A phase-field model of dislocations on parallel slip planes

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Peter Gladbach

aus
Troisdorf

Bonn
2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Sergio Conti

2: Gutachter: Prof. Dr. Stefan Müller

Tag der Promotion: 20.12.2016

Erscheinungsjahr: 2017

Acknowledgements

Firstly, I express my most sincere gratitude to my supervisor Prof. Dr. Sergio Conti for his continued support during my PhD studies. His patience, helpfulness, and immense knowledge helped me greatly during all this time.

Secondly, I want to thank Prof. Dr. Stefan Müller for refereeing this thesis, as well as the other members of the committee.

Thirdly, I thank Prof. Dr. Juan J. L. Velázquez for acting as my mentor.

Additionally, I thank my colleagues for lots of interesting and fruitful discussions, most nobaly Adolfo, Janusz, Johannes, and Luigi.

I thank my girlfriend Eva for all her love and support through these great years. I am most grateful to have her in my life.

Lastly, I thank my family for standing behind me during my studies, and in life.

Summary

We expand the Peierls-Nabarro phase-field model of dislocations on one active slip plane introduced by Koslowski, Cuitiño, and Ortiz to the case of multiple parallel slip planes embedded into a homogeneous anisotropic crystal. We deduce the leading-order behavior of the energy as lattice size and slip plane spacing tend to zero. Under a logarithmic rescaling, the limit energy in the sense of Γ-convergence takes the form of a line-tension functional supported on dislocation lines featuring interactions between parallel dislocation lines in different slip planes. An optimal dislocation configuration is shown to contain a two-scale microstructure.

This is an extension of a result by Conti, Garroni, and Müller. We are able to treat anisotropic materials with possibly nonpositive interaction kernels, and obtain the leading order energy for non-dilute dislocations in a special geometry. We use the theory of functions of bounded variation, the fractional Sobolev space $H^{1/2}$, and linear elasticity theory. We show some new results using iterated mollification and multiscale analysis, and use a modified ball construction for an extension result in BV.

Contents

Nomenclature

$\mathbb{N}$	Natural numbers including 0.
$\mathbb{R}^n$	Euclidean space of dimension $n \in \mathbb{N}$.
$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$	Flat torus of dimension $n \in \mathbb{N}$.
$a \vee b = \max(a,b)$	Maximum of $a, b \in \mathbb{R}$.
$a \wedge b = \min(a,b)$	Minimum of $a, b \in \mathbb{R}$.
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$.
S^{n-1}	Unit sphere in $\mathbb{R}^n$.
$x \cdot y$	Euclidean scalar product of $x, y \in \mathbb{R}^n$.
$x \times y$	Cross product of $x, y \in \mathbb{R}^3$.
$x \otimes y$	For $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, the matrix $A \in \mathbb{R}^{n\times m}$ with entries $A_{ij} = x_i y_j$.
$x^\perp$	The vector $x \in \mathbb{R}^2$ rotated by 90° degrees counterclockwise.
$x \parallel y$	x is parallel to y, i.e. $x, y \in \mathbb{R}^n$ are linearly dependent.
$x \perp y$	$x, y \in \mathbb{R}^n$ are orthogonal, i.e. $x \cdot y = 0$.
$e^{i\theta}$	The vector $(\cos\theta, \sin\theta) \in \mathbb{R}^2$.
$\|A\| = \mathcal{L}^n(A)$	Lebesgue measure of A.
$\mathcal{H}^k(A)$	k-dimensional Hausdorff measure.
$\overline{A}$	Closure of A.
$\mathring{A}$	Interior of A.
∂A	Topological boundary of A.
$\operatorname{conv} A$	Convex hull of $A \subset \mathbb{R}^n$. If $A \subset \mathbb{T}^n$, denotes the smallest set $B \subset \mathbb{T}^n$ containing A and all shortest paths between two points $x, y \in B$.
$[x, y]$	For $x, y \in \mathbb{R}^n$ the closed line segment $\operatorname{conv}(\{a, b\})$.
$\mathbb{1}_A$	Characteristic function of A.
$\int_A f\,dx = \int_A f\,d\mathcal{L}^n$	Lebesgue integral of f.
$\mathcal{D}(U)$	Space of test functions on U.
$\mathcal{S}$	Space of Schwartz functions on $\mathbb{R}^n$.

$\mathcal{D}'(U)$	Space of distributions on U.
$\mathcal{S}'$	Space of tempered distributions on $\mathbb{R}^n$.
$\mathcal{M}(U)$	Class of positive measures on U.
$\|f\|_p = \|f\|_{L^p}$	L^p-norm of f.
$\|f\|_{k,p} = \|f\|_{W^{k,p}}$	Sobolev norm of f.
$\mathcal{F}f(k) = \widehat{f}(k) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(x)e^{-ik\cdot x}\,dx$	Fourier transform of f.
$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(k)e^{ik\cdot x}\,dk$	Inverse Fourier transform of f.

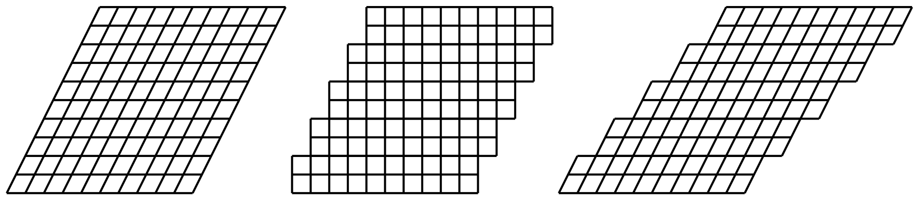

Figure 1:
Left: Purely elastic deformation of a square lattice.
Middle: Purely plastic deformation trough crystallographic slip along parallel slip planes.
Right: Superposition of elastic and plastic deformations.

1 Introduction

1.1 Crystal plasticity

Plasticity describes the effect of permanent deformation of solids under load. Typically, once a certain yield stress is applied, the material starts to deform plastically, accompanied by dissipation of energy into heat. We study in particular plastic deformations of crystalline materials such as metals or minerals. At a microscopic level, the permanent deformation of a single crystal is achieved by plastic slip, the gliding of atoms along a plane to different neighbors. The active slip planes are determined by the crystal structure. In practice, the preferred slip directions are the shortest translation vectors of the crystal's symmetry group, e.g. the unit directions in a simple cubic crystal.

The microscopic kinematics of plastic slip are described by pairs $(\omega, \mathcal{B})$ of slip planes $\omega \subset \mathbb{R}^3$ and discrete Burgers lattices $\mathcal{B} \subset \mathbb{R}^3$, where $\mathcal{B}$ is perpendicular to the slip plane normal ν_ω. We then consider a reference configuration $\Omega \subset \mathbb{R}^3$ intersecting $M \in \mathbb{N}$ slip planes $\omega^1, \dots, \omega^M$, each equipped with a Burgers lattice $\mathcal{B}^1, \dots, \mathcal{B}^M$. The small deformation is described by the displacement field $u : \Omega \to \mathbb{R}^3$, which is allowed to jump along each ω^m by $[u] = b^m : \omega^m \to \mathbb{R}^3$, the jump taking values in or near the Burgers lattice $\mathcal{B}^m$. We naturally decompose the distributional differential $Du = \nabla u \mathcal{L}^3 + \sum_{m=1}^M b^m \otimes \nu^m \mathcal{H}^2 \llcorner \omega^m$ into an elastic distortion $\beta^e = \nabla u \mathcal{L}^3$ and a plastic distortion $\beta^p = \sum_{m=1}^M b^m \otimes \nu^m \mathcal{H}^2 \llcorner \omega^m$. Note that a fitting function space is either a subspace of $SBV^2(\Omega, \mathbb{R}^3)$ with prescribed jump set $J_u \subseteq \bigcup_{m=1}^M \omega^m$, or alternatively $H^1(\Omega \setminus \bigcup_{m=1}^M \omega^m, \mathbb{R}^3)$ as long as $\nabla u \in L^2$, where the jumps appear as differences of the traces of u on ω^m.

1.2 Dislocations

While uniform plastic slip across an entire slip plane leaves the crystal structure intact, in reality, particularly in polycrystals, which are made up of differently aligned crystalline grains, we observe that under increasing load, slip occurs first over a small section of the slip plane, then gradually the slipped region expands. At the boundary line between the slipped and unslipped regions there must necessarily be a crystallographic defect, called a dislocation, see Figure 2. This motion of dislocations governs the propagation of plastic slip.

Wherever dislocations are impeded from moving forward by certain obstructions such as grain boundaries, they tend to pile up, and the repellent interaction between dislocations makes further plastic distortion more difficult. This effect is referred to as strain hardening, or work hardening.

In our model, with $Du = \beta^e + \beta^p$, in general we cannot decompose u into a purely elastic displacement u^e with $Du^e = \beta^e$ and a purely plastic one u^p with $Du^p = \beta^p$. In this linearized setting, this is possible only if $0 = \operatorname{curl}\beta^p = -\operatorname{curl}\beta^e$, where for $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^{3\times3}$ row-wise, $(\operatorname{curl}\beta)_j = \operatorname{curl}\beta_j$. In our case $\beta^p = \sum_{m=1}^{M} b^m \otimes \nu^m \mathcal{H}^2 \llcorner \omega^m$, and

$$\operatorname{curl}\beta^p = \sum_{m=1}^{M} \tilde{D} b^m \times \nu^m. \tag{1}$$

where $\tilde{D}b^m \in \mathcal{D}'(\omega, \mathbb{R}^3 \otimes \omega)$ denotes the in-plane distributional derivative, which is embedded by extension into $\mathcal{D}'(\Omega, \mathbb{R}^{3\times3})$, and the cross product $\tilde{D}b^m \times \nu^m \in \mathbb{R}^{3\times3}$ is taken row-wise.

This means that away from intersections of the slip planes, any nonconstant slip field necessarily induces elastic distortion.

The tensor $\operatorname{curl}\beta^p \in \mathbb{R}^{3\times3}$ is usually referred to as Nye's dislocation density. For a comprehensive study of dislocations, see e.g. [22], [23]. The meaning of the tensor is made clear by Stokes' theorem, in that a closed loop γ in the reference configuration transported elastically to the deformed configuration, i.e. following the crystal structure, ends up as a non-closed loop at an offset

$$\int_\gamma \beta^e \cdot \dot{\gamma}\, d\mathcal{H}^1 = -\int_{A_\gamma} \operatorname{curl}\beta^p \cdot \nu\, d\mathcal{H}^2, \tag{2}$$

where $A_\gamma \subset \Omega$ is a surface bounded by γ and ν its normal. The meaning of $\operatorname{curl}\beta^p \cdot \nu$ is the aggregate offset in a loop normal to ν.

For a single slip plane $\omega = \mathbb{R}^2 \times \{0\}$ with normal e_3 in a simple cubic crystal, we have after normalization that the Burgers lattice is $\mathcal{B} = \mathbb{Z}^2 \times \{0\}$. A typical nonconstant slip field with values in $\mathcal{B}$ is

$$b(x_1, x_2) = \begin{cases} e_1 & \text{, if } x_1 > 0 \\ 0 & \text{, otherwise.} \end{cases} \tag{3}$$

We get $\operatorname{curl}\beta^p = e_1 \otimes e_1 \mathcal{H}^1 \llcorner \mathbb{R}e_1$, a single straight dislocation line. Since $e_1 \parallel e_1$, this is called an edge dislocation.

On the other hand, if

$$b(x_1, x_2) = \begin{cases} e_2 & \text{, if } x_1 > 0 \\ 0 & \text{, otherwise,} \end{cases} \tag{4}$$

we get $\operatorname{curl}\beta^p = e_2 \otimes e_1 \mathcal{H}^1 \llcorner \mathbb{R}e_1$. Since $e_2 \perp e_1$, this is called a screw dislocation. These two types of dislocation were the two first identified by Volterra (see [35]), although more complicated dislocations do exist. The jump in the slip field is called the dislocation's Burgers vector, and it measures the offset of a loop around the dislocation. See Figure 2.

In general, if $b \in BV(\omega, \mathcal{B})$, by the structure theorem for BV functions (see [5]), its jump set forms a rectifiable network of curves $J_b \subset \omega$, with piecewise

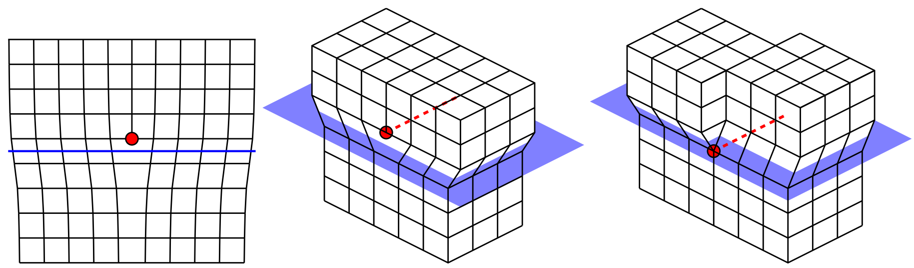

Figure 2:
Left: A point dislocation in a $2D$ square lattice (red). Atoms may change their bonds along the slip line (blue). The dislocation marks the jump in the slip field.
Middle: An edge dislocation (3) in a simple cubic lattice (red). Note that the $2D$ picture is simply extruded in the third dimension. Slip plane in blue.
Right: A screw dislocation (4) in a simple cubic lattice. In this case, the Burgers vector is parallel to the dislocation line.

constant jumps $[b] \in \mathcal{B}$, so that $Db = [b] \otimes \nu_{J_b} \mathcal{H}^1 \llcorner J_b$, and the dislocation density is likewise concentrated on a network of curves

$$\operatorname{curl} \beta^p = [b] \otimes (\nu_{J_b} \times \nu_\omega) \mathcal{H}^1 \llcorner J_b. \tag{5}$$

The general setting of rectifiable dislocation line networks in 3-dimensional space was recently studied in [10] and [12]. The case of parallel straight dislocation lines was treated in the nonlinear setting in [32], [28], and more recently in [19].

1.3 Energy considerations

We shall consider the setting of dislocation networks supported on $M \in \mathbb{N}$ parallel planes via the phase-field approach, where the phase-field consists of the M slip fields $b^m : \omega^m \to \mathbb{R}^3$, and $u : \Omega \to \mathbb{R}^3$ shall be assumed optimal under the jump condition $[u] = b^m$ on ω^m. Since this jump condition automatically determines the gradient decomposition $Du = \beta^e + \beta^p$, we consider the linearized elastic energy

$$\int_\Omega \mathbb{C}\beta^e : \beta^e \, dx, \tag{6}$$

where $\mathbb{C} \in \mathbb{R}^{3\times3\times3\times3}$ denotes the symmetric stiffness tensor, which is not fully coercive but in fact only penalizes the symmetric part $A_{\text{sym}} = (A + A^T)/2$ of a given distortion tensor $A \in \mathbb{R}^{n\times n}$. Given jumps $b^1, \ldots, b^M$, an optimal displacement field then solves $0 = \operatorname{div} \mathbb{C}\beta^e = \operatorname{div} \mathbb{C}\nabla u$ outside of $\bigcup_{m=1}^M \omega^m$. The apparent problem of noncoercivity was famously solved by Korn's inequality

$$\min_{A \in \mathbb{R}^{n\times n}_{\text{skew}}} \int_\Omega |Du - A|^2 \, dx \leq C(\Omega) \int_\Omega |Du_{\text{sym}}|^2, \tag{7}$$

which holds for all open bounded connected Lipschitz domains. Introducing jumps generally makes the situation more complicated, resulting in the space of function of bounded deformation BD (see e.g [34]), but in our case, since the jump planes are fixed, the elastic problem will be shown to be strongly elliptic.

For a slip field containing a single straight dislocation line as in (3) and (4) with Burgers vector $d \in \mathcal{B}$,

$$b(x_1, x_2) = \begin{cases} d & , \text{if } x_1 > 0 \\ 0 & , \text{otherwise,} \end{cases} \tag{8}$$

we try to find the minimal elastic energy per unit length around the dislocation, which we can replace by the squared L^2 norm using Korn's inequality, although some work needs to be done since β^e is not a gradient field, see [32], [28],[19].

We use Stokes' theorem to see that

$$\int_\gamma \beta^e \cdot \dot{\gamma} \, d\mathcal{H}^1 = -d \tag{9}$$

for every clockwise closed curve γ around $\mathbb{R}e_1$. In a hollow cylinder $C_{l,r,R} = \{(x_1, x_2, x_3) : 0 < x_1 < l, r^2 < x_2^2 + x_3^2 < R^2\}$ around $\mathbb{R}e_1$, we get by Hölder's inequality and use of cylindrical coordinates the lower bound for the energy

$$\begin{aligned} &\int_{C_{l,r,R}} |\beta^e|^2 \, dx \\ \geq &\int_0^l \int_r^R \int_{\{x_1\} \times tS^1} |\beta^e \cdot \dot{\gamma}|^2 \, d\mathcal{H}^1 \, dt \, dx_1 \\ \geq &\frac{l|d|^2}{2\pi} \int_r^R \frac{dt}{t} \\ = &\frac{l|d|^2}{2\pi} \log(R/r), \end{aligned} \tag{10}$$

where $\gamma = \gamma_{x_1,t}$ is the clockwise unit speed curve parameterizing the circle $\{x_1\} \times tS^1 \subset \mathbb{R}^3$.

The total elastic energy thus diverges logarithmically around the dislocation line as $r \to 0$. In order to arrive at a finite energy model, different modifications can be made, such as cutting out a small cylinder around the dislocation, mollifying the dislocation density using a smooth kernel, or using a mixed growth model replacing $|\beta^e_{\text{sym}}|^2$ in the energy with $\min(|\beta^e_{\text{sym}}|^2, |\beta^e_{\text{sym}}|^p)$ for some $p < 2$.

In case of the phase-field model, we allow for smooth transitions in the slip field instead of sharp dislocation lines. Deviations of b^m from $\mathcal{B}^m$, which signify a mismatch in the crystal lattice, are penalized by a Peierls potential $W^m(b^m) \approx \text{dist}^2(b^m, \mathcal{B}^m)$. We introduce a length scale $\varepsilon > 0$ denoting the width of the transition, called the core size of the dislocation. The energy we consider for a family of slip fields $b^1, \ldots, b^M$ is then

$$\begin{aligned} &E_\varepsilon(b^1, \ldots, b^M) \\ = &\frac{1}{\varepsilon} \sum_{m=1}^M \int_{\omega^m} \text{dist}^2(b^m, \mathcal{B}^m) \, d\mathcal{H}^2 \\ &+ \min_{u \,:\, [u] = b^m \text{ on } \omega^m} \int_{\Omega \setminus \bigcup_{m=1}^M \omega^m} \mathbb{C}Du : Du \, dx. \end{aligned} \tag{11}$$

A competitor near (8) for a single plane is given by

$$b_\lambda(x_1, x_2) = \begin{cases} d & , \text{ if } x_1 > \lambda \\ x_1 d/\lambda & , \text{ if } 0 \le x_1 \le \lambda \\ 0 & , \text{ if } x_1 < 0, \end{cases} \tag{12}$$

and its energy in the unit square is given by $E_\varepsilon(b_\lambda) \approx \lambda/\varepsilon + |\log \lambda|$, so that the optimal transition width is indeed $\lambda = \varepsilon$, yielding energy on the scale $|\log \varepsilon|$. Note also that the contribution from the Peierls potential remains bounded.

We aim to extract the asymptotic behavior of $E_\varepsilon/|\log \varepsilon|$, in the sense of Γ-convergence, which was introduced in the 1970s by De Giorgi. See [6],[15] for an introduction to the topic.

In general, a family of functionals $F_\varepsilon : X \to [-\infty, \infty]$ on a metric space X is said to Γ-converge to a functional $F : X \to [-\infty, \infty]$ if

i) Whenever $x_\varepsilon \to x$ as $\varepsilon \to 0$, then $F(x) \le \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$.

ii) For every $x \in X$ there is a family x_ε such that $x_\varepsilon \to x$ as $\varepsilon \to 0$ and $\lim_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) = F(x)$.

The first inequality is generally referred to as the lower bound and the second as the upper bound. The family or sequence in (ii) is called a recovery sequence.

A family of energy functionals similar to (11) arises in the theory of phase transitions, where $u : \Omega \to [0, 1]$ describes a mixture of phases, with energy

$$F_\varepsilon(u) = \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon |Du|^2 \, dx, \tag{13}$$

where $W : \mathbb{R} \to \mathbb{R}$ vanishes precisely at 0 and 1. Modica and Mortola showed in [27] that F_ε Γ-converges in the L^1-topology to $F(u) = c|Du| = c \operatorname{Per}(\{u = 1\})$ for $u \in BV(\Omega, \{0, 1\})$. The constant in this case arises from an optimal transition profile and depends explicitly on W.

The singular perturbation arising from the elastic energy in (11) resembles more closely the squared $H^{1/2}$-seminorm than the Dirichlet integral. For a detailed treatment of fractional Sobolev spaces, see e.g. [1]. The model one-dimensional problem

$$G_\varepsilon(u) = \frac{1}{\varepsilon |\log \varepsilon|} \int_\mathbb{R} W(u) \, dx + \frac{1}{|\log \varepsilon|} \int_\mathbb{R} \int_\mathbb{R} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, dx \, dy, \tag{14}$$

with W as before, was treated in [2], where the authors showed Γ-convergence to $2|Du| = 2 \operatorname{Per}(\{u = 1\})$, independently of W, because the main contribution comes from the long-range interaction of the level sets $\{u_\varepsilon \approx 1\}$ and $\{u_\varepsilon \approx 0\}$.

The problem of minimizing the linear elastic energy around a single slip plane $\omega \subseteq \mathbb{R}^2$ was studied in [25], [26], and [17], where it was shown that for isotropic $\mathbb{C}$, there is a positive definite -3-homogeneous kernel $J : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ such that

$$\begin{aligned} &\min_{u\,:\,[u]=b \text{ on } \omega} \int_{\Omega \setminus \omega} \mathbb{C} Du : Du \, dx \\ &= \int_\omega \int_\omega (b(x) - b(y)) J(x - y) (b(x) - b(y)) \, d\mathcal{H}^2(x) \, d\mathcal{H}^2(y). \end{aligned} \tag{15}$$

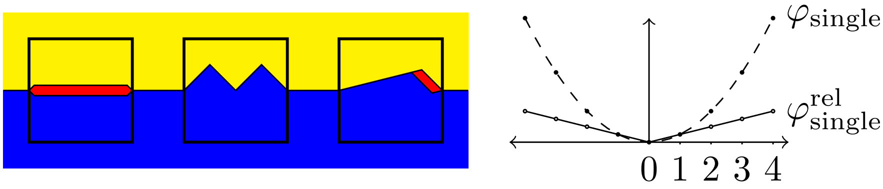

Figure 3: The BV-elliptic envelope is realized by minimizing the original line-tension energy in the square Q_ν with given jump boundary condition. Compare to the definition of quasiconvexity in [14]. Possible microstructures include zig-zag, intermediate phases, or combinations thereof. Since $\varphi_{\text{single}}^{\text{rel}}$ is necessarily subadditive, the quadratic growth of φ_{single} is relaxed to linear growth.

This explicit representation allowed Garroni and Müller to show in [18] for a single slip system and with Conti in [11] for multiple slip systems that the Γ-limit in the L^1-topology of $E_\varepsilon/|\log\varepsilon|$ in the single plane case is finite on $BV(\omega,\mathcal{B})$, where the limit energy I_{single} is of line-tension type

$$I_{\text{single}}(b) = \int_{J_b} \varphi_{\text{single}}^{\text{rel}}([b],\nu)\,d\mathcal{H}^1, \tag{16}$$

where $\varphi_{\text{single}}^{\text{rel}} : \mathcal{B}\times S^1 \to [0,\infty)$ is the BV-elliptic envelope (see [3],[4]) of the function

$$\varphi_{\text{single}}(d,\nu) = \int_{S^1} d\cdot J(x)d\,|x\cdot\nu|\,d\mathcal{H}^1(x), \tag{17}$$

which is the energy density of a straight dislocation line in ω perpendicular to ν with Burgers vector d.

The BV-elliptic envelope is defined as

$$\varphi_{\text{single}}^{\text{rel}}(d,\nu) = \inf\left\{\int_{J_b\cap Q_\nu} \varphi_{\text{single}}([b],\nu)\,d\mathcal{H}^1 \;:\; b = d\mathbb{1}_{\{x\cdot\nu>0\}} \text{ outside of } Q_\nu\right\}. \tag{18}$$

Here Q_ν is the unit square in ω centered at 0 with one side parallel to ν.

The BV-elliptic envelope in the limit energy reflects the formation of dislocation microstructure (see Figure 3), which needs to be appropriately treated in proving the Γ-limit.

1.4 Content of the thesis

In this thesis, we generalize the Peierls-Nabarro model to the case of $M\in\mathbb{N}$ parallel planes. In this case, the bulk elastic energy features interaction between different planes.

In Section 2, we find the minimal Dirichlet energy in $\mathbb{R}^n\setminus\bigcup_{m=1}^M\omega^m$ with given jumps on the ω^m. This result is classical, see e.g. [33], from where we adapted the proof. We show that the space of jumps with finite energy is precisely $H^{1/2}$.

In Section 3, we move from the Dirichlet energy to the vector-valued linear elastic problem. We follow the work in [17] but take some extra steps to show existence, uniqueness, and linearity of the minimizer to the jump problem. We find that, given jumps $b^m \in H^{1/2}(\mathbb{R}^{n-1}\times\{h^m\},\mathbb{R}^n)$, the minimum elastic energy is given by the bilinear form

$$\begin{aligned} &B_{h^1,\dots,h^M}(\boldsymbol{b},\boldsymbol{b}) \\ &= \sum_{m,m'=1}^{M} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \\ &(b^m(x)-b^m(y))J(x-y-(h^{m'}-h^m)e_n)(b^{m'}(x)-b^{m'}(y))\,dx\,dy, \end{aligned} \tag{19}$$

where $\boldsymbol{b} = (b^1,\dots,b^M) \in H^{1/2}(\mathbb{R}^{n-1},\mathbb{R}^{Mn})$ denotes the total slip field. The kernel $J:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}^{n\times n}_{\mathrm{sym}}$ is $-n$-homogeneous and smooth, but not necessarily positive everywhere. In fact, for more than one plane, the interaction is necessarily negative somewhere, and even for one plane in an anisotropic material, we were unable to prove positivity of the kernel. While in [17] it was shown that the kernel is positive for isotropic materials, the use of Fourier transforms of singular functions to define the kernel makes positivity in the anisotropic case difficult to show analytically, even though the problem is finite-dimensional. We were able to find many non-positive kernels yielding an elliptic bilinear form on $H^{1/2}$ in Example 3.21, although those are not necessarily induced by a homogeneous stiffness tensor $\mathbb{C}$.

We go on to show that all the important properties of the kernel still apply if the Euclidean slip plane $\mathbb{R}^{n-1}$ is replaced by the torus $\mathbb{T}^{n-1}$, and provide a decomposition of the interaction kernel into integrable kernels with an inherent length scale. In the rest of the thesis, we mostly restrict ourselves to the torus $\mathbb{T}^2$ for technical reasons.

In Section 4, we proceed to make a few simplifications. Namely, we assume that the slips b^m are indeed tangential, i.e. that $b^m\cdot e_n = 0$, and that the Burgers lattices are all equal to $\mathbb{Z}^2 = \mathbb{Z}^2\times\{0\}$. We also assume that the slip planes are evenly stacked, i.e. that $h^m = mh(\varepsilon)$ for some $h(\varepsilon)$, which may vary. Results for more general configurations can be inferred by a change of variables at the cost of more complicated notation.

we show that for evenly stacked slip planes, i.e. $h^m = h(\varepsilon)m$, the energy $E_{\varepsilon,h(\varepsilon)}/|\log\varepsilon|$ provides compactness in $L^1(\mathbb{T}^2,\mathbb{R}^{nM})$, provided that

$$\limsup_{\varepsilon\to 0}\frac{\log h(\varepsilon)}{\log\varepsilon} < 1, \tag{20}$$

to a limit function $\boldsymbol{b}\in BV(\mathbb{T}^2,\mathbb{Z}^{2M})$.

In this topology, dislocations are free to move within their plane but unable to move between the planes. The jump set $J_{\boldsymbol{b}}\subset\mathbb{T}^2$ is the union of the jump sets $J_{b^m}\subset\mathbb{T}^2$. Whenever only one of the M slip fields jumps, we expect the same limit energy as for a single plane, namely $\varphi((d,0,\dots,0),\nu) = \varphi^{\mathrm{rel}}_{\mathrm{single}}(d,\nu)$. However, whenever multiple of the b^m jump simultaneously, we expect interaction between these dislocations, see Figure 4. We are able to show the following theorem:

Theorem 1.1. *Let $M > 0$. Let $h:(0,\infty)\to[0,\infty)$. Consider the family of energies defined for $\boldsymbol{b}\in H^{1/2}(\mathbb{T}^2,\mathbb{R}^{2M})$,*

$$E_{\varepsilon,h(\varepsilon)}(\boldsymbol{b}) = \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\mathrm{dist}^2(\boldsymbol{b},\mathbb{Z}^{2M}) + B_{h(\varepsilon),\dots,Mh(\varepsilon)}(\boldsymbol{b},\boldsymbol{b}). \tag{21}$$

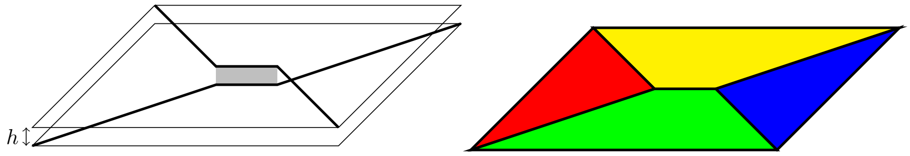

Figure 4: The limit energy is concentrated on the jump set of $\boldsymbol{b}$, which is the union of jump sets of the slip fields. When dislocations in different planes run in parallel, they interact, which may be energetically favorable.

Then if

$$\beta = \lim_{\varepsilon\to 0} 0 \vee \frac{\log h(\varepsilon)}{\log \varepsilon} \wedge 1 \tag{22}$$

exists, we have

i) If $\beta < 1$, then the energies $E_{\varepsilon,h(\varepsilon)}/|\log\varepsilon|$ are equicompact in the $L^1(\mathbb{T}^2,\mathbb{R}^{2M})$ topology, up to constants, with limit functions as $\varepsilon\to 0$ in $BV(\mathbb{T}^2,\mathbb{Z}^{2M})$. The Γ-limit of $E_{\varepsilon,h(\varepsilon)}/|\log\varepsilon|$ is the line-tension energy

$$I(\boldsymbol{b}) = \int_{J_{\boldsymbol{b}}} \left[(1-\beta)\varphi_\infty^{\mathrm{rel}} + \beta\varphi_0\right]^{\mathrm{rel}}([\boldsymbol{b}],\nu)\, d\mathcal{H}^1. \tag{23}$$

Here $\varphi_\infty(\boldsymbol{d},\nu) = \sum_{m=1}^M \varphi_{\mathrm{single}}(d^m,\nu)$ is the self-energy of a dislocation ensemble at distance ∞, i.e. with no interaction between dislocations, and $\varphi_0(\boldsymbol{d},\nu) = \varphi_{\mathrm{single}}\big(\sum_{m=1}^M d^m,\nu\big)$ is the self-energy of a dislocation ensemble at distance 0, i.e. a single dislocation.

ii) If $\beta = 1$, and

$$\limsup_{\varepsilon\to 0} E_{\varepsilon,h(\varepsilon)}(\boldsymbol{b}_\varepsilon)/|\log\varepsilon| < \infty, \tag{24}$$

then the sums $b_\varepsilon^\Sigma = \sum_{m=1}^M b_\varepsilon^m$ are compact up to constants in the $L^1(\mathbb{T}^2,\mathbb{R}^2)$ topology for finite energy sequences $\boldsymbol{b}_\varepsilon$, with limit functions $b^\Sigma \in BV(\mathbb{T}^2,\mathbb{Z}^2)$ and the Γ-limit of $E_{\varepsilon,h(\varepsilon)}/|\log\varepsilon|$ in this topology is given for $\boldsymbol{b}\in BV(\mathbb{T}^2,\mathbb{Z}^{2M})$ by

$$I(\boldsymbol{b}) = \int_{J_{\boldsymbol{b}}} \varphi_0^{\mathrm{rel}}([\boldsymbol{b}],\nu)\, d\mathcal{H}^1. \tag{25}$$

This result was announced in [13], and its physical implications were discussed in [20] and [21].

The compactness statements plus some additional properties are proved in Section 4. The compactness result from Theorem 1.1 is proved in Propositions 4.5 and 4.7. Unlike in [18], [11], we cannot use the kernel representation of $B_{h,\dots,Mh}$ for compactness, because we the kernel is not necessarily positive. Instead we analyze the displacement field $u : \Omega\setminus\bigcup_{m=1}^M \to \mathbb{R}^3$ to show compactness of the jumps of u.

The upper bound for the limit energies is proved in Proposition 5.4 in Section 5, where we also calculate some microstructures, following [13], to show that the double relaxation in the limit is nontrivial.

The rest of this thesis is devoted to the lower bound, which is proved by applying Proposition 8.2 to the results of Proposition 8.1.

In Section 6, we use a modified ball construction, the original having been used in [24],[30], [31] for Ginzburg-Landau theory, to show an extension result for $SBV(\mathbb{R}^2, \mathbb{Z})$ functions defined up to a small error set. This result is used multiple times throughout the thesis to deal with error terms which may be large but localized, by simply covering the regions containing large error terms.

In Section 7, we show some estimates for the family of one-dimensional step functions $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b}\lambda(x \cdot \nu)$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^{2M}$, $\lambda : \mathbb{R} \to \mathbb{Z}$ monotone, and $\nu \in S^1$, which were shown to well-approximate functions in $BV(\mathbb{R}^2, \mathbb{R}^{2M})$ in [11], to replace the action of the kernel with the line-tension energy.

In Section 8, we employ the results from the previous two sections in order to show the lower bound. The proof is based on the one in [11], but special care has to be taken due to the non-positivity of the kernel.

The standard decomposition of a kernel into its actions on annuli, as used in [11] for a single slip plane in an isotropic material, in general does not preserve positivity. Instead we use iterated mollification to decompose any convex positive translation-invariant functional into countably many positive functionals which each have an inherent length scale. We use iterated mollification on the total variation, following [11], as well as on the energy $B_{h(\varepsilon),\ldots,Mh(\varepsilon)}$.

2 The minimal Dirichlet energy for the jump problem

2.1 The space $H^{1/2}(\mathbb{R}^{n-1})$

We start with the following model problem:

Given $\Omega \subset \mathbb{R}^n$ open with piecewise C^1 boundary and boundary values $f \in C^1(\partial\Omega)$, find the minimizer $u \in C^2(\Omega)$ of the Dirichlet energy

$$\inf\left\{\int_\Omega |\nabla u(x)|^2 dx : u = f \text{ on } \partial\Omega\right\}. \tag{26}$$

Assuming a minimizer $u \in H^1(\Omega)$ exists, it must be harmonic.

In case of the half-space $\Omega = \mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty)$, where we write $x = (\tilde{x}, x_n)$, with boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\} =: \mathbb{R}^{n-1}$. We want to find the minimizer $u \in H^1(\mathbb{R}^n_+)$ of the Dirichlet energy whenever one exists and determine its closed form.

Defining the $n-1$-dimensional Fourier transform of u for a fixed $x_n \in [0, \infty)$ as

$$\hat{u}(k, x_n) = \int_{\mathbb{R}^{n-1}} u(\tilde{x}, x_n) e^{-ik\cdot\tilde{x}}\, d\mathcal{H}^{n-1}(\tilde{x}), \tag{27}$$

we calculate the Dirichlet energy

$$\begin{aligned}&\int_{\mathbb{R}^n_+} |\nabla_{\tilde{x}} u(x)|^2 + |\partial_n u(x)|^2\, dx \\ =&\frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} |k^2||\hat{u}(k, x_n)|^2 + |\partial_n \hat{u}(k, x_n)|^2\, dk\, dx_n.\end{aligned} \tag{28}$$

By the Euler-Lagrange equation, Fourier transform of the minimizer solves almost everywhere the ODE $\partial_n^2 \hat{u}(k, x_n) = |k|^2 \hat{u}(k, x_n)$ with initial value $\hat{u}(k, 0) = \hat{f}(k)$. The only solution with finite energy is given by $\hat{u}(k, x_n) = e^{-|k|x_n} \hat{f}(k)$, and (26) reads

$$\begin{aligned} &\frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} 2|k|^2 e^{-2|k|x_n} |\hat{f}(k)|^2 \, dk \, dx_n \\ &= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} |k||\hat{f}(k)|^2 \, dk. \end{aligned} \tag{29}$$

This is also called the squared $H^{1/2}$-seminorm of f, denoted $[f]_{H^{1/2}}^2$, and

$$\|f\|_{H^{1/2}}^2 := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} (1 + |k|)|\hat{f}(k)|^2 \, dk \tag{30}$$

defines the norm of the Hilbert space $H^{1/2}(\mathbb{R}^{n-1}) \subset L^2(\mathbb{R}^{n-1})$.

In real space the minimizer u of (26) is then given by the inverse Fourier transform

$$u(\tilde{x}, x_n) = \mathcal{F}^{-1}\left(e^{-|\cdot|x_n} \hat{f}\right) = \int_{\mathbb{R}^{n-1}} \Phi_{x_n}(\tilde{x} - \tilde{y}) f(\tilde{y}) \, d\tilde{y}, \tag{31}$$

where $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is the Poisson kernel for the half-space and satisfies $\hat{\Phi}_{x_n}(k) = e^{-|k|x_n}$. We can determine Φ explicitly:

Lemma 2.1. *The Poisson kernel for $\mathbb{R}_+^n$ is given by*

$$\Phi_{x_n}(\tilde{x}) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{x_n}{(x_n^2 + |\tilde{x}|^2)^{\frac{n}{2}}} = \frac{2}{n\alpha_n} \frac{x_n}{(x_n^2 + |\tilde{x}|^2)^{\frac{n}{2}}}, \tag{32}$$

denoting by $\alpha_n = |B(0, 1)|$ the volume of the n-unit ball.

Proof. This proof is adapted from [33]. We use the Fourier inversion formula for $\hat{\Phi}_{x_n}$,

$$\Phi_{x_n}(\tilde{x}) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ik\cdot\tilde{x}} \hat{\Phi}_{x_n}(k) dk. \tag{33}$$

From now on we can assume by the scaling property of the Fourier transform that $x_n = 1$. In the case $n = 2$ integrating by parts twice yields

$$\Phi_1(x) = \frac{2}{2\pi} \int_0^\infty \cos(kx) e^{-k} dk = \frac{1}{\pi} \left(1 - x^2 \int_0^\infty \cos(kx) e^{-k} dk\right), \tag{34}$$

and thus

$$\Phi_1(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \tag{35}$$

which is the desired formula.

In higher dimensions we decompose the one-dimensional kernel into Gaussians

$$\frac{1}{1 + x^2} = \int_0^\infty e^{-(1+x^2)u} du = \int_0^\infty e^{-u} e^{-ux^2} du. \tag{36}$$

Taking the one-dimensional Fourier transform on either side and evaluating at $|k|$ for $k \in \mathbb{R}^{n-1}$ yields

$$e^{-|k|} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}} e^{-u} e^{-\frac{|k|^2}{4u}} \, du, \tag{37}$$

since the Fourier transform of a Gaussian is again a Gaussian. Now we apply the $n-1$-dimensional inverse Fourier transform on both sides to obtain

$$\begin{aligned}
\Phi_1(\tilde{x}) &= \frac{1}{2^{n-1}\pi^{n-1+\frac{1}{2}}} \int_0^\infty \frac{1}{\sqrt{u}} e^{-u} \int_{\mathbb{R}^{n-1}} e^{ik\cdot\tilde{x}} e^{-\frac{|k|^2}{4u}} \, dk \, du \\
&= \frac{(4\pi)^{\frac{n-1}{2}}}{2^{n-1}\pi^{n-1+\frac{1}{2}}} \int_0^\infty \frac{u^{\frac{n-1}{2}}}{\sqrt{u}} e^{-u} e^{-u|\tilde{x}|^2} \, du \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_0^\infty u^{\frac{n-2}{2}} e^{-(1+|\tilde{x}|^2)u} \, du \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{1}{(1+|\tilde{x}|^2)^{\frac{n}{2}}},
\end{aligned} \tag{38}$$

where we again used the fact that the Fourier transform of a Gaussian is a Gaussian and switched the order of integration. □

Now that we have an explicit formula for the minimizer of (26), we can find an intrinsic form for its minimum.

Lemma 2.2. *Let $f \in H^{1/2}(\mathbb{R}^{n-1})$. Then*

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} |k||\hat{f}(k)|^2 \, dk = \frac{1}{n\alpha_n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(\tilde{x}) - f(\tilde{y})|^2}{|\tilde{x} - \tilde{y}|^n} \, d\tilde{x} \, d\tilde{y}. \tag{39}$$

Proof. Assume without loss of generality that $f \in C_c^\infty(\mathbb{R}^{n-1})$.

