Class forcing and second-order arithmetic

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Regula Krapf
aus
Zürich, die Schweiz

Bonn, Oktober 2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 12. Januar 2017

Erscheinungsjahr: 2017

Summary

We provide a framework in a generalization of Gödel-Bernays set theory for performing class forcing. The forcing theorem states that the forcing relation is a (definable) class in the ground model (definability lemma) and that every statement that holds in a class-generic extension is forced by a condition in the generic filter (truth lemma). We prove both positive and negative results concerning the forcing theorem. On the one hand, we show that the definability lemma for one atomic formula implies the forcing theorem for all formulae in the language of set theory to hold. Furthermore, we introduce several properties which entail the forcing theorem. On the other hand, we give both counterexamples to the definability lemma and the truth lemma. In set forcing, the forcing theorem can be proved for all forcing notions by constructing a unique Boolean completion. We show that in class forcing the existence of a Boolean completion is essentially equivalent to the forcing theorem and, moreover, Boolean completions need not be unique.

The notion of pretameness was introduced to characterize those forcing notions which preserve the axiom scheme of replacement. We present several new characterizations of pretameness in terms of the forcing theorem, the preservation of separation, the existence of nice names for sets of ordinals and several other properties. Moreover, for each of the aforementioned properties we provide a corresponding characterization of the Ord-chain condition.

Finally, we prove two equiconsistency results which compare models of ZFC (with large cardinal properties) and models of second-order arithmetic with topological regularity properties (and determinacy hypotheses). We apply our previous results on class forcing to show that many important arboreal forcing notions preserve the $\mathbf{\Pi}^1_1$-perfect set property over models of second-order arithmetic and also give an example of a forcing notion which implies the $\mathbf{\Pi}^1_1$-perfect set property to fail in the generic extension.

Zusammenfassung

Wir führen Klassenforcing im axiomatischen Rahmen einer Verallgemeinerung von Gödel-Bernays-Mengenlehre ein. Das Forcing-Theorem besagt, dass die Forcingrelation eine (definierbare) Klasse im Grundmodell ist (Definierbarkeitslemma), und dass jede Aussage in einer generischen Erweiterung von einer Bedingung im generischen Filter erzwungen wird (Wahrheitslemma). Wir beweisen sowohl positive als auch negative Resultate über das Forcing-Theorem. Einerseits zeigen wir, dass das Definierbarkeitslemma für eine einzige atomare Formel reicht, um das Forcing-Theorem für alle Formeln in der Sprache der Mengenlehre zeigen. Außerdem stellen wir mehrere kombinatorische Eigenschaften von Klassenforcings vor, welche das Forcing-Theorem implizieren. Andrerseits präsentieren wir Gegenbeispiele für das Definierbarkeitslemma sowie für das Wahrheitslemma im Kontext von Klassenforcing. Im Mengenforcing ist das Forcing-Theorem eine Konsequenz der Existenz einer eindeutigen Booleschen Vervollständigung. Wir zeigen, dass im Klassenforcing die Existenz einer Booleschen Vervollständigung im Wesentlichen äquivalent zum Forcing-Theorem ist, und dass Boolesche Vervollständiungen im Allgemeinen nicht eindeutig sind.

Pretameness ist eine Eigenschaft von Klassenforcings, welche definiert wurde um die Erhaltung des Ersetzungsaxioms zu charakterisieren. Wir beweisen mehrere neue Charakterisierungen von Pretameness anhand des Forcing-Theorems, der Erhaltung des Aussonderungsaxioms, der Existenz von Nice Names für Mengen von Ordinalzahlen sowie weiteren Eigenschaften von Klassenforcings. Des Weiteren verwenden wir alle diese Eigenschaften um die Ord-Kettenbedingung zu charakterisieren.

Zu guter Letzt geben wir zwei Äquikonsistenzresultate an, welche Modelle von ZFC (mit grossen Kardinalzahlen) und Modelle der zweistufigen Arithmetik mit topologischer Regularität (und Determiniertheit) vergleichen. Wir wenden unsere Resultate über Klassenforcing an um nachzuweisen, dass zahlreiche wichtige Beispiele von Baumforcings die $\mathbf{\Pi}^1_1$-perfekte-Teilmengeneigenschaft über Modelle der zweistufigen Arithmetik erhalten. Andrerseits erläutern wir ein Beispiel eines Klassenforcings, welches die $\mathbf{\Pi}^1_1$-perfekte-Teilmengeneigenschaft in generischen Erweiterungen zerstört.

Acknowledgements

On a general note, I would like to acknowledge the support which has been offered to me by the Hausdorff Center of Mathematics, University of Bonn, and the Bonn International Graduate School of Mathematics (BIGS).

Firstly, I would like to express my gratitude to my advisor Prof. Dr. Peter Koepke for teaching me both in set theory and in formal mathematics as well as for the many inspiring discussions not only about my research but also about foundational and philosophical issues related to set theory and formal mathematics, and for giving me the opportunity to gain teaching experience at University of Bonn. Furthermore, I would like to thank the mathematical logic group of Bonn for their friendship, moral support and many informal coffee-break discussions which have lead us to consider many interesting open problems and fruitful research. In particular, I would like to show my gratitude to Peter Holy and Philipp Schlicht for many intriguing conversations about class forcing which have resulted in several joint papers. Many results presented in this thesis have originated in this research project. It has been a great pleasure to learn from them and work with them. They have also made many suggestions which have helped me to improve this thesis. On a further note, I would also like to thank Philipp Lücke and Ana Njegomir who have also worked on the first part of our project on class forcing. Moreover, Ana has been a wonderful officemate and friend and it has helped me a lot to share both the successful and frustrating moments during our common time as PhD students. Additionally, I would like to mention the further (former) members of the mathematical logic group of Bonn, Thilo Weinert, Ioanna Dimitriou and Aleksandra Kwiatkowska; I have really enjoyed to share common lunch and coffee breaks and discuss about set theory and formal mathematics, but also many other topics which are not related to mathematics.

I would also like to thank Lorenz Halbeisen for his lecture and seminars during my master studies at University of Zürich and ETH Zürich and for arousing my interest in logic and set theory. It was due to him that I was motivated to apply for the PhD program at University of Bonn.

Furthermore, I would like to thank my friends and fellow PhD students in Bonn for many common coffee breaks and enjoyable evenings, in particular Ana, Gennady, Michal, Lisa, Christian and Duygu. For their moral support I would also like to thank my long-term friends from Switzerland Yvonne, Karin, Anita and Eliane. Last but not least, I would like to show gratitude to my parents Johanna and Bernhard Krapf for their support and for always believing in me, as well as my brothers Tobias and Beat. A final thanks goes to my partner Claudius Ruch for the many wonderful moments that we have spent together besides my work as a PhD student.

Contents

Chapter 0

Introduction

This thesis is mainly concerned with class forcing and its applications to second order arithmetic. The starting point of this research project was the unpublished paper [KM07] in which it is shown that within a model of second-order arithmetic (SOA) with sufficient topological regularity, Gödel's constructible universe L is in fact a model of ZFC. Conversely, given a countable transitive model M of ZFC, using class forcing to collapse all cardinals to ω, one obtains a model of SOA in which full projective topological regularity holds; i.e. every projective set of reals satisfies the perfect set property (PSP), the Baire property and is Lebesgue measurable. This is essentially an adaptation of Solovay's proof [Sol70] that by collapsing an inaccessible cardinal, the inner model $\mathsf{L}(\mathbb{R})$ has the topological regularity properties mentioned above. This works, since Ord^M is very similar to an inaccessible cardinal. A very natural question was thus to ask whether the methods used can be generalized to compare models of SOA with additional determinacy hypotheses and models of ZFC with large cardinals. The first step here is to consider $\mathbf{\Pi^1_1}$-determinacy on the one hand, and inner models with sharps for sets of ordinals on the other hand. We have solved this problem by proving that ZFC+"every set of ordinals has a sharp" has the same consistency strength as $\mathsf{SOA} + \mathbf{\Pi^1_1}$-determinacy+$\mathbf{\Pi^1_2}$-$\mathsf{PSP}$.

Since the theory $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{PSP}$ is equiconsistent with ZFC, it can be considered a candidate for an alternative axiomatization of mathematics. Second-order arithmetic provides a very natural theory which – in many ways – can be conceived as closer to "real" mathematics than ZFC which breaks down everything to one predicate, namely the $\in$-relation. This lies in the fact that the focus is laid on the natural numbers and reals, and arithmetic functions such as addition and multiplication are included in the language. However, the method of forcing which has been used to solve many important problems such as Hilbert's first problem[1] (the Continuum Hypothesis), has proven to be a very powerful tool in set theory. Namely, if M is a countable transitive model of ZFC and $M[G]$ is a (set) forcing extension of M, then it is again a model of ZFC. This has motivated me to investigate whether this can be extended to $\mathsf{SOA} + \mathbf{\Pi^1_1}$- PSP. However, in second- order arithmetic it can easily be checked that the only set-sized forcing notion is given by Cohen forcing, and therefore one should consider class forcing instead. This has lead to many foundational questions about class forcing.

We now take one step back and recall the development of the method of forcing. It was

[1]see [Hil00]

first used by Paul Cohen in [Coh63] to construct a model of ZFC where the Continuum Hypothesis fails. The general idea is that one defines some set-sized partial order $\mathbb{P}$ in a model M of set theory and adds a generic filter G, i.e. a filter whose intersection with every dense subset of $\mathbb{P}$ that is an element of M is non-empty. The new model $M[G]$ then satisfies again the axioms of ZFC and has additional properties which are "forced" by the elements of G. However, there are many important statements which cannot be decided using set-sized partial orders. One example is the Generalized Continuum Hypothesis (GCH) which states that for every ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Easton has shown in [Eas70] that using class-sized partial orders instead of set-sized ones, one can force not only the GCH but also its failure by forcing arbitrary values of the continuum function. However, one can easily see that the axioms of ZFC are – in general – not preserved under class forcing. In particular, the axiom schemes of separation and replacement, as well as the power set axiom may fail in class-generic extensions. This easy observation has lead to the question whether and under what conditions other fundamental properties of set forcing carry over to class forcing. This meant opening Pandora's box, since there was hardly any previous research about these foundational topics.

