On a kinetic equation arising in weak turbulence theory for the nonlinear Schrödinger equation

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ARTHUR HUBERTUS MARTINUS KIERKELS

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1. Gutachter: Prof. Dr. Juan J. L. Velázquez
2. Gutachter: Prof. Dr. Barbara Niethammer

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If a man is in any sense a real mathematician, then it is a hundred to one that his mathematics will be far better than anything else he can do, and that he would be silly if he surrendered any decent opportunity of exercising his one talent in order to do undistinguished work in other fields.

G.H. Hardy, in [12]

Abstract

In this thesis we present a study of weak solutions to the following kinetic equation of coagulation-fragmentation type:

$$\dot{G}(x) = \frac{1}{2}\int_0^x \frac{G(x-y)G(y)}{\sqrt{(x-y)y}}\mathrm{d}y - \frac{G(x)}{\sqrt{x}}\int_0^\infty \frac{G(y)}{\sqrt{y}}\mathrm{d}y \\ - \frac{1}{2}\frac{G(x)}{\sqrt{x}}\int_0^x \left[\frac{G(y)}{\sqrt{y}} + \frac{G(x-y)}{\sqrt{x-y}}\right]\mathrm{d}y + \int_0^\infty \frac{G(x+y)}{\sqrt{x+y}}\left[\frac{G(y)}{\sqrt{y}} + \frac{G(x)}{\sqrt{x}}\right]\mathrm{d}y. \tag{QWTE}$$

This quadratic equation is the leading order approximation for long times to the isotropic space-homogeneous weak turbulence equation for the nonlinear Schrödinger equation with defocussing cubic nonlinearity.

We first recall the weak turbulence theory for that nonlinear Schrödinger equation, and we formally derive (QWTE). We then present the general theory of weak solutions to (QWTE), comprising among other things existence of solutions, conservation of mass and energy, and convergence to equilibria. A particularly interesting feature here is the instantaneous onset of a Dirac measure at zero for any nontrivial initial data.

The better part of this thesis is concerned with solutions to (QWTE) that exhibit self-similar behaviour. Due to the two conservation laws it is necessary to introduce a modified notion of self-similarity for (QWTE). In that setting we prove existence of self-similar profiles with finite mass, and either finite or infinite energy. We further present several results on the qualitative behaviour of these profiles, and we pose two conjectures that are backed with consistency analysis and numerics.

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Contents

Chapter 1

Introduction

In the physical literature, theories of weak turbulence, or wave turbulence, are theories that aim to describe the transfer of energy between different spatial frequencies in wave systems with typically weak nonlinearities. The first example can be found in [35], where it was used to describe phonon interactions in anharmonic crystals. Since then, the number of applications has expanded to also include descriptions of waves on fluid surfaces (e.g. [14], [46], [47]), in plasmas (e.g. [44], [45]), in nonlinear optics (cf. [4]), in Bose-Einstein condensates (e.g. [38,39], [40]), in the early universe (cf. [28,29]), or on elastic plates (cf. [3]). For a more exhausting list of examples and references, we recommend the recent overview paper [31], or the book [30].

Any weak turbulence theory originates from a set of nonlinear wave equations, where the nonlinearity is small. Moreover, using the small parameter $\varepsilon > 0$ to quantify the nonlinearity, we need to assume that setting $\varepsilon = 0$ yields a conservative linear system. Supposing further that the wave equations are solved in $\mathbb{R} \times \mathbb{R}^n$, and that they are invariant under translations in space and time, then the linearised problem can be solved using standard Fourier transform methods. Indeed, the space-Fourier transform $\hat{u}$ of a solution u to this linearised problem is formally given by $\hat{u}(t, \mathbf{k}) = \hat{u}(0, \mathbf{k})e^{-i\omega t}$, with $\omega = \omega(\mathbf{k})$ the real-valued dispersion relation.

Now, the object of interest in weak turbulence theory is the density $|\hat{u}(t, \mathbf{k})|^2$ in wave number space. In the linearised case this density is constant in time due to the fact that ω is real, but for $\varepsilon > 0$ the evolution of $|\hat{u}|^2$ is nontrivial due to resonances between specific wave numbers $\mathbf{k}$. Unfortunately, the dynamics of $\hat{u}$ also depends on its phase, so that it is in general not possible to obtain a closed equation for $|\hat{u}|^2$. However, weak turbulence theory hypothesises that if initial data are chosen suitably, then the evolution of $|\hat{u}|^2$ can be approximated by a kinetic equation. Solutions to that equation typically exhibit irreversible behaviour, contrary to the underlying system of equations which is in most cases time-reversible.

Roughly speaking, and restricting to scalar-valued functions u, we suppose our initial data to be of the form $\hat{u}_0(\mathbf{k}) = \sqrt{\alpha_{\mathbf{k}}}\phi_{\mathbf{k}}$, where $\alpha_{\mathbf{k}}$ are i.i.d. random variables according to some probability measure, and where $\phi_{\mathbf{k}}$ are i.i.d. random phases according to the uniform distribution on $\mathbb{S}^1$. We then expect to obtain a good approximation of the evolution of $|\hat{u}|^2$ by averaging over all phases and amplitudes. For a more elaborate road map to get from wave equations to weak turbulence equations, we refer the reader to Part II of [30].

We should note that the precise conditions under which the aforementioned approach is valid have not been rigorously obtained. However, the derivation of kinetic equations as above shows analogies with the formal derivation of the Boltzmann equation from the dynamics of particle systems (cf. [11], [18], [37]). In particular, the assumption of statistical independence of amplitudes and phases of the initial data stands out.

1.1 Weak turbulence theory for NLS

In this section we summarize the theory of weak turbulence for the following nonlinear Schrödinger equation:

$$\big(i\partial_t + \Delta_{\mathbf{x}}\big)u = \varepsilon|u|^2 u, \tag{NLS}$$

with $u = u(t, \mathbf{x}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, and $\varepsilon > 0$ small.

1.1.1 The formal derivation of the weak turbulence equation

Let us first present a formal derivation of the weak turbulence equation for (NLS), where we follow the reasoning in [30]. Applying the space-Fourier transform to (NLS) yields

$$\big(i\partial_t - |\mathbf{k}|^2\big)\hat{u} = \epsilon\ \hat{u} * \hat{u} * \hat{\bar{u}}\ \text{ with } \epsilon/\varepsilon \text{ constant,}$$

so that the function $\tilde{a}(t, \mathbf{k}) = \hat{u}(t, \mathbf{k})e^{i|\mathbf{k}|^2 t}$ satisfies

$$i\dot{\tilde{a}}(\mathbf{k}) = \epsilon \iint_{(\mathbb{R}^3)^2} \tilde{a}(\mathbf{k}_1)\tilde{a}(\mathbf{k}_2)\tilde{a}^*(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})e^{i(|\mathbf{k}|^2+|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}|^2-|\mathbf{k}_1|^2-|\mathbf{k}_2|^2)t}\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2. \tag{1.1}$$

However, we note that

$$\tilde{a}(t, \mathbf{k})\|\tilde{a}(t, \cdot)\|^2_{L^2(\mathbb{R}^3)} = \iint_{(\mathbb{R}^3)^2} \tilde{a}(t, \mathbf{k}_i)\tilde{a}(t, \mathbf{k}_j)\tilde{a}^*(t, \mathbf{k}_i + \mathbf{k}_j - \mathbf{k})\delta_0(\mathbf{k}_i - \mathbf{k})\mathrm{d}\mathbf{k}_i\mathrm{d}\mathbf{k}_j,$$

so that the contribution to the right hand side of (1.1) that comes from the integral over the submanifolds $\{\mathbf{k}_1 = \mathbf{k}\}$ and $\{\mathbf{k}_2 = \mathbf{k}\}$ is purely real, and thus only accounts for a change in the phase of $\tilde{a}$. Noting also that $\|\tilde{a}(t, \cdot)\|_{L^2(\mathbb{R}^3)} = \|\hat{u}(t, \cdot)\|_{L^2(\mathbb{R}^3)} = \|\hat{u}(0, \cdot)\|_{L^2(\mathbb{R}^3)}$, where the second identity is due to Plancherel's theorem, and the conservation laws of solutions to (NLS) (cf. [41]), we may then introduce the function

$$a(t, \mathbf{k}) = \hat{u}(t, \mathbf{k}) \exp\left\{i|\mathbf{k}|^2 t + 2i\epsilon\|\hat{u}(0, \cdot)\|^2_{L^2(\mathbb{R}^3)}t\right\},$$

which satisfies

$$i\dot{a}(\mathbf{k}) = \epsilon \iint_{(\mathbb{R}^3)^2} a(\mathbf{k}_1)a(\mathbf{k}_2)a^*(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})E^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}_{\mathbf{k}_1,\mathbf{k}_2}(t)\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2, \tag{1.2}$$

with the shorthand

$$E^{\mathbf{r}_3,\mathbf{r}_4}_{\mathbf{r}_1,\mathbf{r}_2}(\tau) = \begin{cases} 0 & \text{if } \mathbf{r}_1 = \mathbf{r}_3 \ \&\ \mathbf{r}_2 = \mathbf{r}_4 \text{ or } \mathbf{r}_1 = \mathbf{r}_4 \ \&\ \mathbf{r}_2 = \mathbf{r}_3, \\ e^{i(|\mathbf{r}_3|^2+|\mathbf{r}_4|^2-|\mathbf{r}_1|^2-|\mathbf{r}_2|^2)\tau} & \text{else.} \end{cases} \tag{1.3}$$

Next, we write a as the formal series in ϵ around an initial field b with random phase and amplitude, i.e. we set

$$a(t, \mathbf{k}) = b(\mathbf{k}) + \epsilon a_1(t, \mathbf{k}) + \epsilon^2 a_2(t, \mathbf{k}) + \cdots, \tag{1.4}$$

with $b(\mathbf{k}) = |b(\mathbf{k})|\phi_{\mathbf{k}}$, where $|b(\mathbf{k})|^2$ are i.i.d. according to some probability measure, and where all $\phi_{\mathbf{k}}$ are independent and uniformly distributed over $\mathbb{S}^1$. Introducing further the brackets $\langle\cdot\rangle$ to denote averaging over the probability distributions of $|b(\mathbf{k})|^2$ and $\phi_{\mathbf{k}}$, then using (1.4) we get that

$$\left\langle |a(t,\mathbf{k})|^2 \right\rangle - \left\langle |b(\mathbf{k})|^2 \right\rangle = \epsilon\, 2\Re \left\langle b^*(\mathbf{k}) a_1(t,\mathbf{k}) \right\rangle + \epsilon^2 \Big(\left\langle |a_1(t,\mathbf{k})|^2 \right\rangle + 2\Re \left\langle b^*(\mathbf{k}) a_2(t,\mathbf{k}) \right\rangle \Big) + \cdots . \quad (1.5)$$

Thus, in order to obtain an equation for $\langle |a(t,\mathbf{k})|^2 \rangle$ to leading order in ϵ, we are required to express a_1 and a_2 in terms of b, which is easily done using (1.4) in (1.2):

$$a_1(t,\mathbf{k}) = -i \iint_{(\mathbb{R}^3)^2} b(\mathbf{k}_1) b(\mathbf{k}_2) b^*(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}) \int_0^t E_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(s) \mathrm{d}s \, \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2, \quad (1.6)$$

$$\begin{aligned} a_2(t,\mathbf{k}) = &-i \int_0^t \iint_{(\mathbb{R}^3)^2} b(\mathbf{k}_1) b(\mathbf{k}_2) a_1^*(s,\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}) E_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(s) \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \, \mathrm{d}s \\ &- 2i \int_0^t \iint_{(\mathbb{R}^3)^2} b(\mathbf{k}_1) a_1(s,\mathbf{k}_2) b^*(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}) E_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(s) \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \, \mathrm{d}s, \quad (1.7) \end{aligned}$$

where (1.6) is to be substituted into (1.7). Writing now $\langle \cdot \rangle_\phi$ for the average over only the phases, we observe that

$$\begin{aligned} \left\langle \phi_{\mathbf{r}_1} \phi_{\mathbf{r}_2} \phi_{\mathbf{r}_3}^* \phi_{\mathbf{r}_4}^* \right\rangle_\phi = &\ \delta_0(\mathbf{r}_3-\mathbf{r}_1)\delta_0(\mathbf{r}_4-\mathbf{r}_2) + \delta_0(\mathbf{r}_4-\mathbf{r}_1)\delta_0(\mathbf{r}_3-\mathbf{r}_2) \\ &- \delta_0(\mathbf{r}_2-\mathbf{r}_1)\delta_0(\mathbf{r}_3-\mathbf{r}_2)\delta_0(\mathbf{r}_4-\mathbf{r}_3), \end{aligned}$$

which, using (1.3), implies that

$$\left\langle b(\mathbf{r}_1) b(\mathbf{r}_2) b^*(\mathbf{r}_3) b^*(\mathbf{r}_4) \right\rangle_\phi \times E_{\mathbf{r}_1,\mathbf{r}_2}^{\mathbf{r}_3,\mathbf{r}_4} \equiv 0. \quad (1.8)$$

In particular we thus have $\Re \langle b^*(\mathbf{k}) a_1(t,\mathbf{k}) \rangle_\phi = 0$, whereby the first term on the right hand side of (1.5) vanishes. In order to compute $\langle |a_1(t,\mathbf{k})|^2 \rangle_\phi$ we next find

$$\left\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}^* \phi_{\mathbf{k}_3}^* \phi_{\mathbf{k}_4}^* \phi_{\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}} \right\rangle_\phi \Big|_{\mathbf{k}_i=\mathbf{k}} = \left\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}}^* \phi_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}^* \right\rangle_\phi \ \text{for } i \in \{3,4\}, \quad (1.9)$$

and a similar expression for $i \in \{1,2\}$, whereby, with again (1.8), we conclude that

$$\left\langle |a_1(t,\mathbf{k})|^2 \right\rangle_\phi = 2 \iint_{(\mathbb{R}^3)^2} |b(\mathbf{k}_1)|^2 |b(\mathbf{k}_2)|^2 |b(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})|^2 \left| \int_0^t E_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(s) \mathrm{d}s \right|^2 \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2.$$

For $\langle b^*(\mathbf{k}) a_2(t,\mathbf{k}) \rangle_\phi$ we now write $b^*(\mathbf{k}) a_2(t,\mathbf{k}) = I_1(t,\mathbf{k}) + 2 I_2(t,\mathbf{k})$, where I_1 and I_2 are the products of $b^*(\mathbf{k})$ and, respectively, the first and second integrals on the right hand side of (1.7). Similar to (1.9), we here get

$$\left\langle \phi_{\mathbf{k}}^* \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3}^* \phi_{\mathbf{k}_4}^* \phi_{\mathbf{k}_3+\mathbf{k}_4-(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})} \right\rangle_\phi \Big|_{\mathbf{k}_i=\mathbf{k}} = \left\langle \phi_{\mathbf{k}_3}^* \phi_{\mathbf{k}_4}^* \phi_{\mathbf{k}_j} \phi_{\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_j} \right\rangle_\phi \quad \text{for } i,j \in \{1,2\} \text{ and } i \neq j,$$

and

$$\left\langle \phi_{\mathbf{k}}^* \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \phi_{\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2}^* \phi_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}^* \right\rangle_\phi \Big|_{\mathbf{k}_1=\mathbf{k}} = \left\langle \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \phi_{\mathbf{k}_2}^* \phi_{\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2}^* \right\rangle_\phi ,$$

which, together with (1.8), leads to the expressions

$$\langle I_1(t,\mathbf{k}) \rangle_\phi = 2|b(\mathbf{k})|^2 \iint_{(\mathbb{R}^3)^2} |b(\mathbf{k}_1)|^2 |b(\mathbf{k}_2)|^2 \int_0^t \int_0^s E_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(s-\sigma) \mathrm{d}\sigma \mathrm{d}s \, \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2,$$

and

$$\langle I_2(t,\mathbf{k})\rangle_\phi = -\frac{1}{2}\times 2|b(\mathbf{k})|^2 \iint_{(\mathbb{R}^3)^2} \big(|b(\mathbf{k}_1)|^2+|b(\mathbf{k}_2)|^2\big)|b(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})|^2 \times \int_0^t\int_0^s E_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(s-\sigma)\mathrm{d}\sigma\mathrm{d}s\ \mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2.$$

Observing lastly for $\Omega\in\mathbb{R}$ that

$$\frac{1}{t}\left|\int_0^t e^{i\Omega s}\mathrm{d}s\right|^2 = \frac{1}{t}\times 2\Re\left(\int_0^t\int_0^s e^{i\Omega(s-\sigma)}\mathrm{d}\sigma\mathrm{d}s\right) = \tfrac{1}{t}\left(\tfrac{2}{\Omega}\sin\left(\tfrac{\Omega t}{2}\right)\right)^2 =: \eth_t(\Omega),$$

we thus find that (1.5) to leading order in ϵ equals

$$\left\langle |a(t,\mathbf{k})|^2\right\rangle - \left\langle |b(\mathbf{k})|^2\right\rangle = \epsilon^2 t\left(\eta(t,\mathbf{k}) + \left\langle |b(\mathbf{k})|^2\right\rangle\gamma(t,\mathbf{k})\right), \tag{1.10}$$

with

$$\eta(t,\mathbf{k}) = 2\iiint_{(\mathbb{R}^3)^3} \left\langle |b(\mathbf{k}_1)|^2\right\rangle\left\langle |b(\mathbf{k}_2)|^2\right\rangle\left\langle |b(\mathbf{k}_4)|^2\right\rangle \times \eth_t(|\mathbf{k}|^2+|\mathbf{k}_4|^2-|\mathbf{k}_1|^2-|\mathbf{k}_2|^2)\delta_0(\mathbf{k}+\mathbf{k}_4-\mathbf{k}_1-\mathbf{k}_2)\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_4,$$

and

$$\gamma(t,\mathbf{k}) = 2\iiint_{(\mathbb{R}^3)^3} \Big(\left\langle |b(\mathbf{k}_1)|^2\right\rangle\left\langle |b(\mathbf{k}_2)|^2\right\rangle - \big(\left\langle |b(\mathbf{k}_1)|^2\right\rangle + \left\langle |b(\mathbf{k}_2)|^2\right\rangle\big)\left\langle |b(\mathbf{k}_4)|^2\right\rangle\Big) \times \eth_t(|\mathbf{k}|^2+|\mathbf{k}_4|^2-|\mathbf{k}_1|^2-|\mathbf{k}_2|^2)\delta_0(\mathbf{k}+\mathbf{k}_4-\mathbf{k}_1-\mathbf{k}_2)\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_4.$$

However, the characteristic evolution time of $\langle |a(t,\mathbf{k})|^2\rangle$ is now of order $\frac{1}{\epsilon^2}$, so that for $\epsilon^2 t \ll 1$, replacing the left hand side of (1.10) by $t\,\partial_t\langle |a(t,\mathbf{k})|^2\rangle$, we get

$$\partial_t\left\langle |a(t,\mathbf{k})|^2\right\rangle = \epsilon^2\left(\eta(t,\mathbf{k}) + \left\langle |b(\mathbf{k})|^2\right\rangle\gamma(t,\mathbf{k})\right). \tag{1.11}$$

Moreover, as we are interested in the limit $\epsilon\to 0$, it is reasonable to suppose that (1.11) is valid for all $t\geq 0$. Observing lastly that $\eth_t \to 2\pi\delta_0$ as $t\to\infty$, we finally obtain for $t\gg\epsilon^2$ that the function $n(\mathbf{k}) = n(t,\mathbf{k}) = \lim_{\epsilon\to 0}\langle |a(\frac{t}{\epsilon^2},\mathbf{k})|^2\rangle = \langle |b(\mathbf{k})|^2\rangle$ satisfies

$$\dot{n}(\mathbf{k}) = 4\pi\iiint_{(\mathbb{R}^3)^3} \Big(n(\mathbf{k}_1)n(\mathbf{k}_2)\big(n(\mathbf{k})+n(\mathbf{k}_4)\big) - \big(n(\mathbf{k}_1)+n(\mathbf{k}_2)\big)n(\mathbf{k})n(\mathbf{k}_4)\Big) \times \delta_0(|\mathbf{k}|^2+|\mathbf{k}_4|^2-|\mathbf{k}_1|^2-|\mathbf{k}_2|^2)\delta_0(\mathbf{k}+\mathbf{k}_4-\mathbf{k}_1-\mathbf{k}_2)\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_4, \tag{WTE}$$

the space-homogeneous weak turbulence equation associated to the defocussing nonlinear Schrödinger equation (NLS).

We would like to emphasize again that the above derivation is only formal. Despite the fact that (WTE) has been frequently studied, both in its own right (cf. [4], [30], [48]), and in the context of Bose-Einstein condensation (cf. [15], [17], [36], [38,39], [40]), its rigorous derivation is still a largely open problem. However, for first rigorous results in the case of the discrete nonlinear Schrödinger equation, see [25] and [26,27].

1.1.2 Isotropic solutions to the weak turbulence equation

The paper [9] presents a number of results on isotropic solutions to (WTE), i.e. solutions of the form $n(\mathbf{k}) = f(|\mathbf{k}|^2)$. Now, using this as an ansatz in (WTE), switching to spherical coordinates with $k = |\mathbf{k}|$, and using the integral expression of δ_0, we obtain

$$\dot{f}(k^2) = 4\pi \iiint_{[0,\infty)^3} W\Big(f(k_1^2)f(k_2^2)\big(f(k^2) + f(k_4^2)\big) - \big(f(k_1^2) + f(k_2^2)\big)f(k^2)f(k_4^2)\Big) \times \delta_0(k^2 + k_4^2 - k_1^2 - k_2^2)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_4,$$

with

$$\begin{aligned} W = W(k_1,k_2,k,k_4) &= k_1^2k_2^2k_4^2 \times \iiint_{(\mathbb{S}^2)^3} \left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\mathbf{s}\cdot(\mathbf{k}+\mathbf{k}_4-\mathbf{k}_1-\mathbf{k}_2)}\mathrm{d}\mathbf{s}\right]\mathrm{d}\Omega_1\mathrm{d}\Omega_2\mathrm{d}\Omega_4 \\ &= 8k_1k_2k_4 \times \frac{4\pi}{k}\int_0^\infty \sin(k_1s)\sin(k_2s)\sin(ks)\sin(k_4s)\frac{\mathrm{d}s}{s^2}, \end{aligned}$$

where the integral on the right hand side evaluates to $\frac{\pi}{4}\min\{k_1,k_2,k,k_4\}$ (cf. [39], [7], or Lemma 6 in [Kie16]). Making then the change of variables to $\omega = k^2$, and integrating out the remaining Dirac delta, we find that f must satisfy

$$\begin{aligned} \dot{f}(\omega) = 4\pi^3 \iint_{[0,\infty)^2} \frac{K}{\sqrt{\omega}}\Big(& f(\omega_1)f(\omega_2)\big(f(\omega) + f(\omega_1+\omega_2-\omega)\big) \\ &- \big(f(\omega_1)+f(\omega_2)\big)f(\omega)f(\omega_1+\omega_2-\omega)\Big)\mathrm{d}\omega_1\mathrm{d}\omega_2, \end{aligned} \tag{1.12}$$

with $K = K(\omega_1,\omega_2,\omega) = \min\{\sqrt{\omega_1},\sqrt{\omega_2},\sqrt{\omega},\sqrt{(\omega_1+\omega_2-\omega)_+}\}$. However, we will see that it is actually more convenient to study $g(\omega) = (2\pi)^{3/2}\sqrt{\omega}f(\omega)$, which satisfies

$$\begin{aligned} \dot{g}(\omega) = \frac{1}{2}\iint_{[0,\infty)^2} K\Bigg[& \frac{g(\omega_1)}{\sqrt{\omega_1}}\frac{g(\omega_2)}{\sqrt{\omega_2}}\left(\frac{g(\omega)}{\sqrt{\omega}} + \frac{g(\omega_1+\omega_2-\omega)}{\sqrt{\omega_1+\omega_2-\omega}}\right) \\ &- \left(\frac{g(\omega_1)}{\sqrt{\omega_1}} + \frac{g(\omega_2)}{\sqrt{\omega_2}}\right)\frac{g(\omega)}{\sqrt{\omega}}\frac{g(\omega_1+\omega_2-\omega)}{\sqrt{\omega_1+\omega_2-\omega}}\Bigg]\mathrm{d}\omega_1\mathrm{d}\omega_2, \end{aligned} \tag{CWTE}$$

with K as before. Indeed, integrating (CWTE) against a function φ, using the abbreviated notations $f_i = g(\omega_i)/\sqrt{\omega_i}$ and $\varphi_i = \varphi(\omega_i)$ for $i \in \{1,2,3,4\}$, and exploiting the symmetries of the equation, we find that

$$\begin{aligned} \int_{[0,\infty)} \varphi(\omega)\dot{g}(\omega)\mathrm{d}\omega &= \frac{1}{2}\int\!\cdot\!\cdot\!\int_{[0,\infty)^4} \delta_0(\omega_3+\omega_4-\omega_1-\omega_2)K(\omega_1,\omega_2,\omega_3) \\ &\qquad\times\big[f_1f_2(f_3+f_4) - (f_1+f_2)f_3f_4\big]\varphi_3\,\mathrm{d}\omega_1\cdot\cdot\,\mathrm{d}\omega_4 \\ &= \frac{1}{4}\int\!\cdot\!\cdot\!\int_{[0,\infty)^4} \delta_0K\big[f_1f_2(f_3+f_4) - (f_1+f_2)f_3f_4\big](\varphi_3+\varphi_4)\mathrm{d}\omega_1\cdot\cdot\,\mathrm{d}\omega_4 \\ &= \frac{1}{4}\int\!\cdot\!\cdot\!\int_{[0,\infty)^4} \delta_0K\big[f_1f_2(f_3+f_4)\big](\varphi_3+\varphi_4-\varphi_1-\varphi_2)\mathrm{d}\omega_1\cdot\cdot\,\mathrm{d}\omega_4, \end{aligned} \tag{1.13}$$

where, since we only integrate over the submanifold $\{\omega_3+\omega_4 = \omega_1+\omega_2\}$, the right hand side vanishes for $\varphi \equiv 1$ and $\varphi(\omega) = \omega$. The evolution of (CWTE) thus formally conserves both the integral and the first moment, which we prefer over the conservation of the moments $\frac{1}{2}$ and $\frac{3}{2}$ by the evolution of (1.12). Moreover, integrating (1.13) with respect to the time variable, we obtain

$$\int_{[0,\infty)} \varphi(\omega) g(t,\omega) \mathrm{d}\omega - \int_{[0,\infty)} \varphi(\omega) g(0,\omega) \mathrm{d}\omega = \int_0^t \mathcal{C}_4[g(s,\cdot)](\varphi) \mathrm{d}s \tag{1.14}$$

with

$$\mathcal{C}_4[g](\varphi) = \frac{1}{2} \iiint_{[0,\infty)^3} \frac{K(\omega_1,\omega_2,\omega_3)}{\sqrt{\omega_1\omega_2\omega_3}} \big(\varphi(\omega_3) + \varphi(\omega_1+\omega_2-\omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\big) \times g(\omega_1)g(\omega_2)g(\omega_3)\, \mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3. \tag{1.15}$$

which is well-defined for suitable g and φ (cf. Lemma A.1). It is therefore natural to use (1.14) in order to define the notion of a weak solution to (CWTE), as was done in [9].

The weak formulation allows us to assign a particle interpretation to (CWTE). Indeed, if we suppose $g(t,\omega)$ to be a distribution of particles of sizes $\omega \geq 0$ at time t, then the interaction mechanism is as follows: Two particles of sizes $\omega_1, \omega_2 \geq 0$ interact to produce two particles of sizes $\omega_3, \omega_4 \geq 0$ with $\omega_3+\omega_4 = \omega_1+\omega_2$, where the incidence rate of interaction is proportional to

$$\frac{K(\omega_1,\omega_2,\omega_3)}{\sqrt{\omega_1\omega_2\omega_3}} \times g(t,\omega_1)g(t,\omega_2)g(t,\omega_3).$$

In particular, an interaction can only take place if either of the two resulting particle sizes is already present in the distribution. However, if no particles of size 0 are present, then the interaction $\{\omega_1,\omega_2\} \to \{\omega_1+\omega_2, 0\}$ will not take place, since $K(\omega_1,\omega_2,\omega_1+\omega_2) = 0$.

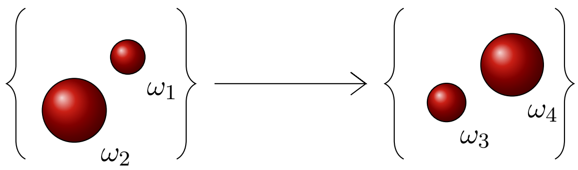

FIGURE 1.1: Collision mechanism in the particle interpretation of (CWTE). Two particles of sizes $\omega_1, \omega_2 \geq 0$ interact to produce two particles of sizes $\omega_3, \omega_4 \geq 0$ with $\omega_3 + \omega_4 = \omega_1 + \omega_2$, where the rate of interaction depends on the amount of particles of sizes ω_1, ω_2, *and* ω_3, that are already present in the distribution.

Besides global existence of weak solutions to (CWTE), it was shown in [9] that the integral and the first moment of a weak solution are invariant. Noting that these conserved quantities correspond to the ones of the linear Schrödinger, being the norms in $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, we will further refer to the integral of a solution as its mass, and to the first moment as its energy. However, it was also shown that all solutions with mass $m \geq 0$ converge in the sense of measures to a Dirac delta with mass m at $\breve{\omega} \geq 0$, where $\breve{\omega}$ denotes the infimum of all sizes of particles that can be obtained by the collision mechanism. Since for most solutions there holds $\breve{\omega} = 0$, and since the energy of a Dirac delta at 0 is zero, this indicates that there has to be some transfer of energy towards infinity. This is the topic of this thesis.

1.1.3 Relation with the bosonic Nordheim equation

Though weak turbulence theory for (NLS) is interesting in its own right, the weak turbulence equation is strongly related to the bosonic Nordheim equation, which was derived by Nordheim as the quantum analogue to the Boltzmann equation (cf. [34]). Its isotropic form is given by

$$\dot{f}(\omega) = \iint_{[0,\infty)^2} \frac{K}{\sqrt{\omega}} \Big(f(\omega_1) f(\omega_2) \big(f(\omega) + f(\omega_1 + \omega_2 - \omega) + 1 \big) - \big(f(\omega_1) + f(\omega_2) + 1 \big) f(\omega) f(\omega_1 + \omega_2 - \omega) \Big) \mathrm{d}\omega_1 \mathrm{d}\omega_2, \quad \text{(BNE)}$$

which differs only from (CWTE) in the occurrence of the regular quadratic Boltzmann terms in the integral on the right hand side. The connection between these equations has been pointed out by several authors (cf. e.g. [4], [17], [30]), and it is thought that the cubic terms should be the dominant ones in certain limits. Indeed, many of the results that were obtained in [7,8] for (BNE) are similar to the ones in [9] for (CWTE), and it is because of this that we refer to the onset of a Dirac delta in (CWTE) as the formation of a condensate. For further results on the quantum Boltzmann equation, we refer the reader to [20–23].

1.2 The quadratic weak turbulence equation

In order to better understand the transfer of energy towards infinity, we suppose that the long time behaviour of weak solutions to (CWTE) can be approximated by the perturbation of a Dirac mass. To that end we consider weak solutions g to (CWTE) of the form $g = \delta_0 + G$, where G is a nonnegative measure-valued function with mass $0 < \varepsilon \ll 1$. In this section we determine the evolution equation for G in the limit $\varepsilon \to 0$, which is the equation of interest in this thesis.

Now, we want to use the ansatz $g(t,\cdot) = \delta_0 + G(t,\cdot)$ in (1.14), to which end we first set $g = \delta_0 + G$ in (1.15). This gives us

$$\mathcal{C}_4[\delta_0 + G](\varphi) = \mathcal{C}_4[G](\varphi) + \mathcal{C}_3^{123}[G](\varphi) + \mathcal{C}_3^{231}[G](\varphi) + \mathcal{C}_3^{312}[G](\varphi) + \mathcal{C}_2^{123}[G](\varphi) + \mathcal{C}_2^{231}[G](\varphi) + \mathcal{C}_2^{312}[G](\varphi) + \mathcal{C}_4[\delta_0](\varphi), \quad (1.16)$$

with

$$\mathcal{C}_3^{ijk}[G](\varphi) = \frac{1}{2} \iiint_{[0,\infty)^3} \frac{K(\omega_1,\omega_2,\omega_3)}{\sqrt{\omega_1\omega_2\omega_3}} \big(\varphi(\omega_3) + \varphi(\omega_1+\omega_2-\omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\big) \times G(\omega_i) G(\omega_j) \delta_0(\omega_k)\, \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3,$$

and

$$\mathcal{C}_2^{ijk}[G](\varphi) = \frac{1}{2} \iiint_{[0,\infty)^3} \frac{K(\omega_1,\omega_2,\omega_3)}{\sqrt{\omega_1\omega_2\omega_3}} \big(\varphi(\omega_3) + \varphi(\omega_1+\omega_2-\omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\big) \times G(\omega_i) \delta_0(\omega_j) \delta_0(\omega_k)\, \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3,$$

where, by symmetries in the integrands, we immediately note that $\mathcal{C}_3^{231}[G] \equiv \mathcal{C}_3^{312}[G]$ and $\mathcal{C}_2^{123}[G] \equiv \mathcal{C}_2^{231}[G]$. We then interpret the last seven terms on the right hand side of (1.16) as particle interactions where one or more of the particles $\omega_1, \omega_2, \omega_3 \geq 0$ have size 0. If all are zero particles, i.e. if $\omega_1 = \omega_2 = \omega_3 = 0$, then $\omega_4 = 0$, which suggests $\mathcal{C}_4[\delta_0] \equiv 0$. Further, $\omega_1 = \omega_2 = 0$ implies $\omega_3 = \omega_4 = 0$, and if $\omega_1 = \omega_3 = 0$ or $\omega_2 = \omega_3 = 0$, then we respectively get $\omega_4 = \omega_2$ or $\omega_4 = \omega_1$, which is an indication that $\mathcal{C}_2^{123}[G] \equiv \mathcal{C}_2^{231}[G] \equiv \mathcal{C}_2^{312}[G] \equiv 0$. This can now be formalized using Lemma A.1, whereby the integrand in the right hand side of (1.15) vanishes on the axes $\{\omega_1 = \omega_2 = 0\}$, $\{\omega_1 = \omega_3 = 0\}$, and $\{\omega_2 = \omega_3 = 0\}$, and we find that

$$\int_{[0,\infty)} \varphi(\omega)G(t,\omega)\mathrm{d}\omega - \int_{[0,\infty)} \varphi(\omega)G(0,\omega)\mathrm{d}\omega \\ = \int_0^t \Big[\mathcal{C}_4[G(s,\cdot)](\varphi) + \mathcal{C}_3^{123}[G(s,\cdot)](\varphi) + 2\,\mathcal{C}_3^{231}[G(s,\cdot)](\varphi)\Big]\mathrm{d}s. \quad (1.17)$$

Integrating out the Dirac deltas, we now get that

$$\mathcal{C}_3^{123}[G](\varphi) = \frac{1}{2}\iint_{[0,\infty)^2} \frac{G(\omega_1)G(\omega_2)}{\sqrt{\omega_1\omega_2}}\big(\varphi(0) + \varphi(\omega_1+\omega_2) - \varphi(\omega_1) - \varphi(\omega_2)\big)\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

and

$$\mathcal{C}_3^{231}[G](\varphi) = \frac{1}{2}\iint_{\{\omega_2\geq\omega_3\geq 0\}} \frac{G(\omega_2)G(\omega_3)}{\sqrt{\omega_2\omega_3}}\big(\varphi(\omega_3) + \varphi(\omega_2-\omega_3) - \varphi(0) - \varphi(\omega_2)\big)\mathrm{d}\omega_2\mathrm{d}\omega_3,$$

which are well-defined for suitable G and φ [cf. (A.2) and (A.3)]. We then combine these terms to obtain

$$\mathcal{C}_3[G](\varphi) := \mathcal{C}_3^{123}[G](\varphi) + 2\,\mathcal{C}_3^{231}[G](\varphi) \\ = \frac{1}{2}\iint_{\mathbb{R}_+^2} \frac{G(x)G(y)}{\sqrt{xy}}\big(\varphi(x+y) + \varphi(|x-y|) - 2\varphi(\max\{x,y\})\big)\mathrm{d}x\mathrm{d}y, \quad (1.18)$$

which in view of Lemma 1.6 is well-defined for suitable G and φ. Normalizing lastly G to a probability measure, we find from (1.17) and (1.18) that $G(t,\cdot) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}G(\frac{t}{\varepsilon},\cdot)$ must satisfy

$$\int_{[0,\infty)} \varphi(\omega)G(t,\omega)\mathrm{d}\omega - \int_{[0,\infty)} \varphi(\omega)G(0,\omega)\mathrm{d}\omega = \int_0^t \mathcal{C}_3[G(s,\cdot)](\varphi)\mathrm{d}s \quad (1.19)$$

which is the motivation for our notion of weak solution (cf. Definition 1.7). It was already noted in [9], assuming sufficient regularity and convergence of integrals, that (1.19) can be seen to be the weak formulation of a quadratic integro-differential equation. This so-called *quadratic weak turbulence equation* was written in [Kie16] as

$$\dot{G}(x) = \frac{1}{2}\int_0^x \frac{G(x-y)G(y)}{\sqrt{(x-y)y}}\mathrm{d}y - \frac{G(x)}{\sqrt{x}}\int_0^\infty \frac{G(y)}{\sqrt{y}}\mathrm{d}y \\ - \frac{1}{2}\frac{G(x)}{\sqrt{x}}\int_0^x \left[\frac{G(y)}{\sqrt{y}} + \frac{G(x-y)}{\sqrt{x-y}}\right]\mathrm{d}y + \int_0^\infty \frac{G(x+y)}{\sqrt{x+y}}\left[\frac{G(y)}{\sqrt{y}} + \frac{G(x)}{\sqrt{x}}\right]\mathrm{d}y, \quad \text{(QWTE)}$$

which allows to easily recognise a connection with coagulation-fragmentation equations.

1.2.1 Connection with coagulation-fragmentation equations

The quadratic weak turbulence equation (QWTE) show analogies with the coagulation-fragmentation equation (cf. [19])

$$\dot{c}(x) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y)c(y)\mathrm{d}y - \int_0^\infty K(x,y)c(x)c(y)\mathrm{d}y \\ - \frac{1}{2}\int_0^x B(x,y)c(x)\mathrm{d}y + \int_0^\infty B(x+y,x)c(x+y)\mathrm{d}y, \quad \text{(CFE)}$$

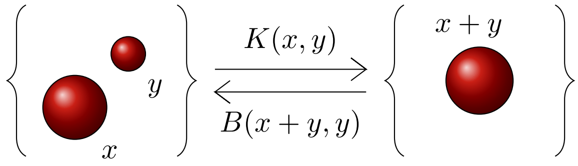

FIGURE 1.2: Binary coagulation and fragmentation of particles. For (CFE) the rate of fragmentation is independent of the particle concentration of the fragments, while for (QWTE) there exists a conditionality on this process.

which describes the time evolution of a particle distribution c, where the symmetric kernel $K(x, y)$ gives the rate of coagulation of two particles of sizes $x, y \geq 0$ into a single particle of size $x + y$, and where the function $B(x, y)$ gives the rate at which a particle of size $x \geq 0$ fragments into two particles of sizes $x - y, y \geq 0$. Indeed, we see here that (QWTE) is an equation of coagulation-fragmentation type, with singular coagulation kernel $K(x, y) = (xy)^{-1/2}$, and some conditional fragmentation rate

$$B(x, y) = K(x, y)c(y) + K(x, x - y)c(x - y). \tag{1.20}$$

Stationary solutions are of particular interest in the study of (CFE), since they are conjectured to be the limiting distributions. In general these are dynamical equilibria, where every process is exactly cancelled by its inverse process. Such solutions should thus satisfy the so-called detailed balance condition

$$K(x, y)c(x)c(y) = B(x + y, y)c(x + y), \tag{1.21}$$

which, depending on K and B, may or may not be recursively solved. Now, using (1.20) in (1.21) with $K(x, y) = (xy)^{-1/2}$, we might expect $G(x) = x^{-1/2}$ to be a stationary solution to (QWTE). However, it was noted in [KV15] that this power law corresponds to a function with constant flux of mass from infinity towards the origin (cf. Section 4.3).

Lastly, we recall that the weak formulation of a pure coagulation equation, i.e. (CFE) with $B \equiv 0$, is given by (cf. [19])

$$\int_{[0,\infty)} \varphi(x)\dot{c}(x)\mathrm{d}x = \frac{1}{2}\iint_{[0,\infty)^2} K(x, y)c(x)c(y)\big[\varphi(x + y) - \varphi(x) - \varphi(y)\big]\mathrm{d}x\mathrm{d}y.$$

In the right hand side we then recognise the collision mechanism of the particle interpretation: Two particles of sizes $x, y \geq 0$ are replaces by one of size $x + y$. Using (1.18) we similarly find a particle interpretation for (QWTE) (cf. Figure 1.3).

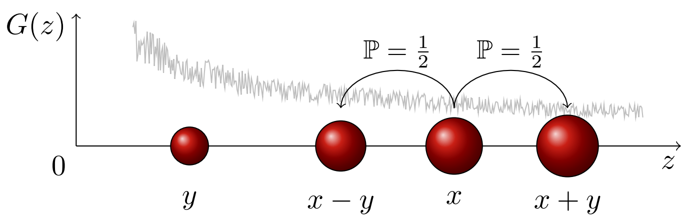

FIGURE 1.3: The nett particle interpretation of (QWTE): If two particles of sizes $x, y \geq 0$ interact, then the larger one disappears from the distribution, and it is replaced, with equal probability, by a particle of size $x+y$ or $|x-y|$.

1.3 Main results, and structure of the thesis

Most of the results that are presented in this thesis have already appeared in the papers [KV15,KV16], and in the preprint [Kie16]. The arguments in these publications have been merged, in particular on the level of notation, and we have tried to simplify our proofs as much as possible.

In Chapter 2 we present the general theory of weak solutions to (QWTE), comprising a proof of existence, the conservation laws of (QWTE), and some monotonicity results. This is essentially the first part of [KV15]. That paper also discusses the phenomenon of instantaneous condensation, i.e. the immediate onset of a Dirac mass at zero for all initial data, and we present a simplified proof here in Chapter 3. As a corollary we then get our first main result, which we may state as

> **Theorem.** *A weak solution to* (QWTE) *has either got a strictly growing condensate at the origin, or it is time-independent.*

A modified notion of self-similar solution to (QWTE) with finite mass was introduced in the final part of [KV15] for finite energy, and then extended in [KV16] to also include solutions with infinite energy. In Section 4.1 we present severely modified proofs of the existence results in those papers (cf. Remark 4.5), which form our second main result:

> **Theorem.** *There are weak solutions to* (QWTE) *that transfer their (possibly infinite) energy towards infinity in a self-similar manner.*

The rigorous results on qualitative behaviour of self-similar profiles, obtained in [KV16], as well as two conjectures from [Kie16], can now be found in Section 4.2.

1.4 Basic notations and definitions

In the final section of this introduction we present notations and definitions that will be used throughout the thesis. The functional analytic setting in which we prove existence of self-similar profiles will be introduced later, on page 35.

We start with some spaces of functions.

Definition 1.1. Given an interval $I \subset [-\infty, \infty]$, we denote by $C(I)$ the space of real-valued functions that are continuous at every point $x \in I$, by $C_c(I)$ the subspace of functions $f \in C(I)$ for which $\operatorname{supp} f \subset I$, and by $C_0(I)$ the closure of $C_c(I)$ in $L^\infty(\mathbb{R})$.

Given further $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, let $C^k(I)$ be the subspace of functions $f \in C(I)$ that are k times differentiable on I with $f^{(\ell)} \in C(I)$ for all $\ell \in \{1, \dots, k\}$, let $C^k_c(I)$ be the subspace of functions $f \in C^k(I)$ for which $\operatorname{supp} f \subset I$, let $C^k_0(I)$ be the closure of $C^k_c(I)$ in $W^{k,\infty}(\mathbb{R})$, let $C^{0,\alpha}(I)$ be the subspace of functions $f \in C(I)$ that are α-Hölder continuous on all compact sets $K \subset I$, and let $C^{k,\alpha}(I)$ be the subspace of functions $f \in C^k(I)$ for which $f^{(k)} \in C^{0,\alpha}(I)$.

Lastly, set $C^\infty(I) = \bigcap_{k\in\mathbb{N}} C^k(I)$, $C^\infty_c(I) = \bigcap_{k\in\mathbb{N}} C^k_c(I)$, and $C^\infty_0(I) = \bigcap_{k\in\mathbb{N}} C^k_0(I)$.

It should be noted that the previous definition is the usual one for $I \subset \mathbb{R}$. The notion of continuity at infinity for a function $f \in C(I)$, where $\{\infty\} \in I$, equates to existence in $\mathbb{R}$ of the limit $\lim_{x\to\infty} f(x)$.

We then define spaces of Radon measures by duality.

Definition 1.2. Given an interval $I \subset [0, \infty]$, we define the space $\mathcal{M}(I)$ of Radon measures on I to be the space of bounded linear functionals on $C_0(I)$, i.e. $\mathcal{M}(I) = (C_0(I))'$. The subspace of nonnegative Radon measures is denoted by $\mathcal{M}_+(I)$, and the space of nonnegative Radon measures with unit measure, i.e. the space of probability measures, is denoted by $\mathcal{P}(I)$.

Furthermore, we write $\mathscr{M}_+(0,\infty)$ for the subset of measures $\mu \in \mathcal{M}_+([0,\infty])$ for which $\mu(\{\infty\}) = 0$.

We will always write $\mu(x)\mathrm{d}x$ to denote integration with respect to a measure μ, even if the measure is not absolutely continuous with respect to the Lebesgue measure. Also, we write $\|\mu\| = \int_I |\mu(x)|\mathrm{d}x$ for any $\mu \in \mathcal{M}(I)$.

The spaces of measures are endowed with their natural coarsest topologies.