By the monotone convergence theorem we can cut off the left-hand side by multiplying the integrand by $e^{-t|k|}$, where t is a small positive number. Then for all $f \in H^{1/2}(\mathbb{R}^{n-1})$

$$\begin{aligned}
&\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} |k||\hat{f}(k)|^2 dk \\
=&\frac{1}{(2\pi)^{n-1}} \lim_{t\downarrow 0} \int_{\mathbb{R}^{n-1}} |k| e^{-t|k|} |\hat{f}(k)|^2 \, dk \\
=&-\frac{1}{(2\pi)^{n-1}} \lim_{t\downarrow 0} \int_{\mathbb{R}^{n-1}} \partial_t e^{-t|k|} |\hat{f}(k)|^2 \, dk \\
=&-\lim_{t\downarrow 0} \int_{\mathbb{R}^{n-1}} f(\tilde{x}) (\partial_t \Phi_t * f)(\tilde{x}) \, d\tilde{x} \\
=&-\lim_{t\downarrow 0} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} f(\tilde{x}) f(\tilde{y}) \partial_t \Phi_t(\tilde{x} - \tilde{y}) \, d\tilde{y} \, d\tilde{x} \\
=&\frac{1}{2} \lim_{t\downarrow 0} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |f(\tilde{x}) - f(\tilde{y})|^2 \partial_t \Phi_t(\tilde{x} - \tilde{y}) \, d\tilde{y} \, d\tilde{x},
\end{aligned} \tag{40}$$

where the last equality is due to the fact that $\int \partial_t \Phi_t(\tilde{x}) \, d\tilde{x} = \partial_t \int \Phi_t(\tilde{x}) \, d\tilde{x} = 0$ and $\Phi_t, \partial_t \Phi_t \in L^1(\mathbb{R}^{n-1})$ for all $t > 0$.

Note that by Lemma 2.1

$$\partial_t \Phi_t(\tilde{x}-\tilde{y}) = \frac{2}{n\alpha_n}\left[\frac{1}{(t^2+|\tilde{x}-\tilde{y}|^2)^{\frac{n}{2}}} - \frac{2t^2}{(t^2+|\tilde{x}-\tilde{y}|^2)^{\frac{n+2}{2}}}\right]. \tag{41}$$

We show that the second term vanishes in the limit of the double integral and only the limit of the first part remains. To see this, take $R > 0$ such that $f \in C_c^\infty(B(0,R))$, where $B(0,R) \subset \mathbb{R}^{n-1}$. Then the domain of integration is $B(0,R) \times B(0,R)$, and $|f(\tilde{x}) - f(\tilde{y})| \leq Lip(f)|\tilde{x}-\tilde{y}|$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |f(\tilde{x})-f(\tilde{y})|^2 \frac{t^2}{(t^2+|\tilde{x}-\tilde{y}|^2)^{\frac{n+2}{2}}}\, d\tilde{y}\, d\tilde{x} \\
\leq &\int_{B(0,R)}\int_{\mathbb{R}^{n-1}} Lip^2(f)\frac{t^2|\tilde{h}|^2}{\left(t^2+|\tilde{h}|^2\right)^{\frac{n+2}{2}}}\, d\tilde{h}\, d\tilde{x} \\
= &\frac{t^4 t^{n-1}}{t^{n+2}}|B(0,R)|Lip^2(f)\int_{\mathbb{R}^{n-1}} \frac{|\tilde{h}|^2}{\left(1+|\tilde{h}|^2\right)^{\frac{n+2}{2}}}\, d\tilde{h} \\
= &t|B(0,R)|Lip^2(f)C_{n-1},
\end{aligned} \tag{42}$$

since the last integral is finite and depends only on n. We conclude that for any $f \in C_c^\infty(\mathbb{R}^{n-1})$

$$\begin{aligned}
&\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} |k||\hat{f}(k)|^2 dk \\
= &\frac{1}{n\alpha_n}\lim_{t\downarrow 0}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|f(\tilde{x})-f(\tilde{y})|^2}{(t^2+|\tilde{x}-\tilde{y}|^2)^{\frac{n}{2}}}\, d\tilde{y}\, d\tilde{x} \\
= &\frac{1}{n\alpha_n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|f(\tilde{x})-f(\tilde{y})|^2}{|\tilde{x}-\tilde{y}|^n}\, d\tilde{y}\, d\tilde{x}
\end{aligned} \tag{43}$$

by the monotone convergence theorem. □

2.2 The Dirichlet energy in terms of parallel jumps

We now consider the case of finitely many parallel interfaces $\omega_{h^m} = \mathbb{R}^{n-1}\times\{h^m\}$, where $h^1 < \ldots < h^M$. We prescribe jumps $b^m \in H^{1/2}(\omega_{h^m})$ at each interface and minimize the Dirichlet energy

$$\begin{aligned}
&E_{int}[b^1,\ldots,b^M,h^1,\ldots,h^M] \\
= &\inf\left\{\int_{\mathbb{R}^n\setminus\bigcup_m \omega_{h^m}} |\nabla u(x)|^2 dx : [u] = b^m \text{ on } \omega_{h^m}\right\}.
\end{aligned} \tag{44}$$

The minimizer $u_{b^1,\ldots,b^M,h^1,\ldots,h^M}$ is harmonic away from $\bigcup_m \omega_{h^m}$, with its jumps $[u] = u^+ - u^- \in H^{1/2}(\omega_{h^m})$ prescribed on the interfaces ω_{h^m}. In the case of one interface, i.e. $M = 1$, $h^1 = 0$, the minimizer u is optimal in both the upper and lower half-space, and the energy depends on its two traces $u^+, u^- \in H^{1/2}(\mathbb{R}^{n-1})$. Since their difference has to be $u^+ - u^- = b$, we pick as the free parameter the lower trace $w = u^-$, so that $u^+ = w + b$. The energy associated with w is by (29)

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} |k| \left(|\hat{w}(k)|^2 + |\hat{w}(k) + \hat{b}(k)|^2 \right) dk. \tag{45}$$

Optimizing this energy yields $\hat{w}(k) = -\frac{1}{2}\hat{b}(k)$ for all $k \neq 0$, i.e. $w = -\frac{1}{2}b + c$ for some $c \in \mathbb{R}$, and thus

$$\begin{aligned} E_{int}[b] &= \frac{1}{2(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} |k||\hat{b}(k)|^2 dk \\ &= \frac{1}{2n\alpha_n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|b(\tilde{x}) - b(\tilde{y})|^2}{|\tilde{x} - \tilde{y}|^n} \, d\tilde{y} \, d\tilde{x}. \end{aligned} \tag{46}$$

If there are multiple interfaces, by Lax-Milgram $u_{b^1,\ldots,b^M,h^1,\ldots,h^M}$ depends linearly on the b^m, so $u_{b^1,\ldots,b^M,h^1,\ldots,h^M} = u^1 + \ldots + u^M$, with $u^m(x)$ the solution for a single jump of b^m at height h^m. More precisely,

$$u^m(x) = \frac{1}{2} \begin{cases} \Phi_{x_n - h^m} * b^m(\tilde{x}) & \text{, if } x_n > h^m \\ -\Phi_{h^m - x_n} * b^m(\tilde{x}) & \text{, if } x_n < h^m. \end{cases}$$

Then the Dirichlet energy for multiple interfaces is given by

$$\begin{aligned} &E_{int}[b^1, \ldots, b^M, h^1, \ldots, h^M] \\ &= \sum_{m=1}^{M} E_{int}[b^m] + 2 \sum_{m<m'} \int_{\mathbb{R}^n \setminus (\omega_{h^m} \cup \omega_{h^{m'}})} \nabla u^m(x) \cdot \nabla u^{m'}(x) dx. \end{aligned} \tag{47}$$

We determine the interaction terms by cutting out the two jump planes of u^m and $u^{m'}$ and integrating by parts. Since both functions are harmonic, this leaves only boundary terms.

To this end fix $m < m'$ and define

$$\Omega_\delta := \mathbb{R}^{n-1} \times \left[(-\infty, h^m - \delta) \cup (h^m + \delta, h^{m'} - \delta) \cup (h^{m'} + \delta, \infty) \right]. \tag{48}$$

Then

$$\begin{aligned} &2 \int_{\Omega_\delta} \nabla u^m(x) \cdot \nabla u^{m'}(x) dx \\ &= - \partial_\delta \int_{\partial\Omega_\delta} u^m(x) u^{m'}(x) d\mathcal{H}^{n-1}(x) \\ &= - \frac{1}{4} \partial_\delta \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} b^m(\tilde{y}) b^{m'}(\tilde{z}) \left[2\Phi_{h^{m'} - h^m + 2\delta}(\tilde{y} - \tilde{z}) - 2\Phi_{h^{m'} - h^m - 2\delta}(\tilde{y} - \tilde{z}) \right] \\ &\quad d\mathcal{H}^{n-1}(\tilde{y}) d\mathcal{H}^{n-1}(\tilde{z}). \end{aligned} \tag{49}$$

Taking the limit as $\delta \to 0$, the left-hand side converges to the integral over $\mathbb{R}^n \setminus (\omega_{h^m} \cup \omega_{h^{m'}})$ by monotone convergence, and the right-hand side converges to

$$\begin{aligned} &- 2 \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} b^m(\tilde{y}) b^m(\tilde{z}) \partial_{x_n} \Phi_{h^{m'} - h^m}(\tilde{y} - \tilde{z}) \, d\mathcal{H}^{n-1}(\tilde{y}) \, d\mathcal{H}^{n-1}(\tilde{z}) \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} (b^m(\tilde{y}) - b^m(\tilde{z}))(b^{m'}(\tilde{y}) - b^{m'}(\tilde{z})) \partial_{x_n} \Phi_{h^{m'} - h^m}(\tilde{y} - \tilde{z}) \\ &\quad d\mathcal{H}^{n-1}(\tilde{y}) \, d\mathcal{H}^{n-1}(\tilde{z}). \end{aligned} \tag{50}$$

Here we used again the fact that $\int \partial_t \Phi_t(x) dx = 0$.

In summary, for $h^1 < \ldots < h^M$ and $b^1, \ldots, b^M \in H^{1/2}(\mathbb{R}^{n-1})$, we can write the minimum of the Dirichlet energy with prescribed jumps b on $\mathbb{R}^{n-1} \times \{h^m\}$ explicitly as

$$\sum_{m,m'=1}^{M} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} (b^m(\tilde{y}) - b^m(\tilde{z})) \partial_{x_n} \Phi_{h^{m'} - h^m}(\tilde{y} - \tilde{z})(b^{m'}(\tilde{y}) - b^{m'}(\tilde{z}))\, d\mathcal{H}^{n-1}(\tilde{y})$$
$$d\mathcal{H}^{n-1}(\tilde{z}). \tag{51}$$

We shall show in the next section that a similar formula can be found for the linear elastic energy.

3 The minimal linear elastic energy for the jump problem

We now investigate the elastic displacement in the whole space around a single slip plane. We shall consider the plane $\omega = \mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$. The slip will be a function $b \in C_c^\infty(\omega, \mathbb{R}^n)$. We want to minimize

$$\int_{\mathbb{R}^3 \setminus \omega} \mathbb{C} Du : Du\, dx \tag{52}$$

among all locally integrable displacement fields $u : \mathbb{R}^n \setminus \omega \to \mathbb{R}^n$ with $Du_{\mathrm{sym}} \in L^2(\mathbb{R}^n \setminus \omega)$ and $[u] = b$ on ω.

From here on out $\mathbb{C} \in \mathbb{R}^{n \times n \times n \times n}$ shall be a fixed fourth-order stiffness tensor with the symmetries $c_{ijkl} = c_{klij} = c_{jikl}$. ($\mathbb{C}$ is a symmetric tensor acting on the space of symmetric matrices $\mathbb{R}^{n\times n}_{\mathrm{sym}}$.) We will assume that $\mathbb{C}$ is positive definite on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, i.e. $A \cdot \mathbb{C}A \geq c|A + A^T|^2$ for all $A \in \mathbb{R}^{n\times n}$, where $c > 0$ is a constant. Note that for $n = 2$, there are 6 degrees of freedom in choosing $\mathbb{C}$, for $n = 3$ there are 21, and in general there are $\frac{n(n+1)(n^2+n+2)}{8}$. If we assume additionally that $\mathbb{C}$ be isotropic, then there are only two degrees of freedom, and $\mathbb{C}A = \frac{\mu}{2}(A + A^T) + \lambda \operatorname{tr} A \operatorname{Id}$.

Alternatively, u can be considered as an $SBV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ function with prescribed gradient decomposition $Du = \nabla u + b \otimes e_n \mathcal{H}^{n-1} \llcorner \omega$. Denoting $Eu = Du_{\mathrm{sym}}$, $\mathbb{C}Du : Du = \mathbb{C}Eu : Eu$ due to the symmetry of $\mathbb{C}$.

We start with some basic facts about this minimization problem:

Lemma 3.1. *Let $u : \mathbb{R}^n_+ \to \mathbb{R}^n$ be locally integrable with $Du_{\mathrm{sym}} \in L^2(\mathbb{R}^n_+, \mathbb{R}^{n\times n}_{\mathrm{sym}})$. Then there is exactly one $A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ with $\int_{\mathbb{R}^n_+} |Du - A|^2\, dx \leq C \int_{\mathbb{R}^n_+} |Du_{\mathrm{sym}}|^2\, dx$, where C depends only on the dimension.*

Remark 3.2. *This result is called Korn's inequality. While every connected open bounded set with Lipschitz boundary and $\mathbb{R}^n$ permit Korn's inequality, for unbounded sets this is generally untrue, as in e.g. $\Omega = \mathbb{R} \times (0, 1)$. The reason it works for the half-space is its self-similarity. For more information on Korn's inequality see e.g. [9].*

Proof. Let C be the Korn's constant for the half-ball B_+. Then for every $k > 0$ there is some $A_k \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that $\int_{B_{2^k+}} |Du - A_k|^2\, dx \leq C \int_{\mathbb{R}^n_+} |Du_{\mathrm{sym}}|^2\, dx$.

By the triangle inequality $|A_k - A_{k+j}|^2 \le C2^{-kn} \int_{B_{2^k +}} |A_k - Du|^2 + |A_{k+j} - Du|^2\,dx \le C2^{-kn} \int_{\mathbb{R}^n_+} |Du_{\mathrm{sym}}|^2\,dx$. As a Cauchy sequence in $\mathbb{R}^{3\times 3}_{\mathrm{skew}}$, the limit matrix A exists and $|A_k - A|^2 \le C2^{-nk} \int_{\mathbb{R}^n_+} |Du_{\mathrm{sym}}|^2\,dx$.

Then for any $k > 0$ we have $\int_{B_{2^k +}} |A - Du|^2\,dx \le 2 \int_{B_{2^k +}} |A_k - Du|^2 + |A_k - A|^2\,dx \le C \int_{\mathbb{R}^n_+} |Du_{\mathrm{sym}}|^2\,dx$. Since the right-hand side is independent of k, we obtain the result for A.

Due to the infinite measure of $\mathbb{R}^n_+$, there is at most one $A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that $\int_{\mathbb{R}^n_+} |Du - A|^2\,dx < \infty$, which proves uniqueness. $\square$

Next we show that if the jump is square-integrable, the two rotations in the upper and lower half-plane coincide. While this does not hold in the general SBD setting, the simple geometry of the single jump plane simplifies the problem.

Lemma 3.3. *Let $u \in SBV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $Du = \nabla u + [u] \otimes e_n \mathcal{H}^{n-1} \llcorner \omega$, with $\nabla u_{\mathrm{sym}} \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and $[u] \in L^2(\omega, \mathbb{R}^n)$, then there is a unique matrix $A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that $\|\nabla u - A\|_{L^2} \le C \|\nabla u_{\mathrm{sym}}\|_{L^2}$.*

Proof. By Lemma 3.1 there are two matrices $A, B \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ in the lower and upper half-space respectively. Consider the blow-down $u_R(x) = u(Rx)/R$ for large R. Then

$$\lim_{R\to\infty} \int_{B_{1-}} |\nabla u_R - A|^2\,dx + \int_{B_{1+}} |\nabla u_R - B|^2\,dx = 0. \tag{53}$$

By Poincaré's inequality in both half-balls, there exist $c_R, d_R \in \mathbb{R}^n$ such that

$$\lim_{R\to\infty} \int_{B_{1-}} |u_R - Ax - c_R|^2\,dx + \int_{B_{1+}} |u_R - Bx - d_R|^2\,dx = 0. \tag{54}$$

By the trace theorem we obtain that

$$\lim_{R\to\infty} \int_{B_1\cap\omega} |u_{R-} - A\tilde{x} - c_R|^2 + |u_{R+} - B\tilde{x} - d_R|^2\,d\mathcal{H}^{n-1}(\tilde{x}) = 0. \tag{55}$$

However the difference of the two traces is the rescaled jump $[u_R(\tilde{x})] = [u(R\tilde{x})]/R$, so by the triangle inequality

$$\begin{aligned}
&\lim_{R\to\infty} \int_{B_1\cap\omega} |A\tilde{x} + c_R - B\tilde{x} - d_R|^2\,d\mathcal{H}^{n-1}(\tilde{x}) \\
\le& 6 \lim_{R\to\infty} \int_{B_1\cap\omega} |u_{R-} - A\tilde{x} - c_R|^2 + |[u_R]|^2 + |u_{R+} - B\tilde{x} - d_R|^2\,d\mathcal{H}^{n-1}(\tilde{x}) \\
=&0.
\end{aligned} \tag{56}$$

It follows that $\lim_{R\to\infty}(c_R - d_R) = 0$ and that $B - A = 0$ on ω. However, since $B - A$ is skew-symmetric, it must be identically 0. Uniqueness of A is obvious. $\square$

With this we can establish the existence of a unique minimizer to the jump problem, up to a constant and a linearized rotation.

Lemma 3.4. *Let $\beta \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^{n\times n})$ be a distribution. Consider the minimization problem*

$$\min \int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : (Du-\beta)\, dx \tag{57}$$

among all $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. Then

i) if some $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |(Du-\beta)_{\mathrm{sym}}|^2\, dx < \infty$ exists, then there is a unique minimizer up to an additive constant and a linearized rotation.

ii) if $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ has $\int_{\mathbb{R}^n} |(Du-\beta)_{\mathrm{sym}}|^2\, dx < \infty$, it solves $\operatorname{div} \mathbb{C}(Du-\beta) = 0$ in $\mathcal{D}'$ if and only if it is a minimizer.

Proof. Assume some $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |(Du-\beta)_{\mathrm{sym}}|^2\, dx < \infty$ exists. Let $u_i \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a minimizing sequence. Let $v_i = u_i - u_1$. Then

$$\begin{aligned} &\int_{\mathbb{R}^n} \mathbb{C}Dv_i : Dv_i\, dx \\ \le& 2\int_{\mathbb{R}^n} \mathbb{C}(Du_i-\beta) : (Du_i-\beta) + \mathbb{C}(Du_1-\beta) : (Du_1-\beta)\, dx \\ \le& C. \end{aligned} \tag{58}$$

By Korn's inequality there are $A_i \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that $Dv_i - A_i$ is bounded in $L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and converges weakly in L^2 (up to a subsequence) to some $Dv \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$. Take $u = v + u_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. Then $(Du_i - \beta)_{\mathrm{sym}} = (Dv_i + Du_1 - \beta)_{\mathrm{sym}} \rightharpoonup (Du-\beta)_{\mathrm{sym}}$ in L^2, and due to lower semicontinuity we get

$$\int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : (Du-\beta)\, dx \le \liminf_{i\to\infty} \int_{\mathbb{R}^n} \mathbb{C}(Du_i-\beta) : (Du_i-\beta)\, dx. \tag{59}$$

This shows that u is in fact a minimizer. Taking the difference between two minimizers gives a function $w \in L^2_{\mathrm{loc}}$ with $\int_{\mathbb{R}^n} \mathbb{C}Dw : Dw\, dx = 0$, so $w = Ax + c$ for some $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$.

If $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is a minimizer, then $u + \phi$ is a competitor for all $\phi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$, and $\int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : D\phi\, dx = 0$, i.e. u solves the equation $\operatorname{div} \mathbb{C}(Du-\beta) = 0$ in $\mathcal{D}'$.

Now let $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a function with $\int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : (Du-\beta)\, dx < \infty$ and $\operatorname{div} \mathbb{C}(Du-\beta) = 0$ in $\mathcal{D}'$. We show that

$$\int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : D\phi\, dx = 0 \tag{60}$$

for all $\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $D\phi_{\mathrm{sym}} \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By density we only need to show this for smooth functions.

Let $\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^n.\mathbb{R}^n) \cap C^\infty$ with $D\phi_{\mathrm{sym}} \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Korn's inequality there is $A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that $D\phi - A \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$. Let $R > 0$. By Poincaré's inequality there is $c_R \in \mathbb{R}^n$ such that

$$\int_{B_{2R}\setminus B_R} |\phi - Ax - c_R|^2\, dx \le CR^2 \int_{B_{2R}\setminus B_R} |D\phi - A|^2\, dx. \tag{61}$$

Let $\eta_R \in C_c^\infty(B_{2R},[0,1])$ be a cutoff function with $\eta_R = 1$ in B_R and $|D\eta_R| \leq 2/R$. Define $\psi_R = (\phi - Ax - c_R)\eta_R$. Note that $\psi_R \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ is a valid test function, so that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : D\psi_R \, dx \\
&= \int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : D\phi \eta_R \, dx \\
&\quad + \int_{\mathbb{R}^n} \mathbb{C}(Du-\beta) : (\phi - Ax - c_R) \otimes D\eta_R \, dx.
\end{aligned}
\tag{62}
$$

By dominated convergence, the first integral converges to the left-hand side of (60), while the absolute value of the second is bounded by

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\mathbb{C}(Du-\beta) : (\phi - Ax - c_R) \otimes D\eta_R| \, dx \\
\leq& \|(Du-\beta)_{\mathrm{sym}}\|_{L^2} \|D\eta_R\|_{L^\infty} \|\phi - Ax - c_R\|_{L^2(B_{2R}\setminus B_R)} \\
\leq& C\, o(R)/R,
\end{aligned}
\tag{63}
$$

due to (61). Letting $R \to \infty$, (60) follows.

To show that u is indeed the minimizer, simply test with $\phi = v - u$, where v is any other competitor with finite energy. □

Remark 3.5. *For $\beta = b \otimes e_n H^{n-1} \llcorner \omega$, with $b \in H^{1/2}(\omega,\mathbb{R}^n)$, the lemma shows existence of a unique minimizer to* (52)*.*

Lemma 3.6. *Let $u \in \mathcal{S}'(\mathbb{R}^n,\mathbb{R}^n)$ be a tempered distribution with $Du_{\mathrm{sym}} \in L^2$. Then $u \in L^2_{\mathrm{loc}}$ and there is $A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that $Du - A \in L^2$. Furthermore there is some $c \in \mathbb{C}^n$ such that $\widehat{u - Ax} - c\delta_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\},\mathbb{C}^n)$ and $\widehat{Du - A} = i\widehat{u - Ax} \otimes k$ almost everywhere.*

Remark 3.7. *This lemma justifies the use of standard Fourier arithmetic for the wide class of tempered distributions with square integrable symmetrized differential.*

Proof. First we show that u is indeed in $L^2_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^n)$. For $\varepsilon > 0$, consider the mollification $u_\varepsilon = u * \phi_\varepsilon \in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Then $u_\varepsilon \rightharpoonup u$ in $\mathcal{S}'$, but also $(Du_\varepsilon)_{\mathrm{sym}} \to Du_{\mathrm{sym}}$ in $L^2(\mathbb{R}^n,\mathbb{R}^{n\times n})$.

By Korn's inequality on $\mathbb{R}^n$, there are $A_\varepsilon \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ such that

$$
\int_{\mathbb{R}^n} |Du_\varepsilon - A_\varepsilon|^2 \, dx \leq C \int_{\mathbb{R}^n} |Du_{\mathrm{sym}}|^2 \, dx. \tag{64}
$$

By Poincaré's inequality on every ball $B(0,N)$, $N \in \mathbb{N}$, there are $c_\varepsilon^N \in \mathbb{R}^n$ such that

$$
\int_{B(0,N)} |u_\varepsilon - A_\varepsilon x - c_\varepsilon^N|^2 \, dx \leq CN^2 \int_{\mathbb{R}^n} |Du_{\mathrm{sym}}|^2 \, dx. \tag{65}
$$

We can then extract a subsequence $\varepsilon_i \to 0$ such that $u_{\varepsilon_i} - A_{\varepsilon_i}x - c_{\varepsilon_i}^N \rightharpoonup v^N$ in $H^1(B(0,N),\mathbb{R}^n)$ for every $N \in \mathbb{N}$. On the other hand $u_{\varepsilon_i} \rightharpoonup u$ in

$\mathcal{D}'(B(0,N),\mathbb{R}^n)$, and $A_{\varepsilon_i}x + c^N_{\varepsilon_i} \rightharpoonup u - v^N$ in $\mathcal{D}'(B(0,N),\mathbb{R}^n)$. However, the distributional limit of an affine sequence must be affine, so there are $A \in \mathbb{R}^{n\times n}_{\text{skew}}$ not depending on N and $c^N \in \mathbb{R}^n$ such that $A_{\varepsilon_i} \to A$ and $c^N_{\varepsilon_i} \to c^N$ for every $N \in \mathbb{N}$, with $u = v^N + Ax + c^N$ in $\mathcal{D}'(B(0,N),\mathbb{R}^n)$. It follows that there is $v \in L^2_{\text{loc}}(\mathbb{R}^n,\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} v \cdot \varphi \, dx = \langle u, \varphi \rangle \tag{66}$$

for every $\varphi \in C^\infty_c(\mathbb{R}^n,\mathbb{R}^n)$. By (65), equality also holds for all Schwartz functions $\varphi \in \mathcal{S}$, so indeed with some abuse of notation $u \in L^2_{\text{loc}}(\mathbb{R}^n,\mathbb{R}^n)$. Since $Du_{\varepsilon_i} - A_{\varepsilon_i} \rightharpoonup Dv = Du + A$ in $L^2(B(0,N),\mathbb{R}^n)$ for every $N \in \mathbb{N}$, we also get

$$\begin{aligned}\int_{B(0,N)} |Du - A|^2 \, dx \leq& \liminf_{i\to\infty} \int_{\mathbb{R}^n} |Du_{\varepsilon_i} - A_{\varepsilon_i}|^2 \, dx \\ \leq& C \int_{\mathbb{R}^n} |Du_{\text{sym}}|^2 \, dx.\end{aligned} \tag{67}$$

Now we assume without loss of generality that $A = 0$, i.e. $Du \in L^2(\mathbb{R}^n,\mathbb{R}^{n\times n})$, and by Plancherel's theorem, $\widehat{Du} \in L^2(\mathbb{R}^n,\mathbb{C}^{n\times n})$. Because $\widehat{Du} = i\widehat{u} \otimes k$ in $\mathcal{S}'$, we get that there is $K \in \mathbb{N}$ such that $\widehat{u} = \sum_{|\alpha|\leq K} a_\alpha \delta_0(\partial^\alpha) + v$, with $a_\alpha \in \mathbb{C}^n$ and $v(k) = -iDu(k)k/|k|^2 \in L^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\},\mathbb{C}^n)$. Now since

$$\begin{aligned}&D\mathcal{F}^{-1}(\sum_{|\alpha|<K} a_\alpha \delta_0(\partial^\alpha)) \\ =&\mathcal{F}^{-1}(i(\widehat{u} - v) \otimes k) \\ =&\mathcal{F}^{-1}(\widehat{Du} - \widehat{Du}) = 0,\end{aligned} \tag{68}$$

$\mathcal{F}^{-1}(\sum_{|\alpha|<K} a_\alpha \delta_0(\partial^\alpha) = c \in \mathbb{C}^n$, so that $\widehat{u} - v = \mathcal{F}(c) = c/(2\pi)^n \delta_0$ for some $c \in \mathbb{C}^n$. □

Now we want to determine the pointwise values of a minimizer to the jump problem. We do this by mollifying the jump measure.

Let $\beta = b \otimes e_n \mathcal{H}^{n-1} \llcorner \omega$ be the $\mathbb{R}^{n\times n}$-valued finite measure associated to a fixed jump function $b \in C^\infty_c(\omega,\mathbb{R}^n)$.

Define $\beta_\varepsilon = \beta * \phi_\varepsilon$ the convolution of β with a standard mollifier $\phi_\varepsilon \in C^\infty_c(B_\varepsilon)$.

Lemma 3.8. *Let $f \in C^\infty_c(\mathbb{R}^n,\mathbb{R}^n)$. Then there is exactly one solution $u \in H^1(\mathbb{R}^n,\mathbb{R}^n)$ to $-\operatorname{div} \mathbb{C}Du = f$, given by $u = G * f$, where $G \in L^1_{\text{loc}}(\mathbb{R}^n,\mathbb{R}^{n\times n})$ is $-(n-2)$-homogeneous, smooth away from 0, and even.*

For the proof, see Theorem 4.1 in [12].

Lemma 3.9. *Let $\beta = b \otimes e_n \mathcal{H}^{n-1} \llcorner \omega$, with $b \in C^\infty_c(\omega,\mathbb{R}^n)$. Then the minimizer of $\int_{\mathbb{R}^n} \mathbb{C}(Du - \beta) : (Du - \beta) \, dx$ is given, up to a rotation and an additive constant, by*

$$u(x) = -\int_\omega \mathbb{C}DG(x - y) e_n b(y) \, d\mathcal{H}^{n-1}(y). \tag{69}$$

Here $\mathbb{C}DGe_n$ is the $n \times n$ matrix with coefficients $(\mathbb{C}DGe_n)_{ij} = c_{inkl}G_{kj,l}$.

Proof. Define u as above. Let $v : \mathbb{R}^n \to \mathbb{R}^n$ be the unique minimizer to (52) with $Dv - \beta \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and $\int_{B(0,1)} v\,dx = 0$. Let $u_\varepsilon = u * \phi_\varepsilon = G * \operatorname{div} \mathbb{C}\beta_\varepsilon$ and $v_\varepsilon = v * \phi_\varepsilon$. Then $Du_\varepsilon, Dv_\varepsilon \in L^2(\mathbb{R}^n, \mathbb{R}^{n\times n})$, and $\operatorname{div} \mathbb{C}Du_\varepsilon = \operatorname{div} \mathbb{C}Dv_\varepsilon = \operatorname{div} \mathbb{C}\beta_\varepsilon$. Due to Lemma 3.4 it follows that $u_\varepsilon - v_\varepsilon$ is constant. Letting ε go to zero, it follows that $u - v$ is also constant. □

Definition 3.10. *Define the Poisson kernel $P : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^{n\times n}$ as $P = -\mathbb{C}DGe_n$, or componentwise $P_{ik} = -c_{jpkn}G_{ij,p}$ so that the minimizer u to the jump problem* (52) *with $b \in C_c^\infty(\omega, \mathbb{R}^n)$ is given by $u(x) = \int_\omega P(x - y)b(y)\,d\mathcal{H}^{n-1}(y)$ whenever $x_n \neq 0$.*

Note that P is odd, smooth away from 0 *and $-(n-1)$-homogeneous.*

Now we define the interaction kernel that appears in the closed form for the elastic energy.

Definition 3.11. *Define the interaction kernel $J : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^{n\times n}$ as $J = \mathbb{C}DPe_n$, or componentwise $J_{kq} = c_{ijkn}P_{iq,j} = -c_{ijkn}c_{mpqn}G_{im,pj}$.*

Note that J is even, smooth away from 0 *and $-n$-homogeneous.*

Also define for $h \in \mathbb{R}$ the bilinear form defined on functions $u, v \in L^2(\omega, \mathbb{R}^n)$

$$
\begin{aligned}
&A_h(u, v) \\
&= \int_\omega \int_\omega (u(x) - u(y))J(x - y - he_n)(v(x) - v(y))\,d\mathcal{H}^{n-1}(x)\,d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{70}
$$

Lemma 3.12. *The bilinear form A_h as the following properties:*

i) A_h is symmetric, and $A_{-h} = A_h$.

ii) For $h \neq 0$ and $u, v \in L^2(\omega, \mathbb{R}^n)$ we have $|A_h(u, v)| \leq C\|u\|_{L^2}\|v\|_{L^2}/|h|$. Also $\int_\omega J(x + he_n)\,d\mathcal{H}^{n-1}(x) = 0$ for all $h \neq 0$.

iii) For all $h \in \mathbb{R}$ and $u, v \in H^{1/2}(\omega, \mathbb{R}^n)$ we have

$$
|A_h(u, v)| \leq C\|u\|_{H^{1/2}}\|v\|_{H^{1/2}}. \tag{71}
$$

Proof. A_h is symmetric because J is a symmetric tensor by definition. Because J is even, $A_{-h} = A_h$ by a change of variables $(x, y) \mapsto (y, x)$ and Fubini's theorem.

The estimate (ii) holds since $|J(x+he_n)| \leq C/(|x|^2+h^2)^{n/2}$, whose integral is $C/|h|$, by the fact that $\int_\omega J(x+he_n)\,d\mathcal{H}^{n-1}(x) = 0$, shown below, and Young's convolution inequality.

To show that $\int_\omega J(x+he_n)\,d\mathcal{H}^{n-1}(x) = 0$, we want to show that $\int_\omega DP(x+he_n)\,d\mathcal{H}^{n-1}(x) = 0$. Clearly all partial derivatives are integrable.

Let $i = 1, \ldots, n-1$ be a tangential direction. Then

$$
\begin{aligned}
&\limsup_{R\to\infty} \left| \int_{B_R\cap\omega} \partial_i P(x + he_n)\,d\mathcal{H}^{n-1} \right| \\
\leq &\limsup_{R\to\infty} \int_{\partial B_R\cap\omega} |P(x + he_n)|\,d\mathcal{H}^{n-2} \\
\leq &\limsup_{R\to\infty} CR^{n-2}/R^{n-1} = 0.
\end{aligned} \tag{72}
$$

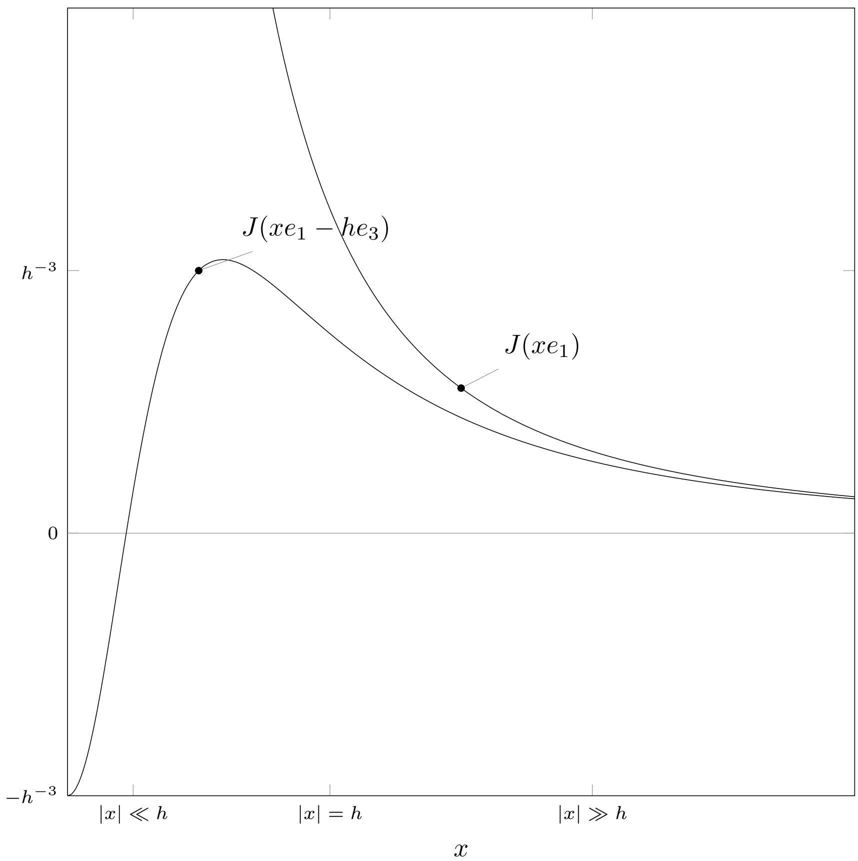

Figure 5: For $n = 3$, the -3-homogeneous kernel $J(x)$ and the translated kernel $J(x - he_3)$, where $x_3 = 0$. The translated kernel is almost equal to the homogeneous one for $|x| \gg h$ and at most Ch^{-3}, so that for $|x| \ll h$ it is dwarfed by the homogeneous one.

For the normal derivative $\partial_n P$, note that since P is $-(n-1)$-homogeneous, the integral

$$\int_{\omega \cap B_{hR}} P(x + he_n)\, d\mathcal{H}^{n-1}(x) \tag{73}$$

is independent of h. Taking the derivative in h, we see that

$$\int_{\omega \cap B_{hR}} \partial_n P(x + he_n)\, d\mathcal{H}^{n-1}(x) = -R \int_{\omega \cap \partial B_{hR}} P(x + he_n)\, d\mathcal{H}^{n-2}(x). \tag{74}$$

Since P is odd, smooth, and $-n-1$-homogeneous, we can rewrite the surface integral

$$\begin{aligned} & R \int_{\omega \cap \partial B_{hR}} P(x + he_n)\, d\mathcal{H}^{n-2}(x) \\ =& \frac{R}{2} \int_{\omega \cap \partial B_{hR}} P(x + he_n) - P(x - he_n)\, d\mathcal{H}^{n-2}(x). \end{aligned} \tag{75}$$

We estimate the integrand using the mean value theorem

$$\frac{R}{2} \int_{\omega \cap \partial B_{hR}} |P(x + he_n) - P(x - he_n)|\, d\mathcal{H}^{n-2}(x) \leq ChR \frac{(hR)^{n-2}}{(hR)^n}. \tag{76}$$

Letting $R \to \infty$, it follows that $\int_\omega \partial_n P(x + he_n)\, d\mathcal{H}^{n-1} = 0$ for all $h > 0$. For $h < 0$ the same argument applies.

The estimate (iii) holds for C_c^∞ functions because $|J(x + he_n)| \leq C/|x|^{n/2}$, which gives the $H^{1/2}$-bilinear form, and the Cauchy-Schwarz inequality. Also A_h has a unique extension to $H^{1/2}(\omega, \mathbb{R}^n)$, for which the same estimate holds. □

We show that the energy (52) is represented by A_0.

Lemma 3.13. *Let $b \in C_c^\infty(\omega, \mathbb{R}^n)$. Then the minimum energy in* (52) *is given by* $\frac{1}{2} A_0(b, b)$.

Proof. By Lemma 3.9, the minimizer to (52) is given by

$$u(\tilde{x}, x_n) = \int_\omega P(x - y) b(y)\, d\mathcal{H}^{n-1}(y). \tag{77}$$

Let $M > 0$ be such that $b = 0$ outside of $B_M \cap \omega$. We define the domains $\Omega_{\varepsilon,R} = \{x = (\tilde{x}, x_n) \in \mathbb{R}^n : |x_n| > \varepsilon, |x| < R\}$ and $\Omega_\varepsilon = \Omega_{\varepsilon,\infty}$. As this is a Lipschitz bounded domain, through integration by parts and the fact that the minimizer u solves $\operatorname{div} \mathbb{C} Du = 0$ in $\Omega_{\varepsilon,R}$ we obtain

$$\int_{\Omega_{\varepsilon,R}} \mathbb{C} Du : Du\, dx = \int_{\partial \Omega_{\varepsilon,R}} \mathbb{C} Du : u \otimes \nu\, d\mathcal{H}^{n-1}. \tag{78}$$

When $|x| = R$ we can estimate $|u(x)| \leq C \int_{B_M \cap \omega} |b(y)|/|x - y|^{n-1}\, d\mathcal{H}^{n-1}(y) \leq C/(R - M)^{n-1}$ and $|Du(x)| \leq C/|R - M|^n$. As $R \to \infty$, the equation above converges to

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \mathbb{C}Du : Du\,dx = \int_{\partial\Omega_\varepsilon} \mathbb{C}Du : u\otimes\nu\,d\mathcal{H}^{n-1} \\
=&\int_{\partial\Omega_\varepsilon}\int_\omega \mathbb{C}DP(x-y)b(y) : u(x)\otimes\nu\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
=&-\int_\omega\int_\omega \mathbb{C}DP(x-y+\varepsilon e_n)b(y) : u(x+\varepsilon e_n)\otimes e_n\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
&+\int_\omega\int_\omega \mathbb{C}DP(x-y-\varepsilon e_n)b(y) : u(x-\varepsilon e_n)\otimes e_n\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
=&-\int_\omega\int_\omega J(x-y+\varepsilon e_n)b(y)\cdot u(x+\varepsilon e_n)\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
&+\int_\omega\int_\omega J(x-y-\varepsilon e_n)b(y)\cdot u(x-\varepsilon e_n)\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x). \qquad (79)
\end{aligned}
$$

Now we use that $DP(\cdot\pm\varepsilon e_n)\in L^1(\omega,\mathbb{R}^{n\times n\times n})$, with $\int_\omega DP(y\pm\varepsilon e_n)\,d\mathcal{H}^{n-1}(y) = 0$ for all $\varepsilon > 0$.

We can then rewrite (79) as

$$
\begin{aligned}
&-\int_\omega\int_\omega J(x-y+\varepsilon e_n)b(y)\cdot u(x+\varepsilon e_n)\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
&+\int_\omega\int_\omega J(x-y-\varepsilon e_n)b(y)\cdot u(x-\varepsilon e_n)\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
=&\frac{1}{2}\int_\omega\int_\omega J(x-y+\varepsilon e_n)(b(y)-b(x))\cdot(u(y+\varepsilon e_n)-u(x+\varepsilon e_n)) \\
&d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x) \\
&-\frac{1}{2}\int_\omega\int_\omega J(x-y-\varepsilon e_n)(b(y)-b(x))\cdot(u(y-\varepsilon e_n)-u(x-\varepsilon e_n)) \\
&d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x), \qquad (80)
\end{aligned}
$$

where we used that $\int J(x\pm\varepsilon e_n)\,d\mathcal{H}^{n-1}(x) = 0$.