As a first step, in a joint project with Peter Holy, Philipp Lücke, Ana Njegomir and Philipp Schlicht, we have gathered many counterexamples to classical results in set forcing. One such result is the so-called *forcing theorem*, which states both that the forcing relation is definable in the ground model (the *definability lemma*) and that every statement which holds in a generic extension is forced by a condition in the generic filter (the *truth lemma*). We have presented counterexamples to both the definability and the truth lemma. Other examples of properties that may fail for class forcing include the existence of a (unique) Boolean completion, the property of producing the same generic extensions as dense suborders and the existence of nice names for sets of ordinals. Most of these results are given in the joint paper [HKL+16]. However, a failure of any of the aforementioned properties is problematic, since the resulting models will fail to satisfy many desirable statements and many classical proofs in set forcing cannot be transferred to class forcing. Therefore, as a second step we have investigated which forcing notions fulfill all of those properties. Sy Friedman has introduced in [Fri00] the notion of *pretameness* which essentially characterizes the preservation of replacement. It has turned out that pretameness can, in fact, be characterized by all fundamental properties of class-sized forcing notions that we have considered so far. The results of this project are presented in the paper [HKS16a] which is joint with Peter Holy and Philipp Schlicht. Further research has focused on the preservation of separation in class forcing extensions and can be found in [HKS16b]. Finally, we use our general results on class forcing to prove that for a large class of forcing notions the $\mathbf{\Pi}^1_1$-PSP is preserved. However, this fails to hold in general.

The outline given above shows how the various topics considered in this thesis are interrelated and which motivation lead us to investigate them. However, this chronological introduction does not correspond to the structure of the thesis. In the following, we briefly summarize the results presented in each chapter. Most results of Chapters 1-4 form part of the joint papers [HKL+16],[HKS16b] and [HKS16a].

The first chapter outlines a general setting for class forcing. Notably, we work in a second-order context which allows for more classes than just the definable ones. We consider countable transitive models of the form $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^- which essentially consists of the axioms of ZF^- for sets, the scheme of collection which allows class parameters, as well as foundation, extensionality and first-order class comprehension for classes. This generalizes the classical setting of models M of (a subtheory of) ZFC equipped with the classes which are definable over M, models of Gödel-Bernays set theory GB (resp GBC, if one assumes the existence of a global well-order) and Kelley-Morse class theory KM. We introduce several strengthenings of our general setting which are required for many proofs in the later chapters, among them the existence of a set-like well-order[2] and a class version of dependent choice. We provide a detailed account on class forcing and present some examples of class-sized forcing notions such as the partial order for forcing a global well-order, the forcing notion $\mathrm{Col}(\omega, \mathrm{Ord})$ which collapses the ordinals to ω and Easton forcing.

In Chapter 2 we are concerned with the forcing theorem. In a slightly informal way, we define the forcing theorem as follows.

Definition. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a model of GB^- and let $\varphi(v_0, \ldots, v_{m-1})$ denote a formula in the language $\mathcal{L}_\in$ of set theory with class name parameters.

(1) We say that a partial order $\mathbb{P} \in \mathcal{C}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$ if

$$\{\langle p, \sigma_0, \ldots, \sigma_{m-1} \rangle \in M \mid p \Vdash \varphi(\sigma_0, \ldots, \sigma_{m-1})\} \in \mathcal{C}.$$

(2) We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$ if for all $\mathbb{P}$-names $\sigma_0, \ldots, \sigma_{m-1}$ and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with $M[G] \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G)$, there is $p \in G$ with $p \Vdash \varphi(\sigma_0, \ldots, \sigma_{m-1})$.

(3) We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$ if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

In Theorem 2.1.5 we show that every notion of class forcing which satisfies the definability lemma for one of the atomic formulae "$v_0 = v_1$" and "$v_0 \in v_1$" already satisfies the forcing theorem for all $\mathcal{L}_\in$-formulae with class parameters. In Sections 2.2, 2.3 and 2.4 we present three properties which imply the forcing theorem to hold. The first one is given by *pretameness* as defined below.

Definition. A partial order $\mathbb{P}$ is said to be *pretame* for $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$, if for every $p \in \mathbb{P}$ and for every sequence of dense classes $\langle D_i \mid i \in I \rangle \in \mathcal{C}$ with $I \in M$, there is $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I \rangle \in M$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense in $\mathbb{P}$ below q.

Pretameness was introduced by Sy Friedman in [Fri00] in order to characterize those partial orders which preserve the axiom scheme of replacement. The second property is the so-called *set decision property* which characterizes those forcing notions which do not add any new sets. Thirdly, *approachability by projections* is a combinatorial property which generalizes the property of being the union of set-sized complete subforcings. Finally, in Section 2.5 we present both failures of the definability lemma (see Corollary 2.5.4) and the truth lemma (see Theorem 2.5.11). The idea is that we add a set-sized binary relation E on ω by forcing such that $\langle \omega, E \rangle$ is isomorphic to the ground model $\langle M, \in \rangle$. The forcing theorem would then imply the existence of a first-order truth predicate which, however, is impossible in the case that the classes are just the definable ones.

[2] i.e. a global well-order whose initial segments are set-sized

An alternative approach to forcing is given by considering Boolean-valued models. This relies on the fact that in set forcing, ever partial order can be embedded into a complete Boolean algebra, its so-called *Boolean completion*, which has the same generic extensions as the original partial order. Moreover, the Boolean completion of a set-sized partial order is unique up to isomorphisms. Chapter 3 investigates to what extent these results carry over to the context of class forcing. In Section 3.2 we show that in class forcing, the existence of a Boolean completion is essentially equivalent to the forcing theorem. In Section 3.3 we prove that in Kelley-Morse class theory KM every class-sized partial order has a Boolean completion. This provides an alternative proof of a result in [Ant15] which states that in KM every class-sized partial order satisfies the forcing theorem. Our proof makes use of the fact that in KM one can perform recursion for proper classes. In Section 3.4 we show that unions of set-sized complete subforcings always have a Boolean completion. Finally, in Section 3.5 we prove that unlike in set forcing, Boolean completions of class-sized partial orders need not be unique.

Chapter 4 contains the main results of this thesis. It shows how one can characterize pretameness in terms of many properties which are always satisfied in the context of set forcing but may fail for class forcing, such as the preservation of the axiom scheme of separation, the forcing theorem, the existence of a Boolean completion and the existence of nice names for sets of ordinals. Moreover, many of the above mentioned properties have a stronger counterpart which is equivalent to the Ord-chain condition, i.e. the statement that all antichains are set-sized.

Notation. Let $\mathbb{M} \models \mathsf{GB}^-$ and let Ψ be some property of a partial order $\mathbb{P} \in \mathcal{C}$ for $\mathbb{M} = \langle M, \mathcal{C} \rangle$. We say that $\mathbb{P}$ *densely* satisfies Ψ if every notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$, for which there is a dense embedding in $\mathcal{C}$ from $\mathbb{P}$ into $\mathbb{Q}$, satisfies the property Ψ.

The following two theorems summarize the results of Chapter 4. Regarding the results of Theorem 1 below, (1) is (in a slightly less general context) due to Sy Friedman (see [Fri00]).

Theorem 1. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^- such that $\mathcal{C}$ contains a set-like well-order of M, and let $\mathbb{P}$ be a class-sized partial order in $\mathcal{C}$. The following properties (over $\mathbb{M}$) are equivalent to the pretameness of $\mathbb{P}$ over $\mathbb{M}$, where we additionally require the non-existence of a first-order truth predicate for (4) and (5), and for (7) we assume that $\mathbb{M} \models \mathsf{KM}$, which is notably incompatible to the assumptions used for (4) and (5).*

(1) *$\mathbb{P}$ preserves GB^-/collection/replacement.*
(2) *$\mathbb{P}$ satisfies the forcing theorem and preserves separation.*
(3) *$\mathbb{P}$ satisfies the forcing theorem and does not add a cofinal/surjective/bijective function from some $\gamma \in \mathrm{Ord}^M$ to Ord^M.*
(4) *$\mathbb{P}$ densely satisfies the forcing theorem.*
(5) *$\mathbb{P}$ densely has a Boolean completion.*
(6) *$\mathbb{P}$ satisfies the forcing theorem and produces the same generic extensions as $\mathbb{Q}$ for every forcing notion $\mathbb{Q}$ such that $\mathcal{C}$ contains a dense embedding from $\mathbb{P}$ into $\mathbb{Q}$.*[3]

[3] More precisely, if $\pi : \mathbb{P} \to \mathbb{Q}$ is a dense embedding in $\mathcal{C}$ and G is $\mathbb{Q}$-generic, then $M[G] = M[\pi^{-1}[G] \cap \mathbb{P}]$.

(7) $\mathbb{P}$ *densely has the property that every set of ordinals in any of its generic extensions has a nice name.*

Theorem 2. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of* GBC *and let $\mathbb{P}$ be a class-sized partial order in $\mathcal{C}$ that satisfies the forcing theorem. The following properties (over $\mathbb{M}$) are equivalent:*

(1) $\mathbb{P}$ *satisfies the* Ord-*cc.*
(2) $\mathbb{P}$ *satisfies the maximality principle.*[4]
(3) $\mathbb{P}$ *densely has a unique Boolean completion.*
(4) $\mathbb{P}$ *has a Boolean completion $\mathbb{B}$ such that every subclass of $\mathbb{B}$ in $\mathcal{C}$ has a supremum in $\mathbb{B}$.*
(5) *If there are $\mathbb{Q}, \pi \in \mathcal{C}$ such that π is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$ and $\sigma \in M^{\mathbb{Q}}$, then there is $\tau \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \pi(\tau)$.*
(6) $\mathbb{P}$ *densely has the property that whenever $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \subseteq \check{\alpha}$ for some $\sigma \in M^{\mathbb{P}}$ and $\alpha \in \mathrm{Ord}^M$ then there is a nice $\mathbb{P}$-name τ such that $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$.*

Chapter 5 is dedicated to comparing second order arithmetic with additional topological regularity and determinacy hypotheses with models of Zermelo-Fraenkel set theory. In Section 5.1 we recall the proof that second order arithmetic is equiconsistent with ZFC^-. In the next section, we show – following [KM07] – that models of $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{PSP}$ have inner models of ZFC, where for a collection Γ of definable classes of reals Γ-PSP is the statement that every uncountable class of reals in Γ has a perfect subclass. Building on this, we prove that models of $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{Det} + \mathbf{\Pi^1_2}$-$\mathsf{PSP}$ have inner models of $\mathsf{ZFC}^\#$, the theory ZFC enhanced with the statement that every set of ordinals has a sharp. In Section 5.3 we use a class version of the Lévy collapse to prove the converse to the previously mentioned results, showing the equiconsistency of $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{PSP}$ with ZFC and $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{Det} + \mathbf{\Pi^1_2}$-$\mathsf{PSP}$ with $\mathsf{ZFC}^\#$. Last but not least, in Section 5.4 we investigate class forcing over models of SOA with dependent choice.[5] In particular, we conclude from results of Castiblanco and Schlicht [CS16] that for a large class of pretame forcing notions, the $\mathbf{\Pi^1_1}$-perfect set property is preserved, but there is also a pretame forcing notion $\mathbb{P}$ such that in every $\mathbb{P}$-generic extension the $\mathbf{\Pi^1_1}$-PSP fails. This essentially uses reshaping and almost disjoint coding.

