Selberg zeta function and relative analytic torsion for hyperbolic odd-dimensional orbifolds

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Ksenia Fedosova

aus

Chelyabinsk

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1. Gutachter: Prof. Dr. Werner Müller

2. Gutachter: Prof. Dr. Werner Ballmann

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Abstract

In this thesis we study the Selberg zeta functions and the analytic torsion of hyperbolic odd-dimensional orbifolds $\Gamma\backslash\mathbb{H}^{2n+1}$. In the first part of the thesis we restrict ourselves to compact orbifolds and establish a version of the Selberg trace formula for non-unitary representations of Γ. We study Selberg zeta functions on $\Gamma\backslash\mathbb{H}^{2n+1}$, prove that these functions admit a meromorphic continuation to $\mathbb{C}$ and describe their singularities. In the second part we define the analytic torsion of a compact orbifold $\Gamma\backslash\mathbb{H}^{2n+1}$ associated to the restriction of a certain representation of G to Γ. Further we investigate the asymptotic behavior of this torsion with respect to special sequences of representations of G. In the third part we extend the results of the second part to hyperbolic odd-dimensional orbifolds of finite volume under the assumption that the orbifold is 3-dimensional.

Our work generalizes the results of Müller to compact orbifolds, results of Bunke and Olbrich to compact orbifolds and non-unitary representations of Γ, and results of Müller and Pfaff to compact and finite-volume 3-dimensional orbifolds.

Chapter 1

Introduction

This thesis deals with two aspects of the geometry and spectral theory on hyperbolic odd-dimensional orbifolds $\Gamma\backslash\mathbb{H}^{2n+1}$. First, it is the Selberg zeta function associated to a possibly non-unitary representation of Γ and a unitary representation of $\mathrm{SO}(2n)$. The second aspect is the analytic torsion of $\Gamma\backslash\mathbb{H}^{2n+1}$ with respect to certain representations of Γ.

Throughout this thesis we let $\mathcal{O}$ be a hyperbolic odd-dimensional orbifold $\mathcal{O} = \Gamma\backslash\mathbb{H}^{2n+1}$; in Chapters 3-5 we assume it is compact, and in Chapter 6 we allow it to be of finite volume, but restrict ourselves to the 3-dimensional case.

1.1 Twisted Selberg trace formula and twisted Selberg zeta function for odd-dimensional compact orbifolds

Our first main topic is the twisted Selberg trace formula and twisted Selberg zeta functions. The Selberg zeta function was introduced by Selberg as an analogue of the Riemann zeta function where the prime numbers are replaced by the lengths of primitive closed geodesics $l(\gamma)$ on a hyperbolic surface:

$$Z(s) := \sum_{\gamma} \sum_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)}), \quad \mathrm{Re}(s) > 1.$$

The Selberg trace formula allows to prove the meromorphic extension of $Z(s)$ to $\mathbb{C}$, a functional equation, formulas for the poles and zeros and an analogue of the Riemann hypothesis [Sel56].

Given an odd-dimensional hyperbolic orbifold $\mathcal{O} = \Gamma\backslash\mathbb{H}^{2n+1}$, we define a more general Selberg zeta function $Z(s, \sigma, \chi)$ twisted by finite-dimensional representations χ of Γ and σ of $\mathrm{SO}(2n)$.

The Selberg zeta function $Z(s, \sigma, \chi)$ is defined by an infinite product which only converges in some half plane, and a crucial step in its investigation is to show that it admits a meromorphic continuation to the entire complex plane.

Theorem 1.1.1. *Suppose $\mathcal{O} = \Gamma\backslash\mathbb{H}^{2n+1}$ is a compact odd-dimensional hyperbolic orbifold, χ is a (possibly non-unitary) finite-dimensional representation of Γ, and σ is a unitary finite-dimensional representation of $SO(2n)$. Then the Selberg zeta function $Z(s, \sigma, \chi)$ admits a meromorphic continuation to $\mathbb{C}$.*

The logarithmic derivative $Z'(s, \sigma, \chi)/Z(s, \sigma, \chi)$ is only defined in some half plane and the key result is that the logarithmic derivative admits a meromorphic extension to $\mathbb{C}$ with integer residues. This implies that the zeta function itself admits a meromorphic extension to $\mathbb{C}$; the idea of the proof goes back to Selberg [Sel56]. If $\mathcal{O}$ is a compact hyperbolic manifold and σ is unitary, this was proven in [BO95]. Later on their result was extended to non-compact finite volume hyperbolic manifolds with cusps in the case when χ is unitary [GP10] and when χ is a restriction of a representation of $SO_0(1, 2n + 1)$ [Pfa12]. Using a slightly different approach, the theorem was proved in [Tsu97] for compact orbifolds when χ and σ are trivial representations. Notably, the theorem does not necessarily hold for non-compact finite volume hyperbolic orbifolds: an example is the Bianchi orbifold of discriminant -3 with χ and σ trivial [Fri05], however, it holds for a certain power of the Selberg zeta function. The approach of [BO95, Pfa12, GP10] is due to Selberg and invokes applying the Selberg trace formula to a certain test function which makes $Z'(s, \sigma, \chi)/Z(s, \sigma, \chi)$ appear as one of the terms in the geometric side of the formula. In order to adopt their approach we need to prove a more general version of the Selberg trace formula.

The Selberg trace formula has a rich history starting from the classical work [Sel56], but has mostly been constrained to unitary representations χ of Γ. The non-unitary case was first studied in [Mül11] under the assumption that Γ contains no non-trivial elements of finite order, also called elliptic elements, which means $\mathcal{O}$ is a compact manifold. We drop this restriction on Γ and prove:

Theorem 1.1.2. *Let G be a connected real semisimple Lie group of non-compact type with finite center, K a maximal compact subgroup of G, and $\Gamma \subset G$ a discrete subgroup such that*

$\mathcal{O} := \Gamma\backslash G/K$ is a compact orbifold. Let χ be a (possibly non-unitary) finite-dimensional representation of Γ, and ν be a unitary finite-dimensional representation of K. For the non-selfadjoint Laplacian $\Delta^{\#}_{\chi,\nu}$ defined in Subsection 3.3.2 and φ belonging to the space of Payley-Wiener functions $PW(\mathbb{C})$ defined in Section 3.2,

$$\sum_{\lambda\in spec(\Delta^{\#}_{\chi,\nu})} m(\lambda)\varphi(\lambda^{1/2}) = \sum_{\{\gamma\}\subset\Gamma} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)E_\gamma(h_\varphi).$$

Above $m(\lambda)$ is the multiplicity of λ; $\{\gamma\}$ denotes the conjugacy class of $\gamma \in \Gamma$; G_γ and Γ_γ are the centralizers of γ in G and Γ, respectively. Finally, $E_\gamma(h_\varphi)$ are the orbital integrals, defined by

$$E_\gamma(h_\varphi) := \int_{G_\gamma\backslash G} \mathrm{tr}\, h_\varphi(g\gamma g^{-1})d\dot{g},$$

where h_φ is the integral kernel of the operator $\varphi(\widetilde{\Delta}_\nu^{1/2})$ defined in (3.30).

To complete the proof of Theorem 1.1.1 we apply Theorem 1.1.2 to the case $G/K \cong \mathbb{H}^{2n+1}$. The major remaining problem is to calculate the orbital integrals $E_\gamma(h_\varphi)$ for the elliptic elements $\gamma \in \Gamma$.

Lemma 1.1.3. *In the above setup $G/K \cong \mathbb{H}^{2n+1}$, the orbital integral $E_\gamma(h_\varphi)$ for elliptic $\gamma \in \Gamma$ equals*

$$E_\gamma(h_\varphi) = \sum_{\sigma'\in\widehat{SO(2n)}} \int_{\mathbb{R}} \Theta_{\sigma',\lambda}(\varphi)P^\gamma_{\sigma'}(i\lambda)d\lambda,$$

where $\widehat{SO(2n)}$ is the unitary dual of $SO(2n)$, $\Theta_{\sigma',\lambda}(\varphi)$ is the character of the unitarily induced representation $\pi_{\sigma',\lambda}$ of G, and $P^\gamma_{\sigma'}(\lambda)$ is a certain even polynomial in λ.

Orbital integrals have so far been computed for $G/K \cong \mathbb{H}^2$, $\mathbb{H}^3$ and $\mathbb{H}^{2n}$ in [GGPS68], [Kna01] and [SW73] respectively. The computation of orbital integrals is not only useful for the proof of Theorem 1.1.1, but also for other applications of the Selberg trace formula.

As a by-product of the proof of Theorem 1.1.2 we obtain the following theorem about the heat trace asymptotics:

Theorem 1.1.4. *Let $E \to \mathcal{O}$ be an orbibundle over a good Riemannian orbifold $\mathcal{O} = G_U\backslash\widetilde{U}$, where $\widetilde{U}$ is a compact manifold and G_U is a finite group of orientation-preserving isometries of $\widetilde{U}$. Let $K(t,x,y)$ be the heat kernel from Definition 4.1.1. Then the following holds: as $t \to 0$,*

$$\int_{\mathcal{O}} \mathrm{tr}\, K(t,x,x)\, d\mathrm{vol}_{\mathcal{O}}(x) \sim I_e(t) + \sum_{\gamma\in G_U,\gamma\neq e} I_\gamma(t),$$

where

$$I_e(t) \sim t^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k t^k, \quad t \to 0,$$

$$I_\gamma(t) \sim t^{-\dim(N_\gamma)/2} \sum_{k=0}^{\infty} a_k^\gamma t^k, \quad t \to 0.$$

Above a_k, a_k^γ *are some coefficients in* $\mathbb{C}$, *and* N_γ *is the fixed point set of* γ *in* $\widetilde{U}$.

1.2 Analytic torsion for compact orbifolds

Our second main result is the following theorem.

Theorem 1.2.1. *Define a pseudopolynomial of degree* p *to be a sum of the following form:*

$$\sum_{j=0}^{p} \sum_{k=0}^{K} C_{j,k} m^j e^{im\phi_{j,k}}.$$

Let $\mathcal{O} = \Gamma \backslash \mathbb{H}^{2n+1}$ *be a compact hyperbolic orbifold. For* $m \in \mathbb{N}$, *let* $\tau(m)$ *be the finite-dimensional irreducible representation of* $\mathrm{SO}_0(1, 2n+1)$ *from Definition 2.6.2 and* $\tau'(m)$ *be the restriction of* $\tau(m)$ *to* Γ. *Let* $E_{\tau(m)'} \to \mathcal{O}$ *be the associated flat vector orbibundle, and denote by* $T_{\mathcal{O}}(\tau(m))$ *its analytic torsion. Then there exists* $C > 0$ *such that*

$$\log T_{\mathcal{O}}(\tau(m)) = PI(m) + PE(m) + O(e^{-Cm}), \quad m \to \infty.$$

Above, $PI(m)$ *is a polynomial in* m *of degree* $\frac{n^2+n+2}{2}$ *and* $PE(m)$ *is a pseudo-polynomial in* m *of degree* $\leq \frac{d^2+d+2}{2}$ *with* $2d+1$ *being the maximal dimension of the fixed point set of* Γ *in* $\mathbb{H}^{2n+1}$.

A similar result was proved for a finite volume hyperbolic manifold [MP11].

Remark 1.2.2. *Our result differs from [MP11] by a term* $PE(m)$ *that does not appear when* $\mathcal{O}$ *is a manifold.*

As a by-product of the proof we obtain the following theorem:

Theorem 1.2.3. *Let* $\mathcal{O} = G_U \backslash \widetilde{U}$ *be a good compact Riemannian orbifold, not necessarily hyperbolic,* $E \to \mathcal{O}$ *an associated flat orbibundle. Pick a Hermitian fiber metric* h *in* E. *Let* $g(u)$, $u \in [0,1]$ *be a smooth family of metrics on* $\mathcal{O}$ *with* $g = g(0)$ *as in Definition 5.2.1 and let* $\Delta^k(u)$ *be the family of Hodge-Laplacians acting on* E*-valued* k*-form. Assume that* $\ker \Delta^k(u) = \emptyset$; *moreover, assume that both* $\mathcal{O}$ *and all the fixed point sets of* G_U *in* $\widetilde{U}$ *are odd-dimensional. Then the analytic torsion* $T_{\mathcal{O}}(h, g(u))$ *does not depend on* u.

Theorem 1.2.3 automatically holds for manifolds by Cheeger-Mueller theorem and of orbifolds is of interest on its own: it is an important problem to understand the relation between the analytic and the Reidemeister torsions for orbifolds. If they were equal, Theorem 1.2.1 would imply the exponential growth of torsion in the cohomology of cocompact arithmetic groups [MP14, MM11]. In turn, any reasonable relation between the analytic and the Reidemeister torsions can be expected to imply that the former does not depend on the variation of the metric; this is shown in Theorem 1.2.3 under certain restrictions.

The proof of Theorem 1.2.1 is based on [MP11] but requires an additional component, namely Theorem 1.2.3. Let $\tau(m)$ and $E_{\tau(m)} \to \mathcal{O}$ be as above; the vector orbibundle $E_{\tau'(m)} \to \mathcal{O}$ can be equipped with a canonical Hermitian fibre metric [MM63, Proposition 3.1]. Let $\Delta_p(\tau(m))$ be the Laplace operator on $E_{\tau'(m)}$-valued p-forms with respect to the metric on $E_{\tau'(m)}$ and the hyperbolic metric on $\mathcal{O}$; its kernel vanishes for sufficiently large m. Denote

$$K(t, \tau(m)) := \sum_{p=0}^{2n+1} (-1)^p p \operatorname{Tr}\left(e^{-t\Delta_p(\tau(m))}\right), \tag{1.1}$$

then the analytic torsion is given by

$$\log T_{\mathcal{O}}(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t,\tau(m))dt\right)\bigg|_{s=0}. \tag{1.2}$$

We will now describe a rough plan of the proof of Theorem 1.2.1. As a simple corollary of Theorem 1.2.3, we can scale the metric g on $\mathcal{O}$ and hence replace $\Delta_p(\tau(m))$ by $\frac{1}{m}\Delta_p(\tau(m))$ in (1.1) and (1.2). Splitting the integral in (1.2) over $[0,\infty)$ into the integrals over $[0,1)$ and $[1,\infty)$, we obtain

$$\log T_{\mathcal{O}}(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}K\left(\frac{t}{m},\tau(m)\right)dt\right)\bigg|_{s=0} + \frac{1}{2}\int_1^\infty t^{s-1}K(t,\tau(m))dt.$$

It follows from [MP11] that the second term is $O(e^{-m/8})$ as $m\to\infty$; to estimate the first term we use the Selberg trace formula. For this we construct a smooth K-finite function $k_t^{\tau(m)}$ on $\mathrm{SO}_0(1,2n+1)$ such that

$$K(t,\tau(m)) = \int_{\Gamma\backslash \mathrm{SO}_0(1,2n+1)} \sum_{\gamma\in\Gamma} k_t^{\tau(m)}(g^{-1}\gamma g)d\dot{g}.$$

By the Selberg trace formula for compact orbifolds,

$$K(t/m,\tau(m)) = I(t/m,\tau(m)) + H(t/m,\tau(m)) + E(t/m,\tau(m)),$$

where $I(t/m, \tau(m))$, $H(t/m, \tau(m))$ and $E(t/m, \tau(m))$ are the contributions from the identity, hyperbolic and elliptic elements of Γ, respectively. Note $E(t, \tau(m))$ vanishes if $\mathcal{O}$ is a smooth manifold. Analogously to [MP11], there exist $m_0, C, c_1 > 0$ such that

$$|H(t/m, \tau(m))| \leq Ce^{-c_1 m}$$

for all $m \geq m_0$ and $0 < t \leq 1$. Recall that $I(t, \tau(m)) = \mathrm{vol}(\mathcal{O})k_t^{\tau(m)}(1)$. We can switch back from t/m to the variable t, then the contribution from the identity element to $\log T_{\mathcal{O}}(\tau(m))$ is given by

$$\frac{\mathrm{vol}(\mathcal{O})}{2} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} k_t^{\tau(m)}(1) dt \right) \bigg|_{s=0},$$

As in [MP11] we apply the Plancherel formula to $k_t^{\tau(m)}(1)$ and use the properties of Plancherel polynomials. We are left with the contribution from the elliptic elements. An important ingredient is now to apply the result of Lemma 1.1.3. Using the properties of $P_\sigma^\gamma(i\nu)$, we obtain Theorem 1.2.1.

We would like to apply this result to study the Reidemeister torsion, but this shall require further work, because there is no Cheeger-Müller theorem for orbifolds. However, there are partial results avaliable [ARS14, Ver14].

1.3 Analytic torsion for finite volume orbifolds

Many important arithmetic groups are not cocompact, for example $\mathrm{SL}(2, \mathbb{Z} \oplus i\mathbb{Z})$, so our third goal is to generalize Theorem 1.2.1 to hyperbolic odd-dimensional 3-orbifolds of finite volume:

Theorem 1.3.1. *Let $\mathcal{O} = \Gamma \setminus \mathbb{H}^3$ be a hyperbolic orbifold of finite volume with $\Gamma \subset \mathrm{SO}_0(1,3)$. For $m \in \mathbb{N}$, let $\tau(m)$ be a finite-dimensional irreducible representation of $\mathrm{SO}_0(1,3)$ from Definition 2.6.2 and $\tau'(m)$ be the restriction of $\tau(m)$ to Γ. Let $E_{\tau(m)'} \to \mathcal{O}$ be the associated flat vector orbibundle, and denote by $T_{\mathcal{O}}(\tau(m))$ its analytic torsion as in Definition 6.4.2. Then*

$$\log T_{\mathcal{O}}(\tau(m)) = -\frac{1}{2\pi} \cdot \mathrm{vol}(\mathcal{O}) \cdot m \cdot \dim(\tau(m)) + O(m \cdot \log(m)) \tag{1.3}$$

as $m \to \infty$.

Let us now describe the proof of Theorem 1.3.1. The first problem is to define the analytic torsion. Let $\Delta_p(\tau(m))$ be the Hodge-Laplacian on the space of $E_{\tau(m)'}$-valued p-forms as in Section 6.4. Since the heat operator $e^{-t\Delta_p(\tau(m))}$ is not of trace class, we cannot define the analytic torsion via the usual zeta function regularization. However we can define a regularized trace $\mathrm{Tr}_{reg} e^{-t\Delta_p(\tau(m))}$ as in [Par09] or [MP12] in the spirit of b-calculus of Melrose. It equals the spectral side of the Selberg trace formula applied to the heat operator $e^{-t\Delta_p(\tau(m))}$. This provides the asymptotic expansions of $\mathrm{Tr}_{reg} e^{-t\Delta_p(\tau(m))}$ of certain type as $t \to +0$ and as $t \to \infty$. The existence of these expansions allows us to define the spectral zeta function $\zeta_p(s;\tau(m))$ as in the case of compact manifolds via the Mellin transform of the regularized trace. Moreover, as in the compact case the zeta function $\zeta_p(s;\tau(m))$ is regular at $s=0$, and we can define the analytic torsion $T_{\mathcal{O}}(\tau(m)) \in \mathbb{C}$ with respect to E_τ by

$$T_{\mathcal{O}}(\tau(m)) := \exp\left(\frac{1}{2} \sum_{p=0}^{3} (-1)^p p \frac{d}{ds} \zeta_p(s;\tau(m)) \Big|_{s=0} \right). \tag{1.4}$$

Note that for sufficiently large m the operator $\Delta_p(\tau(m)) > 0$. Let

$$K(t,\tau(m)) := \sum_{p=0}^{3} (-1)^p \, p \, \mathrm{Tr}_{reg}(e^{-t\Delta_p(\tau(m))}).$$

As the spectral zeta function $\zeta_p(s;\tau(m))$ is expressed via the Mellin transform of the heat kernel $K(t,\tau(m))$, we need to compute the Mellin transform of $K(t,\tau(m))$ at 0 to study the analytic torsion. For this we use the invariant Selberg trace formula [Hof99] to express $K(t,\tau(m))$ as:

$$\begin{gathered} K(t,\tau(m)) = I(t;\tau(m)) + H(t;\tau(m)) + T(t;\tau(m)) + \\ \mathcal{I}(t;\tau(m)) + J(t;\tau(m)) + E(t;\tau(m)) + \mathcal{E}^{cusp}(t;\tau(m)) + \mathcal{J}^{cusp}(t;\tau(m)), \end{gathered} \tag{1.5}$$

where $I(t;\tau(m))$, $H(t;\tau(m))$, and $E(t;\tau(m))$ are the contributions of identity, hyperbolic and elliptic conjugacy classes of Γ, respectively; $T(t;\tau(m))$, $\mathcal{I}(t;\tau(m))$ and $J(t;\tau(m))$ are tempered distributions which are constructed out of the parabolic conjugacy classes of Γ; $\mathcal{E}^{cusp}(t;\tau(m))$ and $\mathcal{J}^{cusp}(t;\tau(m))$ are tempered distributions appearing due to the presence of non-unipotent stabilizers of the cusps of $\mathcal{O}$. Now we evaluate the Mellin transform of each term separately. It turns out that the leading term of the asymptotic expansion (6.23) comes from $MI(\tau(m))$, which is a Mellin transform of $I(t;\tau(m))$ evaluated at zero. It was proved in [MP12] that the contribution of $H(t;\tau(m)) + T(t;\tau(m)) + \mathcal{I}(t;\tau(m)) + J(t;\tau(m))$ to the analytic torsion $T_{\mathcal{O}}(\tau(m))$ is of order $O(m\log(m))$. The contribution of elliptic elements to $T_{\mathcal{O}}(\tau(m))$ was studied in Theorem 5.4.17 and does not affect the leading term of (6.23) as well. We are left with studying $\mathcal{J}^{cusp}(t;\tau(m))$ and $\mathcal{E}^{cusp}(t;\tau(m))$. The former distribution can be treated in a similar way as $J^{cusp}(t;\tau(m))$. The latter distribution is invariant and its Fourier transform was computed explicitly by Hoffmann [Hof97], which allows us to study its Mellin transform.

1.4 Structure of the thesis

This thesis is organized as follows. In Chapter 2, we fix notations and collect some facts about orbifolds, representation theory and Selberg theory. In Chapter 3, we prove the non-unitary Selberg trace formula, calculate the orbital integrals associated to elliptic elements, introduce Selberg zeta functions, establish their convergence in some half plane and prove that they admit meromorphic continuations to the whole complex plane. In Chapter 4, we calculate the heat trace asymptotics on a good orbifold. In Chapter 5, we show the invariance of the analytic torsion on a compact orbifold under variations on the metric and prove the result about the asymptotic behavior of the former. In Chapter 6, we introduce the relative analytic torsion, study the Fourier transform of distributions appearing in the relevant trace formula and prove the result about the asymptotic behavior of the relative analytic torsion.

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Chapter 2

Preliminaries

This chapter contains the preliminary information for the subsequent proofs. The list of dependencies is as follows:

1. to prove Theorem 1.1.1, we need Sections 2.1, 2.2, 2.3.1, 2.4, 2.5, 2.6;
2. to prove Theorem 1.2.1, we additionally need Sections 2.7, 2.11 and 2.12;
3. finally, to prove Theorem 1.3.1, we additionally need Sections 2.3.2, 2.8, 2.9 and 2.10.

2.1 Orbifolds and orbibundles

Definition 2.1.1. *Let $\widetilde{U}$ be a Riemannian manifold and G_U be a discrete group of isometries acting effectively on $\widetilde{U}$. Let G_U act properly discontinuous, that is for any $x, y \in \widetilde{U}$ there exist $U_x \supset x$ and $U_y \supset y$ such that*

$$\{g \in G_U : (g \cdot U_x) \cap U_y \neq \emptyset\}$$

is finite. Then $G_U \backslash \widetilde{U}$ is a good Riemannian orbifold.

Throughout the thesis we assume that the orbifolds we are dealing with are good.

Definition 2.1.2. *Let Q be a topological space. An orbifold chart on Q is a triple $(\widetilde{U}, G_U, \phi_U)$, where $\widetilde{U}$ is a connected open subset in $\mathbb{R}^n$, G_U is a finite group, $\phi_U : \widetilde{U} \to Q$ is a map with an open image $\phi_U(\widetilde{U})$ which induces a homeomorphism from $G_U \backslash \widetilde{U}$ to $\phi_U(\widetilde{U})$. Further we put $U := \phi_U(\widetilde{U})$. In this case, $(\widetilde{U}, G_U, \phi_U)$ is said to uniformize U.*

Definition 2.1.3. *Define λ to be a smooth embedding between two orbifold charts $(\widetilde{U}_1, G_1, \varphi_1)$ and $(\widetilde{U}_2, G_2, \varphi_2)$, if λ is a smooth embedding between $\widetilde{U}_1$ and $\widetilde{U}_2$ such that $\varphi_2 \circ \lambda = \varphi_1$. Two orbifolds charts $(\widetilde{U}_i, G_i, \varphi_i)$ uniformizing U_i, $i = 1, 2$ are called compatible if for any point $x \in U_1 \cap U_2$ there exists an open neighborhood V of x and an orbifold chart $(\widetilde{V}, H, \phi)$ of V such that there are two smooth embeddings $\lambda_i : (\widetilde{V}, H, \phi) \to (\widetilde{U}_i, G_i, \varphi_i)$, $i = 1, 2$.*

Definition 2.1.4. *An orbifold atlas on an orbifold $\mathcal{O}$ is a collection of pairwise compatible orbifold charts $(\widetilde{U}_i, G_i, \varphi_i)$ uniformizing U_i with $i \in I$ such that $\mathcal{O} = \cup_{i \in I} U_i$.*

Definition 2.1.5. *Two orbifolds atlaces are equivalent if their union is an orbifold atlas. An orbifold structure on $\mathcal{O}$ is an equivalent class of orbifold atlaces on $\mathcal{O}$.*

Definition 2.1.6. *Let $\mathcal{O} = G_W \backslash \widetilde{W}$ be an orbifold. The vector orbibundle $E \to \mathcal{O}$, associated to a representation $\rho : G_W \to End(V_\rho)$ is defined as*

$$E := G_U \backslash (\widetilde{W} \times V_\rho) \to G_U \backslash \widetilde{W},$$

where G_W acts on $(w, v) \in \widetilde{W} \times V_\rho$ as follows:

$$g : (w, v) \mapsto (gw, \rho(g)v), \quad g \in G_W.$$

Definition 2.1.7. *We define smooth sections of the orbibundle E as*

$$C^\infty(\mathcal{O}, E) = \{f \in C^\infty(\widetilde{W}, V_\rho), f(\gamma x) = \rho(\gamma) f(x), \quad x \in \widetilde{W}, \gamma \in G_W\}.$$

Definition 2.1.8. *Let F be a fundamental domain for the action of G_U on $\widetilde{W}$. Then*

$$C_c^\infty(\mathcal{O}, E) = \{f \in C^\infty(\mathcal{O}, E), \quad \mathrm{supp}(f|_F) \text{ is compact}\}.$$

Remark 2.1.9. *Note that the definition above does not depend on the choice of a fundamental domain F.*

For more details we refer to [Sch13].

2.2 Lie groups

Let $G = \mathrm{SO}_0(1, 2n+1)$, $K = \mathrm{SO}(2n+1)$. Let $G = NAK$ be an Iwasawa decomposition of G with respect to K. For each $g \in G$ there are uniquely determined elements $n(g) \in N$, $a(g) \in A$, $\kappa(g) \in K$ such that

$$g = n(g)a(g)\kappa(g).$$

Let M be the centralizer of A in K, thus

$$M = \mathrm{SO}(2n).$$

Denote the Lie algebras of G, K, A, M and N by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Define the Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(Y) = -Y^t, \quad Y \in \mathfrak{g},$$

and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Let $H : G \to \mathfrak{a}$ be defined by

$$H(g) := \log a(g). \tag{2.1}$$

There is a G-invariant metric on G/K which is unique up to scaling. Suitably normalized, it is the hyperbolic metric, and G/K is isometric to $\mathbb{H}^{2n+1}$.

Denote by $E_{i,j}$ the matrix in $\mathfrak{g}$ whose (i,j)'th entry is 1 and the other entries are 0. Let

$$\begin{aligned} H_1 &:= E_{1,2} + E_{2,1}, \\ H_j &:= \sqrt{-1} \cdot (E_{2j-1,2j} - E_{2j,2j-1}), \quad j = 2, \dots, n+1. \end{aligned}$$

Then $\mathfrak{a} = \mathbb{R}H_1$, where $\mathfrak{a}$ is from Subsection 2.2.

Definition 2.2.1. *Define*

$$A^+ = \{\exp(tH_1),\, t > 0\}.$$

Let $\mathfrak{b} = \sqrt{-1} \cdot \mathbb{R}H_2 + \ldots + \sqrt{-1} \cdot \mathbb{R}H_{n+1}$ be the standard Cartan subalgebra of $\mathfrak{m}$. Moreover, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{h}_\mathbb{C}, \mathfrak{g}_\mathbb{C}, \mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C}, \mathfrak{k}_\mathbb{C}$ the complexification of $\mathfrak{h}, \mathfrak{g}, \mathfrak{m}, \mathfrak{b}, \mathfrak{k}$, respectively. Define $e_i \in \mathfrak{h}_\mathbb{C}^*$ with $i = 1, \dots, n+1$, by

$$e_i(H_j) = \delta_{i,j},\ 1 \le i, j \le n+1. \tag{2.2}$$

The sets of roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ are given by

$$\begin{aligned} \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) &= \{\pm e_i \pm e_j, 1 \le i < j \le n+1\}, \\ \Delta(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C}) &= \{\pm e_i \pm e_j, 2 \le i < j \le n+1\}. \end{aligned} \tag{2.3}$$

We fix a positive systems of roots by

$$\begin{aligned} \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) &= \{e_i \pm e_j, 1 \le i < j \le n+1\}, \\ \Delta^+(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C}) &= \{e_i \pm e_j, 2 \le i < j \le n+1\}. \end{aligned} \tag{2.4}$$

The half-sum of the positive roots $\Delta^+(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$ equals

$$\rho_M = \sum_{j=2}^{n+1} \rho_j e_j, \quad \rho_j = n + 1 - j. \tag{2.5}$$

Let M' be the normalizer of A in K, and let

$$W(A) = M'/M \tag{2.6}$$

be the restricted Weyl group. It has order 2 and acts on finite-dimensional representations of M [Pfa12, p. 18]. Denote by w_0 the non-identity element of $W(A)$.

2.3 Hyperbolic orbifolds

Consider a discrete subgroup $\Gamma \subset G$, $G = \mathrm{SO}_0(1, 2n+1)$ such that $\Gamma\backslash G$ has finite volume.

2.3.1 Compact hyperbolic orbifolds

In Chapters 3 and 5, we consider compact hyperbolic orbifolds $\Gamma\backslash\mathbb{H}^{2n+1}$. Compactness implies that all non-identity elements of Γ are either hyperbolic or elliptic.

Definition 2.3.1. *An element $\gamma \in \Gamma$ is called hyperbolic if*

$$l(\gamma) := \inf_{x\in\mathbb{H}^{2n+1}} d(x, \gamma x) > 0,$$

where $d(x, y)$ denotes the hyperbolic distance between x and y.

Remark 2.3.2. *Some authors use the term "loxodromic" instead of "hyperbolic".*

Lemma 2.3.3. *[Wal93, Lemma 6.6] For hyperbolic γ there exists $g \in G$, $m_\gamma \in SO(2n)$, $a_\gamma \in A^+$, where A^+ is from Definition 2.2.1, such that*

$$g\gamma g^{-1} = m_\gamma a_\gamma.$$

Here a_γ is unique, and m_γ is determined up to conjugacy in $SO(2n)$.

Definition 2.3.4. *A non-identity element $\gamma \in \Gamma$ is called elliptic if it is of finite order.*

An alternative definition is the following: an element γ is elliptic if and only if it is conjugate to a non-identity element in K, so without loss of generality we may assume γ is of the form:

$$\gamma = \operatorname{diag}\big(\overbrace{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\ldots,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}^{k},\overbrace{R_{\phi_{k+1}},\ldots,R_{\phi_{n+1}}}^{n-k+1}\big), \tag{2.7}$$

where $n-k+1 \neq 0$ and $R_\phi = \left(\begin{smallmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{smallmatrix}\right)$, $\phi \in (0, 2\pi)$. There is an even number of eigenvalue 1 in (2.7), because an element γ should belong to $\mathrm{SO}_0(1, 2n+1)$.

Proposition 2.3.5. *The fixed point set of an elliptic element, acting on $\mathbb{H}^{2n+1}$, is odd-dimensional.*

Proof. It follows from (2.7) that the fixed point set of γ in $\mathbb{R}^{2n+2}$ is even-dimensional; since $\mathbb{H}^{2n+1}$ is of co-dimension one in $\mathbb{R}^{2n+2}$, the fixed point set of γ in $\mathbb{H}^{2n+1}$ is odd-dimensional. □

Definition 2.3.6. *An elliptic element γ is regular if the centralizer G_γ of γ in G is isomorphic to* $\mathrm{SO}_0(1,1) \times \mathrm{SO}(2)^{n-1}$.

Remark 2.3.7. *The calculation of the ordinary and weighted orbital integrals corresponding to elliptic elements appearing in the right hand side of the Selberg trace formula depends on whether an elliptic element is regular or not; see Subsection 3.3.5 and Section 6.2.*

2.3.2 Finite volume hyperbolic orbifolds

In Chapter 6, we consider finite volume hyperbolic 3-dimensional orbifolds that are not necessarily compact. This requires additional preliminary information about the group Γ. Let $\mathfrak{P}$ be a fixed set of representatives of Γ-conjugacy classes of cuspidal parabolic subgroups of G. If $\Gamma\backslash\mathbb{H}^3$ is of finite volume, then the number of cusps $\kappa := \#\mathfrak{P}$ is finite. Without loss of generality we can assume that $P_0 := MAN \in \mathfrak{P}$. For every $P \in \mathfrak{P}$, there exists $k_P \in K$ such that

$$P = N_P A_P M_P \tag{2.8}$$

with $N_P = k_P N k_P^{-1}$, $A_P = k_P A k_P^{-1}$, and $M_P = k_P M k_P^{-1}$.