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus\omega} \mathbb{C}Du : Du\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \mathbb{C}Du : Du\,dx \\
=&\lim_{\varepsilon\to 0}\frac{1}{2}\int_\omega\int_\omega J(x-y+\varepsilon e_n)(b(y)-b(x))\cdot(u(y+\varepsilon e_n)-u(x+\varepsilon e_n)) \\
&-J(x-y-\varepsilon e_n)(b(y)-b(x))\cdot(u(y-\varepsilon e_n)-u(x-\varepsilon e_n)) \\
&d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x). \qquad (81)
\end{aligned}
$$

Note that $|J(y\pm\varepsilon e_n| \le C|y|^{-n}$. Since $b\in H^{1/2}(\omega,\mathbb{R}^n)$, the $u(\cdot\pm\varepsilon e_n)$ are bounded in $H^{1/2}(\omega,\mathbb{R}^n)$ by Korn's inequality, and converge strongly in $H^{1/2}(\omega,\mathbb{R}^n)$ to the traces u^+ and u^- respectively, we can replace $u(\cdot+\varepsilon e_n)$ in the limit with u^+, since

$$
\begin{aligned}
=&\Big| \int_\omega \int_\omega (b(y)-b(x)) J(x-y+\varepsilon e_n) \\
&\cdot \big[(u(y+\varepsilon e_n)-u^+(y)) - (u(x+\varepsilon e_n)-u^+(x))\big] \\
&d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x)\Big| \\
\le& \int_\omega \int_\omega |b(y)-b(x)| \frac{C}{|x-y|^n} \\
&\big|(u(y+\varepsilon e_n)-u^+(y)) - (u(x+\varepsilon e_n)-u^+(x))\big| \\
&d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x) \\
\le& C[b]_{H^{1/2}} [u(\cdot+\varepsilon e_n) - u^+]_{H^{1/2}} \\
&\to_{\varepsilon\downarrow 0} 0,
\end{aligned}
\tag{82}
$$

and similarly we can replace $u(\cdot - \varepsilon e_n)$ with u^-.

Now that we deal only with the double integrals

$$
\begin{aligned}
=\lim_{\varepsilon\to 0} \frac{1}{2}\Big[& \int_\omega \int_\omega (b(y)-b(x)) J(y-x+\varepsilon e_n) \\
&\cdot (u^+(y)-u^+(x))\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x) \\
&- \int_\omega \int_\omega (b(y)-b(x)) J(y-x-\varepsilon e_n) \\
&\cdot (u^-(y)-u^-(x))\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x)\Big],
\end{aligned}
\tag{83}
$$

we can use dominated convergence on both double integrals, since $|J(x-y\pm\varepsilon e_n)| \le C/|x-y|^n$, $J(x-y\pm\varepsilon e_n) \to J(x-y)$ pointwise, and b, u^+, and u^- are all in $H^{1/2}(\omega,\mathbb{R}^n)$, the above limit equals

$$
\begin{aligned}
&\frac{1}{2}\Big[\int_\omega \int_\omega (b(y)-b(x)) J(y-x) \\
&(u^+(y)-u^+(x))\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x) \\
&- \int_\omega \int_\omega (b(y)-b(x)) J(y-x) \\
&(u^-(y)-u^-(x))\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x)\Big] \\
=&\frac{1}{2} \int_\omega \int_\omega (b(y)-b(x)) J(y-x) \\
&(b(y)-b(x))\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x).
\end{aligned}
\tag{84}
$$

□

Example 3.14. *i) For $c_{ijkl} = \delta_{ik}\delta_{jl}$, i.e. $\mathbb{C}Du = Du$, we know that up to constants $G_{ij}(x) = \delta_{ij}|x|^{-(n-2)}$, $P_{ik}(x) = -\partial_n G_{ik}(x) = \delta_{ik} x_n/|x|^n$, and $J_{kq}(x) = \partial_n P_{kq}(x) = \delta_{kq}|x|^{-n} - n x_n^2/|x|^{n+2}$, which for $x_n = 0$ coincides with the kernel* (46) *in the $H^{1/2}$-seminorm.*

ii) For $c_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}$ and $n = 3$, i.e. $\mathbb{C}Du = 2Du_{\text{sym}}$, the isotropic case with Poisson ratio $\sigma = 0$, the Green's function is up to a constant $G_{ij}(x) = 3\delta_{ij}/|x| + x_i x_j/|x|^3$. The kernel J is then given by

$$J_{kq}(x) = \frac{\delta_{kq}}{|x|^3} + 3\delta_{3k}\frac{x_q x_3}{|x|^5} + 3\delta_{3q}\frac{x_k x_3}{|x|^5} - 15\frac{x_3^2 x_k x_q}{|x|^7}, \tag{85}$$

or in matrix form

$$J(x) = \frac{\mathrm{Id}}{|x|^3} + 3x_3\frac{x \otimes e_3 + e_3 \otimes x}{|x|^5} - 15x_3^2\frac{x \otimes x}{|x|^7}, \tag{86}$$

which for $x_3 = 0$ yields $J_{kq}(x) = \delta_{kq}/|x|^3$, as expected.

iii) For the isotropic case with Poisson ratio $\sigma \in (-1/2, 1)$, the kernel J is given, in matrix form, by

$$\begin{aligned} J(x) =&(1-2\sigma)\frac{\mathrm{Id}}{|x|^3} + 2\sigma\frac{e_3 \otimes e_3}{|x|^3} + 3\sigma\frac{x \otimes x}{|x|^5} \\ &+ (3-3\sigma)x_3\frac{x \otimes e_3 + e_3 \otimes x}{|x|^5} + 3\sigma x_3^2\frac{Id}{|x|^5} - 15x_3^2\frac{x \otimes x}{|x|^7}, \end{aligned} \tag{87}$$

which evaluated at $x_3 = 0$ gives

$$J(x) = (1-2\sigma)\frac{\mathrm{Id}}{|x|^3} + 2\sigma\frac{e_3 \otimes e_3}{|x|^3} + 3\sigma\frac{x \otimes x}{|x|^5}, \tag{88}$$

the same kernel as in [11].

3.1 Interaction between parallel planes

For multiple parallel planes, the energy is given by a combination of the A_h. Let $M \in \mathbb{N}$ be the number of jump planes and $h^1 < \ldots < h^M$. We consider the problem of minimizing the elastic elastic energy among deformations with prescribed jumps on the planes $\omega_{h^m} = \mathbb{R}^{n-1} \times \{h^m\} \subset \mathbb{R}^n$:

$$\int_{\mathbb{R}^n \setminus \bigcup_{m=1}^M \omega_{h^m}} \mathbb{C}Du : Du \, dx \tag{89}$$

among all locally integrable functions $u : \mathbb{R}^n \setminus \bigcup_{m=1}^M \omega_{h^m} \to \mathbb{R}^n$ with $Du_{\text{sym}} \in L^2(\mathbb{R}^n \setminus \bigcup_{m=1}^M \omega_{h^m}, \mathbb{R}^n)$ and $[u] = b^m$ on ω_{h^m}.

By Lemma 3.4 we know that (89) has a unique minimizer for every $\boldsymbol{b} = (b^1, \ldots, b^M) \in C_c^\infty(\mathbb{R}^{n-1}, \mathbb{R}^{nM})$ up to a single linearized rotation and a constant, given by $u = \sum_m P_{h^m} * b^m$, where P_{h^m} is the Poisson kernel for the single jump problem on ω_{h^m}. A double-integral form of the elastic energy in (89) follows through integration by parts as in Lemma 3.13.

Lemma 3.15. *Let $h > 0$, $M \in \mathbb{N}$, $b^m \in C_c^\infty(\omega_{h^m}, \mathbb{R}^n)$ for $m = 1, \ldots, M$. Then the minimum of the energy (89) is given by*

$$\frac{1}{2}\sum_{m=1}^M A_0(b^m, b^m) + \sum_{m<m'} A_{h^{m'}-h^m}(b^m, b^{m'}), \tag{90}$$

where we identify the ω_{h^m} with ω by orthogonal projection.

Proof. By Lemma 3.4 the minimizer to (89) is $u = \sum_{m=1}^{M} u^m$, with

$$u^m(\tilde{x}, x_n) = \int_\omega P(\tilde{x} - \tilde{y} - (h^m - x_n)e_n) b^m(y)\, d\mathcal{H}^{n-1}(y). \tag{91}$$

Then

$$\begin{aligned}&\int_{\mathbb{R}^n \setminus \bigcup_{m=1}^{M} \omega_{h^m}} \mathbb{C}Du : Du\, dx \\ =&\frac{1}{2} \sum_{m=1}^{M} A_0(b^m, b^m) \\ &+ 2 \sum_{m<m'} \int_{\mathbb{R}^n \setminus (\omega_{h^m} \cup \omega_{h^{m'}})} \mathbb{C}Du^m : Du^{m'}\, dx.\end{aligned} \tag{92}$$

We now fix $m < m'$ and consider only the interaction term between u^m and $u^{m'}$. We proceed as in the proof of Lemma 3.13, this time splitting the domain into three parts, letting $\varepsilon, R > 0$ and setting

$$\begin{aligned}\Omega^-_{\varepsilon,R} &= \{(\tilde{x}, x_n) \in \mathbb{R}^n \,:\, x_n < h^m - \varepsilon, |\tilde{x}| < R\}, \\ \Omega^0_{\varepsilon,R} &= \{(\tilde{x}, x_n) \in \mathbb{R}^n \,:\, x_n \in (h^m + \varepsilon, h^{m'} - \varepsilon), |\tilde{x}| < R\}, \\ \Omega^+_{\varepsilon,R} &= \{(\tilde{x}, x_n) \in \mathbb{R}^n \,:\, x_n > h^{m'} + \varepsilon, |\tilde{x}| < R\},\end{aligned} \tag{93}$$

we can use the Gauss divergence theorem in each domain, since $\operatorname{div} \mathbb{C}Du^m = \operatorname{div} \mathbb{C}Du^{m'} = 0$ away from $\omega_{h^m} \cup \omega_{h^{m'}}$, giving

$$\int_{\Omega^-_{\varepsilon,R}} \mathbb{C}Du^m : Du^{m'}\, dx = \int_{\partial\Omega^-_{\varepsilon,R}} \mathbb{C}Du^m : u^{m'} \otimes \nu\, d\mathcal{H}^1, \tag{94}$$

and likewise for $\Omega^0_{\varepsilon,R}$ and $\Omega^+_{\varepsilon,R}$. Next we let $R \to \infty$ on both sides of the equation, leaving

$$\begin{aligned}&\int_{\Omega^-_\varepsilon \cup \Omega^0_\varepsilon \cup \Omega^+_\varepsilon} \mathbb{C}Du^m : Du^{m'}\, dx \\ =&-\int_{\omega_{h^{m'}+\varepsilon}} \mathbb{C}Du^m : u^{m'} \otimes \nu\, d\mathcal{H}^{n-1} \\ &+\int_{\omega_{h^{m'}-\varepsilon}} \mathbb{C}Du^m : u^{m'} \otimes \nu\, d\mathcal{H}^{n-1} \\ &-\int_{\omega_{h^m+\varepsilon}} \mathbb{C}Du^m : u^{m'} \otimes \nu\, d\mathcal{H}^{n-1} \\ &+\int_{\omega_{h^m-\varepsilon}} \mathbb{C}Du^m : u^{m'} \otimes \nu\, d\mathcal{H}^{n-1}.\end{aligned} \tag{95}$$

We insert the definition of P and J into the first surface integral, yielding

$$\begin{aligned}&-\int_\omega \int_\omega b^m(x) J(y - x + (h^{m'} - h^m + \varepsilon)e_n) u^{m'}(y + (h^{m'} + \varepsilon)e_n) \\ &d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x) \\ =&\frac{1}{2} \int_\omega \int_\omega (b^m(y) - b^m(x)) J(y - x + (h^{m'} - h^m + \varepsilon)e_n) \\ &(u^{m'}(y + (h^{m'} + \varepsilon)e_n) - u^{m'}(x + (h^{m'} + \varepsilon)e_n))\, d\mathcal{H}^{n-1}(y)\, d\mathcal{H}^{n-1}(x),\end{aligned} \tag{96}$$

where we used that $\int_\omega J(y+he_n)\,d\mathcal{H}^{n-1}(y)=0$ for every $h\neq 0$.

In the limit as $\varepsilon\downarrow 0$, we can replace $u^{m'}(\cdot+(h^{m'}+\varepsilon)e_n)$ by the upper trace $u_+^{m'}(\cdot+h^{m'}e_n)$, since the traces converge strongly in $H^{1/2}(\omega,\mathbb{R}^n)$ and the kernels $J(\cdot+(h^{m'}-h^m+\varepsilon)e_n)$ are uniformly bounded by $C/|\cdot|^n$, allowing to bound the difference using Hölder's inequality by

$$
\begin{aligned}
=&\Big|\int_\omega\int_\omega (b^m(y)-b^m(x))J(y-x+(h^{m'}-h^m+\varepsilon)e_n)\\
&\big[(u^{m'}(y+(h^{m'}+\varepsilon)e_n)-u_+^{m'}(y+h^{m'}e_n))\\
&-(u^{m'}(x+(h^{m'}+\varepsilon)e_n)-u_+^{m'}(x+h^{m'}e_n))\big]\\
&d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x)\Big|\\
\le& C[b^m]_{H^{1/2}}[u^{m'}(\cdot+h^{m'}e_n)-u_+^{m'}(\cdot+h^{m'})]_{H^{1/2}}\\
&\to_{\varepsilon\downarrow 0} 0.
\end{aligned}
\tag{97}
$$

Now that we deal only with the double integral

$$
\begin{aligned}
=&\lim_{\varepsilon\to 0}\frac{1}{2}\int_\omega\int_\omega (b^m(y)-b^m(x))J(y-x+(h^{m'}-h^m+\varepsilon)e_n)\\
&(u_+^{m'}(y+h^{m'}e_n)-u_+^{m'}(x+h^{m'}e_n))\,d\mathcal{H}^{n-1}(y)\,d\mathcal{H}^{n-1}(x),
\end{aligned}
\tag{98}
$$

the kernels $J(\cdot+(h^{m'}-h^m+\varepsilon)e_n)$ converge pointwise to $J(\cdot+(h^{m'}-h^m)e_n)$ and are uniformly bounded by $C/|\cdot|^n$, since both b^m and $u_+^{m'}(\cdot+h^{m'})$ are in $H^{1/2}$, by the dominated converge theorem the double integral converges to $\frac{1}{2}A_{h^{m'}-h^m}(b^m,u_+^{m'}(\cdot+h^{m'}e_n))$. This works also for the other three surface integrals in (95), so that in the limit as $\varepsilon\downarrow 0$ we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus(\omega_{h^m}\cup\omega_{h^{m'}})}\mathbb{C}Du^m:Du^{m'}\,dx\\
=&\frac{1}{2}A_{h^{m'}-h^m}(b^m,u_+^{m'}(\cdot+h^{m'}e_n))\\
&-\frac{1}{2}A_{h^{m'}-h^m}(b^m,u_-^{m'}(\cdot+h^{m'}e_n))\\
&+\frac{1}{2}A_0(b^m,u_+^{m'}(\cdot+h^me_n))\\
&-\frac{1}{2}A_0(b^m,u_-^{m'}(\cdot+h^me_n))\\
=&\frac{1}{2}A_{h^{m'}-h^m}(b^m,b^{m'}),
\end{aligned}
\tag{99}
$$

since A_h is bilinear and the difference of upper and lower trace is the jump of $u^{m'}$, which is $b^{m'}$ and 0 respectively. □

Definition 3.16. *We define for $\boldsymbol{b}=(b^1,\ldots,b^M)\in C_c^\infty(\mathbb{R}^{n-1},\mathbb{R}^n)$ the bilinear form*

$$
B_{h^1,\ldots,h^M}(\boldsymbol{b},\boldsymbol{b})=\sum_{m=1}^M A_0(b^m,b^m)+2\sum_{m<m'}A_{h^{m'}-h^m}(b^m,b^{m'}).
\tag{100}
$$

We also define for $h > 0$ the shorthand notation $B_h = B_{h,2h,\dots,Mh}$.

Note that B_h can be represented by a kernel $\mathbb{J}_h : \mathbb{R}^{n-1} \to \mathbb{R}^{nM\times nM}_{\mathrm{sym}}$, where $\mathbb{J}_{h\,mm'}(z) = \frac{1}{2}(J(z+(m'-m)he_n) + J(z-(m'-m)he_n)) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ for $m, m' = 1, \dots, M$.

Remark 3.17. *Note that $B_0(\boldsymbol{b}, \boldsymbol{b}) = A_0(\sum_{m=1}^M b^m, \sum_{m=1}^M b^m)$. Also if $\boldsymbol{b} \in H^{1/2}(\mathbb{R}^{n-1}, \mathbb{R}^{nM})$ then $h \mapsto B_h(\boldsymbol{b}, \boldsymbol{b})$ is continuous.*

The kernel $\mathbb{J}_h : \mathbb{R}^{n-1} \to \mathbb{R}^{nM\times nM}$ is smooth away from 0 with $\mathbb{J}_{\lambda h}(\lambda x) = \lambda^{-n}\mathbb{J}_h(x)$.

3.2 Positivity of the kernel and iterative mollification

It is unknown in the anisotropic case whether the single-plane kernel J is positive definite. However, through Korn's inequality, we obtain at least

$$\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} (b(x)-b(y))J(x-y)(b(x)-b(y))\,dx\,dy \geq c[b]^2_{H^{1/2}}. \tag{101}$$

This is a strictly weaker condition than a pointwise bound from below, which does however hold for all isotropic $\mathbb{C}$ (see example 3.14).

The multi-plane kernel $\mathbb{J}_h$ in general is not positive semidefinite, as can be seen for $M = 2$ planes ω_0, ω_h, $h > 0$, in $n = 2$ with $\mathbb{C}Du = Du_{\mathrm{sym}}$. Consider the deformation

$$u(x_1, x_2) = \begin{cases} (-2x_1x_2, x_1^2 - 1)^T & \text{in } (-1,1)\times(0,h) \\ (0,0) & \text{elsewhere.} \end{cases} \tag{102}$$

Then the jumps $\boldsymbol{b} = (b^1, b^2) \in H^{1/2}(\mathbb{R}^2, \mathbb{R}^4)$ certainly satisfy

$$\int\int_{|x-y|<h^{3/2}} (\boldsymbol{b}(x)-\boldsymbol{b}(y))\mathbb{J}_h(x-y)(\boldsymbol{b}(x)-\boldsymbol{b}(y))\,dx\,dy \geq ch^{3/2}, \tag{103}$$

since $\mathbb{J}_h(x) \approx \mathrm{Id}/|x|^2$ for $|x| \ll h$, but

$$\int_{\mathbb{R}^2} |Du_{\mathrm{sym}}|^2\,dx = h^2, \tag{104}$$

thus the kernel $\mathbb{J}_h$ must have a significant negative part for $|x| \geq h^{3/2}$. For this reason, a simple dyadic decomposition of the kernel $\mathbb{J}_h$ into its actions on the annuli $|x| \in (2^{-j}, 2^{-j+1})$ does not yield a pointwise positive kernel. However, we can decompose the kernel positively into length scales in other ways.

Let from now on $\phi_l(x) = l^{-(n-1)}\phi(x/l)$ be an even standard mollifier on $\mathbb{R}^n$, i.e. $\phi \in C_c^\infty(B(0,1))$, $\phi \geq 0$, $\int_{\mathbb{R}^{n-1}} \phi\,dx = 1$. Then define a new mollifier as $\tilde{\phi}_l = \circledast_{j=1}^\infty \phi_{2^{-j}l}$, the infinitely iterated convolution of ϕ_l with its rescaled versions. Note that the iterated convolutions $\circledast_{j=1}^M \phi_{2^{-j}l}$ form a Cauchy sequence in any Sobolev space $W^{1,p}(\mathbb{R}^{n-1})$, making $\tilde{\phi}_l \in C_c^\infty(B(0,l)$ again a standard mollifier with the additional property that $\tilde{\phi}_l * \phi_l(x) = \tilde{\phi}_{2l}(x) = 2^{-(n-1)}\tilde{\phi}_l(x/2)$. Then the following lemma allows for a positive decomposition of B_h into dyadic length scales.

Lemma 3.18. *Let $\boldsymbol{b} \in H^{1/2}(\mathbb{R}^{n-1}, \mathbb{R}^{nM})$, $l > 0$. Then* *i)* $B_h(\boldsymbol{b} * \tilde{\phi}_l, \boldsymbol{b} * \tilde{\phi}_l) \leq B_h(\boldsymbol{b} * \phi_l, \boldsymbol{b} * \phi_l) \leq B_h(\boldsymbol{b}, \boldsymbol{b})$.

ii) *The bilinear form* $\boldsymbol{b} \mapsto B_h(\boldsymbol{b} * \phi_l, \boldsymbol{b} * \phi_l)$ *is positive semidefinite, bounded from above by* $C\|\boldsymbol{b}\|_{L^2}^2/l$*, and represented by a double integral with kernel* $z \mapsto -\frac{1}{2}B_h(\phi_l, \phi_l(\cdot - z))$*, which is a* $\mathbb{R}^{nM\times nM}_{\text{sym}}$ *matrix.*

Proof. For the inequality $B_h(\boldsymbol{b}*\phi_l, \boldsymbol{b}*\phi_l) \leq B_h(\boldsymbol{b}, \boldsymbol{b})$, note that B_h is a positive semidefinite bilinear form on $H^{1/2}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ and thus convex. Note that ϕ_l defines a Borel probability measure μ on $H^{1/2}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ supported on the translations $\boldsymbol{b}(\cdot - x)$. Since B_h is translation invariant and by Jensen's inequality,

$$\begin{aligned} &B_h(\boldsymbol{b} * \phi_l, \boldsymbol{b} * \phi_l) \\ =&B_h\Big(\int_{H^{1/2}(\mathbb{R}^{n-1},\mathbb{R}^n)} \boldsymbol{x}\, d\mu(x), \int_{H^{1/2}(\mathbb{R}^{n-1},\mathbb{R}^n)} \boldsymbol{x}\, d\mu(x)\Big) \\ \leq &\int_{H^{1/2}(\mathbb{R}^{n-1},\mathbb{R}^n)} B_h(\boldsymbol{x}, \boldsymbol{x})\, d\mu(x) \\ = &\int_{H^{1/2}(\mathbb{R}^{n-1},\mathbb{R}^n)} B_h(\boldsymbol{b}, \boldsymbol{b})\, d\mu(x) \\ =&B_h(\boldsymbol{b}, \boldsymbol{b}). \end{aligned} \tag{105}$$

The inequality $B_h(\boldsymbol{b} * \tilde{\phi}_l, \boldsymbol{b} * \tilde{\phi}_l) \leq B_h(\boldsymbol{b} * \phi_l, \boldsymbol{b} * \phi_l)$ is the previous inequality applied to the test function $\boldsymbol{b} * \phi_{l/2}$ and the mollifier $\tilde{\phi}_{l/2}$.

To show the kernel representation, use Fubini's theorem to obtain

$$\begin{aligned} &B_h(\boldsymbol{b} * \phi_l, \boldsymbol{b} * \phi_l) \\ =&\int\int \boldsymbol{b}(z) B_h(\phi_l, \phi_l(\cdot - (z - z'))\boldsymbol{b}(z')\, dz\, dz'. \end{aligned} \tag{106}$$

Note that $|B_h(\phi_l, \phi_l(\cdot - z))| \leq C \min(l^{-n}, |z|^{-n})$ by the bounds on $\mathbb{J}_h$. Thus $\int_{\mathbb{R}^{n-1}} |B_h(\phi_l, \phi_l(\cdot - z))|\, dz \leq C/l$ and

$$\int_{\mathbb{R}^{n-1}} B_h(\phi_l, \phi_l(\cdot - z))\, dz = B_h(\phi_l, \int_{\mathbb{R}^{n-1}} \phi_l(\cdot - z)\, dz) = 0. \tag{107}$$

With this we see that

$$\begin{aligned} &B_h(\boldsymbol{b} * \phi_l, \boldsymbol{b} * \phi_l) \\ =&-\frac{1}{2}\int\int (\boldsymbol{b}(z) - \boldsymbol{b}(z')) B_h(\phi_l, \phi_l(\cdot - (z - z'))(\boldsymbol{b}(z) - \boldsymbol{b}(z'))\, dz\, dz'. \end{aligned} \tag{108}$$

The L^2-bound follows from Young's convolution inequality and the L^1-bound on the kernel. □

Now both $B_h(\cdot * \tilde{\phi}_l, \cdot * \tilde{\phi}_l)$ and $B_h - B_h(\cdot * \tilde{\phi}_l, \cdot * \tilde{\phi}_l)$ are positive definite bilinear forms. We can iterate this decomposition as follows:

Definition 3.19. *Define for* $\boldsymbol{b} \in H^{1/2}(\mathbb{R}^{n-1}, \mathbb{R}^{nM})$*,* $h \in [0, \infty]$*, and* $j \in \mathbb{Z}$ *the quadratic form*

$$B_{h,j}(\boldsymbol{b}, \boldsymbol{b}) = B_h(\boldsymbol{b} * \tilde{\phi}_{2^{-j}}, \boldsymbol{b} * \tilde{\phi}_{2^{-j}}) - B_h(\boldsymbol{b} * \tilde{\phi}_{2^{-(j-1)}}, \boldsymbol{b} * \tilde{\phi}_{2^{-(j-1)}}) \tag{109}$$

and the corresponding kernel $\mathbb{J}_{h,j} : \mathbb{R}^{n-1} \to \mathbb{R}^{nM\times nM}_{\text{sym}}$ *as*

$$\mathbb{J}_{h,j}(z) = -\frac{1}{2}\Big(B_h(\tilde{\phi}_{2^{-j}}, \tilde{\phi}_{2^{-j}}(\cdot - z)) - B_h(\tilde{\phi}_{2^{-(j-1)}}, \tilde{\phi}_{2^{-(j-1)}}(\cdot - z))\Big). \tag{110}$$

Lemma 3.20. *i) For every $h > 0, j \in \mathbb{Z}$ $B_{h,j}$ is a positive semidefinite bilinear form on $L^2(\mathbb{R}^{n-1}, \mathbb{R}^{nM})$ represented by the integrable kernel $\mathbb{J}_{h,j}$ with $B_{h,j}(\boldsymbol{b}, \boldsymbol{b}) \le C2^j \|\boldsymbol{b}\|_{L^2}^2$ for all $\boldsymbol{b} \in L^2(\mathbb{R}^{n-1}, \mathbb{R}^{nM})$.*

ii) The kernels $\mathbb{J}_{h,j} : \mathbb{R}^{n-1} \to \mathbb{R}^{nM\times nM}_{\text{sym}}$ obey the following estimates

$$|\mathbb{J}_{h,j}(z)| \le C \begin{cases} 2^{nj} & , \text{ for } |z| \le 2^{-j} \\ 2^{-2j}/|z|^{n+2} & , \text{ for } |z| \ge 2^{-j}. \end{cases} \tag{111}$$

iii) Whenever $2^{-j} \le h$,

$$|\mathbb{J}_{h,j}(z) - \mathbb{J}_{\infty,j}(z)| \le C \begin{cases} 2^{-2j}/h^{n+2} & , \text{ for } |z| \le h \\ 2^{-2j}/|z|^{n+2} & , \text{ for } |z| \ge h. \end{cases} \tag{112}$$

Note that $\mathbb{J}_{\infty,j}$ is the kernel defined without interactions between the planes.

Proof. (i) follows from Lemma 3.18. Define the notation $J_z(x) = J(x + ze_n)$.

To prove (ii), the estimate $|\mathbb{J}_{h,j}(z)| \le 2^{nj}$ also follows from Lemma 3.18. For the other estimate, assume that $|z| > 6 \times 2^{-j}$. Then

$$\begin{aligned} &B_h(\tilde\phi_{2^{-j}}, \tilde\phi_{2^{-j}}(\cdot - z)) \\ =& -2 \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \tilde\phi_{2^{-j}}(x) \mathbb{J}_h(x-y) \tilde\phi_{2^{-j}}(y-z)\, dx\, dy, \end{aligned} \tag{113}$$

so that by a Taylor expansion on $\mathbb{J}_h$ we get since $\tilde\phi_l$ is even,

$$\begin{aligned} &|B_h(\tilde\phi_{2^{-j}}, \tilde\phi_{2^{-j}}(\cdot - z)) + 2\mathbb{J}_h(z)| \\ \le& C2^{-2j} \|D^2 \mathbb{J}_h\|_{L^\infty(B(z,2^{-j}))} \\ \le& C2^{-2j} |z|^{-(n+2)}, \end{aligned} \tag{114}$$

and likewise

$$\begin{aligned} &|B_h(\tilde\phi_{2^{-j+1}}, \tilde\phi_{2^{-j+1}}(\cdot - z)) + 2\mathbb{J}_h(z)| \\ \le& C2^{-2j+2} |z|^{-(n+2)}, \end{aligned} \tag{115}$$

so that $|\mathbb{J}_{h,j} \le C2^{-2j}|z|^{-(n+2)}$ whenever $|z| > 6 \times 2^{-j}$.

For (iii), note that $(B_h - B_\infty)(\boldsymbol{b}, \boldsymbol{b}) = 2\sum_{m<m'} A_{(m'-m)h}(b^m, b^{m'})$. Also, by a Taylor expansion on J_h,

$$|-\frac{1}{2} A_{(m'-m)h}(\tilde\phi_l, \tilde\phi_l(\cdot - z)) - J_{(m'-m)h}(z)| \le C \|D^2 J_{(m'-m)h}\|_{L^\infty(B(z,l))} l^2. \tag{116}$$

The same holds for $\tilde\phi_{2l}$. Since $(h, z) \mapsto J_h(z)$ is smooth and $-n$-homogeneous, its Hessian is smooth and $-n-2$-homogeneous, so for $|z| \le 2h$ we get

$$\|D^2 J_{(m'-m)h}\|_{L^\infty(B(z,l))} \le \frac{C}{h^{n+2}}, \tag{117}$$

and for $z > 2h$

$$\|D^2 J_{(m'-m)h}\|_{L^\infty(B(z,l))} \le \frac{C}{|z|^{n+2}}. \tag{118}$$

□

Note that the kernels $\mathbb{J}_{h,j} : \mathbb{R}^{n-1} \to \mathbb{R}^{nM\times nM}_{\mathrm{sym}}$ are integrable and have integral 0, since

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} B_h(\tilde\phi_{2^{-j}}, \tilde\phi_{2^{-j}}(\cdot - x))\,dx \\
=&\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \\
&(\tilde\phi_{2^{-j}}(y) - \tilde\phi_{2^{-j}}(z))(\tilde\phi_{2^{-j}}(y-x) - \tilde\phi_{2^{-j}}(z-x))\mathbb{J}_h(y-z)\,dz\,dy\,dx \\
=&\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} (\tilde\phi_{2^{-j}}(y) - \tilde\phi_{2^{-j}}(z))\mathbb{J}_h(y-z) \\
&\int_{\mathbb{R}^{n-1}} (\tilde\phi_{2^{-j}}(y-x) - \tilde\phi_{2^{-j}}(z-x))\,dx\,dz\,dy \\
=&\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} 0\,dy\,dz = 0.
\end{aligned}
\tag{119}
$$

It follows that $\mathbb{J}_{h,j}$ does not have a positive sign. In fact, even in the isotropic case with $M = 2$, the kernel $\mathbb{J}_h$ is not positive definite everywhere on $\mathbb{R}^{n-1}$ for $h \neq 0$. For anisotropic $\mathbb{C}$, it is not even known whether the single-plane kernel $J : \mathbb{R}^{n-1} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is positive definite almost everywhere.

However, the quadratic forms associated with these kernels are necessarily nonnegative. This nonnegativity is equivalent for L^1 kernels to nonnegativity of $\hat J(0) - \hat J(k)$ almost everywhere in Fourier space.

We now show that for $n = 3$ there exist smooth -3-homogeneous scalar-valued kernels on $\mathbb{R}^2 \setminus \{0\}$ which are not positive but produce a positive definite, even elliptic, bilinear form on $H^{1/2}$.

Example 3.21. *Let $\alpha, \delta > 0$. Define the kernel $J_{\alpha,\delta} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ as*

$$
J_{\alpha,\delta}(z) = \begin{cases} -\delta|z|^{-3} & , \text{ if } |z_2| < \alpha|z_1| \\ |z|^{-3} & , \text{ otherwise.} \end{cases}
\tag{120}
$$

Then $J_{\alpha,\delta}$ is -3-homogeneous and not positive, but for α, δ small enough we have

$$
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (u(x) - u(y))^2 J_{\alpha,\delta}(x-y)\,dy\,dx \geq c(\alpha,\delta)[u]^2_{H^{1/2}}
\tag{121}
$$

for all $u \in H^{1/2}(\mathbb{R}^2)$.

Proof. For $z \in \mathbb{R}^2$ with $|z_2| < \alpha|z_1|$, we define $z' = z/2 + \alpha z_1 e_2 = (\mathrm{Id}/2 + 2\alpha e_2 \otimes e-1)z$ and $z'' = z - z' = (\mathrm{Id}/2 - 2\alpha e_2 \otimes e-1)z$, and note that $z_2' \in (\alpha z_1', 3\alpha z_1')$, $z_2'' \in (-3\alpha z_1'', (-\alpha z_1'')$, so that $|z'|, |z"| < c(\alpha)|z|$. Then we

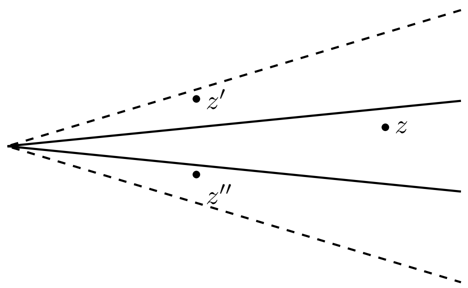

Figure 6: If the difference $z = y - x$ is in the region where $J_{\alpha,\delta} < 0$, z is split up into $z' + z''$, which are both in the region where $J_{\alpha,\delta} > 0$.

estimate for $u \in H^{1/2}(\mathbb{R}^2)$

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\{z \in \mathbb{R}^2 : |z_2| < \alpha |z_1|\}} \frac{(u(x+z) - u(x))^2}{|z|^3} \, dz \, dx \\
\leq & 2 \int_{\mathbb{R}^2} \int_{\{z \in \mathbb{R}^2 : |z_2| < \alpha |z_1|\}} \\
& \frac{(u(x+z'+z'') - u(x+z'))^2 + (u(x+z') - u(x))^2}{|z|^3} \, dz \, dx \\
\leq & 8 c(\alpha)^3 \int_{\mathbb{R}^2} \int_{\{z \in \mathbb{R}^2 : |z_2| \geq \alpha |z_1|\}} \frac{(u(y+z) - u(y))^2}{|z|^3} \, dz \, dy.
\end{aligned} \tag{122}$$

It follows that for $\delta < 1/(16 c(\alpha)^3)$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u(y) - u(x))^2 J_{\alpha,\delta}(y - x) \, dy \, dx \\
\geq & \frac{1}{2} \int_{\mathbb{R}^2} \int_{\{z \in \mathbb{R}^2 : |z_2| \geq \alpha |z_1|\}} \frac{(u(x+z) - u(x))^2}{|z|^3} \, dz \, dx \\
\geq & \frac{1}{2 + 16 c(\alpha)^3} [u]_{H^{1/2}}^2 .
\end{aligned} \tag{123}$$

□

Remark 3.22. *While $J_{\alpha,\delta}$ is not smooth, any smooth (away from 0) -3-homogeneous kernel between $J_{\alpha,\delta}$ and $|\cdot|^{-3}$ has at least the same ellipticity without necessarily being positive.*

After periodification of the kernel (see Section 3.3 below), the above examples will still have a negative part near 0, where the effect of the periodification is negligible.

3.3 Fourier methods

We now change the domain of integration from the Euclidean space $\mathbb{R}^n$ to the partially periodic setting $\mathbb{T}^{n-1} \times \mathbb{R}$, so that the jumps are $(0,1)^{n-1}$-periodic.

Consider the jump problem for M jumps $(b^1, \ldots, b^M) = \boldsymbol{b} \in C^\infty(\mathbb{T}^{n-1}, \mathbb{R}^{nM})$ on M planes $\omega_{h^m} = \mathbb{T}^{n-1} \times \{h^m\}$, where $h^1 < \ldots < h^M$, and $\mathbb{T}^{n-1}$ is the torus

$\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$:

$$\inf\Big\{\int_{\mathbb{T}^{n-1}\times(\mathbb{R}\setminus\{h^1,\dots,h^M\})}\mathbb{C}Du:Du\,dx$$
$$:\ u\in H^1(\mathbb{T}^{n-1}\times(\mathbb{R}\setminus\{h^1,\dots,h^M\}),\mathbb{R}^n),[u]=b^m\text{ on }\omega_{h^m}\Big\}. \quad (124)$$

If we can solve the jump problem in $\mathbb{R}^n\setminus\bigcup_{m=1}^M\omega_{h^m}$, it turns out there is an easy solution for the periodic problem:

Lemma 3.23. *The periodic jump problem* (124) *has a unique (up to constants) minimizer* $u\in H^1(\mathbb{T}^{n-1}\times(\mathbb{R}\setminus\{h^1,\dots,h^M\},\mathbb{R}^n)$ *given by*

$$u(x)=\sum_{m=1}^M\int_{\omega_{h^m}}P^{\mathrm{per}}(x-y)b^m(y)\,d\mathcal{H}^{n-1}(y) \quad (125)$$

with energy

$$\frac{1}{2}\sum_{m=1}^M\int_{\omega_{h^m}}\int_{\omega_{h^m}}(b^m(x)-b^m(y))J^{\mathrm{per}}(x-y)(b^m(x)-b^m(y))\,d\mathcal{H}^{n-1}(x)$$
$$d\mathcal{H}^{n-1}(y)$$
$$+\sum_{m<m'}\int_{\omega_{h^m}}\int_{\omega_{h^{m'}}}(b^m(x)-b^m(y))J^{\mathrm{per}}(x-y)(b^{m'}(x)-b^{m'}(y))$$
$$d\mathcal{H}^{n-1}(x)\,d\mathcal{H}^{n-1}(y), \quad (126)$$

where $P^{\mathrm{per}}:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}^{n\times n}$ *is given by*

$$P^{\mathrm{per}}(x)=\frac{1}{2}\sum_{z\in\mathbb{Z}^{n-1}\times\{0\}}(P(x-z)+P(x+z)). \quad (127)$$

$J^{\mathrm{per}}:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}^{n\times n}_{\mathrm{sym}}$ *is given by*

$$J^{\mathrm{per}}(x)=\sum_{z\in\mathbb{Z}^{n-1}\times\{0\}}J(x-z). \quad (128)$$

Remark 3.24. *Note that in the definition of* P^{per}*, the symmetrization does not formally change the sum, but since in general* $|P(x)|$ *only decays as* $|x|^{-(n-1)}$*, the symmetrization ensures the series' convergence. In the harmonic case of course* $P(x)=Cx_n/|x|^n$*, which is summable without symmetrization.*

Proof. Uniqueness of the minimizer can be shown as in Lemma 3.4. Also $u\in H^1(\mathbb{T}^{n-1}\times(\mathbb{R}\setminus\{h^1,\dots,h^M\},\mathbb{R}^n)$ is a minimizer to (124) if and only if it solves $[u]=b^m$ on $\mathbb{T}^{n-1}\times\{h^m\}$ and $\operatorname{div}\mathbb{C}Du=0$. We now assume that $M=1$, $h=0$, since otherwise we are only dealing with a finite sum of translations of such solutions. Note that there exists $R>0$, $b_0\in C_c^\infty(B_R^{n-1},\mathbb{R}^{nM})$ with $\sum_{z\in\mathbb{Z}^{n-1}}b_0(x-z)=b(x)$. Then the jump problem (52) for b_0 is solved by $u_0=P*b_0$.

We show that $u=\frac{1}{2}\sum_{z\in\mathbb{Z}^{n-1}}(u_0(\cdot-z)+u_0(\cdot+z))$ is in $H^1(\mathbb{T}^{n-1}\times(\mathbb{R}\setminus\{0\},\mathbb{R}^n)$ and solves $[u]=b$ on $\mathbb{T}^{n-1}\times\{0\}$, $\operatorname{div}\mathbb{C}Du=0$.