Definition 1.3. Given an interval $I \subset [0,\infty]$, we endow the space $\mathcal{M}(I)$, and its subspaces $\mathcal{M}_+(I)$ and $\mathcal{P}(I)$, with the weak-$*$ topology, which is the coarsest topology such that for every $\varphi \in C_0(I)$ the mapping $\mu \mapsto \int_I \varphi(x)\mu(x)\mathrm{d}x$ is continuous.

Since further $C_c([0,\infty)) \subset C([0,\infty]) = C_c([0,\infty])$, hence $\mathcal{M}_+([0,\infty]) \subset \mathcal{M}_+([0,\infty))$, we may endow $\mathscr{M}_+(0,\infty)$ with the weak-$*$ topology of $\mathcal{M}_+([0,\infty))$.

We note here that, since $(C_0(I), \|\cdot\|_{L^\infty(I)})$ is a separable Banach space, we can uniquely characterize convergence with respect to the weak-$*$ topology in the following manner: A sequence $\{\mu_n\} \subset \mathcal{M}(I)$ converges with respect to the weak-$*$ topology to $\mu \in \mathcal{M}(I)$, for short $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(I)$, if and only if $\int_I \varphi(x)\mu_n(x)\mathrm{d}x \to \int_I \varphi(x)\mu(x)\mathrm{d}x$ for all $\varphi \in C_0(I)$.

Definition 1.4. We define our frequently used notations.

- Given a function $f \in C(\mathbb{R})$, we will denote by $\Delta_y^2 f(x)$ the second order central difference of size $y \in \mathbb{R}$ at $x \in \mathbb{R}$, i.e. we define

$$\Delta_y^2 f(x) = f(x+y) + f(x-y) - 2f(x).$$

- For $x, y \in \mathbb{R}$ we use the notations $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$, and for $u \in \mathbb{R}$ we write $(u)_+ = \max\{0,u\}$.

⋆ Given a function $\varphi \in C([0,\infty])$, then the first two items of this definition allow us to write $\varphi(x+y) + \varphi(|x-y|) - 2\varphi(\max\{x,y\}) = \Delta_{x\wedge y}^2 \varphi(x \vee y)$ for $x, y \geq 0$.

- We write $\mathbb{R}_+$ for the set of strictly positive real numbers, i.e. $\mathbb{R}_+ = (0,\infty)$.

- Given functions $f, g \in C(\mathbb{R}_+)$, then if there holds $\lim_{z\to\varpi} \frac{f(z)}{g(z)} = 1$ with $\varpi \in \{0,\infty\}$, then we write $f(z) \sim g(z)$ as $z \to \varpi$.

Lastly, we introduce the notion of weak solution to (QWTE). We will need two further lemmas, whose elementary proofs can be found in the appendix.

Lemma 1.5. *Given a function $G \in C([0,\infty) : \mathscr{M}_+(0,\infty))$, then for every $T \geq 0$ there holds*

$$\sup_{t\in[0,T]} \|G(t,\cdot)\| < \infty.$$

Lemma 1.6. *Given a function $\varphi \in C_c([0,\infty)) \cap W^{1,\infty}(0,\infty)$ for which φ' is continuous in a neighbourhood of 0, let $F \in C(\mathbb{R}_+^2)$ be the symmetric function that for $x > y > 0$ satisfies*

$$F(x,y) = \frac{1}{\sqrt{xy}}\big(\varphi(x+y) + \varphi(x-y) - 2\varphi(x)\big),$$

i.e. $F(x,y) = \frac{1}{\sqrt{xy}}\Delta_{x\wedge y}^2 \varphi(x \vee y)$. Then F extends by zero to the closure of $\mathbb{R}_+^2$, i.e. $F \in C_0(\mathbb{R}_+^2)$.

These lemmas allow us to pose the following

Definition 1.7. We call G a *weak solution* to (QWTE) if (i) $G \in C([0,\infty) : \mathscr{M}_+(0,\infty))$, i.e. the mapping $t \mapsto G(t,\cdot)$ is continuous from $[0,\infty)$ into $\mathscr{M}_+(0,\infty)$, endowed with the weak-$*$ topology on $(C_0([0,\infty)))'$; and (ii) for all $t \geq 0$ and $\varphi \in C([0,\infty) : C_c^1([0,\infty))) \cap C^1([0,\infty) : C_c([0,\infty)))$ there holds

$$\int_{[0,\infty)} \varphi(t,x)G(t,x)\mathrm{d}x - \int_{[0,\infty)} \varphi(0,x)G(0,x)\mathrm{d}x - \int_0^t \int_{[0,\infty)} \varphi_s(s,x)G(s,x)\mathrm{d}x \,\mathrm{d}s$$
$$= \int_0^t \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} \Delta^2_{x\wedge y}[\varphi(s,\cdot)](x\vee y)\mathrm{d}x\mathrm{d}y \,\mathrm{d}s \qquad \text{(QWTE)}^{\mathrm{w}}$$

with $\Delta^2_{x\wedge y}[\varphi(s,\cdot)](x\vee y) = \varphi(s,x+y) + \varphi(s,|x-y|) - 2\varphi(s,x\vee y)$ (cf. Definition 1.4).

Chapter 2

General theory

In this chapter we present the general theory of weak solutions to (QWTE). Global existence is shown in Section 2.1, and in Section 2.2 we prove several basic properties of solutions, such as uniform tightness, a scaling result, and their conservation laws. In the final Section 2.3 we introduce the notion of a stationary trivial solution. We further show that the weak-$*$ limit of any weak solution to (QWTE) is trivial in that sense.

2.1 Existence of weak solutions

We will prove the following

Theorem 2.1. *Given $G_0 \in \mathscr{M}_+(0,\infty)$, there exists at least one weak solution G to* (QWTE) *in the sense of Definition 1.7 that satisfies $G(0,\cdot) \equiv G_0$ on $[0,\infty)$.*

The way to prove Theorem 2.1 is quite standard: We replace the kernel $(xy)^{-1/2}$ by a net of bounded approximations, prove existence of solutions to these approximate problems by means of a fixed-point argument, and finally show that in the limit we obtain a weak solution to (QWTE). Our approximating kernels will be $((x+\varepsilon)(y+\varepsilon))^{-1/2}$ with $\varepsilon > 0$, but in view of the product decomposition of this function we prove Lemma 2.2 below, which is slightly more general.

Lemma 2.2. *Given a nonnegative function $k \in C_0([0,\infty))$, then for any $G_0 \in \mathcal{M}_+([0,\infty])$ there exists at least one $G \in C([0,\infty)\colon \mathcal{M}_+([0,\infty]))$ that for all $t \geq 0$ and $\varphi \in C^1([0,\infty)\colon C([0,\infty]))$ satisfies*

$$\int_{[0,\infty]} \varphi(t,x)G(t,x)\mathrm{d}x - \int_{[0,\infty]} \varphi(0,x)G_0(x)\mathrm{d}x - \int_0^t \int_{[0,\infty]} \varphi_s(s,x)G(s,x)\mathrm{d}x\ \mathrm{d}s$$
$$= \int_0^t \frac{1}{2} \iint_{[0,\infty)^2} G(s,x)G(s,y)\ k(x)k(y)\Delta^2_{x\wedge y}[\varphi(s,\cdot)](x\vee y)\ \mathrm{d}x\mathrm{d}y\ \mathrm{d}s. \quad (2.1)$$

In particular, there holds $\|G(t,\cdot)\| = \|G_0\|$ for all $t \geq 0$.

Proof. If either $k \equiv 0$ or $G_0 \equiv 0$, then clearly $G(t,\cdot) \equiv G_0$ satisfies (2.1) for all $t \geq 0$ and $\varphi \in C^1([0,\infty) : C([0,\infty]))$. We thus suppose that $\|k\|_{L^\infty(0,\infty)} =: \kappa > 0$ and $\|G_0\| =: M > 0$, and we fix $T = (8\kappa^2 M)^{-1} > 0$.

For $\varepsilon > 0$ arbitrarily fixed, we now set $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ with $\phi(x) = (1-|x|)_+$. We then first show that there exists at least on $G_\varepsilon \in C([0,T] : \mathcal{M}_+([0,\infty]))$ that for all $t \in [0,T]$ and $\varphi \in C([0,\infty])$ satisfies

$$\int_{[0,\infty]} \varphi(x)G_\varepsilon(t,x)\mathrm{d}x = \int_{[0,\infty]} \varphi(x)e^{-\int_0^t A_\varepsilon[G_\varepsilon(s,\cdot)](x)\mathrm{d}s}G_0(x)\mathrm{d}x$$
$$+ \int_0^t \int_{[0,\infty]} \varphi(x)e^{-\int_\sigma^t A_\varepsilon[G_\varepsilon(s,\cdot)](x)\mathrm{d}s}B_\varepsilon[G_\varepsilon(\sigma,\cdot)](x)\mathrm{d}x\ \mathrm{d}\sigma, \quad (2.2)$$

where $A_\varepsilon : \mathcal{M}_+([0,\infty]) \to C_0([0,\infty))$ is given by

$$A_\varepsilon[G](x) = 2k(x) \int_{[0,\infty)} \int_0^x \phi_\varepsilon(y-z)k(z)\mathrm{d}z\, G(y)\mathrm{d}y \geq 0,$$

and where $B_\varepsilon : \mathcal{M}_+([0,\infty]) \to \mathcal{M}_+([0,\infty])$ is such that for any $\varphi \in C([0,\infty])$ there holds

$$\int_{[0,\infty]} \varphi(x)B_\varepsilon[G](x)\mathrm{d}x \\ = \iint_{[0,\infty)^2} G(x)G(y)\, k(x) \int_0^x \phi_\varepsilon(y-z)k(z)(\varphi(x+z)+\varphi(x-z))\mathrm{d}z\, \mathrm{d}x\mathrm{d}y.$$

Note that both mappings A_ε and B_ε are continuous due to the additional regularization, achieved by convolving with ϕ_ε. Therefore, the right hand side of (2.2) is continuous as a function of t on $[0,T]$ for all $G_\varepsilon \in C([0,T] : \mathcal{M}_+([0,\infty]))$ and $\varphi \in C([0,\infty])$, whence we can define a mapping $\mathcal{T}_\varepsilon$ from $C([0,T] : \mathcal{M}_+([0,\infty]))$ into itself such that for all $G \in C([0,T] : \mathcal{M}_+([0,\infty]))$, $t \in [0,T]$, and $\varphi \in C([0,\infty])$ there holds

$$\int_{[0,\infty]} \varphi(x)\mathcal{T}_\varepsilon[G](t,x)\mathrm{d}x = \int_{[0,\infty]} \varphi(x)e^{-\int_0^t A_\varepsilon[G(s,\cdot)](x)\mathrm{d}s} G_0(x)\mathrm{d}x \\ + \int_0^t \int_{[0,\infty]} \varphi(x)e^{-\int_\sigma^t A_\varepsilon[G(s,\cdot)](x)\mathrm{d}s} B_\varepsilon[G(\sigma,\cdot)](x)\mathrm{d}x\, \mathrm{d}\sigma. \quad (2.3)$$

Using then first $\varphi \equiv 1$ in (2.3), we find for all $t \in [0,T]$ that

$$\|\mathcal{T}_\varepsilon[G](t,\cdot)\| \leq \|G_0\| + 2\kappa^2 \int_0^t \|G(\sigma,\cdot)\|^2 \mathrm{d}\sigma \leq \|G_0\| + 2\kappa^2 \left(\sup_{t\in[0,T]} \|G(t,\cdot)\|\right)^2 T,$$

and it follows with our choice of $T > 0$ that $\mathcal{T}_\varepsilon$ maps $\mathcal{X} = \{G \in C([0,T] : \mathcal{M}_+([0,\infty])) : \sup_{t\in[0,T]} \|G(t,\cdot)\| \leq 2M\}$ into itself. For $G \in \mathcal{X}$, $0 \leq t_1 \leq t_2 \leq T$, and $\varphi \in C([0,\infty])$, we further find that

$$\int_{[0,\infty]} \varphi(x)\mathcal{T}_\varepsilon[G](t_2,x)\mathrm{d}x - \int_{[0,\infty]} \varphi(x)\mathcal{T}_\varepsilon[G](t_1,x)\mathrm{d}x \\ = \int_{[0,\infty]} \left(e^{-\int_{t_1}^{t_2} A_\varepsilon[G(s,\cdot)](x)\mathrm{d}s} - 1\right) \varphi(x)\mathcal{T}_\varepsilon[G](t_1,x)\mathrm{d}x \\ + \int_{t_1}^{t_2} \int_{[0,\infty]} \varphi(x)e^{-\int_\sigma^{t_2} A_\varepsilon[G(s,\cdot)](x)\mathrm{d}s} B_\varepsilon[G(\sigma,\cdot)](x)\mathrm{d}x\, \mathrm{d}\sigma,$$

where the absolute value of the right hand side can be estimated by

$$\int_{[0,\infty]} \left(\int_{t_1}^{t_2} A_\varepsilon[G(s,\cdot)](x)\mathrm{d}s\right) \|\varphi\|_{L^\infty(0,\infty)} \mathcal{T}_\varepsilon[G](t_1,x)\mathrm{d}x \\ + \int_{t_1}^{t_2} \int_{[0,\infty]} \|\varphi\|_{L^\infty(0,\infty)} B_\varepsilon[G(\sigma,\cdot)](x)\mathrm{d}x\, \mathrm{d}\sigma \\ \leq \|\varphi\|_{L^\infty(0,\infty)} \left(\int_{t_1}^{t_2} 2\kappa^2 \|G(s,\cdot)\|\mathrm{d}s\, \|\mathcal{T}_\varepsilon[G](t_1,\cdot)\| + \int_{t_1}^{t_2} 2\kappa^2 \|G(\sigma,\cdot)\|^2 \mathrm{d}\sigma\right).$$

For all $G \in \mathcal{X}$, $t_1, t_2 \in [0,T]$, and $\varphi \in C([0,\infty])$, there thus holds

$$\left| \int_{[0,\infty]} \varphi(x)\mathcal{T}_\varepsilon[G](t_2,x)\mathrm{d}x - \int_{[0,\infty]} \varphi(x)\mathcal{T}_\varepsilon[G](t_1,x)\mathrm{d}x \right| \leq 16\kappa^2 M^2 \|\varphi\|_{L^\infty(0,\infty)} |t_2 - t_1|,$$

hence $\mathcal{T}_\varepsilon[\mathcal{X}] \subset \mathcal{X}$ is precompact, by Arzelà-Ascoli (cf. Chapter 7 in [16]). By Schauder's fixed-point theorem there then indeed exists at least on $G_\varepsilon \in C([0,T] : \mathcal{M}_+([0,\infty]))$ such that $\mathcal{T}_\varepsilon[G_\varepsilon] \equiv G_\varepsilon$, which indeed satisfies (2.2) for all $t \in [0,T]$ and $\varphi \in C([0,\infty])$. We next check that for $t \in [0,T]$ and $\varphi \in C^1([0,T] : C([0,\infty]))$ it further satisfies

$$\begin{aligned} \int_{[0,\infty]} \varphi(t,x)G_\varepsilon(t,x)\mathrm{d}x - \int_{[0,\infty]} \varphi(0,x)G_0(x)\mathrm{d}x - \int_0^t \int_{[0,\infty]} \varphi_s(s,x)G_\varepsilon(s,x)\mathrm{d}x\,\mathrm{d}s \\ = \int_0^t \iint_{[0,\infty)^2} G_\varepsilon(s,x)G_\varepsilon(s,y)\; k(x) \int_0^x \phi_\varepsilon(y-z)k(z)\Delta_z^2[\varphi(s,\cdot)](x)\mathrm{d}z\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Indeed, given $t \in [0,T]$ and $\varphi \in C^1([0,T] : C([0,\infty]))$, it follows from (2.2) that

$$\begin{aligned} \left[\int_{[0,\infty]} \varphi(t,x)G_\varepsilon(t,x)\mathrm{d}x \right]_t &= \int_{[0,\infty]} \varphi_t(t,x)G_\varepsilon(t,x)\mathrm{d}x \\ &\quad - \int_{[0,\infty)} \varphi(t,x)A_\varepsilon[G_\varepsilon(t,\cdot)](x)G_\varepsilon(t,x)\mathrm{d}x + \int_{[0,\infty]} \varphi(t,x)B_\varepsilon[G_\varepsilon(t,\cdot)](x)\mathrm{d}x, \end{aligned}$$

which when integrated yields (2.4).

We now consider a collection $\mathcal{G} = \{G_\varepsilon\}_{\varepsilon>0} \subset \mathcal{X}$, where for every $\varepsilon > 0$ we require that $G_\varepsilon \equiv \mathcal{T}_\varepsilon[G_\varepsilon]$. Since (2.3) is independent of ε, it follows by again Arzelà-Ascoli that there exist a subsequence $\varepsilon \to 0$, and $G \in \mathcal{X}$, such that $G_\varepsilon(t,\cdot) \overset{*}{\rightharpoonup} G(t,\cdot)$, uniformly for all $t \in [0,T]$. For all $t \in [0,T]$ and $\varphi \in C^1([0,T] : C([0,\infty]))$ the left hand side of (2.4) then trivially converges to the left hand side of (2.1). Now, using Fubini, we rewrite the right hand side of (2.4) as

$$\begin{aligned} \int_0^t \frac{1}{2} \iint_{[0,\infty)^2} G_\varepsilon(s,x)G_\varepsilon(s,y) \Bigg[\int_0^x \phi_\varepsilon(y-z)k(x)k(z)\Delta_z^2[\varphi(s,\cdot)](x)\mathrm{d}z \\ + \int_0^y \phi_\varepsilon(x-z)k(y)k(z)\Delta_z^2[\varphi(s,\cdot)](y)\mathrm{d}z \Bigg] \mathrm{d}x\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{2.5}$$

where the term between square brackets converges uniformly for all $x, y \geq 0$ and $t \in [0,T]$ to

$$k(x)k(y)\Delta_{x\wedge y}^2[\varphi(s,\cdot)](x \vee y),$$

(cf. Lemma A.5). Recalling then for all $\varepsilon > 0$ that $\sup_{t\in[0,T]} \|G_\varepsilon(t,\cdot)\| \leq 2\|G_0\|$, it follows that the limit of (2.5) as $\varepsilon \to 0$ coincides with

$$\lim_{\varepsilon\to 0} \int_0^t \frac{1}{2} \iint_{[0,\infty)^2} G_\varepsilon(s,x)G_\varepsilon(s,y)\; k(x)k(y)\Delta_{x\wedge y}^2[\varphi(s,\cdot)](x\vee y)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}s,$$

which, using the decay at infinity of k, can be checked to be equal to the right hand side of (2.1).

For any $G_0 \in \mathcal{M}_+([0,\infty])$ we thus have some $G \in C([0,T] : \mathcal{M}_+([0,\infty]))$ that satisfies (2.1) for all $t \in [0,T]$ and $\varphi \in C^1([0,T] : C([0,\infty]))$, and in particular $\|G(t,\cdot)\| = \|G_0\|$ for all $t \in [0,T]$. Iterating this construction lastly yields the desired function $G \in C([0,\infty) : \mathcal{M}_+([0,\infty]))$ that satisfies (2.1) for all $t \geq 0$ and $\varphi \in C^1([0,\infty) : C([0,\infty]))$. □

We are now set to prove existence of weak solutions to (QWTE).

Proof of Theorem 2.1. Given $G_0 \in \mathscr{M}_+(0,\infty) \subset \mathcal{M}_+([0,\infty])$, then, following Lemma 2.2, for any $\varepsilon > 0$ there exists at least one $G_\varepsilon \in C([0,\infty) : \mathcal{M}_+([0,\infty]))$ that for all $t \geq 0$ and $\varphi \in C^1([0,\infty) : C([0,\infty]))$ satisfies

$$\begin{aligned}\int_{[0,\infty]} \varphi(t,x)G_\varepsilon(t,x)\mathrm{d}x - \int_{[0,\infty]} \varphi(0,x)G_0(x)\mathrm{d}x - \int_0^t \int_{[0,\infty]} \varphi_s(s,x)G_\varepsilon(s,x)\mathrm{d}x\ \mathrm{d}s \\ = \int_0^t \frac{1}{2} \iint_{[0,\infty)^2} G_\varepsilon(s,x)G_\varepsilon(s,y)\ \frac{\Delta^2_{x\wedge y}[\varphi(s,\cdot)](x\vee y)}{\sqrt{(x+\varepsilon)(y+\varepsilon)}}\ \mathrm{d}x\mathrm{d}y\ \mathrm{d}s. \quad (2.6)\end{aligned}$$

We first check that $G_\varepsilon(t,\cdot) \in \mathscr{M}_+(0,\infty)$ for all $\varepsilon > 0$ and $t \geq 0$, which we do by showing that the collection $\mathcal{G} = \{G_\varepsilon(t,\cdot)\}_{\varepsilon>0,t\geq 0}$ is uniformly tight. We thereto fix $\varepsilon > 0$ and $t \geq 0$, and for $R > 0$ we set $\varphi_R(x) = (1 - \frac{x}{R})_+$, which is convex and nonincreasing, and which in particular satisfies $\mathbf{1}_{[0,R)} \geq \varphi_R \geq \varphi_R(r)\mathbf{1}_{[0,r]}$ for any $r \geq 0$. Using then $\varphi \equiv \varphi_R$ as a time-independent test function in (2.6), the right hand side is nonnegative by convexity of φ_R (cf. Lemma A.2), hence there holds

$$\int_{[0,\infty]} \varphi_R(x)G_\varepsilon(t,x)\mathrm{d}x \geq \int_{[0,\infty]} \varphi_R(x)G_0(x)\mathrm{d}x,$$

and in particular

$$\int_{[0,R)} G_\varepsilon(t,x)\mathrm{d}x \geq \left(1 - \frac{1}{\sqrt{R}}\right) \int_{[0,\sqrt{R}]} G_0(x)\mathrm{d}x \text{ for all } R \geq 1.$$

Combining this with the fact that $\|G_\varepsilon(t,\cdot)\| = \|G_0\|$ for all $t \geq 0$ and $\varepsilon > 0$ (cf. Lemma 2.2), it thus follows for $R \geq 1$ that

$$\int_{[R,\infty]} G_\varepsilon(t,x)\mathrm{d}x \leq \int_{[0,\infty)} G_0(x)\mathrm{d}x - \left(1 - \frac{1}{\sqrt{R}}\right) \int_{[0,\sqrt{R}]} G_0(x)\mathrm{d}x$$

where the right hand side vanishes as $R \to \infty$, independently of $\varepsilon > 0$ and $t \geq 0$, and we conclude that $\mathcal{G}$ is a uniformly tight subset of $\mathscr{M}_+(0,\infty)$.

For $\varphi \in C^1([0,\infty])$ we next find that

$$\left| \frac{\Delta^2_{x\wedge y}\varphi(x\vee y)}{\sqrt{(x+\varepsilon)(y+\varepsilon)}} \right| \leq 2\|\varphi\|_{W^{1,\infty}(0,\infty)} \text{ for all } x,y \geq 0 \text{ and } \varepsilon > 0,$$

(cf. Lemma A.3), so using $\varphi \in C^1([0,\infty])$ as a time-independent test function in (2.6), we find for $t_1, t_2 \geq 0$ that

$$\left| \int_{[0,\infty]} \varphi(x)G_\varepsilon(t_2,x)\mathrm{d}x - \int_{[0,\infty]} \varphi(x)G_\varepsilon(t_1,x)\mathrm{d}x \right| \leq \|G_0\|^2 \|\varphi'\|_{L^\infty(0,\infty)}\ |t_2 - t_1|.$$

Now, since $C^1([0,\infty])$ is dense in $C([0,\infty])$, it follows that for any $\varphi \in C([0,\infty])$ the collection of mappings

$$\left\{ t \mapsto \int_{[0,\infty]} \varphi(x)G_\varepsilon(t,x)\mathrm{d}x \right\}_{\varepsilon>0}$$

is equicontinuous. By Arzelà-Ascoli we then obtain existence of a subsequence $\varepsilon \to 0$, and a function $G \in C([0,\infty) : \mathcal{M}_+([0,\infty]))$, such that $G_\varepsilon(t,\cdot) \overset{*}{\rightharpoonup} G(t,\cdot)$, locally uniformly for all $t \in [0,\infty)$. Moreover, it can be checked that there hold $\{G(t,\cdot)\}_{t\geq 0} \subset \mathscr{M}_+(0,\infty)$, and $G(0,\cdot) \equiv G_0$ on $[0,\infty)$.

We complete the proof by checking that G is a weak solution to (QWTE). It is immediate that the left hand side of (2.6) converges to the left hand side of (QWTE)$^{\mathrm{w}}$ for all $t \geq 0$ and $\varphi \in C^1([0,\infty) : C([0,\infty]))$. For any time-independent test function $\varphi \in C^1([0,\infty])$, we further notice that the fraction in the right hand side of (2.6) converges uniformly for all $x, y \geq 0$ to the continuous function in $C_0(\mathbb{R}^2_+)$ that for $x, y > 0$ is given by

$$\frac{\Delta^2_{x\wedge y}\varphi(x\vee y)}{\sqrt{xy}},$$

(cf. Lemma A.6). Combining then the decay at infinity of this limit function with uniform tightness of $\mathcal{G}$, we find for any $t \geq 0$ and $\varphi \in C([0,\infty) : C^1([0,\infty]))$ that the right hand side of (2.6) converges to the right hand side of (QWTE)$^{\mathrm{w}}$, hence G satisfies (QWTE)$^{\mathrm{w}}$ for all $t \geq 0$ and $\varphi \in C([0,\infty) : C^1([0,\infty])) \cap C^1([0,\infty) : C([0,\infty]))$. Thus, restricting to test functions that in space are compactly supported in $[0,\infty)$, and recalling that the weak-$*$ topology on $\mathcal{M}_+([0,\infty))$ is weaker than the one on $\mathcal{M}_+([0,\infty])$, we conclude that G is a weak solution to (QWTE) in the sense of Definition 1.7. ☐

2.2 Selected properties

In this section we prove some elementary properties of weak solutions to (QWTE). The following monotonicity lemma will be useful throughout.

Lemma 2.3. *Let G be weak solution to* (QWTE)*, and let $\varphi \in C_0([0,\infty))$ be convex [concave]. Then the mapping*

$$t \mapsto H[\varphi](t) := \int_{[0,\infty)} \varphi(x) G(t,x)\mathrm{d}x \tag{2.7}$$

is continuous and nondecreasing [nonincreasing] on $[0,\infty)$.

Proof. We may restrict ourselves to proving monotonicity, since continuity follows immediately from the use of the weak-$*$ topology. Moreover, since $H[\varphi] \equiv -H[-\varphi]$, it suffices to consider the case where φ is convex. In that case, there is a sequence $\{\varphi_n\} \subset C^1_c([0,\infty))$ of convex functions such that $\varphi \equiv \sup_n \varphi_n$. By monotone convergence there then holds $H[\varphi] \equiv \sup_n H[\varphi_n]$, hence it suffices to check monotonicity of $H[\varphi_n]$ for all n. We thereto use any φ_n as a time-independent test function in (QWTE)$^{\mathrm{w}}$, which for $t \geq 0$ yields

$$H[\varphi_n](t) = H[\varphi_n](0) + \int_0^t \frac{1}{2} \iint_{\mathbb{R}^2_+} \frac{G(s,x)G(s,y)}{\sqrt{xy}} \Delta^2_{x\wedge y}\varphi_n(x\vee y)\mathrm{d}x\mathrm{d}y\,\mathrm{d}s. \tag{2.8}$$

Since the integrand in the second term on the right hand side of (2.8) is nonnegative by convexity of φ_n (cf. Lemma A.2), it thus follows that $H[\varphi_n]$ is indeed nondecreasing as a function of t on $[0,\infty)$. ☐

In the proof of Theorem 2.1 we saw that $\{G_\varepsilon(t,\cdot)\}_{\varepsilon>0,t\geq 0}$ was a uniformly tight subset of $\mathscr{M}_+(0,\infty)$. The result and its proof carry over naturally to weak solutions to (QWTE):

Proposition 2.4. *Let G be a weak solution to* (QWTE)*, and let $\eta, R > 0$ be arbitrary. Then there holds*

$$\int_{[0,\frac{R}{\eta}]} G(t,x)\mathrm{d}x \geq (1-\eta)\int_{[0,R]} G(0,x)\mathrm{d}x \textit{ for all } t \geq 0.$$

Proof. Using $\varphi(x) = (1-\eta\frac{x}{R})_+$ in Lemma 2.3, the function $H[\varphi]$ is nondecreasing. Furthermore, there holds $\mathbf{1}_{[0,\frac{R}{\eta}]} \geq \varphi \geq (1-\eta)_+\mathbf{1}_{[0,R]}$, so for all $t \geq 0$ we have

$$\int_{[0,\frac{R}{\eta}]} G(t,x)\mathrm{d}x \geq H[\varphi](t) \geq H[\varphi](0) \geq (1-\eta)_+ \int_{[0,R]} G(0,x)\mathrm{d}x,$$

and the claim follows easily. □

Our notion of weak solution to (QWTE) requires only that (QWTE)$^{\mathrm{w}}$ is satisfied for test functions with compact support in $[0,\infty)$. However, one can check that weak solutions satisfy (QWTE)$^{\mathrm{w}}$ for more test functions.

Lemma 2.5. *Let G be a weak solution to* (QWTE). *Then G also satisfies* (QWTE)$^{\mathrm{w}}$ *for all $t \geq 0$ and $\varphi \in C([0,\infty) : C^1([0,\infty])) \cap C^1([0,\infty) : C([0,\infty]))$.*

Proof. For all $n \in \mathbb{N}$, let $\zeta_n(x) = \int_x^\infty \phi(x-n)\mathrm{d}x$ with $\phi(x) = (1-|x|)_+$. Given now $\varphi \in C([0,\infty) : C^1([0,\infty])) \cap C^1([0,\infty) : C([0,\infty]))$, we set $\varphi_n(t,x) = \varphi(t,x)\zeta_n(x)$, which for any $n \in \mathbb{N}$ is an admissible test function in (QWTE)$^{\mathrm{w}}$, and we note that $\varphi_n(t,\cdot)$ and $\partial_t\varphi_n(t,\cdot)$ converge to $\varphi(t,\cdot)$ and $\varphi_t(t,\cdot)$ as $n \to \infty$, pointwise on $[0,\infty)$ and uniformly for all $t \geq 0$. Recalling then Lemma 1.5, for all $t \geq 0$ it is immediate by dominated convergence that

$$\begin{aligned}\int_{[0,\infty)} \varphi(t,x)G(t,x)\mathrm{d}x - \int_{[0,\infty)} \varphi(0,x)G(0,x)\mathrm{d}x - \int_0^t \int_{[0,\infty)} \varphi_s(s,x)G(s,x)\mathrm{d}x\ \mathrm{d}s \\ = \lim_{n\to\infty} \int_0^t \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} \Delta^2_{x\wedge y}[\varphi_n(s,\cdot)](x \vee y)\mathrm{d}x\mathrm{d}y\ \mathrm{d}s, \quad (2.9)\end{aligned}$$

and using also Lemma A.6, convergence of the right hand side of (2.9) follows as well. □

The previous result in particular allows to use $\varphi \equiv 1$ as a test function in order to get conservation of mass. We can further use that result to find that initially finite energies are conserved.

Proposition 2.6. *Given a weak solution G to* (QWTE), *then $\|G(t,\cdot)\| = \|G(0,\cdot)\|$ for all $t \geq 0$. Moreover, if $G(0,\cdot)$ has finite first moment, then there holds*

$$\int_{[0,\infty)} xG(t,x)\mathrm{d}x = \int_{[0,\infty)} xG(0,x)\mathrm{d}x \textit{ for all } t \geq 0. \quad (2.10)$$

Proof. Using $\varphi \equiv 1$ as a time-independent test function in (QWTE)$^{\mathrm{w}}$, which is possible following Lemma 2.5, it immediately follows that $\|G(t,\cdot)\| = \|G(0,\cdot)\|$ for all $t \geq 0$.

For $\varepsilon > 0$, let now $H[\varphi_\varepsilon]$ be given by (2.7) with $\varphi_\varepsilon(x) = \frac{x}{1+\varepsilon x}$, and note that $H[\varphi_\varepsilon]$ is nonincreasing (cf. Lemma 2.3). Moreover, invoking again Lemma 2.5 to use φ_ε as a time-independent test function in (QWTE)$^{\mathrm{w}}$, for $t \geq 0$ there holds

$$0 \leq H[\varphi_\varepsilon](0) - H[\varphi_\varepsilon](t) = \int_0^t \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} \left|\Delta^2_{x\wedge y}\varphi_\varepsilon(x \vee y)\right| \mathrm{d}x\mathrm{d}y\ \mathrm{d}s. \quad (2.11)$$

For $x, y \geq 0$ we further explicitly compute that

$$\left|\Delta^2_{x\wedge y}\varphi_\varepsilon(x \vee y)\right| = \frac{2\varepsilon(x\wedge y)^2}{(1+\varepsilon(x+y))(1+\varepsilon(x\vee y))(1+\varepsilon|x-y|)} \leq 2\varepsilon(x\wedge y)\varphi_\varepsilon(y),$$

and using this estimate we can bound the right hand side of (2.11) from above by

$$\varepsilon \int_0^t \iint_{\mathbb{R}_+^2} G(s,x)G(s,y)\varphi_\varepsilon(y)\mathrm{d}x\mathrm{d}y \;\mathrm{d}s \le \|G(0,\cdot)\| t \;\varepsilon H[\varphi_\varepsilon](0). \tag{2.12}$$

Now, if the first moment of $G(0,\cdot)$ is bounded, then by monotone convergence it is equal to the limit $\lim_{\varepsilon\to 0} H[\varphi_\varepsilon](0)$. For any fixed $t \ge 0$, the right hand side of (2.12) then thus vanishes as $\varepsilon \to 0$, hence so does the right hand side of (2.11), and there holds

$$\lim_{\varepsilon\to 0} \int_{[0,\infty)} \tfrac{x}{1+\varepsilon x} G(t,x)\mathrm{d}x = \int_{[0,\infty)} xG(0,x)\mathrm{d}x. \tag{2.13}$$

We then conclude the claim since the left hand sides of (2.13) and (2.10) coincide by again monotone convergence. □

We end this section with a straightforward scaling result.

Lemma 2.7. *Let G be a weak solution G to* (QWTE)*, and let $\kappa_1, \kappa_2 > 0$ be arbitrary. Then also $\bar{G} \in C([0,\infty) : \mathscr{M}(0,\infty))$, defined to be such that*

$$\int_{[0,\infty)} \varphi(x)\bar{G}(t,x)\mathrm{d}x = \int_{[0,\infty)} \varphi(\tfrac{x}{\kappa_2})\kappa_1 G(\kappa_1\kappa_2 t, x)\mathrm{d}x \text{ for } t \ge 0 \text{ and } \varphi \in C([0,\infty)),$$

is a weak solution to (QWTE)*, and there holds $\|\bar{G}(t,\cdot)\| = \kappa_1 \|G(0,\cdot)\|$ for all $t \ge 0$.*

Proof. Immediate from elementary manipulations. □

2.3 The measure of the origin, and trivial solutions

We now take a closer look at the measure of the origin of a weak solution G to (QWTE). In the particle interpretation (cf. Figure 1.3) we see that only the larger one of two interacting particles is taken from the distribution. A particle of size 0 can thus only disappear due to interaction with another zero-particle. However, the particle is then replaced by a particle of size 0, which doesn't contribute to a nett change in the distribution. (Alternatively, in the particle interpretation of (CWTE), this corresponds to an interaction of three particles $\omega_1, \omega_2, \omega_3 \ge 0$ where two of them are zero-particles, which indeed does not affect the total distribution.) We therefore expect that the number of particles of size 0 cannot decrease, which can be formalized as follows.

Proposition 2.8. *Given a weak solution G to* (QWTE)*, then the mapping*

$$t \mapsto m(t) := \int_{\{0\}} G(t,x)\mathrm{d}x$$

is right-continuous and nondecreasing on $[0,\infty)$.

Proof. Defining $H[\varphi_n]$ by (2.7) with $\varphi_n(x) = (1-nx)_+$ and $n \in \mathbb{N}$, we have $m \equiv \inf_n H[\varphi_n]$. Monotonicity of m is then immediate from monotonicity of $H[\varphi_n]$ (cf. Lemma 2.3), and since for $t \ge 0$ we further observe that

$$m(t) \le \limsup_{s\to t^+} m(s) \le \inf_{n\in\mathbb{N}} \Big(\lim_{s\to t} H[\varphi_n](s) \Big) = \inf_{n\in\mathbb{N}} H[\varphi_n](t) = m(t),$$

right-continuity also follows. □

As an immediate consequence of the monotonicity of the mass of the origin, we obtain the following uniqueness result.

Corollary 2.9. *Given $m \geq 0$, then the weak solution G to* (QWTE) *with $G(0,\cdot) \equiv m\delta_0$ is unique and time-independent, i.e. $G(t,\cdot) \equiv m\delta_0$ for all $t \geq 0$.*

Proof. Given a weak solution G to (QWTE) that satisfies $G(0,\cdot) \equiv m\delta_0$, which exists by Theorem 2.1, then for all $t \geq 0$ there holds

$$0 \leq \int_{(0,\infty)} G(t,x)\mathrm{d}x = m - \int_{\{0\}} G(t,x)\mathrm{d}x, \tag{2.14}$$

(cf. Propositions 2.6). Moreover, since the right hand side of (2.14) is monotonically decreasing as a function of t (cf. Proposition 2.8), it follows that it is bounded from above by 0, and we conclude the claim. □

This now motivates the introduction of the notion of a trivial solution.

Definition 2.10. We say that G is a *trivial solution* to (QWTE) if it is a weak solution in the sense of Definition 1.7 that satisfies $\mathrm{supp}(G(0,\cdot)) \subset \{0\}$. A weak solution to (QWTE) that is not trivial will be called *nontrivial*.

The argumentation preceding Proposition 2.8 seems to suggest that zero-particles are somehow unconnected to the other particles in the distribution. Indeed, we find that we may add and subtract zero-particles to move between weak solutions to (QWTE).

Lemma 2.11. *Let G be a weak solution to* (QWTE)*, and let $m \in \mathbb{R}$ be arbitrary. Then $\bar{G}$, given by*

$$\bar{G}(t,\cdot) \equiv m\delta_0 + G(t,\cdot) \text{ on } [0,\infty) \text{ for } t \geq 0,$$

satisfies (QWTE)$^{\mathrm{w}}$ *for all $t \geq 0$ and $\varphi \in C([0,\infty) : C_c^1([0,\infty))) \cap C^1([0,\infty) : C_c([0,\infty)))$. Moreover, if $m + \int_{\{0\}} G(0,x)\mathrm{d}x \geq 0$, then $\bar{G}$ is actually a weak solution to* (QWTE).

Proof. Since the product measure $\delta_0 \times \delta_0$ is supported outside the domain of integration on the right hand side of (QWTE)$^{\mathrm{w}}$, the claim holds if G is the zero solution $G \equiv 0$. If G is an arbitrary weak solution to (QWTE), then $\bar{G}$ is a linear combination of functions that satisfy (QWTE)$^{\mathrm{w}}$ for all $t \geq 0$ and $\varphi \in C([0,\infty) : C_c^1([0,\infty))) \cap C^1([0,\infty) : C_c([0,\infty)))$. It thus suffices to check that the cross terms vanish, which follows from the observation that the product measures $\delta_0 \times G(t,\cdot)$ with $t \geq 0$ are also supported outside the domain of integration on the right hand side of (QWTE)$^{\mathrm{w}}$. Lastly, due to monotonicity of the measure of the origin (cf. Proposition 2.8) the initial estimate is sufficient to guarantee that $\bar{G}(t,\cdot) \geq 0$ on $[0,\infty)$ for all $t \geq 0$, which makes $\bar{G}$ a weak solution. □

Lastly, it is natural to expect solutions to a kinetic equation to converge in some sense to their equilibria. Despite not having proven uniqueness of the equilibrium solutions to (QWTE) (yet, but see Corollary 3.2), we can show weak-$*$ convergence to trivial solutions.

Proposition 2.12. *Given a weak solution G to* (QWTE)*, then $G(t,\cdot) \overset{*}{\rightharpoonup} \|G(0,\cdot)\|\delta_0$ as $t \to \infty$.*

Proof. Since the claim is immediate for trivial solutions, we suppose without loss of generality that G is nontrivial. Setting then $\varphi(x) = e^{-x}$ in Lemma 2.3, we obtain a continuous and nondecreasing function H that for all $t \geq 0$ satisfies

$$H(t) = H(0) + \int_0^t \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} e^{-(x\vee y)} \left(e^{\frac{1}{2}(x\wedge y)} - e^{-\frac{1}{2}(x\wedge y)}\right)^2 \mathrm{d}x\mathrm{d}y \, \mathrm{d}s, \tag{2.15}$$

and noticing further that

$$H(t) = \int_{[0,\infty)} e^{-x} G(t,x)\mathrm{d}x \le \int_{[0,\infty)} G(t,x)\mathrm{d}x =: M \text{ for all } t \ge 0,$$

(cf. Proposition 2.6), it follows that $\lim_{t\to\infty} H(t) = \sup_{t\ge 0} H(t) =: L \le M$ exists. We will next check that $L = M$, to which end we conversely suppose that $L < M$. Defining then $a = \log(1 + \frac{M-L}{2M+L}) > 0$, we observe for all $s \ge 0$ that

$$\int_{[0,a]} G(s,x)\mathrm{d}x \le e^a \int_{[0,\infty)} e^{-x} G(s,x)\mathrm{d}x \le \left(1 + \frac{M-L}{2M+L}\right) L < L + \tfrac{1}{3}(M-L),$$

hence there holds

$$\int_{(a,\infty)} G(s,x)\mathrm{d}x = M - \int_{[0,a]} G(s,x)\mathrm{d}x > \tfrac{2}{3}(M-L) \text{ for all } s \ge 0. \tag{2.16}$$

We can then further choose some $R > a$ such that

$$\int_{[0,R]} G(0,x)\mathrm{d}x \ge M - \tfrac{1}{6}(M-L) = \frac{5M+L}{6},$$

and setting $\eta = 1 - \frac{4M+2L}{5M+L}$ and $b = \frac{R}{\eta}$, we use Proposition 2.4 to obtain that

$$\int_{[0,b]} G(s,x)\mathrm{d}x \ge \frac{2M+L}{3} = M - \tfrac{1}{3}(M-L) \text{ for all } s \ge 0. \tag{2.17}$$

Combining (2.16) and (2.17), it now follows that

$$\int_{(a,b]} G(s,x)\mathrm{d}x = \int_{[0,b]} G(s,x)\mathrm{d}x + \int_{(a,\infty)} G(s,x)\mathrm{d}x - M > \tfrac{1}{3}(M-L) \text{ for all } s \ge 0, \tag{2.18}$$

and using (2.18) in (2.15) we find for all $t \ge 0$ that

$$\begin{aligned} H(t) &\ge \int_0^t \frac{1}{2} \iint_{(a,b]^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} e^{-(x\vee y)} \left(e^{\frac{1}{2}(x\wedge y)} - e^{-\frac{1}{2}(x\wedge y)}\right)^2 \mathrm{d}x\mathrm{d}y\,\mathrm{d}s \\ &\ge \tfrac{2}{b} e^{-b} \sinh^2(\tfrac{a}{2}) \int_0^t \left(\int_{(a,b]} G(s,x)\mathrm{d}x\right)^2 \mathrm{d}s > \tfrac{1}{9} C(a,b)(M-L)^2\, t. \end{aligned}$$

Since this contradicts the boundedness of H, we thus have $L = M$, and we find that any weak-$*$ limit of G is supported at the origin. Arguing lastly by compactness for existence of such limits, the claim follows. □

Chapter 3

Instantaneous condensation

We have seen that the measure of the origin of a weak solution to (QWTE) is nondecreasing (cf. Proposition 2.8). In this chapter we will prove that this measure is actually strictly increasing as long as the solution does not coincide with a trivial one, i.e. we prove

Theorem 3.1. *Given a weak solution G to* (QWTE) *in the sense of Definition 1.7, then for every $\bar{t} \geq 0$ for which $\int_{(0,\infty)} G(\bar{t}, x)\mathrm{d}x > 0$, there holds*

$$\int_{\{0\}} G(t, x)\mathrm{d}x > \int_{\{0\}} G(\bar{t}, x)\mathrm{d}x \textit{ for all } t > \bar{t}.$$

As a consequence of this result, we obtain the characterization of time-independent weak solutions to (QWTE) as the unique trivial ones.

Corollary 3.2. *A weak solution to* (QWTE) *is time-independent if and only if it is trivial in the sense of Definition 2.10.*

Proof. From Corollary 2.9 we know that trivial solutions are time-independent. Supposing conversely that G is a nontrivial weak solution to (QWTE), then in particular there holds $\int_{(0,\infty)} G(0, x)\mathrm{d}x > 0$, and G cannot be time-independent by Theorem 3.1. □

Let us briefly outline the proof of Theorem 3.1. We note first that (QWTE) is invariant under time-translations, so that we may restrict ourselves to $\bar{t} = 0$. Moreover, in view of Lemma 2.11 it is sufficient to show that, *given a nontrivial weak solution G to* (QWTE) *that satisfies $\int_{\{0\}} G(0, x)\mathrm{d}x = 0$, then there holds*

$$\int_{\{0\}} G(t, x)\mathrm{d}x > 0 \textit{ for all } t > 0. \tag{3.1}$$

Indeed, given any nontrivial weak solution G to (QWTE), by that lemma we can define another weak solution $\bar{G}$ by

$$\bar{G}(t, \cdot) \equiv G(t, \cdot) - \left(\int_{\{0\}} G(0, x)\mathrm{d}x \right) \delta_0 \text{ on } [0, \infty) \text{ for } t \geq 0,$$

which has initially zero mass at the origin. The instantaneous onset of a Dirac delta at the origin for $\bar{G}$ then implies the strict monotonicity of the measure of the origin of G, by the fact that

$$\int_{\{0\}} \bar{G}(t, x)\mathrm{d}x > 0 \quad \Leftrightarrow \quad \int_{\{0\}} G(t, x)\mathrm{d}x > \int_{\{0\}} G(0, x)\mathrm{d}x.$$

We now conversely suppose that (3.1) does not hold, which by monotonicity of the measure of the origin (cf. Proposition 2.8) implies the existence of some finite $T > 0$ for which

$$\int_{\{0\}} G(s, x)\mathrm{d}x = 0 \text{ for all } s \in [0, T],$$

and for which thus

$$\int_{\frac{1}{3}T}^{\frac{2}{3}T}\int_{[0,r]} G(s,x)\mathrm{d}x\ \mathrm{d}s = \int_{\frac{1}{3}T}^{\frac{2}{3}T}\int_{(0,r]} G(s,x)\mathrm{d}x\ \mathrm{d}s \text{ for all } r \geq 0. \tag{3.2}$$

In the following we will prove lower and upper bounds on the left and right hand sides of (3.2) respectively, from which a contradiction will follow. The proofs of these bounds rely heavily on the following two lemmas, as well as on Proposition 2.4.

