 \, \mathrm{d}\mathcal{H}^1 \leq \frac{2}{\rho} \int_{Q_\rho(\bar{x})} |\partial_2 u|^2 \, \mathrm{d}\mathcal{L}^2$$

Proof of i):
The one-dimensional Poincaré estimate on I_z tells us that there is $\tilde{u} \in \mathbb{R}$, depending only on z and $\bar{x}$ such that

$$\int_{I_z} |u - \tilde{u}|^2 \leq \frac{1}{9}\rho^2 \int_{I_z} |\partial_2 u|^2 \, \mathrm{d}\mathcal{H}^1 \leq \rho \int_{Q_\rho(\bar{x})} |\partial_2 u|^2 \, \mathrm{d}\mathcal{L}^2.$$

If we want to get an estimate of $u(y)$ for an arbitrary $y \in Q_\rho(\bar{x})$ we will compare it with the value of u at the point that lies in vertical direction on I_z, see the second picture in Figure 11. Applying the triangle inequality yields that for almost every $y \in Q_\rho(\bar{x})$ and for almost every $t \in (\bar{x}_2 - \rho, \bar{x}_2 + \rho)$ it holds

$$|u(y_1, t) - \tilde{u}| \leq |u(z_1, t) - \tilde{u}| + \int_{J_{(y_1,t)}} |\partial_1 u| \, \mathrm{d}\mathcal{L}^1.$$

Integration in t Hölder's inequality tells us that for almost every $y \in Q_\rho(\bar{x})$ we may estimate

$$\begin{aligned}
\|u - \tilde{u}\|_{L^2(I_y)} &\leq \|u - \tilde{u}\|_{L^2(I_z)} + \left(\int_{I_y} \left(\frac{1}{\rho} \int_{J_{(y_1,t)}} \rho |\partial_1 u| \, \mathrm{d}\mathcal{L}^1 \right)^2 \mathrm{d}\mathcal{L}^1 \right)^{1/2} \\
&\leq \rho^{1/2} \|\partial_2 u\|_{L^2(Q_\rho(\bar{x}))} + \rho^{1/2} \|\partial_1 u\|_{L^2(Q_\rho(\bar{x}))} \\
&\leq 2\rho^{1/2} \|\nabla u\|_{L^2(Q_\rho(\bar{x}))}.
\end{aligned}$$

Proof of ii):
We easily see

$$\begin{aligned}
|u(\bar{x}) - u * \varphi_\rho(\bar{x})| &= \left| \int_{Q_\rho(0)} (u(\bar{x}) - u(\bar{x} - w)) \varphi_\rho(y) \, \mathrm{d}\mathcal{L}^2(w) \right| \\
&\leq C\rho^{-1} \|u(\bar{x}) - u(\bar{x} - \cdot)\|_{L^2(Q_\rho(0))}
\end{aligned} \tag{10}$$

and achieve a pointwise estimate for the integrand of this expression by the same approach as before, see the third picture in Figure 11. For almost every $w \in Q_\rho(0)$ it holds

$$\begin{aligned}
&|u(\bar{x}) - u(\bar{x} - w)| \\
\leq &\int_{J_{\bar{x}}} |\partial_1 u| \, \mathrm{d}\mathcal{H}^1 + \int_{I_z} |\partial_2 u| \, \mathrm{d}\mathcal{H}^1 + \int_{J_{\bar{x}-w}} |\partial_1 u| \, \mathrm{d}\mathcal{H}^1 \\
\leq &2\rho^{1/2} \|\partial_1 u\|_{L^2(J_{\bar{x}})} + 2\rho^{1/2} \|\partial_2 u\|_{L^2(I_z)} + 2\rho^{1/2} \|\partial_1 u\|_{L^2(J_{\bar{x}-w})} \\
\leq &2\rho^{1/2} \|\partial_1 u\|_{L^2(J_{\bar{x}})} + C \|\partial_y u\|_{L^2(Q_\rho(\bar{x}))} + 2\rho^{1/2} \|\partial_1 u\|_{L^2(J_{\bar{x}-w})}.
\end{aligned}$$

Hence we get via integration that for almost every $\bar{x} \in R$ it holds that

$$\begin{aligned}
&\|u(\bar{x}) - u(\bar{x} - \cdot)\|_{L^2(Q_\rho(0))} \\
\leq &C\rho^{3/2} \|\partial_1 u\|_{L^2(J_{\bar{x}})} + C\rho \|\partial_2 u\|_{L^2(Q_\rho(\bar{x}))} + C\rho \left(\int_{(-\rho,\rho)} \int_{(-\rho,\rho)} |\partial_1 u(z - \bar{x}_1, w_2 - \bar{x}_2)|^2 \, \mathrm{d}z \, \mathrm{d}w_2 \right)^{1/2} \\
\leq &C\rho^{3/2} \|\partial_1 u\|_{L^2(J_{\bar{x}})} + C\rho \|\partial_2 u\|_{L^2(Q_\rho(\bar{x}))} + C\rho \|\partial_1 u\|_{L^2(Q_\rho(\bar{x}))}.
\end{aligned}$$

Before concluding, notice that

$$\int_R \chi_{Q_\rho(x)}(y)\,\mathrm{d}\mathcal{L}^2(x) \le \min\{\rho,w\}\cdot\min\{\rho,h\}\chi_{Q_\rho(R)}(x) \qquad \text{and}$$
$$\int_{-w}^{w} \chi_{J_x}(s,x_2)\,\mathrm{d}x_1 = \int_{-w}^{w} \chi_{(s-\rho,s+\rho)}(x_1)\,\mathrm{d}x_1 \le \min\{\rho,w\}\chi_{(-w-\rho,w+\rho)}(s).$$

Now we can complete the proof starting with (10), applying the last two inequalities and Fubini's theorem:

$$\begin{aligned}
&\|u-u*\varphi_\rho\|^2_{L^2(R)}\\
&\le C\rho^{-2}\int_R \|u(x)-u(x-\,.)\|^2_{L^2(Q_\rho(0))}\,\mathrm{d}\mathcal{L}^2(x)\\
&\le C\rho\int_R \|\partial_1 u\|^2_{L^2(J_x)}\,\mathrm{d}\mathcal{L}^2(x) + C\int_R \|\nabla u\|^2_{L^2(Q_\rho(x))}\,\mathrm{d}\mathcal{L}^2(x)\\
&\le C\rho\int_R\int_{\mathbb{R}} \chi_{J_x}(s,x_2)|\partial_1 u(s,x_2)|^2\,\mathrm{d}\mathcal{L}^1(s)\,\mathrm{d}\mathcal{L}^2(x) + C\int_R\int_{\mathbb{R}^2}\chi_{Q_\rho(x)}(w)|\nabla u(w)|^2\,\mathrm{d}\mathcal{L}^2(w)\,\mathrm{d}\mathcal{L}^2(x)\\
&\le C\rho\min\{\rho,w\}\int_{-w-\rho}^{w+\rho}\int_{-h}^{h}|\partial_1 u(s,x_2)|^2\,\mathrm{d}\mathcal{L}^1(x_2)\,\mathrm{d}\mathcal{L}^1(s) + C\rho\min\{\rho,w\}\int_{Q_\rho(R)}|\nabla u(x)|^2\,\mathrm{d}\mathcal{L}^2(x)\\
&\le C\rho\min\{\rho,w\}\|\nabla u\|^2_{L^2(Q_\rho(R))}.
\end{aligned}$$

□

We will now use the first of these estimates to prove the energy of the convolution of u with a standard mollifier φ_ρ.

Lemma 4.4 (Energy of the convolution with a standard mollifier)**.** *Let $r>0$, $\eta\in(0,\frac{1}{8})$ and $\rho\in(2\eta r,\frac{1}{4}r)$. Let $u\in SBV^2_{e_2}(Q_r)$ such that $\mathcal{H}^1(Ju)\le\eta r$ and define $u_\rho = u*\varphi_\rho$.*
Then $u_\rho\in C^\infty(Q_{r-\rho})\cap SBV^2_{e_2}(Q_{r-\rho})$ and

$$\int_{Q_{r-\rho}}|\nabla u_\rho|^2\,d\mathcal{L}^2 \le \left(1+C\frac{\eta r}{\rho}\right)^2\int_{Q_r}|\nabla u|^2\,d\mathcal{L}^2.$$

Proof. It is obvious that the function is in $C^\infty(Q_{r-\rho})$, we only need to prove the estimate for the derivatives.

It is a standard argument for convolutions that we can put the partial derivative onto the u i.e., $\partial_1 u_\rho = \varphi_\rho * \partial_1 u$ almost everywhere, which leads to the estimate for $\partial_1 u_\rho$. This is not possible for $\partial_2 u_\rho$ due to the jumps of u that are located in vertical direction. An argument of this type will, however, be possible on large parts of the domain since the jump set is small in $\mathcal{H}^1$.

Notice first that due to integration by parts for every constant $c\in\mathbb{R}$, every $x\in Q_{r-\rho}$

$$\int_{Q_\rho(x)}\partial_2\varphi_\rho(x-y)c\,\mathrm{d}\mathcal{L}^2(y)=0,$$

hence for every $c \in \mathbb{R}$, $x \in Q_{r-\rho}$ we can add this term to the derivative i.e.,

$$\partial_2 u_\rho(x) = \int_{Q_\rho(x)} \partial_2 \varphi_\rho(x-y)(u(y)-c)\,\mathrm{d}\mathcal{L}^2(y).$$

We fix $\bar{x} \in Q_{r-\rho}$ and want to estimate $\partial_2 u_\rho(\bar{x})$ pointwise.
Let us again denote by $\omega = \{y \in Q_\rho(\bar{x}) \,|\, I_y \cap Ju = \emptyset\}$ the set of points in $Q_\rho(\bar{x})$ for which there is no jump on the horizontal line through the point.
For almost every $y \in \omega$ we can use integration by parts to move the derivative from φ to u i.e.,

$$\int_\omega \partial_2 \varphi_\rho(\bar{x}-y)(u(y)-c)\,\mathrm{d}\mathcal{L}^2(y) = \int_\omega \varphi_\rho(\bar{x}-y)\partial_2 u(y)\,\mathrm{d}\mathcal{L}^2(y). \tag{11}$$

If $y \notin \omega$ we will use the first assertion of Lemma 4.3. For almost every $y \in Q_\rho(\bar{x}) \setminus \omega$ it holds that

$$\begin{aligned}\left|\int_{I_y} \partial_2 \varphi_\rho(x-\cdot)(u-\tilde{u})\,\mathrm{d}\mathcal{H}^1\right| &\le \|\partial_2\varphi_\rho(x-\cdot)\|_{L^2(I_y)}\|u-\tilde{u}\|_{L^2(I_y)} \\ &\le C(\rho^{-6}\rho)^{1/2}\rho^{1/2}\|\nabla u\|_{L^2(Q_\rho(\bar{x}))} \\ &\le C\rho^{-2}\|\nabla u\|_{L^2(Q_\rho(\bar{x}))}.\end{aligned}$$

The number of the slices that do not lie in ω is bounded by the $\mathcal{H}^1$-size of the jump set i.e., $\mathcal{L}^1(\{y_1 \,|\, (y_1, \bar{x}_2) \notin \omega \cap Q_\rho(\bar{x})\}) \le \eta r$. We conclude that

$$\left|\int_{Q_\rho(\bar{x})\setminus\omega} \partial_2\varphi_\rho(\bar{x}-y)(u(y)-\tilde{u})\,\mathrm{d}\mathcal{L}^2(y)\right| \le C\frac{\eta r}{\rho^2}\|\nabla u\|_{L^2(Q_\rho(\bar{x}))}.$$

Together with (11) for $c = \tilde{u}$ we get for almost every $x \in Q_{r-\rho}$ the pointwise estimate

$$\begin{aligned}|\partial_2 u_\rho|(x) &\le \left|\int_\omega \varphi_\rho(x-y)\partial_2 u(y)\,\mathrm{d}\mathcal{L}^2(y)\right| + \left|\int_{Q_\rho(x)\setminus\omega} \partial_2\varphi_\rho(x-y)(u(y)-\tilde{u})\,\mathrm{d}\mathcal{L}^2(y)\right| \\ &\le \varphi_\rho * |\partial_2 u|(x) + C\frac{\eta r}{\rho^2}\|\nabla u\|_{L^2(Q_\rho(x))}.\end{aligned}$$

Due to the fact that $\int_{Q_{r-\rho}}\int_{Q_\rho(x)} f(y)\,\mathrm{d}\mathcal{L}^2(y)\,\mathrm{d}\mathcal{L}^2(x) \sim \rho^2 \int_{Q_r} f(y)\,\mathrm{d}\mathcal{L}^2(y)$, Fubini's theorem and the estimate for convolutions we can finish the proof by estimating

$$\|\partial_2 u_\rho\|^2_{L^2(Q_{r-\rho})} \le \|\partial_2 u\|^2_{L^2(Q_{r-\rho})} + C\left(\frac{\eta r}{\rho^2}\right)^2 \rho^2 \|\nabla u\|^2_{L^2(Q_r)}.$$

□

This estimate of the convolution can only be achieved in a smaller square then the original one, since the mollifier needs values from a larger set and has different boundary values then the original function. We additionally need an interpolation such that we can combine this construction with the function on the rest of the domain.

Proposition 4.5 (Construction for type-I squares)**.** *Let $r > 0$, $\eta \in (0, \frac{1}{96})$, $u \in SBV^2_{e_2}(Q_r)$ such that $\mathcal{H}^1(Ju) \leq \eta r$.*
Then there is $v \in SBV^2_{e_2}(Q_r)$ such that $v \in W^{1,2}(Q_{\frac{6}{7}r})$, $v = u$ on $Q_r \setminus Q_{\frac{13}{14}r}$, $Jv \subseteq Ju$,

$$\|\nabla v\|^2_{L^2(Q_r)} \leq \left(1 + C\eta^{1/2}\right) \|\nabla u\|^2_{L^2(Q_r)}, \qquad |Dv|(Q_r) \leq Cr\|\nabla u\|_{L^2(Q_r)} + |D^J u|(Q_r)$$

and

$$\|v - u\|_{L^1(Q_r)} \leq Cr^2\eta^{1/2}\|\nabla u\|_{L^2(Q_r)}.$$

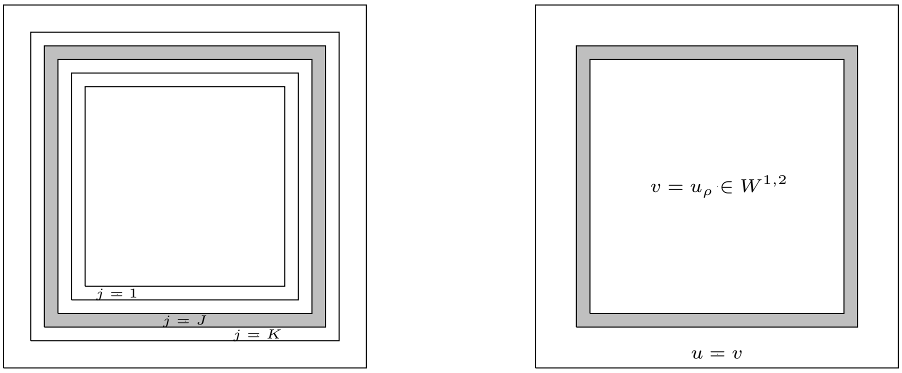

Figure 12: Type-I squares: Finding a good area for the interpolation between construction and data.

Proof. We will define the function v as the convolution with a standard mollifier of length scale ρ i.e., $v = u_\rho = u * \varphi_\rho$, in the inner square $Q_{\frac{6}{7}r}$ and perform the interpolation between u and u_ρ on a layer of thickness 3ρ.
We need to choose this layer in a way such that there is not to much concentration of the gradient in it. Let $K = \lfloor \frac{1}{14}\frac{r}{3\rho} \rfloor$ and define $Q_j = Q_{\frac{13}{14}r - j3\rho}$ for each $j \in \{0, \ldots, K\}$. It holds that the disjoint union $\bigcup_{j=0,\ldots,K-1} Q_j \setminus Q_{j+1} = Q_{\frac{13}{14}r} \setminus Q_K \sim Q_{\frac{13}{14}r} \setminus Q_{\frac{6}{7}r}$ and that $\mathcal{L}^2(Q_j \setminus Q_{j-1}) \sim Cr\rho$. Therefore there exists at least one $J \in \{0, \ldots, K-1\}$ such that

$$\|\nabla u\|_{L^2(Q_J \setminus Q_{J+1})} \leq CK^{-1/2}\|\nabla u\|_{L^2(Q_{\frac{13}{14}r} \setminus Q_K)} \leq CK^{-1/2}\|\nabla u\|_{L^2(Q_r)}.$$

Notice that since $K \geq \frac{13}{14}\frac{r}{3\rho} - 1$ it holds that $\frac{1}{K} \leq C\frac{\rho}{r}$. The interpolation will now be performed in the middle third of the so-chosen layer, such that the gradient of the convolution on this layer of width ρ is controlled by the gradient on the layer $Q_J \setminus Q_{J+1}$.
That is: We choose a standard interpolator $\psi \in C^1_c(Q_r)$ such that

$$\psi(x) = \begin{cases} 1 & x \in Q_{r - J3\rho - 2\rho} =: Q^i \\ \in [0,1] & x \in Q_{r - J3\rho - \rho} \setminus Q_{r - J3\rho - 2\rho} =: L \\ 0 & x \in Q_r \setminus Q_{r - J3\rho - \rho} =: Q^o \end{cases}$$

and $\|\psi'\|_{L^\infty(Q_r)} \leq C\rho^{-1}$. We define the function v by

$$v = \psi u_\rho + (1 - \psi)u.$$

Now v satisfies most of the stated properties. The main point we need to show is the estimate for the L^2-norm of the gradient. Using Lemma 4.4 on the square $Q_{\tilde{r}}$ with $\tilde{r} = r - J3\rho - \rho$ we compute

$$\begin{aligned}
&\|\nabla v\|^2_{L^2(Q_r)} \\
&= \|\nabla\psi(u_\rho - u) + \psi\nabla u_\rho + (1-\psi)\nabla u\|^2_{L^2(Q_r)} \\
&= \|\nabla u_\rho\|^2_{L^2(Q^i)} + \|\nabla\psi(u_\rho - u) + \psi\nabla u_\rho + (1-\psi)\nabla u\|^2_{L^2(L)} + \|\nabla u\|^2_{L^2(Q^o)} \\
&\leq \left(1 + C\frac{\eta r}{\rho}\right)^2 \|\nabla u\|^2_{L^2(Q^i\cup L)} + \|\nabla\psi(u_\rho - u) + \psi\nabla u_\rho + (1-\psi)\nabla u\|^2_{L^2(L)} + \|\nabla u\|^2_{L^2(Q^o)}.
\end{aligned}$$

We still need to estimate the term in the interpolation layer L. It is easy to see that, in similarity to Lemma 4.4, the following estimate holds on L:

$$\|\psi\nabla u_\rho\|^2_{L^2(L)} \leq \|\nabla u_\rho\|^2_{L^2(L)} \leq 2\|\nabla u\|^2_{L^2(Q_J\setminus Q_{J+1})}.$$

Applying Lemma 4.3 ii) gives us

$$\|\nabla\psi(u_\rho - u)\|^2_{L^2(L)} \leq C\rho^{-2}\|u_\rho - u\|^2_{L^2(L)} \leq C\|\nabla u\|^2_{L^2(Q_J\setminus Q_{J+1})}.$$

Putting this together with the choice of J in the beginning of the proof yields

$$\begin{aligned}
\|\nabla v\|^2_{L^2(Q_r)} &\leq \left(1 + C\frac{\eta r}{\rho}\right)^2 \|\nabla u\|^2_{L^2(Q_r)} + \|\nabla u\|^2_{L^2(Q_J\setminus Q_{J+1})} \\
&\leq \left(1 + C\frac{\eta r}{\rho}\right)^2 \|\nabla u\|^2_{L^2(Q_r)} + \frac{1}{K}\|\nabla u\|^2_{L^2(Q_r)} \\
&\leq \|\nabla u\|^2_{L^2(Q_r)} + C\left(\frac{\eta r}{\rho} + \frac{\rho}{r}\right)\|\nabla u\|^2_{L^2(Q_r)}.
\end{aligned}$$

We hence set $\rho = \eta^{1/2} r$ to achieve the stated estimate for the L^2-norm of the gradient of v.
We moreover estimate

$$|Dv|(Q_r) \leq \|\nabla v\|_{L^1(Q_r)} + \int_{Jv} |v^+ - v^-| \, \mathrm{d}\mathcal{H}^1 \leq Cr\|\nabla u\|_{L^2(Q_r)} + \int_{Ju} (u^+ - u^-) \, \mathrm{d}\mathcal{H}^1.$$

Using again Lemma 4.3 and Hölders inequality implies

$$\begin{aligned}
\|u - v\|_{L^1(Q_r)} \leq r\|u - v\|_{L^2(Q_r)} \leq r\|u - u_\rho\|_{L^2(Q_{r-\rho})} &\leq Cr\rho\|\nabla u\|_{L^2(Q_r)} \\
&= Cr^2\eta^{1/2}\|\nabla u\|_{L^2(Q_r)},
\end{aligned}$$

which finishes the proof. □

4.2.2 Construction on type-II squares

We will now perform a construction on a square on which the size of the jump set is arbitrary. The error in energy in this construction will be relatively large but will be controlled in total, since we will not use many of these squares.

We will replace u by a function that is constant on rectangles of the form $(-R,R)\times(y_i,y_{i+1})$. It is a variant of the Poincaré inequality that the L^2-distance of these two functions is small. In Chapter 7 we will present a more complex variant of such a Poincaré-type inequality in a SBD-setting.

Proposition 4.6 (Construction on type-II squares)**.** *Let $R\in(0,1)$ and $u\in SBV^2_{e_2}(Q_R)$. Then there is $v\in SBV^2_{e_2}(Q_R)$ and finitely many values $y_1,\dots,y_K\in(-R,R)$ such that*

$$Jv\cap Q_{\frac{12}{14}R}=\bigcup_{i=0}^{K}(-\frac{12}{14}R,\frac{12}{14}R)\times\{y_i\},\qquad v=u \text{ on } Q_R\setminus Q_{\frac{13}{14}R},$$
$$\|\nabla v\|_{L^2(Q_R)}\le C\|\nabla u\|_{L^2(Q_R)},\qquad \|u-v\|_{L^1(Q_R)}\le CR^2\|\nabla u\|_{L^2(Q_R)}$$
$$\mathcal{H}^1(Jv)\le C\mathcal{H}^1(Ju)+CR\quad\text{and}\qquad |Dv|(Q_R)\le CR\|\nabla u\|_{L^2(Q_R)}+C|D^Ju|(Q_R).$$

It moreover holds that there are $a\in(-\frac{13}{14}R,-\frac{12}{14}R)$, $b\in(\frac{12}{14}R,\frac{13}{14}R)$ such that

$$Jv\cap Q_{\frac{13}{14}R}\setminus Q_{\frac{12}{14}R}=Ju\cap Q_{\frac{13}{14}R}\setminus Q_{\frac{12}{14}R}\cup\bigcup_{i=1}^{K}\Big(\Big(a,-\frac{12}{14}R\Big]\times\{y_i\}\cup\Big[\frac{12}{14}R,b\Big)\times\{y_i\}\Big).$$

The idea of the proof is to choose a vertical segment on which the gradient is small and such that the segment intersect with the right number of jumps. We will interpolate to a piecewise constant function in the interior and create jump of length $\sim 2R$ for each jump point on these lines, see Figure 13.

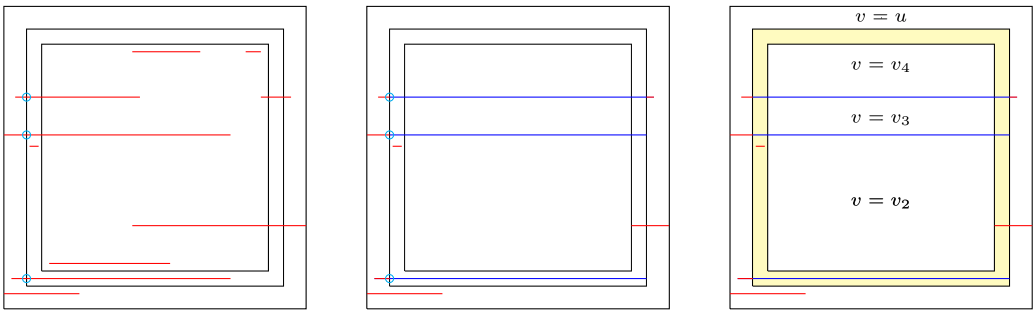

Figure 13: Type-II squares: Finitely many jumps ($C\cdot M=3$) are inserted, indicated by the blue lines. We let the function v be constant between these jumps and interpolate to the original function u in the yellow area.

Proof. Let $M\in\mathbb{N}$ be minimal such that $MR\ge\mathcal{H}^1(Ju)$. We will construct a function that has $C\cdot M\sim CR^{-1}\mathcal{H}^1(Ju)$ many jumps of length $2R$. Let $J=(-\frac{13}{14}R,-\frac{12}{14}R)$ be the interval in which we want to find a good vertical slice and define

$$A=\left\{x\in J\,\Big|\,\int_{-R}^{R}|\partial_y u|^2(x,\cdot)\,\mathrm{d}\mathcal{L}^1(y)\le 42R^{-1}\|\partial_y u\|^2_{L^2(Q)}\right\}.$$

It easily seen that $\mathcal{L}^1(J \setminus A) \leq \frac{R}{42}$ and therefore $\mathcal{L}^1(A) = \frac{1}{14}R - \mathcal{L}^1(J \setminus A) \geq \frac{2}{3} \cdot \frac{1}{14}R$. Let

$$B = \left\{ x \in J \,\middle|\, \mathcal{H}^0(Ju \cap \{x\} \times (-R,R)) \leq 42M \right\} \qquad \text{and}$$

$$D = \left\{ x \in J \,\middle|\, \int_{Ju \cap (\{x\} \times (-R,R))} [u] \, \mathrm{d}\mathcal{H}^0(y) \leq 42R^{-1} \int_{Ju} [u] \, \mathrm{d}\mathcal{H}^1 \right\}.$$

With the same arguments as before it follows that $\mathcal{L}^1(J \setminus B) \leq \frac{R}{42}$ and $\mathcal{L}^1(J \setminus D) \leq \frac{R}{42}$ and therefore $\mathcal{L}^1(B) \geq \frac{2}{3} \cdot \frac{1}{14}$ and $\mathcal{L}^1(D) \geq \frac{2}{3} \cdot \frac{1}{14}$. Hence the intersection of these three sets is nonempty and we can choose $x_- \in A \cap B \cap D \cap J$.

Let $y_1 < y_2 < \cdots < y_I$ be the y-values of the finitely many points at which the jump set of u intersects the axis $\{x_-\} \times (-R,R)$ i.e., $Ju \cap (\{x_-\} \times (-R,R)) = \bigcup_{i=1}^{I} \{x_-, y_i\}$. The point x_- has been chosen such that $I \leq CM \leq CR^{-1}\mathcal{H}^1(Ju)$.

Let $\psi \in C^1(Q_R)$ be a interpolation function such that $\|\psi\|_{L^\infty(Q_R)} \leq 1$, $\psi = 0$ on $Q_{\frac{12}{14}R}$, $\psi = 1$ on $Q_R \setminus Q_{\frac{13}{14}R}$ and $\|\nabla\psi\|_{L^\infty(Q_R)} \leq \frac{C}{R}$.

We define the piecewise constant function $w = \sum_{i=0}^{I} u_i \chi_{R_i}$ where $u_i = \max\{u_{i-1}, u^+(x^-, y_i)\}$ and $R_i = (-R,R) \times (y_i, y_{i+1})$. I.e., w takes the value of u on the lower left edge of each rectangle R_i for this rectangle if this creates a positive jump and retains the formerly chosen value if the created jump would be negative. It holds $w \in SBV^2_{e_2}(Q_R)$. Define

$$v = \psi u + (1-\psi)w = \psi(u-w) + w$$

and notice that also $v \in SBV^2_{e_2}(Q_R)$. The absolutely continuous part of the gradient of v is estimated

$$\begin{aligned}
\|\nabla v\|^2_{L^2(Q_R)} &\leq 2\|\nabla u\|^2_{L^2(Q_R)}\|\psi\|^2_{L^\infty(Q_R)} + 2\|u-w\|^2_{L^2(Q_R)}\|\nabla\psi\|^2_{L^\infty(Q_R)} \\
&\leq 2\|\nabla u\|^2_{L^2(Q_R)} + \frac{C}{R^2}\sum_{i=0}^{I}\|u-u_i\|^2_{L^2(R_i)} \\
&\leq 2\|\nabla u\|^2_{L^2(Q_R)} + \frac{C}{R^2}\sum_{i=0}^{I}\|u-u^+(x_-,y_i)\|^2_{L^2(R_i)} + \frac{C}{R^2}\sum_{i=0}^{I}\|u^+(x_-,y_i)-u_i\|^2_{L^2(R_i)}.
\end{aligned} \tag{12}$$

The third term occurs by the triangle inequality if the value at the lower edge of R_i is smaller then the value at the edge of R_{i-1}.