Finally, in Chapter 6 we present a selection of open questions that we find particularly interesting for further research.

Our set-theoretical notation is standard and follows standard textbooks on set theory such as [Jec03] and [Kan09].

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[4]See Definition 4.2.6.

[5]All models obtained by collapsing the ordinals of a model of ZFC using the class version of the Lévy collapse satisfy dependent choice.

Chapter 1

A general setting for class forcing

In this chapter, we outline the axiomatic frameworks that are adequate for class forcing and introduce the basic concepts of class forcing. In the end, we present several examples of forcing notions which will accompany us throughout this thesis.

1.1 Axiomatic frameworks for class forcing

In the following, we will present several foundational (second-order) frameworks of set theory.

1.1.1 Subsystems of ZFC

Zermelo-Fraenkel set theory (ZFC) is the most commonly used axiomatization of set theory. We will, however, often work with subsystems of ZFC. In particular, we are interested in ZF^-, the theory obtained from ZFC by deleting both the power set axiom and the axiom of choice. In this context, one has to be careful, since without the power set axiom, the different formulations of the axioms of replacement and choice are no longer equivalent. For more details on this topic, consult [GHJ11].

By ZF^- we mean the axioms of extensionality, foundation, pairing, union, infinity and the axiom schema of collection which we will specify below. By $\mathcal{L}_\in$ we denote the language of set theory, i.e. the first-order language with equality and the binary predicate symbol $\in$.

Definition 1.1.1. Let $\varphi(x, x_0, \ldots, x_{n-1}), \psi(x, y, x_0, \ldots, x_{n-1})$ be $\mathcal{L}_\in$-formulae. We define the following *axiom schemata*:

(1) $\forall x_0 \ldots \forall x_n \forall a \exists y \forall x\, [x \in y \leftrightarrow x \in a \wedge \varphi(x, x_0, \ldots, x_{n-1})]$ (*Separation*).
(2) $\forall x_0 \ldots \forall x_n \forall a\, [\forall x \in a \exists! y\, \psi(x, y, x_0, \ldots, x_{n-1}) \rightarrow$
$\exists b \forall x \in a \exists! y \in b\, \psi(x, y, x_0, \ldots, x_{n-1})]$ (*Replacement*).
(3) $\forall x_0 \ldots \forall x_n \forall a\, [\forall x \in a \exists y\, \psi(x, y, x_0, \ldots, x_{n-1}) \rightarrow$
$\exists b \forall x \in a \exists y \in b\, \psi(x, y, x_0, \ldots, x_{n-1})]$ (*Collection*).

It follows from an easy argument that collection implies replacement and replacement implies separation. Moreover, if the power set axiom holds, then replacement implies collection. This implication may, however, fail in the absence of the power set axiom. For examples of models where replacement holds but collection fails, consult [GHJ11]. In order to avoid such pathological models, in ZF^- we therefore include the strongest axiom, namely collection.

We will also be interested in the theory ZFC^-, the theory ZF^- enhanced by adding a choice principle. Here again, we have to be careful which principle we would like to add, since in ZF^- not all commonly used choice principles are equivalent. This is discussed in detail in [Zar82]. For the sake of completeness, we state two of the main choice principles.

Definition 1.1.2. We define the following axioms.

(1) $\forall x\, [\emptyset \notin x \to \exists f : x \to \bigcup x (\forall y \in x\, f(y) \in y)]$ (*axiom of choice*),
(2) *Every set can be well-ordered* (*well-ordering principle*).

Since the well-ordering principle is the strongest form of choice – in particular, it implies the axiom of choice as well as Zorn's lemma – in ZF^-, we denote by ZFC^- the theory obtained from ZF^- by adding the well-ordering principle. Moreover, ZF is the theory ZF^- enhanced by the power set axiom and, correspondingly, ZFC is the theory ZFC^- enhanced by the power set axiom.

When we perform class forcing over a model of ZF^-, we will always restrict ourselves to those classes which are definable over the given model. It will become clear that this is just a special case of the setting presented in 1.1.3.

1.1.2 Second-order arithmetic

Second-order arithmetic (SOA) axiomatizes the natural numbers (first-order part) as well as sets of natural numbers (second-order part) which are considered as reals. The language of SOA (denoted $\mathcal{L}_2$) is a two-sorted language which uses lowercase letters $n, m, \ldots$ for variables denoting natural numbers and capital letters $X, Y, \ldots$ for sets of natural numbers, i.e. reals. Furthermore, there are first-order constants $0, 1$, first-order binary function symbols $+, *$, first-order binary relation symbols $<, =$ and a binary relation symbol $\in$ between first- and second-order variables to denote elementhood.

The axioms of SOA also have a first-order part which consists of the axioms of Robinson arithmetic (see [Rob50]) and the following second-order axioms resp. schemes:

(1) $\forall X\, [0 \in X \wedge \forall n\, (n \in X \to n+1 \in X) \to \forall n\, (n \in X)]$ (*induction axiom*).
(2) For any $\mathcal{L}_2$-formula $\varphi(n)$ such that X is not free in φ:

$$\exists X \forall n\, [n \in X \leftrightarrow \varphi(n)] \quad (\textit{axiom of comprehension})$$

(3) For any $\mathcal{L}_2$-formula $\varphi(n, X)$:

$$\forall n \exists X\, \varphi(n, X) \to \exists X \forall n\, \varphi(n, (X)_n) \quad (\textit{axiom of choice})$$

where $(X)_n = \{m | (n, m) \in X\}$ and $(\, , \,)$ denotes Gödel pairing which is definable in SOA. Clearly, $(X)_n$ exists by comprehension.

Remark 1.1.3. In the scheme of comprehension, the assumption that X does not appear as a free variable of φ is significant because otherwise one could define the set

$$\exists X \forall n (n \in X \leftrightarrow n \notin X)$$

which leads to Russell's paradox.

1.1.3 Gödel-Bernays set theory

The simplest setting to perform class forcing is to work in (a subsystem) of ZFC and consider just the definable classes. However, it is more natural to work in a setting which allows for more second-order objects. This leads us to *Gödel-Bernays set theory* GB and its weakenings and strengthenings. As in the context of ZFC, we are interested in a theory which does not include the power set axiom.

Following [HKL$^+$16] we denote by GB^- the theory in the two-sorted language $\mathcal{L}^2_\in$ with (lowercase) variables for sets and (uppercase) variables for classes, with the set axioms given by ZF^- with class parameters allowed in the axiom schemata of separation and collection, and the class axioms given by extensionality and foundation for classes as well as *first-order class comprehension* as defined below.

Definition 1.1.4. By *first-order class comprehension* we denote the following axiom schema. For every $\mathcal{L}_\in$-formula $\varphi(x, X_0, \ldots, X_{n-1})$ with class parameters $X_0, \ldots, X_{n-1}$ but without class quantifiers,

$$\forall X_0 \ldots \forall X_{n-1} \exists Y \forall x \, [x \in Y \leftrightarrow \varphi(x, X_0, \ldots, X_{n-1})].$$

By GB we denote the theory GB^- enhanced with the power set axiom. Frequently, we need to add choice either for sets of for classes. The weakest form is to just add the set version of choice; the theory obtained from GB^- (resp. GB) by adding the axiom of choice shall simply be denoted GBc^- (resp. GBc). Furthermore, we denote by GBC the theory GB enhanced by the axiom of *global choice*. Here again, there are many different versions which are not all equivalent in GB^-.

Definition 1.1.5. By the principle of *global choice* we mean the statement.

$$\exists F : M \setminus \{\emptyset\} \to M \; \forall x (x \neq \emptyset \to F(x) \in x),$$

where M is the class of all sets.

A variant of the axiom of global choice is the existence of certain global well-orders.

Definition 1.1.6. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a model of GB^-.

(1) By a *global well-order* for $\mathbb{M}$ we mean a well-order $\prec \in \mathcal{C}$ of M.
(2) We say that a global well-order $\prec$ for $\mathbb{M}$ is *set-like*, if for every set $x \in M$, we have $\{y \in M \mid y \prec x\} \in M$.

Note that if $\prec$ is a set-like well-order then its ordertype is Ord^M. Conversely, it is easy to check that every well-order of ordertype Ord^M is set-like.

The following observation is folklore and is also noted in [Ham14b].

Fact 1.1.7. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a model of GB. Then the following statements are equivalent:

(1) $\mathbb{M}$ satisfies global choice.
(2) There is a bijection $M \to \mathrm{Ord}^M$ in $\mathcal{C}$.
(3) $\mathcal{C}$ contains a global well-order.
(4) $\mathcal{C}$ contains a set-like well-order of M.
(5) $\mathcal{C}$ contains a global well-order of ordertype Ord^M.

Proof. We start by proving that (1) implies (2). Assume that $F : M \setminus \{\emptyset\} \to M$ is a global choice function. We recursively define a bijective map $G : \mathrm{Ord}^M \to M$ as follows. Suppose that $G(\beta)$ is given for all $\beta < \alpha$. Now let $\gamma \in \mathrm{Ord}^M$ be the minimal ordinal such that $\mathsf{V}_\gamma \nsubseteq \{G(\beta) \mid \beta < \alpha\}$. Then put $G(\alpha) = F(\mathsf{V}_\gamma \setminus \{G(\beta) \mid \beta < \alpha\})$. By construction, G is injective. To see that G is also surjective, suppose that $x \in M$. But then $x \in \mathsf{V}_\alpha$ for some ordinal α. Now if $x \notin \mathrm{ran}(G)$ then $\mathrm{ran}(G) \subseteq \mathsf{V}_\alpha$, a contradiction. Next we prove that (2) implies (4). Suppose that $F : M \to \mathrm{Ord}^M$ is a bijection. Then we can well-order M by

$$x \prec y \Longleftrightarrow F(x) < F(y).$$

By construction, $\prec$ is set-like. Notice that as remarked above, (4) and (5) are equivalent. Furthermore, (4) implies (3), so it remains to check that (1) follows from (3). Let $\prec$ be a global well-order in $\mathcal{C}$. Then we can define a global choice function F by taking $F(x)$ to be the $\prec$-least element of x for every nonempty set $x \in M$. □

The following folklore result states that the consistency strength of GBC is that of ZFC.