To this end note that for $x \in (0,1)^{n-1} \times \mathbb{R}$ and $z \in \mathbb{Z}^{n-1}$ with $|z| > \sqrt{n}$, we have

$$|P(x+z) + P(x-z)| = |P(z+x) - P(z-x)| \le C\frac{|x|}{|z|^n} \tag{129}$$

since P is odd. It follows that the series in the definition of P^{per} converges absolutely. Also $u(x) = \frac{1}{2}\sum_{z\in\mathbb{Z}^{n-1}} u_0(x-z) + u_0(x+z))$ converges absolutely locally uniformly in $\mathbb{T}^{n-1} \times (\mathbb{R} \setminus \{0\})$, and u is smooth away from its jump plane. By the locally uniform convergence we obtain $\operatorname{div} \mathbb{C}Du = 0$ away from the jump. We have yet to show that $u \in H^1(\mathbb{T}^{n-1} \times (\mathbb{R}\setminus\{0\}), \mathbb{R}^n)$. To this end, we write $u = u_1 + u_2$, with $u_1(x) = \sum_{z\in\mathbb{Z}^{n-1}\cap B_{R+\sqrt{n}}} u_0(x-z)$ and $u_2(x) = \frac{1}{2}\sum_{z\in\mathbb{Z}^{n-1}\setminus B_{R+\sqrt{n}}}(u_0(x-z) + u_0(x+z))$. Note that while u is periodic, u_1 and u_2 are not. We see that $u_1 \in H^1((0,1)^{n-1} \times (\mathbb{R}\setminus\{0\}), \mathbb{R}^n)$, since it is a finite sum of H^1-functions. To show that $u_2 \in H^1((0,1)^{n-1} \times \mathbb{R}, \mathbb{R}^n)$, remember

$$|DP(x)| \le \frac{C}{|x|^n}. \tag{130}$$

Then for $x \in (0,1)^{n-1} \times \mathbb{R}$ we have

$$\begin{aligned}|Du_2(x)| \le& C \sum_{z\in\mathbb{Z}^{n-1}\setminus B_{R+\sqrt{n}}} \|b_0\|_{L^1} \|DP\|_{L^\infty(\mathbb{R}^n\setminus B_{|x-z|-R})} \\ \le& C\|b_0\|_{L^1} \sum_{z\in\mathbb{Z}^{n-1}\setminus B_{R+\sqrt{n}}} \min((|z| - R - \sqrt{n})^{-n}, x_n^{-n}) \\ \le& C(1+|x_n|)^{-1}. \end{aligned}\tag{131}$$

This implies that $Du_2 \in L^2((0,1)^{n-1} \times \mathbb{R}, \mathbb{R}^{n\times n}$. All in all, $u \in H^1$ and has jump $[u] = b$ on $\mathbb{T}^{n-1} \times \{0\}$ by the continuity of the trace operator.

Finally to show the representation of the energy with help of the periodic kernel integrate by parts. □

Since the minimal energy for $\mathbb{T}^{n-1}$ behaves the same way as on $\mathbb{R}^{n-1}$, we introduce a shorthand notation for the case $h^m = mh$ with $h \in [0,\infty]$.

Definition 3.25. *For $\boldsymbol{b} \in C^\infty(\mathbb{T}^{n-1}, \mathbb{R}^{nM})$ and $h > 0$ define for $x \in \mathbb{T}^{n-1}$ the kernel$J_h : \mathbb{T}^{n-1} \to \mathbb{R}^{nM\times nM}_{\text{sym}}$ as*

$$\mathbb{J}^{\text{per}}_{h\,mm'}(x) = \frac{1}{2}\left(J^{\text{per}}(x + (mh - m'h)e_n) + J^{\text{per}}(x - (mh - m'h)e_n)\right). \tag{132}$$

Also define the quadratic form

$$\begin{aligned}&B^{\text{per}}_h(\boldsymbol{b}, \boldsymbol{b}) \\ =& \int_{\mathbb{T}^{n-1}}\int_{\mathbb{T}^{n-1}} (\boldsymbol{b}(x) - \boldsymbol{b}(y))\mathbb{J}^{\text{per}}_h(x-y)(\boldsymbol{b}(x) - \boldsymbol{b}(y)) \\ & d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^{n-1}(y).\end{aligned}\tag{133}$$

Since the torus $\mathbb{T}^{n-1}$ is a group under addition, the convolution of two periodic functions $f, g \in L^2(\mathbb{T}n-1)$ is the periodic function $f*g(x) = \int_{\mathbb{T}^{n-1}} f(y)g(x-y)\, d\mathcal{H}^{n-1}(y)$.

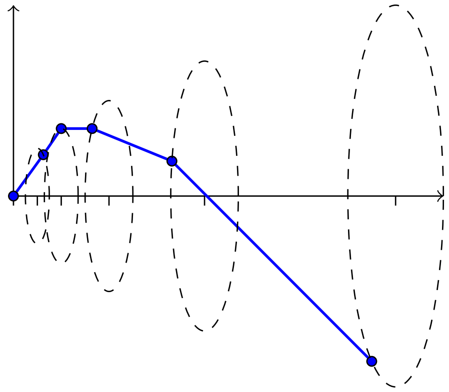

Figure 7: Illustration of Lemma 4.1. The classical one-dimensional Sobolev inequality can be improved upon by a logarithmic prefactor if we are given a choice of dyadic length scales, since the square root function is logarithmically not in $H^1(0,1)$.

Also since $\mathbb{T}^{n-1}$ is locally Euclidean, and $\phi \in C_c^\infty(B^{n-1}(0,1/2))$ is a standard mollifier on $\mathbb{R}^{n-1}$, its periodic extension is a standard mollifier on $\mathbb{T}^{n-1}$. Defining again $\tilde{\phi}_l = \circledast_{j=1}^\infty \phi_{2^{-j}l}$ the infinite convolution, for $l \le \frac{1}{4}$ we get again $\tilde{\phi}_{2l} = \tilde{\phi}_l * \phi_l = 2^{n-1}\tilde{\phi}(2\cdot)$. The decomposition of B_h^{per} from Definition 3.19 can be defined in the same way, and the results from Lemmas 3.18,3.20 hold as well, where in the latter $|z|$ is shorthand notation for the periodic function $\mathrm{dist}(z, \mathbb{Z}^{n-1})$ for $z \in \mathbb{T}^{n-1}$, since there are multiple straight lines from z to 0 in the torus.

4 The phase-field energy and compactness

We consider the following energy functional defined for $\boldsymbol{b} \in H^{1/2}(\mathbb{T}^2, \mathbb{R}^{2M})$, where $\varepsilon, h > 0$.

$$E_{\varepsilon,h}(\boldsymbol{b}) = \frac{1}{\varepsilon}\int_{\mathbb{T}^2} \mathrm{dist}^2(\boldsymbol{b}, \mathbb{Z}^{2M}) + B_h^{\mathrm{per}}(\boldsymbol{b}, \boldsymbol{b}), \tag{134}$$

4.1 Preliminaries

We show the following variant of a Poincaré/Sobolev inequality.

Lemma 4.1. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1} \times [0,\infty))$. Let $N \in \mathbb{N}$, $j_1, \ldots, j_N \in \mathbb{Z}$, $\theta \in (0,1)$. Then for at least $\lfloor (1-\theta)N \rfloor$ many j_i,*

$$\int_{\mathbb{R}^{n-1}} |u(x, 2^{-j}) - u(x,0)|^2 \, d\mathcal{H}^{n-1}(x) \le \frac{16\, 2^{-j}}{\theta N + 1} \int_{\mathbb{R}^{n-1} \times (0,\infty)} |\partial_n u(x)|^2 \, dx. \tag{135}$$

Remark 4.2. *For $N = 1$, $\theta = 0$, this is the classic one-dimensional Sobolev inequality. We will take $N \approx |\log \varepsilon|$, this will allow us to cancel the factor $1/|\log \varepsilon|$ in the energy, in turn discarding some length scales where the inequality may not be satisfied.*

Proof. We prove this by contradiction. Assume by rescaling that there exists $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1} \times [0,\infty))$ with $\int_{\mathbb{R}^{n-1}\times(0,\infty)} |\partial_n u(x)|^2\,dx = 1$ and

$$\int_{\mathbb{R}^{n-1}} |u(\tilde{x}, 2^{-j}) - u(\tilde{x}, 0)|^2 \, d\mathcal{H}^{n-1}(\tilde{x}) \geq \frac{16\, 2^{-j}}{\theta N} \tag{136}$$

for at least $\lceil \theta N \rceil + 1$ many points $x_n = 2^{-j}$.

Fix $\Psi \in L^2(\mathbb{R}^{n-1})$ with $\|\Psi\|_{L^2} = 1$ and consider instead the function $v(\tilde{x}, x_n) = u(\tilde{x}, 0) + \|u(\tilde{x}, x_n) - u(\tilde{x}, 0)\|_{L^2}\Psi(\tilde{x})$. Then (136) is satisfied for v as well, and $\int_{\mathbb{R}^{n-1}\times(0,\infty)} |\partial_n v(x)|^2\,dx \leq 1$. Also $v(\tilde{x}, x_n) = w(x_n)\Psi(\tilde{x})$, with $w(0) = 0$, $|w(2^{-j})|^2 \geq \frac{16\,2^{-j}}{\theta N+1}$ for at least $\lceil \theta N \rceil + 1$ many $j \in \mathbb{Z}$, and $\int_0^\infty |w'(t)|^2\,dt \leq 1$.

We show that such a w cannot exist. We replace w by the piecewise affine interpolation between $(0,0)$, all the points $\left(2^{-j}, \sqrt{\frac{16\,2^{-j}}{\theta N+1}}\right)$ where j is as in (136), and continued constantly after the largest such 2^{-j}. We shall call this function $\tilde{w}$.

Since $\tilde{w}$ is concave, increasing, and linear between interpolation points, its energy is minimal for the obstacle problem (136), in particular lower than that of w. However, we can estimate its energy in each of the at least $\lceil \theta N \rceil + 1$ intervals $I = (2^{-j'}, 2^{-j})$ or $I = (0, 2^{-j})$ where $\tilde{w}$ is affine. On each such interval we have $|\tilde{w}'|^2 \geq \frac{4\,2^j}{\theta N+1}$, whereas the length of I is at least $2^{-j}/2$, giving a contribution $\int_I |\tilde{w}'(t)|^2\,dt \geq \frac{2}{\theta N+1}$. This contradicts the assumption that there are at least $\lceil \theta N \rceil + 1$ such intervals. $\square$

We now show that a series of stacked plates is rigid as long as there is no normal jump between them and at least one of them is thick.

Lemma 4.3. *Let $\omega \subset \mathbb{R}^{n-1}$ be open, or $\omega = \mathbb{T}^{n-1}$. Let $\Omega = \omega \times (-1,1)$. Let $M \in \mathbb{N}$ and $\tilde{\omega} \Subset \omega$ open, bounded, connected with Lipschitz boundary. Then there is a constant $C_{M,\omega} > 0$ such that the following holds: For all $-1 < h^1 < \ldots < h^M < 1$ let $\omega_{h^m} = \omega \times \{h^m\}$. Also define $h_0 = -1$ and $h_{M+1} = 1$. Let $u \in H^1(\Omega \setminus \bigcup_{m=1}^M \omega_{h^m}, \mathbb{R}^n)$ with $[u \cdot e_n] = 0$ on ω_{h^m}. Then for $m = 0, \ldots, M$ there are $A^m \in \mathbb{R}^{n\times n}_{\mathrm{skew}}$ with $A^m e_n = A^{m'} e_n$ for all $m, m' = 1, \ldots, M$ and*

$$\sum_{m=0}^{M} \int_{\tilde{\omega}\times(h^m,h^{m+1})} |Du - A^m|^2\,dx \leq C_{M,\omega} \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2\,dx. \tag{137}$$

In the case $\omega = \mathbb{T}^{n-1}$ we can choose $A^m = 0$.

Remark 4.4. *Note that the constant does not depend on the width of the plates, only on their number. This is due to the lack of jump in the normal direction. If normal jumps are allowed, a thin bent plate between two undeformed blocks violates the inequality. The only lack of rigidity is the rotation of plates against each other. As $M \to \infty$, we proved exponential growth of the Korn's constant. We can show a lower bound of the order M^2 by considering M plates of width $2/M$ and deforming each plate identically by the displacement $u(\tilde{x}, y) = \cos(x_1)e_n + y\sin(x_1)e_1$, with $y = x_n - h^m$ in $\omega \times (h^m, h^{m+1})$. Note that u only jumps in e_1-direction, and that*

$$\int_{\Omega\setminus\bigcup_m \omega_{h^m}} |Eu|^2\,dx \approx M \int_0^{2/M} y^2\,dy \approx 1/M^2, \tag{138}$$

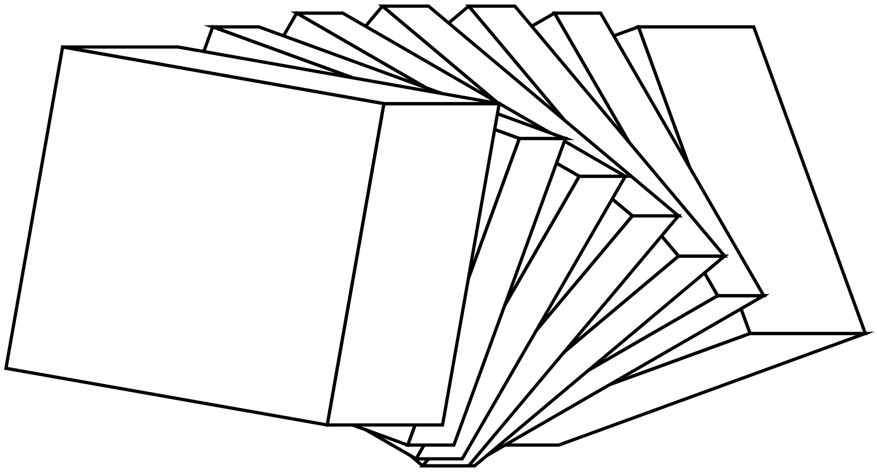

Figure 8: Illustration of Lemma 4.3. Stacked thin plates with no normal displacement jump can rotate against each other but are otherwise rigid.

whereas for any family $(A^m \in \mathbb{R}^{n\times n}_{\text{skew}})_m$ *and any* $\tilde\omega$ *with* $\mathcal{H}^{n-1}(\tilde\omega) \geq \mathcal{H}^{n-1}(\omega)/2$ *we get*

$$\sum_{m=0}^{M} \int_{\tilde\omega\times(h^m,h^{m+1})} |Du - A^m|^2\,dx \approx 1. \tag{139}$$

Proof. Since there are $M+1$ regions without jumps, at least one of them has thickness $h^{m+1} - h^m \geq 2/(M+1)$. Assume without loss of generality that this holds for $m = 0$. Otherwise show the statement for $\Omega \times (h^m, 1)$ and for $\Omega \times (-1, h^{m+1})$, since we no longer need $h^{M+1} - h^0 = 2$.

Let $l = \operatorname{dist}(\tilde\omega, \partial\omega)/\sqrt{n-1})$. We cover $\tilde\omega$ with dyadic $n-1$-cubes $q_{z,l}$ of side length l that lie in ω. We let $\omega' = \bigcup_z q_{z,l}$ and note that $\tilde\omega \subseteq \omega' \subseteq \omega$, and that ω' is connected with Lipschitz boundary.

We obtain $A^0 \in \mathbb{R}^{n\times n}_{\text{skew}}$ by applying Korn's inequality in $\omega' \times (h^0, h^1)$, which is connected with Lipschitz boundary. Note that the Korn's constant only depends on ω' and M. Assume without loss of generality that $A_0 = 0$.

Now assume that $m = 1, \ldots, M+1$ and we have found $A^0, \ldots, A^{m-1} \in \mathbb{R}^{n\times n}_{\text{skew}}$ such that $A^i e_n = 0$ for all $i = 0, \ldots, m-1$ and

$$\sum_{i=0}^{m-1} \int_{\omega'\times(h^i,h^{i+1})} |Du - A^i|^2\,dx \leq C(\Omega, M, m) \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2\,dx. \tag{140}$$

We now take $h = \min(l, h^{m+1} - h^m)$ and cover ω' with cubes $q_{z,h}$ of side length h with finite overlap. This is possible since $h \leq l$, by possibly subdividing each cube making up ω' separately. Now for each cube $q_{z,h}$ there are n-dimensional cubes $Q^+_{z,h} = q_{z,h} \times (h^m, h^m + h)$ and $Q^-_{z,h} = q_{z,h} \times (h^m - h, h^m)$. We now apply Korn's inequality in all the $Q^+_{z,h}$ to obtain a family $(A_z \in \mathbb{R}^{n\times n}_{\text{skew}})_z$ such that

$$\sum_z \int_{Q^+_{z,h}} |Du - A_z|^2\,dx \leq C \int_{\omega'\times(h^m,h^m+h)} |Eu|^2\,dx. \tag{141}$$

By the Poincaré-trace inequality there are $c_z^+ \in \mathbb{R}^n$ with

$$\sum_z \int_{q_{z,h}\times\{h^m\}} |u^+ - A_z x - c_z^+|^2 \, d\mathcal{H}^{n-1} \le Ch \int_{\omega\times(h^m,h^m+h)} |Eu|^2 \, dx. \quad (142)$$

For the lower trace, we note that $u \cdot e_n \in H^1(\omega \times (-1,1))$, and by (140)

$$\int_{\omega'\times(h^0,h^m)} |D(u\cdot e_n)|^2 \, dx \le C(\omega', M, m) \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx. \quad (143)$$

We now apply the Poincaré-trace inequality in all the $Q_{z,h}^-$ to obtain $d_z^- \in \mathbb{R}$ with

$$\sum_z \int_{q_{z,h}\times\{h^m\}} |u^- \cdot e_n - d_z^-|^2 \, d\mathcal{H}^{n-1} \le C(\omega', M, m) \int_{\Omega\setminus\bigcup_{j=1}^M \omega_{h^m}} |Eu|^2 \, dx. \quad (144)$$

Note that $u \cdot e_n$ does not jump on ω_{h^m}.
Taking the difference between the e_n-component of (142) and (144) yields

$$\sum_z \int_{q_{z,h}} |A_z e_n|^2 \, d\mathcal{H}^{n-1} \le \frac{C(\omega', M, m) + C}{h} \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx. \quad (145)$$

We now replace A_z with $\tilde{A}_z \in \mathbb{R}^{(n-1)\times(n-1)}_{\text{skew}}$ by discarding A_z's nth row and column. Then $\tilde{A}_z$ can also be considered a skew-symmetric $n \times n$ matrix by filling it with zeros. Then

$$\sum_z \int_{Q_{z,h}^+} |A_z - \tilde{A}_z|^2 \, dx \le (C + C(\omega', M, m)) \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx \quad (146)$$

and

$$\sum_z \int_{Q_{z,h}^+} |Du - \tilde{A}_z|^2 \, dx \le (C + C(\omega', M, m)) \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx. \quad (147)$$

Now define $v \in H^1(\omega', \mathbb{R}^{n-1})$ by $v(\tilde{x}) = \fint_{h^m}^{h^m+h} \pi \circ u(\tilde{x}, t) \, dt$, where $\pi = \mathrm{Id} - e_n \otimes e_n$. Then by Jensen's inequality

$$\begin{aligned} &\int_{\omega'} |Ev|^2 \, d\mathcal{H}^{n-1} \\ \le &\sum_z \int_{q_{z,h}} |Dv - \tilde{A}_z|^2 \, d\mathcal{H}^{n-1} \\ \le &\frac{C + C(\omega, M, m)}{h} \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx. \end{aligned} \quad (148)$$

We can now use Korn's inequality in ω' to obtain $A \in \mathbb{R}^{(n-1)\times(n-1)}_{\text{skew}}$ with

$$\int_{\omega'} |Dv - A|^2 \, d\mathcal{H}^{n-1} \le \frac{C + C(\omega', M, m)}{h} \int_{\Omega\setminus\bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx. \quad (149)$$

By a triangle inequality, identifying A with an $n \times n$ matrix A^m as before,

$$\int_{\omega' \times (h^m, h^m+h)} |Du - A^m|^2 \, dx \leq \tilde{C} \int_{\Omega \setminus \bigcup_{m=1}^M \omega_{h^m}} |Eu|^2 \, dx. \tag{150}$$

If $h = h^{m+1} - h^m$ this shows the induction step with a new constant $C(\omega', M, m+1) = \tilde{C}$. If $h = l$, we can extend the above inequality to $\omega' \times (h^m, \min(h^m + 2l, h^{m+1}))$ by choosing a larger constant. Repeating this step at most $2/l$ times, we can extend it up to $\omega' \times (h^m, h^{m+1})$.

This proves the statement by restricting from ω' to $\tilde{\omega}$. Note that for $\omega = \mathbb{T}^{n-1}$ we can choose $\omega' = \omega$, allowing us to choose $A^m = 0$ for each m. □

4.2 Compactness results

We now focus only on the case where $n = 3$, $h^m = mh$ for some $h > 0$.

In this situation we show compactness for the energy

$$E_{\varepsilon,h}(\boldsymbol{b}) = \frac{1}{\varepsilon} \int_{\mathbb{T}^2} \operatorname{dist}^2(\boldsymbol{b}, \mathbb{Z}^{2M}) + B_h^{\text{per}}(\boldsymbol{b}, \boldsymbol{b}), \tag{151}$$

where $\boldsymbol{b} = (b^1, \ldots, b^M) \in H^{1/2}(\mathbb{T}^2, \mathbb{R}^{2M})$ is a slip field, i.e. a jump field $\boldsymbol{b} \in H^{1/2}(\mathbb{T}^2, \mathbb{R}^{3M})$ with $b^m \cdot e_3 = 0$ for every $m = 1, \ldots, M$.

Proposition 4.5 (Compactness I)**.** *Let $M > 0$, $\varepsilon_i \downarrow 0$, $h_i = h(\varepsilon_i) \downarrow 0$, with*

$$\liminf_{i \to \infty} \frac{\log h(\varepsilon_i)}{\log \varepsilon_i} \geq 1. \tag{152}$$

Then for every sequence $\boldsymbol{b}_i \in H^{1/2}(\mathbb{T}^2, \mathbb{R}^{2M})$ with

$$\limsup_{i \to \infty} \frac{1}{|\log \varepsilon_i|} E_{\varepsilon_i, h_i}(\boldsymbol{b}_i) \leq T < \infty, \tag{153}$$

there are numbers $\eta_0 \in (0,1)$, $C > 0$ such that for all $\eta \in (0, \eta_0)$ there are sequences $k_i \to \infty$, $v_i^\Sigma \in BV(\mathbb{T}^2, \mathbb{Z}^2)$ with

i) $\lim_{i \to \infty} 2^{k_i} \|v_i^\Sigma - \sum_{m=1}^M b_i^m\|_{L^2} = 0$.

ii) $\sup_i |Dv_i^\Sigma|(\mathbb{T}^2) \leq \frac{CT}{\eta}$.

iii) $\limsup_{i\ to\infty} \frac{1}{k_i} A_0(v_i^\Sigma * \Phi_{2^{-k_i}}, v_i^\Sigma * \Phi_{2^{-k_i}}) \leq (1+\eta)(\log 2) T$.

iv) For every i, there are at least $\lfloor (1-\eta) k_i \rfloor$ many $j \in \{1, \ldots, k_i\}$ with

$$\int \int_{|x-y| \leq 2^{-j}} \frac{|v_i^\Sigma(x) - v_i^\Sigma(y)|^2}{2^{-3j}} \, dx \, dy \leq \frac{CT}{\eta}. \tag{154}$$

Remark 4.6. *This gives us compactness for the sum of slips $\sum_{m=1}^M b^m$ for bounded energy sequences, which (modulo constants and subsequences) will converge in L^1 to a $BV(\mathbb{T}^2, \mathbb{Z}^2)$ function.*

Stronger compactness can not be expected, as the Γ-limit depends only on $\sum_{m=1}^M b^m$ and not the individual slips.

Proposition 4.7 (Compactness II)**.** *Let $M > 0$, $\varepsilon_i \downarrow 0$, $h_i = h(\varepsilon_i) \downarrow 0$, with*

$$\lim_{i\to\infty} \frac{\log h(\varepsilon_i)}{\log \varepsilon_i} = \beta \in [0, 1). \tag{155}$$

Then for every sequence $\boldsymbol{b}_i \in H^{1/2}(\mathbb{T}^2, \mathbb{R}^{2M})$ with

$$\limsup_{i\to\infty} \frac{1}{|\log \varepsilon_i|} E_{\varepsilon_i,h_i}(\boldsymbol{b}_i) \le T < \infty, \tag{156}$$

there are numbers $\eta_0 \in (0, 1)$, $C > 0$ such that for all $\eta \in (0, \eta_0)$ there are sequences $k_i \to \infty$, $\boldsymbol{v}_i \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ with

i) $\lim_{i\to\infty} 2^{k_i} \|\boldsymbol{v}_i - \boldsymbol{b}_i\|_{L^2} = 0$.

ii) $\sup_{i\to\infty} |D\boldsymbol{v}_i|(\mathbb{T}^2) \le \frac{CT}{\eta}$.

iii) $\limsup_{i\to\infty} \frac{1}{k_i} B^{\mathrm{per}}_{h_i}(\boldsymbol{v}_i * \Phi_{2^{-k_i}}, \boldsymbol{v}_i * \Phi_{2^{-k_i}}) \le (1 + \eta)(\log 2)T$.

iv)

$$\limsup_{i\to\infty} \frac{1}{k_i} \left[\sum_{j=\lceil(\beta+\eta)k_i\rceil}^{k_i} B^{\mathrm{per}}_{\infty,j}(\boldsymbol{v}_i, \boldsymbol{v}_i) + \sum_{j=1}^{\lfloor(\beta-\eta)k_i\rfloor} B^{\mathrm{per}}_{0,j}(\boldsymbol{v}_i, \boldsymbol{v}_i) \right]$$
$$\le (1 + \eta)(\log 2)T. \tag{157}$$

v) For every i, there are at least $\lfloor(1 - \beta - \eta)k_i\rfloor$ many $j \in \{\lceil\beta k_i\rceil, \dots, k_i\}$ with

$$\int\int_{|x-y|\le 2^{-j}} \frac{|\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^2}{2^{-3j}} \, dx \, dy \le \frac{CT}{\eta}. \tag{158}$$

vi) For every i, there are at least $\lfloor(\beta - \eta)k_i\rfloor$ many $j \in \{1, \dots, \lfloor\beta k_i\rfloor\}$ with

$$\int\int_{|x-y|\le 2^{-j}} \frac{|\sum_{m=1}^M v_i^m(x) - \sum_{m=1}^M v_i^m(y)|^2}{2^{-3j}} \, dx \, dy \le \frac{CT}{\eta}. \tag{159}$$

Remark 4.8. *Note that the sum $\sum_{m=1}^M v_i^m$ is still bounded in $BV(\mathbb{T}^2, \mathbb{Z}^2)$, but now the full slip vector $\boldsymbol{b}$ converges (modulo constants and subsequences) to a $BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ function in L^1. In fact we get this for bounded energy sequences as long as $\liminf_{i\to\infty} \frac{\log h(\varepsilon_i)}{\log \varepsilon_i} < 1$, as in that case we can extract a subsequence and some $\beta \in [0, 1)$ where the limit is attained.*

Proof of Compactness I. Fix $i \in \mathbb{N}$. Let $u : \mathbb{T}^2 \times \mathbb{R} \setminus \bigcup_{m=1}^M \omega_{mh_i} \to \mathbb{R}^3$ be the minimizer of (52). Then it follows from Lemma 4.3 that

$$\int_{\mathbb{T}^2\times\mathbb{R}\setminus\bigcup_{m=1}^M \omega_{mh_i}} |Du|^2 \, dx \le CT|\log \varepsilon|. \tag{160}$$

Consider all $N = \lfloor(1 - \eta/2)|\log_2 \varepsilon|\rfloor$ length scales $2^{-j} \in (\varepsilon_i^{1-\eta/2}, 1)$. By Lemma 4.1 applied to $u^-(x_1, x_2, x_3) = u(x_1, x_2, h_i - x_3)$ and $u^+(x_1, x_2, x_3) =$

$u(x_1, x_2, x_3 - Mh_i)$ with $\theta = \eta/128$, to obtain $(1-\eta/64)N$ length scales 2^{-j} with

$$\int_{\mathbb{T}^2} |u^\pm(x_1, x_2, 2^{-j}) - u^\pm(x_1, x_2, 0)|^2 \, d\mathcal{H}^2(x_1, x_2) \leq 2^{-j} \frac{C \log 2}{\eta(1-\eta/2)} T. \tag{161}$$

for both u^+ and u^-. Additionally, for at least $(1-\eta/32)N$ of these length scales 2^{-j}, additionally

$$\int_{2^{-j}}^{2^{-j+1}} \int_{\mathbb{T}^2} |Du^\pm|^2 \, dx \leq 64CT/\eta. \tag{162}$$

Now fix one such j and consider the lattice $Z_j = 2^{-j}\mathbb{Z}^2 \cap \mathbb{T}^2$ and for each $z \in Z_j$ the squares $Q_{z,2^{-j+1}}$ and $Q_{z,2^{-j}}$, where $Q_{z,l} \subset \mathbb{T}^2$ denotes the square with center z and side length l. Note that the larger squares have finite overlap and the smaller squares cover $\mathbb{T}^2$. Let

$$w_z = \fint_{Q_{z,2^{-j+1}}} u^+(x_1, x_2, 2^{-j}) - u^-(x_1, x_2, 2^{-j}) \, d\mathcal{H}^2(x_1, x_2). \tag{163}$$

Then by (162) and a Poincaré and trace inequality

$$\begin{aligned} \sum_{z \in Z_j} \int_{Q_{z,2^{-j+1}}} &|u(x_1, x_2, 2^{-j}) - u^-(x_1, x_2, 2^{-j}) - w_z|^2 \, d\mathcal{H}^2(x_1, x_2) \\ \leq &C 2^{-j} T/\eta. \end{aligned} \tag{164}$$

Also by (161) and a triangle inequality

$$\sum_{z \in Z_j} \int_{Q_{z,2^{-j+1}}} |u^+(x_1, x_2, 0) - u^-(x_1, x_2, 0) - w_z|^2 \, d\mathcal{H}^2(x_1, x_2) \leq C 2^{-j} T/\eta, \tag{165}$$

where the $u^\pm$ are understood as traces. Now the difference of the uppermost trace $u^+(x_1, x_2, 0)$ and the lowermost trace $u^-(x_1, x_2, 0)$ at slip planes is similar to the sum of all jumps.

$$\begin{aligned} \int_{\mathbb{T}^2} &\left| u^+(x_1, x_2, 0) - u^-(x_1, x_2, 0) - \sum_{m=1}^M b^m(x_1, x_2) \right|^2 d\mathcal{H}^2(x_1, x_2) \\ \leq &Mh_i \int_{\mathbb{T}^2 \times (0, Mh_i) \setminus \bigcup_{m=1}^M \omega_{mh_i}} |Du|^2 \, dx, \end{aligned} \tag{166}$$

by the one-dimensional Sobolev inequality. Note that $h_i |\log \varepsilon_i| \ll \varepsilon_i^{1-\eta/2} \leq 2^{-j}$ for ε_i small enough. Combining all these estimates and Lemma 4.3 yields

$$\sum_{z \in Z_j} \int_{Q_{z,2^{-j+1}}} |\sum_{m=1}^M b^m(x_1, x_2) - w_z|^2 \, d\mathcal{H}^2(x_1, x_2) \leq C 2^{-j} T/\eta. \tag{167}$$

Now note that $\operatorname{dist}^2(\sum_{m=1}^M b^m, \mathbb{Z}^2) \leq M \operatorname{dist}^2(\boldsymbol{b}, \mathbb{Z}^{2M})$, and also $\varepsilon_i |\log \varepsilon_i| \ll 2^{-j}$. Thus

$$\sum_{z \in Z_j} \int_{Q_{z,2^{-j+1}}} \operatorname{dist}^2(w_z, \mathbb{Z}^2) \, d\mathcal{H}^2(x_1, x_2) \leq C 2^{-j} T/\eta. \tag{168}$$

We can find functions $w_j : \mathbb{T}^2 \to \mathbb{Z}^2$ piecewise constant in all $Q_{z,2^{-j}}$ such that

$$\sum_{z\in\mathbb{Z}} \int_{Q_{z,2^{-j}}} |w_j(x_1,x_2) - w_z|^2 \, d\mathcal{H}^2(x_1,x_2) \leq C2^{-j}T/\eta. \tag{169}$$

Then by a triangle inequality,

$$\int_{\mathbb{T}^2} |w_j(x_1,x_2) - \sum_{m=1}^{M} b^m(x_1,x_2)|^2 \, d\mathcal{H}^2(x_1,x_2) \leq C2^{-j}T/\eta. \tag{170}$$

Also, because the squares $Q_{z,2^{-j+1}}$ and $Q_{z',2^{-j+1}}$ have significant overlap for $z, z' \in Z_j$ nearest neighbors, we also get

$$\sum_{z,z' \text{ n.n.}} \int_{Q_{z,2^{-j+1}} \cap Q_{z',2^{-j+1}}} |w_j(z) - w_j(z')|^2 \, d\mathcal{H}^2(x_1,x_2) \leq C2^{-j}T/\eta, \tag{171}$$

and consequently

$$\sum_{z,z' \text{ n.n.}} \int_{\partial Q_{z,2^{-j+1}} \cap \partial Q_{z',2^{-j+1}}} |[w_j]|^2 \, d\mathcal{H}^1 \leq CT/\eta. \tag{172}$$

Since in $\mathbb{Z}^2$, $|x-y|^2 \geq |x-y|$, this gives us in particular $|Dw_j|(\mathbb{T}^2) \leq CT/\eta$, but also

$$\int \int_{|x-y|\leq 2^{-j}} \frac{|w_j(x) - w_j(y)|^2}{2^{-3j}} \, dx \, dy \leq CT/\eta. \tag{173}$$

Define $k_i = \lfloor (1-\eta) |\log_2 \varepsilon_i| \rfloor$. Note that since there are $(1-\eta/32)N$ length scales 2^{-j}, there has to be at least one with $(1-\eta/4)N \leq j \leq N$, i.e. $\varepsilon_i^{1-\eta/2} \leq 2^{-\tilde{j}} \leq \varepsilon_i^{1-7\eta/8}$ for η small enough. Define $v_i^\Sigma = w_{\tilde{j}}$ for one such $\tilde{j}$. Then we have already shown that (i) and (ii) hold. (iv) holds for the at least $(1-\eta/32-\eta/2)N \geq (1-\eta)k_i$ length scales $j \leq k_i$ because $\|v_i^\Sigma - w_j\|_{L^2}^2 \leq C2^{-j}T/\eta$, and the double integral (173) is a positive semidefinite bilinear form on $L^2(\mathbb{T}^2)$, with

$$\begin{aligned} &\int \int_{|x-y|\leq 2^{-j}} \frac{|v_i^\Sigma(x) - v_i^\Sigma(y)|^2}{2^{-3j}} \, dx \, dy \\ \leq &2 \int \int_{|x-y|\leq 2^{-j}} \frac{|w_j(x) - w_j(y)|^2}{2^{-3j}} \, dx \, dy + C2^j \|v_i^\Sigma - w_j\|_{L^2}^2. \end{aligned} \tag{174}$$

Finally, to show (iii), consider the deformation

$$\tilde{u}(x_1,x_2,x_3) = \begin{cases} u(x_1,x_2,x_3 + Mh_i) & \text{, if } x_3 > 0 \\ u(x_1,x_2,x_3 + h_i) & \text{, if } x_3 < 0. \end{cases} \tag{175}$$

Note that $\tilde{u} \in H^1(\mathbb{T}^2 \times \mathbb{R} \setminus \omega_0, \mathbb{R}^3)$, with jump $[\tilde{u}](x_1,x_2) = u^+(x_1,x_2,0) - u^-(x_1,x_2,0)$, and

$$\int_{\mathbb{T}^2 \times \mathbb{R} \setminus \omega_0} \mathbb{C} D\tilde{u} : D\tilde{u} \, dx \leq B_{h_i}^{\text{per}}(\boldsymbol{b}, \boldsymbol{b}). \tag{176}$$

This implies that $A_0([\tilde{u}] * \tilde{\phi}_{2^{-k_i}}, [\tilde{u}] * \tilde{\phi}_{2^{-k_i}}) \leq A_0([\tilde{u}], [\tilde{u}]) \leq B^{\mathrm{per}}_{h_i}(\boldsymbol{b}, \boldsymbol{b})$. Also, the bilinear form $A_0(\cdot * \tilde{\phi}_{2^{-k_i}}, \cdot * \tilde{\phi}_{2^{-k_i}})$ is positive semidefinite and L^2-regular, meaning that

$$\begin{aligned} &A_0(v_i^\Sigma * \tilde{\phi}_{2^{-k_i}}, v_i^\Sigma * \tilde{\phi}_{2^{-k_i}}) \\ \leq &(1+\eta) A_0([\tilde{u}] * \tilde{\phi}_{2^{-k_i}}, [\tilde{u}] * \tilde{\phi}_{2^{-k_i}}) \\ &+ C(1+\frac{1}{\eta}) 2^{k_i} \|[\tilde{u}] - v_i^\Sigma\|_{L^2}^2, \end{aligned} \tag{177}$$

where the L^2-norm was estimated in (166). Since $2^{-j} \ll 2^{-k_i}$ for i large, the error term tends to 0. This completes the proof. □

To show compactness in the separated regime $\beta \in [0,1)$, we proceed similarly as before, only making sure to retain information about each of the jumps instead of only their sum.

Proof of compactness II. Fix $i \in \mathbb{N}$. Let $u : \mathbb{T}^2 \times \mathbb{R} \setminus \bigcup_{m=1}^M \omega_{mh_i} \to \mathbb{R}^3$ be the minimizer of (52). Then it follows from Lemma 4.3 that

$$\int_{\mathbb{T}^2 \times \mathbb{R} \setminus \bigcup_{m=1}^M \omega_{mh_i}} |Du|^2 \, dx \leq CT |\log \varepsilon_i|. \tag{178}$$

Consider all $N = \lfloor (1-\eta/2) |\log_2 \varepsilon_i| \rfloor$ length scales $2^{-j} \in (\varepsilon_i^{1-\eta/2}, 1)$.

For $m = 1, \ldots, M$ define $u_m^\pm(x_1, x_2, x_3) = u(x_1, x_2, mh_i \pm x_3)$.

By Lemma 4.1 applied to each of the $u_m^\pm$ with $\theta = \frac{\eta}{128M}$, to obtain $(1-\beta-\eta/64)N$ length scales $2^{-j} \leq h_i$ with

$$\int_{\mathbb{T}^2} \|u_m^\pm(x_1, x_2, 2^{-j}) - u_m^\pm(x_1, x_2, 0)\|^2 \, d\mathcal{H}^2(x_1, x_2) \leq 2^{-j} \frac{C \log 2}{\eta(1-\eta/2)} T. \tag{179}$$

for each of the $u_m^\pm$. Additionally, for at least $(1-\beta-\eta/32)N$ of these length scales 2^{-j}, additionally

$$\int_{2^{-j}}^{2^{-j+1}} \int_{\mathbb{T}^2} |Du_m^\pm|^2 \, dx \leq 64CT/\eta. \tag{180}$$

Now fix one such j and consider the lattice $Z_j = 2^{-j}\mathbb{Z}^2 \cap \mathbb{T}^2$ and for each $z \in Z_j$ the squares $Q_{z,2^{-j+1}}$ and $Q_{z,2^{-j}}$, where $Q_{z,l} \subset \mathbb{T}^2$ denotes the square with center z and side length l. Note that the larger squares have finite overlap and the smaller squares cover $\mathbb{T}^2$. Let

$$w_z^m = \fint_{Q_{z,2^{-j+1}}} u_m^+(x_1, x_2, 2^{-j}) - u_m^-(x_1, x_2, 2^{-j}) \, d\mathcal{H}^2(x_1, x_2). \tag{181}$$

Then by (180) and a Poincaré and trace inequality

$$\begin{aligned} &\sum_{z \in Z_j} \int_{Q_{z,2^{-j+1}}} |u_m^+(x_1, x_2, 2^{-j}) - u_m^-(x_1, x_2, 2^{-j}) - w_z^m|^2 \, d\mathcal{H}^2(x_1, x_2) \\ \leq &C 2^{-j} T/\eta. \end{aligned} \tag{182}$$

Also by (179) and a triangle inequality

$$\sum_{z\in Z_j}\int_{Q_{z,2^{-j+1}}} |b^m(x_1,x_2)-w_z^m|^2\,d\mathcal{H}^2(x_1,x_2)\le C2^{-j}T/\eta, \tag{183}$$

since b^m is the difference of the traces $u_m^\pm$.

