Plasticity as the Γ-limit of a Dislocation Energy

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Summary

In this thesis, we derive macroscopic crystal plasticity models from mesoscopic dislocation models by means of Γ-convergence as the interatomic distance tends to zero. Crystal plasticity is the effect of a crystal undergoing an irreversible change of shape in response to applied forces. At the atomic scale, dislocations — which are local defects of the crystalline structure — are considered to play a main role in this effect. We concentrate on reduced two-dimensional models for straight parallel edge dislocations.

Firstly, we consider a model with a nonlinear, rotationally invariant elastic energy density with mixed growth. Under the assumption of well-separateness of dislocations, we identify all scaling regimes of the stored elastic energy with respect to the number of dislocations and prove Γ-convergence in all regimes. As the main mathematical tool to control the non-convexity induced by the rotational invariance of the energy, we prove a generalized rigidity estimate for fields with non-vanishing curl. For a given function with values in $\mathbb{R}^{2\times 2}$, the estimate provides a quantitative bound for the distance to a specific rotation in terms of the distance to the set of rotations and the curl of the function. The most important ingredient for the proof is a fine estimate which shows that in two dimensions a vector-valued function $f \in L^1$ can be decomposed into two parts belonging to certain negative Sobolev spaces with critical exponent such that corresponding estimates depend only on $\operatorname{div} f$ and the L^1-norm of f. This is a generalization of an estimate due to Bourgain and Brézis.

Secondly, we consider a dislocation model in the setting of linearized elasticity. The main difference to the first case above and existing literature is that we do not assume well-separateness of dislocations. In order to prove meaningful lower bounds, we adapt ball construction techniques which have been used successfully in the context of the Ginzburg-Landau functional. The building block for this technique are good lower bounds on annuli. In contrast to the vortices in the Ginzburg-Landau model, in the setting of linear elasticity, a massive loss of rigidity can be observed on thin annuli which leads to inadequate lower bounds. Hence, our analysis focuses on finding thick annuli which carry almost all relevant energy.

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Firstly, it is a great pleasure to thank my supervisor Prof. Dr. Stefan Müller for introducing me to this very interesting topic and for his constant support and help during the three years of my PhD-studies.

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Contents

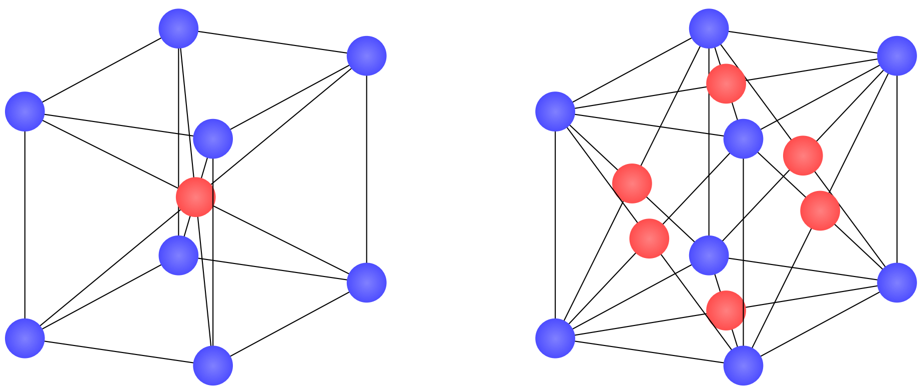
1 Introduction

Plasticity is the effect of a solid undergoing an irreversible change of shape in response to applied forces. However, the underlying mechanisms that lead to this effect depend highly on the considered material. In this thesis, we concentrate on crystals i.e., materials whose atoms form periodic patterns. This includes a large class of important materials, for example metals. In fact, most pure metals have relatively simple crystalline structures, examples include face-centered-cubic structures (copper, nickel, aluminium, etc.) and body-centered-cubic structures (iron, chromium, etc.), see Figure 1.1. For a more detailed discussion of crystalline structures, we refer to [47] or [50].

In the engineering literature, there is a wide variety of phenomenologically derived macroscopic plasticity models. It would also be desirable to derive macroscopic models rigorously as a limit of models at smaller scales. The main cause for plasticity in crystals on an atomic scale is the presence of so-called dislocations, cf. [66, 73]. Dislocations are topological defects of the crystalline structure and will be considered in detail in the following section.

In special situations, first rigorous mathematical derivations of macroscopic plasticity models from mesoscopic and microscopic dislocation models were established, e.g. [21, 23, 30, 38, 59, 71].

The aim of this thesis is to complement these results by deriving a similar macroscopic model, starting from different modeling assumptions at the atomic scale.



(a) Unit-cell of a body-centered-cubic (bcc) crystal: one atom at each corner of the cube (blue) plus one atom centered in the cube (red).

(b) Unit-cell of a face-centered-cubic (fcc) crystal: one atom at each corner of the cube (blue) plus one atom centered at each face of the cube(red).

Figure 1.1: Examples of typical crystalline structures in pure metals.

We start from a reduced two-dimensional model for straight, parallel edge dislocations. This setting will be explained in detail in Section 1.2. Mathematically, we study a variational model of the form

$$\int_\Omega W(\beta)\,dx \quad \text{for } \beta : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^{2\times 2} \text{ subject to } \operatorname{curl}\beta = \sum_i \varepsilon b_i\,\delta_{x_i}$$

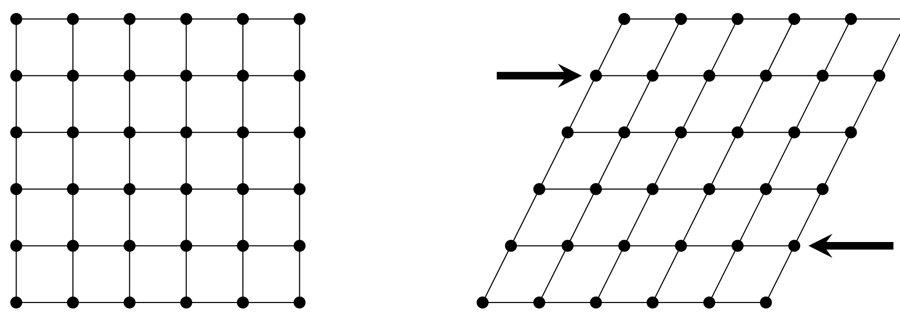
under different assumptions on W. Here, the $b_i \in \mathbb{R}^2$ are constrained to belong to a certain discrete set which depends on the crystalline structure. We identify the different scaling regimes of the energy and the limit of the suitably rescaled energy in the sense of Γ-convergence. In the most interesting regime—the so-called critical regime—, we prove that the limit is a strain-gradient model of the form

$$\int_\Omega \mathcal{C}\beta : \beta \, dx + \int_\Omega \varphi \left(\frac{d \operatorname{curl} \beta}{d |\operatorname{curl} \beta|} \right) d \operatorname{curl} \beta,$$

where $\mathcal{C}$ is a linearized elastic tensor and φ is a 1-homogeneous function.
As a main tool for compactness in the case of a rotationally invariant energy density W with mixed growth and well-separateness of dislocations, we prove a generalized rigidity estimate for fields with non-vanishing curl. The estimate bounds the distance of a function f to a single rotation in terms of the distance of f to the set of rotations and the total variation of the measure $\operatorname{curl} f$. As a major ingredient, we show that a function $f \in L^1$ can be decomposed in two parts belonging to certain negative Sobolev spaces such that corresponding estimates depend only on $\operatorname{div} f$ and the L^1-norm of f. This is a generalization of an estimate due to Bourgain and Brézis, [11].
In the setting of a linearized elastic energy but no well-separateness of dislocations, we prove optimal lower bounds for compactness with the use of ball construction techniques.
A more detailed overview of the main results of this thesis can be found in Section 1.5.

In order to gain an understanding of the mathematical modeling of dislocations, we will first approach the effect of plasticity and the role of dislocations therein phenomenologically. Later, we discuss the continuum mechanical description of dislocations and heuristics for the scaling of the energy, Section 1.2 and Section 1.3. An overview of mathematical contributions to the field is presented in Section 1.4.

1.1 A Phenomenological Approach to Crystal Plasticity and Dislocations

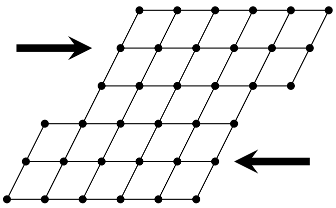


(a) The undisturbed crystal in its equilibrium configuration.

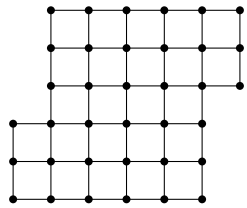
(b) Under a small load, the crystal deforms elastically.

Figure 1.2: Sketch of the elastic deformation of a crystal under a small load.

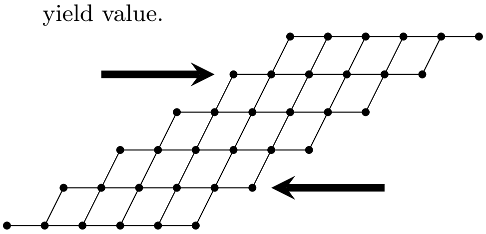
Let us first consider the following idealized two-dimensional example which captures the basic concepts. Suppose that the equilibrium configuration of a given material is a simple cubic lattice. Applying a small shear load as in Figure 1.2 induces a small distortion of the crystal lattice, Figure 1.2b. After unloading, the crystal regains its equilibrium shape, Figure 1.2a. This is called an *elastic* deformation. If we increase the load over a critical value, we observe a slip of the upper atoms in horizontal direction resulting in a so-called *elasto-plastic* deformation, see Figure 1.3a. Note that the bonds between the two rows of atoms which were formerly bonded have broken and rebonded one atom in the direction of the slip. After unloading, the elastic deformation vanishes, Figure 1.3b. As a consequence of the slip of the upper atoms, a permanent plastic deformation remains. A higher load results in a larger slip of atoms in horizontal direction and consequently in a larger permanent deformation after unloading, see Figures 1.3c and 1.3d. If the load becomes too high, the crystal fractures.



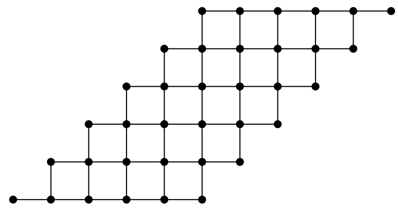
(a) A crystal responding elasto-plastically to a load that is larger than the critical yield value.



(b) After unloading, a permanent plastic deformation remains.



(c) A larger load induces a larger deformation.



(d) Also, the remaining plastic deformation is larger.

Figure 1.3: Under large loads, the crystal deforms elasto-plastically. After removing the load, a permanent plastic deformation remains.

The slipping of rows of atoms is also obtained in practice. See Figure 1.4 for an experimental picture of a cadmium crystal deforming by slip under a tensile load.

In three dimensions, the above considerations correspond to the slip of atoms above a certain plane, the *slip plane*. In our example, this is the plane which includes the horizontal direction and the direction pointing out of the paper. Clearly, the planes and slip directions in which this is possible

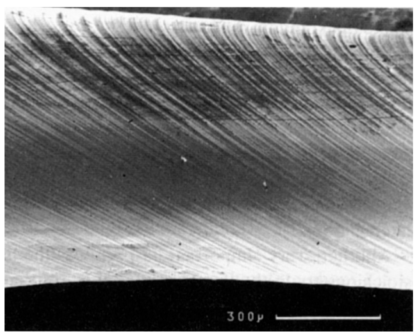


Figure 1.4: A scanning electron micrograph of a single crystal of cadmium deforming by slip as a response to a tensile load in horizontal direction. Unlike in our sketches, the direction of the load does not lie in the slip plane. Picture reprinted by permission of http://www.doitpoms.ac.uk/tlplib/miller_indices/uses.php (date of retrieval: 04/10/2016).

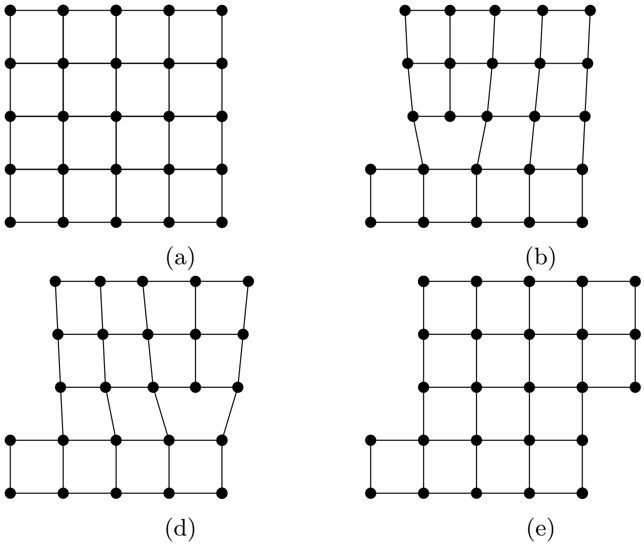


Figure 1.5: Sketch of the motion of a dislocation through a crystal. Once the dislocation has moved through the crystal, a slip remains.

depend on the crystalline structure. Usually, they are described by *slip systems* $(\gamma, m) \in \mathbb{R}^3 \times \mathbb{R}^3$. Here, γ is the direction of the slip and m is a unit normal to the slip plane. As the plastic deformation does not change the shape of the equilibrium lattice locally, the slip systems satisfy the condition $\gamma \cdot m = 0$. Typically, the feasible slip directions are those with the highest number of atoms per length whereas slip planes have the highest number of atoms per area. For a list of slip systems in typical crystallographic lattices, we refer to [47].

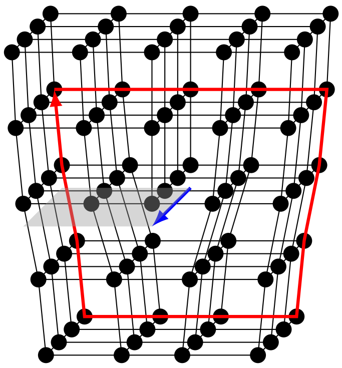
In 1926, Frenkel computed, in a first approximation and a situation similar to the one in Figure 1.3, a theoretical critical shear stress that is needed in order to obtain a permanent plastic deformation via the slip of rows of atoms, [35]. His result states that

$$\tau_{theoretical} \approx \frac{\mu}{2\pi},$$

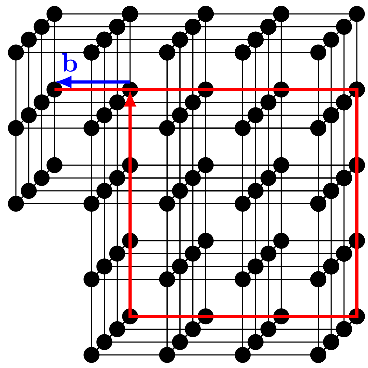
where $\tau_{theoretical}$ denotes the theoretically needed shear stress and μ is the shear modulus of the material. As observed in 1929 in [67], this theoretical result differs from practical observations of the minimal stress needed to obtain a permanent deformation — the yield stress — by orders of magnitude (at least 10^3). In the 1930s, several authors introduced the idea of dislocations as the mechanism for plastic deformations, cf. [66, 73]. The idea is the following. For moving a complete plane of atoms simultaneously, a lot of energy is required. In practice, the plastic flow is not uniform. Instead, one can imagine that first the atoms on the very left slip to the right. Then, this defect — called dislocation — can be transported through the crystal, see Figure 1.5. In particular, as the slip mechanism occurs on a plane, the defect is necessarily concentrated on the so-called dislocation line which lies in the slip plane and separates regions with different slips, see Figure 1.6. In our case, this is the line pointing into the paper and passing through the two-dimensional defect.

In order to describe the dislocation, the two most important quantities are the tangent vector of the dislocation line and the *Burgers vector* which is essentially the difference of the slip of the neighboring regions, cf. [15]. The procedure to compute the Burgers vector consists in drawing a circuit around the defect in the deformed crystal and drawing the same circuit in a perfect reference crystal, see Figure 1.6. Every time we surround a defect in the deformed configuration, the associated circuit in the reference crystal is not closed. The difference of the ending point and the starting point of this path in the reference crystal is the Burgers vector of the dislocation. Note that, by definition, the Burgers vector can only be an integer combination of the basic lattice vectors. The convention is that the Burgers circuit is drawn in the positive sense with respect to the tangent of the corresponding dislocation line, see Figure 1.6a.

There exist two important basic types of dislocations: *Edge dislocations* (the Burgers vector is



(a) Sketch of a Burgers circuit (red), the dislocation line (blue, pointing into the paper), and the slip plane (gray) in the distorted crystal.



(b) The associated path in the perfect reference crystal (red) and the Burgers vector (blue).

Figure 1.6: Sketch of an edge dislocation in a three-dimensional cubic lattice.

perpendicular to the dislocation line; see Figure 1.6 and Figure 1.7 for a picture in the continuous setting) and *screw dislocations* (the Burgers vector is parallel to the dislocation line; see Figure 1.7 for a sketch in the continuous setting). Clearly, a lot of dislocations which appear in practice are of mixed type.

We restrict ourselves to this basic view on dislocations. For a discussion of more complex phenomena involving dislocations, we refer to [47] or [50].

In the next section, we link these basic crystallographic considerations with a continuum mechanical description.

1.2 The Continuum Description of Dislocations

For a general introduction to continuum mechanics, we refer to [45]. We limit ourselves to quickly explaining how dislocations are modeled in this context.

The deformation of a body $\Omega \subset \mathbb{R}^3$ is described by a function $\varphi : \Omega \to \mathbb{R}^3$. In the nonlinear theory (finite plasticity), the elastic energy of the deformed configuration is given by a nonlinear functional depending on φ. In the linearized theory, it is assumed that the deformation is already very close to the identity map. By a Taylor expansion of the elastic energy, the quantity of interest is the displacement field u which is given by $u(x) = \varphi(x) - x$.

Now, let us consider a deformation or displacement of a crystal Ω given by a function $v \in SBV(\Omega; \mathbb{R}^3)$ (for an introduction to functions of bounded variation, see [5]) such that a constant jump of v is concentrated on a hyperplane Σ with a jump height $[v]$ that corresponds to a feasible translation of the crystal lattice. Here, the jump on Σ represents exactly the slip over the slip plane Σ in direction

$[v]$. The classical decomposition for the derivative of a function in $SBV(\Omega;\mathbb{R}^3)$ in this setting is

$$Dv = \nabla v \, d\mathcal{L}^3 + [v] \otimes m \, d\mathcal{H}^2_{|\Sigma\cap\Omega},$$

where m is the normal to Σ.
In the linearized theory of dislocations, one decomposes the strain additively into an elastic and a plastic part, $Dv = \beta_{el} + \beta_{pl}$. Here, the elastic part is exactly represented by the absolutely continuous part of the measure Dv i.e., by $\nabla v \, d\mathcal{L}^3$ whereas the plastic part is given by $[v] \otimes m \, d\mathcal{H}^2_{|\Sigma\cap\Omega}$. As Dv is the derivative of v, it holds in the sense of distributions $\operatorname{curl} Dv = 0$. Since $[v]$ is assumed to be constant, this implies

$$\operatorname{curl} \beta_{el} = -\operatorname{curl} \beta_{pl} = [v] \otimes \tau \, \mathcal{H}^1_{|\partial\Sigma\cap\Omega}, \tag{1.1}$$

where the curl is assumed to act row-wise. Here, $\partial\Sigma$ has to be understood as the one-dimensional boundary of the hyperplane Σ and τ is the unit tangent to $\partial\Sigma$ in the correct orientation. In particular, the dislocations are concentrated on the dislocation lines $\partial\Sigma$ as in the discrete case. The right hand side of (1.1) is usually referred to as *Nye-dislocation-density*, [63], and is denoted by μ. An easy consequence of (1.1) is that a dislocation density μ satisfies $\operatorname{div} \mu = 0$.
Moreover, note that the curl-condition in (1.1) is the continuous counterpart to the discrete circulation condition via the Burgers circuit. Hence, the dislocation measure μ captures the most important quantities of the lattice distortion, precisely the Burgers vector $b = [v]$ and the direction of the dislocation line τ. In general, we should be more precise and write $b = [[v]]$ as the dislocation might separate regions with different slips and not only regions with slip from those without slip. As in the discrete case, edge dislocations are characterized by $b \perp \tau$ whereas screw dislocations correspond to $b \parallel \tau$. A sketch of continuum deformations with an edge or a screw dislocation can be found in Figure 1.7, cf. the discrete case in Figure 1.6a and the deformation of cylinders discussed by Volterra in [76].

The nonlinear theory is observer invariant. Hence, also rotated versions of the feasible Burgers vectors appear in the deformed configuration. The (locally defined) inverse strains correspond to mappings onto the reference configuration in which only the non-rotated lattice exists. Therefore, the considerations above should be formulated in terms of the inverse strains. However, in the following we will neglect this modeling issue and use the inverse strains in the nonlinear theory as if they were the strains. A more detailed discussion of this transference can be found in [60].
In a variational model, one associates to the elastic strain the stored elastic energy, which is of the form

$$\int_\Omega W(\beta_{el}) \, dx$$

for an elastic energy density $W : \mathbb{R}^{3\times3} \to [0,\infty]$. We will quickly discuss the classical assumptions on W; for a general introduction to elasticity theory we refer to [45]. A mathematically rigorous derivation of the linearized theory of elasticity can be found in [28].
In the linearized theory, which is formulated in terms of the displacement, W would be given by a linear strain-stress-correspondence i.e., $W(\beta_{el}) = \mathcal{C}\beta_{el} : \beta_{el}$. Here, the so-called elasticity tensor $\mathcal{C}$ only acts on the symmetric part of a matrix and satisfies $c|F_{sym}|^2 \leq \mathcal{C}F : F \leq C|F_{sym}|^2$ for any matrix $F \in \mathbb{R}^{3\times3}$.

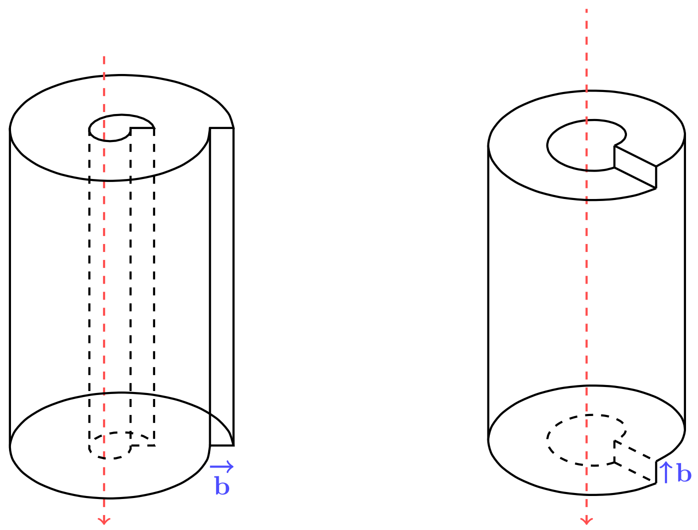


Figure 1.7: Sketch of an edge dislocation (left) and a screw dislocation (right) in a deformed cylinder. The dislocation line is the dashed, red line oriented downwards. The Burgers vector is drawn in blue.

In the nonlinear theory, which is formulated in terms of the deformation, W satisfies the usual assumptions of nonlinear elasticity, precisely

- frame indifference: $W(RF) = W(F)$ for all $R \in SO(3)$,
- stress-free reference configuration: $W(Id) = 0$.

Moreover, one would typically complement these assumptions by a coercivity assumption of the form $W(F) \geq \operatorname{dist}(F, SO(3))^2$.
In both theories, the singularity of the elastic strain (1.1) leads to some inconsistency with this energetic description: let us consider a single straight dislocation line in the x_3-direction with a given Burgers vector $b \in \mathbb{R}^3$ and an associated elastic strain satisfying

$$\operatorname{curl} \beta_{el} = b \otimes e_3 \, \mathcal{H}^1_{|\mathbb{R}\, e_3}.$$

Consider the following cylinder around the dislocation line, see Figure 1.8:

$$B_{R,r,h} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : r^2 \leq x_1^2 + x_2^2 \leq R^2,\, 0 \leq x_3 \leq h \right\}.$$

We show that the energy diverges on these cylinders as $r \to 0$. First, note that by a version of Korn's inequality (see for example [23, Lemma 5.9]) there exists a constant skew-symmetric matrix W such that

$$\int_{B_{R,r,h}} |(\beta_{el})_{sym}|^2 \, dx \geq k \int_{B_{R,r,h}} |\beta_{el} - W|^2 \, dx.$$

In general, the constant k depends on R, r, h but it may be chosen uniformly for R, h fixed and $r \to 0$.

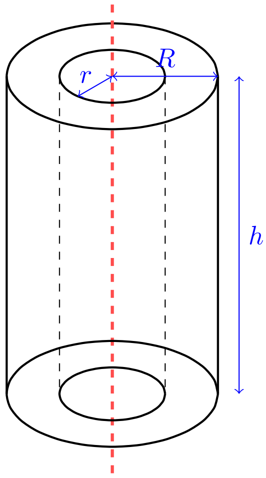


Figure 1.8: The elastic energy on cylinders with height h, outer radius R and inner radius r around a straight dislocation line in vertical direction (red) diverges as the inner radius tends to 0.

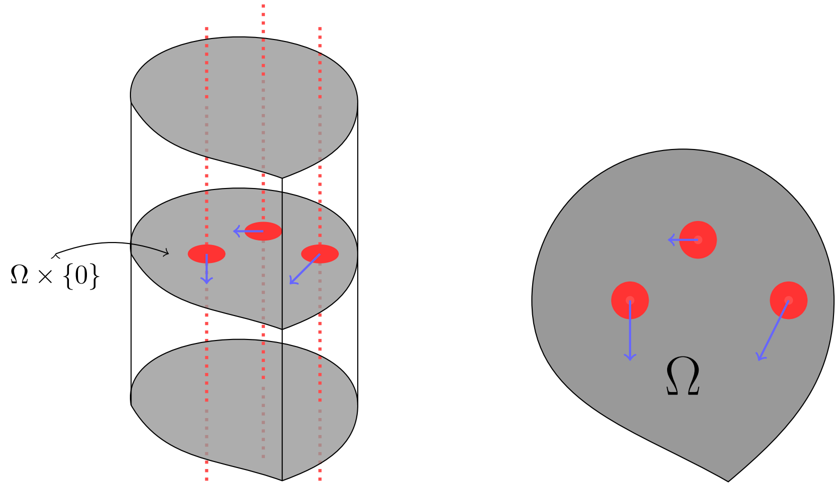
This leads to

$$\begin{aligned}\int_{B_{R,r,h}} W(\beta_{el})\,dx &\geq k\int_{B_{R,r,h}} |\beta_{el}-W|^2\,dx\\ &= k\int_0^h\int_r^R\int_{\{x_1^2+x_2^2=t^2,x_3=s\}} |\beta_{el}-W|^2\,d\mathcal{H}^1\,dt\,ds\\ &\geq k\int_0^h\int_r^R \frac{1}{2\pi t}\left|\int_{\{x_1^2+x_2^2=t^2,x_3=s\}}(\beta_{el}-W)\cdot\begin{pmatrix}x_2\\-x_1\\0\end{pmatrix}\,d\mathcal{H}^1\right|^2\,dt\,ds\\ &= k\int_0^h\int_r^R\frac{|b|^2}{2\pi t}\,dt\,ds\\ &= k\frac{|b|^2}{2\pi}h\,\log\left(\frac{R}{r}\right).\end{aligned}\tag{1.2}$$

In particular, one sees that the energy blows up logarithmically for R and h fixed whereas $r\to 0$. There are different ways of treating this modeling inconsistency. Typically, in these models continuous quantities such as the elastic strain coexist with length scales coming from the discrete picture, e.g. the lattice spacing, which determines the set of admissible Burgers vectors.

In equation (1.1), one could use more regular versions of the dislocation density to gain integrability of β_{el}. Also, a different growth of W could be assumed (at least in the nonlinear case), cf. [71]. In Chapters 3 and 4, we consider a nonlinear energy density with subquadratic growth for large strains. In the *core-radius approach*, one computes the elastic energy on a reduced domain which is obtained by cutting out a neighborhood of the size of the lattice spacing of the support of the dislocation density (the so-called *core*), cf. [7, 47]. This approach is justified by the fact that there can only be finitely many atoms in the cores which should not induce such a high amount of energy. A mathematically rigorous result in the context of screw dislocations can be found in [68]. In Chapter 5 we discuss this approach in the context of a linearized elastic energy.

Another approach would be to consider the slip $[v]$ as the main variable and let it transition between two admissible values at a scale of order of the lattice spacing. These phase-field models were inspired by the classical works by Peierls [65] and Nabarro [61]. For a modern version of this model for dislo-



(a) Sketch of the cylindrical set $\Omega \times (-\infty; \infty)$ and the straight, parallel dislocation lines (red). The core regions of the lines (red) are only drawn in the plane $\{x_3 = 0\}$, the corresponding Burgers vectors are drawn in blue.

(b) Sketch of Ω. The dislocation cores around the intersection with the dislocation lines are drawn in red, the Burgers vectors (which are all in the same plane as Ω) are drawn in blue.

Figure 1.9: Sketch of the geometry in the case of straight, parallel dislocation lines of edge type.

cations, we refer to [54] and references therein.

Next, let us explain how the specific situation of straight, parallel dislocation lines of edge type in a crystal with an infinite cylindrical structure $\Omega \times \mathbb{R}$ can be understood in a reduced two-dimensional model. This model will be the starting point of our analysis. Let us consider vertical dislocation lines and fix the points $x_i \in \Omega$ where the lines intersect the x_1-x_2-plane. We may identify the points x_i with their canonical versions in $\mathbb{R}^3$ if needed. For a sketch of the situation, see Figure 1.9b. Then the dislocation density (recall (1.1)) takes the form

$$\mu = \sum_i b_i \otimes e_3 \, \mathcal{H}^1_{|x_i + \mathbb{R}e_3}.$$

As we consider dislocations of edge type, the Burgers vectors b_i are perpendicular to e_3 and are therefore of the form $b_i = (b_1^i, b_2^i, 0)^T$. This leads to the representation

$$\mu = \sum_i \begin{pmatrix} 0 & 0 & b_1^i \\ 0 & 0 & b_2^i \\ 0 & 0 & 0 \end{pmatrix} \delta_{x_i} \otimes \mathcal{L}^1,$$

where the measure has to be understood as a product measure on $\mathbb{R}^2 \times \mathbb{R}$. By the cylindrical symmetry, we make the ansatz for the deformation $\varphi(x_1, x_2, x_3) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2), x_3)^T$, respectively the displacement has the form $u(x_1, x_2, x_3) = (u_1(x_1, x_2), u_2(x_1, x_2), 0)^T$. For the corresponding elastic strain, it holds that $(\beta_{el})_{ij} = \delta_{33}$ and $(\beta_{el})_{ij} = 0$ for all terms involving at least one index equal to 3. For the other terms, we deduce from (1.1) that

$$\left(\operatorname{curl} \begin{pmatrix} (\beta_{el})_{11} & (\beta_{el})_{12} \\ (\beta_{el})_{21} & (\beta_{el})_{22} \end{pmatrix} \right)_{|\Omega \times \{0\}} = \sum_i \begin{pmatrix} b_1^i \\ b_2^i \end{pmatrix} \delta_{x_i}. \tag{1.3}$$

Consequently, in this situation there is no real dependence on the x_3-coordinate. Hence, it is enough to understand the elastic energy on the two-dimensional slice Ω. In the theory of linear elasticity, which is formulated in terms of the displacement, this leads to the energy

$$\int_\Omega W\left(\begin{pmatrix}(\beta_{el})_{11} & (\beta_{el})_{12} & 0\\ (\beta_{el})_{21} & (\beta_{el})_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\right) dx$$

subject to the constraint (1.3). Here, W is given as the quadratic form of an elasticity tensor as explained before. If the theory is set in the context of nonlinear elasticity, the integral differs only by a one in the lower right entry of the matrix. Moreover, W would be a rotationally invariant energy density. One can check easily that the assumptions of elasticity in the nonlinear or linearized setting for W can be transferred to the corresponding statements in two dimensions for the associated two-dimensional energy densities given by

$$\tilde{W}\left(\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}\right) = W\left(\begin{pmatrix}F_{11} & F_{12} & 0\\ F_{21} & F_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\right), \quad \tilde{W}\left(\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}\right) = W\left(\begin{pmatrix}F_{11} & F_{12} & 0\\ F_{21} & F_{22} & 0\\ 0 & 0 & 1\end{pmatrix}\right).$$

Summarized, we obtain a stored elastic energy of the form

$$\int_\Omega W(\beta)\, dx \quad \text{for } \beta : \Omega \to \mathbb{R}^{2\times 2} \text{ subject to } \operatorname{curl}\beta = \sum_i b_i\, \delta_{x_i}, \tag{1.4}$$

where the $b_i \in \mathbb{R}^2$ are (projected) admissible Burgers vectors and W satisfies the classical assumptions of elasticity (linear or nonlinear) in two dimensions as discussed for three dimensions before.
Also, this two-dimensional energetic description features the same inconsistency of a logarithmically diverging energy close to the singularities induced by the curl-condition; the computation is very similar to (1.2).
In this thesis, we discuss two models: a rotationally invariant energy with mixed growth and a core-radius approach in the setting of linearized elasticity, which corresponds in the two-dimensional setting to eliminating balls of the size of the lattice spacing around the points x_i, see Figure 1.9b. In both settings, we identify the Γ-limit of the suitably rescaled stored energy.

1.3 Heuristics for the Scaling of the Stored Energy

Starting from the two-dimensional model for straight, parallel edge dislocations in (1.4), in this chapter we discuss the scaling of the stored energy.
A computation similar to the one in (1.2) shows in the case of a linearized elastic energy that for a dislocation density of the form $\mu = \sum_{i=1}^M b_i\, \delta_{x_i}$ such that the x_i are separated by a distance of at least $2\varepsilon^\gamma$ for some $0 \leq \gamma < 1$ and an associated elastic strain β satisfying $\operatorname{curl}\beta = \mu$ we find that

$$\sum_{i=1}^M \int_{B_{\varepsilon^\gamma}(x_i)\setminus B_\varepsilon(x_i)} W(\beta)\, dx \geq c \sum_{i=1}^M |b_i|^2\, (1-\gamma)|\log\varepsilon|. \tag{1.5}$$

For a lattice spacing ε, the Burgers vectors are typically of size ε. Hence, the estimate (1.5) leads to the conjecture that the stored energy close to the dislocations scales as $\#\{\text{dislocations}\}\, \varepsilon^2 |\log\varepsilon|$.

Furthermore, notice that the lower bound on the right hand side of (1.5) depends only on the dislocation density. It shows that each dislocation induces a minimal amount of energy depending on its Burgers vector. The full self-energy of each dislocation is distributed in an area of order 1 around the dislocation. However, a fraction of $(1-\gamma)$ of the full self-energy can already be found in a region of radius ε^γ around each dislocation. Hence, most of the self-energy is concentrated in a region that shrinks to the dislocation point as the lattice spacing ε tends to 0. Consequently, depending on the rescaling of the energy, we should expect to find a relict from the self-energy close to the dislocations in the limit. On the other hand, the limit should also capture the elastic energy far from the dislocations.

A more detailed discussion of the heuristics for the scaling, which involves also the interaction of dislocations, can be found in [38] and [60]. It leads to the same result i.e., the expected scaling for N_ε-many dislocations is $N_\varepsilon \varepsilon^2 |\log \varepsilon|$.

On the other hand, consider a dislocation density μ_ε and an associated strain β_ε for the lattice spacing $\varepsilon > 0$ such that the stored energy is of order $N_\varepsilon \varepsilon^2 |\log \varepsilon|$. Estimate (1.5) shows that the dislocation energy μ_ε should be of order $\varepsilon N_\varepsilon$. If we assume that W has quadratic growth, the naive conjecture is that β_ε is of order $\varepsilon \sqrt{N_\varepsilon |\log \varepsilon|}$. One sees that the dislocation density and the associated strain are of the same order if and only if $N_\varepsilon \sim |\log \varepsilon|$. This is the so-called critical regime. The sub-critical and super-critical regime are the regimes corresponding to $N_\varepsilon \ll |\log \varepsilon|$, respectively $N_\varepsilon \gg |\log \varepsilon|$, in which one of the quantities is expected to be much greater than the other.

1.4 Recent Mathematical Contributions to Dislocation Theory

In the past years, there has been extensive research in the mathematical community to understand crystal plasticity at different scales and from different points of view. In [6], Ariza and Ortiz develop a model with fully discrete dislocations. The basis of this model are discrete eigenstrains and ideas from algebraic topology. In [56], Luckhaus and Mugnai present a different fully discrete model for dislocations which is completely set up in the actual configuration and does not need to refer to a global reference configuration. In the context of screw dislocations and antiplane plasticity, Ponsiglione showed in [68] the Γ-convergence of a discrete model to a continuum model (after suitable rescaling). A relation between discrete screw dislocation models, models for spin system, and the Ginzburg-Landau model in two dimensions is discussed in [2]. Building upon this result in [3], Alicandro et al. treat the dynamics of screw dislocations and show the convergence of the time-discrete minimizing movement with respect to a quadratic isotropic dissipation to a gradient flow of the renormalized energy. Choosing a crystalline dissipation that accounts for the specific lattice structure and that is minimal exactly on the preferred slip directions leads to a dynamical model that predicts motion in preferred slip directions, [4].

Another option is to start from continuum (or semi-discrete) models as discussed in Section 1.2. A phase-field model for dislocations based on [54] and inspired by the classical works of Peierls and Nabarro is considered in [39, 40]. In these papers, Müller and Garroni study the Γ-limit of a model for the slip on a single slip plane, on which one slip system is active, subject to pinning conditions in certain areas (e.g. inclusion of a material that restrains slip). The elastic energy induced by a certain slip leads to a nonlocal term involving a singular kernel, which behaves like the $H^{\frac{1}{2}}$-norm of the slip. Depending on the number of obstacles, there exist three different scaling regimes. The most interesting regime is the one in which the number of obstacles scales like $\varepsilon^{-1}|\log \varepsilon|$. Here, the energy converges to a line tension limit i.e., the limit energy involves an energy defined on the dislocation lines

possibly depending on the orientation of the line and the Burgers vector. In [16] and [21], the authors treat the situation with multiple active slip systems on the slip plane without a pinning condition. The logarithmically rescaled energy (to compensate the usual logarithmic convergence) Γ-converges again to a line tension limit as the lattice spacing tends to zero. A rescaling by $|\log\varepsilon|^2$ leads to a strain-gradient model in the limit, [22]. The case of several slip planes and a logarithmic rescaling is considered in [42] by Gladbach. If the planes are well-separated, one recovers essentially the same behavior as for a single plane. On the other hand, if the planes have a distance of order ε^γ for some $1 > \gamma > 0$, the dislocation lines interact, and microstructures at different scales may result in a lower limit energy. Moreover, the author considers also the case of anisotropic elasticity. For a discussion of the results, see also [24].
Recently, a first fully three-dimensional result was established in the setting described at the beginning of Section 1.2 by Conti, Garroni, and Ortiz in [23]. The authors derive a line tension limit from a dislocation model in the setting of linearized elasticity as the lattice spacing tends to zero under some diluteness condition on the dislocation lines. The authors show that a core-radius approach and a regularization of the dislocation densities lead to the same result. Within this framework, it is useful to interpret the dislocation lines as tensor-valued 1-currents. Compactness and lower-semicontinuity of energies defined on 1-currents have been discussed by Conti, Garroni, and Massaccesi in [20].
Many other results are restricted to the situation of plane plasticity as described at the end of Section 1.2, which is also the starting point of the analysis in this thesis. A first result in this setting with a core-radius approach was established in [17]. For a fixed finite number of dislocation positions, Cermelli and Leoni derive an asymptotic expansion of the energy as the lattice spacing goes to zero in the setting of isotropic, linearized elasticity. The term with leading order $|\log\varepsilon|$ is the self-energy of the dislocations whereas the lower order term is considered to be the counterpart to the renormalized energy of vortices in the Ginzburg-Landau model; for a deeper insight to the theory for Ginzburg-Landau vortices, we refer to [9]. DeLuca, Garroni and Ponsiglione derived a line tension limit as the Γ-limit in the setting of linearized elasticity in the subcritical regime without assumptions on the positions of the dislocations, [30]. In order to compute sharp lower bounds, they adapt ball-construction techniques as known from [51, 70] to identify clusters of dislocations which contribute jointly to the energy on certain scales. Under the assumption of well-separateness of the dislocations, this result was generalized by Scardia and Zeppieri in [71] to a nonlinear situation. The authors consider a core-radius approach for a quadratic energy density and a regularization by an energy density with subquadratic growth for large strains. Both approaches lead essentially to the same line tension limit as already found in [30].
In the critical scaling regime (the number of dislocations is of order $|\log\varepsilon|$), Garroni, Leoni, and Ponsiglione derive a strain-gradient plasticity model under the assumption of well-separateness of dislocations, [38]. The counterpart for a quadratic, rotationally invariant energy density and a core-radius regularization was established in [59] and [60] by Müller, Scardia, and Zeppieri.
In elasticity theory, the main tool to obtain compactness is Korn's inequality [36, 52, 53], respectively a geometric rigidity estimate [37], see also [19, 58] for variants with mixed growth. These estimates are valid for gradients. However, the presence of dislocations leads to strains with non-vanishing curl. In the case of a finite number of dislocations, the classical results can still be used to prove good estimates. The transition to a growing number of dislocations is non-trivial. For this reason, in [38, 59, 60] corresponding estimates for fields with non-vanishing curl are developed. A central role in the proofs plays a very fine estimate of the H^{-1}-norm for L^1-vector-fields whose divergence is in H^{-2} in two dimensions, [14] (see also [11, 12]). Related results can be found in [57, 74, 75].

In the following section we present the main results of this thesis.

1.5 Main Results

As already discussed in Section 1.2—see in particular (1.4)—, in this thesis we will focus on a dislocation model for straight, parallel edge dislocations which is formulated in the orthogonal plane. We are interested in the behavior of the stored elastic energy as the lattice spacing ε goes to zero.
First, we consider a nonlinear energy density W with mixed growth to regularize the energy as proposed in [71] i.e., $W \sim \min\{\operatorname{dist}(\cdot, SO(2))^2, \operatorname{dist}(\cdot, SO(2))^p\}$ for $p < 2$. The renormalized stored energy is given by

$$E_\varepsilon(\mu,\beta) = \begin{cases} \frac{1}{\varepsilon^2 N_\varepsilon |\log\varepsilon|} \int_\Omega W(\beta)\,dx & \text{if } \beta \in L^p(\Omega;\mathbb{R}^{2\times2}), \mu = \operatorname{curl}\beta = \sum_i \varepsilon\xi_i\delta_{x_i}, \xi_i \in \mathbb{S}, \\ +\infty & \text{else in } \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times2}), \end{cases} \tag{1.6}$$

where $\mathbb{S}$ is the set of (renormalized) admissible Burgers vectors depending on the crystalline structure. Under the assumption of well-separateness of dislocations, we identify all scaling regimes of the stored energy depending on the number of dislocations N_ε and show Γ-convergence of the energy E_ε. The three different regimes are the subcritical regime $N_\varepsilon \ll |\log\varepsilon|$, the critical regime $N_\varepsilon \sim |\log\varepsilon|$, and the supercritical regime $N_\varepsilon \gg |\log\varepsilon|$. The corresponding limits are given by

- **The subcritical regime:** $0 \ll N_\varepsilon \ll |\log\varepsilon|$**:**

$$E^{sub}(\mu,\beta,R) = \begin{cases} \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx + \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \mu \in \mathcal{M}(\Omega;\mathbb{R}^2), \\ & \operatorname{curl}\beta = 0, R \in SO(2) \\ +\infty & \text{otherwise .} \end{cases}$$

- **The critical regime:** $N_\varepsilon \sim |\log\varepsilon|$**:**

$$E^{crit}(\mu,\beta,R) = \begin{cases} \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx + \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \mu \in H^{-1}(\Omega;\mathbb{R}^2) \cap \mathcal{M}(\Omega;\mathbb{R}^2), \\ & \operatorname{curl}\beta = R^T\mu, R \in SO(2) \\ +\infty & \text{otherwise .} \end{cases}$$

- **The supercritical regime:** $N_\varepsilon \gg |\log\varepsilon|$**:**

$$E^{sup}(\beta) = \begin{cases} \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx & \text{if } \beta_{sym} = \frac{1}{2}(\beta^T + \beta) \in L^2(\Omega, \mathbb{R}^{2\times2}), \\ +\infty & \text{otherwise .} \end{cases}$$

Here, $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(Id)$ and the function φ is given by a cell-formula and a relaxation procedure. The term involving $\mathcal{C}$ measures the stored linearized elastic energy whereas the term involving φ accounts for the self-energy of concentrated dislocations. The rotation R reflects the fact that we derive this linearized model from a nonlinear, rotationally invariant model.
In particular, the critical regime is of interest. Here, the scaling of the strains and the dislocation densities is of the same order. We derive a strain-gradient plasticity model as the Γ-limit. Unlike most macroscopic plasticity models, strain-gradient models are not scale independent but they add

a certain length scale to the problem in order to capture certain size effects. For general insight to strain-gradient plasticity models we refer, for example, to [8,33,34,46,62] and references therein. Note that φ is 1-homogeneous as also proposed in other strain-gradient models, e.g. in [25] the authors choose $\varphi = |\cdot|$. In addition, the limit turns out to be essentially the same as the one derived from a core-radius approach in a linearized, respectively non-linear, setting in [38, 59]. Hence, this thesis complements these results and justifies a-posteriori the usage of an ad-hoc cut-off radius in [38, 59]. Moreover, we prove compactness in the subcritical and critical regime. In the supercritical regime, we construct a counterexample to compactness.

The main tool of our compactness statement is a generalized version of a geometric rigidity estimate with mixed growth for fields with non-vanishing curl. Precisely, we prove that for $p < 2$ and a simply connected set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, there exists a constant $C > 0$ such that for all $\beta \in L^p(\Omega; \mathbb{R}^{2\times 2})$ satisfying that $\mu = \operatorname{curl} \beta$ is a measure there exists a rotation $R \in SO(2)$ such that

$$\int_\Omega \min\{|\beta - R|^2, |\beta - R|^p\}\, dx \le C \left(\int_\Omega \min\{\operatorname{dist}(\beta, SO(2))^2, \operatorname{dist}(\beta, SO(2))^p\} dx + |\mu|(\Omega)^2 \right). \quad (1.7)$$

In the proof, the central point is to derive good estimates for $\operatorname{curl} \beta$ in the space $H^{-1} + W^{-1,p}$. This can be done by a generalization of a fine regularity estimate due to Bourgain, Brézis and, van Schaftingen, [12, 14]. We prove that for an open, bounded set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, $p < 2$, and a vector-valued function $f \in L^1(\Omega; \mathbb{R}^2)$ such that $\operatorname{div} f = a + b \in H^{-2} + W^{-2,p}$, there exist $A \in H^{-1}$ and $B \in W^{-1,p}$ such that $f = A + B$ and

(i) $\|A\|_{H^{-1}} \le C(\|f\|_{L^1} + \|a\|_{H^{-2}})$,

(ii) $\|B\|_{W^{-1,p}} \le C\|b\|_{W^{-2,p}}$.

Second, we consider a core-radius approach which is set in the context of straight, parallel edge dislocations and linearized elasticity with elasticity tensor $\mathcal{C}$. We focus on the critical rescaling by $|\log \varepsilon|^2$. The main difference to existing results (in particular [30]) is that we do *not* assume well-separateness of the dislocations. We prove that the Γ-limit is finite for $\beta \in L^2(\Omega; \mathbb{R}^{2\times 2})$ and $\mu = \operatorname{curl} \beta \in \mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2)$. There, it is given by

$$\int_\Omega \mathcal{C}\beta : \beta \, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|,$$

where again φ is given by a cell-formula and a relaxation procedure.

In order to obtain adequate lower bounds, we adjust a technique known in the theory of the Ginzburg-Landau model as ball-construction technique, see e.g. [51, 70]. Versions of the ball construction technique have also been applied successfully to dislocation problems in the subcritical scaling regime, [30, 68]. The building block for estimates using the ball construction are good lower bounds on annuli. In elasticity theory, there is a massive loss of rigidity on thin annuli which becomes manifest in inadequate lower bounds. Hence, the focus of our analysis is to find thick annuli which carry already most of the energy. Using the established lower bounds, we show compactness and discuss optimality of these results.

This thesis is ordered as follows. In the next section, we introduce notation. Chapter 2 is devoted to prove the generalization of the Bourgain-Brézis type estimate discussed above. In Chapter 3, we use the Bourgain-Brézis type estimate to prove the generalized rigidity estimate for fields with non-vanishing curl in the context of a nonlinear energy density with mixed growth, see (1.7). Armed with the generalized rigidity estimate, we discuss the behavior of the energy E_ε as defined in (1.6). We prove Γ-convergence and compactness in the critical and subcritical regime in Section 4.3 and Section 4.4. In the supercritical regime (Section 4.5), we prove Γ-convergence for E_ε and discuss the non-existence of a compactness result. Finally, in Chapter 5 we discuss a core-radius approach without the assumption of well-separateness of dislocations in the critical scaling regime.

1.6 Notation

In this thesis, we use standard notation for the space $\mathbb{R}^n$. The euclidean norm is denoted by $|\cdot|$. For two scalar values $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices. The identity matrix is denoted by Id. For a given matrix $M \in \mathbb{R}^{n\times n}$, we write M^T for the transposed matrix. Moreover, we use the classical notation $M_{sym} = \frac{1}{2}(M + M^T)$ for the symmetric part of M and $M_{skew} = \frac{1}{2}(M - M^T)$ for the skew-symmetric part of M. The subsets $Sym(n)$, $Skew(n)$, $SO(n)$ of $\mathbb{R}^{n\times n}$ denote the space of symmetric, respectively skew-symmetric matrices, and the set of rotations. For two given vectors $a, b \in \mathbb{R}^n$, we write $a \otimes b \in \mathbb{R}^{n\times n}$ for the rank-one matrix whose (i, j)-th entry is given by $a_i b_j$. In addition, for a matrix-valued function the operators div and curl are always understood to act row-wise.
The n-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ is denoted by $\mathcal{L}^n(A)$ or sometimes just by $|A|$. For the k-dimensional Hausdorff measure we write $\mathcal{H}^k$. More generally, for an open set $\Omega \subset \mathbb{R}^n$ we use the standard notation $\mathcal{M}(\Omega; \mathbb{R}^m)$ for the space of (vector-valued) Radon measures. For a Radon measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$, the quantity $|\mu|$ denotes the associated total variation measure. For a μ-measurable set A, by $\mu_{|A}$ we mean the restriction of the measure μ to the set A, defined by $\mu_{|A}(B) = \mu(A \cap B)$ for any μ-measurable set B. The weak star convergence of a sequence of Radon measures μ_k to μ is indicated by $\mu_k \overset{*}{\rightharpoonup} \mu$. For a general introduction to measure theory, we refer to [32].
Moreover, we use standard notations for Lebesgue spaces. The weak-L^p spaces are denoted by $L^{p,\infty}$ and equipped with the quasi-norm $\|f\|_{L^{p,\infty}} = \inf\{C > 0 : \lambda \mathcal{L}^n(\{|f| > \lambda\})^{\frac{1}{p}} \leq C \text{ for all } \lambda > 0\}$. The notation for Sobolev spaces of order $k \in \mathbb{N}$ on an open set Ω is $W^{k,p}(\Omega; \mathbb{R}^m)$ for $1 \leq p \leq \infty$; in the special case $p = 2$, we write also $H^k(\Omega; \mathbb{R}^m)$. For an open, bounded set Ω with Lipschitz boundary, the space $W_0^{k,p}(\Omega; \mathbb{R}^m)$ denotes all functions in $W^{k,p}(\Omega; \mathbb{R}^m)$ whose derivatives up to order $k - 1$ vanish on the boundary in the sense of traces. The homogeneous norm in $W_0^{k,p}(\Omega; \mathbb{R}^m)$ is given by $\|f\|_{W_0^{k,p}(\Omega;\mathbb{R}^m)} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p}$. On bounded sets Ω, this norm is equivalent to the classical Sobolev norm in $W_0^{k,p}(\Omega; \mathbb{R}^m)$. The topological dual space of $W_0^{k,p}(\Omega; \mathbb{R}^m)$ is denoted by $W^{-k,p'}(\Omega; \mathbb{R}^m)$ where p' is determined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$; in the special case $p = 2$ we write $H^{-k}(\Omega; \mathbb{R}^m)$. For a general introduction to Sobolev spaces, see [1].
For a general pair of a space X and its dual X', we write $< \cdot, \cdot >_{X',X}$ for the dual pairing.
Furthermore, we use the usual notation for Γ-convergence, cf. [13, 27].
Finally, we make use of the convention that if not explicitly stated differently, C denotes a positive constant which may change in a chain of inequalities from line to line.

2 A Bourgain-Brézis type Estimate

This chapter is devoted to prove the following statement which will be needed in the proof of the generalized rigidity estimate for an energy density with mixed growth in chapter 3.

Theorem 2.0.1. *Let $1 < p < 2$ and $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz-boundary. Then there exists a constant $C > 0$ such that for all $f \in L^1(\Omega;\mathbb{R}^2)$ satisfying $\operatorname{div} f = a + b \in H^{-2}(\Omega) + W^{-2,p}(\Omega)$ there exist $A \in H^{-1}(\Omega;\mathbb{R}^2)$ and $B \in W^{-1,p}(\Omega;\mathbb{R}^2)$ such that $f = A + B$,*

$$\|A\|_{H^{-1}} \leq C(\|f\|_{L^1} + \|a\|_{H^{-2}}), \text{ and } \|B\|_{W^{-1,p}} \leq C\|b\|_{W^{-2,p}}.$$

This is a generalization of a statement which has been proved by Bourgain, Brézis, and van Schaftingen, see [14, Lemma 3.3 and Remark 3.3] and [11, 12]. Their statement is used in the proofs of the generalized Korn inequality in [38] and the generalized rigidity estimate in [59]. It states the following:

Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz-boundary. Then there exists a constant $C > 0$ such that for all $f \in L^1(\Omega;\mathbb{R}^2)$ it holds

$$\|f\|_{H^{-1}} \leq C(\|f\|_{L^1} + \|\operatorname{div} f\|_{H^{-2}}).$$

Let us shortly remark the following: the exponents for the Sobolev embedding H^1_0 to L^∞ are critical in two dimensions. The embedding does not hold. If it held, by duality, there would be a bounded embedding $L^1 \to H^{-1}$ which is also not true in general. The statement above gives a positive answer to the question which L^1-functions are elements of H^{-1}.
The general statement by Bourgain, Brézis, and van Schaftingen is also valid in higher dimensions where one has to replace the Sobolev spaces with L^2-integrability by those with L^n-integrability. However, we restrict ourselves to the two-dimensional case.

The proof of Theorem 2.0.1 consists of different steps. The first step is to prove a primal statement from which the result can be derived via dualization. Precisely, we show first (Theorem 2.4.4) that for $2 < q < \infty$ and a function $f \in H^1_0(\Omega;\mathbb{R}^2) \cap W^{1,q}(\Omega;\mathbb{R}^2)$ there exists a decomposition $f = g + \nabla h$ such that

$$\begin{aligned} \|g\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|g\|_{H^1_0(\Omega;\mathbb{R}^2)} + \|h\|_{H^2_0(\Omega)} &\leq C\|f\|_{H^1_0(\Omega;\mathbb{R}^2)}, \\ \|g\|_{W^{1,q}_0(\Omega;\mathbb{R}^2)} + \|h\|_{W^{2,q}_0(\Omega)} &\leq C\|f\|_{W^{1,q}_0(\Omega;\mathbb{R}^2)}. \end{aligned}$$

This reduces to find, for a given function f, a good solution to $\operatorname{curl} g = \operatorname{curl} f$. In two dimensions, the curl-operator differs from the div-operator only by a rotation by 90 degrees. For the sake of a simpler notation, we formulate and prove the results for the div-operator. We show the existence of good solutions to $\operatorname{div} Y = f$ first on the torus (Theorem 2.0.1) and use localization and covering arguments to transport the result for the torus to general Lipschitz domains, Theorem 2.4.1. Then, we establish the main result of this chapter by dualization and scaling in section 2.4.

In the next section, we discuss first some preliminaries which are useful in the proof of the statement on the torus. More precisely, we discuss convolution estimates for special kernels, estimates for very particular Fourier multipliers, and give a brief overview over Littlewood-Paley theory for the torus.

2.1 Preliminaries

The objective of this section is to establish tools from Harmonic Analysis which will turn out to be useful in the proof of the primal Bourgain-Brézis type estimate on the torus. For a general introduction to Harmonic Analysis, we refer to [43, 72].

First, let us introduce some notation. By Π^n we denote the n-dimensional torus which can be identified with $[-\pi, \pi]^d$ together with the measure $\frac{1}{(2\pi)^d}\mathcal{L}^d$. For a function $f \in L^1(\Pi^d)$ and $n \in \mathbb{Z}^d$, we write $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{in\cdot x}\,dx$ for its n-th Fourier coefficient.

For $n \in \mathbb{N}$, the n-th Fejér kernel on the one-dimensional torus $\Pi^1 \simeq [-\pi, \pi]$ is defined as

$$K_n(x) = \sum_{|k|<n} \frac{n-|k|}{n} e^{ikx} = \frac{1}{n}\frac{1-\cos(nx)}{1-\cos(x)} \geq 0,$$

see Figure 2.1. On Π^2 we write $K_n \otimes K_n$ for the kernel given by $K_n \otimes K_n(x, y) = K_n(x)K_n(y)$.

The main property of the Fejér kernel is that K_n is a nonnegative kernel that is localized in Fourier space. Moreover, it holds for any trigonometric polynomial $P = \sum_{|k|<n} a_k\, e^{inx}$ of degree less than n that $P * ((1 + e^{inx} + e^{-inx})K_n) = P$ where the convolution is meant as a convolution on Π^1. In particular, it follows that $|P| \leq 3(|P| * K_n)$ as K_n is nonnegative.

As a first tool for the proof of the Bourgain-Brézis type estimate we show a convolution estimate for the Fejér kernels. First, we prove the existence of symmetrically decreasing majorants for the Fejér kernels with uniformly bounded integrals. This property is useful to bound convolutions with the Fejér kernels in terms of maximal functions which in turn leads to good L^p-estimates.

Lemma 2.1.1. *There exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ there exists a symmetrically decreasing function $G_n : [-\pi, \pi] \to \mathbb{R}$ such that $0 \leq K_n(x) \leq G_n(x)$ and $\int_{-\pi}^{\pi} G_n(x)\,dx \leq C$.*

Proof. Fix $n \in \mathbb{N}$. We construct a majorant function for K_n which is constant on intervals of the type $\left[\frac{k\pi}{n}, \frac{(k+1)\pi}{n}\right]$ where $-n \leq k \leq n-1$. By Taylor's theorem, there exists a constant $c > 0$ such that $1 - \cos(x) \geq cx^2$ for all $|x| \leq \pi$. This inequality implies for $x \geq \frac{k\pi}{n}$, where $1 \leq k \leq n-1$, that

$$K_n(x) = \frac{1}{n}\frac{1-\cos(nx)}{1-\cos(x)} \leq 2\frac{n}{ck^2\pi^2}.$$

Moreover, one can check that it holds $K_n \leq n$. Let us define the function G_n by

$$G_n(x) = \max\left\{n, \frac{2n}{c\pi^2}\right\}\mathbf{1}_{[-\frac{\pi}{n},\frac{\pi}{n}]} + \sum_{k=1}^{n-1} 2\frac{n}{c\pi^2k^2}\mathbf{1}_{[\frac{k\pi}{n},\frac{(k+1)\pi}{n}]\cup[-\frac{(k+1)\pi}{n},\frac{-k\pi}{n}]}.$$

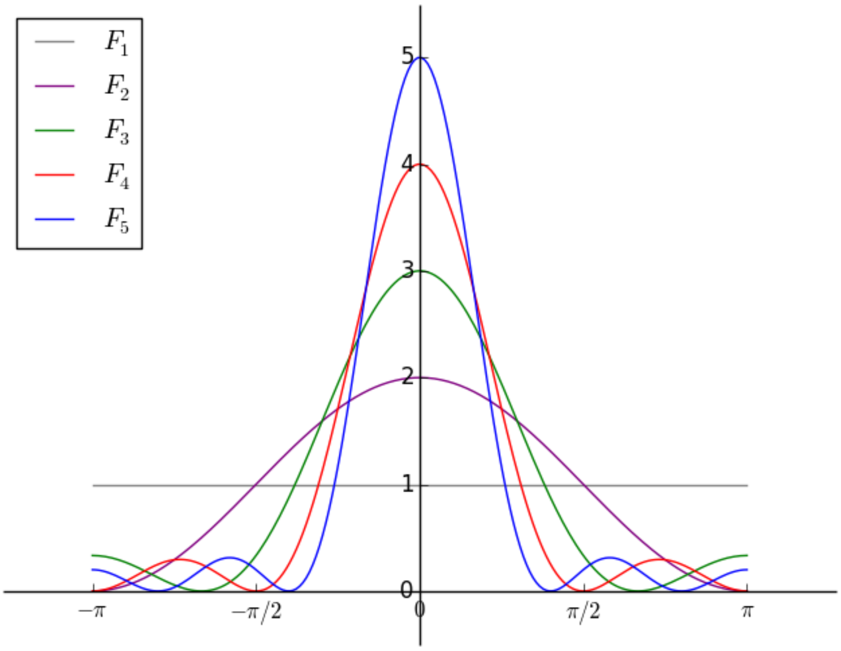


Figure 2.1: The Féjer kernel for $n = 1, \ldots, 5$. Note that the zeros of the n-th Fejér kernel are at $\frac{2k\pi}{n}$.