Definition 2.3.8. *Let Z_Γ be the center of Γ. The group Γ is neat if*

$$\Gamma \cap P = Z_\Gamma \cdot \Gamma \cap N_P$$

for every $P \in \mathfrak{P}$.

Note that the structure of a cusp corresponding to a cuspidal parabolic subgroup $P \in \mathfrak{P}$ depends on $\Gamma \cap P$. If Γ is neat, then the cross-section of a cusp P is a torus. The known results about the analytic torsion of $\Gamma\backslash\mathbb{H}^3$ require that Γ is neat [MP11, Par09]. As this excludes various important arithmetic groups, we would like to drop this condition and allow Γ to be not neat. For example, we allow Γ to have elements of the following type:

Definition 2.3.9. *Let $\gamma \in \Gamma$ be an elliptic element. If there exists $P \in \mathfrak{P}$ such that $\gamma \in \Gamma \cap P$, then γ is called a cuspidal elliptic element.*

Example 2.3.10. *Let $G = \mathrm{SL}_2(\mathbb{C})$, then the group $\Gamma = \mathrm{PSL}_2(\mathbb{Z} \oplus (-1+i\sqrt{3})\mathbb{Z}/2)$ is not neat, as $\begin{pmatrix} (-1+i\sqrt{3})/2 & 0 \\ 0 & (1+i\sqrt{3})/2 \end{pmatrix}$ is a cuspidal elliptic element. The cross-section of the only cusp is an orbifold with 3 singular points of order 3.*

Recall that a Levi component L of P_0 is a centralizer of A in G, thus $L = MA$. In order to formulate the Selberg trace formula for the case of finite volume orbifolds, we need to introduce the following set of elements of Γ:

Definition 2.3.11. *Denote by $\Gamma_M(P_0)$ the set of projections to L of $\Gamma \cap P_0$.*

Remark 2.3.12. *By [War79, p. 5], $\Gamma \cap P_0 \subset MN$, hence $\Gamma_M(P_0) \subset M$. This implies that the set $\Gamma_M(P_0)$ is finite and each of its elements is of finite order.*

We recall the following lemma [Sel56]:

Lemma 2.3.13. *A finitely generated group Γ of matrices over a field of characteristic zero has a normal torsion-free subgroup Γ_0 of finite index.*

2.4 Normalization of measures

We normalize the Haar-measure on K such that K has volume 1. For $t \in \mathbb{R}$, we let

$$a(t) := \exp(tH_1). \tag{2.9}$$

Note that for any $a \in A$ there exists a unique $t \in \mathbb{R}$ such that (2.9) holds. We define the Haar measure on N as follows: first, we note that the Lie algebra $\mathfrak{n}$ of N is isometric to $\mathbb{R}^{2n}$ with respect to the inner product

$$\langle X, Y \rangle_\theta := -\frac{1}{2(d-1)} B(X, \theta(Y)). \tag{2.10}$$

Second, we identify $\mathfrak{n}$ and N by the exponential map; the definition of the measure dn on N follows from the previous two identifications. We normalize the Haar measure on G by setting

$$\int_G f(g)dg = \int_N \int_{\mathbb{R}} \int_K e^{-2nt} f(na(t)k)dkdtdn. \tag{2.11}$$

Remark 2.4.1. *The letter n in e^{-2nt} corresponds to the dimension of $\mathbb{H}^{2n+1}$, whereas the same letter in $f(na(t)k)$ denotes the element of N.*

2.5 Locally homogeneous vector bundles

Let $\nu : K \to GL(V_\nu)$ be a finite-dimensional unitary representation of K on $(V_\nu, \langle \cdot, \cdot \rangle_\nu)$.

Definition 2.5.1. *[Mia80, p.4] Denote by*

$$\widetilde{E}_\nu := (G \times V_\nu)/K \to G/K$$

the associated homogeneous vector bundle, where K acts on $G \times V_\nu$ by

$$(g, v)k = (gk, \nu(k^{-1})v), \quad g \in G,\ k \in K,\ v \in V_\nu.$$

Denote by $C^\infty(G/K, \widetilde{E}_\nu)$ the space of smooth sections of $\widetilde{E}_\nu$. Let $C_0^\infty(G/K, \widetilde{E}_\nu)$ be the sections of $\widetilde{E}_\nu$ with compact support.

Note that $\langle \cdot, \cdot \rangle_\nu$ induces a G-invariant metric on $\widetilde{E}_\nu$. Denote by $L^2(G/K, \widetilde{E}_\nu)$ the space of L^2-sections of $\widetilde{E}_\nu$. Let

$$\begin{aligned} C^\infty(G;\nu) &:= \{f : G \to V_\nu \mid f \in C^\infty, f(gk) = \nu(k^{-1})f(g), \quad g \in G, k \in K\}; \\ C^\infty(\Gamma\backslash G;\nu) &:= \{f \in C^\infty(G;\nu), f(\gamma g) = f(g) \quad g \in G, \gamma \in \Gamma\}. \end{aligned} \tag{2.12}$$

Similarly, we denote by $C_c^\infty(G;\nu)$ the subspace of compactly supported functions in $C^\infty(G;\nu)$ and by $L^2(G;\nu)$ the completion of $C_c^\infty(G;\nu)$ with respect to the inner product

$$\langle f_1, f_2 \rangle = \int_{G/K} \langle f_1(g), f_2(g) \rangle d\dot{g}.$$

Proposition 2.5.2. *[Mia80, p. 4] There is a canonical isomorphism*

$$C^\infty(G/K, \tilde{E}_\nu) \cong C^\infty(G;\nu). \tag{2.13}$$

Similarly, there are isomorphisms $C_c^\infty(G/K, \tilde{E}_\nu) \cong C_c^\infty(G;\nu)$ and $L^2(G/K, \tilde{E}_\nu) \cong L^2(G;\nu)$.

Definition 2.5.3. *Let ∇^ν be the canonical G-invariant connection on $\widetilde{E}_\nu$ defined by*

$$\nabla^\nu_{g_* Y} f(gK) := \frac{d}{dt}\Big|_{t=0} (g\exp(tY))^{-1} f(g\exp(tY)K),$$

where $f \in C^\infty(G;\nu)$ and $Y \in \mathfrak{p}$.

Definition 2.5.4. *Denote by $\widetilde{\Delta}_\nu = (\widetilde{\nabla}^\nu)^* \widetilde{\nabla}^\nu$ the associated Bochner-Laplace operator.*

Then $\tilde{\Delta}_\nu$ is essentially selfadjoint; denote its selfadjoint extension by the same symbol. Note that $\tilde{\Delta}_\nu$ is G-invariant, that is $\tilde{\Delta}_\nu$ commutes with the right action of G on $C^\infty(G/K, \tilde{E}_\nu)$. Let $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ and $\Omega_K \in Z(\mathfrak{k}_\mathbb{C})$ be the Casimir elements of G and K, respectively. Assume that ν is irreducible. Let R denote the right regular representation of G on $C^\infty(G;\nu)$.

Proposition 2.5.5. *[Mia80, Proposition 1.1] With respect to (2.13), we have*

$$\tilde{\Delta}_\nu = -R(\Omega) + \lambda_\nu \mathrm{Id}, \tag{2.14}$$

where $\lambda_\nu = \nu(\Omega_K) \geq 0$ is the Casimir eigenvalue of ν.

Definition 2.5.6. *Let $E_\nu := \Gamma\backslash\tilde{E}_\nu$ be the locally homogeneous vector orbibundle over $\Gamma\backslash G/K$ induced by $\tilde{E}_\nu$.*

Definition 2.5.7. *Let $\widetilde{A}_\nu$ be the differential operator that acts on $C^\infty(G,\nu)$ by $-R(\Omega)$. Let A_ν be its push-forward to $C^\infty(\Gamma\backslash G,\nu)$.*

Proposition 2.5.8. *The operator A_ν admits a self-adjoint extension in $L^2(\Gamma\backslash G,\nu)$.*

Proof. Follows from that $\Gamma\backslash G$ is a complete manifold. □

From now on denote this self-adjoint extension of A_ν by the same letter. Let e^{-tA_ν}, $t > 0$ be the semigroup of A_ν on $L^2(\Gamma\backslash G,\nu)$. Let $H^\nu_t(g)$ be its convolution kernel, and

$$h^\nu_t(g) := \operatorname{tr} H^\nu_t(g), \quad g \in G, \tag{2.15}$$

where tr denotes the trace in $\mathrm{End}(V_\nu)$.

2.6 Representations

Let $\sigma : M \mapsto \mathrm{End}(V_\sigma)$ be a finite-dimensional irreducible representation of M.

Definition 2.6.1. *We define $\mathcal{H}^\sigma$ to be the space of measurable functions $f : K \mapsto V_\sigma$ such that*

1. $f(mk) = \sigma(m)f(k)$ *for all* $k \in K$ *and* $m \in M$*;*
2. $\int_K ||f(k)||^2 dk < \infty$.

Recall $H : G \to \mathfrak{a}$, $\kappa : G \to K$ as in Subsection 2.2, and $e_1 \in \mathfrak{h}_{\mathbb{C}}^*$ is as in Subsection 2.2. For $\lambda \in \mathbb{R}$, define the representation $\pi_{\sigma,\lambda}$ of G on $\mathcal{H}^\sigma$ by the following formula:

$$\pi_{\sigma,\lambda}(g)f(k) := e^{(i\lambda+n)(H(kg))} f(\kappa(kg)), \tag{2.16}$$

where $f \in \mathcal{H}^\sigma$, $g \in G$. Fix $\tau_1, \ldots, \tau_{n+1} \in \mathbb{N}$, such that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_{n+1}$. Recall that $n = \frac{\dim(\Gamma\backslash G/K)-1}{2}$.

Definition 2.6.2. *For $m \in \mathbb{N}$ denote by $\tau(m)$ the finite-dimensional representation of G with highest weight*

$$(m+\tau_1)e_1 + \ldots + \ldots (m+\tau_{n+1})e_{n+1},$$

where e_i, $i = 1, \ldots n+1$ are defined as in (2.2).

Definition 2.6.3. *Let τ be the finite-dimensional irreducible representation of G with highest weight $\tau_1 e_1 + \ldots + \tau_{n+1}e_{n+1}$. Then denote by $\sigma_{\tau,k}$ be the representation of M with highest weight*

$$\Lambda_{\sigma_{\tau,k}} := (\tau_2+1)e_2 + \ldots + (\tau_k+1)e_{k+1} + \tau_{k+2}e_{k+2} + \ldots + \tau_{n+1}e_{n+1}. \tag{2.17}$$

2.7 Admissible metric and Fourier transform

Let (ρ, V_ρ) be a finite-dimensional representation of Γ and let $E_\rho \to \mathcal{O}$ be the associated flat orbibundle.

Let us specify to the case where $\rho = \tau|_\Gamma$ is the restriction to Γ of a finite-dimensional irreducible representation τ of G.

In this case E_ρ can be equipped with a distinguished metric which is unique up to scaling. Namely, E_ρ is canonically isomorphic to the locally homogeneous orbibundle E_τ associated to $\tau|_K$ (by analogy with [MM63, Proposition 3.1]). Moreover, there exists a unique up to scaling inner product $\langle\cdot,\cdot\rangle$ on V_ρ such that

1. $\langle \tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle, \quad Y \in \mathfrak{k}$,

2. $\langle \tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle, \quad Y \in \mathfrak{p}$,

for all $u, v \in V_\rho$. Note that $\tau|_K$ is unitary with respect to this inner product, hence it induces a unique up to scaling metric h on E_τ.

Definition 2.7.1. *Such a metric on E_τ is called admissible.*

From now on fix an admissible metric h. Let $\Delta_p(\tau)$ be the Hodge-Laplacian acting on p-forms with values in E_τ with respect to h.

Lemma 2.7.2. *[MM63, (6.9)] One has*

$$\Delta_p(\tau) = -R(\Omega) + \tau(\Omega)\mathrm{Id}$$

for

$$\tau(\Omega) = \sum_{j=1}^{n+1}(k_j(\tau)+\rho_j)^2 - \sum_{j=1}^{n+1}\rho_j^2,$$

where $k_1(\tau)e_1 + \ldots + k_{n+1}(\tau)e_{n+1}$ is the highest weight of τ and ρ_j is from (2.5).

Definition 2.7.3. *Let $\widetilde{E}_{\nu_p(\tau)} := G \times_{\tau_p(\tau)} \Lambda^p\mathfrak{p}^* \otimes V_\tau$, where*

$$\nu_p(\tau) := \Lambda^p Ad^* \otimes \tau : K \mapsto GL(\Lambda^p\mathfrak{p}^* \otimes V_\tau)$$

and let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $C^\infty(\mathbb{H}^{2n+1}, \widetilde{E}_{\nu_p(\tau)})$.

Definition 2.7.4. *Denote by*

$$H_t^{\tau,p} : G \mapsto End(\Lambda^p\mathfrak{p}^* \otimes V_\tau)$$

the convolution kernel of $e^{-t\widetilde{\Delta}_p(\tau)}$ as in [MP14, p. 16].

Let

$$h_t^{\tau,p}(g) := \mathrm{tr}\, H_t^{\tau,p}(g), \tag{2.18}$$

where tr denotes the trace in $\mathrm{End}(\Lambda^p\mathfrak{p}^* \otimes V_\tau)$. Put

$$k_t^\tau(g) := e^{-t\tau(\Omega)} \sum_{p=1}^{2n+1} (-1)^p\, p\, h_t^{\tau,p}(g). \tag{2.19}$$

We express $k_t^\tau(g)$ in a more convenient way:

Proposition 2.7.5. *[MP12, Proposition 8.2, (8.13)] For $k = 0, \ldots, n$ let*

$$\lambda_{\tau,k} = k_{k+1}(\tau) + n - k, \tag{2.20}$$

where $k_1(\tau)e_1 + \ldots + k_{n+1}(\tau)e_{n+1}$ is the highest weight of τ. Let $\sigma_{\tau_\rho,k}$ be as in Definition 2.6.3, and $h_t^{\sigma_{\tau,k}}$ be as in [MP12, (8.8)]. Then

$$k_t^\tau = \sum_{k=0}^{n} (-1)^{k+1} e^{-t\lambda_{\tau,k}^2} h_t^{\sigma_{\tau,k}}. \tag{2.21}$$

For a principal series representation $\pi_{\sigma',\lambda}$, $\lambda \in \mathbb{R}$, the Fourier transform is:

$$\Theta_{\sigma',\lambda}(h_t^\sigma) = e^{-t\lambda^2} \quad \text{for } \sigma' \in \{\sigma, w_0\sigma\}; \qquad \Theta_{\sigma',\lambda}(h_t^\sigma) = 0, \quad \text{otherwise.} \tag{2.22}$$

2.8 Truncation

The goal of this subsection is to introduce the height function on every cusp. To do so, for each $P \in \mathfrak{P}$ define

$$\iota_P \colon \mathbb{R}^+ \to A_P$$

by $\iota_P(t) := a_P(\log(t))$. For $Y > 0$, let

$$A_P^0[Y] := (\iota_P(Y), \iota(\infty)).$$

There exists a $Y_0 > 0$ such that for every $Y \geq Y_0$ there exists a compact connected subset $C(Y)$ of G such that in the sense of a disjoint union one has

$$G = \Gamma \cdot C(Y) \sqcup \bigsqcup_{P \in \mathfrak{P}} \Gamma \cdot N_P A_P^0[Y] K, \tag{2.23}$$

and such that

$$\gamma \cdot N_P A_P^0[Y] K \cap N_P A_P^0[Y] K \neq \emptyset \Leftrightarrow \gamma \in \Gamma_N. \tag{2.24}$$

Definition 2.8.1. *For $P \in \mathfrak{P}$, let $\chi_{P,Y}$ be the characteristic function of $N_P A_P^0[Y] K \subset G$.*

2.9 Eisenstein series

In this section we recall the definition and main properties of Eisenstein series [War79]. Let Γ be a lattice in $SO_0(1,3)$. Let $\sigma_P \in \widehat{M}_P$, $\nu \in \widehat{K}$ such that $[\nu : \sigma_P] \neq 0$.

Definition 2.9.1. *Let $\mathcal{E}_P(\sigma_P, \nu)$ be the space of continuous functions*

$$\Phi : (\Gamma \cap P)A_P N_P \backslash G \to \mathbb{C}$$

such that

1. *$m \in M_P \mapsto \Phi(xm)$ belongs to σ_P-isotypical component of the right regular representation of M,*
2. *$k \in K \mapsto \Phi(kx)$ belongs to ν-isotypical subspace of right regular representation of K.*

Definition 2.9.2. *We define an inner product $\langle \cdot, \cdot \rangle : \mathcal{E}_P(\sigma_P, \nu) \times \mathcal{E}_P(\sigma_P, \nu) \to \mathbb{C}$ by*

$$\langle \Phi, \Psi \rangle := \int_K \int_{M/\Gamma_M} \Phi(km)\bar{\Psi}(km)\, dk\, dm, \tag{2.25}$$

where $\Gamma_M = \Gamma \cap M \cdot N/\Gamma \cap N$.

Definition 2.9.3. *For $\Phi \in \mathcal{E}_P(\sigma_P, \nu)$, $\lambda \in \mathbb{C}$, define:*

$$\Phi_{P,\lambda}(g) := e^{(\lambda+n)H_P(x)},$$

and for $x \in \Gamma$, $x \in \Gamma g$, define

$$E(\Phi_P : x : \lambda) := \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \Phi_{P,\lambda}(\gamma g).$$

On $\Gamma\backslash G \times \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > n\}$ the series converge absolutely and locally uniformly.

Definition 2.9.4. *For $\Phi = (\Phi_P)_{P\in\mathfrak{P}}$, $\Phi_P \in \mathcal{E}_P(\sigma_P, \nu)$ and $x \in \Gamma\backslash G$, let*

$$E(\Phi : x : \lambda) := \sum_{P \in \mathfrak{B}} E(\Phi_P : x : \lambda).$$

Define the constant term of $E(\Phi_P : - : \lambda)$ along P' as follows:

$$E_{P'}(\Phi_P : g : \lambda) := \frac{1}{\mathrm{vol}(\Gamma \cap P' \backslash N_{P'})} \int_{\Gamma \cap P' \backslash N_{P'}} E(\Phi_P : yg : \lambda)\, dy.$$

Proposition 2.9.5. *Let $w_{P'}$ be the non-trivial element of $W(A_{P'})$. Then there exists a meromorphic function*

$$C_{P|P'}(\nu:\sigma_P:\lambda):\mathcal{E}(\sigma_P,\nu)\to\mathcal{E}(w_{P'}\sigma_{P'},\nu)$$

such that for $P\neq P'$, one has

$$E_{P'}(\Phi_P:g:\lambda)=\left(C_{P|P'}(\nu:\sigma_P:\lambda)\Phi_P\right)_{-\lambda}(g)$$

and such that

$$E_P(\Phi_P:g:\lambda)=\Phi_{P,\lambda}(g)+\left(C_{P|P'}(\nu:\sigma_P:\lambda)\Phi_P\right)_{-\lambda}(g).$$

Definition 2.9.6. *Define*

$$C_{P|P'}(\sigma_P,\lambda):=\bigoplus_{\nu}C_{P|P'}(\nu:\sigma_P:\lambda).$$

Definition 2.9.7. *Let $C(\nu:\sigma:\lambda)$ and $C(\sigma:\lambda)$ be the maps built from $C_{P'|P''}(\sigma_{P'},\nu,\lambda)$ and $C_{P'|P''}(\sigma_{P'},\lambda)$, respectively, where $\sigma_{P'}\in\sigma$.*

Definition 2.9.8. *For $\lambda\in\mathbb{C}$ and $\Phi\in\mathcal{E}_P(\sigma_P,\nu)$, define the representation $\pi_{\Gamma,\sigma,\lambda}$ of G on $\mathcal{E}(\sigma_P,\nu)$ as follows:*

$$\pi_{\Gamma,\sigma_P,\lambda}(g)\Phi(n_Pa_Pk):=e^{(\lambda+n)H_P(kg)}\Phi(kg). \tag{2.26}$$

Above $n_P\in N_P$, $a_P\in A_P$, $k\in K$ and $H(x)$ is from (2.1). Note that $\pi_{\Gamma,\sigma_P,\lambda}$ is unitary for $\lambda\in i\mathbb{R}$.

Definition 2.9.9. *For $\sigma\in\widehat{M}$ define*

$$\boldsymbol{\mathcal{E}}(\sigma):=\bigoplus_{\sigma_{P'}\in\sigma}\bigoplus_{\substack{\nu\in\widehat{K},\\ [\nu:\sigma_{P'}]\neq0}}\mathcal{E}(\sigma_{P'},\nu).$$

Definition 2.9.10. *Define the representation $\boldsymbol{\pi}_{\Gamma,\sigma,\lambda}$ of G on $\boldsymbol{\mathcal{E}}(\sigma)$ as follows:*

$$\boldsymbol{\pi}_{\Gamma,\sigma,\lambda}:=\bigoplus_{\sigma_{P'}\in\sigma}\pi_{\Gamma,\sigma_P,\lambda}. \tag{2.27}$$

Lemma 2.9.11 (Maass-Selberg relations)**.** *Let $\Phi,\Psi\in\boldsymbol{\mathcal{E}}^0$ and $\lambda\in\mathfrak{a}^*$. The lemma below follows from the proof of [Pfa12, Lemma 4.3] with minor changes. Note that the inner product $\langle\cdot,\cdot\rangle$ in this lemma should be understood as in Definition 2.25.*

$$\int_{\Gamma\backslash G}E^Y(\Phi,i\lambda,x)\overline{E^Y}(\Psi,i\lambda,x)\,dx=-\left\langle\mathbf{C}(\sigma:-i\lambda)\frac{d}{dz}\mathbf{C}(\sigma:i\lambda)\Phi,\Psi\right\rangle+$$
$$2\langle\Phi,\Psi\rangle\log Y+\frac{Y^{2i\lambda}}{2i\lambda}\langle\Phi,\mathbf{C}(\sigma:i\lambda)\Psi\rangle-\frac{Y^{-2i\lambda}}{2i\lambda}\langle\mathbf{C}(\sigma:i\lambda)\Phi,\Psi\rangle.$$

2.10 Knapp-Stein intertwining operators

Let $\bar{P}_0 := \bar{N}_0 A_0 M_0$ be the parabolic subgroup opposite to P_0.

Definition 2.10.1. *For $\Phi \in \mathcal{H}^\sigma$ from Definition 2.6.1, define*

$$\Phi_\lambda(nak) := e^{(i\lambda+n)H(a)}\Phi(k),$$

where $H(a)$ is from (2.1).

Definition 2.10.2. *For $Im(\lambda) < 0$ and $\Phi \in (\mathcal{H}^\sigma)^K$ define the intertwining operator*

$$J_{\bar{P}_0|P_0}(\sigma,\lambda)(\Phi)(k) := \int_{\bar{N}} \Phi(\bar{n}k)d\bar{n}.$$

Proposition 2.10.3. *[KS80] The operator $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ extends to an operator between*

$$J_{\bar{P}_0|P_0}(\sigma,\lambda) : \mathcal{H}^\sigma \to \mathcal{H}^\sigma$$

and has a meromorphic continuation to $\mathbb{C}$ in λ. It is regular and invertible on $\mathbb{R} - \{0\}$; moreover, if $\sigma \neq w_0\sigma$, then $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ is regular and invertible on $\mathbb{R}$.

2.11 Selberg trace formula for compact orbifolds and 3-orbifolds of finite volume

Let π_Γ be the right-regular representation of G on $L^2(\Gamma\backslash G)$. Then there exists an orthogonal decomposition

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G) \tag{2.28}$$

into closed π_Γ-invariant subspaces. The restriction of π_Γ to $L^2_c(\Gamma\backslash G)$ is isomorphic to the direct integral over all unitary principle series representations of Γ. The restriction of π_Γ to $L^2_d(\Gamma\backslash G)$ decomposes into the orthogonal direct sum of irreducible unitary representations of Γ.

Definition 2.11.1. *Let $\alpha \in C^\infty(G)$ be a K-finite Schwartz function. Denote by $\pi_\Gamma(\alpha)$ the following operator on $L^2(\Gamma\backslash G)$:*

$$\pi_\Gamma(\alpha)f(x) := \int_G \alpha(g)f(xg)dg. \tag{2.29}$$

Note that relative to (2.28) one has a splitting:

$$\pi_\Gamma(\alpha) = \pi_{\Gamma,d}(\alpha) \oplus \pi_{\Gamma,c}(\alpha).$$

Lemma 2.11.2. *The operator $\pi_{\Gamma,d}$ is of trace class.*

Proof. The estimation of the cuspidal spectrum follows from [Don76, Theorem 9.1]. The residual spectrum is spanned by residues of Eisenstein series for poles in the half plane $\mathrm{Re}(s) > (\dim(\mathcal{O}) - 1)^2/4$. These poles belong all to a finite interval on the real axis and are simple. Moreover, there are only finitely many Eisenstein series, hence the residual spectrum is finite. □

First, recall the Selberg trace formula for cocompact lattices:

Theorem 2.11.3 (Selberg trace formula for compact orbifolds)**.** *Let $\Gamma\backslash\mathbb{H}^{2n+1}$ be compact. Then*

$$L^2_c(\Gamma\backslash G) = \{0\}, \quad \pi_{\Gamma,d}(\alpha) = \pi_\Gamma(\alpha),$$

and for a K-finite Schwartz function $\alpha \in C^\infty(G)$ we have

$$\mathrm{Tr}\,(\pi_\Gamma(\alpha)) = I(\alpha) + H(\alpha) + E(\alpha), \tag{2.30}$$

where

$$I(\alpha) := \mathrm{vol}(\mathcal{O})\alpha(e),$$

$$H(\alpha) := \sum_{\{\gamma\}\,\text{hyperbolic}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \cdot \int_{G_\gamma\backslash G} \alpha(g^{-1}\gamma g)d\dot{g}, \tag{2.31}$$

$$E(\alpha) := \sum_{\{\gamma\}\,\text{elliptic}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)E_\gamma(\alpha), \quad E_\gamma(\alpha) := \int_{G_\gamma\backslash G} \alpha(g^{-1}\gamma g)d\dot{g}, \tag{2.32}$$

where G_γ and Γ_γ denote the centralizers of γ in G and Γ, respectively.

Moreover, there exists an even polynomial $P_\sigma(i\lambda)$ such that [Kna01, Theorem 13.2]

$$I(\alpha) = \mathrm{vol}(\mathcal{O}) \sum_{\sigma\in\widehat{M}} \int_{\mathbb{R}} P_\sigma(i\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda, \tag{2.33}$$

where $\Theta_{\sigma,\lambda}$ is the character of the representation $\pi_{\sigma,\lambda}$ as in (2.16). There also exist even polynomials $P^\gamma_\sigma(i\nu)$ such that

$$E(\alpha) = \sum_{\{\gamma\}\,\text{elliptic}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \sum_{\sigma\in\widehat{M}} \int_{\mathbb{R}} P^\gamma_\sigma(i\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda, \tag{2.34}$$

by Theorem 3.3.25.

Remark 2.11.4. *Although Theorem 3.3.25 was formulated for a specific test function α, the proof does not depend on how the test function looks like.*

Remark 2.11.5. *The polynomials $P_\sigma^\gamma(i\nu)$ and $P_\sigma(i\nu)$ are invariant under the action of $w_o \in W(A)$:*

$$P_\sigma(i\nu) = P_{w_0\sigma}(i\nu), \quad P_\sigma^\gamma(i\nu) = P_{w_0\sigma}^\gamma(i\nu).$$

Now let $\Gamma\backslash G/K$ be a finite volume orbifold of dimension 3. First we introduce the summands that will appear in the Selberg trace formula in Theorem 2.11.18.

Remark 2.11.6. *The main reason why we want to study the Selberg trace formula for orbifolds of finite volume is to obtain the asymptotic expansion from Theorem 1.3.1. During the proof we will apply the trace formula to the following setting: the group Γ is fixed, whereas the vector orbibundle over $\Gamma\backslash\mathbb{H}^3$ is changing. Moreover, during the proof we will figure out that the main term in the asymptotic expansion in Theorem 1.3.1 comes from the contribution from the identity $I(\alpha)$ with $\alpha = k_t^{\tau(m)}$ defined in (2.19); hence for the reason of convenience we will ignore all coefficients at the summands in the trace formula that depend only on the structure of the group Γ, for example, $C(\gamma)$ from Definition 2.11.10 and $C'(\gamma)$ from Remark 2.11.12.*

Definition 2.11.7. *Let $\gamma \in \Gamma_M(P)$, where $\Gamma_M(P)$ is from Definition 2.3.11, and let $\alpha \in C^\infty(G)$ be a K-finite Schwarz function. We define the weighted orbital integral $J_L(\gamma, \alpha)$ by*

$$J_L(\gamma, \alpha) := |D_G(\gamma)|^{1/2} \int_{G/G_\gamma} \alpha(x\gamma x^{-1}) v(x) dx.$$

Above $D_G(\gamma)$ is as in [Hof97, p. 55]; $v(x)$ is a weight function defined in [Hof97, p. 55].

Proposition 2.11.8. *[Hof97, p. 58] The weighted integral in Definition 2.11.7 is not an invariant distribution, but the distribution $I_L(\gamma, \alpha)$ below is invariant:*

$$I_L(\gamma, \alpha) := J_L(\gamma, \alpha) - \frac{1}{2\pi i} \sum_{\sigma \in \widehat{M}} \int_{D_\epsilon} \Theta_{\check{\sigma}_{-\lambda}}(\gamma) \cdot \operatorname{Tr}\left(J_{\bar{P}_0|P_0}(\sigma, z)^{-1} \frac{d}{dz} J_{\bar{P}_0|P_0}(\sigma, z) \pi_{\sigma,z}(\alpha) \right) dz. \tag{2.35}$$

Above $J_{\bar{P}_0|P_0}$ are the intertwining operators from Definition 2.10.2; D_ϵ is the path which is the union of $(-\infty, -\epsilon]$, H_ϵ and $[\epsilon, \infty)$, where H_ϵ is the half-circle from $-\epsilon$ to ϵ in the lower half-plane oriented counter-clockwise for some sufficiently small $\epsilon > 0$; $\pi_{\sigma,z}$ is defined in

Subsection 2.6; $\widehat{M}$ *is the set of equivalence classes of irreducible unitary representations of* M*; and*

$$\Theta_{\breve{\sigma}_{-\lambda}}(m_\gamma a_\gamma) = e^{-i\lambda t} \cdot \Theta_{\breve{\sigma}}(m_\gamma), \quad \Theta_{\breve{\sigma}}(m_\gamma) = \operatorname{tr} \breve{\sigma}(m_\gamma), \tag{2.36}$$

for γ *conjugated to* $m_\gamma a_\gamma$*, where* $a_\gamma = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ *and* $m_\gamma \in M$*. Above* $\breve{\sigma}$ *denotes the contragredient representation of* σ*.*

Remark 2.11.9. *We need to introduce the contour* D_ϵ *only if* $\sigma = w_0\sigma$*, because then the intertwining operator* $J_{\bar{P}_0|P_0}(\sigma, z)$ *has a pole at* $z = 0$ *by Proposition 2.10.3. Otherwise we could have put* $D_\epsilon = \mathbb{R}$*.*

Definition 2.11.10. *Let*

$$E^{cusp}(\alpha) := \sum_{\gamma \in \Gamma_M(P)} C(\gamma) \cdot I_L(\gamma, \alpha),$$

$$\mathcal{J}^{cusp}(\alpha) := \sum_{\gamma \in \Gamma_M(P)} C(\gamma) \cdot (-I_L(\gamma, \alpha) + J_L(\gamma, \alpha)),$$

where $C(\gamma)$ *is a constant depending only on* γ *and* Γ*. It equals*

$$\operatorname{vol}(M_\eta/(\Gamma_M)_\eta) h_p(\eta) |D_L(\eta)|^{-1/2} \tag{2.37}$$

from [Hof99, Theorem 7]; for the definition of the terms occurring in (2.37) we refer to [Hof99]. For the further use, define:

$$\mathcal{J}(\alpha) := -I_L(e, \alpha) + J_L(e, \alpha).$$

Remark 2.11.11. *The information we need about* $C(\gamma)$ *is that it depends only on* γ*.*

Remark 2.11.12. *Recall that* $\Gamma_M(P) \subset M$*, hence* $\Theta_{\breve{\sigma}_{-\lambda}}(\gamma)$ *does not depend on* λ*. Definition 2.11.10 and (2.35) imply*

$$\mathcal{J}^{cusp}(\alpha) = \sum_{\gamma \in \Gamma_M(P)} C'(\gamma) \cdot \mathcal{J}(\alpha)$$

for some new constant $C'(\gamma)$*.*

Definition 2.11.13. *Define*

$$R_\sigma(\alpha) := -\frac{1}{4} \operatorname{Tr} \left(\boldsymbol{C}(\sigma : 0) \boldsymbol{\pi}_{\Gamma, \sigma, 0} \right), \quad \sigma = w_0\sigma,$$

and $R_\sigma(\alpha) := 0$ *otherwise; define*

$$R(\alpha) := \sum_{\sigma \in \hat{M}} R_\sigma.$$

We introduce the following notation in order to be consistent with [Pfa12]:

Definition 2.11.14. *Define*

$$S(\alpha) := J(\alpha) + \sum_{\sigma \in \hat{M}} \frac{1}{4\pi} \int_{\mathbb{R}} \operatorname{Tr}\left(\boldsymbol{\pi}_{\Gamma,\sigma,i\lambda}(\alpha) \boldsymbol{C}(\sigma : i\lambda) \frac{d}{dz} \boldsymbol{C}(\sigma : i\lambda) \right).$$

Now we restrict ourselves to 3-dimensional orbifolds.