Two lemmas

Lemma 3.3. *Let G be a weak solution to* (QWTE), *and for $n \in \mathbb{N}$ let $(z_0, \dots, z_n) \in \mathbb{R}_+^{n+1}$ be such that $z_i - z_{i-1} \in (0, \frac{1}{2}z_0]$ for all $i \in \{1, \dots, n\}$. Then for all $t \geq 0$ there holds*

$$\int_{[0,z_0]} G(t,x)\mathrm{d}x \geq \frac{1}{4}\int_0^t \left[\sum_{i=1}^n \frac{1}{z_i}\left(\int_{(z_{i-1},z_i]} G(s,x)\mathrm{d}x\right)^2\right]\mathrm{d}s.$$

Proof. Let $\varphi \in C_c^1([0,\infty))$ be convex, and such that $\varphi(x) \leq \frac{1}{z_0}(z_0 - x)_+$ for all $x \geq 0$. Using then φ as a time-independent test function in (QWTE)$^{\mathrm{w}}$, we easily find for $t \geq 0$ that

$$\int_{[0,z_0]} G(t,x)\mathrm{d}x \geq \int_0^t \left[\frac{1}{2}\iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}}\Delta_{x\wedge y}^2\varphi(x\vee y)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}s, \tag{3.3}$$

where the integrand in the right hand side is nonnegative as φ is convex (cf. Lemma A.2). Thus, restricting the domain of integration, and using the fact that φ is nonincreasing, we estimate the term between square brackets in the right hand side of (3.3) from below by

$$\frac{1}{2}\iint_{\bigcup_{i=1}^n (z_{i-1},z_i]^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}}\varphi(|x-y|)\mathrm{d}x\mathrm{d}y \geq \frac{\varphi(\frac{1}{2}z_0)}{2}\sum_{i=1}^n \frac{1}{z_i}\left(\int_{(z_{i-1},z_i]} G(s,x)\mathrm{d}x\right)^2.$$

Noting lastly that $\sup_\varphi \varphi(\frac{1}{2}z_0) = \frac{1}{2}$, where the supremum is taken over all φ as specified above, the claim follows. □

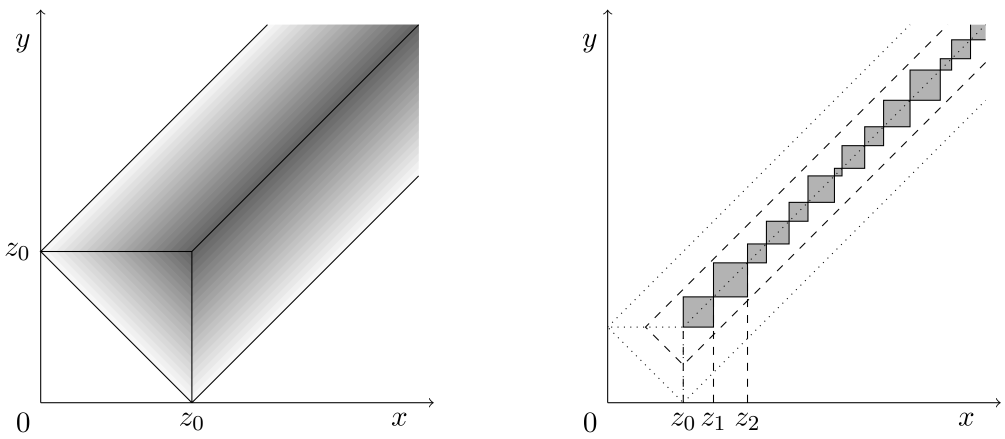

FIGURE 3.1: On the left is the density plot of $\Delta_{x\wedge y}^2\varphi(x\vee y)$ for the function $\varphi(z) = (z_0 - z)_+$, which is $[(x+y-z_0)\wedge(z_0 - |x-y|)]_+$. On the right we have shaded the squares to which we restrict the domain of integration in the proof of Lemma 3.3.

Lemma 3.4 (cf. [13]). *Given sequences $\{a_i\} \subset \mathbb{R}_+$ and $\{b_i\} \subset \mathbb{R}$, then for every $n \in \mathbb{N}$ there holds*

$$\sum_{i=1}^{n} \frac{b_i^2}{a_i} \geq \left(\sum_{i=1}^{n} a_i\right)^{-1} \left(\sum_{i=1}^{n} b_i\right)^{2}.$$

Proof. Immediate from the Cauchy-Schwarz inequality. □

The upper bound

The upper bound on the right hand side of (3.2) is the easier of the two estimates.

Proposition 3.5. *Let G be a weak solution to* (QWTE), *and let $T > 0$ be arbitrary. Then*

$$\int_{\frac{1}{3}T}^{\frac{2}{3}T} \int_{(0,r]} G(s,x)\mathrm{d}x\,\mathrm{d}s \leq \frac{2\sqrt{T\|G(0,\cdot)\|}}{\sqrt{3}-\sqrt{2}} \cdot \sqrt{r} \text{ for all } r \geq 0.$$

Proof. As the claim is trivial for $r = 0$, we fix $r > 0$ arbitrarily. Using the disjoint decomposition $(0,r] = \bigcup_{j=1}^{\infty}((\frac{2}{3})^j r, \frac{3}{2}(\frac{2}{3})^j r]$, we then find by Cauchy-Schwarz that

$$\int_{\frac{1}{3}T}^{\frac{2}{3}T} \int_{(0,r]} G(s,x)\mathrm{d}x\,\mathrm{d}s \leq \sum_{j=1}^{\infty} \sqrt{\tfrac{1}{3}T} \left(\int_{\frac{1}{3}T}^{\frac{2}{3}T} \left(\int_{((\frac{2}{3})^j r, \frac{3}{2}(\frac{2}{3})^j r]} G(s,x)\mathrm{d}x \right)^2 \mathrm{d}s \right)^{\frac{1}{2}}. \tag{3.4}$$

For $j \in \mathbb{N}$, using Lemma 3.3 with $n = 1$ and $z_1 = \frac{3}{2}z_0 = \frac{3}{2}(\frac{2}{3})^j r$, we now further have

$$\int_0^T \left(\int_{((\frac{2}{3})^j r, \frac{3}{2}(\frac{2}{3})^j r]} G(s,x)\mathrm{d}x \right)^2 \mathrm{d}s \leq 4 \cdot \tfrac{3}{2}(\tfrac{2}{3})^j r \cdot \int_{[0,(\frac{2}{3})^j r]} G(T,x)\mathrm{d}x,$$

where the right hand side can be further bounded by $6\|G(0,\cdot)\|(\frac{2}{3})^j r$ (cf. Proposition 2.6). Using lastly this estimate in the right hand side of (3.4), the claim easily follows by evaluating the remaining sum. □

The lower bound

Following Proposition 3.5, it is clear that we can disprove (3.2) by bounding its left hand side from below by a vanishing term of order $\omega(\sqrt{r})$ as $r \to 0$, i.e. by a term that tends to zero as $r \to 0$ strictly slower than the square root. Below we will prove lower bounds of order $O(r^\alpha)$ as $r \to 0$ for all $\alpha \in (0,1)$ (cf. Proposition 3.10), which goes in two steps: First we show that if we have an amount of mass m in the region $[0,R]$, then after a waiting time $\frac{R}{m}T_*$, with $T_* = T_*(\alpha) > 0$, we have a bound of order $O(r^\alpha)$ as $r \to 0$ (cf. Proposition 3.6). In the second step we prove that in arbitrarily small times τ we can get arbitrarily large densities $L = \frac{m}{R}$ (cf. Proposition 3.8).

Proposition 3.6. *Given $\alpha \in (0,1)$, there exists a constant $T_* = T_*(\alpha) > 0$ such that if G is a weak solution to* (QWTE) *for which there exist $m, R > 0$ such that*

$$\int_{[0,R]} G(t,x)\mathrm{d}x \geq m \text{ for all } t \geq 0, \tag{3.5}$$

then

$$\int_{[0,r]} G(t,x)\mathrm{d}x \geq m(\tfrac{r}{2R})^\alpha \text{ for all } r \in [0,R] \text{ and } t \geq \tfrac{R}{m}T_*. \tag{3.6}$$

The proof of Proposition 3.6 will use the following concentration lemma.

Lemma 3.7. *Given $\varepsilon > 0$, there exists a constant $T_0 = T_0(\varepsilon) > 0$ such that if G is a weak solution to* (QWTE) *for which there holds*

$$\int_{[0,1]} G(t,x)\mathrm{d}x \geq 1 \textit{ for all } t \geq 0, \tag{3.7}$$

then

$$\int_{[0,\frac{1}{4}\varepsilon]} G(t_0,x)\mathrm{d}x \geq 1 - \tfrac{1}{2}\varepsilon \textit{ for some } t_0 \in [0,T_0]. \tag{3.8}$$

Proof. For $\varepsilon \geq 2$ we easily see that any $T_0 > 0$ will do, so we fix $\varepsilon \in (0,2)$ arbitrarily, and we let G be a weak solution to (QWTE) for which (3.7) holds. Setting then $z_i = \frac{1}{4}\varepsilon + \frac{1}{8}\varepsilon i$ for all $i \in \mathbb{N}_0$, it follows from Lemmas 3.3 and 3.4 that

$$\int_{[0,\frac{1}{4}\varepsilon]} G(t,x)\mathrm{d}x \geq \frac{1}{4}\frac{1}{nz_n}\int_0^t \left(\int_{(\frac{1}{4}\varepsilon,z_n]} G(s,x)\mathrm{d}x\right)^2 \mathrm{d}s \text{ for all } n \in \mathbb{N} \text{ and } t \geq 0.$$

Fixing now $n \in \mathbb{N}$ such that $z_{n-1} < 1 \leq z_n$, for which we observe that

$$nz_n = n\,\tfrac{\varepsilon}{8}(n+2) < \tfrac{\varepsilon}{8}(n+1)^2 = \tfrac{8}{\varepsilon}z_{n-1}^2 < \tfrac{8}{\varepsilon},$$

we thus in particular have

$$\int_{[0,\frac{1}{4}\varepsilon]} G(t,x)\mathrm{d}x \geq \frac{\varepsilon}{32}\int_0^t \left(\int_{(\frac{1}{4}\varepsilon,1]} G(s,x)\mathrm{d}x\right)^2 \mathrm{d}s \text{ for all } t \geq 0. \tag{3.9}$$

However, supposing for $T_0 > 0$ that (3.8) is false, i.e. supposing that

$$\int_{[0,\frac{1}{4}\varepsilon]} G(t,x)\mathrm{d}x < 1 - \tfrac{1}{2}\varepsilon \text{ for all } t \in [0,T_0], \tag{3.10}$$

it then follows from using (3.7) and (3.10) in (3.9) that

$$1 - \tfrac{1}{2}\varepsilon > \sup_{t\in[0,T_0]} \left\{\frac{\varepsilon}{32}\int_0^t \left(\int_{(\frac{1}{4}\varepsilon,1]} G(s,x)\mathrm{d}x\right)^2 \mathrm{d}s\right\} > T_0 \times \frac{\varepsilon^3}{128},$$

which itself is false for $T_0 = T_0(\varepsilon) = 64\varepsilon^{-3}(2-\varepsilon) > 0$. □

Proof of Proposition 3.6. Let $m, R > 0$ be fixed, and let G be a weak solution to (QWTE) that satisfies (3.5). We then consider $\bar{G}$, defined to be such that

$$\int_{[0,\infty)} \varphi(x)\bar{G}(t,x)\mathrm{d}x = \frac{1}{m}\int_{[0,\infty)} \varphi(\tfrac{x}{R})G(\tfrac{R}{m}t,x)\mathrm{d}x \text{ for } t \geq 0 \text{ and } \varphi \in C([0,\infty)),$$

which is another weak solution to (QWTE) (cf. Lemma 2.7), and we observe that

$$\int_{[0,1]} \bar{G}(t,x)\mathrm{d}x = \frac{1}{m}\int_{[0,R]} G(\tfrac{R}{m}t,x)\mathrm{d}x \geq 1 \text{ for all } t \geq 0.$$

For given $\varepsilon > 0$, and with $T_0 = T_0(\varepsilon) > 0$ as obtained in Lemma 3.7, there then holds

$$\int_{[0,\frac{1}{4}\varepsilon]} \bar{G}(t_0,x)\mathrm{d}x \geq 1 - \tfrac{1}{2}\varepsilon \text{ for some } t_0 \in [0,T_0],$$

so from Proposition 2.4 with $\eta = 2R = \frac{1}{2}\varepsilon$ it follows that

$$\int_{[0,\frac{1}{2}]} \bar{G}(t,x)\mathrm{d}x \geq (1-\tfrac{1}{2}\varepsilon)^2 \geq 1-\varepsilon \text{ for all } t \geq t_0,$$

and coming back to G, we find that (3.5) in particular implies

$$\int_{[0,\frac{1}{2}R]} G(t,x)\mathrm{d}x \geq m(1-\varepsilon) \text{ for all } t \geq \tfrac{R}{m}T_0.$$

Repeating the argumentation above, we next obtain for all $n \in \mathbb{N}$ that

$$\int_{[0,\frac{1}{2^n}R]} G(t,x)\mathrm{d}x \geq m(1-\varepsilon)^n \text{ for all } t \geq \tfrac{R}{m}T_0 \sum_{j=0}^{n-1} \frac{1}{2^j(1-\varepsilon)^j}, \tag{3.11}$$

where the increasing sequence of partial sums in the lower bound on t converges if $\varepsilon < \frac{1}{2}$. Given thus $\alpha \in (0,1)$, setting $\varepsilon = 1-2^{-\alpha}$ in (3.11) we then have

$$\int_{[0,2^{-n}R]} G(t,x)\mathrm{d}x \geq m(\tfrac{2^{1-n}R}{2R})^\alpha \text{ for all } n \in \mathbb{N} \text{ and } t \geq \tfrac{R}{m}T_*,$$

$$\text{with } T_* = T_*(\alpha) = T_0(1-2^{-\alpha}) \times \tfrac{2}{2-2^\alpha},$$

and (3.6) follows, since $[0,2^{-n}R] \subset [0,r]$ and $(2^{1-n}R)^\alpha \geq r^\alpha$ for $r \in (2^{-n}R, 2^{1-n}R]$. □

Now, if we want the contradiction argument in the proof of Theorem 3.1 to succeed, we need the waiting time for formation of a suitable lower bound on the left hand side of (3.2) to be arbitrarily small. Within arbitrarily small times there should thus be $m, R > 0$, with $\frac{m}{R} > 0$ arbitrarily large, such that an estimate of the form (3.5) is valid, which is the aforementioned second step towards the proof of Proposition 3.10.

Proposition 3.8. *Let G be a nonzero weak solution to* (QWTE)*, and let $L, \tau > 0$ be arbitrary. Then there exists $R_0 > 0$ such that*

$$\int_{[0,R_0]} G(t,x)\mathrm{d}x \geq LR_0 \textit{ for all } t \geq \tau.$$

Lemma 3.9. *Let G be a nonzero weak solution to* (QWTE)*, and let $\tau > 0$ be arbitrary. Then there exist $B, R_1 > 0$ such that*

$$\int_{[0,r]} G(t,x)\mathrm{d}x \geq Br \textit{ for all } r \in [0,R_1] \textit{ and } t \geq \tfrac{1}{2}\tau. \tag{3.12}$$

Proof. We first check that for any weak solution G to (QWTE) there holds

$$\int_{[0,r]} G(t,x)\mathrm{d}x \geq \frac{3}{16}\frac{1}{r}\frac{1}{4^n}\int_0^t \left(\int_{(r,2^n r]} G(s,x)\mathrm{d}x\right)^2 \mathrm{d}s \text{ for all } r,t \geq 0 \text{ and } n \in \mathbb{N}. \tag{3.13}$$

Indeed, for arbitrary $r, t \geq 0$ and $n \in \mathbb{N}$, it follows from Lemma 3.3, with $z_i = r + \frac{1}{2}ri$ for $i \in \mathbb{N}_0$, and an appropriate grouping of sums that

$$\int_{[0,r]} G(t,x)\mathrm{d}x \geq \frac{1}{4}\frac{1}{r}\int_0^t \left[\sum_{k=1}^{n}\sum_{j=2^k+1}^{2^{k+1}} \frac{2}{j}\left(\int_{(\frac{1}{2}r(j-1),\frac{1}{2}rj]} G(s,x)\mathrm{d}x\right)^2\right] \mathrm{d}s, \tag{3.14}$$

and (3.13) holds, since by twice using Lemma 3.4 we can bound the term between square brackets on the right hand side of (3.14) by

$$\sum_{k=1}^{n} \frac{1}{4^k} \left(\int_{(2^{k-1}r, 2^k r]} G(s,x) \mathrm{d}x \right)^2 \geq \frac{3}{4} \frac{1}{4^n} \left(\int_{(r, 2^n r]} G(s,x) \mathrm{d}x \right)^2 .$$

Now, noting that the statement of the lemma is immediate for trivial solutions, suppose that G is nontrivial. We can then define $m_0 := \int_{(0,\infty)} G(0,x)\mathrm{d}x > 0$, and

$$R_\ell := \tfrac{1}{3} \inf \left\{ r \geq 0 : \int_{(0,r]} G(0,x) \mathrm{d}x > \tfrac{1}{4} m_0 \right\} > 0,$$
$$R_r := 3 \inf \left\{ r \geq 0 : \int_{(0,r]} G(0,x) \mathrm{d}x \geq \tfrac{3}{4} m_0 \right\} < \infty,$$

so that in particular

$$\int_{(2R_\ell, \frac{1}{2}R_r]} G(0,x) \mathrm{d}x \geq \tfrac{1}{2} m_0.$$

Choosing next $\varphi \in C_c^1([0,\infty))$ such that $0 \leq \varphi \leq 1$, and with $\varphi \equiv 0$ on $(R_\ell, R_r]^c$ and $\varphi \equiv 1$ on $(2R_\ell, \frac{1}{2}R_r]$, and using φ as a time-independent test function in (QWTE)$^{\mathrm{w}}$, we then find for all $t \geq 0$ that

$$\begin{aligned} \int_{(R_\ell, R_r]} G(t,x) \mathrm{d}x - \tfrac{1}{2} m_0 &\geq \int_{[0,\infty)} \varphi(x) G(t,x) \mathrm{d}x - \int_{[0,\infty)} \varphi(x) G(0,x) \mathrm{d}x \\ &\geq - \left| \int_0^t \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} \Delta^2_{x\wedge y} \varphi(x \vee y) \mathrm{d}x \mathrm{d}y \, \mathrm{d}s \right| \\ &\geq - t \times \tfrac{1}{2} m_0^2 \times \sup\nolimits_{x,y>0} \left| \tfrac{1}{\sqrt{xy}} \Delta^2_{x\wedge y} \varphi(x \vee y) \right| =: - t \times \tfrac{1}{2} C m_0^2, \end{aligned}$$

where $C = C(\varphi) > 0$ is finite (cf. Lemma 1.6), hence there holds

$$\int_{(R_\ell, R_r]} G(t,x) \mathrm{d}x \geq \tfrac{1}{4} m_0 \text{ for all } t \in [0, \tfrac{1}{2}(C m_0)^{-1}]. \tag{3.15}$$

From (3.13), with $r \in (0, R_\ell]$ and $n \in \mathbb{N}$ such that $2^n r \in (R_r, 2R_r]$, we further find that

$$\int_{[0,r]} G(t,x) \mathrm{d}x \geq \frac{3}{64} \frac{r}{R_r^2} \int_0^t \left(\int_{(R_\ell, R_r]} G(s,x) \mathrm{d}x \right)^2 \mathrm{d}s \text{ for all } r \in (0, R_\ell] \text{ and } t \geq 0, \tag{3.16}$$

so, combining (3.15) and (3.16), we in particular have

$$\int_{[0,r]} G(\bar{t},x) \mathrm{d}x \geq \tfrac{3}{2^{10}} \tfrac{m_0^2 \bar{t}}{R_r^2} \, r \text{ for all } r \in (0, R_\ell], \text{ and with } \bar{t} = \tfrac{1}{2}(\tau \wedge (C m_0)^{-1}).$$

Applying lastly Proposition 2.4 with $\eta = \frac{1}{2}$ and $R = \frac{1}{2} r$, we thus obtain that

$$\int_{[0,r]} G(t,x) \mathrm{d}x \geq \tfrac{3}{2^{12}} \tfrac{m_0^2 \bar{t}}{R_r^2} \, r \text{ for all } r \in (0, 2R_\ell] \text{ and } t \geq \tfrac{1}{2}\tau,$$

and the claim follows since the estimate is trivial for $r = 0$. □

Proof of Proposition 3.8. Let $B, R_1 > 0$ be as obtained in Lemma 3.9, i.e. such that (3.12) holds, and let $\theta \in (0, \frac{2}{3}]$ be fixed such that $8\theta L \leq B$. If we now suppose that

$$\int_{[0,\theta\bar{r}]} G(\bar{t}, x)\mathrm{d}x \geq \tfrac{1}{2}B\bar{r} \text{ for some } \bar{r} \in (0, R_1] \text{ and } \bar{t} \in [\tfrac{1}{2}\tau, \tau], \tag{3.17}$$

then by Proposition 2.4, with $\eta = \frac{1}{2}$ and $R = \theta\bar{r}$, the claim follows with $R_0 = 2\theta\bar{r}$. Thus, we conversely suppose that (3.17) fails, i.e. [cf. (3.12)] that there holds

$$\int_{(\theta r, r]} G(t, x)\mathrm{d}x > \tfrac{1}{2}Br \text{ for all } r \in (0, R_1] \text{ and } t \in [\tfrac{1}{2}\tau, \tau]. \tag{3.18}$$

For $r \in (0, \theta R_1]$ arbitrarily fixed, we then set $z_{k_j} = \theta^{-j}r$ for $j \in \mathbb{N}_0$, where $k_0 = 0$ and

$$k_j = k_{j-1} + \min\left(\mathbb{N} \cap [2\theta^{-j}(1-\theta), 4\theta^{-j}(1-\theta))\right) \text{ for } j \in \mathbb{N}.$$

These definitions are such that $z_{k_j} - z_{k_{j-1}} = (1-\theta)\theta^{-j}r \leq (k_j - k_{j-1}) \times \frac{r}{2}$, whereby, setting $z_i - z_{i-1} = (z_{k_j} - z_{k_{j-1}})/(k_j - k_{j-1})$ for all $i \in \{k_{j-1}+1, \ldots, k_j\}$ and $j \in \mathbb{N}$, we obtain a sequence $(z_i)_{i\in\mathbb{N}_0} \subset \mathbb{R}_+$ with $z_i - z_{i-1} \in (0, \frac{1}{2}r]$ for all $i \in \mathbb{N}$. Choosing then $n \in \mathbb{N}$ such that $z_{k_n} = \theta^{-n}r \in (\theta R_1, R_1]$, we use the restriction $(z_0, \ldots, z_{k_n}) \in \mathbb{R}_+^{k_n+1}$ in Lemma 3.3 to get

$$\int_{[0,r]} G(t, x)\mathrm{d}x \geq \frac{1}{4}\int_0^t \left[\sum_{j=1}^{n} \sum_{i=k_{j-1}+1}^{k_j} \frac{1}{z_i}\left(\int_{(z_{i-1}, z_i]} G(s, x)\mathrm{d}x\right)^2\right] \mathrm{d}s \text{ for all } t \geq 0.$$

where the term between square bracket can be bounded from below by

$$\sum_{j=1}^{n} \frac{1}{(k_j - k_{j-1})z_{k_j}}\left(\int_{(z_{k_{j-1}}, z_{k_j}]} G(s, x)\mathrm{d}x\right)^2 \geq \frac{r}{4(1-\theta)}\sum_{j=1}^{n}\left(\frac{1}{z_{k_j}}\int_{(\theta z_{k_j}, z_{k_j}]} G(s, x)\mathrm{d}x\right)^2,$$

(cf. Lemma 3.4). Reducing further the domain of integration to $s \in [\frac{1}{2}\tau, \tau]$, it now follows with (3.18) that

$$\begin{aligned}\int_{[0,r]} G(\tau, x)\mathrm{d}x &\geq \frac{1}{4}\frac{r}{4(1-\theta)}\sum_{j=1}^{n}\int_{\frac{1}{2}\tau}^{\tau}\left(\frac{1}{z_{k_j}}\int_{(\theta z_{k_j}, z_{k_j}]} G(s, x)\mathrm{d}x\right)^2 \mathrm{d}s \\ &\geq \frac{r}{16(1-\theta)}\frac{B^2\tau}{8} \times n \geq \frac{B^2\tau}{128(1-\theta)}\frac{\log(\frac{r}{\theta R_1})}{\log\theta} \times r\end{aligned}$$

and applying Proposition 2.4 with $\eta = \frac{1}{2}$ and $R = r$ we obtain

$$\int_{[0,2r]} G(t, x)\mathrm{d}x \geq \frac{B^2\tau}{512(1-\theta)}\frac{\log(\frac{r}{\theta R_1})}{\log\theta} \times 2r \text{ for all } r \in (0, \theta R_1] \text{ and } t \geq \tau.$$

Setting lastly $R_\beta = \theta R_1 e^{-\beta}$, with $\beta > 0$, we thus have

$$\int_{[0,2R_\beta]} G(t, x)\mathrm{d}x \geq \frac{B^2\tau}{512(1-\theta)}\frac{\beta}{|\log\theta|} \times 2R_\beta \text{ for all } t \geq \tau.$$

and, choosing $\beta > 0$ sufficiently large, we conclude the claim with $R_0 = 2R_\beta$. □

To lastly obtain the lower bound on the left hand side of (3.2), we combine Propositions 3.6 and 3.8.

Proposition 3.10. *Let G be a nonzero weak solution to* (QWTE)*, and let $T > 0$ and $\alpha \in (0,1)$ be arbitrary. Then there exists $R_* > 0$ such that*

$$\int_{\frac{1}{3}T}^{\frac{2}{3}T} \int_{[0,r]} G(s,x)\mathrm{d}x \; \mathrm{d}s \geq T_*(2R_*)^{1-\alpha} \cdot r^\alpha \textit{ for all } r \in [0, R_*],$$

with $T_* = T_*(\alpha) > 0$ as obtained in Proposition 3.6.

Proof. From Propositions 3.6 and 3.8 we know that for any two $L, \tau > 0$ fixed, there exists $R_0 > 0$ such that

$$\int_{[0,r]} G(t,x)\mathrm{d}x \geq LR_0(\tfrac{r}{2R_0})^\alpha \text{ for all } r \in [0, R_0] \text{ and } t \geq L^{-1}T_* + \tau.$$

In particular, for $\tau = \frac{1}{6}T$ and $L^{-1} = \frac{1}{6}T/T_*$, there exists $R_* > 0$ such that

$$\int_{[0,r]} G(t,x)\mathrm{d}x \geq \tfrac{3}{T}T_*(2R_*)^{1-\alpha} \cdot r^\alpha \text{ for all } r \in [0, R_*] \text{ and } t \geq \tfrac{1}{3}T,$$

and the claim follows, integrating t over $[\frac{1}{3}T, \frac{2}{3}T]$. □

Finally, we are then able to prove Theorem 3.1.

Proof of Theorem 3.1. In view of the argumentation following Corollary 3.2 at the beginning of this chapter, we restrict ourselves to proving that, for a given nontrivial weak solution G to (QWTE) with initially zero mass at the origin, there holds (3.1). Supposing conversely that there exists some finite $T > 0$ for which (3.2) holds, then following Propositions 3.5 and 3.10 there exists a constant $R_* > 0$ such that

$$T_*(2R_*)^{3/4} \cdot \sqrt[4]{r} \leq \tfrac{2\sqrt{T\|G(0,\cdot)\|}}{\sqrt{3}-\sqrt{2}} \cdot \sqrt{r} \text{ for all } r \in [0, R_*],$$

with $T_* = T_*(\frac{1}{4}) > 0$ as obtained in Proposition 3.6. However, for $r \to 0$ this is absurd, whereby we conclude that (3.1) does hold. □

Chapter 4

Self-similar solutions

As we saw in Chapter 2, given any finite and nonnegative Radon measure G_0, there exists at least one weak solution G to (QWTE) with $G(0,\cdot) \equiv G_0$. Moreover, any such solution converges in the sense of measures to a Dirac measure, supported at zero, with the same mass as the initial data. This poses the same problem that led us to the study of (QWTE) in the first place, since any nontrivial solution, which has nonzero energy, converges weakly to a distribution with no energy.

In this chapter we construct weak solutions to (QWTE) that transfer their energy to infinity in a self-similar manner. However, since weak solutions to (QWTE) formally satisfy two conservation laws, we are required to introduce the following generalized notion of self-similarity.

Definition 4.1. We say that G is a *self-similar solution* to (QWTE) if it is a weak solution in the sense of Definition 1.7 that (i) is not trivial in the sense of Definition 2.10; and that (ii) admits the representation

$$G(t,\cdot) \equiv m(t)\delta_0 + h(t,\cdot) \text{ on } [0,\infty) \text{ for } t \geq 0, \tag{4.1}$$

where $h(t,\cdot)$ has a density with respect to Lebesgue measure that is given by

$$h(t,x) = \lambda(t)^{-1}\Phi\left(\lambda(t)^{-\frac{1}{\rho}}x\right) \tag{4.2}$$

with $\rho \in (1,2]$, with $\lambda(t) = \lambda_1 t + \lambda_0$ for $\lambda_0, \lambda_1 > 0$, and with $\Phi \in L^1(0,\infty)$ nonnegative the *self-similar profile* of G, and where

$$m(t) = M - \lambda(t)^{\frac{1}{\rho}-1}\|\Phi\|_{L^1(0,\infty)} \tag{4.3}$$

with $M \geq \lambda_0^{-(\rho-1)/\rho}\|\Phi\|_{L^1(0,\infty)}$.

Note that this notion of self-similarity can only be introduced by the fact that all information about the energy of a solution to (QWTE) is supported on $(0,\infty)$, whereas its mass is supported on $[0,\infty)$. The energy of a self-similar solution in the sense of Definition 4.1 is self-similar in the classical sense, while conservation of mass is ensured by compensating the loss of mass from the interval $(0,\infty)$ by an increasing mass at zero.

We begin our treatment of self-similar solutions to (QWTE) by giving a necessary and sufficient condition for a nonnegative function $\Phi \in L^1(0,\infty)$ to be the self-similar profile of a solution (cf. Proposition 4.2). The better part of this chapter, that is Section 4.1, is then devoted to the proof of existence of nonnegative functions $\Phi \in L^1(0,\infty)$ that satisfy that condition. In Section 4.2, we consider the asymptotic behaviour of self-similar profiles, presenting our rigorous results, and also two conjectures. Lastly, in Section 4.3, we briefly reflect on the restriction to solutions with finite mass.

Proposition 4.2. *Let $\rho \in (1,2]$ and $\lambda_1 > 0$ be arbitrary, and suppose that there exists a nontrivial and nonnegative function $\Phi \in L^1(0,\infty)$ that for all $\psi \in C_c^1([0,\infty))$ satisfies*

$$\frac{\lambda_1}{\rho}\int_{(0,\infty)} \big(x\psi'(x) - (\rho-1)(\psi(x)-\psi(0))\big)\Phi(x)\mathrm{d}x \\ = \frac{1}{2}\iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y. \quad (4.4)$$

Then Φ is the self-similar profile of a self-similar solution to (QWTE)*. Conversely, if G is a self-similar solution to* (QWTE) *in the sense of Definition 4.1, then its self-similar profile is nontrivial, and it satisfies* (4.4) *for all $\psi \in C_c^1([0,\infty))$.*

Proof. Choosing $\lambda_0 > 0$ and $M \geq \lambda_0^{-(\rho-1)/\rho}\|\Phi\|_{L^1(0,\infty)}$ arbitrarily, we set $\lambda(t) = \lambda_1 t + \lambda_0$, and we let G be given by (4.1) with h given by (4.2), and m given by (4.3). For the first claim, it now suffices to check that this G is a weak solution in the sense of Definition 1.7. Let thereto $\varphi \in C([0,\infty) : C_c^1([0,\infty))) \cap C^1([0,\infty) : C_c([0,\infty))$ be fixed, and let ψ be such that

$$\psi(s,x) = \varphi\left(s, \lambda(s)^{\frac{1}{\rho}}x\right) \text{ for all } s,x \geq 0,$$

for which, noting that $x\psi_x(s,x) = \lambda(s)^{\frac{1}{\rho}}x\,\varphi_x(s,\lambda(s)^{\frac{1}{\rho}}x)$, we easily check that

$$\psi_s(s,x) = \varphi_s\left(s,\lambda(s)^{\frac{1}{\rho}}x\right) + \lambda'(s) \times \tfrac{1}{\rho}\lambda(s)^{-1}\,x\psi_x(s,x). \quad (4.5)$$

Using then (4.5), we get

$$\begin{aligned}\int_{(0,\infty)} \varphi_s(s,x)G(s,x)\mathrm{d}x &= \lambda(s)^{\frac{1}{\rho}-1}\int_{(0,\infty)} \varphi_s\left(s,\lambda(s)^{\frac{1}{\rho}}x\right)\Phi(x)\mathrm{d}x \\ &= \lambda(s)^{\frac{1}{\rho}-1}\int_{(0,\infty)} \psi_s(s,x)\Phi(x)\mathrm{d}x - \lambda(s)^{\frac{1}{\rho}-2} \times \frac{\lambda'(s)}{\rho}\int_{(0,\infty)} x\psi_x(s,x)\Phi(x)\mathrm{d}x \\ &= \partial_s\left[\int_{(0,\infty)} \varphi(s,x)G(s,x)\mathrm{d}x\right] \\ &\qquad - \lambda(s)^{\frac{1}{\rho}-2} \times \frac{\lambda'(s)}{\rho}\int_{(0,\infty)} \big(x\psi_x(s,x) - (\rho-1)\psi(s,x)\big)\Phi(x)\mathrm{d}x \quad (4.6)\end{aligned}$$

where the final identity is due to

$$\begin{aligned}\partial_s\left[\int_{(0,\infty)} \varphi(s,x)G(s,x)\mathrm{d}x\right] &= \partial_s\left[\lambda(s)^{\frac{1}{\rho}-1}\int_{(0,\infty)} \psi(s,x)\Phi(x)\mathrm{d}x\right] \\ &= \lambda(s)^{\frac{1}{\rho}-1}\int_{(0,\infty)} \psi_s(s,x)\Phi(x)\mathrm{d}x - \lambda(s)^{\frac{1}{\rho}-2} \times \frac{\lambda'(s)}{\rho}\int_{(0,\infty)} (\rho-1)\psi(s,x)\Phi(x)\mathrm{d}x,\end{aligned}$$

and we further find that

$$\begin{aligned}\int_{\{0\}} \varphi_s(s,x)G(s,x)\mathrm{d}x &= \varphi_s(s,0)\left(M - \lambda(s)^{\frac{1}{\rho}-1}\|\Phi\|_{L^1(0,\infty)}\right) \\ &= \partial_s\left[\varphi(s,0)\left(M - \lambda(s)^{\frac{1}{\rho}-1}\|\Phi\|_{L^1(0,\infty)}\right)\right] + \partial_s\left[\lambda(s)^{\frac{1}{\rho}-1}\right] \times \varphi(s,0)\|\Phi\|_{L^1(0,\infty)} \\ &= \partial_s\left[\int_{\{0\}} \varphi(s,x)G(s,x)\mathrm{d}x\right] - \lambda(s)^{\frac{1}{\rho}-2} \times \frac{\lambda'(s)}{\rho} \times (\rho-1)\,\varphi(s,0)\|\Phi\|_{L^1(0,\infty)}.\end{aligned} \quad (4.7)$$

Thus, combining (4.6) and (4.7), we have

$$\int_{[0,\infty)} \varphi(t,x)G(t,x)\mathrm{d}x - \int_{[0,\infty)} \varphi(0,x)G(0,x)\mathrm{d}x - \int_0^t \int_{[0,\infty)} \varphi_s(s,x)G(s,x)\mathrm{d}x\ \mathrm{d}s$$
$$= \int_0^t \lambda(s)^{\frac{1}{\rho}-2} \times \frac{\lambda'(s)}{\rho} \int_{(0,\infty)} \big(x\psi_x(s,x) - (\rho-1)(\psi(s,x)-\psi(s,0))\big)\Phi(x)\mathrm{d}x\ \mathrm{d}s, \quad (4.8)$$

while direct computation yields

$$\int_0^t \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{G(s,x)G(s,y)}{\sqrt{xy}} \Delta^2_{x\wedge y}[\varphi(s,\cdot)](x\vee y)\mathrm{d}x\mathrm{d}y\ \mathrm{d}s$$
$$= \int_0^t \lambda(s)^{\frac{1}{\rho}-2} \times \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta^2_{x\wedge y}[\psi(s,\cdot)](x\vee y)\mathrm{d}x\mathrm{d}y\ \mathrm{d}s. \quad (4.9)$$

Recalling lastly that $\lambda'(s) = \lambda_1$, the right hand sides of (4.8) and (4.9) are equal by assumption [cf. (4.4)], and we conclude that G is indeed a self-similar solution to (QWTE).

Conversely, supposing G to be a self-similar solution to (QWTE), we get (4.8) and (4.9) precisely as above. However, now the left hand sides are equal by assumption, whence we conclude that the self-similar profile of G must satisfy (4.4) for all $\psi \in C_c^1([0,\infty))$. □

Note that it follows from the proof of Proposition 4.2 that the scaling function λ in Definition 4.1 must necessarily be affine.

To conclude the preliminary remarks, we state a straightforward scaling result, which reduces the number of parameters in the equation for the self-similar profile to just ρ.

Lemma 4.3. *Given $\rho \in (1,2]$ and $\lambda_1, \lambda_* > 0$ arbitrarily fixed, let $\Phi_* \in L^1(0,\infty)$ be a function that satisfies* (4.4) *for all $\psi \in C_c^1([0,\infty))$, and let $\Phi \in L^1(0,\infty)$ be given by $\Phi(x) = \lambda_1 \Phi_*(\lambda_* x)$. Then for all $\psi \in C_c^1([0,\infty))$ there holds*

$$\frac{1}{\rho} \int_{(0,\infty)} \big(x\psi'(x) - (\rho-1)(\psi(x)-\psi(0))\big)\Phi(x)\mathrm{d}x$$
$$= \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y. \quad (\mathrm{SSPE})^{\mathrm{w}}_\rho$$

Proof. Trivial. □

4.1 Existence of self-similar solutions

In view of Lemma 4.3 and Proposition 4.2, we show existence of self-similar solutions by proving the following

Theorem 4.4. *Given $\rho \in (1,2]$, there exists at least one nontrivial function $\Phi \in L^1(0,\infty)$ that is nonnegative, and that satisfies* $(\mathrm{SSPE})^{\mathrm{w}}_\rho$ *for all $\psi \in C_c^1([0,\infty))$. Moreover, any nontrivial and nonnegative function $\Phi \in L^1(0,\infty)$ that satisfies* $(\mathrm{SSPE})^{\mathrm{w}}_\rho$ *for all $\psi \in C_c^1([0,\infty))$ is smooth and strictly positive on $(0,\infty)$, and is a classical solution on $(0,\infty)$ to*

$$-\tfrac{1}{\rho}x\Phi'(x) - \Phi(x) = \int_0^{x/2} \frac{\Phi(y)}{\sqrt{y}} \left[\frac{\Phi(x+y)}{\sqrt{x+y}} + \frac{\Phi(x-y)}{\sqrt{x-y}} - 2\frac{\Phi(x)}{\sqrt{x}}\right] \mathrm{d}y$$
$$+ \int_{x/2}^\infty \frac{\Phi(y)\Phi(x+y)}{\sqrt{y(x+y)}}\mathrm{d}y - 2\frac{\Phi(x)}{\sqrt{x}} \int_{x/2}^x \frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y. \quad (\mathrm{SSPE})_\rho$$

Remark 4.5. Though Theorem 4.4 combines several results from [KV15, KV16], the proofs that we present in the following differ significantly from the ones given in those papers. There the approach to proving existence was to first get existence of nonnegative measures Ψ that satisfy

$$\frac{1}{\rho}\int_{[0,\infty)}\big(x\vartheta'(x)+(2-\rho)\vartheta(x)\big)\Psi(x)\mathrm{d}x=\frac{1}{2}\iint_{[0,\infty)^2}\frac{\Psi(x)\Psi(y)}{(xy)^{3/2}}\Delta^2_{x\wedge y}[z\vartheta(z)](x\vee y)\mathrm{d}x\mathrm{d}y$$

for suitable test functions ϑ, then to show that these measures Ψ allow to define the finite Radon measures $\Phi(x)=\frac{1}{x}\Psi(x)$ that satisfy $(\mathrm{SSPE})^{\mathrm{w}}_{\rho}$ (using $\vartheta(x)=\frac{1}{x}(\psi(x)-\psi(0))$), and lastly to check that these measures Φ are absolutely continuous with respect to Lebesgue measure. This approach, passing through the equation for the self-similar energy distribution, was chosen in [KV15] to tackle the case of self-similar solutions with finite energy, which made sense as the first step uses a fixed-point argument for an energy-preserving semigroup. However, if $\rho\neq 2$, then self-similar solutions cannot have finite energy, since that energy would not be conserved by the scaling. Nevertheless, the case of self-similar solutions with infinite energy was treated in a similar way in [KV16].

The proof of existence of self-similar solutions with infinite energy, as presented here, differs from the one presented in [KV16] in one major aspect: There the methods in [32], which proves existence of fat-tailed self-similar solutions to Smoluchowski's coagulation equation with locally bounded kernels, were adapted to the equation for the self-similar energy, whereas here we adapt them to the equation for the mass distribution. This way we do not have to concern ourselves with eliminating the presence of a Dirac measure at zero in the energy distribution, as this would disallow the unambiguous definition of the mass distribution. (Indeed, what is $\frac{1}{x}\delta_0(x)\mathrm{d}x$?)

Overall the strategy is the same: To construct a semigroup on a set of measures, such that fixed-points under the action satisfy the weak formulation of the equation for the self-similar profile, and to show that it leaves a suitable subset of measures invariant. A fixed-point theorem then yields existence of a solution measure, which is later shown to be sufficiently regular.

The expected (or desired) decay behaviour of self-similar profiles can be used to determine a possible invariant set. The fat-tailed profiles h from [32] satisfy $h(x)\sim(1-\rho)x^{-\rho}$ as $x\to\infty$ for $\rho\in(0,1)$, which corresponds to $\int_{[0,R]}h(x)\mathrm{d}x\sim R^{1-\rho}$ as $R\to\infty$ for measures. Justifiably, there the invariant sets are thus sets of nonnegative Radon measures h for which there are $R_0,\delta>0$ such that

$$R^{1-\rho}\Big(1-\big(\tfrac{R_0}{R}\big)^{\delta}\Big)_+\leq\int_{[0,R]}h(x)\mathrm{d}x\leq R^{1-\rho}\text{ for all }R>0.$$

Here we look for profiles Φ with finite mass and infinite energy, allowing for power law tails $\Phi(x)\sim Cx^{-\rho}$ as $x\to\infty$ with $\rho\in(1,2)$. Up to a constant, this now suggests the generalization $\int_0^R\int_{[y,\infty)}\Phi(x)\mathrm{d}x\,\mathrm{d}y=\int_{(0,\infty)}(x\wedge R)\Phi(x)\mathrm{d}x\sim R^{2-\rho}$ as $R\to\infty$ for measures, and thus, by analogy to the above, we expect to find an invariant set of nonnegative Radon measures Φ for which there are $R_0,\delta>0$ such that

$$R^{2-\rho}\Big(1-\big(\tfrac{R_0}{R}\big)^{\delta}\Big)_+\leq\int_{[0,\infty]}(x\wedge R)\Phi(x)\mathrm{d}x\leq R^{2-\rho}\text{ for all }R>0.$$

Indeed, using methods adapted from [32], such sets will turn out to be the correct invariant ones here (cf. Definition 4.17, and Proposition 4.19).

Functional analytic setting

Let us present the functional analytic setting of the existence proof. To be able to fully exploit the power of the weak-$*$ topology, we would like to formulate "the set of nonnegative Radon measures μ for which $\sup_{R>0}\left\{R^{\rho-2}\int_{(0,\infty)}(x\wedge R)\mu(x)\mathrm{d}x\right\}<\infty$" as the closed subspace of the topological dual of a separable Banach space of functions.

Definition 4.6. Let $\mathcal{B}$ denote the space of functions $\psi\in C_0((0,\infty])$ for which the right derivative at zero exists. Endowed with norm $\|\psi\|_{\mathcal{B}}=\sup_{z>0}\frac{1+z}{z}|\psi(z)|$, this space is isometrically isomorphic to $(C([0,\infty]),\|\cdot\|_{L^\infty(0,\infty)})$ via the isomorphism $\iota:\mathcal{B}\to C([0,\infty])$ that is given by $\psi\mapsto\iota\psi(z):=\frac{1+z}{z}\psi(z)$.

By analogy to Definition 1.2 we then also define its dual space.