Then one obtains $F_n \leq G_n$. Moreover, G_n is symmetrically decreasing. In addition, we see that

$$\int_{-\pi}^{\pi} G_n(x)\,dx = 2\pi \max\left\{1, \frac{2}{c\pi^2}\right\} + \sum_{k=1}^{n-1} \frac{4}{ck^2\pi} \leq 2\pi \max\left\{1, \frac{2}{c\pi^2}\right\} + \sum_{k=1}^{\infty} \frac{4}{ck^2\pi} < \infty,$$

where the right hand side does not depend on n. □

Armed with these majorants we are able to state and prove the following estimate involving convolutions with the Fejér kernel.

Proposition 2.1.2. *Let* $1 < q < \infty$. *Then there exists a constant* $C > 0$ *such that for every family* $(F_j)_j$ *of* $L^q(\Pi^2)$*-functions and* $\tilde{K}_j = K_j \otimes K_j$ *it holds*

$$\left\| \left(\sum_j |F_j * \tilde{K}_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q}^q \leq C \left\| \left(\sum_j |F_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q}^q.$$

Proof. Let j be arbitrary. Let G_j be the majorant from the Lemma 2.1.1. As the functions G_j are symmetrically decreasing, it follows for all $c \in \mathbb{R}$ that the set $\{x \in [-\pi, \pi] : G_j(x) \geq c\}$ is a centered interval around zero. In the following computations, we identify a function on the torus with its periodic extension to $\mathbb{R}$:

$$
\begin{aligned}
4\pi^2 |F_j * \tilde{K}_j(x)| &\leq \int_{[-\pi,\pi]^2} |F_j(y)| \tilde{K}_j(x-y)\, dy \\
&\leq 4\pi^2 \int_{-\pi}^{\pi} G_j(x_1-y_1) \int_{-\pi}^{\pi} |F_j(y_1,y_2)| G_j(x_2-y_2)\, dy_2\, dy_1 \\
&= \int_{-\pi}^{\pi} G_j(x_1-y_1) \int_0^{\infty} \int_{\{G_j(x_2-\cdot)\geq t\}} |F_j(y_1,s)|\, ds\, dt\, dy_1 \\
&\leq \int_{-\pi}^{\pi} G_j(x_1-y_1) \int_0^{\infty} \mathcal{L}^1(\{G_j(x_2-\cdot)\geq t\}) \left(\sup_{0<r<\pi} \fint_{B_r(x_2)} |F_j(y_1,s)|\, ds \right) dt\, dy_1 \\
&= \|G_j\|_{L^1([-\pi,\pi])} \int_{-\pi}^{\pi} G_j(x_1-y_1) \left(\sup_{0<r<\pi} \fint_{B_r(x_2)} |F_j(y_1,s)|\, ds \right) dy_1 \\
&\leq \|G_j\|^2_{L^1([-\pi,\pi])} \sup_{0<r<\pi} \fint_{B_r(x_1)} \left(\sup_{0<r<\pi} \fint_{B_r(x_2)} |F_j(t,s)|\, ds \right) dt.
\end{aligned}
$$

In view of the last line, we define the operator

$$
T(f)(x) = \sup_{0<r<\pi} \fint_{B_r(x_1)} \left(\sup_{0<r<\pi} \fint_{B_r(x_2)} |f(y_1,y_2)|\, dy_2 \right) dy_1.
$$

This operator T corresponds to first applying the one-dimensional Hardy-Littlewood maximal operator for the torus on $f(x_1,\cdot)$ for each fixed first variable x_1 and then applying the one-dimensional Hardy-Littlewood maximal operator in the first variable to this new function. In this spirit, we write $T(f) = M_1(M_2(f))$ where M_1 and M_2 denote the maximal operators in the first, respectively second variable. The Hardy-Littlewood maximal operator satisfies the following vector-valued inequality on the 1-torus, see for example [44],

$$
\left\| \left(\sum_j |M(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Pi^1)} \leq C \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Pi^1)}.
$$

Together with Fubinii's theorem, this allows us to estimate the quantity of interest:

$$
\begin{aligned}
4^q \pi^{2q} \left\| \left(\sum_j |F_j * \tilde{K}_j|^2 \right)^{\frac{1}{2}} \right\|^q_{L^q([-\pi,\pi]^2)} &\leq C \left\| \left(\sum_j |T(F_j)|^2 \right)^{\frac{1}{2}} \right\|^q_{L^q([-\pi,\pi]^2)} \\
&= C \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\sum_j |M_1(M_2(F_j))(x_1,x_2)|^2 \right)^{\frac{q}{2}} dx_1\, dx_2 \\
&\leq C \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\sum_j |M_2(F_j)(x_1,x_2)|^2 \right)^{\frac{q}{2}} dx_1\, dx_2 \\
&\leq C \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\sum_j |F_j(x_1,x_2)|^2 \right)^{\frac{q}{2}} dx_2\, dx_1
\end{aligned}
$$

$$= C \left\| \left(\sum_j |F_j|^2 \right)^{\frac{1}{2}} \right\|^q_{L^q([-\pi,\pi]^2)} .$$

□

Next, we want to prove multiplier estimates for specific multipliers that appear in the prove of the Bourgain-Brézis type estimate.

Fourier multiplication operators are operators which are given as multiplication operators in Fourier space. More precisely, on the torus, one defines for a function $m : \mathbb{Z}^d \to \mathbb{C}$ the operator T_m by $T_m(f) = \mathcal{F}^{-1}(m\mathcal{F}(f)) = \sum_{k\in\mathbb{Z}^d} m(k)\hat{f}(k)e^{ik\cdot x}$. For a measurable function $m : \mathbb{R}^n \to \mathbb{C}$, one can define the analog for the Fourier transform on $\mathbb{R}^n$. A classical question is whether this operation defines a bounded operator from L^p to L^p. By Parseval's identity, a measurable function m defined on $\mathbb{Z}^n$ or $\mathbb{R}^n$ defines a bounded Fourier multiplication operator on the torus, respectively $\mathbb{R}^n$, from L^2 to L^2 if and only if it is measurable and bounded. Sufficient criteria for the general case $1 < p < \infty$ on $\mathbb{R}^n$ are given by the classical results by Marcinkiewicz and Hörmander-Mikhlin, see for example [43, Chapter 5]. These results also provide estimates on the operator norm of T_m. By transference results, it is possible to link multipliers on $\mathbb{R}^n$ to multipliers on the n-torus, see for example [43, Section 3.6].

We are interested in the operator norm of a very specific Fourier multiplication operator on the 1-torus. Let us consider the following subdivision of $\mathbb{Z} \setminus \{0\}$: define for $k \in \mathbb{N}$ and $\varepsilon > 0$ the following sets of length $\sim \varepsilon 2^{k-1}$:

$$\begin{aligned} I_k^r &= (2^{k-1} + r\varepsilon 2^{k-1}, 2^{k-1} + (r+1)\varepsilon 2^{k-1}] \cap \mathbb{Z}, \\ J_k^r &= [-2^{k-1} - (r+1)\varepsilon 2^k, -2^{k-1} - r\varepsilon 2^{k-1}) \cap \mathbb{Z} \text{ for } r = 0, \dots, \lfloor \varepsilon^{-1} \rfloor - 1, \\ I_k^{\lfloor \varepsilon^{-1} \rfloor} &= \left(2^{k-1} + \lfloor \varepsilon^{-1} \rfloor \varepsilon 2^{k-1}, 2^k\right] \cap \mathbb{Z}, \\ \text{and } J_k^{\lfloor \varepsilon^{-1} \rfloor} &= \left[-2^k, -2^{k-1} - \lfloor \varepsilon^{-1} \rfloor \varepsilon 2^{k-1}\right) \cap \mathbb{Z}. \end{aligned} \tag{2.1}$$

Now, we define the function $m_\varepsilon : \mathbb{Z} \to \mathbb{R}$ which will appear as a Fourier multiplier in the proof of the Bourgain-Brézis type estimate by

$$m_\varepsilon(l) = \begin{cases} \frac{l-(2^{k-1}+r\varepsilon 2^{k-1})}{l} & \text{if } l \in I_k^r, \\ \frac{l-(-2^{k-1}-r\varepsilon 2^{k-1})}{l} & \text{if } l \in J_k^r, \\ 0 & \text{if } l = 0. \end{cases} \tag{2.2}$$

For a sketch of m_ε, see Figure 2.2.

We are interested in the operator norm of the corresponding Fourier multiplication operator T_{m_ε}. In particular, we want to show that it decays faster than $\varepsilon^{\frac{1}{2}}$ as $\varepsilon \to 0$.

The application of the Marcinkiewicz multiplier theorem to the extension of m_ε by linear interpolation and classical transference results show that m_ε is a Fourier multiplier on the 1-torus. Unfortunately, this technique does not provide bounds on the operator norm which decrease as $\varepsilon \to 0$. This is mainly due to the fact that the bounds provided by the Marcinkiewicz multiplier theorem involve essentially the variation of m_ε on each interval of the form $[\pm 2^k, \pm 2^{k+1}]$. This quantity is of order 1.

Fortunately, there exists a multiplier theorem by Coifman, de Francia and Semmes, [18], which involves the so-called q-variation.

Definition 2.1.1. Let $1 \le q < \infty$. Let I be an interval and $m : I \to \mathbb{C}$. We say that m has bounded

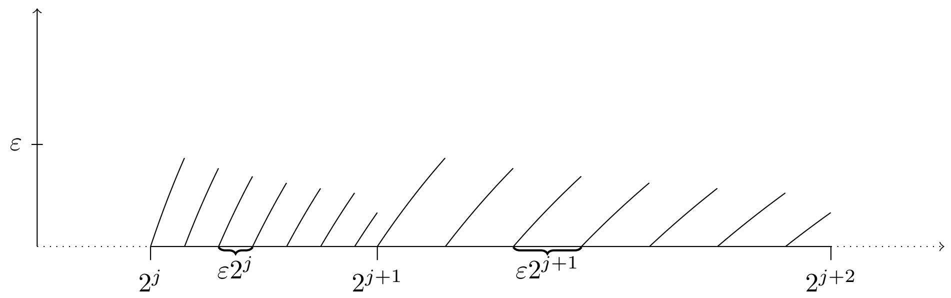


Figure 2.2: Sketch of the function m_ε drawn as a function on $\mathbb{R}$ by extending the definition on the integers to $\mathbb{R}$.

q-variation on I if

$$\|m\|_{V_q(I)} = \sup_{(x_j)_j \subset I, x_j \le x_{j+1}} \left(\sum_{k \ge 0} |m(x_{k+1}) - m(x_k)|^q \right)^{\frac{1}{q}} < \infty.$$

The theorem by Coifman, de Francia and Semmes is the following, [18], see also [55].

Theorem 2.1.3. *Let $1 < p, q < \infty$ such that $\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{1}{q}$. For $k \in \mathbb{Z}$ let $I_k = [2^k, 2^{k+1}]$ and $J_k = [-2^{k+1}, -2^k]$. Then there is a constant C such that for all functions $m : \mathbb{R} \to \mathbb{C}$ it holds that*

$$\|T_m f\|_{L^p(\mathbb{R})} \le C \left(\sup_{k \in \mathbb{Z}} \|m\|_{L^\infty(I_k \cup J_k)} + \|m\|_{V_q(I_k)} + \|m\|_{V_q(J_k)} \right) \|f\|_{L^p(\mathbb{R})}$$

where $T_m f = \mathcal{F}^{-1}(m\mathcal{F}(f))$.

Transference results allow us to use the theorem above for the function m_ε on the torus.

Proposition 2.1.4. *Let $1 < p, q < \infty$ such that $\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{1}{q}$. Let $m : \mathbb{Z} \to \mathbb{C}$ and define for $1 \le k \in \mathbb{N}$ the quantity*

$$\alpha_m^q(k) := \left(\sup_{\substack{x = 2^k, \dots, 2^{k+1} \\ x = -2^{k+1}, \dots, -2^k}} |m(x)| \right) + \left(\sum_{l = 2^k + 1}^{2^{k+1}} |m(l-1) - m(l)|^q \right)^{\frac{1}{q}} + \left(\sum_{l = 2^k + 1}^{2^{k+1}} |m(-l+1) - m(-l)|^q \right)^{\frac{1}{q}} \tag{2.3}$$

and $a_m^q(0) = |m(0)|$. If $\sup_{k \in \mathbb{N}} \alpha_m^q(k) < \infty$, then it holds

$$\left\| \sum_{n \in \mathbb{Z}^d} m(n_1) \hat{f}(n) e^{in \cdot x} \right\|_{L^p(\Pi^d)} \le C \left(\sup_{k \in \mathbb{N}} \alpha_m^q(k) \right) \left\| \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in \cdot x} \right\|_{L^p(\Pi^d)},$$

where C depends only on p and q.

Proof. Extend m piecewise affine to $\mathbb{R}$. We also call the extension m. Then, by the monotonicity of m between integers it holds for each $k \in \mathbb{N}$ the inequality $\|m\|_{V_q(I_k)} + \|m\|_{V_q(J_k)} \le \alpha_m^q(k)$ for the intervals $I_k = [2^k, 2^{k+1}]$ and $J_k = [-2^{k+1}, -2^k]$. Moreover, for $k < 0$ we estimate

$$\|m\|_{V_q(I_k)} + \|m\|_{V_q(J_k)} \leq |m(2^k) - m(2^{k+1})| + |m(-2^k) - m(-2^{k+1})| \leq 4\,\|m\|_{L^\infty} \leq 4 \sup_{k\in\mathbb{N}} \alpha_m^q(k).$$

By the Coifman-de Francia-Semmes multiplier theorem, Theorem 2.1.3, the extended function m is a valid multiplier on $L^p(\mathbb{R})$. The operator norm of the corresponding Fourier multiplication operator is less than $C \sup_{k\in\mathbb{N}} \alpha_m(k)$. Hence, by classical transference results, see for example [43, Theorem 3.6.7], the function m is a valid multiplier on the 1-torus and the operator norm of the Fourier multiplication operator on the torus can be estimated in terms of $\sup_{k\in\mathbb{N}} \alpha_m(k)$ i.e., for every function $g \in L^p(\Pi^1)$ it holds that

$$\left\| \sum_{n\in\mathbb{Z}} m(n)\hat{g}(n)e^{inx} \right\|_{L^p(\Pi^1)} \leq C(\sup_{k\in\mathbb{N}} \alpha_m(k))\, \|g\|_{L^p(\Pi^1)}\,.$$

Now, we can establish the claim of this proposition using Fubinii's Theorem: Let $f \in L^p(\Pi^d)$. Then

$$\begin{aligned}
&\int_{[-\pi,\pi]^d} \left| \sum_{n\in\mathbb{Z}^d} m(n_1)\hat{f}(n)e^{in\cdot x} \right|^p dx \\
=&\int_{[-\pi,\pi]^{d-1}} \int_{-\pi}^{\pi} \left| \sum_{n_1\in\mathbb{Z}} m(n_1) \left(\sum_{n'\in\mathbb{Z}^{d-1}} \hat{f}(n_1,n')e^{in'\cdot x'} \right) e^{in_1x_1} \right|^p dx_1\, dx' \\
\leq& C^p \left(\sup_{k\in\mathbb{N}} \alpha_m^q(k) \right)^p \int_{[-\pi,\pi]^{d-1}} \int_{-\pi}^{\pi} \left| \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{in\cdot x} \right|^p dx \\
=& C^p \left(\sup_{k\in\mathbb{N}} \alpha_m(k) \right)^p \|f\|^p_{L^p([-\pi,\pi]^d)}\,.
\end{aligned}$$

Here, we used the notation x' for the vector in $\mathbb{R}^{d-1}$ which consists of all but the first entry of $x \in \mathbb{R}^d$. □

In view of this proposition, it is enough to prove bounds on $\alpha^q_{m_\varepsilon}$ as defined in (2.3) in order to gain good estimates for the operator norm of the Fourier multiplication operator associated to m_ε. We prove this bound in the following lemma together with a bound for a second multiplier which appears during the proof of the Bourgain-Brézis type estimate.

Lemma 2.1.5. *Let* $1 \leq q < \infty$.

(i) Let $m_\varepsilon : \mathbb{Z} \to \mathbb{C}$ *be defined as in* (2.2). *Then* $\sup_k \alpha^q_{m_\varepsilon}(k) \leq C\varepsilon^{\frac{q-1}{q}}$.

(ii) Let $m(l) = \sum_{k=0}^{\infty} \frac{2^k}{l} \mathbf{1}_{[2^k,2^{k+1})\cup(-2^{k+1},-2^k]}(l)$. *Then* $\sup_k \alpha^q_m(k) < \infty$.

Proof. It can be seen directly that $0 \leq m_\varepsilon \leq \varepsilon$. Fix $k \in \mathbb{N}$ and let us write

$$\sum_{l=2^k+1}^{2^{k+1}} |m_\varepsilon(l-1) - m_\varepsilon(l)|^q = \sum_{r=0}^{\lfloor \varepsilon^{-1} \rfloor} \sum_{l\in I^r_{k+1}} |m_\varepsilon(l-1) - m_\varepsilon(l)|^q. \tag{2.4}$$

Notice that m_ε is monotone in each I^r_k. Hence, we may estimate

$$(2.4) \leq 2(\lfloor \varepsilon^{-1} \rfloor + 1)\varepsilon^q \leq 4\varepsilon^{q-1},$$

where the last inequality is valid for $\varepsilon < 1$. A similar computation can be done for the q-variation in $[-2^{k+1}, -2^k]$. Combining these estimates leads to (i).
For the second estimate, notice that m is monotone on $[2^k, 2^{k+1})$ and $(-2^{k+1}, -2^k]$ and $0 \le m \le 1$. Using these two facts and a similar argument as for m_ε leads to (ii). □

We finish this section by collecting a few classical results about Littlewood-Paley theory, see for example [43].
Let $\varphi \in C_0^\infty(B_{\frac{4}{3}}(0))$ such that $\varphi = 1$ on $B_{\frac{3}{4}}(0)$. Define the function $\psi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\varphi + \sum_{j=1}^{\infty} \psi_j = 1$. One defines associated smooth Fourier projections on the torus by multiplication in Fourier space, namely $P_j(f) = \mathcal{F}^{-1}(\psi_j \mathcal{F}(f)) = \sum_{n\in\mathbb{Z}} \psi_j(n)\, \hat{f}(n)\, e^{in\cdot x}$ for $j \ge 1$ and $P_0(f) = \mathcal{F}^{-1}(\varphi\mathcal{F}(f))$. Essentially, the operator P_j projects in Fourier space to all frequencies $|k| \sim 2^j$. Clearly, by this definition it holds that $Id = \sum_j P_j$. To finish this section, we state the Littlewood-Paley estimates which we need in the next section.
Let $f_j, f \in L^q(\Pi^d)$. Then the following inequalities hold for all $1 < p < \infty$:

1. $c_p \|f\|_{L^p(\Pi^d)} \le \left\| \left(\sum_k |P_k f|^2\right)^{\frac{1}{2}} \right\|_{L^p(\Pi^d)} \le C_p \|f\|_{L^p(\Pi^d)}$,
2. $\left\| \left(\sum_k |P_k \nabla f_k|^2\right)^{\frac{1}{2}} \right\|_{L^p(\Pi^d)} \le C_p \left\| \left(\sum_k |2^k P_k f_k|^2\right)^{\frac{1}{2}} \right\|_{L^p(\Pi^d)}$,
3. $\left\| \left(\sum_k |P_k f_k|^2\right)^{\frac{1}{2}} \right\|_{L^p(\Pi^d)} \le C_p \left\| \left(\sum_k |f_k|^2\right)^{\frac{1}{2}} \right\|_{L^p(\Pi^d)}$.

2.2 The Case of a Torus

In this section, we prove a primal version of the Bourgain-Brézis type estimate on the 2-torus Π^2, which we simply denote by Π in the following. To be precise, we show the following statement.

Theorem 2.2.1. *Let Π be the 2-torus and $2 < q < \infty$. Then there exists a constant $C > 0$ such that for all functions $f \in L^2(\Pi) \cap L^q(\Pi)$ satisfying $\int_\Pi f = 0$ there exists a function $F \in L^\infty(\Pi;\mathbb{R}^2) \cap H^1(\Pi;\mathbb{R}^2) \cap W^{1,q}(\Pi;\mathbb{R}^2)$ such that*

(i) $\operatorname{div} F = f$,

(ii) $\|F\|_{L^\infty} \le C\|f\|_{L^2}$,

(iii) $\|F\|_{H^1} \le C\|f\|_{L^2}$,

(iv) $\|F\|_{W^{1,q}} \le C\|f\|_{L^q}$.

Remark 2.2.1. The result by Bourgain and Brézis in [11, Theorem 1] is the same without the assumption $f \in L^q$ and the resulting estimate for F in L^q. In two dimensions, the same result holds true for the curl-operator as the operators div and curl are linked by a rotation of the vector fields. Hence, the result can be understood as a characterization of the failure of the embedding H^1 to L^∞ in two dimensions. A function in H^1-function can be decomposed such that the part of the decomposition which is not controlled in L^∞ is a gradient.

Remark 2.2.2. Statements of this type in the pure L^2-case hold for a more general class of operators. In [12, Theorem 10] it is shown that it is sufficient that for an operator $S : W^{1,n}(\Pi^n, \mathbb{R}^r) \to X$ with closed range, where X is a Banach space, there exists for each $1 \le s \le r$ an index $1 \le i_s \le d$ such that for all functions $f \in W^{1,n}(\Pi^n, \mathbb{R}^r)$ it holds that

$$\|Sf\| \le C \max_{1\le s\le r} \max_{i\ne i_s} \|\partial_i f_s\|_{L^n}$$

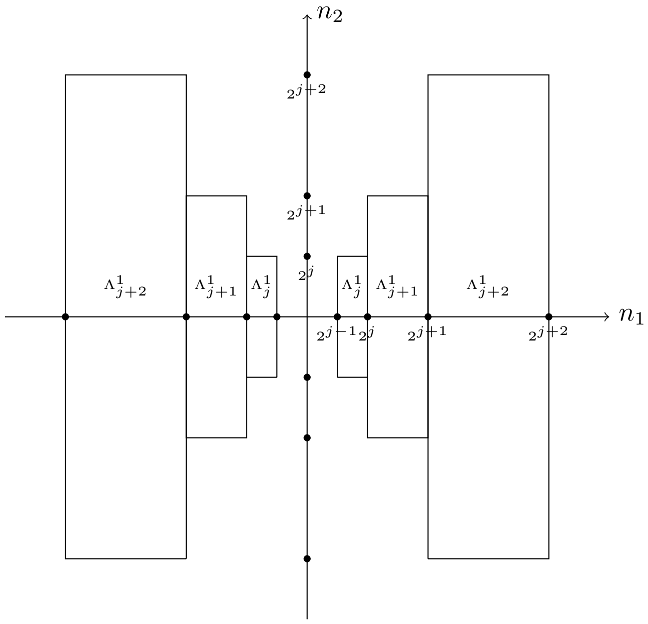


Figure 2.3: Sketch of Λ^1.

to guarantee that for any $f \in W^{1,n}(\Pi^n, \mathbb{R}^r)$ there exists $g \in W^{1,n}(\Pi^n, \mathbb{R}^r) \cap L^\infty(\Pi^n, \mathbb{R}^r)$ satisfying $S(f) = S(g)$ and corresponding bounds. Clearly, this condition holds true for the div-operator. The fact that the operator is blind for the derivatives $\partial_{i_s} f_s$ allows to insert oscillations in this particular direction.

Remark 2.2.3. Moreover, Bourgain and Brézis show that the correspondance of f to a solution $F \in H^1 \cap L^\infty$ of $\operatorname{div} F = f$ which satisfies the bounds of the theorem cannot be linear, see [11, Proposition 2].

We prove this result following the ideas presented in the proof of Theorem 1 in [11].
The main ingredient to prove Theorem 2.2.1 is the following lemma which gives a first approximation to Theorem 2.2.1. It shows that the equation $\operatorname{div} F = f$ can be almost solved by a function F which satisfies estimates with a good linear term and a bad nonlinear term.

Lemma 2.2.2 (Nonlinear approximation)**.** *Let* Π *be the* 2*-torus and* $2 < q < \infty$*. There exists* $c > 0$ *such that for all* $f \in L^2(\Pi) \cap L^q(\Pi)$ *satisfying* $\|f\|_{L^2} \leq c$ *and* $\int_\Pi f = 0$ *the following holds:*
For every $\delta > 0$ *there exist* $C_\delta > 0$ *and* $F \in L^\infty(\Pi; \mathbb{R}^2) \cap H^1(\Pi; \mathbb{R}^2) \cap W^{1,q}(\Pi; \mathbb{R}^2)$ *such that*

(i) $\|F\|_{L^\infty} \leq C_\delta$,

(ii) $\|F\|_{H^1} \leq C_\delta \|f\|_{L^2}$,

(iii) $\|\operatorname{div} F - f\|_{L^2} \leq \delta \|f\|_{L^2} + C_\delta \|f\|^2_{L^2}$,

(iv) $\|F\|_{W^{1,q}} \leq C_\delta \|f\|_{L^q}$,

(v) $\|\operatorname{div} F - f\|_{L^q} \leq \delta \|f\|_{L^2} + C_\delta \|f\|_{L^2} \|f\|_{L^q}$.

Proof. Let $f \in L^2(\Pi) \cap L^q(\Pi)$ such that $\int_\Pi f = 0$ and $\|f\|_{L^2} \leq c$ where $c > 0$ will be fixed later.
Consider the following decomposition of $\mathbb{Z}^2 \setminus \{0\}$, see Figure 2.3,

$$\Lambda^1_j = \left\{2^{j-1} < |n_1| \leq 2^j; |n_2| \leq 2^j\right\} \text{ and } \Lambda^2_j = \left\{2^{j-1} < |n_2| \leq 2^j; |n_1| \leq 2^{j-1}\right\} \text{ for } j \in \mathbb{N}. \tag{2.5}$$

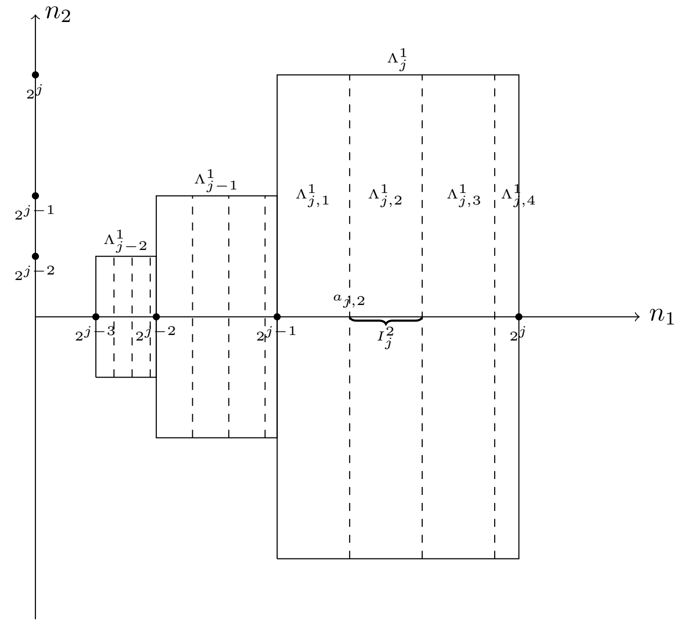


Figure 2.4: Sketch of the subdivision of Λ^1 into the stripes $\Lambda_{j,r}^1$ for positive n_1.

For $\alpha = 1, 2$ set $\Lambda^\alpha = \bigcup_j \Lambda_j^\alpha$. Correspondingly, let $f^\alpha = P_{\Lambda^\alpha} f = \sum_{n\in\Lambda^\alpha} \hat{f}(n)e^{in\cdot x}$ and decompose $f = f^1 + f^2$. In the following, we construct functions $Y_\alpha : \Pi \to \mathbb{R}$ which satisfy

1. $\|Y_\alpha\|_{L^\infty} \le C_\delta$,
2. $\|Y_\alpha\|_{H^1} \le C_\delta \|f\|_{L^2}$,
3. $\|\partial_\alpha Y_\alpha - f^\alpha\|_{L^2} \le \delta\|f\|_{L^2} + C_\delta\|f\|_{L^2}^2$,
4. $\|Y_\alpha\|_{W^{1,q}} \le C_\delta\|f\|_{L^q}$,
5. $\|\partial_\alpha Y_\alpha - f^\alpha\|_{L^q} \le \delta\|f\|_{L^2} + C_\delta\|f\|_{L^2}\|f\|_{L^q}$.

Without loss of generality we may assume that $f = f^1$ and construct only Y_1.
Let us define

$$f_j = P_{\Lambda_j^1} f = \sum_{n\in\Lambda_j^1} \hat{f}(n)\, e^{in\cdot x} \text{ and } F_j = \sum_n \frac{1}{n_1}\hat{f}_j(n)e^{in\cdot x} = \sum_{n\in\Lambda_j^1} \frac{1}{n_1}\hat{f}(n)e^{in\cdot x}.$$

Moreover, fix a small $\varepsilon > 0$ and subdivide Λ_j^1 in stripes of length $\sim \varepsilon 2^{j-1}$ by setting

$$\Lambda_j^1 = \bigcup_{0\le r\le 2\lfloor\varepsilon^{-1}\rfloor+1} \Lambda_{j,r}^1,$$

where for $0 \le r \le \lfloor\varepsilon^{-1}\rfloor$ we set $\Lambda_{j,r}^1 = I_j^r \times [-2^j, 2^j]$ whereas for $\lfloor\varepsilon^{-1}\rfloor + 1 \le r \le 2\lfloor\varepsilon^{-1}\rfloor + 1$ we set $\Lambda_{j,r}^1 = I_j^{r-\lfloor\varepsilon^{-1}\rfloor-1} \times [-2^j, 2^j]$ where I_k^r and J_k^r are defined as in (2.1) in the previous section. For a sketch of the situation, see Figure 2.4.
Next, define

$$\tilde{F}_j(x) = \sum_r \left| \sum_{n\in\Lambda_{j,r}^1} \frac{1}{n_1}\hat{f}_j(n)e^{in\cdot x} \right|.$$

The main property of $\tilde{F}_j$ is the smallness of its partial derivative in x_1-direction.
In fact, we can rewrite $\tilde{F}_j(x) = \sum_r \left|\sum_{n\in\Lambda^1_{j,r}} \frac{1}{n_1}\hat{f}_j(n)e^{in\cdot x}e^{-ia_{j,r}x_1}\right|$ where $a_{j,r}$ is the left endpoint of I^r_j, respectively the right endpoint of $J_j^{r-\lfloor\varepsilon^{-1}\rfloor-1}$. Differentiation leads to

$$|\partial_1\tilde{F}_j| = \sum_r \left|\sum_{n\in\Lambda^1_{j,r}} \frac{n_1 - a_{j,r}}{n_1}\hat{f}_j(n)e^{in\cdot x}\right| = \sum_r \left|\sum_{n\in\Lambda^1_{j,r}} m_\varepsilon(n_1)\hat{f}_j(n)e^{in\cdot x}\right|, \tag{2.6}$$

where m_ε is the the special function defined in (2.2) in the previous section. As $0 \le m_\varepsilon \le \varepsilon$ and using Plancharel's identity and Hölder's inequality for the sum over r, we derive that

$$\begin{aligned}\|\partial_1\tilde{F}_j\|_{L^2} &\le C\varepsilon^{-\frac{1}{2}}\sum_r \left\|\sum_{n\in\Lambda^1_{j,r}} m_\varepsilon(n_1)\hat{f}(n)e^{in\cdot x}\right\|_{L^2} \\ &\le C\varepsilon^{\frac{1}{2}}\|f_j\|_{L^2}.\end{aligned}$$

For an L^q-version of this estimate, we will later use Proposition 2.1.4 and Lemma 2.1.5.
As we also need an appropiate localization in Fourier space of $\tilde{F}_j$, let us recall that the n-th one-dimensional Féjer-kernel is given by

$$K_n(t) = \sum_{|k|<n} \frac{n-|k|}{n}e^{ikt} = \frac{1}{n}\frac{1-\cos(nt)}{1-\cos(t)} \ge 0.$$

If we define

$$G_j = 9\tilde{F}_j * (K_{2^{j+1}} \otimes K_{2^{j+1}}),$$

we obtain by the properties of the Fejéjer kernel discussed in the beginning of the previous section that

$$\operatorname{supp}\hat{G}_j \subset [-2^{j+1}, 2^{j+1}] \times [-2^{j+1}, 2^{j+1}] \subset \{|n| \le C2^j\} \text{ and } |F_j| \le |\tilde{F}_j| \le G_j. \tag{2.7}$$

Moreover, in the proof of [11, Theorem 1] it is shown that

$$\|G_j\|_{L^\infty} \le 9\|\tilde{F}_j\|_\infty \le C\|f_j\|_{L^2}, \tag{2.8}$$

$$\|G_j\|_{L^2} \le C\varepsilon^{-\frac{1}{2}}2^{-j}\|f_j\|_{L^2}, \tag{2.9}$$

$$\|\partial_1 G_j\|_{L^2} \le C\varepsilon^{\frac{1}{2}}\|f_j\|_{L^2}, \tag{2.10}$$

$$\|\nabla G_j\|_{L^2} \le C\varepsilon^{-\frac{1}{2}}\|f_j\|_{L^2}. \tag{2.11}$$

Let us only prove (2.8), the rest can be proved similarly:

$$|\tilde{F}_j(x)| \le \sum_r \sum_{n\in\Lambda^1_{j,r}} |\frac{1}{n_1}\hat{f}(n)| \le 2^{-j+1}\sum_{n\in\Lambda^1_j}|\hat{f}(n)| \le C\left(\sum_{n\in\Lambda^1_j}|\hat{f}(n)|^2\right)^{\frac{1}{2}} = C\,\|f_j\|_{L^2}.$$

As in [11], we define

$$Y_1 = \sum_j F_j \prod_{k>j}(1-G_k).$$

By (2.7) and (2.8), it holds $|F_j| \le C\|f_j\|_{L^2} \le C\,\|f\|_{L^2}$. We assume that $\|f\|_{L^2}$, respectively c in the formulation of the theorem, is so small that $C\|f\|_{L^2} < 1$. Then, one can show that

$$|Y_1| \leq \sum_j |F_j| \prod_{k>j} (1 - |F_k|) \leq 1. \tag{2.12}$$

Another calculation, see [11, equation (5.19)], shows that

$$Y_1 = \sum_j F_j - \sum_j G_j H_j,$$

where

$$H_j = \sum_{k<j} F_k \prod_{k<l<j} (1 - G_l).$$

Thus,

$$\partial_1 Y_1 = \sum_j f_j - \sum_j \partial_1 (G_j H_j) = f - \sum_j \partial_1 (G_j H_j). \tag{2.13}$$

Moreover, by definition of H_j and F_j and (2.7) it can be seen that

$$\begin{aligned} &|H_j| \leq 1, \ \operatorname{supp} \hat{H}_j \subset \{|n| \leq C2^j\}, \ P_k(G_j H_j) = 0 \text{ for all } k > j + m, \\ &\text{respectively } G_j H_j = \sum_{k \leq j+m} P_k(G_j H_j), \end{aligned} \tag{2.14}$$

where the P_k are smooth Littlewood-Paley-projections on $\{|n| \sim 2^k\}$ as discussed at the end of the previous section and m is independent of j. In [11, proof of Theorem 1], Bourgain and Brézis show, using (2.7) - (2.14), that

$$\|\partial_1 Y_1 - f\|_{L^2} \leq C \log(\varepsilon^{-1}) \left(\varepsilon^{\frac{1}{2}} \|f\|_{L^2} + \varepsilon^{-\frac{1}{2}} \|f\|_{L^2}^2 \right) \text{ and } \|Y_1\|_{H^1} \leq C_\varepsilon \|f\|_{L^2}. \tag{2.15}$$

Hence, properties 1.–3. for Y_1 are already shown. In what follows, we adopt the ideas of their proof to show the corresponding estimates in L^q i.e., properties 4. and 5., for Y_1.
First, we estimate

$$\|\nabla Y_1\|_{L^q} \leq \|\nabla \sum_j F_j\|_{L^q} + \|\nabla \sum_j G_j H_j\|_{L^q}. \tag{2.16}$$

For the first term on the right hand side, we observe that

$$\begin{aligned} \left\| \sum_j \nabla F_j \right\|_{L^q} &= \left\| \sum_j \sum_{n \in \Lambda_j^1} \frac{n}{n_1} \hat{f}(n) e^{in\cdot x} \right\|_{L^q} \\ &\leq C \left\| \sum_j \sum_{n \in \Lambda_j^1} \hat{f}(n) e^{in\cdot x} \right\|_{L^q} \\ &= C \|f\|_{L^q}. \end{aligned} \tag{2.17}$$

Note that we used for the first inequality that $\frac{n}{n_1} \mathbf{1}_{\bigcup_j \Lambda_j^1}$ is an L^p-multiplier. This can be shown by multiplier transference and the Marcinkiewicz multiplier theorem (note that in Λ_j the second variable n_2 is controlled by $2n_1$). Next, we estimate the second term of the right hand side of (2.16). Using (2.14) and classical Littlewood-Paley estimates as discussed at the end of the previous section, we obtain

$$\left\| \sum_j \nabla(G_j H_j) \right\|_{L^q} \leq C \left\| \left(\sum_k \left| P_k \sum_j \nabla(G_j H_j) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.18}$$

Note that the operator ∇ can also be seen as a Fourier multiplication operator. Hence, it commutes with the Littlewood-Paley-projections P_k. In particular, the localization in Fourier space for $G_j H_j$ in (2.14) also holds for $\nabla G_j H_j$. The triangle inequality and rewriting with the change of variables $j \to k+s$ yield

$$\leq C \sum_{s \geq -m} \left\| \left(\sum_k |P_k \nabla(G_{k+s} H_{k+s})|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.19}$$

The change $k \to k-s$ leads to

$$= C \sum_{s \geq -m} \left\| \left(\sum_{k \geq s} |P_{k-s} \nabla(G_k H_k)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.20}$$

The Littlewood-Paley inequality for gradients yields

$$\leq C \sum_{s \geq -m} 2^{-s} \left\| \left(\sum_k |2^k G_k \underbrace{H_k}_{|\cdot| \leq 1}|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \tag{2.21}$$

$$\leq C \sum_{s \geq -m} 2^{-s} \left\| \left(\sum_j |2^k G_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.22}$$

By definition, G_k is the convolution of $\tilde{F}_k$ with a Fejér kernel. Applying Proposition 2.1.2 leads to

$$\leq C \sum_{s \geq -m} 2^{-s} \left\| \left(\sum_k |2^k \tilde{F}_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.23}$$

$$= C \sum_{s \geq -m} 2^{-s} \left\| \left(\sum_k \left(\sum_{r \leq 2\lfloor \varepsilon^{-1} \rfloor - 1} \left| \sum_{n \in \Lambda^1_{k,r}} \frac{2^k}{n_1} \hat{f}(n) e^{in \cdot x} \right| \right)^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.24}$$

Using Hölder's inequality for the sum over r yields

$$\leq C\varepsilon^{-\frac{1}{2}} \sum_{s\geq -m} 2^{-s} \left\| \left(\sum_k \sum_{r\leq 2\lfloor \varepsilon^{-1}\rfloor -1} \left| \sum_{n\in\Lambda^1_{k,r}} \frac{2^k}{n_1} \hat{f}(n) e^{in\cdot x} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.25}$$

Now, we use a one-sided Littlewood-Paley-type inequality for non-dyadic decompositions which goes back to Rubio de Francia, [69, Theorem 8.1]. For the case of a torus, see [41, Theorem 2.5] or [10] for the dual statement. The statement is the following: For $q > 2$ there exists a constant $C > 0$ such that for all partitions of $\mathbb{Z}$ into intervals $(I_k)_k$ it holds that

$$\left\| (\sum_k |S_k f|^2)^{\frac{1}{2}} \right\|_{L^q(\Pi^1)} \leq C \left\| f \right\|_{L^q(\Pi^1)} ,$$

where $S_k f = \sum_{l\in I_k} \hat{f}(l) e^{il\cdot x}$. We use this inequality in the first variable for the decomposition of the n_1-axis given by $\Lambda^1_{k,r}$, respectively I^r_k and J^r_k:

$$(2.25) \leq C\varepsilon^{-\frac{1}{2}} \sum_{s\geq -m} 2^{-s} \left\| \sum_{k,r\leq 2\lfloor \varepsilon^{-1}\rfloor -1} \sum_{n\in\Lambda^1_{k,r}} \frac{2^k}{n_1} \hat{f}(n) e^{in\cdot x} \right\|_{L^q} . \tag{2.26}$$

Finally, we use that we know from Proposition 2.1.4 and Lemma 2.1.5 that $\sum_k \frac{2^k}{n_1} \mathbf{1}_{\Lambda^1_k}$ is an L^p-multiplier to obtain

$$\leq C\varepsilon^{-\frac{1}{2}} \sum_{s\geq -m} 2^{-s} \Big\| \underbrace{\sum_{k,r\leq 2\lfloor \varepsilon^{-1}\rfloor -1} \sum_{n\in\Lambda^1_{k,r}}}_{\sum_{n\in\Lambda^1}} \hat{f}(n) e^{in\cdot x} \Big\|_{L^q} \tag{2.27}$$

$$= C\varepsilon^{-\frac{1}{2}} \sum_{s\geq -m} 2^{-s} \left\| f \right\|_{L^q} \tag{2.28}$$

$$\leq C\varepsilon^{-\frac{1}{2}} \left\| f \right\|_{L^q} . \tag{2.29}$$

Collecting (2.16), (2.17), and (2.18) - (2.29) leads to

$$\left\| \nabla Y_1 \right\|_{L^q} \leq C\varepsilon^{-\frac{1}{2}} \left\| f \right\|_{L^q} .$$

As we may assume without loss of generality that $\int_\Pi Y_1 = 0$, this implies

$$\left\| Y \right\|_{W^{1,q}} \leq C\varepsilon^{-\frac{1}{2}} \left\| f \right\|_{L^q} . \tag{2.30}$$

Hence, it is left to prove property 5. for Y_1. By (2.13), it remains to control $\left\| \partial_1 \sum_j (G_j H_j) \right\|_{L^q}$. As in

(2.18)–(2.20), we can estimate

$$\left\| \partial_1 \sum_j G_j H_j \right\|_{L^q} \leq \sum_{s \geq -m} \left\| \left(\sum_j |P_{j-s} \partial_1 (G_j H_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} .$$

Now, fix $s_* \in \mathbb{N}$ and estimate for $s > s_*$ as in (2.20)–(2.28)

$$\left\| \left(\sum_j |P_{j-s} \partial_1 (G_j H_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \varepsilon^{-\frac{1}{2}} 2^{-s} \|f\|_{L^q} . \tag{2.31}$$

For $s \leq s_*$ we estimate, using that $|H_j| \leq 1$,

$$\left\| \left(\sum_j |P_{j-s} \partial_1 (G_j H_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \left\| \left(\sum_j |\partial_1 G_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} + C \left\| \left(\sum_j |G_j \partial_1 H_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} . \tag{2.32}$$

As G_j is the convolution of $\tilde{F}_j$ with a Fejér kernel, we may apply Proposition 2.1.2 to the first term on the right hand side to derive

$$\left\| \left(\sum_j |\partial_1 G_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \left\| \left(\sum_j |\partial_1 \tilde{F}_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} .$$

Equation (2.6) and Hölder's inequality for the sum over r lead to

$$\leq C \varepsilon^{-\frac{1}{2}} \left\| \left(\sum_j \sum_{r \leq 2\lfloor \varepsilon^{-1} \rfloor - 1} \left| \sum_{n \in \Lambda^1_{j,r}} m_\varepsilon(n_1) \hat{f}(n) e^{in \cdot x} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q} .$$

Using the Rubio-de-Francia-inequality for arbitrary intervals in the first variable as in (2.25)–(2.26) yields

$$\leq C \varepsilon^{-\frac{1}{2}} \left\| \sum_j \sum_{r \leq 2\lfloor \varepsilon^{-1} \rfloor - 1} \sum_{n \in \Lambda^1_{j,r}} m_\varepsilon(n_1) \hat{f}(n) e^{in \cdot x} \right\|_{L^q} .$$

By the improvement of the Marcinkiewicz multiplier theorem due to Coifman, de Francia, and Semmes, Proposition 2.1.4 and Lemma 2.1.5, the function m_ε defines a multiplier whose associated operator-norm from L^q to L^q can be estimated by $C_r \varepsilon^{\frac{r-1}{r}}$ for any r such that $|\frac{1}{2} - \frac{1}{q}| < \frac{1}{r}$. In particular, there exists $r > 2$ such that

$$\leq C \varepsilon^{\frac{r-1}{r} - \frac{1}{2}} \left\| \sum_{n \in \Lambda^1} \hat{f}(n) e^{in \cdot x} \right\|_{L^q} = C \varepsilon^{\frac{r-1}{r} - \frac{1}{2}} \|f\|_{L^q} . \tag{2.33}$$

For the second term of the right hand side of (2.32), note that in [11] Bourgain and Brézis show that

$$\|\nabla H_j\|_{L^\infty} \leq 2^j \|f\|_{L^2}.$$

Hence, we can estimate

$$\left\| \left(\sum_j |G_j \partial_1 H_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq \left\| \left(\sum_j |2^j G_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \|f\|_{L^2}.$$

The right hand side can now be treated as in (2.22)–(2.28) to obtain

$$\left\| \left(\sum_j |G_j \partial_1 H_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq C\varepsilon^{-\frac{1}{2}} \|f\|_{L^q} \|f\|_{L^2}. \tag{2.34}$$

Collecting (2.31), (2.33) and (2.34) yields

$$\left\| \partial_1 \sum_j G_j H_j \right\|_{L^q} \leq C 2^{-s_*} \varepsilon^{-\frac{1}{2}} \|f\|_{L^q} + \sum_{-m \leq s \leq s_*} \left(\varepsilon^{-\frac{1}{2}} \|f\|_{L^q} \|f\|_{L^2} + \varepsilon^{\frac{r-1}{r} - \frac{1}{2}} \|f\|_{L^q} \right). \tag{2.35}$$

Eventually, choose s_* such that $2^{-s_*} \sim \varepsilon$. Then (2.35) provides

$$\left\| \partial_1 \sum_j G_j H_j \right\|_{L^q} \leq C \log(\varepsilon^{-1}) \left(\varepsilon^{\frac{r-1}{r} - \frac{1}{2}} \|f\|_{L^q} + \varepsilon^{-\frac{1}{2}} \|f\|_{L^q} \|f\|_{L^2} \right). \tag{2.36}$$

Here, we used that $\varepsilon^{\frac{1}{2}} \leq \varepsilon^{\frac{r-1}{r} - \frac{1}{2}}$ for $\varepsilon < 1$. Notice that $r > 2$ and therefore $\log(\varepsilon^{-1})\varepsilon^{\frac{r-1}{r} - \frac{1}{2}} \to 0$ as $\varepsilon \to 0$. Comparing with (2.15) and (2.30) shows that for given $\delta > 0$ the properties 1.–5. of Y can be achieved for $\varepsilon > 0$ small enough. This finishes the proof. □

Remark 2.2.4. Note that the explicit construction given in the proof is nonlinear which is in accordance with Remark 2.2.3.

From the nonlinear estimate we can now derive a linear estimate.

Lemma 2.2.3 (Linear estimate)**.** *Let Π be the 2-torus and $2 < q < \infty$. Then for every $\delta > 0$ there exists a constant $C_\delta > 0$ such that for every function $f \in L^2(\Pi) \cap L^q(\Pi)$ satisfying $\int_\Pi f = 0$ there exists $F \in L^\infty(\Pi;\mathbb{R}^2) \cap H^1(\Pi;\mathbb{R}^2) \cap W^{1,q}(\Pi;\mathbb{R}^2)$ such that*

(i) $\|F\|_{L^\infty} \leq C_\delta \|f\|_{L^2}$,

(ii) $\|F\|_{H^1} \leq C_\delta \|f\|_{L^2}$,

(iii) $\|\operatorname{div} F - f\|_{L^2} \leq \delta \|f\|_{L^2}$,

(iv) $\|F\|_{W^{1,q}} \leq C_\delta \|f\|_{L^q}$,

(v) $\|\operatorname{div} F - f\|_{L^q} \leq \delta \|f\|_{L^q}$.

Proof. As we want to prove a linear estimate, we may assume without loss of generality that it holds $\|f\|_{L^2} = \delta C_\delta^{-1} < c$ where $c > 0$ is the constant from Lemma 2.2.2. The application of Lemma 2.2.2 provides the existence of $F \in L^\infty(\Pi;\mathbb{R}^2) \cap H^1(\Pi;\mathbb{R}^2) \cap W^{1,q}(\Pi;\mathbb{R}^2)$ such that

(i) $\|F\|_{L^\infty} \le C_\delta = \delta^{-1} C_\delta^2 \|f\|_{L^2}$,

(ii) $\|F\|_{H^1} \le C_\delta \|f\|_{L^2}$,

(iii) $\|\operatorname{div} F - f\|_{L^2} \le \delta \|f\|_{L^2} + C_\delta \|f\|_{L^2}^2 = 2\delta \|f\|_{L^2}$,

(iv) $\|F\|_{W^{1,q}} \le C_\delta \|f\|_{L^q}$,

(v) $\|\operatorname{div} F - f\|_{L^q} \le \delta \|f\|_{L^q} + C_\delta \|f\|_{L^2} \|f\|_{L^q} = 2\delta \|f\|_{L^q}$.

Now, take $\tilde{\delta} = 2\delta$ and $C_{\tilde{\delta}} = \delta^{-1} C_\delta^2$. □

Armed with this approximation we are now able to prove Theorem 2.2.1 by iterating this approximation.

Proof of Theorem 2.2.1. Let $f \in L^2(\Pi) \cap L^q(\Pi)$ such that $\int_\Pi f = 0$. We apply Lemma 2.2.3 for $\delta = \frac{1}{2}$. Hence, there exists F_1 such that

- $\|F_1\|_{L^\infty} \le C_{\frac{1}{2}} \|f\|_{L^2}$,
- $\|F_1\|_{H^1} \le C_{\frac{1}{2}} \|f\|_{L^2}$,
- $\|\operatorname{div} F_1 - f\|_{L^2} \le \frac{1}{2} \|f\|_{L^2}$,
- $\|F_1\|_{W^{1,q}} \le C_{\frac{1}{2}} \|f\|_{L^q}$,
- $\|\operatorname{div} F_1 - f\|_{L^q} \le \frac{1}{2} \|f\|_{L^q}$.

We define F_i for $i \ge 2$ inductively: let $\tilde{f}_i = f - \operatorname{div} \sum_{j=1}^{i-1} F_j$. Note that by the periodicity of the F_j it holds $\int_\Pi \tilde{f}_i = 0$. Reapplication of Lemma 2.2.3 for $\delta = \frac{1}{2}$ and $\tilde{f}_i$ provides the existence of F_i such that

(i) $\|F_i\|_{L^\infty} \le C_{\frac{1}{2}} \|f - \operatorname{div} \sum_{j=1}^{i-1} F_j\|_{L^2} \le C_{\frac{1}{2}} (\frac{1}{2})^{i-1} \|f\|_{L^2}$,

(ii) $\|F_i\|_{H^1} \le C_{\frac{1}{2}} \|f - \operatorname{div} \sum_{j=1}^{i-1} F_j\|_{L^2} \le C_{\frac{1}{2}} (\frac{1}{2})^{i-1} \|f\|_{L^2}$,

(iii) $\|\operatorname{div} F_i + \operatorname{div} \sum_{j=1}^{i-1} F_j - f\|_{L^2} \le \frac{1}{2} \|\operatorname{div} \sum_{j=1}^{i-1} F_j - f\|_{L^2} \le (\frac{1}{2})^i \|f\|_{L^2}$,

(iv) $\|F_i\|_{W^{1,q}} \le C_{\frac{1}{2}} \|f - \operatorname{div} \sum_{j=1}^{i-1} F_j\|_{L^q} \le C_{\frac{1}{2}} (\frac{1}{2})^{i-1} \|f\|_{L^q}$,

(v) $\|\operatorname{div} F_i + \operatorname{div} \sum_{j=1}^{i-1} F_j - f\|_{L^q} \le \frac{1}{2} \|\operatorname{div} \sum_{j=1}^{i-1} F_j - f\|_{L^q} \le (\frac{1}{2})^i \|f\|_{L^q}$.

Define $F = \sum_{j=1}^\infty F_j$. Then, $\operatorname{div} F = f$ and the claimed estimates follow by the triangle inequality with $C = 2C_{\frac{1}{2}}$. □

2.3 Localization

This section is devoted to localize our previous result in the sense that we show that on cubes there exist solutions to $\operatorname{div} Y = f$ satisfying $Y = 0$ on the boundary and bounds in L^∞, H^1, and $W^{1,q}$. The proofs in this chapter follow the lines of the corresponding proofs for solutions in the space $L^\infty(\Omega; \mathbb{R}^2) \cap H_0^1(\Omega; \mathbb{R}^2)$ presented in [11, Section 7].

Proposition 2.3.1. *Let $Q = (0,1)^2$ and $2 < q < \infty$. Then there exists $C > 0$ such that for all functions $f \in L^2(Q) \cap L^q(Q)$ satisfying $\int_Q f = 0$ there exists $Y \in L^\infty(Q; \mathbb{R}^2) \cap H_0^1(Q; \mathbb{R}^2) \cap W_0^{1,q}(Q; \mathbb{R}^2)$ such that* $\operatorname{div} Y = f$,

$$\|Y\|_{L^\infty(Q;\mathbb{R}^2)} + \|Y\|_{H^1(Q;\mathbb{R}^2)} \le C\|f\|_{L^2(Q)}, \text{ and } \|Y\|_{W^{1,q}(Q;\mathbb{R}^2)} \le C\|f\|_{L^q(Q)}.$$

Proof. **Step 1.** *Existence of a solution satisfying* $Y = 0$ *on* $(0,1)\times\{0\}$.
Let $\tilde{Q} = (0,1)\times(-2,1)$ and

$$\tilde{f} = \begin{cases} f & \text{in } Q, \\ 0 & \text{in } \tilde{Q}\setminus Q. \end{cases}$$

We may interpret $\tilde{Q}$ as a torus. Applying Theorem 2.2.1 provides a periodic solution Z to $\operatorname{div} Z = \tilde{f}$ in $\tilde{Q}$ satisfying

$$\|Z\|_{L^\infty(\tilde{Q};\mathbb{R}^2)} + \|Z\|_{H^1(\tilde{Q};\mathbb{R}^2)} \le C\|\tilde{f}\|_{L^2(\tilde{Q})} \text{ and } \|Z\|_{W^{1,q}(\tilde{Q};\mathbb{R}^2)} \le C\|\tilde{f}\|_{L^q(\tilde{Q})}.$$

Next, define the funtion $Y : Q \to \mathbb{R}^2$ by setting for $(x,y)\in Q$

$$Y_1(x,y) = Z_1(x,y) + 3Z_1(x,-y) - 4Z_1(x,-2y) \text{ and } Y_2(x,y) = Z_2(x,y) - 3Z_2(x,-y) + 2Z_2(x,-2y).$$

A straightforward calculation shows that for $(x,y)\in Q$ it holds $\operatorname{div} Y(x,y) = f(x,y)$. Moreover, the function Y inherits the estimates from Z. Precisely,

$$\begin{aligned} &\|Y\|_{L^\infty(Q;\mathbb{R}^2)} + \|Y\|_{H^1(Q;\mathbb{R}^2)} \le C(\|Z\|_{L^\infty(\tilde{Q};\mathbb{R}^2)} + \|Z\|_{H^1(\tilde{Q};\mathbb{R}^2)}) \le C\|\tilde{f}\|_{L^2(\tilde{Q})} = C\|f\|_{L^2(Q)} \\ &\text{and } \|Y\|_{W^{1,q}(Q;\mathbb{R}^2)} \le C\|f\|_{L^q(Q)}. \end{aligned} \tag{2.37}$$

Finally, one checks that for $x\in(0,1)$ it holds that

$$Y_1(x,0) = Z_1(x,0) + 3Z_1(x,0) - 4Z_1(x,0) = 0 \text{ and } Y_2(x,0) = Z_2 - 3Z_2(x,0) + 2Z_2(x,0) = 0.$$

Step 2. *Existence of a solution satisfying* $Y = 0$ *on* $(0,1)\times\{0\}\cup\{0\}\times(0,1)$.
Let $\hat{Q} = (-2,1)\times(0,1)$ and

$$\hat{f} = \begin{cases} f & \text{on } Q, \\ 0 & \text{on } \hat{Q}\setminus Q. \end{cases}$$

With slight changes we can use step 1 to find a function Z satisfying $\operatorname{div} Z = \hat{f}$ on $\hat{Q}$, $Z = 0$ on $(-2,1)\times\{0\}$, and the bounds in (2.37). Then, define similarly to step 1 the function $Y : Q\to\mathbb{R}^2$ by

$$Y_1(x,y) = Z_1(x,y) - 3Z_1(-x,y) + 2Z_1(-2x,y) \text{ and } Y_2(x,y) = Z_2(x,y) + 3Z_2(-2x,y) - 4Z_2(-x,y).$$

Similar to step 1, one can check that $Y = 0$ on $(0,1)\times\{0\}\cup\{0\}\times(0,1)$ and $\operatorname{div} Y = f$. Moreover, Y satisfies the bounds in (2.37).
Step 3. *Conclusion.*
Let x_i, $i = 1,\dots,4$, be the vertices of $(0,1)^2$ and $Q_i = Q\cap B_1(x_i)$. For $i = 1,\dots,4$ let $\psi_i \in C^\infty(Q)$ such that $\psi_i = 0$ on $Q\setminus Q_i$ and $\sum_{i=1}^4 \psi_i = 1$ on Q. By step 2, there exist solutions $Z^1,\dots,Z^4$ such that $\operatorname{div} Z^i = f$ and $Z^i = 0$ on the two edges ending at x_i, see figure 2.5. In addition, the solutions Z_i satisfy the bounds in (2.37).
Now, define

$$Z = \sum_{i=1}^4 \psi_i Z^i.$$

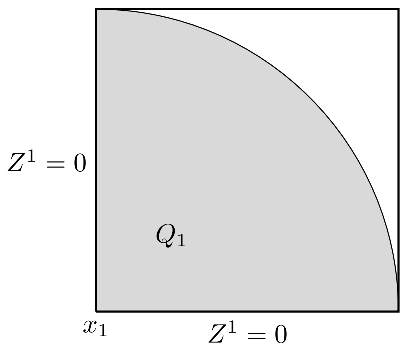


Figure 2.5: Sketch of Q_1 and the boundary values for Z^1 as defined in step 3 of the proof of Proposition 2.3.1.

Then $Z = 0$ on ∂Q and Z satisfies the bounds in (2.37). Moreover,

$$\operatorname{div} Z = f + \sum_{i=1}^{4} Z^i \cdot \nabla \psi_i.$$

Let us write $g = \sum_{i=1}^{4} Z^i \cdot \nabla \psi_i$. It is easily seen that $g = 0$ on ∂Q and

- $\int_Q g \, dx = 0$,
- $\|g\|_{L^\infty(Q)} \leq C \sum_{i=1}^{4} \|Z^i\|_{L^\infty(Q;\mathbb{R}^2)} \leq C\|f\|_{L^2(Q)}$,
- $\|g\|_{H^1(Q)} \leq C \sum_{i=1}^{4} \|Z^i\|_{H^1(Q;\mathbb{R}^2)} \leq C\|f\|_{L^2(Q)}$,
- $\|g\|_{W^{1,q}(Q)} \leq C \sum_{i=1}^{4} \|Z^i\|_{W^{1,q}(Q;\mathbb{R}^2)} \leq C\|f\|_{L^q(Q)}$.

The next lemma shows that there exists a function $R \in L^\infty(Q;\mathbb{R}^2) \cap H_0^1(Q;\mathbb{R}^2) \cap W_0^{1,q}(Q;\mathbb{R}^2)$ such that $\operatorname{div} R = -g$,

$$\|R\|_{L^\infty(Q;\mathbb{R}^2)} \leq C\|g\|_{L^\infty(Q)}, \ \|R\|_{H^1(Q;\mathbb{R}^2)} \leq C\|g\|_{H^1(Q)}, \text{ and } \|R\|_{W^{1,q}(Q;\mathbb{R}^2)} \leq C\|g\|_{W^{1,q}(Q)}.$$

Finally, set $Y = Z + R$. □

Next, we prove the lemma which we used in the proof of Proposition 2.3.1.