Remark 1.1.8. If M is a transitive model of ZFC and $\mathrm{Def}(M)$ denotes the collection of classes which are definable over M, then $\langle M, \mathrm{Def}(M) \rangle$ is a model of GB. This shows that GBC is conservative over ZFC, i.e. every statement that GBC proves about sets can already be proven in ZFC. To see that adding the axiom of global choice to GB does not enhance the consistency strength, note that a global well-order can be added using class forcing. This shall be discussed in Section 1.3.2.

Furthermore, when performing class forcing over models of GB^- we sometimes need additional assumptions on our models. The next definition was introduced in [HKL+16].

Definition 1.1.9. We say that a model $\langle M, \mathcal{C} \rangle$ of GB^- satisfies *representatives choice*, if for every equivalence relation $E \in \mathcal{C}$ there is $A \in \mathcal{C}$ and a surjective map $\pi : \mathrm{dom}(E) \to A$ in $\mathcal{C}$ such that $\langle x, y \rangle \in E$ if and only if $\pi(x) = \pi(y)$.

Representatives choice is useful to obtain partial orders from preorders. More precisely, suppose that representatives choice holds in $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$ and let $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle \in \mathcal{C}$ be a preorder. By considering the equivalence relation

$$p \approx q \quad \text{iff} \quad p \leq_{\mathbb{P}} q \wedge q \leq_{\mathbb{P}} p,$$

we obtain a partial order $\mathbb{Q} \in C$ and a surjective map $\pi : \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ such that for all $p, q \in \mathbb{P}$, $p \approx q$ if and only if $\pi(p) = \pi(q)$.

Clearly, representatives choice follows from the existence of a global well-order. Furthermore, if M satisfies the power set axiom, then we also obtain representatives choice, since by Scott's trick we have sets

$$[p] = \{q \in \mathbb{P} \mid q \approx p \wedge \forall r\, [q \approx r \rightarrow \mathsf{rnk}(q) \leq \mathsf{rnk}(r)]\} \in M$$

for $p \in \mathbb{P}$, where for $x \in M$, $\mathsf{rnk}(x)$ denotes the von Neumann rank of x.

A stronger assumption that we will frequently make generalizes both the existence of a set-like well-order and the power set axiom.

Definition 1.1.10. A model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^- *has a hierarchy* if there is $C \in \mathcal{C}$ such that

(1) $C \subseteq \mathrm{Ord}^M \times M$;
(2) For each $\alpha \in \mathrm{Ord}^M$, $C_\alpha = \{x \in M \mid \exists \beta < \alpha(\langle \beta, x \rangle \in C)\} \in M$;
(3) If $\alpha < \beta$ in Ord^M then $C_\alpha \subseteq C_\beta$;
(4) $M = \bigcup_{\alpha \in \mathrm{Ord}^M} C_\alpha$.

If C defines a hierarchy on M, then the *C-rank* of $x \in M$, denoted $\mathsf{rnk}_C(x)$, is the least $\alpha \in \mathrm{Ord}^M$ such that $x \in C_{\alpha+1}$.

The simplest examples of GB^--models which have a hierarchy are models of GB, i.e. models of GB^- which satisfy the power set axiom. Their hierarchy is provided by the von Neumann hierarchy C. In that case, we simply write $\mathsf{rnk}(x)$ for $\mathsf{rnk}_C(x)$. A second example is given by models $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$ such that $\mathcal{C}$ contains a set-like well-order $\prec$. Then

$$C = \{\langle \alpha, x \rangle \mid \alpha \in \mathrm{Ord}^M \wedge x \prec \alpha\}$$

witnesses that $\mathbb{M}$ has a hierarchy.

Remark 1.1.11. If $\mathbb{M} \models \mathsf{GB}^-$ has a hierarchy and $\mathcal{C}$ contains a well-order of M then $\mathcal{C}$ contains a set-like well-order of M. To see this, let $C \in \mathcal{C}$ be a hierarchy on M and let $\prec \in \mathcal{C}$ be a well-order of M. Then we obtain a set-like well-order $\lhd$ of M by stipulating $x \lhd y$ if and only if $\mathsf{rnk}_C(x) < \mathsf{rnk}_C(y)$ or $\mathsf{rnk}_C(x) = \mathsf{rnk}_C(y) = \alpha$ for some $\alpha \in \mathrm{Ord}^M$ and $x \prec y$.

Remark 1.1.12. Observe that if $\mathbb{M}$ does not have a hierarchy, then the existence of global well-order of M in $\mathcal{C}$ does not imply that $\mathcal{C}$ contains a global well-order of ordertype Ord^M. To see this, consider a countable transitive model N of $\mathsf{ZFC}+\neg\mathsf{CH}$ such that $H(\omega_1)^N$ has a definable well-order. Note that assuming the existence of a countable transitive model of ZFC, the existence of such a model N is guaranteed by [Har77]. Now let $M = H(\omega_1)^N$ and let $\mathcal{C}$ denote $\mathrm{Def}(M)$. Then $\langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$ and $\mathcal{C}$ contains a well-order of M. However, since CH fails in N, $\mathcal{C}$ cannot contain a well-order of M of ordertype Ord^M.

Another choice scheme that we will sometimes require is the following.

Definition 1.1.13. (1) Let κ be a cardinal in M and $\varphi(x, y, X_0, \ldots, X_{n-1})$ an $\mathcal{L}_\in$-formula with class parameters $X_0, \ldots, X_{n-1}$. Then $\mathsf{DC}_\kappa(\varphi)$ states that

$$\forall \vec{x} \exists y\, \varphi(\vec{x}, y, X_0, \ldots, X_{n-1}) \rightarrow \forall z \exists \vec{x}\, [x_0 = z \wedge \forall i < \kappa\, \varphi(\vec{x} \upharpoonright i, x_i, X_0, \ldots, X_{n-1})],$$

where $\vec{x}$ denotes a sequence $\langle x_i \mid i < j \rangle$ for some $j \leq \kappa$.

(2) By DC_κ we denote the scheme that for every $\mathcal{L}_\in$-formula φ with class parameters, $\mathsf{DC}_\kappa(\varphi)$ holds.
(3) The *dependent choice scheme* DC states that for every M-cardinal κ, DC_κ holds.

Note that in [GHJ11] it is shown that in ZFC^-, DC_ω is equivalent to the reflection principle which states that for any formula φ and any set x there is a transitive set y such that $x \subseteq y$ and φ is absolute between M and y. In ZFC, the reflection principle clearly holds witnessed by the von Neumann hierarchy. However, it is an open question whether it holds in ZFC^-, posed by Zarach in [Zar96].

We will denote by GBdc^- the theory GB^- enhanced with the axiom of choice (for sets) and DC.

1.1.4 Kelley-Morse class theory

Gödel-Bernays class theory allows only first-order class comprehension, i.e. the quantification in the comprehension axiom is restricted to sets. If we omit this constraint, we obtain *Kelley-Morse class theory* (KM). The resulting theory is no longer conservative over ZFC and its consistency strength exceeds that of ZFC.

More precisely, the set axioms of KM are given by extensionality, foundation, pairing, union, infinity and power set. Furthermore, KM includes the class axioms of foundation and extensionality for classes, global choice as well as replacement and class comprehension as defined below.

Definition 1.1.14.

(1) If F is a class function and x is a set then, $\{F(y) \mid y \in x\}$ is a set (*Replacement*).
(2) If $\varphi(x, X_0, \ldots, X_{n-1})$ is a second-order formula with class parameters in which quantification is allowed both over sets and classes, then

$$\forall X_0 \ldots \forall X_{n-1} \exists Y \forall x\, [x \in Y \leftrightarrow \varphi(x, X_0, \ldots, X_{n-1})] \quad (\textit{class comprehension}).$$

In the literature there can be found alternative axiomatizations of KM, the most common amongst which replaces the axioms of replacement and global choice by an axiom called *limitation of size* which postulates that a class C is a proper class if and only if it can be mapped injectively into the universe of all sets. However, both axiomatizations can be shown to be equivalent (see [Git14]). The advantage of Kelley-Morse class theory is that it allows full induction and recursion, even for formulae with class quantifiers.

Concerning the consistency strength of KM, it is well known that KM proves $\mathsf{Con}(\mathsf{ZFC})$. The idea of the proof is that in KM one always has a first-order truth predicate.[1] On the other hand, if κ is an inaccessible cardinal then it is straightforward to check that $\langle \mathsf{V}_\kappa, \mathsf{V}_{\kappa+1} \rangle$ is a model of KM.

1.2 Class forcing

We now outline a general setting for class forcing in a second-order context. We will, in general, force over models of GB^-. This approach is convenient, since it generalizes

[1] see [Ham14a]; an alternative proof of the existence of a truth predicate is also given by corollary 3.3.3).

forcing over models of ZFC (with the classes given by the definable ones) as well as models of KM.

From now on, we fix a model V of ZFC. By a *countable transitive model* of GB^- (or some other second-order theory extending GB^-) we mean a model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^- such that M is transitive and both M and $\mathcal{C}$ are countable in V. For any recursively enumerable theory T extending GB^-, we say that "$\mathbb{M}$ is a model of T" to abbreviate the statement that $\mathbb{M}$ satisfies every axiom of T in V with respect to some formalized satisfaction relation (as in [Dra74, Chapter 3.5]). Note that, in general, the assumption that such a model V of ZFC containing a transitive countable model M of T exists is stronger than the consistency of T. However, if $\mathsf{T} \supseteq \mathsf{ZF}^-$, then the results of this thesis can also be proven in the setting of [Kun80, Ch. VII, §9, Approach (1b)], where one works with a language extending $\mathcal{L}_\in$ by a constant symbol $\dot{\mathbb{M}}$ and a model of a theory that extends T by the axiom scheme stating that every axiom of T holds relativized to $\dot{\mathbb{M}}$. The consistency of this theory is equivalent to the consistency of T. Nevertheless, we have chosen our approach because it provides more intuitive arguments and easier formulations.

The following account on class forcing is the one presented in [HKL+16].

We fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^-. By a *notion of class forcing* (for $\mathbb{M}$) we mean a preorder $\mathbb{P} = \langle P, \leq_\mathbb{P} \rangle$ such that $P, \leq_\mathbb{P} \in \mathcal{C}$, i.e. $\mathbb{P}$ is reflexive and transitive. If not mentioned otherwise, we will further assume that $\mathbb{P}$ has a $\leq_\mathbb{P}$-maximal element denoted by $\mathbb{1}_\mathbb{P}$.[2] If $\mathbb{P}$ is additionally antisymmetric, we shall call $\mathbb{P}$ a *partial order*. We will frequently identify $\mathbb{P}$ with its domain P and may thus write $p \in \mathbb{P}$ instead of $p \in P$. We call elements of $\mathbb{P}$ *conditions* of $\mathbb{P}$.

Definition 1.2.1. Let $\mathbb{P} = \langle P, \leq_\mathbb{P} \rangle$ be a notion of class forcing for $\mathbb{M}$ and $p, q \in \mathbb{P}$.

(1) We say that p is *stronger than* q or that p *extends* q, if $p \leq_\mathbb{P} q$.
(2) We say that p and q are *compatible*, denoted $p \parallel_\mathbb{P} q$, if there is $r \in \mathbb{P}$ with $r \leq_\mathbb{P} p, q$.
(3) We say that p and q are *incompatible*, denoted $p \perp_\mathbb{P} q$, if p and q are not compatible.

Definition 1.2.2. Let $\mathbb{P} = \langle P, \leq_\mathbb{P} \rangle$ be a notion of class forcing for $\mathbb{M}$ and let $D \subseteq \mathbb{P}$. Then we say that D is

(1) *dense* in $\mathbb{P}$, if for every $p \in \mathbb{P}$ there is $q \leq_\mathbb{P} p$ with $q \in D$.
(2) *predense* in $\mathbb{P}$, if for every $p \in \mathbb{P}$ there is $q \in D$ which is compatible with p.
(3) *open* in $\mathbb{P}$, if whenever $p \in D$ and $q \leq_\mathbb{P} p$ then $q \in D$.
(4) *dense below* $p \in \mathbb{P}$ (resp. *predense below* p), if for every $q \leq_\mathbb{P} p$ there is $r \in D$ with $r \leq_\mathbb{P} q$ (resp. $r \parallel_\mathbb{P} q$).

Definition 1.2.3. Let $\mathbb{P} = \langle P, \leq_\mathbb{P} \rangle$ be a notion of class forcing.

(1) We say that a class $A \subseteq \mathbb{P}$ is an *antichain*, if all elements of A are incompatible.
(2) An antichain A of $\mathbb{P}$ is said to be *maximal*, if there is no antichain $B \subseteq \mathbb{P}$ in $\mathcal{C}$ with $B \supsetneq A$.

Definition 1.2.4. Let $\mathbb{P} = \langle P, \leq_\mathbb{P} \rangle$ be a notion of class forcing.

[2] Given a preorder $\mathbb{P}$ without a maximal element, we can define another one by placing a new element $\mathbb{1}_\mathbb{P}$ on top. Forcing with this new preorder coincides with forcing with the original one.

(1) A class $G \subseteq \mathbb{P}$ is said to be a *filter*, if it satisfies the following two properties:
   (a) If $p \in G$ and $q \geq_{\mathbb{P}} p$ then $q \in G$ (*upwards closed*).
   (b) If $p, q \in G$ then there is $r \in G$ with $r \leq_{\mathbb{P}} p, q$ (*directed*).

(2) A filter $G \subseteq \mathbb{P}$ is said to be $\mathbb{P}$-*generic over* $\mathbb{M}$, if it meets every dense subclass of $\mathbb{P}$ which is in $\mathcal{C}$.

In the case that $\mathbb{M}$ is of the form $\langle M, \mathrm{Def}(M)\rangle$ for some ZF^--model M, we will simply say that G is $\mathbb{P}$-*generic over* M.

There are many equivalent ways to define genericity of a filter. The following useful fact is the analogue of a folklore result for set forcing over models of ZFC. However, in order to transfer this to class forcing, we need global choice instead of the axiom of choice.

Fact 1.2.5. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ and $G \subseteq \mathbb{P}$ a filter. Then the following statements are equivalent:

(1) G is $\mathbb{P}$-generic over $\mathbb{M}$.
(2) G meets every dense open subclass of $\mathbb{P}$ which is in $\mathcal{C}$.
(3) G meets every predense subclass of $\mathbb{P}$ which is in $\mathcal{C}$.
(4) G meets every subclass of $\mathbb{P}$ in $\mathcal{C}$ which is (pre)dense below some condition $p \in G$.

Moreover, if $\mathbb{M}$ satisfies global choice, then (1)-(4) are also equivalent to

(5) G meets every maximal antichain of $\mathbb{P}$ which is in $\mathcal{C}$.

Proof. The equivalence of (1)-(4) and the implication from (1) to (5) can be shown in the same way as for set forcing. It remains to check that (5) implies (1). Let G be a filter which satisfies (5) and let $D \subseteq \mathbb{P}$ be a dense class which lies in $\mathcal{C}$. Using global choice, we can choose an antichain $A \subseteq D$ which is maximal in D and hence in $\mathbb{P}$. Since G has nonempty intersection with A, it also meets D. □

Note that as in the context of set forcing, $\mathbb{P}$-generic filters never exist in the ground model $\mathbb{M}$. However, since we assume both M and $\mathcal{C}$ to be countable in V, the collection of all dense subclasses of $\mathbb{P}$ which are in $\mathcal{C}$ is countable in V, and so by a standard argument we can find a filter in V which is $\mathbb{P}$-generic over $\mathbb{M}$.

In the following, we fix a notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ for $\mathbb{M}$. We call σ a $\mathbb{P}$-*name* if all elements of σ are of the form $\langle \tau, p\rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Define $M^{\mathbb{P}}$ to be the class of all $\mathbb{P}$-names that are elements of M and define $\mathcal{C}^{\mathbb{P}}$ to be the collection of all $\mathbb{P}$-names that are elements of $\mathcal{C}$. In the following, we will usually call the elements of $M^{\mathbb{P}}$ simply $\mathbb{P}$-*names* and we will call the elements of $\mathcal{C}^{\mathbb{P}}$ *class* $\mathbb{P}$-*names*. If $\sigma \in M^{\mathbb{P}}$ is a $\mathbb{P}$-name, we define

$$\operatorname{rank} \sigma = \sup\{\operatorname{rank} \tau + 1 \mid \exists p \in \mathbb{P}(\langle \tau, p\rangle \in \sigma)\}$$

to be its *name rank*.

Now we show how to obtain class-generic extensions. Given a $\mathbb{P}$-filter G over $\mathbb{M}$ and a $\mathbb{P}$-name σ, we define the G-*evaluation* of σ as

$$\sigma^G = \{\tau^G \mid \exists p \in G(\langle \tau, p\rangle \in \sigma)\},$$

and similarly we define Γ^G for $\Gamma \in \mathcal{C}^{\mathbb{P}}$. Moreover, if G is $\mathbb{P}$-generic over $\mathbb{M}$, then we set $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$ and $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$, and call $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$ a $\mathbb{P}$-*generic extension* of $\mathbb{M}$.

The following basic observation shows that $\mathbb{M}[G]$ is indeed an extension of $M[G]$. For this, we need to define some *canonical names.* If $x \in M$ then we define by recursion

$$\check{x} = \{\langle \check{y}, \mathbb{1}_{\mathbb{P}}\rangle \mid y \in x\}.$$

Similarly, we define $\check{C}$ for $C \in \mathcal{C}$. Furthermore, we put

$$\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}.$$

Fact 1.2.6. The following statements hold for every notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ and for every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$.

(1) If $x \in M$ then $\check{x}^G = x$. In particular, $M \subseteq M[G]$.
(2) If $C \in \mathcal{C}$ then $\check{C}^G = C$ and moreover, $\dot{G}^G = G$. Therefore, $\mathcal{C} \cup \{G\} \subseteq \mathcal{C}[G]$.

Proof. Both statements follow from easy inductive arguments. □

Given an $\mathcal{L}^2_{\in}$-formula $\varphi(v_0, \dots, v_{m-1}, V_0, \dots, V_{n-1})$, set names $\vec{\sigma} \in (M^{\mathbb{P}})^m$ and class names $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$, we write

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \dots, \sigma_{m-1}, \Gamma_0, \dots, \Gamma_{n-1})$$

to denote that for every $\mathbb{P}$-generic filter G over $\mathbb{M}$ with $p \in G$,

$$\mathbb{M}[G] \models \varphi(\sigma_0^G, \dots, \sigma_{m-1}^G, \Gamma_0^G, \dots, \Gamma_{n-1}^G).$$

Whenever the context is clear, we will omit the superscript. In case the model $\mathbb{M}$ is of the form $\langle M, \mathrm{Def}(M)\rangle$, we will often write $p \Vdash^M_{\mathbb{P}} \varphi$ for $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$.

Next we present two easy observations that hold for all notions of class forcing.

Lemma 1.2.7. *Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Then the following statements hold:*

(1) *If $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$ and $q \leq_{\mathbb{P}} p$ then $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$.*
(2) *If G is $\mathbb{P}$-generic over $\mathbb{M}$, then $M[G]$ is transitive and* $\mathrm{Ord}^{M[G]} = \mathrm{Ord}^M$.

Proof. For (1) suppose that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$ and $q \leq_{\mathbb{P}} p$. But then every $\mathbb{P}$-generic filter G with $q \in G$ also contains p by upwards closure and so $\mathbb{M}[G] \models \varphi$.

For the second statement, note that every element of $\mathbb{M}[G]$ is of the form σ^G for some $\sigma \in M^{\mathbb{P}}$. Now if $x \in \sigma^G$ then there is $\langle \tau, p\rangle \in \sigma$ such that $p \in G$ and $x = \tau^G \in M[G]$. This shows that $M[G]$ is transitive. For the second statement, let $\sigma \in M^{\mathbb{P}}$ such that $\sigma^G \in \mathrm{Ord}^{M[G]}$. Observe that $\mathsf{rnk}(\sigma^G) \leq \mathsf{rnk}(\sigma)$, hence we may assume that every element of σ^G is in Ord^M and so $\sigma^G \in \mathrm{Ord}^M$. □

When forcing over models of ZFC, a fundamental result in set forcing is that the axioms of ZFC are always preserved. It will become clear in Section 1.3 that in class forcing this is not the case. However, some axioms are always preserved.