We can estimate the second term by the following Poincaré-type inequality. Define $h_i = y_{i+1} - y_i$, then we can estimate, using fundamental theorem of integration theory, that

$$\|u-u^+(x_-,y_i)\|^2_{L^2(R_i)} \leq 2Rh_i^2\|\partial_y u(x_-,\cdot)\|^2_{L^2((y_i,y_{i+1}))} + 2R^2\|\partial_x u\|^2_{L^2(R_i)}.$$

The second term of (12) is hence estimated, using that $h_i \leq 1$, by

$$\begin{aligned}
\frac{C}{R^2}\sum_{i=0}^{I}\|u-u^+(x_-,y_i)\|^2_{L^2(R_i)} &\leq \frac{C}{R^2}\sum_{i=0}^{I}(Rh_i^2\|\partial_y u(x_-,\cdot)\|^2_{L^2((y_i,y_{i+1}))} + R^2\|\partial_x u\|^2_{L^2(R_i)}) \\
&\leq CR\|\partial_y u(x_-,\cdot)\|^2_{L^2((-R,R))} + C\|\partial_x u\|^2_{L^2(Q_R)}.
\end{aligned}$$

For the third term we notice the following: If $u_i \neq u^+(x_-, y_i)$ then there is a maximal $j < i$ such that $u_j = u^+(x_-, y_j) = u_i$ and it holds that $u^+(x_-, y_j) > u^+(x_-, y_i)$.

Then, since $u^+(x_-, y_k) - u^-(x_-, y_k) > 0$ we have

$$\begin{aligned} |u^+(x_-, y_i) - u_i| = u^+(x_-, y_j) - u^+(x_-, y_i) &\leq u^+(x_-, y_j) - u^-(x_-, y_i) \\ &\leq \sum_{k=j}^{i-1} (u^+(x_-, y_k) - u^-(x_-, y_{k+1})) \\ &= \int_{y_j}^{y_i} |\partial_y u|(x_-, \cdot) \, \mathrm{d}\mathcal{H}^1. \end{aligned}$$

If we insert this in the third term of (12) we get

$$\begin{aligned} \frac{C}{R^2} \sum_{i=0}^{I} \|u^+(x_-, y_i) - u_i\|^2_{L^2(R_i)} &\leq \frac{C}{R^2} \sum_{i=0}^{I} h_i R |u^+(x_-, y_i) - u_i|^2 \\ &\leq C \sum_{i=0}^{I} 2h_i \|\partial_y u(x_-, \cdot)\|^2_{L^2((-R,R))} \\ &\leq CR \|\partial_y u(x_-, \cdot)\|^2_{L^2((-R,R))}, \end{aligned}$$

since $\sum_{j=0}^{I} h_j = R$.

The vertical segment $\{x_-\} \times (-R, R)$ was chosen such that $\|\partial_y u(x_-, \cdot)\|^2_{L^2(y_k, y_{k+1})} \leq \frac{C}{R} \|\partial_y u\|^2_{L^2(Q_R)}$ and so altogether we see that (12) yields

$$\|\nabla v\|^2_{L^2(Q_R)} \leq C \|\nabla u\|^2_{L^2(Q_R)}.$$

Notice that the interpolation between two functions with jumps of positive height has the union of the jump sets of the two functions as a jump set, up to a $\mathcal{H}^1$-null set. Since the jump of w is regular the jump set is even equal to that union pointwise everywhere. If we choose the interpolator ψ to be the product of two one-dimensional interpolators i.e., $\psi(x, y) = \tilde{\psi}(x)\tilde{\psi}(y)$ we have for $\operatorname{supp}(\tilde{\psi}) = (a, b)$

$$Jv = Ju \setminus Q_{\frac{12}{14}R} \cup \bigcup_{i \in \tilde{I}} (a, b) \times \{y_i\}.$$

We have chosen $\tilde{I} \subseteq \{1, \ldots, I\}$ so that w has a jump between R_{i-1} and R_i for $i \in \tilde{I}$. In particular

$$\mathcal{H}^1(Jv) \leq \mathcal{H}^1(Ju) + IR \leq \mathcal{H}^1(Ju) + CMR \leq \mathcal{H}^1(Ju) + C\mathcal{H}^1(Ju) + CR \leq C\mathcal{H}^1(Ju) + CR.$$

The functions u and v coincide on $\{-\frac{13}{14}R\} \times (-R, R)$ and hence the Poincaré inequality yields

$$\|u - v\|_{L^1(Q_R)} \leq CR \|\partial_x u - \partial_x v\|_{L^1(Q_R)} \leq CR^2 (\|\partial_x u\|_{L^2(Q_R)} + \|\partial_x v\|_{L^2(Q_R)}) \leq CR^2 \|\partial_x u\|_{L^2(Q_R)}.$$

To estimate $|Dv|$ it remains to estimate the height of the jumps of v in L^1. Fix some $i \in \tilde{I}$ and let $m < i$ maximal such that $m \in \tilde{I}$. In particular we have $u_m = u^+(x, y_m)$. It follows with the same calculations as before that

$$0 \leq u_i - u_{i-1} = \sum_{j=m+1}^{i} (u^+(x_-, y_j) - u^-(x_-, y_j)) + \int_{y_m}^{y_i} \partial_y u(x_-, \cdot) \, \mathrm{d}\mathcal{H}^1.$$

This equality together with the choice of the set D yields the estimate

$$\begin{aligned} |Dv|(Q_R) &\leq |Du|(Q_R) + |Dw|(Q_R) + \frac{1}{R}\|u - v\|_{L^1(Q_R)} \\ &\leq \|\nabla u\|_{L^1(Q_R)} + |D^J u|(Q_R) + R \sum_{i \in \tilde{I}} (u_i - u_{i-1}) + CR\|\nabla u\|_{L^2(Q_R)} \\ &\leq CR\|\nabla u\|_{L^2(Q_R)} + |D^J u|(Q_R) + R \sum_{i=1}^{I} [u](x, y_i) + R\|\partial_y u(x, \cdot)\|_{L^1((-R,R))} \\ &\leq CR\|\nabla u\|_{L^2(Q_R)} + C|D^J u|(Q_R), \end{aligned}$$

which finishes the proof. □

4.2.3 Construction on type-III squares

Type-III squares are the squares in which u has a jump that has length approximately $2R$ and is nearly straight. The existence of such squares is far less obvious then the existence of type-I and type-II squares. It is a consequence of regularity results for rectifiable sets which are proven with blow-up techniques. We will discuss this in detail in Lemma 4.12.
To be more precise: A type-III square Q_R has a smaller square Q_r in it, such that r is close to R and such that the intersection of jump set and ∂Q_r are exactly two points, each on a different vertical edge of Q_r, whose vertical distance is small. Moreover, the gradient of u on ∂Q_r is comparable to the gradient of u on Q_R.

Lemma 4.7 (Construction for type-III squares)**.** *Let $\varepsilon \in (0, \frac{1}{2})$, $\eta \in (2\varepsilon, \frac{1}{8})$, $R > 0$.*
Let $r \in (R - 2\eta R, R - \eta R)$ and define the squares $Q = Q_R(0)$ and $q = Q_r(0)$. Let $u \in SBV^2_{e_2}(Q)$ such that

$$\|\nabla u\|^2_{L^2(\partial q)} \leq C\eta^{-1}R^{-1}\|\nabla u\|^2_{L^2(Q)}$$

and $Ju \cap \partial q = \{z^+, z^-\}$ where $|z_2^\pm| \leq \varepsilon r$ and $z_1^\pm = \pm r$.
Then there exists $v \in SBV^2_{e_2}(Q)$ such that

i) $\|\nabla v\|^2_{L^2(Q)} \leq C\eta^{-1}\varepsilon^{-1/2}\|\nabla u\|^2_{L^2(Q)}$,

ii) $\|u - v\|_{L^1(Q)} \leq C\frac{r^{3/2}}{\eta^{1/2}\varepsilon^{1/4}}\|\nabla u\|_{L^2(Q)}$,

iii) $u = v$ *on* $Q \setminus q$,

iv) $Jv \cap q$ *is union of two segments,* $\mathcal{H}^1(Jv \cap q) \leq 2R + C\varepsilon^{1/2}R$ *and*

v) $|Dv|(q) \leq CR([u](z^+) + [u](z^-)) + C\frac{r^{1/2}}{\eta^{1/2}\varepsilon^{1/4}}\|\nabla u\|_{L^2(Q)}$.

Proof. Define $y^\pm = z_2^\pm$ be the y-values of the points where the jump set meets the boundary of the square q. Let without loss of generality $y^- < y^+$. Notice that the case $y^+ = y^-$ needs a different, but much easier construction.

Let $\gamma = y^+ - y^-$ and fix $\delta > 0$ which will be chosen later. Divide the square q in an upper and a lower domain R^t and R^b and a parallelogram R^m in the middle, see Figure 14. The upper and lower set are defined as $R^t = \{(x,y) \in q \,|\, g^t(x) \leq y\}$ and $R^b = \{(x,y) \in q \,|\, g^b(x) \geq y\}$ where

$$g^t(x) = \begin{cases} y^- & x \leq 0 \\ y^- + \frac{\gamma}{\delta}x & x \in (0,\delta) \\ y^+ & x \geq \delta \end{cases} \quad \text{and} \quad g^b(x) = \begin{cases} y^- & x \leq 2\delta \\ y^- + \frac{\gamma}{\delta}x & x \in (2\delta,3\delta) \\ y^+ & x \geq 3\delta. \end{cases}$$

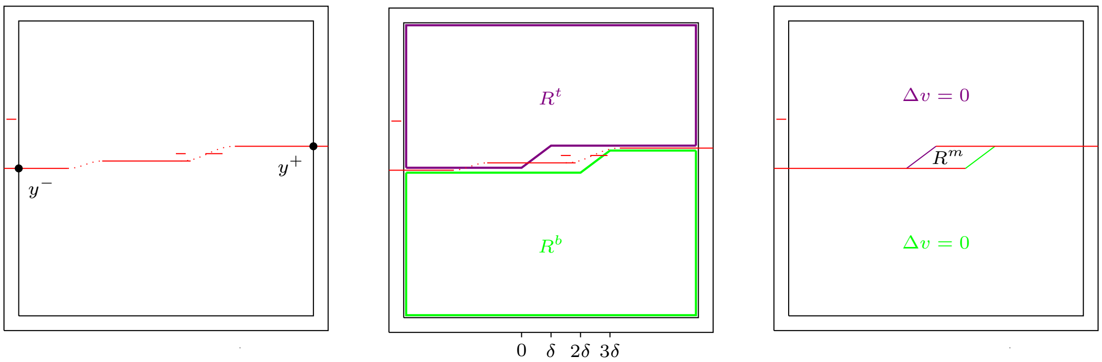

Figure 14: Type-III squares: The jump (red), mostly covered by a C^1 curve (dotted line), is replaced by two large jumps in horizontal direction. We solve Laplace's equation on the upper and lower half of the domain.

The sets R^t and R^b are only slight deformations of the rectangle $(-r,r) \times (0,r)$. Since there is a bijective function $\varphi \in W^{1,\infty}(R, R^t)$ with $\|D\varphi - \mathrm{Id}\,\|_{L^\infty} \leq \frac{\gamma}{\delta}$ we can easily deduce that for a solution w of Laplace's equation on R^t it holds $\|\nabla w\|^2_{L^2(R^t)} \leq (1 + C\frac{\gamma}{\delta})r\|\nabla w \cdot \nu\|^2_{L^2(\partial R^t)}$.

Let the four values at the jump points $z^\pm$ be given by $u_l^\pm = u^\pm(-r,y^-)$ and $u_r^\pm = u^\pm(r,y^+)$. We want to solve Laplace's equation on both domains and need to define the the boundary values on the lower edge of R^t and the upper edge of R^b. It seems reasonable to chose the linear interpolation in x-direction between u_l^+ and u_r^+ for R^t and between u_l^- and u_r^- for R^b.

The linear interpolation is a good choice since the gradient on this edge is then estimated by the gradient on the other three edges, which is by assumption estimated by the gradient of u on Q_r. Moreover, it is ensured that the jump at $\partial R^t \cap \partial R^b$ is of positive height, since it is the linear interpolation in x between the two positive jumps at the boundary of q.

However, we will modify this interpolation slightly to simplify the construction on the middle part R^m. We are performing a linear interpolation on $(-r,0)$, then stop the interpolation for $x \in (0,3\delta)$ and then continue the interpolation on $(3\delta,r)$ slightly faster. Since $\delta \ll r$ this does not change the scaling of the gradient. Define

$$v_t(x,y) = \begin{cases} u_l^+ + \frac{x+r}{2r}(u_l^+ - u_r^+) & x \in (-r,0), y = g^t(x) \\ \frac{1}{2}(u_l^+ + u_r^+) & x \in (0,3\delta), y = g^t(x) \\ \frac{1}{2}(u_l^+ + u_r^+) + \frac{x-3\delta}{2r-6\delta}(u_l^+ - u_r^+) & x \in (3\delta, r), y = g^t(x) \end{cases}$$

and

$$v_b(x,y) = \begin{cases} u_l^- + \frac{x+r}{2r}(u_l^- - u_r^-) & x \in (-r,0), y = g^t(x) \\ \frac{1}{2}(u_l^- + u_r^-) & x \in (0,3\delta), y = g^t(x) \\ \frac{1}{2}(u_l^- + u_r^-) + \frac{x-3\delta}{2r-6\delta}(u_l^- - u_r^-) & x \in (3\delta, r), y = g^t(x). \end{cases}$$

As argued above we see that a solution v^t of Laplace's equation on R^t with this boundary values satisfies

$$\|\nabla v_t\|_{L^2(R_t)} \le (1 + C\frac{\gamma}{\delta})r\left(\|\nabla u\|_{L^2(g_t((-r,r)))} + \|\nabla u\|_{L^2(\partial\mathbb{R}^t \setminus g_t((-r,r)))}\right) \le C(1+\frac{\gamma}{\delta})\frac{1}{\eta}\|\nabla u\|^2_{L^2(Q)}$$

and the similarly constructed solution v_b on R_b satisfies $\|\nabla v_b\|_{L^2(R_b)} \le C(1+\frac{\gamma}{\delta})\frac{1}{\eta}\|\nabla u\|^2_{L^2(Q)}$.

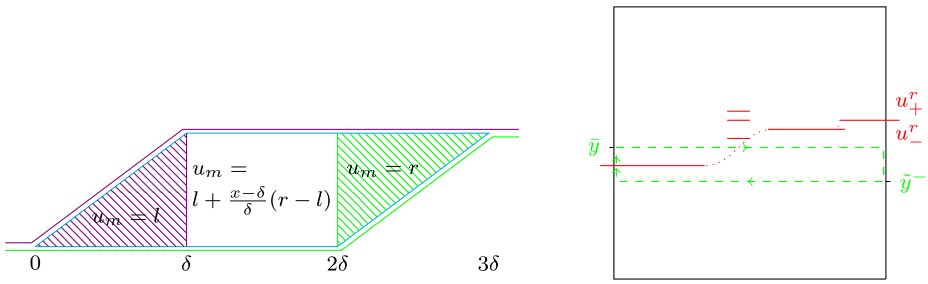

Figure 15: Type-III squares: Interpolation on R^m (left). The height of the jumps at the boundary can be estimated by the gradient since it is possible to avoid the jump set (right).

We may complete the construction with the definition of an interpolating function v on the parallelogram $R^m = \{(x,y)\,|\, x \in (-\varepsilon^{1/2}, \varepsilon^{1/2}), y \in (g^b(x), g^t(x))\}$. The function should fit continuously to the previous constructions on the vertical edges and is allowed to have jumps on the horizontal edges. We therefore use again a linear interpolation in x-direction. Notice that the functions u^t and u^b are constant along the edges of R^m. Since a dependence of the interpolation on the y-variable would enter the energy with a factor of γ^{-1} we want to fill the triangles in the parallelogram with constant values and interpolate between this values in x on the small square, see Figure 15 for a sketch. We define

$$u_m(x,y) = \begin{cases} \frac{1}{2}(u_r^+ + u_l^+) & x \in (0,\delta) \\ \frac{1}{2}((u_r^+ + u_l^+) + \frac{x-\delta}{\delta}(u_r^- + u_l^- - u_r^+ - u_l^+)) & x \in (\delta, 2\delta) \\ \frac{1}{2}(u_r^- + u_l^-) & x \in (2\delta, 3\delta). \end{cases}$$

We easily see that the jumps are of positive height at the horizontal edges of R^m whilst the function is continuous in x-direction. The gradient is estimated by

$$\|\nabla u_m\|^2_{L^2(R^m)} \leq \frac{\gamma}{2\delta}|u_r^- + u_l^- - u_r^+ - u_l^+|^2 \leq \frac{\gamma}{\delta}\left(|u_r^- - u_r^+|^2 + |u_l^- - u_l^+|^2\right).$$

So it remains to estimate the height of the two jumps in dependence of the gradient. This can be done by avoiding the jump, similarly to the proof for the Poincaré inequalities for type-I squares, see Figure 15 for a sketch.
Choose $\bar{y} \in (y^-, y^+)$ such that $u(\cdot, \bar{y}) \in W^{1,2}((-r, r))$ and

$$\int_{-r}^{r} |\partial_x u(\cdot, \bar{y})|^2 \leq 2\gamma^{-1}\|\nabla u\|^2_{L^2(Q)}.$$

There are also $\tilde{y}^- \in (y^- - \frac{r}{2}, y^-)$ and $\tilde{y}^+ \in (y^+, y^+ + \frac{r}{2})$ that satisfy

$$\int_{-r}^{r} |\partial_x u(\cdot, \tilde{y}^\pm)|^2 \leq 4r^{-1}\|\nabla u\|^2_{L^2(Q)}.$$

We have $\eta < 1$ and $\gamma < r$ and can therefore estimate

$$\begin{aligned}|u_l^+ - u_l^-|^2 \leq & 2\gamma\|\partial_y u(-r, \cdot)\|^2_{L^2((-r,r))} + 2r\|\partial_x u(\cdot, \bar{y})\|^2_{L^2((-r,r))} \\ & + 2\gamma\|\partial_y u(r, \cdot)\|^2_{L^2((-r,r))} + 2r\|\partial_x u(\cdot, \tilde{y}^-)\|^2_{L^2((-r,r))} \\ \leq & C\left(\frac{\gamma}{r\eta} + \frac{r}{\gamma} + 1\right)\|\nabla u\|^2_{L^2(Q)} \\ \leq & C\frac{r}{\eta\gamma}\|\nabla u\|^2_{L^2(Q)}.\end{aligned}$$

With a similar argument we have $|u_r^+ - u_r^-|^2 \leq C\frac{r}{\eta\gamma}\|\nabla u\|^2_{L^2(Q)}$.
Hence, defining v as u^m, u^t and u^b on R^m, R^t and R^b and as u on $Q \setminus q$ we conclude, choosing $\delta = r\varepsilon^{1/2}$

$$\|\nabla v\|^2_{L^2(Q)} \leq C(1 + \frac{r}{\eta\delta} + \frac{\gamma}{\eta\delta})\|\nabla u\|^2_{L^2(Q)} \leq C\frac{1}{\eta\varepsilon^{1/2}}\|\nabla u\|^2_{L^2(Q)}.$$

The choice of δ and the fact that that the jump set is $Jv \cap q = (-r, 2\delta) \times \{y^-\} \cup (\delta, r) \times \{y^+\}$ yields $\mathcal{H}^1(Jv \cap q) = 2r + 2r\varepsilon^{1/2}$. Since $u = v$ on ∂q we get due to the Poincaré inequality that

$$\|u - v\|_{L^1(q)} \leq r^{3/2}\|\partial_x(u - v)\|_{L^2(Q_r)} \leq C\frac{r^{3/2}}{\eta^{1/2}\varepsilon^{1/4}}\|\nabla u\|_{L^2(Q)}.$$

The height of the two jumps can also be estimated pointwise by the height of the original jumps at $z^\pm$. It therefore holds that $|Dv|(q) \leq CR([u](z^+) + [u](z^-)) + C\frac{r^{1/2}}{\eta^{1/2}\varepsilon^{1/4}}\|\nabla u\|_{L^2(Q)}$. □

4.3 Two density results

We have now the local constructions ready to prove the approximation results.
We will first approximate an arbitrary function $u \in SBV^2_{e_2,a}$ with functions $u_k \in SBV^2_{e_2,a}$ such that the jump set of each u_k is compact. An similar approximation has already been done for the space $SBV(\Omega) \cap L^\infty(\Omega)$ and an even more general energy by Braides and Chiadò Piat (see [10]) but

since our function space is much more restrictive we need a different approach in our proof. The L^∞ constraint is not crucial but could be easily inserted by a standard cut-off argument. We do, however, perform our constructions without applying such a cut-off since we have in mind that we later would like to generalize to a density result in a SBD-setting where L^∞-bounds can not be achieved that easy.
The main difference is that in their setting one can replace the function on squares on which the jump accumulates by a constant function. This decreases the elastic energy, whilst the length of the jump is only increased by the perimeter of the squares and hence the overall energy is increased by at most the sum over the sidelengths of such squares. They choose a compact set in the jump set that covers most of the jump set. Then the remaining part of the domain is covered with squares, and it follows, that the sum over the sidelengths of these squares that contain much jump set, is small and so is the overall error in the energy.
However, this technique is not applicable in our setting since the main step, replacing the function by a constant on some small domains, creates jump that is in *vertical* direction. We will instead use the construction for type-II squares, introduced in the previous subsection. Since the jump of these constructions is only regular in an inner square we need to cover the set Ω with the inner squares such that the overlap of the large squares is not to large. A similar covering has for example been done by Friesecke, James and Müller in [31].

4.3.1 A covering theorem

We recall a formulation of Whitney's covering theorem (see [48], Chapter VI 1, Theorem 1) which is obtained by a straightforward construction with dyadic cubes.

Proposition 4.8 (Whitney's covering theorem)**.** *For every $U \subseteq \mathbb{R}^n$ open there exists a collection of closed cubes $\mathcal{F} = \{Q_n \,|\, n \in \mathbb{N}\}$ such that:*

i) $\bigcup_{n\in\mathbb{N}} Q_n = U$.

ii) $\mathring{Q}_n \cap \mathring{Q}_m = \emptyset$.

iii) $\operatorname{diam}(Q_n) \le \operatorname{dist}(Q_n, \partial U) \le 4\operatorname{diam}(Q_n)$.

In our construction we will not be able to use Whitney's covering theorem directly, but will apply the following variant which is achieved with an additional application of the Besicovitch covering theorem.
For a cube $Q = Q_r(x)$ we write shortly bQ for the rescaled cube $Q_{br}(x)$.

Proposition 4.9 (Whitney-Besicovitch-type covering theorem)**.** *There is a constant $N = N(n)$ such that for every $U \subseteq \mathbb{R}^n$ open there is a collection of closed cubes $\mathcal{F} = \{Q_n \,|\, n \in \mathbb{N}\}$ such that:*

i) $\bigcup_{n\in\mathbb{N}} \frac{6}{7}\mathring{Q}_n = U = \bigcup_{n\in\mathbb{N}} Q_n$.

ii) There are subcollections $\mathcal{F}_i \subseteq \mathcal{F}$ such that $\mathcal{F} = \bigcup_{i=1,\dots,N} \mathcal{F}_i$ and if $Q_1, Q_2 \in \mathcal{F}_i$ then $Q_1 \cap Q_2 = \emptyset$.

iii) For each point $x \in \Omega$ there is an open set $A \subseteq \Omega$ such that $x \in A$ and A intersects at most N cubes in $\mathcal{F}$.

iv) If two cubes $\hat{Q}$, $\tilde{Q}$ are K-neighbors, then the sidelengths $\hat{r}$, $\tilde{r}$ of these cubes satisfy $\hat{r} \leq b^K \tilde{r}$ and the number of the K-neighbors of $\hat{Q}$ is estimated by a^K.
We call two squares K-neighbors if there are $K-1$ many squares Q^i such that for $Q^0 = \hat{Q}$ and $Q^K = \tilde{Q}$ it holds: $Q^i \cap Q^{i+1} \neq \emptyset$.

Proof. Fix the open set $U \subseteq \mathbb{R}^n$. We use Whitney's covering theorem in the formulation of Proposition 4.8 to obtain a collection of closed cubes $\mathcal{G} = \{Q_i(x_i)\}$ that covers U. Define the family of cubes enlarged by a factor of $\frac{71}{60}$ by $\mathcal{F} := \{\frac{71}{60}Q_i\}$.
Let $E = \bigcup_{i\in\mathbb{N}}\{x_i\}$ be the set of centers of the cubes in $\mathcal{G}$. The family of larger open cubes defined by $\mathring{\mathcal{F}} = \{\frac{72}{60}\mathring{Q}_i\}$ is a Besicovitch-cover for E. Due to the Besicovitch covering theorem (see Theorem 2.17 in [3]) there are N many subcollections $\mathring{\mathcal{F}}_i$ of $\mathring{\mathcal{F}}$ such that every collection is disjoint and the union of all chosen cubes still covers E.
Each $x \in E$ is only contained in one $Q \in \mathcal{G}$. Property iii) together with an easy geometric argument tells us that for each $\varepsilon < \frac{1}{5}$ it holds that the center of a cube $x_i \notin (1+\varepsilon)Q_j$ for $j \neq i$. Hence we know that each cube $\tilde{Q} \in \mathcal{G}$ has a corresponding cube in some $\mathring{\mathcal{F}}_i$.
The corresponding subcollections $\mathcal{F}_i$ of closed cubes are still disjoint since the cubes in $\mathcal{F}$ are smaller then the open cubes in $\mathring{\mathcal{F}}$ and it is an application of the third property of Whitney's covering theorem that the cubes in $\mathcal{F}$ are still contained in U. The cubes in $\mathcal{G}$ cover U and $\frac{6}{7}\frac{71}{60}\mathring{\tilde{Q}} \supseteq \tilde{Q}$ for each $\tilde{Q} \in \mathcal{G}$ so therefore i) and ii) are satisfied.
Fix $\bar{x} \in U$. Then there are at most N many cubes $Q_1, \ldots, Q_N \in \mathcal{G}$ such that $\bar{x} \in \frac{72}{60}\mathring{Q}_i$. Let $\tilde{Q}$ be the smallest of these cubes and denote its radius by $\tilde{r}$. Define the open set $A = B_{\frac{1}{300}\tilde{r}}(\bar{x})$ and notice that $A \subseteq \frac{72}{60}\mathring{Q}_i$ for all of the cubes that contain $\bar{x}$. If there is a cube $\hat{Q} = Q_{\hat{r}}(z) \in \mathcal{G}$ such that $\frac{71}{60}\hat{Q} \cap A \neq \emptyset$ then in particular $\frac{72}{60}\tilde{Q} \cap \frac{72}{60}\hat{Q} \neq \emptyset$. We can again use the third property of Whitney's covering theorem to show that $\bar{r} \geq \frac{1}{5}\tilde{r}$. Then $A \subseteq \frac{72}{60}\hat{Q}$ hence $\bar{x} \in \hat{Q}$ and hence $\hat{Q} = Q_i$ for some $i \in \{1, \ldots N\}$. This is property iii).
Fix a cube $Q \in \mathcal{F}$ with radius r. A neighbor $\hat{Q}$ of Q with radius $\hat{r}$ satisfies: $\frac{1}{5}r \leq \hat{r} \leq 5r$. In particular: All neighbors Q_i of Q are contained in the cube $5Q$. That is: The cubes $\frac{6}{7}Q_i$ are disjoint, contained in $5Q$, and have radius estimated from bellow by $\frac{1}{5}r$. Hence their number is uniformly bounded, independent of r. The estimate for K-neighbors follows by induction over K.