Definition 2.11.15. *Denote by* r *the canonical isomorphism between* $\mathrm{PSL}(2,\mathbb{C})$ *and* $\mathrm{SO}_0(1,3)$*, let* $r_1 := r\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $r_2 = r\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. *Then*

$$T_P(\alpha) := \sum_{\nu \in \Gamma_M(P) \cap \{r_2, +1\}} C(P, \nu, \Gamma) \int_{G/G_{\nu r_1}} \alpha(x \nu r_1 x^{-1})\, dx,$$

$$T(\alpha) := \sum_{P \in \mathfrak{P}} T_P,$$

where $C(P, \nu, \Gamma)$ *is the constant term in the Laurent expansion of some Epstein function, associated to* Γ*. Once again, we will ignore this constant.*

Proposition 2.11.16. *For* $\nu \in \Gamma_M(P) \cap \{r_2, +1\}$,

$$\int_{G/G_{\nu r_1}} \alpha(x \nu r_1 x^{-1})\, dx = \int_K \int_N \alpha(k n k^{-1}) dk\, dn.$$

Proof. Note that $G_{\nu r_1} = N$, and hence $G/G_{\nu r_1} = KA$. Rewrite $x = ka$, then

$$\int_{G/G_{\nu r_1}} \alpha(x \nu r_1 x^{-1})\, dx = \int_K \int_{\mathbb{R}} e^{-2nt} \alpha(k a(t) \nu r_1 a(t)^{-1} k^{-1}) dt dk. \tag{2.38}$$

Note that $n = a(t)\nu r_1 a(t)^{-1}$ ranges over N, while t runs over $\mathbb{R}$; moreover $dn = C_2 e^{-2nt} \cdot dt$, thus the change of variables proves Proposition 2.11.16. □

Proposition 2.11.17. *We have that*

$$T(\alpha) = \sum_{\sigma \in \widehat{M}} C_3(\Gamma) \cdot \dim(\sigma) \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(\alpha) d\lambda$$

for some $C_3(\Gamma) \in \mathbb{R}$.

Proof. Follows from Definition 2.11.15, Proposition 2.11.16 and [Pfa12, (6.9)]. □

Theorem 2.11.18 (Selberg trace formula for finite volume orbifolds)**.** *Let $\Gamma\backslash G/K$ be a hyperbolic orbifold. For a K-finite Schwartz function $\alpha \in C^\infty(G)$ we have*

$$\begin{gathered} \operatorname{Tr}(\pi_{\Gamma,d}(\alpha)) = I(\alpha) + H(\alpha) + T(\alpha) + \mathcal{I}(\alpha) + R(\alpha) + \\ \mathcal{S}(\alpha) + E(\alpha) + E^{cusp}(\alpha) + \mathcal{J}^{cusp}(\alpha). \end{gathered} \tag{2.39}$$

Proof. The first two lines in [Hof99, Theorem 4.2] correspond to summing $I(\alpha) + H(\alpha) + E(\alpha)$. The third line in [Hof99, Theorem 4.2] corresponds to $T(\alpha)$. The fourth line corresponds to $E^{cusp}(\alpha) + \mathcal{J}^{cusp}(\alpha) + \mathcal{I}(\alpha) + J(\alpha)$; the fifth line corresponds to $S(\alpha) - \mathcal{J}(\alpha)$; the sixth line corresponds to $R(\alpha)$. □

2.12 Mellin transform

Let $f(t) \in C^0(\mathbb{R})$. Assume that

$$f(t) = O(e^{-Ct}), \quad t \to \infty$$

for some $C > 0$. Moreover, let

$$f(t) \sim \sum_{j=1}^{\infty} a_j t^{\alpha_j}, \quad t \to 0,$$

where $\alpha_j \in \mathbb{R}$ tend to $+\infty$ as $j \to \infty$.

Definition 2.12.1. *The Mellin transform $\widetilde{f}(s)$ of $f(t)$ is defined by*

$$\widetilde{f}(s) := \int_0^\infty t^{s-1} f(t)\, dt \tag{2.40}$$

for $Re(s) > -min_j \alpha_j$.

Proposition 2.12.2. *[JL94, Theorem 1.1] The Mellin transform defined by (2.40) admits a meromorphic continuation to $\mathbb{C}$ with simple poles of residue a_j as $s = -\alpha_j$ and no other poles.*

Chapter 3

The Selberg trace formula and Selberg zeta function for non-unitary twists

3.1 Pseudodifferential operators on orbibundles

In this subsection we explain why the necessary elements of the classical analysis of pseudodifferential operators can be applied to orbifolds.

Sobolev spaces

To define Sobolev norms on an orbifold $\mathcal{O}$, first define Sobolev norms locally. Let $\widetilde{U}$ and G_U be as in Definition 2.1.2. Note that if G_U is finite, then

$$C_0^\infty(G_U\backslash\widetilde{U}, (G_U\backslash(\widetilde{U}\times\mathbb{R}^k)) \cong C_0^\infty(\widetilde{U}, \widetilde{U}\times\mathbb{R}^k)^{G_U}, \tag{3.1}$$

where $C_0^\infty(\widetilde{U}, \widetilde{U}\times\mathbb{R}^k)^{G_U}$ denotes the space of G_U-equivariant sections of $C_0^\infty(\widetilde{U}, \widetilde{U}\times\mathbb{R}^k)$ and $C_0^\infty(G_U\backslash\widetilde{U}, (G_U\backslash(\widetilde{U}\times\mathbb{R}^k))$ is from Definition 2.1.8. The space $C_0^\infty(\widetilde{U}, \widetilde{U}\times\mathbb{R}^k)$ is equipped with usual Sobolev norm $||\cdot||_s$, and this norm restricts to G_U-invariant sections. We equip $C_0^\infty(\widetilde{U}, \widetilde{U}\times\mathbb{R}^k)^{G_U}$, and hence $C_0^\infty(G_U\backslash\widetilde{U}, (G_U\backslash(\widetilde{U}\times\mathbb{R}^k))$ with the following norm:

$$||f'||_{s;U} := \frac{1}{|G_U|}||f||_s. \tag{3.2}$$

for $f' \in C_0^\infty(G_U\backslash\widetilde{U}, (G_U\backslash(\widetilde{U}\times\mathbb{R}^k))$ and the corresponding element $f \in C_0^\infty(\widetilde{U}, \widetilde{U}\times\mathbb{R}^k)$. Next we use an orbifold atlas and a partition of unity to define the Sobolev norm on the space of smooth sections of an orbibundle $E \to \mathcal{O}$. Sobolev norms defined using equivalent atlases will be themselves equivalent. The space $H^s(\mathcal{O}, E)$ denotes the completion of $C^\infty(\mathcal{O}; E)$ with respect to any of these norms; put $L^2(\mathcal{O}; E) := H^0(\mathcal{O}; E)$.

Remark 3.1.1. *The isomorphism (3.1) does not necessarily hold if G_U is infinite. For example, let γ act on $\mathbb{R}$ by $x \cdot \gamma = x + 1$ and put $G_U = \{\gamma^n : n \in \mathbb{Z}\}$. Then $C_0^\infty(\mathbb{R})^{G_U} = \{0\}$, but $C_0^\infty(G_U \backslash \mathbb{R}) \neq \{0\}$.*

Pseudodifferential operators

We recall some basic facts about pseudodifferential operators on orbibundles. For more details see [Buc99, p. 28], [Kor12, Section 2.2].

Definition 3.1.2. *Let $E \to \mathcal{O}$ be an orbibundle. For any orbifold chart $(\widetilde{U}, G_U, \phi_U)$ of $\mathcal{O}$, let $(\widetilde{U} \times V_\rho, G_U, \widetilde{\phi}_U)$ be a local trivialization of E over $(\widetilde{U}, G_U, \phi_U)$ as in [Kor12, Section 2.2]. A linear mapping $A : C^\infty(\mathcal{O}, E) \to C^\infty(\mathcal{O}, E)$ is a pseudodifferential operator on $E \to \mathcal{O}$ of order m if:*

1. *the Schwartz kernel of A is smooth outside any neighborhood of the diagonal in $\mathcal{O} \times \mathcal{O}$,*
2. *for any $x \in \mathcal{O}$ and for any local trivialization $(\widetilde{U} \times V_\rho, G_U, \widetilde{\phi}_U)$ of E over an orbifold chart $(\widetilde{U}, G_U, \phi_U)$ with $x \in U$, the operator*

$$C_c^\infty(U, E) \ni f \mapsto A(f)|_U \in C^\infty(U, E)$$

is given by the restriction to G_U-invariant functions of a pseudodifferential operator $\widetilde{A}$ of order m on $C^\infty(\widetilde{U}, V_\rho)$ that commutes with the induced G_U-action on $C^\infty(\widetilde{U}, V_\rho)$.

Definition 3.1.3. *A pseudodifferential operator A on $\mathcal{O}$ is elliptic if a pseudodifferential operator $\widetilde{A}$ is elliptic for any choice of orbifold charts.*

The Sobolev embedding and the Kondrachov-Rellich theorem are valid as in the case of manifolds:

Proposition 3.1.4 (Sobolev embedding)**.** *For $s > s'$, the embedding*

$$H^s(\mathcal{O}) \subset H^{s'}(\mathcal{O})$$

is continuous.

Proposition 3.1.5 (Kondrachov-Rellich theorem). *Let $\mathcal{O}$ be compact and $s > s'$, then the embedding*

$$H^s(\mathcal{O}) \subset H^{s'}(\mathcal{O})$$

is compact.

Proof of Propositions 3.1.4 and 3.1.5. Instead of the original proofs [Shu87, p. 60], one chooses a partition of unity and reduces the theorems to their local versions in a single chart. As sections over orbifold charts are G_U-invariant sections over the corresponding smooth charts, the desired proofs are obtained by repeating the local arguments from [Shu87] verbatim for the subspaces of G_U-invariant sections. □

Remark 3.1.6. *For another proof of the Sobolev embedding and the Kondrachov-Rellich theorem on orbifolds, see [Far01].*

Remark 3.1.7. *Let $\mathcal{O}$ be compact. Note that any pseudodifferential operator of order* 0 *extends to a bounded operator in $L^2(\mathcal{O}, E)$; compare [Shu87, Theorem 6.5]. Moreover, the Proposition 3.1.5 implies that any pseudodifferential operator of negative order is compact; compare [Shu87, Corollary 6.2].*

Theorem 3.1.8. *Let H be a second order elliptic pseudodifferential operator acting on sections of an orbibundle E over a compact good orbifold $\mathcal{O}$ with the leading symbol*

$$\sigma(H)(x,\xi) = ||\xi||_x^2 \cdot \mathrm{Id}_{E_x}, \quad x \in \mathcal{O}, \xi \in T_x^*\mathcal{O}. \tag{3.3}$$

For a subset $I \subset [-\pi, \pi]$ let

$$\Lambda_I := \{re^{i\phi} : 0 \leqslant r < \infty,\, \phi \in I\}$$

and

$$B_R(0) := \{x \in \mathbb{C},\, |x| \leq R\}.$$

Then for every $0 < \varepsilon < \pi/2$ there exists $R > 0$ such that the spectrum of H is contained in the set $B_R(0) \cup \Lambda_{[-\varepsilon,\varepsilon]}$. Moreover, the spectrum of H is discrete, and there exists $R \in \mathbb{R}$ such that for $|\lambda| > R$ and $\lambda \notin \Lambda_{[-\varepsilon,\varepsilon]}$,

$$||(H-\lambda)^{-1}|| \leqslant C/|\lambda|.$$

Proof. The proof of theorem is similar to the smooth case for which we refer to [Shu87, Theorem 9.3 and Theorem 8.4], except for the following: in the case of manifolds a partition of unity reduces the proof to $\mathbb{R}^n$, whereas in our case it is $G_U \backslash \mathbb{R}^n$, where G_U is a finite group. □

3.2 Functional analysis

In this section we refine the necessary facts from functional analysis from [Mül11, Section 2] for the case of compact orbifolds. The main difference from the case of compact manifolds is that we replace all theorems involving Sobolev spaces to their orbifold analogues from the previous section. Note that we though we assume our orbifold $\mathcal{O}$ is a good orbifold, it is not a necessary condition till the end of this section. The requirement on $\mathcal{O}$ to be compact is crucial, because we will need Remark 3.1.7.

Let $E \to \mathcal{O}$ be a Hermitian orbibundle, pick a Hermitian metric in E and let ∇ be a covariant derivative in E which is compatible with the Hermitian metric.

Definition 3.2.1. *The operator*

$$\Delta_E = \nabla^* \nabla \tag{3.4}$$

is the Bochner-Laplacian associated to the connection ∇ and the Hermitian fiber metric.

By [Buc99, Theorem 3.5], the Bochner-Laplace operator Δ_E is essentially selfadjoint. We denote its selfadjoint extension by the same symbol. Consider the class of elliptic operators

$$H : C^\infty(\mathcal{O}, E) \to C^\infty(\mathcal{O}, E), \tag{3.5}$$

which are perturbations of the Laplace operator Δ_E by a first order differential operator, i.e.

$$H = \Delta_E + D_1, \tag{3.6}$$

where $D_1 : C^\infty(\mathcal{O}, E) \to C^\infty(\mathcal{O}, E)$ is a first order differential operator.

For every $0 < \epsilon < \pi/2$ there exists $R > 0$ such that the spectrum of H is contained in $B_R(0) \cup \Lambda_{[-\epsilon,+\epsilon]}$ by Theorem 3.1.8. Though H is not self-adjoint in general, it has nice spectral properties. The reason is the following: $D_1(\Delta_E - \lambda)^{-1}$ is a pseudodifferential operator of order -1, and hence by Remark 3.1.7 is compact. This implies [Mar88] that $L^2(\mathcal{O}, E)$ is the closure of the algebraic direct sum of finite-dimensional H-invariant subspaces V_k

$$L^2(\mathcal{O}, E) = \overline{\bigoplus_{k \geqslant 1} V_k}, \tag{3.7}$$

such that the restriction of H to V_k has a unique eigenvalue λ_k, and for each k there exists $N_k \in \mathbb{N}$ such that $(H - \lambda_k \cdot \mathrm{Id})^{N_k} V_k = 0$, and $|\lambda_k| \to \infty$.

Denote by $\operatorname{spec}(H)$ the spectrum of H. Suppose that $0 \notin \operatorname{spec}(H)$. It follows from Theorem 3.1.8 that there exists an Agmon angle θ for H, and we can define the square root $H_\theta^{1/2}$. If θ is fixed, we simply denote $H_\theta^{1/2}$ by $H^{1/2}$. Note that $H^{1/2}$ is a classical pseudodifferential operator with the leading symbol

$$\sigma(H^{1/2})(x,\xi) = ||\xi||_x \cdot \mathrm{Id}_{E_x}. \tag{3.8}$$

By spectral theorem we can define $\Delta_E^{1/2}$. The principal symbols of $H^{1/2}$ and $\Delta_E^{1/2}$ coincide, hence

$$H^{1/2} = \Delta_E^{1/2} + B_0, \tag{3.9}$$

where B_0 is a pseudodifferential operator of order zero.

Lemma 3.2.2. *The resolvent of $H^{1/2}$ is compact, and the spectrum of $H^{1/2}$ is discrete. There exists $b > 0$ and $d \in \mathbb{R}$ such that the spectrum of $H^{1/2}$ is contained in the domain*

$$\Omega_{b,d} := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > d,\ |\mathrm{Im}(\lambda)| < b\}.$$

Proof. The proof follows in the same way as in [Mül11]. First note that $H^{1/2}$ is an elliptic pseudodifferential operator of order 1, hence by Remark 3.1.7 its resolvent is compact, that implies the spectrum of $H^{1/2}$ is discrete. Second, the operator B_0 extends to a bounded operator in $L^2(\mathcal{O}, E)$ by Remark 3.1.7; denote

$$b := 2 \cdot ||B_0||. \tag{3.10}$$

Recall that [Kat66, Chapter V, (3.16)] for $\lambda \notin \mathrm{spec}(\Delta_E^{1/2})$

$$||(\Delta_E^{1/2} - \lambda \cdot \mathrm{Id})^{-1}|| \leq |\mathrm{Im}(\lambda)|^{-1}. \tag{3.11}$$

The equations (3.10) and (3.11) imply

$$||B_0 \cdot (\Delta_E^{1/2} - \lambda \cdot \mathrm{Id})^{-1}|| \leq 1/2, \quad |\mathrm{Im}(\lambda)| \geq b,$$

and hence $I + B_0 \cdot (\Delta_E^{1/2} - \lambda \cdot \mathrm{Id})^{-1}$ is invertible for such λ, and

$$||(I + B_0 \cdot (\Delta_E^{1/2} - \lambda \cdot \mathrm{Id})^{-1})^{-1}|| \leq 2, \quad |\mathrm{Im}(\lambda)| \geq b.$$

Moreover,

$$(H^{1/2} - \lambda \cdot \mathrm{Id})^{-1} = (\Delta_E - \lambda \cdot \mathrm{Id})^{-1} \cdot \Big(I + B_0 \cdot (\Delta_E^{1/2} - \lambda \cdot \mathrm{Id})^{-1}\Big)^{-1},$$

that together with (3.11) implies

$$||(H^{1/2} - \lambda \cdot \mathrm{Id})^{-1}|| \leq 2 \cdot |\mathrm{Im}(\lambda)|^{-1}, \quad |\mathrm{Im}(\lambda)| \geq b,$$

hence the $\mathrm{spec}(H^{1/2}) \subset \Omega_{b,d}$.

□

It follows from the spectral decomposition (3.7) that $H^{1/2}$ has the same spectral decomposition as H with eigenvalues $\lambda^{1/2}$, $\lambda \in \operatorname{spec}(H)$ and multiplicities $m(\lambda^{1/2}) = m(\lambda)$. We need to introduce some class of function for further use.

Definition 3.2.3. *Denote by $PW(\mathbb{C})$ be the space of Paley-Wiener functions on $\mathbb{C}$, that is*

$$PW(\mathbb{C}) = \cup_{R>0} PW^R(\mathbb{C})$$

with the inductive limit topology. Above $PW^R(\mathbb{C})$ is the space of entire functions ϕ on $\mathbb{C}$ such that for every $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\phi(\lambda)| \leqslant C_N (1 + |\lambda|)^{-N} e^{R|Im(\lambda)|}, \quad \lambda \in \mathbb{C}.$$

Proposition 3.2.4. *Given $h \in C_0^\infty((-R, R))$, let*

$$\varphi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(r) e^{-ir\lambda} dr, \quad \lambda \in \mathbb{C}, \tag{3.12}$$

be the Fourier-Laplace transform of h. Then φ satisfies (3.2.3) for every $N \in \mathbb{N}$, that is $\varphi \in PW^R(\mathbb{C})$. Conversely, by the Paley-Wiener theorem, every $\phi \in PW^R(\mathbb{C})$ is the Fourier-Laplace transform of a function in $C_c^\infty((-R, R))$.

Recall that we are assuming $0 \notin \operatorname{spec}(H)$.

Definition 3.2.5. *For $b > 0$ and $d \in \mathbb{R}$ let $\Gamma_{b,d}$ be the contour which is the union of the two half-lines $L_{\pm b,d} := \{z \in \mathbb{C} : Im(z) = \pm b,\ Re(z) \geqslant d\}$ and the semicircle $S = \{d + be^{i\theta} : \pi/2 \leqslant \theta \leqslant 3\pi/2\}$, oriented clockwise.*

By Lemma 3.2.2 there exists $b > 0$, $d \in \mathbb{R}$ such that $\operatorname{spec}(H^{1/2})$ is contained in the interior of $\Gamma_{b,d}$. For an even Paley-Wiener function φ put

$$\varphi(H^{1/2}) := \frac{i}{2\pi} \int_{\Gamma_{b,d}} \varphi(\lambda)(H^{1/2} - \lambda)^{-1} d\lambda. \tag{3.13}$$

Remark 3.2.6. *In the next chapter, the Selberg trace formula will contain $\operatorname{Tr} \varphi(H^{1/2})$ as a spectral side for some H.*

Lemma 3.2.7. *$\varphi(H^{1/2})$ is an integral operator with a smoothing kernel.*

Proof. The proof follows in the same way as in [Mül11, Lemma 2.4]. For $k, l \in \mathbb{N}$ we have

$$H^k \varphi(H^{1/2}) H^l = \frac{i}{2\pi} \int_{\Gamma_{b,d}} \lambda^{2(k+l)} \varphi(\lambda)(H^{1/2} - \lambda)^{-1} d\lambda.$$

The operator $H^k \varphi(H^{1/2}) H^l$ is a bounded operator in $L^2(\mathcal{O}, E)$, since $\lambda \mapsto \lambda^{2(k+l)} \varphi(\lambda)$ is rapidly decreasing on $L_{\pm b,d}$. One can easily observe that as in the case of manifolds, $H^s(\mathcal{O}, E)$ is the completion of $C^\infty(\mathcal{O}, E)$ with respect to the norm $||(\mathrm{Id} + H)^{s/2} f||$, where $|| \cdot ||$ is the L^2-norm. It follows that for all $s, r \in \mathbb{R}$, $\varphi(H^{1/2})$ extends to a bounded operator from $H^s(\mathcal{O}, E)$ to $H^r(\mathcal{O}, E)$ and hence is a smoothing operator. □

Analogously to [Mül11, Lemma 2.2], we obtain:

Theorem 3.2.8 (The Weyl law). *Let*

$$N(r, H) := \sum_{\lambda \in \mathrm{spec}(H), |\lambda| \leq r} m(\lambda)$$

be the counting function of the spectrum of H, where eigenvalues are counted with algebraic multiplicity. Then

$$N(r, H) = \frac{\mathrm{rk}(E) \operatorname{vol}(\mathcal{O})}{(4\pi)^{(\dim \mathcal{O})/2} \Gamma((\dim \mathcal{O})/2 + 1)} r^{(\dim \mathcal{O})/2} + o(r^{(\dim \mathcal{O})/2}), \quad r \to \infty. \tag{3.14}$$

Proof. The Weyl law for Δ_E from Theorem 4.2.7 and the compactness of $D_1 \cdot (\Delta_E - \lambda \cdot \mathrm{Id})^{-1}$ implies (3.14) [Mar88, I, Corollary 8.5]. □

We need to establish an auxiliary result about smoothing operators. The proof of the following lemma literally repeats [Mül11, Proposition 2.5]:

Lemma 3.2.9. *Let*

$$A : L^2(\mathcal{O}, E) \to L^2(\mathcal{O}, E)$$

be an integral operator with a smooth kernel K; denote by $d\mu(x)$ the Riemanian measure on $\mathcal{O}$. Then A is a trace class operator and

$$\mathrm{Tr}(A) = \int_{\mathcal{O}} \mathrm{tr}\, K(x, x) d\mu(x).$$

Proof. The proof generalizes [Lan69, Chapter VII, §1]. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathcal{O},E)$ consisting of eigensections of Δ_E with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. We can expand K in the orthonormal basis as

$$K(x,y) = \sum_{i,j=1}^{\infty} a_{i,j}\phi_i(x) \otimes \phi_j^*(y), \tag{3.15}$$

where

$$a_{i,j} = \langle A\phi_i, \phi_j \rangle. \tag{3.16}$$

Note that

$$(1+\lambda_i+\lambda_j)^N a_{i,j} = \langle (I+\Delta_E \otimes I + I \otimes \Delta_E)^N A, \phi_i \otimes \phi_j^* \rangle,$$

hence for every $N \in \mathbb{N}$ there exists C_N such that

$$|a_{i,j}| \leq C_N(1+\lambda_i+\lambda_j)^{-N}$$

for any $i,j \in \mathbb{N}$. Then by Theorem 4.2.7 the right hand side of (3.15) converges in C^∞-topology.

Definition 3.2.10. *Define $P_{i,j}$, $i,j = 1,\ldots,\infty$ to be the integral operator with kernel $\phi_i \otimes \phi_j^*$. Put*

$$A_1 = \sum_{i,j=1}^{\infty} a_{i,j}(1+\lambda_j)^n P_{i,j}.$$

Definition 3.2.11. *Define P_j, $j = 1,\ldots,\infty$ to be the orthogonal projection of $L^2(\mathcal{O},E)$ onto $\mathbb{C}\phi_j$. Put*

$$A_2 = \sum_{j=1}^{\infty}(1+\lambda_j)^{-n} P_j.$$

By Theorem 4.2.7, both A_1 and A_2 are Hilbert-Schmidt operators, hence $A = A_1A_2$ is of trace class; moreover, by (3.16) and (3.15):

$$\operatorname{Tr} A = \sum_{i=1}^{\infty} a_{i,i} = \sum_{i,j=1}^{\infty} a_{i,j} \int_{\mathcal{O}} \langle \phi_i(x), \phi_i(x) \rangle d\mu(x) = \int_{\mathcal{O}} \operatorname{tr} K(x,x) d\mu(x).$$

□

Now we apply this result to $\varphi(H^{1/2})$, where $\varphi \in PW(\mathbb{C})$ and φ is even. Let $K_\varphi(x,y)$ be the kernel of $\varphi(H^{1/2})$. Then by Lemma 3.2.9, $\varphi(H^{1/2})$ is a trace class operator, and we have

$$\operatorname{Tr}\varphi(H^{1/2}) = \int_{\mathcal{O}} \operatorname{tr} K_\varphi(x,x) d\mu(x). \tag{3.17}$$

Moreover, the following lemma holds:

Lemma 3.2.12. *Let $\varphi \in PW(\mathbb{C})$ be even. Then we have*

$$\sum_{\lambda \in spec(H)} m(\lambda)\varphi(\lambda^{1/2}) = \int_{\mathcal{O}} \operatorname{tr} K_\varphi(x,x)d\mu(x), \tag{3.18}$$

where $m(\lambda)$ is the multiplicity of λ.

Proof. By Lidskii's theorem [GK69, Theorem 8.4], the trace is equal to the sum of the eigenvalues of $\varphi(H^{1/2})$, counted with their algebraic multiplicities. One can show that $\varphi(H^{1/2})$ leaves the decomposition (3.7) invariant and that $\varphi(H^{1/2})|_{V_k}$, has the unique eigenvalue $\varphi(\lambda_k^{1/2})$. Applying Lidskii's theorem and (3.17), we get Lemma 3.2.12. □

3.3 The Selberg trace formula

3.3.1 The wave equation

In this subsection we give a description of the kernel K_φ of the smoothing operator $\varphi(H^{1/2})$ in terms of the solution of the wave equation. For technical reasons we impose some restrictions on the orbifold $\mathcal{O}$, namely assume that $\mathcal{O} = \Gamma\backslash G/K$ where Γ is a discrete subgroup of the isometry group of a symmetric space G/K. For these subgroups the following lemma holds:

Lemma 3.3.1. *[Sel60, Lemma 8] A finitely generated group Γ of matrices over a field of characteristic zero has a normal torsion-free subgroup Γ_0 of finite index.*

Remark 3.3.2. *The restriction that $\mathcal{O} = \Gamma\backslash G/K$ is rather technical; during Subsection 3.3.1 we could have assumed that $\mathcal{O}$ is a good orbifold.*

It follows from Lemma 3.3.1 that $\Gamma_0\backslash G/K$ is a manifold. Let $\rho : \Gamma \to \mathrm{GL}(\mathbb{C}^n)$ be a finite-dimensional representation of Γ, and let

$$E = \Gamma\backslash(G/K \times \mathbb{C}^n) \to \mathcal{O}$$

be the associated vector orbibundle. Let ρ_0 be the restriction of ρ to Γ_0, and denote by

$$E_0 = \Gamma_0\backslash(G/K \times \mathbb{C}^n) \to \Gamma_0\backslash G/K$$

the associated vector bundle. Note that every $f \in C^\infty(\mathcal{O}, E)$ can be pulled back to $f_0 \in C^\infty(\Gamma_0\backslash G/K, E_0)$ as well as every $(\Gamma_0\backslash\Gamma)$-invariant $f_0 \in C^\infty(\Gamma_0\backslash G/K, E_0)$ can be pushed down to $f \in C^\infty(\mathcal{O}, E)$.

Proposition 3.3.3. *Denote by* $||\cdot||_{s;\Gamma_0\backslash G/K}$ *and* $||\cdot||_{s;\mathcal{O}}$ *the s-Sobolev norms on* $H^s(\Gamma_0\backslash G/K, E_0)$ *and* $H^s(\mathcal{O}, E)$*, respectively. Then there exists* $C, c > 0$ *such that for any* f *and* f_0 *as above the following inequality holds:*

$$c \cdot ||f_0||_{s;\Gamma_0\backslash G/K} \leq ||f||_{s;\mathcal{O}} \leq C \cdot ||f_0||_{s;\Gamma_0\backslash G/K}$$

Proof. Follows from that $\Gamma_0\backslash G/K$ is a finite covering of $\mathcal{O}$. □

Consider the wave equation:

$$\left(\partial^2/\partial t^2 + H\right) u = 0, \quad u(0, x; f) = f(x), \quad u_t(0, x; f) = 0 \tag{3.19}$$

for $u(t, x; f) \in C^\infty(\mathbb{R} \times \mathcal{O}, E)$.

Lemma 3.3.4. *For each* $f \in C^\infty(\mathcal{O}, E)$ *there is a unique solution* $u(t, x; f) \in C^\infty(\mathbb{R} \times \mathcal{O}, E)$ *of the wave equation (3.19). Moreover for every* $T > 0$ *and* $s \in \mathbb{R}$ *there exists* $C > 0$ *such that for every* $f \in C^\infty(\mathcal{O}, E)$

$$||u(t, \cdot; f)||_{s;\mathcal{O}} \leqslant C||f||_{s;\mathcal{O}}, \quad |t| \leqslant T \tag{3.20}$$

Proof. The proof follows from [Mül11, Proposition 3.1]: first pull-back the wave equation to $C^\infty(\mathbb{R} \times \Gamma_0\backslash G/K, E_0)$ and solve it there. The solution of the pulled-back wave equation satisfies [Mül11, (3.2)], that is the estimates of type (3.20), but for Sobolev norms $|| \cdot ||_{s;\Gamma_0\backslash G/K}$ instead of $|| \cdot ||_{s;\mathcal{O}}$. Moreover, it is invariant undet $\Gamma_0\backslash\Gamma$, because the pull-back of the initial conditions is invariant under $\Gamma_0\backslash\Gamma$. Second, push the solution down to $C^\infty(\mathcal{O}, E)$ and use Proposition 3.3.3. □

Lemma 3.3.5. *Let* $\varphi \in PW(\mathbb{C})$ *and* $\widehat{\varphi}$ *be the Fourier transform of* $\varphi|_{\mathbb{R}}$*. Then for every* $f \in C^\infty(\mathcal{O}, E)$ *we have*

$$[\varphi(H^{1/2})f](\cdot) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) u(t, \cdot; f) dt. \tag{3.21}$$

Proof. We follow [Mül11, Proposition 3.2]. Let $\Gamma_{b,d}$ be as in Definition 3.2.5, choose $c > 0$ such that

$$\operatorname{spec}(H + c) \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

For $\sigma > 0$, define the operator $\cos(tH^{1/2})e^{-\sigma(H+c)}$ by:

$$\cos(tH^{1/2})e^{-\sigma(H+c)} := \frac{i}{2\pi} \int_{\Gamma_{b,d}} \cos(t\lambda) e^{-\sigma(\lambda^2+c)} (H^{1/2} - \lambda)^{-1} d\lambda.$$

Note that for $f \in C^\infty(\mathcal{O}, E)$

$$\left(\cos(tH^{1/2})\, e^{-\sigma(H+c)} f\right)(x) - u(t,x;f)$$

is the unique solution of the wave equation (3.19) with initial condition $e^{-\sigma(H+c)}f - f$; the proof is by substitution and does not change if $\mathcal{O}$ is an orbifold. The rest of the proof is to show that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) \cos(tH^{1/2}) e^{-\sigma(H+c)} f \, dt = \frac{i}{2\pi} \int_{\Gamma_{b,d}} \varphi(\lambda) e^{-\sigma(\lambda^2+c)} (H^{1/2} - \lambda)^{-1} f \, d\lambda, \qquad (3.22)$$

and that the right hand side of (3.22) converges to $\varphi(H^{1/2})f$ as $\sigma \to 0$, whereas the left hand side converges to $(2\pi)^{-1} \int_{\mathbb{R}} \hat{\varphi}(t) u(t, \cdot; f)\, dt$. Proof of convergence is analogous to the manifold case. □

Now we would like to lift the wave equation once again, but now to G/K. Let

$$\widetilde{E} := (G/K) \times \mathbb{C}^n \to G/K$$

be a lift of E to G/K and let

$$\widetilde{H} : C^\infty(G/K, \widetilde{E}) \to C^\infty(G/K, \widetilde{E})$$

be the lift of H to G/K. Let $\widetilde{u}(u, \widetilde{x}; f)$ and $\widetilde{f}$ be the pull-back to G/K of $u(t,x;f)$ and f, respectively. Then the following holds:

$$\left(\partial^2/\partial t^2 + \widetilde{H}\right) \widetilde{u} = 0, \quad \widetilde{u}(0,x;f) = f(x), \quad \widetilde{u}_t(0,x;f) = 0 \qquad (3.23)$$

As in [Mül11, (3.15)], with the help of the finite propagation speed argument one can show that it does not matter if:

1. either we solve the wave equation (3.19) on $\mathcal{O}$ and then pull the solution back to G/K,

2. or we first pull back the initial condition to G/K and then solve the wave equation (3.23).

Let $d(x,y)$ denote the geodesic distance of $x, y \in G/K$. For $\delta > 0$ define $U_\delta := \{(x,y) \in G/K \times G/K : d(x,y) < \delta\}$.