Lemma 2.3.2. *Let $Q = (0,1)^2$ and $2 < q < \infty$. There exists a constant $C > 0$ such that for every function $f \in L^\infty(Q) \cap H_0^1(Q) \cap W_0^{1,q}(Q)$ satisfying $\int_Q f = 0$ there exists $Y \in L^\infty(Q;\mathbb{R}^2) \cap H_0^1(Q;\mathbb{R}^2) \cap W_0^{1,q}(Q;\mathbb{R}^2)$ such that $\operatorname{div} Y = f$,*

$$\|Y\|_{L^\infty(Q;\mathbb{R}^2)} \leq C\|f\|_{L^\infty}, \ \|Y\|_{H^1(Q;\mathbb{R}^2)} \leq C\|f\|_{H^1(Q)}, \ \textit{and } \|Y\|_{W^{1,q}(Q;\mathbb{R}^2)} \leq C\|f\|_{W^{1,q}(Q)}.$$

Proof. The proof follows a standard construction, which could be applied inductively to establish results in higher dimensions. However, we consider only the two-dimensional case.
Let $f \in L^\infty(Q) \cap H_0^1(Q) \cap W_0^{1,q}(Q)$ such that $\int_Q f = 0$. For $y \in (0,1)$ define

$$g(y) = \int_0^1 f(x,y) \, dx.$$

As $f = 0$ on ∂Q, we obtain that $g(0) = g(1) = 0$. Moreover, it holds $\int_0^1 g(y)\,dy = 0$. Furthermore, one can check, using Jensen's inequality, that

$$\|g\|_{L^\infty(0,1)} \le \|f\|_{L^\infty(Q)},\ \|g\|_{H^1(0,1)} \le \|f\|_{H^1(Q)} \text{ and } \|g\|_{W^{1,q}(0,1)} \le \|f\|_{W^{1,q}(Q)}.$$

In addition, let us define

$$Z(y) = \int_0^y g(t)\,dt.$$

Clearly, $Z(0) = Z(1) = 0$ and Z satisfies the estimates

$$\|Z\|_{L^\infty(0,1)} \le \|g\|_{L^\infty(0,1)} \le \|f\|_{L^\infty(Q)},\ \|Z\|_{H^1(0,1)} \le C\|f\|_{H^1(Q)}, \text{ and } \|Z\|_{W^{1,q}(0,1)} \le \|f\|_{W^{1,q}(Q)}.$$

Furthermore, let $\psi \in C_c^\infty((0,1))$ such that $\int_0^1 \psi\,dx = 1$. This function can be chosen independently from f. We set

$$h(x,y) = \int_0^x f(t,y) - \psi(t)g(y)\,dt.$$

Then $f(x,y) = \partial_x h(x,y) + \psi(x)g(y)$ and $h = 0$ on ∂Q. In addition, one varifies that

$$\|h\|_{L^\infty(Q)} \le C\|f\|_{L^\infty(Q)},\ \|h\|_{H_0^1(Q)} \le C\|f\|_{H_0^1(Q)}, \text{ and } \|h\|_{W_0^{1,q}(Q)} \le C\|f\|_{W_0^{1,q}(Q)}.$$

Finally, set

$$Y(x,y) = (h(x,y), \psi(x)Z(y)).$$

Then $\operatorname{div} Y = f$. The desired estimates follow from those for h and Z. □

2.4 Lipschitz Domains

So far we have shown that on cubes there exist good solutions for the equation $\operatorname{div} Y = f$ subject to $Y = 0$ on the boundary. Later, we are interested in a decomposition result for functions in $H^1 \cap W^{1,q}$ which gives rise to estimates of the $H^{-1} + W^{-1,q}$-norm of an L^1- function. One could first derive the decomposition result for cubes from the previous section and later use covering arguments to obtain this decomposition result for more general domains. However, the decomposition statement involves second derivatives (see Theorem 2.4.4). Hence, the straightforward transformation of this result including a partition of unity needs higher regularity of the boundary i.e., $\partial\Omega \in C^{1,1}$. Therefore, we first extend the result of Proposition 2.3.1 to Lipschitz domains.

Theorem 2.4.1. *Let $2 < q < \infty$ and $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Then there exists a constant $C > 0$ such that for every $f \in L^2(\Omega) \cap L^q(\Omega)$ satisfying $\int_\Omega f\,dx = 0$ there exists $Y \in L^\infty(\Omega;\mathbb{R}^2) \cap H_0^1(\Omega;\mathbb{R}^2) \cap W_0^{1,q}(\Omega;\mathbb{R}^2)$ such that* $\operatorname{div} Y = f$,

$$\|Y\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|Y\|_{H_0^1(\Omega;\mathbb{R}^2)} \le C\|f\|_{L^2(\Omega)}, \text{ and } \|Y\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} \le C\|f\|_{L^q(\Omega)}.$$

For the proof we need two lemmas treating the local situation. The theorem can then be proved by a covering argument. The prove follows the ideas presented in [11, Section 7].

Lemma 2.4.2. *Let $2 < q < \infty$. There exists $\varepsilon_0 > 0$ and a constant $C > 0$ such that for all intervals I and $\psi \in Lip(I)$ with $Lip(\psi) \le \varepsilon_0$ the following holds: Let*

$$U = \{(x,y) \in I \times \mathbb{R} : \psi(x) < y < \psi(x) + \delta\},$$

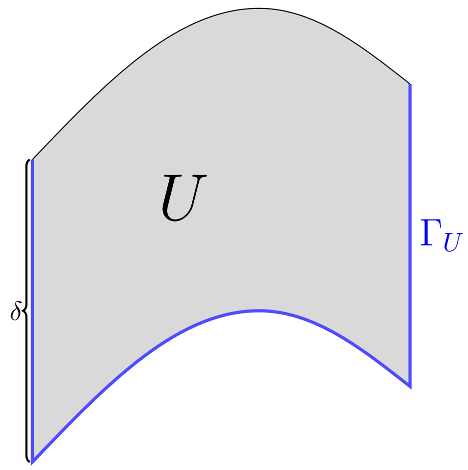


Figure 2.6: Sketch of the situation in Lemma 2.4.2 and Lemma 2.4.3.

where δ is the length of I. Then for every $f \in L^2(U) \cap L^q(U)$ there exists Y such that $\operatorname{div} Y = f$*. Moreover, Y satisfies*

$$\|Y\|_{L^\infty(U;\mathbb{R}^2)} + \|Y\|_{H^1(U;\mathbb{R}^2)} \le C\|f\|_{L^2(U;\mathbb{R}^2)}, \ \|Y\|_{W^{1,q}(U;\mathbb{R}^2)} \le C\|f\|_{L^q(U;\mathbb{R}^2)},$$

and $Y = 0$ on Γ_U where

$$\Gamma_U = \{(x,y) \in \mathbb{R}^2 : x \in I, y = \psi(x)\} \cup \{(x,y) \in \mathbb{R}^2 : x \in \partial I, \psi(x) \le y \le \psi(x) + \delta\}.$$

For a sketch of U and Γ_U, see Figure 2.6.

Proof. Let I be an interval and $\psi \in Lip(I)$ such that $Lip(\psi) \le \varepsilon_0$, where ε_0 will be fixed later. Moreover, let $f \in L^2(U) \cap L^q(U)$ where U is defined as in the statement of the lemma.
For $(x,y) \in I \times (0,\delta) =: Q$ we define

$$\hat{f}(x,y) = f(x, y + \psi(x)).$$

Clearly, $\|\hat{f}\|_{L^2(Q)} = \|f\|_{L^2(U)}$ and $\|\hat{f}\|_{L^q(Q)} = \|f\|_{L^q(U)}$. Next, consider $\tilde{Q} = I \times (0, 2\delta)$ and

$$\tilde{f}(x,y) = \begin{cases} \hat{f}(x,y) & \text{in } I \times (0,\delta), \\ -\hat{f}(x, y-\delta) & \text{in } I \times (\delta, 2\delta). \end{cases}$$

By Proposition 2.3.1 (scale the second variable of $I \times (0, 2\delta)$ by a factor $\frac{1}{2}$ to receive a function defined on a cube) there is a solution $\tilde{Y} \in L^\infty(\tilde{Q};\mathbb{R}^2) \cap H_0^1(\tilde{Q};\mathbb{R}^2) \cap W_0^{1,q}(\tilde{Q};\mathbb{R}^2)$ to $\operatorname{div} \tilde{Y} = \tilde{f}$ in $\tilde{Q}$ which satisfies

$$\|\tilde{Y}\|_{L^\infty(\tilde{Q};\mathbb{R}^2)} + \|\tilde{Y}\|_{H^1(\tilde{Q};\mathbb{R}^2)} \le C\|\tilde{f}\|_{L^2(\tilde{Q})} \le C\|f\|_{L^2(U)} \text{ and } \|\tilde{Y}\|_{W^{1,q}(\tilde{Q};\mathbb{R}^2)} \le C\|f\|_{L^q(U)}.$$

Set for $(x,y) \in U$

$$Z(x,y) = \tilde{Y}(x, y - \psi(x)).$$

Notice that $Z = 0$ on Γ_U. Moreover, one computes for $(x, y) \in U$

$$\begin{aligned}\operatorname{div} Z(x,y) &= \partial_1 \tilde{Y}_1(x, y-\psi(x)) - \partial_2 \tilde{Y}_1(x, y-\psi(x))\psi'(x) + \partial_2 \tilde{Y}_2(x, y-\psi(x)) \\ &= \tilde{f}(x, y-\psi(x)) - \partial_2 \tilde{Y}_1(x, y-\psi(x))\psi'(x) \\ &= f(x,y) - \partial_2 \tilde{Y}_1(x, y-\psi(x))\psi'(x).\end{aligned}$$

Consequently, it holds

$$\| \operatorname{div} Z - f\|_{L^2(U)} \leq \varepsilon_0 \|\tilde{Y}\|_{H^1(\tilde{Q};\mathbb{R}^2)} \leq C\varepsilon_0 \|f\|_{L^2(U)} \text{ and } \| \operatorname{div} Z - f\|_{L^q(U)} \leq C\varepsilon_0 \|f\|_{L^q(U)}.$$

Note that by scaling one can see that the constant does not depend on δ.
In a similar way one verifies that

- $\|Z\|_{L^\infty(U;\mathbb{R}^2)} \leq C\|f\|_{L^2(U)}$,
- $\|Z\|_{H^1(U;\mathbb{R}^2)} \leq C(1+\varepsilon_0)\|f\|_{L^2(U)}$,
- $\|Z\|_{W^{1,q}(U;\mathbb{R}^2)} \leq C(1+\varepsilon_0)\|f\|_{L^q(U)}$.

We can use this construction inductively to approach the desired solution to $\operatorname{div} Y = f$.
Set $f^1 = f$ and $Z^1 = Z$. Inductively, one finds for $k \geq 2$ and $f^k = \operatorname{div} Z^{k-1} - f^{k-1}$ a function $Z^k \in L^\infty(U;\mathbb{R}^2) \cap H^1(U;\mathbb{R}^2) \cap W^{1,q}(U;\mathbb{R}^2)$ such that

- $\| \operatorname{div} Z^k - f^k\|_{L^2(U)} \leq C\varepsilon_0 \|f^k\|_{L^2(U)} = C\varepsilon_0 \| \operatorname{div} Z^{k-1} - f^{k-1}\|_{L^2(U)} \leq (\varepsilon_0 C)^k \|f\|_{L^2(U)}$,
- $\| \operatorname{div} Z^k - f^k\|_{L^q(U)} \leq (\varepsilon_0 C)^k \|f\|_{L^q(U)}$,
- $\|Z^k\|_{L^\infty(U;\mathbb{R}^2)} \leq C\| \operatorname{div} Z^{k-1} - f^{k-1}\|_{L^2(U)} \leq C(\varepsilon_0 C)^{k-1}\|f\|_{L^2(U)}$,
- $\|Z^k\|_{H^1(U;\mathbb{R}^2)} \leq (1+\varepsilon_0)C\| \operatorname{div} Z^{k-1} - f^{k-1}\|_{L^2(U)} \leq C(1+\varepsilon_0)(\varepsilon_0 C)^{k-1}\|f\|_{L^2(U)}$,
- $\|Z\|_{W^{1,q}(U;\mathbb{R}^2)} \leq (1+\varepsilon_0)C\| \operatorname{div} Z^{k-1} - f^{k-1}\|_{L^q(U)} \leq C(1+\varepsilon_0)(\varepsilon_0 C)^{k-1}\|f\|_{L^q(U)}$.

Eventually, choose $\varepsilon_0 > 0$ so small such that $\varepsilon_0 C < \frac{1}{2}$. Then $Y = \sum_{k=1}^\infty Z^k$ fulfills the claim of the lemma. □

In the following lemma we show how one can get rid of the smallness condition $Lip(\psi) \leq \varepsilon_0$ by a scaling argument.

Lemma 2.4.3. *Let $2 < q < \infty$, I an interval, $\psi \in Lip(I)$, and U defined as in Lemma 2.4.2. Then there exists a constant $C > 0$ such that for every $f \in L^2(U) \cap L^q(U)$ there exists a function Y satisfying $Y = 0$ on Γ_U (Γ_U defined as in Lemma 2.4.2), $\operatorname{div} Y = f$,*

$$\|Y\|_{L^\infty(U;\mathbb{R}^2)} + \|Y\|_{H^1(U;\mathbb{R}^2)} \leq C\|f\|_{L^2(U)}, \text{ and } \|Y\|_{W^{1,q}(U;\mathbb{R}^2)} \leq C\|f\|_{L^q(U)}.$$

The constant C may depend on $Lip(\psi)$ but not on I.

Proof. Let I be an interval, δ the length of I, $\psi \in Lip(I)$, and $f \in L^2(U) \cap L^q(U)$.
Define $N = \lceil \frac{Lip(\psi)}{\varepsilon_0} \rceil$ where ε_0 is the constant from Lemma 2.4.2. Moreover, set $\tilde{I} = N \cdot I$. Next, define for $x \in \tilde{I}$ the function $\tilde{\psi}(x) = \psi(\frac{x}{N})$. Then $Lip(\tilde{\psi}) \leq \frac{Lip(\psi)}{N} \leq \varepsilon_0$. Define the function $\tilde{f} : \tilde{U} := \{(x,y) \in \tilde{I} \times \mathbb{R} : \tilde{\psi} < y < \tilde{\psi}(x) + \delta\} \to \mathbb{R}$ by

$$\tilde{f}(x,y) = f\left(\frac{x}{N}, y\right).$$

Subdivide $\tilde{I}$ into N subintervals $\tilde{I}_k$ of length δ. Now, the application of Lemma 2.4.2 to each function $\tilde{f}_{|\tilde{U}_k}$ on the set $\tilde{U}_k = \{(x,y) \in \tilde{I}_k \times \mathbb{R} : \tilde{\psi}(x) < y < \tilde{\psi}(x) + \delta\}$ provides a function Z^k such that $\operatorname{div} Z^k = \tilde{f}_k$ on $\tilde{U}_k$,

$$\|Z^k\|_{L^\infty(\tilde{U}_k;\mathbb{R}^2)} + \|Y\|_{H^1(\tilde{U}_k;\mathbb{R}^2)} \leq C\|\tilde{f}\|_{L^2(\tilde{U}_k)}, \text{ and } \|Z^k\|_{W^{1,q}(\tilde{U}_k;\mathbb{R}^2)} \leq C\|\tilde{f}\|_{L^q(\tilde{U}_k)}.$$

Moreover, it holds that $Z^k = 0$ on $\Gamma_{\tilde{U}_k}$. We define $Z : \tilde{U} \to \mathbb{R}^2$ by $Z(x,y) = Z^k(x,y)$ for $(x,y) \in \tilde{U}_k$. By the boundary values of Z^k on Γ_{U_k}, it holds that $\operatorname{div} Z = \tilde{f}$. Moreover, we obtain that Z is a Sobolev function satisfying the estimates

$$\|Z\|_{L^\infty(\tilde{U};\mathbb{R}^2)} + \|Z\|_{H^1(\tilde{U};\mathbb{R}^2)} \leq C\|\tilde{f}\|_{L^2(\tilde{U};\mathbb{R}^2)} \leq C(N)\|f\|_{L^2(U;\mathbb{R}^2)}$$
$$\text{and } \|Z\|_{W^{1,q}(\tilde{U};\mathbb{R}^2)} \leq C(N)\|f\|_{L^q(U;\mathbb{R}^2)}.$$

Finally, for $(x,y) \in U$ we set $Y(x,y) = (\frac{1}{N} Z_1(Nx,y), Z_2(Nx,y))$. From this definition one can check the desired properties for Y directly. □

Using the two previous lemmas discussing the local situation at the boundary, we are now able to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. Let $f \in L^2(\Omega) \cap L^q(\Omega)$.
By the definition of a Lipschitz boundary, we can find a finite cover of $\partial\Omega$ by sets $(U_i)_{i=1,\ldots,k}$ which have up to rotation the form as in Lemma 2.4.3 such that up to rotation $\partial U_i \cap \partial\Omega = \{(x, \psi_i(x)) : x \in I_i\}$. Let $(\theta_i)_{i=0,\ldots,k}$ be a partition of unity in the following sense. For $i = 1, \ldots, k$ let $\theta_i \in C^\infty(\Omega)$ such that $\theta_i = 0$ on $\Omega \setminus U_i$ and let $\theta_0 \in C_c^\infty(\Omega)$ such that $\sum_{i=0}^k \theta_i = 1$ on Ω. By applying Lemma 2.4.3, we find for $i = 1, \ldots, k$ solutions $Z^i \in L^\infty(U_i;\mathbb{R}^2) \cap H^1(U_i;\mathbb{R}^2) \cap W^{1,q}(U_i;\mathbb{R}^2)$ to $\operatorname{div} Z^i = f$ in U_i such that $Z^i = 0$ on $\partial U_i \cap \partial\Omega$ satisfying the bounds from Lemma 2.4.3. Note that the constant for the bounds depends only on ψ_i and hence on Ω.
Furthermore, let Q be a cube containing Ω. Extend f by 0 to Q and call this extension g. By Proposition 2.3.1, there exists $Z^0 \in L^\infty(Q;\mathbb{R}^2) \cap H_0^1(Q;\mathbb{R}^2) \cap W_0^{1,q}(Q;\mathbb{R}^2)$ such that $\operatorname{div} Z^0 = g$,

$$\|Z^0\|_{L^\infty(Q;\mathbb{R}^2)} + \|Z^0\|_{H^1(Q;\mathbb{R}^2)} \leq C\|f\|_{L^2(\Omega;\mathbb{R}^2)}, \text{ and } \|Z^0\|_{W^{1,q}(Q;\mathbb{R}^2)} \leq C\|f\|_{L^q(\Omega;\mathbb{R}^2)}.$$

Define $Z = \sum_{i=0}^k Z^i\theta_i$. Then $Z = 0$ on $\partial\Omega$. By construction, it holds also that $Z \in L^\infty(\Omega;\mathbb{R}^2) \cap H_0^1(\Omega;\mathbb{R}^2) \cap W_0^{1,q}(\Omega;\mathbb{R}^2)$ satisfies the estimates

$$\|Z\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|Z\|_{H^1(\Omega;\mathbb{R}^2)} \leq C\|f\|_{L^2(\Omega;\mathbb{R}^2)}, \text{ and } \|Z\|_{W^{1,q}(\Omega;\mathbb{R}^2)} \leq C\|f\|_{L^q(\Omega;\mathbb{R}^2)}.$$

Moreover,

$$\operatorname{div} Z = f + \underbrace{\sum_{i=0}^k Z^i \cdot \nabla\theta_i}_{=:h}.$$

Note that $h = 0$ on $\partial\Omega$. In addition, straightforward estimates show that

$$\|h\|_{H^1(\Omega)} \leq C\|Z\|_{H^1(\Omega;\mathbb{R}^2)} \leq C\|f\|_{L^2(\Omega;)} \text{ and } \|h\|_{W^{1,q}(\Omega)} \leq C\|f\|_{L^q(\Omega)}. \tag{2.38}$$

Now, fix $q < r < \infty$. By the Sobolev embedding theorem, we derive from (2.38) that

$$\|h\|_{L^r(\Omega)} \le C\,\|h\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \text{ and } \|h\|_{L^r(\Omega)} \le C\|f\|_{L^q(\Omega)}. \tag{2.39}$$

By [11, Theorem 2'], there exists a function $R \in W_0^{1,r}(\Omega, \mathbb{R}^2)$ such that

$$\operatorname{div} R = h \text{ and } \|R\|_{W^{1,r}(\Omega;\mathbb{R}^2)} \le C\|h\|_{L^r(\Omega)}.$$

As $r > q > 2$ it follows from the Sobolev embedding theorem and (2.39) that

$$\begin{aligned}\|R\|_{H^1(\Omega;\mathbb{R}^2)} + \|R\|_{L^\infty(\Omega;\mathbb{R}^2)} &\le C\,\|R\|_{W^{1,r}(\Omega;\mathbb{R}^2)} \le C\|h\|_{L^r(\Omega)} \le \|f\|_{L^2(\Omega)} \\ \text{and } \|R\|_{W^{1,q}(\Omega;\mathbb{R}^2)} &\le \|f\|_{L^q(\Omega)}.\end{aligned}$$

Eventually, $Y = Z - R$ has the demanded properties. □

As discussed at the beginning of the chapter we use Theorem 2.4.1 to prove a decomposition statement for functions in $H_0^1 \cap W_0^{1,q}$. This statement is the content of the next theorem. This theorem can be understood as the primal version to the Bourgain-Brézis type estimate (Theorem 2.0.1) as the the Bourgain-Brézis type estimate can be derived from Theorem 2.4.4 by dualization.

Theorem 2.4.4 (The primal result)**.** *Let $2 < q < \infty$ and $\Omega \subset \mathbb{R}^2$ open, simply connected, bounded with Lipschitz boundary. Then there exists a constant $C > 0$ such that for every $\varphi \in H_0^1(\Omega;\mathbb{R}^2) \cap W_0^{1,q}(\Omega;\mathbb{R}^2)$ there exist $h \in H_0^2(\Omega) \cap W_0^{2,q}(\Omega)$ and $g \in L^\infty(\Omega;\mathbb{R}^2) \cap H_0^1(\Omega;\mathbb{R}^2) \cap W_0^{1,q}(\Omega;\mathbb{R}^2)$ satisfying*

(i) $\varphi = g + \nabla h$,

(ii) $\|g\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|g\|_{H_0^1(\Omega;\mathbb{R}^2)} + \|h\|_{H_0^2(\Omega)} \le C\|\varphi\|_{H_0^1(\Omega;\mathbb{R}^2)}$,

(iii) $\|g\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} + \|h\|_{W_0^{2,q}(\Omega)} \le C\|\varphi\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)}$.

Proof. Let $\varphi \in H_0^1(\Omega;\mathbb{R}^2) \cap W_0^{1,q}(\Omega)$.
By the boundary values of φ it holds that $\int_\Omega \operatorname{curl}\varphi\,dx = 0$. The application of Theorem 2.4.1 to $\operatorname{curl}\varphi \in L^2(\Omega) \cap L^q(\Omega)$ provides a function $Y \in L^\infty(\Omega;\mathbb{R}^2) \cap H_0^1(\Omega;\mathbb{R}^2) \cap W_0^{1,q}(\Omega;\mathbb{R}^2)$ such that $\operatorname{div} Y = \operatorname{curl}\varphi$ and

$$\|Y\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|Y\|_{H_0^1(\Omega;\mathbb{R}^2)} \le C\|\operatorname{curl}\varphi\|_{L^2(\Omega)} \le C\|\varphi\|_{H^1(\Omega;\mathbb{R}^2)} \text{ and } \|Y\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} \le C\|\varphi\|_{W^{1,q}(\Omega;\mathbb{R}^2)}.$$

Set $g = Y^\perp = (-Y_2, Y_1)$. Then g satisfies the same bounds as Y and $\operatorname{curl} g = \operatorname{div} Y = \operatorname{curl}\varphi$. As Ω is simply-connected, by the Hodge decomposition there exists a vector field $h \in H^2(\Omega) \cap W^{2,q}(\Omega)$ such that $\varphi - g = \nabla h$,

$$\|h\|_{H^2(\Omega)} \le C\|g - \varphi\|_{H_0^1(\Omega;\mathbb{R}^2)} \le C\|\varphi\|_{H^1(\Omega;\mathbb{R}^2)}, \text{ and } \|h\|_{W^{2,q}(\Omega)} \le C\|\varphi\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)}.$$

Moreover, $\nabla h = \varphi - g = 0$ on $\partial\Omega$. Therefore, h is constant on the boundary of Ω and we may assume it is zero. Hence, $h \in H_0^2(\Omega) \cap W_0^{2,q}(\Omega)$. □

Remark 2.4.1. From Theorem 2.4.4 we can derive the corresponding dual statement i.e., a function $f \in L^1(\Omega;\mathbb{R}^2)$ satisfying $\operatorname{div} f = a + b \in H^{-2}(\Omega) + W^{-2,p}(\Omega)$, $p < 2$, is an element of the space $H^{-1}(\Omega;\mathbb{R}^2) + W^{-1,p}(\Omega;\mathbb{R}^2)$ and

$$\|f\|_{H^{-1}(\Omega;\mathbb{R}^2)+W^{-1,p}(\Omega;\mathbb{R}^2)} \le C\left(\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)} + \|b\|_{W^{-2,p}(\Omega)}\right).$$

In fact, let $\varphi \in H_0^1(\Omega;\mathbb{R}^2)\cap W_0^{1,p'}(\Omega;\mathbb{R}^2)$. We use the decomposition $\varphi = g+\nabla h$ from Theorem 2.4.4 and estimate

$$\begin{aligned}\int_\Omega f\varphi\,dx &= \int_\Omega f(g+\nabla h)\,dx\\ &\le C\left(\|f\|_{L^1(\Omega;\mathbb{R}^2)}\|g\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)}\|h\|_{H_0^2(\Omega)} + \|b\|_{W^{-2,p}(\Omega)}\|h\|_{W_0^{2,p'}(\Omega)}\right)\\ &\le C\left(\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)} + \|b\|_{W^{-2,p'}(\Omega)}\right)\max\left(\|\varphi\|_{H_0^1(\Omega;\mathbb{R}^2)}, \|\varphi\|_{W_0^{1,p}(\Omega;\mathbb{R}^2)}\right).\end{aligned}$$

In particular, $f\in (H_0^1(\Omega;\mathbb{R}^2)\cap W_0^{1,p'}(\Omega;\mathbb{R}^2))' = H^{-1}(\Omega;\mathbb{R}^2)+W^{-1,p}(\Omega;\mathbb{R}^2)$. Hence, it can be written as $f = A+B \in H^{-1}(\Omega;\mathbb{R}^2)+W^{-1,p}(\Omega;\mathbb{R}^2)$. The difference to the Bourgain-Brézis type estimate, which is our final goal in this chapter, is that A and B only satisfy a combined estimate, precisely

$$\|A\|_{H^{-1}(\Omega;\mathbb{R}^2)} + \|B\|_{W^{-1,p}(\Omega;\mathbb{R}^2)} \le C(\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)} + \|b\|_{W^{-2,p}(\Omega)}).$$

The point is to use a scaling argument to obtain separate estimates for A and B.

The classical $W^{k,p}$-norm and the homogeneous $W_0^{k,p}$-norm are equivalent norms on the space $W_0^{k,p}$. So far, it has not been important which of these norms we use on $W_0^{k,p}$. Next, we are interested in the scaling of the optimal constant in Theorem 2.4.4. We show the scaling invariance of the optimal constant in Theorem 2.4.4 for the homogeneous $W_0^{k,p}$-norms i.e., $\|f\|_{W_0^{k,p}(\Omega,\mathbb{R}^m)} = \sum_{|\alpha|=k}\|D^\alpha f\|_{L^p(\Omega;\mathbb{R}^m)}$.

Proposition 2.4.5. *Let $2<q<\infty$ and $\Omega\subset\mathbb{R}^2$ open, simply connected, bounded with Lipschitz boundary. Let $R>0$ and $\Omega_R = R\cdot\Omega$. If we denote by $C(\Omega)$, respectively $C(\Omega_R)$, the optimal constant of Theorem 2.4.4 for the domain Ω, respectively Ω_R, then $C(\Omega)=C(\Omega_R)$.*

Proof. Let $\varphi \in W_0^{1,q}\left(\Omega_R;\mathbb{R}^2\right)\cap H_0^1\left(\Omega_R;\mathbb{R}^2\right)$. Define the function $\varphi_{R^{-1}}:\Omega\to\mathbb{R}^2$ for $x\in\Omega$ by $\varphi_{R^{-1}}(x) = \varphi(Rx)$. Then $\varphi_{R^{-1}}\in W_0^{1,q}(\Omega;\mathbb{R}^2)\cap H_0^1(\Omega;\mathbb{R}^2)$ and $\varphi_{R^{-1}}$ fulfills

$$\begin{aligned}\|\varphi_{R^{-1}}\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} &= \sum_{i=1}^2 \|\partial_i\varphi_{R^{-1}}\|_{L^q(\Omega;\mathbb{R}^2)} = R^{1-\frac{2}{q}}\|\varphi\|_{W_0^{1,q}(\Omega_R;\mathbb{R}^2)}\\ \text{and}\ \ \|\varphi_{R^{-1}}\|_{H_0^1(\Omega;\mathbb{R}^2)} &= \|\varphi\|_{H_0^1(\Omega_R;\mathbb{R}^2)}.\end{aligned}$$

By Theorem 2.4.4, there exist $h_{R^{-1}}\in H_0^2(\Omega)\cap W_0^{2,q}(\Omega)$ and $g_{R^{-1}}\in L^\infty(\Omega;\mathbb{R}^2)\cap H_0^1(\Omega;\mathbb{R}^2)\cap W_0^{1,q}(\Omega;\mathbb{R}^2)$ such that $\varphi_{R^{-1}} = g_{R^{-1}}+\nabla h_{R^{-1}}$ and

$$\begin{aligned}\|g_{R^{-1}}\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|g_{R^{-1}}\|_{H_0^1(\Omega;\mathbb{R}^2)} + \|h_{R^{-1}}\|_{H_0^2(\Omega)} &\le C(\Omega)\|\varphi_{R^{-1}}\|_{H_0^1(\Omega;\mathbb{R}^2)}\\ &= C(\Omega)\|\varphi\|_{H_0^1(\Omega_R;\mathbb{R}^2)}, \qquad (2.40)\end{aligned}$$

$$\|g_{R^{-1}}\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} + \|h_{R^{-1}}\|_{W_0^{2,q}(\Omega)} \le C(\Omega)\|\varphi_{R^{-1}}\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} = C(\Omega)R^{1-\frac{2}{q}}\|\varphi\|_{W_0^{1,q}(\Omega_R;\mathbb{R}^2)}. \qquad (2.41)$$

Next, define the functions $g:\Omega_R\to\mathbb{R}^2$ and $h:\Omega_R\to\mathbb{R}$ for $x\in\Omega_R$ by $g(x) = g_{R^{-1}}\left(\frac{x}{R}\right)$ and $h(x) = R\,h_{R^{-1}}\left(\frac{x}{R}\right)$. Then it holds $\varphi = g+\nabla h$. Moreover, by (2.40) and (2.41), it follows that

$$\begin{aligned}\|g\|_{L^\infty(\Omega_R;\mathbb{R}^2)} + \|g\|_{H_0^1(\Omega_R;\mathbb{R}^2)} + \|h\|_{H_0^2(\Omega_R)} &= \|g_{R^{-1}}\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|g_{R^{-1}}\|_{H_0^1(\Omega;\mathbb{R}^2)} + \|h_{R^{-1}}\|_{H_0^2(\Omega)}\\ &\le C(\Omega)\|\varphi\|_{H_0^1(\Omega_R)}\\ \text{and}\ \|g\|_{W_0^{1,q}(\Omega_R;\mathbb{R}^2)} + \|h\|_{W_0^{2,q}(\Omega_R)} &= R^{-1+\frac{2}{q}}(\|g_{R^{-1}}\|_{W_0^{1,q}(\Omega;\mathbb{R}^2)} + \|h_{R^{-1}}\|_{W^{2,q}(\Omega)})\\ &\le C(\Omega)\|\varphi\|_{W_0^{1,q}(\Omega_R)}.\end{aligned}$$

This proves $C(\Omega) \leq C(\Omega_R)$. The reverse inequality follows by $\Omega = (\Omega_R)_{R^{-1}}$. □

Using Proposition 2.4.5, we can finally prove the main result of this chapter by a scaling argument.

Theorem (Bourgain-Brézis type estimate)**.** *Let $1 < p < 2$ and $\Omega \subset \mathbb{R}^2$ open, simply connected, and bounded with Lipschitz boundary. Then there exists a constant $C > 0$ such that for every $f \in L^1(\Omega; \mathbb{R}^2)$ satisfying $\operatorname{div} f = a + b \in H^{-2}(\Omega) + W^{-2,p}(\Omega)$ there exist $A \in H^{-1}(\Omega; \mathbb{R}^2)$ and $B \in W^{-1,p}(\Omega; \mathbb{R}^2)$ such that the following holds:*

(i) $f = A + B$,

(ii) $\|A\|_{H^{-1}(\Omega;\mathbb{R}^2)} \leq C(\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)})$,

(iii) $\|B\|_{W^{-1,p}(\Omega;\mathbb{R}^2)} \leq C\|b\|_{W^{-2,p}(\Omega)}$.

Proof. Let $f \in L^1(\Omega; \mathbb{R}^2)$, $R > 0$, and $\Omega_R = R \cdot \Omega$.
Define the function $f_R : \Omega_R \to \mathbb{R}^2$ by $f_R(x) = f\left(\frac{x}{R}\right)$ for $x \in \Omega_R$. Now, consider a test function $\varphi \in H_0^1(\Omega_R; \mathbb{R}^2) \cap W_0^{1,p'}(\Omega_R; \mathbb{R}^2)$. By Theorem 2.4.4, there exist functions $h \in H_0^2(\Omega_R) \cap W_0^{2,p'}(\Omega_R)$ and $g \in L^\infty(\Omega_R; \mathbb{R}^2) \cap H_0^1(\Omega_R; \mathbb{R}^2) \cap W_0^{1,p'}(\Omega_R; \mathbb{R}^2)$ such that $\varphi = g + \nabla h$,

$$\begin{aligned} \|g\|_{L^\infty(\Omega_R;\mathbb{R}^2)} + \|g\|_{H_0^1(\Omega_R;\mathbb{R}^2)} + \|h\|_{H_0^2(\Omega_R)} &\leq C\|\varphi\|_{H_0^1(\Omega_R;\mathbb{R}^2)}, \\ \text{and } \|g\|_{W_0^{1,p'}(\Omega_R;\mathbb{R}^2)} + \|h\|_{W_0^{2,p'}(\Omega_R)} &\leq C\|\varphi\|_{W_0^{1,p'}(\Omega_R;\mathbb{R}^2)}. \end{aligned}$$

Note that by Proposition 2.4.5 the constant C does not depend on R. Next, notice that

$$\begin{aligned} &\int_{\Omega_R} f_R \cdot \varphi \, dx = \int_{\Omega_R} f_R \cdot (g + \nabla h) \, dx \\ &= < f_R, g >_{L^1(\Omega_R;\mathbb{R}^2), L^\infty(\Omega_R;\mathbb{R}^2)} - < a_R, h >_{H^{-2}(\Omega_R), H_0^2(\Omega_R)} - < b_R, h >_{W^{-2,p}(\Omega_R), W_0^{2,p'}(\Omega_R)}, \end{aligned} \tag{2.42}$$

where a_R is defined by $< a_R, h >_{H^{-2}(\Omega_R), H_0^2(\Omega_R)} = R < a, h_{R^{-1}} >_{H^{-2}(\Omega), H_0^2(\Omega)}$ and b_R is defined by $< b_R, h >_{W^{-2,p}(\Omega_R), W_0^{2,p'}(\Omega_R)} = R < b, h_{R^{-1}} >_{W^{-2,p}(\Omega), W_0^{2,p'}(\Omega)}$ for $h_{R^{-1}}(x) = h(Rx)$. By scaling one sees that

$$\|a_R\|_{H^{-2}(\Omega_R)} = R^2 \|a\|_{H^{-2}(\Omega)} \text{ and } \|b_R\|_{W^{-2,p}(\Omega_R)} = R^{1+\frac{2}{p}} \|b\|_{W^{-2,p}(\Omega)}. \tag{2.43}$$

Moreover, from (2.42) we derive that

$$\begin{aligned} &\left| \int_{\Omega_R} f_R \cdot \varphi \, dx \right| \\ &\leq C \left(\|f_R\|_{L^1(\Omega_R;\mathbb{R}^2)} + \|a_R\|_{H^{-2}(\Omega_R)} + \|b_R\|_{W^{-2,p}(\Omega_R)} \right) \max \left(\|\varphi\|_{H^1(\Omega_R;\mathbb{R}^2)}, \|\varphi\|_{W^{1,p'}(\Omega_R;\mathbb{R}^2)} \right). \end{aligned}$$

The dual space of $H_0^1(\Omega_R; \mathbb{R}^2) \cap W_0^{1,p'}(\Omega_R; \mathbb{R}^2)$ equipped with the norm

$$\|\varphi\|_{H_0^1(\Omega_R;\mathbb{R}^2) \cap W_0^{1,p'}(\Omega_R;\mathbb{R}^2)} = \max \left(\|\varphi\|_{H_0^1(\Omega_R;\mathbb{R}^2)}, \|\varphi\|_{W_0^{1,p'}(\Omega_R;\mathbb{R}^2)} \right)$$

is isomorphic to the space $H^{-1}(\Omega_R; \mathbb{R}^2) + W^{-1,p}(\Omega_R; \mathbb{R}^2)$ endowed with the norm

$$\|F\|_{H^{-1}(\Omega_R;\mathbb{R}^2) + W^{-1,p}(\Omega_R;\mathbb{R}^2)} = \inf\{\|F_1\|_{H^{-1}(\Omega_R;\mathbb{R}^2)} + \|F_2\|_{W^{-1,p}(\Omega_R;\mathbb{R}^2)} : F_1 + F_2 = F\}.$$

Hence, $f_R \in H^{-1}(\Omega_R; \mathbb{R}^2) + W^{-1,p}(\Omega_R; \mathbb{R}^2)$ and

$$\|f_R\|_{H^{-1}(\Omega_R;\mathbb{R}^2) + W^{-1,p}(\Omega_R;\mathbb{R}^2)} \leq C \left(\|f_R\|_{L^1(\Omega_R;\mathbb{R}^2)} + \|a_R\|_{H^{-2}(\Omega_R)} + \|b_R\|_{W^{-2,p}(\Omega_R)} \right).$$

In particular, there exist $A_R \in H^{-1}\left(\Omega_R;\mathbb{R}^2\right), B_R \in W^{-1,p}\left(\Omega_R;\mathbb{R}^2\right)$ such that $f_R = A_R + B_R$ and

$$\|A_R\|_{H^{-1}(\Omega_R)} + \|B_R\|_{W^{-1,p}(\Omega_R)} \leq C\left(\|f_R\|_{L^1(\Omega_R)} + \|a_R\|_{H^{-2}(\Omega_R)} + \|b_R\|_{W^{-2,p}(\Omega_R)}\right). \tag{2.44}$$

We define $A \in H^{-1}(\Omega;\mathbb{R}^2)$ and $B \in W^{-1,p}(\Omega;\mathbb{R}^2)$ by

$$\begin{aligned} &< A, \varphi >_{H^{-1}(\Omega;\mathbb{R}^2),H_0^1(\Omega;\mathbb{R}^2)} = R^{-2} < A_R, \varphi_R >_{H^{-1}(\Omega_R;\mathbb{R}^2),H_0^1(\Omega_R;\mathbb{R}^2)} \\ \text{and } &< B, \varphi >_{W^{-1,p}(\Omega;\mathbb{R}^2),W_0^{1,p'}(\Omega;\mathbb{R}^2)} = R^{-2} < B_R, \varphi_R >_{W^{-1,p}(\Omega_R;\mathbb{R}^2),W_0^{1,p'}(\Omega_R;\mathbb{R}^2)} \end{aligned}$$

for every $\varphi \in C_c^\infty(\Omega)$ and $\varphi_R \in C_c^\infty(\Omega_R)$ given by $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$. Then it holds for every $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega f \cdot \varphi \, dx = R^{-2} \int_{\Omega_R} f_R \cdot \varphi_R \, dx =< A, \varphi >_{H^{-1}(\Omega;\mathbb{R}^2),H_0^1(\Omega;\mathbb{R}^2)} + < B, \varphi >_{W^{-1,p}(\Omega;\mathbb{R}^2),W_0^{1,p'}(\Omega;\mathbb{R}^2)} .$$

Consequently, $f = A + B$. Moreover, by (2.43) and (2.44) we see that

$$\begin{aligned} \|A\|_{H^{-1}(\Omega;\mathbb{R}^2)} &= R^{-2} \|A_R\|_{H^{-1}(\Omega_R;\mathbb{R}^2)} \leq C\left(\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)} + R^{\frac{2}{p}-1} \|b\|_{W^{-2,p}(\Omega)}\right) \\ \|B\|_{W^{-1,p}(\Omega;\mathbb{R}^2)} &= R^{-1-\frac{2}{p}} \|B_R\|_{W^{-1,p}(\Omega_R;\mathbb{R}^2)} \\ &\leq C\left(R^{1-\frac{2}{p}}\left(\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)}\right) + \|b\|_{W^{-2,p}(\Omega)}\right). \end{aligned}$$

Choosing R such that $R^{1-\frac{2}{p}} = \frac{\|b\|_{W^{-2,p}(\Omega)}}{\|f\|_{L^1(\Omega;\mathbb{R}^2)} + \|a\|_{H^{-2}(\Omega)}}$ finishes the proof. □

Remark 2.4.2. Let us remark here that by a similar argumentation this result also holds for Radon measures.

3 A Generalized Rigidity Estimate with Mixed Growth

The goal of this section is to prove a rigidity estimate for fields with non-vanishing curl in the case of a nonlinear energy density with mixed growth. Precisely, we show the following theorem.

Theorem 3.0.1. *Let $1 < p < 2$ and $\Omega \subset \mathbb{R}^2$ open, simply connected, bounded with Lipschitz boundary. There exists a constant $C > 0$ such that for every $\beta \in L^p(\Omega; \mathbb{R}^{2\times 2})$ such that $\operatorname{curl}\beta \in \mathcal{M}(\Omega; \mathbb{R}^2)$ there exists a rotation $R \in SO(2)$ such that*

$$\int_\Omega |\beta - R|^2 \wedge |\beta - R|^p \, dx \leq C \left(\int_\Omega \operatorname{dist}(\beta, SO(2))^2 \wedge \operatorname{dist}(\beta, SO(2))^p dx + |\operatorname{curl}\beta|(\Omega)^2 \right).$$

One of the simplest versions of a rigidity statement is the following. For a given a connected set $\Omega \subset \mathbb{R}^2$ and a function $u \in C^2(\Omega; \mathbb{R}^2)$ such that $\nabla u(x) \in SO(2)$ for all $x \in \Omega$, there exists a rotation $R \in SO(2)$ and vectors $a, b \in \mathbb{R}^2$ such that $u(x) = R(x - a) + b$, see [45].
Intuitively, this means that a deformation that is locally a rotation, is already a global rotation. The same statement is also true for infinitesimal rotations i.e., for the set of skew-symmetric matrices.
First qualitative versions of this statement are the different versions of Korn's inequality, see [36,52,53]. An estimate in the nonlinear case was developed by Friesecke, James and Müller in [37]. Extensions to the case of energy densities with mixed growth include [19, 58].
First results for fields with non-vanishing curl are an L^2-version of our result in [59, Theorem 3.3] and the generalized Korn's inequality in [38, Theorem 11].
The proofs of the statements in [38, 59] make use of the Bourgain-Brézis inequality as stated at the beginning of the previous chapter, see also [14, Lemma 3.3 and Remark 3.3]. The proof in our case is based on its counterpart in the case of mixed growth i.e., Theorem 2.0.1.

Before we are able to prove Theorem 3.0.1, we need to show two simple lemmas and a version of the classical rigidity estimates for gradients in the case of mixed growth which involves the weak L^2-norm, Proposition 3.0.5.
We start proving an easy triangle-inequality for a quantity with mixed growth.

Lemma 3.0.2. *Let $m \in \mathbb{N}$ and $1 < p < 2$. There exists a constant $C > 0$ such that for all $a, b \in \mathbb{R}^m$ it holds*

$$|a + b|^2 \wedge |a + b|^p \leq C \left(|a|^2 \wedge |a|^p + |b|^2 \wedge |b|^p \right).$$

Proof. We can restrict ourselves to the following cases:

1. $|a + b| \leq 1$
   a) $|a|, |b| \leq 1$. Here, the statement follows by the usual triangle inequality.
   b) $|b| > 1$. Then $|a + b|^2 \leq |b|^p \leq |a|^2 \wedge |a|^p + |b|^p = |a|^2 \wedge |a|^p + |b|^2 \wedge |b|^p$.

2. $|a+b| > 1$.

   a) $|a|, |b| \leq 1$. Then $|a+b| \leq 2$ and $|a| \geq \frac{1}{2}$ or $|b| \geq \frac{1}{2}$. Wlog $|b| \geq \frac{1}{2}$. Then $|a+b|^p \leq 2^p \leq 2^{p+2}|b|^2 \leq C\left(|a|^2 + |b|^2\right) = C\left(|a|^2 \wedge |a|^p + |b|^2 \wedge |b|^p\right)$.

   b) $|a| < 1, |b| \geq 1$. Then $|a+b|^p \leq 2^p|b|^p \leq 2^p\left(|a|^2 + |b|^p\right) = 2^p\left(|a|^2 \wedge |a|^p + |b|^2 \wedge |b|^p\right)$.

   c) $|a|, |b| \geq 1$. This follows again from the usual triangle-inequality.

□

Next, we prove a simple decomposition result for the sum of two functions $f \in L^{2,\infty}, g \in L^p$, where $1 < p < 2$, which we need later in the proof of the rigidity statement involving weak L^2-norms.

Lemma 3.0.3. *Let $U \subset \mathbb{R}^n$ and $1 < p < 2$. Then for every $k > 0$ there exists a constant $C(k) > 0$ such that for every two nonnegative functions $f \in L^{2,\infty}(U)$, $g \in L^p(U)$ there exist functions $\tilde{f} \in L^{2,\infty}(U)$ and $\tilde{g} \in L^p(U)$ such that*

(i) $f + g = \tilde{f} + \tilde{g}$,

(ii) $\left\|\tilde{f}\right\|^2_{L^{2,\infty}(U)} + \|\tilde{g}\|^p_{L^p(U)} \leq C(k)\left(\|f\|^2_{L^{2,\infty}(U)} + \|g\|^p_{L^p(U)}\right)$,

(iii) $\tilde{g} \in \{0\} \cup (k,\infty]$ *and* $\tilde{f} \leq k$.

Proof. Let $k > 0$. Define $\tilde{f} = (f+g)\mathbf{1}_{\{f+g\leq k\}}$ and $\tilde{g} = (f+g)\mathbf{1}_{\{f+g>k\}}$. Then (i) and (iii) are clearly satisfied. Moreover, we can estimate

$$\begin{aligned}
\left\|\tilde{f}\right\|^2_{L^{2,\infty}(U)} &\leq 4\|f\|^2_{L^{2,\infty}(U)} + 4\left\|g\mathbf{1}_{\{g\leq k\}}\right\|^2_{L^2(U)} \\
&\leq 4\|f\|^2_{L^{2,\infty}(U)} + 4k^{2-p}\left\|g\mathbf{1}_{\{g\leq k\}}\right\|^p_{L^p(U)} \\
&\leq C(k)\left(\|f\|^2_{L^{2,\infty}(U)} + \|g\|^p_{L^p(U)}\right).
\end{aligned}$$

For $\tilde{g}$, notice that $f\mathbf{1}_{\{f+g>k\}} = f\mathbf{1}_{\{f+g>k\}}\mathbf{1}_{\{f\leq\frac{k}{2}\}} + f\mathbf{1}_{\{f+g>k\}}\mathbf{1}_{\{f>\frac{k}{2}\}} \leq g + f\mathbf{1}_{\{f>\frac{k}{2}\}}$. Thus, we can conclude that $\|\tilde{g}\|^p_{L^p(U)} \leq C\left(\left\|f\mathbf{1}_{\{f>\frac{k}{2}\}}\right\|^p_{L^p(U)} + \|g\|^p_{L^p(U)}\right)$. We estimate

$$\begin{aligned}
\left\|\mathbf{1}_{\{f>\frac{k}{2}\}}f\right\|^p_{L^p(U)} &= \int_0^\infty pt^{p-1}\mathcal{L}^n(\{\mathbf{1}_{\{f>\frac{k}{2}\}}f > t\})\,dt \qquad (3.1)\\
&= \int_0^{\frac{k}{2}} pt^{p-1}\mathcal{L}^n\left(\left\{\mathbf{1}_{\{f>\frac{k}{2}\}}f > t\right\}\right)dt + \int_{\frac{k}{2}}^\infty pt^{p-1}\mathcal{L}^2\left(\left\{\mathbf{1}_{\{f>\frac{k}{2}\}}f > t\right\}\right)dt \\
&\leq C(k)\mathcal{L}^n\left(\left\{\mathbf{1}_{\{f>\frac{k}{2}\}}f > \frac{k}{2}\right\}\right) + \int_{\frac{k}{2}}^\infty pt^{p-3}\|f\|^2_{L^{2,\infty}(U)}\,dt \\
&\leq C(k)\mathcal{L}^n\left(\left\{f > \frac{k}{2}\right\}\right) + C(k)\|f\|^2_{L^{2,\infty}(U)} \\
&\leq C(k)\|f\|^2_{L^{2,\infty}(U)}.
\end{aligned}$$

Hence, $\|\tilde{g}\|^p_{L^p(U)} \leq C(k)\left(\|f\|^2_{L^{2,\infty}(U)} + \|g\|^p_{L^p(U)}\right)$. This finishes the proof. □

As a second ingredient for the proof of the preliminary mixed-growth rigidity result we need the following truncation argument from [37, Proposition A.1].

Proposition 3.0.4. *Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $m \geq 1$. Then there is a constant $c_1 = c_1(U)$ such that for all $u \in W^{1,1}(U, \mathbb{R}^m)$ and all $\lambda > 0$ there exists a measurable set $E \subset U$ such that*

(i) u is $c_1\lambda$-Lipschitz on E,

(ii) $\mathcal{L}^n(U \setminus E) \leq \frac{c_1}{\lambda} \int_{\{|\nabla u| > \lambda\}} |\nabla u| \, dx$.

With the use of this result, we are now able to prove the mixed-growth rigidity estimate involving weak norms. This result will be the last ingredient to prove the main result of this chapter, Theorem 3.0.1. In [19], the authors prove rigidity estimates for fields whose distance to $SO(n)$ is either the sum of an L^p- and an L^q-function, or in a weak space $L^{p,\infty}$. Our result is a combination of these results.

Proposition 3.0.5. *Let $1 < p < 2$, $n \geq 2$, and $U \subset \mathbb{R}^n$ open, simply connected and bounded with Lipschitz boundary. Let $u \in W^{1,1}(U; \mathbb{R}^{n \times n})$ such that there exist $f \in L^{2,\infty}(U)$ and $g \in L^p(U)$ such that*

$$\operatorname{dist}(\nabla u, SO(n)) = f + g.$$

Then there exist matrix fields $F \in L^{2,\infty}(U; \mathbb{R}^{n \times n})$ and $G \in L^p(U; \mathbb{R}^{n \times n})$ and a proper rotation $R \in SO(n)$ such that

$$\nabla u = R + G + F$$

and

$$\|F\|^2_{L^{2,\infty}(U;\mathbb{R}^{n \times n})} + \|G\|^p_{L^p(U;\mathbb{R}^{n \times n})} \leq C(\|f\|^2_{L^{2,\infty}(U)} + \|g\|^p_{L^p(U)}).$$

The constant C does not depend on u, f, g.

Proof. Without loss of generality we may assume that f and g are nonnegative. According to Lemma 3.0.3, we may also assume that $f \leq k$ and $g \in \{0\} \cup (k, \infty)$ where k will be fixed later.
First, we apply Proposition 3.0.4 for $\lambda = 2n$ to obtain a measurable set $E \subset U$ such that u is Lipschitz continuous on E with Lipschitz constant $M = 2c_1 n$. Let u_M be a Lipschitz continuous extension of $u_{|E}$ to U with the same Lipschitz constant. In particular, $u_M = u$ on E. Set $k = 2M$. Then we obtain

$$\operatorname{dist}(\nabla u_M, SO(2)) \leq f + 2M\mathbf{1}_{U \setminus E}. \tag{3.2}$$

Indeed, notice that

$$\operatorname{dist}(\nabla u_M, SO(2)) \leq 2c_1 n + \sqrt{n} \leq 2M. \tag{3.3}$$

Hence, we derive $\operatorname{dist}(\nabla u_M, SO(2)) \leq 2M$ on $U \setminus E$. On E, we obtain that

$$\operatorname{dist}(\nabla u_M, SO(2)) = \operatorname{dist}(\nabla u, SO(2)) = f + g.$$

As we may assume that $g \in \{0\} \cup (2M, \infty]$, in view of equation (3.3), it holds $\operatorname{dist}(\nabla u_M, SO(2)) = f$ on E. This shows (3.2).
By applying the weak-type rigidity estimate for $L^{2,\infty}$ from [19, Corollary 4.1], we find a proper rotation $R \in SO(2)$ such that

$$\begin{aligned} \|\nabla u_M - R\|^2_{L^{2,\infty}(U;\mathbb{R}^{n \times n})} &\leq C \, \|\operatorname{dist}(\nabla u_M, SO(2))\|^2_{L^{2,\infty}(U)} \\ &\leq 4C \, \|f\|^2_{L^{2,\infty}(U)} + 16CM^2 \left\|\mathbf{1}_{U \setminus E}\right\|^2_{L^{2,\infty}(U)} . \end{aligned} \tag{3.4}$$

Next, note that if $|\nabla u| > 2n$, then

$$|\nabla u| \leq \sqrt{n} + \operatorname{dist}(\nabla u, SO(n)) \leq 2 \operatorname{dist}(\nabla u, SO(n)) = 2(f+g) \leq 4 \max\{f, g\}. \tag{3.5}$$

Using Proposition 3.0.4 (ii) and (3.5), we can estimate

$$\begin{aligned}
\mathcal{L}^n(U \setminus E) &\leq \frac{c_1}{2n} \int_{\{|\nabla u| > 2n\}} |\nabla u| \, dx \\
&\leq \frac{c_1}{2n} \int_{\{4f \geq n\}} 4f \, dx + \frac{c_1}{2n} \int_{\{4g \geq n\}} 4g \, dx \\
&\leq \frac{c_1}{2n^p} \int_{\{4f \geq n\}} 4^p f^p \, dx + \frac{c_1}{2n^p} \int_{\{4g \geq n\}} 4^p g^p \, dx \\
&\leq C \left(\left\| f \mathbf{1}_{\{4f \geq n\}} \right\|_{L^p(U)}^p + \|g\|_{L^p(U)}^p \right) \\
&\leq C \left(\|f\|_{L^{2,\infty}(U)}^2 + \|g\|_{L^p(U)}^p \right),
\end{aligned}$$

where we used a similar estimate as in (3.1) for the last inequality. In particular, it follows from (3.4) that

$$\|\nabla u_M - R\|_{L^{2,\infty}(U;\mathbb{R}^{n \times n})}^2 \leq C(\|f\|_{L^{2,\infty}(U)}^2 + \|g\|_{L^p(U)}^p).$$

Hence, we can write $\nabla u - R = \nabla u - \nabla u_M + \nabla u_M - R$ and it remains to control $\nabla u - \nabla u_M$. Clearly, we only have to consider $\nabla u - \nabla u_M$ on $U \setminus E$. On $U \setminus E$, it holds the pointwise estimate

$$|\nabla u - \nabla u_M| \leq |\nabla u| + 2c_1 n \leq \operatorname{dist}(\nabla u, SO(2)) + 2M \mathbf{1}_{U \setminus E} = f + g + 2M \mathbf{1}_{U \setminus E}.$$

As before, we know that $\left\| \mathbf{1}_{U \setminus E} \right\|_{L^{2,\infty}(U)}^2 \leq C(\|f\|_{L^{2,\infty}(U)}^2 + \|g\|_{L^p(U)}^p)$. Therefore, we are able to write $\nabla u - \nabla u_M = h_1 + h_2$ where $\|h_1\|_{L^{2,\infty}(U;\mathbb{R}^{n \times n})}^2, \|h_2\|_{L^p(U;\mathbb{R}^{n \times n})}^p \leq C(\|f\|_{L^{2,\infty}(U)}^2 + \|g\|_{L^p(U)}^p)$. This finishes the proof. □

Armed with this weak-type rigidity estimate for mixed growth we are now able to prove the generalized rigidity estimate for fields with non-vanishing curl in our setting, Theorem 3.0.1. The proof is similar to the one of the corresponding statement in [59, Theorem 3.3] but uses quantities with mixed growth instead of quantities in L^2, in particular the Bourgain-Brézis type estimate for mixed growth, Theorem 2.0.1.

Proof of Theorem 3.0.1. Define $\delta = \left(\int_\Omega \operatorname{dist}(\beta, SO(2))^2 \wedge \operatorname{dist}(\beta, SO(2))^p dx + |\operatorname{curl} \beta|(\Omega)^2 \right)$.
As $1 < p < 2$, the embedding $\mathcal{M}(\Omega; \mathbb{R}^2) \hookrightarrow W^{-1,p}(\Omega; \mathbb{R}^2)$ is bounded. Hence, there exists a unique solution v to the problem

$$\begin{cases} \Delta v = \operatorname{curl} \beta, \\ v \in W_0^{1,p}(\Omega; \mathbb{R}^2). \end{cases} \tag{3.6}$$

Define $\tilde{\beta} = \nabla v J$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Optimal regularity for elliptic equations with measure valued right hand side yields (see e.g. [31])

$$\left\| \tilde{\beta} \right\|_{L^{2,\infty}(U;\mathbb{R}^{2 \times 2})} \leq C \left| \operatorname{curl} \beta \right| (\Omega). \tag{3.7}$$

In addition, we have that $\operatorname{curl} \tilde{\beta} = \operatorname{curl} \beta$. Hence, there exists a function $u \in W^{1,p}(\Omega; \mathbb{R}^2)$ such that $\nabla u = \beta - \tilde{\beta}$. Clearly,

$$|\operatorname{dist}(\nabla u, SO(2))| \leq |\tilde{\beta}| + |\operatorname{dist}(\beta, SO(2))| . \tag{3.8}$$

Notice that

$$\operatorname{dist}(\beta, SO(2)) = \underbrace{\operatorname{dist}(\beta, SO(2))\mathbf{1}_{\{|\operatorname{dist}(\beta,SO(2))|\leq 1\}}}_{=:f_1} + \underbrace{\operatorname{dist}(\beta, SO(2))\mathbf{1}_{\{|\operatorname{dist}(\beta,SO(2))|>1\}}}_{=:f_2},$$

where $\|f_1\|^2_{L^2(\Omega)} \leq \delta$ and $\|f_2\|^p_{L^p(\Omega)} \leq \delta$. Together with (3.7) and (3.8), this proves the existence of functions $g_1 \in L^{2,\infty}(\Omega)$ and $g_2 \in L^p(\Omega)$ such that

$$\begin{aligned} &\operatorname{dist}(\nabla u, SO(2)) = g_1 + g_2 \text{ where} \\ &\|g_1\|^2_{L^{2,\infty}(\Omega)} \leq 4 \left\|\tilde{\beta}\right\|^2_{L^{2,\infty}(\Omega)} + 4 \|f_1\|^2_{L^{2,\infty}(\Omega)} \leq C\delta \text{ and } \|g_2\|^p_{L^p(\Omega)} \leq \|f_2\|^p_{L^p(\Omega)} \leq C\delta. \end{aligned}$$

By Proposition 3.0.5, we derive the existence of a rotation $Q \in SO(2)$ and $G_1 \in L^{2,\infty}(\Omega; \mathbb{R}^{2\times 2})$, $G_2 \in L^p(\Omega; \mathbb{R}^{2\times 2})$ such that

$$\nabla u - Q = G_1 + G_2, \|G_1\|^2_{L^{2,\infty}} \leq C\delta, \text{ and } \|G_2\|^p_{L^p} \leq C\delta.$$

Without loss of generality we may assume that $Q = Id$ (otherwise replace β by $Q^T\beta$).
Next, let $\vartheta : \Omega \to [-\pi, \pi)$ be a measurable function such that the corresponding rotation

$$R(\vartheta) = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}$$

satisfies

$$|\beta(x) - R(\vartheta(x))| = \operatorname{dist}(\beta, SO(2)) \text{ for almost every } x \in \Omega. \tag{3.9}$$

Now, let us decompose

$$\begin{aligned} R(\vartheta(x)) - Id &= R(\vartheta(x)) - \beta + \beta - \nabla u + \nabla u - Id \\ &= R(\vartheta(x)) - \beta + \tilde{\beta} + G_1 + G_2. \end{aligned} \tag{3.10}$$

As $SO(2)$ is a bounded set, it is true that $|Id - R(\vartheta(x))|^2 \leq C|Id - R(\vartheta(x))|^p \wedge |Id - R(\vartheta(x))|^2$. In addition, one can check that $|R(\vartheta(x)) - Id| \geq \frac{|\vartheta(x)|}{2}$. Hence, by (3.9), (3.10), and the triangle inequality in Lemma 3.0.2, we can estimate

$$\frac{|\vartheta(x)|^2}{4} \leq |R(\vartheta(x)) - Id|^2 \leq C \left(\operatorname{dist}(\beta, SO(2))^2 \wedge \operatorname{dist}(\beta, SO(2))^p + |\tilde{\beta}|^2 + |G_1|^2 + |G_2|^p\right).$$

Taking the $L^{1,\infty}$-quasinorm on both sides of the inequality we obtain

$$\|\vartheta\|^2_{L^{2,\infty}(\Omega)} \leq C\delta. \tag{3.11}$$

Following [59, Theorem 3.3], we define the linearized rotation by

$$R_{lin}(\vartheta) = \begin{pmatrix} 1 & -\vartheta \\ \vartheta & 1 \end{pmatrix}.$$

Using [59, Lemma 3.2], we derive from (3.11) that

$$\|R(\vartheta) - R_{lin}(\vartheta)\|_{L^2}^2 \le C\delta.$$

Thus, we can find functions $h_1 \in L^2(\Omega;\mathbb{R}^{2\times 2})$ and $h_2 \in L^p(\Omega;\mathbb{R}^{2\times 2})$ such that

$$\beta - R_{lin}(\vartheta) = \underbrace{\beta - R(\vartheta)}_{\in L^p(\Omega;\mathbb{R}^{2\times 2}) + L^2(\Omega;\mathbb{R}^{2\times 2})} + \underbrace{R(\vartheta) - R_{lin}(\vartheta)}_{\in L^2(\Omega;\mathbb{R}^{2\times 2})} = h_1 + h_2 \tag{3.12}$$

and $\|h_1\|^p_{L^p(\Omega;\mathbb{R}^{2\times 2})}, \|h_2\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \le C\delta$. By definition, we see that $\operatorname{curl} R_{lin}(\vartheta) = -\nabla\vartheta$. Hence,

$$\operatorname{curl}\beta = -\nabla\vartheta + \operatorname{curl} h_1 + \operatorname{curl} h_2,$$

which implies

$$\operatorname{div}\left((\operatorname{curl}\beta)^\perp\right) = \underbrace{\operatorname{div}\left((\operatorname{curl} h_1)^\perp\right)}_{\in W^{-2,p}(\Omega)} + \underbrace{\operatorname{div}\left((\operatorname{curl} h_2)^\perp\right)}_{\in H^{-2}(\Omega)}.$$

Therefore, we can apply Theorem 2.0.1 to obtain $A \in H^{-1}(\Omega;\mathbb{R}^2)$ and $B \in W^{-1,p}(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned}(\operatorname{curl}\beta)^\perp = A + B,\ \|A\|^2_{H^{-1}(\Omega;\mathbb{R}^2)} &\le C(|\operatorname{curl}\beta|(\Omega)^2 + \left\|\operatorname{div}(\operatorname{curl} h_1)^\perp\right\|^2_{H^{-2}(\Omega)}),\\ \text{and } \|B\|^p_{W^{-1,p}(\Omega;\mathbb{R}^2)} &\le C\left\|\operatorname{div}(\operatorname{curl} h_2)^\perp\right\|^p_{W^{-2,p}(\Omega)}.\end{aligned} \tag{3.13}$$

In particular, one derives from (3.12) and (3.13) that

$$\|A\|^2_{H^{-1}(\Omega;\mathbb{R}^2)} \le C(|\operatorname{curl}\beta|(\Omega)^2 + \left\|\operatorname{div}(\operatorname{curl} h_1)^\perp\right\|^2_{H^{-2}(\Omega)}) \le C(\delta + \|h_1\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})}) \le C\delta \tag{3.14}$$

and similarly

$$\|B\|^p_{W^{-1,p}(\Omega;\mathbb{R}^2)} \le C\delta. \tag{3.15}$$

Clearly, the same holds for $\operatorname{curl}\beta$, $-A^\perp$ and $-B^\perp$. According to this decomposition of $\operatorname{curl}\beta$ we can also decompose the solution v to (3.6).