□

With this proposition at hand we can start the proof of the density statements.

4.3.2 Density of functions with compact jump set

At first we want to approximate and arbitrary function in $SBV^2_{e_2,a}$ with functions that have a compact jump set. We divide the proof of this density result in two steps: First we present a construction that does not involve the boundary values. The complete statement is proven in Proposition 4.11. At this point we will only need the construction of type-I and type-II squares.

Proposition 4.10 (Approximation with compact jump set for Mumford-Shah energy). *For each $u \in SBV^2_{e_2,a}$, for each $\delta > 0$ exists $v \in L^1(\Omega)$ such that for all $\gamma > 0$ it holds: $v \in SBV^2_{e_2}((\gamma, 1-\gamma)^2)$, $\mathcal{H}^1((\overline{Jv} \setminus Jv) \cap (\gamma, 1-\gamma)^2) = 0$,*

$$\|v - u\|_{L^1(\Omega)} \leq \delta, \quad |Dv|(\Omega) \leq C(1 + \|\nabla u\|_{L^2(\Omega)} + |Du|(\Omega)) \quad \text{and} \quad J(v) \leq (1+\delta)J(u).$$

The functional J denotes the Mumford-Shah functional.

Proof. Fix $u \in SBV^2_{e_2,a}$, $\delta > 0$ and choose $\varepsilon, \eta > 0$ arbitrary. There exists a compact set $K \subseteq Ju$ such that $\mathcal{H}^1(Ju \setminus K) \leq \varepsilon$ (see [3], Proposition 2.66) and we can choose K such that it is contained in finitely many C^1-curves $\{S_l \,|\, l = 1, \ldots, L\}$.
We use the Whitney-Besicovitch-type covering of Proposition 4.9 for $U = \Omega \setminus K$ to define N collections of squares $\{\mathcal{F}_i \,|\, i = 1, \ldots, N\}$. They are in particular chosen such that the squares in each family $\mathcal{F}_i$ are disjoint, are contained in $\Omega \setminus K$ and are such that $\Omega \setminus K = \bigcup_{i=1}^N \bigcup_{Q \in \mathcal{F}_i} \frac{6}{7} Q$.
In the sense of $\mathcal{H}^1$-measure, most parts of the jump set is already given as a compact set and we only need to regularize the remaining part. For this purpose we subdivide each collection $\mathcal{F}_i$ in two subcollections $\mathcal{F}_i^1$ and $\mathcal{F}_i^2$ where $\mathcal{F}_i^1 = \{Q_r(x) \in \mathcal{F}_i \,|\, \mathcal{H}^1(Ju \cap Q_r(x)) \leq \eta r\}$ and $\mathcal{F}_i^2 = \mathcal{F}_i \setminus \mathcal{F}_i^1$. On $\mathcal{F}_1$ we will perform the type-I construction such that there will be no jump left whilst we will add additional jump with the type-II construction on $\mathcal{F}_2$.
Notice that since the squares in $\mathcal{F}_i^2$ are disjoint we get for each i that the sum over the radii of these squares is small i.e.,

$$\varepsilon \geq \mathcal{H}^1(Ju \setminus K) \geq \mathcal{H}^1 \left(Ju \cap \bigcup_{Q \in \mathcal{F}_i^2} Q \right) \geq \sum_{Q_{r_j}(x_j) \in \mathcal{F}_i^2} \eta \, r_j .$$

The choice of the squares does depend on ε even though this is not indicated by the notation. Define the domains covered by the different families of squares by $G_{\varepsilon,\eta,i} = \bigcup_{Q_{r_j}(x_j) \in \mathcal{F}_i^1} Q_{r_j}(x_j)$ and $E_{\varepsilon,\eta,i} = \bigcup_{Q_{r_j}(x_j) \in \mathcal{F}_i^2} Q_{r_j}(x_j)$ and let additionally $E^N_{\varepsilon,\eta,i}$ be the union of those squares that are N-neighbors to squares in $E_{\varepsilon,\eta,i}$. We can now compute the size of this set since we know that the number of neighbors and their radius is estimated by the original collection. We estimate

$$\begin{aligned} \mathcal{L}^2 \left(E^N_{\varepsilon,\eta,i} \right) &\leq \sum_{k=1}^N \sum_{Q_{r_j}(x_j) \in \mathcal{F}_i^2} a^k (b^k r_j)^2 \leq N \sum_{Q_{r_j}(x_j) \in \mathcal{F}_i^2} a^N b^{2N} r_j \\ &\leq N a^N b^{2N} \sum_{Q_{r_j}(x_j) \in \mathcal{F}_i^2} \frac{\mathcal{H}^1(Ju \cap Q_{r_j}(x_j))}{\eta} \\ &\leq C \frac{\varepsilon}{\eta}. \end{aligned}$$

We will need this domain to be small hence we choose $\eta = \varepsilon^{1/2}$. We use the dominated convergence theorem to choose ε so small that

$$\|\nabla u\|_{L^2(\bigcup_{i=1,\ldots,N} E^N_{\varepsilon,\eta,i})} \leq \delta.$$

At the very end of the proof we will have additional constrains that let us choose ε even smaller.
After this preliminary choices we start using the construction for type-I squares (see Proposition 4.5) recursively on the collection of squares $\mathcal{F}_i^1$. In each $\mathcal{F}_i^1$ the squares are disjoint and the construction does only depend on the values in the interior of each square. For fixed i the construction can therefore be performed simultaneously in $G_{\varepsilon,\eta,i}$. It will depend recursively on the construction for $k < i$ since the different families $\mathcal{F}_j$ do overlap. But since this happens only finitely many times we

are still able to control the error in the energy.

Let v_1 be the function that is created if Proposition 4.5 is applied on each of the disjoint squares of $\mathcal{F}_1^1$.

Then

$$\begin{aligned} &Jv_1 \subseteq Ju, && \|\nabla v_1\|^2_{L^2(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega)} + C\eta^{1/2}\|\nabla u\|^2_{L^2(G_{\varepsilon,\eta,1})} \\ &\|u - v_1\|_{L^1(\Omega)} \le \eta^{1/2}\|\nabla u\|_{L^2(\Omega)}, && |Dv_1|(\Omega) \le C(\|\nabla u\|_{L^2(G_{\varepsilon,\eta,1})} + |Du|(\Omega)) \end{aligned}$$

and $Jv_1 \cap \frac{6}{7}Q = \emptyset$ for each $Q \in \mathcal{F}_1^1$.

Assume now we have already constructed a function v_k on the squares of $\mathcal{F}_k^1$. Since the jump set gets only smaller in the construction, the squares in $\mathcal{F}_{k+1}^1$ still satisfy that $\mathcal{H}^1(Jv_i \cap Q_r(x)) \le r\eta$. Use Proposition 4.5 to create a function v_{k+1} by changing v_k on the disjoint squares of $\mathcal{F}_{k+1}^1$. Then $Jv_{k+1} \subseteq Jv_k$, $Jv_{k+1} \cap \frac{6}{7}Q = \emptyset$ for each $Q \in \bigcup_{i=1}^{k+1} \mathcal{F}_1^i$ and the estimates for the L^1-difference and the gradients can be computed inductively:

$$\begin{aligned} \|\nabla v_{k+1}\|^2_{L^2(\Omega)} &\le (1 + C\eta^{1/2})\|\nabla v_k\|^2_{L^2(\Omega)} \le (1 + C\eta^{1/2})^{k+1}\|\nabla u\|^2_{L^2(\Omega)} \le (1 + \tilde{C}\eta^{1/2})\|\nabla u\|^2_{L^2(\Omega)}, \\ \|u - v_{k+1}\|_{L^1(\Omega)} &\le \|v_k - v_{k+1}\|_{L^1(\Omega)} + \|v_k - v\|_{L^1(\Omega)} \\ &\le \eta^{1/2}\|\nabla v_k\|_{L^2(\Omega)} + \eta^{1/2}\tilde{C}\eta^{1/2}\|\nabla v_k\|_{L^2(\Omega)} \\ &\le \tilde{C}\eta^{1/2}\|\nabla u\|_{L^2(\Omega)} \end{aligned}$$

and $|Dv_{k+1}|(\Omega) \le \tilde{C}(\|\nabla u\|_{L^2(\Omega)} + |Du|(\Omega))$. This process terminates with a finite constant since there are only finitely many disjoint families of squares.

As a next step we want to perform the type-II construction on $\mathcal{F}^2$. At first we want to change v_N on the squares of $\mathcal{F}_1^2$. Even though the property that $\mathcal{H}^1(Jv_N \cap Q_r(x)) \le r\eta$ might not longer be fulfilled we will still apply the type-II constructions. After the first step of recursion, using Proposition 4.6 on each square, we find a function w_1 such that $Jw_1 \cap \frac{6}{7}Q$ is finite union of segments for all $Q \in \mathcal{F}_1^2$,

$$\begin{aligned} \|\nabla w_1\|^2_{L^2(\Omega)} &\le \|\nabla v_N\|^2_{L^2(\Omega)} + C\|\nabla v_N\|^2_{L^2(E_{\varepsilon,\eta,1})} \\ &\le \|\nabla v_N\|^2_{L^2(\Omega)} + C\|\nabla v\|^2_{L^2(E^N_{\varepsilon,\eta,1})} \le \|\nabla v_N\|^2_{L^2(\Omega)} + C\delta, \\ \|v_N - w_1\|_{L^1(\Omega)} &\le \sum_{Q_{r_j}(x_j) \in \mathcal{F}_1^2} Cr_i^{3/2}\|\nabla v_N\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\|\nabla v_N\|_{L^2(\Omega)}, \\ |Dw_1|(\Omega) &\le C(\|\nabla w_N\|_{L^2(\Omega)} + |Dv_N|(\Omega)) \end{aligned}$$

and

$$\mathcal{H}^1(Jw_1) \le \mathcal{H}^1(Jv_N) + \sum_{Q_{r_j}(x_j) \in \mathcal{F}_1^2} \big(C\mathcal{H}^1(Jv_N \cap Q_{r_j}(x_j)) + Cr_j\big) \le \mathcal{H}^1(Jv_N) + C(\varepsilon + \varepsilon^{1/2}).$$

We repeat this argument recursively on the different families of squares $\mathcal{F}_k$ and reach finally a function w_N that satisfies

$$Jw_N \subseteq K \cup \bigcup_{k \in \mathbb{N}} (a_k, b_k) \times \{y_k\}$$

for some values a_k, b_k, $y_k \in (0,1)$. Due to property iii) of the Whitney-Besicovitch-type covering for each $x \in \Omega \setminus K$ there is an open set U such that $x \in U$ and U intersects with only finitely many of the squares. Hence the only possible accumulation points of the jump set lie in K and $\partial\Omega$. In particular it holds that for each $\gamma > 0$ the set $Jw_N \cap Q_{1-\gamma}$ is contained in the union of the compact set K and countably many open segments that do not have accumulation points outside of the compact set. However, we still need to show that the points in K are still jump points of w_N. Since K is chosen as a subset of finitely many C^1 functions $\{S_l \,|\, l = 1, \dots, L\}$ we can for almost every $\bar{x} \in K$ find a small radius r such that for some l it holds: $K \setminus S_l \cap Q_r(\bar{x}) = \emptyset$ and $S_l = \gamma((-r,r))$ for some C^1 function γ. Then in particular $Q^b = \{(x,y) \in Q_r(\bar{x}) \,|\, y \le \gamma(x)\}$ is a Lipschitz domain, hence we can apply the trace theorem and $u^-(\bar{x}) = \mathrm{Tr}(u)(\bar{x})$.
We can also understand w_{i+1} and v_{i+1} as the limit of a sequence w_i^m and v_i^m respectively where the mth element of the sequence differs from the $m-1$th by applying the construction on the mth square of $\mathcal{F}_1^i$ and $\mathcal{F}_2^i$ respectively. The forgoing estimates for the energy imply $v_i^m \overset{*}{\rightharpoonup} v_{i+1}$ and $w_i^m \overset{*}{\rightharpoonup} w_{i+1}$ in $BV(\Omega \setminus K)$.
We can moreover estimate $|Dw_i^{m+1} - Dw_i^m|(\Omega \setminus K) \le C|Dw_{i-1}|(Q_{m+1})$, where Q_m denotes the mth square of $\mathcal{F}_i^2$, and can hence see that w_i^m is a Cauchy-sequence in $BV(\Omega \setminus K)$. It follows that the sequence converges strongly in $BV(\Omega \setminus K)$ and in particular in $BV(Q_r^b(x))$. Since each w_i^m equals to w_{i-1} in a neighborhood of $\bar{x}$ we can conclude, using the trace theorem on $Q_r^b(\bar{x})$ that $w_i^-(\bar{x}) = \mathrm{Tr}_{Q_r^b(\bar{x})}(w_i)(\bar{x}) = \lim_{m\to\infty} \mathrm{Tr}_{Q_r^b(\bar{x})}(w_i^m)(\bar{x}) = \mathrm{Tr}_{Q_r^b(\bar{x})}(w_{i-1})(\bar{x}) = w_{i-1}^-(\bar{x})$. The same argument holds for the v_i's.
Moreover the recursive estimates for the derivatives yield $\mathcal{H}^1(Jw_N) \le \mathcal{H}^1(Ju) + NC(\varepsilon + \varepsilon^{1/2})$,

$$\begin{aligned}
\|\nabla w_N\|_{L^2(\Omega)}^2 &\le \|\nabla u\|_{L^2(\Omega)}^2 + C\varepsilon^{1/4}\|\nabla u\|_{L^2(\Omega)}^2 + C(1 + C\varepsilon^{1/4})\|\nabla u\|_{L^2(\bigcup_{i=1,\dots,N} E_{\varepsilon,\eta,i})}^2 \\
&\le \|\nabla u\|_{L^2(\Omega)}^2 + C\varepsilon^{1/4}\|\nabla u\|_{L^2(\Omega)}^2 + C(1 + C\varepsilon^{1/4})\delta, \\
\|u - w_N\|_{L^1(\Omega)} &\le \|u - v_N\|_{L^1(\Omega)} + \|v_N - w_N\|_{L^1(\Omega)} \\
&\le C\varepsilon^{1/4}\|\nabla u\|_{L^2(\Omega)} + C\varepsilon^{1/2}(1 + \varepsilon^{1/4})\|\nabla u\|_{L^2(\Omega)},
\end{aligned}$$

and

$$|Dw_N|(\Omega) \le \hat{C}(\|\nabla u\|_{L^2(\Omega)}^2 + |Du|(\Omega)).$$

Now we can choose ε so small that the stated inequalities hold for $v = w_N$. □

The foregoing proposition is not the first density result in its full generality. There are two things that still need correction: The approximating functions might have a concentration of jumps at the boundary of Ω and the boundary conditions may not be satisfied. We therefore want to prove

Proposition 4.11 (Approximation with compact jump set and conserved boundary values for Mumford-Shah energy)**.** *For each $u \in SBV_{e_2,a}^2$, for each $\delta > 0$ exists $v \in SBV_{e_2,a}^2$ such that $\mathcal{H}^1(\overline{Jv} \setminus Jv) = 0$, and*

$$\|v - u\|_{L^1(\Omega)} \le \delta, \quad |Dv|(\Omega) \le C(1 + \|\nabla u\|_{L^2(\Omega)} + |Du|(\Omega)) \quad \text{and} \quad J(v) \le (1+\delta)J(u).$$

Proof. The proof will be an application of Proposition 4.10 in a combination of some shifts of the function in vertical and horizontal direction. Let us at first construct a function that has the correct boundary values. Define $u^1 : (-\gamma, 1) \times (0,1) \to \mathbb{R}$ by

$$u^1(x,y) = \begin{cases} u(x,y) & x \geq 0 \\ y & x \leq 0. \end{cases}$$

Then let $\tilde{u}(x,y) : (0,1)^2 \to \mathbb{R}$ be defined as $\tilde{u}(x,y) = u^1(\frac{x+\gamma}{1+\gamma}, y)$. Notice that due to the fundamental theorem of integration theory and some easy computations we conclude

$$\|\tilde{u} - u\|_{L^1(\Omega)} \leq \gamma^{1/2} I(u), \quad I(\tilde{u}) \leq (1 + C\gamma) I(u) \quad \text{and } |D\tilde{u}|(\Omega) \leq C|Du|(\Omega).$$

Now we apply the foregoing proposition on $\tilde{u}$. Since the radius of the squares gets smaller as they get closer to the boundary we can follow that all squares in $(0, \frac{\gamma}{4}) \times (0,1)$ are type-I and that $\tilde{u}$ is already an affine function there. The constructed function $\tilde{v}$ in particular coincides with the affine function on $(0, \gamma/4) \times (0,1)$ since convoluting an affine function with a standard-mollifier does not change that function. This in particular implies that there are no accumulation points of $J\tilde{v}$ close to left edge of the boundary and that the boundary values are satisfied. If we define $v^1(x,y) = \tilde{v}(\frac{x}{1-\gamma}, y)$ then all accumulation points of $\tilde{v}$ at $\{1\} \times (0,1)$ are transported out of the unit square and by fundamental theorem all estimates still hold.
It remains to get rid of the accumulation points of jump at the horizontal edges of the square. Let us without loss of generality only consider the upper edge $(0,1) \times \{1\}$. Similar as at the right vertical edge we want to push the jump that might accumulate at the boundary out of the domain, but due to the jumps we do not have the fundamental theorem available.
We therefore use the following construction: Fix $\eta \in (0,1)$, $\alpha \ll \eta$ and define $\hat{v} : \Omega \to \mathbb{R}$ as

$$\hat{v}(x,y) = \begin{cases} v^1\left(x, \left(1 - \frac{\alpha}{\eta}\right) y + \frac{\alpha - \alpha\eta}{\eta}\right) & y \geq 1 - \eta \\ v^1(x,y) & y \leq 1 - \eta. \end{cases}$$

The function $\hat{v}$ satisfies the boundary values $\hat{v}(x,1) = v^1(x, 1-\alpha)$, $\hat{v}(x, 1-\eta) = v^1(x, 1-\eta)$ and $\mathcal{H}^1(J\hat{v}) = \mathcal{H}^1(Jv^1 \cap (0,1) \times (0, 1-\alpha))$. It holds that:

$$\begin{aligned} \|\hat{v} - v^1\|_{L^1(\Omega)} &\leq \|v^1\|_{L^1((0,1)\times(1-\eta,1))} + \|\hat{v}\|_{L^1((0,1)\times(1-\eta,1))} \\ &\leq \|v^1\|_{L^1((0,1)\times(1-\eta,1))} + \frac{1}{1-\frac{\alpha}{\eta}} \|v^1\|_{L^1((0,1)\times(1-\eta,1-\alpha))} \end{aligned}$$

which is small if η is small. The estimates for the derivative hold by the usual application of the chain-rule. □

4.3.3 Density of functions whose jump set is a finite union of segments

Our goal is to apply the construction of the recovery sequence on jumps that are isolated from all other components of the jump set. This is in general still not true for a compact jump set, since segments of jump may have an accumulation point that is a jump point located in a different segment. We will now prove a stronger density result in which the jump of the approximating functions is

only a finite union of segments.
The difference to the foregoing result is that we will now also use the construction for type-III squares. They are needed in situations where the jump is already nearly a segment but might nevertheless be not isolated. In the first compactness result these parts of the jump were mostly contained in the compact set K and hence we did not perform an explicit construction there.
The following technical lemma provides a covering of the domain with squares of different types. It is inspired by Lemma 4.2 in the work of Cortesani and Toader [18] but bears many differences in the details. The proof again uses the Whitney-Besicovitch-type covering theorem stated before and the usual regularity results for jump sets of BV-functions.
An important point is that the squares for the type-III construction need to be truly disjoint and not only up to finite subfamilies. Also it is necessary to use only finitely many of the different types of squares to cover the jump set.

Lemma 4.12. *Let $u \in SBV^2_{e_2,a}$ such that $\mathcal{H}^1(\overline{Ju} \setminus Ju) = 0$, let $\varepsilon \in (0, \frac{1}{16})$, $\eta \in (2\varepsilon, \frac{1}{8})$. Let N be the constant from the Whitney-Besicovitch-type covering theorem. Then there exists:*

a) A finite family of outer squares $\{Q_{\rho_i}(x_i) \,|\, i \in \{1, \dots, M\}\}$, a finite family of inner squares $Q_{r_i}(x_i) \subseteq Q_{\rho_i}(x_i)$ such that

- *i) the outer squares are disjoint and contained in Ω,*
- *ii) the inner squares have radii approximately ρ_i i.e., $r_i \in (\rho_i - 2\eta\rho_i, \rho_i - \eta\rho_i)$,*
- *iii) the inner squares have gradients on the edges that are comparable with the gradient on the corresponding outer square i.e., $\|\nabla u\|^2_{L^2(\partial Q_{r_i}(x_i))} \leq C\eta^{-1}\rho_i^{-1}\|\nabla u\|^2_{L^2(Q_{\rho_i}(x_i))}$,*
- *iv) the edges of the inner square intersect the jump set only as often as necessary and the height of the jump at the intersection is not to large i.e., $Ju \cap \partial Q_{r_i}(x_i) = \{z^{+,i}, z^{-,i}\}$ where $|z_2^{i,\pm} - x_{i2}| \leq \varepsilon r_i$ and $[u](z^{\pm,i}) \leq C(\varepsilon + \rho_i^{-1} \int_{Ju \cap Q_{\rho_i}(x_i)} [u] \, d\mathcal{H}^1)$,*
- *v) the gradient of u on the outer squares goes sublinearly in $\eta \cdot \varepsilon$ i.e., $\frac{1}{\eta}\|\nabla u\|^2_{L^2(\bigcup_{i=1}^M Q_{\rho_i}(x_i))} \leq \varepsilon^2$ and*
- *vi) the inner squares cover most parts of the jump set whilst the outer squares are only as large as needed i.e., $\mathcal{H}^1(Ju \setminus \bigcup_{i=1}^M Q_{r_i}(x^i)) \leq C\eta\mathcal{H}^1(Ju)$ and $\sum_{i=1}^M 2\rho_i \leq (1 + 2\varepsilon)\mathcal{H}^1(Ju)$.*

b) A family of closed squares $\mathcal{F} = \{P_k \,|\, k \in \mathbb{N}\}$ such that

- *i) the union of the scaled interior of the squares P_k covers all parts of the jump set that is not already covered by the inner squares $Q_{r_i}(x_i)$ of the first family and the family is locally finite i.e., for all $\gamma > 0$ exists $\hat{\mathcal{F}}^\gamma \subseteq \mathcal{F}$ finite such that $\bigcup_{P_k \in \hat{\mathcal{F}}^\gamma} \frac{6}{7}\mathring{P}_k \supseteq Ju \cap Q_{1-\gamma} \setminus \bigcup_{i=1}^M Q_{r_i}(x_i)$,*
- *ii) the squares do not overlap to often i.e., there are subcollections $\{\mathcal{F}_i \,|\, i \in \{1, \dots, N\}\} \subseteq \mathcal{P}(\mathcal{F})$ such that $\mathcal{F} = \bigcup_{i=1,\dots,N} \mathcal{F}_i$ and if $Q_1, Q_2 \in \mathcal{F}_i$ then $Q_1 \cap Q_2 = \emptyset$,*
- *iii) for each point $x \in \Omega$ there is $U \subseteq \Omega$ open such that $x \in U$ and U intersects at most N squares in $\mathcal{F}$ and*
- *iv) the gradient of u is small on the union of those squares in $\mathcal{F}$ that carry much jump set i.e., on $\tilde{\mathcal{F}} = \{P_k = Q_{\tilde{r}_k}(\tilde{x}^k) \subseteq \mathcal{F} \,|\, \mathcal{H}^1(Ju \cap P_k) \geq \tilde{r}_k\eta^{1/2}\}$ it holds: $\|\nabla u\|^2_{L^2(\bigcup \tilde{\mathcal{F}})} \leq \varepsilon$.*

Proof. Fix $\tilde{\varepsilon} < \varepsilon$, $\tilde{\eta} \in (2\tilde{\varepsilon}, \eta)$. We will use the fact that we can choose these values small in the very end of the proof to show the smallness of the gradients of u on the different families of squares.
There are countably many functions $S_k \in C^1(\mathbb{R}, \mathbb{R}^2)$ such that, up to a set of $\mathcal{H}^1$-measure zero, $Ju \subseteq \bigcup_{k\in\mathbb{N}} S_k(\mathbb{R})$. It holds:

- For $\mathcal{H}^1$-almost every $x \in Ju$ exists a k such that $\limsup_{\rho\to 0} \frac{\mathcal{H}^1(Ju\cap Q_\rho(x)\setminus S_k)}{2\rho} = 0$ and we have that $\nu_{S_k}(x) = e_2$(see Theorem 3.5 in [46]) .
- For $\mathcal{H}^1$-almost every $x \in Ju$ it holds: $\limsup_{\rho\to 0} \frac{\mathcal{H}^1(Ju\cap Q_r(x))}{2\rho} = 1$ (see Theorem 3.2.19 in [29]).
- For $\mathcal{H}^1$-almost every $x \in Ju$ it holds: $\limsup_{\rho\to 0} \frac{\|[u]\|_{L^1(Q_\rho(x)\cap Ju, \mathcal{H}^1)}}{\mathcal{H}^1(Q_\rho(x)\cap Ju)} = [u](x)$ (since $u \in L^1(Ju, \mathcal{H}^1)$ and 1.62 in [28]).

Hence there is a set $\tilde{J} \subseteq Ju$ such that $\mathcal{H}^1(Ju \setminus \tilde{J}) = 0$ and such that the set

$$\begin{aligned}\mathcal{G}^{\tilde{\varepsilon}} = \big\{Q_\rho(x) \,|\, x \in \tilde{J}, \rho \le \tilde{\varepsilon}, \rho \text{ s.t. } &\mathcal{H}^1(Ju \cap Q_\rho(x) \setminus S_k) \le 2\tilde{\varepsilon}\rho, |\nabla S_k - e_1| \le \tilde{\varepsilon} \text{ on } Q_\rho(x)\\ &\text{and for all } \tilde{\rho} \le \rho \text{ it holds}\\ &|\mathcal{H}^1(Ju \cap Q_{\tilde{\rho}}(x)) - 2\tilde{\rho}| \le \frac{\varepsilon}{2}\tilde{\rho} \text{ and } |\|[u]\|_{L^1(B_{\tilde{\rho}}(x)\cap Ju, \mathcal{H}^1)} - 2\tilde{\rho}[u](x)| \le \varepsilon\tilde{\rho}\big\}\end{aligned}$$

is a fine Besicovitch cover for $\tilde{J}$. Due to a corollary of the Besicovitch covering theorem we get finitely many disjoint squares $\hat{\mathcal{G}}^{\tilde{\varepsilon}} = \{Q_{\rho_i}(x_i) \,|\, i \in \{1, \dots M\}\} \subseteq \mathcal{G}^{\tilde{\varepsilon}}$ such that

$$\mathcal{H}^1(Ju \setminus \bigcup_{i=1}^{M} Q_{\rho_i}(x_i)) = \mathcal{H}^1(\tilde{J} \setminus \bigcup_{i=1}^{M} Q_{\rho_i}(x_i)) \le \tilde{\varepsilon}. \tag{13}$$

These squares are disjoint and will be, for a specific choice of $\tilde{\varepsilon}$, the family of the larger squares in i). For each of this finitely many squares $Q_{\rho_i}(x_i) \in \hat{\mathcal{G}}^{\tilde{\varepsilon}}$ exists $r_i \in (\rho_i - 2\tilde{\eta}\rho_i, \rho_i - \tilde{\eta}\rho_i)$ such that

$$\|\nabla u\|^2_{L^2(\partial Q_{r_i}(x_i))} \le C\tilde{\eta}^{-1}\rho_i^{-1}\|\nabla u\|^2_{L^2(Q_{\rho_i}(x_i))}$$

and such that $Ju \cap \partial Q_{r_i}(x_i)$ contains exactly two points, $z^{-,i}, z^{+,i} \in S_k$, that satisfy $z_1^{\pm,i} = x_{i1} \pm r_i$ and $z_2^{\pm,i} \in (x_{i2} - \tilde{\varepsilon}\rho_i, x_{i2} + \tilde{\varepsilon}\rho_i)$. This uses the fact that $\tilde{\eta} \ge 2\tilde{\varepsilon}$ and the constraints for the jump set in $Q_{\rho_i}(x_i)$. We additionally choose the points such that $[u](z^{\pm,i}) \le C\rho_i^{-1}\tilde{\eta}^{-1} \int_{Ju\cap Q_{\rho_i}(x_i)\setminus Q_{r_i}(x_i)} [u] \, \mathrm{d}\mathcal{H}^1$.
By the choices in $\mathcal{G}^{\tilde{\varepsilon}}$ we have

$$\begin{aligned}\int_{Ju\cap Q_{\rho_i}(x_i)\setminus Q_{r_i}(x_i)} [u] \, \mathrm{d}\mathcal{H}^1 &\le 2\rho_i[u](x_i) + \varepsilon\rho_i - 2r_i[u](x_i) - \varepsilon r_i\\ &\le 2\varepsilon\tilde{\eta}\rho_i + 2\tilde{\eta}(\|u\|_{L^1(B_{\rho_i}(x_i)\cap Ju, \mathcal{H}^1)} + \varepsilon\rho_i)\end{aligned}$$

and hence

$$[u](z^{\pm,i}) \le C(\varepsilon + \rho^{-1}\|u\|_{L^1(B_{\rho_i}(x_i)\cap Ju, \mathcal{H}^1)}).$$

We see that a)i)-a)iv) is satisfied, independent of the choice of $\tilde{\varepsilon}$.
Define $K = \bigcup_{i=1}^{M} \overline{Q_{(1-\tilde{\varepsilon})r_i}(x^i)}$. The set K is compact and contains most of the small squares $Q_{r_i}(x_i)$ and hence most of the jump set.