Lemma 3.3.6. *There exists $\delta > 0$ and $H_\varphi \in C^\infty(G/K \times G/K, \mathrm{Hom}(\tilde{E}, \tilde{E}))$ with $\operatorname{supp} H_\varphi \subset U_\delta$, such that for all $\psi \in C^\infty(G/K, \tilde{E})$ we have*

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) u(t, \tilde{x}, \psi) dt = \int_{G/K} H_\varphi(\tilde{x}, \tilde{y})(\psi(\tilde{y})) d\tilde{y}.$$

Proof. The proof follows [Mül11, Proposition 3.3] and is based on the finite propagation speed argument, that is valid for orbifolds as well. □

Using Lemmas 3.3.5 and 3.3.6 we obtain

$$[\varphi(H^{1/2})f](\tilde{x}) = \int_{G/K} H_\varphi(\tilde{x},\tilde{y})(\tilde{f}(\tilde{y}))\, d\tilde{y} \tag{3.24}$$

for all $f \in C^\infty(X,E)$. Let $F \subset M$ be a fundamental domain for the action of Γ on G/K. Given $\gamma \in \Gamma$, let

$$R_\gamma : \widetilde{E} \mapsto \widetilde{E} \tag{3.25}$$

be the induced bundle map. Note that

$$\widetilde{f}(\gamma\widetilde{y}) = R_\gamma(\widetilde{f}(\widetilde{y})), \quad \gamma \in \Gamma. \tag{3.26}$$

Arguing as in [Mül11] by rewriting $\int_{G/K}$ as $\sum_{\gamma\in\Gamma}\int_{\gamma F}$ in (3.24) and using (3.26), one can show that the kernel K_φ of $\varphi(H^{1/2})$ is given by

$$K_\varphi(x,y) = \sum_{\gamma\in\Gamma} H_\varphi(\tilde{x},\gamma\tilde{y}) \circ R_\gamma, \tag{3.27}$$

where $\widetilde{x}$, $\widetilde{y}$ are any lifts of x and y to the fundamental domain F. Together with Lemma 3.2.12 we obtain:

Lemma 3.3.7. *[Mül11, Proposition 3.4] Let $\varphi \in PW$ be even. Then we have*

$$\sum_{\lambda\in spec(H)} m(\lambda)\varphi(\lambda^{1/2}) = \sum_{\gamma\in\Gamma}\int_F \operatorname{tr}(H_\varphi(\tilde{x},\gamma\tilde{x})\circ R_\gamma)d\tilde{x}.$$

3.3.2 The twisted Bochner-Laplace operator

In this section we follow [Mül11, Section 4] to introduce the twisted non-selfadjoint Laplacian $\Delta^\#_{E,\chi}$. Let $\mathcal{O}$ be a good orbifold $\mathcal{O} = \Gamma\backslash G/K$, $G = \mathrm{SO}_0(1,2n+1)$, $K = \mathrm{SO}(2n+1)$.

Let $\chi : \Gamma \to GL(V_\chi)$ be a finite-dimensional representation of Γ, and let $F \to \mathcal{O}$ be the associated orbibundle over $\mathcal{O}$; let ∇^F be a canonical flat connection on F. Let E be a Hermitian vector orbibundle over $\mathcal{O}$ with a Hermitian connection ∇^E.

Definition 3.3.8. *We equip $E\otimes F$ with the product connection $\nabla^{E\otimes F}$, defined by*

$$\nabla^{E\otimes F}_Y := \nabla^E_Y \otimes 1 + 1 \otimes \nabla^F_Y$$

for $Y \in C^\infty(G/K, T(G/K))$.

Definition 3.3.9. *The twisted connection Laplacian* $\Delta^{\#}_{E,\chi}$ *associated to* $\nabla^{E\otimes F}$ *is given by*

$$\Delta^{\#}_{E,\chi} := -\operatorname{Tr}(\nabla^{E\otimes F})^2,$$

where $(\nabla^{E\otimes F})^2$ *is the invariant second covariant derivative.*

Definition 3.3.10. *Denote by* Δ_E *the Bochner-Laplace operator* $(\nabla^E)^*\nabla^E$.

Remark 3.3.11. *The principal symbol of* $\Delta^{\#}_{E,\chi}$ *is given by*

$$\sigma(\Delta^{\#}_{E,\chi})(x,\xi) = ||\xi||^2_x \cdot \operatorname{Id}_{(E\otimes F)_x}.$$

Let $\widetilde{E}$ and $\widetilde{F}$ be the pullback to G/K of E and F, respectively. Note that

$$C^\infty(G/K, \widetilde{E}\otimes\widetilde{F}) \cong C^\infty(G/K,\widetilde{E})\otimes V_\chi.$$

Let $\widetilde{\Delta}^{\#}_{E,\chi}$ and $\widetilde{\Delta}_E$ be the lift of $\Delta^{\#}_{E,\chi}$ and Δ_E to G/K, respectively. Note that:

$$\widetilde{\Delta}^{\#}_{E,\chi} = \widetilde{\Delta}_E \otimes \operatorname{Id}, \tag{3.28}$$

where $\widetilde{\Delta}_E\otimes \operatorname{Id}$ acts on $C^\infty(G/K,\widetilde{E})\otimes V$. Then for any $\psi \in C^\infty_c(G/K,\widetilde{E})$, the unique solution of the equation

$$(\partial^2/\partial t^2 + \widetilde{\Delta}^{\#}_{E,\chi})u(t,\cdot;\psi) = 0, \quad u(0,\cdot;\psi) = \psi,\ u_t(0,\cdot;\psi) = 0$$

splits as well and is given by

$$u(t,\cdot;\psi) = \left(\cos(t(\widetilde{\Delta}_E)^{1/2})\otimes \operatorname{Id}\right)\psi(\cdot),$$

where $\cos(t(\widetilde{\Delta}_E)$ is defined by the spectral theorem. Let $\varphi \in PW(\mathbb{C})$ be even and let $k_\varphi(\tilde{x},\tilde{y})$ be the kernel of

$$\varphi\left((\widetilde{\Delta}_E)^{1/2}\right) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\varphi}(t)\cos(t(\widetilde{\Delta}_E)^{1/2})\,dt.$$

Then H_φ from Lemma 3.3.6 is given by $H_\varphi(\tilde{x},\tilde{y}) = k_\varphi(\tilde{x},\tilde{y})\otimes \operatorname{Id}$. Then it follows from (3.27) that the kernel of the operator $\varphi\left((\Delta^{\#}_{E,\chi})^{1/2}\right)$ is given by

$$K_\varphi(x,y) = \sum_{\gamma\in\Gamma} k_\varphi(\tilde{x},\gamma\tilde{y})\circ(R_\gamma\otimes\chi(\gamma)).$$

Lemma 3.3.7 implies the following Lemma:

Lemma 3.3.12. *Let F be a flat vector orbibundle over $\mathcal{O}$, associated to a finite-dimensional complex representation $\chi : \Gamma \to GL(V_\chi)$. Let $\Delta^{\#}_{E,\chi}$ be the twisted connection Laplacian from Definition 3.3.9 acting in $C^\infty(\mathcal{O}, E \otimes F)$. Let $\varphi \in PW(\mathbb{C})$ be even and denote by $k_\varphi(\tilde{x}, \tilde{y})$ the kernel of $\varphi\left((\widetilde{\Delta}_E)^{1/2}\right)$. Then we have*

$$\sum_{\lambda \in spec(\Delta^{\#}_{E,\chi})} m(\lambda)\varphi(\lambda^{1/2}) = \sum_{\gamma \in \Gamma} \operatorname{tr} \chi(\gamma) \int_F \operatorname{tr}(k_\varphi(\tilde{x}, \gamma\tilde{y}) \circ R_\gamma)\, d\tilde{x}.$$

3.3.3 Locally symmetric spaces and the pre-trace formula

In this subsection we specify Lemma 3.3.12 to the case when E is a locally homogeneous orbibundle.

Let $\chi : \Gamma \to GL(V_\chi)$ be a finite-dimensional (possibly non-unitary) complex representation and let $F \to \mathcal{O}$ be the associated flat vector bundle over $\mathcal{O}$ as in previous subsection. Let $\nu : K \to \mathrm{GL}(V_\nu)$ be a unitary representation of K and let $E_\nu \to \mathcal{O}$ be the locally homogeneous orbibundle as in Definition 2.5.6.

Denote by $\Delta^{\#}_{E_\nu,\chi}$ be the twisted connection Laplacian acting in $C^\infty(\mathcal{O}, E_\nu \otimes F)$ as in Definition 3.3.9. To simplify notations denote:

$$\Delta^{\#}_{\nu,\chi} := \Delta^{\#}_{E_\nu,\chi}. \tag{3.29}$$

Let $\widetilde{\Delta_\nu}$ be as in Definition 2.5.5. We are now interested in rewriting k_φ in a different way with respect to the information that E_ν is a locally homogeneous orbibundle. Note that $\varphi(\widetilde{\Delta}_\nu^{1/2})$ is a G-invariant operator. With respect to the isometry (2.13) it can be identified with a compactly supported C^∞ function

$$h_\varphi : G \to \operatorname{End} V_\nu,$$

such that

$$h_\varphi(k_1 g k_2) = \nu(k_1) \circ h_\varphi(g) \circ \nu(k_2), \quad k_1, k_2 \in K.$$

Then $(\widetilde{\Delta}_\nu^{1/2})$ acts by convolution:

$$\left(\varphi(\widetilde{\Delta}_\nu^{1/2}) f\right)(g_1) = \int_G h_\varphi(g_1^{-1} g_2)(f(g_2))dg_2, \tag{3.30}$$

and the kernel K_φ of $\varphi((\Delta^{\#}_{\nu,\chi})^{1/2})$ is given by

$$K_\varphi(g_1 K, g_2 K) = \sum_{\gamma \in \Gamma} h_\varphi(g_1^{-1} \gamma g_2) \otimes \chi(\gamma). \tag{3.31}$$

By Lemma 3.3.12 we get

$$\sum_{\lambda \in \mathrm{spec}(\Delta^{\#}_{E,\chi})} m(\lambda)\varphi(\lambda^{1/2}) = \sum_{\gamma\in\Gamma} \operatorname{tr}\chi(\gamma) \int_{\Gamma\backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g)d\dot{g}. \tag{3.32}$$

Definition 3.3.13. *For $\gamma \in \Gamma$, denote by $\{\gamma\}_\Gamma$ its Γ-conjugacy class.*

Definition 3.3.14. *For $\gamma \in \Gamma$, denote by Γ_γ and G_γ the centralizers of γ in Γ and G, respectively.*

Collect the terms in the right hand side of (3.32) according to their conjugacy classes. Separating $\{e\}_\Gamma$, we obtain a pre-trace formula.

Proposition 3.3.15. *[Pre-trace formula] For all even $\varphi \in PW(\mathbb{C})$ we have:*

$$\begin{aligned} \sum_{\lambda \in spec(\Delta^{\#}_{E,\rho})} m(\lambda)\varphi(\lambda^{1/2}) &= \dim(V_\chi)\mathrm{vol}(\Gamma\backslash G/K) \operatorname{tr} h_\varphi(e) + \\ &+ \sum_{\{\gamma\}_\Gamma \neq \{e\}} \operatorname{tr}\chi(\gamma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g)d\dot{g}. \end{aligned} \tag{3.33}$$

Restrict ourselves to the case $G = \mathrm{SO}_0(1, 2n+1)$, $K = \mathrm{SO}(2n+1)$. In order to make the formula more explicit, we need to evaluate the orbital integrals $\int_{G_\gamma\backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g)d\dot{g}$ on the right hand side of (3.33), that will be done in Subsections 3.3.4 and 3.3.5 for hyperbolic for elliptic γ, respectively.

3.3.4 Orbital integrals for hyperbolic elements

In this section we slightly modify [Wal93, Theorem 6.7] to evaluate the orbital integrals $\int_{G_\gamma\backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g)d\dot{g}$ for hyperbolic γ.

Let Γ_γ and G_γ be the centralizers of γ in Γ and G, respectively.

Definition 3.3.16. *For a hyperbolic $\gamma \in \Gamma$, define its primitive element as an element $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^k$, and for any $\gamma_0' \in \Gamma$ such that $\gamma = (\gamma_0')^n$, it follows that $n \leq k$.*

A primitive element γ_0 is not necessarily unique. It is defined up to

$$\Gamma^1_\gamma := \Gamma \cap G^1_\gamma,$$

where G^1_γ is the maximal compact subgroup of G_γ.

Remark 3.3.17. *Note that*

$$\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) = l(\gamma_0)/|\Gamma_\gamma^1|.$$

Recall that for $\sigma \in \widehat{SO(2n)}$ and $\lambda \in \mathbb{R}$ one can define the unitarily induced representations $\pi_{\sigma,\lambda}$ as in (2.16), let $\Theta_{\sigma,\lambda}$ denote the character of $\pi_{\sigma,\lambda}$. For hyperbolic γ we slightly modify [Wal93, Theorem 6.7] in order to get the following lemma:

Lemma 3.3.18. *Let $\gamma \in \Gamma$ be a hyperbolic element. Then the following holds:*

$$\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g) d\dot{g} =$$
$$\frac{l(\gamma_0)}{2\pi |\Gamma_\gamma^1| D(\gamma)} \cdot \sum_{\sigma \in \widehat{SO(2n)}} \overline{\operatorname{tr} \sigma(\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h_\varphi) \cdot e^{-il(\gamma)\lambda} d\lambda,$$

where

$$D(\gamma) = e^{-nl(\gamma)} \left| \det(\operatorname{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}} - \operatorname{Id}) \right| \tag{3.34}$$

and $l(\gamma_0)$ is from Definition 2.3.1, m_γ and a_γ is from Lemma 2.3.3.

Proof. The modification of [Wal93, Theorem 6.7] is as follows: we take $\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma)$ and change it to $l(\gamma_0)/|\Gamma_\gamma^1|$ by Remark 3.3.17. □

3.3.5 Orbital integrals for elliptic elements

In this subsection we evaluate the orbital integrals

$$E_\gamma(h_\varphi) := \int_{G_\gamma \backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g) d\dot{g} \tag{3.35}$$

for $\gamma \in \Gamma$ elliptic. We may assume γ is of the form:

$$\gamma = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & R_{\phi_{k+1}} & & \\ & & & & \ddots & \\ & & & & & R_{\phi_{n+1}} \end{pmatrix}, \tag{3.36}$$

where $R_\phi = \left(\begin{smallmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{smallmatrix}\right)$, $\phi \in (0, 2\pi)$. Note that in general γ is not a regular element, e.g. if $k > 1$ or some of the angles ϕ_i coincide; see Definition 2.3.6. For further use we want to approximate γ by a sequence of regular elements γ_ε parametrized by ε:

$$\gamma_\varepsilon = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & R_{\varepsilon_2} & & & & \\ & & & \ddots & & & \\ & & & & R_{\varepsilon_k} & & \\ & & & & & R_{\varepsilon_{k+1}} & \\ & & & & & & \ddots \\ & & & & & & & R_{\varepsilon_{n+1}} \end{pmatrix}, \tag{3.37}$$

where sequences $\varepsilon_i \in \mathbb{R}$, $i = 2, \dots, n+1$ are chosen in the following way: all ε_i are pairwise distinct, that is $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$, and

$$\lim_{\varepsilon \to 0} \varepsilon_i = 0, \quad i \leq k,$$
$$\lim_{\varepsilon \to 0} \varepsilon_i = \phi_i, \quad i > k.$$

The strategy for the subsection is the following: first we recall how to calculate $E_{\gamma_\varepsilon}(h_\varphi)$, second we apply a certain element of the symmetric algebra $S(\mathfrak{b}_\mathbb{C})$, and set $\varepsilon = 0$ to obtain $E_\gamma(h_\phi)$. To calculate $E_{\gamma_\varepsilon}(h_\varphi)$, we combine together an adjusted version of [Kna01, Theorem 13.1] together with the following proposition:

Proposition 3.3.19. *Let* $\eta(\gamma) \in \mathrm{Spin}(1, 2n+1)$ *be a lift of* γ. *Let* π *is a canonical projection* $\pi : \mathrm{Spin}(1, 2n+1) \to \mathrm{SO}_0(1, 2n+1)$. *For* $\gamma \in G$, *denote* $\mathrm{Spin}(1, 2n+1)_{\eta(\gamma)}$ *to be the centralizer of* $\eta(\gamma)$ *in* $\mathrm{Spin}(1, 2n+1)$. *Define*

$$\pi^* h_\varphi : \mathrm{Spin}(1, 2n+1) \mapsto \mathbb{C},$$
$$\pi^* h_\varphi(x) := h_\varphi(\pi(x)).$$

Then

$$\int_{\mathrm{Spin}(1,2n+1)_{\eta(\gamma)} \backslash \mathrm{Spin}(1,2n+1)} \pi^* h_\varphi(g^{-1}\eta(\gamma)g)dg = \int_{\mathrm{SO}_0(1,2n+1)_\gamma \backslash \mathrm{SO}_0(1,2n+1)} h_\varphi(g^{-1}\gamma g)dg. \tag{3.38}$$

Proof. Note that $\mathrm{Spin}(1, 2n+1)_{\eta(\gamma)}$ is a 2-fold covering of $\mathrm{SO}_0(1, 2n+1)_\gamma$, hence

$$\pi\left(\mathrm{Spin}(1, 2n+1)_{\eta(\gamma)} \backslash \mathrm{Spin}(1, 2n+1)\right) \cong \mathrm{SO}_0(1, 2n+1)_\gamma \backslash \mathrm{SO}_0(1, 2n+1).$$

Moreover,

$$h_\varphi(\pi(g)^{-1}\gamma\pi(g)) = h_\varphi(\pi(g^{-1}\eta(\gamma)g)) = \pi^* h_\varphi(g^{-1}\eta(\gamma)g).$$

and hence

$$\begin{aligned}
&\int_{\mathrm{Spin}(1,2n+1)_{\eta(\gamma)} \backslash \mathrm{Spin}(1,2n+1)} \pi^* h_\varphi\left(g^{-1}\eta(\gamma)g\right) dg = \\
&\int_{\mathrm{Spin}(1,2n+1)_{\eta(\gamma)} \backslash \mathrm{Spin}(1,2n+1)} h_\varphi\left(\pi(g)^{-1}\gamma\pi(g)\right) dg = \\
&\int_{\pi(\mathrm{Spin}(1,2n+1)_{\eta(\gamma)} \backslash \mathrm{Spin}(1,2n+1))} h_\varphi\left(g^{-1}\gamma g\right) dg = \\
&\int_{SO_0(1,2n+1)_\gamma \backslash SO_0(1,2n+1)} h_\varphi\left(g^{-1}\gamma g\right) dg,
\end{aligned} \tag{3.39}$$

that proves the proposition. □

Lemma 3.3.20. *The orbital integral $E_{\gamma_\varepsilon}(h_\varphi)$ can be expressed as*

$$E_{\gamma_\varepsilon}(h_\varphi) = C \cdot \sum_{\sigma \in \widehat{SO(2n)}} \int_{\mathbb{R}} \sum_{s \in W} \det(s) \left(\xi_{-s(\Lambda(\sigma)+\rho_M)} \otimes e^{-\sqrt{-1}\lambda}(\gamma_\varepsilon) \right) \cdot \Theta_{\sigma,\lambda}(h_\varphi) d\lambda, \tag{3.40}$$

where $C \in \mathbb{R}\backslash\{0\}$ does not depend on ε. The sum in (3.40) is finite, because h_φ is K-finite. Above $\det(s)$ *denotes the determinant of s, and for every*

$$\gamma = \begin{pmatrix} \gamma^1 & 0 \\ 0 & \gamma^2 \end{pmatrix}, \tag{3.41}$$

where $\gamma^1 = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in \mathrm{SO}_0(1,1)$ and $\gamma^2 \in \mathrm{SO}(2n)$, the tensor product acts as:

$$\xi_{-s(\Lambda(\sigma)+\rho_M)} \otimes e^{-\sqrt{-1}\lambda}(\gamma) := \left[\xi_{-s(\Lambda(\sigma)+\rho_M)}(\gamma^2)\right] \cdot [e^{-\sqrt{-1}\lambda t}].$$

Remark 3.3.21. *For γ_ϵ elliptic, $\gamma^1_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and hence*

$$\xi_{-s(\Lambda(\sigma)+\rho_M)} \otimes e^{-\sqrt{-1}\lambda}(\gamma_\varepsilon)$$

does not depend on λ.

Remark 3.3.22. *Our notation differs from [Kna01, Theorem 13.1], namely h, f and $F^T_f(h)$ corresponds to our γ_ϵ, h_φ and $E_{\gamma_\varepsilon}(h_\varphi)$, respectively; $F^T_f(h)$ is defined in [Kna01, p. 349].*

Definition 3.3.23. *For $\alpha \in \Delta^+(\mathfrak{so}_0(1, 2n+1)_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, denote by H_α its coroot.*

Without loss of generality assume that all ϕ_i from (3.36) are different, then the stabilizer G_γ of γ is equal to $\mathrm{SO}(2)^k \times SO_0(1, 2k-1)$. The root system for G_γ can be written as

$$\Delta_\gamma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm e_i \pm e_j, 1 \leqslant i < j \leqslant k\}.$$

We can choose an ordering such that

$$\Delta^+_\gamma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm(e_i + e_j), 1 \leqslant i < j \leqslant k\}.$$

Lemma 3.3.24. *[SW73, (5.2)] There exists $M_\gamma \in \mathbb{R} \setminus \{0\}$ such that*

$$E_\gamma(h_\varphi) = M_\gamma \cdot \lim_{\gamma_\varepsilon \to \gamma} \left(\prod_{\alpha \in \Delta^+_\gamma} H_\alpha \right) E_{\gamma_\varepsilon}(h_\varphi).$$

We are ready to prove the main theorem in this subsection:

Theorem 3.3.25. *There exists an even polynomial $P_\sigma^\gamma(\sqrt{-1}\lambda)$ such that*

$$E_\gamma(h_\varphi) = \sum_{\sigma\in\widehat{SO(2n)}} \int_{\mathbb{R}} P_\sigma^\gamma(\sqrt{-1}\lambda)\Theta_{\sigma,\lambda}(h_\varphi)d\lambda.$$

Proof. Theorem 3.3.25 holds with

$$P_\sigma^\gamma(\sqrt{-1}\lambda) = \prod_{\alpha\in\Delta_\gamma^+} H_\alpha\left(\sum_{s\in W}\det(s)\left(\xi_{-s(\Lambda(\sigma)+\rho_M)}\otimes e^{-\sqrt{-1}\lambda}(\gamma_\varepsilon)\right)\right)\Bigg|_{\epsilon=0} \tag{3.42}$$

by Lemmas 3.3.20 and 3.3.24. We need to show that $P_\sigma^\gamma(\sqrt{-1}\lambda)$ is an even polynomial. Note that every $\alpha\in\Delta_\gamma^+$ is a root with $\langle\alpha,\alpha\rangle = 2$, hence

$$\begin{gathered}\prod_{\alpha\in\Delta_\gamma^+} H_\alpha\left(\xi_{-s(\Lambda(\sigma)+\rho_M)}\otimes e^{-\sqrt{-1}\lambda}(\gamma_\varepsilon)\right) = \\ \left\{\prod_{\alpha\in\Delta_\gamma^+}\langle -s(\Lambda(\sigma)+\rho_M) - \sqrt{-1}\lambda e_1,\alpha\rangle\right\}\left(\xi_{-s(\Lambda(\sigma)+\rho_M)}\otimes e^{-\sqrt{-1}\lambda}(\gamma_\varepsilon)\right).\end{gathered} \tag{3.43}$$

Let $s(\Lambda(\sigma)+\rho_M) = \sum_{2\leqslant i\leqslant n+1} k_i e_i$ with ρ_M as in (2.5). For simplicity assume $k_1 = 0$ and denote by $\delta_{i,j}$ the Kronecker delta. Then

$$\begin{gathered}\prod_{\alpha\in\Delta_\gamma^+}\langle -s(\Lambda(\sigma)+\rho_M) - \sqrt{-1}\lambda e_1,\alpha\rangle = \\ (-1)^{|\Delta_\gamma^+|}\prod_{1\leqslant i'<j'\leqslant k}\langle\sum_{2\leqslant i\leqslant n+1} k_i e_i + \sqrt{-1}\lambda e_1, e_{i'} - e_{j'}\rangle\cdot\langle\sum_{2\leqslant i\leqslant n+1} k_i e_i + \sqrt{-1}\lambda e_1, e_{i'} + e_{j'}\rangle = \\ (-1)^{|\Delta_\gamma^+|}\prod_{1\leqslant i'<j'\leqslant k}\left(\sqrt{-1}\lambda(\delta_{i',1} - \delta_{j',1}) + (k_{i'} - k_{j'})\right)\cdot\left(\sqrt{-1}\lambda(\delta_{i',1} + \delta_{j',1}) + (k_{i'} + k_{j'})\right).\end{gathered} \tag{3.44}$$

Note that above $\delta_{j',1}$ is always equal to 0. Now we would like to study the dependance of (3.47) on λ, for this we split the product above as:

$$\prod_{1\leq i'<j'\leq k} = \prod_{\substack{i'=1,\\ 2\leq j'\leq k}}\cdot\prod_{2\leq i'<j'\leq k},$$

and first notice that

$$\prod_{2\leq i'\leq j'\leq k}\left(\sqrt{-1}\lambda\delta_{i',1} + (k_{i'} - k_{j'})\right)\cdot\left(\sqrt{-1}\lambda\delta_{i',1} + (k_{i'} + k_{j'})\right) = \prod_{2\leq i'<j'\leq k}(k_{i'}^2 - k_{j'}^2) := C(k) \tag{3.45}$$

does not depend on λ. Second,

$$\prod_{\substack{i'=1,\\ 2\leq j'\leq k}} \left(\sqrt{-1}\lambda\delta_{i',1}+(k_{i'}-k_{j'})\right)\cdot\left(\sqrt{-1}\lambda\delta_{i',1}+(k_{i'}+k_{j'})\right)= \prod_{2\leqslant j'\leq k}\left(\sqrt{-1}\lambda-k_{j'}\right)\cdot\left(\sqrt{-1}\lambda+k_{j'}\right)=-\prod_{2\leq j'\leq k}(\lambda^2+k_{j'}^2). \tag{3.46}$$

Putting together (3.44-3.46) gives us

$$\prod_{\alpha\in\Delta_\gamma^+}\langle -s(\Lambda(\sigma)+\rho_M)-\sqrt{-1}\lambda e_1,\alpha\rangle=(-1)^{|\Delta_\gamma^+|+1}\cdot C(k)\cdot\prod_{2\leq j'\leq k}(\lambda^2+k_{j'}^2), \tag{3.47}$$

where $(-1)^{|\Delta_\gamma^+|+1}\cdot C(k)$ does not depend on λ. Note that (3.47) is an even polynomial in λ and by Remark 3.3.21, $\xi_{-s(\Lambda(\sigma)+\rho_M)-\sqrt{-1}e_1\lambda}(\gamma_\epsilon)$ does not depend on λ. Hence, (3.43) and (3.42) are even polynomials in λ as well. □

We would like to mention the resemblance of Theorem 3.3.25 to the following:

Proposition 3.3.26. *[Kna01, Theorem 13.2] There exists an even polynomial $P_{\sigma'}(\sqrt{-1}\lambda)$ such that*

$$\operatorname{tr} h_\varphi(e)=\sum_{\sigma'\in\widehat{M}}\int_{\mathbb{R}}P_{\sigma'}(\sqrt{-1}\lambda)\Theta_{\sigma,\lambda}(h_\varphi)d\lambda.$$

For further use we need to show one property of the polynomial $P_\sigma^\gamma(\sqrt{-1}\lambda)$. Let M' be the normalizer of A in K and let $W(A)=M'/M$ be the restricted Weyl group. It has order 2 and acts on finite-dimensional representations of M [Pfa12, p. 18]. Let σ be a finite-dimensional representation of M with the highest weight

$$\Lambda(\sigma)=\sum_{j=2}^{n+1}\lambda_j(\sigma)e_j, \tag{3.48}$$

then the highest weight of a representation $w_0\sigma$, where w_0 is the non-identity element of $W(A)$ from (2.6), equals

$$\Lambda(w_0\sigma)=\sum_{j=2}^{n}\lambda_j(\sigma)e_j-\lambda_{n+1}(\sigma)e_{n+1}. \tag{3.49}$$

Lemma 3.3.27. *The polynomial P_σ^γ is invariant under the action of $W(A)$:*

$$P_\sigma^\gamma(\sqrt{-1}\lambda)=P_{w_0\sigma}^\gamma(\sqrt{-1}\lambda).$$

Proof. Recall that $s \in W$ acts on the roots by even sign changes and the permutations. Then it follows from (3.48) and (3.49) that if $s(\Lambda(\sigma)+\delta_M) = \sum_{2\leqslant i\leqslant n+1} k_i e_i$ for some $k_i \in \mathbb{Z}$, then $s(\Lambda(\sigma)+\delta_M) = \sum_{2\leqslant i\leqslant n+1} \hat{k}_i e_i$, where $\hat{k}_i = -k_i$ for exactly one i and $\hat{k}_j = k_j$ for all $j \neq i$. It follows that $\hat{k}_i^2 = k_i^2$. By (3.47) the polynomial $P_\sigma^\gamma(\sqrt{-1}\lambda)$ depends only on k_i^2 which completes the proof of Lemma 3.3.27. □

3.4 Selberg zeta function

Let $\sigma \in \widehat{M}$, where $M = \mathrm{SO}(2n)$.

Remark 3.4.1. *Note that ν from the previous section was a unitary representation of $K = \mathrm{SO}(2n+1)$.*

Definition 3.4.2. *The Selberg zeta function is:*

$$Z(s,\sigma,\chi) := \exp\left(- \sum_{\{\gamma\}\ hyperbolic} \frac{\mathrm{tr}(\chi(\gamma))\cdot \mathrm{tr}(\sigma(m_\gamma))\cdot e^{-(s+n)l(\gamma)}}{n_\Gamma \cdot |\Gamma_\gamma^1| \cdot \det(\mathrm{Id} - Ad(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})} \right). \tag{3.50}$$

Proposition 3.4.3. *There exist $c > 0$ such that $Z(s,\sigma,\chi)$ converges absolutely and locally uniformly for $Re(s) > c$.*

Proof. Analogously to [Spi15, Lemma 3.3], there exist $k, K \geq 0$ such that

$$\mathrm{tr}(\chi(\gamma)) \leq K e^{kl(\gamma)}.$$

It follows by definition that $|\Gamma_\gamma^1| \geq 1$, $n_\Gamma \geq 1$ and $\mathrm{tr}(\sigma(m_\gamma)) \leq \dim(\sigma)$. We need the following lemma to estimate the number of closed geodesics:

Lemma 3.4.4. *There exists a constant $C_3 > 0$ such that for all $x \in \mathbb{H}^{2n+1}$, the following estimate holds:*

$$\#\{\gamma \text{ hyperbolic}, \gamma \in \Gamma : \rho(x,\gamma x) \leq R\} \leq C_3 e^{2nR}, \tag{3.51}$$

where $\rho(x,y)$ denotes the hyperbolic distance between x and y.

Proof. Let $x \in \mathbb{H}^{2n+1}$, denote by $B_R(x)$ the hyperbolic ball around x of radius R; note that

$$\mathrm{vol}(B_R(x)) \leq C_2 \cdot e^{2nR}$$

for some $C_2 > 0$. Note that because Γ is cocompact, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \cap \gamma B_\varepsilon(x) = \emptyset, \quad \gamma \in \Gamma,\, \gamma \text{ hyperbolic},\, x \in \mathbb{H}^{2n+1}.$$

Thus

$$\bigsqcup_{\gamma \in \Gamma,\, \gamma \text{ hyperbolic},\, \rho(x,\gamma x) \leq R} \gamma B_\varepsilon(x) \subseteq B_{R+\varepsilon}(x),$$

that implies (3.51). □

Moreover,

$$\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}) \geq (1 - e^{-l(\gamma)})^n,$$

hence there exists a constant C_4 such that for every γ hyperbolic

$$\frac{1}{\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})} < C_4.$$

Collecting all together proves Proposition 3.4.3. □

3.4.1 The symmetric Selberg zeta function

Let $\sigma \in \widehat{M}$ with the highest weight

$$k_2(\sigma)e_2 + \ldots + k(\sigma)_{n+1}e_{n+1}.$$

Definition 3.4.5. *For $Re(s) > c$ with the constant c as in Proposition 3.4.3, we define the symmetric Selberg zeta function by*

$$S(s,\sigma,\chi) = \begin{cases} Z(s,\sigma,\chi)Z(s,w_0\sigma,\chi), & \text{if } \sigma \neq w_0\sigma; \\ Z(s,\sigma,\chi), & \text{if } \sigma = w_0\sigma. \end{cases} \tag{3.52}$$

In this subsection we prove the existence of the meromorphic continuation of the symmetric Selberg zeta function. We follow the approach of [Pfa12] which associates a vector bundle $E(\sigma)$ to every representation $\sigma \in \widehat{SO(2n)}$. This vector bundle is graded and there exists a canonical graded differential operator $A(\sigma,\chi)$ which acts on smooth sections of $E(\sigma)$. The next step is to apply the Selberg trace formula to $A(\sigma,\chi)$ with a certain test function.

First, we construct the bundle $E(\sigma)$ and the operator $A(\sigma,\chi)$.

Definition 3.4.6. *Let $R(K)$ and $R(M)$ be the representation rings over $\mathbb{Z}$ of K and M, respectively.*

Definition 3.4.7. *Denote by $\iota^* : R(K) \to R(M)$ the restriction map induced by the inclusion $\iota : M \hookrightarrow K$.*

By [Pfa12, Prop. 2.17], there exist integers $m_\nu(\sigma) \in \{-1, 0, 1\}$ such that for $\sigma = w_0\sigma$ one has

$$\sigma = \sum_{\nu \in \widehat{K}} m_\nu(\sigma)\iota^*\nu$$

and for $\sigma \neq w_0\sigma$ one has

$$\sigma + w_0\sigma = \sum_{\nu \in \widehat{K}} m_\nu(\sigma)\iota^*\nu.$$

Moreover, $m_\nu(\sigma)$ are zero except for finitely many $\nu \in \widehat{K}$.