Definition 4.7. We write $\mathcal{B}'$ for the topological dual space of $\mathcal{B}$, which we endow with the weak-$*$ topology. Note that a sequence $\{\beta_n\}\subset\mathcal{B}'$ converges with respect to the weak-$*$ topology to $\beta\in\mathcal{B}'$, for short $\beta_n\overset{*}{\rightharpoonup}\beta$ in $\mathcal{B}'$, if and only if $\langle\beta_n,\psi\rangle\to\langle\beta,\psi\rangle$ for all $\psi\in\mathcal{B}$, since $\mathcal{B}$ is a separable Banach space. We say that an element $\beta\in\mathcal{B}'$ is nonnegative if $\langle\beta,\psi\rangle\geq 0$ for all $0\leq\psi\in\mathcal{B}$.

It should be noted that elements in $\mathcal{B}'$ are not always Radon measures. (Indeed, the mapping $\psi\mapsto\psi'(0)$ is a bounded linear functional on $\mathcal{B}$.) Regardless, we now introduce the spaces in which we will prove existence, but see Remark 4.9.

Definition 4.8. Given $\rho\in(1,2]$, define

$$\mathcal{X}_\rho=\Big\{\beta\in\mathcal{B}':\beta\circ\iota^{-1}\in\mathcal{M}_+([0,\infty])\text{ and }\sup_{R>0}\left\{R^{\rho-2}\langle\beta,(\cdot\wedge R)\rangle\right\}=:\|\beta\|_\rho<\infty\Big\}.$$

We then write $\mathcal{U}_\rho$ for the closed unit ball in $\mathcal{X}_\rho$, i.e. $\mathcal{U}_\rho=\mathcal{X}_\rho\cap\{\|\beta\|_\rho\leq 1\}$, and $\mathcal{S}_\rho$ for the unit sphere in $\mathcal{X}_\rho$, i.e. $\mathcal{S}_\rho=\mathcal{X}_\rho\cap\{\|\beta\|_\rho=1\}$, and we further let $\mathcal{X}_1$ denote the set of measures $\mu\in\mathcal{M}_+([0,\infty])$ for which $\mu(\{0\})=\mu(\{\infty\})=0$, with $\|\mu\|_1:=\int_{(0,\infty)}\mu(x)\mathrm{d}x$.

Remark 4.9. Given $\rho\in(1,2)$, then for any $\beta\in\mathcal{X}_\rho$ there holds

$$(\beta\circ\iota^{-1})(\{0\})\leq\langle\beta,(\cdot\wedge R)\rangle\leq R^{2-\rho}\|\beta\|_\rho\text{ for all }R>0,$$

hence $(\beta\circ\iota^{-1})(\{0\})=0$, and we conclude that elements in $\mathcal{X}_\rho$ are measures. Noting further that $(\beta\circ\iota^{-1})(\{\infty\})\leq\inf_{R>0}\{R^{1-\rho}\|\beta\|_\rho\}=0$, we may also write pairings $\langle\beta,\psi\rangle$ of elements $\beta\in\mathcal{X}_\rho$ and $\psi\in\mathcal{B}$ as integrals over $(0,\infty)$, and in particular there holds

$$\|\beta\|_\rho=\sup_{R>0}\left\{R^{\rho-2}\int_{(0,\infty)}(x\wedge R)|\beta(x)|\mathrm{d}x\right\}.$$

We emphasize that this argument fails in the case $\rho=2$, and $\mathcal{X}_2$ still contains elements that are not measures.

We finish with two useful results, the proofs of which are postponed to the appendix. One is a consequence of Banach-Alaoglu, and the other is a fixed-point theorem from [6].

Lemma 4.10. *Given $\rho\in(1,2]$, then $\mathcal{U}_\rho$ is compact with respect to the weak-$*$ topology on $\mathcal{B}'$.*

Lemma 4.11. *Let X be a locally convex topological vector space, let $Y\subset X$ be nonempty, convex, and compact, let $(S(t))_{t\geq0}$ be a continuous semigroup on Y, (i.e. let $t\mapsto S(t):Y\to Y$ be continuous for $t\geq0$, with $S(t_1+t_2)=S(t_1)S(t_2)$ for all $t_1,t_2\geq0$), and suppose that for every $t\geq0$ the mapping $y\mapsto S(t)y$ is continuous. Then there exists at least one $y\in Y$ that is a fixed-point for $(S(t))_{t\geq0}$, i.e. an element $y\in Y$ such that $S(t)y=y$ for all $t\geq0$.*

Remark 4.12. As we define self-similar solutions to have self-similar profiles in $L^1(0,\infty)$, we call $\Phi \in \mathcal{X}_1$ a *candidate self-similar profile* if it satisfies $(\text{SSPE})^{\text{w}}_\rho$ for all $\psi \in C^1_c([0,\infty))$.

4.1.1 Candidate self-similar profiles for $\rho \in (1,2)$

Despite all efforts to unify the existence part of Theorem 4.4, a separate treatment of the cases $\rho \in (1,2)$ and $\rho = 2$ has turned out to be unavoidable in the construction of candidate profiles. Using the tailor-made machinery that was introduced above, we address the case $\rho \in (1,2)$ first.

Proposition 4.13. *Given $\rho \in (1,2)$, there exists at least one $\Phi \in \mathcal{S}_\rho \cap \mathcal{X}_1$ that satisfies* $(\text{SSPE})^{\text{w}}_\rho$ *for all $\psi \in C^1_c([0,\infty))$.*

Construction of a semigroup

We first construct a semigroup, fixed-points of which satisfy an approximation to $(\text{SSPE})^{\text{w}}_\rho$. To that end we prove local existence of certain mild solutions (cf. Lemma 4.14), which we then show to be weak solutions, with semigroup property, in Proposition 4.15. Proving lastly continuous dependence on the initial data (cf. Lemma 4.16), we thus have a semigroup as in the statement of Lemma 4.11.

Lemma 4.14. *Given $\rho \in (1,2)$ and $\varepsilon_0 > 2\varepsilon > 0$, there exists $T > 0$ such that for every $\Phi_0 \in \mathcal{U}_\rho$ there is a unique function $F \in C([0,T] : \mathcal{X}_\rho)$ that for all $t \in [0,T]$ and $\varphi \in \mathcal{B}$ satisfies*

$$\begin{aligned}\int_{(0,\infty)} \varphi(x) F(t,x)\mathrm{d}x = &\int_{(0,\infty)} \varphi(x) e^{-\int_0^t A(s)[F(s,\cdot)](x)\mathrm{d}s} \Phi_0(x)\mathrm{d}x \\ &+ \int_0^t \int_{(0,\infty)} \varphi(x) e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s} B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\, \mathrm{d}\sigma, \end{aligned} \tag{4.10}$$

where for $s \geq 0$ the mapping $A(s) : \mathcal{X}_\rho \to C([0,\infty])$ is given by

$$A(s)[F](x) = \int_{(0,\infty)} \frac{2xy}{\left(x + e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}} \int_0^x \frac{\frac{1}{e^{\frac{s}{\rho}}\varepsilon}\left(1 - \left|\frac{y-z}{e^{\frac{s}{\rho}}\varepsilon}\right|\right)_+}{\left(z + e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}} \mathrm{d}z\, F(y)\mathrm{d}y - 1,$$

and where $B(s) : \mathcal{X}_\rho \to \mathcal{X}_\rho$ is such that for any $\varphi \in \mathcal{B}$ there holds

$$\begin{aligned}&\int_{(0,\infty)} \varphi(x) B(s)[F](x)\mathrm{d}x \\ &= \iint_{\mathbb{R}^2_+} F(x)F(y) \frac{xy}{\left(x + e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}} \int_0^x \frac{\frac{1}{e^{\frac{s}{\rho}}\varepsilon}\left(1 - \left|\frac{y-z}{e^{\frac{s}{\rho}}\varepsilon}\right|\right)_+}{\left(z + e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}} \begin{bmatrix} \varphi(x+z) \\ +\varphi(x-z) \end{bmatrix} \mathrm{d}z\, \mathrm{d}x\mathrm{d}y.\end{aligned}$$

Proof. For arbitrarily fixed $\Phi_0 \in \mathcal{U}_\rho$, we show that there exists $T = T(\rho, \varepsilon_0) > 0$ such that the operator $\mathcal{T} : C([0,T] : \mathcal{X}_\rho) \to C([0,T] : \mathcal{X}_\rho)$, defined to be such that for all $t \in [0,T]$ and $\varphi \in \mathcal{B}$ there holds

$$\begin{aligned}\int_{(0,\infty)} \varphi(x) \mathcal{T}[F](t,x)\mathrm{d}x = &\int_{(0,\infty)} \varphi(x) e^{-\int_0^t A(s)[F(s,\cdot)](x)\mathrm{d}s} \Phi_0(x)\mathrm{d}x \\ &+ \int_0^t \int_{(0,\infty)} \varphi(x) e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s} B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\, \mathrm{d}\sigma,\end{aligned}$$

is a contraction on $C([0,T] : 2\mathcal{U}_\rho) = \{\beta \in C([0,T] : \mathcal{X}_\rho) : \sup_{t\in[0,T]} \|\beta(t,\cdot)\|_\rho \leq 2\}$. The claim then follows by Banach's fixed-point theorem.

In order to check that $\mathcal{T}$ is well-defined, it is sufficient to show for $s \geq 0$ and $F \in \mathcal{X}_\rho$ that $B(s)[F] \in \mathcal{X}_\rho$. To that end we first note for $\varphi \in \mathcal{B}$ and $x \geq z \geq 0$ that

$$|\varphi(x+z)+\varphi(x-z)| \leq \|\varphi\|_{\mathcal{B}} \left(\tfrac{x+z}{1+x+z}+\tfrac{x-z}{1+x-z}\right) \leq 2\|\varphi\|_{\mathcal{B}}\, \tfrac{x}{1+x},$$

and for $s \geq 0$ and $x, y > 0$ that

$$\begin{aligned}\frac{xy}{\left(x+e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}}\int_0^x \frac{\frac{1}{e^{\frac{s}{\rho}}\varepsilon}\left(1-\left|\frac{y-z}{e^{\frac{s}{\rho}}\varepsilon}\right|\right)_+}{\left(z+e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}}\mathrm{d}z &\leq \frac{y}{e^{\frac{s}{\rho}}\varepsilon_0}\int_0^x \frac{\frac{1}{e^{\frac{s}{\rho}}\varepsilon}\left(1-\left|\frac{y-z}{e^{\frac{s}{\rho}}\varepsilon}\right|\right)_+}{e^{\frac{s}{\rho}}\varepsilon_0-(y-z)+y}\mathrm{d}z \\ &\leq \frac{y}{e^{\frac{s}{\rho}}\varepsilon_0}\frac{\int_{\mathbb{R}}(1-|\zeta|)_+\mathrm{d}\zeta}{e^{\frac{s}{\rho}}(\varepsilon_0-\varepsilon)+y} \leq \frac{1}{e^{\frac{s}{\rho}}\varepsilon_0}\frac{y}{e^{\frac{s}{\rho}}\frac{\varepsilon_0}{2}+y} \leq \frac{1}{2}\Big(\tfrac{2}{e^{\frac{s}{\rho}}\varepsilon_0}\Big)^2\Big(y\wedge \tfrac{e^{\frac{s}{\rho}}\varepsilon_0}{2}\Big), \quad (4.11)\end{aligned}$$

from which it follows that $B(s)[F] \in \mathcal{B}'$. It is further straightforward to see that $B(s)[F]$ is nonnegative, and using (4.11) and the fact that $((x+z)\wedge R)+((x-z)\wedge R) \leq 2(x\wedge R)$ it follows that

$$\begin{aligned}\|B(s)[F]\|_\rho &\leq \sup_{R>0}\left\{R^{\rho-2}\int_{(0,\infty)}(x\wedge R)F(x)\mathrm{d}x\right\} \times \Big(\tfrac{2}{e^{\frac{s}{\rho}}\varepsilon_0}\Big)^2\int_{(0,\infty)}\Big(y\wedge\tfrac{e^{\frac{s}{\rho}}\varepsilon_0}{2}\Big)F(y)\mathrm{d}y \\ &\leq \Big(\tfrac{2}{e^{\frac{s}{\rho}}\varepsilon_0}\Big)^\rho\|F\|_\rho \times \Big(\tfrac{e^{\frac{s}{\rho}}\varepsilon_0}{2}\Big)^{\rho-2}\int_{(0,\infty)}\Big(y\wedge\tfrac{e^{\frac{s}{\rho}}\varepsilon_0}{2}\Big)F(y)\mathrm{d}y \leq \tfrac{2^\rho}{\varepsilon_0^\rho}e^{-s}\|F\|_\rho^2. \quad (4.12)\end{aligned}$$

Moreover, with (4.12), and since $A(s)[F](x)+1 \geq 0$ for all $s \geq 0$, $F \in \mathcal{X}_\rho$ and $x > 0$, we find for all $F \in C([0,T]:\mathcal{X}_\rho)$ and $t \in [0,T]$ that

$$\begin{aligned}\|\mathcal{T}[F](t,\cdot)\|_\rho &\leq e^t\|\Phi_0\|_\rho + \int_0^t e^{t-\sigma}\|B(\sigma)[F(\sigma,\cdot)]\|_\rho\mathrm{d}\sigma \\ &\leq e^t\left(1+\frac{2^\rho}{\varepsilon_0^\rho}\int_0^t e^{-2\sigma}\mathrm{d}\sigma \times \sup\nolimits_{\sigma\in[0,T]}\|F(\sigma,\cdot)\|_\rho^2\right),\end{aligned}$$

and it follows that $\mathcal{T}$ maps $C([0,T]:2\mathcal{U}_\rho)$ into itself if $T>0$ is small enough, depending only on ρ and ε_0.

Now, to check that $\mathcal{T}$ is contractive on $C([0,T]:2\mathcal{U}_\rho)$ for sufficiently small $T>0$, we note for $t \in [0,T]$ and any two $F_1, F_2 \in C([0,T]:\mathcal{X}_\rho)$ that

$$\begin{aligned}&\|\mathcal{T}[F_1](t,\cdot)-\mathcal{T}[F_2](t,\cdot)\|_\rho \\ &\quad= \sup_{R>0}\left\{R^{\rho-2}\times\sup\nolimits_{\varphi\in\mathcal{B},|\varphi(x)|\leq(x\wedge R)}\int_{(0,\infty)}\varphi(x)(\mathcal{T}[F_1](t,x)-\mathcal{T}[F_2](t,x))\mathrm{d}x\right\}.\end{aligned}$$

We then note that for any $\varphi \in \mathcal{B}$ there holds

$$\begin{aligned}&\int_{(0,\infty)}\varphi(x)(\mathcal{T}[F_1](t,x)-\mathcal{T}[F_2](t,x))\mathrm{d}x \\ &\quad\leq \int_{(0,\infty)}|\varphi(x)|\left|e^{-\int_0^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s}-e^{-\int_0^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s}\right|\Phi_0(x)\mathrm{d}x \\ &\qquad+\int_0^t\left|\int_{(0,\infty)}\varphi(x)e^{-\int_\sigma^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F_1(\sigma,\cdot)](x)\mathrm{d}x\right. \\ &\qquad\qquad\left.-\int_{(0,\infty)}\varphi(x)e^{-\int_\sigma^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F_2(\sigma,\cdot)](x)\mathrm{d}x\right|\mathrm{d}\sigma,\end{aligned}$$

where the integrand of the second integral on the right hand side can be estimated by

$$\frac{1}{2}\sum_{i=1}^{2}\int_{(0,\infty)}|\varphi(x)|\left|e^{-\int_\sigma^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s}-e^{-\int_\sigma^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s}\right|B(\sigma)[F_i(\sigma,\cdot)](x)\mathrm{d}x$$
$$+\frac{1}{2}\sum_{i=1}^{2}\left|\int_{(0,\infty)}\varphi(x)e^{-\int_\sigma^t A(s)[F_i(s,\cdot)](x)\mathrm{d}s}\left(B(\sigma)[F_1(\sigma,\cdot)](x)-B(\sigma)[F_2(\sigma,\cdot)](x)\right)\mathrm{d}x\right|,$$

and using further arguments similar to the ones used to obtain (4.12), we obtain the estimate

$$\|\mathcal{T}[F_1](t,\cdot)-\mathcal{T}[F_2](t,\cdot)\|_\rho\le\|\Phi_0\|_\rho\times\sup_{x>0}\left|e^{-\int_0^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s}-e^{-\int_0^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s}\right|$$
$$+\frac{1}{2}\sum_{i=1}^{2}\int_0^t\left[\|B(\sigma)[F_i(\sigma,\cdot)]\|_\rho\times\sup_{x>0}\left|e^{-\int_\sigma^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s}-e^{-\int_\sigma^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s}\right|\right]\mathrm{d}\sigma$$
$$+\int_0^t\left[\tfrac{2\rho}{\varepsilon_0^\rho}e^{t-2\sigma}\left(\|F_1(\sigma,\cdot)\|_\rho+\|F_2(\sigma,\cdot)\|_\rho\right)\|F_1(\sigma,\cdot)-F_2(\sigma,\cdot)\|_\rho\right]\mathrm{d}\sigma. \quad (4.13)$$

Moreover, with (4.11) we find for $s\ge 0$ and $F_1,F_2\in\mathcal{X}_\rho$ that

$$|A(s)[F_1](x)-A(s)[F_2](x)|\le\tfrac{2\rho}{\varepsilon_0^\rho}e^{-s}\|F_1-F_2\|_\rho,$$

so recalling the fact that $|e^{x-x_1}-e^{x-x_2}|\le e^x|x_1-x_2|$ for $x_1,x_2\ge 0$ and $x\in\mathbb{R}$, we get for $t\ge\sigma\ge 0$ and $F_1,F_2\in C([0,\infty):\mathcal{X}_\rho)$ that

$$\sup_{x>0}\left|e^{-\int_\sigma^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s}-e^{-\int_\sigma^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s}\right|$$
$$\le e^{t-\sigma}\int_\sigma^t\left[\sup_{x>0}|A(s)[F_1(s,\cdot)](x)-A(s)[F_2(s,\cdot)](x)|\right]\mathrm{d}s$$
$$\le\tfrac{2\rho}{\varepsilon_0^\rho}e^{t-\sigma}(e^{-\sigma}-e^{-t})\times\sup_{s\in[0,t]}\|F_1(s,\cdot)-F_2(s,\cdot)\|_\rho. \quad (4.14)$$

Using lastly (4.12) and (4.14) in (4.13), it follows for $t\in[0,T]$ and $F_1,F_2\in C([0,T]:2\mathcal{U}_\rho)$ that

$$\|\mathcal{T}[F_1](t,\cdot)-\mathcal{T}[F_2](t,\cdot)\|_\rho\le\tfrac{2\rho}{\varepsilon_0^\rho}\left(5+4\tfrac{2\rho}{\varepsilon_0^\rho}\right)te^t\times\sup_{s\in[0,t]}\|F_1(s,\cdot)-F_2(s,\cdot)\|_\rho,$$

and we conclude that there exists some $T=T(\rho,\varepsilon_0)>0$ such that $\mathcal{T}$ is a contraction on $C([0,T]:2\mathcal{U}_\rho)$. □

Indeed, we are now able to introduce a semigroup.

Proposition 4.15. *Given $\rho\in(1,2)$ and $\varepsilon_0>2\varepsilon>0$, let $T>0$ be as obtained in Lemma 4.14, and for every $t\in[0,T]$ let $S(t):\mathcal{U}_\rho\to\mathcal{X}_\rho$ be such that for any given $\Phi_0\in\mathcal{U}_\rho$ there holds*

$$\int_{(0,\infty)}\psi(x)S(t)\Phi_0(x)\mathrm{d}x=\int_{(0,\infty)}e^{-\frac{t}{\rho}}\psi\left(e^{-\frac{t}{\rho}}x\right)F(t,x)\mathrm{d}x\text{ for all }\psi\in\mathcal{B}, \quad (4.15)$$

where $F\in C([0,T]:\mathcal{X}_\rho)$ is the unique function that satisfies (4.10) for all $t\in[0,T]$ and $\varphi\in\mathcal{B}$ (cf. Lemma 4.14). Then $(S(t))_{t\in[0,T]}$ is a family of endomorphisms of $\mathcal{U}_\rho$ with the additional properties that (i) $S(0)=I_{\mathcal{U}_\rho}$, the identity on $\mathcal{U}_\rho$; (ii) $S(t_1+t_2)=S(t_1)S(t_2)$ for all $t_1,t_2\ge 0$ with $t_1+t_2\le T$; and (iii) for any $\Phi_0\in\mathcal{U}_\rho$ the mapping $t\mapsto S(t)\Phi_0$ is weakly-$*$ continuous on $[0,T]$.

Moreover, the family $(S(t))_{t\in[0,T]}$ extends to a weakly-$*$ continuous semigroup $(S(t))_{t\geq 0}$ on $\mathcal{U}_\rho$, and writing Φ_t for $S(t)\Phi_0$, then for all $t \geq 0$ and $\varphi \in C^1([0,\infty) : \mathcal{B})$ there holds

$$\begin{aligned}\int_{(0,\infty)} e^{\frac{t}{\rho}}\varphi\left(t, e^{\frac{t}{\rho}}x\right)\Phi_t(x)\mathrm{d}x - \int_{(0,\infty)}\varphi(0,x)\Phi_0(x)\mathrm{d}x \\ - \int_0^t \int_{(0,\infty)}\left(e^{\frac{s}{\rho}}\varphi_s\left(s, e^{\frac{s}{\rho}}x\right) + e^{\frac{s}{\rho}}\varphi\left(s, e^{\frac{s}{\rho}}x\right)\right)\Phi_s(x)\mathrm{d}x\ \mathrm{d}s \\ = \int_0^t \iint_{\mathbb{R}^2_+}\Phi_s(x)\Phi_s(y)\ \frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}}\int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}}\Delta_z^2\left[e^{\frac{s}{\rho}}\varphi\left(s, e^{\frac{s}{\rho}}\ \cdot\right)\right](x)\mathrm{d}z\ \mathrm{d}x\mathrm{d}y\ \mathrm{d}s,\end{aligned} \tag{4.16}$$

where $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ with $\phi(x) = (1-|x|)_+$.

Proof. Throughout the proof we let $\Phi_0 \in \mathcal{U}_\rho$ be fixed arbitrarily, and we let F be its associated solution to (4.10), i.e. $F \in C([0,T] : \mathcal{X}_\rho)$ is the unique function that satisfies (4.10) for all $t \in [0,T]$ and $\varphi \in \mathcal{B}$ (cf. Lemma 4.14). Given $\varphi \in C^1([0,T] : \mathcal{B})$, then for all $t \in [0,T]$ there holds

$$\begin{aligned}\int_{(0,\infty)}\varphi(t,x)F(t,x)\mathrm{d}x = \int_{(0,\infty)}\varphi(t,x)e^{-\int_0^t A(s)[F(s,\cdot)](x)\mathrm{d}s}\Phi_0(x)\mathrm{d}x \\ + \int_0^t\int_{(0,\infty)}\varphi(t,x)e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\ \mathrm{d}\sigma,\end{aligned} \tag{4.17}$$

and taking the derivative with respect to t yields

$$\begin{aligned}\partial_t\left[\int_{(0,\infty)}\varphi(t,x)F(t,x)\mathrm{d}x\right] - \int_{(0,\infty)}\left(\varphi_t(t,x)+\varphi(t,x)\right)F(t,x)\mathrm{d}x \\ = \int_{(0,\infty)}\varphi(t,x)B(t)[F(t,\cdot)](x)\mathrm{d}x - \int_{(0,\infty)}\varphi(t,x)\left(A(t)[F(t,\cdot)](x)+1\right)F(t,x)\mathrm{d}x,\end{aligned}$$

where the right hand side can be rewritten as

$$\iint_{\mathbb{R}^2_+}F(t,x)F(t,y)\ \frac{xy}{\left(x+e^{\frac{t}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}}\int_0^x \frac{\frac{1}{e^{\frac{t}{\rho}}\varepsilon}\left(1-\left|\frac{y-z}{e^{\frac{t}{\rho}}\varepsilon}\right|\right)_+}{\left(z+e^{\frac{t}{\rho}}\varepsilon_0\right)^{\frac{3}{2}}}\Delta_z^2[\varphi(t,\cdot)](x)\mathrm{d}z\ \mathrm{d}x\mathrm{d}y.$$

Using now the shorthand Φ_s for $S(s)\Phi_0$ [cf. (4.15)], it is straightforward that (4.16) holds for all $t \in [0,T]$ and $\varphi \in C^1([0,T] : \mathcal{B})$.

Next we set $\varphi_R(x) = (x \wedge R)$, for $R > 0$, and we use φ_R as a time-independent test function in (4.16). Then the right hand side is nonpositive, since φ_R is concave, hence

$$\begin{aligned}\int_{(0,\infty)} e^{\frac{t}{\rho}}\left(\left(e^{\frac{t}{\rho}}x\right)\wedge R\right)\Phi_t(x)\mathrm{d}x - \int_{(0,\infty)}(x\wedge R)\,\Phi_0(x)\mathrm{d}x \\ \leq \int_0^t\int_{(0,\infty)} e^{\frac{s}{\rho}}\left(\left(e^{\frac{s}{\rho}}x\right)\wedge R\right)\Phi_s(x)\mathrm{d}x\ \mathrm{d}s \text{ for all } t\in[0,T].\end{aligned}$$

By Gronwall, and a rearrangement of terms, it then follows for all $t \in [0,T]$ that

$$\left(e^{-\frac{t}{\rho}}R\right)^{\rho-2}\int_{(0,\infty)}\left(x\wedge\left(e^{-\frac{t}{\rho}}R\right)\right)\Phi_t(x)\mathrm{d}x \leq R^{\rho-2}\int_{(0,\infty)}(x\wedge R)\,\Phi_0(x)\mathrm{d}x \text{ for all } R>0,$$

and, taking the supremum over $R > 0$, we find that $\|S(t)\Phi_0\|_\rho \le \|\Phi_0\|_\rho \le 1$. We thus conclude, for arbitrary $t \in [0,T]$, that $S(t)$ is an endomorphism of $\mathcal{U}_\rho$, and properties (i) and (iii) are immediately satisfied by $(S(t))_{t\in[0,T]}$.

In order to check property (ii), let now $\tau \in [0,T]$ be fixed arbitrarily. For $t \in [\tau,T]$ and $\varphi \in \mathcal{B}$ there then holds [cf. (4.17)]

$$\int_{(0,\infty)} \varphi(x)F(t,x)\mathrm{d}x = \int_{(0,\infty)} \varphi(x)e^{-\int_\tau^t A(s)[F(s,\cdot)](x)\mathrm{d}s}F(\tau,x)\mathrm{d}x + \int_\tau^t \int_{(0,\infty)} \varphi(x)e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma,$$

which, using (4.15), we may rewrite as

$$\int_{(0,\infty)} \varphi(x)F(t,x)\mathrm{d}x = \int_{(0,\infty)} e^{\frac{\tau}{\rho}}\varphi\left(e^{\frac{\tau}{\rho}}x\right)e^{-\int_0^{t-\tau} A(\tau+s)[F(\tau+s,\cdot)](e^{\tau/\rho}x)\mathrm{d}s}S(\tau)\Phi_0(x)\mathrm{d}x + \int_0^{t-\tau}\int_{(0,\infty)} \varphi(x)e^{-\int_\sigma^{t-\tau} A(\tau+s)[F(\tau+s,\cdot)](x)\mathrm{d}s}B(\tau+\sigma)[F(\tau+\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma. \tag{4.18}$$

Next, we define $F_* \in C([0,T-\tau] : \mathcal{X}_\rho)$ to be such that for $t \in [\tau,T]$ there holds

$$\int_{(0,\infty)} \varphi(x)F_*(t-\tau,x)\mathrm{d}x = \int_{(0,\infty)} e^{-\frac{\tau}{\rho}}\varphi\left(e^{-\frac{\tau}{\rho}}x\right)F(t,x)\mathrm{d}x \text{ for all } \varphi\in\mathcal{B}, \tag{4.19}$$

and we compute, for $s \in [0,T-\tau]$ and $x > 0$, that

$$\begin{aligned}
&A(\tau+s)[F(\tau+s,\cdot)](e^{\tau/\rho}x)\\
&= \int_{(0,\infty)} \frac{2e^{\frac{\tau}{\rho}}xy}{\left(e^{\frac{\tau}{\rho}}x + e^{\frac{\tau+s}{\rho}}\varepsilon_0\right)^{\frac32}} \int_0^{e^{\frac{\tau}{\rho}}x} \frac{\frac{1}{e^{\frac{\tau+s}{\rho}}\varepsilon}\left(1-\left|\frac{y-z}{e^{\frac{\tau+s}{\rho}}\varepsilon}\right|\right)_+}{\left(z+e^{\frac{\tau+s}{\rho}}\varepsilon_0\right)^{\frac32}}\mathrm{d}z\, F(\tau+s,y)\mathrm{d}y - 1\\
&= \int_{(0,\infty)} e^{-\frac{\tau}{\rho}}\frac{2xe^{-\frac{\tau}{\rho}}y}{\left(x+e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac32}} \int_0^x \frac{\frac{1}{e^{\frac{s}{\rho}}\varepsilon}\left(1-\left|\frac{e^{-\frac{\tau}{\rho}}y-z}{e^{\frac{s}{\rho}}\varepsilon}\right|\right)_+}{\left(z+e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac32}}\mathrm{d}z\, F(\tau+s,y)\mathrm{d}y - 1\\
&= \int_{(0,\infty)} \frac{2xy}{\left(x+e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac32}} \int_0^x \frac{\frac{1}{e^{\frac{s}{\rho}}\varepsilon}\left(1-\left|\frac{y-z}{e^{\frac{s}{\rho}}\varepsilon}\right|\right)_+}{\left(z+e^{\frac{s}{\rho}}\varepsilon_0\right)^{\frac32}}\mathrm{d}z\, F_*(s,y)\mathrm{d}y - 1\\
&= A(s)[F_*(s,\cdot)](x),
\end{aligned}$$

and similarly, for $\sigma \in [0,T-\tau]$ and $\varphi \in \mathcal{B}$, that

$$\int_{(0,\infty)} e^{-\frac{\tau}{\rho}}\varphi\left(e^{-\frac{\tau}{\rho}}x\right)B(\tau+\sigma)[F(\tau+\sigma,\cdot)](x)\mathrm{d}x = \int_{(0,\infty)} \varphi(x)B(\sigma)[F_*(\sigma,\cdot)](x)\mathrm{d}x.$$

This now allows us to rewrite (4.18) as

$$\int_{(0,\infty)} e^{\frac{\tau}{\rho}}\varphi\left(e^{\frac{\tau}{\rho}}x\right)F_*(t-\tau,x)\mathrm{d}x = \int_{(0,\infty)} e^{\frac{\tau}{\rho}}\varphi\left(e^{\frac{\tau}{\rho}}x\right)e^{-\int_0^{t-\tau} A(s)[F_*(s,\cdot)](x)\mathrm{d}s}S(\tau)\Phi_0(x)\mathrm{d}x + \int_0^{t-\tau}\int_{(0,\infty)} e^{\frac{\tau}{\rho}}\varphi\left(e^{\frac{\tau}{\rho}}x\right)e^{-\int_\sigma^{t-\tau} A(s)[F_*(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F_*(\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma,$$

hence F_* is the unique function that for all $t \in [0, T-\tau]$ and $\varphi \in \mathcal{B}$ satisfies (4.10) with Φ_0 replaced by $S(\tau)\Phi_0$ (cf. Lemma 4.14). Using lastly (4.15) and (4.19) it then follows for all $t \in [\tau, T]$ that

$$\int_{(0,\infty)} \psi(x) S(t-\tau)S(\tau)\Phi_0(x)\mathrm{d}x = \int_{(0,\infty)} e^{-\frac{t-\tau}{\rho}} \psi\left(e^{-\frac{t-\tau}{\rho}} x\right) F_*(t-\tau, x)\mathrm{d}x$$
$$= \int_{(0,\infty)} e^{-\frac{t}{\rho}} \psi\left(e^{-\frac{t}{\rho}} x\right) F(t,x)\mathrm{d}x = \int_{(0,\infty)} \psi(x) S(t)\Phi_0(x)\mathrm{d}x \text{ for all } \psi \in \mathcal{B},$$

and we conclude that property (ii) holds.

Now, for $t > T$ we define the mappings $S(t) : \mathcal{U}_\rho \to \mathcal{U}_\rho$ recursively to be such that

$$S(t) = S(t-nT)S(nT) \text{ if } t \in (nT, (n+1)T] \text{ with } n \in \mathbb{N},$$

or equivalently such that

$$S(t) = S(t-mT)S(mT) \text{ for all } m \in \mathbb{N} \text{ with } m < \tfrac{t}{T}.$$

To show that $(S(t))_{t\geq 0}$ is then a weakly-$*$ continuous semigroup on $\mathcal{U}_\rho$, it suffices to check the semigroup property. Let thereto $n \in \mathbb{N}$ be such that there holds

$$S(t_1 + t_2) = S(t_1)S(t_2) \text{ for all } t_1, t_2 \geq 0 \text{ with } t_1 + t_2 \leq nT, \tag{4.20}$$

and let $t \in (nT, (n+1)T]$. For $\tau \in [nT, t]$ there then holds

$$S(t) = S(t-nT)S(nT) = S(t-\tau)S(\tau - nT)S(nT) = S(t-\tau)S(\tau),$$

while for $\tau \in [(m-1)T, mT)$, with $m \in \{1, \ldots, n\}$, we have

$$S(t) = S(t-mT)S(mT) = S(t-mT)S(mT-\tau)S(\tau),$$

where for $t-\tau \leq nT$ it is immediate that $S(t-mT)S(mT-\tau) = S(t-\tau)$. For $t-\tau > nT$, in which case $m = 1$, we need an additional step to observe that $S(t-T)S(T-\tau) = S(t-\tau-T)S(\tau)S(T-\tau) = S(t-\tau-T)S(T) = S(t-\tau)$, whereby we finally conclude that

$$S(t) = S(t-\tau)S(\tau) \text{ for all } 0 \leq \tau \leq t \leq (n+1)T.$$

The semigroup property then follows by induction, since (4.20) holds for $n = 1$.

Lastly, using the shorthand Φ_s for $S(s)\Phi_0$, it is again an easy computation to see that (4.16) holds for all $t \geq 0$ and $\varphi \in C^1([0,\infty) : \mathcal{B})$. □

We conclude the construction of the suitable semigroup with a continuity result.

Lemma 4.16. *Given $\rho \in (1,2)$ and $\varepsilon_0 > 2\varepsilon > 0$, let $(S(t))_{t\geq 0}$ be the semigroup on $\mathcal{U}_\rho$ that was obtained in Proposition 4.15. Then for every $t \geq 0$ the mapping $S(t) : \mathcal{U}_\rho \to \mathcal{U}_\rho$ is weakly-$*$ continuous.*

Proof. Let $t > 0$ be fixed, and let $\Phi_0^1, \Phi_0^2 \in \mathcal{U}_\rho$ be arbitrary. The goal is now to show that for any pair $(\psi, \epsilon) \in \mathcal{B} \times \mathbb{R}_+$ there exists an open set $\mathcal{O} = \mathcal{O}(\psi, \epsilon)$ in the weak-$*$ topology of $\mathcal{B}'$ such that if $\Phi_0^1 - \Phi_0^2 \in \mathcal{O}$, then $|\langle S(t)\Phi_0^1 - S(t)\Phi_0^2, \psi\rangle| < \epsilon$. For $s \in [0,t]$ and $i \in \{1,2\}$ we thereto let $F_s^i \in \mathcal{X}_\rho$ be such that

$$\int_{(0,\infty)} \varphi(x) F_s^i(x)\mathrm{d}x = \int_{(0,\infty)} e^{\frac{s}{\rho}} \varphi\left(e^{\frac{s}{\rho}} x\right) S(s)\Phi_0^i(x)\mathrm{d}x \text{ for all } \varphi \in \mathcal{B}. \tag{4.21}$$

For $i \in \{1,2\}$ and $\varphi \in C^1([0,t]:\mathcal{B})$ there then holds [cf. (4.16) and (4.21)]

$$\begin{aligned}\big\langle F_t^i, \varphi(t,\cdot)\big\rangle - \big\langle \Phi_0^i, \varphi(0,\cdot)\big\rangle - \int_0^t \big\langle F_s^i, \varphi_s(s,\cdot)+\varphi(s,\cdot)\big\rangle \,\mathrm{d}s \\ = \int_0^t \iint_{\mathbb{R}_+^2} F_s^i(x)F_s^i(y)\, \frac{xy}{(x+\varepsilon_0(s))^{\frac{3}{2}}} \int_0^x \frac{\phi_{\varepsilon(s)}(y-z)}{(z+\varepsilon_0(s))^{\frac{3}{2}}} \Delta_z^2[\varphi(s,\cdot)](x)\mathrm{d}z\, \mathrm{d}x\mathrm{d}y\, \mathrm{d}s,\end{aligned}$$

with $\varepsilon_0(s) = e^{\frac{s}{\rho}}\varepsilon_0$ and $\varepsilon(s) = e^{\frac{s}{\rho}}\varepsilon$, and taking the difference we obtain

$$\begin{aligned}&\big\langle F_t^1 - F_t^2, \varphi(t,\cdot)\big\rangle - \big\langle \Phi_0^1 - \Phi_0^2, \varphi(0,\cdot)\big\rangle \\ &\quad = \int_0^t \big\langle F_s^1 - F_s^2, \varphi_s(s,\cdot) + \varphi(s,\cdot) + \mathcal{L}_1(s)[\varphi(s,\cdot)] + \mathcal{L}_2(s)[\varphi(s,\cdot)]\big\rangle \,\mathrm{d}s, \quad (4.22)\end{aligned}$$

where the mappings $s \mapsto \mathcal{L}_1(s)$ and $s \mapsto \mathcal{L}_2(s)$, given by

$$\mathcal{L}_1(s)[\varphi](x) = \frac{1}{2}\sum_{i=1}^2 \int_{(0,\infty)} \frac{xy}{(x+\varepsilon_0(s))^{\frac{3}{2}}} \int_0^x \frac{\phi_{\varepsilon(s)}(y-z)}{(z+\varepsilon_0(s))^{\frac{3}{2}}} \Delta_z^2\varphi(x)\mathrm{d}z\, F_s^i(y)\mathrm{d}y,$$

and

$$\mathcal{L}_2(s)[\varphi](x) = \frac{1}{2}\sum_{i=1}^2 \int_{(0,\infty)} \frac{yx}{(y+\varepsilon_0(s))^{\frac{3}{2}}} \int_0^y \frac{\phi_{\varepsilon(s)}(x-z)}{(z+\varepsilon_0(s))^{\frac{3}{2}}} \Delta_z^2\varphi(y)\mathrm{d}z\, F_s^i(y)\mathrm{d}y,$$

are continuous from $[0,t]$ into the space of bounded linear operators on $\mathcal{B}$. Indeed, noting that for $x \ge z \ge 0$ and $\varphi \in \mathcal{B}$ there holds

$$|\Delta_z^2\varphi(x)| \le \left(\tfrac{x+z}{1+x+z} + \tfrac{x-z}{1+x-z} + 2\tfrac{x}{1+x}\right)\|\varphi\|_{\mathcal{B}} \le \tfrac{x}{1+x} \times 4\|\varphi\|_{\mathcal{B}},$$

and recalling the estimate (4.11), we check for all $s \in [0,t]$ and $\varphi \in \mathcal{B}$ that

$$\begin{aligned}\|\mathcal{L}_1(s)[\varphi]\|_{\mathcal{B}} &\le \frac{1}{2}\sum_{i=1}^2 \frac{1}{2}\Big(\tfrac{2}{\varepsilon_0(s)}\Big)^2 \int_{(0,\infty)} \Big(y \wedge \tfrac{\varepsilon_0(s)}{2}\Big)F_s^i(y)\mathrm{d}y \times 4\|\varphi\|_{\mathcal{B}} \\ &= \sum_{i=1}^2 \big(\tfrac{\varepsilon_0}{2}\big)^{\rho-2} \int_{(0,\infty)} \big(y \wedge \tfrac{\varepsilon_0}{2}\big)\, S(s)\Phi_0^i(y)\mathrm{d}y \times \tfrac{2^\rho}{\varepsilon_0^\rho}\|\varphi\|_{\mathcal{B}} \le 4\tfrac{2^\rho}{\varepsilon_0^\rho}\|\varphi\|_{\mathcal{B}},\end{aligned}$$

and similarly we obtain

$$\begin{aligned}&\|\mathcal{L}_2(s)[\varphi]\|_{\mathcal{B}} \\ &\le \frac{1}{2}\sum_{i=1}^2 \int_{(0,\infty)} \frac{1}{\varepsilon_0(s)}\frac{y}{y+\varepsilon_0(s)}\, F_s^i(y)\mathrm{d}y \times \sup_{x,y>0}\,(1+x)\int_0^y \frac{\phi_{\varepsilon(s)}(x-z)}{z+\varepsilon_0(s)}\mathrm{d}z \times 4\|\varphi\|_{\mathcal{B}} \\ &\le \sum_{i=1}^2 \varepsilon_0^{\rho-2} \int_{(0,\infty)} (y\wedge\varepsilon_0)\, S(s)\Phi_0^i(y)\mathrm{d}y \times \sup_{x>0} \tfrac{1+x}{x+\frac{1}{2}\varepsilon_0} \times \tfrac{2}{\varepsilon_0^\rho}\|\varphi\|_{\mathcal{B}} \le \tfrac{8}{\varepsilon_0^\rho}\Big(1+\tfrac{2}{\varepsilon_0}\Big)\|\varphi\|_{\mathcal{B}}.\end{aligned}$$

As a consequence (cf. [10]) there exists a unique solution $\varphi \in C^1([0,t]:\mathcal{B})$ to the problem

$$\begin{cases} \varphi_s(s,x) = -\varphi(s,x) - \mathcal{L}_1(s)[\varphi(s,\cdot)](x) - \mathcal{L}_2(s)[\varphi(s,\cdot)](x), \\ \varphi(t,x) = e^{-\frac{t}{\rho}}\psi\left(e^{-\frac{t}{\rho}}x\right), \end{cases}$$

and using this solution in (4.22), it follows with (4.21) that

$$\left\langle \Phi_0^1 - \Phi_0^2, \varphi(0,\cdot)\right\rangle = \left\langle F_t^1 - F_t^2, \varphi(t,\cdot)\right\rangle = \left\langle S(t)\Phi_0^1 - S(t)\Phi_0^2, \psi\right\rangle .$$

We thus conclude that $\mathcal{O} = \{\beta \in \mathcal{B}' : |\langle \beta, \varphi(0,\cdot)\rangle| < \epsilon\}$ satisfies the set requirements. □

An invariant set

The second element in the statement of Lemma 4.11 is a suitable subset that is invariant under the evolution of the semigroup. In view of Remark 4.5, we introduce

Definition 4.17. Given $\rho \in (1,2)$ and $R_0 > 0$, let $\mathcal{Y}_\rho(R_0)$ denote the set of elements $\beta \in \mathcal{U}_\rho$ for which there holds

$$\int_{(0,\infty)} (x \wedge R)\beta(x)\mathrm{d}x \geq R^{2-\rho}\ell_\rho\big(\tfrac{R}{R_0}\big) \text{ for all } R > 0,$$

with $\ell_\rho(x) = (1 - |x|^{-(2-\rho)/2})_+$.

Lemma 4.18. *Given $\rho \in (1,2)$ and $R_0 > 0$, then $\mathcal{Y}_\rho(R_0)$ is nonempty, convex, and compact with respect to the weak-$*$ topology on $\mathcal{B}'$.*

Proof. Trivial with the observation that $(2-\rho)(\rho-1)x^{-\rho}\mathrm{d}x \in \mathcal{Y}_\rho(R_0)$. □

In the following we will show that there exist constants $R_\rho = R_0(\rho) > 0$, independent of the regularizing parameters $\varepsilon_0 > 2\varepsilon > 0$, such that the sets $\mathcal{Y}_\rho(R_\rho)$ are invariant under all semigroups as obtained in Proposition 4.15.

Proposition 4.19. *Given $\rho \in (1,2)$, there exists a constant $R_\rho > 0$ such that $\mathcal{Y}_\rho(R_\rho)$ is positively invariant under any semigroup $(S(t))_{t\geq 0}$ on $\mathcal{U}_\rho$ as obtained in Proposition 4.15, i.e. such that if the semigroup $(S(t))_{t\geq 0}$ on $\mathcal{U}_\rho$ is as obtained in Proposition 4.15, with $\varepsilon_0 > 2\varepsilon > 0$ an arbitrarily fixed pair, then for all $t \geq 0$ there holds $S(t)\mathcal{Y}_\rho(R_\rho) \subset \mathcal{Y}_\rho(R_\rho)$.*

Similar to the proof of existence of an invariant set in [32], the proof of Proposition 4.19 relies on a comparison argument (cf. Lemma 4.21). Here the argument involves a solution to the fractional heat equation, for which reason we state the following result. Its proof is included in the appendix for the sake of completeness.

Lemma 4.20. *Given $\rho \in (1,2)$ and an odd function $\psi \in C(\mathbb{R}) \cap L^1(\mathbb{R}; |x|^{-\rho-1}\mathrm{d}x)$, then there exists a unique function $u \in C^1(\mathbb{R}_+ : C^\infty(\mathbb{R})) \cap C([0,\infty) : C_{\mathrm{odd}}(\mathbb{R}) \cap L^1(\mathbb{R}; |x|^{-\rho-1}\mathrm{d}x))$ that satisfies*

$$u_\tau(\tau,\xi) = \int_{\mathbb{R}_+} \zeta^{-\rho-1}\Delta_\zeta^2[u(\tau,\cdot)](\xi)\mathrm{d}\zeta \text{ for all } \tau > 0 \text{ and } \xi \in \mathbb{R}, \tag{4.23}$$

and $u(0,\cdot) \equiv \psi$ on $\mathbb{R}$. For $\tau > 0$ and $\xi \in \mathbb{R}$, this unique solution u is given by

$$u(\tau,\xi) = \int_{\mathbb{R}} \psi(\zeta)\, v\Big(\tfrac{\zeta-\xi}{\tau^{1/\rho}}\Big)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}}, \tag{4.24}$$

where v is the unique probability density function that has characteristic function $\exp(-c_\rho|k|^\rho)$ with $c_\rho = 2\int_{\mathbb{R}} |y|^{-\rho-1}\sin^2(\frac{y}{2})\mathrm{d}y$. In particular, this v is smooth, even, and nonincreasing on $\mathbb{R}_+$, and there holds $\lim_{z\to\infty} z^{\rho+1}v(z) = 1$. Moreover, the following two statements hold true.