As a next step we choose a countable family of squares $\mathcal{F} = \{P_k \,|\, k \in \mathbb{N}\}$ to cover the remaining parts of the domain. We apply the Whitney-Besicovitch-type covering theorem as formulated in Proposition 4.9 for the open set $\Omega \setminus K$.
The so-defined family satisfies b)ii), b)iii) and

$$\bigcup_{k\in\mathbb{N}} \frac{6}{7}\mathring{P}_k = \Omega \setminus K = \bigcup_{k\in\mathbb{N}} P_k.$$

Hence $\{\frac{6}{7}\mathring{P}_k\}$ it is also a covering of the smaller set $\Omega \setminus \bigcup_i Q_{(1-\frac{\tilde{\varepsilon}}{2})r_i}(x_i)$ and thus for all $\gamma > 0$ an open covering of the compact set $\overline{Q}_{1-\gamma} \cap \overline{Ju} \setminus \bigcup_i Q_{(1-\frac{\tilde{\varepsilon}}{2})r_i}(x_i)$. We therefore know that there exists a finite subcovering $\{\frac{6}{7}\mathring{P}_k \,|\, k \in \{1,\dots,L\}\}$ of these squares.
This finite family is in particular a covering of the even smaller set $Q_{1-\gamma} \cap Ju \setminus \bigcup_i Q_{r_i}(x_i)$, hence b)i) is satisfied if we define $\hat{\mathcal{F}}^{\tilde{\varepsilon},\gamma} = \{P_k \,|\, k \in \{1,\dots,L\}\}$ as the family of the larger squares.
It remains to estimate the gradients on the different families. Notice that we can without loss of generality assume that S_k is a C^1 graph with $|S_k'(t)| \le \tilde{\varepsilon}$ for all $t \in (x_{i1} - \rho_i, x_{i1} + \rho_i)$. It in particular follows that the length of S_k on an interval I is estimated by $\mathcal{L}^1(I)(1+\tilde{\varepsilon})$. One of the constraints in the definition of $\mathcal{G}^{\tilde{\varepsilon}}$ yields $\sum_{i=1}^M \rho_i \le \sum_{i=1}^M \mathcal{H}^1(Ju \cap Q_{\rho_i}(x_i)) \le \mathcal{H}^1(Ju)$.
We will use this estimate twice. First to prove the first estimate in a)vi) assuming $\tilde{\eta}$ is small enough. We estimate

$$\begin{aligned}
\mathcal{H}^1(Ju \setminus \bigcup_{i=1}^M Q_{r_i} \le& \mathcal{H}^1(Ju \cap \bigcup_{i=1}^M (Q_{\rho_i}(x_i) \setminus Q_{r_i}(x^i))) + \tilde{\eta} \\
\le& \sum_{i=1}^M \left(\mathcal{H}^1(S_k \cap (Q_{\rho_i}(x^i) \setminus Q_{r_i}(x_i))) + \mathcal{H}^1(((Ju \setminus S_k) \cap Q_{\rho_i}(x_i))\right) + \tilde{\eta} \\
\le& \sum_{i=1}^M ((1+\varepsilon)\tilde{\eta}\rho_i + 2\tilde{\varepsilon}\rho_i) + \tilde{\eta} \\
\le& C\tilde{\eta}\mathcal{H}^1(Ju) + \tilde{\eta}.
\end{aligned}$$

We use it afterwards to prove that the P_k's do not have to cover much jump. It holds:

$$\begin{aligned}
\mathcal{H}^1(Ju \cap \bigcup_{k=1}^\infty P_k) &= \mathcal{H}^1(Ju \setminus K) = \mathcal{H}^1(Ju \setminus \bigcup_{i=1}^M Q_{\rho_i}(x_i)) + \sum_{i=1}^M \mathcal{H}^1(Ju \cap (Q_{\rho_i}(x_i) \setminus Q_{(1-\tilde{\varepsilon})r_i}(x_i))) \\
&\le \tilde{\varepsilon} + \sum_{i=1}^M ((\tilde{\varepsilon}\rho_i + \tilde{\eta}\rho_i)(1+\tilde{\varepsilon}) + 2\tilde{\varepsilon}\rho_i) \\
&\le \tilde{\varepsilon} + 2\tilde{\eta}\mathcal{H}^1(Ju).
\end{aligned}$$

In b)iv) we need to estimate the gradient of u on those squares P_k on which there is much jump, that is on $\tilde{\mathcal{F}}^{\tilde{\varepsilon}} = \{P_k = Q_{\tilde{r}_k}(\tilde{x}_k) \subseteq \mathcal{F}^{\tilde{\varepsilon}} \,|\, \mathcal{H}^1(Ju \cap P_k) \ge \tilde{r}_k \eta^{1/2}\}$. Since $r_k \le 1$ we estimate the volume of the union of these squares by

$$\mathcal{L}^2\left(\bigcup_{P_k \in \tilde{\mathcal{F}}^{\tilde{\varepsilon}}} P_k\right) \le \sum_{Q_{\tilde{r}_k}(\tilde{x}_k) \in \tilde{\mathcal{F}}^{\tilde{\varepsilon}}} 4\tilde{r}_k \le 4N\frac{1}{\eta^{1/2}}\mathcal{H}^1(Ju \cap \bigcup_{k=1}^\infty P_k) \le C\tilde{\eta}^{1/2}\mathcal{H}^1(Ju) + C\tilde{\varepsilon}\eta^{-1/2}.$$

All squares $Q_{\rho_i}(x_i)$ of the first family satisfy $\rho_i \leq \tilde{\varepsilon}$. We hence estimate, using again the fact that $\sum_{i=1}^M \rho_i \leq \mathcal{H}^1(Ju)$,

$$\mathcal{L}^2\left(\bigcup \hat{\mathcal{G}}^{\tilde{\varepsilon}}\right) = \mathcal{L}^2\left(\bigcup_{i=1}^M Q_{\rho_i}(x_i)\right) = 4\sum_{i=1}^M r_i^2 \leq 4\tilde{\varepsilon}\mathcal{H}^1(Ju).$$

Applying the dominant convergence theorem we can choose the parameters $\tilde{\varepsilon}$ and $\tilde{\eta}$ so small that $\|\nabla u\|^2_{L^2(\bigcup \hat{\mathcal{G}}^{\tilde{\varepsilon}})} \leq \eta\varepsilon^2$ and $\|\nabla u\|^2_{L^2(\bigcup \tilde{\mathcal{F}}^{\tilde{\varepsilon}})} \leq \varepsilon$ and hence a)v) and b)iv) are satisfied.
Similarly it is easy to see that

$$\mathcal{H}^1(Ju \cap Q_{\rho_i(x_i)}) \geq (1-\tilde{\varepsilon})2\rho_i \quad \text{and hence} \quad \sum_{i=1}^M 2\rho_i \leq \mathcal{H}^1(Ju) + 2\tilde{\varepsilon}\mathcal{H}^1(Ju).$$

We see that a)vi) is satisfied and the proof is finished. □

Proposition 4.13 (Approximation with finite segments for the Mumford-Shah energy). *For each $u \in SBV^2_{e_2,a}$ with $\mathcal{H}^1(\overline{Ju} \setminus Ju) = 0$, for each $\delta > 0$ exists $v \in SBV^2_{e_2,a}$ such that Jv is finite union of segments $(a_i, b_i) \times \{y_i\}$ and*

$$\|v-u\|_{L^1(\Omega)} \leq C\delta, \qquad |Dv|(\Omega) \leq C(1 + |Du|(\Omega) + \|\nabla u\|_{L^2(\Omega)}) \quad \textit{and} \quad J(v) \leq (1+C\delta)J(u).$$

Moreover it holds that the segments are isolated and that the jumps satisfy a growth condition around zero i.e., if $y_i = y_j$ then $a_i \neq b_j$ and for all $\lambda > 0$ there exists $\beta > 0$ such that the height of the jumps satisfies: if $[u](x) \leq \beta$ then $\operatorname{dist}(\{x\}, \{(a_i, y_i)\} \cup \{(b_i, y_i)\}) \leq \lambda$.

Proof. Let $\varepsilon > 0$, $\eta \in (2\varepsilon, \frac{1}{8})$.
We use the covering from Lemma 4.12 and apply the construction for type-III squares (see Proposition 4.7) on each of the inner squares $Q_{r_i}(x_i)$. We can use the same idea as in the foregoing density result to avoid problems at the boundary so assume without loss of generality $\gamma = 0$.
Since a)i)-a)iii) hold we know that the type-III construction can be applied. We obtain a function $\bar{v} \in SBV^2_{e_2,a}$ such that all jumps in $Q_{r_i}(x^i)$ are the result of a type-III construction and it holds, using av), avi) and the estimates of the construction that

$$\begin{aligned} \|\nabla \bar{v}\|^2_{L^2(\Omega)} &\leq (1 + C\varepsilon^{3/2})\|\nabla u\|^2_{L^2(\Omega)}, & \|u - \bar{v}\|_{L^1(\Omega)} &\leq C\varepsilon^{3/4} \\ \mathcal{H}^1(J\bar{v}) &\leq (1 + C\varepsilon^{1/2})\mathcal{H}^1(Ju) \quad \text{and} & |D\bar{v}|(\Omega) &\leq C|Du|(\Omega). \end{aligned}$$

Now, since we have chosen $\gamma = 0$, there are finitely many squares $\{\frac{6}{7}P_k \,|\, k = 1, \ldots, L\}$ that cover the remaining parts of the jump. With the same method as in the proof of Proposition 4.10 we divide them in type-I and type-II squares and apply the constructions with the same recursive pattern, using the function $\bar{v}$ as initial data of the construction. This time, assumption b)iv) ensures that there are not too many type-II squares hence the function results in a function $v \in SBV^2_{e_2,a}$ who, after choosing $\eta > 0$ and $\varepsilon \leq \eta$ small enough, satisfies all the stated inequalities.
As a next step we want to show that if we denote the finitely many segments as $(a_i, b_i) \times \{y_i\}$ that then $y_i = y_j$ implies $a_i \neq b_j$. Each endpoint of a jump (a_i, y_i) is either the result of a type-I square mollifying an endpoint of a jump created by a type-III square or a type-II square interpolating a

jump with something that was there before. If the former is the case then it is clear that this jump is isolated. If the later is the case then the last type-II square operating at this point interpolates either with an open set where there is no jump (and hence the jump point would clearly be isolated) or with some other jump, hence it would not be an endpoint.

Denote the height of a jump on a segment $(a_i, b_i) \times \{y_i\}$ by $h = (u^+ - u^-)|_{(a_i,b_i)\times\{y_i\}}$. It is clear that there is some constant $K > 0$ which is smaller then the minimal value of all the jumps created in the interior of the different squares, such that for all $x \in Ju$ it holds: If $h(x) < K$ then it follows $h(x) = \eta_i(x - a_i)(A_i(x - a_i) + d_i)$ where η_i is a positive standard interpolator, or the same with b_i instead of a_i.

Then for all $\lambda > 0$ for all $\beta < K$ it holds: $h(x) \le \beta$ then $h(x) = \eta_i(x - a_i)(A_i(x - a_i) + d_i)$. The function h is positive and strictly monotone in this neighborhood, hence if one chooses β small enough in dependence of λ it follows that $h(x) < \beta$ implies $x - a_i \le \lambda$. □

4.3.4 Density with higher regularity

As a last step we want to gain some additional regularity for the function we want to approximate. This will be accomplished by a uniform mollification away from the jump.

Lemma 4.14. *Let $\delta > 0$. Let $u \in SBV^2_{e_2,a}$ such that $Ju = \bigcup_{i=1}^k (a_i, b_i) \times \{y_i\}$ is a finite union of segments such that the segments are isolated and satisfy a growth condition at zero i.e., if $y_i = y_j$ then $a_i \neq b_j$ and for all $\lambda > 0$ there exists $\beta > 0$ such that the height of the jumps satisfies: if $[u](x) \le \beta$ then $\operatorname{dist}(\{x\}, \{(a_i, y_i)\} \cup \{(b_i, y_i)\}) \le \lambda$.*

Then there exists $v \in SBV^2_{e_2,a} \cap C^\infty(\Omega \setminus Jv)$ such that Jv is finite union of segments,

$$\|v - u\|_{L^1(\Omega)} \le C\delta, \qquad |Dv|(\Omega) \le C|Du|(\Omega), \qquad J(v) \le (1 + C\delta)J(u)$$

and such that $\|v\|_{L^\infty} + \|Dv\|_{L^\infty} + \|D^2v\|_{L^\infty}$ is finite.

Proof. Let us without loss of generality assume that the jump set contains just a single segment i.e., $Ju = (a, b) \times \{\bar{y}\}$.

If the slice would divide the square into two halfs i.e., if $(a, b) = (0, 1)$, the statement could be achieved by a convolution with a standard mollifier using an extension theorem for $(0, 1) \times (\bar{y}, 1)$ and $(0, 1) \times (0, \bar{y})$. If this is not the case, the construction will differ in a neighborhood of the edges $(a, \bar{y})$, $(b, \bar{y})$. For the sake of simplicity we will only prove the statement for $a = 0$ and $b \in (0, 1)$. The affine boundary values at the left edge can be threated with a linear interpolation together with the same scaling techniques as presented in the proof of Proposition 4.11.

Fix $\eta > 0$. We want to convolute u with a standard mollifier of constant lengthscale $\rho \ll \eta$. At the boundary of the domain we can extend the function but we need to perform a more complicated construction along the jump set.

We will mirror the function along the jump symmetrically and interpolate on scale η between the mirrored function and the original one, see Figure 16 for a sketch. Let therefore $\varphi \in C^\infty([0, 1], \mathbb{R}^+)$ such that $\varphi(x) = 1$ for $x \le b$, $\varphi(x) = 0$ for $x \ge b + \eta$ and such that $|\varphi'| \le \frac{2}{\eta}$. Define u_t for the upper

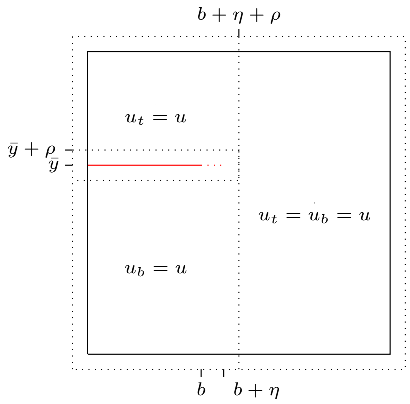

Figure 16: Mirroring a function at the jump set and the boundary before mollifying.

part of the domain and u_b for the lower part of the domain via

$$u_t(x,y) = \begin{cases} u(x,y) & y \geq \bar{y} \\ \varphi(x)u(x,2\bar{y}-y) + (1-\varphi(x))u(x,y) & y \leq \bar{y} \end{cases}$$

and

$$u_b(x,y) = \begin{cases} \varphi(x)u(x,2\bar{y}-y) + (1-\varphi(x))u(x,y) & y \leq \bar{y} \\ u(x,y) & y \geq \bar{y}. \end{cases}$$

The functions u_t and u_b are continuous, coincide with u if $x \geq b+\eta$ and can be extended to a square of radius $1+\rho$ without having large error terms as long as $\rho \leq \min\{\bar{y}, 1-\bar{y}\}$. We can then easily define

$$v(x,y) = \begin{cases} (u_t * \varphi_\rho)(x,y) & y \geq \bar{y} \\ (u_b * \varphi_\rho)(x,y) & y \leq \bar{y} \end{cases}.$$

Obviously $v \in C^\infty(\Omega \backslash (0, b+\eta+\rho) \times \{\bar{y}\})$ and each derivative is bounded in $L^\infty(\Omega)$. In particular we see that $\mathcal{H}^1(Jv) \leq \mathcal{H}^1(Ju) + \eta + \rho$. It is a standard argument that $u_t * \varphi_\rho \to u_t$ in $W^{1,2}((0,1)\times(\bar{y},1))$ as $\rho \to 0$ and since

$$\|\nabla u - \nabla u_t\|^2_{L^2((0,1)\times(\bar{y},1))} \leq C\frac{1}{\eta^2}\|\nabla u\|^2_{L^2((0,1)\times(\bar{y},\bar{y}+\rho))}$$

we can choose $\rho \ll \eta$ and $\eta < \frac{1}{2}\delta$ such that

$$J(v) \leq (1+C\delta)J(u), \quad \|v-u\|_{L^1(\Omega)} \leq C\delta, \quad \text{and} \quad |Dv| \leq C|Du|.$$

However, the proof is not finished yet, since the height of the jump may have lost its positivity during the process. This might happen if the gradient on one side of the jump is very large and points in the wrong direction. We will provide a small modification of the function v that will solve this issue.

Fix $\lambda > 0$. Let $\beta > 0$ such that $[u](x,\bar{y}) \leq \beta$ implies $x \geq b - \lambda$. It follows that for ρ and η small enough the jump of v is positive on $(0, b-\lambda)$. In fact: Using fundamental theorem of integration for u^t and u^b we see that for $x \le b-\lambda$ it follows

$$\begin{aligned}
&v^+(x,\bar y) - v^-(x,\bar y)\\
=&\int_{B_\rho((x,\bar y))} \varphi_\rho((x,\bar y)-z)(u_t(z)-u_b(z))\,\mathrm{d}\mathcal{L}^2(z)\\
=&\int_{B_\rho((x,\bar y))} \varphi_\rho((x,\bar y)-z)\left([u](z_1,\bar y)+\int_{\bar y}^{z_2}\partial_y u(z_1,w)\,\mathrm{d}\mathcal{L}^1(w)-\int_{z_2}^{\bar y}\partial_y u(z_1,w)\,\mathrm{d}\mathcal{L}^1(w)\right)\\
\ge&\beta - C\rho^{-2}\int_{B_\rho((x,\bar y))}\|\partial_y u(z_1,\cdot)\|_{L^2(\bar y-\rho,\bar y+\rho)}\,\mathrm{d}\mathcal{L}^2(z)\\
\ge&\beta - C\|\partial_y u\|_{L^2((-\rho,b-\lambda+\rho)\times(\bar y-\rho,\bar y+\rho))}\,\mathrm{d}\mathcal{L}^2(z)
\end{aligned}$$

which is positive if ρ is small enough.

We can similarly prove that for $x \in (b-\lambda, b+\eta+\rho)$

$$\begin{aligned}
v^+(x,\bar y) - v^-(x,\bar y) &\ge -C\|\partial_y u\|_{L^2((b-\lambda-\rho,b+\eta+2\rho)\times(\bar y-\rho,\bar y+\rho))}\,\mathrm{d}\mathcal{L}^2(z)\\
&\ge -C\|\partial_y u\|_{L^2((0,1)\times(\bar y-\rho,\bar y+\rho))}\,\mathrm{d}\mathcal{L}^2(z)\\
&=: L(\rho).
\end{aligned}$$

Now chose $\psi \in C_c^\infty((-1,1))$ such that $\psi(x)\in[0,1]$ and $\psi \ge \frac12$ on $(-\frac12,\frac12)$ and define, since we choose $\eta+\rho\le\beta$,

$$K(x,y) = \begin{cases} 2L(\rho)\left(\frac{1-y}{1-\bar y}\right)\psi\left(\frac{x-b}{2\beta}\right) & y \ge \bar y\\ 0 & y \le \bar y. \end{cases}$$

Then $K \in SBV^2_{e_2,0}$ and it holds that $K(x,\bar y) = 2L(\rho)\psi\left(\frac{x-b}{2\beta}\right)$ and thus $K \ge M(\rho)$ on $(b-\beta, b+\eta+\rho)$. The gradient of K satisfies

$$\|\nabla K\|_\infty \le \frac{C}{\beta} l(\rho)$$

and since $L(\rho)\to 0$ as $\rho\to 0$ we can choose ρ so small such that $v+K$ satisfies all the desired properties. □

The following theorem is the direct application of the foregoing results.

Theorem 4.15 (Main density result). *Let $u \in SBV^2_{e_2,a}$ and $\delta > 0$.*

Then there exists $v \in SBV^2_{e_2,a}\cap C^\infty(\Omega\setminus Jv)$ such that Jv is finite union of segments,

$$\|v-u\|_{L^1(\Omega)} \le C\delta,\quad |Dv|(\Omega) \le C(1+|Du|(\Omega)+\|\nabla u\|_{L^2(\Omega)})\quad \textit{and}\quad J(v) \le (1+C\delta)J(u)$$

and such that $\|v\|_{L^\infty}+\|Dv\|_{L^\infty}+\|D^2 v\|_{L^\infty}$ is finite.

4.4 The recovery sequence for functions whose jump set is a finite union of segments

Having the density result at hand we can now rigorously construct the recovery sequence for functions whose jump set is a finite union of segments.

Proposition 4.16. *Let $\delta > 0$ and let $u \in SBV^2_{e_2,0} \cap C^\infty(\Omega \setminus \overline{Ju})$ such that Ju is finite union of isolated segments and $\|u\|_\infty + \|Du\|_\infty + \|D^2 u\|_\infty$ is finite.*
Then there exists $\theta_0 > 0$ such that for all $\theta \in (0, \theta_0)$ there exists a function $v \in \mathcal{A}$ such that

$$I^\theta(v) \leq (1+\delta)I(u) \qquad \|u - v\|_{L^1(\Omega)} \leq \delta \quad \text{and} \quad |Dv|(\Omega) \leq C(1 + |Du|(\Omega) + \|\nabla u\|_{L^2(\Omega)}).$$

A sketch of the construction is indicated in Figure 17.

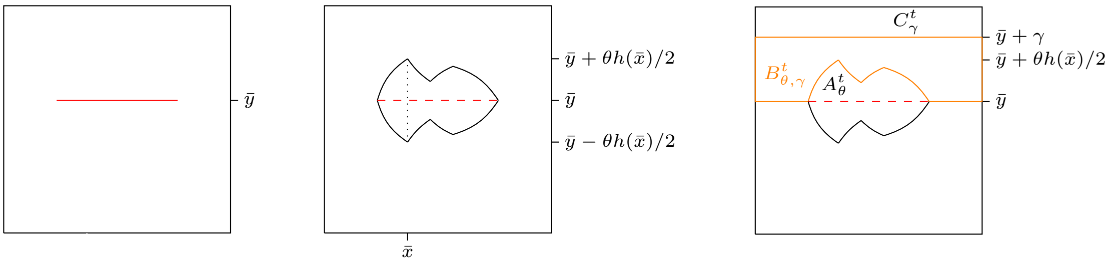

Figure 17: The construction of the recovery sequence if the jump set is a single segment.

Proof. Let us without loss of generality assume that the jump set is given as a single segment i.e., $Ju = (a, b) \times \{\bar{y}\}$. Define $h(x) = [u](x, \bar{y})$. Obviously the area in which we interpolate should still be a subset of Ω, we therefore choose $\theta_0 \leq \frac{1}{4}\|h\|_\infty \min\{\bar{y}, 1 - \bar{y}\}$ and $\theta \leq \theta_0$. Let moreover $\gamma \in (\|h\|_\infty \theta, \min\{\bar{y}, 1 - \bar{y}\})$ and construct v in a way such that $v = u$ on $\Omega \setminus (0,1) \times (\bar{y} - \gamma, \bar{y} + \gamma)$.
We define the function v as

$$v(x,y) = \begin{cases} u(x,y) & y \in (\bar{y}+\gamma, 1) \\ u(x, f(x,y)) & y \in (\bar{y} + \frac{\theta}{2}h(x), \bar{y}+\gamma) \\ \frac{1}{\theta}\left(y - \bar{y} + \frac{\theta}{2}h(x)\right) + u^-(x,\bar{y}) & y \in (\bar{y} - \frac{\theta}{2}h(x), \bar{y} + \frac{\theta}{2}h(x)) \\ u(x, \tilde{f}(x,y)) & y \in (\bar{y} - \gamma, \bar{y} - \frac{\theta}{2}h(x)) \\ u(x,y) & y \in (0, \bar{y} - \gamma). \end{cases}$$

At every $x \in (a, b)$ we are trying to fill the gap between $u^+(x, \bar{y})$ and $u^-(x, \bar{y})$ by introducing a linear function with slope $\frac{1}{\theta}$ on a length $\theta h(x)$. Since $u(x, \bar{y} + \theta h(x)/2)$ and $u^+(x, \bar{y})$ do in general not coincide we need to stretch u in y-direction, using a linear interpolation f.
We will do all computations in the following for $y \geq \bar{y}$ since the other case follows from symmetry.
Let us divide the set $\Omega^t = (0,1) \times (\bar{y}, 1)$ into

$$A^t_\theta = \{(x,y) \,|\, y \in (\bar{y}, \bar{y} + \frac{\theta}{2}h(x))\}, \qquad B^t_{\gamma,\theta} = \{(x,y) \,|\, y \in (\bar{y} + \frac{\theta}{2}h(x), \bar{y}+\gamma)\}$$
$$\text{and} \qquad C^t_\gamma = \{(x,y) \,|\, y \in (\bar{y}+\gamma, 1)\}.$$

We are using the linear function f that satisfies $f(x,\gamma+\bar y)=\gamma+\bar y$ and $f(x,\bar y+\frac{\theta}{2}h(x))=\bar y$ for the interpolation. This function is given by

$$f(x,y)=\frac{1}{\gamma-\frac{\theta}{2}h(x)}\left(\gamma y-(\gamma+\bar y)\frac{\theta}{2}h(x)\right).$$

It is some straightforward computation using that $\gamma\geq 2\theta\|h\|_\infty$ that

$$\begin{aligned}
\|f\|_\infty &\leq C, & |f(x,y)-y| &\leq C\|h\|_\infty\theta,\\
\|1-\partial_y f\|_\infty &\leq C\|h\|_\infty\theta\gamma^{-1}, & \|\partial_x f\|_\infty &\leq C\|h'\|_\infty\theta\gamma^{-1}\\
\text{and} & & \|D^2 f\|_\infty &\leq C(\|h'\|_\infty\theta\gamma^{-1}+\|h'\|_\infty^2\theta^2\gamma^{-2}+\|h''\|_\infty\theta\gamma^{-1}).
\end{aligned}\tag{14}$$

We hence set $\gamma=\theta^{1/2}$, provided that $\theta_0\leq\frac{1}{4\|u\|_\infty}$. We compute

$$\begin{aligned}
&\|\partial_x v\|^2_{L^2(\Omega^t)}\\
\leq&\|\partial_x u\|^2_{L^2(C^t_\gamma)}+\|(\nabla u)(x,f(x,y))\cdot(1,\partial_x f(x,y))\|^2_{L^2(B^t_{\gamma,\theta})}+\|\frac{1}{2}h'(x)+\partial_x u^-(x,\bar y)\|^2_{L^2(A^t_\theta)}\\
\leq&\|\partial_x u\|^2_{L^2(\Omega^t)}+C\|Du\|_\infty\theta^{1/2}+C\|Du\|_\infty\theta\|u\|_\infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|\min\{\partial_y v-1,\partial_y v+\frac{1}{\theta}\}\|^2_{L^2(\Omega^t)}\\
\leq&\|\partial_y u-1\|^2_{L^2(C^t_\gamma)}+\|(\nabla u)(x,f(x,y))\cdot(1,\partial_y f(x,y))-1\|^2_{L^2(B^t_{\theta^{1/2},\theta})}\\
\leq&\||\partial_y u-1|\|^2_{L^2(\Omega^t)}+C(\|Du\|_\infty+1)\theta^{1/2}.
\end{aligned}$$

The most important part is to estimate the second derivatives of v multiplied with θ. We define $\Gamma=\partial A^t_\gamma\cap(0,1)\times(\bar y,1)$ and $\Lambda=(\partial B^t_\gamma\backslash\Gamma)\cap(0,1)\times(\bar y,1)$ and write

$$D^2v=\int_\Gamma(\nabla v^+-\nabla v^-)\cdot\nu\,\mathrm{d}\mathcal{H}^1+\int_\Lambda(\nabla v^+-\nabla v^-)\cdot\nu\,\mathrm{d}\mathcal{H}^1+\nabla^2 v.$$

Due to chain rule and transformation formula we know that

$$\begin{aligned}
|\nabla^2 v|(\Omega^t)&\leq\|\nabla f\|^2_\infty\|\partial_y f\|_\infty\|D^2u\|_{L^1(\Omega^t)}+\|D^2 f\|_\infty\|\partial_y f\|_\infty\|\nabla u\|_{L^1(\Omega_t)}+\|\partial_{xx}u^-+h''\|_\infty\|h\|_\infty\theta\\
&\leq C\|D^2u\|_\infty+C
\end{aligned}$$

and

$$\Big|\int_\Lambda(\nabla v^+-\nabla v^-)\cdot\nu\,\mathrm{d}\mathcal{H}^1\Big|=\Big|\int_\Lambda\partial_y u(1-\partial_y f)\,\mathrm{d}\mathcal{H}^1\Big|\leq\|Du\|_\infty\|u\|_\infty\theta^{1/2}.$$

Moreover we see that

$$\int_\Gamma (\nabla v^+ - \nabla v^-)\cdot\nu \,\mathrm{d}\mathcal{H}^1$$
$$\leq \int_\Gamma \begin{pmatrix} 0 \\ -\frac{1}{\theta}\end{pmatrix}\cdot\nu\,\mathrm{d}\mathcal{H}^1 + \int_\Gamma \begin{pmatrix} (\partial_x u)\circ f + (d_y u)\circ f\partial_x f - \frac{1}{2}\partial_x u^+(\cdot,\bar{y}) + \frac{1}{2}\partial_x u^-(\cdot,\bar{y}) \\ (\partial_y u)\circ f\partial_y f\end{pmatrix}\cdot\nu\,\mathrm{d}\mathcal{H}^1.$$

Hence altogether

$$\theta|D^2 v|(\Omega) \leq \theta\Big|\int_\Gamma \begin{pmatrix} 0 \\ \frac{1}{\theta}\end{pmatrix}\cdot\frac{1}{\sqrt{1+(h')^2}}\begin{pmatrix}\frac{\theta}{2}h' \\ 1\end{pmatrix}\,\mathrm{d}\mathcal{H}^1\Big| + C\theta = (b-a) + C\theta = \mathcal{H}^1(Ju) + C\theta.$$

The estimates for the derivatives show in particular that

$$|Dv|(\Omega^t) \leq \frac{1}{\theta}\mathcal{L}^2(A^t_\theta) + \|\partial_y v\|_{L^2(B^t_{\theta,\gamma}\cup C^t_\theta)} + \|\partial_x v\|_{L^2(\Omega^t)} \leq C(1+\|h\|_{L^1(Ju)} + \|\nabla u\|_{L^2(\Omega)})$$

and we can conclude that

$$\|u-v\|_{L^1(\Omega_t)} \leq 2\|u\|_\infty \mathcal{L}^2(B^t_{\theta,\gamma}\cup A^t_\theta) \leq 2\|u\|_\infty \theta^{1/2}.$$

If we chose θ_0 in dependence of u and δ small enough the statement holds. □

4.5 The main result

We only need to put all the foregoing results together to prove the recovery sequence.