Let $E_{\nu,\chi}$ be the orbibundle associated to $\nu \in \widehat{K}$, $\chi : \Gamma \to GL(V)$ as in Subsection 3.3.3.

Definition 3.4.8. *Let $E(\sigma)$ be the orbibundle*

$$E(\sigma) := \bigoplus_{\nu:\, m_\nu(\sigma) \neq 0} E_{\nu,\chi}.$$

For every $\nu \in \widehat{K}$ let $A_{\nu,\chi}$ be the operator defined by

$$A_{\nu,\chi} := \Delta^{\#}_{\nu,\chi} + c(\sigma) - \nu(\Omega_K),$$

where $\Delta^{\#}_{\nu,\chi}$ is as in Subsection 3.3.2, $\nu(\Omega_K)$ is as in (2.14) and

$$c(\sigma) = \sum_{j=1}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2.$$

Let $A(\sigma, \chi)$ be the operator acting on $C^\infty(\mathcal{O}, E(\sigma))$ defined by

$$A(\sigma, \chi) := \bigoplus_{\nu:\, m_\nu(\sigma) \neq 0} A_{\nu,\chi}.$$

Let $\tilde{E}(\sigma) := \bigoplus_{\nu:\, m_\nu(\sigma) \neq 0} \tilde{E}_{\nu,\chi}$ be the lift of $E(\sigma)$ to $\mathbb{H}^{2n+1}$, and let $\tilde{A}(\sigma, \chi)$ be the lift of $A(\sigma, \chi)$ to $\tilde{E}(\sigma)$. Note that by (3.28),

$$\tilde{A}(\sigma, \chi) = \bigoplus_{\nu:\, m_\nu(\sigma) \neq 0} \Big(\tilde{\Delta}_\nu + c(\sigma) - \nu(\Omega_k)\Big) \otimes \mathrm{Id}_{E_\chi}.$$

Together with (2.14) it gives

$$\tilde{A}(\sigma,\chi) = \bigoplus_{\nu:\, m_\nu(\sigma)\neq 0} (-R(\Omega) + c(\sigma)) \otimes \mathrm{Id}_{E_\chi}.$$

Second, we wish to apply the Selberg trace formula to $A(\sigma,\chi)$. For this let

$$h_t^\sigma(g) := e^{-tc(\sigma)} \sum_{\nu:\, m_\nu(\sigma)\neq 0} m_\nu(\sigma) h_t^\nu(g), \tag{3.53}$$

where $h_t^\nu := \operatorname{tr} H_t^\nu$, and H_t^ν is the integral kernel of $e^{-t\tilde{\Delta}_\nu}$.

Lemma 3.4.9. *[MP12, Section 4]* $\Theta_{\sigma',\lambda}(h_t^\sigma) = e^{-t\lambda^2}$ *for* $\sigma' \in \{\sigma, w_0\sigma\}$ *and equals zero otherwise.*

We are almost ready to apply the Selberg trace formula.

Definition 3.4.10. *Let* B_ν*,* $\nu \in \hat{K}$*, be trace class operators acting on sections of* E_ν*. Let*

$$B = \bigoplus_{\nu: m_\nu(\sigma)\neq 0} B_\nu.$$

Then define

$$\mathrm{Tr}_s\, B := \sum_{\nu:\, m_\nu(\sigma)\neq 0} m_\nu(\sigma) \,\mathrm{Tr}\, B_\nu. \tag{3.54}$$

Proposition 3.3.15, Lemma 3.3.18, Theorem 3.3.25 and Proposition 3.3.26 imply:

Theorem 3.4.11. *We have*

$$\begin{aligned}
\mathrm{Tr}_s(e^{-tA(\sigma,\chi)}) &= \mathrm{vol}(\mathcal{O}) \dim(V_\chi) \sum_{\sigma'\in\hat{M}} \int_{\mathbb{R}} P_{\sigma'}(i\lambda)\Theta_{\sigma',\lambda}(h_t^\sigma) d\lambda + \\
&\sum_{\sigma'\in\hat{M}} \sum_{\{\gamma\}\,\mathrm{elliptic}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \mathrm{tr}(\chi(\gamma)) \sum_{\sigma'\in\hat{M}} \int_{\mathbb{R}} P^\gamma_{\sigma'}(i\lambda)\Theta_{\sigma',\lambda}(h_t^\sigma) d\lambda + \\
&\sum_{\sigma'\in\hat{M}} \sum_{\{\gamma\}\,\mathrm{hyperbolic}} \frac{\mathrm{tr}(\chi(\gamma))\, l(\gamma_0)}{2\pi D(\gamma)} \overline{\mathrm{tr}(\sigma'(\gamma))} \int_{\mathbb{R}} \Theta_{\sigma',\lambda}(h_t^\sigma) e^{-l(\gamma)\lambda} d\lambda.
\end{aligned}$$

Let

$$\epsilon(\sigma) = \begin{cases} 2, & \text{if } \sigma \neq w_0\sigma; \\ 1, & \text{if } \sigma = w_0\sigma. \end{cases} \tag{3.55}$$

Denote

$$
\begin{aligned}
I(t) &:= \epsilon(\sigma)\mathrm{vol}(\mathcal{O})\dim(V_\chi)\int_{\mathbb{R}} P_\sigma(i\lambda)e^{-t\lambda^2}dt,\\
E(t) &:= \epsilon(\sigma)\sum_{\{\gamma\}\,\text{elliptic}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathrm{tr}(\chi(\gamma))\int_{\mathbb{R}} P_\sigma^\gamma(i\lambda)e^{-t\lambda^2}dt,\\
H(t) &:= \sum_{\{\gamma\}\,\text{hyperbolic}} \frac{\mathrm{tr}(\chi(\gamma))\,l(\gamma_0)}{2\pi D(\gamma)|\Gamma_\gamma^1|}\left(\overline{\mathrm{tr}(\sigma(\gamma))}+\overline{\mathrm{tr}(w_0\sigma(\gamma))}\right)\int_{\mathbb{R}} e^{-t\lambda^2}e^{-l(\gamma)\lambda}d\lambda.
\end{aligned}
\tag{3.56}
$$

Then Lemma 3.3.27, Theorem 3.4.11, Lemma 3.4.9 together with (3.55) and (3.56) imply

$$\mathrm{Tr}_s(e^{-tA(\sigma,\chi)}) = I(t)+E(t)+H(t). \tag{3.57}$$

Denote $(A(\sigma,\chi)+s^2)^{-1} =: R(s^2)$ for $s\in\mathbb{C}$, $s\notin \mathrm{spec}(A(\sigma,\chi))$. Note that for $\mathrm{Re}(s^2+A(\sigma,\chi))>0$,

$$R(s^2) = \int_0^\infty e^{-ts^2}e^{-tA(\sigma,\chi)}dt. \tag{3.58}$$

The operator $R(s^2)$ is not a trace class operator, but we will now improve it.

Lemma 3.4.12. *[BO95, Lemma 3.5] Let $s_1,\dots,s_N\in\mathbb{C}$ such that $s_i^2\neq s_j^2$ for $i\neq j$. Then for every $z\in\mathbb{C}\setminus\{-s_1^2,\dots,-s_N^2\}$ one has*

$$\sum_{i=1}^N \frac{1}{s_i^2+z}\prod_{j=1,j\neq i}^N \frac{1}{s_j^2-s_i^2} = \prod_{i=1}^N \frac{1}{s_i^2+z},$$

hence for $c_i=\prod_{j=1,j\neq i}^N \frac{1}{s_j^2-s_i^2}$, $i=1,\dots,N$,

$$\sum_{j=1}^N c_j R(s_j^2) = \prod_{j=1}^N R(s_j^2). \tag{3.59}$$

Lemma 3.4.13. *The operator $\prod_{j=1}^N R(s_j^2)$ is of trace class.*

Proof. In [BO95] Lemma 3.4.12 was proven for manifolds by the following argument: each of the factors is a pseudodifferential operator of order $-2/(2n+1)$, hence their product is a pseudodifferential operator of order $-2N/(2n+1)$ that is of trace class for sufficiently large N by the Weyl law.

Let R be the value of the resolvent of a self-adjoint Laplacian on $E(\sigma)$ at a point $x\in\mathbb{R}$, $x<0$, that is not in its spectrum. Note that it is self-adjoint and non-negative.

Lemma 3.4.14. *The operator R^N is of trace class for $N>(2n+1)/2$.*

Proof. Let λ_k be the k-th eigenvalue of R. Then by Theorem 3.2.8, $\lambda_k = O(k^{-2/(2n+1)})$ as $k \to \infty$. Note that λ_k^N is the k-th eigenvalue of R^N, hence $\lambda_k^N = O(k^{-2N/(2n+1)})$ as $k \to \infty$. □

Lemma 3.4.15. *The operator $R^{-N} \cdot \prod_{j=1}^N R(s_j^2)$ is bounded.*

Proof. It is a pseudodifferential operator of order 0 and hence bounded by Remark 3.1.7. □

By the above two lemmas,

$$\prod_{j=1}^N R(s_j^2) = R^N \cdot \left(R^{-N} \cdot \prod_{j=1}^N R(s_j^2) \right)$$

is of trace class. □

From now on let all s_j, $j = 1, \ldots, N$ satisfy $\operatorname{Re}(s_j + A(\sigma,\chi)^2) > 0$. We can choose such s_j, because the real parts of eigenvalues of $A(\sigma,\chi)$ are bounded from below. Put $s := s_1$ and $c_j := c_j'/c_1$ for $1 \le j \le N$ in Lemmas 3.4.12 and 3.4.13, then $R(s^2) + \sum_{j=2}^N c_j R(s_j^2)$ is of trace class, and by (3.58)

$$\operatorname{Tr}_s \left(R(s^2) + \sum_{j=2}^N c_j R(s_j^2) \right) = \int_0^\infty \left(e^{-ts^2} + \sum_{j=2}^N c_j e^{-ts_j^2} \right) \cdot \operatorname{Tr}_s \left(e^{-tA(\sigma,\chi)} \right) dt. \tag{3.60}$$

We would like to apply Theorem 3.4.11 to the right hand side. By analogy with [Pfa12, pp. 68-70],

$$\begin{aligned}
\int_0^\infty \left(e^{-ts^2} + \sum_j c_j e^{-ts_j^2} \right) \cdot I(t) dt &= \epsilon(\sigma) \operatorname{vol}(\mathcal{O}) \dim(V_\chi) \times \\
&\quad \left(\frac{\pi}{s} P_\sigma(s) + \sum_j \frac{c_j \pi}{s_j} P_\sigma(s_j) \right), \\
\int_0^\infty \left(e^{-ts^2} + \sum_j c_j e^{-ts_j^2} \right) \cdot E(t) dt &= \sum_{\{\gamma\} \text{ elliptic}} \epsilon(\sigma) \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \operatorname{tr}(\chi(\gamma)) \times \\
&\quad \left(\frac{\pi}{s} P_\sigma^\gamma(s) + \sum_j \frac{c_j \pi}{s_j} P_\sigma^\gamma(s_j) \right), \\
\int_0^\infty \left(e^{-ts^2} + \sum_j c_j e^{-ts_j^2} \right) \cdot H(t) dt &= \frac{1}{2s} \frac{S'(s,\sigma,\chi)}{S(s,\sigma,\chi)} + \\
&\quad \sum_j \frac{c_j}{2s_j} \frac{S'(s_j,\sigma,\chi)}{S(s_j,\sigma,\chi)}.
\end{aligned} \tag{3.61}$$

Note that we are crucially using that $P^{\gamma}_{\sigma}(\nu)$ and $P_{\sigma}(\nu)$ are even polynomials in ν. Thus we get

$$\begin{aligned}\operatorname{Tr}_s\left(R(s^2)+\sum_j c_jR(s_j^2)\right) = \frac{1}{2s}\frac{S'(s,\sigma)}{S(s,\sigma)}+\sum_j\frac{c_j}{2s_j}\frac{S'(s_j,\sigma)}{S(s_j,\sigma)}+\\ \epsilon(\sigma)\mathrm{vol}(\mathcal{O})\dim(V_\chi)\cdot\left(\frac{\pi}{s}P_\sigma(s)+\sum_j\frac{c_j\pi}{s_j}P_\sigma(s_j)\right)+\\ \sum_{\{\gamma\}\text{ elliptic}}\epsilon(\sigma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathrm{tr}(\chi(\gamma))\cdot\left(\frac{\pi}{s}P^{\gamma}_\sigma(s)+\sum_j\frac{c_j\pi}{s_j}P^{\gamma}_\sigma(s_j)\right).\end{aligned}$$

Put

$$\begin{aligned}\Xi(s,\sigma,\chi) := \exp\left(-2\pi\epsilon(\sigma)\mathrm{vol}(\mathcal{O})\dim(V_\chi)\int_0^s P_\sigma(r)dr - 2\epsilon(\sigma)\sum_{\{\gamma\}\text{ elliptic}}\mathrm{tr}(\chi(\gamma))\int_0^s P^{\gamma}_\sigma(r)dr\right)\times\\ S(s,\sigma,\chi)\end{aligned} \tag{3.62}$$

Then (3.4.1) can be rewritten as

$$\operatorname{Tr}_s\left(R(s^2)+\sum_{j=1}^N c_jR(s_j^2)\right) = \frac{1}{2s}\frac{\Xi'(s,\sigma,\chi)}{\Xi(s,\sigma,\chi)}+\sum_{j=1}^N\frac{c_j}{2s_j}\frac{\Xi'(s_j,\sigma,\chi)}{\Xi(s_j,\sigma,\chi)}, \tag{3.63}$$

where $\Xi'(s,\sigma)$ denotes the differentiation with respect to the first variable. It follows from (3.62) and (3.63), that $S(s,\sigma)$ extends meromorphically to $\mathbb{C}$ if and only if $\Xi(s,\sigma)$ does, moreover, its singularities coincide. Let λ_i, $i=1,2,\dots$ be the eigenvalues of $A(\sigma,\chi)$. For each λ_i let $\mathcal{E}(\lambda_i)$ be the eigenspace of $A(\sigma)$ with eigenvalue λ_i. Put

$$m_s(\lambda_i,\sigma) := \sum_{\nu\colon m_\nu(\sigma)\neq 0}(-1)^{m_\nu(\sigma)+1}\dim\mathcal{E}_\nu(\lambda_i), \tag{3.64}$$

where $\mathcal{E}_\nu(\lambda_i)$ is the eigenspace of $A_{\nu,\chi}$ with eigenvalue λ_i. Put

$$s_i^{\pm} = \pm\sqrt{-1}\cdot\sqrt{\lambda_i},\, j\in\mathbb{N},$$

where $\sqrt{\lambda_i}$ is chosen to have the non-negative imaginary part. Note that $\frac{1}{\lambda_i+s^2}$ and $\frac{c_j}{\lambda_i+s_j^2}$ are the eigenvalues of $R(s^2)$ and $c_jR(s_j^2)$, hence by (3.64) and Lidskii's theorem,

$$\operatorname{Tr}_s\left(R(s^2)+\sum_{j=1}^N c_jR(s_j^2)\right) = \sum_{i=1}^{\infty}\left(\frac{m_s(\lambda_i,\sigma)}{\lambda_i+s^2}+\sum_{j=1}^N\frac{c_j\cdot m_s(\lambda_i,\sigma)}{\lambda_i+c_j^2}\right). \tag{3.65}$$

Note that (3.65) and

$$\frac{2s}{\lambda_i+s^2} = \frac{1}{s+s_i^+}+\frac{1}{s-s_i^-}$$

imply that all residues of $2s \cdot \mathrm{Tr}_s \left(R(s^2) + \sum_{j=1}^{N} c_j R(c_j^2) \right)$ in s are integers. Hence by (3.63), $\Xi(s,\sigma)$ admits a meromorphic extension to $\mathbb{C}$. Together with (3.62)

Theorem 3.4.16. *The symmetrised Selberg zeta function $S(\sigma, s, \chi)$ has a meromorphic extension to $\mathbb{C}$. The set of singularities of $S(s,\sigma,\chi)$ equals $\{s_i^{\pm} : i \in \mathbb{N}\}$. If $\lambda_i \neq 0$, then the order of $S(s,\sigma,\chi)$ at both $s_i i^{+}$ and s_i^{-} is equal to $m_s(\lambda_i,\sigma)$. The order of the singularity at $s = 0$ is $2m_s(0,\sigma)$.*

3.4.2 Antisymmetric Selberg zeta function

Suppose that $\sigma \neq w_0\sigma$, otherwise the symmetric Selberg zeta function equals the Selberg zeta function and this section can be skipped. For $\mathrm{Re}(s) > c$ with the constant c as in Proposition 3.4.3 we define the antisymmetric Selberg zeta function as

$$S_a(s,\sigma,\chi) := Z(s,\sigma,\chi)/Z(s,w_0\sigma,\chi). \tag{3.66}$$

In this subsection we prove the meromorphic continuation of antisymmetric Selberg zeta function $S_a(s,\sigma,\chi)$.

Dirac bundles and twisted Dirac operators

Let $\mathrm{Cl}(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to the scalar product on $\mathfrak{p}$. Let κ be the spin-representation of K and put $\Delta_{2n} := \mathbb{C}^{2^n}$; denote by $\tilde{S} = G \times_\kappa \Delta^{2n}$ be the spinor bundle on $\mathbb{H}^{2n+1}$ and equip it with a connection ∇^S.

Let $\sigma \in \hat{M}$. By [BO95, Proposition 1.1], there a unique $\nu(\sigma) \in R(K)$ such that

$$\nu(\sigma) \otimes \kappa = \nu^{+}(\sigma) \oplus \nu^{-}(\sigma) =: \nu_\kappa(\sigma),$$

where $\nu^{\pm}(\sigma) \in \hat{K}$. Define $\widetilde{E}_{\nu_\kappa(\sigma)}$ to be the locally homogeneous vector bundle over $\mathbb{H}^{2n+1}$:

$$\widetilde{E}_{\nu_\kappa(\sigma)} := G \times_{\nu_\kappa(\sigma)} (V_{\nu(\sigma)} \otimes \Delta^{2n}) \to \mathbb{H}^{2n+1}.$$

Remark 3.4.17. *Note that*

$$\widetilde{E}_{\nu_\kappa(\sigma)} = \widetilde{E}_{\nu(\sigma)} \times \tilde{S},$$

that allows us to equip $\widetilde{E}_{\nu_\kappa(\sigma)}$ with the product connection $\nabla^\sigma := \nabla^{\nu(\sigma)} \otimes 1 + 1 \otimes \nabla^S$.

Define $E_\sigma := \Gamma\backslash\widetilde{E}_{\nu_\kappa(\sigma)}$ and denote the pull-back of $\nabla^{\nu_\kappa(\sigma)}$ by the same symbol. Let $\chi : \Gamma \to GL(V_\chi)$ be a finite-dimensional (possibly non-unitary) complex representation and let $E_\chi \to \mathcal{O}$ be the associated flat vector bundle over $\mathcal{O}$ as in Subsecton 3.3.2, equiped with the flat connection ∇^χ. Define

$$E_{\sigma,\chi} := E_\sigma \otimes E_\chi,$$

equip it with the connection

$$\nabla^{\sigma,\chi} := \nabla^\sigma \otimes 1 + 1 \otimes \nabla^\chi.$$

Let

$$\cdot : \mathfrak{p} \otimes \Delta_{2n} \to \Delta_{2n}$$

denote the Clifford multiplication. We extend its action to $V_{\nu(\sigma)} \otimes \Delta_{2n}$ and $(V_{\nu(\sigma)} \otimes \Delta_{2n}) \otimes V_\chi$ as follows:

$$e \cdot_\sigma (w \otimes s) := w \otimes (e \cdot s), \quad w \in V_{\nu(\sigma)}, e \in \mathrm{Cl}(\mathfrak{p}), s \in \Delta_{2n}.$$

$$e \cdot_{\sigma,\chi} ((w \otimes s) \otimes v) := (w \otimes (e \cdot s)) \otimes v, \quad w \in V_{\nu(\sigma)}, e \in \mathrm{Cl}(\mathfrak{p}), v \in V_\chi, s \in \Delta_{2n},$$

Consider an open subset U of $\mathcal{O}$ such that $E_\chi|_U$ is trivial. Then $E_{\sigma,\chi}|_U$ is isomorphic to the direct sum of $\mathrm{rank}(E_\chi)$ copies of $E_\sigma|_U$. Let v_j be the basis of flat sections of $E_\chi|_U$, then each $\varphi \in C^\infty(U, E_{\sigma,\chi}|_U)$ can be written as:

$$\varphi = \sum_{j=1}^{\mathrm{rank}(E_\chi)} \phi_j \otimes v_j,$$

where $\phi_j \in C^\infty(U, E_\sigma|_U)$. The Dirac operator $D(\sigma, \chi)$ acting on sections of $E_{\sigma,\chi}$ is defined as follows: for each φ as above,

$$D(\sigma,\chi)\varphi = \sum_{i=1}^{\dim \mathcal{O}} \sum_{j=1}^{\mathrm{rank}(E_\chi)} e_i \cdot_{\sigma,\chi} \left(\nabla^\sigma_{e_i} \phi_j \otimes v_j \right).$$

The Dirac operator $\widetilde{D}(\sigma)$ acting on sections of $\widetilde{E}_{\nu_k(\sigma)}$ is defined as follow:

$$\widetilde{D}(\sigma) f = \sum_{i=1}^{\dim \mathcal{O}} e_i \cdot_\sigma \nabla^\sigma_{e_i} f,$$

where $f \in C^\infty(\mathbb{H}^{2n+1}, V_{\nu(\sigma)} \otimes \Delta_{2n})$.

Note that $D(\sigma,\chi)^2$ a second order elliptic differential operator and by Theorem 3.1.8, its spectrum is discrete and there exist $R \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\mathrm{spec}(D(\sigma,\chi)^2) \in L := \Lambda_{[-\varepsilon,\varepsilon]} \cup B(R). \tag{3.67}$$

Selberg trace formula

In this subsection we verify that the Selberg trace formula can be applied to the operator $D(\sigma,\chi)e^{-tD(\sigma,\chi)^2}$. We define the operator $D(\sigma,\chi)e^{-tD(\sigma,\chi)^2}$ via the integral

$$D(\sigma,\chi)e^{-tD(\sigma,\chi)^2} := \frac{i}{2\pi}\int_B e^{-t\lambda}D(\sigma,\chi)(D(\sigma,\chi)^2-\lambda)^{-1}d\lambda \tag{3.68}$$

for $B=\partial L$ with L as in (3.67).

Proposition 3.4.18. *The right hand side of (3.68) converges.*

Proof. Follows from Theorem 3.1.8. □

Note that the lift of $D(\sigma,\chi)$ to $\mathbb{H}^{2n+1}$ splits into $\widetilde{D}(\sigma)\otimes \mathrm{Id}$ by the same arguments as in Sections 3.3.2 and 3.4.1. Also the operator $D(\sigma,\chi)e^{-tD(\sigma,\chi)}$ is an integral operator with smooth kernel, because $e^{-tD(\sigma,\chi)}$ is. By an analogy with the previous calculations we obtain:

Lemma 3.4.19. *Denote by $k_t^\sigma(\cdot)$ the convolution kernel of $\widetilde{D}(\sigma)e^{-t\widetilde{D}^2(\sigma)}$. Then we have*

$$\begin{aligned}\sum_{\lambda\in spec(D)}\lambda e^{-t\lambda^2} &= \dim(V_\chi)\mathrm{vol}(\Gamma\backslash G/K)\,\mathrm{tr}\,k_t^\sigma(e)+\\ &\sum_{\{\gamma\}\neq\{e\}}\mathrm{tr}\chi(\gamma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G}\mathrm{tr}k_t^\sigma(g^{-1}\gamma g)d\dot{g}.\end{aligned} \tag{3.69}$$

Analogously to Theorem 3.4.11, we get

$$\begin{aligned}\mathrm{Tr}_s(D(\sigma,\chi)e^{-tD(\sigma,\chi)^2}) &= \mathrm{vol}(\mathcal{O})\dim(V_\chi)\sum_{\sigma'\in\hat{M}}\int_{\mathbb{R}}P_{\sigma'}(i\lambda)\Theta_{\sigma',\lambda}(k_t^\sigma)d\lambda+\\ &\sum_{\sigma'\in\hat{M}}\sum_{[\gamma]\ \mathrm{elliptic}}\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathrm{tr}(\chi(\gamma))\sum_{\sigma'\in\hat{M}}\int_{\mathbb{R}}P^\gamma_{\sigma'}(i\lambda)\Theta_{\sigma',\lambda}(k_t^\sigma)d\lambda+\\ &\sum_{\sigma'\in\hat{M}}\sum_{[\gamma]\ \mathrm{hyperbolic}}\frac{\mathrm{tr}(\chi(\gamma))l(\gamma_0)}{2\pi D(\gamma)|\Gamma^1_\gamma|}\overline{\mathrm{tr}(\sigma'(\gamma))}\int_{\mathbb{R}}\Theta_{\sigma',\lambda}(k_t^\sigma)e^{-l(\gamma)\lambda}d\lambda.\end{aligned} \tag{3.70}$$

Proposition 3.4.20. *[Pfa13, Proposition 8.2], [MS89] Let $\sigma\in\hat{M}$, $k_{n+1}(\sigma)>0$. Then for $\lambda\in\mathbb{R}$ one has*

$$\Theta_{\sigma,\lambda}(k)=(-1)^n\lambda e^{-t\lambda^2},\quad \Theta_{w_0\sigma,\lambda}(k)=(-1)^{n+1}\lambda e^{-t\lambda^2}.$$

Moreover, if $\sigma'\in\hat{M}$ and $\sigma'\neq\{\sigma,w_0\sigma\}$, for every $\lambda\in\mathbb{R}$ one has $\Theta_{\sigma',\lambda}(k)=0$.

Applying Proposition 3.4.20 to (3.70), we get

$$\begin{aligned}(-1)^n \mathrm{Tr}_s\left(De^{-tD^2}\right) &= \mathrm{vol}(\mathcal{O})\dim(V_\chi)\int_{\mathbb{R}}(P_\sigma(i\lambda)-P_{w_0\sigma}(i\lambda))\lambda e^{-t\lambda^2}d\lambda+\\ &+\sum_{[\gamma]\text{ elliptic}}\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{\mathbb{R}}(P^\gamma_\sigma(i\lambda)-P^\gamma_{w_0\sigma}(i\lambda))\lambda e^{-t\lambda^2}d\lambda+\\ &+\sum_{[\gamma]\text{ hyperbolic}}C_2(\gamma)\frac{l(\gamma_0)}{2\pi}\left(L(\gamma,\sigma)-L(\gamma,w_0\sigma)\right)\int_{\mathbb{R}}\lambda e^{-t\lambda^2}e^{-l(\gamma)\lambda}d\lambda,\end{aligned} \tag{3.71}$$

Moreover, the first and the second summand in the right hand side of (3.71) vanish by the following two remarks.

Remark 3.4.21. *By [MP12, (2.22)],*

$$P_\sigma(i\nu)-P_{w_0\sigma}(i\nu)=0.$$

Remark 3.4.22. *By Lemma 3.3.27,*

$$P^\gamma_\sigma(i\nu)-P^\gamma_{w_0\sigma}(i\nu)=0.$$

We proceed as in Section 3.4.1. The operator $D(\sigma,\chi)\cdot(D(\sigma,\chi)^2+s^2)^{-1}$ is not of trace class, but we can choose coefficients c_j and s_j such that $D(\sigma,\chi)\cdot(D(\sigma,\chi)^2+s^2)^{-1}+\sum_j c_j D(\sigma,\chi)\cdot(D(\sigma,\chi)^2+s_j^2)^{-1}$ is of trace class. By the same arguments as in (3.60)-(3.62) and Remarks 3.4.21 and 3.4.22, we obtain

$$\begin{aligned}\mathrm{Tr}\left(D(\sigma,\chi)\cdot(D(\sigma,\chi)^2+s^2)^{-1}+\sum_j c_j D(\sigma,\chi)\cdot(D(\sigma,\chi)^2+s_j^2)^{-1}\right)=\\ \frac{1}{2s}\frac{S'_a(s,\sigma,\chi)}{S_a(s,\sigma,\chi)}+\sum_j\frac{c_j}{2s_j}\frac{S'_a(s_j,\sigma,\chi)}{S_a(s_j,\sigma,\chi)}\end{aligned} \tag{3.72}$$

The theorem below follows.

Theorem 3.4.23. *The antisymmetric Selberg zeta $S_a(s,\sigma,\chi)$ function has a meromorphic extension to $\mathbb{C}$. It has singularities at the points $\pm i\mu_k$ of order $\frac{1}{2}(d(\pm\mu_k,\sigma)-d(\mp\mu_k,\sigma))$, where μ_k is a non-zero eigenvalue of $D(\sigma,\chi)$ of multiplicity $d(\mu_k,\sigma)$.*

Using that $Z(s,\sigma,\chi)=S(s,\sigma,\chi)S_a(s,\sigma,\chi)$, we obtain

Theorem 3.4.24. *The Selberg zeta function has an meromorphic extension to $\mathbb{C}$. It has the following singularities:*

- *If* $\sigma = w_0\sigma$, *a sigularity at the points* $\pm i\sqrt{\lambda_k}$ *of order* $m_s(\lambda_k, \sigma)$, *where* λ_k *is a non-zero eigenvalue of* $A(\sigma, \chi)$ *and* $m_s(\lambda_k, \sigma)$ *is the graded dimension of the corresponding eigenspace.*

- *If* $\sigma \neq w_0\sigma$, *a singularity at the points* $\pm i\mu_k$ *of order* $\frac{1}{2}(m_s(\mu_k^2, \sigma) + d(\pm\mu_k, \sigma) - d(\mp\mu_k, \sigma))$. *Here* μ_k *is a non-zero eigenvalue of* $D(\sigma, \chi)$ *of multiplicity* $d(\mu_k, \sigma)$ *and* $m_s(\mu_k^2, \sigma)$ *is the graded dimension of the eigenspace* $A(\sigma, \chi)$ *corresponding to the eigenvalue* μ_k^2.

- *At the point* $s = 0$ *a singularity of order* $2m_s(0, \sigma)$ *if* $\sigma = w_0\sigma$ *and of order* $m_s(0, \sigma)$ *if* $\sigma \neq w_0\sigma$.

Chapter 4

The heat kernel

Let $E \to \mathcal{O}$ be from Definition 2.1.6, equip it with a Hermitian fibre metric and let Δ be the Bochner-Laplace operator acting on sections of E; assume that E is a complex orbibundle of rank k. Our goal is to construct and study the heat kernel for Δ. The main idea is to take an approximate solution of the heat equation from [BGV92, Theorem 2.26] and use the construction of a parametrix for the heat operator as in [DGGW08].

The results of this section are widely used throughout the thesis:

1. In Theorem 3.2.8, we needed the Weyl law from Theorem 4.2.7 for self-adjoint operators in order to establish the Weyl law for non self-adjoint operators ,

2. Lemma 3.2.9 used the Weyl law from Theorem 4.2.7 to establish the fast decay of certain coefficients,

3. The definition of spectral zeta function from Definition 5.1.3 uses the heat trace expansion from Lemma 4.2.1,

4. To establish the independence of the analytic torsion on the variation of metric in Corollary 5.2.12 we need to show that the constant term in the heat trace asymptotics in Lemma 4.2.5 vanishes.

4.1 Existence and uniqueness of the heat kernel

Definition 4.1.1. *We say that $K \in \Gamma((0,\infty) \times \mathcal{O} \times \mathcal{O}, E \boxtimes E^*)$ is a heat kernel, if it satisfies:*

1. *K is C^0 in all three variables, C^1 in the first, and C^2 in the second,*

2. *$(\frac{\partial}{\partial t} + \Delta_x)K(t, x, y) = 0$, where Δ_x denotes Δ that acts on the second variable,*

3. *$\lim_{t\to+0} K(t, x, \cdot) = \delta_x$ for all $x \in \mathcal{O}$, where δ_x is the Dirac distribution.*

Let $(\widetilde{U}_\alpha, G_U, \pi_U)$ be a chart of $\mathcal{O}$ of radius no less than $\epsilon > 0$. Denote

$$U_\epsilon := \{(x, y) \in \widetilde{U}_\alpha \times \widetilde{U}_\alpha \mid d(x, y) < \epsilon\},$$

where $d(x, y)$ is the distance between x and y. The following lemma follows from [BGV92, Theorem 2.26]:

Lemma 4.1.2. *For $l \in \mathbb{N} \cup \{0\}$ there exist $u_i(x, y) \in C^\infty(U_\epsilon, E \boxtimes E^*)$ with $i = 0, \ldots, l$, such that*

$$\left(\frac{\partial}{\partial t} + \Delta_x\right)\left[(4\pi t)^{-n/2} \cdot e^{-d(x,y)^2/4t} \cdot \sum_{i=0}^{l} u_i(x, y)t^i\right] = (4\pi)^{-n/2} t^{l-n/2} \cdot e^{-d(x,y)^2/4t} \cdot \Delta_x u_l(x, y). \tag{4.1}$$

Remark 4.1.3. *In [BGV92] the functions u_i were constructed for $\hat{H} : \Gamma(M, E \otimes |\Lambda|^{1/2}) \to \Gamma(M, E \otimes |\Lambda|^{1/2})$ instead of $H : \Gamma(M, E) \to \Gamma(M, E)$, where $|\Lambda|$ is density and M is a manifold.*

Remark 4.1.4. *In Lemma 4.1.2 we are still in the manifold setup.*

Remark 4.1.5. *As Δ_x commutes with isometries of U_ϵ and $d(\gamma x, \gamma y) = d(x, y)$ for $\gamma \in G_\alpha$, the following equality holds:*

$$u_i(\gamma x, y) = R_\gamma \circ u_i(x, \gamma^{-1} y) \circ R_{\gamma^{-1}}, \tag{4.2}$$

where R_γ is from (3.25).