In fact, as v is the unique solution to the linear problem (3.6), in view of (3.13), (3.14) and (3.15) there exists a decomposition $v = v_1 + v_2$ where $\|v_1\|^2_{H^1(\Omega;\mathbb{R}^2)} \le C\delta$ and $\|v_2\|^p_{W^{1,p}(\Omega;\mathbb{R}^2)} \le C\delta$. Following the notation from the beginning of the proof, we define

$$\tilde\beta_1 = \nabla v_1 J \text{ and } \tilde\beta_2 = \nabla v_2 J.$$

Then $\nabla u = \beta - \tilde\beta = \beta - \tilde\beta_1 - \tilde\beta_2$. Now, using the classical mixed growth rigidity estimate from [58, Proposition 2.3], there exists a proper rotation $R \in SO(2)$ such that

$$\int_\Omega |\nabla u - R|^2 \wedge |\nabla u - R|^p \,\mathrm{d}x \le C\int_\Omega \operatorname{dist}(\nabla u, SO(2))^2 \wedge \operatorname{dist}(\nabla u, SO(2))^p \,\mathrm{d}x.$$

Eventually, we obtain with the use of Lemma 3.0.2 the following chain of inequalities

$$\begin{aligned}&\int_\Omega |\beta - R|^2 \wedge |\beta - R|^p \,\mathrm{d}x\\ &\le C\left(\int_\Omega |\nabla u - R|^2 \wedge |\nabla u - R|^p \,\mathrm{d}x + \left\|\tilde\beta_1\right\|^2_{L^2} + \left\|\tilde\beta_2\right\|^p_{L^p}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\int_\Omega \operatorname{dist}(\nabla u, SO(2))^2 \wedge \operatorname{dist}(\nabla u, SO(2))^p \ \mathrm{d}x + \delta\right) \\
&\leq C\left(\int_\Omega \operatorname{dist}(\beta, SO(2))^2 \wedge \operatorname{dist}(\beta, SO(2))^p \ \mathrm{d}x + \left\|\tilde{\beta}_1\right\|_{L^2}^2 + \left\|\tilde{\beta}_2\right\|_{L^p}^p + \delta\right) \\
&\leq C\delta,
\end{aligned}
$$

which finishes the proof. □

4 Plasticity as the Γ-Limit of a Nonlinear Dislocation Energy with Mixed Growth and the Assumption of Diluteness

In section 1.2, we discussed how to model the behavior of an infinite cylindrical body in which only straight, parallel edge dislocations appear. In this chapter, we investigate the behavior of the stored energy as the interatomic distance goes to zero under the assumption of well-separateness of dislocations. We focus on the situation of a nonlinear energy with subquadratic growth for large strains. This allows us to compute the stored energy without introducing an ad-hoc cut-off radius, see Section 1.3.
We characterize the Γ-limit of the suitably rescaled energy in all existing scaling regimes, Theorem 4.3.2, Theorem 4.4.2 and Theorem 4.5.1. In particular, in the so-called critical scaling regime we derive the same strain-gradient plasticity model as the authors in [38, 59] who started from models involving an ad-hoc cut-off radius around the dislocations. Hence, our result justifies a-posteriori their cut-off approach in this regime. Moreover, we discuss compactness properties in the different regimes, Theorem 4.3.1 and Theorem 4.4.1.

4.1 Setting of the Problem

In this section, we introduce the mathematical setting of the problem. For a discussion of the physical situation, see Section 1.2 and in particular Figure 1.9b.
We consider $\Omega \subset \mathbb{R}^2$ to be a simply-connected, bounded domain with Lipschitz boundary representing the cross section of an infinite cylindrical crystal. The set of (normalized) minimal Burgers vectors for the given crystal is denoted by $S = \{b_1, b_2\}$ for two linearly independent vectors $b_1, b_2 \in \mathbb{R}^2$. Moreover, we write

$$\mathbb{S} = \operatorname{span}_{\mathbb{Z}} S = \{\lambda_1 b_1 + \lambda_2 b_2 : \lambda_1, \lambda_2 \in \mathbb{Z}\}$$

for the set of (renormalized) admissible Burgers vectors.
Let $\varepsilon > 0$ the interatomic distance for the given crystal. The set of admissible dislocation densities is defined as

$$X_\varepsilon = \left\{ \mu \in \mathcal{M}(\Omega; \mathbb{R}^2) : \mu = \sum_{i=1}^{M} \varepsilon \xi_i \delta_{x_i},\ M \in \mathbb{N}, B_{\rho_\varepsilon}(x_i) \subset \Omega,\ |x_j - x_k| \geq 2\rho_\varepsilon \text{ for } j \neq k,\ 0 \neq \xi_i \in \mathbb{S} \right\},$$

where we assume that ρ_ε satisfies

1) $\lim_{\varepsilon \to 0} \rho_\varepsilon / \varepsilon^s = \infty$ for all fixed $s \in (0, 1)$ and

2) $\lim_{\varepsilon \to 0} |\log \varepsilon| \rho_\varepsilon^2 = 0$.

This means that we assume the dislocations to be separated on an intermediate scale $\varepsilon \ll \rho_\varepsilon \to 0$. Furthermore, we define the set of admissible strains generating $\mu \in X_\varepsilon$ by

$$\mathcal{AS}_\varepsilon(\mu) = \left\{\beta \in L^p(\Omega, \mathbb{R}^{2\times 2}) : \operatorname{curl}\beta = \mu \text{ in the sense of distributions}\right\}. \tag{4.1}$$

The energy density $W : \mathbb{R}^{2\times 2} \to [0,\infty)$ satisfies the usual assumptions of nonlinear elasticity:

(i) $W \in C^0(\mathbb{R}^{2\times 2})$ and $W \in C^2$ in a neighbourhood of $SO(2)$;

(ii) stress-free reference configuration: $W(Id) = 0$;

(iii) frame indifference: $W(RF) = W(F)$ for all $F \in \mathbb{R}^{2\times 2}$ and $R \in SO(2)$.

In addition, we assume that W satisfies the following growth condition:

(iv) there exists $\mathbf{1} < \mathbf{p} < \mathbf{2}$ and $0 < c \leq C$ such that for every $F \in \mathbb{R}^{2\times 2}$ it holds

$$c\left(\operatorname{dist}(F, SO(2))^2 \wedge \operatorname{dist}(F, SO(2))^p\right) \leq W(F) \leq C\left(\operatorname{dist}(F, SO(2))^2 \wedge \operatorname{dist}(F, SO(2))^p\right). \tag{4.2}$$

According to the scaling heuristics discussed in Section 1.3, we define the rescaled energy for $\varepsilon > 0$ by

$$E_\varepsilon(\mu,\beta) = \begin{cases} \frac{1}{\varepsilon^2 N_\varepsilon |\log\varepsilon|}\int_\Omega W(\beta)\,dx & \text{if } (\mu,\beta) \in X_\varepsilon \times \mathcal{AS}_\varepsilon(\mu), \\ +\infty & \text{else in } \mathcal{M}(\Omega;\mathbb{R}^2)\times L^p(\Omega;\mathbb{R}^{2\times 2}), \end{cases} \tag{4.3}$$

where we used the usual trick of extending the energy by $+\infty$ to non-admissible strains and dislocation densities.
Our goal is to determine the Γ-limit of E_ε as $\varepsilon \to 0$.
The behavior of the energy depends highly on the scaling of N_ε with respect to ε. As discussed in Section 1.3, the different scaling regimes are: the subcritical regime $N_\varepsilon \ll |\log\varepsilon|$, the critical regime $N_\varepsilon \sim |\log\varepsilon|$, and the supercritical regime $N_\varepsilon \gg |\log\varepsilon|$.

Before we can state the different Γ-convergence results, we introduce the self-energy of a dislocation, which corresponds to the minimal energy that a single dislocation induces. As discussed in Section 1.3, the self-energy of the dislocations is expected to contribute to the limit in the subcritical and the critical regime.

4.2 The Self-Energy

For proofs of the statements in this section, we refer to [38].
Let $0 < r_1 < r_2$ and $\xi \in \mathbb{R}^2$. We define

$$\mathcal{AS}_{r_1,r_2}(\xi) = \left\{\eta \in L^2\left(B_{r_2}(0)\setminus B_{r_1}(0);\mathbb{R}^{2\times 2}\right) : \operatorname{curl}\eta = 0 \text{ and } \int_{\partial B_{r_1}(0)} \eta\cdot t = \xi\right\}.$$

Here, τ denotes the unit tangent to $\partial B_{r_1}(0)$. The circulation condition has to be understood in the sense of traces. For a function $\eta \in L^2\left(B_{r_2}(0)\setminus B_{r_1}(0);\mathbb{R}^{2\times 2}\right)$ which is curl-free the tangential boundary values are well-defined in $H^{-\frac{1}{2}}\left(B_{r_2}(0)\setminus B_{r_1}(0);\mathbb{R}^2\right)$, cf. [29, Theorem 2]. The integral is

then understood as testing with the constant 1-function.
Note that this definition of admissible strains $\mathcal{AS}_{r_1,r_2}(\xi)$ is defined by a circulation condition and not by a curl-condition as in the previous section. Clearly, the two formulations are linked via Stoke's theorem.
Next, we set

$$\psi_{r_1,r_2}(\xi) = \min \left\{ \frac{1}{2} \int_{B_{r_2}(0)\setminus B_{r_1}(0)} \mathcal{C}\eta : \eta \, dx : \eta \in \mathcal{AS}_{r_1,r_2}(\xi) \right\}, \tag{4.4}$$

where $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(Id)$. Note that by scaling it holds that $\psi_{r_1,r_2}(\xi) = \psi_{\frac{r_1}{r_2},1}(\xi)$. The special case $r_2 = 1$ will be denoted by

$$\psi(\xi, \delta) = \min \left\{ \frac{1}{2} \int_{B_1(0)\setminus B_\delta(0)} \mathcal{C}\eta : \eta \, dx : \eta \in \mathcal{AS}_{1,\delta}(\xi) \right\}. \tag{4.5}$$

Observe that for fixed $\delta > 0$ the function $\psi(\cdot, \delta)$ is convex and 2-homogeneous.

We state here the following result from [38, Corollary 6 and Remark 7].

Proposition 4.2.1. *Let $\xi \in \mathbb{R}^2$, $\delta \in (0,1)$ and let $\psi(\xi, \delta)$ be defined as in* (4.5)*. Then for every $\xi \in \mathbb{R}^2$ it holds*

$$\lim_{\delta \to 0} \frac{\psi(\xi, \delta)}{|\log \delta|} = \psi(\xi),$$

where $\psi : \mathbb{R}^2 \to [0, \infty)$ is defined by

$$\psi(\xi) = \lim_{\delta \to 0} \frac{1}{|\log \delta|} \frac{1}{2} \int_{B_1(0)\setminus B_\delta(0)} \mathcal{C}\eta_0 : \eta_0 \, dx \tag{4.6}$$

and $\eta_0 : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is a fixed distributional solution to

$$\begin{cases} \operatorname{curl} \eta_0 = \xi \delta_0 & \text{in } \mathbb{R}^2, \\ \operatorname{div} \eta_0 = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

In particular, both limits exist. Moreover, there exists a constant $K > 0$ such that for all $\delta > 0$ small enough and $\xi \in \mathbb{R}^2$ it holds

$$\left| \frac{\psi(\xi, \delta)}{|\log \delta|} - \psi(\xi) \right| \leq K \frac{|\xi|^2}{|\log \delta|}.$$

Remark 4.2.1. Note that the function ψ is 2-homogeneous and convex.

Remark 4.2.2. In [38, Proposition 8], the authors show the following extension of the result above. Let $0 < r_\delta \to 0$ such that $\frac{\log(r_\delta)}{\log(\delta)} \to 0$. Define for $\xi \in \mathbb{R}^2$ the function $\tilde{\psi}(\cdot, \delta)$ by

$$\tilde{\psi}(\xi, \delta) = \min \left\{ \int_{B_{r_\delta}(0)\setminus B_\delta(0)} \frac{1}{2} \mathcal{C}\eta : \eta \, dx : \eta \in \mathcal{AS}_{r_\delta,\delta}(\xi) \right\}.$$

Then $\frac{\tilde{\psi}(\xi,\delta)}{|\log \delta|} = \frac{\psi(\xi,\delta)}{|\log \delta|}(1 + o(1))$ where $o(1) \to 0$ as $\delta \to 0$.

The function ψ is the (renormalized) limit self-energy of a single dislocation with Burgers vector ξ. The well-separateness condition on the dislocations does not prevent dislocations from merging to a single dislocation in the limit. This could lead to a smaller limit energy per dislocation than ψ. The classical way to capture this energetic behavior is to define the limit self-energy density φ through a relaxation procedure.

Definition 4.2.1. We define the function $\varphi : SO(2) \times \mathbb{R}^2 \to [0, \infty)$ by

$$\varphi(R, \xi) = \min \left\{ \sum_{k=1}^{M} \lambda_k \psi(R^T \xi_k) : \sum_{k=1}^{M} \lambda_k \xi_k,\ M \in \mathbb{N},\ \lambda_k \geq 0,\ \xi_k \in \mathbb{S} \right\}. \tag{4.7}$$

Remark 4.2.3. Indeed, it can be seen by the 2-homogeneity of ψ that the min in the definition of φ exists.

Remark 4.2.4. Note that $\varphi(R, -)$ is convex and 1-homogenous.

Remark 4.2.5. The dependence of the function φ on R reflects the fact that we start from a rotational invariant model and will end up with a linearized model.

4.3 The Critical Regime

In this section, we treat the case $N_\varepsilon \sim |\log \varepsilon|$; for the sake of a simpler notation we assume that $N_\varepsilon = |\log \varepsilon|$. We show that the energy E_ε as defined in (4.3) converges in the sense of Γ-convergence to a strain-gradient plasticity model of the form (see Theorem 4.3.2)

$$\int_\Omega \mathcal{C}\beta : \beta \, dx + \int_\Omega \varphi \left(R, \frac{d\mu}{d|\mu|} \right) d|\mu|,$$

where $\mu = R \operatorname{curl} \beta$ and R is the limit of a sequence of constant rotations provided by the generalized rigidity estimate (Theorem 3.0.1). Moreover, we show that for sequences of uniformly bounded energy $E_\varepsilon(\mu_\varepsilon, \beta_\varepsilon)$ there exists a subsequence such that suitably rescaled versions of μ_ε and β_ε converge in an appropriate sense.

Compactness

In this paragraph, we prove the compactness statement in the critical regime. The main ingredient in the proof will be the generalized rigidity estimate from Theorem 3.0.1. The result is the following.

Theorem 4.3.1 (Compactness)**.** *Let $\varepsilon_j \to 0$ and $N_{\varepsilon_j} = |\log \varepsilon_j|$. Let $(\beta_j, \mu_j)_j \subset L^p(\Omega, \mathbb{R}^{2\times 2}) \times \mathcal{M}(\Omega; \mathbb{R}^2)$ be a sequence such that $\sup_j E_{\varepsilon_j}(\mu_j, \beta_j) < \infty$. Then there exist a subsequence (not relabeled), a sequence $(R_j) \subset SO(2)$, a rotation $R \in SO(2)$, a measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2)$, and a function $\beta \in L^2(\Omega; \mathbb{R}^{2\times 2})$ such that*

(i) $\frac{\mu_j}{\varepsilon_j |\log \varepsilon_j|} \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}(\Omega; \mathbb{R}^2)$,

(ii) $\frac{R_j^T \beta_j - Id}{\varepsilon_j |\log \varepsilon_j|} \rightharpoonup \beta$ *in* $L^p(\Omega; \mathbb{R}^{2\times 2})$,

(iii) $R_j \to R$,

(iv) $\operatorname{curl} \beta = R^T \mu$.

Proof. **Step 1.** *Weak convergence of the scaled dislocation measures.*
In this step, it is our objective to show that there exists a constant $C > 0$ such that

$$\frac{|\mu_j|(\Omega)}{\varepsilon_j |\log \varepsilon_j|} \leq C.$$

Let us fix $\alpha \in (0,1)$. By the finiteness of the energy for (μ_j,β_j), it follows $\mu_j \in X_{\varepsilon_j}$; we write $\mu_j = \sum_{i=1}^{M_j} \varepsilon_j \xi_{i,j}\, \delta_{x_{i,j}}$ for appropriate $\xi_{i,j} \in \mathbb{S}$ and $x_{i,j} \in \mathbb{R}^2$. The uniform boundedness of the energy of (μ_j,β_j) and the well-separateness assumption on the dislocations guarantee that for j large enough it holds

$$C \geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{\varepsilon_j^\alpha}(x_{i,j}) \setminus B_{\varepsilon_j}(x_{i,j})} W(\beta_j)\, dx. \tag{4.8}$$

Furthermore, let $L_{i,j} = x_{i,j} + (\varepsilon_j, \varepsilon_j^\alpha) \times \{0\}$ and write $A_{i,j} = \Big(B_{\varepsilon_j^\alpha}(x_{i,j}) \setminus B_{\varepsilon_j}(x_{i,j})\Big) \setminus L_{i,j}$. As $A_{i,j}$ is simply connected, we can find functions $v_{i,j} \in W^{1,p}(A_{i,j};\mathbb{R}^{2\times 2})$ such that $\beta_j = \nabla v_{i,j}$ in $A_{i,j}$. A simple covering argument shows that all rigidity estimates hold also on the domain $A_{i,j}$ although it has no Lipschitz boundary. By applying the mixed-growth rigidity estimate for curl-free fields from [58, Proposition 2.3], we find rotations $R_{i,j} \in SO(2)$ such that for all $1 \leq i \leq M_j$ and $j \in \mathbb{N}$ it holds

$$\int_{A_{i,j}} |\nabla v_{i,j} - R_{i,j}|^2 \wedge |\nabla v_{i,j} - R_{i,j}|^p\, dx \leq C \int_{A_{i,j}} \operatorname{dist}(\nabla v_{i,j}, SO(2))^2 \wedge \operatorname{dist}(\nabla v_{i,j}, SO(2))^p\, dx. \tag{4.9}$$

Note that as the ratio of the $A_{i,j}$ is uniformly bounded from below, we can choose the constant C in the estimate above uniformly in i and j. Furthermore, using Jensen's inequality for third inequality, we find

$$\begin{aligned}
&\int_{A_{i,j}} |\nabla v_{i,j} - R_{i,j}|^2 \wedge |\nabla v_{i,j} - R_{i,j}|^p\, dx \\
\geq &\int_{\varepsilon_j}^{\varepsilon_j^\alpha} \int_{\partial B_t(x_{i,j})} \frac{|\beta_j - R_{i,j}|^2}{2} \wedge \frac{|\beta_j - R_{i,j}|^p}{p}\, d\mathcal{H}^1\, dt \\
\geq &\int_{\varepsilon_j}^{\varepsilon_j^\alpha} 2\pi t \fint_{\partial B_t(x_{i,j})} \frac{|(\beta_j - R_{i,j})\cdot\tau|^2}{2} \wedge \frac{|(\beta_j - R_{i,j})\cdot\tau|^p}{p}\, d\mathcal{H}^1\, dt \\
\geq &\int_{\varepsilon_j}^{\varepsilon_j^\alpha} 2\pi t \left(\frac{1}{2}\left| \fint_{\partial B_t(x_{i,j})} (\beta_j - R_{i,j})\cdot\tau d\mathcal{H}^1 \right|^2 \wedge \frac{1}{p}\left| \fint_{\partial B_t(x_{i,j})} (\beta_j - R_{i,j})\cdot\tau d\mathcal{H}^1 \right|^p \right) dt \\
\geq &\int_{\varepsilon_j}^{\varepsilon_j^\alpha} \pi t \left(\left| \frac{\varepsilon_j\, \xi_{i,j}}{2\pi t} \right|^2 \wedge \left| \frac{\varepsilon_j\, \xi_{i,j}}{2\pi t} \right|^p \right) dt.
\end{aligned} \tag{4.10}$$

Here, τ denotes the tangent to $\partial B_t(x_{i,j})$.

Claim: Let $\alpha < \gamma < 1$. Then it holds $\varepsilon_j |\xi_{i,j}| \leq \varepsilon_j^\gamma$ for all $1 \leq i \leq M_j$ and $j \in \mathbb{N}$ large enough.
Assume this is not the case i.e., there exists a subsequence (not relabeled) and indices $1 \leq i_j \leq M_j$ such that $\varepsilon_j |\xi_{i_j,j}| \geq \varepsilon_j^\gamma$. Combining (4.8), (4.9), and (4.10), we derive for j large enough that

$$\begin{aligned}
C &\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\varepsilon_j}^{\varepsilon_j^\alpha} \pi t \left(\left| \frac{\varepsilon_j\, \xi_{i_j,j}}{2\pi t} \right|^2 \wedge \left| \frac{\varepsilon_j\, \xi_{i_j,j}}{2\pi t} \right|^p \right) dt \\
&\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\varepsilon_j}^{\frac{\varepsilon_j^\gamma}{2\pi}} \pi t \left| \frac{\varepsilon_j\, \xi_{i_j,j}}{2\pi t} \right|^p dt
\end{aligned}$$

$$= \frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\varepsilon_j^p|\xi_{i_j,j}|^p 2^{-p}\pi^{1-p}(2-p)^{-1}\left(\frac{\varepsilon_j^{(2-p)\gamma}}{(2\pi)^{2-p}} - \varepsilon_j^{(2-p)}\right)$$

As we assume that $\varepsilon_j|\xi_{i_j,j}| \geq \varepsilon_j^\gamma$, we derive from the estimate above

$$C \geq 2^{-p}\pi^{1-p}(2-p)^{-1}\frac{1}{|\log\varepsilon_j|^2}\left(\frac{\varepsilon_j^{2(\gamma-1)}}{(2\pi)^{2-p}} - \varepsilon_j^{p(\gamma-1)}\right) \to \infty$$

since $2(\gamma-1) < p(\gamma-1) < 0$. Contradiction!

Fix $\alpha < \gamma < 1$. Using the claim, (4.8), (4.9), and (4.10), we can estimate

$$\begin{aligned} C &\geq \sum_{i=1}^{M_j}\frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\int_{\varepsilon_j}^{\varepsilon_j^\alpha}\pi t\left(\left|\frac{\varepsilon_j\,\xi_{i,j}}{2\pi t}\right|^2 \wedge \left|\frac{\varepsilon_j\,\xi_{i_j,j}}{2\pi t}\right|^p\right)dt \\ &\geq \frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{\varepsilon_j^\gamma}^{\varepsilon_j^\alpha}\pi t\left|\frac{\varepsilon_j\,\xi_{i,j}}{2\pi t}\right|^2 dt \\ &= \frac{1}{4\pi|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}|\xi_{i,j}|^2(\gamma-\alpha)|\log\varepsilon_j|. \end{aligned} \tag{4.11}$$

As the non-zero elements of $\mathbb{S}$ are bounded away from zero, we may derive from (4.11) that

$$C \geq \frac{1}{|\log\varepsilon_j|}\sum_{i=1}^{M_j}|\xi_{i,j}| = \frac{|\mu_j|(\Omega)}{\varepsilon_j|\log\varepsilon_j|}.$$

Hence, there exists a subsequence (not relabeled) and $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2)$ such that $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$.

Step 2. *Weak convergence of the scailed strains.*
By the finiteness of $E_{\varepsilon_j}(\mu_j,\beta_j)$, it follows that $\beta_j \in \mathcal{AS}_{\varepsilon_j}(\mu_j)$; in particular, $\operatorname{curl}\beta_j = \mu_j$. By the generalized rigidity estimate in Theorem 3.0.1, there exist rotations $R_j \in SO(2)$ such that

$$\int_\Omega |\beta_j - R_j|^2 \wedge |\beta_j - R_j|^p\,dx \leq C\left(\int_\Omega \operatorname{dist}(\beta_j,R)^2 \wedge \operatorname{dist}(\beta_j,R)^p\,dx + |\mu_j|(\Omega)^2\right).$$

From the lower bound on W (see (iv) in Section 4.1) and step 1 it follows

$$\int_\Omega |\beta_j - R_j|^2 \wedge |\beta_j - R_j|^p\,dx \leq C\varepsilon_j^2|\log\varepsilon_j|^2. \tag{4.12}$$

We set $G_j = \frac{R_j^T\beta_j - Id}{\varepsilon_j|\log\varepsilon_j|}$. Then $\varepsilon_j|\log\varepsilon_j||G_j| = |\beta_j - R|$. In particular, it holds

$$\int_\Omega |G_j|^2 \wedge \frac{|G_j|^p}{\varepsilon_j^{2-p}|\log\varepsilon_j|^{2-p}}\,dx \leq C. \tag{4.13}$$

This implies that $(G_j)_j$ is a bounded sequence in $L^p(\Omega;\mathbb{R}^{2\times2})$. Hence, there exists a subsequence (again denoted by G_j) which converges weakly in $L^p(\Omega;\mathbb{R}^{2\times2})$ to some function $\beta \in L^p(\Omega;\mathbb{R}^{2\times2})$. Next, we show that $\beta \in L^2(\Omega;\mathbb{R}^{2\times2})$.

Consider the decomposition of Ω into the two sets

$$A_j^2 = \left\{ x \in \Omega : |G_j(x)|^2 \leq \frac{|G_j(x)|^p}{\varepsilon_j^{2-p} |\log \varepsilon_j|^{2-p}} \right\} \text{ and } A_j^p = \left\{ x \in \Omega : |G_j(x)|^2 > \frac{|G_j(x)|^p}{\varepsilon_j^{2-p} |\log \varepsilon_j|^{2-p}} \right\}.$$

By (4.13), the sequence $|G_j|\mathbf{1}_{A_j^2}$ is bounded in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Consequently, up to taking a further subsequence, the sequence converges weakly in $L^2(\Omega;\mathbb{R}^{2\times 2})$ to a function $\tilde{\beta} \in L^2(\Omega;\mathbb{R}^{2\times 2})$. It remains to show that $\beta = \tilde{\beta}$.
By the definition of G_j, one can verify that

$$A_j^p = \left\{ x \in \Omega : |\beta_j - R_j|^2 > |\beta_j - R_j|^p \right\} = \{ x \in \Omega : |\beta_j - R_j| > 1 \}.$$

But, (4.12) implies

$$\left|A_j^p\right| \leq \int_{A_j^p} |\beta_j - R_j|^p \, dx \leq C\varepsilon_j^2 |\log \varepsilon_j|^2 \to 0.$$

Thus, $\mathbf{1}_{A_j^2} \to 1$ boundedly in measure. Together with $G_j \rightharpoonup \beta$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$, this ensures that

$$G_j \mathbf{1}_{A_j^2} \rightharpoonup \beta \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}).$$

Hence, $\beta = \tilde{\beta} \in L^2(\Omega;\mathbb{R}^{2\times 2})$.

Step 3. $\mu \in H^{-1}(\Omega;\mathbb{R}^2)$ *and* $\operatorname{curl}\beta = R^T\mu$.
As $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$, it is clear that $\operatorname{curl}\beta \in H^{-1}(\Omega;\mathbb{R}^2)$. Moreover, one computes for $\varphi \in C_c^\infty(\Omega;\mathbb{R}^2)$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ that

$$< \mu, \varphi >_{\mathcal{D}',\mathcal{D}} = \lim_j \frac{1}{\varepsilon_j |\log \varepsilon_j|} < \mu_j, \varphi >_{\mathcal{D}',\mathcal{D}} = \lim_j \frac{1}{\varepsilon_j |\log \varepsilon_j|} < \operatorname{curl}(\beta_j - R_j), \varphi >_{\mathcal{D}',\mathcal{D}}$$
$$= -\lim_j \frac{1}{\varepsilon_j |\log \varepsilon_j|} < \beta_j - R_j, J\nabla\varphi >_{\mathcal{D}',\mathcal{D}} = - < R\beta, J\nabla\varphi >_{\mathcal{D}',\mathcal{D}} = < \operatorname{curl}(R\beta), \varphi >_{\mathcal{D}',\mathcal{D}} .$$

As $\operatorname{curl}(R\beta) = R\operatorname{curl}\beta$, it follows that $R^T\mu = \operatorname{curl}\beta$. □

The Γ-convergence Result

This paragraph is devoted to state and prove the Γ-convergence result for the energy E_ε as defined in (4.3) in the critical regime $N_\varepsilon \sim |\log \varepsilon|$. First, we need to specify which topology we use in $\mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2})$ for the Γ-convergence result. In view of the compactness result in Theorem 4.3.1, we define the following notion of convergence.

Definition 4.3.1. Let $\varepsilon \to 0$. We say that a sequence $(\mu_\varepsilon, \beta_\varepsilon) \subset \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2})$ converges to a triplet $(\mu, \beta, R) \in \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ if there exists a sequence $(R_\varepsilon)_\varepsilon \subset SO(2)$ such that

$$\frac{\mu_\varepsilon}{\varepsilon|\log \varepsilon|} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2), \tag{4.14}$$

$$\frac{R_\varepsilon^T \beta_\varepsilon - I}{\varepsilon|\log \varepsilon|} \rightharpoonup \beta \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}), \text{ and } R_\varepsilon \to R. \tag{4.15}$$

With respect to this convergence we can now state the Γ-convergence result.

Theorem 4.3.2. *Let $N_\varepsilon = |\log \varepsilon|$. The energy functional E_ε defined as in* (4.3) *Γ-converges with respect to the notion of convergence given in Definition 4.3.1 to the functional E^{crit} defined on $\mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times2}) \times SO(2)$ as*

$$E^{crit}(\mu,\beta,R) = \begin{cases} \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx + \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \mu \in H^{-1}(\Omega;\mathbb{R}^2) \cap \mathcal{M}(\Omega;\mathbb{R}^2), \\ & \beta \in L^2(\Omega;\mathbb{R}^{2\times2}), \text{ and } \operatorname{curl}\beta = R^T\mu, \\ +\infty & \text{otherwise}, \end{cases}$$

where $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(Id)$ and φ is the relaxed self-energy density as defined in (4.7).

The proof will be given in the next two propositions.

Proposition 4.3.3 (The lim inf-inequality)**.** *Let $\varepsilon_j \to 0$ and $N_{\varepsilon_j} = |\log \varepsilon_j|$. Let $(\mu_j, \beta_j) \subset \mathcal{M}(\Omega;\mathbb{R}^2)\times L^p(\Omega;\mathbb{R}^{2\times2})$ be a sequence that converges to a triplet $(\mu,\beta,R) \in \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times2}) \times SO(2)$ in the sense of Definition 4.3.1. Then*

$$\liminf_{j\to\infty} E_{\varepsilon_j}(\mu_j,\beta_j) \geq E^{crit}(\mu,\beta,R).$$

Proof. We may assume that $\liminf_{j\to\infty} E_{\varepsilon_j}(\mu_j,\beta_j) = \lim_{j\to\infty} E_{\varepsilon_j}(\mu_j,\beta_j)$. Moreover, we may assume that $\sup_j E_{\varepsilon_j}(\mu_j,\beta_j) < \infty$. This implies that $\mu_j \in X_{\varepsilon_j}$ and $\beta_j \in \mathcal{AS}_{\varepsilon_j}(\mu_j)$ for all j. In particular, the dislocation density μ_j is of the form $\mu_j = \sum_{i=1}^{M_j} \varepsilon_j \xi_{i,j}\,\delta_{x_{i,j}}$ for some $0 \neq \xi_{i,j} \in \mathbb{S}$ and $x_{i,j} \in \Omega$. A straightforward argument shows that the rotations provided by the application of the generalized rigidity estimate in the proof of the compactness result also converge to R. In the following, we assume that the R_j are those from the compactness result. Then it follows that $\beta \in L^2(\Omega;\mathbb{R}^{2\times2})$, $\mu \in H^{-1}(\Omega;\mathbb{R}^2) \cap \mathcal{M}(\Omega;\mathbb{R}^2)$, and $\operatorname{curl}\beta = R^T\mu$.

In order to prove the lower bound, we subdivide the energy $E_{\varepsilon_j}(\mu_j,\beta_j)$ into a part far from the dislocations and a contribution close to the dislocations (see also Figure 4.1) i.e.,

$$E_{\varepsilon_j}(\mu_j,\beta_j) = \frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\int_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} W(\beta_j)\,dx + \frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{B_{\rho_{\varepsilon_j}}(x_{i,j})} W(\beta_j)\,dx,$$

where we define for $r > 0$ the set $\Omega_r(\mu_j) = \Omega \setminus \bigcup_{i=1}^{M_j} B_r(x_{i,j})$. The two contributions will be treated separately.
Recalling the heuristics from Section 1.3, the second term on the right hand side should include the self-energies of the dislocations. The first term on the right hand side is the elastic interaction energy of the dislocations. By the rescaling, this term should just linearize in the limit.

Lower bound far from the dislocations. We will perform a second order Taylor expansion at scale $\varepsilon_j|\log\varepsilon_j|$ of the function W. As the energy density has a minimum at the identity matrix, there exists a function $\sigma : \mathbb{R}^{2\times2} \to \mathbb{R}$ such that for all $F \in \mathbb{R}^{2\times2}$ it holds

$$W(Id + F) = \frac{1}{2}\mathcal{C}F : F + \sigma(F)$$

and $\sigma(F)/|F|^2 \to 0$ as $|F| \to 0$. Set $\omega(t) = \sup_{|F|\leq t}|\sigma(F)|$. Notice that also $\omega(t)/t^2 \to 0$ as $t \to 0$.

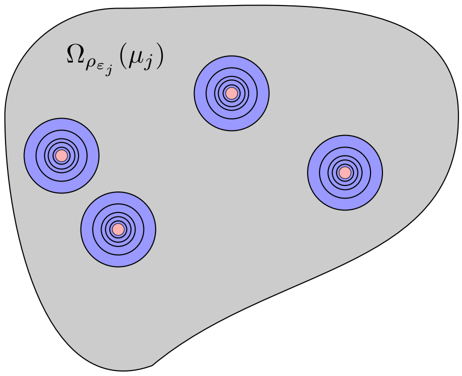


Figure 4.1: Sketch of the situation in the proof of the lim inf-inequality. The reduced domain $\Omega_{\rho_{\varepsilon_j}}(\mu_j)$ is drawn in gray, the balls around the dislocations with radius $\delta\varepsilon_j^\alpha$ are drawn in red. The annuli $B_{\rho_{\varepsilon_j}}(x_{i,j}) \setminus B_{\delta\varepsilon_j^\alpha}(x_{i,j})$ are drawn in blue and subdivided into annuli with constant ratio δ^{-1}.

Moreover, we obtain for all $F \in \mathbb{R}^{2\times 2}$ that

$$W(Id + \varepsilon_j|\log\varepsilon_j|F) \geq \frac{1}{2}\varepsilon_j^2|\log\varepsilon_j|^2\mathcal{C}F : F - \omega(\varepsilon_j|\log\varepsilon_j||F|). \tag{4.16}$$

Next, define

$$G_j = \frac{R_j^T\beta_j - Id}{\varepsilon_j|\log\varepsilon_j|}.$$

and

$$A^2_{\varepsilon_j} = \left\{ x \in \Omega : |G_j(x)|^2 \leq \frac{|G_j|^p}{\varepsilon_j^{2-p}|\log\varepsilon_j|^{2-p}} \right\}.$$

As in the proof of the compactness (Theorem 4.3.1), it can be shown that $G_j\mathbf{1}_{A^2_{\varepsilon_j}} \rightharpoonup \beta$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$ and $\mathbf{1}_{A^2_{\varepsilon_j}} \to 1$ boundedly in measure. Furthermore, define the set

$$B_{\varepsilon_j} = \left\{ x \in \Omega : |G_j| \leq \varepsilon_j^{-\frac{1}{2}} \right\}.$$

The boundedness of the sequence $(G_j)_j$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$ yields that $\mathbf{1}_{B_{\varepsilon_j}} \to 1$ boundedly in measure. In the proof of the compactness result, it was shown that $\frac{|\mu_j|(\Omega)}{\varepsilon_j|\log\varepsilon_j|} \leq C$. As the non-zero elements of $\mathbb{S}$ are bounded away from zero, it follows for the number of dislocations M_j that

$$M_j \leq C\sum_{i=1}^{M_j}|\xi_{i,j}| = C\frac{|\mu_j|(\Omega)}{\varepsilon_j} \leq C|\log\varepsilon_j|.$$

Hence, by the assumptions on ρ_{ε_j} it holds

$$|\Omega \setminus \Omega_{\rho_{\varepsilon_j}}(\mu_j)| \leq C|\log\varepsilon_j|\rho^2_{\varepsilon_j} \to 0.$$

Consequently, $\mathbf{1}_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} \to 1$ boundedly in measure.

Eventually, we define the function

$$\chi_{\varepsilon_j}(x) = \begin{cases} 1 & \text{if } x \in \Omega_{\rho_{\varepsilon_j}}(\mu_j) \cap A^2_{\varepsilon_j} \cap B_{\varepsilon_j}, \\ 0 & \text{else.} \end{cases}$$

By the considerations above, we conclude that $\chi_{\varepsilon_j} \to 1$ boundedly in measure. As $G_j \mathbf{1}_{A^2_{\varepsilon_j}} \rightharpoonup \beta$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$, we derive that also

$$G_j \chi_{\varepsilon_j} = G_j \mathbf{1}_{A^2_{\varepsilon_j}} \chi_{\varepsilon_j} \rightharpoonup \beta \text{ in } L^2(\Omega; \mathbb{R}^{2\times 2}). \tag{4.17}$$

Using the frame indifference and (4.16), we can estimate

$$\begin{aligned} \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} W(\beta_j)\, dx &= \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} W(R_j^T \beta_j)\, dx \\ &\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_\Omega \chi_{\varepsilon_j} W(R_j^T \beta_j)\, dx \\ &= \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_\Omega \chi_{\varepsilon_j} W(Id + \varepsilon_j |\log \varepsilon_j| G_j)\, dx \\ &\geq \int_\Omega \frac{1}{2} \mathcal{C} \left(\chi_{\varepsilon_j} G_j\right) : \left(\chi_{\varepsilon_j} G_j\right) - \chi_{\varepsilon_j} \frac{\omega(\varepsilon_j |\log \varepsilon_j| |G_j|)}{\varepsilon_j^2 |\log \varepsilon_j|^2}\, dx \\ &= \int_\Omega \frac{1}{2} \mathcal{C} \left(\chi_{\varepsilon_j} G_j\right) : \left(\chi_{\varepsilon_j} G_j\right) - |\chi_{\varepsilon_j} G_j|^2 \frac{\omega(\varepsilon_j |\log \varepsilon_j| |G_j|)}{|G_j|^2 \varepsilon_j^2 |\log \varepsilon_j|^2}\, dx. \end{aligned} \tag{4.18}$$

Now, recall (4.17) and notice that the first term in (4.18) is lower semi-continuous with respect to weak convergence in $L^2(\Omega; \mathbb{R}^{2\times 2})$. For the second term in (4.18), recall that $(\chi_{\varepsilon_j} G_j)_{\varepsilon_j}$ is a bounded sequence in $L^2(\Omega; \mathbb{R}^{2\times 2})$. On the other hand, notice that whenever $\chi_{\varepsilon_j}(x) = 1$ it also holds that $\varepsilon_j |\log \varepsilon_j| |G_j(x)| \leq \varepsilon_j^{\frac{1}{2}} |\log \varepsilon_j| \to 0$. Hence, by the properties of ω we find that

$$\chi_{\varepsilon_j} \frac{\omega(\varepsilon_j |\log \varepsilon_j| |G_j|)}{|G_j|^2 \varepsilon_j^2 |\log \varepsilon_j|^2} \to 0 \text{ in } L^\infty(\Omega).$$

Thus,

$$\int_\Omega |\chi_{\varepsilon_j} G_j|^2 \frac{\omega(\varepsilon_j |\log \varepsilon_j| |G_j|)}{|G_j|^2 \varepsilon_j^2 |\log \varepsilon_j|^2}\, dx \to 0 \text{ as } \varepsilon_j \to 0.$$

Eventually, we derive

$$\liminf_{j\to\infty} \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} W(\beta_j)\, dx \geq \int_\Omega \frac{1}{2} \mathcal{C}\beta : \beta\, dx.$$

Lower bound close the dislocations. Fix $\alpha, \delta \in (0,1)$. We subdivide for each $i \in \{1, \ldots, M_j\}$ the annulus $B_{\rho_{\varepsilon_j}}(x_{i,j}) \setminus B_{\delta \varepsilon_j^\alpha}(x_{i,j})$ around the dislocation point $x_{i,j}$ into annuli with constant ratio δ^{-1}, see Figure 4.1. Precisely, we define

$$C_j^{k,i} = B_{\delta^{k-1} \rho_{\varepsilon_j}}(x_{i,j}) \setminus B_{\delta^k \rho_{\varepsilon_j}}(x_{i,j}) \tag{4.19}$$

for $k \in \{1, \ldots, \tilde{k}_j\}$ where

$$\tilde{k}_j = \left\lfloor \alpha \frac{|\log \varepsilon_j|}{|\log \delta|} - \frac{|\log \rho_{\varepsilon_j}|}{|\log \delta|} \right\rfloor + 1. \tag{4.20}$$

Notice that for $k \le \tilde{k}_j$ it holds $\delta^k \rho_{\varepsilon_j} \ge \delta^{\tilde{k}_j}\rho_{\varepsilon_j} \ge \delta\varepsilon_j^\alpha$. Hence, for every $j \in \mathbb{N}$ and $i \in \{1, \dots, M_j\}$ we have the estimate

$$\int_{B_{\rho_{\varepsilon_j}}(x_{i,j})\setminus B_{\delta\varepsilon_j^\alpha}(x_{i,j})} \frac{W(\beta_j)}{\varepsilon_j^2} \ge \sum_{k=1}^{\tilde{k}_j} \int_{C_j^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2}\, dx. \tag{4.21}$$

Similar to the proof of [71, Proposition 3.11], the key estimate for the lower bound close to the dislocations is the following result.

Lemma 4.3.4. *In the situation above, the following holds true. There exists a sequence* $\sigma_j \overset{j\to\infty}{\to} 0$ *such that for all* $j \in \mathbb{N}$*,* $i \in \{1, \dots, M_j\}$*, and* $k \in \{1, \dots, \tilde{k}_j\}$ *it holds*

$$\int_{C_j^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2}\, dx \ge \psi(R^T\xi_{i,j}, \delta) - \sigma_j |\xi_{i,j}|^2, \tag{4.22}$$

where $\psi(\cdot, \delta)$ *is defined as in* (4.5)*.*

Proof. The claim of the lemma is equivalent to

$$\liminf_{j\to\infty} \left(\sup_{k=1,\dots,\tilde{k}_j,\, i=1,\dots,M_j} \int_{C_j^{k,i}} \frac{W(\beta_j)}{|\xi_{i,j}|^2\varepsilon_j^2}\, dx - \psi\left(R^T\frac{\xi_{i,j}}{|\xi_{i,j}|}, \delta\right)\right) \ge 0,$$

where we used the 2-homogeneity of $\psi(\cdot, \delta)$.
Assume this is not the case i.e.,

$$\liminf_{j\to\infty} \sup_{k=1,\dots,\tilde{k}_j,\, i=1,\dots,M_j} \left(\int_{C_j^{k,i}} \frac{W(\beta_j)}{|\xi_{i,j}|^2\varepsilon_j^2}\, dx - \psi\left(R^T\frac{\xi_{i,j}}{|\xi_{i,j}|}, \delta\right)\right) < 0. \tag{4.23}$$

Up to extracting a subsequence, we may assume that the lim inf above is a limit.
For every $j \in \mathbb{N}$ we denote by $k_j \in \{1, \dots, \tilde{k}_j\}$ and $i_j \in \{1, \dots, M_j\}$ the indices that maximize $\int_{C_j^{k,i}} \frac{W(\beta_j)}{|\xi_{i,j}|^2\varepsilon_j^2}\, dx - \psi\left(R^T\frac{\xi_{i,j}}{|\xi_{i,j}|}, \delta\right)$ among all $k \in \{1, \dots, \tilde{k}_j\}$ and $i \in \{1, \dots, M_j\}$.
The assumption (4.23) implies for $j \in \mathbb{N}$ large enough the bound

$$\int_{C_j^{k_j,i_j}} \frac{W(\beta_j)}{|\xi_{i_j,j}|^2\varepsilon_j^2}\, dx \le \psi\left(R^T\frac{\xi_{i_j,j}}{|\xi_{i_j,j}|^2}, \delta\right) \le C,$$

where we used that $\sup_{|\xi|=1}\psi(\xi, \delta) < \infty$ for the second inequality. This follows from convexity and pointwise finiteness of $\psi(\cdot, \delta)$. Next, we apply the classical mixed-growth rigidity estimate [58, Proposition 2.3] on $C_j^{k_j,i_j}$ to obtain rotations $\tilde{R}_j \in SO(2)$ such that

$$\begin{aligned}\int_{C_j^{k_j,i_j}} \frac{\left|\beta_j - \tilde{R}_j\right|^2 \wedge \left|\beta_j - \tilde{R}_j\right|^p}{|\xi_{i_j,j}|^2\varepsilon_j^2}\, dx &\le C\int_{C_j^{k_j,i_j}} \frac{\operatorname{dist}(\beta_j, SO(2))^2 \wedge \operatorname{dist}(\beta_j, SO(2))^p}{|\xi_{i_j,j}|^2\varepsilon_j^2}\, dx \\ &\le C\int_{C_j^{k_j,i_j}} \frac{W(\beta_j)}{|\xi_{i_j,j}|^2\varepsilon_j^2}\, dx.\end{aligned} \tag{4.24}$$

Here, we used the lower bound on the energy density W, (iv) in Section 4.1, for the second inequality. As the ratio of all $C_j^{k_j,i_j}$ is the same for all j, the constant on the right hand side can be chosen uniformly for all j.
As the sequence of rotations R_j comes from the application of the generalized rigidity estimate in the

proof of the compactness result, it follows that

$$\int_{C_j^{k_j,i_j}} |\beta_j - R_j|^2 \wedge |\beta_j - R_j|^p \, dx \leq C\varepsilon_j^2 |\log \varepsilon_j|^2.$$

Together with (4.24), this implies for the difference of the rotations R_j and $\tilde{R}_j$ that

$$\begin{aligned} |R_j - \tilde{R}_j|^2 &\leq \frac{C}{\mathcal{L}^2(C_j^{k_j,i_j})} \left(\int_{C_j^{k_j,i_j}} |\beta_j - R_j|^2 \wedge |\beta_j - R_j|^p \, dx + \int_{C_j^{k_j,i_j}} \left|\beta_j - \tilde{R}_j\right|^2 \wedge \left|\beta_j - \tilde{R}_j\right|^p \, dx \right) \\ &\leq \frac{C}{\delta^{2\tilde{k}_j} \rho_{\varepsilon_j}^2} (\varepsilon_j^2 |\xi_{i_j,j}|^2 + \varepsilon_j^2 |\log \varepsilon_j|^2) \leq \frac{C}{\delta^2 \varepsilon_j^{2\alpha}} \varepsilon_j^2 |\log \varepsilon|^2 \overset{j\to\infty}{\longrightarrow} 0, \end{aligned}$$

where we used that $\varepsilon_j |\xi_{i_j,j}| \leq |\mu_j|(\Omega) \leq C\varepsilon_j |\log \varepsilon_j|$. It follows $\lim_{j\to\infty} \tilde{R}_j = \lim_{j\to\infty} R_j = R$.
Next, define on $B_1(0) \setminus B_\delta(0)$ the function

$$\eta_j(x) = \frac{\tilde{R}_j^T \beta_j \left(\rho_{\varepsilon_j} \delta^{k_j - 1} x + x_{i_j,j}\right) - Id}{\varepsilon_j |\xi_{i_j,j}|} \rho_{\varepsilon_j} \delta^{k_j-1}.$$

Then $\operatorname{curl} \eta_j = 0$ in $B_1(0) \setminus B_\delta(0)$ and $\int_{\partial B_\delta(0)} \eta_j \cdot \tau \, d\mathcal{H}^1 = \tilde{R}_j^T \frac{\xi_{i_j,j}}{|\xi_{i_j,j}|}$ where the integral has to be understood in the sense of traces (see the comment about the trace of curl-free L^p-functions in Section 4.2). Furthermore, by (4.24) a change of variables yields

$$\int_{B_1(0)\setminus B_\delta(0)} |\eta_j|^2 \wedge \frac{|\eta_j|^p \, (\rho_{\varepsilon_j} \delta^{k_j-1})^{2-p}}{\varepsilon_j^{2-p} |\xi_{i_j,j}|^{2-p}} \, dx = \int_{C_j^{k_j,i_j}} \frac{\left|\tilde{R}_j^T \beta_j(x) - Id\right|^2 \wedge \left|\tilde{R}_j^T \beta_j(x) - Id\right|^p}{\varepsilon_j^2 |\xi_{i_j,j}|^2} \, dx \leq C. \tag{4.25}$$

Let us define the set $U_j = \left\{ x \in B_1(0) \setminus B_\delta(0) : |\eta_j(x)|^2 \geq \frac{|\eta_j(x)|^p (\rho_{\varepsilon_j} \delta^{k_j-1})^{2-p}}{\varepsilon_j^{2-p} |\xi_{i_j,j}|^{2-p}} \right\}$. As it holds that $\frac{\rho_{\varepsilon_j} \delta^{k_j-1}}{\varepsilon_j |\xi_{i_j,j}|} \geq \frac{\varepsilon_j^\alpha}{\varepsilon_j |\log \varepsilon_j|} \to \infty$, one derives from (4.25) that $\mathcal{L}^2(U_j) \to 0$.
Moreover, from (4.25) one sees that η_j is a bounded sequence in $L^p(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$ and $\eta_j \mathbf{1}_{U_j}$ is a bounded sequence in $L^2(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$. Hence, up to taking a subsequence, there exists $\eta \in L^p(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$ such that $\eta_j \rightharpoonup \eta$ in $L^p(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$. By standard arguments it also holds that $\eta_j \mathbf{1}_{U_j}$ has the same week limit in $L^2(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$. In particular, it follows that $\eta \in L^2(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$. In addition,

$$\operatorname{curl} \eta_j = 0 \text{ in } B_1(0) \setminus B_\delta(0) \text{ and } \xi := \lim_{j\to\infty} \int_{\partial B_\delta(0)} \eta_j \cdot \tau \, d\mathcal{H}^1 \in S^1 \text{ exists.} \tag{4.26}$$

For the second statement, one uses that taking the tangential boundary values is continuous from $\{\beta \in L^p(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2}) : \operatorname{curl} \beta \in L^p B_1(0) \setminus B_\delta(0)\}$ to $W^{-1,p}(\partial(B_1(0) \setminus B_\delta(0)))$, cf. [29, Theorem 2]. The statement then follows by testing with the constant function with value 1.
Now, define $A_j = \{x \in B_1(0) \setminus B_\delta(0) : |\eta_j| \leq \varepsilon_j^{-\frac{1-\alpha}{2}}\}$. By the boundedness of η_j in L^p, it follows that $|A_j| \to 0$. In addition, define $\chi_j = \mathbf{1}_{A_j} \mathbf{1}_{U_j}$ and observe that $\chi_j \eta_j \rightharpoonup \eta$ in $L^2(B_1(0) \setminus B_\delta(0); \mathbb{R}^{2\times 2})$. We estimate, with the use of the frame-indifference of W and a Taylor expansion for W, similarly to

(4.18)

$$\int_{C_j^{k_j,i_j}} \frac{W(\beta_j)}{|\xi_{i_j,j}|^2\varepsilon_j^2}\,dx \geq \int_{B_1(0)\setminus B_\delta(0)} \chi_j \frac{W\left(Id + \frac{\varepsilon_j|\xi_{i_j,j}|}{\rho_{\varepsilon_j}\delta^{k_j-1}}\eta_j\right)}{\varepsilon_j^2|\xi_{i_j,j}|^2}\rho_{\varepsilon_j}^2\delta^{2k_j-2}\,dx$$
$$\geq \int_{B_1(0)\setminus B_\delta(0)} \mathcal{C}\eta_j\chi_j : \eta_j\chi_j - \chi_j\frac{\omega\left(\eta_j\frac{\varepsilon_j|\xi_{i_j,j}|}{\rho_{\varepsilon_j}\delta^{k_j-1}}\right)}{\varepsilon_j^2|\xi_{i_j,j}|^2}\rho_{\varepsilon_j}^2\delta^{2k_j-2}\,dx, \tag{4.27}$$

where $\omega(t) \in o(t^2)$. Notice that

$$\chi_j|\eta_j|\frac{\varepsilon_j|\xi_{i_j,j}|}{\rho_{\varepsilon_j}\delta^{k_j-1}} \leq \varepsilon_j^{-\frac{1-\alpha}{2}}\varepsilon_j|\xi_{i_j,j}|\varepsilon_j^{-\alpha} = |\xi_{i_j,j}|\varepsilon_j^{\frac{1-\alpha}{2}} \leq C|\log\varepsilon_j|\varepsilon_j^{\frac{1-\alpha}{2}} \to 0. \tag{4.28}$$

The second integrand in (4.27) can be written as

$$\chi_j|\eta_j|^2\frac{\omega\left(\eta_j\frac{\varepsilon_j|\xi_{i_j,j}|}{\rho_{\varepsilon_j}\delta^{k_j-1}}\right)}{|\eta_j|^2\varepsilon_j^2|\xi_{i_j,j}|^2}\rho_{\varepsilon_j}^2\delta^{2k_j-2},$$

which is by the computation in (4.28) the product of a function bounded in $L^1(B_1(0)\setminus B_\delta(0))$ and a function converging uniformly to 0. Hence, the second term in (4.27) vanishes in the limit.
By lower-semicontinuity of the first term in (4.27) with respect to weak convergence in the space $L^2(B_1(0)\setminus B_\delta(0);\mathbb{R}^{2\times 2})$, we observe that

$$\liminf_{j\to\infty}\int_{C_j^{k_j,i_j}}\frac{W(\beta_j)}{|\xi_{i_j,j}|^2\varepsilon_j^2}\,dx \geq \int_{B_1(0)\setminus B_\delta(0)}\mathcal{C}\eta:\eta\,dx \geq \psi(\xi,\delta), \tag{4.29}$$

where we used (4.26) and the definition of $\psi(\cdot,\delta)$ for the second inequality.
Moreover, from the convexity and finiteness of $\psi(\cdot,\delta)$ we derive the continuity of the function $\psi(\cdot,\delta)$. Thus, (4.26) yields $\lim_{j\to\infty}\psi\left(\frac{R^T\xi_{i,j}}{|\xi_{i_j,j}|},\delta\right) = \psi(\xi,\delta)$. Together with (4.29), this implies

$$\liminf_{j\to\infty}\int_{C_j^{k_j,i_j}}\frac{W(\beta_j)}{|\xi_{i_j,j}|^2\varepsilon_j^2}\,dx - \psi\left(R^T\frac{\xi_{i_j,j}}{|\xi_{i_j,j}|^2},\delta\right) \geq 0,$$

which contradicts (4.23). This proves the lemma. □

Using the previous lemma together with (4.20) and (4.21), we can estimate

$$\begin{aligned}
&\frac{1}{|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{B_{\rho_{\varepsilon_j}}(x_{i,j})}\frac{W(\beta_j)}{\varepsilon_j^2}\\
\geq&\frac{1}{|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\sum_{k=1}^{\tilde{k}_j}\int_{C_j^{k,i}}\frac{W(\beta_j)}{\varepsilon_j^2}\,dx\\
\geq&\frac{1}{|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\tilde{k}_j\left(\psi(R^T\xi_{i,j},\delta)-\sigma_j|\xi_{i,j}|^2\right)\\
\geq&\frac{1}{|\log\varepsilon_j|}\sum_{i=1}^{M_j}\left(\alpha-\frac{|\log\rho_{\varepsilon_j}|}{|\log\varepsilon_j|}\right)\left(\frac{\psi(R^T\xi_{i,j},\delta)}{|\log\delta|}-\frac{\sigma_j|\xi_{i,j}|^2}{|\log\delta|}\right).
\end{aligned} \tag{4.30}$$

From Proposition 4.2.1 we know that there exists $K > 0$ (which does not depend on δ) such that for every $\xi \in \mathbb{R}^2$ it holds

$$\left| \frac{\psi(\xi,\delta)}{|\log \delta|} - \psi(\xi) \right| \leq \frac{K|\xi|^2}{|\log \delta|}.$$

Together with (4.30), this yields

$$\begin{aligned} &\frac{1}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{i,j})} \frac{W(\beta_j)}{\varepsilon_j^2} \\ \geq &\frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \left(\psi(R^T \xi_{i,j}) - \frac{K|\xi_{i,j}|^2}{|\log \delta|} - \frac{\sigma_j |\xi_{i,j}|^2}{|\log \delta|} \right). \end{aligned}$$

Arguing as in the proof of the compactness, we can find similarly to (4.11) that

$$\frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} |\xi_{i,j}|^2 \leq C$$

and hence

$$\frac{1}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{i,j})} \frac{W(\beta_j)}{\varepsilon_j^2} \geq \frac{1}{|\log \varepsilon_j|} \left[\sum_{i=1}^{M_j} \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \psi(R^T \xi_{i,j}) \right] \tag{4.31}$$

$$- C \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \left(\frac{1 + \sigma_j}{|\log \delta|} \right). \tag{4.32}$$

Now, write $\tilde{\mu}_j = \frac{\mu_j}{\varepsilon_j |\log \varepsilon_j|}$. Using $\psi \geq \varphi$ and the 1-homogeneity of φ, cf. the definition of φ in (4.7) and the remark below, we can estimate

$$\frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \psi(R^T \xi_{i,j}) \geq \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \int_\Omega \varphi \left(R, \frac{d\tilde{\mu}_j}{d|\tilde{\mu}_j|} \right) d|\tilde{\mu}_j|. \tag{4.33}$$

By the definition of the convergence of (μ_j, β_j) to the triple (β, μ, R), it holds in particular that $\tilde{\mu}_{\varepsilon_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^2)$. As φ is a continuous, convex, and 1-homogeneous function, we can apply Reshetnyak's theorem to derive from (4.33) that

$$\liminf_{j \to \infty} \frac{1}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \psi(R^T \xi_{i.j}) \geq \alpha \int_\Omega \varphi \left(R, \frac{d\mu}{d|\mu|} \right) d|\mu|. \tag{4.34}$$

Combining (4.32) and (4.34) yields

$$\begin{aligned} \liminf_{j \to \infty} \frac{1}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{i,j})} \frac{W(\beta_j)}{\varepsilon_j^2} &\geq \alpha \int_\Omega \varphi \left(R, \frac{d\mu}{d|\mu|} \right) d|\mu| - \limsup_{j \to \infty} C \left(\alpha - \frac{|\log \rho_{\varepsilon_j}|}{|\log \varepsilon_j|} \right) \left(\frac{1 + \sigma_j}{|\log \delta|} \right) \\ &= \alpha \int_\Omega \varphi \left(R, \frac{d\mu}{d|\mu|} \right) d|\mu| - \frac{C\alpha}{|\log \delta|}. \end{aligned}$$

Letting $\alpha \to 1$ and $\delta \to 0$ finishes the proof of the lower bound close to the dislocations.
Combining the estimates close and far from the dislocations shows the claimed lim inf-inequality. □

In the following, we prove the existence of a recovery sequence for the energy E^{crit}. We will use

that the limit energy E^{crit} is the same as in [38] and [59]. In particular, we make use of the density result in [38] that allows us to restrict ourselves to the case that μ is locally constant and absolutely continuous with respect to the Lebesgue measure.
The construction is then closely related to the one for the nonlinear energy with quadratic growth in [59].