Theorem 4.17 (Recovery sequence)**.** *Let $u\in SBV^2_{e_2,0}$ and $\theta_k \searrow 0$. Then there exits $v_k\in\mathcal{A}$ such that*

$$\limsup_{k\in\mathbb{N}} I^{\theta_k}(v_k) \leq I(u) \qquad \text{and } v_k \overset{*}{\rightharpoonup} u \text{ in } BV((0,1)\times(\delta,1-\delta)) \text{ for all } \delta>0.$$

Proof. Notice that due to Lemma 4.1 we can assume $u\in SBV^2_{e_2,0}\cap SBV(\Omega)$. We will use Remark 4.2 to apply the density result of Theorem 4.15 on functions in $SBV^2_{e_2,0}$. Notice that the θ_0 chosen in Proposition 4.16 depends on u and δ we therefore use a diagonal argument in the following way:
By Theorem 4.15 there exists a sequence $u_m\in SBV^2_{e_2,0}$ such that $\|u_m-u\|\leq\frac{1}{m}$, $I(u_m)\leq I(u)+\frac{1}{m}$ and such that u satisfies the assumptions of Proposition 4.16.
Let θ_0^1 be such that the proposition can be applied to u_1 for $\delta=1$ and choose $v_k=0$ for $\theta_k>\theta_0^1$.
Let $\theta_0^2\leq\frac{1}{2}$ be such that the proposition can be applied to u_2 for $\delta=\frac{1}{2}$ and let for all k such that $\theta_k\in(\theta_0^2,\theta_0^1)$ the function v_k be the result of the construction for u_2, θ_k and $\delta=\frac{1}{2}$.
Let $\theta_0^m\leq\frac{1}{m}$ be such that the proposition can be applied to u_m for $\delta=\frac{1}{m}$ and let for all k such that $\theta_k\in(\theta_0^m,\theta_0^{m-1})$ the function v_k be the result of the construction for u_m, θ_k and $\delta=\frac{1}{m}$.
Then $I^{\theta_k}(v_k)\leq I(u)+\frac{1}{m}$ and $\|v_k-u\|_{L^1(\Omega)}\leq\frac{1}{m}$ and $m\to\infty$ as $k\to\infty$.
It is a result of the compactness statement proven in Theorem 3.1 that for all $\delta>0$ we can estimate $\|v_k\|_{W^{1,1}((0,1)\times(\delta,1-\delta))}\leq C(u)(1+\frac{1}{\delta})$ and hence $v_k\overset{*}{\rightharpoonup} u$ in $BV((0,1)\times(\delta,1-\delta))$. □

This finishes the discussion of the scalar-valued problem. The Γ-limit is proven as a corollary of this theorem and the results of the foregoing chapter.

5 Compactness and $\liminf$-inequality for the vector-valued problem

The scalar-valued energy is a simplified version of the original problem. In general, we are interested in deformations mapping subsets of $\mathbb{R}^3$ to $\mathbb{R}^3$. As we already pointed out in the introduction, one of the coordinates in both, domain and codomain, is negligible in the setting of two rank-one connected energy wells. We set $\Omega = (0,1)^2$ and consider the following energy for functions $u : \Omega \to \mathbb{R}^2$:

$$E^{\theta,p}(u) = \begin{cases} \int_\Omega \min\{|e(u + ye_1)|_p^p, |e(u - \frac{y}{\theta}e_1)|_p^p\} \, \mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2u|(\Omega) & u \in \mathcal{W} \\ \infty & \text{otherwise} \end{cases}$$

where the space $\mathcal{W}$ is given by

$$\mathcal{W} = \{u \in W^{1,p}(\Omega, \mathbb{R}^2) \,|\, \partial_x u, \partial_y u \in BV(\Omega, \mathbb{R}^2), \, u_1(0, y) = \frac{\theta^2}{1+\theta}y, \, u_2(\cdot, 0) = 0\}.$$

For some function $v \in BD(\Omega)$ we denote by $e(v)$ the absolute continuous part of the symmetric part of the gradient. The expression $\min\{|e(u + ye_1)|_p^p, |e(u - \frac{y}{\theta}e_1)|_p^p\}$ is thereby an abbreviation of $|\partial_x u_1|^p + |\partial_y u_2|^p + \min\{|\partial_y u_1 + \partial_x u_2 + 1|^p, |\partial_y u_1 + \partial_x u_2 - \frac{1}{\theta}|^p\}$.
In the nonlinear energy we considered rotated versions of $\mathrm{Id} + \theta e_1 \otimes e_2$ and $\mathrm{Id} + (1-\theta)e_1 \otimes e_2$ as energy wells. The first term in the energy $E^{\theta,2}$ was derived as the rescaled, linearized version of the bulk term close to the unrotated matrices $\mathrm{Id} + \theta e_1 \otimes e_2$ and $\mathrm{Id} + (1-\theta)e_1 \otimes e_2$. The terms $e_1 \odot e_2$ and $\frac{1}{\theta}e_1 \odot e_2$ represent infinitesimal pertubations of these matrices with rotations. We will see that in the limiting energy the phase where $(Du)_{sym} \sim \frac{1}{\theta}(e_1 \odot e_2)$ will decouple to subsets of one-dimensional hypersurfaces where either $D^J u = |[u]|e_1$ or $D^J u = |[u]|e_2$. This is were the martensite-martensite interfaces are now located. The direction e_2 represents the normal coming from the rank-one connection between $\mathrm{Id} + \theta e_1 \otimes e_2$ and $\mathrm{Id} + (1-\theta)e_1 \otimes e_2$. The second normal e_1 is present since a linearization at the two matrices $\mathrm{Id} + \theta e_2 \otimes e_1$ and $\mathrm{Id} + (1-\theta)e_2 \otimes e_1$ leads also to the energy $E^{\theta,2}$.
In this geometrically linearized setting the original energy wells of the elastic energy $e_1 \otimes e_2$ and $\frac{1}{\theta}(e_1 \otimes e_2)$ are transformed to $e_1 \odot e_2$ and $\frac{1}{\theta}(e_1 \odot e_2)$. We will see that in the limiting energy the phase where $(Du)_{sym} \sim \frac{1}{\theta}(e_1 \odot e_2)$ will decouple to phases where either $D^J u = |[u]|e_1$ or $D^J u = |[u]|e_2$. That is: The only possible normals at a martensite-martensite interface are e_2 and e_1.
We define the candidate for the limiting energy by

$$E^p(u) = \begin{cases} \int_\Omega |e(u + ye_1)|_p^p \, \mathrm{d}\mathcal{L}^2 + 2\sigma\mathcal{H}^1(Ju) & u \in SBD^p_{e_2 \odot e_1, 0} \\ \infty & \text{otherwise} \end{cases}$$

where the space $SBD^p_{e_2 \odot e_1, 0}$ is given by

$$SBD^p_{e_2 \odot e_1, 0} = \{u \in SBD_{loc}(\Omega, \mathbb{R}^2) \,|\, e(u) \in L^p(\Omega; \mathbb{R}^{2\times 2}), \, u \in L^p(\Omega, \mathbb{R}^2), \, \mathcal{H}^1(Ju) < \infty, \, u_1(0, \cdot) = 0, \\ u_2(\cdot, 0) = 0, \, [u_1]\nu_{Ju} \in [0,\infty)e_2, \, [u_2]\nu_{Ju} \in [0,\infty)e_1 \; \mathcal{H}^1\text{-a.e.}\}.$$

We need to make sure that the boundary values in this definition are well-defined since this is not the case for general functions in SBD_{loc}. However, we can easily see that the first component of such a function is an element of $W^{1,1}((0,1);L^p(0,1))$ in the sense that $x \mapsto u_1(x,\cdot)$ and so is the second component via $y \mapsto u_2(\cdot,y)$. A trace theorem does hold for $W^{1,1}((0,1);L^p(0,1))$ and hence the definition above makes sense.
For the sake of generality we consider L^p-norms in the gradient instead of the simpler L^2-norms.

5.1 Preliminaries

Before actually starting to prove the results let us first comment on the structure of the jump set of functions in $SBD^p_{e_2\odot e_1,a}$. The differences between Ju and $Ju_1 \cup Ju_2$ are in general more subtle for functions of bounded deformation then for the usual vector-valued functions of bounded variation. However, the regularity of $\partial_x u_1$ and $\partial_y u_2$ gives us additional informations.
Let $v \in L^p(\Omega)$. Remember that $x \in \Omega$ is an element of Jv iff there exists a normal $\nu \in S^1$ and values v^+ and v^- such that for $v_0(w) = \begin{cases} v^+ & \langle w,\nu\rangle > 0 \\ v^- & \langle w,\nu\rangle < 0 \end{cases}$ and $v_\rho(w) = v(x+\rho w)$ it holds: $v_\rho \to v_o$ in $L^1(B_1(0))$.
Imagine the function v in the following lemma to be the first component of a function in $SBD^p_{e_2\odot e_1,0}$.

Lemma 5.1. *Let $p \in (1,\infty)$, $v \in L^p(\Omega)$ such that $Dv \cdot e_1 \in L^p(\Omega)$ and such that Jv is $\mathcal{H}^1$-rectifiable.*
Then for $\mathcal{H}^1$-almost every $z \in Jv$ it holds: $\nu_{Jv}(z) = e_2$.
If $p \geq 2$ we even have that $\nu_{Jv}(z) = e_2$ for every $z \in Jv$.

Proof. Fix $z \in Jv$ and define $v_\rho(w) = v(z+\rho w)$ and $v_0(w) = \begin{cases} v^+(z) & \langle w,\nu_{Jv}(z)\rangle > 0 \\ v^-(z) & \langle w,\nu_{Jv}(z)\rangle < 0. \end{cases}$
Then by the definition of Jv it holds that $v_\rho \to v_0$ in $L^1(B_1(0))$. It moreover holds

$$\|\partial_x v_\rho\|_{L^1(B_1(0))} = \rho^{-1}\|\partial_x v\|_{L^1(B_\rho(x))} \leq \rho^{-1+\frac{2}{p'}}\|\partial_x v\|_{L^p(B_\rho(x))} = \left(\rho^{p-2}\int_{B_\rho(x)} |\partial_x v|^p \,\mathrm{d}\mathcal{L}^2\right)^{1/p}.$$

If $p \geq 2$ this converges to zero for every $z \in Jv$. Since Jv is rectifiable one can also show that this converges to zero for $\mathcal{H}^1$-almost every $z \in Jv$ as long as $2-p<1$ which is true for every $p>1$. We can conclude that $Dv_0 \cdot e_1 = 0$ and since $v^+(z) \neq v^-(z)$ this implies $\nu_{Jv}(z) = e_2$. □

Remark. The jump set of a BD function is rectifiable. The foregoing lemma therefore implies: If $u \in SBD^p_{e_2\otimes e_1,0}$ then $Ju = Ju_1 \cup Ju_2$ up to a set of $\mathcal{H}^1$-measure zero.

5.2 Compactness and lim inf-inequality

We show that from the point of the lim inf-inequality our candidate for the limiting energy E^p is correct. Moreover we present a fitting compactness result.
Following the considerations in the scalar-valued setting we will choose strong L^1-convergence together with weak-$*$ BD_{loc}-convergence as notions of convergence for the problem.

Theorem 5.2 (Compactness and $\liminf$-inequality). *Let $p \in (1,\infty)$, $\{u^k \mid k \in \mathbb{N}\} \subseteq \mathcal{W}$ and let $\theta_k \searrow 0$ such that $E^{\theta_k,p}(u^k) \leq M$. Then there is a subsequence $\{k_l \mid l \in \mathbb{N}\} \subseteq \mathbb{N}$ and a function $u \in SBD^p_{e_2\odot e_1,0}$ such that $u^{k_l} \overset{*}{\rightharpoonup} u$ in $BD_{loc}(\Omega,\mathbb{R}^2)$ and $u^{k_l} \to u$ in $L^1(\Omega,\mathbb{R}^2)$.*

Moreover: $E^p(u) \leq \liminf E^{\theta_k,p}(u^k)$.

Proof. We initially fix a subsequence that realizes the $\liminf$ such that further subsequences can be chosen without loss of generality. We never relabel subsequences.

Step 1: Convergence
Obviously $\|\partial_x u_1^k\|_{L^1(\Omega)} + \|\partial_y u_2^k\|_{L^1(\Omega)} \leq M^{1/p}$ and using the boundary values we immediately get $\|u^k\|_{L^1(\Omega)} \leq M^{1/p}$. Our first goal is an estimate for $\|e(u^k)\|_{L^1((\gamma,1-\gamma)^2)}$ where $\gamma > 0$.

For the estimate of the third term in $e(u)$ we want to investigate the functional on diagonal slices. We define $\Delta = \{(s,1-s) \mid s \in (0,1)\}$ and for each $z \in \Delta$ the one-dimensional, scalar-valued function $u^{k,z}_{(1,1)}(t) = u_1^k\left(z + t(1,1)\right) + u_2^k\left(z + t(1,1)\right)$. Notice that $z + t(1,1) \in \Omega$ iff $|t| \leq z_1 \wedge z_2 := \min\{z_1, z_2\}$. By transformation formula, chain rule and Fubini's theorem we estimate

$$
\begin{aligned}
&\int_\Delta \int_{-(z_1\wedge z_2)}^{z_1\wedge z_2} \min\{|\partial_t u^{k,z}_{(1,1)} + 1|^p, |\partial_t u^z_{(1,1)} - \frac{1}{\theta_k}|^p\} \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^1(z) \\
=&\sqrt{2}\int_\Omega \min\{|\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k + 1|^p, |\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k - \frac{1}{\theta_k}|^p\} \,\mathrm{d}\mathcal{L}^2 \qquad (15)\\
\leq& C\int_\Omega \min\{|e(u^k + ye_1)|_p^p, |e(u^k - \frac{y}{\theta_k}e_1)|_p^p\} \,\mathrm{d}\mathcal{L}^2.
\end{aligned}
$$

Fix $\gamma > 0$. We can estimate, using the same arguments as in the foregoing equality and the triangle inequality,

$$
\begin{aligned}
\|\partial_y u_1^k + \partial_x u_2^k\|_{L^1((\gamma,1-\gamma)^2)} \leq& C\int_{\Delta\cap(\gamma,1-\gamma)^2} \|\partial_t u^{k,z}_{(1,1)}\|_{L^1((-(z_1\wedge z_2)+\gamma/2, z_1\wedge z_2-\gamma/2))} \,\mathrm{d}\mathcal{H}^1(z) \\
&+ \|\partial_x u_1^k\|_{L^1(\Omega)} + \|\partial_y u_2^k\|_{L^1((\Omega)}.
\end{aligned}
$$

Notice that for each $z \in \Delta\cap(\gamma,1-\gamma)$ the length of the segment $(-(z_1\wedge z_2)+\gamma/2, z_1\wedge z_2-\gamma/2)$ is at least γ and that a slightly larger domain than $(\gamma,1-\gamma)^2$ is covered by the union of these segments. The length of the whole segment in $(0,1)^2$ is given by $2(z_1\wedge z_2)$ and has length at least 2γ.

We use the one-dimensional compactness result in Lemma 2.4 with $\lambda = 2(z_1\wedge z_2)$ and $\delta = \gamma/2$ to get that for $\mathcal{H}^1$-almost every $z \in \Delta\cap(\gamma,1-\gamma)^2$

$$
\begin{aligned}
&\|\partial_t u^{k,z}_{(1,1)}\|_{L^1((-z_1\wedge z_2+\gamma/2, z_1\wedge z_2-\gamma/2))} \\
\leq& C\Big(1 + \gamma^{(p-1)/p}\|\min\{|\partial_t u^{k,z}_{(1,1)} + 1|, |\partial_t u^z_{(1,1)} - \frac{1}{\theta_k}|\}\|_{L^p((-(z_1\wedge z_2), z_1\wedge z_2))} \\
&+ \frac{\|u_k\|_{L^1((-(z_1\wedge z_2), z_1\wedge z_2))}}{\gamma}\Big).
\end{aligned}
$$

Putting things together yields

$$\begin{aligned}&\|\partial_y u_1^k + \partial_x u_2^k\|_{L^1((\gamma,1-\gamma)^2)}\\ \leq &C + C(1+\gamma^{(p-1)/p}+\gamma^{-1})\left(\int_\Omega \min\{|e(u^k+ye_1)|_p^p, |e(u^k-\frac{y}{\theta_k}e_1)|_p^p\}\,\mathrm{d}\mathcal{L}^2\right)^{1/p}\end{aligned}$$

and hence there is $u \in BD_{loc}(\Omega)$ such that $u^k \overset{*}{\rightharpoonup} u$ in $BD_{loc}(\Omega)$. The boundary values and the boundedness of $\partial_x u_1$ and $\partial_y u_2$ in $L^p(\Omega)$ yield that moreover $u_k \to u$ in $L^p(\Omega)$ and hence also in $L^1(\Omega)$.

Step 2: $u \in SBD^p_{e_2\odot e_1,0}$

Since $\partial_x u_1^k$ and $\partial_y u_2^k$ have subsequences in L^p that are converging against $\partial_x u_1$ and $\partial_y u_2$ weakly-$*$ in measure, we gain that $Du_1\cdot e_1 = \partial_x u_1 \in L^p(\Omega)$ and $Du_2 \cdot e_2 = \partial_y u_2 \in L^p(\Omega)$.

We also mentioned above that the trace theorem holds for $W^{1,1}((0,1);L^p(0,1))$. Since $u_1^k(0,\cdot)\to 0$ in C^0 and $u_2^k(\cdot,y)=0$ it immediately follows that $u_1(0,\cdot)=0$ and $u_2(\cdot,0)=0$.

We notice that, due to Fubini's theorem, transformation formula and Fatou's lemma we have that for $\mathcal{H}^1$-almost every $z\in\Delta$

$$\begin{aligned}&\liminf_{k\in\mathbb{N}}\Big(\int_{-(z_1\wedge z_2)}^{z_1\wedge z_2}\min\{|\partial_t u^{k,z}_{(1,1)}+1|^p, |\partial_t u^z_{(1,1)}-\frac{1}{\theta_k}|^p\} + |u^{k,z}_{(1,1)}|\,\mathrm{d}\mathcal{L}^1\\ &\qquad + |\partial_t\partial_t u^{k,z}_{(1,1)}|(-(z_1\wedge z_2), z_1\wedge z_2)\Big)\\ &\qquad\text{is finite and for all } \gamma>0 \text{ and } \mathcal{H}^1\text{-almost every } z\in\Delta\cap(\gamma,1-\gamma)^2 \qquad (16)\\ &\liminf_{k\in\mathbb{N}}\Big(\int_{-(z_1\wedge z_2)+\gamma/2}^{z_1\wedge z_2-\gamma/2}|\partial_t u^{k,z}_{(1,1)}|\,\mathrm{d}\mathcal{L}^2(1)\Big)<\infty.\end{aligned}$$

We know that $u^k \to u$ in $L^1(\Omega)$. It is a consequence of Fubini's theorem that for $\mathcal{H}^1$-almost every $z\in\Delta\cap(\gamma,1-\gamma)^2$ it holds $u^{k,z}_{(1,1)}\to u^z_{(1,1)}$ in $L^1(-(z_1\wedge z_2), z_1\wedge z_2)$ and moreover $u^{k,z}_{(1,1)} \overset{*}{\rightharpoonup} u^z_{(1,1)}$ weakly-$*$ in $BV_{loc}(-(z_1\wedge z_2), z_1\wedge z_2)$. The one-dimensional lim inf-inequality (see Lemma 2.2) yields among others: $u^z_{(1,1)}\in SBV(-(z_1\wedge z_2), z_1\wedge z_2)$.

It is a result of the slicing technique for BD-functions, introduced by Ambrosio, Coscia and Dal Maso (see [1]), together with the already proven regularity results for $Du_1\cdot e_1$ and $Du_2\cdot e_2$, that $u\in SBD_{loc}(\Omega)$.

It is an easy computation, performed in Lemma 5.1 above, that $\nu_{J_{u_1}}=e_2$ and $\nu_{J_{u_2}}=e_1$ $\mathcal{H}^1$-almost everywhere. Hence

$$\begin{aligned}((Du)_{sym})^J &= \int_{Ju}[u]\odot\nu\,\mathrm{d}\mathcal{H}^1 = \int_{Ju_1}[u]\odot e_2\,\mathrm{d}\mathcal{H}^1 + \int_{Ju_2}[u]\odot e_1\,\mathrm{d}\mathcal{H}^1\\ &= \int_{Ju_1}\begin{pmatrix}0 & \frac12[u_1]\\ \frac12[u_1] & [u_2]\end{pmatrix}\mathrm{d}\mathcal{H}^1 + \int_{Ju_2}\begin{pmatrix}[u_1] & \frac12[u_2]\\ \frac12[u_2] & 0\end{pmatrix}\mathrm{d}\mathcal{H}^1\\ &= \int_{Ju_1}\begin{pmatrix}0 & \frac12[u_1]\\ \frac12[u_1] & 0\end{pmatrix}\mathrm{d}\mathcal{H}^1 + \int_{Ju_2}\begin{pmatrix}0 & \frac12[u_2]\\ \frac12[u_2] & 0\end{pmatrix}\mathrm{d}\mathcal{H}^1,\end{aligned}$$

where we used that $[u_1]=0$ $\mathcal{H}^1$-almost everywhere on Ju_2 and vice versa.

We define

$$A_k = \{|\partial_y u_1^k + \partial_x u_2^k + 1| \leq |\partial_y u_1^k + \partial_x u_2^k - \frac{1}{\theta_k}|\}, \quad f_k = \chi_{A_k}(\partial_y u_1^k + \partial_x u_2^k) \quad \text{and}$$
$$g_k = \partial_y u_1^k + \partial_x u_2^k - f_k.$$

Then $\|f_k\|_{L^p(\Omega)}^p \leq M$ and hence after taking a subsequence $f_k \rightharpoonup f$ in $L^p(\Omega)$. We also know that $\partial_y u_1^k + \partial_x u_2^k \overset{*}{\rightharpoonup} Du_1 \cdot e_2 + Du_2 \cdot e_1$ locally as measures and hence

$$0 \leq g_k \overset{*}{\rightharpoonup} Du_1 \cdot e_2 + Du_2 \cdot e_1 - f = (Du_1 \cdot e_2 + Du_2 \cdot e_1)^{ac} + (Du_1 \cdot e_2 + Du_2 \cdot e_1)^J - f.$$

Since $(Du_1 \cdot e_2 + Du_2 \cdot e_1)^{ac} - f \perp (Du_1 \cdot e_2 + Du_2 \cdot e_1)^J$, we conclude that the locally finite measure $D^J u_1 \cdot e_2 + D^J u_2 \cdot e_1$ is nonnegative on every measurable set and hence $[u_1]\nu_{Ju} \in [0,\infty)e_2$ and $[u_2]\nu_{Ju} \in [0,\infty)e_1$ $\mathcal{H}^1$-almost everywhere. The finiteness of $\mathcal{H}^1(Ju)$ and $\|e(u)\|_{L^p(\Omega)}$ will be a consequence of the lim inf-inequality.