Definition 4.1.6. *$F \in \Gamma((0, \infty) \times \mathcal{O} \times \mathcal{O}, E \boxtimes E^*)$ is a parametrix for the heat operator if:*

1. *F is C^∞-smooth,*

2. *$[(\frac{\partial}{\partial t} + \Delta_x)F](t, x, y) \in \Gamma((0, \infty) \times \mathcal{O} \times \mathcal{O}, E \boxtimes E^*)$ is a C^0-function in all three variables,*

3. *$\lim_{t\to 0} F(t, x, \cdot) = \delta_x$ for all $x \in \mathcal{O}$.*

The construction of a parametrix is as follows. Fix $\varepsilon > 0$ such that for each $x \in \mathcal{O}$, there exists a coordinate chart of radius ε centered at x. Cover $\mathcal{O}$ with finitely many such charts $(\widetilde{U}_\alpha, G_\alpha, \phi_\alpha)$, $\alpha \in I$. Identify $\widetilde{U}_\alpha$ with the unit ball, let $\widetilde{p}_\alpha$ be its center. Let

$$\widetilde{W}_\alpha := \{u \in \widetilde{U}_\alpha : d(u, \widetilde{p}_\alpha) \leq \epsilon/4\},$$

$$\widetilde{V}_\alpha := \{u \in \widetilde{U}_\alpha : d(u, \widetilde{p}_\alpha) \leq \epsilon/2\}.$$

Let p_α be the center of U_α, and let

$$W_\alpha := \{u \in U_\alpha : d(u, p_\alpha) \leq \epsilon/4\},$$

$$V_\alpha := \{u \in U_\alpha : d(u, p_\alpha) \leq \epsilon/2\}.$$

Definition 4.1.7. *For each $\alpha \in I$ and each non-negative integer m, define $\widetilde{H}_\alpha^{(m)} \in \Gamma(\mathbb{R}_+ \times \widetilde{U}_\alpha \times \widetilde{U}_\alpha, E \boxtimes E^*)$ by*

$$\widetilde{H_\alpha}^{(m)}(t, \widetilde{x}, \widetilde{y}) := (4\pi t)^{-n/2} e^{-d^2(\tilde{x},\tilde{y})/4t} \sum_{i=0}^{m} t^i u_i(\tilde{x}, \tilde{y}),$$

where the u_i are from Lemma 4.1.2.

Proposition 4.1.8. *The sum*

$$H_\alpha^{(m)}(t, \tilde{x}, \tilde{y}) := \sum_{\gamma \in G_\alpha} \widetilde{H_\alpha}^{(m)}(t, \tilde{x}, \gamma\tilde{y}) \circ R_\gamma \tag{4.3}$$

descends to a function $H_\alpha^{(m)} \in \Gamma(\mathbb{R}_+ \times U_\alpha \times U_\alpha, E \boxtimes E^*)$.

Remark 4.1.9. *As we will see further, the sum (4.3) resembles (4.6).*

Proof of Proposition 4.1.8. In order to prove the proposition, we need to show that $H_\alpha^{(m)}(t, \tilde{x}, \tilde{y})$ is G_α-invariant in the second and the third variable.

$$H_\alpha^{(m)}(t, \tilde{x}, \gamma'\tilde{y}) = \sum_{\gamma \in G_\alpha} \widetilde{H_\alpha}^{(m)}(t, \tilde{x}, \gamma\gamma'\tilde{y}) \circ R_\gamma = \left(\sum_{g \in G_\alpha} \widetilde{H_\alpha}^{(m)}(t, \tilde{x}, g\tilde{y}) \circ R_g \right) \circ R_{\gamma'^{-1}} = H_\alpha^{(m)}(t, \tilde{x}, \tilde{y}) \circ R_{\gamma'^{-1}}. \tag{4.4}$$

Above the second equality is due to the change of variables $g = \gamma\gamma'$.

$$H_\alpha^{(m)}(t, \gamma'\tilde{x}, \tilde{y}) = \sum_{\gamma \in G_\alpha} \widetilde{H_\alpha}^{(m)}(t, \gamma'\tilde{x}, \gamma\tilde{y}) \circ R_\gamma =$$

$$\sum_{\gamma \in G_\alpha} R_{\gamma'} \circ \widetilde{H_\alpha}^{(m)}(t, \tilde{x}, \gamma'^{-1}\gamma\tilde{y}) \circ R_{\gamma'^{-1}} \circ R_\gamma = R_{\gamma'} \circ \left(\sum_{g \in G_\alpha} \widetilde{H_\alpha}^{(m)}(t, \tilde{x}, g\tilde{y}) \circ R_g \right) =$$

$$R_{\gamma'} \circ H_\alpha^{(m)}(t, \tilde{x}, \tilde{y}).$$

Above the second equality is due to (4.2), and the third follows from the change of variables $g = \gamma'^{-1}\gamma$. $\square$

Let $\psi_\alpha : \mathcal{O} \to \mathbb{R}$ be a C^∞ cut-off function, which is identically one on V_α and is supported in W_α. Let $\{\eta_\alpha\}$ be a partition of unity on $\mathcal{O}$ with $\operatorname{supp}(\eta_\alpha) \subset \overline{U_\alpha}$.

Definition 4.1.10. *Define $H^{(m)} \in \Gamma(\mathbb{R}_+ \times \mathcal{O} \times \mathcal{O}, E \boxtimes E^*)$ by*

$$H^{(m)}(t,x,y) := \sum_\alpha \psi_\alpha(x)\eta_\alpha(y)H_\alpha^{(m)}(t,x,y).$$

The following lemma is implied by [DGGW08, p. 13]:

Lemma 4.1.11. *$H^{(m)}$ is a parametrix for the heat operator on $\mathcal{O}$ if $m > n/2$.*

From this point, the construction of the heat kernel from the parametrix $H^{(m)}$ is carried out as in [BGM71, p. 210]. The uniqueness of the heat kernel follows from [Don76, Theorem 3.3].

4.2 Computation of the heat asymptotics

Lemma 4.2.1. *Let $E \to \mathcal{O}$ be an orbibundle over a good Riemannian orbifold; moreover, assume that $\mathcal{O} = \Gamma\backslash M$, where M is a compact manifold and Γ is a finite group of orientation-preserving isometries of M. Denote by $\widetilde{E}$ the lift of E to M, and let $K(t,\widetilde{x},\widetilde{y})$ and $K^{\mathcal{O}}(t,x,y)$ be the heat kernels on $\widetilde{E}$ and E, respectively. Then*

$$\int_{\mathcal{O}} \operatorname{tr} K^{\mathcal{O}}(t,x,x)\, d\mathrm{vol}_{\mathcal{O}}(x) \sim I_e(t) + \sum_{\gamma\in\Gamma,\gamma\neq e} I_\gamma(t), \quad t \to 0, \tag{4.5}$$

where

$$I_e(t) \sim t^{-\dim(\mathcal{O})/2}\sum_{k=0}^{\infty} a_k t^k, \quad t \to 0,$$
$$I_\gamma(t) \sim t^{-\dim(N_\gamma)/2}\sum_{k=0}^{\infty} a_k^\gamma t^k, \quad t \to 0.$$

Above a_k, a_k^γ are some coefficients in $\mathbb{C}$, and N_γ is the fixed point set of γ in M.

Proof. Let $\pi : M \mapsto \mathcal{O}$ be a natural projection. Then

$$K^{\mathcal{O}}(t,x,y) = \sum_{\gamma\in\Gamma} K(t,\tilde{x},\gamma\tilde{y}) \circ R_\gamma, \tag{4.6}$$

where $\tilde{x}$ and $\tilde{y}$ are elements of $\pi^{-1}(x)$ and $\pi^{-1}(y)$, respectively. Then

$$\begin{aligned}\int_{\mathcal{O}} \operatorname{tr} K^{\mathcal{O}}(t,x,x)\, \mathrm{dvol}_{\mathcal{O}}(x) = \frac{1}{|\Gamma|}\int_M \operatorname{tr} K(t,\tilde{x},\tilde{x})\, \mathrm{dvol}_M(\widetilde{x}) + \\ \frac{1}{|\Gamma|}\sum_{e\neq\gamma\in\Gamma}\int_M \operatorname{tr}\left(K(t,\tilde{x},\gamma(\tilde{x}))\circ R_\gamma\right)\, \mathrm{dvol}_M(\widetilde{x}).\end{aligned} \tag{4.7}$$

where $d\mathrm{vol}_M$ and $d\mathrm{vol}_{\mathcal{O}}$ denotes the Riemannian measure on M and $\mathcal{O}$, respectively. We study the asymptotic behavior of (4.7) following [Gil95]. The asymptotic expansion of the first summand in the right hand side of (4.7) follows from the following theorem:

Theorem 4.2.2. *[Gil95, Theorem 1.7.6] There exist $a_k(x) : M \to \mathbb{C}$, $k = 0,\dots,\infty$ such that*

$$\int_M \operatorname{tr} K(t,x,x)dx \sim \sum_{k=0}^{\infty} t^{(k-\dim(M))/2}\int_M a_k(x)\, \mathrm{dvol}_M(x), \quad t\to 0.$$

Remark 4.2.3. *The leading coefficient is given by $a_0(x) = (4\pi)^{-\dim(M)/2}$.*

The asymptotic expansion of the second summand of the right hand side of (4.7) follows with minor modification from [Gil95, Lemma 1.8.2]:

Theorem 4.2.4. *There exist $a_n^\gamma(x) : N_\gamma \to \mathbb{C}$, $k = 0,\dots,\infty$ such that*

$$\int_M \operatorname{tr}\left(K(t,\widetilde{x},\gamma\widetilde{x})\circ R_\gamma\right)\, \mathrm{dvol}_M(\widetilde{x}) \sim \sum_{n=0}^{\infty} t^{(n-\dim(N_\gamma))/2}\int_{N_\gamma} a_n^\gamma(x)\mathrm{dvol}_{N_\gamma}(x),$$

where $\mathrm{dvol}_{N_\gamma}(x)$ denotes the Riemannian measure on N_γ.

Proof. By [Gil95, Lemma 1.8.2],

$$\int_M \operatorname{tr}\left(R_\gamma^{-1}\circ K(t,\gamma\widetilde{x},\widetilde{x})\right)\, \mathrm{dvol}_M(\widetilde{x}) \sim \sum_{n=0}^{\infty} t^{(n-\dim(N_\gamma))/2}\int_{N_\gamma} a_n^\gamma(x)\mathrm{dvol}_{N_\gamma}(x). \tag{4.8}$$

Note that

$$\begin{aligned}\operatorname{tr}\left(R_\gamma^{-1}\circ K(t,\gamma\widetilde{x},\widetilde{x})\right) = \operatorname{tr}\left(R_\gamma\circ R_\gamma^{-1}\circ K(t,\gamma\widetilde{x},\widetilde{x})\circ R_\gamma^{-1}\right) = \\ \operatorname{tr}\left(K(t,\gamma\widetilde{x},\widetilde{x})\circ R_\gamma^{-1}\right).\end{aligned} \tag{4.9}$$

Let $\widetilde{y} = \gamma\widetilde{x}$, then

$$\int_M \operatorname{tr}\left(K(t,\gamma\widetilde{x},\widetilde{x})\circ R_\gamma^{-1}\right)\mathrm{dvol}_M(\widetilde{x}) = \int_M \operatorname{tr}\left(K(t,\widetilde{y},\gamma^{-1}\widetilde{y})\circ R_{\gamma^{-1}}\right)\mathrm{dvol}_M(\widetilde{y}). \tag{4.10}$$

Putting together (4.8), (4.9) and (4.10) implies Theorem 4.2.4. ☐

Applying Theorems 4.2.2 and 4.2.4 finishes the proof of Lemma 4.2.1. ☐

Lemma 4.2.5. *Let $E \to \mathcal{O}$ be as above, let B_0 be a zeroth order pseudodifferential operator and let Δ_p be the Bochner-Laplacian acting on p-forms with coefficients in E. Then*

$$\operatorname{Tr}\left(B_0 e^{-t\Delta_p}\right) \sim t^{-\dim(\mathcal{O})/2} \sum_{k=0}^{\infty} a_k t^k + \sum_{\gamma \in \Gamma, \gamma \neq e} t^{-\dim(N_\gamma)/2} \sum_{k=0}^{\infty} a_k^\gamma t^k, \quad t \to 0$$

for some $a_k^\gamma, a_k \in \mathbb{C}$, $k = 0, \ldots, \infty$.

Proof. Analogously to the proof of Lemma 4.2.1. □

Remark 4.2.6. *For a similar result, see [LR91, pp. 438-440].*

As usual, we establish the Weyl law.

Theorem 4.2.7. *The counting function $N(r, \Delta)$ for Δ satisfies*

$$N(r, \Delta) = \frac{\operatorname{rk}(E)\operatorname{vol}(\mathcal{O})}{(4\pi)^{(\dim \mathcal{O})/2}\Gamma((\dim \mathcal{O})/2 + 1)} r^{(\dim \mathcal{O})/2} + o(r^{(\dim \mathcal{O})/2}), \quad r \to \infty.$$

Proof. Follows from the Tauberian theorem, Lemma 4.2.1 and Remark 4.2.3. □

Chapter 5

Analytic torsion of compact orbifolds

5.1 Definition of the analytic torsion

Let $E \to \mathcal{O}$ be a flat vector orbibundle over a compact good Riemannian orbifold $(\mathcal{O}, g)$; pick a Hermitian fiber metric h in E. Denote by $\Delta_p(h,g)$ the Bochner Laplace operator acting on E-valued p-forms on $\mathcal{O}$ with respect to h. By [Buc99, Theorem 3.5], it is essentially selfadjoint; another possibility is just to take the Friedrichs extension. We denote its selfadjoint extension by the same symbol. Note that $\mathrm{spec}(\Delta_p(h,g))$ is semi-bounded, hence we can define $e^{-t\Delta_p(h,g)}$ by a suitable Dunford integral as in [Gil95].

Definition 5.1.1. *The zeta function $\zeta_p(s;h,g)$ is*

$$\zeta_p(s;h,g) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta_p(h,g)}(1-P))dt,$$

where P is the orthogonal projection to $\ker \Delta_p(h,g)$ and $\Gamma(s)$ is the gamma-function.

Assumption 5.1.2. *From now on and till the end of the thesis assume that $\Delta_p(h,g) > 0$ and, hence, $P = 0$.*

Note that by Lemma 4.2.1, $\mathrm{Tr}\left(e^{-t\Delta_p(h,g)}\right)$ admits the following asymptotic expansion as $t \to +0$:

$$\mathrm{Tr}\left(e^{-t\Delta_p(h,g)}\right) \sim t^{-\dim(\mathcal{O})/2} \sum_{k=0}^\infty a_k t^k + \sum_{\gamma\in\Gamma, \gamma\neq e} t^{-\dim(N_\gamma)/2} \sum_{k=0}^\infty a_k^\gamma t^k.$$

By Assumption 5.1.2,

$$\mathrm{Tr}\left(e^{-t\Delta_p(h,g)}\right) \sim O(e^{-ct}), \quad t \to \infty$$

for some $c > 0$. Hence by Proposition 2.12.2, the zeta function $\zeta_p(s; h, g)$ admits a meromorphic extension to $\mathbb{C}$; moreover, it is regular at $s = 0$.

Definition 5.1.3. *The analytic torsion $T_{\mathcal{O}}(h, g)$ is defined as*

$$\log T_{\mathcal{O}}(h, g) := \frac{1}{2} \sum_{p=1}^{\dim \mathcal{O}} (-1)^p p \frac{d}{ds} \, \zeta_p(s; h, \rho)\big|_{s=0} \, .$$

Fix an admissible metric h from Definition 2.7.1.

Definition 5.1.4. *Further denote $T_{\mathcal{O}}(\rho) := T_{\mathcal{O}}(h, g)$; here ρ indicates the dependence of the analytic torsion on the orbibundle E_ρ, and $T_{\mathcal{O}}(h, g)$ is from Definition 5.1.3.*

5.2 Analytic torsion under metric variation

In this section we study the Ray-Singer analytic torsion $T_{\mathcal{O}}(h, g)$ of a compact odd-dimensional good Riemannian orbifold $(\mathcal{O}, g)$, where $\mathcal{O} = G_U \backslash \widetilde{U}$ for a finite group G_U and a compact manifold $\widetilde{U}$. The main goal is to establish the invariance of the analytic torsion under certain deformations of the metric g, which we will now specify.

5.2.1 Deformations of the metric

For a moment we let $\mathcal{O} = G_U \backslash \widetilde{U}$ be a compact orbifold, where $\widetilde{U}$ is a (not necessarily compact) manifold, and G_U is a (not necessarily finite) group. Consider an orbifold atlas of $\mathcal{O}$ consisting of charts $(\widetilde{U}_\alpha, G_\alpha, \phi_\alpha)$ as in Definition 2.1.2.

Definition 5.2.1. *By a smooth family of metrics on $\mathcal{O}$ we mean a collection of G_U-invariant metrics $g(u)$, $u \in [0, 1]$ on $\widetilde{U}$, depending smoothly on u.*

Example 5.2.2. *Let g be a metric on an orbifold $\mathcal{O}$ and $m \in \mathbb{R}$. The family of metrics $\lambda \cdot g$, $\lambda \in [1, m]$ is a smooth family of metrics.*

5.2.2 Deformation of the analytic torsion

Let $\mathcal{O}$ be an orbifold equipped with a smooth family of metrics $g(u)$, $u \in [0, 1]$ from Definition 5.2.1, and let $E \to \mathcal{O}$ be a flat orbibundle equipped with a Hermitian metric h.

Definition 5.2.3. *Denote by $\Delta_k(u)$ the Bochner-Laplace operator acting on E-valued k-forms of $\mathcal{O}$.*

Definition 5.2.4. *To simplify the notations, put*

$$L^2\Omega^k(u)(\mathcal{O}) := L^2\Omega^k(u)(\mathcal{O},E), \quad H^2\Omega^k(u)(\mathcal{O}) := H^2\Omega^k(u)(\mathcal{O},E).$$

Note that for different values of u the Laplacians $\Delta_k(u)$ act on different Hilbert spaces $L^2\Omega^k(u)(\mathcal{O})$. However, we can identify these spaces by a natural isometry

$$T(u) : L^2\Omega^k(u)(\mathcal{O}) \mapsto L^2\Omega^k(0)(\mathcal{O}), \quad T(u) : f(x) \mapsto \left(\frac{\det g(0)(x)}{\det g(u)(x)}\right)^{\frac{1}{4}} \cdot f(x)$$

for $x \in \mathcal{O}$ and $u \in [0,1]$.

Proposition 5.2.5. *The operator $T(u)$ maps $H^2\Omega^k(u)(\mathcal{O})$ to $H^2\Omega^k(0)(\mathcal{O})$.*

Proof. Take a partition of unity; then Definition 5.2.1 together with the same result for manifolds imply the proposition. □

Remark 5.2.6. *The domain of $\Delta_k(u)$ is*

$$\mathrm{dom}\,(\Delta_k(u)) = H^2\Omega^k(u)(\mathcal{O})$$

Together Proposition 5.2.5 and Remark 5.2.6 imply:

Proposition 5.2.7. *The operator $T(u)$ is a well-defined map from $\mathrm{dom}\,(\Delta_k(u))$ to $\mathrm{dom}\,(\Delta_k(0))$.*

Define the self-adjoint operators

$$H_k(u) := T(u) \circ \Delta_k(u) \circ T(u)^{-1} \tag{5.1}$$

with the fixed domain $\mathrm{dom}\, H_k(u) = H^2\Omega^k(0)(\Omega)$. To establish the independence of the analytic torsion on u, we follow [MV10, Subsection 3.2], the key steps of whose proof we repeat here for the reader's convenience.

Lemma 5.2.8. *We have that:*

$$\frac{\partial}{\partial u}\mathrm{Tr}\left(e^{-t\Delta_k(u)}\right)\Big|_{u=u_0} = -t\cdot\mathrm{Tr}\left(\frac{\partial\Delta_k(u)}{\partial u}\Big|_{u=u_0} \circ e^{-t\Delta_k(u_0)}\right).$$

Proof. From the semigroups properties it follows that for any $u, u_0 \in [0,1]$,

$$\frac{e^{-tH_k(u)} - e^{-tH_k(u_0)}}{u-u_0} = \int_0^t \frac{\partial}{\partial s}\left(\frac{e^{-(t-s)H_k(u_0)}e^{-sH_k(u)}}{u-u_0}\right) ds =$$
$$\int_0^t e^{-(t-s)H_k(u_0)} \cdot \frac{H_k(u_0) - H_k(u)}{u-u_0} \cdot e^{-sH_k(u)} ds.$$

Taking $u \to u_0$ gives

$$\frac{\partial}{\partial u}\operatorname{Tr} e^{-tH_k(u)}\bigg|_{u=u_0} = -\int_0^t \operatorname{Tr}\left(e^{-(t-s)H_k(u_0)} \cdot \frac{\partial H_k(u)}{\partial u}\bigg|_{u=u_0} \cdot e^{-sH_k(u_0)}\right) ds = -t \cdot \operatorname{Tr}\left(\frac{\partial H_k(u)}{\partial u}\bigg|_{u=u_0} \cdot e^{-tH_k(u_0)}\right).$$

The previous equation and (5.1) imply:

$$\frac{\partial}{\partial u}\operatorname{Tr}\left(e^{-tH_k(u)}\right)\bigg|_{u=u_0} = -t \cdot \operatorname{Tr}\left(\frac{\partial T(u)}{\partial u}\bigg|_{u=u_0} \circ \Delta_k(u_0) \circ e^{-t\Delta_k(u_0)} \circ T^{-1}(u_0)\right) - t \cdot \operatorname{Tr}\left(\Delta_k(u_0) \circ \left(\frac{\partial T(u)}{\partial u}\bigg|_{u=u_0}\right)^{-1} \circ T(u_0) \circ e^{-t\Delta_k(u_0)}\right) - t \cdot \operatorname{Tr}\left(\frac{\partial \Delta_k(u)}{\partial u}\bigg|_{u=u_0} \circ e^{-t\Delta_k(u_0)}\right). \tag{5.2}$$

Remark 5.2.9. *Although $\Delta_k(u)$ have different domains for different u, the operator on the right hand side of (5.2) is well-defined. The reason is as follows: in the mentioned equation $\Delta_k(u)$ acts on the range of $e^{-t\Delta_k(u_0)}$, that is a smoothing operator.*

The first and the second summand in the right hand side of (5.2) cancel, as

$$\begin{aligned} &\operatorname{Tr}\left(\Delta_k(u_0) \circ \left(\frac{\partial T(u)}{\partial u}\bigg|_{u=u_0}\right)^{-1} T(u_0) \circ e^{-t\Delta_k(u_0)}\right) = \\ &\operatorname{Tr}\left(e^{-(t/2)\Delta_k(u_0)}\Delta_k(u_0) \circ \left(\frac{\partial T(u)}{\partial u}\bigg|_{u=u_0}\right)^{-1} T(u_0) \circ e^{-(t/2)\Delta_k(u_0)}\right) = \\ &\operatorname{Tr}\left(\left(\frac{\partial T(u)}{\partial u}\bigg|_{u=u_0}\right)^{-1} T(u_0) \circ \Delta_k(u_0) \circ e^{-t\Delta_k(u_0)}\right) = \\ &-\operatorname{Tr}\left(T^{-1}(u_0)\,\frac{\partial T(u)}{\partial u}\bigg|_{u=u_0} \circ \Delta_k(u_0) \circ e^{-t\Delta_k(u_0)}\right). \end{aligned} \tag{5.3}$$

Together (5.2) and (5.3) imply the statement of the lemma. □

Let $*_u$ denote the Hodge-star operator associated with $g(u)$, and put $\alpha_u^k := *_u^{-1} \cdot \frac{\partial *}{\partial u}$, where k denotes the restriction to the forms of degree k. By the arguments as in [RS71, p.153], Lemma 5.2.8 implies

$$\sum_{k=0}^{\dim \mathcal{O}} (-1)^k \cdot k \cdot \frac{\partial}{\partial u}\operatorname{Tr}\left(e^{-t\Delta_k(u)}\right)\bigg|_{u=u_0} = t \cdot \frac{\partial}{\partial t}\sum_{k=0}^{\dim \mathcal{O}} (-1)^k \cdot \operatorname{Tr}\left(\alpha_u^k e^{-t\Delta_k(u)}\right). \tag{5.4}$$

Suppose that $\ker \Delta_k(u) = \{0\}$ for all k and u and put

$$f(u,s) := \frac{1}{2} \sum_{k=0}^{\dim \mathcal{O}} (-1)^k \cdot k \cdot \frac{1}{\Gamma(k)} \int_0^\infty t^{s-1} \cdot \mathrm{Tr}\left(\alpha_u^k e^{-t\Delta_k(u)}\right) dt. \tag{5.5}$$

Remark 5.2.10. *By definition of the analytic torsion,*

$$\log T_{\mathcal{O}}(h, g(u)) = \left.\frac{\partial}{\partial s}\right|_{s=0} f(u,s). \tag{5.6}$$

By the exponential decay of the heat trace we can differentiate the right hand side of (5.5) with respect to u by differentiating inside the integral sign. Together with (5.4), it implies:

$$\frac{\partial f(u,s)}{\partial u} = \frac{1}{2} \sum_{k=0}^{\dim \mathcal{O}} (-1)^k \cdot \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \mathrm{Tr}\,(\alpha_u^k e^{-t\Delta_k(u)}) dt. \tag{5.7}$$

Differentiating by parts, we obtain

$$\frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \mathrm{Tr}\,(\alpha_u^k e^{-t\Delta_k(u)}) dt = (-s) \cdot \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \cdot \mathrm{Tr}\,(\alpha_u^k e^{-t\Delta_k(u)}) dt \tag{5.8}$$

By Lemma 4.2.5,

$$\mathrm{Tr}\left(\alpha_u^k e^{-t\Delta_k(u)}\right) \sim \sum_{k=0}^{\infty} c_k t^{-\dim(\mathcal{O})/2+k} + \sum_{e \neq \gamma \in G_U} \sum_{k=0}^{\infty} d_k t^{-\dim N_\gamma/2+k} \tag{5.9}$$

for some $c_k, d_k \in \mathbb{C}$ as $t \to 0$. Putting together (5.7), (5.6), (5.8), (5.9) and proceeding as in [RS71], we obtain the following statement:

Theorem 5.2.11. *Let $\mathcal{O}$ be a good Riemannian orbifold, and let $g(u), u \in [0,1]$ be a smooth family of metrics on $\mathcal{O}$. Suppose that $\ker(\Delta_k(u)) = \{0\}$ for all $u \in [0,1]$ and $k = 1, \ldots \dim \mathcal{O}$. Furthermore, let $l_k(u)$ denote the constant term of the asymptotic expansion (5.9). Then*

$$\frac{\partial}{\partial u} \log T_{\mathcal{O}}(h, g(u)) = -1/2 \sum_{k=0}^{\dim \mathcal{O}} (-1)^k \, l_k(u).$$

Corollary 5.2.12. *Assume $\mathcal{O} = G_U \backslash \widetilde{U}$, $\dim \mathcal{O}$ is odd and the fixed point set of every element $\gamma \in G_U$ in $\widetilde{U}$ is odd-dimensional, for example $\mathcal{O} = \Gamma \backslash \mathbb{H}^{2n+1}$; see Proposition 2.3.5. Then $l_k(u) = 0$ for $k = 0, \ldots, \dim \mathcal{O}$, and $\log T_{\mathcal{O}}(h, g(u))$ does not depend on u.*

Further we will apply Corollary 5.2.12 to the families of metrics $g(u)$ from Example 5.2.2.

5.3 L^2-torsion

In this section we recall the notion of L^2-torsion. Our main goal is to establish the analytic behavior of the analytic torsion, and as a by-product we will study the asymptotic behavior of the L^2-torsion.

Let $\mathcal{O} = \Gamma\backslash\mathbb{H}^{2n+1}$ be a compact hyperbolic orbifold, let $\widetilde{\Delta}_p(\tau)$ be as in Definition 2.7.3 and $\tau(m)$ be as in Definition 2.6.2. Denote by $\mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\tau(m))})$ the Γ-trace of $e^{-t\widetilde{\Delta}_p(\tau(m))}$ on $\mathbb{H}^{2n+1}$ as in [Lot92]. It follows that

$$\mathrm{Tr}_\Gamma\, e^{-t\widetilde{\Delta}_p(\tau(m))} = \mathrm{vol}(\mathcal{O}) h_t^{\tau(m),p}(1),$$

where $h_t^{\tau(m),p}$ is as in (2.18); see [MP11]. Therefore, we have that

$$\sum_{p=1}^{2n+1} (-1)^p p \cdot \mathrm{Tr}_\Gamma\, e^{-t\widetilde{\Delta}_p(\tau(m))} = I(t, \tau(m)),$$

where $I(t,\tau(m)) := \mathrm{vol}(\mathcal{O}) \cdot k_t^{\tau(m)}(e)$, where $k_t^{\tau(m)}(e)$ is as in (2.19) with the representation $\tau = \tau(m)$ from Definition 2.6.2.

Definition 5.3.1. *The L^2-torsion is defined by*

$$\log T_{\mathcal{O}}^{(2)}(\tau(m)) := \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_{\mathbb{R}} t^{s-1} \sum_{p=1}^{2n+1} (-1)^p p \cdot \mathrm{Tr}_\Gamma\, e^{-t\widetilde{\Delta}_p(\tau(m))} \right)\Bigg|_{s=0}.$$

5.4 Asymptotic behavior of analytic and L^2-torsion

In this section we establish the asymptotic behavior of the analytic and the L^2-torsion for odd-dimensional hyperbolic orbifolds $\mathcal{O} = \Gamma\backslash\mathbb{H}^{2n+1}$. We will refer to some technical lemmas from Section 5.5. Our main results are Theorems 5.4.2 and 5.4.3.

Definition 5.4.1. *A pseudopolynomial $PE(m)$ of degree $\leq \frac{d^2+d+2}{2}$ is a sum of type*

$$\sum_{j=0}^{\frac{d^2+d+2}{2}} \sum_{k=0}^{K} C_{j,k} m^j e^{im\phi_{j,k}},$$

where $C_{j,k}, \phi_{j,k} \in \mathbb{R}$, $K \in \mathbb{N}$ are constants depending on elliptic elements of Γ.

Theorem 5.4.2. *Let $\tau(m)$ be as in Definition 2.6.2. Then there exists $c_1 > 0$ such that*

$$\log T_{\mathcal{O}}(\tau(m)) = PI(m) + PE(m) + O(e^{-c_1 m}), \quad m \to \infty.$$

Above $PI(m)$ is a polynomial in m of degree $\frac{n^2+n+2}{2}$; $PE(m)$ is a pseudopolynomial in m of degree $\leq \frac{d^2+d+2}{2}$ with $2d+1$ being the maximal dimension of fixed point sets of elements $\gamma \in \Gamma$, $\gamma \neq e$ in $\mathbb{H}^{2n+1}$.

Theorem 5.4.3. *We have that*

$$\log T^{(2)}_{\mathcal{O}}(\tau(m)) = PI(m), \quad m \to \infty,$$

where $PI(m)$ is the polynomial from Theorem 5.4.2.

Definition 5.4.4. *By [MP11, Proposition 5.5], the Mellin transform*

$$\int_0^\infty t^{s-1} I(t, \tau(m)) dt$$

of $I(t,\tau(m))$ is a meromorphic functions of $s \in \mathbb{C}$, which is regular at $s = 0$. Denote by $MI(\tau(m))$ its value at $s = 0$.

We would like to investigate the Mellin transform

$$\int_0^\infty t^{s-1} E(t, \tau(m)) dt$$

of $E(t,\tau)$ and examine its behavior at $s = 0$. To do this, we use the calculation of the orbital integral from Lemma 1.1.3.

Remark 5.4.5. *Although the set-up, in which we obtained Lemma 1.1.3, deals with the Selberg trace formula applied to certain non-selfadjoint operators, the non self-adjointness does not matter as long as we consider only the orbital integrals.*

Lemma 1.1.3 and [MP11, Proposition 5.5] imply the following proposition:

Proposition 5.4.6. *The Mellin transform*

$$\int_0^\infty t^{s-1} E(t, \tau(m)) dt$$

of $E(t,\tau(m))$ is is a meromorphic function of $s \in \mathbb{C}$, which is regular at $s = 0$.

Definition 5.4.7. *Denote by $ME(\tau(m)))$ its its value at $s = 0$.*

Proof of Proposition 5.4.6. The crucial step of [MP11, Proposition 5.5] is to use that $P_{\sigma'}(i\lambda)$ from Theorem 3.3.26 are polynomials in λ. By Lemma 1.1.3, we have that $P^{\gamma}_{\sigma'}(i\lambda)$ are polynomials in λ, that analogously to [MP11, Proposition 5.5] proves the proposition. □

Lemma 5.4.8.

$$\log T_{\mathcal{O}}(\tau(m)) = \frac{1}{2}\left(MI(\tau(m)) + ME(\tau(m))\right) + O(e^{-c_1 m})$$

as $m \to \infty$.