- Maximum principle. *If $\psi \geq 0$ on $\mathbb{R}_+$, then $u(\tau,\cdot) \geq 0$ on $\mathbb{R}_+$ for every $\tau > 0$.*
- *If ψ is concave on $\mathbb{R}_+$, then $u(\tau,\cdot)$ is concave on $\mathbb{R}_+$ for every $\tau > 0$, and in particular*

$$\Delta_y^2[u(\tau,\cdot)](x) \leq 0 \text{ for all } x \geq 0,\, y \in \mathbb{R}, \text{ and } \tau \geq 0. \tag{4.25}$$

The key element in the proof of Proposition 4.19 is now the following comparison result.

Lemma 4.21. *Given $\rho \in (1,2)$ and $\varepsilon_0 > 2\varepsilon > 0$, let $(S(t))_{t\geq 0}$ be the semigroup on $\mathcal{U}_\rho$ that was obtained in Proposition 4.15, and let further $\psi \in C(\mathbb{R})$ be an odd function for which $\psi|_{[0,\infty]} \in \mathcal{B}$ is concave. For all $\Phi_0 \in \mathcal{U}_\rho$ and $t \geq 0$ there then holds*

$$\int_{(0,\infty)} \psi(x) S(t)\Phi_0(x)\mathrm{d}x \geq e^{(\rho-1)\frac{t}{\rho}} \int_{(0,\infty)} u\left(\rho t, e^{-\frac{t}{\rho}} x\right) \Phi_0(x)\mathrm{d}x, \tag{4.26}$$

where u is the unique solution to (4.23) *with $u(0,\cdot) \equiv \psi$ (cf. Lemma 4.20).*

Proof. Throughout the proof we let $\Phi_0 \in \mathcal{U}_\rho$ and $t > 0$ be fixed arbitrarily, for $s \in [0,t]$ we write Φ_s instead of $S(s)\Phi_0$, and we define

$$\varphi(s,x) = e^{t-s} e^{-\frac{t}{\rho}} u\left(\rho(t-s), e^{-\frac{t}{\rho}} x\right) \text{ for } s \in [0,t] \text{ and } x \in \mathbb{R},$$

where we note that $e^{\frac{t}{\rho}}\varphi(t, e^{\frac{t}{\rho}}x) = u(0,x) = \psi(x)$ and $\varphi(0,x) = e^{(\rho-1)\frac{t}{\rho}} u(\rho t, e^{-\frac{t}{\rho}}x)$. Thus, using this function in (4.16), we find that (4.26) is equivalent to

$$\int_0^t \Bigg[\int_{(0,\infty)} \left(e^{\frac{s}{\rho}}\varphi_s\left(s, e^{\frac{s}{\rho}}x\right) + e^{\frac{s}{\rho}}\varphi\left(s, e^{\frac{s}{\rho}}x\right)\right)\Phi_s(x)\mathrm{d}x$$
$$+ \iint_{\mathbb{R}_+^2} \Phi_s(x)\Phi_s(y) \frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}} \int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}} \Delta_z^2\Big[e^{\frac{s}{\rho}}\varphi\Big(s, e^{\frac{s}{\rho}}\ \cdot\ \Big)\Big](x)\mathrm{d}z\ \mathrm{d}x\mathrm{d}y\Bigg]\mathrm{d}s \geq 0,$$

which in particular holds if

$$\iint_{\mathbb{R}_+^2} \Phi_s(x)\Phi_s(y) \frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}} \int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}} \Delta_z^2\Big[e^{\frac{s}{\rho}}\varphi\Big(s, e^{\frac{s}{\rho}}\ \cdot\ \Big)\Big](x)\mathrm{d}z\ \mathrm{d}x\mathrm{d}y$$
$$\geq -\int_{(0,\infty)} \left(e^{\frac{s}{\rho}}\varphi_s\left(s, e^{\frac{s}{\rho}}x\right) + e^{\frac{s}{\rho}}\varphi\left(s, e^{\frac{s}{\rho}}x\right)\right)\Phi_s(x)\mathrm{d}x \text{ for almost all } s \in [0,t]. \tag{4.27}$$

Moreover, since u satisfies (4.23), we have for $s \in [0,t)$ and $x \in \mathbb{R}$ that

$$\varphi_s(s,x) + \varphi(s,x) = -\rho e^{t-s} e^{-\frac{t}{\rho}} \int_{\mathbb{R}_+} \zeta^{-\rho-1}\Delta_\zeta^2\left[u(\rho(t-s),\cdot)\right]\left(e^{-\frac{t}{\rho}}x\right)\mathrm{d}\zeta,$$

hence

$$e^{\frac{s}{\rho}}\varphi_s\left(s, e^{\frac{s}{\rho}}x\right) + e^{\frac{s}{\rho}}\varphi\left(s, e^{\frac{s}{\rho}}x\right) = -\rho e^{t-s} e^{-\frac{t-s}{\rho}} \int_{\mathbb{R}_+} \zeta^{-\rho-1}\Delta_\zeta^2\left[u(\rho(t-s),\cdot)\right]\left(e^{-\frac{t-s}{\rho}}x\right)\mathrm{d}\zeta$$
$$= -\rho e^{t-s} \int_{\mathbb{R}_+} z^{-\rho-1}\Delta_z^2\Big[e^{t-s}e^{-\frac{t-s}{\rho}} u\Big(\rho(t-s), e^{-\frac{t-s}{\rho}}\ \cdot\ \Big)\Big](x)\mathrm{d}z$$
$$= -\rho e^{t-s} \int_{\mathbb{R}_+} z^{-\rho-1}\Delta_z^2\Big[e^{\frac{s}{\rho}}\varphi\Big(s, e^{\frac{s}{\rho}}\ \cdot\ \Big)\Big](x)\mathrm{d}z,$$

whereby, using the shorthand $U(s,x) = e^{\frac{s}{\rho}}\varphi(s, e^{\frac{s}{\rho}}x)$, we find that (4.27) becomes

$$\int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} \Phi_s(y) \frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}} \int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}} \Delta_z^2[U(s,\cdot)](x)\mathrm{d}z\ \mathrm{d}y\right]\Phi_s(x)\mathrm{d}x$$
$$\geq \int_{\mathbb{R}_+} \left[\rho e^{t-s} \int_{\mathbb{R}_+} z^{-\rho-1}\Delta_z^2[U(s,\cdot)](x)\mathrm{d}z\right]\Phi_s(x)\mathrm{d}x \text{ for almost all } s \in [0,t),$$

and it thus suffices to check that

$$\int_{\mathbb{R}_+} \Phi_s(y)\ \frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}} \int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}} \Delta_z^2[U(s,\cdot)](x)\mathrm{d}z\ \mathrm{d}y$$
$$\geq \rho e^{t-s} \int_{\mathbb{R}_+} z^{-\rho-1}\Delta_z^2[U(s,\cdot)](x)\mathrm{d}z \text{ for a.a. } s\in[0,t) \text{ and } x\in\mathbb{R}_+. \quad (4.28)$$

Fixing now $s \in [0,t)$ and $x \in \mathbb{R}_+$, we note that $\Delta_z^2[U(s,\cdot)](x) \leq 0$ for $z \geq 0$ (with Lemma 4.20), so that

$$\frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}} \int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}} \Delta_z^2[U(s,\cdot)](x)\mathrm{d}z \geq y \int_{\mathbb{R}_+} \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^2} \Delta_z^2[U(s,\cdot)](x)\mathrm{d}z. \quad (4.29)$$

Integrating by parts, we further find that

$$\int_{\mathbb{R}_+} \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^2} \Delta_z^2[U(s,\cdot)](x)\mathrm{d}z = \int_{\mathbb{R}_+} \int_z^\infty \frac{\phi_\varepsilon(y-\zeta)}{(\zeta+\varepsilon_0)^2}\mathrm{d}\zeta \left(\int_{x-z}^{x+z} U_{\xi\xi}(s,\xi)\mathrm{d}\xi \right) \mathrm{d}z, \quad (4.30)$$

where the term between brackets on the right hand side is nonpositive (cf. Lemma A.4), so, using (4.29) and (4.30), we can bound the left hand side of (4.28) from below by

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_z^\infty \frac{\phi_\varepsilon(y-\zeta)}{(\zeta+\varepsilon_0)^2}\mathrm{d}\zeta\ y\Phi_s(y)\mathrm{d}y \left(\int_{x-z}^{x+z} U_{\xi\xi}(s,\xi)\mathrm{d}\xi \right) \mathrm{d}z. \quad (4.31)$$

Noting then for $y, z > 0$ that

$$\int_z^\infty \frac{\phi_\varepsilon(y-\zeta)}{(\zeta+\varepsilon_0)^2}\mathrm{d}\zeta \leq \frac{1}{z^2}\int_z^\infty \left(1 \wedge \tfrac{z}{\zeta+\varepsilon_0}\right)\phi_\varepsilon(y-\zeta)\mathrm{d}\zeta \leq \frac{1}{z^2}\left(1 \wedge \tfrac{z}{y+\varepsilon_0-\varepsilon}\right) \leq \frac{1}{z^2}\left(1 \wedge \tfrac{z}{y}\right),$$

then by the definition of the norm $\|\cdot\|_\rho$, and by the fact that $\Phi_s \in \mathcal{U}_\rho$, we find that

$$\int_{\mathbb{R}_+} \int_z^\infty \frac{\phi_\varepsilon(y-\zeta)}{(\zeta+\varepsilon_0)^2}\mathrm{d}\zeta\ y\Phi_s(y)\mathrm{d}y \leq z^{-\rho} \times z^{\rho-2} \int_{\mathbb{R}_+} (y \wedge z)\Phi_s(y)\mathrm{d}y \leq z^{-\rho},$$

whereby, recalling the nonpositivity of the term between brackets, we bound (4.31) from below by

$$\int_{\mathbb{R}_+} z^{-\rho} \left(\int_{x-z}^{x+z} U_{\xi\xi}(s,\xi)\mathrm{d}\xi \right) \mathrm{d}z = \rho \int_{\mathbb{R}_+} z^{-\rho-1}\Delta_z^2[U(s,\cdot)](x)\mathrm{d}z. \quad (4.32)$$

The claim then follows, as the right hand side of (4.32) is bigger than the right hand side of (4.28) (by $\Delta_z^2[U(s,\cdot)](x) \leq 0$ for $z \geq 0$). □

It may then be clear that we kick off the proof of Proposition 4.19 by using Lemma 4.21 with ψ the odd extensions of $\psi(x) = (x \wedge R)$ for $x \geq 0$ and $R > 0$. The following two technical lemmas, whose proofs have been postponed to the appendix, will serve to swiftly move from the initial estimate to the core of the argument.

Lemma 4.22. *Let $\rho \in (1,2)$ and $\Phi \in \mathcal{X}_\rho$ be arbitrary, and let $\Theta \in W^{2,\infty}(\mathbb{R})$ be an odd function that satisfies $\lim_{x\to\infty} \Theta'(x)x^{2-\rho} = 0$. Then there holds*

$$\int_{(0,\infty)} \Theta(x)\Phi(x)\mathrm{d}x = -\int_{(0,\infty)} \Theta''(x) \int_{(0,\infty)} (z \wedge x)\Phi(z)\mathrm{d}z\ \mathrm{d}x.$$

Lemma 4.23. *Let $\rho \in (1,2)$, let $v \in C^\infty(\mathbb{R})$ be the self-similar profile associated to the fundamental solution to* (4.23) *(cf. Lemma 4.20), let $\theta_1, \theta_2 > 0$ be arbitrary, and define*

$$\Theta(x) = \int_{\mathbb{R}} y \left(1 \wedge \left|\tfrac{\theta_1}{y}\right|\right) v\left(\tfrac{x-y}{\theta_2}\right) \tfrac{\mathrm{d}y}{\theta_2}.$$

Then Θ is odd and smooth, it satisfies $\lim_{x\to\infty} \Theta'(x) x^{2-\rho} = 0$, and there holds

$$-\Theta''(x) = \left(v\left(\tfrac{x-\theta_1}{\theta_2}\right) - v\left(\tfrac{x+\theta_1}{\theta_2}\right)\right) \tfrac{1}{\theta_2} \geq 0 \text{ for } x \geq 0. \tag{4.33}$$

Proof of Proposition 4.19. Throughout the proof, let $\varepsilon_0 > 2\varepsilon > 0$ be arbitrarily fixed, and let $(S(t))_{t\geq 0}$ be their associated semigroup on $\mathcal{U}_\rho$ as obtained in Proposition 4.15. We will show that there exists $R_0 = R_0(\rho) > 0$, independent of the regularizing parameters, such that if $\Phi_0 \in \mathcal{Y}_\rho = \mathcal{Y}_\rho(R_0)$, then $S(t)\Phi_0 \in \mathcal{Y}_\rho$ for all $t \geq 0$.

To that end, let $R_0 > 0$ be to be fixed, and let $R > 0$, $\Phi_0 \in \mathcal{U}_\rho$, and $t \geq 0$ be arbitrary. By Lemma 4.21 with $\psi(x) = x(1 \wedge |\frac{R}{x}|)$, and by a change of variables, we first find that

$$\begin{aligned}\int_{(0,\infty)} (x \wedge R) S(t)\Phi_0(x)\mathrm{d}x &\geq e^{(\rho-1)\frac{t}{\rho}} \int_{(0,\infty)} \int_{\mathbb{R}} \zeta\big(1 \wedge |\tfrac{R}{\zeta}|\big)\, v\Big(\tfrac{\zeta - e^{-t/\rho}x}{(\rho t)^{1/\rho}}\Big) \tfrac{\mathrm{d}\zeta}{(\rho t)^{1/\rho}}\, \Phi_0(x)\mathrm{d}x \\ &= e^{(\rho-2)\frac{t}{\rho}} \int_{(0,\infty)} \int_{\mathbb{R}} y \left(1 \wedge \left|\tfrac{e^{t/\rho}R}{y}\right|\right) v\Big(\tfrac{y-x}{(\rho t e^t)^{1/\rho}}\Big) \tfrac{\mathrm{d}y}{(\rho t e^t)^{1/\rho}}\, \Phi_0(x)\mathrm{d}x, \end{aligned} \tag{4.34}$$

where v is the self-similar profile associated to the fundamental solution to (4.23) (cf. Lemma 4.20). Using then Lemmas 4.22 and 4.23, we may rewrite the integral in the right hand side of (4.34) as

$$\begin{aligned}&\int_{(0,\infty)} \left(v\Big(\tfrac{x-e^{t/\rho}R}{(\rho t e^t)^{1/\rho}}\Big) - v\Big(\tfrac{x+e^{t/\rho}R}{(\rho t e^t)^{1/\rho}}\Big)\right) \int_{(0,\infty)} (z\wedge x)\Phi_0(z)\mathrm{d}z\, \tfrac{\mathrm{d}x}{(\rho t e^t)^{1/\rho}} \\ &\qquad = \int_{\mathbb{R}} v\Big(\tfrac{x-e^{t/\rho}R}{(\rho t e^t)^{1/\rho}}\Big) \frac{x}{|x|} \int_{(0,\infty)} (z \wedge |x|)\Phi_0(z)\mathrm{d}z\, \tfrac{\mathrm{d}x}{(\rho t e^t)^{1/\rho}},\end{aligned}$$

whereby we find, for $R > 0$, $\Phi_0 \in \mathcal{U}_\rho$, and $t \geq 0$, that

$$R^{\rho-2} \int_{(0,\infty)} (x\wedge R) S(t)\Phi_0(x)\mathrm{d}x \geq \Big(e^{t/\rho}R\Big)^{\rho-2} u\Big(\rho t e^t, e^{t/\rho}R\Big),$$

with u the solution to (4.23) with

$$u(0,\cdot) \equiv \mathrm{sgn}(\cdot)\int_{(0,\infty)} (z \wedge |\cdot|)\Phi_0(z)\mathrm{d}z.$$

However, if we suppose that $\Phi_0 \in \mathcal{Y}_\rho(R_0)$, then $u(0,x) \geq x|x|^{1-\rho}\ell_\rho(\frac{x}{R_0})$ for $x \geq 0$, so that by the maximum principle from Lemma 4.20 we have

$$u(\tau,\xi) \geq \int_{\mathbb{R}} \zeta|\zeta|^{1-\rho}\ell_\rho(\tfrac{\zeta}{R_0})\, v\Big(\tfrac{\zeta-\xi}{\tau^{1/\rho}}\Big)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}} = R_0^{2-\rho} \int_{\mathbb{R}} z|z|^{1-\rho}\ell_\rho(z)\, v\Big(\tfrac{z-\xi R_0^{-1}}{(\tau R_0^{-\rho})^{1/\rho}}\Big)\tfrac{\mathrm{d}z}{(\tau R_0^{-\rho})^{1/\rho}},$$

and it follows, for $R > 0$, $\Phi_0 \in \mathcal{U}_\rho$, and $t \geq 0$, that

$$R^{\rho-2} \int_{(0,\infty)} (x\wedge R) S(t)\Phi_0(x)\mathrm{d}x \geq \Big(\tfrac{e^{t/\rho}R}{R_0}\Big)^{\rho-2} u^*\Big(\tfrac{\rho t e^t}{R_0^\rho}, \tfrac{e^{t/\rho}R}{R_0}\Big), \tag{4.35}$$

where u^* is the solution to (4.23) with $u^*(0,x) = x|x|^{1-\rho}\ell_\rho(x)$ for $x \in \mathbb{R}$. As a consequence, as the right hand side of (4.35) only depends on the lower bound on elements in $\mathcal{Y}_\rho(R_0)$, it now remains to check that there is some $R_0 = R_0(\rho) > 0$ such that for all $r \geq 1$ there holds

$$\left(e^{t/\rho} r\right)^{\rho-2} u^*\left(\tfrac{\rho t e^t}{R_0^\rho}, e^{t/\rho} r\right) \geq \ell_\rho(r) \text{ as } t \to 0. \tag{4.36}$$

Let thereto u_* be the solution to (4.23) with $u_*(0,x) = x|x|^{1-\rho}(1-|x|^{-(2-\rho)/2})$ for $x \in \mathbb{R}$, for which, by again the maximum principle in Lemma 4.20, we find that $u^*(\tau,\xi) \geq u_*(\tau,\xi)$ for all $\tau, \xi \geq 0$. Working out this solution, we find for $\tau > 0$ and $\xi > 1$ that [cf. (4.24)]

$$\begin{aligned} \xi^{\rho-2} u_*(\tau,\xi) &= \xi^{\rho-2} \int_{\mathbb{R}} (\xi-\zeta)|\xi-\zeta|^{1-\rho}\left(1-|\xi-\zeta|^{-(2-\rho)/2}\right) v\left(\tfrac{\zeta}{\tau^{1/\rho}}\right)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}} \\ &= \left(1-\xi^{-(2-\rho)/2}\right) + \int_{\mathbb{R}} \left(\Lambda(\xi,\tfrac{\zeta}{\xi}) - \Lambda(\xi,0)\right) v\left(\tfrac{\zeta}{\tau^{1/\rho}}\right)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}}, \end{aligned} \tag{4.37}$$

with

$$\Lambda(x,y) = (1-y)|1-y|^{1-\rho}\left(1 - x^{-(2-\rho)/2}|1-y|^{-(2-\rho)/2}\right),$$

where we have used the fact that $\int_{\mathbb{R}} v(z)\mathrm{d}z = 1$. Moreover, as v is even, we note that

$$\int_{\mathbb{R}} \left(\Lambda(\xi,\tfrac{\zeta}{\xi}) - \Lambda(\xi,0)\right) v\left(\tfrac{\zeta}{\tau^{1/\rho}}\right)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}} = \int_{\mathbb{R}} \left(\Lambda(\xi,\tfrac{\zeta}{\xi}) - \Lambda(\xi,0) - \tfrac{\zeta}{\xi}\Lambda_y(\xi,0)\right) v\left(\tfrac{\zeta}{\tau^{1/\rho}}\right)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}},$$

and, since $\sup_{z\in\mathbb{R}} |z|^{\rho+1} v(z) =: \kappa_\rho < \infty$ by the fact that $v(z) \sim |z|^{-\rho-1}$ as $z \to \infty$, we now obtain the estimate

$$\begin{aligned} &\left|\int_{\mathbb{R}} \left(\Lambda(\xi,\tfrac{\zeta}{\xi}) - \Lambda(\xi,0)\right) v\left(\tfrac{\zeta}{\tau^{1/\rho}}\right)\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}}\right| \\ &\qquad \leq \kappa_\rho \int_{\mathbb{R}} \left|\Lambda(\xi,\tfrac{\zeta}{\xi}) - \Lambda(\xi,0) - \tfrac{\zeta}{\xi}\Lambda_y(\xi,0)\right| \left|\tfrac{\zeta}{\tau^{1/\rho}}\right|^{-\rho-1}\tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}}, \\ &\qquad \leq \frac{\tau\kappa_\rho}{\xi^\rho} \int_{\mathbb{R}} |\Lambda(\xi,y) - \Lambda(\xi,0) - y\Lambda_y(\xi,0)|\,|y|^{-\rho-1}\mathrm{d}y, \end{aligned}$$

where the integral on the right hand side can be bounded uniformly for all $\xi > 1$. Thereby there then exists another constant $\kappa = \kappa(\rho) > 0$ such that [cf. (4.37)]

$$\xi^{\rho-2}u^*(\tau,\xi) \geq \xi^{\rho-2}u_*(\tau,\xi) \geq \left(1-\xi^{-(2-\rho)/2}\right) - \frac{\tau\kappa}{\xi^\rho} \text{ for all } \tau > 0 \text{ and } \xi > 1,$$

which for $r \geq 1$ in particular implies that

$$\begin{aligned} \left(e^{t/\rho} r\right)^{\rho-2} u^*\left(\tfrac{\rho t e^t}{R_0^\rho}, e^{t/\rho} r\right) &\geq \left(1 - \left(e^{t/\rho} r\right)^{-(2-\rho)/2}\right) - \frac{\frac{\rho t e^t}{R_0^\rho}\kappa}{(e^{t/\rho} r)^\rho} \\ &= \left(1 - r^{-(2-\rho)/2}\right) + \left(1 - e^{-\frac{2-\rho}{2\rho}t}\right) r^{-(2-\rho)/2} - t\tfrac{\rho\kappa}{R_0^\rho} r^{-\rho} \\ &\geq \left(1 - r^{-(2-\rho)/2}\right) + t\left(\tfrac{1}{2}\tfrac{2-\rho}{2\rho} r^{-(2-\rho)/2} - \tfrac{\rho\kappa}{R_0^\rho} r^{-\rho}\right) \text{ as } t \to 0, \end{aligned}$$

and fixing $R_0 = R_0(\rho) = (\frac{1}{2}\frac{2-\rho}{2\rho})^{-\frac{1}{\rho}}(\rho\kappa)^{\frac{1}{\rho}} > 0$, we thus conclude that (4.36) holds. □

To summarize, we now have continuous semigroups, fixed-points of which are approximate solutions to $(\text{SSPE})^{\text{w}}_\rho$, and which all leave a nonempty, convex, and compact set invariant. The proof of Proposition 4.13 will now follow, using Lemma 4.11, and the compactness of the invariant set.

Proof of Proposition 4.13. Let $R_\rho = R_0(\rho) > 0$ be as obtained in Proposition 4.19, and set $\mathcal{Y}_\rho = \mathcal{Y}_\rho(R_\rho)$. Let further $\varepsilon_0 > 2\varepsilon > 0$ be arbitrary, and let $(S(t))_{t\geq 0}$ be their associated semigroup on $\mathcal{U}_\rho$ as obtained in Proposition 4.15. In view of Lemma 4.16 and Proposition 4.19, it then follows with Lemma 4.11 that there exists at least one fixed-point $\Phi^{\varepsilon_0,\varepsilon} \in \mathcal{Y}_\rho$ under $(S(t))_{t\geq 0}$. Moreover, the time-independent function $\Phi_t = \Phi^{\varepsilon_0,\varepsilon}$ satisfies (4.16) for all $t \geq 0$ and $\varphi \in C([0,\infty) : \mathcal{B})$, and using in particular

$$\varphi(t,x) = e^{-\frac{t}{\rho}}\left(\psi\left(e^{-\frac{t}{\rho}}x\right) - \psi(0)\right),$$

for any $\psi \in C_c^1([0,\infty))$, we find that $\Phi^{\varepsilon_0,\varepsilon}$ satisfies

$$\begin{aligned}\frac{1}{\rho}\int_{(0,\infty)} \big(x\psi'(x) - (\rho-1)(\psi(x)-\psi(0))\big)\Phi^{\varepsilon_0,\varepsilon}(x)\mathrm{d}x \\ = \iint_{\mathbb{R}_+^2} \Phi^{\varepsilon_0,\varepsilon}(x)\Phi^{\varepsilon_0,\varepsilon}(y)\,\frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}}\int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}}\Delta_z^2\psi(x)\mathrm{d}z\,\mathrm{d}x\mathrm{d}y. \quad (4.38)\end{aligned}$$

Now, for $\varepsilon_0 > 0$ fixed, consider the family $\{\Phi^{\varepsilon_0,\varepsilon}\}_{\varepsilon_0>2\varepsilon>0} \subset \mathcal{Y}_\rho$ of measures that satisfy (4.38) for all $\psi \in C_c^1([0,\infty))$. By compactness of $\mathcal{Y}_\rho$ there then exist a decreasing sequence $\varepsilon \to 0$, and a measure $\Phi^{\varepsilon_0} \in \mathcal{Y}_\rho$, such that $\Phi^{\varepsilon_0,\varepsilon} \stackrel{*}{\rightharpoonup} \Phi^{\varepsilon_0}$ in $\mathcal{B}'$, and writing

$$\begin{aligned}\frac{1}{2}\iint_{\mathbb{R}_+^2} \Phi^{\varepsilon_0,\varepsilon}(x)\Phi^{\varepsilon_0,\varepsilon}(y)\Bigg[\frac{xy}{(x+\varepsilon_0)^{\frac{3}{2}}}\int_0^x \frac{\phi_\varepsilon(y-z)}{(z+\varepsilon_0)^{\frac{3}{2}}}\Delta_z^2\psi(x)\mathrm{d}z \\ + \frac{yx}{(y+\varepsilon_0)^{\frac{3}{2}}}\int_0^y \frac{\phi_\varepsilon(x-z)}{(z+\varepsilon_0)^{\frac{3}{2}}}\Delta_z^2\psi(y)\mathrm{d}z\Bigg]\mathrm{d}x\mathrm{d}y,\end{aligned}$$

for the right hand side of (4.38), it follows as in the proof of Lemma 2.2, i.e. with Lemma A.5, that Φ^{ε_0} for all $\psi \in C_c^1([0,\infty))$ satisfies

$$\begin{aligned}\frac{1}{\rho}\int_{(0,\infty)} \big(x\psi'(x) - (\rho-1)(\psi(x)-\psi(0))\big)\Phi^{\varepsilon_0}(x)\mathrm{d}x \\ = \frac{1}{2}\iint_{\mathbb{R}_+^2} \frac{x\Phi^{\varepsilon_0}(x)}{(x+\varepsilon_0)^{\frac{3}{2}}}\frac{y\Phi^{\varepsilon_0}(y)}{(y+\varepsilon_0)^{\frac{3}{2}}}\Delta_{x\wedge y}^2\psi(x\vee y)\mathrm{d}x\mathrm{d}y. \quad (4.39)\end{aligned}$$

We further check that there exists a constant $K = K(\rho) > 0$ such that

$$\int_{(0,z]} \left(\tfrac{x}{x+\varepsilon_0}\right)^{\frac{3}{2}}\Phi^{\varepsilon_0}(x)\mathrm{d}x \leq K\, z^{1-\frac{\rho}{2}} \text{ for all } z > 0. \quad (4.40)$$

We thereto let $z > 0$ be arbitrary, and we let $\psi \in C_c^1([0,\infty))$ be convex and nonincreasing such that (i) $\psi(x) = (z-x)$ if $z > 2x > 0$; and (ii) $\psi(x) = 0$ if $2x > 3z$. For $x > 0$ this gives

$$\tfrac{1}{\rho}\big(x\psi'(x) - (\rho-1)(\psi(x)-\psi(0))\big) \leq \tfrac{\rho-1}{\rho}(z-\psi(x)) \leq \tfrac{\rho-1}{\rho}(x\wedge z),$$

whereby the left hand side of (4.39) is smaller than $\frac{\rho-1}{\rho}\, z^{2-\rho}$. Moreover, we have

$$\Delta_{x\wedge y}^2\psi(x\vee y) = (z-|x-y|) \geq \tfrac{z}{2} \geq \tfrac{1}{3}\sqrt{xy} \text{ for } x,y \in (\tfrac{3}{2}z, 2z],$$

so as the integrand in the right hand side of (4.39) is nonnegative on $\mathbb{R}_+^2$, there thus holds

$$\frac{1}{6}\iint_{(\frac{3}{2}z,2z]^2} \left(\tfrac{x}{x+\varepsilon_0}\right)^{\frac{3}{2}}\Phi^{\varepsilon_0}(x)\left(\tfrac{y}{y+\varepsilon_0}\right)^{\frac{3}{2}}\Phi^{\varepsilon_0}(y)\mathrm{d}x\mathrm{d}y \leq \tfrac{\rho-1}{\rho}\, z^{2-\rho}.$$

It thereby follows that

$$\int_{(\frac{3}{4}z,z]} \left(\tfrac{x}{x+\varepsilon_0}\right)^{\frac{3}{2}} \Phi^{\varepsilon_0}(x)\mathrm{d}x \le \sqrt{6\,\tfrac{\rho-1}{\rho}\,2^{\rho-2}}\,z^{1-\frac{\rho}{2}} \text{ for all } z>0,$$

and, using the decomposition $(0,z] = \bigcup_{j=0}^{\infty}(\frac{3}{4}(\frac{3}{4})^j z, (\frac{3}{4})^j z]$, we conclude that (4.40) holds for some $K>0$ that only depends on ρ.

Then, again by compactness of $\mathcal{Y}_\rho$, there are a decreasing sequence $\varepsilon_0 \to 0$, and a measure $\Phi \in \mathcal{Y}_\rho$, such that $\Phi^{\varepsilon_0} \xrightarrow{*} \Phi$ in $\mathcal{B}'$. We first show that Φ is a finite measure, i.e. that $\Phi \in \mathcal{S}_\rho \cap \mathcal{X}_1$, for which it suffices to check that (4.40) carries over to the limit: Setting $\eta_\epsilon(x) = \eta(\frac{x}{\epsilon})$, for $\epsilon>0$, and with $\eta(x) = 1 \wedge (x-1)_+$, we find that for any $z>0$ that

$$\begin{aligned}\int_{(0,z]} \Phi(x)\mathrm{d}x &= \lim_{\epsilon\to 0}\int_{(0,\infty)} (\eta_\epsilon(x) - \eta_\epsilon(x-z))\Phi(x)\mathrm{d}x \\ &= \lim_{\epsilon\to 0}\lim_{\varepsilon_0\to 0}\int_{(0,\infty)} (\eta_\epsilon(x) - \eta_\epsilon(x-z))\left(\tfrac{x}{x+\varepsilon_0}\right)^{\frac{3}{2}}\Phi^{\varepsilon_0}(x)\mathrm{d}x \le \lim_{\epsilon\to 0} K(z+2\epsilon)^{1-\frac{\rho}{2}}.\end{aligned}$$

It then only remains to check that Φ satisfies $(\mathrm{SSPE})^{\mathrm{w}}_\rho$ for all $\psi \in C_c^1([0,\infty))$. Fixing thus $\psi \in C_c^1([0,\infty))$ arbitrarily, we immediately note that the left hand side of (4.39) converges trivially to the one of $(\mathrm{SSPE})^{\mathrm{w}}_\rho$. For the right hand side we take η_ϵ as above, and we remark that

$$\left|\iint_{\mathbb{R}_+^2} \frac{x\Phi^{\varepsilon_0}(x)}{(x+\varepsilon_0)^{\frac{3}{2}}}\frac{y\Phi^{\varepsilon_0}(y)}{(y+\varepsilon_0)^{\frac{3}{2}}}\Delta^2_{x\wedge y}\psi(x\vee y)(1-\eta_\epsilon(x)\eta_\epsilon(y))\mathrm{d}x\mathrm{d}y\right| \le O(\epsilon^{2-\rho}) \text{ as } \epsilon \to 0,$$

independently of $\varepsilon_0 > 0$, while for any $\epsilon > 0$, we find by dominated convergence that

$$\begin{aligned}\lim_{\varepsilon_0\to 0}\iint_{\mathbb{R}_+^2} &\frac{x\Phi^{\varepsilon_0}(x)}{(x+\varepsilon_0)^{\frac{3}{2}}}\frac{y\Phi^{\varepsilon_0}(y)}{(y+\varepsilon_0)^{\frac{3}{2}}}\Delta^2_{x\wedge y}\psi(x\vee y)\eta_\epsilon(x)\eta_\epsilon(y)\mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathbb{R}_+^2}\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\Delta^2_{x\wedge y}\psi(x\vee y)\eta_\epsilon(x)\eta_\epsilon(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The result then follows by again dominated convergence in the limit $\epsilon \to 0$. □

4.1.2 A candidate self-similar profile in the case $\rho = 2$

We mentioned before that the space $\mathcal{X}_2$ contains elements that are not measures (cf. Remark 4.9). Moreover, the invariant sets that were constructed in the previous subsection require the constant c_ρ in Lemma 4.20 to be finite, yet $\lim_{\rho\to 2^-} c_\rho = \infty$. These are some of the reasons why we have not been able to extend the construction of candidate profiles for $\rho \in (1,2)$ to $\rho \in (1,2]$. The fact that self-similar profiles exhibit different qualitative behaviour in the cases $\rho = 2$ and $\rho \ne 2$ (cf. Section 4.2), is perhaps the most convincing indication that a different approach must be taken. Anyhow, in this subsection we will prove the following

Proposition 4.24. *There exists at least one $\Phi \in \mathcal{S}_2 \cap \mathcal{X}_1$ that for all $\psi \in C_c^1([0,\infty))$ satisfies*

$$\int_{(0,\infty)} \big(x\psi'(x) - \psi(x) + \psi(0)\big)\Phi(x)\mathrm{d}x = \iint_{\mathbb{R}_+^2}\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y. \qquad (\mathrm{SSPE})^{\mathrm{w}}_2$$

Our approach to prove Proposition 4.24 differs slightly from the one in [KV15]. There, as it is the energy that exhibits self-similar behaviour in the classical sense, it was thought useful to consider the equation for $\Psi(x) = x\Phi(x)$, to then perform all arguments in the shell of probability measures, and to finally move back to the equation for Φ (cf. Remark 4.5). Here we still move to the space of probability measures to do a fixed-point argument for a regularized equation for the energy profile, but we then immediately return to the self-similar mass distribution, using compactness properties in $\mathcal{B}'$ to complete the proof.

Remark 4.25. Given $\epsilon > 0$, then $\mathcal{M}([\epsilon,\infty])$ is a subspace of $\mathcal{B}'$ via the natural pairing

$$\langle \mu, \psi\rangle = \int_{[\epsilon,\infty]} \psi(x)\mu(x)\mathrm{d}x \text{ for } \mu \in \mathcal{M}([\epsilon,\infty]) \text{ and } \psi \in \mathcal{B}.$$

Moreover, the weak-$*$ topology on $\mathcal{M}([\epsilon,\infty])$ coincides with the topology that is induced by the weak-$*$ topology on $\mathcal{B}'$.

The main auxiliary result is now the following

Proposition 4.26. *Given $0 < 2\epsilon < 1$, there exists at least one $\Phi^\epsilon \in \mathcal{M}_+([\epsilon,\infty])$ with energy $\int_{[\epsilon,\infty]} x\Phi^\epsilon(x)\mathrm{d}x = 1$ and $\int_{[\epsilon,\infty]}(x-1)_+^2\Phi^\epsilon(x)\mathrm{d}x \le 100$, that for all $\psi \in C_c^1([0,\infty))$ satisfies*

$$\begin{aligned}\int_{[\epsilon,\infty]} \eta_\epsilon(x)\big(x\psi'(x) - \psi(x) + \psi(0)\big)\Phi^\epsilon(x)\mathrm{d}x &= \iint_{[\epsilon,\infty]^2} \frac{\Phi^\epsilon(x)\Phi^\epsilon(y)}{\sqrt{xy}} \Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y \\ &+ 2\iint_{\{x>y\ge\epsilon\}} \frac{\Phi^\epsilon(x)\Phi^\epsilon(y)}{\sqrt{xy}}(1-\eta_\epsilon(x-y))\begin{bmatrix}\psi(2x) - \psi(x+y) \\ -\psi(x-y) + \psi(0)\end{bmatrix}\mathrm{d}x\mathrm{d}y, \quad (4.41)\end{aligned}$$

where η_ϵ is as in Definition 4.27 below.

Definition 4.27. Let $\eta \in C^\infty(\mathbb{R})$ be monotone with $\eta \equiv 0$ on $(-\infty, 1]$, $\eta \equiv 1$ on $[2,\infty)$, and $\eta' \le \frac{3}{2}$ on $\mathbb{R}$, and let $\eta_\epsilon(x) = \eta(\frac{x}{\epsilon})$ for $\epsilon > 0$.

The idea behind Proposition 4.26, and its proof, is that we "thicken" the origin. For weak solutions to (QWTE), we have seen before that mass at the origin is trapped, and, perhaps more importantly, that mass at the origin has no influence on the dynamics of the rest of the mass distribution (cf. Lemma 2.11). The particle interpretation of the collision kernel on the right hand side of (4.41) is as follows: If two particles of sizes $x, y \ge \epsilon$ interact, then the larger of the two is replaced, with probability $\frac{1}{2}\eta_\epsilon(|x-y|)$, by a particle of size $x+y$ or $|x-y|$, or, with probability $\frac{1}{2}(1-\eta_\epsilon(|x-y|))$, by either a zero-particle, or a particle twice its size. This process formally conserves the energy in $[\epsilon,\infty]$, but the number of particles in that region decreases due to the artificial removal of particles of sizes smaller than ϵ. The use of the cut-off function in the left hand side of (4.41) further avoids loss of energy from the interval $[\epsilon,\infty]$, whereby this equation should define the evolution of an energy-preserving semigroup. However, since we look for a measure Φ^ϵ with first moment equal to 1, it makes sense to consider the equation for the probability measure $x\Phi^\epsilon(x)$. We will therefore prove the alternative

Proposition 4.28. *Given $0 < 2\epsilon < 1$, there exists at least one probability measure $\Pi \in \mathcal{P}([\epsilon,\infty])$ with $\int_{[\epsilon,\infty]} \frac{1}{x}(x-1)_+^2\Pi(x)\mathrm{d}x \le 100$, that for all $\vartheta \in C^1([\epsilon,\infty])$ with $z\vartheta'(z) \in C([\epsilon,\infty])$ satisfies*

$$\int_{[\epsilon,\infty]} \tfrac{1}{2}\eta_\epsilon(x)x\vartheta'(x)\Pi(x)\mathrm{d}x = \frac{1}{2}\iint_{[\epsilon,\infty]^2} \frac{\Pi(x)\Pi(y)}{(xy)^{3/2}}\,\Xi^\epsilon[\vartheta](x,y)\mathrm{d}x\mathrm{d}y, \qquad (4.42)$$

where η_ϵ is as in Definition 4.27, and where $\Xi^\epsilon[\vartheta]$ is continuous and symmetric such that

$$\begin{aligned}\Xi^\epsilon[\vartheta](x,y) = \Delta_y^2\psi(x) + (1-\eta_\epsilon(x-y))\big(\psi(2x) - \psi(x+y) - \psi(x-y)\big) \\ \textit{for } x \ge y \ge \epsilon \textit{ and } x \ge \epsilon > y \ge 0, \textit{ with } \psi(z) = z\vartheta(z).\end{aligned}$$

Proposition 4.26 is then an immediate consequence:

Proof of Proposition 4.26. Corollary of Proposition 4.28, using $\vartheta(x) = \frac{1}{x}(\psi(x) - \psi(0))$. □

Now, in order to prove Proposition 4.28, we will once again use Lemma 4.11 to obtain approximate solutions $\Pi_\varepsilon \in \mathcal{P}([\epsilon, \infty])$ to (4.42) in the compact subset of probability measures with $\int_{[\epsilon,\infty]} \frac{1}{x}(x-1)_+ \Pi_\varepsilon(x)\mathrm{d}x \le 100$. It is then fairly standard to remove the regularization, and to prove the result.

In the **construction of the semigroups**, fixed-points of which are approximate solutions to (4.42), we will integrate out the transport term on the left hand side. To that end, we introduce the following

Definition 4.29. For $\epsilon > 0$, and with η_ϵ as in Definition 4.27, let $\xi^\epsilon \in C^\infty(\mathbb{R}^2)$ be the unique solution to

$$\begin{cases} \xi^\epsilon_t(t,x) = \frac{1}{2}\eta_\epsilon(\xi^\epsilon(t,x))\xi^\epsilon(t,x), \\ \xi^\epsilon(0,x) = x. \end{cases}$$

Moreover, for $\epsilon > 2\varepsilon > 0$, let $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ with $\phi(x) = (1-|x|)_+$.

We feel compelled to emphasize that the parameter ε has the same role here, as it had in the proof of Lemma 2.2, and in Section 4.1.1. That is, it acts as an auxiliary regularization to make sure that all terms in the fixed-point argument are continuous. It should not be confused with the true regularizing parameter ϵ.

Remark 4.30. For the reader's convenience, let us note that the functions $\xi^\epsilon \in C^\infty(\mathbb{R}^2)$ as introduced in Definition 4.29 satisfy

$$\begin{aligned} &\xi^\epsilon(t_1+t_2,x) = \xi^\epsilon(t_1,\xi^\epsilon(t_2,x)), \text{ and} \\ &\xi^\epsilon_x(t_1+t_2,x) = \xi^\epsilon_x(t_2,x)\xi^\epsilon_x(t_1,\xi^\epsilon(t_2,x)), \text{ for } t_1,t_2,x \in \mathbb{R}. \end{aligned}$$

Moreover, for $t, x \in \mathbb{R}$ there holds

$$\partial_t\big[\xi^\epsilon_x(t,x)\big] = \tfrac{1}{2}\big(\eta'_\epsilon(\xi^\epsilon(t,x))\xi^\epsilon(t,x) + \eta_\epsilon(\xi^\epsilon(t,x))\big)\xi^\epsilon_x(t,x),$$

where the sum between brackets is bounded by 4, and since further $\xi^\epsilon_x(0,\cdot) \equiv 1$ we conclude for $t \ge 0$ that $\xi^\epsilon_x(-t,\cdot) \le 1$ and $\xi^\epsilon_x(t,\cdot) \le e^{2t}$.

The following Lemma 4.31, Proposition 4.32, and Lemma 4.33, are now full analogues of Lemma 4.14, Proposition 4.15, and Lemma 4.16, respectively, and might be skipped.

Lemma 4.31. *Given $\epsilon > 2\varepsilon > 0$, there exists $T > 0$ such that for every $\Pi_0 \in \mathcal{P}([\epsilon,\infty])$ there is a unique function $F \in C([0,T] : \mathcal{M}_+([\epsilon,\infty]))$ that for all $t \in [0,T]$ and $\varphi \in C([\epsilon,\infty])$ satisfies*

$$\begin{aligned} \int_{[\epsilon,\infty]} \varphi(x)F(t,x)\mathrm{d}x = &\int_{[\epsilon,\infty]} \varphi(x)e^{-\int_0^t A(s)[F(s,\cdot)](x)\mathrm{d}s}\Pi_0(x)\mathrm{d}x \\ &+ \int_0^t \int_{[\epsilon,\infty]} \varphi(x)e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma, \end{aligned} \tag{4.43}$$

where for $s \ge 0$ the mapping $A(s) : \mathcal{M}_+([\epsilon,\infty]) \to C([\epsilon,\infty])$ is given by

$$\begin{aligned} A(s)[F](x) = 2\int_{[\epsilon,\infty]} \int_0^{\xi^\epsilon(-s,x)} &\frac{\phi_\varepsilon(\xi^\epsilon(-s,y)-z)}{\sqrt{\xi^\epsilon(-s,x)}z^{3/2}}\mathrm{d}z\; F(y)\xi^\epsilon_y(-s,y)\mathrm{d}y \\ &- \tfrac{1}{2}\big(\eta'_\epsilon(\xi^\epsilon(-s,x))\xi^\epsilon(-s,x) + \eta_\epsilon(\xi^\epsilon(-s,x))\big), \end{aligned}$$

and where $B(s) : \mathcal{M}_+([\epsilon,\infty]) \to \mathcal{M}_+([\epsilon,\infty])$ *is such that for any* $\varphi \in C([\epsilon,\infty])$ *there holds*

$$\int_{[\epsilon,\infty]} \varphi(x)B(s)[F](x)\mathrm{d}x = \iint_{[\epsilon,\infty]^2} F(x)F(y)\xi^\epsilon_x(-s,x)\xi^\epsilon_y(-s,y) \\ \times \int_0^{\xi^\epsilon(-s,x)} \frac{\phi_\varepsilon(\xi^\epsilon(-s,y)-z)}{(\xi^\epsilon(-s,x)z)^{3/2}}\,\mathfrak{X}^\epsilon(s)[\varphi](\xi^\epsilon(-s,x),z)\mathrm{d}z\,\mathrm{d}x\mathrm{d}y,$$

with

$$\mathfrak{X}^\epsilon(s)[\varphi](X,z) = (1-\eta_\epsilon(X-z))2X\varphi(\xi^\epsilon(s,2X))\xi^\epsilon_x(s,2X) \\ + \eta_\epsilon(X-z)\big[(X+z)\varphi(\xi^\epsilon(s,X+z))\xi^\epsilon_x(s,X+z) \\ + (X-z)\varphi(\xi^\epsilon(s,X-z))\xi^\epsilon_x(s,X-z)\big].$$

Proof. For $\Pi_0 \in \mathcal{P}([\epsilon,\infty])$ arbitrarily fixed, we will show that there exists $T = T(\epsilon) > 0$ such that $\mathcal{T} : C([0,T] : \mathcal{M}_+([\epsilon,\infty])) \to C([0,T] : \mathcal{M}_+([\epsilon,\infty]))$, with $\int_{[\epsilon,\infty]} \varphi(x)\mathcal{T}[F](t,x)\mathrm{d}x$ given by the right hand side of (4.43), is a contraction on

$$\mathcal{C} := \left\{\mu \in C([0,T] : \mathcal{M}_+([\epsilon,\infty])) : \sup_{t\in[0,T]}\|\mu(t,\cdot)\| \le 2\right\},$$

where $\|\mu\| = \int_{[\epsilon,\infty]} \mu(x)\mathrm{d}x$. The claim then follows by Banach's fixed-point theorem.