Step 3: lim inf*-inequality:*
Recall that from (16) we deduced that for $\mathcal{H}^1$-almost every $z \in \Delta \cap (\gamma, 1-\gamma)^2$ it holds $u_{(1,1)}^{k,z} \to u_{(1,1)}^z$ in $L^1(-(z_1 \wedge z_2), z_1 \wedge z_2)$ and moreover $u_{(1,1)}^{k,z} \overset{*}{\rightharpoonup} u_{(1,1)}^z$ weakly-$*$ in $BV_{loc}(-(z_1 \wedge z_2), z_1 \wedge z_2)$.
The one-dimensional lim inf-inequality (see Lemma 2.2) tells us that for all $z \in \Delta \cap (\gamma, 1-\gamma)^2$ it holds

$$2\sigma\mathcal{H}^0(Ju_{(1,1)}^z) \leq \sigma \liminf_{k\in\mathbb{N}} \theta_k |\partial_t\partial_t u_{(1,1)}^{k,z}|(-(z_1 \wedge z_2), z_1 \wedge z_2)).$$

Since $\gamma > 0$ is chosen arbitrary, we have that this inequality holds for all $z \in \Delta$ and hence, using Fatou's lemma, slicing results and the orientation of the normals we can conclude that $2\sigma\mathcal{H}^1(Ju) \leq \liminf_{k\in\mathbb{N}} \sigma\theta_k|D^2 u_k|(\Omega)$.
Notice that due to (15) we know that

$$\int_\Omega |\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k + 1|^p \chi_{\{\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k \leq \frac{1}{\theta_k} - \frac{1}{2}\}} \,\mathrm{d}\mathcal{L}^2$$

is uniformly bounded. Again Lemma 2.2 and the fact that $\gamma > 0$ is arbitrary tells us that for $\mathcal{H}^1$-almost every $z \in \Delta$

$$\begin{aligned}
&(\partial_t u_{(1,1)}^{k,z} + 1)\chi_{\{\partial_t u_{(1,1)}^{k,z} \leq \frac{1}{\theta_k} - \frac{1}{2}\}} \rightharpoonup (\partial_t u_{(1,1)}^z + 1) \quad \text{in } L^p((-(z_1 \wedge z_2), z_1 \wedge z_2)) \quad \text{and hence} \\
&(\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k + 1)\chi_{\{\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k \leq \frac{1}{\theta_k} - \frac{1}{2}\}} \\
&\rightharpoonup \partial_x u_1 + \partial_y u_1 + \partial_x u_2 + \partial_y u_2 + 1 \qquad \text{in } L^p(\Omega).
\end{aligned} \tag{17}$$

The one-dimensional compactness result from Lemma 2.4 tells us that

$$\begin{aligned}
&\mathcal{L}^2\Big(\Big\{\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k \leq \frac{1}{\theta_k} - \frac{1}{2}\Big\} \cap (\gamma, 1-\gamma)^2\Big) \\
\leq &\theta_k(C + C\gamma^{(p-1)/p}\| \min\{|e(u^k + ye_1)|_p, |e(u^k - \frac{y}{\theta_k}e_1)|_p\|_{L^p(\Omega)} + C\gamma^{-1}\|u_k\|_{L^1(\Omega)}).
\end{aligned} \tag{18}$$

We already know that $\partial_x u_1^k \rightharpoonup \partial_x u_1$ and $\partial_y u_2^k \rightharpoonup \partial_y u_2$ in $L^p(\Omega)$. From (18) we may follow that also

$$(\partial_x u_1^k + \partial_y u_2^k)\chi_{\{\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k \leq \frac{1}{\theta_k} - \frac{1}{2}\}} \rightharpoonup \partial_x u_1 + \partial_y u_2^k \text{ in } L^p(\Omega).$$

Then (17) yields

$$(\partial_y u_1^k + \partial_x u_2^k + 1)\chi_{\{\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k \leq \frac{1}{\theta_k} - \frac{1}{2}\}} \rightharpoonup \partial_y u_1 + \partial_x u_2 + 1 \quad \text{in } L^p(\Omega)$$

and by the lower semicontinuity of weak L^p convergence we conclude

$$\begin{aligned}\|\partial_y u_1 + \partial_x u_2 + 1\|_{L^p(\Omega)}^p &\leq \liminf_{k\in\mathbb{N}} \|\partial_y u_1^k + \partial_x u_2^k + 1\|_{L^p(\{\partial_x u_1^k + \partial_y u_1^k + \partial_x u_2^k + \partial_y u_2^k \leq \frac{1}{\theta_k} - \frac{1}{2}\})}^p \\ &\leq \liminf_{k\in\mathbb{N}} \|\partial_y u_1^k + \partial_x u_2^k + 1\|_{L^p(\Omega)}^p.\end{aligned}$$

This finishes the proof.

□

6 The recovery sequence for the vector-valued problem

In the previous chapter we have identified a possible limiting energy and function space and achieved the $\liminf$-inequality. As a next step one would like to prove a matching recovery sequence. Although we are not able to recover arbitrary functions we are able to construct a recovery sequence for regular functions whose jump set is a finite union of segments. This strongly suggests that our choice of the limiting function E^p is correct.
The missing part is a density result for $SBD^p_{e_2\odot e_1,0}$ with respect to the energy E^p. We discuss the problems arising in the proof of such a density result at the end of this chapter.

6.1 Preliminaries

As in the scalar valued case we perform our considerations on a transformed energy on a function space with different boundary values. In the same spirit we define a local version of the space for a set $A \subset\subset \Omega$ by

$$\begin{aligned} SBD^p_{e_2\odot e_1}(A) = \{u \in SBD(A,\mathbb{R}^2) \,|\, e(u) \in L^p(A;\mathbb{R}^{2\times 2}), \mathcal{H}^1(Ju) < \infty, \\ [u_1]\nu_{Ju} \in [0,\infty)e_2,\ [u_2]\nu_{Ju} \in [0,\infty)e_1\ \mathcal{H}^1\text{-a.e.}\} \end{aligned}$$

and a version with affine boundary values by

$$\begin{aligned} SBD^p_{e_2\odot e_1,a} = \{u \in SBD(\Omega,\mathbb{R}^2) \,|\, e(u) \in L^p(\Omega;\mathbb{R}^{2\times 2}), \mathcal{H}^1(Ju) < \infty, \\ [u_1]\nu_{Ju} \in [0,\infty)e_2,\ [u_2]\nu_{Ju} \in [0,\infty)e_1\ \mathcal{H}^1\text{-a.e.} \\ u_1(0,y) = y \text{ and } u_2(x,0) = 0\, \mathcal{H}^1\text{-a.e.}\}. \end{aligned}$$

We introduce the transformed energy for the limiting space

$$\tilde{E}^p(u) = \int_\Omega |e(u)|_p^p \,\mathrm{d}\mathcal{L}^2 + 2\sigma\mathcal{H}^1(Ju)$$

and

$$\tilde{E}^{p,\theta}(u) = \int_\Omega \min\{|e(u)|_p^p, |e(u + (1-\frac{1}{\theta})ye_1|_p^p\} \,\mathrm{d}\mathcal{L}^2 + \sigma\theta|D^2u|(\Omega)$$

as the transformed energy for the approximating functions.

Remark 6.1. Let $g(x,y) = ye_1$. Then for each function $u \in SBD^p_{e_2\odot e_1,a}$ it holds: $u - g \in SBD^p_{e_2\odot e_1,0}$ and $\tilde{E}^p(u) = E^p(u-g)$. For a sequence of functions $v^k \in W^{1,2}(\Omega)$ with $Dv_k \in BV(\Omega)$, $v^k(x,0) = 0$ and $v^k(0,y) = y$ it holds: $v_k - g \in \mathcal{W}$ and $E^{\theta_k,p}(v_k - g) = \tilde{E}^{\theta_k,p}(v_k)$. The opposite direction is also true, up to the local finiteness of $|(Du)^J_{sym}|(\Omega)$. This issue can be solved in a similar way as presented in Lemma 4.1
The recovery sequences in the two different settings can therefore be transformed into each other.

We want to divide the construction of the recovery sequence in small building blocks. Obviously, if the jump set of the function is just a single segment of one of the jump sets, say Ju_1, the construction presented in Theorem 4.16 is still applicable without relevant changes. The main difficulty lies in the treatment of crossings.

6.2 The recovery sequence for functions whose jump set is a single crossing of segments for $p < 2$

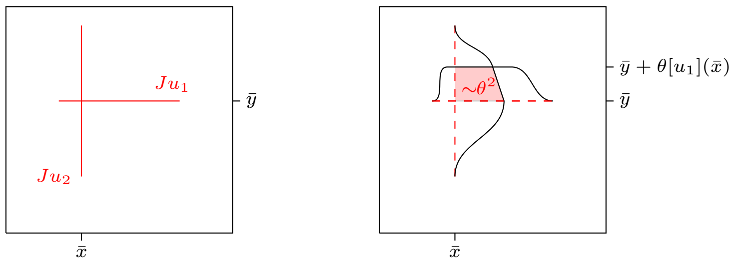

Figure 18: The naïve idea for a recovery sequence in BD for a crossing of jump sets. In the red area, the pointwise error in the energy is $\frac{1}{\theta^p}$. The energy is hence only recovered for $p < 2$.

In the case for $p < 2$, crossings can be recovered easily. The main idea as indicated in Figure 18 is to insert the scalar valued construction for u_1 and u_2 independently. Then $\partial_y u_1 \sim \frac{1}{\theta}$ and $\partial_x u_2 \sim \frac{1}{\theta}$ on the intersection of the two sets A_1 and A_2. The pointwise error in the energy on this intersection is given by $\frac{1}{\theta}$ whilst the area of the intersection is given by $\theta^2[u_1][u_2]$. Obviously, this construction does not bear any energy if $p < 2$, adds a finite term to the energy if $p = 2$ and bears infinite energy if $p > 2$. We will provide a more involved construction that will be used in recovery sequences if $p \geq 2$.

Let us first compute the energy of this construction for $p < 2$ rigorously.

Proposition 6.2. *Let $p \in (1, 2)$ and let $u \in SBD^p_{e_2 \odot e_1, a}$ with $Ju_1 = (a_1, a_2) \times \{\bar{y}\}$, $Ju_2 = \{\bar{x}\} \times (b_1, b_2)$, $[u_i] \in C^2(Ju_i)$ and $u \in W^{2,1}(\Omega \setminus Ju)$.*
Then for all $\theta_k \searrow 0$ there exists a sequence $u^k \in W^{2,1}(\Omega)$ with $u_1^k(0, y) = y + \frac{\theta_k^2}{1-\theta_k} y$ and $u_2^k(x, 0) = 0$ such that $\tilde{E}^{\theta_k, p}(u^k) \to \tilde{E}(u)$.

In this vector-valued context we want to avoid changing u on parts of the domain that are far away from the jump, since interactions of the different components might cause trouble. We hence do not follow the construction for the scalar-valued case of Chapter 4, but the one presented in the author's joint work with Conti and Zwicknagl, see [16].

Proof. Define the two positive functions $h^1(x) = [u_1](x, \bar{y})$ and $h^2(y) = [u_2](\bar{x}, y)$ and let without loss of generality be $\theta_k(\|h_1\|_\infty + \|h_2\|_\infty) \leq \min\{\bar{y}, \bar{x}, 1 - \bar{y}, 1 - \bar{x}\}$.

We define the approximating function via

$$u_1^k(x,y) = \begin{cases} u_1(x,y) + \left(\frac{y-\bar{y}}{\theta_k} - h^1(x)\right) & y \in (\bar{y}, \bar{y} + \theta_k h^1(x)) \\ u_1(x,y) & \text{otherwise} \end{cases}$$

and

$$u_2^k(x,y) = \begin{cases} u_1(x,y) + \left(\frac{x-\bar{x}}{\theta_k} - h^2(y)\right) & x \in (\bar{x}, \bar{x} + \theta_k h^2(y)) \\ u_1(x,y) & \text{otherwise .} \end{cases}$$

Obviously $u_1^k(0,y) = u_1(0,y) = y$ and $u_2^k(x,0) = u_2(x,0) = 0$. The addition of a small, linear term on a small scale yields the correct boundary values for $u_1^k(0,\cdot)$. In fact: Let us define the sequence $w_k(x,y) = \frac{\theta_k^2}{1+\theta_k} y \frac{\theta_k - x}{\theta_k} \chi_{\{x \leq \theta_k\}}$. We easily compute that $\|\nabla w_k\|_\infty \leq \theta_k$ and $|D^2 w_k|(\Omega) \leq C\theta_k$. The sequence $u^k + (w_k, 0)$ satisfies the boundary conditions and has the same limit in BD and in energy as u^k.

We define $A_1^k = \{(x,y) \,|\, y \in (\bar{y}, \bar{y} + \theta_k h^1(x))\}$ and $A_2^k = \{(x,y) \,|\, x \in (\bar{x}, \bar{x} + \theta_k h^2(y))\}$. The function u coincides with u^k outside of these two sets hence

$$\|u^k - u\|_{L^1(\Omega)} \leq \|h^1\|_{L^\infty} \mathcal{L}^1(A_1^k) + \|h^2\|_{L^\infty} \mathcal{L}^1(A_2^k) \leq (\|h^1\|_{L^\infty}^2 + \|h^2\|_{L^\infty}^2)\theta_k \to 0.$$

Notice that it holds

$$Du_1^k(x,y) = \begin{cases} Du_1(x,y) + \begin{pmatrix} -\partial_x h^1(x) \\ \frac{1}{\theta_k} \end{pmatrix} & y \in (\bar{y}, \bar{y} + \theta_k h^1(x)) \\ Du_1(x,y) & \text{otherwise} \end{cases}$$

and since the symmetric result is true for Du_2^k we have

$$e(u^k) = \begin{cases} e(u) + \begin{pmatrix} -\partial_x h^1(x) & \frac{1}{\theta_k} \\ \frac{1}{\theta_k} & -\partial_y h^2(y) \end{pmatrix} & \text{in } A_1^k \cap A_2^k \\ e(u) + \begin{pmatrix} -\partial_x h^1(x) & \frac{1}{2\theta_k} \\ \frac{1}{2\theta_k} & 0 \end{pmatrix} & \text{in } A_1^k \setminus A_2^k \\ e(u) + \begin{pmatrix} 0 & \frac{1}{2\theta_k} \\ \frac{1}{2\theta_k} & -\partial_y h^2(y) \end{pmatrix} & \text{in } A_2^k \setminus A_1^k \\ e(u) & \text{otherwise.} \end{cases}$$

It follows, using the triangle inequality, that

$$\begin{aligned} &\| \min\{|e(u^k)|_p^p, |e(u^k + (1 - \frac{1}{\theta_k})ye_1)\}|_p^p\|_{L^1(\Omega)} \\ \leq &\||e(u)|_p^p\|_{L^1(\Omega)} + (|\partial_x h^1|_{L^\infty}^p + 1)\mathcal{L}^1(A_1^k \setminus A_2^k) + (|\partial_y h^2|_{L^\infty}^p + 1)\mathcal{L}^1(A_2^k \setminus A_1^k) \\ &+ \left(|\partial_y h^2|_{L^\infty}^p + |\partial_x h^1|_{L^\infty}^p + \left(1 + \frac{1}{\theta_k}\right)^p\right) \mathcal{L}^1(A_1^k \cap A_2^k). \end{aligned}$$

Notice that $\mathcal{L}^1(A_1^k \cap A_2^k) \leq |\partial_y h^2|_{L^\infty} |\partial_x h^1|_{L^\infty} \theta_k^2$. Hence for $p < 2$ it follows

$$\lim_{k\to\infty} \| \min\{|e(u^k)|_p^p, |e(u^k + (1 - \frac{1}{\theta_k})ye_1)\}|_p^p\|_{L^1(\Omega)} = \||e(u)|_p^p\|_{L^1(\Omega)}.$$

Moreover it holds that

$$\begin{aligned}\theta_k|D^2u^k|(\Omega) \leq &\theta_k\Bigg(\|D^2u\|_{L^1(\Omega)} + \|\partial_x\partial_x h^1\|_\infty \mathcal{L}^2(A_1^k) + \|\partial_y\partial_y h^2\|_\infty \mathcal{L}^2(A_2^k) \\ &+ \int_{\partial A_1^k} \left| \begin{pmatrix} -\partial_x h^1(x) \\ \frac{1}{\theta_k} \end{pmatrix} \cdot \nu \right| \, \mathrm{d}\mathcal{H}^1(x,y) + \int_{\partial A_2^k} \left| \begin{pmatrix} \frac{1}{\theta_k} \\ -\partial_y h^2(y) \end{pmatrix} \cdot \nu \right| \, \mathrm{d}\mathcal{H}^1(x,y) \Bigg) \\ \leq &\theta_k C + \mathcal{H}^1(A_k^1) + \mathcal{H}^1(A_k^2) \\ &\to 2\mathcal{H}^1(Ju).\end{aligned}$$

So we know that

$$\limsup_{k\in\mathbb{N}} \tilde{E}^{p,\theta_k}(u_k) \leq \tilde{E}(u) \qquad \text{and} \qquad u_k \to u \text{ in } L^1(\Omega).$$

The compactness result yields the BD_{loc}-convergence of a subsequence. □

6.3 The recovery sequence for functions whose jump set is a single crossing of segments for $p \geq 2$

It is easy to see that the naïve construction presented in Theorem 6.2 will bear energy if applied for $p \geq 2$. We now construct a sequence of function that is able to recover crossings for arbitrary p.

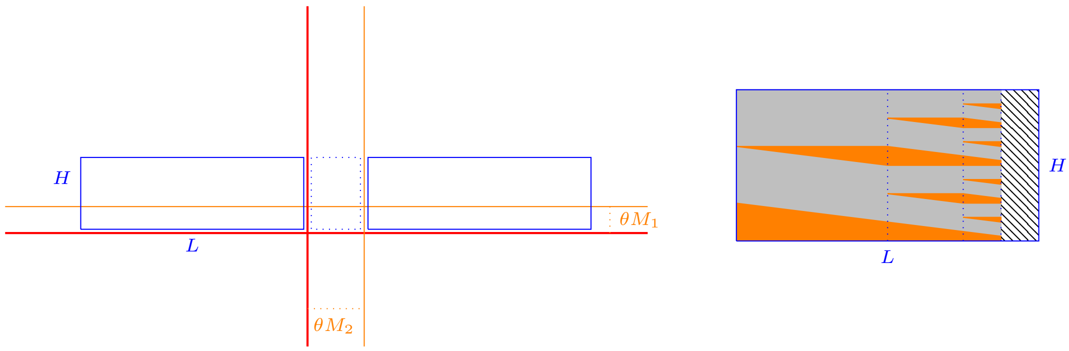

Figure 19: Crossing of jumps for $p \geq 2$: Away from the crossing we recover the jumps(red lines) by the usual tunnels(orange). We enlarge the tunnel for the first component on the blue rectangle(left). To realize this, we perform a Kohn-Müller twinning construction(right).

For simplicity, assume that we want to recover a function that only has two jumps of constant heights and no absolute continuous part of the gradient, that is $u_1(x,y) = M_1\chi_{\{y>\frac{1}{2}\}}$ and $u_2(x,y) = M_2\chi_{\{x>\frac{1}{2}\}}$. Away from the crossing, each of the jumps will be replaced by a tunnel of height θM_i, in which an affine function with slope $\frac{1}{\theta}$ is inserted. Close to the crossing, the vertical tunnel widens, via a twinning construction, to a height of $H > \theta M_1$until it starts passing the other tunnel, and interpolates to an affine function with slope $\frac{M_1}{H}$. The error in energy is then $\frac{M_1^p}{H^p} \cdot \theta M_2 \cdot H$ which will be small for $H^{p-1} \gg \theta M_2 M_1^p$. A sketch of this idea is given in Figure 19.

As one might notice, the structure of the twinning construction indicated in the second picture of Figure 19 is the well-known construction introduced by Kohn and Müller [34]. We will need to compute the energy of the construction adapted to our boundary values to make sure that the length scales can be chosen in a way, that the overall energy in the twinning gets small. We therefore compute the energy of a 'building block' of the energy, see Figure 20.

We will only construct the first component of v, which will be a $SBV^p_{e_2}$ function.

Proposition 6.3 (Energy of a building block)**.** *Let $p \in (1, \infty)$, $h, l > 0$ and $R_{l,h} = (0,l) \times (0,h)$. There is $C > 0$ and a function $v \in SBV^p_{e_2}(R_{l,h})$ such that the following boundary values hold: $v(x,0) = 0$, $v(x,h) = \tilde{M}h$,*

$$v(0,y) = \begin{cases} \frac{y}{\theta} & y \in (0, \theta\tilde{M}h) \\ \tilde{M}h & y \in (\theta\tilde{M}h, h) \end{cases} \quad \textit{and} \quad v(l,y) = \begin{cases} \frac{y}{\theta} & y \in (0, \theta\tilde{M}\frac{h}{2}) \\ \tilde{M}\frac{h}{2} & y \in (\theta\tilde{M}h, \frac{h}{2}) \\ \tilde{M}\frac{h}{2} + \frac{y-\frac{h}{2}}{\theta} & y \in (\frac{h}{2}, \frac{h}{2}(1+\theta\tilde{M})) \\ \tilde{M}h & y \in (\frac{h}{2}(1+\theta\tilde{M}), h). \end{cases}$$

The following estimates hold for the derivatives of v:

$$\min\{|\partial_y v - \frac{1}{\theta}|, |\partial_y v|\} = 0, \qquad \|\partial_x v\|^p_{L^p(R_{l,h})} \leq \frac{\tilde{M}^p h^{p+1}}{l^{p-1}},$$
$$\sigma\theta|\partial_y\partial_y v|(R_{l,h}) \leq C\sigma l, \qquad \sigma\theta|\partial_x\partial_y v|(R_{l,h}) \leq C\sigma h$$
$$\textit{and} \qquad \sigma\theta|D\partial_x v|(R_{l,h}) \leq C\sigma\theta(l+h)\frac{\tilde{M}h}{l}.$$

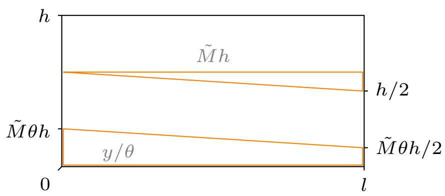

Figure 20: A building block for the twinning construction.

Proof. We chose the function, as indicated in Figure 20, to be

$$v(x,y) = \begin{cases} \frac{y}{\theta} & y \in (0, \theta\tilde{M}h - \frac{x}{l}\theta\tilde{M}\frac{h}{2})) \\ \tilde{M}h - \frac{x}{l}\tilde{M}\frac{h}{2} & y \in (\theta\tilde{M}h - \frac{x}{l}\theta\tilde{M}\frac{h}{2}, h/2 + \theta\tilde{M}\frac{h}{2} - \frac{x}{l}\theta\tilde{M}\frac{h}{2}) \\ \frac{y}{\theta} - \frac{h}{2\theta} + \tilde{M}\frac{h}{2} & y \in (h/2 + \theta\tilde{M}\frac{h}{2} - \frac{x}{l}\theta\tilde{M}\frac{h}{2}, \frac{h}{2} + \theta\tilde{M}\frac{h}{2}) \\ \tilde{M}h & y \in (\frac{h}{2} + \theta\tilde{M}\frac{h}{2}, h). \end{cases}$$

We easily see that elastic energy is only stored in the x-derivative, on the second part of the area, that is on a domain of volume $\sim hl$. We therefore estimate $\|\partial_x v\|^p_{L^p(R_{l,h})} \leq lh\left(\frac{\tilde{M}h}{2l}\right)^p \leq C\frac{\tilde{M}^p h^{p+1}}{l^{p-1}}$.

The jump of the derivatives is constant, hence the surface energy is given by

$$|\partial_y\partial_y v|(R_{l,h}) \leq Cl\frac{1}{\theta}, \quad |\partial_x\partial_y v|(R_{l,h}) \leq Ch\frac{1}{\theta} \quad \text{and} \quad |D\partial_x v|(R_{l,h}) \leq C(l+h)\frac{\tilde{M}h}{l}.$$

□

We now combine these building blocks to a twinning construction, as indicated in Figure 19. Since we are working with the full second gradient and not only with $\partial_y\partial_y u$ as part of the energy, we will need to stop the twinning at some point and continue with linear interpolation.

Proposition 6.4. *Let $p \in (1,\infty)$, $M_1 > 0$ and $\theta \in (0, M_1)$. Let $1 > H, L > 0$ such that $H \geq M_1\theta$ and $L \gg H$ and let $R_{L,H} = (0,L)\times(0,H)$. There is a function $w \in SBV^p_{e_2}(R_{L,H})$ such that the following boundary values hold: $w(x,0) = 0$, $w(x,H) = M_1$,*

$$w(0,y) = \begin{cases} \frac{y}{\theta} & y \in (0,\theta M_1) \\ M_1 & y \in (\theta M_1, H) \end{cases} \qquad \text{and} \qquad w(L,y) = \frac{y}{H}M_1.$$

The following estimate holds for the energy of w:

$$\begin{aligned}&\|\min\{|\partial_y w - \frac{1}{\theta}|, |\partial_y w|\}\|^p_{L^p(R_{L,H})} + \|\partial_x w\|^p_{L^p(R_{L,H})} + \sigma\theta|D^2 w|(R_{L,H}) \\ &\leq C(M_1^p H L^{-p+1} + \sigma L + \sigma\theta H^2 M_1^{-1}).\end{aligned}$$

Moreover we have $\|Dv\|_{L^1(R_{L,H})} \leq CM_1$.

The proof is a standard variant of the original construction in [34], similar to the ones given in [13], [26] or [54].

Proof. We want to perform a self-similar twinning construction with the functions from Proposition 6.3 as building blocks. We therefore construct vertical rows, starting at left edge, in which 2^k building blocks with identical sidelengths are used. We have a vertical length of $h_k = 2^{-k}H$ in the kth row and choose the horizontal length to be $l_k = L(1-\gamma)\gamma^k \sim L\gamma^k$ for some $\gamma \in (0,1)$. We set $\tilde{M} = \frac{M_1}{H}$.
If we move horizontally through the row we add the constant $\tilde{M}h_k = \frac{M_1}{2^k}$ in each step, in order to produce a continuous function, without changing the derivatives. In vertical direction, this construction is only performed up to the Kth step, for some $K \in \mathbb{N}$ that is chosen later.
The y-derivative is either 0 or $\frac{1}{\theta}$ in this construction, for the x-derivative we compute

$$\begin{aligned}\|\partial_x w\|^p_{L^p((0,L(1-\gamma^K))\times(0,H))} &\leq C\sum_{k=0}^{K} 2^k \tilde{M}^p \frac{(2^{-k}H)^{p+1}}{(L\gamma^k)^{p-1}} \\ &\leq CM_1^p H L^{-p+1}\sum_{k=0}^{K}\left(\frac{1}{2^p\gamma^{p-1}}\right)^k. \end{aligned} \tag{19}$$

The sum over the second derivatives in y direction is estimated by

$$\sigma\theta|\partial_y\partial_y w|((0, L(1-\gamma^K) \times (0,H)) \leq C\sigma \sum_{k=0}^{K} 2^k L\gamma^k = C\sigma L \sum_{k=0}^{K} (2\gamma)^k.$$

We fix γ such that $\gamma < \frac{1}{2}$ and $\gamma > \frac{1}{2^{p/(p-1)}}$. Then both sums converge even for $K = \infty$ and are hence finite for arbitrary K. This implies

$$\|\partial_x w\|^p_{L^p((0,L(1-\gamma^K))\times(0,H))} + \sigma\theta|\partial_y\partial_y w|((0, L(1-\gamma^K)) \times (0,H)) \leq CM_1^p HL^{-p+1} + C\sigma L.$$

It is a direct consequence that we need $L \gg H$. There are additional terms in the Hessian that force us to stop the construction at some step K. One of the terms is estimated by

$$\sigma\theta|\partial_x\partial_y w|((0, L(1-\gamma^K)) \times (0,H)) \leq C\sigma \sum_{k=0}^{K} 2^k H 2^{-k} = C\sigma K H.$$

We will therefore choose $K \leq \frac{L}{H}$. We also estimate the remaining part of the Hessian by

$$\begin{aligned}\sigma\theta|D\partial_x w|((0, L(1-\gamma^K)) \times (0,H)) \leq& C\sigma\theta \sum_{k=0}^{K} 2^k (l_k + h_k)\tilde{M}\frac{h_k}{l_k} \\ \leq& C\sigma\theta \sum_{k=0}^{K} 2^k (L\gamma^k + 2^{-k}H)\frac{M_1}{L2^k\gamma^k} \\ \leq& C\sigma\theta \sum_{k=0}^{K} \left(M_1 + \frac{HM_1}{(2\gamma)^k}\right) \\ \leq& C\sigma\theta \left(KM_1 + HM_1 \frac{1-(2\gamma)^{-K-1}}{1-(2\gamma)^{-1}}\right) \\ \leq& C\sigma\theta(KM_1 + HM_1(2\gamma)^{-K}).\end{aligned}$$

The last estimate holds since $(2\gamma)^{-K} \gg 1$.

For the sake of simplicity we want all surface terms to scale identically and will hence choose $\theta HM_1(2\gamma)^{-K} \leq L$ and $K \leq \frac{L}{M_1\theta}$, in addition to the previous property $K \leq \frac{L}{H}$. The first property can be written as $K \leq \ln\left(\frac{L}{HM_1\theta}\right)\ln((2\gamma^{-1}))^{-1}$. Since $H \geq M_1\theta$, $L \gg H$ and $\ln((2\gamma^{-1}))^{-1} > 0$, we have $C\ln\left(\frac{L}{H}\frac{1}{M_1\theta}\right) \leq \frac{L}{H} \leq \frac{L}{M_1\theta}$.