Now note that $\varepsilon_i|\log\varepsilon_i|\ll 2^{-j}$. By the Pythagorean identity and the triangle inequality the vector $\boldsymbol{w}_z=(b_z^1,\dots,b_z^M)$ satisfies

$$\sum_{z\in Z_j}\int_{Q_{z,2^{-j+1}}} \operatorname{dist}^2(\boldsymbol{w}_z,\mathbb{Z}^{2M})\,d\mathcal{H}^2(x_1,x_2)\le C2^{-j}T/\eta. \tag{184}$$

We can now find functions $\boldsymbol{w}_j:\mathbb{T}^2\to\mathbb{Z}^{2M}$ piecewise constant in all $Q_{z,2^{-j}}$ such that

$$\sum_{z\in\mathbb{Z}}\int_{Q_{z,2^{-j}}} |\boldsymbol{w}_j(x_1,x_2)-\boldsymbol{w}_z|^2\,d\mathcal{H}^2(x_1,x_2)\le C2^{-j}T/\eta. \tag{185}$$

Then by a triangle inequality,

$$\int_{\mathbb{T}^2} |\boldsymbol{w}_j(x_1,x_2)-\boldsymbol{b}(x_1,x_2)|^2\,d\mathcal{H}^2(x_1,x_2)\le C2^{-j}T/\eta. \tag{186}$$

Because the squares $Q_{z,2^{-j+1}}$ and $Q_{z',2^{-j+1}}$ have significant overlap for $z,z'\in Z_j$ nearest neighbors, we also get

$$\sum_{z,z'\ \text{n.n.}}\int_{Q_{z,2^{-j+1}}\cap Q_{z',2^{-j+1}}} |\boldsymbol{w}_j(z)-\boldsymbol{w}_j(z')|^2\,d\mathcal{H}^2(x_1,x_2)\le C2^{-j}T/\eta, \tag{187}$$

and consequently

$$\sum_{z,z'\ \text{n.n.}}\int_{\partial Q_{z,2^{-j+1}}\cap\partial Q_{z',2^{-j+1}}} |[\boldsymbol{w}_j]|^2\,d\mathcal{H}^1\le CT/\eta. \tag{188}$$

Since in $\mathbb{Z}^{2M}$, $|x-y|^2\ge|x-y|$, this gives us in particular $|D\boldsymbol{w}_j|(\mathbb{T}^2)\le CT/\eta$, but also

$$\int\int_{|x-y|\le 2^{-j}}\frac{|\boldsymbol{w}_j(x)-\boldsymbol{w}_j(y)|^2}{2^{-3j}}\,dx\,dy\le CT/\eta. \tag{189}$$

Now define $k_i=\lfloor(1-\eta)|\log_2\varepsilon_i|\rfloor$. Note that since there are $(1-\beta-\eta/32)N$ length scales $2^{-j}\le h_i$, for η small enough there has to be at least one with $(1-\eta/4)N\le j\le N$, i.e. $\varepsilon_i^{1-\eta/2}\le 2^{-\tilde{j}}\le\varepsilon_i^{1-7\eta/8}$. Define $\boldsymbol{v}_i=\boldsymbol{w}_{\tilde{j}}$ for one such $\tilde{j}$. Then we have already shown that (i) and (ii) hold. (v) holds for the at least $(1-\beta-\eta/32-\eta/2)N\ge(1-\beta-\eta)k_i$ length scales $j\le k_i$ because $\|\boldsymbol{v}_i-\boldsymbol{w}_j\|_{L^2}^2\le C2^{-j}T/\eta$, and the double integral (189) is a positive semidefinite bilinear form on $L^2(\mathbb{T}^2)$, with

$$\begin{aligned}&\int\int_{|x-y|\le 2^{-j}}\frac{|\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)|^2}{2^{-3j}}\,dx\,dy\\ \le\,&2\int\int_{|x-y|\le 2^{-j}}\frac{|\boldsymbol{w}_j(x)-\boldsymbol{w}_j(y)|^2}{2^{-3j}}\,dx\,dy+C2^j\|\boldsymbol{v}_i-\boldsymbol{w}_j\|_{L^2}^2.\end{aligned} \tag{190}$$

To show (iii), use that the bilinear form $B_{h_i}^{\mathrm{per}}(\cdot * \tilde{\phi}_{2^{-k_i}}, \cdot * \tilde{\phi}_{2^{-k_i}})$ is positive semidefinite and L^2-regular, meaning that

$$\begin{aligned} &B_{h_i}^{\mathrm{per}}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-k_i}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-k_i}}) \\ \leq &(1+\eta) B_{h_i}^{\mathrm{per}}(\boldsymbol{b} * \tilde{\phi}_{2^{-k_i}}, \boldsymbol{b} * \tilde{\phi}_{2^{-k_i}}) \\ &+ C(1+\frac{1}{\eta}) 2^{k_i} \|\boldsymbol{b} - \boldsymbol{v}_i\|_{L^2}^2. \end{aligned} \tag{191}$$

Since $2^{-\tilde{j}} \ll 2^{-k_i}$ for i large, the error term tends to 0. This leaves us to prove (iv) and (vi).

We first estimate

$$\begin{aligned} &B_{h_i}^{\mathrm{per}}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-k_i}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-k_i}}) \\ \geq &B_{h_i}^{\mathrm{per}}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-k_i}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-k_i}}) \\ &- B_{h_i}^{\mathrm{per}}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil+1}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil+1}}) \\ &+ B_{h_i}^{\mathrm{per}}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}) \\ = &\sum_{j=\lceil(\beta+\eta)k_i\rceil}^{k_i} B_{j,h_i}^{\mathrm{per}}(\boldsymbol{v}_i, \boldsymbol{v}_i) + B_{h_i}^{\mathrm{per}}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}). \end{aligned} \tag{192}$$

Note that $2^{-\lceil(\beta+\eta)k_i\rceil} \ll h_i$. We now wish to replace h_i by ∞ in the first term and by 0 in the second. By Lemma 3.20,

$$\begin{aligned} &\left| \sum_{j=\lceil(\beta+\eta)k_i\rceil}^{k_i} |B_{j,h_i}^{\mathrm{per}}(\boldsymbol{v}_i, \boldsymbol{v}_i) - B_{j,\infty}^{\mathrm{per}}(\boldsymbol{v}_i, \boldsymbol{v}_i) \right| \\ \leq &C \int\int_{|x-y|\leq h_i} \frac{2^{-2\lceil(\beta+\eta)k_i\rceil} |\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^2}{h_i^5} \, dx \, dy \\ &+ C \int\int_{|x-y|> h_i} \frac{2^{-2\lceil(\beta+\eta)k_i\rceil} |\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^2}{|x-y|^5} \, dx \, dy. \end{aligned} \tag{193}$$

Now we note that for $x, y \in \mathbb{T}^2$ and $l \in (0, 1/4)$, the ball $B(x, 2l)$ is Euclidean, and

$$\begin{aligned} &\int\int_{|x-y|\leq 2l} |\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^2 \, dx \, dy \\ \leq &2 \int\int_{|x-y|\leq 2l} |\boldsymbol{v}_i(x) - \boldsymbol{v}_i(\frac{x+y}{2})|^2 + |\boldsymbol{v}_i(y) - \boldsymbol{v}_i(\frac{x+y}{2})|^2 \, dx \, dy \\ = &4 \int\int_{|x-y|\leq 2l} |\boldsymbol{v}_i(x) - \boldsymbol{v}_i(\frac{x+y}{2})|^2 \, dx \, dy \\ = &16 \int\int_{|x-y|\leq l} |\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^2 \, dx \, dy. \end{aligned} \tag{194}$$

Note that this inequality also holds for $l > 1/4$. Thus,

$$\begin{aligned}
&\int\int_{|x-y|>h_i} \frac{2^{-2\lceil(\beta+\eta)k_i\rceil}|\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)|^2}{|x-y|^5}\,dx\,dy\\
\leq& 2^{-2\lceil(\beta+\eta)k_i\rceil}\sum_{j=1}^{\infty}\int\int_{|x-y|\leq 2^j h_i} \frac{|\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)|^2}{2^{5j}h_i^5}\,dx\,dy\\
\leq& 2^{-2\lceil(\beta+\eta)k_i\rceil}\sum_{j=1}^{\infty}2^{-j}\int\int_{|x-y|\leq h_i} \frac{|\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)|^2}{h_i^5}\,dx\,dy\\
=& 2^{-2\lceil(\beta+\eta)k_i\rceil}\int\int_{|x-y|\leq h_i} \frac{|\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)|^2}{h_i^5}\,dx\,dy. \qquad (195)
\end{aligned}$$

Now note that there is a $j_1 \in \lceil(\beta+\eta/2)k_i\rceil, \ldots, \lceil(\beta+\eta)k_i\rceil$ with

$$\int\int_{|x-y|\leq 2^{-j_1}} \frac{\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)}{2^{-3j_1}}\,dx\,dy \leq CT/\eta. \qquad (196)$$

Using (194), we get that

$$\begin{aligned}
&2^{-2\lceil(\beta+\eta)k_i\rceil}\int\int_{|x-y|\leq h_i} \frac{|\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)|^2}{h_i^5}\,dx\,dy\\
\leq& \frac{2^{-j_1}}{h_i}2^{-2\lceil(\beta+\eta)k_i\rceil}\int\int_{|x-y|\leq 2^{-j_1}} \frac{\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)}{2^{-5j_1}}\,dx\,dy\\
\leq& 2^{-\eta k_i/2}2^{-2j_1}\int\int_{|x-y|\leq 2^{-j_1}} \frac{\boldsymbol{v}_i(x)-\boldsymbol{v}_i(y)}{2^{-5j_1}}\,dx\,dy\\
\leq& C2^{-\eta k_i/2}T/\eta. \qquad (197)
\end{aligned}$$

This tends to 0 as $k_i \to \infty$, so we can replace $\sum_{j=\lceil(\beta+\eta)k_i\rceil}^{k_i} B^{\mathrm{per}}_{j,h_i}(\boldsymbol{v}_i,\boldsymbol{v}_i)$ with $\sum_{j=\lceil(\beta+\eta)k_i\rceil}^{k_i} B^{\mathrm{per}}_{j,\infty}(\boldsymbol{v}_i,\boldsymbol{v}_i)$.

We have yet to replace h_i by 0 in $\sum_{j=1}^{\lfloor(\beta-\eta)k_i\rfloor} B^{\mathrm{per}}_{j,h_i}(\boldsymbol{v}_i,\boldsymbol{v}_i)$. If $\beta = 0$ we are already done because the sum is empty. Otherwise note that

$$\begin{aligned}
&E_{h_i,h_i}(\boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}})\\
=& \frac{1}{|\log h_i|}B^{\mathrm{per}}_{h_i}(\boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}})\\
&+ \frac{1}{h_i|\log h_i|}\int_{\mathbb{T}^2}\mathrm{dist}^2(\boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \mathbb{Z}^{2M})\,dx\\
\leq& \frac{1}{|\log h_i|}B^{\mathrm{per}}_{h_i}(\boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}})\\
&+ \frac{1}{h_i|\log h_i|}\int_{\mathbb{T}^2}|\boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}} - \boldsymbol{v}_i|\,dx\\
\leq& \frac{1}{|\log h_i|}B^{\mathrm{per}}_{h_i}(\boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde\phi_{2^{-\lceil(\beta+\eta)k_i\rceil}})\\
&+ C\frac{2^{-(\beta+\eta)k_i}}{h_i|\log h_i|}\frac{T}{\eta}. \qquad (198)
\end{aligned}$$

Letting $i \to \infty$, we use that $\frac{|\log h_i|}{|\log \varepsilon_i|} \to \beta$, so that

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{1}{|\log h_i|} E_{h_i,h_i}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}) \\
\leq & \frac{1+\eta}{(\log 2)\beta k_i} \limsup_{i\to\infty} B^{\text{per}}_{h_i}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}). \qquad (199)
\end{aligned}$$

We can apply Proposition 4.5 to the sequence $\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}$, giving us $\tilde{k}_i \in (\beta-\eta)k_i, \beta k_i)$ and a function $v_i^\Sigma \in BV(\mathbb{T}^2, \mathbb{Z}^2)$ with

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{1}{\tilde{k}_i} \sum_{j=1}^{\lfloor(\beta-\eta)k_i\rfloor} A_{j,0}(v_i^\Sigma, v_i^\Sigma) \\
\leq & \limsup_{i\to\infty} \frac{(1+\eta)^2}{\beta k_i} B^{\text{per}}_{h_i}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}). \qquad (200)
\end{aligned}$$

Using the L^2-estimate (i) from Proposition 4.5 and the fact that $B_0^{\text{per}}(\boldsymbol{b}, \boldsymbol{b}) = A_0(\sum_{m=1}^M b^m, \sum_{m=1}^M b^m)$, it follows that

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{1}{k_i} \sum_{j=1}^{\lfloor(\beta-\eta)k_i\rfloor} B^{\text{per}}_{j,0}(\boldsymbol{v}_i, \boldsymbol{v}_i) \\
\leq & \limsup_{i\to\infty} \frac{1+C\eta}{k_i} B^{\text{per}}_{h_i}(\boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}, \boldsymbol{v}_i * \tilde{\phi}_{2^{-\lceil(\beta+\eta)k_i\rceil}}), \qquad (201)
\end{aligned}$$

Proposition 4.5 also gives us the $(\beta-\eta)k_i$ length scales $j \in (1, \lfloor, \beta k \rfloor)$ with

$$\int \int_{|x-y|\leq 2^{-j}} \frac{|v_i^\Sigma(x) - v_i^\Sigma(y)|^2}{2^{-3j}} \, dx \, dy \leq CT/\eta. \qquad (202)$$

Again by (i) from Proposition 4.5 we obtain the same bound for $\sum_{m=1}^M v_i^m$. This ends the proof. □

5 The limit energy and upper bound

By the compactness result, the limit energy, in the sense of Γ-convergence, is finite only for functions $\boldsymbol{u} \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$, if $\beta < 1$, or for $\sum_{m=1}^M u^m \in BV(\mathbb{T}^2, \mathbb{Z}^2)$, if $\beta \geq 1$. We shall show that the limit energy is of line-tension type, i.e. there is $\varphi : \mathbb{Z}^{2M} \times S^1 \to [0, \infty)$ such that the Γ-limit of $E_{\varepsilon, h(\varepsilon)}/|\log \varepsilon|$ in the L^1-topology is given by $I(\boldsymbol{u}, \mathbb{T}^2)$, where for any Borel set $A \subset \mathbb{T}^2$

$$I(\boldsymbol{u}, A) = \int_{J_{\boldsymbol{u}} \cap A} \varphi([\boldsymbol{u}], \nu) \, d\mathcal{H}^1, \qquad (203)$$

where $J_{\boldsymbol{u}} \subset \mathbb{T}^2$ is the jump set of $\boldsymbol{u}$, $[\boldsymbol{u}] \in \mathbb{Z}^{2M}$ the jump, and $\nu \in S^1$ the normal to the jump set, which exist $\mathcal{H}^1$-almost everywhere.

We can obtain an upper bound on φ by considering for a function $\boldsymbol{u} \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ with a jump set consisting of finitely many line segments the competitor $\boldsymbol{u}_\varepsilon = \boldsymbol{u} * \phi_\varepsilon$.

Then whenever $\beta = \lim_{\varepsilon\to 0} \frac{\log h(\varepsilon)}{\log \varepsilon} \in [0,1]$ exists, we can calculate

$$\lim_{\varepsilon\to 0} \frac{E_{\varepsilon,h(\varepsilon)}(\boldsymbol{u}_\varepsilon)}{|\log\varepsilon|} = (1-\beta) I_\infty(\boldsymbol{u}, \mathbb{T}^2) + \beta I_0(\boldsymbol{u}, \mathbb{T}^2), \tag{204}$$

where I_∞ is the line-tension energy with energy density

$$\varphi_\infty(\boldsymbol{b}, \nu) = \sum_{m=1}^{M} \varphi_{\text{single}}(b^m, \nu), \tag{205}$$

I_0 is the line-tension energy with energy density

$$\varphi_0(\boldsymbol{b}, \nu) = \varphi_{\text{single}}(\sum_{m=1}^{M} b^m, \nu), \tag{206}$$

and $\varphi_{\text{single}} : \mathbb{Z}^2 \times S^1 \to [0,\infty)$ is given by

$$\varphi_{\text{single}}(b, \nu) = 2 \int_{\{x\in\mathbb{R}^2 \,:\, x\cdot\nu = 1\}} bJ(x)b \, d\mathcal{H}^1(x). \tag{207}$$

The energy density φ_{single} is so named because it appears naturally for dislocations in a single plane. I_∞ and I_0 are so called because they arise naturally for distance between planes of ∞ or 0 respectively. Note that for $\limsup_{\varepsilon\to 0} \frac{\log h(\varepsilon)}{\log\varepsilon} \leq 0$ we also get that the limit energy of $\boldsymbol{u}_\varepsilon$ is I_∞, corresponding to $\beta = 0$ and if $\liminf_{\varepsilon\to 0} \frac{\log h(\varepsilon)}{\log\varepsilon} \geq 1$ the limit energy of $\boldsymbol{u}_\varepsilon$ is I_0, corresponding to $\beta = 1$.

One could now assume that $(1-\beta)I_\infty + \beta I_0$ is the Γ-limit. However, neither I_∞ nor I_0 are lower semicontinuous. In fact, lower semicontinuity of line-tension functionals is equivalent to the following condition, see [3], [4].

Definition 5.1. *A function $\varphi : \mathbb{Z}^{2M} \times S^1 \to [0,\infty)$ is called BV-elliptic if for any $\boldsymbol{b} \in \mathbb{Z}^{2M}$ and any $\nu \in S^1$, defining $\boldsymbol{u}_{\boldsymbol{b},\nu}(x) = \boldsymbol{b}\mathbb{1}_{\{x\cdot\nu>0\}}$, we have*

$$\varphi(\boldsymbol{b}, \nu) \leq \inf\left\{ \int_{J_{\boldsymbol{u}}\cap\overline{Q_\nu}} \varphi(\boldsymbol{u}, \nu)\, d\mathcal{H}^1 \,:\, \boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{b},\nu} \text{ outside of } Q_\nu \right\}, \tag{208}$$

where $Q_\nu \subset \mathbb{R}^2$ is a unit square with one side parallel to ν.

Given a function $\varphi : \mathbb{Z}^{2M} \times S^1 \to [0,\infty)$, define the BV-elliptic envelope of φ as

$$\varphi^{\text{rel}}(\boldsymbol{b}, \nu) = \inf\left\{ \int_{J_{\boldsymbol{u}}\cap\overline{Q_\nu}} \varphi(\boldsymbol{u}, \nu)\, d\mathcal{H}^1 \,:\, \boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{b},\nu} \text{ outside of } Q_\nu \right\}. \tag{209}$$

Note that the BV-elliptic envelope is indeed BV-elliptic, and the relaxation of I in (203) is given by

$$I^{\text{rel}}(\boldsymbol{u}, A) = \int_{J_{\boldsymbol{u}}\cap A} \varphi^{\text{rel}}([\boldsymbol{u}], \nu)\, d\mathcal{H}^1. \tag{210}$$

We deduce that the Γ-limit of $E_{\varepsilon,h(\varepsilon)}/|\log\varepsilon|$ is at most

$$[(1-\beta)I_\infty + \beta I_0]^{\text{rel}}, \tag{211}$$

and we will later find matching lower bounds for the cases $\beta = 0, 1$, but not for $\beta \in (0, 1)$. Note that

$$\varphi_\infty^{\text{rel}}(\boldsymbol{b}, \nu) = \sum_{m=1}^{M} \varphi_{\text{single}}^{\text{rel}}(b^m, \nu) \tag{212}$$

and

$$\varphi_0^{\text{rel}}(\boldsymbol{b}, \nu) = \varphi_{\text{single}}^{\text{rel}}\left(\sum_{m=1}^{M} b^m, \nu\right), \tag{213}$$

by taking as competitor an almost optimal competitor to the cell problem for b^m in each component m in the first case, whereas in the second case we take $u^m = u_{b^m,\nu}$ for $m = 1, \dots, M-1$, and for $m = M$ we take an almost optimal competitor for $\sum_{m=1}^{M} b^m$ minus $\sum_{m=1}^{M-1} u_{b^m,\nu}$.

We now define polyhedral functions.

Definition 5.2. *A function $\boldsymbol{u} \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ is called polyhedral if $\boldsymbol{u} = \sum_{i=1}^{N} \boldsymbol{b}_i \mathbb{1}_{A_i}$ for some $N \in \mathbb{N}$, some $\boldsymbol{b}_1, \dots, \boldsymbol{b}_N \in \mathbb{Z}^{2M}$ and some polyhedra $A_1, \dots, A_N \subset \mathbb{T}^2$. A set $A \subset \mathbb{T}^2$ is called a polyhedron if its boundary consists of finitely many line segments.*

It turns out that polyhedral functions are energy dense for all line-tension energies.

Lemma 5.3. *Let $\varphi : \mathbb{Z}^{2M} \times S^1 \to [0, \infty)$ be Borel with $\varphi(\boldsymbol{z}, \nu) \leq C\varphi(\boldsymbol{b}, \nu)$ whenever $\nu \in S^1$, $\boldsymbol{b} \in \mathbb{Z}^{2M}$, and $\boldsymbol{z} \in \mathbb{Z}^{2M}$ with $\boldsymbol{z}_i \leq \boldsymbol{b}_i$ for $i = 1, \dots, 2M$. Let $\boldsymbol{u} \in GSBV(\mathbb{T}^2, \mathbb{Z}^{2M}) \cap L^1(\mathbb{T}^2, \mathbb{Z}^{2M})$. Then there are polyhedral functions $\boldsymbol{u}_k$ converging to $\boldsymbol{u}$ in L^1 such that for all $A \subset \mathbb{T}^2$ open*

$$\limsup_{k\to\infty} \int_{J_{\boldsymbol{u}_k} \cap A} \varphi([\boldsymbol{u}_k], \nu) \, d\mathcal{H}^1 \leq \int_{J_{\boldsymbol{u}} \cap A} \varphi([\boldsymbol{u}], \nu) \, d\mathcal{H}^1. \tag{214}$$

The growth condition is satisfied in particular by φ_{single},φ_∞, φ_0, their relaxations, and all their convex combinations. To prove this lemma, we use Corollary 1.2 from [7].

Proof. First take $M \in \mathbb{N}$, define $\boldsymbol{u}_M$ componentwise as $(\boldsymbol{u}_M)_i = -M \vee \boldsymbol{u}_i \wedge M$. Then $\boldsymbol{u}_M \to \boldsymbol{u}$ in L^1 as $M \to \infty$, $J_{\boldsymbol{u}_M} \subset J_{\boldsymbol{u}}$, and $[\boldsymbol{u}_m] \to [\boldsymbol{u}]$ pointwise almost everywhere on $J_{\boldsymbol{u}}$. By the growth condition on φ we can use the dominated convergence theorem,

$$\lim_{M\to\infty} \int_{J_{\boldsymbol{u}_M} \cap A} \varphi([\boldsymbol{u}_M], \nu) \, d\mathcal{H}^1 = \int_{J_{\boldsymbol{u}} \cap A} \varphi([\boldsymbol{u}], \nu) \, d\mathcal{H}^1. \tag{215}$$

This shows that we need only consider $\boldsymbol{u} \in (BV \cap L^\infty)(\mathbb{T}^2, \mathbb{Z}^{2M})$. Let $\varphi^{**} : \mathbb{Z}^{2M} \times \mathbb{R}^2 \to [0, \infty)$ denote the lower semicontinuous convex envelope in ν of the 1-homogeneous-in-ν extension of φ to $\mathbb{Z}^{2M} \times \mathbb{R}^2$. Then we use Corollary 1.2 from [7] to obtain a sequence $\boldsymbol{v}_k \to \boldsymbol{u}$ in L^1 of polyhedral functions with

$$\limsup_{k\to\infty} \int_{J_{\boldsymbol{v}_k} \cap A} \varphi^{**}([\boldsymbol{v}_k], \nu) \, d\mathcal{H}^1 \leq \int_{J_{\boldsymbol{u}} \cap A} \varphi([\boldsymbol{u}], \nu) \, d\mathcal{H}^1. \tag{216}$$

Finally, by Carathéodory's theorem for convex functions, see [29], for each $\boldsymbol{b} \in \mathbb{Z}^{2M}$ and each $\nu \in \mathbb{R}^2$ and each $\delta > 0$ there are $\nu_1, \nu_2, \nu_3 \in S^1$ and $t_1, t_2, t_3 > 0$ with $t_1\nu_1 + t_2\nu_2 + t_3\nu_3 = \nu$ and

$$\varphi^{**}(\boldsymbol{b}, \nu) \leq \sum_{i=1}^{3} t_i \varphi(\boldsymbol{b}, \nu_i) + \delta. \tag{217}$$

To obtain $\boldsymbol{u}_k$, simply replace all maximal line segments in $J_{\boldsymbol{v}_k}$ with normal ν and jump $[\boldsymbol{b}]$ with nonintersecting pairwise disjoint polygonal curves with the same endpoints and normals ν_1, ν_2, ν_3 as above. □

5.1 The upper bound

We now show that in the intermediate case $\beta \in (0,1)$, we can improve on (211).

Proposition 5.4. *Assume that* $\beta = \lim_{\varepsilon \to 0} \frac{\log h(\varepsilon)}{\log \varepsilon} \in [0,1]$ *exists. Let* $\boldsymbol{u} \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$*. Then for every* ε *there is* $\boldsymbol{u}_\varepsilon \in H^{1/2}(\mathbb{T}^2, \mathbb{R}^{2M})$ *such that* $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ *in* L^1 *and*

$$\left[(1-\beta) I_\infty^{\mathrm{rel}} + \beta I_0\right]^{\mathrm{rel}} (\boldsymbol{u}, \mathbb{T}^2) \leq \liminf_{\varepsilon \to 0} \frac{E_{\varepsilon, h(\varepsilon)}(\boldsymbol{u}_\varepsilon)}{|\log \varepsilon|}. \tag{218}$$

For $\beta = 1$*, if instead of* $\boldsymbol{u} \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ *we only assume* $\sum_{m=1}^M u^m \in BV(\mathbb{T}^2, \mathbb{Z}^2)$ *then we can find* $\boldsymbol{u}_\varepsilon \in H^{1/2}(\mathbb{T}^2, \mathbb{Z}^{2M})$ *such that* $\sum_{m=1}^M u_\varepsilon^m \to \sum_{m=1}^M u^m$ *in* $L^1(\mathbb{T}^2, \mathbb{R}^2)$ *and*

$$I_0^{\mathrm{rel}}(\boldsymbol{u}, \mathbb{T}^2) \leq \liminf_{\varepsilon \to 0} \frac{E_{\varepsilon, h(\varepsilon)}(\boldsymbol{u}_\varepsilon)}{|\log \varepsilon|}. \tag{219}$$

Proof. Assume by energy density that $\boldsymbol{u}$ is polyhedral. Assume that $\beta \in (0,1)$, since otherwise there is nothing left to prove. Let $0 < \eta < \min(\beta, 1-\beta)/2$. For every line-segment $\gamma \subset \mathbb{T}^2$ making up $J_{\boldsymbol{u}}$ with normal ν and jump $\boldsymbol{b} \in \mathbb{Z}^{2M}$, find a polyhedral competitor $\boldsymbol{v} \in BV(\overline{Q_\nu}, \mathbb{Z}^{2M})$ to the cell problem in Definition 5.1 with

$$I_\infty(\boldsymbol{v}, \overline{Q_\nu}) \leq (1+\eta) \varphi_\infty^{\mathrm{rel}}(\boldsymbol{b}, \boldsymbol{\nu}). \tag{220}$$

Then define for $\varepsilon > 0$ a global function $\boldsymbol{v}_{\varepsilon^\beta}$ by replacing $\boldsymbol{u}$ with a translation of $\boldsymbol{v}(\cdot/\varepsilon^\beta)$ in a family of dyadic squares $x + \varepsilon^\beta Q_\nu$ centered on γ, for each γ, such that the squares are disjoint and

$$I_\infty(\boldsymbol{v}_{\varepsilon^\beta}, \mathbb{T}^2) \leq (1+2\eta) I_\infty^{\mathrm{rel}}(\boldsymbol{u}, \mathbb{T}^2). \tag{221}$$

Then as a competitor to the energy $E_{\varepsilon, h(\varepsilon)}/|\log \varepsilon|$ take $\boldsymbol{u}_\varepsilon = \boldsymbol{v}_{\varepsilon^\beta} * \phi_{\varepsilon/2}$. We shall decompose $\mathbb{T}^2$ into the following five disjoint sets depending on ε and η:

$$A_1 = \left\{x \in \mathbb{T}^2 \,:\, \mathrm{dist}(x, J_{\boldsymbol{v}_{\varepsilon^\beta}}) \leq \varepsilon\right\}, \tag{222}$$

$$A_2 = \left\{x \in \mathbb{T}^2 \,:\, \mathrm{dist}(x, J_{\boldsymbol{v}_{\varepsilon^\beta}}) \in (\varepsilon, \varepsilon^{\beta+\eta})\right\}, \tag{223}$$

$$A_3 = \left\{x \in \mathbb{T}^2 \,:\, \mathrm{dist}(x, J_{\boldsymbol{v}_{\varepsilon^\beta}}) \geq \varepsilon^{\beta+\eta}, \mathrm{dist}(x, J_{\boldsymbol{u}}) \leq \varepsilon^{\beta-\eta}\right\}, \tag{224}$$

$$A_4 = \left\{x \in \mathbb{T}^2 \,:\, \mathrm{dist}(x, J_{\boldsymbol{u}}) \in (\varepsilon^{\beta-\eta}, \varepsilon^{\eta})\right\}, \tag{225}$$

$$A_5 = \left\{x \in \mathbb{T}^2 \,:\, \mathrm{dist}(x, J_{\boldsymbol{u}}) \geq \varepsilon^{\eta}\right\}. \tag{226}$$

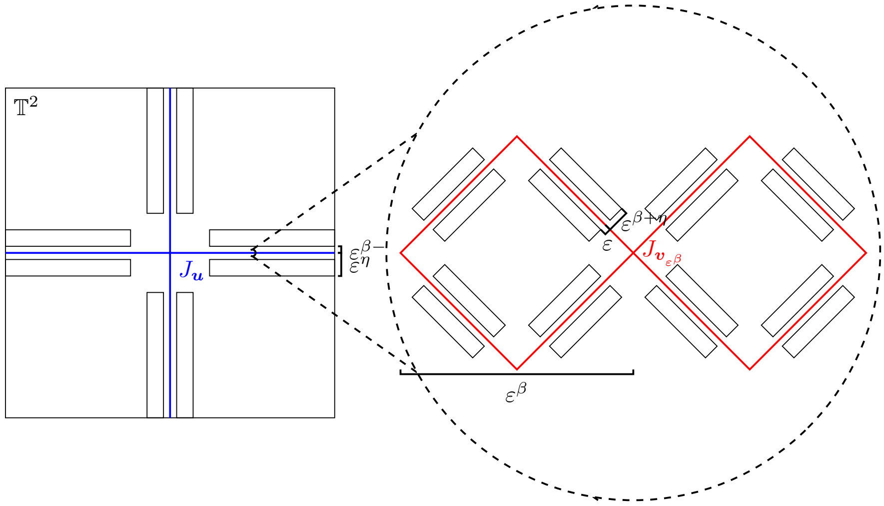

Figure 9: Construction of the recovery sequence for $(1-\beta)I_\infty^{\mathrm{rel}} + \beta I_0$. The main contribution to the energy comes from the rectangles around the jump lines at scales ε to $\varepsilon^{\beta+\eta}$ and $\varepsilon^{\beta-\eta}$ to ε^{η}.

We note immediately that $\boldsymbol{u}_\varepsilon = \boldsymbol{v}_{\varepsilon^\beta}$ outside of A_1 and $\boldsymbol{u}_\varepsilon = \boldsymbol{u}$ in $A_4 \cup A_5$.

We start by estimating the nonconvex part of the energy,

$$\int_{\mathbb{T}^2} \operatorname{dist}^2(\boldsymbol{u}_\varepsilon, \mathbb{Z}^{2M})\, dx \le |A_1| \le C\varepsilon, \tag{227}$$

showing that this quantity vanishes in the limit when rescaled by $1/\varepsilon|\log\varepsilon|$.

To estimate the nonlocal part, we first note that $\|\boldsymbol{u}_\varepsilon\|_{L^\infty} \le C$ independently of ε, and that $\boldsymbol{u}_\varepsilon$ is C/ε-Lipschitz, so that for all $x \in \mathbb{T}^2$ we get

$$\begin{aligned}&\int_{\mathbb{T}^2} \frac{|\boldsymbol{u}_\varepsilon(x) - \boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\, dy \\ \le &\int_{B(x,\varepsilon)} \frac{C|x-y|^2}{\varepsilon^2|x-y|^3}\, dy + \int_{\mathbb{T}^2\setminus B(x,\varepsilon)} \frac{C}{|x-y|^3}\, dy \\ \le & C/\varepsilon,\end{aligned} \tag{228}$$

and if $\operatorname{dist}(x, J_{\boldsymbol{v}_{\varepsilon^\beta}}) > \varepsilon$, we even get

$$\begin{aligned}&\int_{\mathbb{T}^2} \frac{|\boldsymbol{u}_\varepsilon(x) - \boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\, dy \\ \le &\int_{\mathbb{T}^2\setminus B(x,\operatorname{dist}(x,J_{\boldsymbol{v}_{\varepsilon^\beta}})/2)} \frac{C}{|x-y|^3}\, dy \\ \le & C/\operatorname{dist}(x, J_{\boldsymbol{v}_{\varepsilon^\beta}}),\end{aligned} \tag{229}$$

since $\boldsymbol{u}_\varepsilon(y) = \boldsymbol{u}_\varepsilon(x)$ otherwise. The first estimate is enough to show that

$$\begin{aligned}&\int_{A_1}\int_{\mathbb{T}^2} \frac{|\boldsymbol{u}_\varepsilon(x) - \boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\, dy\, dx \\ \le & C|A_1|/\varepsilon \le C,\end{aligned} \tag{230}$$

which upon rescaling logarithmically tends to zero. Now since $\boldsymbol{v}_{\varepsilon^\beta}$ is polyhedral, we can use the coarea formula to bound the contributions to the double integral from A_3 by

$$
\begin{aligned}
&\int_{A_3}\int_{\mathbb{T}^2} \frac{|\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\,dy\,dx\\
\leq& C\int_{A_3} \frac{C}{\operatorname{dist}(x,J_{\boldsymbol{v}_{\varepsilon^\beta}})}\,dx\\
\leq& C\int_{\varepsilon^{\beta+\eta}}^{2\varepsilon^{\beta-\eta}} \frac{C}{t}\\
\leq& C\eta|\log\varepsilon|, \qquad (231)
\end{aligned}
$$

which upon rescaling is at most $C\eta$. Here we used the fact that to be $\varepsilon^{\beta-\eta}$-close to $J_{\boldsymbol{u}}$ means being at most distance $2\varepsilon^{\beta-\eta}$ from $J_{\boldsymbol{v}_{\varepsilon^\beta}}$. The contribution from A_5 is estimated the same way as

$$
\begin{aligned}
&\int_{A_5}\int_{\mathbb{T}^2} \frac{|\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\,dy\,dx\\
\leq& C\int_{A_5} \frac{C}{\operatorname{dist}(x,J_{\boldsymbol{v}_{\varepsilon^\beta}})}\,dx\\
\leq& C\int_{\varepsilon^{\eta}}^{1} \frac{C}{t}\\
\leq& C\eta|\log\varepsilon|. \qquad (232)
\end{aligned}
$$

This makes it clear that the main contribution to the energy comes from A_2 and A_4.

We first focus on the contribution from A_2, where

$$
\int_{A_2}\int_{\mathbb{T}^2\setminus B(x,\varepsilon^{\beta+\eta})} \frac{|\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\,dy\,dx \leq \frac{C|A_2|}{\varepsilon^{\beta+\eta}} \leq C. \qquad (233)
$$

Define for every maximal line segment $\gamma=[a,b]\subset J_{\boldsymbol{v}_{\varepsilon^\beta}}$ with normal $\nu=(b-a)^\perp/|b-a|$ and constant jump $\boldsymbol{b}\in\mathbb{Z}^{2M}$ the set

$$
B_\gamma = \{x+s\nu\,:\,x\in\gamma, |s|\in(\varepsilon,\varepsilon^{\beta+\eta}), B(x,2\varepsilon^{\beta+\eta})\cap J_{\boldsymbol{v}_{\varepsilon^\beta}}\setminus\gamma=\emptyset\}, \qquad (234)
$$

which consists of two rectangles to either side of γ, so that the different B_γ are all contained in A_2 and at distance at least $\varepsilon^{\beta+\eta}$ from each other. Note that due to the construction of $\boldsymbol{v}_{\varepsilon^\beta}$, we have

$$
\lim_{\varepsilon\to0}\sup_{t\in(\varepsilon,\varepsilon^{\beta+\eta})} \mathcal{H}^1(\{x\in\mathbb{T}^2\,:\,\operatorname{dist}(x,J_{\boldsymbol{v}_{\varepsilon^\beta}})=t\}\setminus\bigcup_\gamma B_\gamma)=0. \qquad (235)
$$

Thus by the coarea formula and since the above sup not only tends to 0 but is also uniformly bounded,

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\int_{A_2\setminus\bigcup_\gamma B_\gamma}\int_{B(x,\varepsilon^{\beta+\eta})} \frac{|\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y)|^2}{|x-y|^3}\,dy\,dx\\
\leq& \lim_{\varepsilon\to0}\int_{\varepsilon}^{\varepsilon^{\beta+\eta}} \frac{\mathcal{H}^1(\{x\in\mathbb{T}^2\,:\,\operatorname{dist}(x,J_{\boldsymbol{v}_{\varepsilon^\beta}})=t\}\setminus\bigcup_\gamma B_\gamma)}{t|\log\varepsilon|}\,dt\\
=&0. \qquad (236)
\end{aligned}
$$

The only contribution from A_2 that is left in the limit is

$$\frac{1}{|\log\varepsilon|}\sum_{\gamma}\int_{B_\gamma}\int_{A_2}(\boldsymbol{v}_{\varepsilon^\beta}(x)-\boldsymbol{v}_{\varepsilon^\beta}(y))\mathbb{J}_h^{\mathrm{per}}(\varepsilon)(\boldsymbol{v}_{\varepsilon^\beta}(x)-\boldsymbol{v}_{\varepsilon^\beta}(y))$$
$$\mathbb{1}_{|x-y|<\varepsilon^{\beta+\eta}}\,dy\,dx. \tag{237}$$

At this point we can replace $\mathbb{J}_{h(\varepsilon)}^{\mathrm{per}}$ with the -3-homogeneous kernel $\mathbb{J}_\infty$, since for $|x-y|<\varepsilon^{\beta+\eta}$ we have $|\mathbb{J}_h^{\mathrm{per}}(\varepsilon)(x-y)-\mathbb{J}_\infty(x-y)|\le C/h^3 \ll |x-y|^{-3}$, since $h(\varepsilon)\approx\varepsilon^\beta$.

We note that if $x\in B_\gamma$, $y\in A_2$ with $|x-y|<\varepsilon^{\beta+\eta}$, then $\boldsymbol{u}_\varepsilon(y)-\boldsymbol{u}_\varepsilon(x)$ is either 0 or $\boldsymbol{b}$, depending on whether x and y are on equal or opposite sides of γ. This finally allows us to explicitly calculate, with L denoting the length of B_γ,

$$\begin{aligned}
&\int_{B_\gamma}\int_{A_2}(\boldsymbol{v}_{\varepsilon^\beta}(x)-\boldsymbol{v}_{\varepsilon^\beta}(y))\mathbb{J}_\infty(x-y)(\boldsymbol{v}_{\varepsilon^\beta}(x)-\boldsymbol{v}_{\varepsilon^\beta}(y))\\
&\mathbb{1}_{|x-y|<\varepsilon^{\beta+\eta}}\,dy\,dx\\
=&2L\int_{-\varepsilon^{\beta+\eta}}^{-\varepsilon}\int_{\{y\in\mathbb{T}^2\,:\,y\cdot\nu>\varepsilon,|y-t\nu|<\varepsilon^{\beta+\eta}\}}\boldsymbol{b}\mathbb{J}_\infty(x-y)\boldsymbol{b}\,dy\,dt\\
=&2L\int_{-\varepsilon^{\beta+\eta}}^{-\varepsilon}\int_{\{x\in S^1\,:\,x\cdot\nu>0\}}\int_{\frac{\varepsilon-t}{x\cdot\nu}}^{\varepsilon^{\beta+\eta}}\frac{r\boldsymbol{b}\mathbb{J}_\infty(x)\boldsymbol{b}}{r^3}\,dr\,dx\,dt\\
=&2L\int_{-\varepsilon^{\beta+\eta}}^{-\varepsilon}\int_{\{x\in S^1\,:\,x\cdot\nu>0\}}\boldsymbol{b}\mathbb{J}_\infty(x)\boldsymbol{b}\left(\frac{x\cdot\nu}{\varepsilon-t}-\varepsilon^{-\beta-\eta}\right)\\
\le&L\int_{S^1}\boldsymbol{b}\mathbb{J}_\infty(x)\boldsymbol{b}x\cdot\nu\,((1-\beta-\eta)|\log\varepsilon|+C)\\
\le&((1-\beta-\eta)|\log\varepsilon|+C)\,\mathcal{H}^1(\gamma)\varphi_\infty(\boldsymbol{b},\nu).
\end{aligned}\tag{238}$$

Rescaling logarithmically and summing over all γ yields

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\int_{A_2}\int_{\mathbb{T}^2}(\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y))\mathbb{J}_{h(\varepsilon)}^{\mathrm{per}}(\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y))\,dy\,dx\\
\le&(1-\beta-\eta)I_\infty(\boldsymbol{v}_{\varepsilon^\beta},\mathbb{T}^2)+C\eta.
\end{aligned}\tag{239}$$

We can do the same construction in A_4, where we note for $|x-y|\in(\varepsilon^{\beta_\eta},\varepsilon^\eta)$ that $|\mathbb{J}_{h(\varepsilon)}^{\mathrm{per}}(x-y)-\mathbb{J}_0(x-y)|\le Ch(\varepsilon)^2/|x-y|^5+C\ll|x-y|^{-3}$, that

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\int_{A_4}\int_{\mathbb{T}^2}(\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y))\mathbb{J}_{h(\varepsilon)}^{\mathrm{per}}(\boldsymbol{u}_\varepsilon(x)-\boldsymbol{u}_\varepsilon(y))\,dy\,dx\\
\le&(\beta-\eta)I_0(\boldsymbol{u},\mathbb{T}^2)+C\eta.
\end{aligned}\tag{240}$$

Combining all estimates and since η was arbitrary, we obtain that

$$\Gamma\text{-}\limsup_{\varepsilon\to0}\frac{E_{\varepsilon,h(\varepsilon)}}{|\log\varepsilon|}\le(1-\beta)I_\infty^{\mathrm{rel}}+\beta I_0,\tag{241}$$

and by energy density of polyhedral functions and lower semicontinuity of the Γ-$\limsup$ that

$$\Gamma\text{-}\limsup_{\varepsilon\to0}\frac{E_{\varepsilon,h(\varepsilon)}}{|\log\varepsilon|}\le((1-\beta)I_\infty^{\mathrm{rel}}+\beta I_0)^{\mathrm{rel}}\tag{242}$$

for all $u\in BV(\mathbb{T}^2,\mathbb{Z}^{2M})$. □

5.2 Some properties of the double relaxation

In this section we calculate the limit energy in the case of two planes in a simple cubic lattice, with distance $h(\varepsilon) = \varepsilon^{1/2}$. In this case, the limit energy density for a jump $\boldsymbol{d} = (d^1, d^2) \in \mathbb{Z}^4$ with normal $\nu \in S^1$ is given by

$$\frac{1}{2}(\varphi_\infty^{\mathrm{rel}} + \varphi_0)^{\mathrm{rel}}(\boldsymbol{d}, \nu)). \tag{243}$$

For ease of notation we shall ignore the factor $\frac{1}{2}$ in this section. Then we get ,since $\varphi^{\mathrm{rel}} \leq \varphi$ for all line-tension energy densities, that

$$\varphi_\infty^{\mathrm{rel}} + \varphi_0^{\mathrm{rel}} \leq (\varphi_\infty^{\mathrm{rel}} + \varphi_0)^{\mathrm{rel}} \leq (\varphi_\infty + \varphi_0)^{\mathrm{rel}}. \tag{244}$$

All three energy densities in this inequality are BV-elliptic. The left one appears if a single microstructure minimizing both energies at once exists, the middle one appears in our situation, where small-scale microstructure has to follow large-scale microstructure, and the right one is the best energy achieved by using a single microstructure.