Proposition 4.3.5 (The lim sup-inequality). *Let $\varepsilon_j \to 0$ and $N_{\varepsilon_j} = |\log \varepsilon_j|$. Let $R \in SO(2)$, $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ such that $\operatorname{curl}\beta = R^T\mu \in \mathcal{M}(\Omega;\mathbb{R}^2)$. Then there exists a sequence of dislocation measures and associated strains $(\mu_j,\beta_j)_j \subset \mathcal{M}(\Omega;\mathbb{R}^2)\times L^p(\Omega,\mathbb{R}^{2\times 2})$ converging to (μ,β,R) in the sense of Definition 4.3.1 such that*

$$\limsup_{j\to\infty} E_{\varepsilon_j}(\mu_j,\beta_j) \le E^{crit}(\mu,\beta,R).$$

Proof. **Step 1.** *$\mu = \xi\, dx$ for some $\xi \in \mathbb{R}^2$.*
Let $\lambda_1,\dots,\lambda_M > 0$ and $\xi_1,\dots,\xi_M \in \mathbb{S}$ such that $\xi = \sum_{k=1}^M \lambda_k\xi_k$ and $\varphi(R,\xi) = \sum_{k=1}^M \lambda_k\psi(R^T\xi_k)$. Moreover, set

$$\Lambda = \sum_{k=1}^M \lambda_k \text{ and } r_{\varepsilon_j} = \frac{1}{2\sqrt{\Lambda N_{\varepsilon_j}}}.$$

Then, by the assumptions on ρ_{ε_j}, it holds that $\frac{\rho_{\varepsilon_j}}{r_{\varepsilon_j}} = 2\sqrt{\Lambda}\sqrt{\rho_{\varepsilon_j}^2|\log\varepsilon_j|} \to 0$. According to [38, Lemma 11], there exists a sequence of measures $\mu_j = \sum_{k=1}^M \varepsilon_j\xi_k\mu_j^k$ with μ_j^k of the type $\sum_{l=1}^{M_j^k}\delta_{x_{l,j}^k}$ for some $x_{l,j}^k \in \Omega$ such that for all $x,y\in\operatorname{supp}(\mu_j)$ it holds $B_{r_{\varepsilon_j}}(x)\subset\Omega$ and $|x-y| \ge 2r_{\varepsilon_j}$. Moreover, the construction of μ_j in [38, Lemma 11] implies that

$$\frac{|\mu_j^k|(\Omega)}{|\log\varepsilon_j|} \to \lambda_k|\Omega| \text{ and } \frac{\mu_j}{\varepsilon_j|\log\varepsilon_j|} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2). \tag{4.35}$$

The basic concept behind the construction of μ_j is the following. First, consider a Burgers vector ξ such that $\psi(R^T\xi) = \varphi(R,\xi)$. Cover Ω with cubes of side length $\sqrt{N_{\varepsilon_j}}^{-1}$. In all balls which lie completely in Ω put a Dirac measure with weight ξ. For a sketch of the construction, see Figure 4.2. In the general case, first approximate $\frac{\xi}{\Lambda}\, dx$ by measures that are locally constant with values $\xi_1,\dots,\xi_M$ on volume fractions of size $\frac{\lambda_1}{\Lambda},\dots,\frac{\lambda_M}{\Lambda}$. On the sets where an approximating measure is constant use the construction described above for squares with side length $\sqrt{\Lambda N_{\varepsilon_j}}^{-1}$. Then take a diagonal sequence.

Note that by construction it holds $\mu_j \in X_{\varepsilon_j}$.
It is useful to combine the two summations in the definition of μ_j into

$$\mu_j = \sum_{i=1}^{M_j}\varepsilon_j\xi_{i,j}\delta_{x_{i,j}}$$

for appropriate $\xi_{i,j}\in\mathbb{S}$ and $x_{i,j}\in\Omega$. In particular, by the well-separateness of dislocations of μ_j on the scale r_{ε_j} it follows $M_j \le Cr_{\varepsilon_j}^{-2}$.
In [7], it is shown that for every $i = 1,\dots,M_j$ there exists a strain field $\eta_i^j : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ of the form $\eta_i^j = \frac{1}{|x-x_{i,j}|}\Gamma_{R^T\xi_{i,j}}\left(\frac{x-x_{i,j}}{|x-x_{i,j}|}\right)$ where the function $\Gamma_{R^T\xi_{i,j}} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ depends on the linearized

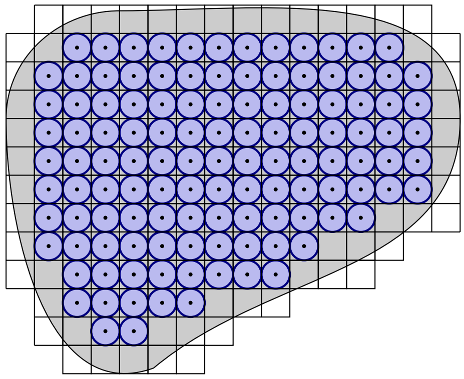


Figure 4.2: Sketch of the construction of the measures μ_j, $\tilde{\mu}_j^{r_{\varepsilon_j}}$, and $\hat{\mu}_j^{r_{\varepsilon_j}}$ for a Burgers vector ξ such that $\varphi(R,\xi) = \psi(R^T\xi)$: cover Ω with squares of side length $\sqrt{N_{\varepsilon_j}}^{-1}$. In every square that is included in Ω put a Dirac mass with weight ξ (black dot) for μ_j, a continuously distributed mass of ξ on the circle of diameter $\sqrt{N_{\varepsilon_j}}^{-1}$ (blue circle) for $\tilde{\mu}_j^{r_{\varepsilon_j}}$, and a measure of mass ξ distributed on the boundary of that circle (dark blue) for $\hat{\mu}_j^{r_{\varepsilon_j}}$.

elasticity tensor $\mathcal{C}$ such that $\Gamma_{R^T\xi_{i,j}}\left(\frac{x-x_{i,j}}{|x-x_{i,j}|}\right) \leq C$ and η_i^j solves

$$\begin{cases} \operatorname{curl} \eta_i^j = R^T \xi_{i,j} \delta_{x_{i,j}} & \text{in } \mathbb{R}^2, \\ \operatorname{div} \mathcal{C}\eta_i^j = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

We define

$$\eta^j = \sum_{i=1}^{M_j} \varepsilon_j \eta_i^j \, \mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})}. \tag{4.36}$$

Then $\operatorname{curl} \eta^j$ equals $R^T \mu_j$ up to an error term arising from $\mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})}$, precisely

$$\begin{aligned} \operatorname{curl} \eta^j &= \sum_{i=1}^{M_j} \varepsilon_j R^T \xi_{i,j} \delta_{x_{i,j}} - \varepsilon_j \eta_i^j(x) \frac{(x-x_{i,j})^\perp}{|(x-x_{i,j})|} \, d\mathcal{H}^1_{|\partial B_{r_{\varepsilon_j}}(x_{i,j})} \\ &= R^T \mu_j - \sum_{i=1}^{M_j} \varepsilon_j \frac{\eta_i^j(x)\,(x-x_{i,j})^\perp}{r_{\varepsilon_j}} \, d\mathcal{H}^1_{|\partial B_{r_{\varepsilon_j}}(x_{i,j})} \\ &=: R^T \mu_j - R^T \hat{\mu}_j^{r_{\varepsilon_j}}. \end{aligned} \tag{4.37}$$

Note that $\hat{\mu}_j^{r_{\varepsilon_j}} \in H^{-1}$.

Moreover, we define the auxiliary measure

$$\tilde{\mu}_j^{r_{\varepsilon_j}} = R \sum_{i=1}^{M_j} 2\varepsilon_j \frac{\eta_i^j(x)\,(x-x_{i,j})^\perp}{r_{\varepsilon_j}^2} \mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})} \, dx. \tag{4.38}$$

For a sketch of the measures $\tilde{\mu}_j^{r_{\varepsilon_j}}$ and $\hat{\mu}_j^{r_{\varepsilon_j}}$, see Figure 4.2.

A straightforward computation shows that for all $i \in \{1, \dots, M_j\}$ it holds

$$\tilde{\mu}_j^{r_{\varepsilon_j}}(B_{r_{\varepsilon_j}}(x_{i,j})) = \hat{\mu}_j^{r_{\varepsilon_j}}(\partial B_{r_{\varepsilon_j}}(x_{i,j})) = \varepsilon_j \xi_{i,j}.$$

In [38, Lemma 11], it is also shown that

$$\begin{aligned} &\frac{\hat{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j|\log\varepsilon_j|} \overset{*}{\rightharpoonup} R^T\mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2), \ \frac{\tilde{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j|\log\varepsilon_j|} \overset{*}{\rightharpoonup} R^T\mu \text{ in } L^\infty(\Omega;\mathbb{R}^2), \\ &\frac{\tilde{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j|\log\varepsilon_j|} \to R^T\mu \text{ in } H^{-1}(\Omega;\mathbb{R}^2), \text{ and } \frac{\tilde{\mu}_j^{r_{\varepsilon_j}} - \hat{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j|\log\varepsilon_j|} \to 0 \text{ in } H^{-1}(\Omega;\mathbb{R}^2). \end{aligned} \tag{4.39}$$

In order to define the recovery sequence, we introduce the auxiliary strain

$$\tilde{K}_{\mu_j}^{r_{\varepsilon_j}} = \frac{\varepsilon_j}{r_{\varepsilon_j}^2}\sum_{i=1}^{M_j}\eta_i^j\,|x-x_{i,j}|^2\,\mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})}. \tag{4.40}$$

A straightforward calculation shows that $\operatorname{curl}\tilde{K}_{\mu_j}^{r_{\varepsilon_j}} = R^T(\tilde{\mu}_j^{r_{\varepsilon_j}} - \hat{\mu}_j^{r_{\varepsilon_j}})$.
Now, we define the approximating strains as

$$\beta_j = R\left(Id + \varepsilon_j|\log\varepsilon_j|\beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j\right), \tag{4.41}$$

where $\tilde{\beta}_j = \nabla w_j J$ for $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and w_j is the solution to

$$\begin{cases} -\Delta w_j = \varepsilon_j|\log\varepsilon_j|R^T\mu - R^T\tilde{\mu}_j^{r_{\varepsilon_j}} & \text{in } \Omega, \\ w_j \in H_0^1(\Omega;\mathbb{R}^2). \end{cases} \tag{4.42}$$

Then $\beta_j \in \mathcal{AS}_{\varepsilon_j}(\mu_j)$. Indeed, one can show by a direct computation that η^j and $K_{\mu_j}^{r_{\varepsilon_j}}$ are in $L^p(\Omega;\mathbb{R}^{2\times 2})$; the function $\tilde{\beta}_j$ belongs to the space $L^2(\Omega;\mathbb{R}^{2\times 2})$ by definition. Hence, each summand in the definition of β_j is in the space $L^p(\Omega;\mathbb{R}^{2\times 2})$. Furthermore,

$$\operatorname{curl}\beta_j = \varepsilon_j|\log\varepsilon_j|\mu + \mu_j - \hat{\mu}_j^{r_{\varepsilon_j}} - \tilde{\mu}_j^{r_{\varepsilon_j}} + \hat{\mu}_j^{r_{\varepsilon_j}} - \varepsilon_j|\log\varepsilon_j|\mu + \tilde{\mu}_j^{r_{\varepsilon_j}} = \mu_j.$$

As in [59, Γ-limsup inequality], it can be shown for $\Omega_{\varepsilon_j}(\mu_j) = \Omega\setminus\bigcup_{x\in\operatorname{supp}(\mu_j)}B_{\varepsilon_j}(x)$ that

a) $\frac{\eta^j\mathbf{1}_{\Omega_{\varepsilon_j}(\mu_j)}}{\varepsilon_j|\log\varepsilon_j|} \rightharpoonup 0$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$,

b) $\frac{\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}}{\varepsilon_j|\log\varepsilon_j|} \to 0$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$,

c) $\frac{\tilde{\beta}_j}{\varepsilon_j|\log\varepsilon_j|} \to 0$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$.

The boundedness in $L^2(\Omega;\mathbb{R}^{2\times 2})$ of the function in a) is a straightforward computation. The identification of the weak limit can be done in $L^p(\Omega;\mathbb{R}^{2\times 2})$. For b) notice that $|\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}| \le C\varepsilon_j\sqrt{|\log\varepsilon_j|}$. In view of (4.42) and (4.39), the last statement follows by classical elliptic estimates.
Furthermore, it can be shown that

d) $\frac{\eta^j}{\varepsilon_j|\log\varepsilon_j|} \to 0$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$.

In fact,

$$\begin{aligned}
\int_\Omega \left| \frac{\eta^j}{\varepsilon_j |\log \varepsilon_j|} \right|^p dx &= \frac{1}{|\log \varepsilon_j|^p} \sum_{i=1}^{M_j} \int_{B_{r_{\varepsilon_j}}(x_{i,j})} |\eta_i^j|^p \, dx \\
&\leq \frac{C}{|\log \varepsilon_j|^p} \sum_{i=1}^{M_j} \int_0^{r_{\varepsilon_j}} r^{1-p} \, dr \\
&\leq C(2-p)^{-1} \frac{M_j}{|\log \varepsilon_j|^p} r_{\varepsilon_j}^{2-p} \\
&\leq C(2-p)^{-1} |\log \varepsilon_j|^{-p} r_{\varepsilon_j}^{-p} \leq C(2-p)^{-1} |\log \varepsilon_j|^{-\frac{p}{2}} \to 0.
\end{aligned}$$

Hence, (μ_j, β_j) converges to (μ, β, R) in the sense of Definition 4.3.1 with $R_{\varepsilon_j} = R$.

Next, we will show the lim sup-inequality for the energies. For this purpose, fix $\alpha \in (0,1)$. We split the energy as follows

$$E_{\varepsilon_j}(\mu_j, \beta_j) = \underbrace{\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} W(\beta_j) \, dx}_{=:I^1_{\varepsilon_j}} + \underbrace{\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{\varepsilon_j^\alpha}(x_{i,j})} W(\beta_j) \, dx}_{=:I^2_{\varepsilon_j}} . \tag{4.43}$$

First, we show that

$$\limsup_{\varepsilon_j \to 0} I^1_{\varepsilon_j} \leq \int_\Omega \frac{1}{2} \mathcal{C}\beta : \beta \, dx + \alpha \int_\Omega \varphi(R, \xi) \, dx = \int_\Omega \frac{1}{2} \mathcal{C}\beta : \beta \, dx + \alpha \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu|. \tag{4.44}$$

Using a second order Taylor expansion and the frame indifference of the energy density W, we obtain similarly to the lim inf-inequality, (4.18), that

$$\begin{aligned}
I^1_{\varepsilon_j} = &\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} \frac{1}{2} \mathcal{C}(\varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j) : (\varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j) \, dx \\
&+ \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} \sigma(\varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j) \, dx,
\end{aligned}$$

where $\frac{\sigma(F)}{|F|^2} \to 0$ as $F \to 0$.

By a) – c), in the first integral all mixed terms and the quadratic terms involving $\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}$ or $\tilde{\beta}_j$ vanish in the limit. In addition, one derives from the non-negativity of $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(Id)$ that

$$\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} \frac{1}{2} \mathcal{C}(\varepsilon_j |\log \varepsilon_j| \beta) : (\varepsilon_j |\log \varepsilon_j| \beta) \, dx \leq \int_\Omega \frac{1}{2} \mathcal{C}\beta : \beta \, dx.$$

Hence, we still have to consider the term involving η^j. Using the special form of the η_i^j for the second equality and Proposition 4.2.1 (in particular (4.6)) for the inequality, we find that

$$\frac{1}{\varepsilon_j |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} \frac{1}{2} \mathcal{C}\eta^j : \eta^j \, dx = \frac{1}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \int_{B_{r_{\varepsilon_j}}(x_{i,j}) \setminus B_{\varepsilon_j^\alpha}(x_{i,j})} \frac{1}{2} \mathcal{C}\eta_i^j : \eta_i^j \, dx$$

$$
\begin{aligned}
&= \frac{|\log \frac{\varepsilon_j^\alpha}{r_{\varepsilon_j}}|}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} \frac{1}{|\log \frac{\varepsilon_j^\alpha}{r_{\varepsilon_j}}|} \int_{B_1(x_{i,j})\setminus B_{\frac{\varepsilon_j^\alpha}{r_{\varepsilon_j}}}(x_{i,j})} \frac{1}{2}\mathcal{C}\eta_i^j : \eta_i^j \, dx \\
&\leq \frac{|\log \frac{\varepsilon_j^\alpha}{r_{\varepsilon_j}}|}{|\log \varepsilon_j|^2} \sum_{i=1}^{M_j} (\psi(R^T\xi_{i,j}) + o(1)). \\
&= \frac{\alpha + o(1)}{|\log \varepsilon_j|} \sum_{i=1}^{M_j} (\psi(R^T\xi_{i,j}) + o(1)),
\end{aligned}
$$

where $o(1) \to 0$ as $\varepsilon_j \to 0$ (notice that we deal only with finitely many values of $\xi_{i,j}$). By definition of μ_j^k, this equals

$$
= \sum_{k=1}^{M} (\alpha + o(1)) \frac{|\mu_{\varepsilon_j}^k|(\Omega)}{|\log \varepsilon_j|} (\psi(R^T\xi_k) + o(1)).
$$

Using (4.35), this yields in the limit

$$
\limsup_{\varepsilon_j \to 0} \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} \frac{1}{2}\mathcal{C}\eta^j : \eta^j \, dx \leq \alpha |\Omega| \sum_{k=1}^{M} \lambda_k \psi(R^T\xi_k) = \alpha \int_\Omega \varphi(R,\xi) \, dx.
$$

To establish (4.44) we still have to show that

$$
\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)} \sigma(\varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j) \, dx \to 0. \tag{4.45}
$$

First, we observe that for $x \in \Omega_{\varepsilon_j^\alpha}(\mu_j)$ it holds

$$
\begin{aligned}
|\eta^j(x)| &\leq \sup_{i=1,\dots,M_j} \varepsilon_j |\mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})} \eta_i^j| \leq C \varepsilon_j^{1-\alpha} \\
\text{and } |\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}(x)| &\leq \sup_{i=1,\dots,M_j} |\mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})} \eta_i^j \, |x - x_{i,j}|^2| \leq C \frac{\varepsilon_j}{r_{\varepsilon_j}}.
\end{aligned}
$$

Hence, $\mathbf{1}_{\Omega_{\varepsilon_j^\alpha}(\mu_j)}(\eta^j + \tilde{K}_{\mu_j}^{r_{\varepsilon_j}})$ converges uniformly to zero. To compensate the lack of uniform convergence of $\tilde{\beta}_j$ and $\varepsilon_j |\log \varepsilon_j| \beta$, fix $L > 0$ and define the set

$$
U_{\varepsilon_j}^L = \left\{ x \in \Omega_{\varepsilon_j^\alpha(\mu_j)} : |\tilde{\beta}_j(x)| \leq |\eta^j(x) + \tilde{K}_{\mu_j}^{r_{\varepsilon_j}}(x)| \text{ and } |\beta(x)| \leq L \right\}.
$$

Then $\mathbf{1}_{U_{\varepsilon_j}^L}(\varepsilon_j |\log \varepsilon_j| \beta + \eta^j + \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_{\varepsilon_j})$ converges to zero uniformly. Set $\omega(t) = \sup_{|F| \leq t} |\sigma(F)|$ and notice that $\frac{\omega(t)}{t^2} \to 0$ as $t \to 0$. By definition of ω it holds

$$
\begin{aligned}
&\frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \left| \int_{U_{\varepsilon_j}^L} \sigma(\varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j) \, dx \right| \\
&\leq \int_{U_{\varepsilon_j}^L} \frac{\omega(\varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j)}{\left| \varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j \right|^2} \frac{\left| \varepsilon_j |\log \varepsilon_j| \beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j \right|^2}{\varepsilon_j^2 |\log \varepsilon_j|^2} \, dx \to 0
\end{aligned} \tag{4.46}
$$

as the first term converges to zero uniformly and the second is bounded in L^1 by a) – c).

For the integral on $\Omega_{\varepsilon_j^\alpha}(\mu_j) \setminus U^L_{\varepsilon_j}$, we need the following bound on ω:

$$\omega(t) = \sup_{|F|\le t} |\sigma(F)| = \sup_{|F|\le t} \left| W(Id+F) - \frac{1}{2}\mathcal{C}F : F \right| \le C \sup_{|F|\le t} |F|^2 \le Ct^2.$$

Notice that on $\Omega_{\varepsilon_j^\alpha}(\mu_j) \setminus U^L_{\varepsilon_j}$ it holds $|\tilde\beta| \ge |\eta^j(x) + \tilde K^{r_{\varepsilon_j}}_{\mu_j}(x)|$. Hence,

$$\begin{aligned}
&\frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\left|\int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)\setminus U^L_{\varepsilon_j}} \sigma(\varepsilon_j|\log\varepsilon_j|\beta + \eta^j - \tilde K^{r_{\varepsilon_j}}_{\mu_j} + \tilde\beta_j)\,dx\right| \\
\le& C\int_{\Omega_{\varepsilon_j^\alpha}(\mu_j)\setminus U^L_{\varepsilon_j}} \frac{\left|\varepsilon_j|\log\varepsilon_j|\beta + \eta^j - \tilde K^{r_{\varepsilon_j}}_{\mu_j} + \tilde\beta_j\right|^2}{\varepsilon_j^2|\log\varepsilon_j|^2}\,dx \\
\le& C\int_{\{|\beta|>L\}} \frac{|\tilde\beta_{\varepsilon_j}|^2}{\varepsilon_j^2|\log\varepsilon_j|^2} + |\beta|^2\,dx \overset{j\to\infty}{\longrightarrow} \int_{\{|\beta|>L\}} |\beta|^2\,dx \overset{L\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we used c) for the convergence in j. Together with (4.46), this establishes (4.45) which finishes the proof of (4.44).

Next, we control $I_2^{\varepsilon_j}$ from (4.43). Notice that

$$\begin{aligned}
&\frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{B_{\varepsilon_j^\alpha}(x_{i,j})} W(\beta_j)\,dx \\
\le& C\frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{B_{\varepsilon_j^\alpha}(x_{i,j})} \operatorname{dist}(\beta_j, SO(2))^2 \wedge \operatorname{dist}(\beta_j, SO(2))^p\,dx \\
\le& C\frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{B_{\varepsilon_j^\alpha}(x_{i,j})} \varepsilon_j^2|\log\varepsilon_j|^2|\beta|^2 + |\eta^j|^2\wedge|\eta^j|^p + |\tilde K^{r_{\varepsilon_j}}_{\mu_j}|^2 + |\tilde\beta_j|^2\,dx.
\end{aligned}$$

Due to b) and c), the terms involving $\tilde K^{r_{\varepsilon_j}}_{\mu_j}$ and $\tilde\beta_j$ vanish in the limit. Moreover, as

$$\mathcal{L}^2\left(\bigcup_{i=1}^{M_j} B_{\varepsilon_j^\alpha}(x_{i,j})\right) = M_j\pi\varepsilon_j^{2\alpha} \le C|\log\varepsilon_j|\varepsilon_j^{2\alpha} \to 0,$$

also the term involving β vanishes in the limit. Lastly, we estimate

$$\begin{aligned}
&\frac{1}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\int_{B_{\varepsilon_j^\alpha}(x_{i,j})} |\eta^j|^2\wedge|\eta^j|^p\,dx \\
\le&\frac{C}{\varepsilon_j^2|\log\varepsilon_j|^2}\sum_{i=1}^{M_j}\left(\int_{B_{\varepsilon_j^\alpha}(x_{i,j})\setminus B_{\varepsilon_j}(x_{i,j})} |\eta^j|^2\,dx + \int_{B_{\varepsilon_j}(x_{i,j})} |\eta^j|^p\,dx\right) \\
\le&\frac{CM_j}{|\log\varepsilon_j|^2}\int_{\varepsilon_j}^{\varepsilon_j^\alpha} r^{-1}\,dr + \frac{CM_j}{\varepsilon_j^{2-p}|\log\varepsilon_j|^2}\int_0^{\varepsilon_j} r^{1-p}\,dr \\
\le& C(1-\alpha) + C|\log\varepsilon_j|^{-1}.
\end{aligned}$$

Hence, $\limsup_{\varepsilon_j\to 0} I^2_{\varepsilon_j} \le C(1-\alpha)$. Together with (4.44), this implies

$$\limsup_{\varepsilon_j\to 0} E_{\varepsilon_j}(\beta_j,\mu_j) \le \int_\Omega \frac{1}{2}\mathcal{C}\beta : \beta\, dx + \alpha \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) + C(1-\alpha).$$

Letting $\alpha \to 1$ finishes step 1.
For step 2, it is useful to notice here that $\eta^j = \tilde{K}^{r_{\varepsilon_j}}_{\mu_j} = 0$ on $\partial\Omega$ and therefore we find by (4.39) and c) that (cf. [29, Theorem 2])

$$\left(\frac{R^T\beta_j - Id}{\varepsilon_j|\log\varepsilon_j|} - \beta\right)\cdot\tau = \left(\frac{\eta^j - \tilde{K}^{r_{\varepsilon_j}}_{\mu_j} + \tilde{\beta}_j}{\varepsilon_j|\log\varepsilon_j|}\right)\cdot\tau = \frac{\tilde{\beta}_j}{\varepsilon_j|\log\varepsilon_j|}\cdot\tau \to 0 \text{ strongly in } H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^2), \quad (4.47)$$

where τ denotes the unit tangent to $\partial\Omega$.

Step 2. *$\mu = \sum_{l=1}^L \xi^l\, d\mathcal{L}^2_{|\Omega^l}$ where $\xi^l \in \mathbb{R}^2$ and $\Omega^l \subset \Omega$ are pairwise disjoint Lipschitz-domains such that $\mathcal{L}^2\left(\Omega \setminus \bigcup_{l=1}^L \Omega^l\right) = 0$.*
We make use of the recovery sequence of step 1 on each Ω^l. For this, we define $\beta^l = \beta\mathbf{1}_{\Omega^l}$ and $\mu^l = \mu_{|\Omega^l}$. For each $l = 1,\dots,L$ let $(\mu^l_j,\beta^l_j)_j$ be the recovery sequence from step 1 for (μ^l,β^l,R) on Ω^l. Now, we define

$$\tilde{\beta}_j = \sum_{l=1}^L \beta^l_j \mathbf{1}_{\Omega^l}.$$

Then, $\operatorname{curl}\tilde{\beta}_j \notin X_{\varepsilon_j}$, precisely

$$\operatorname{curl}\tilde{\beta}_j = \sum_{l=1}^L \mu^l_j - (\beta^l_j\cdot\tau_{\partial\Omega^l})\, d\mathcal{H}^1_{|\partial\Omega^l\cap\Omega} \text{ in } \mathcal{D}'(\Omega),$$

where $\tau_{\partial\Omega^l}$ is the unit tangent to $\partial\Omega^l$. Note that for two neighboring regions Ω^l the corresponding tangents have opposite signs. By (4.47), we find

$$\begin{aligned}\left\|\frac{\operatorname{curl}\tilde{\beta}_j - \sum_{l=1}^L \mu^l_j}{\varepsilon_j|\log\varepsilon_j|}\right\|_{H^{-1}(\Omega;\mathbb{R}^2)} &= \left\|\sum_{l=1}^L\left(\frac{\beta^l_j - R}{\varepsilon_j|\log\varepsilon_j|} - R\beta\right)\cdot\tau_{\partial\Omega^l\cap\Omega}\, d\mathcal{H}^1_{|\partial\Omega^l}\right\|_{H^{-1}(\Omega;\mathbb{R}^2)} \\ &\le C\sum_{l=1}^L\left\|\left(\frac{\beta^l_j - R}{\varepsilon_j|\log\varepsilon_j|} - R\beta\right)\cdot\tau_{\partial\Omega^l}\right\|_{H^{-\frac{1}{2}}(\partial\Omega^l;\mathbb{R}^2)} \to 0 \text{ as } j\to\infty.\end{aligned}$$

For the last inequality we used that for a Lipschitz domain U the trace space for $H^1(U;\mathbb{R}^2)$ is $H^{\frac{1}{2}}(\partial U;\mathbb{R}^2)$. By this estimate in H^{-1}, we can find a sequence of functions $f_j \in L^2(\Omega;\mathbb{R}^{2\times 2})$ such that $\operatorname{curl} f_j = \operatorname{curl}\tilde{\beta}_j - \sum_{l=1}^L \mu^l_j$ and $\frac{1}{\varepsilon_j|\log\varepsilon_j|}\|f_j\|_{L^2} \to 0$. Now, define the recovery sequence as

$$\beta_j = \tilde{\beta}_j - f_j \text{ and } \mu_j = \sum_{l=1}^L \mu^l_j.$$

Then $\mu_j \in X_{\varepsilon_j}$ and $\beta_j \in \mathcal{AS}_{\varepsilon_j}(\mu_j)$. From the construction of the μ^l_j in step 1 it can be seen that $\frac{\mu_j}{\varepsilon_j|\log\varepsilon_j|} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$. Moreover, in the proof of step 1 it can be seen that although we subtract the vanishing sequence f_{ε_j} it still holds for all $l = 1, \dots, L$

$$\limsup_{\varepsilon_j \to 0} \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|^2} \int_{\Omega^l} W(\beta_j)\, dx \leq \int_{\Omega^l} \mathcal{C}\beta : \beta\, dx + \int_{\Omega^l} \varphi\left(R, \frac{d\mu^l}{d|\mu^l|}\right) d|\mu^l|. \tag{4.48}$$

Summing over (4.48) finishes step 2.

Step 3. $\mu \in H^{-1}(\Omega; \mathbb{R}^2) \cap \mathcal{M}(\Omega; \mathbb{R}^2)$.
As our limit energy is the same, we can argue as in [38, Theorem 12, step 3] to reduce the general case to step 2. Let us shortly sketch the argument for the sake of completeness. By reflection arguments and mollification, the authors show that there exists a sequence of smooth functions β_j such that $\beta_j \to \beta$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$, $\operatorname{curl} \beta_j \overset{*}{\rightharpoonup} \operatorname{curl} \beta$ in $\mathcal{M}(\Omega; \mathbb{R}^2)$, and $|\operatorname{curl} \beta_j|(\Omega) \to |\operatorname{curl} \beta|(\Omega)$. The energy E^{crit} is continuous with respect to this convergence. Then, the authors carefully approximate $\operatorname{curl} \beta$ by piecewise constant functions and correct the corresponding error in the curl by a vanishing sequence in $L^2(\Omega; \mathbb{R}^{2\times 2})$. □

Remark 4.3.1. In [60], the authors construct a recovery sequence β_j which fulfills $\det \beta_j > 0$. This construction could also be used in our case. Most computations in the proof would remain the same. Using this construction, we could weaken our assumptions on W in the sense that we would need the upper bound $W(F) \leq C \operatorname{dist}(F, SO(2))^2 \wedge \operatorname{dist}(F, SO(2))^p$ only for F such that $\det(F) > 0$.

4.4 The Subcritical Regime

In this section, we consider the scaling regime $1 \ll N_\varepsilon \ll |\log \varepsilon|$, which corresponds heuristically to few dislocations. In [71], the authors consider $N_\varepsilon \leq C$. In the critical regime, the coupling between μ_ε and β_ε survives the limiting procedure as both quantities live on the same scale. In the subcritical regime, we expect the distance of the strain β_ε to $SO(2)$ to be of scale $\varepsilon\, (|\log \varepsilon| N_\varepsilon)^{\frac{1}{2}}$ whereas the dislocation density $\operatorname{curl} \beta_\varepsilon = \mu_\varepsilon$ is of scale $\varepsilon N_\varepsilon$ (see Section 1.3). Hence, the limit variables are decoupled. This is made rigorous in the compactness result. The limit of E_ε will be given by

$$\int_\Omega \frac{1}{2} \mathcal{C}\beta : \beta\, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|,$$

where $\operatorname{curl} \beta = 0$ and $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2)$. The same line-tension limit has also been derived in the subcritical regime from a core-radius approach in the setting of linearized elasticity in [38] and [30].

Compactness

The compactness result in the subcritical regime is the following.

Theorem 4.4.1 (Compactness). *Let $\varepsilon_j \to 0$ and $1 \ll N_{\varepsilon_j} \ll |\log \varepsilon_j|$. Let $(\mu_j, \beta_j)_j \subset X_{\varepsilon_j} \times \mathcal{AS}_{\varepsilon_j}(\mu_{\varepsilon_j})$ such that $\sup_j E_{\varepsilon_j}(\mu_j, \beta_j) < \infty$ where the energy E_{ε_j} is defined as in (4.3). Then there exist a subsequence (not relabeled), a sequence of rotations $(R_j)_j \subset SO(2)$, $R \in SO(2)$, $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2)$, and $\beta \in L^2\left(\Omega; \mathbb{R}^{2\times 2}\right)$ such that*

(i) $\frac{\mu_j}{\varepsilon_j N_{\varepsilon_j}} \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}(\Omega; \mathbb{R}^2)$,

(ii) $\frac{R_j^T \beta_j - Id}{\varepsilon_j (N_{\varepsilon_j} |\log \varepsilon_j|)^{\frac{1}{2}}} \rightharpoonup \beta$ *in* $L^p(\Omega; \mathbb{R}^{2\times 2})$,

(iii) $R_j \to R$,

(iv) $\operatorname{curl}\beta = 0$.

Proof. Arguing as in step 1 of the proof of the compactness result in the critical regime (Proposition 4.3.1), one finds

$$\frac{|\mu_j|(\Omega)}{\varepsilon_j N_{\varepsilon_j}} \leq C E_{\varepsilon_j}(\mu_j, \beta_j).$$

For a subsequence follows (i).
An application of the generalized rigidity estimate, Theorem 3.0.1, together with the lower bound on the energy density W, (iv) in Section 4.1, provides $R_j \in SO(2)$ such that

$$\begin{aligned}\frac{1}{\varepsilon_j^2 N_{\varepsilon_j}|\log \varepsilon_j|}\int_\Omega |\beta_j - R_j|^2 \wedge |\beta_j - R_j|^p \,\mathrm{d}x &\leq C E_{\varepsilon_j}(\mu_j, \beta_j) + \frac{C}{\varepsilon_j^2 N_{\varepsilon_j}|\log \varepsilon_j|}|\mu_j|(\Omega)^2 \\ &\leq C + \frac{C}{\varepsilon_j^2 N_{\varepsilon_j}^2}|\mu_j|(\Omega)^2 \leq C.\end{aligned}$$

It follows that $\frac{R_j^T\beta_j - Id}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}$ is bounded in $L^p(\Omega;\mathbb{R}^{2\times 2})$. Hence, there exist a subsequence (not relabeled) and a function $\beta \in L^p(\Omega;\mathbb{R}^{2\times 2})$ such that

$$\frac{R_j^T\beta_j - Id}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} \rightharpoonup \beta \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}).$$

Thus, (ii). Up to taking a further subsequence, (iii) can also be satisfied. Arguing as in step 2 of the proof of the compactness statement in the critical regime, it can be shown that $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$. Finally, we show (iv). Let $\varphi \in C_c^\infty(\Omega;\mathbb{R}^2)$. We compute

$$\begin{aligned}&< \operatorname{curl}\beta, \varphi >_{\mathcal{D}',\mathcal{D}} = \lim_{j\to\infty} < \operatorname{curl}\frac{R_j^T\beta_j - Id}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}, \varphi >_{\mathcal{D}',\mathcal{D}} = \lim_{j\to\infty} < \frac{R_j^T\mu_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}, \varphi >_{\mathcal{D}',\mathcal{D}} \\ &= \lim_{j\to\infty} \underbrace{\sqrt{\frac{N_{\varepsilon_j}}{|\log \varepsilon_j|}}}_{\to 0} < \underbrace{\frac{R_j^T\mu_j}{\varepsilon_j N_{\varepsilon_j}}}_{\overset{*}{\rightharpoonup} R^T\mu}, \varphi >_{\mathcal{D}',\mathcal{D}} = 0.\end{aligned}$$

Hence, $\operatorname{curl}\beta = 0$. □

The Γ-convergence Result

In the spirit of the compactness result, we define for the subcritical regime the following notion of convergence.

Definition 4.4.1. Let $\varepsilon \to 0$. We say that a sequence $(\mu_\varepsilon, \beta_\varepsilon) \subset \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2})$ converges to a triplet $(\mu, \beta, R) \in \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ if there exists a sequence $(R_\varepsilon)_\varepsilon \subset SO(2)$ such that

$$\frac{\mu_\varepsilon}{\varepsilon N_\varepsilon} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2),$$

$$\frac{R_\varepsilon^T\beta_\varepsilon - Id}{\varepsilon\sqrt{N_\varepsilon|\log \varepsilon|}} \rightharpoonup \beta \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}), \text{ and } R_\varepsilon \to R.$$

With this notion of convergence we can state the Γ-convergence result in the subcritical regime.

Theorem 4.4.2. *Let $1 \ll N_{\varepsilon_j} \ll |\log \varepsilon|$. The energy functional E_ε as defined in* (4.3) *Γ-converges with respect to the notion of convergence given in Definition 4.4.1 to the functional E^{sub} defined on $\mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ as*

$$E^{sub}(\mu,\beta,R) = \begin{cases} \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx + \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu| & \text{if } \mu \in \mathcal{M}(\Omega;\mathbb{R}^2), \beta \in L^2(\Omega;\mathbb{R}^{2\times 2}), \\ & \text{and } \operatorname{curl}\beta = 0, \\ +\infty & \text{otherwise}, \end{cases}$$

where $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(Id)$ and φ is the relaxed self-energy density defined as in (4.7).

Remark 4.4.1. This regime is sometimes also called self-energy regime. Note that by the decoupling of the limit variables one can essentially minimize out the dependence from the strains in the definition of E_{ε_j} and obtain only the self-energy of μ in the limit, cf. [30].

The proof of the Γ-convergence result is given in the following two propositions treating the lim inf-inequality and the construction of a recovery sequence, respectively.

Proposition 4.4.3 (The lim inf-inequality)**.** *Let $\varepsilon_j \to 0$ and $1 \ll N_{\varepsilon_j} \ll |\log \varepsilon_j|$. Let $(\mu_j,\beta_j) \subset \mathcal{M}(\Omega;\mathbb{R}^2) \times L^p(\Omega;\mathbb{R}^{2\times 2})$ be a sequence which converges in the sense of Definition 4.4.1 to a triplet $(\mu,\beta,R) \in \mathcal{M}(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$. Then*

$$\liminf_{j\to\infty} E_{\varepsilon_j}(\mu_j,\beta_j) \geq E^{sub}(\mu,\beta,R).$$

Proof. Essentially, the proof works as the proof in the critical regime.
Without loss of generality we may assume that $\liminf_{\varepsilon_j\to 0} E_{\varepsilon_j}(\mu_j,\beta_j) = \lim_{j\to\infty} E_{\varepsilon_j}(\mu_j,\beta_j) < \infty$. For j large enough we derive that $\mu_j \in X_{\varepsilon_j}$ and write $\mu_j = \sum_{k=1}^{M_j} \varepsilon_j \xi_{k,j}\delta_{x_{k,j}}$ for appropriate $x_{k,j} \in \Omega$ and $\xi_{k,j} \in \mathbb{S}$. As in the proof in the critical regime, we divide the energy into a contribution far from the dislocations and a part close to the dislocations:

$$E_{\varepsilon_j}(\mu_j,\beta_j) = \frac{1}{\varepsilon_j^2|\log\varepsilon_j|N_{\varepsilon_j}} \int_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} W(\beta_j)\,dx + \frac{1}{\varepsilon_j^2|\log\varepsilon_j|N_{\varepsilon_j}} \sum_{k=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{k,j})} W(\beta_j)\,dx,$$

where $\Omega_{\rho_{\varepsilon_j}}(\mu_j) = \Omega \setminus \bigcup_{k=1}^{M_j} B_{\rho_{\varepsilon_j}}(x_{k,j})$.
The first term on the right hand side can be treated essentially as in the critical case. The only difference is that we use Taylor's theorem on the scale $\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}$ instead of $\varepsilon_j|\log\varepsilon_j|$. We obtain

$$\liminf_{j\to\infty} \frac{1}{\varepsilon_j^2|\log\varepsilon_j|N_{\varepsilon_j}} \int_{\Omega_{\rho_{\varepsilon_j}}(\mu_j)} W(\beta_j)\ \mathrm{d}x \geq \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx.$$

For the contribution close to the dislocations, we argue as in the critical regime (cf. (4.30) and (4.33)) to obtain for $\alpha,\delta \in (0,1)$ the estimate

$$\begin{aligned} \frac{1}{\varepsilon_j^2|\log\varepsilon_j|N_{\varepsilon_j}} \sum_{k=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{k,j})} W(\beta_j)\,dx \geq{}& \left(\alpha - \frac{|\log\rho_{\varepsilon_j}|}{|\log\varepsilon_j|}\right) \int_\Omega \varphi\left(R, \frac{d\tilde\mu_j}{d|\tilde\mu_j|}\right) d|\tilde\mu_j| \\ & - \left(\alpha - \frac{|\log\rho_{\varepsilon_j}|}{|\log\varepsilon_j|}\right)\left(\frac{C}{|\log\delta|} + \frac{C\sigma_{\varepsilon_j}}{|\log\delta|}\right), \end{aligned}$$

where $\tilde\mu_j = \frac{\mu_j}{\varepsilon_j N_{\varepsilon_j}}$. By the assumptions on the convergence of μ_j, it follows $\tilde\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$.

Convexity and 1-homogeneity of φ and Reshetnyak's theorem allow to conclude

$$\liminf_{\varepsilon_j\to 0} \frac{1}{\varepsilon_j^2|\log\varepsilon_j|N_{\varepsilon_j}} \sum_{k=1}^{M_j} \int_{B_{\rho_{\varepsilon_j}}(x_{k,j})} W(\beta_j)\,dx \geq \alpha \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu| - \frac{C\alpha}{|\log\delta|}.$$

Sending $\alpha \to 1$ and $\delta \to 0$ finishes the proof. □

Lastly, we prove the existence of a recovery sequence for the energy E^{sub}.

Proposition 4.4.4 (The lim sup-inequality)**.** *Let $\varepsilon_j \to 0$ and $1 \ll N_{\varepsilon_j} \ll |\log\varepsilon_j|$. Let $R \in SO(2)$, $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$, and $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2)$. Then there exists a sequence $(\mu_j,\beta_j)_j \subset \mathcal{M}(\Omega;\mathbb{R}^2)\times L^p(\Omega,\mathbb{R}^{2\times 2})$ converging to (μ,β,R) in the sense of Definition 4.4.1 such that*

$$\limsup_{\varepsilon_j\to 0} E_{\varepsilon_j}(\mu_j,\beta_j) \leq E^{sub}(\mu,\beta,R).$$

Proof. We may assume that $E^{sub}(\mu,\beta,R) < \infty$ and hence $\operatorname{curl}\beta = 0$.
As in the critical case, we can restrict ourselves to $\mu = \xi\, d\mathcal{L}^2$ for some fixed $\xi \in \mathbb{R}^2$. Note that the corresponding energy density result is even easier since β and μ are now decoupled. Moreover, we assume that $R = Id$.
Let $\xi = \sum_{k=1}^M \lambda_k\xi_k$ such that $\xi_k \in \mathbb{S}$ and $\varphi(Id,\xi) = \sum_{k=1}^M \lambda_k\psi(\xi)$. According to [38, Lemma 11], there exists $\mu_j = \sum_{k=1}^M \varepsilon_j\xi_k\mu_j^k$ with $\mu_j^k = \sum_{l=1}^{M_j^k} \delta_{x_{j,l}^k}$ such that $\mu_j \in X_{\varepsilon_j}$ and

$$\frac{|\mu_j^k|(\Omega)}{N_{\varepsilon_j}} \to \lambda_k|\Omega| \text{ and } \frac{\mu_j}{\varepsilon_j N_{\varepsilon_j}} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2).$$

For a sketch of the construction, see Figure 4.2 in Section 4.3. Combining the summations in the definition of μ_j we write $\mu_j = \sum_{k=1}^{M_j} \xi_{k,j}\,\delta_{x_{k,j}}$. As in the critical regime, we define the function $\eta^j = \sum_{i=1}^{M_j} \varepsilon_j\eta_i^j \mathbf{1}_{B_{r_{\varepsilon_j}}(x_{i,j})}$ where the η_i^j are special solutions of

$$\begin{cases} \operatorname{curl}\eta_i^j = \xi_{i,j}\delta_{x_{i,j}} & \text{in } \mathbb{R}^2, \\ \operatorname{div}\mathcal{C}\eta_i^j = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

Following the proof in the critical case we introduce the auxiliary measures $\tilde{\mu}_j^{r_{\varepsilon_j}}$, $\hat{\mu}_j^{r_{\varepsilon_j}}$ and the auxiliary strain $\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}$ as defined in (4.37), (4.38), and (4.40) for $r_{\varepsilon_j} = \frac{1}{2}\sqrt{\Lambda N_{\varepsilon_j}}^{-1}$ and $\Lambda = \sum_{k=1}^M \lambda_k$. Moreover, let $\tilde{\beta}_j$ be a solution to $\operatorname{curl}\tilde{\beta}_j = \tilde{\mu}_j^{r_{\varepsilon_j}}$ such that $\left\|\tilde{\beta}_j\right\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq C\left\|\tilde{\mu}_j^{r_{\varepsilon_j}}\right\|_{H^{-1}(\Omega;\mathbb{R}^2)}$. Then, define

$$\beta_j = Id + \varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}\beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j.$$

It follows $\operatorname{curl}\beta_j = \mu_j$ (recall that $\operatorname{curl}\beta = 0$).
According to [38, Lemma 11], it holds that $\frac{\tilde{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j N_{\varepsilon_j}} \rightharpoonup \xi\, d\mathcal{L}^2$ in $H^{-1}(\Omega;\mathbb{R}^2)$. Consequently, we derive

$$\left\|\frac{\tilde{\beta}_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}}\right\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq \left\|\frac{\tilde{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}}\right\|_{H^{-1}(\Omega;\mathbb{R}^2)} \to 0.$$

Moreover, similar to the critical case one can prove that

a) $\frac{\eta^j \mathbf{1}_{\Omega_{\varepsilon_j}(\mu_j)}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \rightharpoonup 0$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$,

b) $\frac{\tilde{K}^{r\varepsilon_j}_{\mu_j}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \to 0$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$,

c) $\frac{\eta^j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \rightharpoonup 0$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$.

Therefore, $\frac{\beta_j - Id}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \rightharpoonup \beta$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$. Hence, the sequence $(\mu_j,\beta_j)_j$ converges in the sense of Definition 4.4.1 to (μ,β,Id).
The desired lim sup-inequality for the energies can be shown analogously to the the lim sup-inequality in the critical regime by replacing $|\log\varepsilon_j|$ by N_{ε_j} at the correct places. □

4.5 The Supercritical Regime

The supercritical regime corresponds to the scaling $|\log\varepsilon| \ll N_\varepsilon$. In this regime, the distance of the strain to $SO(2)$ is expected to be of order $\varepsilon\sqrt{|\log\varepsilon|N_\varepsilon}$ whereas the energy bound in Section 1.3 show that the dislocation density $\operatorname{curl}\beta_\varepsilon = \mu_\varepsilon$ is expected to be of scale $\varepsilon N_\varepsilon$. Hence, we cannot control $\frac{|\mu_\varepsilon|(\Omega)}{\varepsilon\sqrt{|\log\varepsilon|N_\varepsilon}}$ and thus the generalized rigidity cannot be used to obtain compactness. Even more, we will show that there cannot be a compactness result as in the other regimes.
On the other hand, μ_ε depends on β_ε which scales like $\varepsilon\sqrt{|\log\varepsilon|N_\varepsilon}$. At least in a negative Sobolev norm this gives an upper bound for the scaling of μ_ε by $\varepsilon\sqrt{|\log\varepsilon|N_\varepsilon}$ which is much smaller than the natural scaling that comes from the energy. Hence, the elastic energy should be dominant and the self-energy vanishes in the limit. A Γ-convergence result which does only include β can still be shown. The limit will simply be a linearized elastic energy, precisely

$$\int_\Omega \frac{1}{2}\mathcal{C}\beta : \beta\, dx.$$

The notion of convergence we use is the following.

Definition 4.5.1. We say that a sequence $(\beta_j)_j \subset L^p(\Omega;\mathbb{R}^{2\times 2})$ converges to $\beta \in L^p(\Omega;\mathbb{R}^{2\times 2})$ if there exists a sequence $(R_j)_j \subset SO(2)$ such that

$$\frac{R_j^T\beta_j - Id}{\varepsilon_j\sqrt{|\log\varepsilon_j N_{\varepsilon_j}}} \rightharpoonup \beta \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}).$$

As discussed above, the dislocation density does not appear in the limit. For this reason, we eliminate it from the definition of the energies. The Γ-convergence result is the following.

Theorem 4.5.1. *Let $N_\varepsilon \gg |\log\varepsilon|$ such that $N_\varepsilon|\log\varepsilon| \ll \rho_\varepsilon^{-4}$. Then energy functional E^{sup}_ε defined as*

$$E^{sup}_\varepsilon(\beta) = \begin{cases} \frac{1}{\varepsilon^2|\log\varepsilon|N_\varepsilon}\int_\Omega W(\beta)\,dx & \text{if } (\operatorname{curl}\beta,\beta) \in X_\varepsilon \times \mathcal{AS}_\varepsilon(\operatorname{curl}\beta),\\ +\infty & \text{else in } L^p(\Omega;\mathbb{R}^{2\times 2}),\end{cases} \tag{4.49}$$

Γ-converges with respect to the notion of convergence given in Definition 4.5.1 to the functional E^{sup} defined on $L^p(\Omega;\mathbb{R}^{2\times 2})$ as

$$E^{sup}(\beta) = \begin{cases} \frac{1}{2}\int_\Omega \mathcal{C}\beta : \beta\,dx & \text{if } \beta_{sym} \in L^2(\Omega,\mathbb{R}^{2\times 2}),\\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^{2\times 2}),\end{cases}$$

where $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(I)$.

The proof will be given in the following two propositions. First, we show the lim inf-inequality.

Proposition 4.5.2 (The lim inf-inequality). *Let $\varepsilon_j \to 0$ and $N_{\varepsilon_j} \gg |\log \varepsilon_j|$ such that it holds $N_{\varepsilon_j}|\log \varepsilon_j| \ll \rho_{\varepsilon_j}^{-4}$. Let $(\beta_j)_j \subset L^p(\Omega;\mathbb{R}^{2\times 2})$ be a sequence which converges to $\beta \in L^p(\Omega;\mathbb{R}^{2\times 2})$ in the sense of Definition 4.5.1. Then*

$$\liminf_j E^{sup}_{\varepsilon_j}(\beta_j) \geq E^{sup}(\beta).$$

Proof. Without loss of generality we may assume that $\liminf_j E^{sup}_{\varepsilon_j}(\beta_j) = \lim_j E^{sup}_{\varepsilon_j}(\beta_j) < \infty$ and $\sup E^{sup}_{\varepsilon_j}(\beta_j) < \infty$. As we obtain the lower bound by linearizing at Id, we may assume without loss of generality that the sequence $(R_j)_j \subset SO(2)$ from the definition of convergence of β_j converges to Id. Let $S_j : \Omega \to SO(2)$ be a measurable function such that $\operatorname{dist}(\beta_j, SO(2)) = |\beta_j - S_j|$. By the boundedness of the energy $E^{sup}_{\varepsilon_j}(\beta_j)$ and the lower bound of the energy density W, one sees that $\frac{\beta_j - S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}$ is bounded in $L^p(\Omega;\mathbb{R}^{2\times 2})$. Moreover, $\mathbf{1}_{A_{\varepsilon_j}} \frac{\beta_j - S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}$ is bounded in $L^2(\Omega;\mathbb{R}^{2\times 2})$ where $A_{\varepsilon_j} = \{x \in \Omega : \operatorname{dist}(\beta_j, SO(2)) \leq 1\}$. In addition,

$$|\Omega \setminus A_{\varepsilon_j}| \leq \int_{\Omega\setminus A_{\varepsilon_j}} \operatorname{dist}(\beta_j, SO(2))^p \, dx \leq C\varepsilon_j^2 |\log \varepsilon_j| N_{\varepsilon_j} \, E^{sup}_{\varepsilon_j}(\beta_j) \leq C\varepsilon_j^2|\log \varepsilon_j|N_{\varepsilon_j} \to 0.$$

For the convergence, notice that $\varepsilon_j^2|\log \varepsilon_j|N_{\varepsilon_j} \ll \varepsilon_j^2 \rho_{\varepsilon_j}^{-4} \to 0$. In particular, it follows that $\mathbf{1}_{A_{\varepsilon_j}} \to 1$ boundedly in measure. Hence, there exists $\tilde{\beta} \in L^2(\Omega;\mathbb{R}^{2\times 2})$ such that (up to taking a subsequence) it holds

$$\frac{\beta_j - S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} \rightharpoonup \tilde{\beta} \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}) \text{ and } \mathbf{1}_{A_{\varepsilon_j}} \frac{\beta_j - S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} \rightharpoonup \tilde{\beta} \text{ in } L^2(\Omega;\mathbb{R}^{2\times 2}).$$

Consequently,

$$\frac{Id - R_j^T S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} = \frac{Id - R_j^T \beta_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} + R_j^T \frac{\beta_j - S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} \rightharpoonup -\beta + \tilde{\beta} \text{ in } L^p(\Omega;\mathbb{R}^{2\times 2}). \tag{4.50}$$

This implies that $Id - R_j^T S_j$ converges to 0 boundedly in measure.
Moreover, by the structure of $SO(2)$ as a manifold and the fact that $T_{Id}SO(2) = Skew(2)$, there is a map $T_j : \Omega \to Skew(2)$ such that

$$\frac{Id - R_j^T S_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} = \frac{T_j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} + \frac{\mathcal{O}(|Id - R_j^T S_j|^2)}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}.$$

By (4.50) and the convergence of $Id - R_j^T S_j$ to 0 boundedly in measure it holds that $\frac{\mathcal{O}(|Id - R_j^T S_j|^2)}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}} \rightharpoonup 0$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$. Notice that the space of functions in $L^p(\Omega;\mathbb{R}^{2\times 2})$ with values in $Skew(2)$ almost everywhere is strongly closed in $L^p(\Omega;\mathbb{R}^{2\times 2})$. Hence, by Mazur's lemma the weak limit of $\frac{Id - R_{\varepsilon_j}^T S_{\varepsilon_j}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}$ takes values in $Skew(2)$ almost everywhere. By (4.50), it follows that $\beta - \tilde{\beta} \in Skew(2)$ almost everywhere.
As the energy density W is rotationally invariant, the quadratic form induced by $\mathcal{C} = \frac{\partial^2 W}{\partial^2 F}(Id)$ acts only on the symmetric part of a matrix and thus $\mathcal{C}\beta : \beta = \mathcal{C}\tilde{\beta} : \tilde{\beta}$. Hence, it suffices to show that $\liminf_j E_{\varepsilon_j}(\beta_j) \geq \frac{1}{2}\int_\Omega \mathcal{C}\tilde{\beta} : \tilde{\beta} \, dx$.
Let us define the function $G_j = \frac{\beta_j - S_{\varepsilon_j}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}$ and $U_{\varepsilon_j} = A_{\varepsilon_j} \cap \left\{ x \in \Omega : |G_j| \leq \sqrt{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log \varepsilon_j|}}^{\,-1} \right\}$.

Notice that $\mathbf{1}_{U_{\varepsilon_j}} \to 1$ boundedly in measure and that $\varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|}\, \mathbf{1}_{U_{\varepsilon_j}} G_j$ converges to zero uniformly. Moreover, $\mathbf{1}_{U_{\varepsilon_j}} G_j = \mathbf{1}_{U_{\varepsilon_j}} \mathbf{1}_{A_{\varepsilon_j}} G_j \rightharpoonup \tilde{\beta}$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$. As $S_j \to Id$ boundedly in measure it holds furthermore that $\mathbf{1}_{U_{\varepsilon_j}} S_j^T G_j \rightharpoonup \tilde{\beta}$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$.
We estimate using Taylor's theorem

$$\begin{aligned}
\frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega W(\beta_j)\, dx =& \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega W(S_j^T \beta_j)\, dx \\
\geq & \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_{U_{\varepsilon_j}} W\left(Id + \varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|} S_j^T G_j\right) dx \\
\geq & \int_\Omega \mathcal{C}\left(\mathbf{1}_{U_{\varepsilon_j}} S_j^T G_j\right) : \left(\mathbf{1}_{U_{\varepsilon_j}} S_j^T G_j\right) dx \\
& - \int_{U_{\varepsilon_j}} \frac{\omega\left(\left|\varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|} S_j^T G_j\right|\right)}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|\, |G_j|^2} |G_j|^2\, dx,
\end{aligned}$$

where $\frac{\omega(t)}{t^2} \to 0$ for $t \searrow 0$, cf. (4.18).
The error term in the last line of the above estimate goes to zero as $|G_j|^2 \mathbf{1}_{U_{\varepsilon_j}}$ is bounded in L^1 and $\varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|} S_{\varepsilon_j}^T G_j$ goes to zero uniformly on U_{ε_j}.
For the first term in the last line of the estimate, we use that the quadratic form induced by $\mathcal{C}$ is lower semi-continuous with respect to weak convergence in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Hence, we obtain

$$\liminf_j \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega W(\beta_j)\, dx \geq \int_\Omega \mathcal{C}\tilde{\beta} : \tilde{\beta}\, dx.$$

This finishes the proof. □

Next, we shortly sketch the proof of the upper bound. It is much easier than the one in the critical regime since we do not have to perform the careful analysis in order to recover the self-energy. Instead, we simply have to recover the linearized elastic energy using Taylor's theorem. The statement is the following.

Proposition 4.5.3 (The lim sup-inequality)**.** *Let $\varepsilon_j \to 0$ and $N_{\varepsilon_j} \gg |\log \varepsilon_j|$ such that it still holds $N_{\varepsilon_j} |\log \varepsilon_j| \ll \rho_{\varepsilon_j}^{-4}$. Let $\beta \in L^p(\Omega;\mathbb{R}^{2\times 2})$. Then there exists a sequence $(\beta_j)_j \subset L^p(\Omega;\mathbb{R}^{2\times 2})$ such that β_j converges to β in the sense of Definition 4.5.1 and* $\limsup_j E_{\varepsilon_j}^{sup}(\beta_j) \leq E^{sup}(\beta)$.