Lemma 5.4.9. *There exists* $m' \in \mathbb{N}$ *such that for any* $m > m'$, *the representation* $\tau(m)$ *satisfies Assumption 5.1.2, that is*

$$\Delta^p(\rho(m)) > 0.$$

Proof of Lemma 5.4.9. The proof follows from the following two facts. First, [Pfa12, Lemma 9.2] proves the same result for manifolds; and second, by the Selberg lemma, our orbifold $\Gamma\backslash\mathbb{H}^{2n+1}$ is finitely covered by a manifold. □

Proof of Lemma 5.4.8. Lemma 5.4.9 together with Corollary 5.2.12 the analytic torsion $T_{\mathcal{O}}(\tau(m); h, g)$ is invariant under smooth deformation of the metric g. As in Example 5.2.2 we can rescale the metric by $\sqrt{m}$ or, equivalently, replace $\Delta_p(\tau(m))$ by $\frac{1}{m}\Delta_p(\tau(m))$, so that (1.2) becomes

$$\log T_{\mathcal{O}}(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\bigg|_{s=0}. \tag{5.10}$$

Note that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_1^{\infty} t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\bigg|_{s=0} = \int_1^{\infty} t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt. \tag{5.11}$$

To continue the proof, we need to introduce the following lemma:

Lemma 5.4.10. *[MP11, Proposition 5.3] Let* H^0_t *be the heat kernel of the Laplacian* $\widetilde{\Delta}_0$ *on* $C^{\infty}(\mathbb{H}^{2n+1})$. *Then there exists* $m_0 \in \mathbb{N}$ *and* $C > 0$ *such that for all* $m \geq m_0$, $g \in G$, $t \in (0, \infty)$ *and* $p = \{0, \dots, 2n+1\}$, *one has*

$$\left|h_t^{\tau(m),p}(g)\right| \leq C \cdot \dim(\tau(m)) \cdot e^{-tm^2/2} \cdot H^0_t(g).$$

Proposition 5.4.11. *We have that*

$$\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt = O(e^{-m/8}), \quad m \to \infty. \tag{5.12}$$

Proof of Proposition 5.4.11. By Lemma 5.4.10, applied to $h_{t/m}^{\tau(m),p}(g)$ instead of $h_t^{\tau(m),p}(g)$, and the Selberg trace formula,

$$\begin{aligned}\left|K\left(\frac{t}{m}, \tau(m)\right)\right| &\le C \cdot e^{-mt/2} \cdot \dim(\tau(m)) \cdot \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} H^0_{t/m}(g^{-1}\gamma g) d\dot{g} \\ &= C \cdot e^{-\frac{m}{2}t} \cdot \dim(\tau(m)) \cdot \operatorname{Tr}(e^{-\frac{t}{m}\Delta_0}),\end{aligned} \tag{5.13}$$

where Δ_0 denotes the Laplacian on $C^\infty(\Gamma\backslash\mathbb{H}^{2n+1})$. By Lemma 4.2.1 and Remark 4.2.3,

$$\operatorname{Tr}(e^{-\frac{t}{m}\Delta_0}) = C_1 \cdot \operatorname{vol}(\Gamma\backslash\mathbb{H}^{2n+1}) \cdot m^{(2n+1)/2} + O(m^{(2n-1)/2}), \quad m \to \infty.$$

The last equation and (5.13) imply

$$\left|K\left(\frac{t}{m}, \tau(m)\right)\right| \le C_2 \cdot \tau(m) \cdot e^{-mt/2}, \quad t \ge 1$$

and

$$\left|\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right| \le C_3 \cdot \dim \tau(m) \cdot e^{-m/4} \int_1^\infty t^{-1} e^{-m^2 t/4} dt. \tag{5.14}$$

Recall that by a consequence of Weyl's dimension formula,

$$\dim(\tau(m)) = C \cdot m^{\frac{n(n+1)}{2}} + O(m^{\frac{n(n+1)}{2}-1}), \quad m \to \infty,$$

that together with (5.14) imply (5.12). □

Putting together (5.10), (5.11) and (5.12), we obtain

$$\log T_{\mathcal{O}}(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\Bigg|_{s=0} + O(e^{-m/8}). \tag{5.15}$$

We need to estimate $K(t/m, \tau(m))$ for $0 < t \le 1$.

Proposition 5.4.12. *We have that*

$$K\left(\frac{t}{m}, \tau(m)\right) = I\left(\frac{t}{m}, \tau(m)\right) + E\left(\frac{t}{m}, \tau(m)\right) + H\left(\frac{t}{m}, \tau(m)\right). \tag{5.16}$$

Proof. Let $k_{t/m}^{\tau(m)}$ be as in (2.19). Substituting $\alpha = k_{t/m}^{\tau(m)}$ to (2.30) implies the proposition. □

Proposition 5.4.13. *The contribution from $H\left(\frac{t}{m}, \tau(m)\right)$ to the first summand of the right hand side of (5.15) decays exponentially:*

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}H\left(\frac{t}{m},\tau(m)\right)dt\right)\bigg|_{s=0} = O(e^{-c_2 m})m \quad m\to\infty.$$

Proof of Proposition 5.4.13. The Proposition follows from [MP14, p. 23] with minor modification; for the convenience of the reader we include the key steps of the proof. First note that

$$H(t,\tau(m)) := \int_{\Gamma\backslash G}\sum_{\gamma \text{ hyperbolic},\gamma\in\Gamma} k_t^{\tau(m)}(g^{-1}\gamma g)d\dot{g}.$$

By Lemma 5.4.10,

$$\sum_{\gamma \text{ hyperbolic}}\left|k_t^{\tau(m)}(g\gamma g^{-1})\right| \leq C\cdot e^{-tm^2/2}\cdot\dim(\tau(m))\cdot\sum_{\gamma \text{ hyperbolic},\gamma\in\Gamma} H_t^0(g^{-1}\gamma g).$$

To finish the proof, we need to prove that

$$\sum_{\gamma \text{ hyperbolic},\gamma\in\Gamma} H_t^0(g^{-1}\gamma g)\leq Ce^{-c/t} \tag{5.17}$$

for some $C>0$.

Remark 5.4.14. *The inequality (5.17) corresponds to [MP11, Proposition 3.2].*

It follows from the construction of a fundamental solution of the heat equation (see Section 4.1) that there exists $C>0$ such that for all $g\in G$, $t\in(0,1]$, one has

$$H_t^0(g)\leq C\cdot t^{-\dim(\mathcal{O})/2}\cdot e^{-\frac{\rho^2(gK,1K)}{4\pi}}, \tag{5.18}$$

where $\rho(x,y)$ denotes the hyperbolic distance between x and y. Together (5.18) and (3.51) imply (5.17), that finishes the proof. □

From (2.33), (2.34) and Remark 2.11.5 we obtain

$$\begin{aligned} I(t,\tau(m)) &= 2\operatorname{vol}(\mathcal{O})\sum_{k=0}^{n}(-1)^{k+1}e^{-t\lambda^2_{\tau(m),k}}\int_{\mathbb{R}} e^{-t\lambda^2}P_{\sigma_{\tau(m),k}}(i\lambda)d\lambda,\\ E(t,\tau(m)) &= 2\sum_{\{\gamma\}\text{ elliptic}}\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\sum_{k=0}^{n}(-1)^{k+1}e^{-t\lambda^2_{\tau(m),k}}\int_{\mathbb{R}} e^{-t\lambda^2}P^{\gamma}_{\sigma_{\tau(m),k}}(i\lambda)d\lambda. \end{aligned} \tag{5.19}$$

For further estimates of (5.15) we need

Lemma 5.4.15. *There exists $C > 0$ such that*

$$\int_0^1 t^{-1} I\left(\frac{t}{m}, \tau(m)\right) dt = \int_0^\infty t^{-1} I\left(\frac{t}{m}, \tau(m)\right) dt + O(e^{-Cm}), \tag{5.20}$$

$$\int_0^1 t^{-1} E\left(\frac{t}{m}, \tau(m)\right) dt = \int_0^\infty t^{-1} E\left(\frac{t}{m}, \tau(m)\right) dt + O(e^{-Cm}), \tag{5.21}$$

as $m \to \infty$.

Proof. We repeat the estimate of (5.20) for the reader's convenience [MP11, p. 25]. Recall that the polynomial

$$P_{\sigma(m),k}(t) = \sum_{i=0}^n a_{k,i}(m) t^{2i}$$

and there exists $C > 0$ such that

$$|a_{k,i}| \le C m^{2n+n(n+1)/2}$$

for all $k, i = 0, \dots, n$ and $m \in \mathbb{N}$. Applying this estimate to (5.19) and using that

$$\lambda_{\tau(m),i} \ge m$$

for $i = 0, \dots, m$, we get

$$\left| I\left(\frac{t}{m}, \tau(m)\right) \right| \le C e^{-c(m+t)}, \quad t \ge 1 \tag{5.22}$$

for some $c > 0$, and hence

$$\int_1^\infty t^{-1} I\left(\frac{t}{m}, \tau(m)\right) dt = O(e^{-cm}), \quad m \to \infty. \tag{5.23}$$

By Lemma 5.5.8 we get the same estimates for $\int_1^\infty t^{-1} E\left(\frac{t}{m}, \tau(m)\right) dt$, that implies (5.21).

□

Using (5.23) and the arguments after [MP11, Proposition 5.5], we obtain Lemma 5.4.8.

□

Lemma 5.4.16. *[MP11, (5.17)] We have that $\log T_{\mathcal{O}}^{(2)}(\tau(m)) = \frac{1}{2}\left(MI(\tau(m))\right)$.* □

Theorem 5.4.17. *We have that*

$$MI(\tau(m)) = PI(m), \quad ME(\tau(m)) = PE(m),$$

where $PI(m)$ is the polynomial in m of degree $\frac{n^2+n+2}{2}$, $PE(m)$ is the pseudopolynomial in m of degree $\frac{d^2-d+1}{2}$ with $2d-1$ being the maximal dimension the fixed point sets of $\gamma \in \Gamma, \gamma \neq e$.

Remark 5.4.18. *In Theorem 5.4.17, we let the maximal dimension the fixed point sets of $\gamma \in \Gamma, \gamma \neq e$ be equal to $2d-1$, but not $2d+1$ as in Theorem 5.4.2. In order to obtain Theorem 5.4.2, set $d' = d-1$, then $d^2 - d + 1 = (d'+1)^2 - (d'+1) + 1 = d'^2 + d' + 1$ and $2d - 1 = 2d' + 1$.*

Proof of Theorem 5.4.17. Note that

$$MI(\tau(m)) = \mathrm{vol}(\Gamma\backslash G)\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt,$$

$$ME(\tau(m)) = \sum_{k=0}^{n}\sum_{\{\gamma\}\text{ elliptic}}(-1)^k \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_0^{\lambda_{\tau(m),k}} P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt.$$

As the estimate of $MI(\tau(m))$ does not depend on the structure of the group Γ, [MP11, Corollary 5.7] implies:

$$\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt = c(n)m\dim\tau(m) + O(m^{\frac{n(n+1)}{2}}),$$

where $c(n)$ is as in [MP11, (2.21)]. It remains to estimate

$$\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt.$$

Let γ be an elliptic element as in (2.7). Recall that by definition of $\lambda_{\tau(m),k}$,

$$\lambda_{\tau(m),0} > \lambda_{\tau(m),1} > \ldots > \lambda_{\tau(m),n}. \tag{5.24}$$

Split the integrals

$$\begin{gathered}\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt = \\ \int_0^{\lambda_{\tau(m),n}}\sum_{k=0}^{n}(-1)^k P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt + \sum_{k=0}^{n}(-1)^k\int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt.\end{gathered} \tag{5.25}$$

We calculate the first summand in the right hand side of (5.25). By Lemma 5.5.3, it follows that

$$\sum_{k=0}^{n}(-1)^k P^{\gamma}_{\sigma_{\tau(m),k}}(t)$$

does not depend on t, hence

$$\sum_{k=0}^{n}(-1)^k\int_0^{\lambda_{\tau(m),n}} P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt = \lambda_{\tau(m),n}\cdot\sum_{k=0}^{n-1}(-1)^k P^{\gamma}_{\sigma_{\tau(m),k}}(t) = (\tau_{n+1}+m)\cdot\sum_{k=0}^{n-1}(-1)^k P^{\gamma}_{\sigma_{\tau(m),k}}(t). \tag{5.26}$$

The second equality is due to (2.20). Hence by Lemma 5.5.6 the expression in the right hand side of (5.26) is a pseudopolynomial of order $\leqslant \frac{d^2-d+2}{2}$. We are left to understand $\int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} P^{\gamma}_{\sigma_{\tau(m),k}}(t)dt$. By Lemma 5.5.10,

$$\left| m^{d(d-1)/2}\int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \frac{P^{\gamma}_{\sigma_{\tau(m),k}}(t)}{m^{d(d-1)/2}}dt\right| \leqslant m^{d(d-1)/2}\cdot(\lambda_{\tau(m),k}-\lambda_{\tau(m),n})\cdot O(1) = O(m^{(d^2-d)/2}),$$

that proves the theorem. □

Theorems 5.4.2 and 5.4.3 follow from Lemmas 5.4.8, 5.4.16 and Theorem 5.4.17.

5.5 Some technical lemmas

In this section we collect the technical lemmas which have been used in the proof of Theorem 5.4.17, namely Lemmas 5.5.3, 5.5.6, 5.5.8 and 5.5.10.

Proposition 5.5.1. *Let $\gamma \in \Gamma$ be from (3.37) with $d := k$ and let $\sigma \in \hat{M}$. It follows from Theorem 3.3.25 that:*

$$P^{\gamma}_{\sigma}(\nu) = \sum_{s\in W}\det(s)\prod_{\alpha\in\Delta^+_{\gamma}}\langle\alpha, -s(\Lambda(\sigma)+\rho_M) - i\nu e_1\rangle\zeta_{-s(\Lambda(\sigma)+\rho_M)}(\gamma), \quad \nu\in\mathbb{C}. \tag{5.27}$$

Remark 5.5.2. *Note that γ fixes in $\mathbb{H}^{2n+1}$ the point set of dimension $2d-1$; compare with $\gamma = e$, that corresponds to $d = n+1$ and fixes $\mathbb{H}^{2n+1}$.*

For $\Lambda \in \mathfrak{h}^*_{\mathbb{C}}$, we denote

$$A(\Lambda,\nu) := \prod_{\alpha\in\Delta^+_{\gamma}}\langle\alpha, -\Lambda - \sqrt{-1}\nu e_1\rangle, \quad B(\Lambda) = \zeta_{-\Lambda}(\gamma). \tag{5.28}$$

Note that if $\Lambda = v_2e_2 + \ldots v_{n+1}e_{n+1}$, then (5.28) becomes:

$$A(\Lambda,\nu) = \prod_{2\le j\le d}(-\nu^2 - v_j^2)\prod_{2\le i<j\le d}(v_i^2 - v_j^2), \quad B(\Lambda) = e^{-\sqrt{-1}(v_{d+1}\phi_{d+1}+\ldots v_{n+1}\phi_{n+1})}. \tag{5.29}$$

Note that $A(\Lambda,\nu)$ is an even polynomial in ν of order $2(d-1)$, and that $B(\Lambda)$ does not depend on ν. First we present two lemmas needed for the estimate (5.26).

Lemma 5.5.3. *The sum*

$$\sum_{k=0}^{n}(-1)^k P^{\gamma}_{\tau(m),k}(\nu) \tag{5.30}$$

does not depend on ν.

Remark 5.5.4. *Compare Lemma 5.5.3 with the same result for* $\sum_{k=0}^{n}(-1)^k P_{\tau(m),k}(\nu)$ *[Pfa12, Corollary 9.9].*

Remark 5.5.5. *Every summand in (5.30) is a polynomial of order* $2(d-1)$ *in* ν *by (5.27-5.29), but the whole sum does not depend on* ν.

Lemma 5.5.6. *The sum* $\sum_{k=0}^{n}(-1)^k P^{\gamma}_{\tau(m),k}(\nu)$ *is a pseudopolynomial in* m *of order* $\le \frac{d(d-1)}{2}$.

Remark 5.5.7. *Note that* $\gamma = \mathrm{id}$ *is equivalent to* $d = n+1$. *Then the sum*

$$\sum_{k=0}^{n}(-1)^k P_{\tau(m),k}(\nu) = \sum_{k=0}^{n}(-1)^k P^{\gamma}_{\tau(m),k}(\nu)$$

is a polynomial in m *of order* $n(n+1)/2$; *compare [MP11, Corollary 1.4].*

The following two lemmas are needed to prove Lemma 5.4.15.

Lemma 5.5.8. *Let*

$$P^{\gamma}_{\tau(m),k}(\nu) = \sum_{i=0}^{n} a^{\gamma}_{k,i}(m)\nu^{2i},$$

then there exists $C > 0$ *such that*

$$|a^{\gamma}_{k,i}| \leqslant C m^{2(d-1)+d(d-1)/2}.$$

Definition 5.5.9. *For brevity put* $\lambda_i := \lambda_{\tau(m),i}$ *for* $i = 0, \dots, n$, *where* $\lambda_{\tau(m),i}$ *is (2.20).*

Lemma 5.5.10. *There exists* $C > 0$ *such that for any* $k = 0, \dots, n$ *and* $\nu \in [\lambda_n, \lambda_k]$

$$P^{\gamma}_{\tau(m),k}(\nu) m^{-\frac{d(d-1)}{2}} < C.$$

Then by Definition 2.6.2, (2.5) and (2.20)

$$\Lambda(\sigma_{\tau(m),k}) + \rho_M = \sum_{i=2}^{k+1} \lambda_{i-2} e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1} e_i. \tag{5.31}$$

Substituting (5.28) to (5.27), we obtain

$$\sum_{k=0}^{n}(-1)^k P^{\gamma}_{\tau(m),k}(\nu) = \sum_{k=0}^{n}\sum_{s\in W}(-1)^k \det(s) A(s\cdot\Lambda(\sigma_{\tau(m),k})+\rho_M),\nu)B(s\cdot\Lambda(\sigma_{\tau(m),k})+\rho_M). \tag{5.32}$$

To prove Lemma 5.5.3, we need the following two propositions:

Definition 5.5.11. *Fix some* $(k,s)\in\{0,\dots,n\}\times W$. *Denote*

$$U(k,s) := \left\{(k',s')\in\{0,\dots,n\}\times W \;\middle|\; B(s\cdot(\Lambda(\sigma_{\tau(m),k})+\rho_M)) = B(s'\cdot(\Lambda(\sigma_{\tau(m),k'})+\rho_M))\right\}.$$

Proposition 5.5.12. *The sum*

$$\sum_{(k',s')\in U(k,s)} A(s'\cdot(\Lambda(\sigma_{\tau(m),k'})+\rho_M),\nu)$$

does not depend on ν.

Proof of Proposition 5.5.12. Without loss of generality assume that W contains only the permutation elements, but not the sign changes.

Fix $k\in\{0,\dots,n\}$ and fix $\lambda_{\eta(d+1)},\dots,\lambda_{\eta(n+1)}\in\{\lambda_0,\lambda_1,\dots,\widehat{\lambda}_k,\dots,\lambda_n\}$, such that $\eta(d+i)\neq\eta(d+j)$ for $i\neq j$; above $\widehat{\lambda_k}$ means, that the element λ_k is omitted. Without loss of generality assume

$$\eta(d+1)<\eta(d+2)<\ldots<\eta(n+1).$$

Denote by

$$\Lambda^k_{\mathrm{stan}} := \lambda_0 e_2 + \lambda_1 e_3 + \ldots + \widehat{\lambda}_k e_{k+2} + \ldots + \lambda_n e_{n+1}.$$

Let $l(i)\in\mathbb{N}$, $i=2,\dots,d$ be as follows:

1. $\{l(i)\}_{i=2}^{d}\cap\{\eta(j)\}_{j=d+1}^{n+1}=\emptyset$,
2. $\{l(i)\}_{i=2}^{d}\cup\{\eta(j)\}_{j=d+1}^{n+1}=\{i\}_{i=0}^{n}\backslash\{k\}$,
3. $l(2)<l(3)<\ldots<l(d)$.

Define

$$l := \lambda_{l(2)}e_2+\ldots+\lambda_{l(d)}e_d,$$

$$\eta := \lambda_{\eta(d+1)}e_{d+1}+\ldots+\lambda_{\eta(n+1)}e_{n+1},$$

and let

$$K := \{0, 1, \ldots, n\} \setminus \cup_{i=d+1}^{n+1} \{\eta(i)\}$$

Let $w \in W$ be the element such that

$$w \cdot \Lambda^k_{\text{stan}} = l + \eta.$$

It follows from (5.29), that

$$B(w \cdot \Lambda^k_{\text{stan}}) = B(s \cdot \Lambda^\kappa)$$

holds if and only if $s \cdot \Lambda^\kappa = s' \cdot v + \eta$, where

1. $\kappa \in K$,

2. $v = \lambda_{v(2)} e_2 + \ldots + \lambda_{v(d)} e_d$, where $v(i) \in K \setminus \{k\}$ and $v(i) < v(j)$ for $i < j$,

3. s' is a restriction of W that maps the first $d-1$ coordinates into themselves, i.e. w permutes the coefficients at $e_1, \ldots, e_d$ and keeps the coefficients at $e_{d+1}, \ldots, e_{n+1}$.

Note that once $w \cdot \Lambda^k_{\text{stan}}$ and $\kappa \in K$ is fixed, the choice of v is unique, though s' can still be any element of W, satisfying (3). Thus grouping together summands as in Proposition 5.5.12 would result into a group

$$\begin{aligned} &\left(\sum_{\kappa \in K} \sum_{s'} \det(w) A(s' \cdot v + \eta, \nu) \right) \cdot B(w \cdot \Lambda^k_{\text{stan}}) = \\ &\left(\sum_{\kappa \in K} \sum_{s'} \det(w) \det(s') \cdot A(v, \nu) \right) \cdot B(w \cdot \Lambda^k_{\text{stan}}). \end{aligned} \tag{5.33}$$

The equality above follows from (5.29) and

$$A(s' \cdot v + \eta, \nu) = \det(s') \cdot A(v, \nu).$$

Every $A(v, \nu)$ is a polynomial in ν of order $2(d-1)$, moreover,

$$A(v, \nu) = \prod_{i \in K, i \neq k} (-\nu^2 - \lambda_i^2) \prod_{0 \leq i < j \leq n,\, i,j \in K \setminus \{k\}} (\lambda_i^2 - \lambda_j^2).$$

We are interested in its values in $2d$ points $\nu = \pm\sqrt{-1}\lambda_j$, $j \in K$. For convenience put $\eta(d) := -1$ and $\eta(n+2) := n+2$. Let

$$\eta(d + B) < \kappa < \eta(d + B + 1) \tag{5.34}$$

for some $B = 0, \dots, n-d+1$.

$$A(v, \pm i\lambda_j) = 0, \quad j \in K\backslash\{\kappa\},$$
$$A(v, \pm i\lambda_\kappa) = (-1)^{k-B} \prod_{0\le i<j\le n,\, i,j\in K} (\lambda_i^2 - \lambda_j^2). \tag{5.35}$$

Note that $w^{-1} \cdot s'$ maps $v + \nu$ to Λ^k_{stan}. We know the exact formulas for both, hence we can calculate the amount of permutations needed to map one into another, to obtain

$$\det(w \cdot s') = \det(w^{-1} \cdot s') = (-1)^{\kappa+B}, \tag{5.36}$$

where B is defined by (5.34). Putting together (5.33), (5.35) and (5.36), we obtain:

$$\sum_{\kappa\in K}\sum_{s'} \det(w) A(s'\cdot\nu + \eta, \nu)\Big|_{\nu=\pm i\lambda_k} = \prod_{0\le 1<j\le n; i,j\in K} (\lambda_i^2 - \lambda_j^2), \quad k \in K.$$

By definition, it is a polynomial in ν of order $2(d-1)$. As we have shown, its value coincides the same as $2d$ points, hence it is a constant not depending on ν, that finishes the proof.

□

Proof of Lemma 5.5.3. We collect the summands in the right hand side of (5.32) into groups as in Definition 5.5.11; Lemma 5.5.3 follows from Proposition 5.5.12, which states that each group separately does not depend on ν. □

Proof of Lemma 5.5.6. By the proof of the previous lemma, it suffices to consider $A(v, \nu)$ for any ν; choose $\nu = \sqrt{-1}\lambda_0$:

$$A(v, \sqrt{-1}\lambda_0) = \prod_{0\le i<j\le n;\, i,j\in K} (\lambda_i^2 - \lambda_j^2). \tag{5.37}$$

We need to estimate its growth as $m \to \infty$. Recall that $\lambda_i = m + \tau_{i+1} + n - i$, thus $\lambda_i^2 - \lambda_j^2$ has a linear growth in m. The set K consists of d elements, hence (5.37) is the product of $\frac{d(d-1)}{2}$ factors of linear growth, hence $A(\Lambda_0, \sqrt{-1}\lambda_0) = O(m^{\frac{d(d-1)}{2}}), \quad m \to \infty.$ □

Proof of Lemma 5.5.8. To prove the lemma, it suffices to estimate

$$\prod_{2\le j\le d} (-\nu^2 - v_j^2) \prod_{2\le i<j\le d} (v_i^2 - v_j^2), \quad m \to \infty$$

where every v_i equals λ_k for some $k \in [0, n]$. Note that

$$v_i^2 - v_j^2 = O(m), \qquad \prod_{2\le i<j\le d} (v_i^2 - v_j^2) = O(m^{d(d-1)/2}), \quad m \to \infty,$$

and

$$\prod_{2\leq j\leq d} v_j^2 = O(m^{2(d-1)}),$$

that implies Lemma 5.5.8. □

Proof of Lemma 5.5.10. To prove the lemma, we need to estimate

$$\frac{\prod_{1\leqslant j\leqslant n; j\in K\setminus\{k\}}(-\nu^2-\lambda_j^2)\prod_{1\leqslant i<j\leqslant n, i,j\in K\setminus\{k\}}(\lambda_i^2-\lambda_j^2)}{m^{d(d-1)/2}}.$$

First note that

$$\frac{\prod_{1\leqslant i<j\leqslant n, i,j\in K\setminus\{k\}}(\lambda_i^2-\lambda_j^2)}{m^{\frac{(d-1)(d-2)}{2}}} = \prod_{1\leqslant i<j\leqslant n; i,j\in K}\frac{(\lambda_i^2-\lambda_j^2)}{m}$$

is bounded for every K and m; the equality above follows from $|K\setminus\{k\}| = d-1$. Second,

$$\frac{\prod_{1\leqslant j\leqslant n; j\in K}(-\nu^2-\lambda_j^2)}{m^d} = \prod_{1\leqslant j\leqslant n, j\in K}\frac{(-\nu^2-\lambda_j^2)}{m}.$$

To estimate the latter note that $\lambda_n^2-\lambda_j^2 \leqslant \nu^2-\lambda_j^2 \leqslant \lambda_k^2-\lambda_j^2, \quad 1\leqslant j\leqslant n, j\in K$, hence

$$\frac{|\nu^2-\lambda_j^2|}{m} \leqslant \frac{|\lambda_k^2-\lambda_j^2|+|\lambda_k^2-\lambda_n^2|}{m}.$$

Note that this expression is bounded for every j and m; this proves the lemma. □

Chapter 6

Analytic torsion for finite volume orbifolds

The main goal of this chapter is to prove Theorem 1.3.1. We restrict ourselves to 3-dimensional hyperbolic orbifolds $\Gamma\backslash\mathbb{H}^3$ of finite volume.

6.1 Trace formula and trace regularization

In this section we define the regularized trace $\mathrm{Tr}_{reg}\left(e^{-tA_\nu}\right)$ and express it as the spectral side of the Selberg trace formula.

Let $\mathcal{O}$ be a hyperbolic manifold. Note that an operator A_ν has a non-empty continuous spectrum and hence the heat operator is not of trace class. To overcome this problem, various authors have used a regularization of the heat trace. A natural definition of the regularized trace is as follows: one needs to integrate the pointwise trace of the heat kernel over the truncated manifold and show that the integral admits an expansion in the truncation parameter:

Theorem 6.1.1. *[MP12] There exists the following asymptotic expansion:*

$$\int_{C(Y)} h^\nu(t;x,x)\,dx = C_1(t)\log(Y) + C_2(t) + o(1), \quad Y\to\infty,$$

where $C(Y)$ is from (2.23) and $h^\nu(t;x,x')$ is the kernel of $\pi_\Gamma(h_t^\nu)$, where h_t^ν is from (2.15). Define the regularized trace as

$$\mathrm{Tr}_{reg}(e^{-tA_\nu}) := C_2(t).$$

Moreover, it follows from [MP12], that the regularized trace of e^{-tA_ν} equals the spectral side of the Selberg trace formula applied to a test function h_t^ν. We define the regularized trace on an orbifold in the same spirit:

Definition 6.1.2. *[MP12] The regularized trace* $\mathrm{Tr}_{reg}(e^{-tA_\nu})$ *equals the spectral side of the Selberg trace formula applied to* $\exp(-tA_\nu)$*, and hence*

$$\mathrm{Tr}_{reg}(e^{-tA_\nu}) = I(h_t^\nu) + H(h_t^\nu) + T(h_t^\nu) + \mathcal{I}(h_t^\nu) + J(h_t^\nu) + E^{cusp}(h_t^\nu) + \mathcal{J}^{cusp}(h_t^\nu). \quad (6.1)$$

6.2 Fourier transform of the weighted orbital integrals

In this section we recall the Fourier transform of the distributions $\mathcal{I}(\alpha)$ and $E^{cusp}(\alpha)$ [Hof97].

6.2.1 Fourier transform of $\mathcal{I}(\alpha)$.

Definition 6.2.1. *For* $\sigma \in \hat{M}$ *with highest weight*

$$k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$$

and $\lambda \in \mathbb{R}$*, define* $\lambda_\sigma \in (\mathfrak{h})^*_{\mathbb{C}}$ *by*

$$\lambda_\sigma := i\lambda e_1 + \sum_{j=2}^{n+1}(k_j(\sigma) + \rho_j)e_j,$$

where ρ_j *is from (2.5).*

Let $S(\mathfrak{b}_{\mathbb{C}})$ be the symmetric algebra of $\mathfrak{b}_{\mathbb{C}}$. Define $\Pi \in S(\mathfrak{b}_{\mathbb{C}})$ by

$$\Pi := \prod_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} H_\alpha. \quad (6.2)$$

The restriction of the Killing form to $\mathfrak{h}_{\mathbb{C}}$ defines a non-degenerate symmetric bilinear form; thus we can identify $\mathfrak{h}^*_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}$ via this form; denote the induced symmetric bilinear form on $\mathfrak{h}^*_{\mathbb{C}}$ by $\langle \cdot, \cdot \rangle$. Define the reflection

$$s_\alpha : \mathfrak{h}^*_{\mathbb{C}} \to \mathfrak{h}^*_{\mathbb{C}}$$

by

$$s_\alpha(x) := x - 2\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$$

for $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Theorem 6.2.2. *[Hof97, Corollary on p.96] For every K-finite $\alpha \in C^2(G)$ one has*

$$\mathcal{I}(\alpha) = \frac{\kappa}{4\pi}\sum_{\sigma\in\hat{M}}\int_\mathbb{R}\Omega(\check{\sigma}, -\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda,$$

where

$$\Omega(\sigma,\lambda) := -2\dim(\sigma)\gamma - \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_\alpha)) + \psi(1-\lambda_\sigma(H_\alpha))\right)$$

and $\check{\sigma}$ is the contragredient representation of σ. □

6.2.2 Fourier transform of $E^{cusp}(\alpha)$ on $\mathrm{SO}_0(1,3)$.

Note that for $G = \mathrm{SO}_0(1,3)$, λ_σ from Definition 6.2.1 reads:

$$\lambda_\sigma = i\lambda e_1 + k_2(\sigma)e_2. \tag{6.3}$$

Note that the non-identity element w_0 of the Weyl group W from Subsection 2.2 acts on λ_σ as

$$w_0\lambda_\sigma = -i\lambda e_1 - k_2(\sigma)e_2. \tag{6.4}$$

Recall that $\Sigma_P^+ = \{e_1 - e_2\} \cup \{e_1 + e_2\}$, and

$$\begin{aligned} \lambda_\sigma(H_{e_1-e_2}) = i\lambda - k_2(\sigma), \quad \lambda_\sigma(H_{e_1+e_2}) = i\lambda + k_2(\sigma), \\ \lambda_{w_0\sigma}(H_{e_1-e_2}) = -i\lambda + k_2(\sigma), \quad \lambda_{w_0\sigma}(H_{e_1+e_2}) = -i\lambda - k_2(\sigma). \end{aligned} \tag{6.5}$$

Let

$$\gamma = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \\ & \begin{smallmatrix} \cos(2\phi) & \sin(2\phi) \\ -\sin(2\phi) & \cos(2\phi) \end{smallmatrix} \end{pmatrix}, \tag{6.6}$$

where $\phi \in [0, 2\pi)$, then

$$\gamma^{n(e_1-e_2)} = e^{-2in\phi}, \quad \gamma^{n(e_1+e_2)} = e^{2in\phi},$$

and

$$\gamma^{\lambda_\sigma} = e^{2i\phi\, k_2(\sigma)}, \quad \gamma^{w_0\lambda_\sigma} = e^{-2i\phi\, k_2(\sigma)}.$$

Then [Hof97, Theorem 1] reads

Theorem 6.2.3. *Let $\gamma \in \Gamma_M(G)$, then*

$$I_L(\gamma,\alpha) = \frac{1}{2\pi i}\sum_{\sigma\in\widehat{M}}\int_{\mathbb{R}}\Omega(\gamma,\check{\sigma})\Theta_{\sigma,\lambda}(\alpha)d\lambda, \tag{6.7}$$

where

$$\begin{aligned}\Omega(\gamma,\sigma) = \frac{1}{2}\Bigg[&e^{2i\phi k_2(\sigma)}\left(\sum_{n=1}^{\infty}\frac{e^{2in\phi}}{n+i\lambda-k_2(\sigma)}+\sum_{n=1}^{\infty}\frac{e^{-2in\phi}}{n+i\lambda+k_2(\sigma)}\right)+\\ &e^{-2i\phi k_2(\sigma)}\left(\sum_{n=1}^{\infty}\frac{e^{2in\phi}}{n-i\lambda+k_2(\sigma)}+\sum_{n=1}^{\infty}\frac{e^{-2in\phi}}{n-i\lambda-k_2(\sigma)}\right)\Bigg].\end{aligned} \tag{6.8}$$

For convenience, express $\Omega(\gamma,\sigma)$ in terms of the digamma function. For this denote

$$b(s,z) := \sum_{n=1}^{\infty}\frac{z^n}{n+s}, \tag{6.9}$$

which is absolutely convergent for $s, z \in \mathbb{C}$ with $s \neq \{-1,-2,\ldots\}$, $|z| < 1$, and conditionally convergent for $|z| = 1$, $z \neq 1$.