We now show, following [13], that both inequalities in (244) can be strict.

To this end we use the explicit form of $\varphi_{\mathrm{single}}$ and $\varphi_{\mathrm{single}}^{\mathrm{rel}}$ for small Burgers vectors found in [8]. There it was found that up to a constant

$$\varphi_{\mathrm{single}}(d, \nu) = |d|^2 - \frac{\tilde{\nu}}{1 - \tilde{\nu}}(d \cdot \nu)^2, \tag{245}$$

where $\tilde{\nu} \in (-1, 1/2)$ is the material's Poisson ratio. It follows by a truncation argument that for $i = 1, 2$

$$\varphi_{\mathrm{single}}^{\mathrm{rel}}(e_i, \nu) = \varphi_{\mathrm{single}}(e_i, \nu), \tag{246}$$

with only straight dislocations realizing the relaxation. In fact, for every $\eta > 0$ there is $\delta = \delta(\eta) > 0$ such that whenever $u \in BV_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{Z}^2)$ is a competitor to the cell problem for e_i, ν, i.e. $u(x) = e_i \mathbb{1}_{\{x \cdot \nu > 0\}}$ outside of Q_ν, with

$$\int_{J_u \cap Q_\nu} \varphi_{\mathrm{single}}^{\mathrm{rel}}([u], \nu_{J_u})\, d\mathcal{H}^1 \leq (1 + \delta)\varphi_{\mathrm{single}}(e_i, \nu), \tag{247}$$

then $|D(u \cdot e_i)|(Q_\nu) \leq 1 + \eta$ and $|D(u \cdot e_i^\perp)|(Q_\nu) \leq \eta$.

On the other hand, for $d = e_1 + e_2$ and $\nu = e_1$, the authors found that

$$\varphi_{\mathrm{single}}^{\mathrm{rel}}(e_1 + e_2, e_1) < \varphi_{\mathrm{single}}(e_1 + e_2, e_1) = \varphi_{\mathrm{single}}(e_1, e_1) + \varphi_{\mathrm{single}}(e_2, e_1), \tag{248}$$

with an almost optimal microstructure requiring deviations from the straight line, i.e. there is $\delta_0 > 0$ such that whenever $u \in BV_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{Z}^2)$ with $u(x) = (e_1 + e_2)\mathbb{1}_{\{x \cdot e_1 > 0\}}$ outside of Q has

$$\int_{J_u \cap Q} \varphi_{\mathrm{single}}([u], \nu_{J_u})\, d\mathcal{H}^1 \leq (1 + \delta_0)\varphi_{\mathrm{single}}^{\mathrm{rel}}(e_1 + e_2, e_1), \tag{249}$$

then $(|D(u \cdot e_1)| + |D(u \cdot e_2)|)(Q) \geq 2 + \delta_0$.

We can then show that this incompatability in microstructure can make both the inequalities in (244) strict.

First we show that for $\boldsymbol{d} = (e_1, e_2)$, $\nu = e_1$, we have

$$(\varphi_\infty^{\mathrm{rel}} + \varphi_0^{\mathrm{rel}})(\boldsymbol{d}, \nu) < (\varphi_\infty^{\mathrm{rel}} + \varphi_0)^{\mathrm{rel}}(\boldsymbol{d}, \nu). \tag{250}$$

The left-hand side evaluates by (246) to

$$\begin{aligned}&(\varphi_\infty^{\mathrm{rel}} + \varphi_0^{\mathrm{rel}})(\boldsymbol{d}, e_1)\\ =&\varphi_{\mathrm{single}}(e_1, e_1) + \varphi_{\mathrm{single}}(e_2, e_1) + \varphi_{\mathrm{single}}(e_1 + e_2, e_1).\end{aligned} \tag{251}$$

Now assume that inequality holds and consider a competitor $\boldsymbol{u} = (u^1, u^2) \in BV_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{Z}^4)$ to the cell problem on the right with energy in the square $Q = [-1/2, 1/2]^2$

$$\begin{aligned}&\int_{J_{\boldsymbol{u}} \cap Q} (\varphi_\infty^{\mathrm{rel}} + \varphi_0)([\boldsymbol{u}], \nu_{J_{\boldsymbol{u}}})\, d\mathcal{H}^1\\ \leq&(1 + \tilde{\delta})(\varphi_\infty^{\mathrm{rel}} + \varphi_0^{\mathrm{rel}})(\boldsymbol{d}, e_1),\end{aligned} \tag{252}$$

with $\tilde{\delta} = \min(\delta_0, \delta(\delta_0/5))$, so that

$$|D(u^1 \cdot e_1)|(Q) \leq 1 + \frac{\delta_0}{5}, \tag{253}$$

$$|D(u^1 \cdot e_2)|(Q) \leq \frac{\delta_0}{5}, \tag{254}$$

$$|D(u^2 \cdot e_1)|(Q) \leq \frac{\delta_0}{5}, \tag{255}$$

$$|D(u^2 \cdot e_2)|(Q) \leq 1 + \frac{\delta_0}{5}, \tag{256}$$

and

$$\begin{aligned}&2 + \delta_0\\ \leq&(|D((u^1 + u^2) \cdot e_1)| + |D((u^1 + u^2) \cdot e_2)|)(Q)\\ \leq&(|D(u^1 \cdot e_1)| + |D(u^1 \cdot e_2)| + |D(u^2 \cdot e_1)| + |D(u^2 \cdot e_2)|)(Q)\\ \leq&2 + \frac{4\delta_0}{5},\end{aligned} \tag{257}$$

a contradiction.

Similarly, we can show that for $\boldsymbol{d} = (e_1 + e_2, -e_2), \nu = e_1$, we have

$$(\varphi_\infty^{\mathrm{rel}} + \varphi_0)(\boldsymbol{d}, \nu) < (\varphi_\infty + \varphi_0)^{\mathrm{rel}}(\boldsymbol{d}, \nu), \tag{258}$$

and the same inequality automatically holds for the BV-elliptic envelope of the left-hand side.

Evaluating the left-hand side yields by (246)

$$(\varphi_\infty^{\mathrm{rel}} + \varphi_0)(\boldsymbol{d}, e_1) = \varphi_{\mathrm{single}}^{\mathrm{rel}}(e_1 + e_2, e_1) + \varphi_{\mathrm{single}}^{\mathrm{rel}}(e_2, e_1) + \varphi_{\mathrm{single}}^{\mathrm{rel}}(e_1, e_1). \tag{259}$$

We again assume equality holds and consider a competitor $\boldsymbol{u} = (u^1, u^2) \in BV_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{Z}^4)$ to the cell problem on the right with energy

$$\begin{aligned}&\int_{J_{\boldsymbol{u}} \cap Q} (\varphi_\infty + \varphi_0)([\boldsymbol{u}], \nu_{J_{\boldsymbol{u}}})\, d\mathcal{H}^1\\ \leq&(1 + \tilde{\delta})(\varphi_\infty^{\mathrm{rel}} + \varphi_0)^{\mathrm{rel}}(\boldsymbol{d}, e_1),\end{aligned} \tag{260}$$

where again $\tilde{\delta} = \min(\delta_0, \delta(\delta_0/5))$, so that

$$|D((u^1 + u^2) \cdot e_1)|(Q) \leq 1 + \frac{\delta_0}{5}, \tag{261}$$

$$|D((u^1 + u^2) \cdot e_2)|(Q) \leq \frac{\delta_0}{5}, \tag{262}$$

$$|D(u^2 \cdot e_1)|(Q) \leq \frac{\delta_0}{5}, \tag{263}$$

$$|D(u^2 \cdot e_2)|(Q) \leq 1 + \frac{\delta_0}{5}. \tag{264}$$

But now

$$\begin{aligned}
&2 + \delta_0 \\
\leq &(|D(u^1 \cdot e_1)| + |D(u^1 \cdot e_2)|)(Q) \\
\leq &|D((u^1 + u^2) \cdot e_1)| + |D((u^1 + u^2) \cdot e_2)| \\
&+ |D(u^2 \cdot e_1)| + |D(u^2 \cdot e_2)|)(Q) \\
\leq &2 + \frac{4\delta_0}{5},
\end{aligned} \tag{265}$$

a contradiction.

6 Extension of BV functions in a perforated domain

In this section, we develop a type of ball construction as seen in [24],[30], [31].

6.1 The boundaries of bounded convex sets in the plane

Lemma 6.1. *Let $U \subseteq \mathbb{R}^2$ be open, bounded, and convex. Then the measure-theoretic boundary of U is $\partial_* U = \partial U$, and there are Lipschitz closed curves $\gamma : S^1 \to \partial U$, $\theta : S^1 \to S^1$ such that both are surjective and "monotone" and $\gamma(t) + s\theta(t) + r\theta^\perp(t)$ is not in U for all $t \in S^1$, $s \geq 0$, and $r \in \mathbb{R}$.*

Here a continuous map $f : A \to B$ between topological spaces is called "monotone" if $f^{-1}(V)$ is connected for all $V \subset B$ connected. Note that for $f : \mathbb{R} \to \mathbb{R}$ continuous, this coincides with the normal notion of monotonicity.

Proof. Clearly $\partial_* U \subseteq \partial U$. Conversely, let $x \in \partial U$. Take some $B(y, r) \Subset U$.

By convexity and openness, the interior of the cone $\mathrm{conv}(\{x\} \cup B(y, r))$ is contained in U. For any $\rho \in (0, |x - y| - r)$ the limit cone takes up the same volume fraction $\theta \in (0, 1)$ of $B(x, \rho)$. Thus $|B(x, \rho) \cap U| \geq \limsup_{n\to\infty} |B(x, \rho) \cap \mathrm{conv}(\{x_n\} \cup B(y, r))| = |B(x, \rho) \cap \mathrm{conv}(\{x\} \cup B(y, r))| = \theta |B(x, \rho)|$. On the other hand, there is a supporting hyperplane, and $|B(x, \rho) \cap U| \leq |B(x, \rho)|/2$. Thus $x \in \partial_* U$.

In order to parametrize ∂U, assume by a scaling and translation that $B(0, 1) \subseteq U \subseteq B(0, R)$ for some $R > 0$. Consider the map $\tilde{\gamma}(v) := \sup\{t : tv \in U\} v$ from S^1 to ∂U. $\tilde{\gamma}$ is well-defined since U is open and bounded. It is automatically injective and its image is contained in ∂U.

Assume that there is $x \in \partial U \backslash \tilde{\gamma}$. Then x lies in the interior of the line segment $(0, \gamma(x/|x|))$. But then x lies in the interior of the cone $\mathrm{conv}(\gamma(x/|x|) \cup B(0, 1))$,

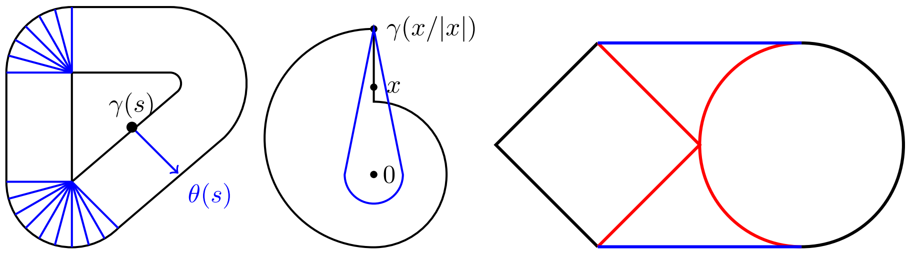

Figure 10:
Left: The boundary of a bounded convex set can be parametrized from the circle. For every boundary point $\gamma(s)$, there may be multiple normal vectors $\theta(s)$. The perimeter of the ball around the convex set grows linearly.
Middle: The map γ in the proof of Lemma 6.1 hits the entire boundary, because any covered boundary point x lies in the blue cone, and thus in the interior of the convex set.
Right: If two convex sets touch, the convex hull of their union has minimal perimeter among all sets covering both. The blue lines are shorter than the red lines they replace.

which is contained in U. Thus x is not in ∂U, a contradiction (see Figure 10 middle).

$\tilde{\gamma}$ is Lipschitz because the gauge function $v \mapsto \inf\{s > 0 : v \in sU\}$ is convex on $\mathbb{R}^2$ and thus Lipschitz on all compact subsets, and in particular bounded away from 0 on S^1, so that $\tilde{\gamma}(v) = v/\inf\{s > 0 : v \in sU\}$ is also Lipschitz.

We take $\tilde{\theta} := -\tilde{\gamma}'^{\perp}/|\tilde{\gamma}'|$ wherever it exists. Due to the convexity of U, $\tilde{\theta} : S^1 \to S^1$ is "monotone" and thus rectifiable. By parameterizing both by their combined arc length and interpolating the jumps of $\tilde{\theta}$ we obtain Lipschitz versions γ, θ of both. □

Lemma 6.2. *Let $U \subset \mathbb{R}^n$ be open and convex. Then $U_r = B(U, r)$ is open and convex, and $\mathcal{H}^1(\partial U_r) = \mathcal{H}^1(\partial U) + 2\pi r$.*

Proof. Openness and convexity are clear. We shall use γ and θ from Lemma 6.1 to parametrize ∂U_r. If $x \in \partial U_r$, there is $v \in S^1$ with $x = \gamma(v) + r\theta(v)$. Thus

$$\begin{aligned} &\mathcal{H}^1(\partial U_r) \\ =&\int_{S^1} |\gamma' + r\theta'| \, d\mathcal{H}^1 \\ =&\int_{S^1} |\gamma'| + r|\theta'| \, d\mathcal{H}^1 \\ =&\mathcal{H}^1(\partial U) + 2\pi r, \end{aligned} \tag{266}$$

since θ' and γ' are parallel with equal signs almost everywhere. □

Lemma 6.3. *Let $U_1, U_2 \subset \mathbb{R}^2$ be open, convex, and bounded, with* $\operatorname{dist}(U_1, U_2) = 0$*. Then $\mathcal{H}^1(\partial \operatorname{conv}(U_1 \cup U_2)) \leq \mathcal{H}^1(\partial U_1) + \mathcal{H}^1(\partial U_2)$.*

This is true in a much more general context. For a proof see e.g. [16]. For a graphical argument, see Figure 10 right. The analogous statement in higher dimensions is false.

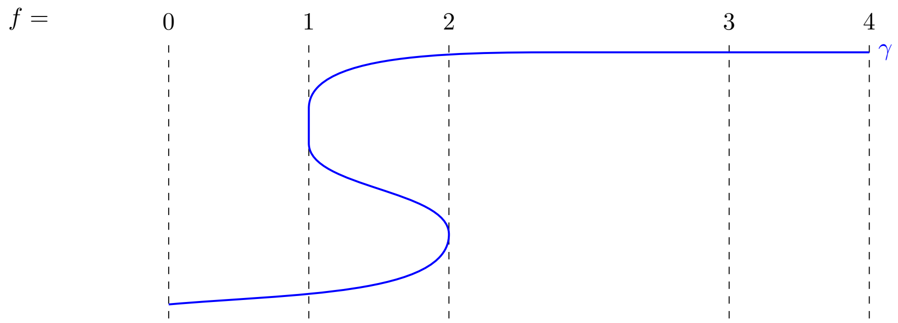

Figure 11: For a measure μ concentrated on a curve γ, we can bound $\mu(\Omega)$ by an integral over the level sets of a C^1 function f. Note that most level sets $\{f = t\}$ only intersect γ countably often. The inequality in Lemma 6.4 is due to an angular mismatch and can be made precise.

6.2 A ball construction using increasing convex sets

We show a coarea formula for measures concentrated on rectifiable curves.

Lemma 6.4. *Let $\Omega \subset \mathbb{R}^2$ be open. Let $\mu \in \mathcal{M}(\Omega)$ be a Radon measure concentrated on a countable family of rectifiable C^1 curves, i.e. $\mu = \sum_{i\in\mathbb{N}} g_i \mathcal{H}^1 \llcorner \gamma_i$, with $g_i : \gamma_i \to [0, \infty)$ Borel measurable. Let $f \in C^1(\Omega)$. Then*

$$\int_{\mathbb{R}} \sum_{i\in\mathbb{N}} \sum_{x\in f^{-1}(t)\cap\gamma_i} \frac{g_i(x)}{|\nabla f(x)|}\, dt \le \mu(\Omega). \tag{267}$$

In particular, the integrand is well-defined for almost every $t \in \mathbb{R}$.

Proof. Since both sides of the equation are additive with respect to curves, we can assume without loss of generality that $\mu = g\mathcal{H}^1 \llcorner \gamma$, with $\gamma \in C^1([0, L], \Omega)$ parametrized by arc length. Then $\mu(\Omega) = \int_0^L (g \circ \gamma)(s)\, ds$.

Note that by Sard's Lemma, $\mathcal{L}^1(\{f \circ \gamma(s) : (f \circ \gamma)'(s) = 0\}) = 0$, and by localization, we can assume that $(f \circ \gamma)' \neq 0$. By the chain rule, $|(f \circ \gamma)'(s)| = |\nabla f(\gamma(s))\cdot\dot{\gamma}(s)| \le |\nabla f(\gamma(s))|$, where we also used the Cauchy-Schwarz inequality for the Riemannian metric.

Now locally $(f \circ \gamma)^{-1} \in C^1(\mathbb{R})$ by the inverse function theorem, and

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_{x\in f^{-1}(t)\cap\gamma} \frac{g(x)}{|\nabla f(x)|}\, dt \\
&= \int_{(f\circ\gamma)([0,L])} \frac{(g \circ \gamma)(f \circ \gamma)^{-1}(t)}{|\nabla f((f \circ \gamma)^{-1}(t))|}\, dt \\
&\le \int_{(f\circ\gamma)([0,L])} (g \circ \gamma)(f \circ \gamma)^{-1}(t)|((f \circ \gamma)^{-1})'(t)| \\
&= \int_0^L (g \circ \gamma)(s)\, ds = \mu(\Omega).
\end{aligned} \tag{268}$$

□

Using this formula, we show that, given a measure concentrated on a family of rectifiable curves, such as $|D\boldsymbol{u}|$ for $u \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$, and a small set with very small perimeter ω, there is a set $\tilde{\omega} \supset \omega$ with small perimeter and a small amount of measure on the boundary.

Lemma 6.5. *For any $\delta > 0$ there exists $\eta > 0$ such that for any $\omega \subset \mathbb{T}^2$ with $|\omega| < 1/2$ and $\mathcal{H}^1(\partial\omega) \leq \eta$ and any Radon measure concentrated on a countable family of rectifiable C^1-curves $\mu = \sum_{j\in\mathbb{N}} g_j \mathcal{H}^1 \llcorner \gamma_j$, there is a finite family $\omega_1, \dots, \omega_N \subset \mathbb{T}^2$ of pairwise disjoint convex closed sets, covering ω, with $|\tilde{\omega}_i| < 1/2$, $\sum_{i=1}^N \mathcal{H}^1(\partial\omega_i) < \delta$, and*

$$\sum_{i=1}^{N} \left(\mathcal{H}^1(\partial\omega_i) \sum_{j\in\mathbb{N}} \sum_{x\in\partial\omega_i\cap\gamma_j} g_j(x) \right) \leq \delta\mu(\mathbb{T}^2 \setminus \omega). \tag{269}$$

Also each $\omega_i \cap \omega \neq \emptyset$ and $\max_i \sup\{|x-y| : x \in \omega_i, y \in \omega\} < \delta$.

Note that here, $\partial\omega$ denotes the topological boundary, which is generally much larger than the reduced boundary $\partial^*\omega$ and the measure-theoretic boundary $\partial_*\omega$.

We shall prove this by applying Lemma 6.4 with a carefully chosen f. This is essentially a ball construction as used in [24], [30], [31] for Ginzburg-Landau energies, except we use general convex sets instead of just balls for the covering.

Proof. We start by covering $\partial\omega$ with finitely many balls $\overline{B(x_k, r_k)}$, $k = 1, \dots, K$, with $\sum_{k=1}^K r_k \leq C\eta$. Now we define a family of at most K convex closed sets by starting with $\{\overline{B(x_k, r_k)} : k = 1, \dots, K\}$ and iteratively replacing two nondisjoint convex open sets with the convex hull of their union until all remaining sets are pairwise disjoint.

We end up with the pairwise disjoint family $\omega_1^0, \dots, \omega_{K(0)}^0$ of convex closed sets covering $\partial\omega$, and because $|\omega| < 1/2$ also ω, with $\sum_{k=1}^{K(0)} \mathcal{H}^1(\partial\omega_k^0) \leq C\eta$ due to Lemma 6.3.

We then define functions $r_k : [0, \infty) \to [0, \infty)$, $k = 1, \dots, K(0)$ through $r_k(0) =$, $r_k' = \mathcal{H}^1(\partial B(\omega_k, r_k))$, and sets $\omega_k^t = B(\omega_k^0, r_k(t))$. Then by Lemma 6.2

$$\frac{d}{dt}\mathcal{H}^1(\partial\omega_k^t) = 2\pi\mathcal{H}^1(\partial\omega_k^t) \tag{270}$$

and

$$\frac{d}{dt}\sum_{k=1}^{K(0)} \mathcal{H}^1(\partial\omega_k^t) = 2\pi \sum_{k=1}^{K(0)} \mathcal{H}^1(\partial\omega_k^t). \tag{271}$$

We keep enlarging the convex sets until the collision time $t_1 = \inf\{t > 0 : \text{the family} \omega_1^t, \dots, \omega_{K(0)}^t \text{ is not disjoint}\}$, at which point we again iteratively replace any nondisjoint pair $\omega_k^{t_1}$, $\omega_{k'}^{t_1}$ with $\operatorname{conv}(\omega_k^{t_1} \cup \omega_{k'}^{t_1})$ until the family is disjoint. At this point we reduce $K(t)$ to the new number of disjoint closed convex sets $\omega_1^{t_1}, \dots, \omega_{K(t_1)}^{t_1}$.

We then again let $r_k : [t_1, \infty) \to [0, \infty)$, $k = 1, \dots, K(t_1)$ solve $r_k(t_1) = \mathcal{H}^1(\partial\omega_k^{t_1})$, $r_k' = \mathcal{H}^1(\partial B(\omega_k^{t_1}, r_k)$ and set $\omega_k^t = B(\omega_k^{t_1}, r_k(t))$ for $t \geq t_1$.

We grow the convex sets up to the next collision time, and then iterate again. Since we started with finitely many sets, there are only finitely many collision times, and the evolution of $K(t)$ and the family $\omega_1^t, \ldots, \omega_{K(t)}^t$ is well-defined for all $t \geq 0$, satisfying

$$\frac{d}{dt}\mathcal{H}^1(\partial\omega_k^t) = 2\pi\mathcal{H}^1(\partial\omega_k^t) \tag{272}$$

for all non-collision times t and all $k = 1, \ldots, K(t)$, and by Gronwall's Lemma and Lemma 6.3

$$\sum_{k=1}^{K(t)} \mathcal{H}^1(\partial\omega_k^t) \leq C\eta e^{2\pi t}. \tag{273}$$

Now we define $f(x) = \inf\{t \geq 0 : x \in \bigcup_{k=1}^{K(t)} \omega_k^t\}$. Now whenever $f(x)$ is not a collision time, then $x \in \partial\omega_k^{t(x)}$ for some $k = 1, \ldots, K(t(x))$, and

$$\nabla f(x) = \frac{\nu}{2\pi\mathcal{H}^1(\partial\omega_k^{t(x)}}, \tag{274}$$

where $\nu \in S^1$ is the outer normal to $\omega_k^{t(x)}$ at x.

Now if

$$\eta < \frac{\delta}{2\pi C e^{2\pi/\delta}} \tag{275}$$

we get for $t \leq 2\pi/\delta$ that

$$\sum_{k=1}^{K(t)} \mathcal{H}^1(\partial\omega_k^t) \leq \delta, \tag{276}$$

and by Lemma 6.4 that

$$\begin{aligned}&\int_0^{2\pi/\delta} \sum_{k=1}^{K(t)} \left(\sum_{x \in \partial\omega_k^t \cap \bigcup_{j\in\mathbb{N}} \gamma_j} 2\pi g_j(x)\mathcal{H}^1(\partial\omega_k^t) \right) dt \\ \leq & \mu(\mathbb{T}^2 \setminus \omega).\end{aligned} \tag{277}$$

The result then follows by choosing a non-collision time $t \in (0, 2\pi/\delta)$ such that

$$\begin{aligned}&\sum_{k=1}^{K(t)} \left(\sum_{x \in \partial\omega_k^t \cap \bigcup_{j\in\mathbb{N}} \gamma_j} g_j(x)\mathcal{H}^1(\partial\omega_k^t) \right) \\ \leq & \delta\mu(\mathbb{T}^2 \setminus \omega).\end{aligned} \tag{278}$$

□

Corollary 6.6. *For any $\delta > 0$ there exists $\eta > 0$ such that for any open $\Omega \subset \mathbb{T}^2$, for any Borel $\omega \Subset \Omega$ with $\mathcal{H}^1(\partial\omega) \leq \eta$ and $\mathrm{dist}(\omega, \partial\Omega) \geq \delta$, and any $\boldsymbol{u} \in BV(\Omega\setminus\omega, \mathbb{Z}^{2M})$ with a rectifiable, piecewise C^1 jump set, there is a function $\boldsymbol{w} \in BV(\Omega, \mathbb{Z}^{2M})$ with $\boldsymbol{w} = \boldsymbol{u}$ on $\Omega\setminus B(\omega, \delta)$ and $|D\boldsymbol{w}|(\overline{\{w \neq u\}}) \leq \delta|Du|(\Omega\setminus\omega)$.*

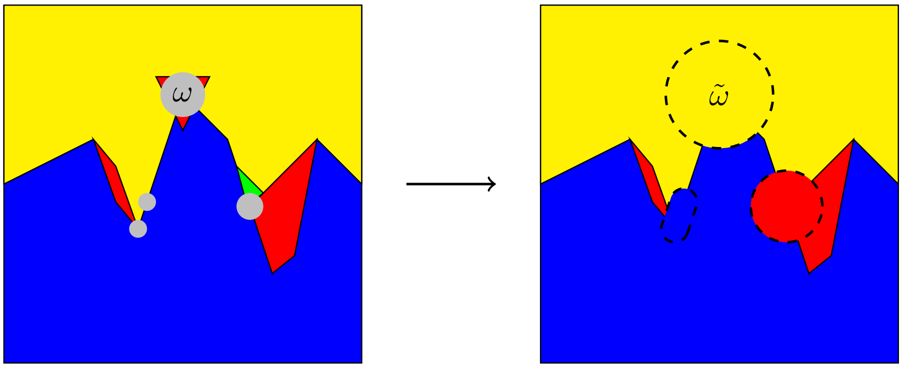

Figure 12: Sketch of Corollary 6.6. The missing set ω is enlarged to $\tilde{\omega}$, where $\boldsymbol{u}$ is extended from its boundary values. The extra total variation is small if $\mathcal{H}^1(\omega)$ is small.

Proof. Apply Lemma 6.5 to $\mu = |D\boldsymbol{u}|\llcorner\Omega$. Define

$$\boldsymbol{w} = \begin{cases} \boldsymbol{z}_i & , \text{ in } \omega_i \\ \boldsymbol{u} & , \text{ in } \Omega \setminus \bigcup_{i=1}^N \omega_i, \end{cases} \tag{279}$$

where $\boldsymbol{z}_i \in \mathbb{Z}^{2M}$ is one of the values taken by the trace of $\boldsymbol{u}$ on $\partial\omega_i$. Then

$$\begin{aligned} & |D\boldsymbol{w}|(\{\boldsymbol{w} \neq \boldsymbol{u}\}) \\ =& \sum_{i=1}^N \int_{\partial\omega_i} |\boldsymbol{u} - \boldsymbol{z}_i| \, d\mathcal{H}^1 \\ \leq& \sum_{i=1}^N \left(\mathcal{H}^1(\partial\omega_i) \sum_{x \in J_{\boldsymbol{u}} \cap \partial\omega_i} [\boldsymbol{u}](x) \right) \\ \leq& \delta |D\boldsymbol{u}|(\Omega \setminus \omega), \end{aligned} \tag{280}$$

where we used the Poincaré inequality on each of the $\partial\omega_i$. $\square$

7 Slip fields with parallel straight dislocation lines

In this section we introduce the subclasses $\mathcal{S}$ and $\mathcal{S}_1$ of $BV(\mathbb{R}^2, \mathbb{Z}^{2M})$ of functions which jump on parallel straight lines, with jumps positive multiples of the same Burgers vector.

Definition 7.1. *Define the class of one-dimensional step functions as*

$$\begin{aligned} \mathcal{S} := \{ & \boldsymbol{u} \in BV_{\text{loc}}(\mathbb{R}^2, \mathbb{Z}^{2M}) : \boldsymbol{u}(x) = a + b\lambda(x \cdot \nu) \\ & \textit{for some } a, b \in \mathbb{Z}^{2M}, \nu \in S^1, \lambda : \mathbb{R} \to \mathbb{Z} \textit{ monotone} \}. \end{aligned} \tag{281}$$

We also define the class of one-dimensional single-step functions as

$$\mathcal{S}_1 := \{ u(x) = a + b\lambda(x \cdot \nu) \in \mathcal{S} : \sharp J_\lambda \leq 1 \}. \tag{282}$$

These functions were used in [11] to show the lower bound, since they allow one to pass from the nonlocal energies $B^{\mathrm{per}}_{h,j}$ to the line-tension energy appearing in the limit. Because the kernel we use is not positive, we have to take care that the jump lines are far enough apart, see Lemma 7.6.

7.1 Energy estimates for positive kernels

Here we focus on estimates for kernels $K_j : \mathbb{T}^2 \to \mathbb{R}$, $j \in \mathbb{N}$, of the type

$$K_j(x) = \begin{cases} 2^{3j} & , \text{ if } |x| \leq 2^{-j} \\ \frac{2^{-2j}}{|x|^5} & , \text{ if } |x| > 2^{-j}. \end{cases} \tag{283}$$

Note that by Lemma 3.20, we have $|\mathbb{J}^{\mathrm{per}}_{h,j}| \leq CK_j$ for all $j \in \mathbb{N}, h \in [0, \infty]$. Here $|x| = \mathrm{dist}(x, \mathbb{Z}^2)$.

The next lemma shows that long-range effects in the double integral with kernel K_j are much smaller than effects up to length 2^{-j}.

Lemma 7.2. *Let $A \subset \mathbb{T}^2$ be convex, in the sense that for $x, y \in A$, any shortest path connecting x, y is also in A. Let $u \in L^2(A)$, $1 \leq j, i \in \mathbb{N}$. Then*

$$\begin{aligned}&\int_A\int_A (u(x)-u(y))^2 K_j(x-y)\mathbb{1}_{\{|x-y|\in(2^{-j-i},2^{-j-i+1})\}}\,dx\,dy\\ \leq &2^{-i}\int_A\int_A (u(x)-u(y))^2 K_j(x-y)\mathbb{1}_{\{|x-y|\leq 2^{-j}\}}\,dx\,dy.\end{aligned} \tag{284}$$

Note that if $A \subset \mathbb{T}^2$ has $\mathrm{diam}\, A \leq \frac{1}{2}$, it is Euclidean, and there is exactly one shortest path. In practice we shall consider only A with small diameter or $A = \mathbb{T}^2$.

Proof. Note that due to the convexity of A, we can change variables and estimate

$$\begin{aligned}&\int_A\int_A (u(x)-u(y))^2 \mathbb{1}_{\{|x-y|\in(2^{-j-i},2^{-j-i+1})\}}\,dx\,dy\\ \leq\ &2\int_A\int_A \left[(u(x)-u(z(x,y)))^2 + (u(y)-u(z(x,y)))^2\right]\\ &\mathbb{1}_{\{|x-y|\in(2^{-j-i},2^{-j-i+1})\}}\,dx\,dy\\ =\ &16\int_A\int_A (u(x)-u(y))^2 \mathbb{1}_{\{|x-y|\in(2^{-j-i-1},2^{-j-i})\}}\,dx\,dy,\end{aligned} \tag{285}$$

where $(x, y) \mapsto z$ is a measurable almost everywhere differentiable map assigning to two points an intermediate point. Note that $\nabla_x z = \nabla_y z = \frac{1}{2}\mathrm{Id}$ at all points of differentiability. Iterating this inequality and multiplying by a constant yields

$$\begin{aligned}&\int_A\int_A (u(x)-u(y))^2 \frac{2^{-2j}}{2^{5(-j-i)}}\mathbb{1}_{\{|x-y|\in(2^{-j-i},2^{-j-i+1})\}}\,dx\,dy\\ \leq &2^{-i}\int_A\int_A (u(x)-u(y))^2 2^{3j}\mathbb{1}_{\{|x-y|\in(2^{-j-1},2^{-j})\}}\,dx\,dy.\end{aligned} \tag{286}$$

The result then follows because $K_j(x) \leq \frac{2^{-2j}}{2^{5(-j-i}}$ for $|x|$ in $(2^{-j-i}, 2^{-j-i+1})$ and $K_j(x) = 2^{3j}$ for $|x|$ in $(0, 2^{-j})$. $\square$

We can also control the double integral with kernel K_j by the squared jump length $\int_{J_w}[w]^2\,d\mathcal{H}^1$ for $w\in BV(\mathbb{T}^2,\mathbb{Z})$ provided the jump set is not too dense:

Lemma 7.3. *Let $1<i<j\in\mathbb{N}$. Let $A,B\subseteq\mathbb{T}^2$ be such that $[x,y]\subset A\cup B$ whenever $x\in A$, $y\in B$, with $|x-y|\le 2^{-j+i}$. Let $w\in BV(A\cup B,\mathbb{Z})$ have a jump set consisting of countably many line segments along which $[w]$ is constant, such that every line segment $[x,y]$ as above only intersects J_w at most N times. Then*

$$\begin{aligned}&\int_A\int_B (w(x)-w(y))^2K_j(x-y)\mathbb{1}_{\{|x-y|\le 2^{-j+i}\}}\,dy\,dx\\ \le& CN\int_{J_w\cap(A\cup B)}[w]^2\,d\mathcal{H}^1,\end{aligned}\tag{287}$$

for some universal constant C.

Remark 7.4. *Note that if $A=B$ is convex, or if A is convex and $B=B(A,R)\setminus A$, then any line segment $[x,y]$ is indeed contained in $A\cup B$. If $w\in\mathcal{S}$ has lots of parallel jumps, then the statement is clearly false, as $(w(x)-w(y))^2=(\sum_{z\in J_w\cap[x,y]}[w](z))^2\gg\sum_{z\in J_w\cap[x,y]}[w]^2(z)$.*

Proof. If $x\in A,y\in B,|x-y|\le 2^{-j+i}$, then

$$\begin{aligned}&(w(x)-w(y))^2\\ \le& N\sum_{z\in J_w\cap[x,y]}[w]^2(z)\\ =& N\sum_{[a,b]\subset J_w:[x,y]\cap[a,b]\neq\emptyset}[w]^2_{[a,b]}\end{aligned}\tag{288}$$

for $\mathcal{L}^4$-almost every (x,y), since two short line segments in the torus intersect nowhere, once, or everywhere. Thus,

$$\begin{aligned}&\int_A\int_B (w(x)-w(y))^2K_j(x-y)\mathbb{1}_{\{|x-y|\le 2^{-j+i}\}}\,dy\,dx\\ \le&\sum_{[a,b]\subset J_w}[w]^2_{[a,b]}\int_{B([a,b],2^{-j+i})}\int_{U_x}K_j(x-y)\,dy\,dx.\end{aligned}\tag{289}$$

Here $U_x=\{y\in B(x,2^{-j+i}):[x,y]\cap[a,b]\neq\emptyset\}$ is the shadow of $[a,b]$ behind x.

We now restrict ourselves to one segment $[a,b]\subset J_w$ and assume without loss of generality that $-a=b=\frac{L}{2}e_1$, with $L>0$.

We consider three cases:

Case 1: $L\ge 2^{-j+i}$. In this case , if $|x_2|\le 2^{-j}$, then

$$\int_{U_x}K_j(y-x)\,dy\le\int_{\mathbb{T}^2}K_j(y-x)\,dy\le C2^j.\tag{290}$$

If $|x_2|>2^{-j}$ we get

$$\int_{U_x}K(y-x)\,dy\le\int_{\mathbb{T}^2\setminus B(x,|x_2|)}K_j(y-x)\,dy\le C\frac{2^{-2j}}{|x_2|^3},\tag{291}$$

and integrating over $B([a,b],2^{-j+i})$ yields

$$\int_{B([a,b],2^{-j+i})}\int_{U_x} K_j(y-x)\,dy\,dx \leq CL\int_0^{2^{-j+i}}\left(2^j \wedge \frac{2^{-2j}}{t^3}\right)dt \\ \leq CL. \tag{292}$$

Case 2: $L \leq 2^{-j}$. In this case, if $|x| < L$, we estimate

$$\int_{U_x} K_j(y-x)\,dy \leq 2^j. \tag{293}$$

If $L < |x| < 2^{-j}$, then, since $[a,b] \subset B(0,L)$, we see that U_x is contained in a cone around x with angle $\frac{L}{|x|}$, and

$$\int_{U_x} K_j(y-x)\,dy \leq \frac{L}{|x|}\int_{\mathbb{T}^2} K_j(y-x)\,dy \leq C\frac{L}{|x|}2^j. \tag{294}$$

If $|x| \geq 2^{-j}$, U_x is still contained in the same cone, and

$$\int_{U_x} K_j(y-x)\,dy \leq \frac{L}{|x|}\int_{\mathbb{T}^2\setminus B(x,|x|)} K_j(y-x)\,dy \\ \leq C\frac{L}{|x|}\frac{2^{-2j}}{|x|^3}. \tag{295}$$

Integrating over $B([a,b],2^{-j+i})$, which is contained in $B(0,2^{-j+i+1})$, we see that

$$\int_{B(0,2^{-j+i+1})}\int_{U_x} K_j(y-x)\,dy \\ \leq \int_0^L Ct2^j\,dt + \int_L^{2^{-j}} Ct\frac{L}{t}2^j\,dt + \int_{2^{-j}}^{2^{-j-i+1}} Ct\frac{L}{t}\frac{2^{-2j}}{t^3}\,dt \\ \leq CL^2 2^j + CL + CL2^{-2j}2^{2j} \leq CL. \tag{296}$$

Case 3: $2^{-j} < L < 2^{-j+i}$. If $|x_2| \leq 2^{-j}$ and $|x_1| \leq L$, then

$$\int_{U_x} K_j(y-x)\,dy \leq C2^j. \tag{297}$$

If $2^{-j} < |x_2| < L$ and $|x_1| \leq L$, we get

$$\int_{U_x} K_j(y-x)\,dy \leq \int_{\mathbb{T}^2\setminus B(x,|x_2|)} K_j(y-x)\,dy \leq C\frac{2^{-2j}}{|x_2|^3}. \tag{298}$$

If $|x_2| \geq L$ or $|x_1| \geq L$, then $|x| \geq L$, and we can again use the cone estimate

$$\int_{U_x} K_j(y-x)\,dy \leq C\frac{L}{|x|}\frac{2^{-2j}}{|x|^3}. \tag{299}$$

Integrating over the first domain yields twice

$$\int_{[-L,L]\times[0,2^{-j}]}\int_{U_x} K_j(y-x)\,dy \leq CL. \tag{300}$$

Integrating over the second domain yields twice

$$\int_{[-L,L]\times[2^{-j},L]}\int_{U_x} K_j(y-x)\,dy \le CL\int_{2^{-j}}^{L}\frac{2^{-2j}}{t^3}\,dt \le CL. \tag{301}$$

Integrating over the third domain, using the inclusion $B([a,b],2^{-j+i})\subset B(0,2^{-j-i+1})$, we get that

$$\begin{aligned}&\int_{B(0,2^{-j-i+1})\setminus B(0,L)}\int_{U_x} K_j(y-x)\,dy\\ \le&\int_{L}^{2^{-j-i+1}} Ct\frac{L}{t}\frac{2^{-2j}}{t^3}\,dt\\ \le& CL\frac{2^{-2j}}{L^2}\le CL.\end{aligned} \tag{302}$$

All in all we always get that the total integral is bounded by a constant times the length of the segment $[a,b]$. Summing up over all segments making up J_w yields the result. ☐

7.2 Energy estimates for step functions

One-dimensional step functions allow for replacing the nonlocal energies $B_{j,\infty}^{\mathrm{per}}$ and $B_{j,0}^{\mathrm{per}}$ with a line-tension energy.