Definition 1.2.8. Since we will frequently use names for ordered pairs, we introduce the notation

$$\mathsf{op}(\sigma,\tau)=\{\langle\{\langle\sigma,\mathbb{1}_{\mathbb{P}}\rangle\},\mathbb{1}_{\mathbb{P}}\rangle,\langle\{\langle\sigma,\mathbb{1}_{\mathbb{P}}\rangle,\langle\tau,\mathbb{1}_{\mathbb{P}}\rangle\},\mathbb{1}_{\mathbb{P}}\rangle\}$$

for $\sigma,\tau\in M^{\mathbb{P}}$ and $\alpha\in\mathrm{Ord}$. Clearly, $\mathsf{op}(\sigma,\tau)$ is the canonical name for the ordered pair $\langle\sigma^G,\tau^G\rangle\in M[G]$.

Lemma 1.2.9. *Any class-generic extension of a model of* GB^- *satisfies all single axioms of* ZF^-*, that is all axioms of* ZF^- *except for possibly instances of separation, replacement and collection, and it satisfies extensionality and foundation for classes.*

Proof. Extensionality and foundation are clear both for sets and classes, since $\mathbb{M}[G]$ is an element of $\mathsf{V}\models\mathsf{ZFC}$. Infinity is also obvious and pairing is clear by Definition 1.2.8 above. Regarding unions, we consider *weak union* as the relevant axiom of ZF^-, that is the existence of a superset of $\bigcup x$ for every x. The existence of $\bigcup x$ is then an instance of the axiom of separation[3]. To verify weak union, let $\mathbb{P}$ be a notion of class forcing and let $\sigma\in M^{\mathbb{P}}$ be a $\mathbb{P}$-name. Let G be $\mathbb{P}$-generic over $\mathbb{M}$. Consider $\tau=\bigcup\mathrm{dom}(\sigma)$. Then we are done, since $\sigma^G\supseteq\bigcup\tau^G$. □

In the context of set forcing, we often encounter the notion of *complete* and *dense embeddings* of forcing notions because many properties of forcing notions are preserved under such embeddings. We will analyze which results can be generalized to class forcing.

Definition 1.2.10. Let $\mathbb{P}$ and $\mathbb{Q}$ be notions of class forcing and let $\pi:\mathbb{P}\to\mathbb{Q}$ be a map in $\mathcal{C}$. We say that π is an

(1) *embedding*, if for all $p,q\in\mathbb{P}$, if $p\leq_{\mathbb{P}}q$ then $\pi(p)\leq_{\mathbb{Q}}\pi(q)$, and if $p\perp_{\mathbb{P}}q$ then $\pi(p)\perp_{\mathbb{Q}}\pi(q)$.
(2) *complete embedding*, if it is an embedding with the property that for every maximal antichain $A\in\mathcal{C}$ of $\mathbb{P}$, $\pi''A$ is a maximal antichain of $\mathbb{Q}$.
(3) *dense embedding*, if it is an embedding and $\pi''\mathbb{P}$ is dense in $\mathbb{Q}$.

In particular, if π is the identity map, we call $\mathbb{P}$ a *subforcing* of $\mathbb{Q}$.

Note that every dense embedding is complete. Given an embedding $\pi:\mathbb{P}\to\mathbb{Q}$ in $\mathcal{C}$, we can lift it to a map $\pi^*:M^{\mathbb{P}}\to M^{\mathbb{Q}}$ defined recursively by

$$\pi^*(\sigma)=\{\langle\pi^*(\tau),\pi(p)\rangle\mid\langle\tau,p\rangle\in\sigma\}.$$

The next result is the same as for set forcing and their proof shall therefore be omitted.

Lemma 1.2.11. *Suppose that* $\pi:\mathbb{P}\to\mathbb{Q}$ *is an embedding in* $\mathcal{C}$ *of notions of class forcing for* $\mathbb{M}$.

(1) *If* π *is a complete embedding and* H *is* $\mathbb{Q}$*-generic over* $\mathbb{M}$*, then*

$$\pi^{-1}[H]=\{p\in\mathbb{P}\mid\pi(p)\in H\}$$

is $\mathbb{P}$*-generic over* $\mathbb{M}$*.*

[3] We do not know whether this instance can fail in a generic extension.

(2) *If* π *is a dense embedding and* G *is* $\mathbb{P}$*-generic over* $\mathbb{M}$*, then the* upwards closure

$$H = \{q \in \mathbb{Q} \mid \exists p \in G(\pi(p) \leq_{\mathbb{P}} q)\}$$

of $\pi''G$ *in* $\mathbb{Q}$ *is* $\mathbb{Q}$*-generic over* $\mathbb{M}$. □

In the context of set forcing, it holds that if $\pi : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then $\mathbb{P}$ and $\mathbb{Q}$ produce the same generic extensions. In class forcing, however, this may fail (see Section 4.5). Nevertheless, we are often interested in which properties of forcing notions are preserved under dense embeddings.

Notation. Let $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ and let Ψ be some property of a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$. We say that $\mathbb{P}$ *densely* satisfies Ψ if every notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$, for which there is a dense embedding in $\mathcal{C}$ from $\mathbb{P}$ into $\mathbb{Q}$, satisfies the property Ψ.

Frequently, in particular in Chapter 4, we need the following assumption on forcing notions.

Definition 1.2.12. We say that a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is *separative*, if for all $p, q \in \mathbb{P}$ with $p \not\leq_{\mathbb{P}} q$ there is $r \leq_{\mathbb{P}} p$ such that r is incompatible with q.

In Section 3.1.3 we will show that, given a non-separative notion of class forcing $\mathbb{P}$, we can map $\mathbb{P}$ surjectively to its so-called *separative quotient* $\mathbb{S}(\mathbb{P})$ in such a way that $\mathbb{P}$ and $\mathbb{S}(\mathbb{P})$ have the same generic extensions.

Our choice of considering preorders rather than partial orders is due to the reason that in the case of a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ of notions of class forcing, as defined in [Fri00] (see also Chapter Lemma 2.2.15 of this thesis), we will have conditions of the form $\langle p, \dot{q}\rangle$ for $p \in \mathbb{P}$ and p forcing that $\dot{q} \in \dot{\mathbb{Q}}$. In general, there will be distinct pairs $\langle p, \dot{q}_0\rangle$ and $\langle p, \dot{q}_1\rangle$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q}_0 = \dot{q}_1$, i.e. one naturally obtains a preorder that is not antisymmetric. However, in some contexts it will become crucial for our orderings to be antisymmetric. In that case we will usually assume that the ground model $\mathbb{M}$ satisfies representatives choice, since given a preorder $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle \in \mathcal{C}$, by considering the equivalence relation

$$p \approx q \iff p \leq_{\mathbb{P}} q \wedge q \leq_{\mathbb{P}} p,$$

we obtain a partial order $\mathbb{Q} \in \mathcal{C}$ and a surjective map $\pi : \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ such that for all $p, q \in \mathbb{P}$, $p \approx q$ if and only if $\pi(p) = \pi(q)$.

1.3 Examples

In this section we introduce several notions of class forcing which we will use in the upcoming chapters. First we need the following notation.

Notation. Let $\mathbb{P}$ be a separative notion of class forcing, let σ be a $\mathbb{P}$-name and let p be a condition in $\mathbb{P}$. Then we define the *p-evaluation of* σ to be

$$\sigma^p \ = \ \{\tau^p \mid \exists q \in \mathbb{P}\,(\langle \tau, q\rangle \in \sigma \ \wedge \ p \leq_{\mathbb{P}} q)\}.$$

Observe that we will generalize this notation in Section 2.3 to non-separative forcing notions.

1.3.1 Variants of collapse forcings

The following examples of notions of class forcing will frequently be used to motivate our results, especially in Chapter 4.

Definition 1.3.1. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^-.

(1) Let $\mathrm{Col}(\omega, \mathrm{Ord})^M$ denote the partial order $\mathrm{Col}(\omega, \mathrm{Ord}^M)$, i.e. $\mathrm{Col}(\omega, \mathrm{Ord})^M$ is the partial order whose conditions are finite partial functions $p : \omega \xrightarrow{par} \mathrm{Ord}^M$ ordered by reverse inclusion.
(2) Define $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ to be the (dense) suborder of $\mathrm{Col}(\omega, \mathrm{Ord})^M$ consisting of all conditions p with $\mathrm{dom}(p) \in \omega$.
(3) Let $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ be the notion of forcing whose conditions are finite partial functions $p : \omega \xrightarrow{par} \mathrm{Ord}^M \cup \{\geq \alpha \mid \alpha \in \mathrm{Ord}^M\}$, where $\geq \alpha$ is an element of M which is not in Ord^M for every $\alpha \in \mathrm{Ord}^M$, and whose ordering is given by $p \leq q$ if and only if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for every $n \in \mathrm{dom}(q)$, either
   - $p(n) = q(n)$ or
   - $q(n)$ is $\geq \alpha$ for some $\alpha \in \mathrm{Ord}^M$ and there is an ordinal $\beta \geq \alpha$ such that $p(n) \in \{\beta, \geq \beta\}$.

In the case that the context is clear, we will omit the superscripts.

Note that all of these partial orders are definable over the corresponding model M. The next lemma gives some basic properties of the different collapse forcings defined above.

Lemma 1.3.2. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^-.*

(1) *If G is a $\mathrm{Col}(\omega, \mathrm{Ord})^M$-generic filter over M, then for every ordinal in M there is a surjection from a subset of ω onto that ordinal in $M[G]$.*
(2) *If G is a $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$-generic filter over M, then $M = M[G]$.*
(3) *No non-trivial maximal antichain of $\mathrm{Col}(\omega, \mathrm{Ord})^M$ or $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ is an element of M.*
(4) *If M is a model of ZFC, then no non-trivial complete suborder of $\mathrm{Col}(\omega, \mathrm{Ord})^M$ or of $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ is an element of M.*
(5) *$\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ is the union of Ord^M-many set-sized complete subforcings.*

Proof. To see (1), pick $\lambda \in \mathrm{Ord}^M$. Given $\alpha \in \mathrm{Ord}^M$, define

$$D_\alpha \;=\; \{p \in \mathrm{Col}(\omega, \mathrm{Ord})^M \mid \exists n \in \mathrm{dom}(p)\, [p(n) = \alpha]\}.$$

Then each D_α is dense in $\mathrm{Col}(\omega, \mathrm{Ord})^M$ and definable over M and hence $D_\alpha \in \mathcal{C}$. This implies that if G is $\mathrm{Col}(\omega, \mathrm{Ord})^M$-generic over M, then for every $\alpha \in M \cap \mathrm{Ord}$ there is an $n \in \omega$ with $\{\langle n, \alpha \rangle\} \in G$. This shows that

$$\sigma \;=\; \{\langle \mathsf{op}(\check{n}, \check{\alpha}), \{\langle n, \alpha \rangle\} \rangle \mid \alpha < \lambda,\ n \in \omega\}$$

is a name for a surjection from a subset of ω onto λ.