Proof. As in [38], by a convolution argument we may assume without loss of generality $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ and $\mu = \operatorname{curl} \beta \in C^0(\overline{\Omega},\mathbb{R}^2) \subset H^{-1}(\Omega;\mathbb{R}^2)$. Using a similar construction as in the critical regime for $r_{\varepsilon_j} \sim (N_{\varepsilon_j} |\log \varepsilon_j|)^{\frac{1}{4}}$, there exists a sequence of measure $\mu_j = \sum_{k=1}^{M_j} \varepsilon_j \xi_{k,j} \delta_{x_{k,j}} \in X_{\varepsilon_j}$ (the assumed growth restriction on N_{ε_j} guarantees that $\frac{\rho_{\varepsilon_j}}{r_{\varepsilon_j}} \to 0$ and hence well-separateness of the dislocations, cf. the sketched construction of μ_j in the lim sup-inequality in the critical regime) such that it holds $\frac{\mu_j}{\varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|}} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$ and the corresponding measures $\tilde{\mu}_j^{r_{\varepsilon_j}}$ as defined in (4.38) satisfy $\frac{\tilde{\mu}_j^{r_{\varepsilon_j}}}{\varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|}} \to \mu$ in $H^{-1}(\Omega;\mathbb{R}^2)$. Furthermore, this construction can be done such that $|\xi_{k,j}| \leq C \left\| \operatorname{curl} \beta \right\|_{L^\infty(\Omega;\mathbb{R}^2)}$.
Notice that the measures μ_j approximate $\operatorname{curl} \beta$ on the scale $\varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|}$ whereas the scale of approximation in the critical and subcritical regime is $\varepsilon_j N_{\varepsilon_j} \gg \varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|}$. Hence, the measure μ_j is smaller than usually which leads to the fact that it does not contribute to the limit.
Using the notation η^j, $\hat{\mu}_j^{r_{\varepsilon_j}}$, and $\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}$ as defined in (4.36), (4.37), and (4.40) we define the recovery

sequence by

$$\beta_j = Id + \varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}\beta + \eta^j - \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j,$$

where $\tilde{\beta}$ satisfies the equation $\operatorname{curl}\tilde{\beta}_j = -\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}\operatorname{curl}\beta + \tilde{\mu}_j^{r_{\varepsilon_j}}$ and the corresponding bound $\left\|\tilde{\beta}_{\varepsilon_j}\right\|_{L^2(\Omega;\mathbb{R}^{2\times2})} \leq C\left\|\tilde{\mu}_j^{r_{\varepsilon_j}} - \varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}\operatorname{curl}\beta\right\|_{H^{-1}(\Omega;\mathbb{R}^2)}$. It follows that $\operatorname{curl}\beta_j = \mu_j \in X_{\varepsilon_j}$. Moreover, one can check that (note that the main difference to the critical regimes is the strong convergence in (a))

a) $\frac{\eta^j \mathbf{1}_{\Omega_{\varepsilon_j}(\mu_j)}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \to 0$ in $L^2(\Omega;\mathbb{R}^{2\times2})$,

b) $\frac{\eta^j}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \to 0$ in $L^p(\Omega;\mathbb{R}^{2\times2})$,

c) $\frac{\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \to 0$ in $L^2(\Omega;\mathbb{R}^{2\times2})$,

d) $\frac{\tilde{\beta}_{\varepsilon_j}}{\varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}} \to 0$ in $L^2(\Omega;\mathbb{R}^{2\times2})$.

Let us briefly prove a). It holds

$$\begin{aligned}\int_{\Omega_{\varepsilon_j}(\mu_j)} |\eta^j|^2\,dx &\leq C\sum_{i=1}^{M_j}\varepsilon_j^2|\xi_{i,j}|^2|\log\varepsilon_j| \\ &\leq C\,\|\operatorname{curl}\beta\|^2_{L^\infty(\Omega;\mathbb{R}^2)}\,\varepsilon_j^2\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|^3} \ll \varepsilon_j^2 N_{\varepsilon_j}|\log\varepsilon_j|.\end{aligned}$$

Following the arguments in the lim sup-inequality in the critical regime, this leads to the fact that the functions η^j do not induce self-energy in the limit.
The desired estimate follows from a) – d) by copying the arguments from the critical regime. □

As already discussed in the beginning of this section, there is no compactness result in this regime. To conclude the discussion of the supercritical regime, we provide a counterexample to compactness in the supercritical regime.

Example (A counterexample to compactness)**.** Let $\varepsilon_j \to 0$ and $N_{\varepsilon_j} \gg |\log\varepsilon_j|$ such that $N_{\varepsilon_j} \ll \rho_{\varepsilon_j}^{-2}$. For simplicity, let $\Omega = (-1,1)^2 \subset \mathbb{R}^2$.
We define $\alpha_{\varepsilon_j} = \varepsilon_j\sqrt{N_{\varepsilon_j}|\log\varepsilon_j|}$, $a_{\varepsilon_j} = \varepsilon_j\left(N_{\varepsilon_j}^2|\log\varepsilon_j|\right)^{\frac{1}{3}}$ and $\delta_{\varepsilon_j} = \frac{\alpha_{\varepsilon_j}^2}{a_{\varepsilon_j}^2}$. As $N_{\varepsilon_j} \gg |\log\varepsilon_j|$, it holds $\alpha_{\varepsilon_j} \ll a_{\varepsilon_j}$ and hence $\delta_{\varepsilon_j} \to 0$ as $\varepsilon_j \to 0$.
Let $T \in Skew(2)$. We may assume that $T \cdot e_2 \in \mathbb{S}$ (otherwise scale T and rotate Ω such that $T \cdot b \in \mathbb{S}$ for b one of the basic vectors of the rotated cube). As $T_{Id}SO(2) = Skew(2)$ there exist rotations $R_j \in SO(2)$ such that $R_j - Id = a_{\varepsilon_j}T + \mathcal{O}(a_{\varepsilon_j}^2)$.
We define the function $\varphi_j : \Omega \to \mathbb{R}^{2\times2}$ by $\varphi_j = Id + (R_j - Id)\psi_j$ where

$$\psi_j(x,y) = \begin{cases} 0 & \text{if } x \leq -\frac{\delta_{\varepsilon_j}}{2}, \\ \frac{x}{\delta_{\varepsilon_j}} + \frac{1}{2} & \text{if } -\frac{\delta_{\varepsilon_j}}{2} \leq x \leq \frac{\delta_{\varepsilon_j}}{2}, \\ 1 & \text{if } \frac{\delta_{\varepsilon_j}}{2} \leq x. \end{cases}$$

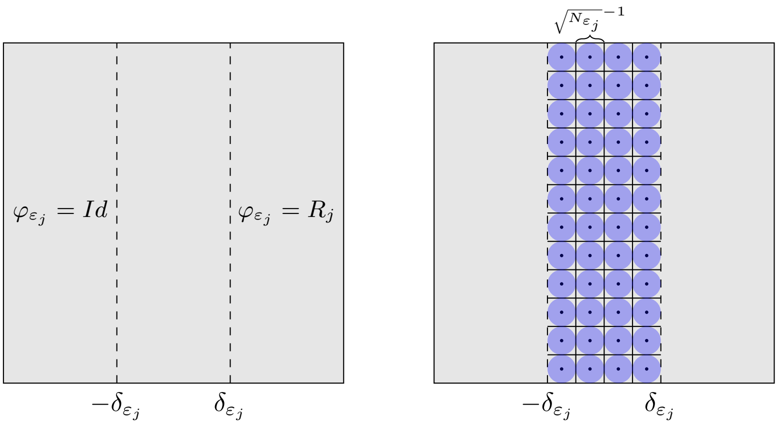


Figure 4.3: Left: the construction of φ_j. Right: illustration of μ_j and $\tilde{\mu}_{\varepsilon_j}^{r_{\varepsilon_j}}$: the black dots correspond to Dirac masses of mass $\varepsilon_j T \cdot e_2$, each blue circle corresponds to the mass $\varepsilon_j T \cdot e_2$ continuously distributed on the circle of radius $\sqrt{N_{\varepsilon_j}}^{-1}$.

Then

$$
\begin{aligned}
\operatorname{curl} \varphi_j &= (R_j - Id) \cdot e_2 \frac{1}{\delta_{\varepsilon_j}} \mathcal{L}^2_{|(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)} \\
&= \frac{a_{\varepsilon_j}}{\delta_{\varepsilon_j}} T \cdot e_2 \, \mathcal{L}^2_{|(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)} + \mathcal{O}\left(\frac{a^2_{\varepsilon_j}}{\delta_{\varepsilon_j}} \right) \mathcal{L}^2_{|(\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)} \\
&= \varepsilon_j N_{\varepsilon_j} T \cdot e_2 \, \mathcal{L}^2_{|(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)} + \mathcal{O}\left(\frac{a^2_{\varepsilon_j}}{\delta_{\varepsilon_j}} \right) \mathcal{L}^2_{|(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)}.
\end{aligned}
$$

Now, construct $\mu_j \in X_{\varepsilon_j}$ as in the construction of the recovery sequences in the different regimes i.e., cover $(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)$ with squares with side length $\frac{1}{\sqrt{N_{\varepsilon_j}}}$ and put a Dirac mass with mass $\varepsilon_j T \cdot e_2$ in the center $x_{k,j}$ of each of these squares, see Figure 4.3. Note that the growth assumptions on N_ε guarantee well-separateness of dislocations. Denote by $\tilde{\mu}_{\varepsilon_j}^{r_{\varepsilon_j}}$ the corresponding regularized measure on the ball of radius $\frac{1}{2}\sqrt{N_{\varepsilon_j}}^{-1}$ as defined in (4.38) for $r_{\varepsilon_j} = \frac{1}{2}\sqrt{N_{\varepsilon_j}}^{-1}$.
The application of Lemma 4.5.4 for $r = \frac{1}{2\sqrt{N_{\varepsilon_j}}}$, $U = (-\delta_{\varepsilon_j}, \delta_{\varepsilon_j}) \times (-1,1)$ and $f_r = \tilde{\mu}_j^{r_{\varepsilon_j}} - \varepsilon_j N_{\varepsilon_j} T \cdot e_2$ (here, we identify the measure $\tilde{\mu}_j^{r_{\varepsilon_j}}$ and its density with respect to the Lebesgue measure) provides the following estimate

$$
\left\| \tilde{\mu}_j^{r_{\varepsilon_j}} - \varepsilon_j N_{\varepsilon_j} T \cdot e_2 \, \mathcal{L}^2_{|(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)} \right\|_{H^{-1}} \leq C \frac{1}{\sqrt{N_{\varepsilon_j}}} \varepsilon_j N_{\varepsilon_j} |T \cdot e_2| \sqrt{\delta_{\varepsilon_j}} \leq C \varepsilon_j \sqrt{N_{\varepsilon_j}}.
$$

As $\left\| \mathcal{O}\left(\frac{a^2_{\varepsilon_j}}{2\delta_{\varepsilon_j}} \right) \mathcal{L}^2_{|(-\delta_{\varepsilon_j}, \delta_{\varepsilon_j}) \times (-1,1)} \right\|_{H^{-1}} \leq C \frac{a^2_{\varepsilon_j}}{2\delta_{\varepsilon_j}} \sqrt{\delta_{\varepsilon_j}} = \frac{C}{2} \frac{a^3_{\varepsilon_j}}{\alpha_{\varepsilon_j}}$, we find that

$$
\frac{\left\| \tilde{\mu}_j^{r_{\varepsilon_j}} - \operatorname{curl} \varphi_j \right\|_{H^{-1}}}{\alpha_{\varepsilon_j}} \leq C \left(\frac{\varepsilon_j \sqrt{N_{\varepsilon_j}}}{\alpha_{\varepsilon_j}} + \frac{a^3_{\varepsilon_j}}{\alpha^2_{\varepsilon_j}} \right) = C \left(\frac{1}{\sqrt{|\log \varepsilon_j|}} + \varepsilon_j N_{\varepsilon_j} \right) \to 0.
$$

Now, we can construct the counterexample similarly to the construction of the recovery sequence. Let $\beta_j = \varphi_j + \eta^j + \tilde{K}_{\mu_j}^{r_{\varepsilon_j}} + \tilde{\beta}_j$ where $\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}$ is defined as in (4.40) and $\operatorname{curl} \tilde{\beta}_j = \tilde{\mu}_j^{r_{\varepsilon_j}} - \operatorname{curl} \varphi_j$ such that $\left\| \tilde{\beta}_j \right\|_{L^2} \leq C \left\| \tilde{\mu}_j^{r_{\varepsilon_j}} - \operatorname{curl} \varphi_j \right\|_{H^{-1}}$. In particular, it follows $\left\| \frac{\tilde{\beta}_j}{\alpha_{\varepsilon_j}} \right\|_{L^2} \to 0$. As in the construction of the recovery sequence in the supercritical regime, one can derive from $\frac{|\mu_j|(\Omega)}{\varepsilon_j N_{\varepsilon_j}} \to 0$ that (cf. a) – d) in the proof of the lim sup-inequality in the supercritical regime)

$$\frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega |\eta^j|^2 \wedge |\eta^j|^p + |\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}|^2 \, dx \to 0.$$

As $\operatorname{curl} \beta_j = \mu_j \in X_{\varepsilon_j}$, we obtain that

$$\begin{aligned} E_{\varepsilon_j}^{sup}(\beta_j) \le & C \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega \operatorname{dist}(\beta_j, SO(2))^2 \wedge \operatorname{dist}(\beta_j, SO(2))^p \, dx \\ \le & \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega |\eta^j|^2 \wedge |\eta^j|^p + |\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}|^2 + |\tilde{\beta}_j|^2 \, dx \\ & + C \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_{(-\frac{\delta_{\varepsilon_j}}{2}, \frac{\delta_{\varepsilon_j}}{2}) \times (-1,1)} |Id - R_j|^2 \\ \le & \frac{1}{\varepsilon_j^2 N_{\varepsilon_j} |\log \varepsilon_j|} \int_\Omega |\eta^j|^2 \wedge |\eta^j|^p + |\tilde{K}_{\mu_j}^{r_{\varepsilon_j}}|^2 + |\tilde{\beta}_j|^2 \, dx + C2 \underbrace{\frac{\delta_{\varepsilon_j} a_{\varepsilon_j}^2}{\alpha_{\varepsilon_j}^2}}_{=1} \\ \le & C. \end{aligned}$$

On the other hand, it is clear that there cannot be a sequence $(S_j)_j \subset SO(2)$ such that up to a subsequence $\frac{S_j^T \beta_j - Id}{\alpha_{\varepsilon_j}}$ converges weakly in $L^p(\Omega; \mathbb{R}^{2\times 2})$ because the only relevant part of β_j on scale $\alpha_{\varepsilon_j} = \varepsilon_j \sqrt{N_{\varepsilon_j} |\log \varepsilon_j|}$ is φ_j which is essentially either Id or R_j. These rotations are separated on scale $a_{\varepsilon_j} \gg \alpha_{\varepsilon_j}$ and live both on sets of order 1.
Note that $\frac{|\mu_j|(\Omega)}{\alpha_{\varepsilon_j}} \sim \frac{\varepsilon_j N_{\varepsilon_j} \delta_{\varepsilon_j}}{\alpha_{\varepsilon_j}} = N_{\varepsilon_j}^{\frac{1}{6}} |\log \varepsilon_j|^{-\frac{1}{6}} \to \infty$. This illustrates why we cannot use the generalized rigidity estimate to obtain compactness in the supercritical regime.

Remark 4.5.1. A similar construction could be done on any Lipschitz domain using cubes to separate the domain in a left and a right part. Then, use the construction above on the cubes and extend β_j constantly as Id, respectively R_j, to the left and the right of the cubes.

Finally, we prove the scaling estimate in H^{-1} that we used in the construction of the counterexample.

Lemma 4.5.4. *Let Q be the unit cube and $f \in L^2(Q; \mathbb{R}^2)$ with $\int_Q f = 0$. Let $r > 0$, $U \subset \mathbb{R}^2$ bounded and $(Q_k)_k$ a family of scaled copies of Q with side length r and center x_k such that $\dot{\bigcup}_k Q_k \subset U$ and $U \subset \bigcup_k \overline{Q_k}$. Define $f_r(x) = f\left(\frac{x - x_k}{r}\right)$ on Q_k. Then it holds*

$$\|f_r\|_{H^{-1}(U)} \le Cr \, \|f\|_{L^2(Q)} \, \mathcal{L}^2(U)^{\frac{1}{2}},$$

where C does not depend on f, r and U.

Proof. Let $\varphi \in H^1(U; \mathbb{R}^2)$ and write $< \varphi >_{Q_k} = ⨍_{Q_k} \varphi \, dx$. We estimate, using Hölder's and Poincaré's inequality (recall that Poincaré's constant scales like r on domains rescaled by r),

$$\begin{aligned}\int_U f_r \cdot \varphi \, dx &= \sum_k \int_{Q_k} f_r \cdot (\varphi - < \varphi >_{Q_k}) \, dx \\ &\leq \sum_k \|f_r\|_{L^2(Q_k;\mathbb{R}^2)} \|\varphi - < \varphi >_{Q_k}\|_{L^2(Q_k;\mathbb{R}^2)} \\ &\leq Cr^2 \sum_k \|f\|_{L^2(Q)} \|\nabla\varphi\|_{L^2(Q_k;\mathbb{R}^{2\times 2})} \\ &\leq Cr^2 \|f\|_{L^2(Q;\mathbb{R}^2)} \|\nabla\varphi\|_{L^2(U;\mathbb{R}^{2\times 2})} \sqrt{\#\text{ number of cubes } Q_k} \\ &\leq Cr \|f\|_{L^2(Q;\mathbb{R}^2)} \mathcal{L}^2(U)^{\frac{1}{2}} \|\varphi\|_{H^1(U;\mathbb{R}^2)} \, .\end{aligned}$$

□

5 Plasticity as the Γ-limit of a Dislocation Energy without the Assumption of Diluteness

In this chapter, we consider a core-radius approach for straight, parallel edge dislocations in the context of the linearized theory as described in Section 1.3. In particular, we compute the stored elastic energy on a reduced domain which does not include the dislocation cores. The second main difference to the model discussed in Chapter 4 and other models of this type (cf. [38, 59, 71]) is that in this chapter we drop the assumption of well-separateness of dislocations (cf. the definition of the set of admissible dislocation densities in Section 4.1). In the proofs of the Γ-convergence results in the previous chapter, it has been of enormous importance that we could compute the self-energy of each dislocation separately and relax this energy in a second step on a larger scale. Without the assumption of well-separateness this is not possible anymore. On a technical level, this leads also to the fact that one cannot expect to obtain upper bounds on the total variation of the dislocation density. For example, two dislocations of different sign, which are very close, should essentially be seen as no dislocation and therefore not contribute significant self-energy. The existence of many of those dipoles could then prevent a compactness statement in the sense of weak$*$-convergence in measures to hold. Therefore, we need to weaken the notion of convergence of dislocation densities in a way that allows those dipoles to vanish in the limit. The solution will be to consider strong convergence in the dual space of Lipschitz functions which vanish on the boundary. This convergence is sometimes also called flat convergence and was used successfully in the treatment of the subcritical regime, cf. [30]. A main tool to prove bounds on the dislocation densities which imply compactness in the flat topology will be ball construction techniques, which are also known in the context of vortices in Ginzburg-Landau energies, cf. [51, 70]. The building block of energy estimates using the ball construction techniques are energy bounds on annuli. In the context of elasticity, one obtains a massive loss of rigidity on thin annuli which leads to inadequate lower bounds on thin annuli. Mathematically, this phenomenon becomes manifest in the explosion of Korn's constant for thin annuli, see Section 5.A. This will be one of the major problems we will face in order to prove meaningful lower bounds.

In this chapter, we focus only on the critical regime; the subcritical regime has already been discussed by de Luca, Garroni, and Ponsiglione in [30]. The supercritical regime can essentially be treated as in Section 4.5. We identify the Γ-limit of the rescaled stored energy to be essentially the same as in Section 4.3 i.e., a strain-gradient plasticity model of the form (see Theorem 5.2.1)

$$\int_\Omega \mathcal{C}\beta : \beta \, dx + \int_\Omega \varphi\left(R, \frac{d\mu}{d|\mu|}\right) d|\mu|,$$

where $\mathcal{C}$ is the elasticity tensor and φ is the relaxed self-energy density for dislocations as defined in (4.7) without the dependence of a global rotation (as we already start from a linearized model). Moreover, we prove a compactness result, see Theorem 5.2.2 and Section 5.5 for the proof, and discuss its optimality.

The chapter is ordered as follows. First, we state the precise mathematical setting of the problem and the main results in Section 5.1 and Section 5.2. In section 5.3, we revisit the ball construction technique as it is known, for example, from [51] and discuss the particular difficulties in the context of elasticity theory. Next, we prove the key lower bounds for compactness and the Γ-convergence result in Section 5.4. In Section 5.5, we prove compactness. Then, we discuss the proof of the Γ-convergence result in the Sections 5.6 and 5.7. Finally, we discuss briefly the scaling of Korn's constant on thin annuli in 5.A.

5.1 Setting of the Problem

Throughout this chapter we consider $\Omega \subset \mathbb{R}^2$ to be a bounded, simply connected Lipschitz domain which represents the horizontal cross section of a cylindrical crystal, see Section 1.2. We denote by $\varepsilon > 0$ the lattice spacing.

As in the mixed growth case, we consider the set of *normalized* minimal Burgers vectors in the horizontal plane to be $S = \{b_1, b_2\}$ for two linearly independent vectors $b_1, b_2 \in \mathbb{R}^2$. The set of (normalized) admissible Burgers vectors is then given by $\mathbb{S} = \operatorname{span}_{\mathbb{Z}} S$. We consider the following space of admissible dislocation distributions.

$$X(\Omega) = \left\{ \mu \in \mathcal{M}(\Omega;\mathbb{R}^2) : \mu = \sum_{i=1}^{N} \xi_i \delta_{x_i} \text{ for some } N \in \mathbb{N}, 0 \neq \xi_i \in \mathbb{S}, \text{ and } x_i \in \Omega \right\}.$$

Note that dealing with a linearized energy density allows us to scale out the dependence of the admissible Burgers vectors from the lattice spacing. Associated to $\mu \in X(\Omega)$, we consider the strains generating μ. In contrast to the mixed growth case, in the geometrically linearized setting strains typically create an infinite energy in a core-radius around each dislocation. In particular, strains satisfying $\operatorname{curl} \beta = \mu$ for some $\mu \in X(\Omega)$ cannot be in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Hence, we cut out a core-radius of order ε around each dislocation and work on a reduced domain, precisely

$$\Omega_\varepsilon(\mu) = \Omega \setminus \bigcup_{x \in \operatorname{supp}(\mu)} B_\varepsilon(x).$$

In general, we write $\Omega_r(\mu) = \Omega \setminus \bigcup_{x\in\operatorname{supp}(\mu)} B_r(x)$ for some $r > 0$.

The curl-condition in (4.1) is then replaced by a circulation condition around the cores. We define the admissible strains as

$$\mathcal{AS}_\varepsilon^{lin}(\mu) = \Big\{ \beta \in L^2(\Omega;\mathbb{R}^{2\times 2}) : \beta = 0 \text{ in } \Omega \setminus \Omega_\varepsilon(\mu), \operatorname{curl}\beta = 0 \text{ in } \Omega_\varepsilon(\mu), \text{ and for every smoothly bounded open set } A \subset \Omega \text{ such that } \partial A \subset \Omega_\varepsilon(\mu) \text{ it holds that } \int_{\partial A} \beta \cdot \tau \, d\mathcal{H}^1 = \mu(A) \Big\}.$$

Here, $\beta \cdot \tau$ has to be understood in the sense of traces, see [29, Theorem 2] and the discussion in Section 4.2. Note that if the core $B_\varepsilon(x_i)$ of a dislocation with Burgers vector ξ does not intersect any other core, the definition of $\mathcal{AS}_\varepsilon^{lin}$ implies that

$$\int_{\partial B_\varepsilon(x_i)} \beta \cdot \tau \, d\mathcal{H}^1 = \xi.$$

Instead of this circulation condition, one could also consider the set $X(\Omega)$ to consist of more regular measures such as

$$\frac{\xi}{\pi\varepsilon^2}\,\mathcal{L}^2_{|B_\varepsilon(x)},\ \frac{\xi}{2\pi\varepsilon}\,\mathcal{H}^1_{|\partial B_\varepsilon(x)} \text{ or } \xi\delta_x * \rho_\varepsilon \text{ where } \rho_\varepsilon \text{ is a standard mollifier}$$

and a strict curl-condition for the admissible strains. These other possibilities are not equivalent but turn out to produce the same limit energy.

As we focus on the critical regime, we define the rescaled energy $F_\varepsilon : \mathcal{M}(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^{2\times 2}) \to [0,\infty]$ as

$$F_\varepsilon(\mu,\beta) = \begin{cases} \frac{1}{|\log\varepsilon|^2}\left(\int_{\Omega_\varepsilon(\mu)} \frac{1}{2}\mathcal{C}\beta : \beta\,dx + |\mu|(\Omega)\right) & \text{if } \mu \in X(\Omega) \text{ and } \beta \in \mathcal{AS}^{lin}_\varepsilon(\mu), \\ +\infty & \text{else,} \end{cases}$$

for an elasticity tensor $\mathcal{C} \in \mathbb{R}^{2\times2\times2\times2}$ which acts only on the symmetric part of a matrix and satisfies

$$l|F_{sym}|^2 \le \mathcal{C}F : F \le L|F_{sym}|^2 \text{ for all } F \in \mathbb{R}^{2\times 2} \tag{5.1}$$

for some constants $l, L > 0$.
Hence, the energy consists of a linearized elastic part and an energy associated to the core of each dislocation. The core penalization is expected not to contribute in the limit as the dislocation densities are expected to be of order $|\log\varepsilon|$. In [68] it is shown that in a discrete setting the energy of screw dislocations inside the core is indeed of order 1. The same penalization was also used in [30] in the subcritical regime. On a technical level, the main reason for this penalization is to avoid that the whole domain is covered with cores of dislocations i.e., $\Omega_\varepsilon(\mu_\varepsilon) = \emptyset$.
Finally, we introduce notation for local versions of $X(\Omega), \mathcal{AS}^{lin}_\varepsilon$, and the energy F_ε. Let $U \subset \Omega$ be measurable. In the following, we write $X(U)$ for the admissible dislocation densities on U (simply replace Ω in the definition by U). For $\mu \in X(U)$, we denote by $\mathcal{AS}^{lin}_\varepsilon(\mu, U)$ the strains generating μ in U (again replace Ω by U in the definition of $\mathcal{AS}^{lin}_\varepsilon$). Finally, we write $F_\varepsilon(\cdot,\cdot,U)$ for the functional defined analogously to F_ε where Ω is replaced by U.

5.2 The Main Results

We define the limit energy $F : \mathcal{M}(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^{2\times2}) \to [0,\infty]$ as

$$F(\mu,\beta) = \begin{cases} \int_\Omega \frac{1}{2}\mathcal{C}\beta : \beta\,dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right)\,d|\mu| & \text{if } \mu \in \mathcal{M}(\Omega;\mathbb{R}^2)\cap H^{-1}(\Omega;\mathbb{R}^2), \\ & \beta \in L^2(\Omega;\mathbb{R}^{2\times2}), \text{ and } \operatorname{curl}\beta = \mu, \\ +\infty & \text{else .} \end{cases}$$

Here, φ is the relaxed self-energy density as defined in (4.7) for the constant rotation $R = Id$. This is the same limit as also obtained in [38] for a linearized model *with* the assumption of well-separateness of dislocations. The only difference to the limit of the nonlinear case (see Section 4.3) is that we do not have to keep track of constant rotations.
Before we define the topology which we use to prove the Γ-convergence of F_ε to F, we introduce the

flat norm. Given a measure $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2)$, we define the flat norm by

$$\|\mu\|_{flat} = \sup_{\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^2); Lip(\varphi)\leq 1} \int_\Omega \varphi \, d\mu.$$

Note that by the Arzelà-Ascoli theorem the embedding $W_0^{1,\infty}(\Omega;\mathbb{R}^2) \hookrightarrow C_0^0(\Omega;\mathbb{R}^2)$ is compact. This implies in particular that for a sequence of measures $\mu_k \in \mathcal{M}(\Omega;\mathbb{R}^2)$ converging to $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2)$ in the sense of weak$*$-convergence of measures, it follows the convergence with respect to the flat norm. Now, we define the convergence of dislocation densities and strains that we use in the Γ-convergence result.

Definition 5.2.1. Let $\varepsilon_k \to 0$. We say that a sequence $(\mu_k, \beta_k)_k \subset \mathcal{M}(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^{2\times 2})$ converges to $(\mu,\beta) \in \mathcal{M}(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^{2\times 2})$ if

$$\frac{\mu_k}{|\log \varepsilon_k|} \to \mu \text{ in the flat topology and } \frac{\beta_k}{|\log \varepsilon_k|} \rightharpoonup \beta \text{ in } L^2(\Omega;\mathbb{R}^{2\times 2}).$$

Remark 5.2.1. In view of the convergence, in the critical regime in the mixed growth situation (Definition 4.3.1) one could expect to define the convergence in this context such that $\frac{\beta_k - W_k}{|\log \varepsilon_k|}$ converges weakly to β for a sequence of skew-symmetric matrices W_k. The compactness result will involve a statement of this type, see Theorem 5.2.2. However, it is not possible to derive exactly the weak convergence on all of Ω but only local versions of it. As a $\liminf$-inequality is still valid for the convergence of the compactness result, this convergence could be seen as the most natural one for the problem. For the sake of a simpler notation, we stick to the convergence defined as above, in particular because the additive appearance of the skew-symmetric matrices leaves no footprint in the limit (this is different for the multiplication of rotations in the nonlinear case).

The Γ-convergence result with respect to the convergence defined above is the following.

Theorem 5.2.1. *Let $\varepsilon_k \to 0$. With respect to the convergence in Definition 5.2.1 it holds*

$$F_{\varepsilon_k} \xrightarrow{\Gamma} F.$$

The proof will be given in the Sections 5.6 and 5.7.
Moreover, we prove the following compactness statement in Section 5.5.

Theorem 5.2.2. *Let $\Omega \subset \mathbb{R}^2$ a bounded, simply connected Lipschitz domain. Let $\varepsilon_k \to 0$ and consider a sequence $(\mu_k, \beta_k) \in X(\Omega) \times \mathcal{AS}_\varepsilon^{lin}(\mu_k)$ such that $\sup_k F_{\varepsilon_k}(\mu_k,\beta_k) < \infty$. Then there exist a function $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$, a vector-valued Radon measure $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2) \cap H^{-1}(\Omega;\mathbb{R}^2)$, and a sequence of skew-symmetric matrices $W_k \in Skew(2)$ such that for a (not relabeled) subsequence it holds*

(i) $\frac{\mu_k}{|\log \varepsilon_k|} \to \mu$ *in the flat topology,*

(ii) for all $1 > \gamma > 0$ *and* $U \subset\subset \Omega$ *we have* $\frac{\beta_k - W_k}{|\log \varepsilon_k|} \mathbf{1}_{\Omega_{\varepsilon^\gamma}(\mu_k)} \rightharpoonup \beta$ *in* $L^2(U;\mathbb{R}^{2\times 2})$,

(iii) $\frac{(\beta_k)_{sym}}{|\log \varepsilon_k|} \rightharpoonup (\beta)_{sym}$ *in* $L^2(\Omega;\mathbb{R}^{2\times 2})$,

(iv) $\operatorname{curl} \beta = \mu$.

Finally, the obtained convergence is enough to prove the $\liminf$*-inequality*

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k,\beta_k,\Omega) \geq \int_\Omega \mathcal{C}\beta : \beta \, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

Remark 5.2.2. Notice that we need the localized weak convergence only if we want to control the full strains β_k. The symmetric parts $(\beta_k)_{sym}$ converge weakly on the full domain.

5.3 Ball Construction Technique Revisited

In order to prove compactness or a lim inf-inequality, we need to prove bounds for (modified versions of) the dislocation densities μ_ε in terms of the energy F_ε. The only information one can use is the circulation condition of a corresponding strain $\beta_\varepsilon \in \mathcal{AS}_\varepsilon^{lin}(\mu_\varepsilon)$. On a technical level, this circulation condition shares structural properties with the approximation of vortices in the Ginzburg-Landau model. A prominent role in proving lower bounds for the Ginzburg-Landau energy play ball constructions, see for example [51, 70]. The main ingredient for proving lower bounds, by the use of a ball construction, is a bound of the energy on annuli. These estimates are based on the fact that a non-zero circulation around an annulus induces a certain minimal amount of energy. As we deal with linearized elasticity, we control only the symmetric part of the strains. The use of Korn's inequality allows us to get a lower bound of the energy in terms of the circulation of the strain, see Proposition 5.3.2. As Korn's constant blows up for thin annuli, we need to avoid carefully annuli whose radii go below a certain ratio, see the proofs of Proposition 5.4.1 and Proposition 5.4.2. In the dilute regime this can be done by a combinatorial argument, see Section 5.3.1 for heuristics and [30] for the full treatment of the dilute regime.

Now, let us define what we mean by a *ball construction*. For a visualization of the construction, see Figure 5.1.
Ball construction:

Fix $c > 1$. Given a finite family of closed balls $(B_i)_{i\in I}$ with radii R_i, let us perform the following construction.

Preparation: Find a family of disjoint closed balls $(B_i(0))_{i\in I(0)}$ such that for each $i \in I$ there exists $j \in I(0)$ with the following properties: $B_i \subset B_j(0)$ and $\operatorname{diam}(B_j(0)) \leq \sum_{i:B_i\subset B_j(0)} \operatorname{diam}(B_i)$.
It is not difficult to see that this is always possible.

Expansion: Define for $t > 0$ and $i \in I(0)$ the radii $R_i(t) = c^t R_i(0)$ and consider the family of closed balls $(B_i(t))_{i\in I(0)}$ where $B_i(t)$ is the ball with the same center as $B_i(0)$ and radius $R_i(t)$. Moreover, let $I(t) = I(0)$. We perform this expansion as long as the balls $(B_i(t))_{i\in I(t)}$ are pairwise disjoint. For the first $t > 0$ such that the family $(B_i(t))_{i\in I(t)}$ is not disjoint anymore, perform the merging below.

Merging: If the family $(B_i(t))_{i\in I(t)}$ is not disjoint, find similarly to the preparation step a disjoint family of balls $(B_j(t))_{j\in J}$ such that for each $i \in I(t)$ there exists an index $j \in J$ which fulfills $B_i(t) \subset B_j$ and $\operatorname{diam}(B_j) \leq 2\sum_{i:B_i(t)\subset B_j} R_i(t)$. For notational simplicity, let us assume that the index $i \in I(t)$ of a ball $B_i(t)$ that is not affected during the described procedure remains the same i.e., it holds $i \in J$ and B_i is the same ball as $B_i(t)$.
Then, replace $I(t)$ by J, $(B_i(t))_{i\in I(t)}$ by $(B_j)_{j\in J}$ and the radii $R_i(t)$ by the corresponding $\frac{1}{2}\operatorname{diam}(B_j)$. The time t is called a merging time. After the merging, we continue with the expansion below.

Expansion II: Let $\tau > 0$ be a merging time. For $t > \tau$, we define the new radii $R_i(t) = c^{t-\tau} R_i(\tau)$ and

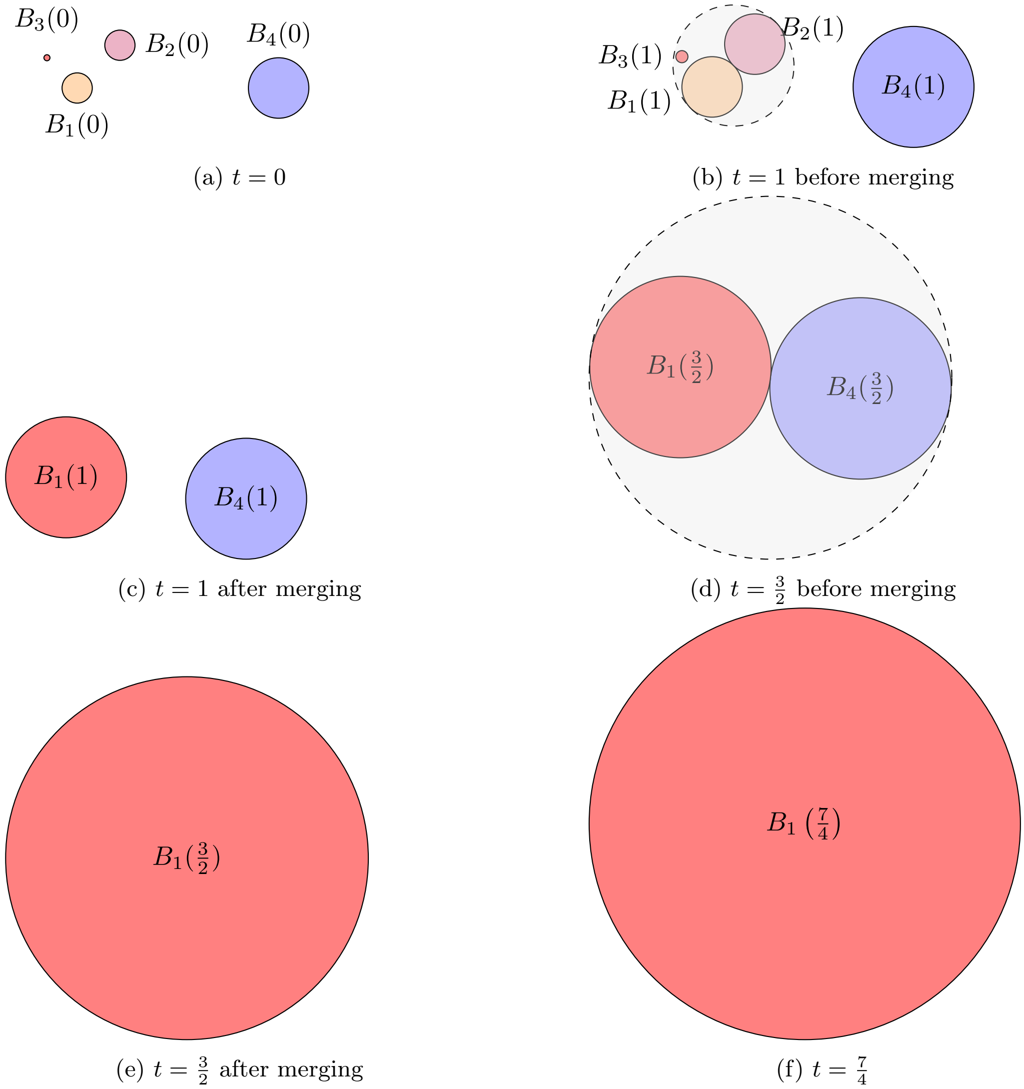


Figure 5.1: Sketch of ball construction for four balls with $c = 2$.

$I(t) = I(\tau)$. Moreover, for $i \in I(t)$ set $B_i(t)$ to be the ball with the same center as $B_i(\tau)$ and radius $R_i(t)$. Perform this expansion as long as the family $(B_i(t))_{i \in I(t)}$ is disjoint. At the first $t > \tau$ such that this is not the case anymore, perform a merging as described above.

We refer to the family $(I(t), (B_i(t))_{i\in I(t)}, (R_i(t))_{i\in I(t)})_t$ constructed as above as as the *ball construction* starting with $(B_i)_{i\in I}$ and associated to c.
By the *discrete ball construction* starting with $(B_i)_{i\in I}$ and associated to c we mean the family $(I(n), (B_i(n))_{i\in I(n)}, (R_i(n))_{i\in I(n)})_{n\in\mathbb{N}}$.

Moreover, we introduce the following notation to link a ball in the construction at time t with its past and future in the construction: For $s_2 > t > s_1 > 0$ and $i \in I(t)$ let us define

$$P_i^t(s_1) = \{B_j(s_1) : j \in I(s_1) \text{ and } B_j(s_1) \subset B_i(t)\} \text{ and} \tag{5.2}$$

$$F_i^t(s_2) = B_j(t_s) \text{ the unique ball at time } s_2 \text{ such that } B_i(t) \subset B_j(s_2). \tag{5.3}$$

Next, let us make the following observations.

Lemma 5.3.1. *Let $(B_i)_{i\in I}$ be a finite family of balls with radii R_i and $c>1$. For the corresponding (discrete) ball construction it holds that:*

i) $R_i(t)\le c^t\sum_{j:B_j\subset B_i(t)} R_j$ *for all* $i\in I(t)$,

ii) The construction is monotone in the following sense. Let $t>s\ge 0$. Then for every $i\in I(s)$ there exists $j\in I(t)$ such that $B_i(s)\subset B_j(t)$. In particular, it holds that $R_i(s)\le R_j(t)$ and $\bigcup_{i\in I} B_i\subset\bigcup_{i\in I(s)} B_i(s)\subset\bigcup_{i\in I(t)} B_i(t)$.

Proof. Property i) is true for $t=0$. It is easily seen that the the expansion and merging steps preserve this property for growing t.
Property ii) is also immediate from the construction. □

The main idea of the ball construction is to obtain lower bounds during the expansion phase on the growing annuli. Hence, estimates on annuli play the role of building blocks for the bounds obtained through the ball construction. The main difference to the classical ball construction estimates in Ginzburg-Landau theory is that in the theory of linearized elastic energy there is a significant loss of rigidity on thin annuli which is expressed mathematically by the appearance of Korn's constant in the estimate. The following estimate was already proven in [30]. We state and prove it here for convenience of the reader.

Lemma 5.3.2. *Let $R>r>0$ and $\beta\in L^2(B_R(0)\setminus B_r(0);\mathbb{R}^{2\times 2})$ such that $\operatorname{curl}\beta=0$ in $B_R(0)\setminus B_r(0)$. Then it holds that*

(i) $\int_{B_R(0)\setminus B_r(0)}\mathcal{C}\beta:\beta\,dx\ge\frac{1}{K(\frac{R}{r})\,2\pi}|\xi|^2\log\left(\frac{R}{r}\right)$,

(ii) $\int_{B_R(0)\setminus B_r(0)}|\beta|^2\,dx\ge\frac{1}{2\pi}|\xi|^2\log\left(\frac{R}{r}\right)$.

Here, $\xi=\int_{\partial B_r(0)}\beta\cdot\tau\,d\mathcal{H}^1$ where τ denotes the unit tangent to $\partial B_r(0)$ and $K(\frac{R}{r})$ is Korn's constant for the annulus $B_R(0)\setminus B_r(0)$.

Proof. Let $\beta\in L^2(B_R(0)\setminus B_r(0);\mathbb{R}^{2\times 2})$. By a density argument, we may assume that it holds $\beta\in C^0(\overline{B_R(0)\setminus B_r(0)};\mathbb{R}^{2\times 2})$. Korn's inequality provides a skew-symmetric matrix $W\in\mathbb{R}^{2\times 2}$ such that

$$\int_{B_R(0)\setminus B_r(0)}|\beta-W|^2\,dx\le K\left(\frac{R}{r}\right)\int_{B_R(0)\setminus B_r(0)}\mathcal{C}\beta:\beta\,dx.$$

Using a change of variables one can further estimate

$$\int_{B_R(0)\setminus B_r(0)}|\beta-W|^2\,dx=\int_r^R\int_{\partial B_t(0)}|\beta-W|^2\,d\mathcal{H}^1\,dt\ge\int_r^R\int_{\partial B_t(0)}|(\beta-W)\cdot\tau|^2\,d\mathcal{H}^1\,dt.$$

Here, τ denotes the tangent to the corresponding $\partial B_t(0)$.
Jensen's inequality yields

$$\int_r^R\int_{\partial B_t(0)}|(\beta-W)\cdot\tau|^2\,d\mathcal{H}^1\,dt\ge\int_r^R\frac{1}{2\pi t}\left|\int_{\partial B_t(0)}(\beta-W)\cdot\tau\,d\mathcal{H}^1\right|^2dt=\frac{1}{2\pi}\log\left(\frac{R}{r}\right)|\xi|^2.$$

Combining the estimates, we find (i). The last two estimates for $W=0$ show (ii). □

5.3.1 Heuristics and Difficulties of Using Ball Constructions in Dislocation Models

In this section, we explain which difficulties appear in the application of the ball construction technique in the critical regime. In particular, we compare the situation to the subcritical regime in [30]. To show compactness for the dislocations densities, one typically wants to derive a bound of the form

$$\frac{|\tilde{\mu}_\varepsilon|(\Omega)}{|\log\varepsilon|} \leq CF_\varepsilon(\mu_\varepsilon, \beta_\varepsilon), \tag{5.4}$$

where $\tilde{\mu}_\varepsilon$ is a modified version of the dislocation density μ_ε which converges to μ_ε in the flat norm. The combination leads to precompactness with respect to the flat norm. The modified version $\tilde{\mu}_\varepsilon$ is typically constructed by finding the correct clusters of dislocations and weighting the cluster by the accumulated Burgers vector of the dislocations. This fits very well to the structure of ball constructions. Starting with the cores of the dislocations, the merging steps provide a natural way to subdivide the dislocations into groups.

Finding lower bounds with the use of a ball construction could work as follows, cf. [51]. Let us fix $c > 1$ and consider the ball construction associated to c starting with the cores of a system of dislocations. Between two consecutive merging times $\tau_1 < \tau_2$, one can estimate the energy on all expanding balls through a statement that estimates the minimal energy on an annulus, in our case Lemma 5.3.2. Let $\tilde{B}_i(\tau_2)$ be the version of $B_i(\tau_1)$ which expanded by the factor $c^{\tau_2-\tau_1}$. Then it holds

$$\int_{\tilde{B}_i(\tau_2)\setminus B_i(\tau_1))} \frac{1}{2}\mathcal{C}\beta_\varepsilon : \beta_\varepsilon \, dx \geq K(c^{\tau_2-\tau_1})(\tau_2-\tau_1)\log(c)|\mu_\varepsilon(B_i(\tau_1))|^2.$$

Here, $K(c^{\tau_2-\tau_1})$ is Korn's constant for the annulus with ratio $c^{\tau_2-\tau_1}$.

These energy estimates can then be combined using subadditivity. Let $(B_i(\tau_1))_i$ balls whose expanded versions $(\tilde{B}_i(\tau_2))_i$ at time τ_2 merge to the ball $B_j(\tau_2)$. Then, we can relate the energy which we found in the expansion phase to the newly emerged ball at time τ_2 by

$$\begin{aligned} |\log\varepsilon|^2 F_\varepsilon\left(\mu_\varepsilon, \beta_\varepsilon, \bigcup_i \left(\tilde{B}_i(\tau_2)\setminus B_i(\tau_1)\right)\right) &\geq K(c^{\tau_2-\tau_1}) \sum_i |\mu_\varepsilon(B_i(\tau_1))|^2 \,(\tau_2-\tau_1)\log c \\ &\geq K(c^{\tau_2-\tau_1}) \sum_i k\,|\mu_\varepsilon(B_i(\tau_1))|\,(\tau_2-\tau_1)\log c \\ &\geq kK(c^{\tau_2-\tau_1}) \left|\mu_\varepsilon\left(\bigcup_i B_i(\tau_1)\right)\right| (\tau_2-\tau_1)\log c \\ &= kK(c^{\tau_2-\tau_1})\,|\mu_\varepsilon(B_j(\tau_2))|\,(\tau_2-\tau_1)\log c. \end{aligned}$$

Here, we used that μ_ε is a measure with values in $\mathbb{S}$ whose non-zero elements are bounded away from zero. If we ignore for a moment Korn's constant, we can sum over all merging times up to a time t (which is chosen such that all constructed balls up to time t lie in Ω) and obtain a bound of the form

$$\sum_{i\in I(t)} |\mu_\varepsilon(B_i(t))|\, t\log c \leq CF_\varepsilon(\mu_\varepsilon, \beta_\varepsilon)|\log\varepsilon|^2.$$

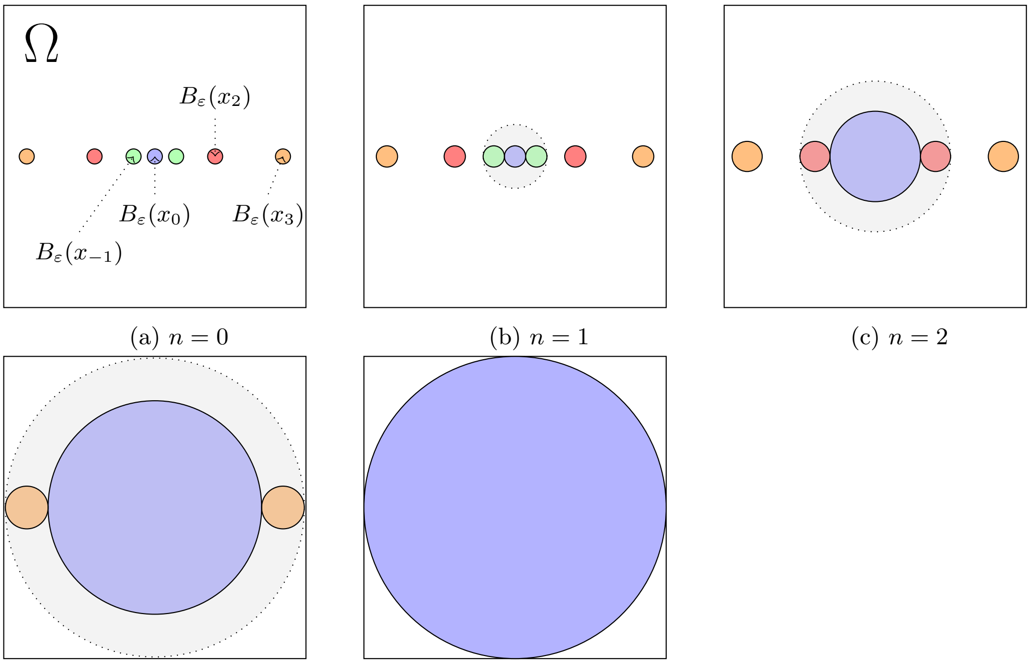
The modified dislocation density would then be defined as $\tilde{\mu}_\varepsilon = \sum_{i\in I(t)} \mu_\varepsilon(B_i(t))\,\delta_{x_i}$ where the x_i are the centers of the balls $B_i(t)$. If we could choose a time $t \sim \frac{|\log\varepsilon|}{2\log c}$, this would provide the right bound for $\tilde{\mu}_\varepsilon$. As we start our ball construction with at most $C|\log\varepsilon|^2$- many balls of radius ε, by Lemma 5.3.1

this choice of t would correspond to balls with a maximal radius $\lesssim |\log\varepsilon|^2\varepsilon^{\frac{1}{2}}$ in the construction at time t. In particular, this maximal radius tends to zero as $\varepsilon \to 0$. Moreover, this bound corresponds to the idea that the energy in a ball of radius ε^γ around a dislocation is of order $(1-\gamma)|\log\varepsilon|$. Therefore, up to some technicalities close to the boundary of Ω (a $|\log\varepsilon|^2\varepsilon^{\frac{1}{2}}$-neighborhood of $\partial\Omega$ in this case), we could find a nice lower bound for a modified version of the dislocation density μ_ε. The part of the dislocation density which is supported in a shrinking neighborhood of the boundary can be shown to vanish in the flat norm in the limit.
As Korn's constant blows up for thin annuli, see section 5.A, Lemma 5.3.2 does not provide a uniform lower bound if there is no lower bound on the distance of two consecutive merging times. The idea presented in [30] to overcome this difficulty is to consider the discrete ball construction. A discrete time step is called expansion step if no ball in the construction merges within this step. Therefore, in an expansion step, it is guaranteed that all balls grow exactly by the factor c. If one uses again the lower bound from Lemma 5.3.2, one obtains at time $n \in \mathbb{N}$ the lower bound

$$K(c) \sum_{i\in I(n)} |\mu_\varepsilon(B_i(n))|\,\#\{\text{expansion steps up to time } n\}\log c \leq F_\varepsilon(\mu_\varepsilon,\beta_\varepsilon)|\log\varepsilon|^2.$$

In order to make this estimate meaningful, it is crucial to find a lower bound for the number of expansion steps. Let us make the following observations. As we want to estimate the energy in balls around the dislocations of a radius of say $\varepsilon^{\frac{1}{2}}$, there will be approximately $\frac{|\log\varepsilon|}{2\log c}$ steps until a construction starting with balls of radius ε and expansion factor c leaves this neighborhood. In the subcritical regime (the stored energy is rescaled by the factor $|\log\varepsilon|$), the number of dislocations for a sequence of bounded energy is uniformly bounded by $C|\log\varepsilon|$. Hence, by choosing c small but uniformly in ε one can guarantee that along this sequence of dislocation densities one can find at least $\frac{|\log\varepsilon|}{4\log c}$ expansion steps in the ball construction. The appearing factor $K(c)$ in the corresponding estimate may be small but it is uniform in ε. This provides the desired lower bound.
In the critical regime, we are confronted with up to $|\log\varepsilon|^2$ dislocations. This changes the situation drastically. In particular, the considerations for the dilute regime above do not longer apply. As the following example shows, in general it is not longer possible to find a fixed expansion factor c which provides uniformly in ε enough expansion steps during an associated discrete ball construction starting with the dislocation cores.

Example. For simplicity, let $\Omega = (-1,1)^2$. Let $\varepsilon > 0$ and $c > 1$.
Consider the points $x_i = (2i\varepsilon\, c^i, 0)$ for $i = -L_\varepsilon^c, \ldots, L_\varepsilon^c$ where L_ε^c is the largest natural number such that for $n = L_\varepsilon^c - 1$ it still holds $n\log c + \log(2n+1) < |\log\varepsilon|$.
In the following, we will see that there is no expansion step in the discrete ball construction associated to c starting with the closed balls $(\overline{B_\varepsilon(x_i)})_{i=-L_\varepsilon^c,\ldots,L_\varepsilon^c}$ until one of the balls in the construction meets $\partial\Omega$. For a visualization, see Figure 5.2. This behavior is not linked to effects coming from dislocations located very close to the boundary. In fact, we see that essentially the ball located at 0 causes the problem.
At time $n = 1$, the balls with centers in x_{-1}, x_0 and x_1 and radii $c\varepsilon$ merge to a ball with radius $3c\,\varepsilon$ and center $(0,0)$, see Figure 5.2b. Hence, the first discrete step is not an expansion step. At time $n = 2$, the expanded version of the new ball with center in $(0,0)$ and radius $3c^2\,\varepsilon$ merges with the balls with radii $c^2\varepsilon$ and centers x_{-2}, x_2 to a ball with center $(0,0)$ and radius $5c^2\varepsilon$. Hence, also the second discrete step is not an expansion step, see Figure 5.2c. By induction one can show that at time $n \leq L_\varepsilon^c$ the ball with center $(0,0)$ and radius $(2n-1)c^n\varepsilon$ merges with the balls with centers



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

(d) $n = 3$ (e) The ball meets $\partial\Omega$ before the next discrete step could be executed.

Figure 5.2: A sketch describing the specific ball construction in the example.

x_{-n}, x_n and radii $c^n\varepsilon$ to a ball with center $(0,0)$ and radius $(2n+1)c^n\,\varepsilon$ which shows that none of these steps can be an expansion step. Moreover, by the defining property of L^c_ε none of the balls in the construction up to time $L^c_\varepsilon - 1$ intersects the boundary of Ω. On the other hand, in the discrete step from $L^c_\varepsilon - 1$ to L^c_ε the ball centered in $(0,0)$ merges with the two balls centered in $x_{-L^c_\varepsilon}$ and $x_{L^c_\varepsilon}$. The new ball is centered in $(0,0)$ and has the radius $(2L^c_\varepsilon + 1)c^{L^c_\varepsilon}\varepsilon \geq 1$. In particular, the intersection with $\partial\Omega$ is not empty. Therefore, no step before a ball in the construction intersects the boundary of Ω is an expansion step.
This construction starts with $2L^c_\varepsilon + 1$-many balls. Notice that by definition $L^c_\varepsilon - 1 \leq \frac{|\log\varepsilon|}{\log c}$. If we consider a dislocation density μ_ε which has a uniformly bounded Burgers vector in every x_i, we observe that $|\mu_\varepsilon|(\Omega) \leq C\frac{|\log\varepsilon|}{\log c}$.
In order to obtain a uniformly bounded energy F_ε, it is in principal allowed that $|\mu_\varepsilon|(\Omega) \leq C|\log\varepsilon|^2$. Hence, we are free to choose $c = c(\varepsilon)$ such that $\log c = |\log\varepsilon|^{-1}$ in order to obtain dislocation densities which satisfy this necessary bound. Note that the definition of $c(\varepsilon)$ implies that $c(\varepsilon) \searrow 1$.
Considering this sequence of dislocation measures μ_ε it is clear that one cannot find a universal $\tilde{c} > 1$ such that the corresponding discrete ball constructions starting with the cores of μ_ε has any expansion step for ε small enough (precisely for $\log\tilde{c} \geq |\log\varepsilon|^{-1}$).
This example discusses only a possible structure of the dislocation densities. The existence of corresponding strains such that the couples satisfy uniform energy estimates is not discussed here. However, the construction illustrates that controlling the number of dislocations by $|\log\varepsilon|^2$ is not enough to proceed as in the subcritical regime.

On the other hand, in this construction it can be seen that the outer balls (centered at x_i for $|i| \geq \frac{L^c_\varepsilon}{2}$) expand over a significant time span. Hence, they could be used to estimate at least a part of the energy. These combinatorics are essential for the treatment of the critical regime and are elaborated in the proofs of Proposition 5.4.1 and 5.4.2 in the next section.

5.4 The Main Ingredients for Lower Bounds

As discussed in the section above, the main difficulty in a regime with more than $|\log \varepsilon|$ dislocations is that in a ball construction argument one cannot avoid the combinatorics of distinguishing balls which expand for a certain minimal time and therefore induce a relevant energy to the system from those that merge so frequently that they do not allow to estimate their corresponding energy uniformly due to the blow-up of Korn's constants on thin annuli, see Section 5.A.
In the following proposition, we show how to reduce the general situation in the critical regime to a situation that is easier to analyze. Essentially, we prove that in a neighborhood of the dislocations of order ε^γ we can change the strain β_ε slightly. The total variation of the curl of the new strain $\bar{\beta}_\varepsilon$ is controlled in terms of $|\log \varepsilon|$ and the curl is concentrated in at most $C|\log \varepsilon|$ balls with a radius that is much smaller than ε^γ for some fixed $0 < \gamma < 1$.

Proposition 5.4.1. *Let $1 > \alpha > \gamma > 0$, $\delta \geq 0$, and $K > 0$. There exists $\varepsilon_0 = \varepsilon_0(\alpha, \gamma, K)$ such that for all $0 < \varepsilon < \varepsilon_0$ it holds the following:*
Let $A_\varepsilon \subset \mathbb{R}^2$. Let $\mu_\varepsilon \in X(A_\varepsilon)$ and $\beta_\varepsilon \in \mathcal{AS}_\varepsilon^{lin}(\mu_\varepsilon, A_\varepsilon)$ such that $\operatorname{dist}(\operatorname{supp}(\mu_\varepsilon), \partial A_\varepsilon) \geq \varepsilon^\gamma$ and $F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, A_\varepsilon) \leq K|\log \varepsilon|^{-\delta}$. Then there exist a family of disjoint closed balls $(D_i^\varepsilon)_{i \in I_\varepsilon}$ and a function $\bar{\beta}_\varepsilon : A_\varepsilon \to \mathbb{R}^{2\times 2}$ such that

(i) $\operatorname{diam} D_i^\varepsilon \leq \varepsilon^\alpha$ *and* $D_i^\varepsilon \cap \operatorname{supp}(\mu_\varepsilon) \neq \emptyset$ *for all* $i \in I_\varepsilon$,

(ii) $|I_\varepsilon| \leq C(\alpha, K)|\log \varepsilon|^{1-\delta}$,

(iii) $\bar{\beta}_\varepsilon = \beta_\varepsilon$ *on* $A_\varepsilon \setminus \bigcup_{x \in \operatorname{supp}(\mu_\varepsilon)} B_{\varepsilon^\alpha}(x)$,

(iv) $\operatorname{curl} \bar{\beta}_\varepsilon \in \mathcal{M}(A_\varepsilon; \mathbb{R}^2)$,

(v) $\operatorname{supp}(\operatorname{curl} \bar{\beta}_\varepsilon) \subset \bigcup_{i \in I_\varepsilon} D_i^\varepsilon$ *and* $(\operatorname{curl} \bar{\beta}_\varepsilon)(U) = \mu_\varepsilon(U)$ *for each connected component U of A_ε*,

(vi) $|\operatorname{curl} \bar{\beta}_\varepsilon|(A_\varepsilon)| \leq C(\alpha, K)|\log \varepsilon|^{1-\delta}$,

(vii) $\frac{1}{|\log \varepsilon|^2} \int_{A_\varepsilon} \frac{1}{2}\mathcal{C}\bar{\beta}_\varepsilon : \bar{\beta}_\varepsilon \, dx \leq F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, A_\varepsilon) + C(\alpha, K)\frac{F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, A_\varepsilon)}{|\log \varepsilon|}$.

Proof. Let $\sigma = \frac{1-\alpha}{3}$ and fix $c > 1$. Let $\varepsilon > 0$.
The prove is subdivided in three steps. It is based on a ball construction starting with the balls of radius ε around the dislocation points. First, we estimate the number of balls, whose μ_ε-measure is non-zero, at some time in the ball construction that corresponds to balls of the intermediate radius $\varepsilon^{\alpha+2\sigma}$. Secondly, at a later point in the construction we bound the number of balls whose accumulated Burgers vector is zero by deleting dipoles without creating too much energy nor changing the strains on a large set, see (vii) and (iii). Combining the estimates leads to (ii). In a third step, we modify the strains slightly in order to obtain a strain with a curl that is still related to μ_ε but whose total variation is bounded in terms of $|\log \varepsilon|$, see (v) and (vi).