Lemma 7.5. *Let $\boldsymbol{u}\in\mathcal{S}_1$. Let $j\in\mathbb{Z}$, $s<t\in\mathbb{N}$. Then for any square $Q_{z,2^{-j}}$, defining the box $A=B(J_{\boldsymbol{u}},2^{-j-t+s})\cap Q_{z,2^{-j}}$, we have*

$$\begin{aligned}&(\log 2)I_\infty(u,Q_{x,2^{-j}})\\ \le&\int_A\int_A(\boldsymbol{u}(x)-\boldsymbol{u}(y))\mathbb{J}_{j+t,\infty}^{\mathrm{per}}(x-y)(\boldsymbol{u}(x)-\boldsymbol{u}(y))\,dx\,dy\\ &+c2^{j-s}(|D\boldsymbol{u}|(Q_{z,2\times2^{-j}}))^2.\end{aligned} \tag{303}$$

The proof of this lemma involves a lengthy calculation.

Proof. We write $\boldsymbol{u}=\boldsymbol{a}+\boldsymbol{b}\lambda(x\cdot\nu)$, with $\boldsymbol{a},\boldsymbol{b}\in\mathbb{Z}^{2M}$, $\lambda:\mathbb{R}\to\{0,1\}$ monotone, and assume by density that $\nu\in\mathbb{Q}^2$. Consider the periodic function $\boldsymbol{u}_0\in BV(\mathbb{T}^2,\mathbb{Z}^{2M})$ defined by $\boldsymbol{u}_0(x)=\boldsymbol{a}+\boldsymbol{b}\Lambda(x\cdot N\nu)$, where $\Lambda:\mathbb{T}^1\to\{0,1\}$ is the function jumping up at $\mathbb{Z}$ and down at $\mathbb{Z}+\frac{1}{2}$ and $N\in\mathbb{N}$ is such that $N\nu\in\mathbb{Z}^2$.

If u is constant, all terms are 0. We can therefore assume that $\boldsymbol{u}$ has exactly one jump line in Q, making A either a triangle, if the jump happens in a corner of $Q_{z,2^{-j}}$, or a polygon with four to six corners.

Now we simplify the geometry of A. Assume by translation that $J_\lambda=\{0\}$ and that $J_{\boldsymbol{u}}\cap Q_{z,2^{-j}}=\{x\in\mathbb{R}^2\,:\,x\cdot\nu=0, x\cdot\frac{\nu^\perp}{|\nu|}\in(0,\mathcal{H}^1(J_{\boldsymbol{u}}))\}$. We now replace

$$\int_A\int_A(\boldsymbol{u}(x)-\boldsymbol{u}(y))\mathbb{J}_{j+t,\infty}^{\mathrm{per}}(x-y)(\boldsymbol{u}(x)-\boldsymbol{u}(y))\,dx\,dy \tag{304}$$

by

$$2\mathcal{H}^1(J_u \cap Q_{z,2^{-j}}) \int_{(-2^{-j-t+s},0)\frac{\nu}{|\nu|}} \int_{(0,\infty)\nu+\nu^\perp} \boldsymbol{b}\mathbb{J}_{j+t,\infty}(x-y)\boldsymbol{b}$$
$$\mathbb{1}_{|x-y|\le 2^{-j-t+s}}\, dx\, d\mathcal{H}^1(y). \quad (305)$$

Note that we have replaced the periodic kernel $\mathbb{J}^{\text{per}}_{j,\infty}$ by $\mathbb{J}_{j,\infty}$ and added or removed some near the boundary of A. To see how much has changed, note that

$$|\mathbb{J}^{\text{per}}_{j,\infty} - \mathbb{J}_{j,\infty}| \le C \sum_{z\in\mathbb{Z}^2\setminus\{0\}} \frac{2^{-2j-2t}}{|z|^5} \le C2^{-2j-2t}. \quad (306)$$

The corner terms due to the change in geometry we can bound by $|b|^2 2^{-j-t}$.

Now note that the double integral in (305) only depends on s and not on j or t, since we replaced $\mathbb{J}^{\text{per}}_{j+t,\infty}$ with $\mathbb{J}_{j+t,\infty}$.

On the other hand, a term similar to (305) appears also in $B^{\text{per}}_{k,\infty}(\boldsymbol{u}_0,\boldsymbol{u}_0)$. Here we have

$$\begin{aligned} &B^{\text{per}}_{k,\infty}(\boldsymbol{u}_0,\boldsymbol{u}_0)\\ \le& 2\mathcal{H}^1(J_{\boldsymbol{u}_0}) \int_{(-2^{-k+s},0)\frac{\nu}{|\nu|}} \int_{(0,\infty)\nu+\nu^\perp} \boldsymbol{b}\mathbb{J}_{k,\infty}(x-y)\boldsymbol{b}\\ &\mathbb{1}_{|x-y|\le 2^{-k+s}}\, dx\, d\mathcal{H}^1(y)\\ &+ C|\boldsymbol{b}|^2\mathcal{H}^1(J_{\boldsymbol{u}_0})(2^{-s}+2^{-k}), \end{aligned} \quad (307)$$

where error terms arise again from long-range effects and from replacing the periodic kernel with the Euclidean one.

Finally, we can easily find that

$$\begin{aligned} &B^{\text{per}}_{k,\infty}(\boldsymbol{u}_0,\boldsymbol{u}_0)\\ =&B^{\text{per}}_{\infty}(\boldsymbol{u}_0 * \tilde\phi_{2^{-k}}, \boldsymbol{u}_0 * \tilde\phi_{2^{-k}}) - B^{\text{per}}_{\infty}(\boldsymbol{u}_0 * \tilde\phi_{2^{-k+1}}, \boldsymbol{u}_0 * \tilde\phi_{2^{-k+1}})\\ =&(\log 2) I_\infty(\boldsymbol{u}_0) + O(2^{-k}). \end{aligned} \quad (308)$$

Summing up all error terms, we find that

$$\begin{aligned} &\int_A\int_A (\boldsymbol{u}(x)-\boldsymbol{u}(y))\mathbb{J}^{\text{per}}_{j+t,\infty}(x-y)(\boldsymbol{u}(x)-\boldsymbol{u}(y))\, dx\, dy\\ \ge&\mathcal{H}^1(J_{\boldsymbol{u}}\cap Q)\varphi_\infty(\boldsymbol{b},\nu) - c|\boldsymbol{b}|^2 2^{-j}(2^{-s}+2^{-t}). \end{aligned} \quad (309)$$

This is the result of the lemma, since $(|D\boldsymbol{u}|(Q_{z,2\times 2^{-j}}))^2 \le 4|\boldsymbol{b}|^2 2^{-2j}$. □

We can generalize this estimate to step functions with multiple jumps if these jumps are far enough apart.

Lemma 7.6. *Let $j\in\mathbb{Z}$, $t,s\in\mathbb{N}$, $Q_{z,2^{-j}}\subset\mathbb{T}^2$ some square. Let $\boldsymbol{w}\in\mathcal{S}$ with no two jumps within $(0,2^{-j-t+s})$ of each other. Then*

$$\begin{aligned} &(\log 2) I_\infty(\boldsymbol{w}, Q_{z,2^{-j}})\\ \le& \int_{Q_{z,2^{-j}}}\int_{Q_{z,2^{-j}}} (\boldsymbol{w}(x)-\boldsymbol{w}(y))\mathbb{J}^{\text{per}}_{j+t,\infty}(x-y)(\boldsymbol{w}(x)-\boldsymbol{w}(y))\, dx\, dy\\ &+ c2^{j-s}(|D\boldsymbol{w}|(Q_{z,2\times 2^{-j}}))^2. \end{aligned} \quad (310)$$

Proof. Write $\boldsymbol{w}(x) = \boldsymbol{a} + \boldsymbol{b}\lambda(x \cdot \nu)$, with $\nu \in S^1$.

By Fubini's theorem

$$\begin{aligned}
&\int_{Q_{z,2^{-j}}} \int_{Q_{z,2^{-j}}} |\boldsymbol{w}(x) - \boldsymbol{w}(y)|^2 K_{j+t}(x-y) \mathbb{1}_{\{|x-y|<2^{-j-t}\}} \, dx \, dy \\
\le & C2^{-j}|\boldsymbol{b}|^2 \int_{\mathbb{R}} \int_{\mathbb{R}} 2^{2j+2t} |\lambda(t) - \lambda(s)|^2 \mathbb{1}_{\{|t-s|<2^{-j-t}\}} \, dt \, ds \\
\le & C2^{-j}|\boldsymbol{b}|^2 \int_{\mathbb{R}} 2^{j+t} |\lambda(s+2^{-j-t}) - \lambda(s)|^2 \, ds \\
\le & C2^{-j}|\boldsymbol{b}|^2 (|D\lambda|(\mathbb{R}))^2 \le C2^j (|D\boldsymbol{w}|(Q_{z,2\times 2^{-j}}))^2. \qquad (311)
\end{aligned}$$

By Lemma 7.2, we can neglect long-range effects for the kernel K_{j+t} and by extension, for $\mathbb{J}^{\mathrm{per}}_{j+t}$, more precisely,

$$\begin{aligned}
&\left| \int_{Q_{z,2^{-j}}} \int_{Q_{z,2^{-j}}} (\boldsymbol{w}(x) - \boldsymbol{w}(y)) \mathbb{J}^{\mathrm{per}}_{j+t}(x-y) (\boldsymbol{w}(x) - \boldsymbol{w}(y)) \mathbb{1}_{\{|x-y|>2^{-j-t+s}\}} \, dx \, dy \right| \\
\le & c2^{-s} \int_{Q_{z,2^{-j}}} \int_{Q_{z,2^{-j}}} |\boldsymbol{w}(x) - \boldsymbol{w}(y)|^2 K(x-y) \mathbb{1}_{\{|x-y|<2^{-j-t}\}} \, dx \, dy \\
\le & c2^{j-s} (|D\boldsymbol{w}|(Q_{z,2\times 2^{-j}}))^2. \qquad (312)
\end{aligned}$$

We can now apply Lemma 7.5 to boxes A_i around each jump of $\boldsymbol{w}$, where $\boldsymbol{w}|_{A_i} = \boldsymbol{w}_i \in \mathcal{S}_1$ and obtain the result since

$$\sum_i (|D\boldsymbol{w}_i|(Q_{z,2\times 2^{-j}}))^2 \le (|D\boldsymbol{w}|(Q_{z,2\times 2^{-j}}))^2. \qquad (313)$$

□

Lemma 7.7. *Let $t, s \in \mathbb{N}$. Let $\delta > 0$. Then there is $\eta > 0$ such that the following holds: Let $j \in \mathbb{N} \setminus \{0\}$,, $Q_{z,2^{-j}} \subset \mathbb{T}^2$ some square. Let $\boldsymbol{w} \in \mathcal{S}$ have no two jumps within $(0, 2^{-j-t+s})$ of each other. Let $B \subset Q_{z,2^{-j}}$ be open with Lipschitz boundary, $|B| \le |Q_{z,2^{-j}}|/2$, and $\mathcal{H}^1(\partial B) \le \eta 2^{-j}$. Let $\boldsymbol{v} \in BV(Q \setminus B, \mathbb{Z}^{2M})$. Then*

$$\begin{aligned}
I^{\mathrm{rel}}_\infty(\boldsymbol{w}, Q) \le & (1+\delta) I^{\mathrm{rel}}_\infty(\boldsymbol{v}, Q \setminus B) \\
& + c2^{j+t-s} \|\boldsymbol{w} - \boldsymbol{v}\|_{L^1(Q\setminus B)} + c2^{-t} |D\boldsymbol{w}|(Q_{z,2\times 2^{-j}}). \qquad (314)
\end{aligned}$$

Here c depends only on I^{rel}_∞.

Note that Q might as well be a subset of $\mathbb{R}^2$, since $j \ge 1$.

Proof. Assume without loss of generality that $Q_{z,2^{-j}} \subset \mathbb{R}^2$, $j = 0$. Also note that the estimate depends only on $t - s$, and we shall assume that $s = 0$. We denote $Q_r = Q_{z,r}$ for $r \in (0, 2)$.

If η is small enough, there is $r \in (1 - 2^{-t}, 1 - 2^{-t-1})$ such that $\partial Q_r \cap \overline{B} = \emptyset$, $\mathcal{H}^1(\partial Q_r \cap J_{\boldsymbol{w}}) = 0$, and

$$\|\boldsymbol{w} - \boldsymbol{v}\|_{L^1(\partial Q_r)} \le c2^t \|_{L^1(Q_1 \setminus B)}. \qquad (315)$$

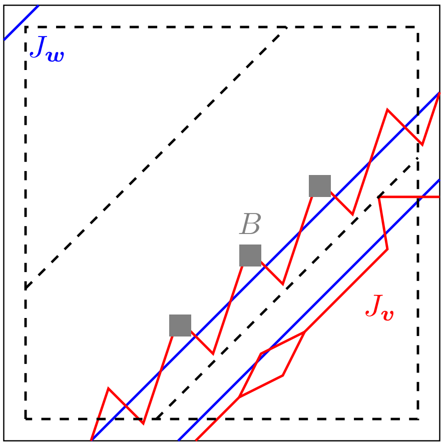

Figure 13: Sketch of the proof of Lemma 7.7. In each of the convex dashed regions, $\boldsymbol{v}$ is almost a competitor to $\boldsymbol{w}$ for the cell problem for I_∞, for which $\boldsymbol{w}$ is optimal. Jumps of $\boldsymbol{w}$ outside of the dashed square, as in the upper left, produce a small error term.

Now define $\tilde{\boldsymbol{u}} \in BV(Q_1 \setminus (B \cap Q_r), \mathbb{Z}^{2M})$ as

$$\tilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{w} & , \text{ in } Q_2 \setminus Q_r \\ \boldsymbol{v} & , \text{ in } Q_r \setminus B. \end{cases} \tag{316}$$

Write $\boldsymbol{w}(x) = a + b\lambda(x \cdot \nu)$ with $\nu \in S^1$. Let $t_1 < \ldots < t_N$ be the jump points of λ. Note that by the assumption, $t_{i+1} - t_i \geq 2^{-t}$. If η is small enough, we can now find $s_i \in (t_i, t_{i+1})$ for $i = 1, \ldots, N-1$ such that $\{x \cdot \nu = s_i\} \cap \partial B = \emptyset$, $\mathcal{H}^1(\{x \cdot \nu = s_i\} \cap J_{\tilde{\boldsymbol{u}}}) = 0$, and

$$\sum_{i=1}^{N-1} \|\tilde{\boldsymbol{u}} - \boldsymbol{w}\|_{L^1(\{x \cdot \nu = s_i\} \cap Q_1)} \leq c 2^t \|v - w\|_{L^1(Q_1 \setminus B)}. \tag{317}$$

See Figure 13 for a visualization.

Knowing this, we define the functions $\tilde{\boldsymbol{u}}_i \in BV(Q_1 \setminus (B \cap Q_r), \mathbb{Z}^{2M})$ as

$$\tilde{\boldsymbol{u}}_i(x) = \begin{cases} a + b\lambda^+(t_i) & , \text{ if } x \cdot \nu \geq s_i \\ \tilde{\boldsymbol{u}} & , \text{ if } x \cdot \nu \in (s_{i-1}, s_i) \\ a + b\lambda^-(t_i) & , \text{ if } x \cdot \nu \leq s_{i-1}, \end{cases} \tag{318}$$

with $s_0 = -\infty$, $s_N = \infty$. Finally, if η is small enough, we can use Lemma 6.5 to extend each $\tilde{\boldsymbol{u}}_i$ to a function $\boldsymbol{u}_i \in BV(Q_1, \mathbb{Z}^{2M})$ which outside of Q_1 has a single jump line, making it a competitor to a cell problem. Then

$$
\begin{aligned}
I_\infty^{\mathrm{rel}}(\boldsymbol{w}, Q_1) \leq & \sum_{i=1}^N I_\infty^{\mathrm{rel}}(\boldsymbol{u}_i, Q_1) \\
\leq & (1+\delta) \sum_{i=1}^N I_\infty^{\mathrm{rel}}(\tilde{\boldsymbol{u}}_i, Q_1 \setminus (B \cap Q_r)) \\
\leq & (1+\delta) I_\infty^{\mathrm{rel}}(\boldsymbol{v}, Q_r \setminus B) \\
& + c 2^t \|\boldsymbol{v} - \boldsymbol{w}\|_{L^1(Q_1 \setminus B)} \\
& + c 2^{-t} |D\boldsymbol{w}|(Q_2).
\end{aligned} \tag{319}
$$

Here we used the fact that I_∞^{rel} is BV-elliptic and in particular $I_\infty^{\mathrm{rel}}(u, A) \leq c|Du|(A)$.

□

7.3 Approximation by step functions

Next we show that BV-functions can be well-approximated by one-dimensional step functions assuming a low decay in total variation under mollification.

Proposition 7.8. *Let $l > 0$, $\phi_l : \mathbb{R}^2 \to [0, \infty)$ a standard mollifier with* $\operatorname{supp} \phi_l \subseteq B(0, l)$ *and $\phi_l \geq \frac{1}{l^2}$ on $B(0, l/2)$. Define the square $Q_L := (-L/2, L/2)^2$ for all $L > 0$. Let $M > 0$. Then there is a constant C_M such that for all $u \in W^{1,1}(Q_{9l}, \mathbb{R}^N)$, defining*

$$
\begin{aligned}
\eta_1 := & \frac{1}{l^2} \int_{Q_l} \operatorname{dist}(u, \mathbb{Z}^N) \, \mathrm{d}x, \\
\eta_2 := & \frac{1}{l} |Du|(Q_{9l}), \\
\eta_3 := & \frac{1}{l} \left(\int_{\mathbb{R}^2} |Du|(\phi_{4l} * \chi_{Q_l}) \, \mathrm{d}x - \int_{Q_l} |D(u * \phi_{4l})| \, \mathrm{d}x \right),
\end{aligned}
$$

whenever $\eta_2 \leq M$, there is $u_0 \in \mathcal{S}$ with

$$
\frac{1}{l} \|u - u_0\|_{L^2(Q_{l/2})} \leq C_M \left(\eta_1 + \eta_2^{5/6} \eta_3^{1/6} \right).
$$

and $\|u_0 - a\|_{L^\infty(Q_{l/2})} \leq \eta_2$ for some $a \in \mathbb{Z}^N$.

The proof of this proposition can be found in [11].

Here a mollification radius of $4l$ is needed so that for all $x, y \in Q_l$ we get $\phi_{4l}(x - y) \geq \frac{c}{l^2}$. Also $\chi_{Q_l} * \phi_{4l} \leq \chi_{Q_{9l}}$.

Remark 7.9. *Note that since $\eta_1 \leq 1$ and $\eta_3 \leq \eta_2 \leq M$, we also have for such u that*

$$
\frac{1}{l^2} \|u - u_0\|_{L^2(Q_{l/2})}^2 \leq C_M \left(\eta_1 + \eta_2^{5/6} \eta_3^{1/6} \right).
$$

8 The lower bound

We now show the lower bound for the energy $\frac{1}{|\log \varepsilon|} E_{\varepsilon, h(\varepsilon)}$.

Proposition 8.1. *Let $M > 0$, $\beta \in (0,1)$, $\eta > 0$, $k_i \to \infty$, $\boldsymbol{v}_i \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ be as in the results of Proposition 4.7. Let $\delta > 0$. Then there is a sequence of open sets $B_i \subset \mathbb{T}^2$ with $\limsup_{i\to\infty} \mathcal{H}^1(\partial B_i) \le \delta$, $\limsup_{i\to\infty} |B_i| \le \frac{\delta^2}{4\pi}$, and a sequence $\boldsymbol{w}_i \in BV(\mathbb{T}^2 \setminus B_i, \mathbb{Z}^{2M})$ with*

$$\begin{aligned}&\liminf_{i\to\infty} (\log 2) \left[(1-\beta) I_\infty^{\mathrm{rel}} + \beta I_0\right]^{\mathrm{rel}} (\boldsymbol{w}_i, \mathbb{T}^2 \setminus B_i)\\ \le&(1 + C\sqrt{\eta} + \delta) \liminf_{i\to\infty} \frac{1}{k_i} \left[\sum_{j=\lceil(\beta+\eta)k_i\rceil}^{k_i} B_{\infty,j}^{\mathrm{per}}(\boldsymbol{v}_i, \boldsymbol{v}_i) + \sum_{j=1}^{\lfloor(\beta-\eta)k_i\rfloor} B_{0,j}^{\mathrm{per}}(\boldsymbol{v}_i, \boldsymbol{v}_i) \right]\end{aligned} \tag{320}$$

and $\limsup_{i\to\infty} \|\boldsymbol{w}_i - \boldsymbol{v}_i\|_{L^1(\mathbb{T}^2 \setminus B_i)} = 0$.

We will follow the proof from [11], choosing two length scales $j_\infty, j_0 \in \{1, \dots, k_i\}$ instead of one at which to modify the function $\boldsymbol{v}_i$ according to Proposition 7.8. This is made more difficult by the non-positivity of the kernels in $B_{\infty,j}^{\mathrm{per}}$. We finally compare the line-tension energy of the small-scale competitor to that of the large-scale competitor using Lemma 7.7.

Proof. In the proof we will consider k fixed and suppress the index i. We also write $\boldsymbol{v} = \boldsymbol{v}_k$. We introduce the large cutoff parameters $M_\infty, M_0 > 0$, the very large iterative mollification step sizes $m_\infty, m_0 \in \mathbb{N}$, the minimum and maximum length scale gaps between square size and kernel $T_\infty^+ > T_\infty^- \in \mathbb{N}$, $T_0^+ > T_0^- \in \mathbb{N}$, and the jump distance bandwidths $s_\infty, s_0 \in \mathbb{N}$, all of which will be chosen later.

Step 1: Iterative mollification and choice of length scale.

We define the functions $\boldsymbol{v}_j : \mathbb{T}^2 \to \mathbb{R}^{2M}$ for $j \in \mathbb{N}$ as follows: For $j \ge k$ take $\boldsymbol{v}_j = \boldsymbol{v}$. For $j < k$ define iteratively $\boldsymbol{v}_j = \boldsymbol{v}_{j+m_\infty} * \Phi_{2^{-j+4}}$. We note first of all that $\|\operatorname{dist}(\boldsymbol{v}_j, \mathbb{Z}^{2M}\|_{L^1(\mathbb{T}^2)} \le \|\boldsymbol{v}_j - \boldsymbol{v}\|_{L^1(\mathbb{T}^2)} \le C2^{-j}/\eta$ by the Poincaré inequality. Also since $B_{\infty,\tilde{j}}^{\mathrm{per}}$ is convex and invariant under translations (see Lemma 3.18) we observe that $B_{\infty,\tilde{j}}^{\mathrm{per}}(\boldsymbol{v}_j, \boldsymbol{v}_j) \le B_{\infty,\tilde{j}}^{\mathrm{per}}(\boldsymbol{v}, \boldsymbol{v})$ for all $j, \tilde{j} \in \mathbb{N}$. Finally by a telescopic sum,

$$\sum_{j=1}^{k} |D\boldsymbol{v}_{j+m_\infty}|(\mathbb{T}^2) - |D\boldsymbol{v}_j|(\mathbb{T}^2) \le \frac{m_\infty C}{\eta}, \tag{321}$$

and each term in the sum is nonnegative.

Now we pick a length scale $j_\infty \in \{\lceil(\beta+\eta)k\rceil, \dots, k\}$ such that the following hold:

i)

$$\frac{1}{T_\infty^+ - T_\infty^-} \sum_{t=T_\infty^-}^{T_\infty^+} B_{\infty,j_\infty+t}^{\mathrm{per}}(\boldsymbol{v}, \boldsymbol{v}) \le \frac{1 + 2\sqrt{\eta}}{\lfloor(1-\beta-\eta)k\rfloor} \sum_{j=\lceil(\beta+\eta)k\rceil}^{k} B_{\infty,j}^{\mathrm{per}}(\boldsymbol{v}, \boldsymbol{v}). \tag{322}$$

ii) $|D\boldsymbol{v}_{j_\infty+m_\infty}|(\mathbb{T}^2) - |D\boldsymbol{v}_{j_\infty+m_\infty}|(\mathbb{T}^2) \le \frac{Cm_\infty}{\sqrt{\eta}k}$.

iii) There are at least $(1-\sqrt{\eta})(T_\infty^+ - T_\infty^-)$ many t in $\{T_\infty^-, \dots, T_\infty^+\}$ such that

$$\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |\boldsymbol{v}(x) - \boldsymbol{v}(y)|^2 K_{j_\infty+t}(x-y)\, dx\, dy \le \frac{C}{\eta}. \tag{323}$$

Since the first condition is not satisfied by at most $(1-2\sqrt{\eta})\lfloor(1-\beta-\eta)k\rfloor$ many $j \in \{\lceil(\beta+\eta)k\rceil, \dots, k\}$. The second condition is not satisfied by at most $\sqrt{\eta}\lfloor(1-\beta-\eta)k\rfloor$ many j. By Lemma 7.2 the third condition is not satisfied by $\frac{1}{2}\sqrt{\eta}\lfloor(1-\beta-\eta)k\rfloor$ many j, so that for k large enough there is at least one such j_∞.

Step 2: The first local modification and choice of t_∞.

We now consider the following family of squares: Let $Z_\infty = 2^{-j_\infty}\mathbb{Z}^2/\mathbb{Z}^2 \subset \mathbb{T}^2$. Then in each square $Q_{z,8\times 2^{-j}}$, $z \in Z_\infty$, we consider the three quantities

$$\begin{aligned}
\eta_1^z =& 2^{2j_\infty} \int_{Q_{z,8\times 2^{-j}}} |\boldsymbol{v}_{j_\infty+m_\infty} - \boldsymbol{v}|\, dx,\\
\eta_2^z =& 2^{j_\infty} |D\boldsymbol{v}_{j_\infty+m_\infty}|(Q_{z,8\times 2^{-j}}),\\
\eta_3^z =& 2^{j_\infty} \Bigg(\int_{\mathbb{T}^2} |D\boldsymbol{v}_{j_\infty+m_\infty}|(\mathbb{1}_{Q_{z,8\times 2^{-j_\infty}}} * \Phi_{2^{-j_\infty+4}})\, dx\\
&- |D(\boldsymbol{v}_{j_\infty+m_\infty} * \Phi_{2^{-j_\infty+4}})|(Q_{z,8\times 2^{-j_\infty}}) \Bigg).
\end{aligned} \tag{324}$$

Now, as noted, $\sum_z \eta_1^z \le C2^{j_\infty - m_\infty}/\eta$, $\sum_z \eta_2^z \le C2^{j_\infty}/\eta$, and $\sum_z \eta_3^z \le \frac{Cm_\infty}{\sqrt{\eta}k} 2^{j_\infty}$.

We shall identify two families of bad squares $B_\infty^1, B_\infty^2 \subset Z_\infty$ where we are unable to produce good estimates, where we leave the function $\boldsymbol{v}_{j_\infty+m_\infty}$ untouched. Later we will have to deal with these missing squares.

We define the set $B_\infty^1 = \{z \in Z_\infty : \eta_2^z > M_\infty\}$ and note that $2^{-j_\infty}\sharp B_\infty^1 \le \frac{C}{\eta M_\infty}$ and apply Proposition 7.8 to all squares $Q_{z,8\times 2^{-j_\infty}}$ with $z \notin B_\infty^1$, yielding modifications $\boldsymbol{u}_z \in \mathcal{S}$ so that by Remark 7.9

$$\sum_{z \in Z_\infty \setminus B_\infty^1} \|\boldsymbol{u}_z - \boldsymbol{v}\|_{L^1(Q_{z,4\times 2^{-j}})} \le C_{M_\infty} 2^{-j_\infty} \left(2^{-m_\infty} + \frac{Cm_\infty}{\eta k^{1/6}}\right), \tag{325}$$

and

$$\sum_{z \in Z_\infty \setminus B_\infty^1} \|\boldsymbol{u}_z - \boldsymbol{v}_{j_\infty+m_\infty}\|^2_{L^2(Q_{z,4\times 2^{-j}})} \le C_{M_\infty} 2^{-j_\infty} \left(2^{-m_\infty} + \frac{Cm_\infty^{1/6}}{\eta k^{1/6}}\right). \tag{326}$$

Additionally,

$$\sum_{z \in Z_\infty \setminus B_\infty^1} |D\boldsymbol{u}_z|(Q_{z,4\times 2^{-j_\infty}}) \le \frac{C}{\eta}, \tag{327}$$

and for all $t \in \{T_\infty^-, \dots, T_\infty^+\}$ for which (323) holds, we get

$$\begin{aligned}
&\sum_{z \in Z_\infty \setminus B_\infty^1} \int_{Q_{z,4\times 2^{-j_\infty}}} \int_{Q_{z,4\times 2^{-j_\infty}}} |\boldsymbol{u}_z(x) - \boldsymbol{u}_z(y)|^2 K_{j_\infty+t}(x-y)\, dx\, dy\\
\le& \frac{C}{\eta}\left(1 + 2^t C_{M_\infty}\left(2^{-m_\infty} + \frac{Cm_\infty}{\eta k^{1/6}}\right)\right).
\end{aligned} \tag{328}$$

From this, we can infer that the squared jumps of $\boldsymbol{u}_z$ in smaller squares are integrable, i.e.

$$\begin{aligned}&\sum_{z\in Z_\infty\cap B^1_\infty}\int_{J_{\boldsymbol{u}_z}\cap Q_{z,2\times 2^{-j_\infty}}}|[\boldsymbol{u}_z]|^2\,d\mathcal{H}^1\\ \le&\frac{C}{\eta}\left(1+2^tC_{M_\infty}\left(2^{-m_\infty}+\frac{Cm_\infty}{\eta k^{1/6}}\right)\right).\end{aligned}\tag{329}$$

Finally, note that in every square $Q_{z,4\times 2^{-j_\infty}}$, $z\in Z_\infty\setminus B^1_\infty$, there are at most M_∞ parallel jump lines, and the sum of lengths of the jumps is bounded by C/η.

We are now ready to pick the length scale $t_\infty\in\{T^-_\infty,\dots,T^+_\infty\}$ such that

iv)

$$\begin{aligned}&B^{\text{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{v},\boldsymbol{v})\\ \le&\frac{1+6\sqrt{\eta}}{T^+_\infty-T^-_\infty}\sum_{t=T^-_\infty}^{T^+_\infty}B^{\text{per}}_{\infty,j_\infty+t}(\boldsymbol{v},\boldsymbol{v})\\ \le&\frac{1+C\sqrt{\eta}}{(1-\beta-\eta)k}\sum_{j=\lceil(\beta+\eta)k\rceil}^{k}B^{\text{per}}_{\infty,j}(\boldsymbol{v},\boldsymbol{v}).\end{aligned}\tag{330}$$

v)

$$\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}|\boldsymbol{v}(x)-\boldsymbol{v}(y)|^2K_{j_\infty+t_\infty}(x-y)\,dx\,dy\le C/\eta.\tag{331}$$

vi) Letting $B^2_\infty=\{z\in Z_\infty\setminus B^1_\infty\,:\,u_z$ has two jump lines with distance between $2^{-j_\infty-t_\infty-s_\infty}$ and $2^{-j_\infty-t_\infty+s_\infty}\}$, we have

$$2^{-j}\sharp(B^2_\infty)\le\frac{2s_\infty M}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\sum_{z\in Z_\infty\setminus B^1_\infty}|D\boldsymbol{u}_z|(Q_{z,4\times 2^{-j_\infty}}).\tag{332}$$

Such a t_∞ exists because the first inequality is not satisfied by at most $\frac{T^+_\infty-T^-_\infty}{1+6\sqrt{\eta}}\le(1-3\sqrt{\eta})(T^+_\infty-T^-_\infty)$ for η small enough, the second by at most $\sqrt{\eta}(T^+_\infty-T^-_\infty)$ due to (ii), and the third by at most $\sqrt{\eta}(T^+_\infty-T^-_\infty)$, since

$$\begin{aligned}&2^{-j_\infty}\sum_{t=T^-_\infty}^{T^+_\infty}\sharp(\{z:\boldsymbol{u}_z\text{ has jumps within }(2^{-j_\infty-t-s_\infty},2^{-j_\infty-t+s_\infty})\}\\ \le&2sM\sum_{z\in Z_\infty\setminus B^1_\infty}|D\boldsymbol{u}_z|(Q_{z,4\times 2^{-j_\infty}}).\end{aligned}\tag{333}$$

Step 3: The second local modification

Wen now modify all $\boldsymbol{u}_z$ for $z\in Z_\infty\setminus B^1_\infty\setminus B^2_\infty$ such that all jumps are far apart. If $\boldsymbol{u}_z(x)=a+b\lambda(x\cdot\nu)$, with $\nu\in S^1$, define $\boldsymbol{w}_z(x)=a+b\Lambda(x\cdot\nu)$, where $\Lambda:\mathbb{R}\to\mathbb{Z}$ is a function obtained by modifying λ between a maximal group of jump points $t_1<\ldots<t_N\in J_\lambda$ with $|t_N-t_1|<2^{-j_\infty-t_\infty+s_\infty}$. Note that these maximal groups are unique and have distance at least $2^{-j_\infty - t_\infty + s_\infty}$ as long as $s_\infty \geq 1$, because otherwise $z \in B^2_\infty$. For each such group, we set

$$\Lambda(t) = \begin{cases} \lambda^+(t_N) & , \text{if } t > t_1, \\ \lambda^-(t_1) & , \text{if } t \leq t_1. \end{cases} \tag{334}$$

Now the jumps of $\boldsymbol{w}_z$ are at least $2^{-j_\infty - t_\infty + s_\infty}$ apart, and we can easily see that

vii)

$$\begin{aligned} &\sum_{z \in Z_\infty \setminus B^1_\infty \setminus B^2_\infty} \|\boldsymbol{w}_z - \boldsymbol{u}_z\|^2_{L^2(Q_{z,3\times 2^{-j_\infty}})} \\ \leq& 2^{-t_\infty - s_\infty} \sum_{z \in Z_\infty \setminus B^1_\infty \setminus B^2_\infty} (|D\boldsymbol{u}_z|(Q_{z,4\times 2^{-j_\infty}}))^2 \\ \leq& \frac{CM_\infty}{\eta} 2^{-j_\infty - t_\infty - s_\infty} \end{aligned} \tag{335}$$

and

viii) $|D\boldsymbol{w}_z|(Q_{z,3\times 2^{-j_\infty}}) \leq |D\boldsymbol{u}_z|(Q_{z,4\times 2^{-j_\infty}})$ for all $z \in Z_\infty \setminus B^1_\infty \setminus B^2_\infty$.

Finally, for $z \in B^1_\infty \cup B^2_\infty$, set $\boldsymbol{w}_z = \boldsymbol{v}_{j_\infty + m_\infty}$.

Step 4: The first global competitor $\boldsymbol{w}_\infty$

We now eliminate the overlap of the squares $Q_{z,3\times 2^{-j_\infty}}$ to construct a global function $\boldsymbol{w}_\infty$ for which we control the energy I_∞ in most of $\mathbb{T}^2$.

We pick an $a \in Q_{0,2^{-j_\infty}}$ such that

ix) The squared difference along the edges is integrable, i.e.

$$\begin{aligned} &\sum_{z,z' \in Z_\infty} \int_{\partial Q_{z+a,2^{-j_\infty}} \cap \partial Q_{z+a,2^{-j_\infty}}} |\boldsymbol{w}_z - \boldsymbol{w}_{z'}|^2 \, d\mathcal{H}^1 \\ \leq& C_{M_\infty} \left(2^{-m_\infty} + \frac{m_\infty^{1/6}}{\eta k^{1/6}} + \frac{1}{\eta} 2^{-t_\infty - s_\infty} \right) \end{aligned} \tag{336}$$

and

x) The double integral with the positive kernel restricted to a small strip near the boundary of the squares is small, i.e.

$$\begin{aligned} &\sum_{z \in Z_\infty} \int_{Q_{z+a,2^{-j_\infty}} \setminus Q_{z+a,2^{-j_\infty}(1-2^{-t_\infty+s_\infty})}} \int_{\mathbb{T}^2} \\ &|\boldsymbol{w}_z(x) - \boldsymbol{v}_{j_\infty+m_\infty}(y)|^2 K_{j_\infty+t_\infty}(x-y) \, dy \, dx \\ \leq& 2^{-t_\infty+s_\infty} \left(\frac{C}{\eta} + C_{M_\infty} 2^{t_\infty} \left(2^{-m_\infty} + \frac{Cm_\infty^{1/6}}{\eta k^{1/6}} + + \frac{1}{\eta} 2^{-t_\infty - s_\infty} \right) \right). \end{aligned} \tag{337}$$

Such an a exists due to estimates (326) and (335), where we used the inequality $|p-r|^2 \leq 2|p-q|^2 + 2|q-r|^2$ on both integrands. More precisely, for the double integral near the boundary of the squares

$$
\begin{aligned}
&\int_{Q_{0,2^{-j_\infty}}} \sum_{z\in Z_\infty} \int_{Q_{z+a,2^{-j_\infty}}\setminus Q_{z+a,2^{-j_\infty}(1-2^{-t_\infty+s_\infty})}} \int_{\mathbb{T}^2} \\
&|\boldsymbol{w}_z(x) - \boldsymbol{v}_{j_\infty+m_\infty}(y)|^2 K_{j_\infty+t_\infty}(x-y)\,dy\,dx\,da \\
\le & 2\int_{Q_{0,2^{-j_\infty}}} \sum_{z\in Z_\infty} \int_{Q_{z+a,2^{-j_\infty}}\setminus Q_{z+a,2^{-j_\infty}(1-2^{-t_\infty+s_\infty})}} \int_{\mathbb{T}^2} \\
&|\boldsymbol{w}_z(x) - \boldsymbol{v}_{j_\infty+m_\infty}(x)|^2 K_{j_\infty+t_\infty}(x-y)\,dy\,dx\,da \\
&+ 2\int_{Q_{0,2^{-j_\infty}}} \sum_{z\in Z_\infty\setminus B^1_\infty} \int_{Q_{z+a,2^{-j_\infty}}\setminus Q_{z+a,2^{-j_\infty}(1-2^{-t_\infty+s_\infty})}} \int_{\mathbb{T}^2} \\
&|\boldsymbol{v}_{j_\infty+m_\infty}(x) - \boldsymbol{v}_{j_\infty+m_\infty}(y)|^2 K_{j_\infty+t_\infty}(x-y)\,dy\,dx\,da \\
\le & 2^{-2j_\infty-t_\infty+s_\infty}\left(C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{Cm_\infty^{1/6}}{\eta k^{1/6}}+\frac{1}{\eta}2^{-t_\infty-s_\infty}\right)+\frac{C}{\eta}\right),
\end{aligned} \tag{338}
$$

whereas for the edges

$$
\begin{aligned}
&\int_{Q_{0,2^{-j_\infty}}} \sum_{z,z'\in Z_\infty} \int_{\partial Q_{z+a,2^{-j_\infty}}\cap \partial Q_{z+a,2^{-j_\infty}}} |\boldsymbol{u}_z - \boldsymbol{u}_{z'}|^2\,d\mathcal{H}^1\,da \\
\le & C2^{-j_\infty}\sum_{z\in Z_\infty}\int_{Q_{z,2\times 2^{-j_\infty}}} |\boldsymbol{u}_z - \boldsymbol{v}_{j_\infty+m_\infty}|^2\,dx \\
\le & 2^{-2j_\infty}C_{M_\infty}\left(2^{-m_\infty}+\frac{Cm_\infty^{1/6}}{\eta k^{1/6}}\right).
\end{aligned} \tag{339}
$$

Here a can be any point where no integrand is larger two times its mean. Now we can define the global function $\boldsymbol{w}_\infty : \mathbb{T}^2 \to \mathbb{R}^{2M}$ as

$$
\boldsymbol{w}_\infty = \boldsymbol{w}_z, \text{ in } Q_{z+a,2^{-j_\infty}}. \tag{340}
$$

Then by the previous estimates,

$$
\|\boldsymbol{w}_\infty - \boldsymbol{v}_{j_\infty+m_\infty}\|^2_{L^2(\mathbb{T}^2)} \le C_{M_\infty}2^{-j_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+\frac{1}{\eta}2^{-t_\infty-s_\infty}\right), \tag{341}
$$

so that

$$
\begin{aligned}
&B^{\text{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_\infty,\boldsymbol{w}_\infty) \\
\le & (1+\eta)B^{\text{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{v},\boldsymbol{v}) + \frac{C_{M_\infty}}{\eta}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+\frac{1}{\eta}2^{-t_\infty-s_\infty}\right).
\end{aligned} \tag{342}
$$

Additionally we get the finer estimates

$$
\begin{aligned}
&\sum_{z\in Z_\infty} \int_{Q_{z+a,2^{-j_\infty}}\setminus Q_{z+a,2^{-j_\infty}(1-2^{-t_\infty+s_\infty})}} \int_{\mathbb{T}^2} \\
&|\boldsymbol{w}_\infty(x) - \boldsymbol{w}_\infty(y)|^2 K_{j_\infty+t_\infty}(x-y)\,dy\,dx \\
\le & 2^{-t_\infty+s_\infty}\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+\frac{1}{\eta}2^{-t_\infty-s_\infty}\right)\right),
\end{aligned} \tag{343}
$$

and

$$\int_{J_{\boldsymbol{w}_\infty}\setminus\bigcup_{z\in B^1_\infty\cup B^2_\infty}\overline{Q_{z+a,2^{-j_\infty}}}}|[\boldsymbol{w}_\infty]|^2\,d\mathcal{H}^1$$

$$\leq\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+\frac{1}{\eta}2^{-t_\infty+s_\infty}\right). \tag{344}$$

from (viii) and (vii) respectively.