Next we prove (2). Let σ be a $\mathrm{Col}_*(\omega,\mathrm{Ord})^M$-name in M. Then $\mathrm{ran}(p) \subseteq \mathrm{rank}(\sigma)$ holds for every condition p in $\mathrm{tc}(\sigma) \cap \mathrm{Col}_*(\omega,\mathrm{Ord})^M$. If we define

$$D \;=\; \{p \in \mathrm{Col}_*(\omega,\mathrm{Ord})^M \mid \mathrm{rank}(\sigma) \in \mathrm{ran}(p)\},$$

then D is dense in $\mathrm{Col}_*(\omega,\mathrm{Ord})^M$ and definable over M. Moreover, by the above observation, we have $\sigma^G = \sigma^p \in M$, whenever G is an $\mathrm{Col}_*(\omega,\mathrm{Ord})^M$-generic filter over $\mathbb{M}$ and $p \in D \cap G$, because such p either extends or is incompatible to any condition in $\mathrm{tc}(\sigma)$.

For (3), let Col denote either $\mathrm{Col}(\omega,\mathrm{Ord})^M$ or $\mathrm{Col}_*(\omega,\mathrm{Ord})^M$. Assume that $A \in M$ is an antichain of Col which is not equal to $\{\mathbb{1}\}$. Pick $a \in A$. Now for any $b \in A \setminus \{a\}$, the domains of a and b cannot be disjoint by incompatibility. Define $c \in \mathrm{Col}$ with $\mathrm{dom}(c) = \mathrm{dom}(a)$ and for every $n \in \mathrm{dom}(c)$, let $c(n) = \sup\{b(n) \mid b \in A\} + 1$. Hence c is incompatible with every element of A, showing that A is not maximal.

Condition (4) follows from the above results because our assumptions imply that set-sized partial orders in M contain non-trivial maximal antichains.

Finally, we verify (5). Let for every $\alpha \in \mathrm{Ord}^M$, $\mathrm{Col}_{\geq}(\omega,\alpha)$ denote the subforcing of $\mathrm{Col}_{\geq}(\omega,\mathrm{Ord}^M)$ consisting of finite partial functions $p : \omega \xrightarrow{par} \alpha \cup \{\geq \beta \mid \beta \leq \alpha\}$ with the induced ordering. Clearly,

$$\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathrm{Col}_{\geq}(\omega,\alpha).$$

It remains to check that for every $\alpha \in \mathrm{Ord}^M$, $\mathrm{Col}_{\geq}(\omega,\alpha)$ is a complete subforcing of $\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$. Let A be a maximal antichain of $\mathrm{Col}_{\geq}(\omega,\alpha)$ and let $p \in \mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$. Consider the condition $\bar{p} \in \mathrm{Col}_{\geq}(\omega,\alpha)$ which is obtained from p by replacing $p(n)$ by $\geq \alpha$ whenever $p(n) \geq \alpha$ or $p(n)$ is of the form $\geq \beta$ for some $\beta > \alpha$. Since A is a maximal antichain, there is $a \in A$ such that a and $\bar{p}$ are compatible. Let $\bar{q} \in \mathrm{Col}_{\geq}(\omega,\alpha)$ be a common strengthening of $\bar{p}$ and a. But then the condition q obtained from $\bar{q}$ by replacing $\bar{q}(n)$ by $p(n)$ for every $n \in \mathrm{dom}(p)$ such that $\bar{q}(n)$ is of the form $\geq \alpha$ witnesses that p and a are compatible. □

The collapse forcings will play an important role throughout this thesis, since they are simple counterexamples to many properties satisfied by set-sized forcing notions but not class-sized ones. Examples include the existence of a unique Boolean completion, the property of generating the same extensions as dense subforcings or the existence of nice names for sets of ordinals.

1.3.2 Forcing a global well-order

Given a countable transitive model $\mathbb{M}$ of GB, we can extend $\mathbb{M}$ to a model of GBC without adding any new sets. There are several simple ways to achieve this; we will, however, present only one of them.

For a countable transitive model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^-, let $\mathbb{W}^M$ denote the forcing notion consisting of injective functions $p : \alpha \to M$ for some ordinal α, ordered by reverse inclusion. Note that $\mathbb{W}^M$ is definable over M. As usual, we may omit the superscript if it is clear from the context which model is referred to.

Lemma 1.3.3. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of* GB^-*.*

(1) $\mathbb{W}^M$ *is* $< \kappa$*-closed for every* M*-cardinal* κ[4]*.*
(2) *If* G *is* $\mathbb{W}^M$*-generic over* $\mathbb{M}$ *then* $M[G] = M$*.*
(3) *If* G *is* $\mathbb{W}^M$*-generic over* $\mathbb{M}$ *then* $\bigcup G$ *is a bijection between* Ord^M *and* M*.*

Proof. For (1), let κ be an M-cardinal and $\langle p_i \mid i < \kappa \rangle$ a $\leq_{\mathbb{W}}$-descending sequence of conditions. Then $\bigcup_{i<\kappa} p_i$ is also in $\mathbb{W}^M$. The second condition follows in a similar way as property (2) in Lemma 1.3.2 using the density of

$$D = \{p \in \mathbb{W}^M \mid \mathrm{rank}(\sigma) \in \mathrm{dom}(p)\}$$

for a given $\mathbb{W}^M$-name because if $p \in D \cap G$ then $\sigma^G = \sigma^p$, since the domain of every condition in $\mathrm{tc}(\sigma) \cap \mathbb{W}^M$ is smaller than $\mathrm{dom}(p)$. The last statement is a consequence of standard density arguments. □

Remark 1.3.4. The proof of property (2) in Lemma 1.3.3 shows that we can define the forcing relation of $\mathbb{W}^M$ over M by

$$p \Vdash^{\mathbb{M}}_{\mathbb{W}} \sigma \in \tau \iff \forall q \leq_{\mathbb{W}} p\, (\mathrm{rank}(\sigma \cup \tau) \in \mathrm{dom}(q) \to \sigma^q \in \tau^q).$$

This idea will be generalized in Section 2.3 in order to prove that every forcing notion which doesn't add any new sets satisfies the forcing theorem[5].

Other ways to force a set-like well-order of M include adding a Cohen subset of Ord^M or adding a set-like well-order using initial segments as conditions.

1.3.3 Friedman's forcing $\mathbb{F}$

The following notion of class forcing due to Sy Friedman is mentioned in [Sta03, Remark 1.8]. It will be crucial for the proofs for the failure of the forcing theorem in Section 2.5.

Definition 1.3.5. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^-. Define $\mathbb{F}^M$ to be the partial order whose conditions are triples $p = \langle d_p, e_p, f_p \rangle$ satisfying

(1) d_p is a finite subset of ω,
(2) e_p is a binary acyclic relation on d_p,
(3) f_p is an injective function with $\mathrm{dom}(f_p) \in \{\emptyset, d_p\}$ and $\mathrm{ran}(f_p) \subseteq M$,
(4) if $\mathrm{dom}(f_p) = d_p$ and $i, j \in d_p$, then we have $i \; e_p \; j$ if and only if $f_p(i) \in f_p(j)$,

and whose ordering is given by

$$p \leq_{\mathbb{F}^M} q \iff d_q \subseteq d_p \ \wedge\ e_p \cap (d_q \times d_q) = e_q \ \wedge\ f_q \subseteq f_p.$$

Note that $\mathbb{F}^M$ is definable over M.

Lemma 1.3.6. *The set of all conditions* p *in* $\mathbb{F}^M$ *with* $\mathrm{dom}(f_p) = d_p$ *is dense.*

[4] A notion of (class) forcing $\mathbb{P}$ is $< \kappa$*-closed* for some M-cardinal κ, if for every $\leq_{\mathbb{P}}$-descending sequence $\langle p_i \mid i < \lambda \rangle$ with $\lambda < \kappa$ there is $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p_i$ for all $i < \lambda$.

[5] See Definition 2.1.1.

Proof. Pick $p \in \mathbb{F}^M$ with $\mathrm{dom}(f_p) = \emptyset \neq d_p$. We inductively define (using that e_p is acyclic) a function f as follows. For every $j \in d_p$ let

$$f(j) = \{f(i) \mid i \; e_p \; j\} \cup \{\{\emptyset, j\}\}.$$

Using that $\emptyset \notin \mathrm{range}(f)$, it is easy to verify inductively that $\bar{p} = \langle d_p, e_p, f\rangle$ satisfies conditions (3) and (4) above, and hence is an extension of p in $\mathbb{F}^M$ with $\mathrm{dom}(f_{\bar{p}}) = d_{\bar{p}}$. □

Remark 1.3.7. The reason why the restriction on $\mathrm{dom}(f)$ in property (3) of Definition 1.3.5 is necessary is because otherwise Lemma 1.3.6 fails: Consider

$$p = \langle \{i, j, k\}, \{\langle i, j\rangle, \langle j, k\rangle\}, \{\langle i, 0\rangle, \langle k, 1\rangle\}\rangle.$$

In order to extend p such that the third coordinate has domain $\{i, j, k\}$, one would have to map j to some x such that $0 \in x$ and $x \in 1$ which is clearly absurd.