Clearly, nonnegativity is preserved by $\mathcal{T}$, and noting that

$$|\mathfrak{X}^\epsilon(s)[\varphi](X,z)| \le 2X\|\xi^\epsilon_x(s,\cdot)\|_{L^\infty(\epsilon,\infty)} \times \|\varphi\|_{C([\epsilon,\infty])},$$

we consecutively find, also with Remark 4.30, that

$$\int_0^X \frac{\phi_\varepsilon(Y-z)}{(Xz)^{3/2}}\,\mathfrak{X}^\epsilon(s)[\varphi](X,z)\mathrm{d}z \le \frac{2e^{2s}}{\sqrt{\epsilon}}\int_0^X \frac{\phi_\varepsilon(Y-z)}{z^{3/2}}\mathrm{d}z \times \|\varphi\|_{C([\epsilon,\infty])},$$

and

$$\left|\int_{[\epsilon,\infty]} \varphi(x)B(s)[F](x)\mathrm{d}x\right| \le \tfrac{2\sqrt{2}}{\epsilon^2}e^{2s}\|F\|^2 \times \|\varphi\|_{C([\epsilon,\infty])}, \tag{4.44}$$

whereby we conclude that $\mathcal{T}$ is well-defined. Moreover, as $e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s} \le e^{2(t-\sigma)}$, and using $\varphi \equiv 1$ in the right hand side or (4.43), we find by the previous estimates that

$$\|\mathcal{T}[F](t,\cdot)\| \le e^{2t}\|\Pi_0\| + \int_0^t e^{2(t-\sigma)}\|B(s)[F(\sigma,\cdot)]\|\mathrm{d}\sigma \\ \le e^{2t}\left(1 + \tfrac{2\sqrt{2}}{\epsilon^2}\,t \times \sup_{\sigma\in[0,t]}\|F(\sigma,\cdot)\|^2\right),$$

from which it follows that $\mathcal{T}$ maps $\mathcal{C}$ into itself for sufficiently small $T > 0$, depending only on ϵ.

To finally show that $\mathcal{T}$ is contractive on $\mathcal{C}$ for $T > 0$ small enough, we first note for $F_1, F_2 \in \mathcal{M}_+([\epsilon,\infty])$ that

$$\|A(s)[F_1] - A(s)[F_2]\|_{L^\infty(\epsilon,\infty)} \le \frac{2}{\sqrt{\epsilon}}\int_{[\epsilon,\infty]}\int_0^\infty \frac{\phi_\varepsilon(\xi^\epsilon(-s,y)-z)}{z^{3/2}}\mathrm{d}z\,\big|F_1-F_2\big|(y)\mathrm{d}y \\ \le \tfrac{4\sqrt{2}}{\epsilon^2}\|F_1-F_2\|,$$

so, using also that $|e^{x-x_1} - e^{x-x_2}| \le e^x |x_1 - x_2|$ for $x_1, x_2 \ge 0$, we find for $F_1, F_2 \in C([0,T] : \mathcal{M}_+([\epsilon,\infty]))$, and $T \ge t \ge \sigma \ge 0$, that

$$\sup_{x\ge\epsilon} \left| e^{-\int_\sigma^t A(s)[F_1(s,\cdot)](x)\mathrm{d}s} - e^{-\int_\sigma^t A(s)[F_2(s,\cdot)](x)\mathrm{d}s} \right| \\ \le e^{2(t-\sigma)} \int_\sigma^t \|A(s)[F_1(s,\cdot)] - A(s)[F_2(s,\cdot)]\|_{L^\infty(\epsilon,\infty)} \mathrm{d}s \\ \le \tfrac{4\sqrt{2}}{\epsilon^2}(t-\sigma)e^{2(t-\sigma)} \times \sup_{s\in[0,t]} \|F_1(s,\cdot) - F_2(s,\cdot)\|.$$

Thus, as in the proof of Lemma 4.14 [cf. (4.13)], we obtain

$$\|\mathcal{T}[F_1](t,\cdot) - \mathcal{T}[F_2](t,\cdot)\| \le \tfrac{4\sqrt{2}}{\epsilon^2}\, te^{2t} \times \sup_{s\in[0,t]} \|F_1(s,\cdot) - F_2(s,\cdot)\| \\ + \int_0^t \tfrac{4\sqrt{2}}{\epsilon^2}(t-\sigma)e^{2(t-\sigma)} \left[\frac{1}{2}\sum_{i=1}^2 \|B(s)[F_i(\sigma,\cdot)]\|\right] \mathrm{d}\sigma \times \sup_{s\in[0,t]} \|F_1(s,\cdot) - F_2(s,\cdot)\| \\ + \int_0^t e^{2(t-\sigma)} \|B(s)[F_1(\sigma,\cdot)] - B(s)[F_2(\sigma,\cdot)]\| \mathrm{d}\sigma,$$

where further, by similar arguments as were used to obtain (4.44), we have

$$\|B(s)[F_1(\sigma,\cdot)] - B(s)[F_2(\sigma,\cdot)]\| \le \tfrac{4\sqrt{2}}{\epsilon^2} e^{2\sigma}(\|F_1(\sigma,\cdot)\| + \|F_1(\sigma,\cdot)\|)\|F_1(\sigma,\cdot) - F_2(\sigma,\cdot)\|.$$

For $F_1, F_2 \in \mathcal{C}$ we thereby find that

$$\|\mathcal{T}[F_1](t,\cdot) - \mathcal{T}[F_2](t,\cdot)\| \le \tfrac{4\sqrt{2}}{\epsilon^2}\, te^{2t} \left(1 + \tfrac{4\sqrt{2}}{\epsilon^2}\, t + 4\right) \times \sup_{s\in[0,t]} \|F_1(s,\cdot) - F_2(s,\cdot)\|,$$

and we conclude the claim. □

Proposition 4.32. *Given $\epsilon > 2\varepsilon > 0$, let $T > 0$ be as obtained in Lemma 4.31, and for every $t \in [0,T]$ let $S(t) : \mathcal{P}([\epsilon,\infty]) \to \mathcal{M}_+([\epsilon,\infty])$ be such that for $\Pi_0 \in \mathcal{P}([\epsilon,\infty])$ there holds*

$$\int_{[\epsilon,\infty]} \vartheta(x)S(t)\Pi_0(x)\mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(-t,x)\vartheta(\xi^\epsilon(-t,x))F(t,x)\mathrm{d}x \textit{ for all } \vartheta \in C([\epsilon,\infty]), \tag{4.45}$$

with $F \in C([0,T] : \mathcal{M}_+([\epsilon,\infty])$ the unique function that satisfies (4.43) *for all $t \in [0,T]$ and $\varphi \in C([\epsilon,\infty])$ (cf. Lemma 4.31). Then $(S(t))_{t\in[0,T]}$ is a family of endomorphisms of $\mathcal{P}([\epsilon,\infty])$ with the additional properties that (i) $S(0) = I$, the identity; (ii) $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \ge 0$ with $t_1 + t_2 \le T$; and (iii) for any $\Pi_0 \in \mathcal{P}([\epsilon,\infty])$ the mapping $t \mapsto S(t)\Pi_0$ is weakly-$*$ continuous on $[0,T]$. Moreover, the family $(S(t))_{t\in[0,T]}$ extends to a weakly-$*$ continuous semigroup $(S(t))_{t\ge0}$ on $\mathcal{P}([\epsilon,\infty])$, and writing Π_t instead of $S(t)\Pi_0$, then for all $t \ge 0$ and any $\vartheta \in C^1([\epsilon,\infty])$ with $z\vartheta'(z) \in C([\epsilon,\infty])$ there holds*

$$\int_{[\epsilon,\infty]} \vartheta(x)\Pi_t(x)\mathrm{d}x - \int_{[\epsilon,\infty]} \vartheta(x)\Pi_0(x)\mathrm{d}x + \int_0^t \int_{[\epsilon,\infty]} \tfrac{1}{2}\eta_\epsilon(x)x\vartheta'(x)\Pi_s(x)\mathrm{d}x\, \mathrm{d}s \\ = \int_0^t \iint_{[\epsilon,\infty]^2} \Pi_s(x)\Pi_s(y) \int_0^x \frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}}\, \Xi^\epsilon[\vartheta](x,z)\mathrm{d}z\, \mathrm{d}x\mathrm{d}y\, \mathrm{d}s, \tag{4.46}$$

where η_ϵ and ϕ_ε are as in Definitions 4.27 and 4.29, and where $\Xi^\epsilon[\vartheta]$ is as in Proposition 4.28.

Proof. Let $\Pi_0 \in \mathcal{P}([\epsilon,\infty])$ be arbitrarily fixed, and let $F \in C([0,T] : \mathcal{M}_+([\epsilon,\infty]))$ be the unique function that satisfies (4.43) for all $t \in [0,T]$ and $\varphi \in C([\epsilon,\infty])$ (cf. Lemma 4.31).

For $\varphi \in C^1([0,T] : C([\epsilon,\infty]))$ and $t \in [0,T]$ we then have

$$\int_{[\epsilon,\infty]} \varphi(t,x)F(t,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \varphi(t,x)e^{-\int_0^t A(s)[F(s,\cdot)](x)\mathrm{d}s}\Pi_0(x)\mathrm{d}x + \int_0^t \int_{[\epsilon,\infty]} \varphi(t,x)e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma,$$

from which, after differentiating with respect to t, and integrating the resulting equation again, it follows that

$$\int_{[\epsilon,\infty]} \varphi(t,x)F(t,x)\mathrm{d}x - \int_{[\epsilon,\infty]} \varphi(0,x)F(0,x)\mathrm{d}x - \int_0^t \int_{[\epsilon,\infty]} \varphi_s(s,x)F(s,x)\mathrm{d}x\,\mathrm{d}s = \int_0^t \left[\int_{[\epsilon,\infty]} \varphi(s,x)B(s)[F(s,\cdot)](x)\mathrm{d}x - \int_{[\epsilon,\infty]} \varphi(s,x)A(s)[F(s,\cdot)](x)F(s,x)\mathrm{d}x\right]\mathrm{d}s. \tag{4.47}$$

Now, given $\vartheta \in C^1([\epsilon,\infty])$ with $z\vartheta'(z) \in C([\epsilon,\infty])$, we set $\varphi(s,x) = \xi^\epsilon_x(-s,x)\vartheta(\xi^\epsilon(-s,x))$, for which, using Remark 4.30, we compute that

$$\varphi_s(s,x) = -\xi^\epsilon_x(-s,x)\tfrac{1}{2}\big(\eta'_\epsilon(\xi^\epsilon(-s,x))\xi^\epsilon(-s,x) + \eta_\epsilon(\xi^\epsilon(-s,x))\big) \times \vartheta(\xi^\epsilon(-s,x)) - \xi^\epsilon_x(-s,x) \times \tfrac{1}{2}\eta_\epsilon(\xi^\epsilon(-s,x))\xi^\epsilon(-s,x)\vartheta'(\xi^\epsilon(-s,x)).$$

This is thus an admissible test function in (4.47), and, by definition [cf. (4.45)], we find that the left hand side of (4.47) now equals

$$\int_{[\epsilon,\infty]} \vartheta(x)S(t)\Pi_0(x)\mathrm{d}x - \int_{[\epsilon,\infty]} \vartheta(x)\Pi_0(x)\mathrm{d}x + \int_0^t \int_{[\epsilon,\infty]} \tfrac{1}{2}\eta_\epsilon(x)x\vartheta'(x)S(s)\Pi_0(x)\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{[\epsilon,\infty]} \tfrac{1}{2}\big(\eta'_\epsilon(x)x + \eta_\epsilon(x)\big)\vartheta(x)S(s)\Pi_0(x)\mathrm{d}x\,\mathrm{d}s.$$

Meanwhile, for the right hand side we notice that $\varphi(s,\xi^\epsilon(s,x))\xi^\epsilon_x(s,x) = \vartheta(x)$, so that

$$\begin{aligned}\mathfrak{X}^\epsilon(s)[\varphi(s,\cdot)](X,z) &= (1-\eta_\epsilon(X-z))2X\vartheta(2X) + \eta_\epsilon(X-z)\big[(X+z)\vartheta(X+z) + (X-z)\vartheta(X-z)\big] \\ &= \Xi^\epsilon[\vartheta](X,z) + 2X\vartheta(X) \text{ for } X \geq \epsilon \text{ and } X \geq z \geq 0,\end{aligned}$$

whereby, using again also (4.45), we have

$$\int_{[\epsilon,\infty]} \varphi(s,x)B(s)[F(s,\cdot)](x)\mathrm{d}x = \iint_{[\epsilon,\infty]^2} S(s)\Pi_0(x)S(s)\Pi_0(y) \times \int_0^x \frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}}\big(\Xi^\epsilon[\vartheta](x,z) + 2x\vartheta(x)\big)\mathrm{d}z\,\mathrm{d}x\mathrm{d}y.$$

Moreover, writing Π_s instead of $S(s)\Pi_0$, we see that

$$\int_{[\epsilon,\infty]} \varphi(s,x)A(s)[F(s,\cdot)](x)F(s,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \vartheta(x)\left[2\int_{[\epsilon,\infty]}\int_0^x \frac{\phi_\varepsilon(y-z)}{\sqrt{x}z^{3/2}}\mathrm{d}z\;\Pi_s(y)\mathrm{d}y - \tfrac{1}{2}\big(\eta'_\epsilon(x)x + \eta_\epsilon(x)\big)\right]\Pi_s(x)\mathrm{d}x,$$

whereby we conclude that (4.46) indeed holds for all $t \in [0,T]$ and $\vartheta \in C^1([\epsilon,\infty])$ with $z\vartheta'(z) \in C([\epsilon,\infty])$. Further, since the constant function $\vartheta \equiv 1$ is admissible, and since $\Xi^\epsilon[1] \equiv 0$, it trivially follows that $S(t)$ is an endomorphism of $\mathcal{P}([\epsilon,\infty])$ for any $t \in [0,T]$.

It only remains to check that $S(t_1+t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \geq 0$ with $t_1+t_2 \leq T$, since the extension to a semigroup $(S(t))_{t\geq 0}$ on $\mathcal{P}([\epsilon,\infty])$ is as in the proof of Proposition 4.15. Let thus $\tau \in [0,T]$ be fixed, so that for any $t \in [\tau,T]$ and $\varphi \in C([\epsilon,\infty])$ there holds

$$\int_{[\epsilon,\infty]} \varphi(x)F(t,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \varphi(x)e^{-\int_\tau^t A(s)[F(s,\cdot)](x)\mathrm{d}s}F(\tau,x)\mathrm{d}x + \int_\tau^t \int_{[\epsilon,\infty]} \varphi(x)e^{-\int_\sigma^t A(s)[F(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F(\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma,$$

which we rewrite as

$$\int_{[\epsilon,\infty]} \varphi(x)F(t,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(\tau,x)\varphi(\xi^\epsilon(\tau,x))e^{-\int_0^{t-\tau} A(\tau+s)[F(\tau+s,\cdot)](\xi^\epsilon(\tau,x))\mathrm{d}s}\Pi_\tau(x)\mathrm{d}x + \int_0^{t-\tau}\int_{[\epsilon,\infty]} \varphi(x)e^{-\int_\sigma^{t-\tau} A(\tau+s)[F(\tau+s,\cdot)](x)\mathrm{d}s}B(\tau+\sigma)[F(\tau+\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma. \tag{4.48}$$

Defining then $F_* \in C([0,T-\tau]:\mathcal{M}_+([\epsilon,\infty]))$ to be such that for $t \in [\tau,T]$ there holds

$$\int_{[\epsilon,\infty]} \varphi(x)F_*(t-\tau,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(-\tau,x)\varphi(\xi^\epsilon(-\tau,x))F(t,x)\mathrm{d}x \text{ for all } \varphi \in C([\epsilon,\infty]), \tag{4.49}$$

it can be checked that

$$A(\tau+s)[F(\tau+s,\cdot)](\xi^\epsilon(\tau,x)) = A(s)[F_*(s,\cdot)](x) \text{ for } s \in [0,T-\tau] \text{ and } x > 0,$$

and

$$\int_{(0,\infty)} \varphi(x)B(\tau+\sigma)[F(\tau+\sigma,\cdot)](x)\mathrm{d}x = \int_{(0,\infty)} \xi_x^\epsilon(\tau,x)\varphi(\xi^\epsilon(\tau,x))B(\sigma)[F_*(\sigma,\cdot)](x)\mathrm{d}x \quad \text{for } \sigma \in [0,T-\tau] \text{ and } \varphi \in C([\epsilon,\infty]),$$

which now allows us to write (4.48) as

$$\int_{[\epsilon,\infty]} \varphi(x)F(t,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(\tau,x)\varphi(\xi^\epsilon(\tau,x))e^{-\int_0^{t-\tau} A(s)[F_*(s,\cdot)](x)\mathrm{d}s}\Pi_\tau(x)\mathrm{d}x + \int_0^{t-\tau}\int_{[\epsilon,\infty]} \xi_x^\epsilon(\tau,x)\varphi(\xi^\epsilon(\tau,x))e^{-\int_\sigma^{t-\tau} A(s)[F_*(s,\cdot)](x)\mathrm{d}s}B(\sigma)[F_*(\sigma,\cdot)](x)\mathrm{d}x\,\mathrm{d}\sigma. \tag{4.50}$$

Moreover, replacing φ in (4.50) by $\xi_x^\epsilon(-\tau,\cdot)\tilde{\varphi}(\xi^\epsilon(-\tau,\cdot))$, and using (4.49) for the left hand side, we find that F_* is the unique function that for all $t \in [0,T-\tau]$ and $\varphi \in C([\epsilon,\infty])$ satisfies (4.43) with Π_0 replaced by $\Pi_\tau = S(\tau)\Pi_0$. Lastly, using both (4.45) and (4.49), we thus find for $t \in [0,T-\tau]$ and $\vartheta \in C([\epsilon,\infty])$ that

$$\int_{[\epsilon,\infty]} \vartheta(x)S(t-\tau)S(\tau)\Pi_0(x)\mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(\tau-t,x)\vartheta(\xi^\epsilon(\tau-t,x))F_*(t-\tau,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(-t,x)\vartheta(\xi^\epsilon(-t,x))F(t,x)\mathrm{d}x = \int_{[\epsilon,\infty]} \vartheta(x)S(t)\Pi_0(x)\mathrm{d}x,$$

and we conclude the claim. □

Lemma 4.33. *Given $\epsilon > 2\varepsilon > 0$, let $(S(t))_{t\geq 0}$ be the semigroup on $\mathcal{P}([\epsilon,\infty])$ that was obtained in Proposition 4.32. Then for every $t \geq 0$ the mapping $S(t) : \mathcal{P}([\epsilon,\infty]) \to \mathcal{P}([\epsilon,\infty])$ is weakly-$*$ continuous.*

Proof. Let $t > 0$ be fixed, and let $\Pi_0^1, \Pi_0^2 \in \mathcal{P}([\epsilon,\infty])$ be arbitrary. We will show that for any pair $(\vartheta, c) \in C([\epsilon,\infty]) \times \mathbb{R}_+$ there exists an open set $\mathcal{O} = \mathcal{O}(\vartheta, c)$ in the weak-$*$ topology of $\mathcal{M}([\epsilon,\infty])$ such that if $\Pi_0^1 - \Pi_0^2 \in \mathcal{O}$, then

$$\left| \int_{[\epsilon,\infty]} \vartheta(x) \big(S(t)\Pi_0^1 - S(t)\Pi_0^2\big)(x) \mathrm{d}x \right| < c.$$

We thereto fix $\vartheta \in C([\epsilon,\infty])$, we write Π_s^i instead of $S(s)\Pi_0^i$, for $s \in [0,t]$ and $i \in \{1,2\}$, and we let $F_s^i \in \mathcal{M}_+([\epsilon,\infty])$ be such that

$$\int_{[\epsilon,\infty]} \varphi(x) F_s^i(x) \mathrm{d}x = \int_{[\epsilon,\infty]} \xi_x^\epsilon(s,x) \varphi(\xi^\epsilon(s,x)) \Pi_s^i(x) \mathrm{d}x \text{ for all } \varphi \in C([\epsilon,\infty]). \tag{4.51}$$

With these definitions in place, the first step is to prove existence of a unique solution $\varphi \in C^1([0,t] : C([\epsilon,\infty]))$ to the problem

$$\begin{cases} \varphi_s(s,x) = \mathcal{L}(s)[\varphi(s,\cdot)](x) \\ \qquad\qquad := -\frac{1}{2}\big(\eta_\epsilon'(\xi^\epsilon(-s,x))\xi^\epsilon(-s,x) + \eta_\epsilon(\xi^\epsilon(-s,x))\big)\varphi(s,x) \\ \qquad\qquad\quad - \xi_x^\epsilon(-s,x)\mathcal{L}_1(s)[\varphi(s,\cdot)](x) - \xi_x^\epsilon(-s,x)\mathcal{L}_2(s)[\varphi(s,\cdot)](x), \\ \varphi(t,x) = \xi_x^\epsilon(-t,x)\vartheta(\xi^\epsilon(-t,x)), \end{cases} \tag{4.52}$$

where for $s \in [0,t]$, $\varphi \in C([\epsilon,\infty])$, and $x \geq \epsilon$ we have

$$\mathcal{L}_1(s)[\varphi](x) = \int_{[\epsilon,\infty]} \int_0^{\xi^\epsilon(-s,x)} \frac{\phi_\varepsilon(y-z)}{(\xi^\epsilon(-s,x)z)^{3/2}} \Xi^\epsilon[\xi_x^\epsilon(s,\cdot)\varphi(\xi^\epsilon(s,\cdot))](\xi^\epsilon(-s,x),z) \mathrm{d}z \\ \times \tfrac{1}{2}\big(\Pi_s^1 + \Pi_s^2\big)(y)\mathrm{d}y,$$

and

$$\mathcal{L}_2(s)[\varphi](x) = \int_{[\epsilon,\infty]} \int_0^{y} \frac{\phi_\varepsilon(\xi^\epsilon(-s,x)-z)}{(yz)^{3/2}} \Xi^\epsilon[\xi_x^\epsilon(s,\cdot)\varphi(\xi^\epsilon(s,\cdot))](y,z) \mathrm{d}z \\ \times \tfrac{1}{2}\big(\Pi_s^1 + \Pi_s^2\big)(y)\mathrm{d}y.$$

To that end we observe, for $s \in [0,t]$ and $\varphi \in C([\epsilon,\infty])$, that

$$\begin{aligned} &\big\|\mathcal{L}_1(s)[\xi_x^\epsilon(-s,\cdot)\varphi(\xi^\epsilon(-s,\cdot))]\big\|_{C([\epsilon,\infty])} \\ &\quad = \sup_{x\geq\epsilon} \left| \int_{[\epsilon,\infty]} \int_0^x \frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}} \Xi^\epsilon[\varphi](x,z)\mathrm{d}z \ \tfrac{1}{2}\big(\Pi_s^1 + \Pi_s^2\big)(y)\mathrm{d}y \right| \leq \tfrac{8\sqrt{2}}{\epsilon^2}\|\varphi\|_{C([\epsilon,\infty])}, \end{aligned}$$

where we have used that $\Xi^\epsilon[\varphi](x,z) \leq 4x\|\varphi\|_{C([\epsilon,\infty])}$, and similarly

$$\begin{aligned} &\big\|\mathcal{L}_2(s)[\xi_x^\epsilon(-s,\cdot)\varphi(\xi^\epsilon(-s,\cdot))]\big\|_{C([\epsilon,\infty])} \\ &\quad = \sup_{x\geq\epsilon} \left| \int_{[\epsilon,\infty]} \int_0^y \frac{\phi_\varepsilon(x-z)}{(yz)^{3/2}} \Xi^\epsilon[\varphi](y,z)\mathrm{d}z \ \tfrac{1}{2}\big(\Pi_s^1 + \Pi_s^2\big)(y)\mathrm{d}y \right| \leq \tfrac{8\sqrt{2}}{\epsilon^2}\|\varphi\|_{C([\epsilon,\infty])}. \end{aligned}$$

By also the fact that ξ^ϵ is smooth, we thus find that $\mathcal{L}(s)$, for any $s \in [0,t]$, is a bounded operator from $C([\epsilon,\infty])$ into itself, from which unique existence follows easily (cf. [10]).

Then, with $\varphi \in C^1([0,t] : C([\epsilon,\infty]))$ the unique solution to (4.52), and using (4.51), we find for $s \in [0,t]$ that

$$\begin{aligned}
-\int_{[\epsilon,\infty]} &\Big(\varphi_s(s,x) + \tfrac{1}{2}\big(\eta_\epsilon'(\xi^\epsilon(-s,x))\xi^\epsilon(-s,x) + \eta_\epsilon(\xi^\epsilon(-s,x))\big)\varphi(s,x)\Big)\big(F_s^1 - F_s^2\big)(x)\mathrm{d}x \\
&= \int_{[\epsilon,\infty]} \Big(\mathcal{L}_1(s)[\varphi(s,\cdot)](\xi^\epsilon(s,x)) + \mathcal{L}_2(s)[\varphi(s,\cdot)](\xi^\epsilon(s,x))\Big)\big(\Pi_s^1 - \Pi_s^2\big)(x)\mathrm{d}x \\
&= \iint_{[\epsilon,\infty]^2} \big(\Pi_s^1(x)\Pi_s^1(y) - \Pi_s^2(x)\Pi_s^2(y)\big) \\
&\qquad \times \int_0^x \frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}}\, \Xi^\epsilon[\xi_x^\epsilon(s,\cdot)\varphi(s,\xi^\epsilon(s,\cdot))](x,z)\mathrm{d}z\ \mathrm{d}x\mathrm{d}y. \qquad (4.53)
\end{aligned}$$

Moreover, noting that

$$\Xi^\epsilon[\xi_x^\epsilon(s,\cdot)\varphi(s,\xi^\epsilon(s,\cdot))](x,z) = \mathfrak{X}^\epsilon(s)[\varphi(s,\cdot)](x,z) - 2\xi_x^\epsilon(s,x)\xi^\epsilon(s,x)\varphi(s,\xi^\epsilon(s,x)),$$

we see that the right hand side of (4.53) is the difference of the nonlinear terms in

$$\int_{[\epsilon,\infty]} \varphi(s,x)B(s)[F_s^i](x)\mathrm{d}x - \int_{[\epsilon,\infty]} \varphi(s,x)A(s)[F_s^i](x)F_s^i(x)\mathrm{d}x \text{ with } i \in \{1,2\}.$$

Therefore, supposing that $t \leq T$, and using φ in (4.47), it thus follows by (4.52) that

$$\int_{[\epsilon,\infty]} \varphi(t,x)\big(F_t^1 - F_t^2\big)(x)\mathrm{d}x = \int_{[\epsilon,\infty]} \varphi(0,x)\big(\Pi_0^1 - \Pi_0^2\big)(x)\mathrm{d}x,$$

and, since the left hand side is equal to the term we wish to bound [cf. (4.51)], it follows that the set

$$\mathcal{O} = \left\{\mu \in \mathcal{M}([\epsilon,\infty]) : \int_{[\epsilon,\infty]} \varphi(0,x)\mu(x)\mathrm{d}x < c\right\},$$

is as required. Lastly, for $t > T$, we iterate the above procedure to conclude the result. □

It then remains to prove existence of **invariant subsets**.

Lemma 4.34. *Given $0 < 2\epsilon < 1$, then the space $\mathcal{Y}^\epsilon$ of probability measures $\Pi \in \mathcal{P}([\epsilon,\infty])$ for which there holds*

$$\int_{[\epsilon,\infty]} \tfrac{1}{x}(x-1)_+^2\Pi(x)\mathrm{d}x \leq 100, \qquad (4.54)$$

is positively invariant under any semigroup $(S(t))_{t\geq 0}$ on $\mathcal{P}([\epsilon,\infty])$ as obtained in Proposition 4.32, i.e. if $(S(t))_{t\geq 0}$ is as obtained in Proposition 4.32, with $\epsilon > 2\varepsilon > 0$ arbitrarily fixed, then for all $t \geq 0$ there holds $S(t)\mathcal{Y}^\epsilon \subset \mathcal{Y}^\epsilon$.

Proof. For $c > 0$ we define $\vartheta_c(x) = \frac{1}{1+cx}\frac{1}{x}(x-1)_+^2$, for which we have

$$\begin{aligned}
\eta_\epsilon(x)x\vartheta_c'(x) &= -\tfrac{cx}{1+cx}\vartheta_c(x) - \vartheta_c(x) + 2\tfrac{1}{1+cx}\tfrac{1}{x}(x-1)_+ \times (x-1+1) \\
&= -\tfrac{cx}{1+cx}\vartheta_c(x) + \vartheta_c(x) + 2\tfrac{1}{1+cx}\tfrac{1}{x}(x-1)_+ \geq \vartheta_c(x) - \tfrac{cx}{1+cx}\vartheta_c(x) \geq 0.
\end{aligned}$$

Moreover, since the mapping $x \mapsto x^2 - x\vartheta_c(x)$ is convex, there holds

$$\begin{aligned}
\Xi^\epsilon[\vartheta_c](x,z) &\leq 2z^2 + 2(1-\eta_\epsilon(x-z))(x^2-z^2) \\
&\leq 2\eta_\epsilon(x-z)z^2 + 2(1-\eta_\epsilon(x-z))(5z)^2 \leq 50z^2 \text{ for } x \geq z \geq \tfrac{\epsilon}{2},
\end{aligned}$$

which, by also the fact that $\Xi^\epsilon[\vartheta_c] \equiv 0$ on $[\epsilon, \frac{1}{2}]^2$, gives

$$\int_0^x \frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}}\,\Xi^\epsilon[\vartheta_c](x,z)\mathrm{d}z \le 50 \times \mathbf{1}_{\{x\ge\frac{1}{2}\}}(x,y)\frac{1}{x}\int_0^x \phi_\varepsilon(y-z)\mathrm{d}z$$
$$\le 100\int_0^x \phi_\varepsilon(y-z)\mathrm{d}z \text{ for } x,y\ge\epsilon.$$

Using thus ϑ_c in (4.46), we then estimate the right hand side by

$$\int_0^t 50\iint_{[\epsilon,\infty]^2}\Pi_s(x)\Pi_s(y)\Big[\int_0^x \phi_\varepsilon(y-z)\mathrm{d}z + \int_0^y \phi_\varepsilon(x-z)\mathrm{d}z\Big]\mathrm{d}x\mathrm{d}y\ \mathrm{d}s \le 50t,$$

[cf. (A.5)] so that for $\Pi_0 \in \mathcal{Y}^\epsilon$ we find that

$$\int_{[\epsilon,\infty]}\vartheta_c(x)\Pi_t(x)\mathrm{d}x \le \int_{[\epsilon,\infty]}\vartheta_c(x)\Pi_0(x)\mathrm{d}x + 50t \le 50(2+T) \text{ for all } t\in[0,T],$$

with any $T>0$. By dominated convergence this then allows us to take the limit $c\to 0$, i.e. to use ϑ_0 in (4.46) directly, to obtain that

$$\int_{[\epsilon,\infty]}\vartheta_0(x)\Pi_t(x)\mathrm{d}x - \int_{[\epsilon,\infty]}\vartheta_0(x)\Pi_0(x)\mathrm{d}x \le -\frac{1}{2}\int_0^t\Big(\int_{[\epsilon,\infty]}\vartheta_0(x)\Pi_s(x)\mathrm{d}x - 100\Big)\mathrm{d}s.$$

hence, using Gronwall's inequality, for any $t\ge 0$ we have

$$\int_{[\epsilon,\infty]}\tfrac{1}{x}(x-1)_+^2\Pi_t(x)\mathrm{d}x \le 100 + e^{-\frac{t}{2}}\Big(\int_{[\epsilon,\infty]}\tfrac{1}{x}(x-1)_+^2\Pi_0(x)\mathrm{d}x - 100\Big),$$

and we conclude the result. □

With these results, we may once again use Lemma 4.11, and a compactness argument, to prove Proposition 4.28.

Proof of Proposition 4.28. For $\epsilon > 2\varepsilon > 0$, let $(S(t))_{t\ge0}$ be the semigroup on $\mathcal{P}([\epsilon,\infty])$ as obtained in Proposition 4.32. In view of Lemmas 4.33 and 4.34, it then follows with Lemma 4.11 that there is at least one fixed-point $\Pi_\varepsilon \in \mathcal{Y}^\epsilon$ under $(S(t))_{t\ge0}$, where $\mathcal{Y}^\epsilon$ is the weakly-$*$ compact subset of probability measures Π on $[\epsilon,\infty]$ that satisfy (4.54), and there holds

$$\int_{[\epsilon,\infty]}\tfrac{1}{2}\eta_\epsilon(x)x\vartheta'(x)\Pi_\varepsilon(x)\mathrm{d}x = \iint_{[\epsilon,\infty]^2}\Pi_\varepsilon(x)\Pi_\varepsilon(y)\int_0^x\frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}}\,\Xi^\epsilon[\vartheta](x,z)\mathrm{d}z\ \mathrm{d}x\mathrm{d}y, \tag{4.55}$$

for all $\vartheta\in C^1([\epsilon,\infty])$ with $z\vartheta'(z)\in C([\epsilon,\infty])$.

Moreover, by compactness, there now exist a subsequence $\varepsilon\to 0$, and $\Pi\in\mathcal{Y}^\epsilon$, such that $\Pi_\varepsilon \xrightarrow{*} \Pi$ in $\mathcal{M}([\epsilon,\infty])$, and the left hand side of (4.55) converges trivially to the one of (4.42). For the remaining part we argue as in the proofs of Lemma 2.2 and Proposition 4.13: Observing that the right hand side of (4.55) can be written as

$$\frac{1}{2}\iint_{[\epsilon,\infty]^2}\Pi_\varepsilon(x)\Pi_\varepsilon(y)\Big[\int_0^x\frac{\phi_\varepsilon(y-z)}{(xz)^{3/2}}\,\Xi^\epsilon[\vartheta](x,z)\mathrm{d}z + \int_0^y\frac{\phi_\varepsilon(x-z)}{(zy)^{3/2}}\,\Xi^\epsilon[\vartheta](z,y)\mathrm{d}z\Big]\mathrm{d}x\mathrm{d}y,$$

where the term between square brackets converges strongly to $(xy)^{-3/2}\Xi^\epsilon(x,y)$ as $\varepsilon\to 0$ (cf. Lemma A.5), the claim follows. □

To finally prove Proposition 4.24 we require a final lemma, the proof of which is postponed to the appendix in order to not break the flow of the argument.

Lemma 4.35. *Given $\mu \in \mathcal{M}_+([0,\infty])$ with $\mu(\{0\}) = 0$, there exists some finite $z > 0$ such that*

$$\left(\frac{1}{4}\int_{(0,1]} x\mu(x)\mathrm{d}x\right)^2 \le \iint_{\mathbb{R}_+^2} \frac{\mu(x)\mu(y)}{\sqrt{xy}}\big[(x+y-z)\wedge(z-|x-y|)\big]_+\mathrm{d}x\mathrm{d}y. \tag{4.56}$$

Proof of Proposition 4.24. For $0 < 2\epsilon < 1$, let $\Phi^\epsilon \in \mathcal{M}_+([\epsilon,\infty])$ be as obtained in Proposition 4.26, i.e. so that its first moment is equal to 1, its tightness estimate is independent of ϵ, and so that (4.41) is satisfied for all $\psi \in C_c^1([0,\infty))$. By inclusion of these measures in the weakly-$*$ compact set $\mathcal{S}_2$ (cf. Remark 4.25, and Lemma 4.10), there then exist a decreasing sequence $\epsilon \to 0$ and an element $\Phi \in \mathcal{S}_2$ such that $\Phi^\epsilon \overset{*}{\rightharpoonup} \Phi$ in $\mathcal{B}'$.

We first show that actually $\Phi \in \mathcal{S}_2 \cap \mathcal{X}_1$, for which we immediately see that

$$\int_{[R,\infty]} x\Phi^\epsilon(x)\mathrm{d}x \le \frac{R}{(R-1)^2}\int_{[\epsilon,\infty]}(x-1)_+^2\Phi^\epsilon(x)\mathrm{d}x \to 0 \text{ as } R\to\infty,$$

uniformly in ϵ. Furthermore, using Lemma 4.35, we find for any $r > 0$ that

$$\left(\frac{1}{4}\int_{(0,r]} x\Phi^\epsilon(x)\mathrm{d}x\right)^2 \le r^2\iint_{\mathbb{R}_+^2}\frac{\Phi^\epsilon(x)\Phi^\epsilon(y)}{\sqrt{xy}}\big[(x+y-rz_\epsilon)\wedge(rz_\epsilon-|x-y|)\big]_+\mathrm{d}x\mathrm{d}y$$

for certain finite $z_\epsilon > 0$. Approximating then $\psi(x) = (rz_\epsilon - x)_+$ in (4.41) by a sequence of nonincreasing admissible test functions (cf. Lemma A.6), it follows that

$$\iint_{\mathbb{R}_+^2}\frac{\Phi^\epsilon(x)\Phi^\epsilon(y)}{\sqrt{xy}}\big[(x+y-rz_\epsilon)\wedge(rz_\epsilon-|x-y|)\big]_+\mathrm{d}x\mathrm{d}y \le \int_{(0,\infty)}(x\wedge rz_\epsilon)\Phi^\epsilon(x)\mathrm{d}x \le 1,$$

whereby we have

$$\int_{(0,r]} x\Phi^\epsilon(x)\mathrm{d}x \le 4r \text{ for all } r > 0, \tag{4.57}$$

and we conclude that the limit Φ is a measure with first moment equal to 1. It remains to check that $\|\Phi\|_1 < \infty$, to which end we first use (4.57) to find that

$$\int_{[\sigma,1]}\Phi^\epsilon(x)\mathrm{d}x \le \sum_{j=0}^{n} 2^{j+1}\int_{(2^{-j-1},2^{-j}]} x\Phi^\epsilon(x)\mathrm{d}x \le 8\left|\log_2(\tfrac{\sigma}{2})\right| \text{ for all } \sigma\in(0,1),$$

(with $n = \lfloor -\log_2\sigma\rfloor$). Approximating next $\psi(x) = (\sigma - x)_+$ in (4.41) by convex test functions, then we actually obtain

$$\begin{aligned}\frac{1}{3}\left(\int_{(\sigma,\frac{3}{2}\sigma]}\Phi^\epsilon(x)\mathrm{d}x\right)^2 &\le \iint_{(\sigma,\frac{3}{2}\sigma]^2}\frac{\Phi^\epsilon(x)\Phi^\epsilon(y)}{\sqrt{xy}}\big[(x+y-\sigma)\wedge(\sigma-|x-y|)\big]_+\mathrm{d}x\mathrm{d}y\\ &\le \sigma\int_{[\sigma,\infty)}\Phi^\epsilon(x)\mathrm{d}x \le 8\sigma\left|\log_2(\tfrac{\sigma}{4})\right| \text{ for all } \sigma\in(0,1).\end{aligned}$$

Bounding now the right hand side by $C(\alpha)\sigma^{2\alpha}$ with $\alpha \in (0,\frac{1}{2})$, it thus follows that

$$\int_{(0,r]}\Phi^\epsilon(x)\mathrm{d}x \le \sqrt{3C(\alpha)}\,r^\alpha \text{ for all } r > 0, \tag{4.58}$$

and we conclude that indeed $\Phi \in \mathcal{S}_1 \cap \mathcal{X}_1$.

Lastly, we show that Φ satisfies $(\mathrm{SSPE})_2^{\mathrm{w}}$ for any $\psi \in C_c^1([0,\infty))$ fixed. Since we have

$$\sup_{x\in(0,2\epsilon)} \left|\tfrac{1+x}{x}\left(x\psi'(x)-\psi(x)+\psi(0)\right)\right| \le 2 \sup_{x\in(0,2\epsilon)} \left|\psi'(x)-\tfrac{\psi(x)-\psi(0)}{x}\right| \to 0 \text{ as } \epsilon\to 0,$$

we find that the left hand side of (4.41) converges to the one of $(\mathrm{SSPE})_2^{\mathrm{w}}$ as $\epsilon \to 0$, by strong times weak-$*$ convergence. For convergence of the right hand side, we multiply the integrands in the double integrals by $\eta_r(x)\eta_r(y) + (1-\eta_r(x)\eta_r(y))$ with $r > 0$. Using then (4.58), we see that the right hand side converges as $\epsilon \to 0$ to

$$\iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta^2_{x\wedge y}\psi(x\vee y)\eta_r(x)\eta_r(y)\mathrm{d}x\mathrm{d}y + o(1),$$

which by dominated convergence tends to the right hand side of $(\mathrm{SSPE})_2^{\mathrm{w}}$ as $r \to 0$, and we conclude the result. □

4.1.3 Regularity of candidate profiles

To complete the existence part of Theorem 4.4, it suffices to check that the candidate profiles, as obtained in Propositions 4.13 and 4.24, are absolutely continuous with respect to Lebesgue measure. Better still, in Proposition 4.37 below, we argue by duality to already prove Hölder-regularity of their Radon-Nykodim derivatives. A bootstrap argument will then finally yield smoothness in the actual proof. These results come straight from [KV16].

The following result will be useful throughout the remainder of this section.

Lemma 4.36. *Given $\rho \in (1,2]$, if $\Phi \in \mathcal{X}_\rho \cap \mathcal{X}_1$ satisfies* $(\mathrm{SSPE})_\rho^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$, then*

$$\int_{(0,Z]} \Phi(x)\mathrm{d}x \le \tfrac{2\sqrt{3}}{\sqrt{3}-\sqrt{2}} \sqrt{\tfrac{\rho-1}{\rho}\|\Phi\|_1} \times \sqrt{Z} \text{ for all } Z > 0.$$

Proof. Let $z > 0$ be fixed arbitrarily, and let $\psi \in C_c^1([0,\infty))$ be a convex function such that $\psi(x) \le (z-x)_+$ for all $x \ge 0$. In particular, there then hold $\psi \ge 0$ and $\psi' \le 0$ on $\mathbb{R}_+$, hence the left hand side of $(\mathrm{SSPE})_\rho^{\mathrm{w}}$ can be bounded from above by $\frac{\rho-1}{\rho}\|\Phi\|_1 \cdot \psi(0)$. Furthermore, as the integrand in the right hand side is nonnegative by convexity of ψ (cf. Lemma A.2), we may restrict the domain of integration to $(z, \frac{3}{2}z]^2$ to obtain

$$\tfrac{\rho-1}{\rho}\|\Phi\|_1 \cdot \psi(0) \ge \frac{\psi(\frac{1}{2}z)}{3z}\left(\int_{(z,\frac{3}{2}z]} \Phi(x)\mathrm{d}x\right)^2.$$

Taking the supremum over all such test functions, i.e. setting $\psi(0) = z$ and $\psi(\frac{1}{2}z) = \frac{z}{2}$, and recalling that $z > 0$ was arbitrary, we now thus get

$$\int_{(z,\frac{3}{2}z]} \Phi(x)\mathrm{d}x \le \sqrt{\tfrac{\rho-1}{\rho}\|\Phi\|_1}\,\sqrt{6} \times \sqrt{z} \text{ for all } z > 0,$$

and the lemma follows by the decomposition $(0,Z] = \bigcup_{j=1}^{\infty}((\frac{2}{3})^j Z, \frac{3}{2}(\frac{2}{3})^j Z]$. □

Proposition 4.37. *Given $\rho \in (1,2]$, then any measure $\Phi \in \mathcal{X}_\rho \cap \mathcal{X}_1$ that satisfies* $(\mathrm{SSPE})_\rho^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$ is absolutely continuous with respect to Lebesgue measure. Moreover, its Radon-Nykodim derivative is locally α-Hölder continuous on $(0,\infty)$ for all $\alpha \in (0,\frac{1}{2})$.*

Proof. Throughout this proof, let $\Phi \in \mathcal{X}_\rho \cap \mathcal{X}_1$ be fixed such that for all $\psi \in C_c^1([0,\infty))$ there holds $(\mathrm{SSPE})_\rho^{\mathrm{w}}$.