Step 5: Localization

We wish to replace $B^{\mathrm{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_\infty,\boldsymbol{w}_\infty)$ with the sum

$$\sum_{z\in Z_\infty\setminus B^1_\infty\setminus B^2_\infty}\int_{Q_{z+a,2^{-j_\infty}}}\int_{Q_{z+a,2^{-j_\infty}}}(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\mathrm{per}}_{j_\infty+t_\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy \tag{345}$$

in order to apply Lemma 7.6. However, to do so we would have to discard all interactions among the bad squares and interactions between bad and good squares. All short-range (shorter than $2^{-j_\infty-t_\infty+s_\infty}$) interactions between squares are small by (343), while all long-range (longer than $2^{-j_\infty-t_\infty+s_\infty}$) interactions are small by (341). This leaves interactions among the bad squares. We now show that these terms can be made mostly positive even if the kernel is not positive by enlarging the bad squares.

To this end, we define the measure $\mu\in\mathcal{M}(\mathbb{T}^2)$ defined by

$$\mu=|[\boldsymbol{w}_\infty]|^2\mathcal{H}^1\llcorner J_{\boldsymbol{w}_\infty}+\sum_{z\in Z_\infty}2^{j_\infty}\int_{J_{\boldsymbol{w}_\infty}\cap Q_{z,3\times2^{-j}}}|[\boldsymbol{w}_\infty]|^2\,d\mathcal{H}^1\mathcal{H}^1\llcorner\partial Q_{z,3\times2^{-j_\infty}}. \tag{346}$$

Then defining $\tilde\omega=\bigcup_{z\in B^1_\infty\cup B^2_\infty}Q_{z,8\times2^{-j_\infty}}$, we see that

$$\mathcal{H}^1(\partial\tilde\omega)\leq C2^{-j_\infty}\sharp(B^1_\infty\cup B^2_\infty)\leq\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}, \tag{347}$$

that $|\tilde\omega|\leq1/2$, and that

$$\mu(\mathbb{T}^2\setminus\tilde\omega)\leq C\int_{J_{\boldsymbol{w}_\infty}\setminus\tilde\omega}|[\boldsymbol{w}_\infty]|^2\,d\mathcal{H}^1\leq\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right). \tag{348}$$

Then by Lemma 6.5 there are pairwise disjoint convex open sets $\omega_1,\ldots,\omega_N\subset\mathbb{T}^2$ with $\sum_{i=1}^N\mathcal{H}^1(\partial\omega_i)\leq\delta\left(C_{M_\infty}(2^{-m_\infty}+\frac{Cm_\infty^{1/6}}{\eta k^{1/6}})+\frac{M_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\right)$ covering all bad squares $Q_{z+a,2^{-j_\infty}}$, $z\in B^1_\infty\cup B^2_\infty$, such that

$$\sum_{i=1}^{N}\mathcal{H}^1(\partial\omega_i)\sum_{x\in\partial\omega_i\cap J_{\boldsymbol{w}_\infty}}|[\boldsymbol{w}_\infty(x)]|^2$$
$$\leq\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T_\infty^+-T_\infty^-)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right), \tag{349}$$

and

$$\sum_{i=1}^{N}\sum_{z\in Z^\infty:Q_{z,3\times 2^{-j_\infty}}\cap\partial\omega_i\neq\emptyset}\int_{J_{\boldsymbol{w}_\infty}\cap Q_{z,2^{-j_\infty}}}|[\boldsymbol{w}_\infty]|^2\,d\mathcal{H}^1$$
$$\leq\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T_\infty^+-T_\infty^-)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right), \tag{350}$$

because $\mathcal{H}^1(\partial\omega_i)\geq 2^{-j_\infty}$.

Here $\delta:[0,\infty)\to[0,\infty)$ is a continuous function with $\delta(0)=0$, by Lemma 6.5.

Now define for every $i=1,\dots,N$ a function $\tilde{\boldsymbol{w}}_i:\partial\omega_i\to\mathbb{Z}^{2M}$, by partitioning $\partial\omega_i$ into finitely many connected curve segments of intrinsic length between $2^{-j_\infty-t_\infty+s_\infty}$ and $2^{-j_\infty-t_\infty+s_\infty+1}$ and setting $\tilde{\boldsymbol{w}}_i=\boldsymbol{w}_\infty(x)$ for some x within that segment. Then the jumps of $\tilde{\boldsymbol{w}}_i$ are at least $2^{-j_\infty-t_\infty+s_\infty}$ apart.

In each such segment $\boldsymbol{w}_\infty\llcorner\partial\omega_i$ jumps at most 16 times (at most twice in each of the four squares $Q_{z+a,2^{-j_\infty}}$ the segment touches and at most once for each time it crosses an edge, which will happen at most eight times). This implies that

$$\sum_{x\in J_{\tilde{\boldsymbol{w}}_i}}|[\tilde{\boldsymbol{w}}_i(x)]|^2\leq 16\sum_{x\in J_{\boldsymbol{w}_\infty}\cap\partial\omega_i}|[\boldsymbol{w}_\infty(x)]|^2, \tag{351}$$

and

$$\int_{\partial\omega_i}|\tilde{\boldsymbol{w}}_i-\boldsymbol{w}_\infty|^2\,d\mathcal{H}^1\leq 16\times 2^{-j_\infty-t_\infty+s_\infty}\sum_{x\in J_{\boldsymbol{w}_\infty}\cap\partial\omega_i}|[\boldsymbol{w}_\infty(x)]|^2. \tag{352}$$

Since the jumps of $\tilde{\boldsymbol{w}}_i$ are at least $2^{-j_\infty-t_\infty+s_\infty}$ apart, there are at most $S=\lfloor\mathcal{H}^1(\partial\omega_i)2^{j_\infty+t_\infty-s_\infty}\rfloor$ of them, and $\tilde{\boldsymbol{w}}_i$ takes at most S different values in Z^{2M}.

We now define a function $\boldsymbol{w}_i:\mathbb{T}^2\to\mathbb{R}^{2M}$, with $\boldsymbol{w}_i=\boldsymbol{w}_\infty$ in ω_i, and extending each of its $2M$ components as follows:

Let $\tilde{w}_i:\partial\omega_i\to\mathbb{Z}$ denote one of the $2M$ components of the function $\tilde{\boldsymbol{w}}_i$. Then its range can be written $\{y_1,\dots,y_S\}$, with $y_s\leq y_{s+1}$ for all $s=1,\dots,S-1$. We then define $\omega_i^0=\omega_i$, $\omega_i^s=B(\omega_i,s2^{-j_\infty-t_\infty+s_\infty})$ for $s=1,\dots,S$.

Note that all the ω_i^s with $s\leq S$ are convex open sets, with $\mathcal{H}^1(\omega_i^s)\leq(1+2\pi)\mathcal{H}^1(\omega_i)$.

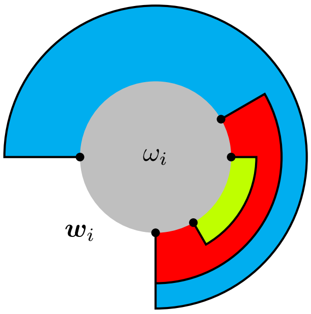

Figure 14: Construction of the function $\boldsymbol{w}_i : \mathbb{T}^2 \to \mathbb{R}^{2M}$ outside of ω_i. The number of phases is reduced going outward until only one is left. The absolute value of the nonlocal energy outside of ω_i is bounded by $c\mathcal{H}^1(\partial\omega_i)|D\boldsymbol{w}|(\partial\omega_i)$.

Let $p : \mathbb{T}^2 \setminus \omega_i : \partial\omega_i$ be the orthogonal projection onto the boundary. We define $w_i(x) = \tilde{w}_i(p(x)) \vee y_s$ in $\omega_i^s \setminus \omega_i^{s-1}$, for $s = 1, \ldots, S$, and $w_i = y_S$ in $\mathbb{T}^2 \setminus \omega_i^S$. See Figure 14.

We see that

$$\begin{aligned}&\sum_{i=1}^N \int_{J_{w_i}\setminus\omega_i} |[\boldsymbol{w}_i]|^2 \, d\mathcal{H}^1 \\ \leq& C\sum_{i=1}^N \mathcal{H}^1(\partial\omega_i) \sum_{x\in\partial\omega_i\cap J_{\boldsymbol{w}_\infty}} |[\boldsymbol{w}_\infty(x)]|^2. \end{aligned} \tag{353}$$

There are two types of jumps of $\boldsymbol{w}_i$: Jumps along rays outward from ω_i, and jumps along the $\partial\omega_i^s$. The length of the rays is at most $S2^{-j_\infty - t_\infty + s_\infty} \leq \mathcal{H}^1(\partial\omega_i)$. To see the bound for jumps along the $\partial\omega_i^s$, note that

$$\begin{aligned}&\mathcal{H}^1(\partial\omega_i) \sum_{x\in J_{\tilde{w}_i}} |[\tilde{w}_i \vee y_{s+1}]|^2 + \int_{\partial\omega_i} |(\tilde{w}_i \vee y_{s+1}) - (\tilde{w}_i \vee y_s)|^2 \, d\mathcal{H}^1 \\ \leq& \mathcal{H}^1(\partial\omega_i) \sum_{x\in J_{\tilde{w}_i}} |[\tilde{w}_i \vee y_s]|^2, \end{aligned} \tag{354}$$

and by induction over s

$$\sum_{s=0}^{S-1} \int_{\partial\omega_i} |(\tilde{w}_i \vee y_{s+1}) - (\tilde{w}_i \vee y_s)|^2 \, d\mathcal{H}^1 \leq \mathcal{H}^1(\partial\omega_i) \sum_{x\in J_{\tilde{w}_i}} |[\tilde{w}_i]|^2. \tag{355}$$

We can now apply Lemma 7.3 to each $\boldsymbol{w}_i$ and the pairs of domains $A = \omega_i^{s+1} \setminus \omega_i^s$, $B = \omega_i^s$ and $A = \omega_i^1 \setminus \omega_i$, $B = \bigcup_{z\in Z_\infty : Q_{z,3\times 2^{-j_\infty}} \cap \omega_i \neq \emptyset} Q_{z,3\times 2^{-j_\infty}}$, with j from the lemma replaced by $j_\infty + t_\infty$ and i from the lemma replaced by s_∞. Note that each line segment from A to B does indeed remain in $A \cup B$ and only intersects $J_{\boldsymbol{w}_i}$ at most 20 times, due to the geometry of the construction. Adding up all contributions, we realize that

$$\sum_{i=1}^N \int_{\mathbb{T}^2\setminus\omega_i}\int_{\mathbb{T}^2} |\boldsymbol{w}_i(x)-\boldsymbol{w}_i(y)|^2 K_{j_\infty+t_\infty}(x-y)\mathbb{1}_{\{|x-y|\le 2^{-j_\infty-t_\infty+s_\infty}\}}\,dy\,dx$$
$$\le\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T_\infty^+-T_\infty^-)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right). \tag{356}$$

Since each $\boldsymbol{w}_i$ is in L^2 and a competitor for the positive semidefinite bilinear form $B^{\mathrm{per}}_{\infty,j_\infty+t_\infty}$, it follows that

$$\begin{aligned}
0\le&\sum_{i=1}^N B^{\mathrm{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_i,\boldsymbol{w}_i)\\
\le&\sum_{i=1}^N\Big[\int_{\omega_i}\int_{\omega_i}(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\mathrm{per}}_{j_\infty+t_\infty,\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy\\
&+C\int_{\mathbb{T}^2\setminus\omega_i}\int_{\mathbb{T}^2}|\boldsymbol{w}_i(x)-\boldsymbol{w}_i(y)|^2K_{j_\infty+t_\infty}(x-y)\mathbb{1}_{\{|x-y|\le 2^{-j_\infty-t_\infty+s_\infty}\}}\,dy\,dx\\
&+C\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}|\boldsymbol{w}_i(x)-\boldsymbol{w}_i(y)|^2K_{j_\infty+t_\infty}(x-y)\mathbb{1}_{\{|x-y|>2^{-j_\infty-t_\infty+s_\infty}\}}\,dy\,dx\Big],
\end{aligned} \tag{357}$$

allowing us to bound from below

$$\sum_{i=1}^N\int_{\omega_i}\int_{\omega_i}(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\mathrm{per}}_{j_\infty+t_\infty,\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy$$
$$\ge-\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T_\infty^+-T_\infty^-)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right). \tag{358}$$

This allows us to discard the self-actions of each ω_i from the double integral $B^{\mathrm{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_\infty,\boldsymbol{w}_\infty)$. We can also bound the short-range interaction between ω_i and $\mathbb{T}^2\setminus\omega_i$ by applying Lemma 7.3 to $\boldsymbol{w}_\infty$, $A=B(\partial\omega_i,2^{-j_\infty-t_\infty+s_\infty})\cap\omega_i$, and $B=B(\partial\omega_i,2^{-j_\infty-t_\infty+s_\infty})\setminus\omega_i$, which again satisfy the conditions with at most eight intersections of the jump set of $\boldsymbol{w}_\infty$ (once in the interior of A, once on $\partial\omega_i$, at most once in each of the three squares touched in ω_i, and at most three times for the edges of the squares), and since $A\cup B\subset\bigcup_{z\in Z^\infty:Q_{z,3\times2^{-j_\infty}}\cap\partial\omega_i\ne\emptyset}Q_{z,3\times2^{-j_\infty}}$ we have

$$\sum_{i=1}^N\int_{\omega_i}\int_{\mathbb{T}^2\setminus\omega_i}|\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y)|^2K_{j_\infty+t_\infty}(x-y)\mathbb{1}_{\{|x-y|\le2^{-j_\infty-t_\infty+s_\infty}\}}\,dy\,dx$$
$$\le C\sum_{z\in Z^\infty:Q_{z,3\times2^{-j_\infty}}\cap\partial\omega_i\ne\emptyset}\int_{J_{\boldsymbol{w}_\infty}\cap Q_{z,2^{-j_\infty}}}|[\boldsymbol{w}_\infty]|^2\,d\mathcal{H}^1$$
$$\le\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T_\infty^+-T_\infty^-)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right). \tag{359}$$

Combining (358), (359), and Lemma 7.2 we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^2\setminus\bigcup_{i=1}^N \omega_i}\int_{\mathbb{T}^2\setminus\bigcup_{i=1}^N \omega_i} (\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\mathrm{per}}_{j_\infty+t_\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy\\
\leq & B^{\mathrm{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_\infty,\boldsymbol{w}_\infty)\\
&-\sum_{i=1}^N\int_{\omega_i}\int_{\omega_i}(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\mathrm{per}}_{j_\infty+t_\infty,\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy\\
&+C\sum_{i=1}^N\int_{\omega_i}\int_{\mathbb{T}^2\setminus\omega_i}|\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y)|^2K_{j_\infty+t_\infty}(x-y)\mathbb{1}_{\{|x-y|\leq 2^{-j_\infty-t_\infty+s_\infty}\}}\,dy\,dx\\
&+C\sum_{i=1}^N\int_{\omega_i}\int_{\mathbb{T}^2\setminus\omega_i}|\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y)|^2K_{j_\infty+t_\infty}(x-y)\mathbb{1}_{\{|x-y|> 2^{-j_\infty-t_\infty+s_\infty}\}}\,dy\,dx\\
\leq & B^{\mathrm{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_\infty,\boldsymbol{w}_\infty)\\
&+\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right)\\
&+C2^{-s_\infty}\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right). \qquad (360)
\end{aligned}$$

Finally we define $\omega_\infty=\bigcup_{z\in Z_\infty:Q_{z+a,2^{-j_\infty}}\cap\bigcup_{i=1}^N\omega_i\neq\emptyset}Q_{z+a,2^{-j_\infty}}$, which consists of squares, has measure $|\omega_\infty|\leq 1/2$, and perimeter

$$\begin{aligned}
\mathcal{H}^1(\partial\omega_\infty)\leq&\sum_{i=1}^N C(\mathcal{H}^1(\omega_\infty)+2^{-j_\infty})\\
\leq& C\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\right). \qquad (361)
\end{aligned}$$

Each of the ω_i contains at least one square of side length 2^{-j_∞} and has at least that diameter, and the total length of all exposed lower edges of squares touching ω_i is at most its diameter plus 2^{-j_∞}, and likewise for all upper, left, and right edges.

Again using Lemma 7.2 and estimate (350), we see that

$$\begin{aligned}
&\int_{\omega_\infty\setminus\bigcup_{i=1}^N\omega_i}\int_{\mathbb{T}^2}|\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y)|^2K_{j_\infty+t_\infty}(x-y)\,dy\,dx\\
\leq&\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right)\\
&+C2^{-s_\infty}\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right), \qquad (362)
\end{aligned}$$

so that

$$\int_{\mathbb{T}^2\setminus\omega_\infty}\int_{\mathbb{T}^2\setminus\omega_\infty}(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\text{per}}_{j_\infty+t_\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy$$
$$\leq B^{\text{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{w}_\infty,\boldsymbol{w}_\infty)$$
$$+\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right)$$
$$+C2^{-s_\infty}\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right). \tag{363}$$

Finally, using Lemma 7.2 and the estimate (343), we can reduce the double integral to the sum of double integrals over the good squares making up $\mathbb{T}^2\setminus\omega_\infty$, so that

$$\sum_{z\in Z_\infty:Q_{z+a,2^{-j_\infty}}\subset\mathbb{T}^2\setminus\omega_\infty}\int_{Q_{z+a,2^{-j_\infty}}}\int_{Q_{z+a,2^{-j_\infty}}}$$
$$(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\text{per}}_{j_\infty+t_\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy$$
$$\leq\delta\left(\frac{C}{\eta M_\infty}+\frac{CM_\infty s_\infty}{\sqrt{\eta}(T^+_\infty-T^-_\infty)}\right)\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right)$$
$$+C2^{-s_\infty}\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+2^{-t_\infty-s_\infty}\right)\right)$$
$$+2^{-t_\infty+s_\infty}\left(\frac{C}{\eta}+C_{M_\infty}2^{t_\infty}\left(2^{-m_\infty}+\frac{m_\infty^{1/6}}{\eta k^{1/6}}+\frac{1}{\eta}2^{-t_\infty-s_\infty}\right)\right). \tag{364}$$

We then use Lemma 7.6 in each of the squares applied to the function $\boldsymbol{w}_\infty\llcorner Q_{z+a,2^{-j_\infty}}\in\mathcal{S}$, which since $z\notin B^1_\infty$ have

$$|D\boldsymbol{w}_\infty\llcorner Q_{z+a,2^{-j_\infty}}|(Q_{z+a,2\times2^{-\infty}})\leq C2^{-j_\infty}M_\infty, \tag{365}$$

yielding

$$(\log 2)\sum_{z\in Z_\infty:Q_{z+a,2^{-j_\infty}}\subset\mathbb{T}^2\setminus\omega_\infty}I_\infty(\boldsymbol{w}_\infty,Q_{z+a,2^{-j_\infty}})$$
$$\leq\sum_{z\in Z_\infty:Q_{z+a,2^{-j_\infty}}\subset\mathbb{T}^2\setminus\omega_\infty}\int_{Q_{z+a,2^{-j_\infty}}}\int_{Q_{z+a,2^{-j_\infty}}}$$
$$(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\mathbb{J}^{\text{per}}_{j_\infty+t_\infty}(x-y)(\boldsymbol{w}_\infty(x)-\boldsymbol{w}_\infty(y))\,dx\,dy$$
$$+c2^{-s_\infty}M_\infty\frac{C}{\eta}. \tag{366}$$

For the jumps on edges between squares, we now only have to use estimate

(336), so that

$$
\begin{aligned}
&(\log 2) I_\infty(\boldsymbol{w}_\infty, \mathbb{T}^2 \setminus \omega_\infty)\\
\le &(1+\eta) B^{\mathrm{per}}_{\infty, j_\infty + t_\infty}(\boldsymbol{v}, \boldsymbol{v}) + \frac{C_{M_\infty}}{\eta} 2^{t_\infty} \left(2^{-m_\infty} + \frac{m_\infty^{1/6}}{\eta k^{1/6}} + \frac{1}{\eta} 2^{-t_\infty - s_\infty} \right)\\
&+ \delta \left(\frac{C}{\eta M_\infty} + \frac{C M_\infty s_\infty}{\sqrt{\eta}(T_\infty^+ - T_\infty^-)} \right)\\
&\times \left(\frac{C}{\eta} + C_{M_\infty} 2^{t_\infty} \left(2^{-m_\infty} + \frac{m_\infty^{1/6}}{\eta k^{1/6}} + 2^{-t_\infty - s_\infty} \right) \right)\\
&+ C 2^{-s_\infty} \left(\frac{C}{\eta} + C_{M_\infty} 2^{t_\infty} \left(2^{-m_\infty} + \frac{m_\infty^{1/6}}{\eta k^{1/6}} + 2^{-t_\infty - s_\infty} \right) \right)\\
&+ 2^{-t_\infty + s_\infty} \left(\frac{C}{\eta} + C_{M_\infty} 2^{t_\infty} \left(2^{-m_\infty} + \frac{m_\infty^{1/6}}{\eta k^{1/6}} + \frac{1}{\eta} 2^{-t_\infty - s_\infty} \right) \right)\\
&+ c 2^{-s_\infty} M_\infty \frac{C}{\eta}\\
&+ C_{M_\infty} \left(2^{-m_\infty} + \frac{m_\infty^{1/6}}{\eta k^{1/6}} + \frac{1}{\eta} 2^{-t_\infty - s_\infty} \right)
\end{aligned}
\tag{367}
$$

where we also replaced $B^{\mathrm{per}}_{\infty, j_\infty + t_\infty}(\boldsymbol{w}_\infty, \boldsymbol{w}_\infty)$ with $B^{\mathrm{per}}_{\infty, j_\infty + t_\infty}(\boldsymbol{v}, \boldsymbol{v})$ using (342).

We finish this part by fixing $1 \ll M_\infty \ll s_\infty \ll T_\infty^- \ll T_\infty^+ \ll m_\infty \ll k$, such that for a given $\delta_\infty > 0$ we have

xi)

$$
(\log 2) I_\infty(\boldsymbol{w}_\infty, \mathbb{T}^2 \setminus \omega_\infty) \le (1 + \delta_\infty + \eta) B^{\mathrm{per}}_{\infty, j_\infty + t_\infty}(\boldsymbol{v}, \boldsymbol{v}), \tag{368}
$$

xii)

$$
\|\boldsymbol{w}_\infty - \boldsymbol{v}\|_{L^1(\mathbb{T}^2 \setminus \omega_\infty, \mathbb{R}^{2M})} \le \delta_\infty 2^{-j_\infty - t_\infty} \le 2^{-\beta k}, \tag{369}
$$

and

xiii) The set $\omega_\infty \subset \mathbb{T}^2$ is closed, with $|\omega_\infty| \le 1/2$ and $\mathcal{H}^1(\partial \omega_\infty) \le \delta_\infty$.

Step 6: The competitor $\boldsymbol{w}_0$.

We now repeat the previous five steps to find $j_0 \in \{\lfloor \eta k \rfloor, \dots, \lfloor (\beta - \eta) k \rfloor\}$ and $t_0 \in \{T_0^-, \dots, T_0^+\}$ along with a function $\boldsymbol{w}_0 : \mathbb{T}^2 \to \mathbb{R}^{2M}$ that is locally in $\mathcal{S}$ in all squares $Q_{z+a, 2^{-j_0}}$, with $z \in Z_0 \setminus B_0^1$, and $a \in Q_{0, 2^{-j_0}}$, with jumps within each square $Q_{z+a, 2^{j_0}}$ for $z \in Z_0 \setminus B_0^1 \setminus B_0^2$ at least $2^{-j_0 - t_0 + s_0}$ apart. In the localization step, when defining $\tilde{\omega}$, we add the squares $Q_{z, 8 \times 2^{-j_0}}$ for which z is in

$$
B_0^3 = \{ z \ inZ_0 \setminus B_0^1 : \mathcal{H}^1(\partial \omega_\infty \cap Q_{z+a, 2^{j_0}}) > \rho 2^{-j_0} \}, \tag{370}
$$

where $\rho > 0$ is some small parameter. We see that

$$
2^{-j_0} \sharp B_0^3 \le \frac{C \delta_\infty}{\rho \eta}, \tag{371}
$$

which is small for δ_∞ small. After growing the set

$$\tilde{\omega}_0 = \bigcup_{z\in B_0^1\cup B_0^2\cup B_0^3} Q_{z,8\times 2^{j_0}} \tag{372}$$

into a set $\omega_0 \subset \mathbb{T}^2$ with $|\omega_0| \leq 1/2$ and $\mathcal{H}^1(\partial\omega_0) \leq \delta(\mathcal{H}^1(\partial\tilde{\omega}_0))$, we get in the remaining squares if $1 \ll M_0 \ll s_0 \ll T_0^- \ll T_0^+ \ll m_0 \ll k$ that

xiv)

$$(\log 2) I_0(\boldsymbol{w}_\infty, \mathbb{T}^2 \setminus \omega_\infty) \leq (1+\delta_0+\eta) B^{\text{per}}_{0,j_0+t_0}(\boldsymbol{v},\boldsymbol{v}), \tag{373}$$

xv)

$$\|\boldsymbol{w}_0 - \boldsymbol{v}\|_{L^1(\mathbb{T}^2\setminus\omega_0,\mathbb{R}^{2M})} \leq \delta_0 2^{-j_0-t_0}, \tag{374}$$

and

xvi) The set $\omega_0 \subset \mathbb{T}^2$ is closed, with $|\omega_0| \leq 1/2$ and $\mathcal{H}^1(\partial\omega_0) \leq \delta_0$.

Now we can use Lemma 7.7 in each of the squares $Q_{z+a,2^{-j_0}}$ making up $\mathbb{T}^2 \setminus \omega_0$ so that

$$\begin{aligned}
&I^{\text{rel}}_\infty(\boldsymbol{w}_0, Q_{z+a,2^{-j_0}}) \\
&\leq (1+\delta'(\rho))\, I^{\text{rel}}_\infty(\boldsymbol{w}_\infty, Q_{z+a,2^{-j_0}} \setminus \omega_\infty) \\
&\quad + c2^{j_0+t_0-s_0}\|\boldsymbol{w}_0 - \boldsymbol{w}_\infty\|_{L^1(Q_{z+a,2^{-j_0}}\setminus\omega_\infty)} + c2^{-t_0}|D\boldsymbol{w}_0|(Q_{z,2\times 2^{-j}}).
\end{aligned} \tag{375}$$

The last error term can be estimated as before, leaving us after summation over all squares with

$$\begin{aligned}
&\sum_{z\in Z_\infty : Q_{z+a,2^{-j_0}}\cap\omega_0=\emptyset} I^{\text{rel}}_\infty(\boldsymbol{w}_0, Q_{z+a,2^{-j_0}}) \\
\leq &(1+\delta'(\rho)) I^{\text{rel}}_\infty(\boldsymbol{w}_\infty, \mathbb{T}^2\setminus\omega_\infty) + \delta_0.
\end{aligned} \tag{376}$$

Here $\delta' : [0,\infty) \to [0,\infty)$ is the modulus of continuity in Lemma 7.7, and since for every $\rho > 0$ we only had to choose δ_∞ small enough, this error term can be made arbitrarily small.

The jumps across the edges of squares are small for $\boldsymbol{w}_0$ as well, as in (336), so that

$$\begin{aligned}
&(\log 2)\left[(1-\beta)I^{\text{rel}}_\infty + \beta I_0\right](\boldsymbol{w}_0, \mathbb{T}^2\setminus\omega_0) \\
\leq &(1+\delta_0)\left[(1-\beta)B^{\text{per}}_{\infty,j_\infty+t_\infty}(\boldsymbol{v},\boldsymbol{v}) + \beta B^{\text{per}}_{0,j_0+t_0}(\boldsymbol{v},\boldsymbol{v})\right] + \delta_0.
\end{aligned} \tag{377}$$

Now we turn to the characterizations of j_∞, t_∞, j_0, and t_0, such that

$$\begin{aligned}
&(\log 2)\left[(1-\beta)I^{\text{rel}}_\infty + \beta I_0\right](\boldsymbol{w}_0, \mathbb{T}^2\setminus\omega_0) \\
\leq &(1+\delta_0+C\sqrt{\eta})\frac{1}{k}\left[\sum_{j=\lceil(\beta+\eta)k\rceil}^{k} B^{\text{per}}_{\infty,j}(\boldsymbol{v},\boldsymbol{v}) + \sum_{j=1}^{\lfloor(\beta-\eta)k\rfloor} B^{\text{per}}_{0,j}(\boldsymbol{v},\boldsymbol{v})\right] + \delta_0.
\end{aligned} \tag{378}$$

We see that this is the statement of the lemma if δ_0 is small enough, with $\boldsymbol{w} = \boldsymbol{w}_0$ and $\omega = \omega_0$, if k is large enough. □

We now apply Lemma 6.5 one final time to pass to the lower bound for $u_0 \in BV(\Omega, \mathbb{Z}^N)$, using the lower semicontinuity of $I^{\mathrm{rel}} = [(1-\beta)I^{\mathrm{rel}}_\infty + \beta I_0]^{\mathrm{rel}}$.

Proposition 8.2. *Let* $I^{\mathrm{rel}}(\boldsymbol{u}, \Omega) = \int_{J_{\boldsymbol{u}} \cap \Omega} \varphi^{\mathrm{rel}}([\boldsymbol{u}], \nu)\, d\mathcal{H}^1$ *on* $BV(\Omega, \mathbb{Z}^{2M})$ *for* $\Omega \subset \mathbb{T}^2$ *open be a lower semicontinuous energy of line-tension type. Let* $\boldsymbol{u}_0 \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$.

Let $\delta > 0$. *Then there is* $\eta > 0$ *such that the following holds:*

Let $\omega_k \subseteq \mathbb{T}^2$ *be a sequence of open bounded sets with* $\mathcal{H}^1(\partial\omega_k) \le \eta$, $|\omega_k| \le 1/2$, $\boldsymbol{u}_k \in BV(\mathbb{T}^2 \setminus \omega_k, \mathbb{Z}^{2M})$ *a sequence with* $\sup_k |D\boldsymbol{u}_k|(\Omega \setminus \omega_k) < \infty$ *and* $\|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{L^1(\mathbb{T}^2 \setminus \omega_k)} \to 0$.

Then

$$(1+\delta) \liminf_{k\to\infty} I^{\mathrm{rel}}(\boldsymbol{u}_k, \mathbb{T}^2 \setminus \omega_k) \ge I^{\mathrm{rel}}(\boldsymbol{u}_0, \mathbb{T}^2).$$

We shall construct from $\boldsymbol{u}_k$ a new sequence $\boldsymbol{w}_k \in BV(\mathbb{T}^2, \mathbb{Z}^{2M})$ using only slightly more energy and ending up in a small L^1-ball around u_0.

Proof. Let $\varepsilon = 2^{-j} > 0$ be some small parameter. We shall consider the grid $L_{\varepsilon,p} = \{x \in \mathbb{T}^2 : x_1 - p_1 \in \varepsilon\mathbb{Z}/\mathbb{Z} \text{ or } x_2 - p_2 \in \varepsilon\mathbb{Z}/\mathbb{Z}\}$ centered at $p \in \mathbb{T}^2$ and its neighborhood $U_{\varepsilon,p} = B(L_{\varepsilon,p}, \varepsilon^2)$. We want to find for every $k \in \mathbb{N}$ some $p_k \in [0,\varepsilon)^2 \subset \mathbb{Z}^2$ such that

i) $|D\boldsymbol{u}_0|(U_{\varepsilon,p_k}) \le c\varepsilon |D\boldsymbol{u}_0|(\mathbb{T}^2)$,

ii) $\omega_k \cap \partial U_{\varepsilon,p_k} = \emptyset$,

iii) $\|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{L^1(\partial U_{\varepsilon,p_k})} \le \frac{c}{\varepsilon} \|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{L^1(\Omega\setminus\omega_k)}$.

Once we have found such a p_k, we define $\tilde{\omega}_k = \omega_k \setminus U_{\varepsilon,p_k}$ and $\boldsymbol{v}_k \in BV(\mathbb{T}^2 \setminus \tilde{\omega}_k, \mathbb{Z}^{2M})$ as

$$\boldsymbol{v}_k = \begin{cases} \boldsymbol{u}_k & \text{, in } \mathbb{T}^2 \setminus \omega_k \setminus U_{\varepsilon,p_k}), \\ \boldsymbol{u}_0 & \text{, in } \mathbb{T}^2 \cap U_{\varepsilon,p_k}. \end{cases} \tag{379}$$

Note that

$$|D\boldsymbol{v}_k|(\mathbb{T}^2 \setminus \tilde{\omega}_k) \le |D\boldsymbol{u}_k|(\mathbb{T}^2 \setminus \omega_k) + c\varepsilon |D\boldsymbol{u}_0|(\mathbb{T}^2) + \frac{c}{\varepsilon}\|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{L^1(\mathbb{T}^2\setminus\omega_k)} \le C \tag{380}$$

for k large enough.

We then apply Lemma 6.5 to $\boldsymbol{v}_k$ in every square $Q = p_k + z + (0,\varepsilon)^2 \subseteq \mathbb{T}^2$ with $z \in \varepsilon\mathbb{Z}^2/\mathbb{Z}^2$, with parameter $\delta = \varepsilon^2$.Then for η small enough with respect to ε, we obtain a function $\boldsymbol{w}_k \in BV(\bigcup Q, \mathbb{Z}^{2M})$ with $\boldsymbol{w}_k = \boldsymbol{u}_0$ on $L_{\varepsilon,p_k} \cap \bigcup Q$. Then by Poincaré inequality in each square we get that

$$\|\boldsymbol{w}_k - \boldsymbol{u}_0\|_{L^1(\bigcup Q)} \le c\varepsilon(|D\boldsymbol{w}_k|(\mathbb{T}^2) + |D\boldsymbol{u}_0|(\mathbb{T}^2) \le C\varepsilon \tag{381}$$

for k large enough.

The energy of $\boldsymbol{w}_k$ can then be bounded by

$$\begin{aligned} I^{\mathrm{rel}}(\boldsymbol{w}_k, \mathbb{T}^2) \le& I^{\mathrm{rel}}(\boldsymbol{v}_k, \mathbb{T}^2 \setminus \tilde{\omega}_k) + c|D\boldsymbol{w}_k|(\{\boldsymbol{w}_k \ne \boldsymbol{v}_k\}) \\ \le& I^{\mathrm{rel}}(\boldsymbol{u}_k, \mathbb{T}^2 \setminus \omega_k) + c|D\boldsymbol{u}_0|(U_{\varepsilon,p_k}) + c\|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{L^1(\partial U_{\varepsilon,p_k})} + C\varepsilon \\ \le& I^{\mathrm{rel}}(\boldsymbol{u}_k, \mathbb{T}^2 \setminus \omega_k) + C\varepsilon + \frac{C}{\varepsilon}\|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{L^1(\mathbb{T}^2\setminus\omega_k)}. \end{aligned} \tag{382}$$

Now we only need to pick ε small enough. In fact, by the lower semicontinuity of I^{rel} there exists $\varepsilon > 0$ such that

$$(1+\delta/2)\inf\{I^{\mathrm{rel}}(\boldsymbol{w},\mathbb{T}^2) : \|\boldsymbol{w}-\boldsymbol{u}_0\|_{L^1(\mathbb{T}^2)} \le C\varepsilon\} \ge I^{\mathrm{rel}}(\boldsymbol{u}_0,\mathbb{T}^2). \tag{383}$$

Inserting $\boldsymbol{w}_k$ for k large enough into the left hand side yields, for ε small enough,

$$\begin{aligned} I^{\mathrm{rel}}(\boldsymbol{u}_0,\mathbb{T}^2) \le& (1+\delta/2)\liminf_{k\to\infty} I^{\mathrm{rel}}(\boldsymbol{w}_k,\mathbb{T}^2) \\ \le& (1+\delta)\liminf_{k\to\infty} I^{\mathrm{rel}}(\boldsymbol{u}_k,\mathbb{T}^2\setminus\omega_k). \end{aligned} \tag{384}$$

□

Combining the previous two propositions, as well as the fact that $k_i \ge (1-\eta)\log\varepsilon_i/\log 2$, we arrive at the lower bound for Theorem 1.1 whenever $\boldsymbol{u}_i \to \boldsymbol{u}_0$ in $L^1(\mathbb{T}^2)$, since then $\|\boldsymbol{w}_i - \boldsymbol{u}_0\|_{L^1(\mathbb{T}^2\setminus\omega_i)} \to 0$.

References

[1] R. A. Adams and J. J. Fournier. *Sobolev spaces.* Academic press, 2003.

[2] G. Alberti, G. Bouchitté, and P. Seppecher. Un résultat de perturbations singulières avec la norme $H^{1/2}$. *C. R. Acad. Sci. Paris Sér. I Math.*, 319:333–338, 1994.

[3] L. Ambrosio and A. Braides. Functionals defined on partitions in sets of finite perimeter. I. Integral representation and Γ-convergence. *J. Math. Pures Appl. (9)*, 69:285–305, 1990.

[4] L. Ambrosio and A. Braides. Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization. *J. Math. Pures Appl. (9)*, 69:307–333, 1990.

[5] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Mathematical Monographs. Oxford University Press, 2000.

[6] A. Braides. Γ*-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2002.

[7] A. Braides, S. Conti, and A. Garroni. Density of polyhedral functions. *in preparation*, 2016.

[8] S. Cacace and A. Garroni. A multi-phase transition model for the dislocations with interfacial microstructure. *Interfaces Free Bound.*, 11:291–316, 2009.

[9] C. Ciarlet. *Mathematical Elasticity, Vol. I : Three-Dimensional Elasticity.* Studies in Mathematics and its Applications. Elsevier Science Inc, 1988.

[10] S. Conti, A. Garroni, and A. Massaccesi. Modeling of dislocations and relaxation of functionals on 1-currents with discrete multiplicity. *Calculus of Variations and Partial Differential Equations*, 54(2):1847–1874, 2015.

[11] S. Conti, A. Garroni, and S. Müller. Singular kernels, multiscale decomposition of microstructure, and dislocation models. *Arch. Ration. Mech. Anal.*, 199:779–819, 2011.

[12] S. Conti, A. Garroni, and M. Ortiz. The line-tension approximation as the dilute limit of linear-elastic dislocations. *Archive for Rational Mechanics and Analysis*, 218(2):699–755, 2015.

[13] S. Conti and P. Gladbach. A line-tension model of dislocation networks on several slip planes. *Mechanics of Materials*, 90:140-147, 2015.

[14] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78. Springer Science & Business Media, 2007.

[15] G. Dal Maso. *An introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[16] A. Fusco and N. Ferriero. A note on sets of finite perimeter in the plane. *Discrete Cont. Dyn. Syst. Ser. B*, 2009.

[17] A. Garroni and S. Müller. Γ-limit of a phase-field model of dislocations. *SIAM J. Math. Anal.*, 36:1943–1964, 2005.

[18] A. Garroni and S. Müller. A variational model for dislocations in the line tension limit. *Arch. Ration. Mech. Anal.*, 181:535–578, 2006.

[19] J. Ginster. *Plasticity as the* Γ*-limit of a Dislocation Energy* . PhD thesis, Universität Bonn, 2016.

[20] P. Gladbach. A Phase-field Model of Dislocations on Parallel Slip Planes. In *PAMM*, volume 15, pages 545–546, 2015.

[21] P. Gladbach. A phase-field model of dislocations on several slip planes. In *Oberwolfach Reports*, volume 17, 2016.

[22] J. Hirth and J. Lothe. *Theory of dislocations.* New York: McGraw-Hill, 1968.

[23] D. Hull and D. Bacon. *Introduction to dislocations*, volume 37 of *International Series on Materials Science and Technology.* Elsevier Science Inc, 1984.

[24] R. Jerrard. Lower bounds for generalized Ginzburg-Landau functionals. *SIAM Journal on Mathematical Analysis*, 30.4:721–746, 1999.

[25] M. Koslowski, A. M. Cuitiño, and M. Ortiz. A phase-field theory of dislocation dynamics, strain hardening and hysteresis in ductile single crystal. *J. Mech. Phys. Solids*, 50:2597–2635, 2002.

[26] M. Koslowski and M. Ortiz. A multi-phase field model of planar dislocation networks. *Model. Simul. Mat. Sci. Eng.*, 12:1087–1097, 2004.

[27] L. Modica and S. Mortola. Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. (5)*, 14-B:285–299, 1977.

[28] S. Müller, L. Scardia, and C. I. Zeppieri. Geometric rigidity for incompatible fields and an application to strain-gradient plasticity. *Indiana univ. Math. J.*, 63:1365–1396, 2014.

[29] R. T. Rockafellar. *Convex analysis.* Princeton university press, 2015.

[30] E. Sandier. Lower bounds for the energy of unit vector fields and applications. *Journal of functional analysis*, 152(2):379–403, 1998.

[31] E. Sandier and S. Serfaty. *Vortices in the magnetic Ginzburg-Landau model*, volume 70. Springer Science & Business Media, 2008.

[32] L. Scardia and C. I. Zeppieri. Line-tension model for plasticity as the Γ-limit of a nonlinear dislocation energy. *SIAM J. Math. Anal.*, 44:2372–2400, 2012.

[33] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.

[34] P. M. Suquet. Un espace fonctionnel pour les équations de la plasticité. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 1(1):77–87, 1979.

[35] V. Volterra. Sur l'équilibre des corps élastiques multiplement connexes. *Annales scientifiques de l'École Normale Supérieure*, 24:401–517, 1907.