We want K to be as large as possible with this property, that is $K \sim \ln\left(\frac{L}{H}\frac{1}{M_1\theta}\right)\ln((2\gamma^{-1}))^{-1}$, or in other terms, $(2\gamma)^K \sim C\theta HM_1 L^{-1}$. The following estimate for the energy of the twinning until step K holds:

$$\|\partial_x w\|^p_{L^p((0,L(1-\gamma^K))\times(0,H))} + \sigma\theta|D^2 w|((0, L(1-\gamma^K)) \times (0,H)) \leq CM_1^p HL^{-p+1} + C\sigma L.$$

To finish the construction we will perform a linear interpolation in x-direction between the values of the construction at $x = L(1-\gamma^K)$ and the boundary values at $x = L$. This interpolation takes place on 2^{K+1} many building blocks of the form $(L(1-\gamma^K), L) \times (0, 2^{-K-1}H)$. We define

$$w(x,y) = \left(\frac{-x}{L\gamma^K} + \frac{1}{\gamma^K}\right) w((1-\gamma^K)L, y) + \left(1 + \frac{x}{L\gamma^K} - \frac{1}{\gamma^K}\right) y \frac{H}{M_1}.$$

The y derivative of w is the linear interpolation in x between the y derivative of $w((1-\gamma^K)L, y)$ and $\frac{H}{M_1}$. Since $\frac{1}{M_1} \leq \frac{1}{\theta}$ we have the very rough estimate

$$\| \min\{|\partial_y w - \frac{1}{\theta}|, |\partial_y w|\}\|^p_{L^p((L(1-\gamma^K),L)\times(0,2^{-K-1}H))} \leq \frac{H^p}{\theta^p}\gamma^K L 2^{-K-1} H.$$

For the x derivative we need to compute the L^∞-distance of the two functions we want to interpolate between. It is maximal at $y = \theta M_1 2^{-K}$ and given by $2^{-K} M_1(1 - M_1\theta H^{-1})$.
We therefore estimate, using $\frac{M_1\theta}{H} \in (0,1)$,

$$\begin{aligned}\|\partial_x w\|^p_{L^p((L(1-\gamma^K),L)\times(0,2^{-K-1}H))} &\leq \left(\frac{1}{\gamma^K L} 2^{-K} M_1(1 - M_1\theta H^{-1})\right)^p \gamma^K L 2^{-K-1} H \\ &\leq C M_1^p \gamma^{K(1-p)} 2^{-K(1+p)} L^{1-p} H. \qquad (20)\end{aligned}$$

We put the different building blocks together and derive, using $(2\gamma)^{-K} \sim \frac{L}{H\theta M_1}$ and $H < 1$, that

$$\begin{aligned}&\|\partial_x w\|^p_{L^p((L(1-\gamma^K),L)\times(0,H))} + \| \min\{|\partial_y w - \frac{1}{\theta}|, |\partial_y w|\}\|^p_{L^p((L(1-\gamma^K),L)\times(0,H))} \\ \leq& C(M_1^p \gamma^K ((2\gamma)^{-K})^p L^{1-p} H + \frac{1}{\theta^p}\gamma^K L H^{p+1}) \\ \leq& C\theta^{-p}\gamma^K L H (H^{-p} + H^p) \qquad (21) \\ \leq& C\theta^{-p}\gamma^K L H^{1-p}.\end{aligned}$$

We know that K is large and $\gamma \leq \frac{1}{2}$, hence γ^K is small. We need to quantify this smallness and hence use the usual rules for the logarithm to compute

$$\gamma^K = \gamma^{\ln(H^{-1}M_1^{-1}\theta^{-1}L)(\ln((2\gamma)^{-1}))^{-1}} = (H^{-1}M_1^{-1}\theta^{-1}L)^{\ln(\gamma)\ln((2\gamma)^{-1}))^{-1}}.$$

We remember that $\gamma > 2^{-p/(p-1)}$ and hence

$$\ln(\gamma)\ln(((2\gamma)^{-1}))^{-1} \geq \ln(2^{-p/(p-1)})(\ln((2^{1-p/(p-1)})^{-1}))^{-1} = \ln(2^{-p/(p-1)})(\ln(2^{1/(p-1)}))^{-1} = -p.$$

The estimate in (21) then reduces to

$$\begin{aligned}&\|\partial_x w\|^p_{L^p((L(1-\gamma^K),L)\times(0,H))} + \| \min\{|\partial_y w - \frac{1}{\theta}|, |\partial_y w|\}\|^p_{L^p((L(1-\gamma^K),L)\times(0,2H))} \\ \leq& C(HM_1\theta L^{-1})^p \theta^{-p} L H^{1-p} \\ \leq& C M_1^p L^{1-p} H\end{aligned}$$

which is the same energy as the elastic energy of the twinning construction.
In the linear interpolation, the only jumps of the derivatives are the jumps in y direction. We have

the jump term estimated by

$$|(\partial_y\partial_y)^J w|((L(1-\gamma^K), L) \times (0, 2^{-K-1}H)) \leq CL\gamma^K \frac{1}{\theta}.$$

However, there is also a mixed term of the absolute continuous part of the gradient present, which is estimated by

$$\begin{aligned}\|(\partial_x\partial_y)^{ac} w\|_{L^1((L(1-\gamma^K),L)\times(0,2^{-K-1}H))} &\leq \theta^{-1}L^{-1}\gamma^{-K}L\gamma^K\theta M_1 2^{-K} + HM_1^{-1}L^{-1}\gamma^{-K}L\gamma^K H2^{-K} \\ &\leq M_1 2^{-K} + H^2 M_1^{-1}2^{-K}.\end{aligned}$$

Putting together the different building blocks we have the following estimate for the surface energy, using that $\theta M_1 \leq H \leq L$:

$$\begin{aligned}\sigma\theta|D^2 w|((L(1-\gamma^K)), L \times (0,H)) &\leq C\sigma(L\gamma^K 2^K + \theta H^2 M_1^{-1} + \theta M_1) \\ &\leq C\sigma(L + \theta H^2 M_1^{-1}).\end{aligned}$$

Altogether, the different terms add up to the following estimate for the energy

$$\begin{aligned}&\|\min\{|\partial_y w - \frac{1}{\theta}|, |\partial_y w|\}\|^p_{L^p(R_{L,H})} + \|\partial_x w\|^p_{L^p(R_{L,H})} + \sigma\theta|D^2 w|(R_{L,H}) \\ \leq& C(M_1^p H L^{-p+1} + \sigma L + \sigma\theta H^2 M_1^{-1}).\end{aligned}$$

For each $x \leq 2^K$ we have that $\partial_y w(x,\cdot) = \frac{1}{\theta}$ on a length of θM_1. In the interpolation region we have $\partial_y w(x,\cdot) \leq \frac{M_1}{H}$ up to a set of length θM_1 on which we have $\partial_y w(x,\cdot) \leq \frac{1}{\theta}$. It immediately follows that

$$\|\partial_y w\|_{L^1((0,L)\times(0,H))} \leq M_1 + HL\gamma^K \frac{M_1}{H} \leq CM_1.$$

For the x-derivative we get, considering the arguments leading to 19 and 20,

$$\|\partial_x w\|_{L^1((0,L)\times(0,H))} \leq CM_1 H$$

Hence the function w fulfills all stated properties. □

We have hence computed how the energies of the different building blocks add up if they are combined as indicated in Figure 6.4. We will use this twinning construction to achieve a recovery sequence for a single crossing. Notice that for the sake of simplicity we only recover a crossing of jumps of constant height on a small square. For more general crossings we will provide a blow-up technique at a later part of this chapter.

Proposition 6.5. *Let* $p \in (1,\infty)$, $r > 0$, $\theta \in (0,\theta_0)$ *and let* $u \in SBD^p_{e_2\odot e_1}(Q_r)$ *be the function* $u(x,y) = (M_1\chi_{\{y>0\}}, M_2\chi_{\{x>0\}})$.
Then there is $v \in W^{1,2}(Q_r)$ *such that* $Dv \in BV(\Omega,\mathbb{R}^4)$ *and such that the following estimates hold:*

$$\tilde{E}^{p,\theta}(v) \leq 8\sigma r + C(M_1,M_2,\sigma,p)\theta^{1/(p^2-p+1)}, \quad \|v-u\|_{L^1(Q_r)} \leq C(M_1,M_2)(\theta + \theta^{(1+p)/(p^2-p+1)})$$

and $$|Dv|(Q_r) \leq C(M_1,M_2).$$

Moreover, the following boundary values are satisfied:

$$v_1(x,-r)=0, \quad v_1(x,r)=M_1, \quad v_1(-r,y)=v_1(r,y)=\begin{cases}0 & y\le 0\\ \frac{y}{\theta} & y\in(0,\theta M_1)\\ M_1 & y\ge M_1\theta,\end{cases}$$

$$v_2(-r,-y)=0, \quad v_2(r,y)=M_2, \quad v_2(x,-r)=v_2(x,r)=\begin{cases}0 & x\le 0\\ \frac{x}{\theta} & x\in(0,\theta M_2)\\ M_2 & x\ge M_2\theta.\end{cases}$$

Proof. We perform the construction, sketched in Figure 19 and explained in the beginning of this section. That is for $H>\theta M_1$, $L\gg H$ we define:

$$v_2(x,y)=\begin{cases}0 & x\le 0\\ \frac{x}{\theta} & x\in(0,\theta M_2)\\ M_2 & x\ge M_2\theta\end{cases} \quad \text{and} \quad v_1(x,y)=\begin{cases}0 & y\le 0, x\notin(-L,\theta M_2+L)\\ \frac{y}{\theta} & y\in(0,\theta M_1), x\notin(-L,\theta M_2+L)\\ M_1 & y\ge M_1\theta, x\notin(-L,\theta M_2+L).\end{cases}$$

We also set $v_1(x,y)=M_1$ for $y\ge H$ and $v_1(x,y)=0$ for $y\le 0$. We define v to be the construction of Proposition 6.4 on $(-L,0)\times(0,H)$ and the mirrored construction on $(\theta M_2,\theta M_2+L)\times(0,H)$. Finally we set $v(x,y)=\frac{yM_1}{H}$ for $x\in(0,M_2\theta)$, $y\in(0,H)$.

The energy is then estimated via

$$\tilde{E}^{p,\theta}(v)\le 8\sigma r+C(M_1^pHL^{-p+1}+\sigma L+\sigma\theta H^2M_1^{-1})+M_1^p\theta M_2H^{1-p}.$$

Notice, that for the sake of simplicity we inserted a slope of $\frac{1}{\theta}$ in the intermediate region, although the energy $\tilde{E}^{p,\theta}$ prefers a slope of $\frac{1}{\theta}+1$. This does only creates an error of θrM_1, which is neglectable.

We choose $H=\theta^{p/(p^2-p+1)}$ and $L=\theta^{1/(p^2-p+1)}$ such that the relevant terms scale similar and estimate the energy by

$$\tilde{E}^{p,\theta}(v)\le 8\sigma r+C(M_1,M_2,\sigma,p)\theta^{1/(p^2-p+1)}.$$

For the gradient of u we get, using the estimates given in Proposition 6.4 an the fact that the x derivative is only present in that construction,

$$|Dv|(Q_r)\le(M_1+M_2)\theta\frac{1}{\theta}+CM_1+\theta M_2H\frac{1}{H}M_1=C(M_1,M_2).$$

We have that u and v are bounded in L^∞ and only different on the tunnels and the crossing. Hence

$$\|u-v\|_{L^1(Q_r)}\le(M_1+M_2)^2\theta+2(M_1+M_2)HL\le C(M_1,M_2)(\theta+\theta^{(1+p)/(p^2-p+1)}).$$

□

6.4 The recovery sequence for functions whose jump set is a finite union of segments

We can now put the things together and construct a recovery sequence for a regular function whose jump set is a finite union of segments.

The main difficulty of the following construction - performing a recovery for crossings - has been solved in the previous section. We can now prove:

Theorem 6.6 (Recovery sequence for isolated segments)**.** *Let $p \in (1,\infty)$ and $u \in SBD^p_{e_2\odot e_1,a}$ such that it fulfills $u_i \in W^{2,\infty}(\Omega \setminus Ju_i)$, $Ju_1 = \bigcup_{i=0}^I (a_i,b_i)\times\{y_i\}$, $Ju_2 = \bigcup_{j=0}^J \{x_j\}\times(d_j,e_j)$ and such that $[u_i] \in C^2(Ju_i)$. Let $\theta_k \searrow 0$.*

Then there is a sequence $v_k \in W^{1,2}(\Omega)$ such that $Dv_k \in BV(\Omega,\mathbb{R}^4)$, $v_k(0,y) = \frac{\theta^2}{1+\theta}y$, $v_k(\cdot,0) = 0$ and

$$\tilde{E}^{p,\theta_k}(v_k) \to \tilde{E}^p(u) \qquad \textit{and} \qquad v_k \overset{*}{\rightharpoonup} u \textit{ in } BD.$$

Proof. The families of horizontal jump sets and of vertical jump sets are both finite. Hence there is a $\gamma > 0$ such that $\operatorname{dist}((a_i,b_i)\times\{y_i\},(a_k,b_k)\times\{y_k\}) \geq \gamma$ and $\operatorname{dist}(\{x_j\}\times(d_j,e_j),\{x_l\}\times(d_l,e_l)) \geq \gamma$. We can without loss of generality choose $\theta_k \leq (\max\{[u_i](x)\,|\,x\in Ju_i\})^{-1}\gamma$.

Away from the crossings we want to insert tunnels in which the slope of the function is approximately $\frac{1}{\theta_k}$. These tunnels do not interact and the energy of this construction converges, see the computations in the proof of Proposition 6.2 for the details. At the crossings we want to implement the construction developed in Proposition 6.5. This construction does, however, start with a function that is constant on both sides of the jump. We show in the following lemma that one can perform a local blow-up of the function at the crossing with small energy. A combination of these results finishes the proof. □

Lemma 6.7. *Let $u \in SBD^p_{e_2\odot e_1}(Q_R)$ such that $Ju_1 = (-R,R)\times\{0\}$, $Ju_2 = \{0\}\times(-R,R)$, $u_i \in W^{2,\infty}(Q_R\setminus Ju_i)$ and such that $[u_i]\in C^2(Ju_i)$.*

Then for $r \in (0,\frac{R}{4})$, there is $v \in SBD^p_{e_2\odot e_1}(Q_R)$ such that

$$v = u \text{ on } \partial Q_R, \qquad v_1\chi_{Q_R} = u_1^+(0)\chi_{Q_R\cap\{y>0\}} + u_1^-(0)\chi_{Q_R\cap\{y<0\}}$$
$$\text{and} \qquad v_2\chi_{Q_R} = u_2^+(0)\chi_{Q_R\cap\{x>0\}} + u_2^-(0)\chi_{Q_R\cap\{x<0\}}.$$

It moreover holds

$$\|v-u\|_{L^1(Q_R)} \leq 4R^2\|u\|_{L^\infty}, \qquad I(v) \leq I(u) + CR^2\|Du\|^p_{L^\infty}$$
$$\text{and} \qquad |Dv|(Q_R) \leq C(R^2\|Du\|_\infty + R\|u\|_\infty).$$

Proof. We only perform the construction for u_1 in the upper half of Q_R, the rest follow similarly and by the triangle inequality. We will use a blow up in radial direction. This is surprisingly simple since the jump is only concentrated in the two fixed segments and we allow a large error in the gradient. We set

$$v_1(z) = \begin{cases} u_1^+(0) & |z| \le 2r \\ u_1\left(\frac{|z|-2r}{R-2r} z\right) & 2r \le |z| \le R \\ u_1(z) & |z| \ge R. \end{cases}$$

and apply the rotated construction for v_2.
It immediately follows that $v \in SBD^p_{e_2 \odot e_1}(Q_R)$, $\|v-u\|_{L^1(Q_R)} \le \|u\|_{L^\infty} 4r^2$, $\mathcal{H}^1(Ju) = \mathcal{H}^1(Jv)$. Since

$$|\nabla v_1(z)| \le \begin{cases} 0 & |z| \le 2r \\ 2\|\nabla u_1\|_\infty \frac{R}{R-2r} & 2r \le |z| \le R \\ |\nabla u_1(z)| & |z| \ge R. \end{cases}$$

we can easily estimate

$$\|\nabla v\|_{L^1(Q_R)} \le CR^2 \|Du\|_{L^\infty} \left(1 + \frac{R}{R-2r}\right) \text{ and } \|Dv\|^p_{L^p(Q_R)} \le CR^2 \|Du\|_{L^\infty} \left(1 + \frac{R}{R-2r}\right)^p.$$

Using the fact that $4r \le R$ yields $\|e(v)\|^p_{L^p(Q_R)} \le CR^2 \|Du\|^p_{L^\infty}$ and all other estimates do also hold.
□

6.5 The density result: An open problem and its difficulties

We have yet not been able to prove an appropriate density result for $SBD^p_{e_2 \odot e_1}$. Recalling the construction in Chapter 4 we expect the difficulty not to lie in the regularity of the function, but in the geometry of the jump set. The desired result would be the following:

Conjecture 6.8. *Let $p \in (1,\infty)$, $u \in SBD^p_{e_2 \odot e_1, a}$ and let $\delta > 0$. There is $v \in SBD^p_{e_2 \odot e_1, a}$ such that Jv_1 is a finite union of horizontal segments, Jv_2 is a finite union of vertical segments, $v_i \in W^{2,\infty}(\Omega \setminus Ju_i)$,*

$$\|v-u\|_{L^1(\Omega)} \le C\delta, \quad |(Dv)_{sym}|(\Omega) \le C(1 + E(u) + E^{1/p}(u)) \quad \textit{and} \quad E(v) \le (1 + C\delta) E(u).$$

At first, it seems straightforward to adapt the proof done in Chapter 4 carefully. We will now discuss up to what point this is possible and where the difficulties lie.
We would like to cover the domain with a collection of small squares and perform different local constructions in dependence of the amount of jump and its regularity in the different squares. All constructions we have in mind need an interpolation layer in a neighborhood of the squares in which the function is also changed. We therefore strongly expect the covering of the domain to be a Whitney-type covering, such that one can ensure that the overlap of these neighborhoods is finite. A Whitney-type covering has the disadvantage that one can not choose the radii of the squares in dependence of the local properties of the jump, they are somehow fixed a priori.
In Chapter 4 we divided the family of squares in different types, the ones with a small amount of jump, the ones with high regularity and the rest. This would also be possible in the $SBD^p_{e_2 \odot e_1}$ setting. For the type-I squares with small amount of jump, again convoluting with a standard-mollifier would give a similar scaling as in the SBV-setting. For this purpose one could use a variant of Proposition 3 in a recent work of Chambolle, Conti and Francfort [12]. We show in the

following chapter how one can adapt their techniques to our setting and achieve such a result. For the type-III squares, in which the jump is close to a C^1 curve and there is no crossing of jumps, a linear interpolation in one of the two directions as performed in Chapter 4 would lead to an appropriate estimate for the energy.

The main problem are the type-II squares in which the length of the jump is larger then ηr_i, where r_i is the sidelength of the ith square, for some universal constant $\eta > 0$. We only need to cover jump of total length ε with these squares, hence we know that $\sum r_i \leq \varepsilon\eta^{-1}$. In analogy to the work for SBV-functions we want to replace the function by a piecewise constant function.

For this function the number of jumps between the constants should be bounded by the number of jumps present in the original function. A Poincaré-type inequality would ensure the closeness of this two functions in L^p. We have been able to prove a Poincaré-type inequality in exactly this setting, the next chapter is dedicated to the proof.

The result is that given a function u there is a piecewise constant function v such that $\|u - v\|^p_{L^p(Q_{r_i})} \leq Cr^p\|e(u)\|^p_{L^p(Q_r)}\left(1 + \frac{(\mathcal{H}^1(Ju))^p}{r^p}\right)$. We also provide an example that suggests that this estimate is optimal. If we use a standard-interpolator on scale δ_i we would get, that for a disjoint family of type-II squares the following estimate holds:

$$\begin{aligned}\|e(v)\|^p_{L^p(\bigcup_i Q_{r_i})} &\leq \frac{1}{\delta_i^p}\|u - v\|^p_{L^p(\bigcup_i Q_{r_i})} + \text{h.o.t.} \\ &\leq \sum_i \frac{1}{\delta_i^p} r_i^p \|e(u)\|^p_{L^p(Q_{r_i})} \frac{(\mathcal{H}^1(Ju \cap Q_{r_i}))^p}{r_i^p} + \text{h.o.t.}\,.\end{aligned} \tag{22}$$

We see, that for this term it is optimal to have $\delta_i \sim r_i$ and that since $\mathcal{L}^2(\bigcup_i Q_{r_i}) \leq \sum r_i \leq \varepsilon\eta^{-1}$, we can let $\sum_i \|e(u)\|^p_{L^p(Q_{r_i})}$ be small for small enough ε.

We do not expect an estimate for $\sum_i \frac{\|e(u)\|^p_{L^p(Q_{r_i})}}{r_i^\alpha}$ to hold since for any α even one single term in this sum might get arbitrary large, if the square is chosen badly.

However, it seems also difficult to control the term $\frac{(\mathcal{H}^1(Ju\cap Q_{r_i}))^p}{r_i^p}$ in this setting, since this would need an adaption of the Whitney-type covering to the local density of the jump set. Only if one had that $\mathcal{H}^1(Ju \cap Q_{r_i}) \leq Mr_i$ one could choose the parameters in a way that (22) gets small.

A first idea would therefore be to introduce a different construction for the squares in which $\mathcal{H}^1(Ju) \geq Mr_i$ for some $M \gg 1$. It is yet not obvious how the additional information that large amounts of jump are present would help to avoid the need of applying a Poincaré-type estimate.

7 Korn-Poincaré-type inequalities for $SBD^p_{e_2\odot e_1}$

There are functions of bounded deformation that are not of bounded variation. An example of Ornstein presents a function with integrable symmetric gradient whose gradient is not integrable, see [42]. In particular this shows that Korn's inequality does not hold for $p = 1$.
However, for all $u \in BD(\Omega)$ there is a global, linear, skew-symmetric map R and a constant c such that a Poincaré-type inequality of the form $\|u - R - c\|_{L^1(\Omega)} \leq C|(Du)_{sym}|(\Omega)$ holds, see [1]. There is ongoing research on finding a stronger estimate for functions that are not only in BD but in SBD^p. In particular: Can one neglect the jump part on the right-hand side of the estimate and use only the absolute continuous part? Can one additionally gain a L^p-estimate for the function using a L^p-norm on the absolute continuous part of the derivative? There are positive results, for example from Conti, Chambolle and Francfort or Friedrich, see [12] and [30], but they only answer the question partially. We will adapt the results of the former to get a stronger result for our smaller space $SBD^p_{e_2\odot e_1}$.
Our Proposition 7.1 is an improvement of Proposition 2 in the work of Chambolle, Conti and Francfort [12]. For SBD^p functions with small jump set they prove an estimate for the L^p distance of the function to an affine, skew-symmetric function, but only up to an exceptional set of small measure. We are able to use the special geometry of the jump set of $SBD^p_{e_2\odot e_1}$-functions to get an estimate for the complete domain.
In Theorem 7 we can also get rid of the assumption of a small jump set. The main idea is to chose an affine line of such a small slope, that it does not intersect the jump set of the first component. Applying slicing techniques and the one-dimensional Poincaré estimate on this line gives us a piecewise Poincaré-type estimate for the second component alone. This values can be transported on the complete domain using the $\partial_y u_2$ term. The estimate for u_1 follows with a symmetric argument.
We indicate in Remark 7.4 that our result is optimal, in the sense that the $\mathcal{H}^1(Ju)$ term needs to enter with the given scaling.

Proposition 7.1 (Korn-Poincaré-type inequality for functions with small jump set)**.** *Let $p \in (1,\infty)$. There are constants $C > 0$, $c_o > 0$ such that for every $u \in SBD^p_{e_2\odot e_1}(Q_r)$ with $\mathcal{H}^1(Ju) \leq c_0 r$ it holds: There is an skew-symmetric matrix $\tilde{R}$ and a constant $c \in \mathbb{R}^2$ such that for $R(x,y) = \tilde{R}\cdot(x,y)^T$ it holds:*

$$\|u - R - c\|^p_{L^p(Q_r)} \leq Cr^p\|e(u)\|^p_{L^p(Q_r)}.$$

Here and in the following we let $\tilde{R}$ be a matrix in $\mathbb{R}^{n\times n}$ and R the linear function that has $\nabla R = \tilde{R}$.
We will first prove an easy lemma, that shows how the constraint on the geometry of the jump set in $SBD^p_{e_2\odot e_1}$ can be used if the exceptional set has some special structure. Afterwards we prove a variant of Proposition 7.1, in which only an exceptional set with that special structure is removed. The proof of Proposition 7.1 follows as an immediate corollary of these two results.

Lemma 7.2. *Let $p \in (1,\infty)$, $u \in SBD^p_{e_2\odot e_1}(Q_r)$, $\tilde{R} \in \mathbb{R}^{2\times 2}$ skew-symmetric, $c \in \mathbb{R}^2$, $C > 0$ and an exceptional set $\omega \subseteq Q_r$ such that $\mathcal{L}^2(\omega) \leq Cr\mathcal{H}^1(Ju)$, $\|u - R - c\|^p_{L^p(Q_r\setminus\omega)} \leq Cr^p\|e(u)\|^p_{L^p(Q_r)}$ and for almost every $x \in (-r,r)$, $y \in (-r,r)$ it holds:*

$$\mathcal{H}^1(\omega\cap\{x\}\times(-r,r)) \leq \frac{3}{2}r \quad \textit{and} \quad \mathcal{H}^1(\omega\cap(-r,r)\times\{y\}) \leq \frac{3}{2}r. \tag{23}$$

Then

$$\|u - R - c\|^p_{L^p(Q_r)} \leq \tilde{C} r^p \|e(u)\|^p_{L^p(Q_r)},$$

where $\tilde{C}$ only depends on the given constant C and p.

Proof. Let without loss of generality $r = 1$ and define $v = u - R - c$. This simplifies notation since $e(u) = e(v)$.
Fix $\bar{y} \in (-1, 1)$. We can choose $\tilde{x} \in (-1, 1)$ such that

$$(\tilde{x}, \bar{y}) \notin \omega \qquad \text{and} \qquad |v(\tilde{x}, \bar{y})|^p \leq \tilde{C} \|v\|^p_{L^p((-1,1)\times\{\bar{y}\}\setminus\omega)}. \tag{24}$$

We have used (23) to get the inequality in (24) with the removed set ω in the L^p-norm. The triangle inequality yields that for almost every $x \in (-1, 1)$ it holds:

$$|v_1(x, \bar{y})|^p \leq \tilde{C} |v_1(\tilde{x}, \bar{y})|^p + \tilde{C} \|\partial_x v_1\|^p_{L^p((-1,1)\times\{\bar{y}\})}.$$

Since $\bar{y}$ was arbitrary we can integrate this inequality and conclude

$$\begin{aligned} \int_{-1}^{1} \int_{-1}^{1} |v_1(x, \bar{y})|^p \, \mathrm{d}\mathcal{L}^1(x) \, \mathrm{d}\mathcal{L}^1(\bar{y}) &\leq \tilde{C} \int_{-1}^{1} |v_1(\tilde{x}, \bar{y})|^p \, \mathrm{d}\mathcal{L}^1(\bar{y}) + \tilde{C} \|\partial_x v_1\|^p_{L^p(Q_1)} \\ &\leq \tilde{C} \|v_1\|^p_{L^p(\omega^C)} + \tilde{C} \|\partial_x v_1\|^p_{L^p(Q_1)} \\ &\leq \tilde{C} \|e(v)\|^p_{L^p(Q_1)}. \end{aligned}$$

The estimate for v_2 holds due to the symmetry of the problem. □

We will apply this lemma after having proven the following statement:

Proposition 7.3 (Variant of Conti-Chambolle-Francfort). *Let $p \in (1, \infty)$. There is a constant $c_o > 0$ such that for every $u \in SBD^p(Q_r)$ with $\mathcal{H}^1(Ju) \leq c_0 r$ it holds :*
There is a constant $c \in \mathbb{R}^2$, a skew-symmetric matrix $\tilde{R}$ and an exceptional set $\omega \subseteq Q_r$ such that $\mathcal{L}^2(\omega) \leq Cr\mathcal{H}^1(Ju)$, $\|u - R - c\|^p_{L^p(Q_r\setminus\omega)} \leq Cr^p \|e(u)\|^p_{L^p(Q_r)}$ and such that almost every slice in the upper right quadrant intersects the exceptional set only on a fraction of its length, that is for $x \in (0, r)$, $y \in (0, r)$ it holds:

$$\mathcal{H}^1(\omega \cap \{x\} \times (0, r)) \leq \frac{3}{4} r \quad \textit{and} \quad \mathcal{H}^1(\omega \cap (0, r) \times \{y\}) \leq \frac{3}{4} r.$$

Remark. Notice, that the foregoing result is not only proven for functions in $SBD^p_{e_2\odot e_1}(Q_r)$ but even in the larger space $SBD^p(Q_r)$. However, it is not possible to use this structure to gain a better Korn-Poincaré-type estimate in the general SBD^p-setting, but only for $SBD^p_{e_2\odot e_1}$.
Moreover it is possible to get the additional structure not only in the upper right quadrant but for every slice, see the proof of Proposition 7.1 below. We omit this in this formulation of the proposition to keep the structure and notation simple.