Step 1. *Estimation of number of balls such that $\mu_\varepsilon(B) \neq 0$.*
Let $B_i^\varepsilon = B_\varepsilon(x_i)$ where $\operatorname{supp} \mu_\varepsilon = \{x_1, \dots, x_{N_\varepsilon}\}$. As the elements in $\mathbb{S}$ are bounded away from zero, we may deduce from the assumed energy bound that $N_\varepsilon \leq k|\mu_\varepsilon| \leq kK|\log \varepsilon|^{2-\delta}$.
Now, perform a continuous ball construction starting with the balls $(B_i^\varepsilon)_{i=1,\dots,N_\varepsilon}$ and denote its

output by $(I_\varepsilon(t),(B_i^\varepsilon(t))_t,(R_i^\varepsilon(t))_t)_t$. In this first step, we consider only times $t>0$ such that $\sum_{i\in I_\varepsilon(t)} R_i^\varepsilon(t)\le \varepsilon^{\alpha+2\sigma}$. Using Lemma 5.3.1, we can compute a lower bound on $t_1^\varepsilon>0$ which we define to be the first time t such that $\sum_{i\in I_\varepsilon(t)} R_i^\varepsilon(t)=\varepsilon^{\alpha+2\sigma}$:

$$\varepsilon^{\alpha+2\sigma}=\sum_{i\in I_\varepsilon(t_1^\varepsilon)} R_i^\varepsilon(t)\le c^{t_1^\varepsilon}\sum_{i\in I_\varepsilon(t_1^\varepsilon)}\varepsilon\,\#\{B_j^\varepsilon: B_j^\varepsilon\subset B_i(t_1^\varepsilon)\}\le kK\,c^{t_1^\varepsilon}\varepsilon|\log\varepsilon|^2.$$

From this estimate one derives directly

$$t_1^\varepsilon\ge\sigma\frac{|\log\varepsilon|}{\log c}-\frac{\log(kK)-2\log|\log\varepsilon|}{\log c}.$$

In particular, for $\varepsilon>0$ small enough (depending on K and σ) we obtain that $t_1^\varepsilon\ge\frac{\sigma}{2}\frac{|\log\varepsilon|}{\log c}+1$. Let us consider the balls $(B_i^\varepsilon(s_1^\varepsilon))_{i\in I(s_1^\varepsilon)}$ of the the ball construction at time $s_1^\varepsilon=\lceil\frac{\sigma}{2}\frac{|\log\varepsilon|}{\log c}\rceil$. Note that all balls in $(B_i^\varepsilon(s_1^\varepsilon))_{i\in I(s_1^\varepsilon)}$ have a radius which is smaller than $\varepsilon^{\alpha+2\sigma}$.
We subdivide the family of balls $(B_i^\varepsilon(s_1^\varepsilon))_{i\in I_\varepsilon(s_1^\varepsilon)}$ into the subset of balls that evolve from few mergings and those that originate from many mergings:

$$\mathcal{F}_\varepsilon(s_1^\varepsilon)=\left\{B_i^\varepsilon(s_1^\varepsilon):\#P_i^{s_1^\varepsilon}(0)\le\frac{s_1^\varepsilon}{2}\right\}\text{ and }\mathcal{M}_\varepsilon(s_1^\varepsilon)=\left\{B_i^\varepsilon(s_1^\varepsilon):\#P_i^{s_1^\varepsilon}(0)>\frac{s_1^\varepsilon}{2}\right\}.$$

Recall that by definition the set $P_i^{s_1^\varepsilon}(0)$ contains the balls at time zero which are included in the ball $B_i^\varepsilon(s_1^\varepsilon)$, see (5.2).
Let us first estimate the number of balls in $\mathcal{M}_\varepsilon(s_1^\varepsilon)$. By definition, every ball $B_i^\varepsilon(s_1^\varepsilon)\in\mathcal{M}_\varepsilon(s_1^\varepsilon)$ originates from at least $\frac{s_1^\varepsilon}{2}$ starting balls. Consequently,

$$\#\mathcal{M}_\varepsilon(s_1^\varepsilon)\le 2\frac{N_\varepsilon}{s_1^\varepsilon}\le 2kK\frac{|\log\varepsilon|^{2-\delta}}{\frac{\sigma}{2}\frac{|\log\varepsilon|}{\log c}}=\frac{4kK}{\sigma}\log(c)\,|\log\varepsilon|^{1-\delta}. \tag{5.5}$$

The next objective is to estimate the number of balls in $\mathcal{F}_\varepsilon(s_1^\varepsilon)$ which have an accumulated Burgers vector which is not zero.
Fix $B_i^\varepsilon(s_1^\varepsilon)\in\mathcal{F}_\varepsilon(s_1^\varepsilon)$. By definition of $\mathcal{F}_\varepsilon(s_1^\varepsilon)$, the ball $B_i^\varepsilon(s_1^\varepsilon)$ includes at most $\frac{s_1^\varepsilon}{2}$ starting balls. Hence, it evolves from at most $\frac{s_1^\varepsilon}{2}$ mergings. By the pigeonhole principle, there exist natural numbers $0\le n_1<\dots<n_{L_i}\le s_1^\varepsilon-1$ such that for all $k=1,\dots,L_i\ge\lfloor\frac{s_1^\varepsilon}{2}\rfloor$ every ball $B_j^\varepsilon(n_k)\in P_i^{s_1^\varepsilon}(n_k)$ purely expands in the time interval $(n_k,n_k+1]$. Recall that by the definition of the ball construction this means that $B_j^\varepsilon(n_k+1)$ has the same center as $B_j^\varepsilon(n_k)$ and the radius $R_j^\varepsilon(n_k+1)=cR_j^\varepsilon(n_k)$. Moreover, we know that $\operatorname{curl}\beta_\varepsilon=0$ in $B_j(n_k+1)\setminus B_j(n_k)$ (remember that $\operatorname{supp}\mu_\varepsilon\subset\bigcup_{i\in I_\varepsilon}B_i^\varepsilon\subset\bigcup_{i\in I_\varepsilon(t)}B_i^\varepsilon(t)$ for all $t>0$). Hence, we can apply Lemma 5.3.2 to $B_j^\varepsilon(n_k+1)\setminus B_j^\varepsilon(n_k)$ to obtain, by summing over all these disjoint annuli,

$$\int_{B_i^\varepsilon(s_1^\varepsilon)}\mathcal{C}\beta_\varepsilon:\beta_\varepsilon\,dx\ge\sum_{k=1}^{L_i}\sum_{B_j^\varepsilon(n_k)\in P_i^{s_1^\varepsilon}(n_k)}\int_{B_j^\varepsilon(n_k+1)\setminus B_j^\varepsilon(n_k)}\mathcal{C}\beta_\varepsilon:\beta_\varepsilon\,dx \tag{5.6}$$

$$\ge\frac{1}{2\pi K(c)}\log(c)\sum_{k=1}^{L_i}\sum_{B_j^\varepsilon(n_k)\in P_i^{s_1^\varepsilon}(n_k)}|\mu_\varepsilon(B_j^\varepsilon(n_k))|^2. \tag{5.7}$$

As $\mu_\varepsilon(B_j^\varepsilon(n_k)) \in \mathbb{S}$ and the non-zero elements in $\mathbb{S}$ are bounded away from zero, we can further estimate

$$
\begin{aligned}
(5.7) \geq & \frac{k}{2\pi K(c)} \log(c) \sum_{k=1}^{L_i} \sum_{B_j^\varepsilon(n_k) \in P_i^{s_1^\varepsilon}(n_k)} |\mu_\varepsilon(B_j^\varepsilon(n_k))| \\
\geq & \frac{k}{2\pi K(c)} \log(c) \sum_{k=1}^{L_i} |\mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon))| \\
\geq & \frac{k}{2\pi K(c)} \log(c)\, L_i\, |\mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon))| \\
\geq & \frac{k}{2\pi K(c)} \log(c) \left\lfloor \frac{s_1^\varepsilon}{2} \right\rfloor |\mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon))| \\
\geq & \frac{\sigma\, k}{16\pi K(c)} |\log \varepsilon|\, |\mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon))|. \qquad (5.8)
\end{aligned}
$$

For the second inequality, we used that $\mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon)) = \sum_{B_j^\varepsilon(n_k) \in P_i^{s_1^\varepsilon}(n_k)} \mu_\varepsilon(B_j^\varepsilon(n_k))$; for the last inequality we used that for ε small enough it holds that $\left\lfloor \frac{s_1^\varepsilon}{2} \right\rfloor \geq \frac{\sigma |\log \varepsilon|}{8 \log c}$.
By summing over all $B_i^\varepsilon(s_1^\varepsilon) \in \mathcal{F}_\varepsilon(s_1^\varepsilon)$, we deduce from the energy bound on $F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, A_\varepsilon)$ that

$$
\sum_{B_i^\varepsilon(s_1^\varepsilon) \in \mathcal{F}_\varepsilon(s_1^\varepsilon)} |\mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon))| \leq \frac{\sigma\, k}{16\pi K(c)} K |\log \varepsilon|^{1-\delta}.
$$

In particular, we obtain the bound (recall that the non-zero elements of $\mathbb{S}$ are bounded away from zero)

$$
\#\{B_i^\varepsilon(s_1^\varepsilon) \in \mathcal{F}_\varepsilon(s_1^\varepsilon) : \mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon)) \neq 0\} \leq C(\alpha, K, c) |\log \varepsilon|^{1-\delta}. \qquad (5.9)
$$

Combining the bounds (5.5) and (5.9) provides the estimate

$$
\#\{B_i^\varepsilon(s_1^\varepsilon) : i \in I_\varepsilon(s_1^\varepsilon) \text{ and } \mu_\varepsilon(B_i^\varepsilon(s_1^\varepsilon)) \neq 0\} \leq \tilde{C}(\alpha, K, c) |\log \varepsilon|^{1-\delta}.
$$

Step 2. *Reduction of number of balls such that $\mu_\varepsilon(B) = 0$.*
In this step, we reduce the number of balls such that $\mu_\varepsilon(B) = 0$ by further growing the balls from step 1 and replacing β_ε by local gradients on balls with μ_ε-mass 0.
Let $\tilde{I}_\varepsilon = I_\varepsilon(s_1^\varepsilon)$ and $\tilde{B}_i^\varepsilon = B_i^\varepsilon(s_1^\varepsilon)$ for all $i \in \tilde{I}_\varepsilon$ where $I_\varepsilon(s_1^\varepsilon)$ and $B_i^\varepsilon(s_1^\varepsilon)$ are from step 1. Consider a new ball construction associated to c starting with the balls $(\tilde{B}_i^\varepsilon)_{i \in \tilde{I}_\varepsilon}$. With a little abuse of notation we call the output of this ball construction again $(I_\varepsilon(t), (B_i^\varepsilon(t))_{i \in I_\varepsilon(t)}, (R_i^\varepsilon)_{i \in I_\varepsilon(t)})_t$.
As the starting balls have — by construction in step 1 — a radius less than $\varepsilon^{\alpha+2\sigma}$, we can argue as in step 1 to obtain that for $\varepsilon > 0$ small enough the inequality $\sum_{i \in I_\varepsilon(t)} R_i^\varepsilon(t) \leq \varepsilon^{\alpha+\sigma}$ holds true for all $t \leq \lceil \frac{\sigma}{2} \frac{|\log \varepsilon|}{\log c} \rceil =: s_2^\varepsilon$.
We define the following partition of the set $\{B_i^\varepsilon(s_2^\varepsilon) : i \in I_\varepsilon(s_2^\varepsilon)\}$ (see Figure 5.3):

$$
A_1^\varepsilon(s_2^\varepsilon) = \left\{ B_i^\varepsilon(s_2^\varepsilon) : \text{ there exists a ball } B_j^\varepsilon(0) \in P_i^{s_2^\varepsilon}(0) \text{ such that } \mu_\varepsilon(B_j^\varepsilon(0)) \neq 0 \right\},
$$

$$
A_2^\varepsilon(s_2^\varepsilon) = \left\{ B_i^\varepsilon(s_2^\varepsilon) : \text{for all } B_j^\varepsilon(0) \in P_i^{s_2^\varepsilon}(0) \text{ it holds } \mu_\varepsilon(B_j^\varepsilon(0)) = 0 \text{ and } \#P_i^{s_2^\varepsilon}(0) > \frac{s_2^\varepsilon}{2} \right\},
$$

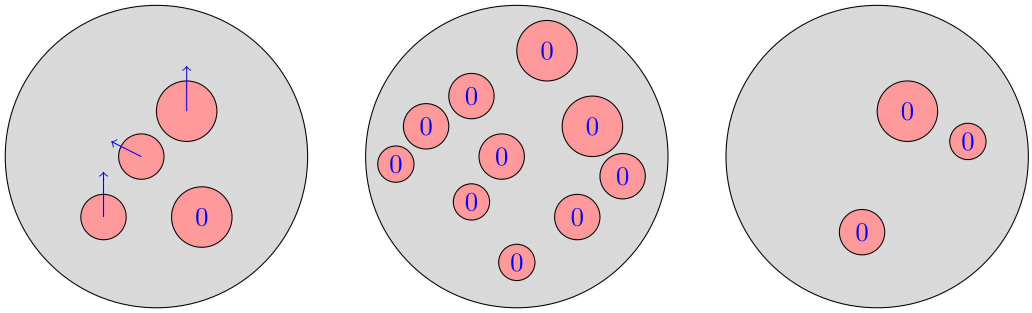


Figure 5.3: Prototypical examples of balls in the sets $A_1^\varepsilon(s_2^\varepsilon)$ (left), $A_2^\varepsilon(s_2^\varepsilon)$ (middle) and $A_3^\varepsilon(s_2^\varepsilon)$ (right). The balls $B_i^\varepsilon(s_2^\varepsilon)$ are drawn in gray, the corresponding balls in $P_i^{s_2^\varepsilon}(0)$ are drawn in red, and their (accumulated) Burgers vectors are drawn in blue, a blue 0 indicates that the μ_ε-mass of this ball is 0 .

$$A_3^\varepsilon(s_2^\varepsilon) = \left\{ B_i^\varepsilon(s_2^\varepsilon) : \text{for all } B_j^\varepsilon(0) \in P_i^{s_2^\varepsilon}(0) \text{ it holds } \mu_\varepsilon(B_j^\varepsilon(0)) = 0 \text{ and } \# P_i^{s_2^\varepsilon}(0) \leq \frac{s_2^\varepsilon}{2} \right\}.$$

Let us make clear that the sets $P_i^{s_2^\varepsilon}(0)$ are meant with respect to the ball construction introduced in the beginning of step 2.
Note that a ball can only be in $A_1^\varepsilon(s_2^\varepsilon)$ if it includes one of the balls from step 1 with non-zero μ_ε-mass. The number of these balls was controlled in step 1. As the balls in $A_1^\varepsilon(s_2^\varepsilon)$ are by definition of the ball construction disjoint, it follows $\# A_1^\varepsilon(s_2^\varepsilon) \leq C(\alpha, K, c)|\log \varepsilon|^{1-\delta}$. In addition, we can argue as in step 1 for the set $\mathcal{M}_\varepsilon(s_1^\varepsilon)$ to obtain that $\# A_2^\varepsilon(s_2^\varepsilon) \leq C(\alpha, K, c)|\log \varepsilon|^{1-\delta}$.
We cannot control the number of balls in $A_3^\varepsilon(s_2^\varepsilon)$. Instead, we will construct a new strain with only slightly more elastic energy and no singularities in elements of $A_3^\varepsilon(s_2^\varepsilon)$ by replacing β_ε by local gradients inside these balls. A similar construction has already been used in [30](also to delete dipoles) and [59](to extend strains into the cores).
Let us pick a ball $B_i^\varepsilon(s_2^\varepsilon) \in A_3^\varepsilon(s_2^\varepsilon), i \in I_\varepsilon(s_2^\varepsilon)$. By definition of the set $A_3^\varepsilon(s_2^\varepsilon)$, there exist natural numbers $0 \leq n_1 < \cdots < n_{L_i} \leq s_2^\varepsilon - 1$, where $L_i \geq \lfloor \frac{s_2^\varepsilon}{2} \rfloor$, such that for every $k = 1, \ldots, L_i$ every $B_j^\varepsilon(n_k) \in P_i^{s_2^\varepsilon}(n_k)$ does not merge in the time interval $(n_k, n_k+1]$. Note that the annuli $B_j^\varepsilon(n_k+1) \setminus B_j^\varepsilon(n_k)$, where $B_j^\varepsilon(n_k) \in P_i^{s_2^\varepsilon}(n_k)$, are pairwise disjoint and contained in $B_i^\varepsilon(s_2^\varepsilon) \setminus \bigcup_{B \in P_i^{s_2^\varepsilon}(0)} B$. Consequently, it holds

$$\sum_{k=1}^{L_i} \sum_{B_j^\varepsilon(n_k) \in P_i^{s_2^\varepsilon}(n_k)} \int_{B_j^\varepsilon(n_k+1)\setminus B_j^\varepsilon(n_k)} \mathcal{C}\beta_\varepsilon : \beta_\varepsilon \, dx \leq \int_{B_i(s_2^\varepsilon)\setminus \bigcup_{B\in P_i^{s_2^\varepsilon}(0)} B} \mathcal{C}\beta_\varepsilon : \beta_\varepsilon \, dx$$

By the mean value theorem, we may choose $k_i \in \mathbb{N}$ such that

$$\begin{aligned} \sum_{B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})} \int_{B_j^\varepsilon(n_{k_i}+1)\setminus B_j^\varepsilon(n_{k_i})} \mathcal{C}\beta_\varepsilon : \beta_\varepsilon \, dx &\leq \frac{1}{L_i} |\log \varepsilon|^2 \, F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, B_i^\varepsilon(s_2^\varepsilon)) \, dx \\ &\leq \frac{4}{s_2^\varepsilon} |\log \varepsilon|^2 \, F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, B_i^\varepsilon(s_2^\varepsilon)), \end{aligned}$$

where the last inequality holds for $\varepsilon > 0$ small enough.
Now, fix $B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})$. By construction ,we have $\operatorname{curl} \beta_\varepsilon = 0$ in $B_j^\varepsilon(n_{k_i}+1) \setminus B_j^\varepsilon(n_{k_i}) =: C_{i,j}^\varepsilon$. Moreover, notice that by definition of $A_3^\varepsilon(s_2)$ it holds that $\mu_\varepsilon(B_j^\varepsilon(n_{k_i})) = 0$ (as the ball $B_j^\varepsilon(n_{k_i})$ evolves

from balls with this property) and therefore

$$\int_{\partial B_j^\varepsilon(n_{k_i})} \beta_\varepsilon \cdot \tau \, d\mathcal{H}^1 = 0,$$

where τ denotes the unit tangent to $\partial B_j^\varepsilon(n_{k_i})$. By standard theory there exists $u_{i,j}^\varepsilon \in H^1(C_{i,j}^\varepsilon;\mathbb{R}^2)$ such that $\beta_\varepsilon = \nabla u_{i,j}^\varepsilon$ on $C_{i,j}^\varepsilon$. Korn's inequality for the annulus applied to $C_{i,j}^\varepsilon$ guarantees the existence of a skew-symmetric matrix $W_{i,j}^\varepsilon \in Skew(2)$ such that

$$\int_{C_{i,j}^\varepsilon} |\nabla u_{i,j}^\varepsilon - W_{i,j}^\varepsilon|^2 \leq K(c) \int_{C_{i,j}^\varepsilon} |(\nabla u_{i,j}^\varepsilon)_{sym}|^2 \, dx.$$

Note that Korn's constant on the right hand side depends only on the ratio of the radii of the annuli $C_{i,j}^\varepsilon$ which equals c by construction. In particular, this constant is independent from ε.
By standard extension results for Sobolev functions there exists a function $v_{i,j}^\varepsilon \in H^1(B_j^\varepsilon(n_{k_i}+1);\mathbb{R}^2)$ such that $\nabla v_{i,j}^\varepsilon = \nabla u_{i,j}^\varepsilon - W_{i,j}^\varepsilon$ on $C_{i,j}^\varepsilon$ and

$$\int_{B_j^\varepsilon(n_{k_i}+1)} |\nabla v_{i,j}^\varepsilon|^2 \, dx \leq C(c) \int_{C_{i,j}^\varepsilon} |\nabla u_{i,j}^\varepsilon - W_{i,j}^\varepsilon|^2 \, dx.$$

Note that by scaling the constant for the extension depends again only on the ratio of the annulus $C_{i,j}^\varepsilon$.
Now, we can estimate the elastic energy of $\nabla v_{i,j}^\varepsilon$ on $B_i^\varepsilon(s_2^\varepsilon)$ by combining the previous two estimates and summing over all balls in $P_i^{s_2^\varepsilon}(n_{k_i})$:

$$\begin{aligned}
\sum_{B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})} \int_{B_j^\varepsilon(n_{k_i})} \mathcal{C}\nabla v_{i,j}^\varepsilon : \nabla v_{i,j}^\varepsilon \, dx &\leq C \sum_{B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})} \int_{B_j^\varepsilon(n_{k_i})} |\nabla v_{i,j}^\varepsilon i|^2 \, dx \qquad (5.10)\\
&\leq C(c) \sum_{B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})} \int_{C_{i,j}^\varepsilon} |\nabla u_{i,j}^\varepsilon - W_{i,j}^\varepsilon|^2 \, dx\\
&\leq C(c) \sum_{B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})} \int_{C_{i,j}^\varepsilon} |(\nabla u_{i,j}^\varepsilon)_{sym}|^2 \, dx\\
&\leq C(c) \sum_{B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i})} \int_{C_{i,j}^\varepsilon} \mathcal{C}\beta_\varepsilon : \beta_\varepsilon \, dx\\
&\leq C(c) \frac{4}{s_2^\varepsilon} |\log\varepsilon|^2 F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, B_i^\varepsilon(s_2^\varepsilon))\\
&\leq C(c) \frac{8}{\sigma} \frac{\log(c)\, F_\varepsilon(\mu_\varepsilon, \beta_\varepsilon, B_i^\varepsilon(s_2^\varepsilon))}{|\log\varepsilon|} |\log\varepsilon|^2, \qquad (5.11)
\end{aligned}$$

where the constant $C(c)$ may change from line to line but depends only on c and global parameters (such as the coercivity constant for $\mathcal{C}$ on symmetric matrices).
Let us define the function $\tilde{\beta}_\varepsilon : A_\varepsilon \to \mathbb{R}^{2\times 2}$ by

$$\tilde{\beta}_\varepsilon(x) = \begin{cases} \nabla v_{i,j}^\varepsilon(x) + W_{i,j}^\varepsilon & \text{if } x \in B_j^\varepsilon(n_{k_i}) \in P_i^{s_2^\varepsilon}(n_{k_i}) \text{ for } B_i^\varepsilon(s_2^\varepsilon) \in A_3^\varepsilon(s_2^\varepsilon),\\ \beta_\varepsilon(x) & \text{else.} \end{cases}$$

Note that on the annuli $C_{i,j}^\varepsilon$ it holds $\nabla v_{i,j}^\varepsilon + W_{i,j}^\varepsilon = \beta_\varepsilon$. Hence, $\tilde{\beta}$ does not create any extra curl on

$\partial B_j^\varepsilon(n_{k_i})$. Therefore, $\operatorname{curl}\tilde\beta_\varepsilon = 0$ on $A_\varepsilon \setminus \bigcup_{B\in A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)} B$ where $A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)$ consists of disjoint balls with a radius less than $\varepsilon^{\alpha+\sigma}$ and $\#(A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)) \le C(\alpha,K,c)|\log\varepsilon|^{1-\delta}$. In particular, the strain $\tilde\beta_\varepsilon$ satisfies for every open $A\subset A_\varepsilon\setminus\bigcup_{B\in A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)} B$ with smooth boundary the circulation condition

$$\int_{\partial A}\tilde\beta_\varepsilon\cdot\tau\,d\mathcal{H}^1 = \tilde\mu_\varepsilon(A),$$

where τ denotes the unit tangent to ∂A and $\tilde\mu_\varepsilon = (\mu_\varepsilon)_{|\bigcup_{B\in A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)} B}$. Note that $\tilde\mu_\varepsilon(U) = \mu_\varepsilon(U)$ for any connected component U of A_ε as we only deleted connected dipoles.
Moreover, in view of (5.10) - (5.11) it holds

$$\frac{1}{|\log\varepsilon|^2}\int_{A_\varepsilon\setminus\left(\bigcup_{B\in A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)} B\right)}\mathcal{C}\tilde\beta_\varepsilon:\tilde\beta_\varepsilon\,dx \le F_\varepsilon(\mu_\varepsilon,\beta_\varepsilon,A_\varepsilon) + C(c,\alpha)\frac{F_\varepsilon(\mu_\varepsilon,\beta_\varepsilon,A_\varepsilon)}{|\log\varepsilon|}. \tag{5.12}$$

Eventually, note that $\tilde\beta_\varepsilon = \beta_\varepsilon$ outside the balls in $A_3^\varepsilon(s_2^\varepsilon)$. On the other hand, these balls are all included in $\bigcup_{x\in\operatorname{supp}(\mu_\varepsilon)} B_{\varepsilon^\alpha}(x)$.

Step 3. *Replacing the circulation condition by a measure-valued* curl.
We know from Step 2 that $\#(A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)) \le C(\alpha,K,c)|\log\varepsilon|^{1-\delta}$. Now, choose $c_1 = c_1(\alpha,K,c) > 1$ such that $\log c_1 = \frac{1}{8}\frac{\sigma}{C(\alpha,K,c)}$ where $c>1$ is the universal expanding factor of the ball constructions in step 1 and 2.
Consider a ball construction associated to c_1 starting with the balls in $A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_3^\varepsilon)$. Again, denote its output by $(I_\varepsilon(t),(B_i^\varepsilon(t))_{i\in I_\varepsilon(t)},(R_i^\varepsilon)_{i\in I_\varepsilon(t)})_t$.
From step 2 we know that for every ball $B\in A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)$ it holds $\operatorname{diam} B \le 2\varepsilon^{\alpha+\sigma}$. Arguing as in step 1 and 2, we obtain that for $\varepsilon>0$ small enough it holds that for all $t \le \lceil\frac{\sigma}{2}\frac{|\log\varepsilon|}{\log c_1}\rceil =: s_3^\varepsilon$ we have $\sum_{i\in I_\varepsilon(t)} R_i(t) \le \varepsilon^\alpha$. During the construction, the number of merging times is definitely bounded by the number of starting balls i.e., less than $C(\alpha,K,c)|\log\varepsilon|^{1-\delta}$. Hence, there are at least $s_3^\varepsilon - C(\alpha,K,c)|\log\varepsilon|^{1-\delta}$ natural numbers $n\le s_3^\varepsilon - 1$ such that there is no merging time in the interval $(n,n+1]$. A direct computation shows that for $\varepsilon>0$ small enough it holds

$$s_3^\varepsilon - C(\alpha,K,k,c)|\log\varepsilon|^{1-\delta} \ge \frac{2}{3}s_3^\varepsilon. \tag{5.13}$$

In particular, there exist natural numbers $\frac{s_3^\varepsilon}{2}\le n_1<\dots<n_L\le s_3^\varepsilon-1$, $L\ge\frac{s_3^\varepsilon}{7}$, such that none of the balls $(B_i^\varepsilon(n_k))_{i\in I_\varepsilon(n_k)}$ merges in the time interval $(n_k,n_k+1]$. As in step 2, by the mean value theorem, we can find a natural number $\frac{s_3^\varepsilon}{2}\le n_k, 1\le k\le L$, which satisfies in addition

$$\sum_{i\in I_\varepsilon(n_k)}\int_{B_i^\varepsilon(n_k+1)\setminus B_i^\varepsilon(n_k)}\mathcal{C}\tilde\beta_\varepsilon:\tilde\beta_\varepsilon\,dx \le \frac{7}{s_3^\varepsilon}\int_{A_\varepsilon\setminus\left(\bigcup_{B\in A_1^\varepsilon(s_2)\cup A_2^\varepsilon(s_2)} B\right)}\mathcal{C}\tilde\beta_\varepsilon:\tilde\beta_\varepsilon\,dx.$$

For $i\in I_\varepsilon(n_k)$ we perform the following construction. Let $\xi_i = \tilde\mu_\varepsilon(B_i^\varepsilon(n_k))$, where $\tilde\mu_\varepsilon$ is defined as in step 2, and define the function

$$K_i(x) = \frac{1}{2\pi}\frac{\xi_i\otimes J(x-x_i^\varepsilon)}{|x-x_i^\varepsilon|^2}.$$

Here, x_i^ε is the center of the ball $B_i^\varepsilon(n_k)$ and J is the clockwise rotation by $\frac{\pi}{2}$. A straightforward computation shows that $\operatorname{curl} K_i = 0$ on $B_i^\varepsilon(n_k+1)\setminus B_i^\varepsilon(n_k) =: C_i^\varepsilon(n_k)$ and

$$\int_{C_i^\varepsilon(n_k)}|K_i|^2\,dx = |\xi_i|^2\frac{\log(c_1)}{2\pi} \le C(c_1)\int_{C_i^\varepsilon(n_k)}\mathcal{C}\tilde\beta_\varepsilon:\tilde\beta_\varepsilon\,dx.$$

For the inequality, we used Lemma 5.3.2.
Moreover, we notice that

$$\operatorname{curl}(\tilde{\beta}_\varepsilon - K_i) = 0 \text{ in } C_i^\varepsilon(n_k) \text{ and } \int_{\partial B_i^\varepsilon(n_k)} (\tilde{\beta}_\varepsilon - K_i) \cdot \tau \, dx = 0.$$

Consequently, there exists a function $u_i^\varepsilon \in H^1(C_i^\varepsilon(n_k);\mathbb{R}^2)$ such that $\nabla u_i^\varepsilon = \tilde{\beta}_\varepsilon - K_i$ on $C_i^\varepsilon(n_k)$. Similar to step 2, we can apply Korn's inequality for the annulus on $C_i^\varepsilon(n_k)$ to obtain a skew-symmetric matrix $W_i^\varepsilon \in Skew(2)$ such that

$$\int_{C_i^\varepsilon(n_k)} |\nabla u_i^\varepsilon - W_i^\varepsilon|^2 \, dx \leq C(c_1) \int_{C_i^\varepsilon(n_k)} |(\nabla u_i^\varepsilon)_{sym}|^2 \, dx.$$

Note again that the constant depends only on the ratio of the annulus $C_i^\varepsilon(n_k)$ which is by construction c_1.
In addition, by classical extension results, there exists a function $v_i^\varepsilon \in H^1(B_i^\varepsilon(n_k+1);\mathbb{R}^2)$ such that $\nabla v_i^\varepsilon = \nabla u_i - W_i^\varepsilon$ and

$$\int_{B_i^\varepsilon(n_k+1)} |\nabla v_i^\varepsilon|^2 \leq C(c_1) \int_{C_i^\varepsilon(n_k)} |\nabla u_i^\varepsilon - W_i^\varepsilon|^2 \, dx.$$

By scaling, also the constant on the right hand side of this inequality depends only on c_1.
Combining the last four estimates and summing over $i \in I_\varepsilon(n_k)$ yields the following chain of inequalities

$$\begin{aligned}
\sum_{i \in I_\varepsilon(n_k)} \int_{B_i^\varepsilon(n_k+1)} \mathcal{C}\nabla v_i^\varepsilon : \nabla v_i^\varepsilon \, dx &\leq C \sum_{i \in I_\varepsilon(n_k)} \int_{B_i^\varepsilon(n_k+1)} |\nabla v_i^\varepsilon|^2 \, dx \qquad (5.14)\\
&\leq C(c_1) \sum_{i \in I_\varepsilon(n_k)} \int_{C_i^\varepsilon(n_k)} |\nabla u_i^\varepsilon - W_i|^2 \, dx \\
&\leq C(c_1) \sum_{i \in I_\varepsilon(n_k)} \int_{C_i^\varepsilon(n_k)} |(\nabla u_i^\varepsilon)_{sym}|^2 \, dx \\
&= C(c_1) \sum_{i \in I_\varepsilon(n_k)} \int_{C_i^\varepsilon(n_k)} |(\tilde{\beta}_\varepsilon - K_i)_{sym}|^2 \, dx \\
&\leq C(c_1) \sum_{i \in I_\varepsilon(n_k)} \int_{C_i^\varepsilon(n_k)} \mathcal{C}\tilde{\beta}_\varepsilon : \tilde{\beta}_\varepsilon \, dx + \int_{C_i^\varepsilon(n_k)} |K_i|^2 \, dx \\
&\leq C(c_1) \sum_{i \in I_\varepsilon(n_k)} \int_{C_i^\varepsilon(n_k)} \mathcal{C}\tilde{\beta}_\varepsilon : \tilde{\beta}_\varepsilon \, dx \\
&\leq C(c_1) \frac{7}{s_3^\varepsilon} \int_{A_\varepsilon \setminus \left(\bigcup_{B \in A_1^\varepsilon(s_2^\varepsilon) \cup A_2^\varepsilon(s_2^\varepsilon)} B \right)} \mathcal{C}\tilde{\beta}_\varepsilon : \tilde{\beta}_\varepsilon \, dx \\
&\leq \frac{C(\alpha, c_1)}{|\log \varepsilon|} \int_{A_\varepsilon \setminus \left(\bigcup_{B \in A_1^\varepsilon(s_2^\varepsilon) \cup A_2^\varepsilon(s_2^\varepsilon)} B \right)} \mathcal{C}\tilde{\beta}_\varepsilon : \tilde{\beta}_\varepsilon \, dx. \qquad (5.15)
\end{aligned}$$

Here, the constant $C(c_1)$ changed from line to line but it depends only on c_1 and global parameters.
Now, define the strain $\bar{\beta}_\varepsilon : A_\varepsilon \to \mathbb{R}^{2\times 2}$ by

$$\bar{\beta}_\varepsilon(x) = \begin{cases} \nabla v_i^\varepsilon(x) + W_i & \text{if } x \in B_i^\varepsilon(n_k+1) \text{ for some } i \in I_\varepsilon(n_k), \\ \tilde{\beta}_\varepsilon(x) & \text{else.} \end{cases}$$

Note that as the balls $(B_i^\varepsilon(n_k+1))_{i\in I_\varepsilon(n_k+1)}$ are disjoint, $\bar\beta_\varepsilon$ is well-defined. Moreover, from (5.14) – (5.15) and (5.12) in step 2 we derive that

$$\frac{1}{|\log\varepsilon|^2}\int_{A_\varepsilon}\frac{1}{2}\mathcal{C}\bar\beta_\varepsilon:\bar\beta_\varepsilon\,dx \le \frac{1}{|\log\varepsilon|^2}\left(1+\frac{C(\alpha,c_1)}{|\log\varepsilon|}\right)\int_{A_\varepsilon\setminus\bigcup_{B\in A_1^\varepsilon(s_2^\varepsilon)\cup A_2^\varepsilon(s_2^\varepsilon)}B}\frac{1}{2}\mathcal{C}\tilde\beta_\varepsilon:\tilde\beta_\varepsilon\,dx$$
$$\le\left(1+\frac{C(\alpha,c)}{|\log\varepsilon|}\right)\left(1+\frac{C(\alpha,c_1)}{|\log\varepsilon|}\right)F_\varepsilon(\mu_\varepsilon,\beta_\varepsilon,A_\varepsilon).$$

In addition, it holds $\operatorname{curl}\bar\beta_\varepsilon=\sum_{i\in I_\varepsilon(n_k)}(K_i\cdot\tau)\,\mathcal{H}^1_{|\partial B_i^\varepsilon(n_k+1)}$ where τ is the unit tangent to $\partial B_i^\varepsilon(n_k+1)$,

$$|\operatorname{curl}\bar\beta_\varepsilon|=\sum_{i\in I_\varepsilon(n_k)}|K_i|\,\mathcal{H}^1_{|\partial B_i^\varepsilon(n_k+1)},\ \text{and}\ \int_{\partial B_i^\varepsilon(n_k+1)}|K_i|\,d\mathcal{H}^1=|\xi_i|=|\tilde\mu_\varepsilon(B_i^\varepsilon(n_k+1))|.$$

Finally, set $I_\varepsilon=I_\varepsilon(n_k+1)$ and $(D_i^\varepsilon)_{i\in I_\varepsilon}=(B_i^\varepsilon(n_k+1))_{i\in I_\varepsilon(n_k+1)}$. Then (i), (ii), (iv), (v), and (vii) are fulfilled. As also in the third step we changed the function from step 2 only in $\bigcup_{x\in\operatorname{supp}(\mu_\varepsilon)}B_{\varepsilon^\alpha}(x)$, it follows (iii).
Hence, it is left to show (vi). Recall that $n_k\ge\frac{s_3^\varepsilon}{2}$. By (5.13), there exist at least $\frac{s_3^\varepsilon}{7}$ natural numbers n below n_k-1 such that there is no merging time between n and $n+1$. A similar computation to (5.6) - (5.8) in step 1 shows that

$$|\operatorname{curl}\bar\beta_\varepsilon|(A_\varepsilon)=\sum_{i\in I_\varepsilon}|\tilde\mu_\varepsilon(D_i^\varepsilon)|\le C(\alpha,K,c_1)|\log\varepsilon|^{1-\delta},$$

which is (vi).
Eventually, note that c_1 depends only on α,K, and c where c is a fixed universal parameter. □

The Proposition above allows us to reduce the complicated situation with at most $|\log\varepsilon|^2$ dislocations to a simpler one. After applying the previous proposition, there are only $\sim|\log\varepsilon|$ balls in which the curl of the modified strain is concentrated. This will be enough to obtain compactness. For the lim inf-inequality, one needs to compute self-energies of dislocations. The self-energy density ψ as defined in (4.6) is the renormalized limit of energies computed for curl-free functions on annuli with larger and larger ratios. As we want to derive the same quantities also in this situation, it is necessary that we are able to find annuli around the dislocation cores with growing ratios in which the strain (respectively the modified strain in the sense of the previous proposition) is curl-free. The previous proposition for $\delta=0$ guarantees essentially only the existence of annuli with a fixed ratio uniformly in ε.
The next proposition shows that either most of the dislocations allow growing ratios in a ball construction or the accumulated Burgers vector is small and the previous proposition allows to reduce the situation to less than $|\log\varepsilon|^{1-\delta}$ dislocation balls for $\delta>0$. The latter case leads in average to growing differences between consecutive merging times in a ball construction. This is enough to obtain annuli with growing ratio in this ball construction.

Proposition 5.4.2. *Let $1>\alpha>\gamma>0$, $K>0, l>0$, and $\frac{1}{5}>\delta>0$. Then there exist $c>1$ and $\varepsilon_0=\varepsilon_0(\alpha,\gamma,\delta,K,l,c)$ such that for all $0<\varepsilon<\varepsilon_0$ the following holds:*

Let $A_\varepsilon\subset\mathbb{R}^2$. Let $(B_i^\varepsilon)_{i\in I_\varepsilon}$ be a family of disjoint balls in A_ε such that

- $\operatorname{diam}B_i^\varepsilon\le\varepsilon^\alpha$ *for all $i\in I_\varepsilon$,*

- $|I_\varepsilon| \leq K|\log\varepsilon|$,
- $\operatorname{dist}(B_i^\varepsilon, \partial A_\varepsilon) \geq l\varepsilon^\gamma$.

Let $\beta_\varepsilon : A_\varepsilon \to \mathbb{R}^{2\times 2}$ such that $\tilde{\mu}_\varepsilon = \operatorname{curl}(\beta_\varepsilon) \in \mathcal{M}(A_\varepsilon;\mathbb{R}^2)$ and $\operatorname{supp}(\operatorname{curl}\beta_\varepsilon) \subset \bigcup_{i\in I_\varepsilon} B_i^\varepsilon$. Moreover, assume that $\int_{A_\varepsilon} \mathcal{C}\beta_\varepsilon : \beta_\varepsilon\, dx + |\operatorname{curl}\beta_\varepsilon|(A_\varepsilon)^2 \leq K|\log\varepsilon|^2$. Then at least one of the following options holds true:

(i) $|(\operatorname{curl}\beta_\varepsilon)(A_\varepsilon)| \leq |\log\varepsilon|^{1-\delta}$,

(ii) Consider a ball construction associated to c starting with the balls $(B_i^\varepsilon)_{i\in I_\varepsilon}$ and the time t_s^ε which is defined to be the first time such that a ball in the ball construction intersects ∂A_ε. Then there exists a subset $\tilde{I}_\varepsilon \subset I_\varepsilon(t_s^\varepsilon)$ such that for any $i \in \tilde{I}_\varepsilon$ there exist at most $|\log\varepsilon|^\delta$-many times $n \in \mathbb{N}$, $n \leq t_s^\varepsilon - 1$, such that at least one ball in $P_i^{t_s^\varepsilon}(n)$ merges in the time interval $(n, n+1]$. Moreover, it holds

$$\left|\sum_{i\in\tilde{I}_\varepsilon} \tilde{\mu}_\varepsilon(B_i^\varepsilon) - \tilde{\mu}_\varepsilon(A_\varepsilon)\right| \leq \delta|\tilde{\mu}_\varepsilon(A_\varepsilon)|. \tag{5.16}$$

Proof. Let $\frac{1}{5} > \delta > 0$. Fix $c > 1$. Let us perform a ball construction associated to c starting with the balls $(B_i^\varepsilon)_{i\in I_\varepsilon}$. As in the previous proof, we denote the output of the construction by $(I_\varepsilon(t), (B_i^\varepsilon(t))_{i\in I_\varepsilon(t)}, (R_i^\varepsilon(t))_{i\in I_\varepsilon(t)})_t$.

Let t_s^ε be the first time at which one of the balls in $(B_i^\varepsilon(t_s^\varepsilon))_{i\in I_\varepsilon(t_s^\varepsilon)}$ intersects ∂A_ε. If t_s^ε is a merging time, still denote by $(B_i^\varepsilon(t_s^\varepsilon))_{i\in I_\varepsilon(t_s^\varepsilon)}$ the unmerged balls whose pairwise intersection is a set of $\mathcal{L}^2$-measure zero. As the balls $(B_i^\varepsilon)_{i\in I_\varepsilon}$ have radii not larger than ε^α and a distance of at least $l\varepsilon^\gamma$ to ∂A_ε, we can argue as in the previous proof to obtain that for ε small enough it holds that $t_s^\varepsilon \geq \lceil \frac{\alpha-\gamma}{2}\frac{|\log\varepsilon|}{\log c}\rceil \gg |\log\varepsilon|^{1-\delta}$. Let us define the set of balls which are affected by at most $|\log\varepsilon|^{1-\delta}$ discrete merging steps by

$$\begin{aligned}\mathcal{G}_\varepsilon = \{B_i^\varepsilon(t_s^\varepsilon) : i \in I_\varepsilon(t_s^\varepsilon) &\text{ and there exist more than } t_s^\varepsilon - |\log\varepsilon|^{1-\delta} \text{ natural numbers}\\ &0 \leq n_1 < \dots < n_{L_i} \leq t_s^\varepsilon - 1 \text{ such that for all } 1 \leq k \leq L_i \text{ none of the balls } B_j^\varepsilon(n_k) \in P_i^{t_s^\varepsilon}(n_k)\\ &\text{merges in the interval } (n_k, n_k+1]\}\end{aligned}$$

and its parents at time $0 < t < t_s^\varepsilon$ by

$$\mathcal{G}_\varepsilon(t) = \{B_j^\varepsilon(t) : j \in I_\varepsilon(t) \text{ and there is } i \in I_\varepsilon(t_s^\varepsilon) \text{ such that } B_j^\varepsilon(t) \subset B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{G}_\varepsilon\}.$$

Analogously we denote the set of balls that are involved in mergings in many discrete steps by $\mathcal{B}_\varepsilon = \{B_i^\varepsilon(t_s^\varepsilon) : i \in I_\varepsilon(t_s^\varepsilon)\} \setminus \mathcal{G}_\varepsilon$ and its parents $\mathcal{B}_\varepsilon(t)$ at time $t > 0$.

In the following, we will show that if the balls in $\mathcal{G}_\varepsilon$ do not carry most of the mass of $\tilde{\mu}_\varepsilon$, then $\tilde{\mu}_\varepsilon(A_\varepsilon)$ has to be small itself.

Claim: If $\left|\sum_{B\in\mathcal{G}_\varepsilon} \tilde{\mu}_\varepsilon(B) - \tilde{\mu}_\varepsilon(A_\varepsilon)\right| \geq \delta|\tilde{\mu}_\varepsilon|(A_\varepsilon)$, then $|\tilde{\mu}_\varepsilon(A_\varepsilon)| \leq |\log\varepsilon|^{1-\delta}$ for ε small enough depending on c, δ, γ and α.

Let $\left|\sum_{B\in\mathcal{G}_\varepsilon} \tilde{\mu}_\varepsilon(B) - \tilde{\mu}_\varepsilon(A_\varepsilon)\right| \geq \delta|\tilde{\mu}_\varepsilon|(A_\varepsilon)$ but let us assume that $|\tilde{\mu}(A_\varepsilon)| > |\log\varepsilon|^{1-\delta}$.

First, we apply the generalized Korn inequality (see [38, Theorem 11]) for any $B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon$ i.e., for any ball $B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon$ there exists a skew-symmetric $W_i^\varepsilon \in Skew(2)$ such that

$$\int_{B_i^\varepsilon(t_s^\varepsilon)} |\beta_\varepsilon - W_i^\varepsilon|^2\, dx \le C \left(\int_{B_i^\varepsilon(t_s^\varepsilon)} |(\beta_\varepsilon)_{sym}|^2\, dx + (|\tilde{\mu}_\varepsilon|(B_i^\varepsilon(t_s^\varepsilon)))^2 \right).$$

Note that by scaling the constant does not depend on the size of the ball.
Summing over all $i \in I_\varepsilon(t_s^\varepsilon)$ such that $B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon$ yields (recall that by construction the pairwise intersections of balls in $(B_i^\varepsilon(t_s^\varepsilon))_{i \in I_\varepsilon(t_s^\varepsilon)}$ are of negligible Lebesgue measure)

$$\sum_{B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon} \int_{B_i^\varepsilon(t_s^\varepsilon)} |\beta_\varepsilon - W_i^\varepsilon|^2\, dx \le C \left(\int_{A_\varepsilon} \mathcal{C}\beta_\varepsilon : \beta_\varepsilon\, dx + \sum_{B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon} (|\tilde{\mu}_\varepsilon|(B_i^\varepsilon(t_s^\varepsilon)))^2 \right) \le CK|\log \varepsilon|^2. \tag{5.17}$$

For the last inequality, we used the simple estimate (recall that $\tilde{\mu}_\varepsilon = \operatorname{curl} \beta_\varepsilon$ is concentrated in the family $(B_i^\varepsilon)_{i \in I_\varepsilon}$ which consists of much smaller balls than the ones in $(B_i^\varepsilon(t_s^\varepsilon))_{i \in I_\varepsilon(t_s^\varepsilon)}$)

$$\sum_{B_i(t_s^\varepsilon) \in \mathcal{B}_\varepsilon} (|\tilde{\mu}_\varepsilon|(B_i^\varepsilon(t_s^\varepsilon)))^2 \le \left(\sum_{B_i^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon} |\tilde{\mu}_\varepsilon|(B_i^\varepsilon(t_s^\varepsilon)) \right)^2 \le (|\tilde{\mu}_\varepsilon|(A_\varepsilon))^2.$$

In the following, we find a lower bound for the energy concentrated on the balls of $\mathcal{B}_\varepsilon$.
First, notice that the balls in $\mathcal{B}_\varepsilon$ emerge from mergings which are distributed over at least $|\log \varepsilon|^{1-\delta}$ time steps of the form $(n, n+1]$ for some $n \le t_s^\varepsilon - 1$. Arguing as for the sets $\mathcal{M}_\varepsilon(s_1^\varepsilon)$ in step 1 of the proof of the previous proposition, we can obtain that for $t_s^\varepsilon \ge t \ge t_s^\varepsilon - \frac{|\log \varepsilon|^{1-\delta}}{2}$ it holds that

$$\#\mathcal{B}_\varepsilon(t) \le C \frac{\#I_\varepsilon}{|\log \varepsilon|^{1-\delta}} \le CK|\log \varepsilon|^\delta. \tag{5.18}$$

Let us denote by $\tau_1^\varepsilon < \cdots < \tau_{L_\varepsilon}^\varepsilon$ the merging times between $t_s^\varepsilon - \frac{|\log \varepsilon|^{1-\delta}}{2}$ and t_s^ε. Moreover, write $\tau_0^\varepsilon = t_s^\varepsilon - \frac{|\log \varepsilon|^{1-\delta}}{2}$ and $\tau_{L_\varepsilon+1}^\varepsilon = t_s^\varepsilon$. From estimate (5.18), we derive that that for any $0 \le k \le L_\varepsilon$ there exists $i_k \in I_\varepsilon(\tau_k^\varepsilon)$ such that

$$B_{i_k}^\varepsilon(\tau_k^\varepsilon) \in \mathcal{B}_\varepsilon(\tau_k^\varepsilon) \text{ and } |\tilde{\mu}_\varepsilon(B_{i_k}^\varepsilon(\tau_k^\varepsilon))| \ge \frac{\delta}{2K}|\log \varepsilon|^{1-2\delta}. \tag{5.19}$$

Here, we used that $\left|\sum_{B_i^\varepsilon(\tau_k^\varepsilon) \in \mathcal{B}_\varepsilon(\tau_k^\varepsilon)} \tilde{\mu}_\varepsilon(B_i^\varepsilon(\tau_k^\varepsilon))\right| = \left|\sum_{B \in \mathcal{G}_\varepsilon} \tilde{\mu}_\varepsilon(B) - \tilde{\mu}_\varepsilon(A_\varepsilon)\right| > \delta|\log \varepsilon|^{1-\delta}$. By Lemma 5.3.2 and (5.19), we estimate for $j \in I_\varepsilon(t_s^\varepsilon)$ such that $B_{i_k}^\varepsilon(\tau_k^\varepsilon) \subset B_j^\varepsilon(t_s^\varepsilon) \in \mathcal{B}_\varepsilon$ the following:

$$\begin{aligned} \int_{B_{i_k}^\varepsilon(\tau_{k+1}^\varepsilon) \setminus B_{i_k}^\varepsilon(\tau_k^\varepsilon)} |\beta_\varepsilon - W_j^\varepsilon|^2\, dx &\ge \frac{1}{2\pi}(\tau_{k+1}^\varepsilon - \tau_k^\varepsilon)(\log c)\, (|\mu_\varepsilon(B_{i_k}^\varepsilon(\tau_k^\varepsilon))|)^2 \\ &\ge (\tau_{k+1}^\varepsilon - \tau_k^\varepsilon)(\log c)\, \frac{\delta^2}{8\pi K^2}|\log \varepsilon|^{2-4\delta}. \end{aligned}$$

Summing over all merging times between $t_s^\varepsilon - \frac{|\log \varepsilon|^{1-\delta}}{2}$ and t_s^ε provides the estimate

$$\sum_{k=0}^{L_\varepsilon} \int_{B_{i_k}^\varepsilon(\tau_{k+1}^\varepsilon) \setminus B_{i_k}^\varepsilon(\tau_k^\varepsilon)} |\beta_\varepsilon - W_j^\varepsilon|^2\, dx \ge \frac{\delta^2}{16\pi K^2}(\log c)\; |\log \varepsilon|^{3-5\delta}.$$

Together with (5.17), this implies

$$\frac{\delta^2}{16\pi K^2}\log(c)\,|\log\varepsilon|^{3-5\delta} \le 2CK|\log\varepsilon|^2,$$

which is a contradiction for $\delta < \frac{1}{5}$ and $\varepsilon > 0$ small enough depending on δ and the occurring constants. Hence, the claim is proven.
As the balls in $\mathcal{G}_\varepsilon$ have the claimed property (ii), this finishes the proof. □

Remark 5.4.1. Note that both propositions in this section hold also if one replaces the elastic tensor by a nonlinear energy density with quadratic growth. In the proof one only needs to replace Korn's inequality by its non-linear counterpart.

5.5 Compactness

In this section, we prove Theorem 5.2.2. A uniform bound on the energy is enough to achieve compactness for the dislocation measures in the flat topology. On the other hand, compactness for the strains can only be shown in a weaker topology than weak convergence in L^2, more precisely in a weak L^2_{loc}-topology excluding neighborhoods around the dislocations. We can show that the appearance of the weak convergence in L^2_{loc} is due to a loss of rigidity close to the dislocations and the boundary. In fact, we can prove weak convergence in L^2 on the whole domain for the symmetric part of the strains. Later in this section, we construct an example which shows that the compactness statement for the strains cannot be improved to weak convergence on the whole domain. However, the weak convergence in L^2_{loc} is enough to show a lim inf-inequality. As this topology allows compactness and the lim inf-inequality, it is in this sense more natural than the weak convergence in L^2 on the whole domain.

Theorem (Compactness)**.** *Let $\Omega \subset \mathbb{R}^2$ open, simply connected and with Lipschitz boundary. Let $\varepsilon_k \to 0$ and consider a sequence $(\mu_k, \beta_k) \in X(\Omega) \times \mathcal{AS}^{lin}_{\varepsilon_k}(\mu_k)$ such that $\sup_k F_{\varepsilon_k}(\mu_k, \beta_k) < \infty$. Then there exist a function $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$, a vector-valued Radon measure $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2) \cap H^{-1}(\Omega;\mathbb{R}^2)$, and a sequence of skew-symmetric matrices $W_k \in Skew(2)$ such that for a (not relabeled) subsequence it holds*

(i) $\frac{\mu_k}{|\log\varepsilon_k|} \to \mu$ *in the flat topology,*

(ii) for all $0 < \gamma < 1$ *and all* $U \subset\subset \Omega$ *we have* $\frac{\beta_k - W_k}{|\log\varepsilon_k|}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \rightharpoonup \beta$ *in* $L^2(U;\mathbb{R}^{2\times 2})$,

(iii) $\frac{(\beta_k)_{sym}}{|\log\varepsilon_k|} \rightharpoonup (\beta)_{sym}$ *in* $L^2(\Omega;\mathbb{R}^{2\times 2})$,

(iv) $\operatorname{curl}\beta = \mu$.

Finally, the obtained convergence is enough to prove the lim inf*-inequality*

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k,\beta_k) \ge \int_\Omega \mathcal{C}\beta : \beta\, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|,$$

where φ is the relaxed self-energy density defined as in (4.7) *for $R = Id$.*

Proof. **Step 1.** *Compactness of the dislocation measures.*
Fix $1 > \gamma > 0$ and define $\alpha = \gamma + \frac{1-\gamma}{2}$. Then $\gamma < \alpha < 1$.
Denote by $(A^j_{\varepsilon_k})_{j\in J_{\varepsilon_k}}$ the connected components of $\bigcup_{x\in\operatorname{supp}\mu_k} B_{\varepsilon_k^\gamma}(x)$ that do not intersect $\partial\Omega$. Define

$U_{\varepsilon_k} = \bigcup_{j\in J_{\varepsilon_k}} A^j_{\varepsilon_k}$.
We apply Proposition 5.4.1 on U_{ε_k} to μ_k, β_k, and $\delta = 0$. The proposition provides a strain $\tilde{\beta}_k : U_{\varepsilon_k} \to \mathbb{R}^{2\times 2}$ which satisfies

$$|\operatorname{curl} \tilde{\beta}_k|(U_{\varepsilon_k}) \leq C(\gamma)|\log \varepsilon_k|.$$

Moreover, by (iii) of Proposition 5.4.1 we can extend $\tilde{\beta}_k$ by β_k to $\Omega \setminus U_{\varepsilon_k}$ without creating additional curl on $\partial U_{\varepsilon_k}$. In the following, we call this extended function also $\tilde{\beta}_k$.
Let us write $\tilde{\mu}_k = (\operatorname{curl} \tilde{\beta}_k)_{|U_{\varepsilon_k}} \in \mathcal{M}(\Omega;\mathbb{R}^2)$ which fulfills $\frac{|\tilde{\mu}_k|}{|\log \varepsilon_k|} \leq C(\gamma)$. Hence, there exist a measure $\mu \in \mathcal{M}(\Omega;\mathbb{R}^2)$ and a (not relabeled) subsequence such that $\frac{\tilde{\mu}_k}{|\log \varepsilon_k|} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$. In particular, it holds true that $\frac{\tilde{\mu}_k}{|\log \varepsilon_k|} \to \mu$ in the flat topology. It remains to show that $\frac{\mu_k - \tilde{\mu}_k}{|\log \varepsilon_k|} \to 0$ in the flat topology. A similar computation to what follows can also be found in [30].
By the definition of flat convergence we need to show that

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|} \sup_{\varphi \in W^{1,\infty}_0(\Omega;\mathbb{R}^2): Lip(\varphi)\leq 1} \int_\Omega \varphi \, d(\mu_k - \tilde{\mu}_k) = 0.$$

Let $\varphi \in W^{1,\infty}_0(\Omega;\mathbb{R}^2)$ with $Lip(\varphi) \leq 1$. Obviously, $\operatorname{supp}(\mu_k - \tilde{\mu}_k) \subset \bigcup_{x\in \operatorname{supp}\mu_k} B_{\varepsilon_k^\gamma}(x)$ and by definition $\tilde{\mu}_k = 0$ outside U_{ε_k}. Hence, we can write

$$\int_\Omega \varphi \, d(\mu_k - \tilde{\mu}_k) = \int_{\operatorname{supp}(\mu_k)\setminus U_{\varepsilon_k}} \varphi \, d\mu_k + \sum_{j\in J_{\varepsilon_k}} \int_{A^j_{\varepsilon_k}} \varphi \, d(\mu_k - \tilde{\mu}_k). \tag{5.20}$$

Let us shortly notice the following. As the energy $F_{\varepsilon_k}(\mu_k, \beta_k)$ is uniformly bounded and the non-zero elements in $\mathbb{S}$ are bounded away from zero, the number of dislocations is bounded in terms of $|\log \varepsilon|^2$. Hence, each connected component of $\bigcup_{x\in \operatorname{supp}\mu_k} B_{\varepsilon_k^\gamma}(x)$ has a diameter less than $C|\log \varepsilon_k|^2 2\varepsilon_k^\gamma$ where C is a universal constant.
First, we consider the first integral of the right hand side of (5.20). Note that if $x \in \operatorname{supp}\mu_k \setminus U_{\varepsilon_k}$, the corresponding connected component of $\bigcup_{x\in \operatorname{supp}\mu_k} B_{\varepsilon_k^\gamma}(x)$ intersects $\partial\Omega$.
As $Lip(\varphi) \leq 1$ and φ vanishes on $\partial\Omega$, we obtain that $|\varphi| \leq C|\log \varepsilon|^2 2\varepsilon_k^\gamma$ in $B_{\varepsilon_k^\gamma}(x)$ and therefore

$$\left| \int_{\operatorname{supp}(\mu_k)\setminus U_{\varepsilon_k}} \varphi \, d\mu_k \right| \leq 2C|\log \varepsilon_k|^2 \varepsilon_k^\gamma \, |\mu_k|(\Omega) \leq \tilde{C}|\log \varepsilon_k|^4 \varepsilon_k^\gamma. \tag{5.21}$$

This shows that the first integral on the right hand side of (5.20) vanishes as ε_k tends to 0 uniformly for all $\varphi \in W^{1,\infty}_0(\Omega)$ with $Lip(\varphi) \leq 1$.
Next, let us consider $j \in J_{\varepsilon_k}$. By (v) of Proposition 5.4.1 it holds $\mu_k(A^j_{\varepsilon_k}) = \tilde{\mu}_k(A^j_{\varepsilon_k})$ which allows us to write

$$\int_{A^j_{\varepsilon_k}} \varphi \, d(\mu_k - \tilde{\mu}_k) = \int_{A^j_{\varepsilon_k}} \left(\varphi - <\varphi>_{A^j_{\varepsilon_k}}\right) d(\mu_k - \tilde{\mu_k}),$$

where $<\varphi>_{A^j_{\varepsilon_k}} = \fint_{A^j_{\varepsilon_k}} \varphi \, dx$. As $Lip(\varphi) \leq 1$, it holds the estimate $|\varphi(x) - <\varphi>_{A^j_{\varepsilon_k}}| \leq \operatorname{diam}(A^j_{\varepsilon_k})$ for all $x \in A^j_{\varepsilon_k}$. Thus,

$$\left| \int_{A^j_{\varepsilon_k}} \varphi \, d(\mu_k - \tilde{\mu_k}) \right| \leq C|\log \varepsilon_k|^2 \varepsilon_k^\gamma \left(|\mu_k|(A^j_{\varepsilon_k}) + |\tilde{\mu}_k|(A^j_{\varepsilon_k})\right). \tag{5.22}$$

Summing over all j in the estimate (5.22) and combining the resulting estimate with (5.21) yields the existence of a constant $C(\gamma)$ such that

$$\left|\int_{\Omega} \varphi \, d(\mu_k - \tilde{\mu_k})\right| = \left|\int_{\operatorname{supp}(\mu_k)\setminus U_{\varepsilon_k}} \varphi \, d\mu_k + \sum_{j\in J_{\varepsilon_k}} \int_{A^j_{\varepsilon_k}} \varphi \, d(\mu_k - \tilde{\mu_k})\right|$$
$$\leq \tilde{C}|\log \varepsilon_k|^4 \varepsilon_k^\gamma + \sum_{j\in J_{\varepsilon_k}} 2C|\log \varepsilon_k|^2 \varepsilon_k^\gamma (|\mu_k|(A^j_{\varepsilon_k}) + |\tilde{\mu}_k|(A^j_{\varepsilon_k}))$$
$$\leq C(\gamma)|\log \varepsilon_k|^4 \varepsilon_k^\gamma.$$

Hence,

$$\frac{1}{|\log \varepsilon_k|} \sup_{\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^2):Lip(\varphi)\leq 1} \int_\Omega \varphi \, d(\mu_k - \tilde{\mu}_k) \leq C(\gamma)|\log \varepsilon_k|^3 \varepsilon_k^\gamma \to 0 \text{ as } k \to \infty.$$

Thus, we established that $\frac{\mu_k}{|\log \varepsilon_k|} \to \mu$ in the flat topology which is (i).
Note that for all $0 < \gamma < 1$ the corresponding measure $\tilde{\mu}_k$ satisfies $|\tilde{\mu}_k|(\Omega) \leq C(\gamma)|\log \varepsilon_k|$ for some constant $C(\gamma)$ depending on γ. A similar argument to the one above shows that the difference of the measures $\tilde{\mu}_k$ for two different values of γ converges to zero weakly∗ in $\mathcal{M}(\Omega;\mathbb{R}^2)$. Hence, the chosen subsequence satisfies that for all $0 < \gamma < 1$ it holds $\frac{\tilde{\mu}_k}{|\log \varepsilon_k|} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$.