Lemma 6.2.4. *For $m \in \mathbb{N}$,*

$$b(s, e^{i\frac{2\pi}{m}}) = -\frac{1}{m}\sum_{k=0}^{m-1} e^{\frac{2\pi i}{m}(m-k)}\psi\left(\frac{s-k}{m}+1\right),$$

where ψ is the digamma function.

Every $\gamma \in \Gamma_M(P)$ is of finite order by Remark 2.3.12, thus Theorem 6.2.3, Lemma 6.2.4 and (6.9) imply the following corollary:

Corollary 6.2.5. *For $\Omega(\gamma,\sigma)$ as in (6.8), we have:*

$$\Omega(\gamma,\sigma) = \sum_{j\in J} c_j\cdot\psi\big(a_j + ib_j\cdot\lambda\big),$$

where J and a_j do not depend on $k_2(\sigma)$; $c_j = O(1)$, $b_j = O(k_2(\sigma))$ as $k_2(\sigma)\to\infty$; and $\Omega(\gamma,\sigma)$ has no pole at $\lambda = 0$.

We will present two proofs of Lemma 6.2.4: the second one was kindly proposed by Werner Hoffmann.

A baby-proof of Lemma 6.2.4. We will consider the case for $m = 2$. The proof for $m \neq 2, m \in \mathbb{N}$ is done by analogy, but with more technicalities, therefore we omit it. Recall that

$$\psi(1+z) = \lim_{n\to\infty}\left(\ln n - \frac{1}{z+1} - \ldots - \frac{1}{z+n}\right), \quad z \neq -1, -2, \ldots.$$

Hence

$$\psi\left(\frac{s-1}{2}+1\right) = \lim_{n\to\infty} \underbrace{\left(\ln n - \frac{2}{s+1} - \frac{2}{s+3} - \ldots - \frac{2}{s-1+2n}\right)}_{=:A_n(s)}, \tag{6.10}$$

$$\psi\left(\frac{s}{2}+1\right) = \lim_{n\to\infty} \underbrace{\left(\ln n - \frac{2}{s+2} - \frac{2}{s+4} - \ldots - \frac{2}{s+2n}\right)}_{=:B_n(s)}. \tag{6.11}$$

Put

$$C_n(s) := -\frac{1}{s+1} + \frac{1}{s+2} - \ldots + (-1)^n \frac{1}{s+n}, \tag{6.12}$$

then

$$b(s,-1) = \lim_{n\to\infty} C_n(s).$$

It follows from (6.10), (6.11) and (6.12) that

$$C_{2n} = \frac{A_n(s) - B_n(s)}{2}, \quad C_{2n+1} = \frac{A_{n+1}(s) - B_n(s)}{2},$$

hence

$$\lim_{n\to\infty} C_{2n}(s) = \lim_{n\to\infty} C_{2n+1}(s) = \lim_{n\to\infty} \frac{A_n(s) - B_n(s)}{2}$$

and

$$b(s,-1) = \lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{A_n(s) - B_n(s)}{2} = \frac{\psi\left(\frac{s}{2}+\frac{1}{2}\right) - \psi\left(\frac{s}{2}+1\right)}{2},$$

that proves Lemma 6.2.4 for $m = 2$. □

Proof of Lemma 6.2.4. Recall that for $\mathrm{Re}(z) > -1$,

$$b(s,z) = z\int_0^1 x^s(1-zx)^{-1}dx,$$

and for $\mathrm{Re}(z) > 0$ [AS64, 6.3.22],

$$\psi(z+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x}dx,$$

where γ is the Euler-Mascheroni constant. Note that

$$\frac{1}{m}\sum_{k=0}^{m-1} e^{\frac{2\pi i}{m}(m-k)}\psi\left(\frac{s-k}{m}+1\right) = \frac{1}{m}\cdot\int_0^1 \sum_{k=0}^{m} e^{2\pi i\cdot(m-k)/m}\cdot\frac{1-x^{\frac{s-k}{m}}}{1-x}\,dx = \int_0^1 \frac{e^{2\pi i/m}}{m}\cdot\frac{x^{-1+\frac{1+s}{m}}}{-1+e^{2\pi i/m}\cdot x^{1/m}}\,dx. \tag{6.13}$$

Let $y = x^{1/m}$, then the right hand side of the equation above equals:

$$\int_0^1 \frac{e^{2\pi i/m}\cdot y^s}{-1+e^{2\pi i/m}\cdot y}\,dy = -b(s, e^{2\pi i/m}),$$

that proves the statement. □

6.2.3 Fourier transform of $E^{cusp}(\alpha)$ on $\mathrm{SO}_0(1,2n+1)$.

Lemma 6.2.4 together with [Hof97, Theorem 1] implies:

Theorem 6.2.6. *For every K-finite $\alpha \in C^2(G)$, one has*

$$\mathcal{E}^{cusp}(\alpha) = \sum_{\sigma\in\hat{M}}\int_{\mathbb{R}} \Omega^{cusp}(\check{\sigma}, -\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda,$$

where

$$\Omega^{cusp}(\sigma,\lambda) := \sum_{j\in I} c_j\psi(a_j + b_j i\lambda).$$

Above I is a finite set; $a_j, b_j \in \mathbb{R}$; $c_j \in \mathbb{C}$; ψ is a digamma function; and $\Omega^{cusp}(\sigma,\lambda)$ is regular at $\lambda = 0$. Let $(\tau_2+m)e_1 + \ldots + (\tau_{n+1}+m)e_{n+1}$ be the highest weight of σ. Then $a_j = O(1)$, $c_j = O(1)$, $|I| = O(1)$ and $b_j = O(m)$ when $m \to \infty$ or $m \to -\infty$. □

6.3 Asymptotic expansion of the regularized trace

In order to define the analytic torsion via the Mellin transform of the regularized trace $\mathrm{Tr}_{reg}e^{-tA_\nu}$, we need to show that the latter admits a certain type of asymptotic expansion as $t \to +0$. For this we need the following lemmas:

Lemma 6.3.1. *Let $\phi(t) := \int_{\mathbb{R}} \frac{e^{-t\lambda^2}}{\lambda+c}\,d\lambda$, where $c \neq 0$, $c \in i\mathbb{R}$. Then there exist $a'_j \in \mathbb{C}$ such that*

$$\phi(t) \sim \sum_{j=0}^{\infty} a'_j t^{j/2}, \quad t \to 0.$$

Proof. Note that

$$\int_{\mathbb{R}} \frac{e^{-t\lambda^2}}{\lambda+c} d\lambda = \int_{\mathbb{R}} \frac{e^{-t\lambda^2}\lambda}{\lambda^2-c^2} d\lambda - c\int_{\mathbb{R}} \frac{e^{-t\lambda^2}}{\lambda^2-c^2} d\lambda = -ce^{-tc^2}\int_{\mathbb{R}} \frac{e^{-t\lambda^2+tc^2}}{\lambda^2-c^2} d\lambda.$$

and (correcting a mistake in [MP14, Lemma 6.6]),

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{e^{-t\lambda^2+tc^2}}{\lambda^2-c^2} d\lambda = -\frac{\sqrt{\pi}}{\sqrt{t}} e^{tc^2}.$$

It follows from the previous equation that

$$\int_{\mathbb{R}} \frac{e^{-t\lambda^2+tc^2}}{\lambda^2-c^2} d\lambda = -C_1 \cdot \operatorname{erfc}(\sqrt{t}) + C_2$$

for some C_1 and C_2. Expanding $\operatorname{erfc}(\sqrt{t})$ in power series implies Lemma 6.3.1. □

Lemma 6.3.2. *Let $\phi_2(t) := \int_{\mathbb{R}} e^{-t\lambda^2}\psi(a+i\lambda)d\lambda$, where $a \in (0,1]$ and ψ is the digamma function. Then there exist a'_j, b'_j, $c'_j \in \mathbb{C}$ such that as $t \to 0$, there is an asymptotic expansion*

$$\phi_2(t) \sim \sum_{j=0}^{\infty} a'_j t^{j-1/2} + \sum_{j=0}^{\infty} b'_j t^{j-1/2}\log t + \sum_{j=0}^{\infty} c'_j t^j, \quad t \to 0.$$

Proof. Follows [MP12, Lemma 6.7] with minor modifications. □

Corollary 6.3.3. *Let $\phi_3(t) := \int_{\mathbb{R}} e^{-t\lambda^2}\psi(a+ib\lambda)d\lambda$, where $a, b \in \mathbb{R}$, $a \neq 0$ and ψ is the digamma function. Then there exist a'_j, b'_j, $c'_j \in \mathbb{C}$ such that as $t \to 0$, there is an asymptotic expansion*

$$\phi_2(t) \sim \sum_{j=0}^{\infty} a'_j t^{j-1/2} + \sum_{j=0}^{\infty} b'_j t^{j-1/2}\log t + \sum_{j=0}^{\infty} c'_j t^j, \quad t \to 0.$$

Proof. Without loss of generality we can assume that $b > 0$ and, moreover, $b = 1$. Then Corollary 6.3.3 follows from Lemmas 6.3.1 and 6.3.2, taking into account that $\phi(z+1) = \phi(z) + \frac{1}{z}$. □

The main result of the subsection is the following proposition:

Proposition 6.3.4. *There exist coefficients a'_j, b'_j, c'_j, where $j \in \mathbb{N}$ such that*

$$\operatorname{Tr}_{reg}(e^{-tA_\nu}) \sim \sum_{j=0}^{\infty} a'_j t^{j-d/2} + \sum_{j=0}^{\infty} b'_j t^{j-1/2}\log t + \sum_{j=0}^{\infty} c'_j t^j,$$

as $t \to +0$, where $\operatorname{Tr}_{reg}\left(e^{-tA_\nu}\right)$ is from Theorem 6.1.1.

Proof. It is sufficient to show that the summands on the right hand side of (6.1) admit an asymptotic expansion as $t \to +0$. The summands $I(h_t^\nu), H(h_t^\nu), T(h_t^\nu), \mathcal{I}(h_t^\nu), J(h_t^\nu)$ were treated in [MP12, Proposition 6.9]. The terms $\mathcal{J}^{cusp}(h_t^\nu)$ and $E(h_t^\nu)$ are treated in the same way as $J(h_t^\nu)$ and $I(h_t^\nu)$, respectively. The asymptotic expansion of $\mathcal{E}^{cusp}(h_t^\nu)$ follows from Corollary 6.3.3 and Theorem 6.2.6. □

6.4 Analytic torsion

In this section we define the analytic torsion on finite volume orbifolds and prove Theorem 1.3.1.

In order to define the spectral zeta function we need to study the asymptotic behavior of $\mathrm{Tr}_{reg} e^{-t\Delta_p(\tau)}$ as $t \to 0$ and $t \to \infty$. It follows from Lemma 2.7.2 and Proposition 6.3.4 that there exist coefficients a_j', b_j', c_j', $j \in \mathbb{N}$ such that

$$\mathrm{Tr}_{reg}(e^{-t\Delta_p(\tau)}) \sim \sum_{j=0}^{\infty} a_j' t^{j-d/2} + \sum_{j=0}^{\infty} b_j' t^{j-1/2} \log t + \sum_{j=0}^{\infty} c_j' t^j, \tag{6.14}$$

as $t \to +0$. As in [MP12, (7.10)] it follows that for some $c_j \in \mathbb{N}$

$$\mathrm{Tr}_{reg}\left(e^{-t\Delta_p(\tau)}\right) \sim h_p(\tau) + \sum_{j=1}^{\infty} c_j t^{-j/2}, \quad t \to \infty, \tag{6.15}$$

where $h_p(\tau) := \dim(\ker \Delta_p(\tau) \cap L^2)$.

Definition 6.4.1. *The spectral zeta function is defined as:*

$$\zeta_p(s;\tau) := \frac{1}{\Gamma(s)} \int_0^1 + \int_1^\infty t^{s-1} \mathrm{Tr}_{reg}\left(e^{-t\Delta_p(\tau)} - h_p(\tau)\right) dt.$$

The first integral is defined for $Re(s) > d/2$, the second one is defined for $Re(s) < 1/2$.

By (6.14) and (6.15) both integrals admit meromorphic continuations to $\mathbb{C}$ that are regular at $s = 0$, and hence we can define:

Definition 6.4.2. *The analytic torsion $T_{\mathcal{O}}(\tau)$ associated with a flat vector bundle E_τ equipped with the admissible metric from Definition 2.7.1, is defined as*

$$T_{\mathcal{O}}(\tau) = \exp\left(\frac{1}{2} \sum_{p=0}^{2n+1} (-1)^p p \frac{d}{ds} \zeta_p(s;\tau)\bigg|_{s=0}\right).$$

Let

$$K(t,\tau) := \sum_{p=0}^{2n+1} (-1)^p \, p \operatorname{Tr}_{reg} e^{-t\Delta_p(\tau)}$$

Note that if $h_p(\tau) = 0$, the analytic torsion is given by:

$$\log T_{\mathcal{O}}(\tau) = \frac{1}{2}\frac{d}{ds}\bigg|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t,\tau)\, dt \right). \tag{6.16}$$

Remark 6.4.3. *If $\tau = \tau(m)$, then $h_p(\tau(m)) = 0$ for sufficiently large m [MP12, Lemma 7.3].*

It follows from Theorem 6.1.1 that

$$K(t,\tau) = (I + H + T + \mathcal{I} + J + E + \mathcal{E}^{cusp} + \mathcal{J}^{cusp})\,(k_t^\tau). \tag{6.17}$$

By (6.16) and (6.17) in order to study the analytic torsion $T_{\mathcal{O}}(\tau)$ we need to study the Mellin transform of the right hand side of (6.17) at zero.

6.5 Asymptotic behavior of the analytic torsion

From now on let $\tau(m)$ be the ray of representations of G from in Definition 2.6.2. Denote by the same symbol its restriction to Γ. Let $\mathcal{M}I(\tau(m))$, $\mathcal{M}H(\tau(m))$, $\mathcal{M}T(\tau(m))$, $\mathcal{M}\mathcal{I}(\tau(m))$ $\mathcal{M}J(\tau(m))$ be as in [MP12] and $\mathcal{M}E(\tau(m))$ be as in Definition 5.4.4. Roughly speaking, they equal the value of Mellin transforms at zero of the corresponding terms in the right hand side of (6.17) with $\tau = \tau(m)$.

Theorem 6.5.1. *There exists a constant C such that for m sufficiently large one has*

$$\mathcal{M}I(t,\tau(m)) = C(n)\mathrm{vol}(X) m \dim \tau(m) + O(m^{\frac{n(n+1)}{2}}),$$
$$|\mathcal{M}H(\tau(m))| \le Cm^{\frac{n(n-1)}{2}}, \quad |\mathcal{M}T(\tau(m))| \le Cm^{\frac{n(n+1)}{2}},$$
$$|\mathcal{M}\mathcal{I}(\tau(m))| \le Cm^{\frac{n(n+1)}{2}}, \quad |\mathcal{M}J(\tau(m))| \le Cm^{\frac{n(n+1)}{2}} \log m,$$
$$|\mathcal{M}E(t,\tau(m))| \le Cm^{\frac{n(n-1)}{2}}.$$

Proof. Follows from [MP12, Propositions 10.1, 10.3, 10.4, 10.10, 10.14] and Theorem 5.4.17. □

Theorem 6.5.2. *There exists a constant C such that for m sufficiently large one has*

$$|\mathcal{M}J^{cusp}(\tau(m))| \le Cm^{\frac{n(n+1)}{2}} \log m.$$

Proof. Follows with minor modifications from [MP12, Proposition 10.14], keeping in mind Remark 2.11.12. □

It follows from Corollary 6.3.3 that $\mathcal{E}^{cusp}(k_t^{\tau(m)})$ admits an asymptotic expansion

$$\mathcal{E}^{cusp}(k_t^{\tau(m)}) \sim \sum_{j=0}^{\infty} a'_j t^{j-1/2} + \sum_{j=0}^{\infty} b'_j t^{j-1/2} \log t + \sum_{j=0}^{\infty} c'_j t^j$$

for some a'_j, b'_j, $c'_j \in \mathbb{C}$. Moreover, $\mathcal{E}^{cusp}(k_t^{\tau(m)}) = O(e^{-tm^2})$ as $m \to \infty$. Hence

$$M\mathcal{E}^{cusp}(s;\tau(m)) := \int_0^{\infty} t^{s-1} \mathcal{E}^{cusp}(k_t^{\tau(m)})\, dt$$

converges for $\mathrm{Re}(s) > \frac{d-1}{2}$, admits a meromorphic continuation to $\mathbb{C}$ and has at most simple pole at $s = 0$. Denote

$$M\mathcal{E}^{cusp}(\tau(m)) := \frac{d}{ds} \left. \frac{M\mathcal{E}^{cusp}(s;\tau(m))}{\Gamma(s)} \right|_{s=0}.$$

Theorem 6.5.3. *There exists a constant C such that for m sufficiently large one has*

$$|\mathcal{M}\mathcal{E}^{cusp}(\tau(m))| \leq C \cdot m \log(m).$$

In order to prove Theorem 6.5.3 we need the following technical lemmas.

Lemma 6.5.4. *For $c \in (0,\infty)$, $s \in \mathbb{C}$, $Re(s) > 0$, $e_j, d_j > 0$ let*

$$\zeta_c(s) := \frac{1}{\pi} \int_0^{\infty} t^{s-1} e^{-tc^2} \int_{\infty} \frac{e^{-tz^2}}{ie_j z + d_j}\, dz\, dt.$$

Then $\zeta_c(s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at 0. Moreover, one has

$$\left. \frac{d}{ds} \right|_{s=0} \frac{\zeta_c(s)}{\Gamma(s)} = -\frac{2}{e_j} \log\left(c + d_j/e_j\right).$$

Proof. Follows with minor modifications from [MP12, Lemma 10.5]. □

Lemma 6.5.5. *Let $c \in \mathbb{R}^+$, $s \in \mathbb{C}$, $Re(s) > 1/2$, $a_j, b_j > 0$. Define*

$$\tilde{\zeta}_c(s) := \frac{1}{\pi} \int_0^{\infty} t^{s-1} e^{-tc^2} \int_{\mathbb{R}} e^{-t\lambda^2} \psi\left(a_j + ib_j\lambda\right) d\lambda\, dt.$$

Then $\tilde{\zeta}_c(s)$ has a meromorphic continuation to $s \in \mathbb{C}$ with at most a simple pole at $s = 0$. Moreover, there exists a constant $C(\psi)$ which is independent of c, a_j and b_j such that

$$\left. \frac{d}{ds} \right|_{s=0} \frac{\tilde{\zeta}_c(s)}{\Gamma(s)} = -\frac{2}{b_j} \log \Gamma(a_j + cb_j) + C(\psi).$$

Proof. Follows from [MP12, Lemma 10.6] with minor modifications. □

Proof of Theorem 6.5.3. Let

$$\mathcal{ME}^{cusp}(s;\sigma_{\tau(m),k}) := \int_0^\infty t^{s-1}e^{-t\lambda^2_{\tau(m),k}}\mathcal{E}^{cusp}(h_t^{\sigma_{\tau(m),k}})\,dt.$$

As above it follows that the integral converges for $\mathrm{Re}(s) > (d-2)/2$ and admits a meromorphic continuation to $\mathbb{C}$ with at most a simple pole at $s = 0$. By [MP12, Proposition 8.2],

$$\mathcal{ME}^{cusp}(\tau(m)) = \sum_{k=0}^{n}(-1)^{k+1}\frac{d}{ds}\bigg|_{s=0}\frac{\mathcal{ME}^{cusp}(s;\sigma_{\tau(m),k})}{\Gamma(s)}.$$

In order to prove Theorem 6.5.3 it suffices to consider

$$\frac{d}{ds}\bigg|_{s=0}\frac{\mathcal{ME}^{cusp}(s;\sigma_{\tau(m),k})}{\Gamma(s)} = \frac{d}{ds}\left(\int_0^\infty t^{s-1}e^{-t\lambda^2_{\tau(m),k}}\mathcal{E}^{cusp}(h_t^{\sigma_{\tau(m),k}})dt\right)\bigg|_{s=0} \tag{6.18}$$

as $m \to \infty$. By Theorem 6.2.6 we can rewrite the left hand side of (6.18) as

$$\frac{d}{ds}\bigg|_{s=0}\int_0^\infty t^{s-1}e^{-t\lambda^2_{\tau(m),k}}\int_{\mathbb{R}}\left(\Omega(\sigma_{\tau(m),k},\lambda)+\Omega(w_0\sigma_{\tau(m),k},\lambda)\right)e^{-t\lambda^2}\,d\lambda\,dt,$$

where

$$\Omega(\sigma_{\tau(m),k},\lambda) = \sum_{j\in J} c_j\cdot\psi(a_j+i\lambda b_j)$$

for some $a_j \in \mathbb{R}$, growing not faster than linearly in m. Moreover, c_j, that is bounded by a constant as m grows; $b_j \in \mathbb{R}$ does not depend on m. The set J is finite and does not depend on m either. Note that

$$\int_{\mathbb{R}}\psi(a_j+ib_j\lambda)e^{-t\lambda^2}d\lambda = \int_{\mathbb{R}}\psi(a_j-ib_j\lambda)e^{-t\lambda^2}d\lambda,$$

hence we can assume that all $b_j > 0$. As $\psi(z+1) = \psi(z) + \frac{1}{z}$, we can we can rewrite the right hand side of (6.18) as

$$\frac{d}{ds}\bigg|_{s=0}\int_0^\infty t^{s-1}e^{-t\lambda^2_{\tau(m),k}}\int_{\mathbb{R}}\left(\sum_{j\in J'}c_j\cdot\psi(a_j+i\lambda b_j)+\sum_{j\in J''}\frac{1}{ie_j\lambda+d_j}\right)e^{-t\lambda^2}\,d\lambda\,dt. \tag{6.19}$$

Above $a_j, b_j, c_j, d_j > 0$, J and J' are finite sets; b_j, e_j and $|J'|$ do not depend on m; a_j, d_j and $|J''|$ grow not faster than linearly in m; and c_j is bounded by a constant as m grows.

By (6.19), Lemmas 6.5.4 and 6.5.5

$$\frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{ME}^{cusp}(s;\sigma_{\tau(m),k})}{\Gamma(s)} = -\sum_{j\in J'}\left(\frac{2c_j}{b_j}\log\Gamma(a_j+b_j\lambda_{\tau(m),k})+c_j\cdot C_j(\psi)\right) - \sum_{j\in J''}\frac{2c_j}{e_j}\log(d_j/e_j+\lambda_{\tau(m),k}). \tag{6.20}$$

By (2.20),

$$a_j+b_j\lambda_{\tau(m),k}=O(m),$$

hence

$$\log\Gamma(a_j+b_j\lambda_{\tau(m),k})=O(m\cdot\log m) \tag{6.21}$$

by the Stirling's formula. On the other hand,

$$\sum_{j\in J''}\frac{2}{e_j}\log(d_j/e_j+\lambda_{\tau(m),k})=O(m\cdot\log m). \tag{6.22}$$

Putting together (6.18), (6.20), (6.21) and (6.22) proves Theorem 1.3.1. □

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Nomenclature

α a K-finite Schwartz function, see p. 24

χ a (possibly non-unitary) representation of Γ; $\chi : \Gamma \to V_\chi$, see p. 42

$\chi_{P,Y}$ truncation function, see p. 21

$\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ set of roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, see p. 13

$\Delta(u)_k$ the Bochner-Laplace operator with respect to the metric $g(u)$, see p. 5.2.3

Δ_E the Bochner-Laplace operator acting on $C^\infty(\mathcal{O}, E)$, see p. 34

$\Delta_E^{1/2}$ a square root of Δ_E; see p. 35

$\Delta_{\nu,\chi}^{\#}$ see p. 44

$\Delta_{E,\chi}^{\#}$ the twisted non-selfadjoint Laplacian acting on $C^\infty(\mathcal{O}, E \otimes F)$, where F is a flat orbibundle and E is a Hermitian orbibundle, see p. 43

γ an element of Γ

Γ_γ the centralizer of γ in Γ

Γ_γ^1 see p. 45

$\Gamma_{b,d}$ the contour of integration, see p. 36

κ number of cusps, see p.15

λ_σ see p. 89

Λ_I subset of $\mathbb{C}$; see p. 33

$\Lambda_{\sigma_{\tau,k}}$ see p. 19

$\mathcal{J}^{cusp}(\alpha)$ see p. 27

$\mathfrak{h}$ Cartan subalgebra of $\mathfrak{g}$, see p. 13

$\mathfrak{P}$ a fixed set of representatives of Γ-nonequivalent proper cuspidal subgroups of G, see p.15

∇ a covariant derivative, see p. 34

∇^{ν} the canonical connection on $\widetilde{E}_{\nu}$, see p. 18

∇^{F} a flat connection on $F \to \mathcal{O}$, see p. 42

$\nabla^{E\otimes F}$ a product connection, see p. 42

ν a finite-dimensional unitary representation of K; $\nu : K \to \mathrm{GL}(V_{\nu})$, see p. 17

$\Omega_{b,d}$ subset of $\mathbb{C}$; see p. 35

π_{Γ} the right-regular representation of G on $L^2(\Gamma\backslash G)$, see p. 24

$\pi_{\Gamma}(\alpha)$ the operator on $L^2(\Gamma\backslash G)$, see p. 24

$\pi_{\Gamma,c}(\alpha)$ see p. 25

$\pi_{\Gamma,d}(\alpha)$ see p. 25

$\pi_{\sigma,\lambda}$ principle series, see p. 2.16

$\sigma(H)(x,\xi)$ principal symbol of an operator H, see p. 33

$\mathrm{spec}(H)$ spectrum of H; see p. 35

$\mathrm{Tr}\, A$ trace of a trace class operator A

$\mathrm{tr}\, B$ the matrix trace of B

θ an Agmon angle for H; see p. 35

$\Theta_{\check{\sigma}_{-\lambda}}(m_{\gamma}a_{\gamma})$ see p. 27

$\Theta_{\sigma,\lambda}$ the character of $\pi_{\sigma,\lambda}$, see p. 45

φ a Paley-Wiener function, see 36 and $PW(\mathbb{C})$

$\varphi(H^{1/2})$ a smoothing operator, see p. 36

$\widetilde{\Delta}_\nu$ the Bochner-Laplace operator on $\widetilde{E}_\nu$, see p. 18

$\widetilde{A}_\nu$ a second-order differential operator, p. 18

$\widetilde{E}_\nu$ homogeneous vector bundle, see p. 17

$\zeta_p(s; h, g)$ the spectral zeta function, see p. 69

$\{\gamma\}_\Gamma$ Γ-conjugacy class of γ, see p. 45

A a usual notation for an operator, see e.g. p. 37

A see p. 12

a_γ see p. 14

A_ν a second-order differential operator, p. 18

$b(s, z)$ see p. 90

B_0 a pseudodifferential operator of order 0, see p. 35

$B_R(0)$ the unit ball in $\mathbb{C}$ of radius R and 0 as a center; see p. 33

$D(\gamma)$ see p. 3.34

D_1 A first order differential operator, see p. 34

$E(\sigma)$ the graded orbibundle, see p. 53

e^{-tA_ν} the semigroup of A_ν on $L^2(\Gamma\backslash G, \nu)$, see p. 18

$E^{cusp}(\alpha)$ see p. 27

$E_\gamma(h_\varphi)$ Orbital integral for γ, see p. 46

E_ν locally homogeneous vector orbibundle, p. 17

e_j elements of $\mathfrak{h}_\mathbb{C}^*$, see p. 12

$F \to \mathcal{O}$ a flat vector orbibundle, see p. 42

G a semisimple Lie group with finite center, see p. 12

G_γ the centralizer of γ in G, see p. 15

G_γ the centralizer of γ in G

G_γ^1 see p. 45

H a second order elliptic operator, possibly non-selfadjoint, see p. 34

$H^s(\mathcal{O}, E)$ s-Sobolev space; see p. 31

$H^{1/2}$ $H_\theta^{1/2}$ with a fixed Agmon angle θ, see p. 35

$H_\theta^{1/2}$ a square root of H, defined with respect to an Agmon angle θ, see p. 35

H_j, $j = 1, \ldots, n+1$ see p. 12

$H_t^\nu(g)$ the convolution kernel of A_ν, see p. 18

$h_t^\nu(g)$ see p. 18

$I_\gamma(t)$ see p.66

$I_e(t)$ see p.66

$I_L(\gamma, \alpha)$ see p. 26

$J_L(\gamma, \alpha)$ see p. 26

K a compact subgroup in G, see p. 12

K an integral kernel of an operator A

$K_\varphi(x, y)$ integral kernel of $\varphi(H^{1/2})$, see p. 38

$l(\gamma)$ the length of the closed geodesic, associated to γ, see p. 14

$L^2(\mathcal{O}, E)$ square-integrable sections of E; see p. 31

$L_c^2(\Gamma \backslash G)$ see p. 24

$L_d^2(\Gamma \backslash G)$ see p. 24

M the centralizer of A in K, see p. 12

m_γ see p. 14

N see p. 12

$N(r,H)$ counting function for an operator H, see p. 37

$PW(\mathbb{C})$ the space of Paley-Wiener functions on $\mathbb{C}$, see p. 36

$R(K)$ the representation ring over $\mathbb{Z}$ of K, see p. 53

$R(M)$ the representation ring over $\mathbb{Z}$ of M, see p. 53

$S_a(s,\sigma,\chi)$ The antisymmetric Selberg zeta function, defined on p. 58

V_χ see χ

V_ν see ν

V_k finite-dimensional H-invariant subspaces V_k; see p. 34

$Z(s,\sigma,\chi)$ The Selberg zeta function, defined on p. 51

Summary

This thesis deals with two aspects of the geometry and the spectral theory of hyperbolic orbifolds, that is, quotients of a hyperbolic space by discrete groups actions.

Our first topic is the twisted Selberg trace formula and the twisted Selberg zeta function. The latter was introduced by Selberg as an analogue of the Riemann zeta function where the prime numbers are replaced by the lengths of primitive closed geodesics $l(\gamma)$ on a hyperbolic surface:

$$Z(s) := \sum_{\gamma} \sum_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)}), \quad \mathrm{Re}(s) > 1.$$

Analogously to the Riemann zeta function, it admits a meromorphic continuation to the whole complex plane and satisfies a certain functional equation; one of the ways to prove it is by using the non-commutative analogue of the Poisson summation formula, called the Selberg trace formula. Moreover, the latter implies that the zeroes of the Selberg zeta function correspond to the eigenvalues of the Laplace operator. This may be regarded as an analogue of the Riemann hypothesis. The main result of the first part of the thesis is as follows: we prove a more general Selberg trace formula for hyperbolic orbifolds $\Gamma\backslash\mathbb{H}^{2n+1}$, which is twisted by representations χ and σ of Γ and $\mathrm{SO}(2n)$, respectively. Further we apply this trace formula to study the meromorphic continuation of a generalization $Z(s,\sigma,\chi)$ of the Selberg zeta function, twisted by the aforementioned representations, and describe its singularities.

Theorem A. *The Selberg zeta function $Z(s,\sigma,\chi)$ admits a meromorphic continuation to $\mathbb{C}$; its singularities correspond to the eigenvalues of a certain Laplace-type and Dirac-type operators, depending on σ and χ.*

In the second and the third part of the thesis, we deal with the analytic torsion defined as follows. Let X be a Riemannian manifold, E a vector bundle over X, and Δ_k the Laplace

operator acting on the k-forms with values in E. Using the method of zeta function regularization, we define the regularized determinant $\det(\Delta_k)$; formally, it is the regularization of the product of its non-zero eigenvalues. Then the analytic torsion is defined as follows:

$$T_X := \prod_{k=0}^{\dim X} (\det(\Delta_k))^{(-1)^{k+1}k/2}.$$

The analytic torsion is the spectral counterpart of a topological invariant called the Reidemeister torsion. Because these invariants have very different natures, their equality has applications in topology, number theory, and mathematical physics.

Now fix a hyperbolic orbifold $\Gamma\backslash\mathbb{H}^{2n+1}$ and associate a flat orbibundle to every finite-dimensional representation of Γ. We vary a representation and consider a varying orbibundle over a fixed hyperbolic orbifold. The main question of the rest of the thesis is: how does the analytic torsion of the orbibundle change, and what kind of geometric information about the orbifold can we obtain from it? In the second and the third part of the thesis, we answer the question for compact and finite volume odd-dimensional hyperbolic orbifolds.

Theorem B. *Let $\Gamma\backslash\mathbb{H}^{2n+1}$ be a compact hyperbolic orbifold and let $E_{\tau(m)} \to \Gamma\backslash\mathbb{H}^{2n+1}$, $m \in \mathbb{N}$ be a certain sequence of flat vector orbibundles; denote by $T_{\mathcal{O}}(\tau(m))$ their analytic torsions. Then there exists $C > 0$, $\phi_{j,k} \in \mathbb{R}$, $K \in \mathbb{N} \cup \{0\}$ and $C_{j,k}, C_j \in \mathbb{R}$ such that*

$$\log T_{\mathcal{O}}(\tau(m)) = \sum_{j=0}^{\frac{d^2+d+2}{2}} \sum_{k=0}^{K} C_{j,k} m^j e^{im\phi_{j,k}} + \sum_{j=0}^{\frac{n^2+n+2}{2}} C_j m^j + O(e^{-Cm}), \quad m \to \infty.$$

Above, $2d+1$ is the maximal dimension of the fixed point set of Γ in $\mathbb{H}^{2n+1}$.

Theorem C. *Let $\Gamma\backslash\mathbb{H}^3$ be a hyperbolic orbifold of finite volume and let $T_{\mathcal{O}}(\tau(m))$ be as above. Then*

$$\log T_{\mathcal{O}}(\tau(m)) = -\frac{1}{2\pi} \cdot \operatorname{vol}(\mathcal{O}) \cdot m^2 + O(m \cdot \log(m)), \quad m \to \infty. \tag{6.23}$$

Our main method is the Selberg trace formula; we show that though its geometric side cannot be used to calculate the analytic torsion for a given orbibundle explicitly, it suits well for studying its asymptotic behavior.