The following proof closely follows the proof given in [12], but is adapted in some parts to gain the additional structure of the exceptional set.

Proof. Let without loss of generality be $r = 1$.

Define the characteristic function $T : \mathbb{R}^2 \times S^1 \times \mathbb{R}$ for the slices on which fundamental theorem does not hold by

$$T(z,\xi,t) = \begin{cases} 1 & z \in Q_1, z + t\xi \in Q_1 \text{ and } \xi \cdot (u(z+t\xi) - u(z)) \neq t \int_0^1 \xi \cdot e(u)(z+ts\xi) \cdot \xi \, \mathrm{d}\mathcal{L}^1(s) \\ 0 & \text{otherwise.} \end{cases}$$

We notice that for almost every (z,ξ,t) it holds: $\xi \cdot (u(z+t\xi) - u(z)) \neq t \int_0^1 \xi \cdot e(u)(z+ts\xi) \cdot \xi \, \mathrm{d}\mathcal{L}^1(s)$ if and only if there is $s \in (0,t)$ such that $z + s\xi \in Ju$. In [12] three points z_0, z_1, z_2 are chosen such that along the edges of the so-created triangle, fundamental theorem is applicable and such that the 'shadow' of the jump set from these points - which will be the exceptional set - is small enough. Moreover, the points are chosen in a way such that there is an affine function a such that $a(z_i) = u(z_i)$,

$$\int_{S^1} \int_{\mathbb{R}} T(z_i,\xi,t) \, \mathrm{d}\mathcal{L}^1(t) \, \mathrm{d}\mathcal{H}^1(\xi) \leq C\mathcal{H}^1(Ju) \leq Cc_0 \tag{25}$$
$$\text{and} \qquad |Da + Da^T| \leq C\|e(u)\|_{L^1(Q_1)}.$$

Having this at hand, an estimate for $\|u-a\|_{L^p(Q_1\setminus\omega)}$ is established, see the publication for the details. The exceptional set ω is defined by the union of the 'shadows' of the jump set, starting from the three points z_i, that is

$$\omega_i = \{z \in Q_1 \,|\, z = z_i + t\xi, T(z,\xi,t) = 1\} \quad \text{and} \quad \omega = \omega_1 \cup \omega_2 \cup \omega_3,$$

see also the green area indicated in Figure 21 for an idea of this set.

We only need to modify these arguments slightly.

We choose the points z_i such that additionally $z_i \in (-\frac{3}{4}, -\frac{1}{2})^2$. This only increases the constants in (25) but the estimates still hold. Let $d_i = \mathrm{dist}(z_i, \partial Q)$ and deduce that $\frac{1}{4} \leq d_i \leq \frac{1}{2}$. We have for almost every ξ and almost every $t \in (\frac{d_i}{2}, d)$: If $T(z_i,\xi,t) = 1$ then $T(z_i,\xi,\frac{d_i}{2}) = 1$. It follows, using (25), that

$$\int_{S^1} T\left(z_i,\xi,\frac{d_i}{2}\right) \, \mathrm{d}\mathcal{H}^1(\xi) \leq Cc_0.$$

Our claim is, assuming c_0 small enough, that for almost every $y \geq 0$ it holds

$$\mathcal{H}^1(\omega_i \cap (-1,1) \times \{y\}) \leq \frac{1}{4}.$$

We therefore fix $y \in (0,1)$. There are essentially two possibilities how $\omega_i \cap (-1,1) \times \{y\}$ might get large. Either large parts of jump lie between $B_{d_i/2}(z_i)$ and $(-1,1) \times \{y\}$ or a small jump set in the ball $B_{d_i/2}(z_i)$ is projected on a large subset of $\omega_i \cap (-1,1) \times \{y\}$, see Figure 21 for a sketch of both variants. We have, however, chosen the points z_i in a way such that in both cases the projection of the jump set is not to large.

We divide $\omega_i \cap (-1,1) \times \{y\}$ into two subsets, up to a null set. Let

$$\omega_i^1 = \{(x,y) \mid \text{ there is } z \in Ju \cap B_{d_i/2}(z_i), \xi \in S^1 \text{ such that } (x,y) \in z_i + \mathbb{R}\xi, z \in z_i + \mathbb{R}\xi\} \quad \text{and}$$
$$\omega_i^2 = \{(x,y) \mid \text{ there is } z \in Ju \setminus B_{d_i/2}(z_i), z_2 \le y, \xi \in S^1 \text{ such that } (x,y) \in z_i + \mathbb{R}\xi, z \in z_i + \mathbb{R}\xi\}.$$

Indeed, for almost every $z \in B_{d_i/2}(z_i) \cap Ju$ we have that for $\tilde{\xi} = \frac{z - z_i}{|z - z_i|}$ and $\tilde{z} = z_i + \frac{d_i}{2}\tilde{\xi}$ it holds: $\tilde{z} \in \partial B_{d_i/2}(z_i) \cap \omega_i$. Hence for almost every $x \in (-1,1)$ with $(x,y) \in \omega_i^1$ we have a corresponding $\tilde{z} \in \partial B_{d_i/2}(z_i) \cap \omega_i$. The projection of the circle $\partial B_{d_i/2}(z_i)$ onto the line $(-1,1) \times \{y\}$ along the directions ξ is a Lipschitz-map and hence we have

$$\mathcal{H}^1(\omega_i^1) \le L \int_{S^1} T\left(z_i, \xi, \frac{d_i}{2}\right) \, \mathrm{d}\mathcal{H}^1(\xi) \le LCc_0.$$

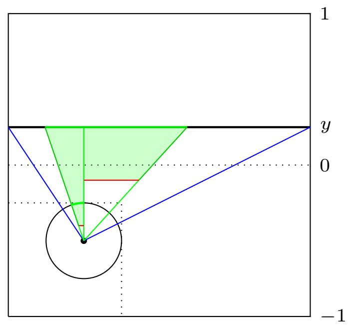

Figure 21: Since the sphere $\partial B_{d_i/2}(z_i)$ is contained in $(-1, -\frac{1}{4})^2$ we have that the maximal angles of the projection on the line $(-1,1) \times \{y\}$ are bounded (blue lines). Parts of the jump set with a distance to z_i that is larger then $d_i/2$ create only shadow on $(-1,1) \times \{y\}$, that is comparable to the original size of the jump set.

Since the circle $\partial B_{d_i/2}(z_i)$ is contained in the square $(-1, -\frac{1}{4})^2$, the radius d_i is bounded from below by a constant and since we are considering only strictly positive values y we have that the Lipschitz constant L is bounded from above, independent of the choice of y and z_i, see Figure 21.
On the other hand it is a consequence of the intercept theorem that the part of the jump set, that has distance larger then d_i to z_i, is not projected on a large subset of $(-1,1) \times \{y\}$. In fact:

$$\mathcal{H}^1(\omega_i^2) \le \frac{1}{d_i} \mathcal{H}^1(Ju) \le Cc_0.$$

We hence see that for c_0 small enough we have $\mathcal{H}^1(\omega \cap (-1,1) \times \{y\}) \le \frac{3}{4}$ and since all arguments were symmetric we also have $\mathcal{H}^1(\omega \cap \{x\} \times (-1,1)) \le \frac{3}{4}$. □

We are now able to put things together and finish the proof of Proposition 7.1.

Proof. (of Proposition 7.1) Obviously it is possible to get the foregoing result with the structure for the jump set in each of the four quadrants of $Q_r = (-r,r)^2$. If we apply Lemma 7.2 on each of these quadrants we get four linear functions R_i and constants c_i such that

$$\|u - R_i - c_i\|_{L^p(Q^i)} \le Cr^p \|e(u)\|^p_{L^p(Q_r)},$$

where $Q^1 = (0,r)^2$, $Q^2 = (-r,0) \times (0,r)$, $Q^3 = (-r,r)^2$ and $Q^4 = (0,r) \times (-r,0)$.
It moreover holds that there are exceptional sets ω_i for each affine function $R_i + c_i$ that satisfy $\mathcal{L}^2(\omega_i) \le Cr\mathcal{H}^1(Ju)$ and such that for the other quadrants we have

$$\|u - R_i - c_i\|^p_{L^p(Q^j \setminus \omega_i)} \le Cr^p \|e(u)\|^p_{L^p(Q_r)}.$$

Applying the triangle inequality yields

$$\begin{aligned} |R_i + c_i - R_k - c_k|^p_p &\le \frac{C}{r^2 - \mathcal{L}^2(\omega_i)} \left(\|u - R_i - c_i\|^p_{L^p(Q^j\setminus\omega_i)} + \|u - R_k - c_k\|^p_{L^p(Q^j\setminus\omega_i)} \right) \\ &\le Cr^{p-2}\|e(u)\|^p_{L^p(Q_r)}. \end{aligned}$$

On each square Q^i the Poincaré inequality with $R_i + c_i$ holds without exceptional set. We can exchange the different $R_i + c_i$'s on the different Q^i's with small error and hence the proposition follows. □

In our main result we also want to neglect the assumption of the smallness of the jump set.
After considering the function $u(x,y) = (M\chi_{\{y>0\}}, 0)$ on Q_1 it is clear that a Poincaré-type inequality with one global constant can not hold. A reasonable inequality will include different local constants for the L^p-distance on different parts of the domain. We will not only be able to control the number of these constants by the size of the jump set, but also the geometry of the sets on which each constant is present. Moreover, only one rotation will be chosen for the whole square.
The motivation for this result comes from the analog statement for SBV functions as used in Chapter 6. A similar result for SBD^p or any subspace of SBD^p that is larger then $W^{1,p}$ is not known to the author.

Theorem (Korn-Poincaré-type inequality)**.** *Let $p \in (1,\infty)$ and $u \in SBD^p_{e_2\odot e_1}(Q_r)$. There is a constant C, a partition $x_1 < \cdots < x_I$, a partition $y_1 < \cdots < y_J$, values $a_i < a_{i+1}$, $b_j < b_{j+1}$ and a skew-symmetric matrix $\tilde{R}$ such that $I \le \frac{C}{r}\mathcal{H}^1(Ju_2)$, $J \le \frac{C}{r}\mathcal{H}^1(Ju_1)$ and for $R(x,y) = \tilde{R}\cdot(x,y)^T$ it holds*

$$\left\| u - R - \sum_{i,j=0}^{I,J} \begin{pmatrix} a_i \\ b_j \end{pmatrix} \chi_{(x_i,x_{i+1})\times(y_j,y_{j+1})} \right\|^p_{L^p(Q_r)} \le Cr^p \|e(u)\|^p_{L^p(Q_r)} \left(1 + \frac{(\mathcal{H}^1(Ju))^p}{r^p}\right),$$

where $x_0 = y_0 = -r$ and $x_{I+1} = y_{J+1} = r$.

Notice that the function $v := R + \sum_{i,j=0}^{I,J} \begin{pmatrix} a_i \\ b_j \end{pmatrix} \chi_{(x_i,x_{i+1})\times(y_j,y_{j+1})}$ is an element of $SBD^p_{e_2\odot e_1}(Q_r)$ with $\tilde{E}^p(v) = r(I+J)$.

Remark 7.4. We want to point out that this inequality is optimal, in the sense that $\mathcal{H}^1(Ju)$ needs to enter as a multiplicative factor with an exponent p. Consider the following construction for $r = 1$ and notice that the usual slicing results finish the argument:
Let $N \in \mathbb{N}$ arbitrary and let $R^i_N = (0,1) \times (\frac{i}{N}, \frac{i+1}{N})$ a partition of the domain in N equisized rectangles. Let $v : R^0_N \to \mathbb{R}^2$ be a function that realizes a large constant in Korn's inequality on thin rectangles, for example $v(x,y) = (-2xy, x^2)$.

We can easily compute $\|v\|^p_{L^p(R^0_N)} \sim N^p \|e(v)\|^p_{L^p(R^0_N)}$ and the scaling does not change by subtracting any affine, skew-symmetric function. For $y \in (\frac{i}{N}, \frac{i+1}{N})$ we define $u(x,y) = v(x, y - \frac{i}{N}) + c_i$, where $c_i = i\|v\|_\infty$. It follows:

$$\|v - c_i \chi_{R^i_N}\|^p_{L^p(Q_1)} \sim N\left(N^p \|e(v)\|^p_{L^p(R^0_N)}\right) \sim N^p \|e(u)\|^p_{L^p(Q_1)} \sim (\mathcal{H}^1(Ju))^p \|e(u)\|^p_{L^p(Q_1)}.$$

Proof. (of Theorem 7) Due to the usual scaling techniques we can assume $r = 1$ and denote $Q = Q_1 = (-1,1)^2$. We can moreover without loss of generality assume that $\mathcal{H}^1(Ju) \geq c_0$ since we can otherwise apply Proposition 7.1 and set $I = J = 0$.

Let $\lambda \in (0, \frac{1}{4})$ to be chosen later. By a standard combinatorial argument there is a constant $C \gg 1$ such that we can choose vertical and horizontal stripes $S^v = (\bar{x}, \bar{x}+\lambda) \times (0,1)$, $S^h = (0,1) \times (\bar{y}, \bar{y}+\lambda)$ of width λ in such a way, such that for $q = S^v \cap S^h$ we have

$$\begin{aligned} \|e(u)\|^p_{L^p(S^{h,v})} &\leq C\lambda \|e(u)\|^p_{L^p(Q)}, & \|e(u)\|^p_{L^p(q)} &\leq C\lambda^2 \|e(u)\|^p_{L^p(Q)}, \\ \mathcal{H}^1(Ju_i \cap S^{h,v}) &\leq C\lambda \mathcal{H}^1(Ju) \quad \text{and} & \mathcal{H}^1(Ju \cap q) &\leq C\lambda^2 \mathcal{H}^1(Ju). \end{aligned} \tag{26}$$

Hence, if we choose $\lambda = \frac{c_0}{C\mathcal{H}^1(Ju)}$ and use Proposition 7.1 on the small square q, we may conclude that there is a skew-symmetric matrix $\tilde{R}$ and a constant $c \in \mathbb{R}^2$ such that

$$\|u - R - c\|^p_{L^p(q)} \leq C\lambda^p \|e(u)\|^p_{L^p(q)}. \tag{27}$$

We will subtract this rotation, coming from the Korn-Poincaré-type inequality on the small square q, from u on the whole square Q. By redefining u we choose R and c to be equal to zero to simplify notation in the sequel.

The L^p-estimate for u_1 on the small square q can be transported on the horizontal stripe S^h via the symmetrized gradient: Choose $\tilde{x} \in (\bar{x}, \bar{x}+\lambda)$ such that $\|u_1(\tilde{x}, \cdot)\|^p_{L^p((\bar{y}, \bar{y}+\lambda))} \leq \lambda^{-1} \|u_1\|^p_{L^p(q)}$. Then by fundamental theorem, (27) and (26) we find

$$\|u_1\|^p_{L^p(S^h)} \leq C\|u_1(\tilde{x}, \cdot)\|^p_{L^p(\bar{y}, \bar{y}+\lambda)} + C\|\partial_x u_1\|^p_{L^p(S^h)} \leq C\lambda \|e(u)\|^p_{L^p(Q)}. \tag{28}$$

The central task is to find an estimate for $\|u_2\|_{L^p}$ on this horizontal stripe. The idea is to choose an affine line in S^h with small slope α that does not intersect Ju_1. We will then apply the one-dimensional Poincaré inequality on this line.

Fix $\xi = (\sqrt{1-\alpha^2}, \alpha)$, $\Pi_\xi = \mathbb{R}\xi^\perp$ and define for $y \in \Pi_\xi$ the affine line in direction ξ through y as $T^y_\xi = y + \mathbb{R}\xi \cap Q$.

Let $T^h = \{T^y_\xi \,|\, y \in \Pi_\xi, T^y_\xi \subseteq S^h\}$ and define $\mathcal{T}^h = \{z \in T^y_\xi \,|\, T^y_\xi \in T^h\}$, see Figure 22 for a sketch. If $\alpha \leq \lambda/2$ then at least half of the volume of S^h is covered by $\mathcal{T}^h$. We show below, using the usual slicing techniques as introduced by Ambrosio, Coscia and Dal Maso (see [1]), that we can choose

$T \in T^h$ such that

$$\begin{aligned} \|u'_\xi\|^p_{L^p(T)} &\leq C\lambda^{-1}\|e(u)\|^p_{L^p(S_h)} & \|u_1\|^p_{L^p(T)} &\leq C\lambda^{-1}\|u_1\|^p_{L^p(S_h)} \\ \mathcal{H}^0(Ju_2 \cap T) &\leq C\lambda^{-1}\mathcal{H}^1(Ju_2 \cap S^h) \qquad \text{and} & \mathcal{H}^0(Ju_1 \cap T) &= 0. \end{aligned} \tag{29}$$

The function $u_\xi \in BV(T)$ is defined by $u_\xi(t) = u(\tilde{y} + t\xi) \cdot \xi$ where $\tilde{y} \in T \cap \Pi_\xi$. We will denote the absolute continuous part of the derivative of u_ξ by u'_ξ. We will understand u_ξ as both a function of an interval and of the one-dimensional slice T and exchange these meanings. Details about the application of the slicing techniques in our setting have already been given in Chapter 5.

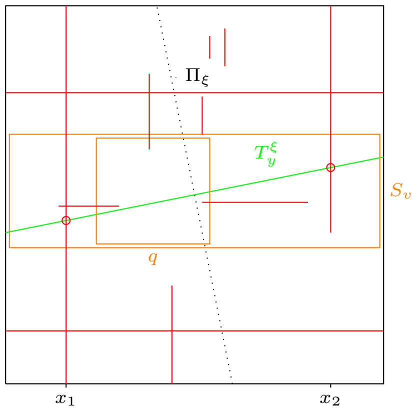

Figure 22: The square q is chosen such that there is not much jump inside. The line T^y_ξ is chosen such that the Poincaré inequality holds on it, up to the $I = 2$ many jumps of u_2. The horizontal jumps of u_1 are completely avoided.

We want to use the generalized coarea formula to achieve the last equation in (29) (see for example Theorem 3.2.22 in [29]). The set $Ju_1 \cap \mathcal{T}^h$ is rectifiable and that the normal and the tangent satisfy $\nu = e_2$ and $\tau = e_1$ $\mathcal{H}^1$-almost everywhere. It follows:

$$\begin{aligned} \int_{\Pi_\xi} \mathcal{H}^0(Ju_1 \cap \mathcal{T}^h \cap T^y_\xi)\, \mathrm{d}\mathcal{H}^1(y) = \int_{Ju_1 \cap \mathcal{T}^h} |(\mathrm{Id} - \xi \otimes \xi) \cdot \tau|\, \mathrm{d}\mathcal{H}^1 &= \int_{Ju_1 \cap \mathcal{T}^h} |\xi \cdot \nu|\, \mathrm{d}\mathcal{H}^1 \\ &= \alpha \mathcal{H}^1(Ju_1 \cap \mathcal{T}^h). \end{aligned}$$

We have $\mathcal{H}^1(Ju \cap \mathcal{T}^h) \leq \mathcal{H}^1(Ju \cap S^h) \leq \alpha c_0$. So if we choose $\alpha = \frac{\lambda}{4c_0}$ we know that at least half of the lines T^y_ξ can not intersect Ju_1. We can therefore indeed choose a line such that (29) holds.
We define $\{w_i \,|\, i \in \{1, \dots, I\}\} = Ju_2 \cap T$ and notice that $I \leq C\mathcal{H}^1(J(u_2))$. We denote the first components of the vectors $w_i \in \mathbb{R}^2$ by $x_i = w_{i1}$. This points will form the partition of the x-axis we are aiming at.
For $a, b \in \mathbb{R}^2$ we denote by $\overline{ab}$ the segment with endpoints a and b. We see that $u_\xi \in W^{1,p}(\overline{w_i w_{i+1}})$ and that $u^+_\xi(w_i) < u^-_\xi(w_i)$. We want to find $I + 1$ constants $\tilde{b}_i$ such that $\tilde{b}_i < \tilde{b}_{i+1}$ and

$$\|u_\xi - \tilde{b}_i\|^p_{L^p(\overline{w_i w_{i+1}})} \leq (x_{i+1} - x_i)^p \|u'_\xi\|^p_{L^p(\overline{w_i w_{i+1}})} \leq \|u'_\xi\|^p_{L^p(\overline{w_i w_{i+1}})}. \tag{30}$$

By the Poincaré inequality, (30) holds for $\tilde{b}_i = \langle u_\xi \rangle_{\overline{w_i w_{i+1}}}$. However, in general we might have $\langle u_\xi \rangle_{\overline{w_i w_{i+1}}} \geq \langle u_\xi \rangle_{\overline{w_{i+1} w_{i+2}}}$ and hence some modifications must be done: Define w as the function u_ξ

with removed jumps i.e., $w(t) = \int_0^t u'_\xi(s)\, \mathrm{d}\mathcal{L}^1(s)$. Then $w \in W^{1,p}(T)$ and $u_\xi - w = \sum_{i=1}^I [u_\xi](w_i)\delta_{w_i}$. Again by the Poincaré inequality we have a constant $\tilde{c}$ such that

$$\|w - \tilde{c}\|^p_{L^p(T)} \le C\|w'\|^p_{L^p(T)} = C\|u'_\xi\|^p_{L^p(T)}.$$

We define $\tilde{b}_i := \tilde{c} + \sum_{j=1}^i [u_\xi](w_j)$. It then holds that

$$\|u_\xi - \tilde{b}_i\|^p_{L^p(\overline{w_i, w_{i+1}})} = \|w - \tilde{c}\|^p_{L^p(\overline{w_i, w_{i+1}})} \tag{31}$$

and $\tilde{b}_i < \tilde{b}_{i+1}$.

At this point of the proof we are interested in an estimate for the second component of u_2 alone. We therefore define $b_i = \tilde{b}_i/\alpha$ and derive

$$\begin{aligned}\|u_2 - b_i\|^p_{L^p(\overline{w_i w_{i+1}})} &\le C\frac{1}{\alpha^p}\|\xi_1 u_1 + \alpha u_2 - \tilde{b}_i\|^p_{L^p(\overline{w_i w_{i+1}})} + C\left(\frac{\xi_1}{\alpha}\right)^p \|u_1\|^p_{L^p(\overline{w_i w_{i+1}})} \\ &\le C\frac{1}{\alpha^p}\left(\|u_\xi - \tilde{b}_i\|^p_{L^p(\overline{w_i w_{i+1}})} + \|u_1\|^p_{L^p(\overline{w_i w_{i+1}})}\right). \end{aligned}\tag{32}$$

Using fundamental theorem in y-direction, (32) and (31) we get that

$$\begin{aligned}\|u_2 - \sum_{i=0}^I b_i \chi_{(x_i, x_{i+1})\times(0,1)}\|^p_{L^p((0,1)^2)} &\le C\sum_{i=0}^I \left(\|u_2 - b_i\|^p_{L^p(\overline{w_i w_{i+1}})} + \|\partial_y u_2\|^p_{L^p((x_i,x_{i+1})\times(0,1))}\right) \\ &\le C\frac{1}{\alpha^p}\left(\|w - \tilde{c}\|^p_{L^p(T)} + \|u_1\|^p_{L^p(T)}\right) + C\|e(u)\|_{L^p(Q)}.\end{aligned}$$

Putting things together and using (31), (29), (26) and (28) we get

$$\begin{aligned}&\|u_2 - \sum_{i=0}^I b_i \chi_{(x_i, x_{i+1})\times(0,1)}\|^p_{L^p(Q)} \\ &\le \frac{C}{\alpha^p}\left(\|u'_\xi\|^p_{L^p(T)} + \|u_1\|^p_{L^p(T)}\right) + C\|e(u)\|_{L^p(Q)} \\ &\le \frac{C}{\alpha^p}\left(\lambda^{-1}\|e(u)\|^p_{L^p(S_h)} + \lambda^{-1}\|u_1\|^p_{L^p(S_h)}\right) + C\|e(u)\|_{L^p(Q)} \\ &\le C\|e(u)\|_{L^p(Q)}\left(1 + \frac{1}{\alpha^p}\right),\end{aligned}$$

which is the stated inequality for the second component.

The problem is symmetric and hence we can easily conclude a similar result for u_1. □

This finishes the discussion of the Korn-Poincaré-type inequalities for $SBD^p_{e_2\odot e_1}$. We have already discussed in 7.4 that a stronger result is not expected.

Glossary

$\tilde{E}^p(u)$	Translated version of the limiting energy for a vector-valued function $u \in SBD^p_{e_1 \otimes e_2, a}$. 65
$\tilde{E}^{p,\theta}(u)$	Translated version of the energy for a vector-valued function u. 65
$E^p(u)$	Limiting energy for a vector-valued function $u \in SBD^p_{e_1 \otimes e_2, 0}$. 59
$E^{\theta,p}(u)$	Energy for a vector-valued function $u \in \mathcal{W}$. 7
$I(u)$	Limiting energy for a scalar-valued function $u \in SBV^2_{e_1,0}$. 21
$I^\theta(u)$	Energy for a scalar-valued function $u \in \mathcal{A}$. 7
$I^\theta_{e_2}(v)$	One-dimensional slicing of the energy for a one-dimensional scalar-valued function$v \in \mathcal{B}_1$. 12
$I_{e_2}(v)$	Limiting energy of the slicing for a one-dimensional, scalar-valued function $v \in SBV_{+,1}$. 12
$J(u)$	The Mumford-Shah functional for a function $u \in SBV^2_{e_1,a}$. 28
$Q_R(x)$	The two-dimensional square with sidelength $2R$ centered at x. 29
$SBV^2_{e_2,0}$	$SBV^2_{loc}((0,1)^2)$-functions whose jump is positive and lies in horizontal slices with zero boundary values on the left edge. 21
$SBV^2_{e_2,a}$	$SBV^2((0,1)^2)$-functions whose jump is positive and lies in horizontal slices with affine boundary values on the left edge. 28
$SBV^2_{e_2}(A)$	$SBV^2(A)$-functions whose jump is positive and lies in horizontal slices without boundary values. 28
$SBV_{+,\lambda}$	One-dimensional $SBV((0,\lambda))$-functions whose jumps are of positive height. 12
Ω	The unit square $(0,1)^2$ in two dimensions. 6
$\mathcal{A}$	Two-dimensional $W^{1,2}((0,1)^2)$-functions whose derivatives are of bounded variation with some boundary values. 21
$\mathcal{B}_\lambda$	One-dimensional $W^{1,2}((0,1))$-functions whose derivatives are of bounded variation. 12
$\mathcal{W}$	Two-dimensional, vector-valued $W^{1,2}((0,1)^2, \mathbb{R}^2)$-functions whose derivatives are of bounded variation with some boundary values. 59
$SBD^p_{e_2 \odot e_1, 0}$	$SBD^p_{loc}((0,1)^2)$-functions whose jump has some orientation with zero boundary values on the lower and left edge. 59
$SBD^p_{e_2 \odot e_1, a}$	$SBD^p((0,1)^2)$-functions whose jump has some orientation with affine boundary values on the lower and left edge. 65
$SBD^p_{e_2 \odot e_1}(A)$	$SBD^p(A)$-functions whose jump has some orientation without boundary values. 65

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