Arguing by duality, we first check that $\Phi \in L^1(0,\infty)\cap L^p_{\mathrm{loc}}((0,\infty])$ for all $p>1$. To that end, let $p,p^*>1$ be fixed such that $\frac{1}{p}+\frac{1}{p^*}=1$, and let $\chi \in C_c^\infty((0,\infty)) \subset \bigcap_{r\ge 1} L^r(0,\infty)$ be arbitrarily fixed. Setting then $\psi(x) = -\int_x^\infty \frac{1}{z}\chi(z)\mathrm{d}z$, we find by $(\mathrm{SSPE})_\rho^{\mathrm{w}}$ that

$$\int_{(0,\infty)} \chi(x)\Phi(x)\mathrm{d}x = (\rho-1)\int_{(0,\infty)}\int_0^x \tfrac{1}{z}\chi(z)\mathrm{d}z\ \Phi(x)\mathrm{d}x \\ + \frac{\rho}{2}\iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\left(\int_{x\vee y}^{x+y}\tfrac{1}{z}\chi(z)\mathrm{d}z - \int_{|x-y|}^{x\vee y}\tfrac{1}{z}\chi(z)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y, \quad (4.59)$$

and, introducing the abbreviations $\Sigma_\chi = \mathrm{supp}(\chi)$ and $\varsigma_\chi = \frac{1}{2}\min(\Sigma_\chi)$, we easily obtain

$$\left|\int_{(0,\infty)}\int_0^x \tfrac{1}{z}\chi(z)\mathrm{d}z\ \Phi(x)\mathrm{d}x\right| \le \left(\int_{(2\varsigma_\chi,\infty)}\Phi(x)\mathrm{d}x \times \left\|\tfrac{1}{z}\right\|_{L^p(\Sigma_\chi)}\right)\|\chi\|_{L^{p^*}(\Sigma_\chi)}, \quad (4.60)$$

and

$$\left|\iint_{(\varsigma_\chi,\infty)^2}\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\left(\int_{x\vee y}^{x+y}\tfrac{1}{z}\chi(z)\mathrm{d}z - \int_{|x-y|}^{x\vee y}\tfrac{1}{z}\chi(z)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\right| \\ \le \left(\frac{2}{\varsigma_\chi}\left(\int_{(\varsigma_\chi,\infty)}\Phi(x)\mathrm{d}x\right)^2 \times \left\|\tfrac{1}{z}\right\|_{L^p(\Sigma_\chi)}\right)\|\chi\|_{L^{p^*}(\Sigma_\chi)}. \quad (4.61)$$

Recalling furthermore that the term between brackets in the double integral on the right hand side of (4.59) vanishes on $(0,\varsigma_\chi]^2$, it remains by symmetry only to estimate the integral over $(x,y)\in(\varsigma_\chi,\infty)\times(0,\varsigma_\chi]$. Let thereto $r>p$ and $q>1$ be such that $\frac{1}{r}+\frac{1}{q}+\frac{1}{p^*}=1$, so that for $(x,y)\in(\varsigma_\chi,\infty)\times(0,\varsigma_\chi]$ there holds

$$\left|\int_x^{x+y}\tfrac{1}{z}\chi(z)\mathrm{d}z - \int_{x-y}^{x}\tfrac{1}{z}\chi(z)\mathrm{d}z\right| \le 2y^{\frac{1}{r}} \times \left\|\tfrac{1}{z}\right\|_{L^q(\Sigma_\chi)} \times \|\chi\|_{L^{p^*}(\Sigma_\chi)}.$$

Thus, we find that

$$\left|\iint_{(\varsigma_\chi,\infty)\times(0,\varsigma_\chi]}\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\left(\int_{x\vee y}^{x+y}\tfrac{1}{z}\chi(z)\mathrm{d}z - \int_{|x-y|}^{x\vee y}\tfrac{1}{z}\chi(z)\mathrm{d}z\right)\mathrm{d}(x,y)\right| \\ \le \left(\frac{2}{\sqrt{\varsigma_\chi}}\int_{(\varsigma_\chi,\infty)}\Phi(x)\mathrm{d}x \times \int_{(0,\varsigma_\chi]} y^{\frac{1}{r}-\frac{1}{2}}\Phi(y)\mathrm{d}y \times \left\|\tfrac{1}{z}\right\|_{L^q(\Sigma_\chi)}\right)\|\chi\|_{L^{p^*}(\Sigma_\chi)}, \quad (4.62)$$

where the second integral between brackets on the right hand side can be bounded, using Lemma 4.36 and a dyadic decomposition of the interval $(0,\varsigma_\chi]$. Note furthermore that the dependence on χ of the terms between brackets on the right hand sides of (4.60), (4.61), and (4.62), is limited to dependence on ς_χ, so that for $k>0$ we have

$$\left|\int_{[k,\infty)}\chi(x)\Phi(x)\mathrm{d}x\right| \le C(\rho,\Phi,p^*,k)\times\|\chi\|_{L^{p^*}(k,\infty)} \text{ for all } \chi\in L^{p^*}(k,\infty),$$

hence $\Phi \in L^p_{\mathrm{loc}}((0,\infty])$ by duality, and $\Phi\in L^1(0,\infty)$ since $\|\Phi\|_1<\infty$. Moreover, from the estimates above we can deduce that $C(\rho,\Phi,p^*,k)\le C(\rho,\Phi,p^*)\times k^{-1/p^*}\|\Phi\|_{L^1(k,\infty)}$ as $k\to\infty$, whereby we have the following useful estimate

$$\|\Phi\|_{L^p(r,\infty)} \le O\left(r^{\frac{1}{p}-1}\|\Phi\|_{L^1(r,\infty)}\right) \text{ as } r\to\infty. \quad (4.63)$$

For the remaining continuity claim, let $\gamma \in (0, \frac{1}{2})$ and $[k_-, k_+] \subset (0, \infty)$ be arbitrarily fixed. For $\psi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \subset [k_-, k_+]$, we now first check that

$$\left| \int_{(0,\infty)} \psi'(x) x \Phi(x) \mathrm{d}x \right| \leq C(\rho, \Phi, \gamma, k_-, k_+) \times \|\psi\|_{H^\gamma(\mathbb{R})}. \tag{4.64}$$

Using ψ in $(\mathrm{SSPE})_\rho^{\mathrm{w}}$, and applying Hölder's inequality, we then find that

$$\left| \int_{(0,\infty)} \psi'(x) x \Phi(x) \mathrm{d}x \right| \leq (\rho - 1) \|\Phi\|_{L^2(k_-,\infty)} \|\psi\|_{L^2(\mathbb{R})} + \frac{\rho}{2} \left| \iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta_{x\wedge y}^2 \psi(x \vee y) \mathrm{d}x \mathrm{d}y \right|,$$

where the first term on the right hand side can be estimated as desired, since $H^\gamma(\mathbb{R})$ is continuously embedded in $L^2(\mathbb{R})$. Furthermore, we note that

$$\frac{1}{2} \left| \iint_{(k_+,\infty)^2} \Big[\cdots \Big] \mathrm{d}x \mathrm{d}y \right| \leq \frac{1}{k_+} \int_{(k_+,\infty)} \Phi(x) \int_{(k_+,x)} \Phi(y) |\psi(x-y)| \mathrm{d}y \, \mathrm{d}x \leq \tfrac{1}{k_+} \|\Phi\|_{L^2(k_+,\infty)} \|\Phi\|_{L^1(0,\infty)} \|\psi\|_{L^2(\mathbb{R})},$$

where we have used both Hölder's inequality, and Young's inequality for convolutions. It thus remains to estimate the double integral over $\mathbb{R}_+^2 \setminus (k_+, \infty)^2$, to which end we observe for $y \in (0, k_+)$ that

$$\left| \int_{y \vee (k_- - y)}^{k_+ + y} \Phi(x) \Delta_y^2 \psi(x) \mathrm{d}x \right| \leq \|\Phi\|_{L^2(\frac{1}{2}k_-, 2k_+)} \|\Delta_y^2 \psi\|_{L^2(\mathbb{R})} \leq C(\Phi, \gamma, k_-, k_+) \times y^\gamma \times \|\psi\|_{H^\gamma(\mathbb{R})}$$

(cf. [42, 43]). It then follows that

$$\frac{\rho}{2} \left| \iint_{\mathbb{R}_+^2 \setminus (k_+,\infty)^2} \Big[\cdots \Big] \mathrm{d}x \mathrm{d}y \right| \leq C(\rho, \Phi, \gamma, k_-, k_+) \times \int_{(0,k_+)} y^{\gamma - \frac{1}{2}} \Phi(y) \mathrm{d}y \times \|\psi\|_{H^\gamma(\mathbb{R})},$$

where the integral on the right hand side is finite, as before, by Lemma 4.36 and a dyadic decomposition, hence we conclude that (4.64) indeed holds.

To complete the proof, let $\zeta \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\zeta) \subset [k_-, k_+]$ be such that $\zeta(x)x = 1$ for all $x \in [\frac{2}{3}k_- + \frac{1}{3}k_+, \frac{1}{3}k_- + \frac{2}{3}k_+]$, and set $\Theta(x) = \zeta(x) x \Phi(x)$. For any $\psi \in C_c^\infty(\mathbb{R})$ we then have

$$\left| \int_{\mathbb{R}} \psi'(x) \Theta(x) \mathrm{d}x \right| \leq \left| \int_{\mathbb{R}} (\psi\zeta)'(x) x \Phi(x) \mathrm{d}x \right| + \left| \int_{\mathbb{R}} \zeta'(x) x \psi(x) \Phi(x) \mathrm{d}x \right| \leq C(\rho, \Phi, \gamma, k_-, k_+) \|\psi\zeta\|_{H^\gamma(\mathbb{R})} + \|\zeta'(x)x\|_{L^\infty(\mathbb{R})} \|\Phi\|_{L^2(k_-,k_+)} \|\psi\|_{L^2(\mathbb{R})},$$

from which we further deduce

$$\left| \int_{\mathbb{R}} \psi'(x) \Theta(x) \mathrm{d}x \right| \leq C(\rho, \Phi, \gamma, k_-, k_+, \zeta) \times \|\psi\|_{H^\gamma(\mathbb{R})} \text{ for all } \psi \in H^\gamma(\mathbb{R}).$$

By duality we thus get $\Theta' \in H^{-\gamma}(\mathbb{R}) = (H^\gamma(\mathbb{R}))'$, whereby $\Theta \in H^{1-\gamma}(\mathbb{R}) \subset C^{0,\frac{1}{2}-\gamma}(\mathbb{R})$ (cf. [2], [42]). Moreover, since $\Theta \equiv \Phi$ on an interval $I = I(k_-, k_+)$, there holds $\Phi \in C^{0,\alpha}(I)$ with $\alpha = \frac{1}{2} - \gamma$, and since every compact set $K \subset (0, \infty)$ can be covered by such intervals, and since $\gamma \in (0, \frac{1}{2})$ was chosen arbitrarily, we conclude the result. □

Finally, we are now able to prove existence of smooth self-similar profiles for (QWTE).

Proof of Theorem 4.4. For $\rho \in (1,2]$ arbitrarily fixed, by Proposition 4.13 or 4.24, there exists at least one $\Phi \in \mathcal{S}_\rho \cap \mathcal{X}_1$ that satisfies $(\text{SSPE})^{\text{w}}_\rho$ for all $\psi \in C^1_c([0,\infty))$. Furthermore, this measure is absolutely continuous with respect to Lebesgue measure, i.e. $\Phi \in L^1(0,\infty)$ (cf. Proposition 4.37), and it follows from $\|\Phi\|_\rho = 1$ that Φ is nontrivial.

Given now any nonnegative function $\Phi \in L^1(0,\infty)$ that satisfies $(\text{SSPE})^{\text{w}}_\rho$ for all $\psi \in C^1_c([0,\infty))$, then, by Lemmas 4.38 and 4.41 below, there actually holds $\Phi \in \mathcal{X}_\rho \cap \mathcal{X}_1$, so we may again invoke Proposition 4.37 to get $\Phi \in \bigcap_{2\alpha<1} C^{0,\alpha}((0,\infty))$. We next check that Φ is a solution to $(\text{SSPE})_\rho$ in the sense of distributions on $(0,\infty)$, i.e. that for all $\psi \in C^\infty_c((0,\infty))$ there holds

$$\begin{aligned}\int_{(0,\infty)} &\left(\tfrac{1}{\rho}[x\psi(x)]_x - \psi(x)\right)\Phi(x)\mathrm{d}x \\ &= \int_{(0,\infty)} \Bigg[\int_0^{x/2} \frac{\Phi(y)}{\sqrt{y}}\left[\frac{\Phi(x+y)}{\sqrt{x+y}} + \frac{\Phi(x-y)}{\sqrt{x-y}} - 2\frac{\Phi(x)}{\sqrt{x}}\right]\mathrm{d}y \\ &\qquad + \int_{x/2}^\infty \frac{\Phi(y)\Phi(x+y)}{\sqrt{y(x+y)}}\mathrm{d}y - 2\frac{\Phi(x)}{\sqrt{x}}\int_{x/2}^x \frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y\Bigg]\psi(x)\mathrm{d}x. \end{aligned} \tag{4.65}$$

We thereto fix $\psi \in C^\infty_c((0,\infty))$ arbitrarily, for which we immediately note that the left hand sides of $(\text{SSPE})^{\text{w}}_\rho$ and (4.65) coincide. Setting further $\eta_\epsilon(x) = \eta(\frac{x}{\epsilon})$, for $\epsilon > 0$, and with $\eta(x) = 1 \wedge (x-1)_+$, it is elementary to compute for $\epsilon \ll \inf(\operatorname{supp}(\psi))$ that

$$\begin{aligned}\iint_{\{x>y>0\}} &\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\Delta_y^2\psi(x) \times \eta_\epsilon(x)\eta_\epsilon(y)\mathrm{d}x\mathrm{d}y \\ &= \int_{(0,\infty)} \Bigg[\int_0^{x/2} \eta_\epsilon(y) \times \frac{\Phi(y)}{\sqrt{y}}\left[\frac{\Phi(x-y)}{\sqrt{x-y}} + \frac{\Phi(x+y)}{\sqrt{x+y}} - 2\frac{\Phi(x)}{\sqrt{x}}\right]\mathrm{d}y \\ &\qquad + \int_{x/2}^\infty \frac{\Phi(x+y)\Phi(y)}{\sqrt{(x+y)y}}\mathrm{d}y - 2\frac{\Phi(x)}{\sqrt{x}}\int_{x/2}^x \frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y\Bigg]\psi(x)\mathrm{d}x, \end{aligned} \tag{4.66}$$

where the left hand side converges to the right hand side of $(\text{SSPE})^{\text{w}}_\rho$ as $\epsilon \to 0$. Using then the local Hölder regularity of Φ, and Lemma 4.36, we may take the limit $\epsilon \to 0$ under the integral on the right hand side of (4.66), obtaining (4.65).

We next show that $\Phi \in \bigcap_{2\alpha<1} C^{k,\alpha}((0,\infty))$ for all $k \in \mathbb{N}$, hence $\Phi \in C^\infty((0,\infty))$, which we do by a bootstrap argument: Given arbitrary $k \in \mathbb{N}$ and $\alpha \in (0,\frac{1}{2})$, and supposing that $\Phi \in C^{k-1,\alpha}((0,\infty))$, we check that the right hand side of $(\text{SSPE})_\rho$ is in $C^{k-1,\alpha-\epsilon}((0,\infty))$ for every $\epsilon > 0$, implying $\Phi \in \bigcap_{\epsilon>0} C^{k,\alpha-\epsilon}((0,\infty))$. The induction step is then easily completed, as for any $\alpha^* \in (0,\frac{1}{2})$ there are $\alpha \in (\alpha^*,\frac{1}{2})$ and $\epsilon > 0$ such that $\alpha^* = \alpha - \epsilon$.

Now, if indeed $\Phi \in C^{k-1,\alpha}((0,\infty))$ for arbitrary $k \in \mathbb{N}$ and $\alpha \in (0,\frac{1}{2})$, then clearly the last two terms on the right hand side of $(\text{SSPE})_\rho$ are sufficiently regular, and we find that

$$f(\tfrac{1}{2}x)\Big[f^{(\ell)}(\tfrac{3}{2}x) + f^{(\ell)}(\tfrac{1}{2}x) - 2f^{(\ell)}(x)\Big] \in C^{k-1-\ell,\alpha}((0,\infty)) \text{ for all } \ell = 0,\dots,k-1,$$

where $f(x) = \frac{\Phi(x)}{\sqrt{x}}$; Thereby it is actually sufficient to show that if $\Phi \in C^{0,\alpha}((0,\infty))$, then

$$\int_0^{x/2} \frac{\Phi(y)}{\sqrt{y}}\left[\frac{\Phi(x+y)}{\sqrt{x+y}} + \frac{\Phi(x-y)}{\sqrt{x-y}} - 2\frac{\Phi(x)}{\sqrt{x}}\right]\mathrm{d}y =: F(x) \in C^{0,\alpha-\epsilon}((0,\infty)). \tag{4.67}$$

Let thereto $k_+ > k_- > 0$ be fixed arbitrarily, and let $\kappa = \kappa(\alpha) > 0$ be the Hölder coefficient for $f(x) := \Phi(x)/\sqrt{x}$ on $[\frac{1}{2}k_-, 2k_+]$. For $k_- \le x_1 \le x_2 \le k_+$ we then have

$$|F(x_2) - F(x_1)| \le \left| \int_{x_1/2}^{x_2/2} f(y)\Delta_y^2 f(x_2)\mathrm{d}y \right| + \int_0^{x_1/2} f(y) \left| \Delta_y^2 f(x_2) - \Delta_y^2 f(x_1) \right| \mathrm{d}y,$$

where the first term on the right hand side is easily bounded by a constant times $|x_2 - x_1|$, and writing $\xi = (x_1/4) \wedge |x_2 - x_1|$, we estimate the second term on the right hand side by

$$4\kappa \int_0^{\xi} f(y)y^{\alpha}\mathrm{d}y + 4\kappa|x_2 - x_1|^{\alpha} \int_{\xi}^{x_1/2} f(y)\mathrm{d}y. \tag{4.68}$$

Using further Lemma 4.36, we find that

$$\int_0^{\xi} f(y)y^{\alpha}\mathrm{d}y = \sum_{j=0}^{\infty} \int_{2^{-j-1}\xi}^{2^{-j}\xi} \Phi(x)x^{\alpha-\frac{1}{2}}\mathrm{d}x \le \sum_{j=0}^{\infty} \frac{C(\rho,\Phi) \times \sqrt{2^{-j}\xi}}{(2^{-j-1}\xi)^{\frac{1}{2}-\alpha}} = \frac{C(\rho,\Phi)2^{\frac{1}{2}-\alpha}}{1-2^{-\alpha}} \times \xi^{\alpha},$$

and, with $n \in \mathbb{N}$ such that $\xi \in (2^{-n-1}x_1, 2^{-n}x_1]$, that

$$\int_{\xi}^{x_1/2} f(y)\mathrm{d}y \le \sum_{j=1}^{n} \int_{2^{-j-1}x_1}^{2^{-j}x_1} \frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y \le \sum_{j=0}^{n} \frac{C(\rho,\Phi) \times \sqrt{2^{-j}\xi}}{\sqrt{2^{-j-1}\xi}} \le C(\rho,\Phi) \times \log\left(\tfrac{x_1}{\xi}\right),$$

hence (4.68) is bounded by a term of order $O(\xi^{\alpha}(1+\log(\xi))) \le O(\xi^{\alpha-\epsilon})$ as $\xi \to 0$, whereby we conclude that (4.67) holds.

Lastly, to prove strict positivity of nontrivial solutions, suppose that for a given nonnegative solution $\Phi \in C^1((0,\infty))$ to $(\mathrm{SSPE})_\rho$ there exists some $x_0 > 0$ such that $\Phi(x_0) = 0$. By nonnegativity there must then also hold $\Phi'(x_0) = 0$, so from $(\mathrm{SSPE})_\rho$, and by continuity, we obtain $\Phi(y)\Phi(x_0-y) = 0$ for all $y \in [0, \frac{1}{2}x_0]$, and in particular $\Phi(\frac{1}{2}x_0) = 0$. Iterating the argument, we thus find that $\Phi(2^{-n}x_0) = 0$ for every $n \in \mathbb{N}$. Furthermore, if we have $\Phi(x_*) = \Phi'(x_*) = 0$ for some $x_* > 0$, then we similarly get $\Phi(y)\Phi(x_*+y) = 0$ for all $y > 0$, whereby there thus holds

$$\Phi(y)\Phi(2^{-n}x_0 + y) = 0 \text{ for all } y > 0 \text{ and } n \in \mathbb{N}.$$

By continuity, this now implies that $\Phi(y) = 0$ for every $y > 0$, and we therefore conclude that if Φ has a root in $(0,\infty)$, then Φ is trivial. □

4.2 Properties of self-similar profiles

In the previous section we have shown existence of self-similar solutions to (QWTE) in the sense of Definition 4.1, proving that for every $\rho \in (1,2]$ there is at least one self-similar profile. It was noted in Remark 4.5 that any self-similar solution with finite energy must satisfy $\rho = 2$, which prompted the separate treatment of the cases $\rho = 2$ and $\rho < 2$. Here we will see that any self-similar profile of a self-similar solution to (QWTE) with $\rho \in (1,2)$ behaves asymptotically like a power law with infinite energy (cf. Proposition 4.40), and that profiles with $\rho = 2$ are bounded pointwise by an exponential (cf. Proposition 4.43), indicating that the two cases are inherently different. However, before we continue the separate treatment of the cases $\rho \in (1,2)$ and $\rho = 2$, let us start with a reformulation result that will be useful throughout this section.

Lemma 4.38. *Given $\rho \in (1,2]$, if $\Phi \in L^1(0,\infty)$ is a nonnegative function that satisfies* $(\mathrm{SSPE})^{\mathrm{w}}_{\rho}$ *for all $\psi \in C^1_c([0,\infty))$, then for all $z > 0$ there holds*

$$\frac{\rho-1}{\rho}\, z \int_{(z,\infty)} \Phi(x)\mathrm{d}x - \frac{2-\rho}{\rho} \int_{(0,z)} x\Phi(x)\mathrm{d}x = \mathcal{I}[\Phi](z), \tag{4.69}$$

where

$$\mathcal{I}[\Phi](z) = \frac{1}{2} \iint_{\mathbb{R}^2_+} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \big[(x+y-z) \wedge (z-|x-y|)\big]_+ \mathrm{d}x\mathrm{d}y. \tag{4.70}$$

Moreover, if $\rho \in (1,2)$, then there holds

$$\frac{1}{\rho}\, R^{\rho-2} \int_{(0,\infty)} (x \wedge R)\Phi(x)\mathrm{d}x = \int_0^R \mathcal{I}[\Phi](z) z^{\rho-3}\mathrm{d}z \text{ for all } R > 0, \tag{4.71}$$

and the right hand side is nondecreasing and bounded as a function of R on $\mathbb{R}_+$.

Proof. For $z > 0$ arbitrarily fixed, by an approximation argument involving Lemmas 1.6 and A.6, we find that we can simply use $\psi(x) = (z-x)_+$ in $(\mathrm{SSPE})^{\mathrm{w}}_{\rho}$ to obtain (4.69).

Moreover, for $\rho \in (1,2)$, multiplying (4.69) by $z^{\rho-3}$ yields

$$\mathcal{I}[\Phi](z) z^{\rho-3} = \frac{1}{\rho}\left((\rho-1) z^{\rho-2} \int_{(z,\infty)} \Phi(x)\mathrm{d}x + (\rho-2) z^{\rho-3} \int_{(0,z)} x\Phi(x)\mathrm{d}x \right) \stackrel{!}{=} \tfrac{1}{\rho} f'(z),$$

where $f(z) = z^{\rho-2} \int_{(0,\infty)} (x \wedge z)\Phi(x)\mathrm{d}x$, which implies (4.71) by the fundamental theorem of calculus. Monotonicity of the right hand side is then immediate from nonnegativity of $\mathcal{I}[\Phi]$, while boundedness follows from $\mathcal{I}[\Phi] \le \frac{1}{2}\|\Phi\|^2_{L^1(0,\infty)}$, and integrability of $z^{3-\rho}$ at infinity. ☐

4.2.1 Fat tails for $\rho \in (1,2)$

The proof of pointwise power law behaviour at infinity of self-similar profiles for (QWTE) with $\rho \in (1,2)$ comprises two steps. We first make sure that any such profile has the right behaviour in an integrated sense.

Lemma 4.39. *Given $\rho \in (1,2)$, if $\Phi \in L^1(0,\infty)$ is a nonnegative function that satisfies* $(\mathrm{SSPE})^{\mathrm{w}}_{\rho}$ *for all $\psi \in C^1_c([0,\infty))$, then there hold*

$$\lim_{R\to\infty} \frac{R^{\rho-1}}{2-\rho} \int_{(R,\infty)} \Phi(x)\mathrm{d}x = \|\Phi\|_{\rho} \quad \textit{and} \quad \lim_{R\to\infty} \frac{R^{\rho-2}}{\rho-1} \int_{(0,R)} x\Phi(x)\mathrm{d}x = \|\Phi\|_{\rho}. \tag{4.72}$$

Proof. For arbitrary $R > 0$, then following Lemma 4.38, we find by (4.71) that

$$\|\Phi\|_{\rho} = R^{\rho-2} \int_{(0,\infty)} (x \wedge R)\Phi(x)\mathrm{d}x + \rho \int_R^{\infty} \mathcal{I}[\Phi](z) z^{\rho-3}\mathrm{d}z, \tag{4.73}$$

and by (4.69) that

$$\begin{aligned} &R^{\rho-2} \int_{(0,\infty)} (x \wedge R)\Phi(x)\mathrm{d}x - \frac{R^{\rho-1}}{2-\rho} \int_{(R,\infty)} \Phi(x)\mathrm{d}x \\ &= \frac{R^{\rho-2}}{2-\rho}\left((2-\rho) \int_{(0,\infty)} (x \wedge R)\Phi(x)\mathrm{d}x - R \int_{(R,\infty)} \Phi(x)\mathrm{d}x \right) = -\rho \times \frac{R^{\rho-2}\,\mathcal{I}[\Phi](R)}{2-\rho}. \end{aligned} \tag{4.74}$$

Thus, combining (4.73) and (4.74), we have

$$\frac{R^{\rho-1}}{2-\rho}\int_{(R,\infty)}\Phi(x)\mathrm{d}x - \|\Phi\|_\rho = \rho\left(\frac{R^{\rho-2}\mathcal{I}[\Phi](R)}{2-\rho} - \int_R^\infty \mathcal{I}[\Phi](z)z^{\rho-3}\mathrm{d}z\right), \tag{4.75}$$

and the first limit in (4.72) follows since the right hand side of (4.75) vanishes as $R \to \infty$. Arguing similarly we also obtain the second limit in (4.72). □

We then use this result to obtain the pointwise decay behaviour.

Proposition 4.40. *Given $\rho \in (1,2)$, if $\Phi \in L^1(0,\infty)$ is a nonnegative function that satisfies* (SSPE)$^{\mathrm{w}}_\rho$ *for all $\psi \in C_c^1([0,\infty))$, then there holds*

$$\Phi(z) \sim (2-\rho)(\rho-1)\|\Phi\|_\rho\, z^{-\rho} \text{ as } z \to \infty. \tag{4.76}$$

Proof. Note that $\|\Phi\|_\rho = 0$ implies $\Phi \equiv 0$, in which case (4.76) trivially holds, so that for the remaining part of the proof we may suppose that $\|\Phi\|_\rho > 0$. In that case Φ is continuous and strictly positive on $(0,\infty)$ (cf. Theorem 4.4), and for all $z > 0$ we have

$$\begin{aligned}\left|\frac{\Phi(z)}{(2-\rho)(\rho-1)\|\Phi\|_\rho\, z^{-\rho}} - 1\right| &\le \left|\frac{z\Phi(z)}{(\rho-1)\int_{(z,\infty)}\Phi(x)\mathrm{d}x} - 1\right| \\ &\quad + \frac{z\Phi(z)}{(\rho-1)\int_{(z,\infty)}\Phi(x)\mathrm{d}x}\frac{1}{\|\Phi\|_\rho}\left|\frac{z^{\rho-1}}{2-\rho}\int_{(z,\infty)}\Phi(x)\mathrm{d}x - \|\Phi\|_\rho\right|. \end{aligned} \tag{4.77}$$

With Lemma 4.39 it is then sufficient to show that the first term on the right hand side of (4.77) vanishes as $z \to \infty$. Moreover, recall that Φ satisfies (4.69) for all $z > 0$, which we may differentiate to obtain

$$(\rho-1)\int_{(z,\infty)}\Phi(x)\mathrm{d}x - z\Phi(z) = \frac{\rho}{2}\iint_{\mathbb{R}_+^2}\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\begin{bmatrix}\mathbf{1}_{\{|x-y|<z\}\cap\{(x\vee y)>z\}} \\ \quad - \mathbf{1}_{\{x+y>z\}\cap\{x,y<z\}}\end{bmatrix}\mathrm{d}x\mathrm{d}y. \tag{4.78}$$

If we now show the right hand side of (4.78) to be $o(z^{1-\rho})$ as $z \to \infty$, the result follows by again Lemma 4.39. To that end, we note that

$$\iint_{\substack{\{x+y>z\}\cap\\ \{|x-y|<z\}}}\frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\mathrm{d}x\mathrm{d}y \le \tfrac{4}{z}\|\Phi\|^2_{L^1(\frac14 z,\infty)} + 2\int_{\frac34 z}^{\frac54 z}\frac{\Phi(x)}{\sqrt{x}}\left(\int_{|z-x|}^{\frac12 z}\frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y\right)\mathrm{d}x, \tag{4.79}$$

where the first term on the right hand side is already $O(z^{2(1-\rho)-1})$. Further, let $\zeta \in (0, \frac14 z)$ be arbitrary, and let n be the largest integer strictly less than $\log_2(\frac{z}{\zeta})$, so that with Lemma 4.36 we find that

$$\int_\zeta^{\frac12 z}\frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y \le \sum_{j=1}^n\int_{2^{-j-1}z}^{2^{-j}z}\frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y \le \sum_{j=1}^n\frac{C(\rho,\Phi)\times\sqrt{2^{-j}z}}{\sqrt{2^{-j-1}z}} \le \tfrac{C(\rho,\Phi)\sqrt{2}}{\log(2)}\times\log(\tfrac{z}{\zeta}). \tag{4.80}$$

Using this estimate, and also Hölder's inequality, we can thus bound the second term on the right hand side of (4.79) from above by a constant times

$$\int_{\frac34 z}^{\frac54 z}\frac{\Phi(x)}{\sqrt{x}}\log\left|\tfrac{z}{z-x}\right|\mathrm{d}x \le \|\Phi\|_{L^2(\frac34 z,\infty)}\left(\int_{\frac34 z}^{\frac54 z}\tfrac{1}{x}\left(\log\left|\tfrac{z}{z-x}\right|\right)^2\mathrm{d}x\right)^{\frac12}, \tag{4.81}$$

and with (4.63) it follows that the right hand side of (4.81) is or order less than $O(z^{\frac12-\rho})$, whereby we conclude the claim. □

4.2.2 Exponential decay for $\rho = 2$

As in the previous subsection, the first step in the proof of the pointwise estimate on self-similar profiles for (QWTE) with $\rho = 2$, is to show the corresponding estimate on the mass of the tail. Based on an idea in [33], but with a simpler execution, we thereto inductively prove a qualitative estimate on the higher moments of a profile, from which we then deduce exponential decay of the tail-mass (cf. Corollary 4.42).

Lemma 4.41. *Given a nonnegative function $\Phi \in L^1(0,\infty)$ that satisfies* $(\mathrm{SSPE})_2^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$, then there exists a finite constant $A > 0$ such that*

$$\int_{(0,\infty)} x^\gamma \Phi(x)\mathrm{d}x \le \gamma^\gamma A^{\gamma+1} \text{ for all } \gamma > 0.$$

Proof. For $r > 0$ arbitrarily fixed, and with $m_\gamma = \int_{(0,r)} x^\gamma \Phi(x)\mathrm{d}x$ for $\gamma \ge 0$, we show that there exists a finite constant $A > 0$, independent of r, such that $m_\gamma \le \gamma^\gamma A^{\gamma+1}$ for all $\gamma > 0$. The result thereby follows.

Recalling Lemma 4.38, there holds [cf. (4.69)]

$$\int_{(r,\infty)} \Phi(x)\mathrm{d}x = \tfrac{2}{r}\,\mathcal{I}[\Phi](r), \tag{4.82}$$

with $\mathcal{I}$ given by (4.70), and, arguing as in that lemma, i.e. with Lemmas 1.6 and A.6, we use $\psi(x) = (r^\gamma - x^\gamma)_+$ with $\gamma > 1$ in $(\mathrm{SSPE})_2^{\mathrm{w}}$ to obtain

$$r^\gamma \int_{(r,\infty)} \Phi(x)\mathrm{d}x - (\gamma-1)\int_{(0,r)} x^\gamma \Phi(x)\mathrm{d}x = \iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta^2_{x\wedge y}\psi(x \vee y)\mathrm{d}x\mathrm{d}y. \tag{4.83}$$

We then observe that

$$\Delta^2_{x\wedge y}\psi(x\vee y) = \begin{cases} ((x+y)^\gamma - r^\gamma)_+ - \Delta^2_{x\wedge y}[(\cdot)^\gamma](x\vee y) & \text{if } x \le r \text{ and } y \le r, \\ (r^\gamma - |x-y|^\gamma)_+ & \text{otherwise,} \end{cases}$$

and, since $((x+y)^\gamma - r^\gamma)_+ = r^\gamma((\frac{x+y}{r})^\gamma - 1)_+ \ge r^\gamma(\frac{x+y}{r} - 1)_+ = r^{\gamma-1}(x+y-r)_+$, and as similarly $(r^\gamma - |x-y|^\gamma)_+ \ge r^{\gamma-1}(r - |x-y|)_+$, we get for $x, y > 0$ that

$$\Delta^2_{x\wedge y}\psi(x\vee y) \ge r^{\gamma-1}\big[(x+y-r)\wedge(r-|x-y|)\big]_+ - \mathbf{1}_{(0,r)^2} \times \Delta^2_{x\wedge y}[(\cdot)^\gamma](x\vee y). \tag{4.84}$$

Combining now (4.82), (4.83), and (4.84), and recalling (4.70), it thus follows that

$$\int_{(0,r)} x^\gamma\Phi(x)\mathrm{d}x \le \frac{2}{\gamma-1}\iint_{\{0<y<x<r\}} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\Delta_y^2[(\cdot)^\gamma](x)\mathrm{d}x\mathrm{d}y \text{ for all } \gamma > 1. \tag{4.85}$$

Moreover, since for $n \in \mathbb{N}$ and $x > y > 0$ we have

$$\Delta_y^2[(\cdot)^n](x) = \sum_{j=1}^n (1+(-1)^j) \times \binom{n}{j} x^{n-j}y^j \le 2\sum_{j=2}^n \binom{n}{j} x^{n-j}y^{j-1} \times \sqrt{xy},$$

we get with (4.85) that

$$m_n \le \frac{4}{n-1}\sum_{j=2}^n \binom{n}{j} m_{n-j}m_{j-1} \text{ for all } n \in \mathbb{N}\cap(1,\infty), \tag{4.86}$$

and in particular $m_2 \le 4m_0m_1$.

Now, using twice Hölder's inequality, once (4.86), and once the fact that $\gamma^\gamma > \frac{1}{4}$, it follows for $\gamma \in [0,2]$ that

$$m_\gamma \le m_0^{1-\frac{\gamma}{2}} m_2^{\frac{\gamma}{2}} = m_0^{1-\frac{\gamma}{2}} m_2^{\gamma} m_2^{-\frac{\gamma}{2}} \le m_0^{1-\frac{\gamma}{2}} (4m_0 m_1)^\gamma m_2^{-\frac{\gamma}{2}} = 4^\gamma m_0^{1+\frac{\gamma}{2}} m_1^\gamma m_2^{-\frac{\gamma}{2}}$$
$$\le 4^\gamma m_0^{1+\frac{\gamma}{2}} \left(m_0^{\frac{1}{2}} m_2^{\frac{1}{2}}\right)^\gamma m_2^{-\frac{\gamma}{2}} = \tfrac{1}{4}(4m_0)^{\gamma+1} \le \gamma^\gamma A^{\gamma+1},$$

for any $A \ge 4\|\Phi\|_{L^1(0,\infty)}$. Furthermore, supposing that $m_\gamma \le \gamma^\gamma A^{\gamma+1}$ for all $\gamma \in \mathbb{N} \cap [0,n)$, then (4.86) implies

$$m_n \le \left(\frac{4}{n-1} \sum_{j=2}^{n} \binom{n}{j} (n-j)^{n-j} (j-1)^{j-1} \right) A^{n+1},$$

and, since $\binom{n}{j}(n-j)^{n-j} \le \frac{1}{j^j} \sum_{\ell=0}^{n} \binom{n}{\ell} (n-j)^{n-\ell} j^\ell = \frac{n^n}{j^j}$, there thus holds

$$m_n \le \left(\frac{4}{n-1} \sum_{j=2}^{n} \frac{(j-1)^{j-1}}{j^j} \right) n^n A^{n+1} \overset{!}{\le} n^n A^{n+1},$$

where the second inequality holds, as the term between brackets is an average of terms smaller than 1. We thus conclude by induction that $m_\gamma \le \gamma^\gamma A^{\gamma+1}$ for all $\gamma \in [0,2] \cup \mathbb{N}$ and any $A \ge 4\|\Phi\|_{L^1(0,\infty)}$. Lastly, for arbitrary $\gamma > 2$, then denoting by n the smallest integer greater than or equal to γ, it follows by Hölder's inequality, and the above, that

$$m_\gamma \le m_0^{1-\frac{\gamma}{n}} m_n^{\frac{\gamma}{n}} \le A^{1-\frac{\gamma}{n}} (n^n A^{n+1})^{\frac{\gamma}{n}} = \gamma^\gamma (\tfrac{n}{\gamma})^\gamma A^{\gamma+1} \le \gamma^\gamma (\tfrac{3}{2} A)^{\gamma+1},$$

whereby there holds $m_\gamma \le \gamma^\gamma A^{\gamma+1}$ for all $\gamma > 0$ and any $A \ge 6\|\Phi\|_{L^1(0,\infty)}$. □

Exponential decay of the tail-mass is now an easy consequence.

Corollary 4.42. *Given a nonnegative function $\Phi \in L^1(0,\infty)$ that satisfies* (SSPE)$_2^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$, then for all $z > 0$ there holds $\|\Phi\|_{L^1(z,\infty)} \le Ae^{-\frac{z}{Ae}}$, with $A > 0$ as in Lemma 4.41.*

Proof. With $A > 0$ as obtained in Lemma 4.41, then for any $z > 0$ we have

$$\|\Phi\|_{L^1(z,\infty)} \le z^{-\gamma} \int_{(0,\infty)} x^\gamma \Phi(x) \mathrm{d}x \le A \exp\left(\gamma \log\left(\tfrac{\gamma A}{z}\right)\right) \text{ for all } \gamma > 0,$$

and setting $\gamma = \frac{z}{Ae} > 0$ in the right hand side yields the result. □

This result then enables us to obtain a pointwise exponential upper bound.

Proposition 4.43. *If $\Phi \in L^1(0,\infty)$ is a nonnegative function that satisfies* (SSPE)$_2^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$, then there exists a constant $a > 0$ such that $\|e^{az}\Phi(z)\|_{L^\infty(1,\infty)} < \infty$.*

Proof. Recalling that Φ is smooth on $(0,\infty)$, then, as in the proof of Proposition 4.40, we find that Φ satisfies (4.78) with $\rho = 2$ for all $z > 0$. Moreover, estimating the double integral over $\{|x-y| < z\} \cap \{(x \vee y) > z\}$ by zero, and using the symmetry of the integrand, we now have

$$z\Phi(z) \le \int_{(z,\infty)} \Phi(x)\mathrm{d}x + 2 \iint_{\{x+y>z\}\cap\{0<y<x<z\}} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \mathrm{d}x\mathrm{d}y \text{ for all } z > 0. \tag{4.87}$$

For $z > 1$ we then bound the double integral on the right hand side of (4.87) by

$$\iint_{\{x>\frac{z}{2},y>\frac{1}{4}\}} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}}\mathrm{d}x\mathrm{d}y + \int_{z-\frac{1}{4}}^{z} \frac{\Phi(x)}{\sqrt{x}}\left(\int_{z-x}^{\frac{1}{2}} \frac{\Phi(y)}{\sqrt{y}}\mathrm{d}y\right)\mathrm{d}x$$
$$\leq \tfrac{2\sqrt{2}}{\sqrt{z}}\|\Phi\|_{L^1(0,\infty)} \times \|\Phi\|_{L^1(\frac{1}{2}z,\infty)} + \tfrac{2C(\Phi)}{\sqrt{3z}} \times \int_{z-\frac{1}{4}}^{z} \Phi(x)\big|\log(z-x)\big|\mathrm{d}x,$$

where the inequality in the second term follows by a similar estimate as (4.80), and we compute for any $a > 0$ that

$$e^{az}\int_{z-\frac{1}{4}}^{z} \Phi(x)\big|\log(z-x)\big|\mathrm{d}x \leq \int_0^{\frac{1}{4}} e^{ax}|\log(x)|\mathrm{d}x \times \|e^{ay}\Phi(y)\|_{L^\infty(z-\frac{1}{4},z)}.$$

Thus, fixing $A > 0$ as obtained in Lemma 4.41, and setting $a = \frac{1}{2}\frac{1}{Ae} > 0$, it follows from using these estimates in (4.87), and from using Corollary 4.42, that we can find some finite constant $Z = Z(a, A) > 1$ such that

$$z\Phi(z) \leq Ae^{-2az} + Ae^{-az} + e^{-az}\|e^{ay}\Phi(y)\|_{L^\infty(z-\frac{1}{4},z)}$$
$$\leq e^{-az}\Big(2A + \|e^{ay}\Phi(y)\|_{L^\infty(1,z)}\Big) \text{ for all } z > Z.$$

For $z > Z$ this then implies

$$\|e^{az}\Phi(z)\|_{L^\infty(1,z)} \leq \|e^{az}\Phi(z)\|_{L^\infty(1,Z)} + \|e^{az}\Phi(z)\|_{L^\infty(Z,z)}$$
$$\leq \|e^{az}\Phi(z)\|_{L^\infty(1,Z)} + \tfrac{1}{Z}\Big(2A + \|e^{az}\Phi(z)\|_{L^\infty(1,z)}\Big),$$

and, by a basic rearrangement of terms, there thus holds

$$\|e^{az}\Phi(z)\|_{L^\infty(1,z)} \leq (Z-1)^{-1}\Big(Z\,\|e^{az}\Phi(z)\|_{L^\infty(1,Z)} + 2A\Big) \text{ for all } z > Z,$$

from which we conclude the claim. □

A natural continuation would now be to prove a pointwise exponential lower bound, as done in [33] for self-similar profiles of solutions to Smoluchowski's coagulation equation with certain kernels of homogeneity zero. There, it was first shown that the amount of mass in the intervals $(R, R+1)$ is exponentially bounded from below, form which the pointwise estimate followed almost immediately. With significantly more involved methods, we have been able to obtain the following

Lemma 4.44 (cf. Section 6.2 in [KV16]). *If $\Phi \in L^1(0,\infty)$ is a nontrivial and nonnegative function that satisfies* (SSPE)$_2^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$, then there exists a constant $B > 0$ such that*

$$\inf_{R>0}\left\{e^{BR}\int_{(R,R+1)}\Phi(x)\mathrm{d}x\right\} > 0.$$

However, because the right hand side of (4.78) does not have a sign, the step from the averaged to the pointwise result is not as straightforward as in [33]. Moreover, this step has not been obtained so far, for which reason we have chosen to omit the proof of Lemma 4.44 from this work. Instead, we will focus in the Section 4.2.3 on what one might expect from the asymptotic behaviour of self-similar profiles of solutions to (QWTE) with finite energy, and on what might be done to achieve that result.

4.2.3 Two conjectures, backed with consistency analysis and numerics

Concerning the asymptotic behaviour of self-similar profiles of solutions to (QWTE) with finite energy, we pose the following

Conjecture 4.45. *Given a positive classical solution $\Phi \in C^\infty((0,\infty)) \cap L^1(0,\infty)$ to* (SSPE)$_2$*, then there exists a constant $a > 0$ such that*

$$\Phi(z) \sim \tfrac{8}{\pi} a z e^{-az} \text{ as } z \to \infty. \tag{4.88}$$

Let us try to motivate this claim. In view of the similarity between the results in Section 4.2.2, and the first part of [33], it is not unreasonable to expect other results from [33] to carry over as well. In particular, we might expect the limit $\lim_{z\to\infty} -\frac{1}{z}\log(\Phi(z))$ to exist in $\mathbb{R}_+$, and, denoting it by $a > 0$, we could then suppose that $\Phi(z) \sim C z^\alpha e^{-az}$ as $z \to \infty$, with $C > 0$ and $\alpha \in \mathbb{R}$ to be determined. Substituting this asymptotic behaviour into the left hand side of (SSPE)$_2$, we now find

$$-\tfrac{1}{2} x \Phi'(x) - \Phi(x) \sim \tfrac{1}{2} a C \times x^{\alpha+1} e^{-ax} \text{ as } x \to \infty.$$

Moreover, for $c \gg 1$ fixed sufficiently large, we find that

$$\begin{aligned} \int_c^{x/2} \frac{\Phi(y)\Phi(x-y)}{\sqrt{y(x-y)}} \mathrm{d}y &\sim C^2 \int_c^{x/2} (y(x-y))^{\alpha-\frac{1}{2}} \mathrm{d}y \times e^{-ax} \\ &\sim \tfrac{1}{2} C^2 \int_0^1 (y(1-y))^{\alpha-\frac{1}{2}} \mathrm{d}y \times x^{2\alpha} e^{-ax} \text{ as } x \to \infty, \end{aligned} \tag{4.89}$$

where the integral on the right hand side converges for $\alpha > -\frac{1}{2}$, and it is clear that if this is the leading order term on the right hand side of (SSPE)$_2$, then the conjecture is consistent (using $\int_0^1 \sqrt{y(1-y)}\mathrm{d}y = \frac{\pi}{8}$). To that end we first note that

$$\int_c^x \frac{\Phi(y)\Phi(x)}{\sqrt{yx}} \mathrm{d}y \sim C^2 \int_c^\infty y^{\alpha-\frac{1}{2}} e^{-ay} \mathrm{d}y \times x^{\alpha-\frac{1}{2}} e^{-ax} \text{ as } x \to \infty,$$

and

$$\int_c^\infty \frac{\Phi(y)\Phi(x+y)}{\sqrt{y(x+y)}} \mathrm{d}y \sim C^2 \times \omega(x) e^{-ax} \text{ as } x \to \infty,$$
$$\text{where } \omega(x) = \int_c^\infty (y(x+y))^{\alpha-\frac{1}{2}} e^{-2ay} \mathrm{d}y = O(x^{\alpha-\frac{1}{2}}),$$

which are both asymptotically negligible compared to (4.89). Moreover, we find that

$$\begin{aligned} &\left(x^{\alpha-\frac{1}{2}} e^{-ax}\right)^{-1} \left|(x+y)^{\alpha-\frac{1}{2}} e^{-a(x+y)} + (x-y)^{\alpha-\frac{1}{2}} e^{-a(x-y)} - 2x^{\alpha-\frac{1}{2}} e^{-ax}\right| \\ &\quad = \left|(1+\tfrac{y}{x})^{\alpha-\frac{1}{2}} e^{-ay} + (1-\tfrac{y}{x})^{\alpha-\frac{1}{2}} e^{ay} - 2\right| \le 4\sinh^2\left(\tfrac{ay}{2}\right) + O(\tfrac{y}{x}) \text{ as } \tfrac{y}{x} \to 0, \end{aligned}$$

whereby it follows that

$$\int_0^c \frac{\Phi(y)}{\sqrt{y}} \left[\frac{\Phi(x+y)}{\sqrt{x+y}} + \frac{\Phi(x-y)}{\sqrt{x-y}} - 2\frac{\Phi(x)}{\sqrt{x}}\right] \mathrm{d}y \sim 4C \int_0^c \frac{\Phi(y)}{\sqrt{y}} \sinh^2(\tfrac{ay}{2}) \mathrm{d}y \times x^{\alpha-\frac{1}{2}} e^{-ax},$$

as $x \to \infty$, and whereby we conclude consistency of Conjecture 4.45.