Step 2. *Finding an L^2_{loc}-limit for the strains.*
Let $0 < \gamma < 1$ and $\alpha = \gamma + \frac{1-\gamma}{2}$. As Ω is simply connected, one can find, with the use of Sard's theorem, a monotone sequence of compactly contained subsets $(\Omega_l)_{l\in\mathbb{N}}$ of Ω such that each Ω_l is simply connected with Lipschitz boundary and $\Omega_l \nearrow \Omega$. In step 1, we applied Proposition 5.4.1 to U_{ε_k}, β_k, and μ_k and obtained a modified strain $\tilde{\beta}_k$ which agrees with β_k outside $\bigcup_{x\in \operatorname{supp} \mu_k} B_{\varepsilon_k^\gamma}(x)$. In particular, $\operatorname{curl} \tilde{\beta}_k = 0$ in $\Omega_{\varepsilon_k^\gamma}(\mu_k)$. Moreover, $|\operatorname{curl} \tilde{\beta}_k|(U_{\varepsilon_k}) \leq C(\gamma)|\log \varepsilon_k|$. As $\Omega_l \subset\subset \Omega$, it is clear that $\operatorname{dist}(\Omega_l, \partial\Omega) > 0$. Hence, for ε_k small enough we find that

$$\bigcup_{x\in \operatorname{supp} \mu_k} B_{\varepsilon_k^\gamma}(x) \cap \Omega_l \subset U_{\varepsilon_k}.$$

The application of the generalized Korn's inequality (see [38, Theorem 11]) provides a sequence of skew-symmetric matrices W_k^l such that

$$\int_{\Omega_l} |\tilde{\beta}_k - W_k^l|^2 \, dx \leq C(\Omega_l) \left(\int_{U_l} |(\tilde{\beta}_k)_{sym}|^2 \, dx + |\operatorname{curl} \tilde{\beta}_k|(\Omega_l)^2 \right). \tag{5.23}$$

From (vii) of Proposition 5.4.1 and the bound on $\operatorname{curl} \tilde{\beta}_k$ we derive that $\frac{\tilde{\beta}_k - W_k^l}{|\log \varepsilon_k|}$ is a bounded sequence in the space $L^2(\Omega_l;\mathbb{R}^{2\times 2})$ where the bound depends on γ. In the following, we use a standard argument to show that the skew-symmetric matrices W_k^l can be chosen independently from l.
Let us fix $l > 1$. In addition, let W_k^l and W_1^l be the skew-symmetric matrices from above. We may estimate

$$\mathcal{L}^2(\Omega_1)|W_k^1 - W_k^l|^2 \leq 2 \left(\int_{\Omega_1} |W_k^1 - \tilde{\beta}_k|^2 \, dx + \int_{\Omega_l} |W_k^l - \tilde{\beta}_k|^2 \, dx \right) \leq C(\gamma, l)|\log \varepsilon|^2.$$

Thus, $|W_k^1 - W_k^l| \leq \tilde{C}(\gamma, l)|\log \varepsilon_k|$ which implies that also $\frac{\tilde{\beta}_k - W_k^1}{|\log \varepsilon_k|}$ is a bounded sequence in the space $L^2(\Omega_l;\mathbb{R}^{2\times 2})$. Let us write $W_k = W_k^1$.

As β_k agrees with $\tilde{\beta}_k$ on $\Omega_l \cap \Omega_{\varepsilon_k^\gamma}(\mu_k)$, we obtain that also $\frac{\beta_k - W_k}{|\log \varepsilon_k|}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)}$ is a bounded sequence in $L^2(\Omega_l;\mathbb{R}^{2\times 2})$.
A similar argument to the one above shows that the matrices W_k can also be chosen independently from γ.
Next, let us consider $1 > \gamma_1 > \gamma_2 > 0$. Assume that $\frac{\beta_k - W_k}{|\log \varepsilon_k|}\mathbf{1}_{\Omega_{\varepsilon^{\gamma_1}}(\mu_k)} \rightharpoonup \beta$ in $L^2(\Omega_l;\mathbb{R}^{2\times 2})$ for some fixed l. As $\mathbf{1}_{\Omega_{\varepsilon^{\gamma_2}}(\mu_k)} \to 1$ boundedly in measure and $\mathbf{1}_{\Omega_{\varepsilon^{\gamma_1}}(\mu_k)}\mathbf{1}_{\Omega_{\varepsilon^{\gamma_2}}(\mu_k)} = \mathbf{1}_{\Omega_{\varepsilon^{\gamma_2}}(\mu_k)}$, we deduce that also $\mathbf{1}_{\Omega_{\varepsilon^{\gamma_2}}(\mu_k)}\frac{\beta_k - W_k}{|\log \varepsilon_k|} \rightharpoonup \beta$ in $L^2(\Omega_l;\mathbb{R}^{2\times 2})$.
On the other hand, it is clear that if for $l_1 > l_2$ we have $\frac{\beta_k - W_k}{|\log \varepsilon_k|}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \rightharpoonup \beta$ in $L^2(U_{l_1};\mathbb{R}^{2\times 2})$; the same holds in $L^2(U_{l_2};\mathbb{R}^{2\times 2})$.
As weak convergence in L^2 is metrizable on bounded sets in L^2, we can find by a diagonal argument (in γ and l) a subsequence and a function $\beta \in L^2_{loc}(\Omega;\mathbb{R}^{2\times 2})$ such that $\frac{\beta_k - W_k}{|\log \varepsilon_k|}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \rightharpoonup \beta$ in $L^2(\Omega_l;\mathbb{R}^{2\times 2})$ for all $1 > \gamma > 0$ and $l \in \mathbb{N}$. Since $\Omega_l \nearrow \Omega$, this proves the convergence in (ii). It still needs to be shown that $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$.

Step 3. $\operatorname{curl}\beta = \mu$.
Fix some $1 > \gamma > 0$. In step 1 we saw that $\frac{(\operatorname{curl}\tilde{\beta}_k)_{|U_{\varepsilon_k}}}{|\log \varepsilon_k|} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^2)$. A similar argument as the one in (5.21) shows that

$$\frac{\operatorname{curl}\tilde{\beta}_k}{|\log \varepsilon_k|} \to \mu \text{ in the flat topology on } \Omega.$$

This implies convergence in $\mathcal{D}'(\Omega)$. On the other hand, we can deduce from step 2 that $\frac{\tilde{\beta}_k - W_k}{|\log \varepsilon_k|} \rightharpoonup \beta$ in $L^2_{loc}(\Omega;\mathbb{R}^{2\times 2})$ (notice that all arguments were based on considerations for $\tilde{\beta}_k$ and $\tilde{\beta}_k$ equals β_k on a set which converges in measure to Ω). Combining these two facts shows that for $\varphi \in C_c^\infty(\Omega;\mathbb{R}^2)$ it holds

$$\begin{aligned} < \operatorname{curl}\beta, \varphi >_{\mathcal{D}',\mathcal{D}} &= \lim_k \frac{1}{|\log \varepsilon_k|} < \operatorname{curl}(\beta_k - W_k), \varphi >_{\mathcal{D}',\mathcal{D}} \\ &= \lim_k \frac{1}{|\log \varepsilon_k|} < \beta_k - W_k, J\nabla\varphi >_{\mathcal{D}',\mathcal{D}} = < \beta, J\nabla\varphi >_{\mathcal{D}',\mathcal{D}}, \end{aligned}$$

where J is the clockwise rotation by $\frac{\pi}{2}$. Consequently, $\operatorname{curl}\beta = \mu$ which is (iv).

Step 4. *The limit β is in $L^2(\Omega;\mathbb{R}^{2\times 2})$.*
We show a simplified version of a lim inf-inequality for β which includes β_{sym} and $|\mu|(\Omega)$ from which we can conclude the square-integrability of β by the generalized Korn's inequality.
Let $U \subset\subset \Omega$ and $1 > \gamma > 0$ fixed. From step 1 we know that

$$\int_U |(\tilde{\beta}_k)_{sym}|^2\,dx + |(\operatorname{curl}\tilde{\beta}_k)_{|U_{\varepsilon_k}}|(\Omega)^2 \le C(\gamma)|\log \varepsilon_k|^2. \tag{5.24}$$

Moreover, from step 1 and step 2 we know that

$$\frac{(\operatorname{curl}\tilde{\beta}_k)_{|U_{\varepsilon_k}}}{|\log \varepsilon_k|} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega;\mathbb{R}^2) \text{ and } \frac{\tilde{\beta}_k - W_k}{|\log \varepsilon_k|} \rightharpoonup \beta \text{ in } L^2(U;\mathbb{R}^{2\times 2}).$$

Hence, we can derive by the usual lim inf-inequalities for weak, respectively weak∗, convergence from (5.24) that

$$\int_U |\beta_{sym}|^2\,dx + |\mu|(\Omega)^2 \le C(\gamma).$$

Taking the supremum over all $U \subset\subset \Omega$ gives

$$\int_\Omega |\beta_{sym}|^2 \, dx + |\mu|(\Omega)^2 \leq C(\gamma).$$

By the generalized Korn's inequality (see [38, Theorem 11]), there exists a matrix $W \in Skew(2)$ such that $\beta - W \in L^2(\Omega; \mathbb{R}^{2\times 2})$. As Ω has finite measure, this implies that $\beta \in L^2(\Omega; \mathbb{R}^{2\times 2})$.

Step 5. *Weak convergence of the symmetric part of the strains.*
Fix $1 > \gamma > 0$. As the matrices W_k are skew-symmetric, it is clear that the symmetric part of $(\beta_k - W_k)\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)}$ is $(\beta_k)_{sym}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)}$. Since taking the symmetric part of a matrix is a linear operation, we may derive from step 2 that

$$(\beta_k)_{sym}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \rightharpoonup \beta_{sym} \text{ in } L^2(U; \mathbb{R}^{2\times 2}) \text{ for all } U \subset\subset \Omega.$$

From the bound on the energy we may deduce directly that $\frac{(\beta_k)_{sym}}{|\log \varepsilon_k|}$ is a bounded sequence in $L^2(\Omega; \mathbb{R}^{2\times 2})$. As $(1 - \mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)}) \to 0$ boundedly in measure, this implies that

$$\frac{(\beta_k)_{sym}}{|\log \varepsilon_k|} \rightharpoonup \beta_{sym} \text{ in } L^2(U; \mathbb{R}^{2\times 2}) \text{ for all } U \subset\subset \Omega.$$

From the uniform bound of $(\beta_k)_{sym}$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$ we conclude (iii).

Step 6. *The* $\liminf$*-inequality.*

In step 4, we have already shown a poor man's version of the lim inf-inequality. For the real lim inf-inequality we refer to the proof of the lim inf-inequality of the Γ-convergence result (Proposition 5.6.1) in which it can be seen that the convergences established in step 1 and 2 are enough to show that for all $1 > \gamma > 0$ it holds that

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k, \beta_k) \geq \int_\Omega \mathcal{C}\beta : \beta \, dx + (1-\gamma)\int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

In fact, the estimate for the self-energy works exactly as in the lim inf-inequality of the Γ-convergence result. It is computed on the set U_{ε_k}. For the energy on $\Omega \setminus U_{\varepsilon_k}$ notice that, by step 5, it holds $\mathbf{1}_{U_{\varepsilon_k}}(\beta_k)_{sym} \rightharpoonup \beta_{sym}$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$. Then the estimate follows by classical lower semi-continuity and the fact that $\mathcal{C}$ only acts on the symmetric part of matrices.
Sending $\gamma \to 0$ yields the desired lim inf-inequality. □

Remark 5.5.1. It can be seen that it is not possible to neglect the reduction to the set $\Omega_{\varepsilon_k^\gamma}(\mu_k)$ in the result above. Consider the following example.

Let $\varepsilon_k \to 0$ and $x \in int\,\Omega$ fixed. We define the following set, for a sketch see Figure 5.4,

$$S_{\varepsilon_k} = x + \left\{ \varepsilon_k |\log \varepsilon_k|^2 \left(\cos\left(j \frac{2\pi}{8\lceil |\log \varepsilon_k|^2 \rceil} \right), \sin\left(j \frac{2\pi}{8\lceil |\log \varepsilon_k|^2 \rceil} \right) \right) : j = 1, \dots, 8\lceil |\log \varepsilon_k|^2 \rceil \right\}.$$

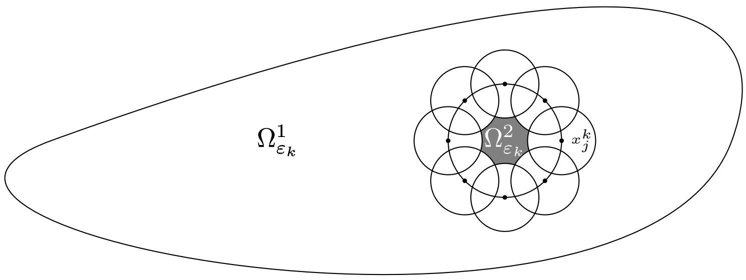


Figure 5.4: A sketch of the situation in Remark 5.5.1.

Let us write for $j = 1, \dots, 8\lceil |\log \varepsilon_k|^2 \rceil$

$$x_j^k = x + \varepsilon_k |\log \varepsilon_k|^2 \left(\cos\left(j \frac{2\pi}{8\lceil |\log \varepsilon_k|^2 \rceil} \right), \sin\left(j \frac{2\pi}{8\lceil |\log \varepsilon_k|^2 \rceil} \right) \right).$$

Moreover, we define for some fixed $\xi \in \mathbb{S}$ the measures

$$\mu_k = \sum_{\substack{j=1 \\ j \text{ even}}}^{8\lceil |\log \varepsilon_k|^2} \xi \, \delta_{x_j^k} - \sum_{\substack{j=1 \\ j \text{ odd}}}^{8\lceil |\log \varepsilon_k|^2} \xi \, \delta_{x_j^k}.$$

Note that $\mu_k(U) = 0$ for all sets U that include S_{ε_k}. It can be seen that for ε_k small enough $\Omega_{\varepsilon_k}(\mu_k)$ has exactly two connected components $\Omega^1_{\varepsilon_k}, \Omega^2_{\varepsilon_k} \subset \Omega$ where $\Omega^2_{\varepsilon_k} \subset B_{\varepsilon_k |\log \varepsilon_k|^2}(x)$ and $\Omega^1_{\varepsilon_k} \subset \Omega \setminus B_{\varepsilon_k |\log \varepsilon_k|^2}(x)$, see Figure 5.4.
We define the strain $\beta_k : \Omega \to \mathbb{R}^{2\times 2}$ by

$$\beta_k(y) = \begin{cases} W_{\varepsilon_k} & \text{if } y \in \Omega^2_{\varepsilon_k}, \\ 0 & \text{else in } \Omega, \end{cases} \quad \text{where } W_{\varepsilon_k} = \frac{1}{\varepsilon_k^2 |\log \varepsilon_k|^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $\mu_k \in X(\Omega)$.
Let $A \subset \Omega$ an open set with smooth boundary in $\Omega_{\varepsilon_k}(\mu_k)$. As $\Omega_{\varepsilon_k}(\mu_k)$ has exactly two connected components, it holds either $\partial A \subset \Omega^1_{\varepsilon_k}$ or $\partial A \subset \Omega^2_{\varepsilon_k}$, see Figure 5.4. Consequently, it holds either $S_{\varepsilon_k} \subset A$ or $S_{\varepsilon_k} \cap A = \emptyset$. In particular, $\mu_k(A) = 0$.
As β_k is constant on $\Omega^1_{\varepsilon_k}$ and $\Omega^1_{\varepsilon_k}$, we derive that $\beta_k \in \mathcal{AS}^{lin}_{\varepsilon_k}(\mu_k)$.
In addition, the energy of the pair (μ_k, β_k) is $F_{\varepsilon_k}(\mu_k, \beta_k) = \frac{|\mu_k|(\Omega)}{|\log \varepsilon_k|^2} \leq C$.
We claim that there cannot be sequence of skew-symmetric matrices W_k such that $\frac{\beta_k - W_k}{|\log \varepsilon_k|}$ has a weakly converging subsequence in $L^2(\Omega; \mathbb{R}^{2\times 2})$. In order to derive a contradiction, let us assume that there exists a subsequence (not relabeled) of β_k and skew-symmetric matrices W_k such that $\frac{\beta_k - W_k}{|\log \varepsilon_k|}$ is bounded in $L^2(\Omega; \mathbb{R}^{2\times 2})$. It follows that $|W_k| \leq C|\log \varepsilon_k|$ and $|W_{\varepsilon_k} - W_k| \leq \frac{C}{\varepsilon_k |\log \varepsilon_k|}$ (note that $\mathcal{L}^2(\Omega^2_{\varepsilon_k}) \sim \varepsilon_k^2 |\log \varepsilon_k|^4$). This implies that $|W_{\varepsilon_k}| \leq C|\log \varepsilon_k| + \frac{C}{\varepsilon_k |\log \varepsilon_k|}$; a contradiction.

In the construction, it is not crucial that one is allowed to use $|\log \varepsilon_k|^2$-many dislocations. Any allowed maximal number of dislocations that grows to infinity for $\varepsilon_k \to 0$ leads to an example of the above type. The main point is that one can disconnect small parts from the rest of Ω by the use of dislocation cores. On each of the resulting connected components of $\Omega_{\varepsilon_k}(\mu_k)$ one can put any constant skew-symmetric matrix without inducing any elastic energy nor violating the definition of $\mathcal{AS}_{\varepsilon_k}^{lin}$.

5.6 The lim inf-inequality

In this section, we prove the lim inf-inequality of the Γ-convergence result in Theorem 5.2.1. The key ingredients for the lower bound for the part of the energy close to the dislocations will be the Propositions 5.4.1 and 5.4.2.

Proposition 5.6.1 (The lim inf-inequality)**.** *Let* $\Omega \subset \mathbb{R}^2$ *an open, bounded set with Lipschitz boundary. Let* $\varepsilon_k \to 0$*. Let* $(\mu, \beta), (\mu_k, \beta_k) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2\times 2})$ *such that*

$$\frac{\mu_k}{|\log \varepsilon_k|} \to \mu \text{ in the flat topology and } \frac{\beta_k}{|\log \varepsilon_k|} \rightharpoonup \beta \text{ in } L^2(\Omega; \mathbb{R}^{2\times 2}).$$

Then it holds

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k, \beta_k) \geq F(\mu, \beta).$$

Proof. Clearly, we only have to consider (μ_k, β_k) such that $\liminf_k F_{\varepsilon_k}(\mu_k, \beta_k) < \infty$. Moreover, up to subsequences we may assume that the lim inf is a lim and $\sup_k F_{\varepsilon_k}(\mu_k, \beta_k) \leq M < \infty$. Then the compactness result, Theorem 5.2.2, yields that $\operatorname{curl} \beta = \mu$.
Fix $1 > \alpha > \gamma > 0$.
Let us consider the set $U_{\varepsilon_k} = \bigcup_{x\in \operatorname{supp}(\mu_k)} B_{\varepsilon_k^\gamma}(x)$. As in the case of well-separated dislocations, we split the elastic energy into a part close to the dislocations and a part far away from the dislocations, precisely

$$F_{\varepsilon_k}(\mu_k, \beta_k) = \int_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \frac{1}{2}\mathcal{C}\beta : \beta \, dx + F_{\varepsilon_k}(\mu_k, \beta_k, U_{\varepsilon_k}),$$

where we recall that $\Omega_{\varepsilon_k^\gamma}(\mu_k) = \Omega \setminus \bigcup_{x\in \operatorname{supp}(\mu_k)} B_{\varepsilon_k^\gamma}(x)$.

Lower bound far from the dislocations. First, notice that $\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \to 1$ boundedly in measure. As $\frac{\beta_k}{|\log \varepsilon_k|} \rightharpoonup \beta$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$, this implies $\frac{\beta_k}{|\log \varepsilon_k|}\mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \rightharpoonup \beta$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$. By the classical lower semi-continuity property of functionals with convex integrands we obtain

$$\liminf_{k\to\infty} \int_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \mathcal{C}\beta_k : \beta_k \, dx = \liminf_{k\to\infty} \int_\Omega \mathcal{C}\beta_k \mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} : \beta_k \mathbf{1}_{\Omega_{\varepsilon_k^\gamma}(\mu_k)} \, dx \geq \int_\Omega \mathcal{C}\beta : \beta \, dx.$$

This establishes the lower bound of the part of the energy that is not induced by the occurrence of dislocations.

Lower bound close to the dislocations. In this step, we estimate the energy close to the dislocations in terms of the relaxed self-energy density φ and the dislocation density μ_k.
Denote by $(A^j_{\varepsilon_k})_{j\in J_{\varepsilon_k}}$ the connected components of $U_{\varepsilon_k} = \bigcup_{x\in \operatorname{supp} \mu_k} B_{\varepsilon_k^\gamma}(x)$ that do not intersect $\partial\Omega$. We apply Proposition 5.4.1 on each of the $A^j_{\varepsilon_k}$ to β_k, α, and $\delta = 0$. For each $j \in J_{\varepsilon_k}$, we obtain a family of balls $(\tilde{B}_i^{j,\varepsilon_k})_{i\in \tilde{I}^j_{\varepsilon_k}}$ and a function $\tilde{\beta}_k : A^j_{\varepsilon_k} \to \mathbb{R}^{2\times 2}$ with the properties (i) - (vii) from

Proposition 5.4.1. In particular, the modified strains satisfy

$$\frac{1}{|\log \varepsilon_k|^2} \int_{A^j_{\varepsilon_k}} \frac{1}{2}\mathcal{C}\tilde{\beta}_k : \tilde{\beta}_k \, dx \leq \left(1 + \frac{C(\alpha, M)}{|\log \varepsilon|}\right) F_{\varepsilon_k}(\mu_k, \beta_k, A^j_{\varepsilon_k}). \tag{5.25}$$

Fix $\frac{1}{5} > \delta > 0$ small enough. One can check that for $\varepsilon_k > 0$ small enough, for each $j \in J_{\varepsilon_k}$ the sets $A^j_{\varepsilon_k}$, the modified strains $\tilde{\beta}_k$, and the balls $(\tilde{B}^{j,\varepsilon_k}_i)_{i\in \tilde{I}^j_{\varepsilon_k}}$ satisfy the assumptions of Proposition 5.4.2 for $l = \frac{1}{2}$ and $K = 2M$. Let us write $\nu_k = \operatorname{curl} \tilde{\beta}_k$ and recall that by (vi) of Proposition 5.4.1 it holds $\nu_k(A^j_{\varepsilon_k}) = \mu_k(A^j_{\varepsilon_k})$.
The application of Proposition 5.4.2 yields that for all ε_k small enough for every $j \in J_{\varepsilon_k}$ we have that at least one of the options in the conclusion of Proposition 5.4.2 holds.

Claim: There exists a constant $C(\alpha, M)$ such that for all $\eta > 0$ there exists $L \in \mathbb{N}$ satisfying that for all $k \geq L$ it holds for all $j \in J_{\varepsilon_k}$ that

$$\left(1 + \frac{C(\alpha, M)}{|\log \varepsilon_k|}\right) F_{\varepsilon_k}(\mu_k, \beta_k, A^j_{\varepsilon_k}) \geq \frac{\alpha - \gamma - \eta - \tilde{\delta}}{|\log \varepsilon_k|} \varphi(\mu_k(A^j_{\varepsilon_k})), \tag{5.26}$$

where $\tilde{\delta} = \delta \frac{\max_{\xi\in S^1} \varphi(\xi)}{\min_{\xi \in S^1} \varphi(\xi)}$.

The strategy to prove this claim will slightly differ depending whether the first or second conclusion in Lemma 5.4.2 holds on $A^j_{\varepsilon_k}$.
Clearly, we may assume that $(\alpha - \gamma - \eta - \tilde{\delta}) > 0$.

Case 1: Conclusion (ii) of Proposition 5.4.2 holds for $A^j_{\varepsilon_k}$.
Recall (ii) of Proposition 5.4.2: there exists a universal $c > 1$ with the following property. Consider a ball construction associated to c starting with the balls $(\tilde{B}^{j,\varepsilon_k}_i)_{i\in \tilde{I}^j_{\varepsilon_k}}$, which are the output of Proposition 5.4.1 on $A^j_{\varepsilon_k}$, and the time $s^{\varepsilon_k}_j$ which is defined to be the first time such that a ball in the ball construction intersects $\partial A^j_{\varepsilon_k}$. We call its output $(\tilde{I}^j_{\varepsilon_k}(t), (B^{j,\varepsilon_k}_i(t))_{i\in\tilde{I}^j_{\varepsilon_k}(t)}, (R^{j,\varepsilon_k}_i(t))_{i\in \tilde{I}^j_{\varepsilon_k}(t)})_t$. Then there exists a subset $I^j_{\varepsilon_k} \subset \tilde{I}^j_{\varepsilon_k}(s^{\varepsilon_k}_j)$ such that for each ball $B^{j,\varepsilon_k}_i(s^{\varepsilon_k}_j)$, $i \in I^j_{\varepsilon_k}$, there exist a least $(s^{\varepsilon_k}_j - |\log \varepsilon_k|^{1-\delta} - 1)$ natural numbers $0 \leq n_1 < \cdots < n_L \leq s^{\varepsilon_k}_j - 1$ such that for all $k = 1, \ldots, L$ no ball in $P^{s^{\varepsilon_k}_j}_i(n_k)$ merges between n_k and $n_k + 1$. Moreover, it holds

$$\left| \sum_{i\in \tilde{I}^j_{\varepsilon_k}} \nu_k(B^{j,\varepsilon_k}_i) - \nu_k(A^j_{\varepsilon_k}) \right| \leq \delta |\nu_k(A^j_{\varepsilon_k})|. \tag{5.27}$$

Notice here that $\tilde{\beta}_k$ is curl-free outside the balls $(\tilde{B}^{j,\varepsilon_k}_i)_{i\in\tilde{I}^j_{\varepsilon_k}}$.
Let $N \in \mathbb{N}$ and define the times $t^{\varepsilon_k}_l = l \frac{s^{\varepsilon_k}_j}{N|\log \varepsilon_k|^{1-\delta}}$ for $l = 0, \ldots, \lfloor N|\log \varepsilon_k|^{1-\delta}\rfloor$. As the starting balls $(\tilde{B}^{j,\varepsilon_k}_i)_{i\in\tilde{I}^j_{\varepsilon_k}}$ of the ball construction have radii less than ε^α_k but distance of at least $\frac{1}{2}\varepsilon^\gamma_k$ to the boundary of $A^j_{\varepsilon_k}$, we can argue as in the proof of Proposition 5.4.1 to obtain that for ε_k small enough it holds that $s^{\varepsilon_k}_j \geq (\alpha - \gamma - \frac{\eta}{4})\frac{|\log \varepsilon_k|}{\log c}$ and consequently

$$\frac{s^{\varepsilon_k}_j}{N|\log \varepsilon_k|^{1-\delta}} \geq \frac{\alpha - \gamma - \frac{\eta}{4}}{N} \frac{|\log \varepsilon_k|^\delta}{\log c}. \tag{5.28}$$

Next, notice that by Proposition 4.2.1 it holds for $\psi_{R,r}(\xi)$ as defined in (4.4) that for all $\xi \in \mathbb{R}^2$ and

$R > r > 0$ we have that

$$\psi_{R,r}(\xi) \geq \log\left(\frac{R}{r}\right)\psi(\xi) - \frac{C|\xi|^2}{\log\left(\frac{R}{r}\right)},$$

where $C > 0$ is a universal constant. Using the 2-homogeneity and continuity (by convexity) of ψ, this implies together with (5.28) that for ε_k small enough we may derive for all $\xi \in \mathbb{R}^2$ that

$$\psi_{c^{t_{l+1}^{\varepsilon_k}}, c^{t_l^{\varepsilon_k}}}(\xi) \geq \left(1 - \frac{\eta}{4}\right)(t_{l+1}^{\varepsilon_k} - t_l^{\varepsilon_k})(\log c)\,\psi(\xi). \tag{5.29}$$

By the properties of $I_{\varepsilon_k}^j$, it is moreover true that if ε_k is small enough, for every $i \in I_{\varepsilon_k}^j$ there exists a subset $J_i^{j,\varepsilon_k} \subset \{1, \ldots, \lfloor N|\log\varepsilon|^{1-\delta}\rfloor\}$ such that $\#J_i^{j,\varepsilon_k} \geq (N-2)|\log\varepsilon|^{1-\delta}$ and for each $l \in J_i^{j,\varepsilon_k}$ none of the balls in $P_i^{s_j^{\varepsilon_k}}(t_l^{\varepsilon_k})$ merges between $t_l^{\varepsilon_k}$ and $t_{l+1}^{\varepsilon_k}$. For a visualization, see Figure 5.5. Hence, we may estimate for $i \in I_{\varepsilon_k}^j$ (note that the occurring annuli are pairwise disjoint by construction)

$$\begin{aligned}\int_{B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})} \frac{1}{2}\mathcal{C}\tilde{\beta}_k : \tilde{\beta}_k\,dx &\geq \sum_{l \in J_i^{j,\varepsilon_k}} \sum_{B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k}) \in P_i^{s_j^{\varepsilon_k}}(t_l^{\varepsilon_k})} \int_{B_m^{j,\varepsilon_k}(t_{l+1}^{\varepsilon_k}) \setminus B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k})} \frac{1}{2}\mathcal{C}\tilde{\beta}_k : \tilde{\beta}_k\,dx \\ &\geq \sum_{l \in J_i^{j,\varepsilon_k}} \sum_{B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k}) \in P_i^{s_j^{\varepsilon_k}}(t_l^{\varepsilon_k})} \psi_{R_m^{\varepsilon_k}(t_{l+1}^{\varepsilon_k}), R_m^{\varepsilon_k}(t_l^{\varepsilon_k})}(\nu_k(B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k}))) \\ &\geq \sum_{l \in J_i^{j,\varepsilon_k}} \sum_{B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k}) \in P_i^{s_j^{\varepsilon_k}}(t_l^{\varepsilon_k})} \left(1 - \frac{\eta}{4}\right)(t_{l+1}^{\varepsilon_k} - t_l^{\varepsilon_k})(\log c)\,\psi(\nu_k(B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k}))).\end{aligned} \tag{5.30}$$

For the last inequality, we used (5.29) and that by construction it holds $\frac{R_m^{\varepsilon_k}(t_{l+1}^{\varepsilon_k})}{R_m^{\varepsilon_k}(t_l^{\varepsilon_k})} = c^{t_{l+1}^{\varepsilon_k} - t_l^{\varepsilon_k}}$. Next, note that $\sum_{B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k}) \in P_i^{s_j^{\varepsilon_k}}(t_l^{\varepsilon_k})} \nu_k(B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k})) = \nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))$. Hence, by the definition of φ and the fact that $\nu_k(B_m^{j,\varepsilon_k}(t_l^{\varepsilon_k})) \in \mathbb{S}$ we can further estimate

$$\begin{aligned}&\geq \left(1 - \frac{\eta}{4}\right) \sum_{l \in J_i^{j,\varepsilon_k}} \frac{s_j^{\varepsilon_k}}{N|\log\varepsilon_k|^{1-\delta}}(\log c)\,\varphi(\nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))) \\ &\geq \left(1 - \frac{\eta}{4}\right)(N-2)|\log\varepsilon_k|^{1-\delta} \frac{s_j^{\varepsilon_k}}{N|\log\varepsilon_k|^{1-\delta}}\,\varphi(\nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))) \\ &\geq \left(1 - \frac{\eta}{4}\right)\frac{N-2}{N}\left(\alpha - \gamma - \frac{\eta}{4}\right)|\log\varepsilon_k|\,\varphi(\nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))).\end{aligned}$$

The simple estimate $\left(1 - \frac{\eta}{4}\right)x \geq x - \frac{\eta}{4}$ for $0 < x < 1$ yields

$$\geq \frac{N-2}{N}\left(\alpha - \gamma - \frac{\eta}{2}\right)|\log\varepsilon_k|\,\varphi(\nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))). \tag{5.31}$$

Finally, we choose N so large that $\frac{N-2}{N}(\alpha - \gamma - \frac{\eta}{2}) \geq (\alpha - \gamma - \eta)$.
As the family $(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))_{i \in I_{\varepsilon_k}^j}$ consists of pairwise disjoint balls, we can sum over the estimate in (5.30) - (5.31) to find that

$$\begin{aligned}&\int_{A_{\varepsilon_k}^j} \frac{1}{2}\mathcal{C}\tilde{\beta}_k : \tilde{\beta}_k\,dx \\ \geq &(\alpha - \gamma - \eta)|\log\varepsilon_k| \sum_{i \in I_{\varepsilon_k}^j} \varphi(\nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})))\end{aligned} \tag{5.32}$$

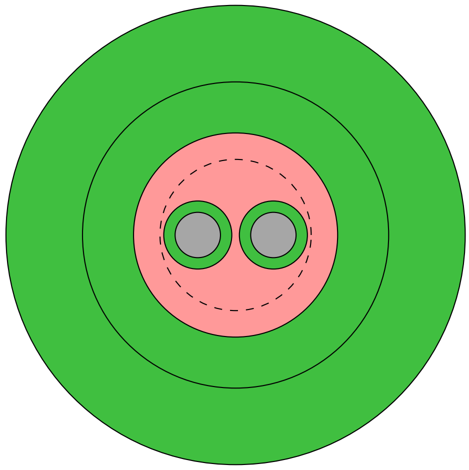


Figure 5.5: Sketch of the situation in case 1 in the proof of Proposition 5.6.1 for a ball $B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})$ such that $i \in I^j_{\varepsilon_k}$. The starting balls included in $B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})$ are drawn in gray. Every dark line marks the state of the ball construction for a certain time $t_l^{\varepsilon_k}$. If there is no merging for a ball included in $B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})$ between $t_l^{\varepsilon_k}$ and $t_{l+1}^{\varepsilon_k}$ the corresponding annuli are drawn in green, otherwise in red. The dashed line indicates a merging.

$$\begin{aligned}
&\geq (\alpha-\gamma-\eta)|\log\varepsilon_k|\,\varphi\left(\sum_{i\in I^j_{\varepsilon_k}} \nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k}))\right)\\
&\geq (\alpha-\gamma-\eta)|\log\varepsilon_k|\left(\varphi(\nu_k(A^j_{\varepsilon_k})) - \varphi\left(\sum_{i\in I^j_{\varepsilon_k}} \nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})) - \nu_k(A^j_{\varepsilon_k})\right)\right). \qquad (5.33)
\end{aligned}$$

Here, we used the subadditivity of φ for the last and last but second inequality. Now, note that by the 1-homogeneity of φ and the properties of $I^j_{\varepsilon_k}$ we may derive that

$$\begin{aligned}
\varphi\left(\sum_{i\in I^j_{\varepsilon_k}} \nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})) - \nu_k(A^j_{\varepsilon_k})\right) &\leq \left(\max_{\xi\in S^1}\varphi(\xi)\right)\left|\sum_{i\in I^j_{\varepsilon_k}} \nu_k(B_i^{j,\varepsilon_k}(s_j^{\varepsilon_k})) - \nu_k(A^j_{\varepsilon_k})\right| \qquad (5.34)\\
&\leq \left(\max_{\xi\in S^1}\varphi(\xi)\right)\delta\,|\nu_k(A^j_{\varepsilon_k})|\\
&\leq \delta\,\frac{\max_{\xi\in S^1}\varphi(\xi)}{\min_{\xi\in S^1}\varphi(\xi)}\,\varphi(\nu_k(A^j_{\varepsilon_k})). \qquad (5.35)
\end{aligned}$$

Combining (5.32) - (5.33), (5.34) - (5.35), and(5.25), using the inequality $(\alpha-\gamma-\eta)(1-\tilde{\delta}) \geq \alpha-\gamma-\eta-\tilde{\delta}$, and recalling that $\nu_k(A^j_{\varepsilon_k}) = \mu_k(A^j_{\varepsilon_k})$ proves the claim in the first case.

Case 2 : $|\mu_k(A^j_{\varepsilon_k})| = |\nu_k(A^j_{\varepsilon_k})| \leq |\log\varepsilon|^{1-\delta}$.
We only need to consider those $A^j_{\varepsilon_k}$ such that $F_{\varepsilon_k}(\mu_k,\beta_k,A_{\varepsilon_k}j) < \frac{\alpha-\gamma-\eta-\tilde{\delta}}{|\log\varepsilon_k|}\varphi(\mu_k(A^j_{\varepsilon_k}))$ (otherwise the desired lower bound is immediate). The 1-homogeneity and continuity of φ yields that

$$F_{\varepsilon_k}(\mu_k,\beta_k,A^j_{\varepsilon_k}) \leq C\,\frac{\alpha-\gamma-\eta-\tilde{\delta}}{|\log\varepsilon_k|}\,|\mu_k(A^j_{\varepsilon_k})| \leq C(\alpha-\gamma-\eta-\tilde{\delta})\,|\log\varepsilon_k|^{-\delta}$$

Hence, we can apply Proposition 5.4.1 to $\mu_k,\beta_k,\ A^j_{\varepsilon_k},\ \alpha,\gamma,\delta$ as fixed before and $K = C$ where C is the universal constant from the estimate above. We obtain a function $\bar{\beta}_k : A^j_{\varepsilon_k} \to \mathbb{R}^{2\times 2}$ and a family of balls $(D_i^{j,\varepsilon_k})_{i\in I^j_{\varepsilon_k}}$ satisfying the conclusions of Proposition 5.4.1. In particular, the radius of the balls in $(D_i^{j,\varepsilon_k})_{i\in I^j_{\varepsilon_k}}$ is less than ε^α,

$$\# I^j_{\varepsilon_k} \leq K(\alpha)|\log \varepsilon_k|^{1-\delta}, \text{ and } \operatorname{curl} \bar{\beta}_k = 0 \text{ on } A^j_{\varepsilon_k} \setminus \bigcup_{i\in I^j_{\varepsilon_k}} D_i^{j,\varepsilon_k}.$$

Moreover, the strains $\bar{\beta}_k$ satisfy

$$\frac{1}{|\log \varepsilon|^2} \int_{A^j_{\varepsilon_k}} \frac{1}{2}\mathcal{C}\bar{\beta}_k : \bar{\beta}_k \, dx \leq \left(1 + \frac{C(\alpha)}{|\log \varepsilon_k|}\right) F_{\varepsilon_k}(\mu_k, \beta_k, A^j_{\varepsilon_k}).$$

Let us consider a ball construction associated to some $c > 1$ not depending on ε_k or j starting with the balls $(D_i^{j,\varepsilon_k})_{i\in I^j_{\varepsilon_k}}$ as long as for the constructed balls it holds that $B_i^{j,\varepsilon_k}(t) \cap \partial A^j_{\varepsilon_k} = \emptyset$. As the number of starting balls is bounded by $K(\alpha)|\log \varepsilon_k|^{1-\delta}$, obviously the number of occurring merging times during the ball construction is also bounded by $K(\alpha)|\log \varepsilon_k|^{1-\delta}$. Hence, we can argue as in case 1 until (5.30) - (5.31) to prove the claim also in this case (in this case we do not need the additional $\tilde{\delta}$ on the right hand side of the desired estimate).

Armed with the statement of the claim we can now prove the lower bound close to the dislocations.

Let $x^j_{\varepsilon_k} \in A^j_{\varepsilon_k}$ and define the measure $\tilde{\mu}_k = \frac{1}{|\log \varepsilon_k|} \sum_{j\in J_{\varepsilon_k}} \mu_k(A^j_{\varepsilon_k})\delta_{x^j_{\varepsilon_k}}$. By the statement of the claim and the 1-homogeneity of φ, it is clear that $\tilde{\mu}_k$ is a bounded sequence of measures. Analogously to the proof of the compactness theorem in Section 5.5 one can show that $\tilde{\mu}_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^2)$. Writing the estimate (5.26) of the claim in terms of $\tilde{\mu}_k$ leads to

$$\left(1 + \frac{C(\alpha, M)}{|\log \varepsilon|}\right) \sum_{j\in J_{\varepsilon_k}} F_{\varepsilon_k}(\mu_k, \beta_k, A^j_{\varepsilon_k}) \geq (\alpha - \gamma - \eta - \tilde{\delta}) \int_\Omega \varphi\left(\frac{d\tilde{\mu}_k}{d|\tilde{\mu}_k|}\right) d|\tilde{\mu}_k|,$$

which implies that

$$\left(1 + \frac{C(\alpha, M)}{|\log \varepsilon|}\right) F_{\varepsilon_k}(\mu_k, \beta_k, U_{\varepsilon_k}) \geq (\alpha - \gamma - \eta - \tilde{\delta}) \int_\Omega \varphi\left(\frac{d\tilde{\mu}_k}{d|\tilde{\mu}_k|}\right) d|\tilde{\mu}_k|.$$

It follows from Reshetnyak's theorem that

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k, \beta_k, U_{\varepsilon_k}) \geq (\alpha - \gamma - \eta - \tilde{\delta}) \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

Letting $\alpha \to 1, \eta \to 0$, and $\delta \to 0$ yields

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k, \beta_k, U_{\varepsilon_k}) \geq (1 - \gamma) \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

Combining the bounds far and close to the dislocations we find

$$\liminf_{k\to\infty} F_{\varepsilon_k}(\mu_k, \beta_k) \geq \int_\Omega \frac{1}{2}\mathcal{C}\beta : \beta \, dx + (1 - \gamma) \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

Finally, $\gamma \to 0$ finishes the proof of the lower bound. □

5.7 The lim sup-inequality

In this section, we prove the lim sup-inequality of the Γ-convergence result in Theorem 5.2.1. In the case of well-separated dislocations and linearized elasticity, the existence of a recovery sequence for the energy F is known, see [38, Theorem 12]. The difference to our setting is that the approximating energies in our case F_{ε_k} carry the extra term $\frac{|\mu|(\Omega)}{|\log \varepsilon|^2}$. Hence, the strategy in the proof of the lim sup-inequality is to show that for the recovery sequences from the case of well-separated dislocations the term associated to the total variation penalization vanishes in the limit.

Proposition 5.7.1 (The lim sup-inequality)**.** *Let* $\varepsilon_k \to 0$ *and* $(\mu, \beta) \in \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2\times 2})$. *There exists* $(\mu_k, \beta_k)_k \subset \mathcal{M}(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^{2\times 2})$ *such that*

(i) $\frac{\mu_k}{|\log \varepsilon_k|} \to \mu$ *in the flat topology and* $\frac{\beta_k}{|\log \varepsilon_k|} \rightharpoonup \beta$ *in* $L^2(\Omega; \mathbb{R}^{2\times 2})$,

(ii) $\limsup_{k\to\infty} F_{\varepsilon_k}(\mu_k, \beta_k) \le F(\mu, \beta)$.

Proof. Let $(\mu_k, \beta_k)_k$ be the recovery sequence from the well-separated case, see [38, Proof of Theorem 12]. To get an idea for the construction one can also imagine our construction for $R = Id$ in the well-separated, nonlinear case in section 4.3, Proposition 4.3.5, and divide the constructed sequences by ε.

Note that for the measures μ_k it holds by construction that $\frac{\mu_k}{|\log \varepsilon_k|}$ converges weakly∗ in the sense of measures to μ. This implies flat convergence. Moreover, the sequence (μ_k, β_k) satisfies that $\mu_k \in X(\Omega)$, $\beta_k \in \mathcal{AS}^{lin}_{\varepsilon_k}(\Omega)$, and

$$\limsup_{k\to\infty} \frac{1}{|\log \varepsilon_k|^2} \int_{\Omega_{\varepsilon_k}(\mu_k)} \frac{1}{2}\mathcal{C}\beta_k : \beta_k \, dx \le F(\mu, \beta).$$

Hence, we only need to show that $\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|^2} |\mu_k|(\Omega) = 0$.

Of course, we only have to consider (μ, β) such that $F(\mu, \beta) < \infty$. In this situation, the well-separateness of dislocations along the recovery sequence allows to prove with the help of Lemma 5.3.2 that for $\mu_k = \sum_{i=1}^{L_k} \xi_i^k \, \delta_{x_i^k}$ for $\xi_i^k \in \mathbb{S}$ and $x_i^k \in \Omega$ it holds that the quantity $\frac{1}{|\log \varepsilon|} \sum_{i=1}^{L_k} |\xi_i^k|^2$ is uniformly bounded in k. As the non-zero elements in $\mathbb{S}$ are bounded away from zero, this implies immediately that

$$\frac{1}{|\log \varepsilon|^2} |\mu|(\Omega) \to 0.$$

This finishes the proof. □

5.A Scaling of Korn's Constant for Singular Fields on Thin Annuli

In this section, we prove what was already discussed in section 5.3, namely the blow-up of Korn's constant on thin annuli in two dimensions. The optimal constant for the classical Korn's inequality for an annulus $B_R \setminus B_r$ of ratio $\frac{R}{r}$ was computed in [26] by analyzing the solvability of the corresponding Euler-Lagrange equations. It is given by a rather complicated expression that converges to the optimal constant for the disc (4, see [64]) as $\frac{R}{r} \to 0$ and behaves like $\left(1 - \frac{R}{r}\right)^{-2}$ as $\frac{R}{r} \to 1$. In this section, we show that the optimal constant for the inequality for curl-free fields with a circulation

condition has the same scaling as $\frac{R}{r} \to 1$.
As the constant for Korn's inequality is invariant under dilations of the domain, we will only consider the case $R = 1, r < 1$.

The technique to prove an upper bound for the constant is a covering argument which is classical for proving Korn's inequality for thin domains. We show first a uniform bound on the constant for special subsets of the annulus.

Lemma 5.A.1. *For $r < 1$ define the set $U_r = \{t(\sin\theta, -\cos\theta) : r < 1 < 1, |\theta| < \frac{1-r}{2}\}$. There exist $r_0 < 1$ and a constant $C > 0$ such that for every $r_0 < r < 1$ and every $u \in H^1(U_r;\mathbb{R}^2)$ there exists a skew-symmetric matrix W such that*

$$\int_{U_r} |\nabla u - W|^2\, dx \leq C \int_{U_r} |(\nabla u)_{sym}|^2\, dx.$$

Proof. Let us define the square $\tilde{U}_r = \left(\frac{r-1}{2}, \frac{1-r}{2}\right) \times (0, 1-r)$ and the function $f_r : \tilde{U}_r \to U_r$ by $f_r(\theta, t) = (1-t)(\sin\theta, -\cos\theta)$, see figure 5.6. Clearly, f_r is a diffeomorphism and its derivative is

$$\nabla f_r(t,\theta) = \begin{pmatrix} (1-t)\cos\theta & (1-t)\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

It holds $|\nabla f_r(t,\theta) - Id| \leq \mathcal{O}(r)$ where $\mathcal{O}(r) \to 0$ as $r \to 1$.
Now, let $u \in H^1(U_r;\mathbb{R}^2)$ such that $\int_{U_r} (\nabla u)_{skew}\, dx = 0$. Define $\tilde{u} : \tilde{U}_r \to \mathbb{R}^2$ by $\tilde{u}(\theta,t) = u(f_r(\theta,t))$. As ∇f_r is almost the identity matrix, the symmetric part of $\nabla\tilde{u}$ is approximately $(\nabla u)_{sym} \circ f_r$. Indeed, by the chain rule we find

$$|(\nabla\tilde{u})_{sym} - (\nabla u)_{sym} \circ f_r| \leq \mathcal{O}(r)|((\nabla u) \circ f_r)_{sym}| \leq \mathcal{O}(r)|(\nabla u) \circ f_r|. \tag{5.36}$$

By Korn's inequality applied to $\tilde{u}$, there exists a skew-symmetric matrix W and a constant $K \geq 1$ such that

$$\int_{\tilde{U}_r} |\nabla\tilde{u} - W|^2\, dx \leq K \int_{\tilde{U}_r} |(\nabla\tilde{u})_{sym}|^2\, dx.$$

Note that by scaling the constant does not depend on r. By (5.36), we find further

$$\begin{aligned}\int_{\tilde{U}_r} |(\nabla\tilde{u})_{sym}|^2\, dx &\leq 2\int_{\tilde{U}_r} |(\nabla u)_{sym} \circ f_r|^2 + \mathcal{O}(r)^2|(\nabla u) \circ f_r|^2\, dx \\ &= 2\int_{U_r} (|(\nabla u)_{sym}|^2 + \mathcal{O}(r)^2|\nabla u|^2)|det\nabla f_r^{-1}|.\end{aligned}$$

On the other hand, one proves similarly that

$$\int_{\tilde{U}_r} |\nabla\tilde{u} - W|^2\, dx \geq \frac{1}{2}\int_{U_r} (|\nabla u|^2 - \mathcal{O}(r)^2|\nabla u|^2)|det\nabla f_r^{-1}|\, dx,$$

where one uses that from $\int_{U_r} (\nabla u)_{skew}\, dx = 0$ it follows $\int_{U_r} |\nabla u - W|^2\, dx \geq \int_{U_r} |\nabla u|^2\, dx$.
Notice that for r close to 1 the gradient ∇f_r is uniformly close to the identity. The same holds for ∇f_r^{-1}. Choose r_0 so close to 1 such that for all smaller r it holds that $|det\nabla f_r^{-1} - 1| \leq \frac{1}{2}$ and

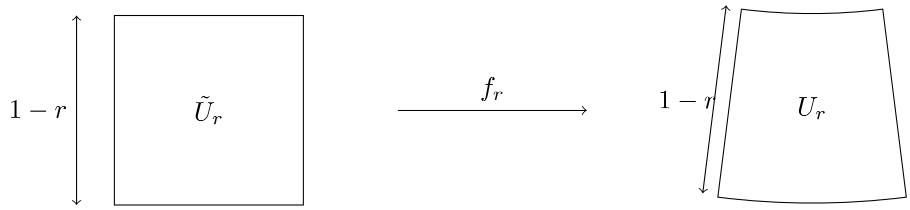


Figure 5.6: Sketch of the situation in Lemma 5.A.1.

$\mathcal{O}(r) \leq \frac{1}{4\sqrt{K}}$. Then, one obtains, by combining the previous inequalities,

$$\int_{U_r} |\nabla u|^2 \, dx \leq 12K \int_{U_r} |(\nabla u)_{sym}|^2 \, dx + \frac{13}{16} \int_{\Omega} |\nabla u|^2 \, dx.$$

Absorbing the very right term to the left hand side ends the proof. □

Remark 5.A.1. The same holds true for any rotated version of U_r with the same constant since Korn's constant does not depend on the choice of coordinates.

Remark 5.A.2. For an explicit bound on the constructed constant in the previous lemma, note that an upper bound for the optimal constant for the square was computed in [49], namely $8+4\sqrt{2}$. In [48], it is conjectured that the optimal constant for a square is 7.

In the following lemma, we state an upper bound on Korn's constant for the annulus $B_1(0) \setminus B_r(0)$ for curl-free fields.

Lemma 5.A.2. *There exists $r_0 < 1$ such that for all $r_0 < r < 1$ Korn's constant $K(r)$ for $B_1(0) \setminus B_r(0)$ is less or equal than $C\,(1-r)^{-2}$ where $C > 0$ is a universal constant. Precisely, for every function $\beta \in L^2(B_1(0) \setminus B_r(0); \mathbb{R}^{2\times 2})$ satisfying $\operatorname{curl} \beta = 0$ there exists a skew-symmetric matrix $W \in \mathbb{R}^{2\times 2}$ such that*

$$\int_{B_1(0)\setminus B_r(0)} |\beta - W|^2 \, dx \leq C\,(1-r)^{-2} \int_{B_1(0)\setminus B_r(0)} |\beta_{sym}|^2 \, dx.$$

Proof. Let r_0 as in Lemma 5.A.1 and $0 < r < r_0$.
In analogy to the previous lemma, we define the sets

$$U_r^k = \left\{ t(\cos\theta, \sin\theta) : r < t < 1 \text{ and } \frac{k-1}{2}(1-r) < \theta < \frac{k+1}{2}(1-r) \right\} \text{ where } k = 1, \dots, \left\lfloor \frac{4\pi}{1-r} \right\rfloor = L_r.$$

As $\frac{\lfloor \frac{4\pi}{1-r} \rfloor + 1}{2}(1-r) > 2\pi$, it follows $B_1(0) \setminus B_r(0) \subset \bigcup_{k=1}^{L_r} U_r^k$. Moreover, notice that for all $k = 1, \dots, L_r - 1$ it holds $\frac{|U_r^k|}{|U_r^k \cap U_r^{k+1}|} = 2$ and $\sum_{k=1}^{L_r} \mathbf{1}_{U_k^r} \leq 2$.
Now, let $\beta \in L^2(B_1(0) \setminus B_r(0); \mathbb{R}^{2\times 2})$ such that $\operatorname{curl} \beta = 0$. In particular, β can be written as a gradient on each U_k^r. For each $k = 1, \dots, L_r$, we apply Lemma 5.A.1 and Remark 5.A.1 on U_r^k to obtain a skew-symmetric matrix W_k such that

$$\int_{U_r^k} |\beta - W_k|^2 \, dx \leq C \int_{U_r^k} |\beta_{sym}|^2 \, dx. \tag{5.37}$$

Note that $C > 0$ does not depend on k, r nor β.
For every $k = 1, \dots, L_r - 1$, the distance between W_k and W_{k+1} can be estimated as follows

$$|W_k - W_{k+1}|^2 \leq 2 \fint_{U_r^k \cap U_r^{k+1}} |W_k - \beta|^2 + |W_{k+1} - \beta|^2 \, dx \leq \frac{4C}{|U_r^k \cap U_r^{k+1}|} \int_{U_r^k \cup U_r^{k+1}} |\beta_{sym}|^2 \, dx.$$

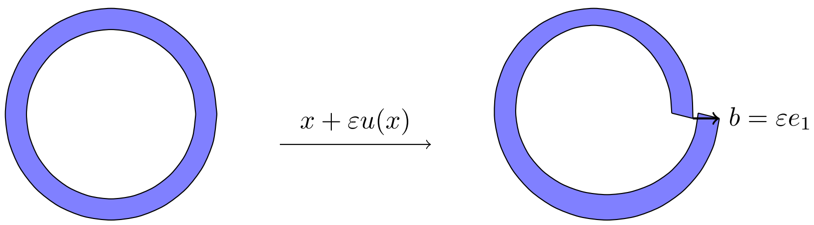


Figure 5.7: Sketch of the almost optimal displacement constructed in the proof of Lemma 5.A.3 for $\xi = e_1$, $\varepsilon = \frac{1}{4}$ and $r = \frac{4}{5}$.

Consequently, we obtain for $1 \le k < l \le L_r$

$$|W_k - W_l|^2 = \left|\sum_{i=k}^{l-1} W_i - W_{i+1}\right|^2 \le (l-k)\sum_{i=k}^{l-1} |W_i - W_{i+1}|^2 \le L_r \sum_{i=1}^{L_r-1} \frac{4C}{|U_r^i \cap U_r^{i+1}|} \int_{U_r^i \cup U_r^{i+1}} |\beta_{sym}|^2 \, dx. \tag{5.38}$$

We define $W = W_1$. Then, we derive from (5.37) and (5.38) the following chain of inequalities

$$\begin{aligned}\int_{B_1(0)\setminus B_r(0)} |\beta - W|^2 &\le 2\sum_{k=1}^{L_r} \int_{U_r^k} |\beta - W_k|^2 + |W - W_k|^2 \, dx \\ &\le 2\sum_{k=1}^{L_r} \left(C\int_{U_r^k} |\beta_{sym}|^2 \, dx + L_r \sum_{i=1}^{L_r-1} 4C \frac{|U_r^i|}{|U_r^i \cap U_r^{i+1}|} \int_{U_r^i \cup U_r^{i+1}} |\beta_{sym}|^2 \, dx \right) \\ &\le 4C \int_{B_1(0)\setminus B_r(0)} |\beta_{sym}|^2 \, dx + 16C\, L_r^2 \sum_{i=1}^{L_r-1} \int_{U_r^i \cup U_r^{i+1}} |\beta_{sym}|^2 \, dx \\ &\le C(4 + 64L_r^2) \int_{B_1(0)\setminus B_r(0)} |\beta_{sym}|^2 \, dx.\end{aligned}$$

Note that by definition $L_r^2 \le \frac{16\pi^2}{(1-r)^2}$ and C is a universal constant. □

Finally, we construct for each $\xi \in \mathbb{R}^2$ a function which is curl-free, whose circulation is exactly ξ, and whose elastic energy is optimal in scaling, see Figure 5.7.

Lemma 5.A.3. *Let $\xi \in \mathbb{R}^2$. There exists $0 < r_0 < 1$ such that for all $r_0 < r < 1$ and $\xi \in \mathbb{R}^2$ there exists a function $\beta : B_1(0) \setminus B_r(0) \to \mathbb{R}^{2\times 2}$ such that $\operatorname{curl}\beta = 0$ and $\int_{\partial B_1} \beta \cdot \tau \, d\mathcal{H}^1 = \xi$ and*

$$\min_{W \in Skew(2)} \int_{B_1(0)\setminus B_r(0)} |\beta - W|^2 \, dx \ge c\,(1-r)^{-2} \int_{B_1(0)\setminus B_r(0)} |\beta_{sym}|^2 \, dx. \tag{5.39}$$

The constant c does not depend on r, ξ or β.

Proof. Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Moreover, we write $e_\rho(\theta) = (\cos\theta, \sin\theta)$ and $e_\theta(\theta) = (-\sin\theta, \cos\theta)$. We define the function $u : B_1(0) \setminus B_r(0) \to \mathbb{R}^2$ in polar coordinates by

$$u(\rho e_\rho(\theta)) = \frac{1}{\pi}\left(\int_0^\theta (\xi_1\cos\varphi + \xi_2 \sin\varphi) e_\rho(\varphi)\, d\varphi + (1-\rho)(\xi_1 \cos\theta + \xi_2 \sin\theta) e_\theta(\theta) \right).$$

For a visualization, see Figure 5.7.

The function u has a jump on the line $\{\theta = 0\}$ of height ξ. Hence, the absolutely continuous part of the derivative of u, $\beta = \nabla u$, satisfies the circulation condition $\int_{B_1} \beta \cdot \tau \, d\mathcal{H}^1 = \xi$ and $\operatorname{curl} \beta = 0$ in $B_1(0) \setminus B_r(0)$. In addition, we can compute explicitly

$$
\begin{aligned}
\beta =& \frac{1}{\pi} \Bigg(\frac{1}{\rho} (\xi_1 \cos\theta + \xi_2 \sin\theta) \, e_\theta(\theta) \otimes e_\rho(\theta) - (\xi_1 \cos\theta + \xi_2 \sin\theta) \, e_\rho(\theta) \otimes e_\theta(\theta) \\
& - \frac{1-\rho}{\rho} (\xi_1 \cos\theta + \xi_2 \sin\theta) \, e_\theta(\theta) \otimes e_\rho(\theta) + \frac{1-\rho}{\rho} (-\xi_1 \sin\theta + \xi_2 \cos\theta) \, e_\theta(\theta) \otimes e_\theta(\theta) \Bigg) \\
=& \frac{1}{\pi} \Bigg(- (\xi_1 \cos\theta + \xi_2 \sin\theta) \, e_\rho(\theta) \otimes e_\theta(\theta) + (\xi_1 \cos\theta + \xi_2 \sin\theta) \, e_\theta(\theta) \otimes e_\rho(\theta) \Bigg) \\
& + \frac{1}{\pi} \frac{1-\rho}{\rho} (-\xi_1 \sin\theta + \xi_2 \cos\theta) \, e_\theta(\theta) \otimes e_\theta(\theta) \\
=& \beta_{skew} + \beta_{sym}.
\end{aligned}
$$

One sees directly that $\int_{B_1(0)\setminus B_r(0)} \beta_{skew} \, dx = 0$. Hence, $\min_{W \in Skew(2)} \int_{B_1(0)\setminus B_r(0)} |\beta - W|^2 \, dx = \int_{B_1(0)\setminus B_r(0)} |\beta|^2 \, dx$. In addition, a straightforward computation show that as $r \to 1$ we obtain

$$
\begin{aligned}
& \int_{B_1(0)\setminus B_r(0)} |\beta_{skew}|^2 \, dx = (1 - r^2) \, \frac{1}{\pi} |\xi|^2 \sim 2(1-r) \frac{1}{\pi} |\xi|^2 \\
\text{and } & \int_{B_1(0)\setminus B_r(0)} |\beta_{sym}|^2 \, dx = \left(-\log(r) - 2(1-r) + \frac{1}{2}(1-r^2) \right) \frac{1}{\pi} |\xi|^2 \sim \frac{1}{3} (1-r)^3 \, \frac{1}{\pi} |\xi|^2,
\end{aligned}
$$

which proves the claim for r close enough to 1 independently of ξ.

□

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