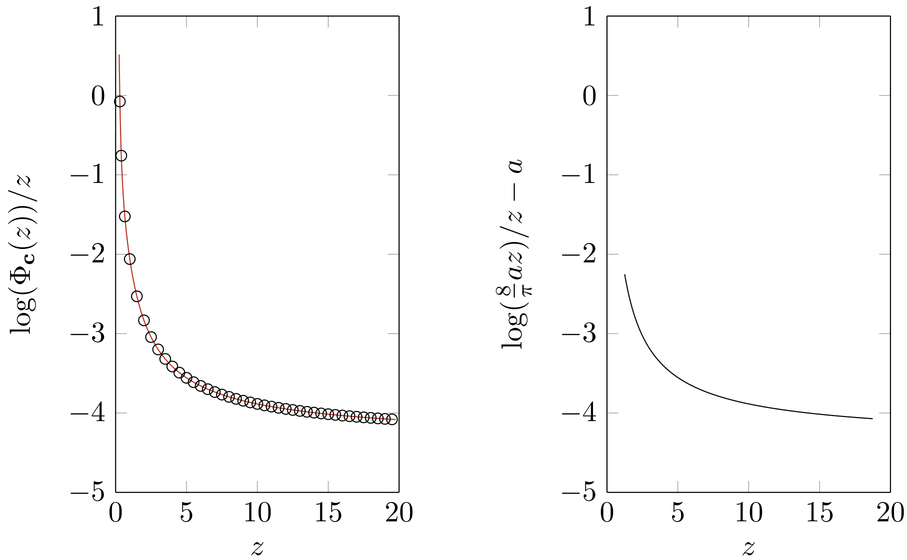

FIGURE 4.1: In the left picture we see the numerical approximation $\Phi_{\mathbf{c}}$ of a solution to $(\mathrm{SSPE})_2$. The graph on the right shows the conjectured asymptotic behaviour, with a determined by a least squares fit on the data points z in $(8, 16]$.

Now, to get from the averaged estimate in Lemma 4.44 to a pointwise lower bound on self-similar profiles of solutions to (QWTE) with finite energy, it would be useful to proceed via the dual problem. Indeed, it is fairly straightforward to see that if φ satisfies

$$\varphi_s(s,x) \leq \frac{2}{\sqrt{x}} \int_0^x \frac{\Phi(y)}{\sqrt{y}} \Delta_y^2[\varphi(s,\cdot)](x)\mathrm{d}y - \big(x\varphi_x(s,x) - \varphi(s,x) + \varphi(s,0)\big), \tag{4.90}$$

for almost all $s \in (0,T)$ and $x \geq 0$, then there holds

$$\int_{\mathbb{R}_+} \varphi(t,x)\Phi(x)\mathrm{d}x \leq \int_{\mathbb{R}_+} \varphi(0,x)\Phi(x)\mathrm{d}x \text{ for all } t \in [0,T], \tag{4.91}$$

with equality in (4.91) only if (4.90) is an equality for almost all $s \in (0,t)$ and $x \geq 0$, and if we were to have such a function with $\varphi(0,x) = \delta_z(x)$, then we would have

$$\Phi(z) \geq \int_{\mathbb{R}_+} \varphi(t,x)\Phi(x)\mathrm{d}x \text{ for all } t \in [0,T].$$

Thus, considering that

$$\frac{1}{t}\int_0^t \int_{\mathbb{R}_+} e^s \delta_z(e^{-s}x)\Phi(x)\mathrm{d}x\,\mathrm{d}s = \frac{1}{t}\int_0^t e^s\Phi(e^s z)\mathrm{d}s = \frac{1}{zt}\int_z^{e^t z} \Phi(x)\mathrm{d}x \overset{t\sim\frac{1}{z}}{\gtrsim} \int_z^{z+1} \Phi(x)\mathrm{d}x,$$

it seems that we require more information on the effect of the integral operator on the right hand side of (4.90). In particular, we would like to know whether a Dirac distribution is instantaneously dissolved or not. However, for this we need to know more about the behaviour of Φ near the origin. Presently, it is only known that

$$\sup_{R>0} \left\{ \frac{1}{\sqrt{R}} \int_{(0,R)} \Phi(x)\mathrm{d}x \right\} < \infty,$$

but it is reasonable to expect solutions to be well-behaved near zero. We therefore have

Conjecture 4.46. *Given $\rho \in (1,2]$, let $\Phi \in L^1(0,\infty)$ be a nonnegative function that satisfies* $(\mathrm{SSPE})_\rho^{\mathrm{w}}$ *for all $\psi \in C_c^1([0,\infty))$. Then there holds*

$$\Phi(z) \sim \frac{A}{\sqrt{z}} \text{ as } z \to 0, \quad \text{with } A = \sqrt{\tfrac{6}{\pi^2}\tfrac{2}{\rho}(\rho-1)\|\Phi\|_{L^1(0,\infty)}}. \tag{4.92}$$

It is clear that this result holds if and only if

$$\lim_{\lambda \to 0^+} f_\lambda(x) = \frac{A}{x} \text{ for all } x > 0, \quad \text{where } f_\lambda(x) = \lambda \frac{\Phi(\lambda x)}{\sqrt{\lambda x}},$$

which is still an open problem. However, using $(\mathrm{SSPE})_\rho^{\mathrm{w}}$, we note for $\psi \in C_c^1([0,\infty))$ that

$$\iint_{\mathbb{R}_+^2} f_\lambda(x) f_\lambda(y) \Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta^2_{x\wedge y}\left[\psi(\tfrac{\cdot}{\lambda})\right](x\vee y)\mathrm{d}x\mathrm{d}y$$
$$= \frac{2}{\rho}\int_{(0,\infty)} \left(\tfrac{x}{\lambda}\psi'(\tfrac{x}{\lambda}) - (\rho-1)(\psi(\tfrac{x}{\lambda}) - \psi(0))\right)\Phi(x)\mathrm{d}x \to \tfrac{2}{\rho}(\rho-1)\|\Phi\|_{L^1(0,\infty)} \times \psi(0),$$

as $\lambda \to 0^+$. In view of Lemma 4.47 below, we thus have

$$\lim_{\lambda\to 0^+} \iint_{\mathbb{R}_+^2} f_\lambda(x) f_\lambda(y) \Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}_+^2} \frac{A}{x}\frac{A}{y} \Delta^2_{x\wedge y}\psi(x\vee y)\mathrm{d}x\mathrm{d}y,$$

but so far we have not been unable to deduce pointwise convergence.

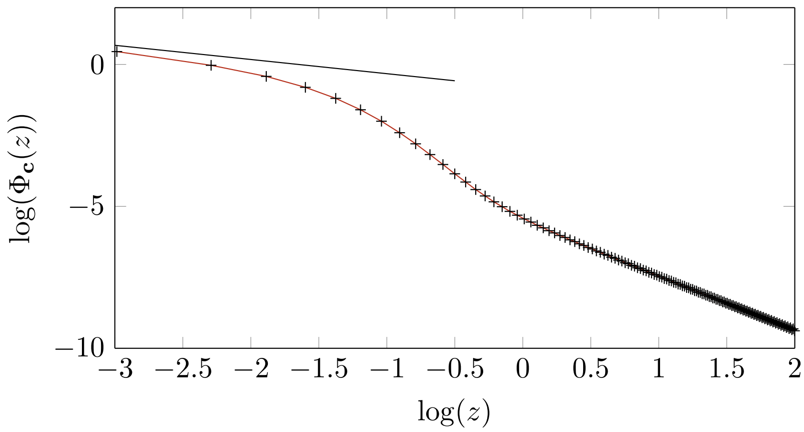

FIGURE 4.2: The numerical approximation $\Phi_{\mathbf{c}}$ of a solution to $(\mathrm{SSPE})_\rho$ with $\rho = 1.9$. The solid line shows the conjectured asymptotic behaviour near the origin. Moreover, the slope of the computed solution approximates $-\rho$ for $\log(z) \gtrsim 0$, indicating agreement between numerics and the theoretical decay (cf. Proposition 4.40).

Note that both Conjectures 4.45 and 4.46 are consistent with the scaling properties of solutions to $(\mathrm{SSPE})_\rho$ (cf. Lemma 4.3). Indeed, if Φ is a solution to $(\mathrm{SSPE})_\rho$, then so are the functions in $\{\Phi_\lambda\}_{\lambda>0}$, with $\Phi_\lambda(x) = \Phi(\lambda x)$, for which $\|\Phi_\lambda\|_{L^1(0,\infty)} = \frac{1}{\lambda}\|\Phi\|_{L^1(0,\infty)}$. It is now easily checked that if Φ satisfies (4.92), then Φ_λ satisfies (4.92) with Φ replaced by Φ_λ. Moreover, if Φ satisfies (4.88), then

$$\Phi_\lambda(z) = \Phi(\lambda z) \sim \tfrac{8}{\pi} a\lambda z e^{-a\lambda z} =: \tfrac{8}{\pi} a_\lambda z e^{-a_\lambda z} \text{ as } z \to \infty,$$

suggesting an inverse proportionality between $a := \lim_{z\to\infty} -\frac{1}{z}\log(\Phi(z))$ and $\|\Phi\|_{L^1(0,\infty)}$, which is actually visible in our numerical treatment of $(\mathrm{SSPE})_2$ (cf. Figure 4.3).

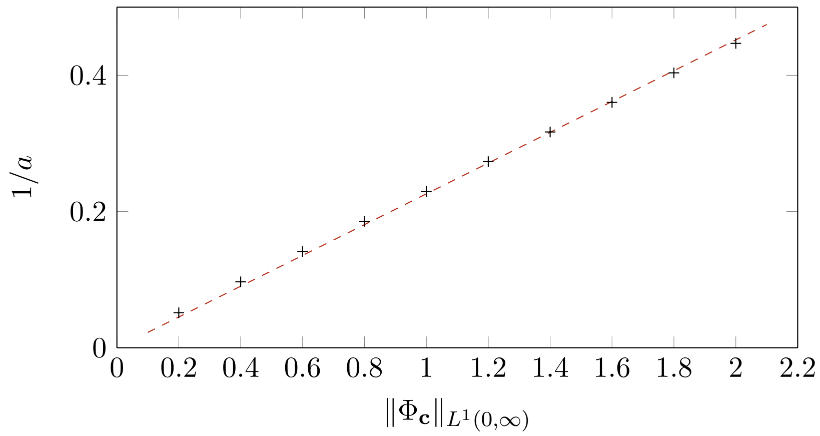

FIGURE 4.3: For numerical approximations $\Phi_{\mathbf{c}}$ of solutions to $(\text{SSPE})_2$, we have determined a from a least squares fit to the conjectured tail behaviour. The observed inversely proportional relation between a and $\|\Phi_{\mathbf{c}}\|_{L^1(0,\infty)}$ is consistent with the scaling property of solutions to $(\text{SSPE})_2$.

On the numerical approximation of solutions to $(\text{SSPE})_\rho$

We have implemented a numerical scheme to obtain approximate solutions to $(\text{SSPE})_\rho$. The decay of exact solutions to $(\text{SSPE})_\rho$ (cf. Propositions 4.40 and 4.43) justifies an approximation with compact support in $[0,\infty)$, and the weak formulation $(\text{SSPE})_\rho^{\text{w}}$ provides the natural starting point for a finite element approach with base functions $\psi_n(x) = (x_n - x)_+$, $(x_n) \in \mathbb{R}_+^N$ (cf. Figure 3.1). Indeed, using Lemma A.6, it can be checked that solutions Φ to $(\text{SSPE})_\rho$ satisfy $F_n[\Phi] = 0$ with

$$F_n[\Phi] = \frac{1}{2} \iint_{\mathbb{R}_+^2} \frac{\Phi(x)\Phi(y)}{\sqrt{xy}} \Delta^2_{x\wedge y} \psi_n(x \vee y) \mathrm{d}x\mathrm{d}y \\ - \frac{1}{\rho} \int_{(0,\infty)} \big(x\psi_n'(x) - (\rho-1)(\psi_n(x) - \psi_n(0))\big)\Phi(x)\mathrm{d}x.$$

Introducing then the approximation $\Phi_{\mathbf{c}}(x) = \Phi[\mathbf{c}](x) = \sum_{n=1}^N c_n \psi_n(x)$, it makes sense to consider the nonlinear mapping $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$ that is given by

$$\mathbf{F}[\mathbf{c}] = \mathbf{A}[\mathbf{c}] - \mathbf{B}\mathbf{c}$$

where

$$\mathbf{B}_{ij} = \begin{cases} \frac{\rho-1}{\rho}\frac{1}{2}x_i(x_j - x_i)^2 - \frac{2-\rho}{\rho}\frac{1}{6}x_i^3\big(3\frac{x_j}{x_i} - 2\big) & \text{if } x_j > x_i, \\ -\frac{2-\rho}{\rho}\frac{1}{6}x_j^3 & \text{else,} \end{cases}$$

and where $A_n[\mathbf{c}] = \mathbf{c}^T \mathbf{A}^n \mathbf{c}$, with $\mathbf{A}^n = (a_{ij}^n) \in \mathbb{R}_{\text{sym}}^{N\times N}$ given by

$$a_{ij}^n = \frac{1}{2} \iint_{\mathbb{R}_+^2} (xy)^{-1/2}(x_i - x)_+(x_j - y)_+\big[(x + y - x_n) \wedge (x_n - |x - y|)\big]_+ \mathrm{d}x\mathrm{d}y.$$

Adding lastly a Lagrange multiplier to prescribe the integral of the approximation $\Phi_{\mathbf{c}}$, this yields an $(N+1)$-dimensional system of equations that can be solved by Newton's method.

4.3 Solutions with infinite mass

Consider the following

Lemma 4.47. *Given a function $\psi \in C_c([0,\infty)) \cap W^{1,\infty}(0,\infty)$ for which ψ' is continuous in a neighbourhood of 0, then there holds*

$$\iint_{\mathbb{R}_+^2} \frac{1}{xy} \Delta^2_{x\wedge y}\psi(x \vee y)\mathrm{d}x\mathrm{d}y = \tfrac{\pi^2}{6}\,\psi(0). \tag{4.93}$$

Proof. Note first of all that the left hand side of (4.93) is well-defined for any ψ as in the statement of the lemma. Indeed, integrability near the axes $\{x = 0\}$ and $\{y = 0\}$ follows with Lemma 1.6, and integrability towards infinity follows by the fact that the second difference is supported in a strip of fixed width along the diagonal $\{x = y\}$. Moreover, by the same observations, it follows from dominated convergence that

$$\iint_{\mathbb{R}_+^2} \frac{1}{xy} \Delta^2_{x\wedge y}\psi(x \vee y)\mathrm{d}x\mathrm{d}y = \lim_{\varepsilon\to 0^+} \iint_{\mathbb{R}_+^2} \frac{\Delta^2_{x\wedge y}\psi(x \vee y)}{(x+\varepsilon)(y+\varepsilon)}\mathrm{d}x\mathrm{d}y.$$

Using further Fubini, we find that

$$\begin{aligned}\iint_{\mathbb{R}_+^2} \frac{\Delta^2_{x\wedge y}\psi(x \vee y)}{(x+\varepsilon)(y+\varepsilon)}\mathrm{d}x\mathrm{d}y &= 2\int_{\mathbb{R}_+}\int_0^{y/2} \frac{\mathrm{d}y}{(x-y+\varepsilon)(y+\varepsilon)}\;\psi(x)\mathrm{d}x \\ &+2\int_{\mathbb{R}_+}\int_0^{\infty} \frac{\mathrm{d}y}{(x+y+\varepsilon)(y+\varepsilon)}\;\psi(x)\mathrm{d}x - 4\int_{\mathbb{R}_+}\int_0^{x} \frac{\mathrm{d}y}{(x+\varepsilon)(y+\varepsilon)}\;\psi(x)\mathrm{d}x,\end{aligned}$$

and evaluating the integrals with respect to y we get

$$\iint_{\mathbb{R}_+^2} \frac{\Delta^2_{x\wedge y}\psi(x \vee y)}{(x+\varepsilon)(y+\varepsilon)}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}_+} \tfrac{1}{\varepsilon}\phi(\tfrac{x}{\varepsilon})\;\psi(x)\mathrm{d}x, \quad \text{with } \phi(\xi) = \frac{4\log(\xi+1)}{\xi(\xi+1)(\xi+2)}.$$

The result then follows by the fact that $\|\phi\|_{L^1(0,\infty)} = \frac{\pi^2}{6}$. □

This result implies that the function $G \in C([0,\infty) : \mathcal{M}_+([0,\infty)))$, given by

$$G(t,x)\mathrm{d}x = \tfrac{\pi^2}{12}\,t \times \delta_0(x)\mathrm{d}x + \frac{\mathrm{d}x}{\sqrt{x}}, \tag{4.94}$$

satisfies (QWTE)$^{\mathrm{w}}$ for all $t \geq 0$ and $\varphi \in C([0,\infty) : C_c^1([0,\infty))) \cap C^1([0,\infty) : C_c([0,\infty)))$ (cf. Remark 3.14 in [KV15]). Moreover, its restriction to $x \in \mathbb{R}_+$ is time-independent, in agreement with the observation in Section 1.2.1. However, since G has infinite mass, it is not a weak solution to (QWTE) in the sense of Definition 1.7.

Our restriction in Definition 1.7 to functions with finite mass makes sense in view of the formal derivation in Section 1.2, since there the cubic term was eliminated under the assumption that the mass outside of the condensate was negligible compared to the (finite) mass in the condensate. For the same reason, we cannot reasonably expect the long time asymptotic behaviour of solutions to (CWTE) with infinite mass to be described well by functions that satisfy (QWTE)$^{\mathrm{w}}$. In order to see what we should expect, let us now suppose that there exists a solution $g \in C([0,\infty) : \mathcal{M}_+([0,\infty)))$ to (CWTE) for which we can write

$$g(t,x)\mathrm{d}x = m(t)\delta_0(x)\mathrm{d}x + \frac{1}{\lambda(t)}\Phi\left(\frac{x}{\lambda(t)^\alpha}\right)\mathrm{d}x, \tag{4.95}$$

where $m \geq 0$ and $\lambda > 0$ are increasing, with $\lim_{t\to\infty}\lambda(t) = \infty$, where $\Phi \in L^1_{\mathrm{loc}}([0,\infty))$ is nonnegative, and where $\alpha > 0$. Using our computations from Section 1.2, we then find from the weak formulation [cf. (1.14)] that for every $\psi \in C_c^2([0,\infty))$ there holds

$$m'(t)\psi(0) + \lambda'(t)\lambda(t)^{\alpha-2}\int_{\mathbb{R}_+} \big(\alpha x\psi'(x) - (1-\alpha)\psi(x)\big)\Phi(x)\mathrm{d}x \\ = m(t)\lambda(t)^{\alpha-2}\mathcal{C}_3[\Phi](\psi) + \lambda(t)^{2\alpha-3}\mathcal{C}_4[\Phi](\psi), \quad (4.96)$$

with $\mathcal{C}_3$ and $\mathcal{C}_4$ given by (1.18) and (1.15) respectively. For functions g with finite mass, we then further have

$$m'(t) = -\partial_t\left[\int_{\mathbb{R}_+} g(t,x)\mathrm{d}x\right] = (1-\alpha)\lambda'(t)\lambda(t)^{\alpha-2}\int_{\mathbb{R}_+}\Phi(x)\mathrm{d}x, \quad (4.97)$$

whereby (4.96) simplifies to

$$\lambda'(t)\int_{\mathbb{R}_+}\big(\alpha x\psi'(x) - (1-\alpha)(\psi(x)-\psi(0))\big)\Phi(x)\mathrm{d}x \\ = m(t)\mathcal{C}_3[\Phi](\psi) + \lambda(t)^{\alpha-1}\mathcal{C}_4[\Phi](\psi). \quad (4.98)$$

Moreover, as we then assume m to be increasing towards a finite limit, we require $\alpha < 1$ [cf. (4.97)], from which it follows that the second term on the right hand side of (4.98) is of lower order as $t \to \infty$. We therefore conclude that it is reasonable to expect self-similar solutions to (QWTE) to be good approximations for long times of weak solutions to (CWTE) with finite mass.

Conversely, if g has infinite mass, then there is no equivalent expression to (4.97) that is readily available. Let us therefore suppose that $m(t) \simeq t^\gamma$ and $\lambda(t) \simeq t^\beta$ with $\beta,\gamma > 0$, where for given functions $f, g \in C(\mathbb{R}_+)$ we write $f(t) \simeq g(t)$ if $f(t) = O(g(t))$ as $t \to \infty$. The time-dependences of the four terms in (4.96) are then as follows:

$$m'(t) \simeq t^{\gamma-1} \qquad m(t)\lambda(t)^{\alpha-2} \simeq t^{\gamma+\beta(\alpha-2)}$$
$$\lambda'(t)\lambda(t)^{\alpha-2} \simeq t^{\beta(\alpha-1)-1} \qquad \lambda(t)^{2\alpha-3} \simeq t^{\beta(\alpha-1)+\beta(\alpha-2)}$$

Supposing first that all terms are of equal order, then we find that $\beta = \frac{1}{2-\alpha}$ and $\gamma = \frac{\alpha-1}{2-\alpha}$, which additionally yields the requirement that $\alpha \in (1,2)$. (Note that this is precisely what you get by solving the system $m'(t) = \lambda'(t)\lambda(t)^{\alpha-2} = \lambda(t)^{2\alpha-3} = m(t)\lambda(t)^{\alpha-2}$.) Using this in (4.95) would thus give rise to a family of solutions to (CWTE) with

$$g(t,x)\mathrm{d}x \simeq t^{\frac{\alpha-1}{2-\alpha}}\delta_0(x)\mathrm{d}x + \frac{1}{t^{\frac{1}{2-\alpha}}}\Phi\left(\frac{x}{t^{\frac{\alpha}{2-\alpha}}}\right)\mathrm{d}x,$$

where $\Phi \in L^1_{\mathrm{loc}}([0,\infty))$ is a distributional solution to

$$A_0\delta_0 = A_1(\alpha x\Phi' + \Phi) + A_2\mathcal{C}_3[\Phi] + A_3\mathcal{C}_4[\Phi] \text{ with } A_i > 0, \text{ and } A_0 \times A_3 = A_1 \times A_2.$$

Moreover, we would expect the self-similar profile to satisfy $\Phi(z) \sim C(\alpha)z^{-\frac{1}{\alpha}}$ as $z \to \infty$, in agreement with the results of formal computations in [9] and [KV16]. However, it is by no means obvious that the mass of the condensate should grow as the power law $t^{\frac{\alpha-1}{2-\alpha}}$, as was noted in [KV16].

Appendix

The purpose of this appendix is twofold. In Sections A.1 we have collected six lemmas, one of which was used in the formal introduction. The remaining five results have multiple applications in this thesis, and are presented here together.

Section A.2 contains the proofs of several lemmas from Chapters 1 and 4, which were postponed in order to not break the flow of the main text.

A.1 Six useful lemmas

The following lemma in this section is a reformulation result, used in the formal derivations of the weak turbulence equation for (NLS) in Section 1.1, and of the quadratic weak turbulence equation (QWTE) in Section 1.2. Due to the formal nature of the derivation, we thought it improper to include this lemma in the main text.

Lemma A.1. *Given $\varphi \in C_c^2([0,\infty))$, then for $\omega_1, \omega_2, \omega_3 \geq 0$ with $\omega_1 + \omega_2 - \omega_3 \geq 0$ there holds*

$$\begin{aligned}
&\varphi(\omega_3) + \varphi(\omega_1 + \omega_2 - \omega_3) - \varphi(\omega_1) - \varphi(\omega_2) \\
&\quad = (\omega_3 - \omega_1)(\omega_3 - \omega_2) \int_0^1 \int_0^1 \varphi''(\omega_1 + \omega_2 - \omega_3 + s_1(\omega_3 - \omega_1) + s_2(\omega_3 - \omega_2)) \mathrm{d}s_1 \mathrm{d}s_2,
\end{aligned} \tag{A.1}$$

and the mapping

$$F : (\omega_1, \omega_2, \omega_3) \mapsto \frac{K(\omega_1, \omega_2, \omega_3)}{\sqrt{\omega_1 \omega_2 \omega_3}} \big(\varphi(\omega_3) + \varphi(\omega_1 + \omega_2 - \omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\big),$$

with $K(\omega_1, \omega_2, \omega_3) = \min\{\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{(\omega_1 + \omega_2 - \omega_3)_+}\}$ extends to $F \in C_0([0,\infty)^3)$.

Proof. To check (A.1) is an easy exercise with the fundamental theorem of calculus, while the continuous extension of F to $[0,\infty)^3$ follows with the observations that

$$\begin{aligned}
F(\omega_1, \omega_2, \omega_3) &\sim \frac{1}{\sqrt{\omega_1 \omega_2}} \big(\varphi(\omega_3) + \varphi(\omega_1 + \omega_2 - \omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\big) \\
&\sim \sqrt{\omega_1 \omega_2} \int_0^1 \int_0^1 \varphi''(\omega_1 + \omega_2 - s_1 \omega_1 - s_2 \omega_2) \mathrm{d}s_1 \mathrm{d}s_2 \text{ as } \omega_3 \to 0,
\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
F(\omega_1, \omega_2, \omega_3) &\sim \frac{\mathbf{1}_{\{\omega_i > \omega_3\}}}{\sqrt{\omega_i \omega_3}} \big(\varphi(\omega_3) + \varphi(\omega_1 + \omega_2 - \omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\big) \\
&\sim -\sqrt{\tfrac{\omega_3}{\omega_i}} (\omega_i - \omega_3)_+ \int_0^1 \int_0^1 \varphi''(\omega_i - \omega_3 + s_i(\omega_3 - \omega_i) + s_j \omega_3) \mathrm{d}s_i \mathrm{d}s_j \text{ as } \omega_j \to 0,
\end{aligned} \tag{A.3}$$

with $i, j \in \{1, 2\}$ and $i \neq j$. □

Three lemmas on the second difference

Lemma A.2. *Given a convex function $\varphi \in C([0,\infty])$, then for $x \geq y \geq 0$ there holds $\Delta_y^2\varphi(x) \geq 0$.*

Proof. Immediate from the definition of convexity. □

Lemma A.3. *Given $\varphi \in C^1([0,\infty])$, then for $x \geq y \geq 0$ there holds*

$$\big|\Delta_y^2\varphi(x)\big| \leq \min\big\{4\|\varphi\|_{L^\infty(0,\infty)}, 2y\|\varphi\|_{W^{1,\infty}(0,\infty)}\big\}.$$

Proof. This is immediate from the observation that

$$\Delta_y^2\varphi(x) = \int_x^{x+y} \varphi'(z)\mathrm{d}z - \int_{x-y}^{x} \varphi'(z)\mathrm{d}z, \tag{A.4}$$

which holds by the fundamental theorem of calculus. □

Lemma A.4. *Given an odd function $\varphi \in C^2(\mathbb{R})$ that is concave on $\mathbb{R}_+$, then for $x, z \geq 0$ there holds*

$$\partial_z\big[\Delta_z^2\varphi(x)\big] = \int_{x-z}^{x+z} \varphi''(\xi)\mathrm{d}\xi \leq 0.$$

Proof. The identity follows by applying the fundamental theorem of calculus to the right hand side of $\partial_z[\Delta_z^2\varphi(x)] = \varphi'(x+z) - \varphi'(x-z)$. For the inequality we note that the second derivative of an odd function is also odd, so that $\int_{x-z}^{x+z} \varphi''(\xi)\mathrm{d}\xi = \int_{|x-z|}^{x+z} \varphi''(\xi)\mathrm{d}\xi \leq 0$. □

Two convergence results

Lemma A.5. *Let $F \in C_0(\mathbb{R}_+^2)$ be symmetric, and for $\varepsilon > 0$ let*

$$F_\varepsilon(x,y) = \int_0^x \phi_\varepsilon(y-z)F(x,z)\mathrm{d}z + \int_0^y \phi_\varepsilon(x-z)F(z,y)\mathrm{d}z \text{ for } x, y \geq 0,$$

where $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ with $\phi(x) = (1-|x|)_+$. Then $F_\varepsilon(x,y) \to F(x,y)$ as $\varepsilon \to 0$, uniformly for all $x, y \geq 0$.

Proof. Let us first note that there holds

$$\int_{-\infty}^x \phi_\varepsilon(y-z)\mathrm{d}z + \int_{-\infty}^y \phi_\varepsilon(x-z)\mathrm{d}z = 1 \text{ for all } x, y \geq 0. \tag{A.5}$$

Indeed, by the facts that ϕ_ε is even, and had unit integral, there holds

$$1 = \int_{-\infty}^x \phi_\varepsilon(y-z)\mathrm{d}z + \int_x^\infty \phi_\varepsilon(z-y)\mathrm{d}z,$$

where changing variables $z \rightsquigarrow x+y-z$ in the second integral yields (A.5). Extending F continuously to $\mathbb{R}^2$ by setting $F \equiv 0$ on $\mathbb{R}^2 \setminus \mathbb{R}_+^2$, we now find for $x, y \geq 0$ that

$$\begin{aligned} &F_\varepsilon(x,y) - F(x,y) \\ &\quad = \int_{-\infty}^x \phi_\varepsilon(y-z)\big(F(x,z) - F(x,y)\big)\mathrm{d}z + \int_{-\infty}^y \phi_\varepsilon(x-z)\big(F(z,y) - F(x,y)\big)\mathrm{d}z, \end{aligned}$$

hence

$$|F_\varepsilon(x,y) - F(x,y)| \leq \sup\nolimits_{(x-z_x)^2+(y-z_y)^2<\varepsilon^2} |F(z_x,z_y) - F(x,y)|,$$

and the claim follows since the extended function F is uniformly continuous on $\mathbb{R}^2$. □

Lemma A.6. *Let $\{\varphi_n\} \subset C_c^1([0,\infty))$ be a bounded sequence for which $\varphi_n \to \varphi \in C_c([0,\infty)) \cap W^{1,\infty}(0,\infty)$ in $L^\infty(0,\infty)$, and for which $\varphi_n' \to \varphi'$ uniformly in a neighbourhood of 0. Then the sequence $\{F_n\} \subset C_0(\mathbb{R}_+^2)$, where $F_n(x,y) = \frac{1}{\sqrt{xy}}\Delta_{x\wedge y}^2 \varphi_n(x \vee y)$, converges uniformly to*

$$F \in C_0(\mathbb{R}_+^2) : \mathbb{R}_+^2 \ni (x,y) \mapsto \frac{1}{\sqrt{xy}}\Delta_{x\wedge y}^2 \varphi(x \vee y).$$

Proof. Since $\varphi_n' \to \varphi'$ uniformly near 0, we know that φ' is continuous in a neighbourhood of 0, so that indeed $F \in C_0(\mathbb{R}_+^2)$ (cf. Lemma 1.6). Moreover, since the convergence $\varphi_n \to \varphi$ is uniform, it trivially follows that $F_n \to F$ locally uniformly on $\mathbb{R}_+^2$, i.e. away from the axes $\{x = 0\}$ and $\{y = 0\}$. Restricting to $x \geq y > 0$, and using (A.4), we now obtain

$$\big|F_n(x,y) - F(x,y)\big| \leq \frac{1}{\sqrt{xy}} \int_x^{x+y} \big|\varphi_n'(z) - \varphi'(z) - \varphi_n'(z-y) + \varphi'(z-y)\big| \mathrm{d}z. \tag{A.6}$$

We then estimate the right hand side of (A.6) by

$$\frac{1}{\sqrt{xy}} \times y \times 2\big\|\varphi_n' - \varphi'\big\|_{L^\infty(0,2x)} \leq 2\big\|\varphi_n' - \varphi'\big\|_{L^\infty(0,2x)},$$

where the right hand side vanishes as $n \to \infty$, uniformly for $x > 0$ small, and it follows that $F_n \to F$, uniformly in a neighbourhood of $(0,0)$. Lastly, for $\epsilon > 0$ small, we find by again (A.6) that

$$\big\|F_n - F\big\|_{L^\infty([\sqrt{\epsilon},\infty)\times[0,\epsilon])} \leq \sqrt[4]{\epsilon} \times 2\big(\|\varphi_n'\|_{L^\infty(0,\infty)} + \|\varphi'\|_{L^\infty(0,\infty)}\big),$$

where the term between brackets is bounded uniformly in n by assumption. Choosing thus a suitable covering, we conclude that the convergence $F_n \to F$ is uniform on $\mathbb{R}_+^2$. □

A.2 Postponed proofs from Chapters 1 and 4

Proofs of Lemmas 1.5 and 1.6

Proof of Lemma 1.5. With $T \geq 0$ fixed arbitrarily, for any $\varphi \in C_0([0,\infty))$ there holds

$$\sup_{t\in[0,T]} \left\{ \int_{[0,\infty)} \varphi(x) G(t,x) \mathrm{d}x \right\} < \infty.$$

By Banach-Steinhaus (cf. [1]) it then follows that

$$\sup_{t\in[0,T]} \left\{ \sup_{\varphi\in C_0([0,\infty):[-1,1])} \left| \int_{[0,\infty)} \varphi(x) G(t,x) \mathrm{d}x \right| \right\} < \infty,$$

which was to be shown. □

Proof of Lemma 1.6. As in the proof of Lemma A.6, using (A.4), we find for $x > y > 0$ that

$$\begin{aligned} |F(x,y)| &= \frac{1}{\sqrt{xy}} \left| \int_x^{x+y} \big(\varphi'(z) - \varphi'(z-y)\big) \mathrm{d}z \right| \\ &\leq \sqrt{\tfrac{y}{x}} \times \min\left\{ 2\|\varphi'\|_{L^\infty(0,\infty)}, \big\|\varphi'(\cdot) - \varphi'(\cdot - y)\big\|_{L^\infty(x,x+y)} \right\}. \end{aligned}$$

The right hand side now vanishes as $y \to 0$ due to the square root for $x > 0$ fixed, while for $x \to 0$ we estimate the root by 1, and we obtain convergence with the second term in the minimum, by the locally uniform continuity of φ'. □

Proofs of Lemmas 4.10 and 4.11

Proof of Lemma 4.10. Since the closed unit ball in the dual norm on $\mathcal{B}'$ is weakly-$*$ compact, by Banach-Alaoglu (cf. [1]), it suffices to show that $\mathcal{U}_\rho$ is weakly-$*$ closed therein. To that end we note for $\psi \in \mathcal{B}$ and $x > 0$ that $\psi(x) \le \frac{x}{1+x}\|\psi\|_{\mathcal{B}} \le (x \wedge 1)\|\psi\|_{\mathcal{B}}$, hence for $\beta \in \mathcal{X}_\rho$ there holds

$$\|\beta\|_{\mathcal{B}'} = \sup_{\|\psi\|_{\mathcal{B}}\le 1} \langle \beta, \psi\rangle = \langle \beta, (\,\cdot\, \wedge 1)\rangle + \sup_{\|\psi\|_{\mathcal{B}}\le 1} \langle \beta, \psi - (\,\cdot\, \wedge 1)\rangle \le \langle \beta, (\,\cdot\, \wedge 1)\rangle \le \|\beta\|_\rho,$$

which proves the inclusion. Given further a sequence $\{\beta_n\} \subset \mathcal{U}_\rho$, and $\beta \in \mathcal{B}'$, such that $\beta_n \overset{*}{\rightharpoonup} \beta$ in $\mathcal{B}'$, then β is clearly nonnegative, and there holds

$$\langle \beta, (\,\cdot\, \wedge R)\rangle = \lim_{n\to\infty} \langle \beta_n, (\,\cdot\, \wedge R)\rangle \le R^{2-\rho} \text{ for all } R > 0,$$

from which we conclude that $\beta \in \mathcal{U}_\rho$. □

Proof of Lemma 4.11. For every $t \ge 0$ there exists at least one element $y(t) \in Y$ such that $S(t)y(t) = y(t)$, by the Schauder-Tychonoff fixed-point theorem (cf. [5]). In particular, for every $n \in \mathbb{N}$ we have at least one $y_n \in Y$ such that $S(2^{-n})y_n = y_n$, which further satisfies

$$S(i2^{-j})y_n = y_n \text{ for } i, j \in \mathbb{N} \text{ with } j \le n.$$

By compactness there then exist an element $y \in Y$ and a convergent subsequence $y_{n_k} \to y$ in Y, and by continuity of the maps $S(t)$ there holds $S(t)y = y$ for all dyadic $t > 0$. Lastly, we thus conclude that $S(t)y = y$ for all $t \ge 0$, since the dyadic numbers are dense in $\mathbb{R}$ and the mapping $t \mapsto S(t)$ is continuous. □

Proofs of Lemmas 4.20, 4.22, and 4.23

Proof of Lemma 4.20. The unique existence of solutions to (4.23) follows by computation of the fundamental solution. Indeed, taking the space-Fourier transform of (4.23) yields

$$\hat{u}_\tau(\tau, k) = -c_\rho |k|^\rho \hat{u}(\tau, k),$$

with $c_\rho > 0$ as in the statement of the lemma, hence we have $\hat{u}(\tau, k) = \hat{u}(0, k)e^{-c_\rho|k|^\rho \tau}$ for $\tau > 0$ and $k \in \mathbb{R}$. Transforming back, there then holds $u(\tau, \xi) = [u(0, \cdot) * u^*(\tau, \cdot)](\xi)$ with

$$u^*(\tau, x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx} e^{-c_\rho|k|^\rho\tau}\mathrm{d}k = \frac{1}{\tau^{1/\rho}} v\Big(\frac{x}{\tau^{1/\rho}}\Big), \text{ where } v(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikz}e^{-c_\rho|k|^\rho}\mathrm{d}k,$$

so we conclude that the unique solution to (4.23) with $u(0, \cdot) \equiv \psi$ is given by (4.24), and that it is odd, as it is the convolution of an odd and a smooth and even function. That v is nonincreasing on $\mathbb{R}_+$ further follows from the fact that $\exp(-c_\rho|k|^\rho)$ is the characteristic function of a symmetric stable probability distribution (cf. [24]), and the asymptotics of v follow, via a standard contour deformation argument, by the expansion

$$v(z) = \sum_{j=0}^{n-1} \frac{-(-c_\rho)^j}{\pi} \frac{\Gamma(\rho j + 1)}{\Gamma(j+1)} \sin\big(\tfrac{\pi\rho j}{2}\big) z^{-\rho j - 1} + O(z^{-\rho n - 1}) \text{ as } z \to \infty,$$

(cf. Section 5.9 in [24]).

Now, for arbitrary $\tau > 0$ and $\xi \geq 0$, and for ψ odd, we rewrite (4.24) as

$$u(\tau,\xi) = \int_{\mathbb{R}_+} \psi(\zeta) \left[v\left(\tfrac{\zeta-\xi}{\tau^{1/\rho}}\right) - v\left(\tfrac{\zeta+\xi}{\tau^{1/\rho}}\right) \right] \tfrac{\mathrm{d}\zeta}{\tau^{1/\rho}}.$$

By the monotonicity of v on $\mathbb{R}_+$, and by the fact that v is even, the term between square brackets is then nonnegative, and the maximum principle follows.

In order to check the last claim, it is sufficient to show that (4.25) holds, to which end we first check that $\Delta_y^2\psi(x) \leq 0$ for all $x \geq 0$ and $y \in \mathbb{R}$. This is immediate from concavity if $|y| \leq x$ (cf. Lemma A.2), while for $|y| > x \geq 0$ we use the fact that ψ is odd to note that

$$\begin{aligned} \Delta_y^2\psi(x) &= \Delta_x^2\psi(|y|) + 2\psi(|y|) - 2\psi(|y|-x) - 2\psi(x) \\ &\leq 2\big(\psi(|y|) - \psi(|y|-x) - \psi(x) + \psi(0)\big) \leq 0. \end{aligned}$$

Remarking lastly that the second difference operator commutes with the integral operator on the right hand side of (4.23), we conclude (4.25) from the maximum principle. □

Proof of Lemma 4.22. Since there holds

$$\int_{(0,\infty)} (z \wedge x)\Phi(z)\mathrm{d}z = \int_0^x \int_y^\infty \Phi(z)\mathrm{d}z\mathrm{d}y,$$

the result follows by integration by parts if the boundary values vanish. The assumption that Θ is odd with bounded first derivative now implies that $\Theta(x) = \Theta'(0)x + o(x^2)$ as $x \to 0$, so we have

$$\begin{aligned} \left|\Theta(x)\int_x^\infty \Phi(z)\mathrm{d}z\right| &\leq |\Theta(x)| \times \frac{1}{x}\int_{(0,\infty)} (z \wedge x)\Phi(z)\mathrm{d}z \\ &\leq 2|\Theta'(0)|x \times x^{1-\rho}\|\Phi\|_\rho \to 0 \text{ as } x \to 0. \end{aligned}$$

Furthermore, we have

$$\left|\Theta'(x)\int_{(0,\infty)} (z \wedge x)\Phi(z)\mathrm{d}z\right| \leq |\Theta'(x)| \times x^{2-\rho}\|\Phi\|_\rho,$$

where the right hand side vanishes as $x \to \infty$, since $\lim_{x\to\infty} \Theta'(x)x^{2-\rho} = 0$ by assumption. As the remaining boundary terms vanish trivially, we conclude the lemma. □

Proof of Lemma 4.23. Being the convolution of an odd function and a smooth even function, it is immediate that Θ is odd and smooth. Then, differentiating under the integral, and integrating by parts, we find

$$\Theta'(x) = -\int_{\mathbb{R}} y\left(1 \wedge \left|\tfrac{\theta_1}{y}\right|\right)\left[v\left(\tfrac{x-y}{\theta_2}\right)\right]_y \tfrac{\mathrm{d}y}{\theta_2} = \int_{-\theta_1}^{\theta_1} v\left(\tfrac{x-y}{\theta_2}\right)\tfrac{\mathrm{d}y}{\theta_2}, \tag{A.7}$$

and differentiating (A.7) once more yields the equality in (4.33), while the nonnegativity follows from the symmetry and monotonicity properties of v. Lastly, using the symmetry and the asymptotic behaviour of v (cf. Lemma 4.20), we get

$$\Theta'(x) = \int_{(x-\theta_1)/\theta_2}^{(x+\theta_1)/\theta_2} v(z)\mathrm{d}z \leq \int_{(x-\theta_1)/\theta_2}^{\infty} v(z)\mathrm{d}z \sim \frac{1}{\rho}\frac{\theta_2^\rho}{x^\rho} \text{ as } x \to \infty,$$

hence $|\Theta'(x)x^{2-\rho}| \leq \frac{1}{\rho}\theta_2^\rho \times x^{2(1-\rho)} \to 0$ as $x \to \infty$, by which the lemma follows. □

Proof of Lemma 4.35. Without loss of generality we restrict ourselves measures μ for which the left hand side of (4.56) is strictly positive. Fixing μ as such, there exists a maximal integer $n \in \mathbb{N}$ for which there holds

$$\int_{\left(0,(\frac{2}{3})^j\right]} x\mu(x)\mathrm{d}x \geq (\tfrac{2}{3})^j \int_{(0,1]} x\mu(x)\mathrm{d}x \text{ for all } j = 0,\ldots,n-1,$$

since otherwise we would have

$$\int_{\left(0,(\frac{2}{3})^j\right]} \mu(x)\mathrm{d}x \geq (\tfrac{3}{2})^j \int_{\left(0,(\frac{2}{3})^j\right]} x\mu(x)\mathrm{d}x \geq \int_{(0,1]} x\mu(x)\mathrm{d}x > 0 \text{ for all } j \in \mathbb{N},$$

where the left hand side vanishes as $j \to \infty$. It then follows that

$$\int_{\left((\frac{2}{3})^n,(\frac{2}{3})^{n-1}\right]} x\mu(x)\mathrm{d}x > \tfrac{1}{2}(\tfrac{2}{3})^n \int_{(0,1]} x\mu(x)\mathrm{d}x,$$

hence

$$\int_{\left((\frac{2}{3})^n,\frac{3}{2}(\frac{2}{3})^n\right]} \mu(x)\mathrm{d}x > \frac{1}{2}\int_{(0,1]} x\mu(x)\mathrm{d}x.$$

Setting then $z = (\frac{2}{3})^n > 0$, we bound the left hand side of (4.56) from below by

$$\iint_{\left((\frac{2}{3})^n,\frac{3}{2}(\frac{2}{3})^n\right]^2} \frac{\mu(x)\mu(y)}{\frac{3}{2}(\frac{2}{3})^n}\left[(\tfrac{2}{3})^n - (\tfrac{3}{2}-1)(\tfrac{2}{3})^n\right]\mathrm{d}x\mathrm{d}y > \frac{1}{3}\left(\frac{1}{2}\int_{(0,1]} x\mu(x)\mathrm{d}x\right)^2,$$

and we conclude the result. □

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