Lemma 1.3.8. *If $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- and G is an $\mathbb{F}^M$-generic filter over $\mathbb{M}$, then there is a binary relation E on ω such that $E \in M[G]$ and the models $\langle \omega, E\rangle$ and $\langle M, \in\rangle$ are isomorphic in V.*

Proof. Define an $\mathbb{F}^M$-name $\dot{E} \in M$ by setting

$$\dot{E} \; = \; \{\langle \mathsf{op}(\check{i}, \check{j}), p_{i,j}\rangle \mid i, j \in \omega, \; i \neq j\},$$

where $p_{i,j}$ denotes the condition in $\mathbb{F}^M$ with $d_{p_{i,j}} = \{i, j\}$, $e_{p_{i,j}} = \{\langle i, j\rangle\}$ and $f_{p_{i,j}} = \emptyset$. Let G be an $\mathbb{F}^M$-generic filter over $\mathbb{M}$ and put $E = \dot{E}^G \in M[G]$. Note that $E = \bigcup\{e_p \mid p \in G\}$. Define $F = \bigcup\{f_p \mid p \in G\}$. By Lemma 1.3.6, the sets $D_n = \{p \in \mathbb{F}^M \mid n \in \mathrm{dom}(f_p)\}$ are dense in $\mathbb{F}^M$. Since these sets are definable over M, we can conclude that F is injective and that $\mathrm{dom}(F) = \omega$. In order to see that F is surjective, we claim that for every $x \in M$, the set $\{p \in \mathbb{F}^M \mid x \in \mathrm{ran}(f_p)\}$ is dense. In order to show this, let $p = \langle d_p, e_p, f_p\rangle \in \mathbb{F}^M$ such that $x \notin \mathrm{ran}(f_p)$. Using Lemma 1.3.6, we may assume that $\mathrm{dom}(f_p) = d_p$. Choose $j \in \omega \setminus d_p$ and define $d_q = d_p \cup \{j\}$, $e_q = e_p \cup \{\langle i, j\rangle \mid f_p(i) \in x\} \cup \{\langle j, i\rangle \mid x \in f_p(i)\}$ and $f_q = f_p \cup \{\langle j, x\rangle\}$. Then $q = \langle d_q, e_q, f_q\rangle$ is an extension of p with $x \in \mathrm{ran}(f_q)$.

It remains to check that F is an isomorphism between the models $\langle \omega, E\rangle$ and $\langle M, \in\rangle$. Take $i, j \in \omega$ such that $\langle i, j\rangle \in E$, i.e. $p_{i,j} \in G$. By the above computations, there is a condition $p \in G$ with $i, j \in \mathrm{dom}(f_p)$. We then have $\langle i, j\rangle \in e_p$ and by (4) in Definition 1.3.5 we have $F(i) = f_p(i) \in f_p(j) = F(j)$. For the converse, suppose that $x, y \in M$ such that $x \in y$. By the above computations, there is a condition $p \in G$ and $i, j \in d_p$ with $F(i) = f_p(i) = x \in y = f_p(j) = F(j)$. By (4) in Definition 1.3.5, this implies $\langle i, j\rangle \in e_p$ and therefore $\langle i, j\rangle \in E$. □

1.3.4 The class version of the Lévy collapse

The next forcing notion that we present is a class version of the Lévy collapse. Recall that the Lévy collapse as described in [Kan09, Chapter 10] is used to construct Solovay's model [Sol70] where every set of reals is Lebesgue measurable, has the Baire property and the perfect set property. This is essentially achieved by collapsing an inaccessible cardinal to ω_1. The Lévy collapse that we introduce here is an analogue, where the inaccessible is replaced by the height of the model Ord^M.

Let M be a countable transitive model of ZFC. We will force over the GBc-model $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$.

Definition 1.3.9. For $\gamma \in \mathrm{Ord}^M \cup \{\mathrm{Ord}^M\}$ we denote by $\mathrm{Col}(\omega, <\gamma)^M$ the partial order whose conditions are functions $p : \mathrm{dom}(p) \to \gamma$ satisfying

- $\mathrm{dom}(p)$ is a finite subset of $\gamma \times \omega$
- for all $\langle \alpha, n\rangle \in \mathrm{dom}(p)$, $p(\alpha, n) < \alpha$,

ordered by reverse inclusion. Note that for $\gamma = \mathrm{Ord}^M$, $\mathrm{Col}(\omega, <\gamma)^M$ is a proper class forcing notion, and for $\gamma \in \mathrm{Ord}^M$ it is set-sized.

$\mathrm{Col}(\omega, <\mathrm{Ord})^M$ has similar properties as its set-sized analogue.

Lemma 1.3.10. *Let $\mathbb{P} = \mathrm{Col}(\omega, <\mathrm{Ord})^M$. Then the following statements hold:*

(1) *For $\gamma \in \mathrm{Ord}^M$, $\mathbb{P}_\gamma = \mathrm{Col}(\omega, <\gamma)$ is a complete subforcing of $\mathbb{P}$.*
(2) *If G is $\mathbb{P}$-generic over M, then $M[G]$ contains a surjection $f_\gamma : \omega \twoheadrightarrow \gamma$ for each $\gamma \in \mathrm{Ord}^M$.*
(3) *If G is $\mathbb{P}$-generic over M, then $M[G] = \bigcup_{\gamma \in \mathrm{Ord}^M} M[G_\gamma]$, where $G_\gamma = G \cap \mathbb{P}_\gamma$ is the induced $\mathbb{P}_\gamma$-generic filter.*

Proof. For (1), note that if A is a maximal antichain in $\mathbb{P}_\gamma$ and $p \in \mathbb{P}$, then $\bar{p} = p \upharpoonright \gamma \times \omega \in \mathbb{P}_\gamma$. In particular, there is $a \in A$ which is compatible with $\bar{p}$, witnessed by $\bar{q} \in \mathbb{P}_\gamma$. But then $q = \bar{q} \cup p \upharpoonright [\gamma, \mathrm{Ord}^M) \times \omega$ witnesses that p and a are compatible.

Let $\gamma \in \mathrm{Ord}^M$. We claim that

$$\dot{f}_\gamma = \{\langle \mathsf{op}(\check{n}, \check{\alpha}), p^\gamma_{n,\alpha}\rangle \mid n \in \omega, \alpha < \gamma\} \in M^{\mathbb{P}},$$

where $p^\gamma_{n,\alpha} = \{\langle\langle \gamma, n\rangle, \alpha\rangle\}$, is a name for a surjection from ω onto γ. Let G be $\mathbb{P}$-generic over M. That $\dot{f}^G_\gamma$ is functional is clear, since for $\alpha \neq \beta$, $p^\gamma_{n,\alpha}$ and $p^\gamma_{n,\beta}$ are incompatible. Note that for each $n \in \omega$, the class

$$D_n = \{p \in \mathbb{P} \mid \langle \alpha, n\rangle \in \mathrm{dom}(p)\} \in \mathrm{Def}(M)$$

is dense in $\mathbb{P}$. But then there is $p \in G \cap D_n$, so $p(\gamma, n) = \alpha$ for some $\alpha < \gamma$ and so $p \leq_{\mathbb{P}} p^\gamma_{n,\alpha}$. In particular, $\langle n, \alpha\rangle \in \dot{f}^G_\gamma$. A similar argument using the density of

$$D_\alpha = \{p \in \mathbb{P} \mid \exists n \in \omega\, [\langle \gamma, n\rangle \in \mathrm{dom}(p) \wedge p(\gamma, n) = \beta]\} \in \mathcal{C}$$

for $\alpha < \gamma$, shows that $\dot{f}^G_\alpha$ is surjective.

For the third statement, note that $\mathbb{P} = \bigcup_{\gamma \in \mathrm{Ord}^M} \mathbb{P}_\gamma$. If G is $\mathbb{P}$-generic over $\mathbb{M}$, then by (1) and Lemma 1.2.11, G_γ is $\mathbb{P}_\gamma$-generic. In particular, every $\mathbb{P}$-name is a $\mathbb{P}_\gamma$-name for some ordinal γ, and $\sigma^G = \sigma^{G_\gamma}$. □

Using Lemma 1.3.10 we may conclude that if G is $\mathbb{P}$-generic over M for $\mathbb{P} = \mathrm{Col}(\omega, <\mathrm{Ord})^M$, then in $M[G]$ the power set axiom fails. To see this, suppose otherwise. Then there is a $\mathbb{P}$-name σ for the power set of ω. By condition (3) in Lemma 1.3.10, σ is a $\mathbb{P}_\gamma$-name for some M-cardinal γ. Since $\mathbb{P}_\gamma$ satisfies the γ-cc by standard arguments, it preserves all cardinals $\geq \gamma$. But by (2), $\mathbb{P}$ adds a surjection from ω onto γ and so in $M[G]$ there is a well-order of ω of ordertype γ which can be coded by a real x. But x cannot be an element of σ^G, a contradiction.

1.3.5 Easton forcing

Easton forcing is the forcing notion introduced by Easton in [Eas70] in order to force almost arbitrary values of the continuum function. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a fixed countable transitive model of GBc with $M \models \mathsf{GCH}$.

Definition 1.3.11. An *Easton function* for $\mathbb{M}$ is a class function $F : \mathrm{dom}(F) \to \mathrm{Card}^M$ in $\mathcal{C}$ such that for all $\kappa, \lambda \in \mathrm{dom}(F)$,

(1) κ is a regular cardinal,
(2) $\mathrm{cf}(F(\kappa)) > \kappa$,
(3) if $\kappa < \lambda$ then $F(\kappa) \leq F(\lambda)$.

Definition 1.3.12. Let $F \in \mathcal{C}$ be an Easton function. The associated *Easton forcing* $\mathbb{P}_F$ is the notion of class forcing whose conditions are functions p with $\mathrm{dom}(p) \subseteq \mathrm{dom}(F)$ such that for each $\kappa \in \mathrm{dom}(p)$, $p(\kappa) \in \mathrm{Add}(\kappa, F(\kappa))$ and for all regular cardinals λ, $|\mathrm{dom}(p) \cap \lambda| < \lambda$, where $\mathrm{Add}(\kappa, F(\kappa))$ is the forcing notion for adding $F(\kappa)$-many Cohen subsets of κ.[6]

Definition 1.3.13. Let F be an Easton function for $\mathbb{M}$, $p \in \mathbb{P}_F$ and λ a regular cardinal. Then we define

$$p^{\leq\lambda} = p \restriction \lambda^+ \quad \text{and} \quad p^{>\lambda} = p \restriction \mathrm{Ord} \setminus \lambda^+.$$

Furthermore, let

$$\mathbb{P}_F^{\leq\lambda} = \{p^{\leq\lambda} \mid p \in \mathbb{P}_F\} \quad \text{and} \quad \mathbb{P}_F^{>\lambda} = \{p^{>\lambda} \mid p \in \mathbb{P}_F\}.$$

Remark 1.3.14. If $p \in \mathbb{P}_F$ then $p = p^{\leq\lambda} \cup p^{>\lambda}$. In particular, this shows that $\mathbb{P}_F$ is isomorphic to $\mathbb{P}_F^{\leq\lambda} \times \mathbb{P}_F^{>\lambda}$.

Lemma 1.3.15. *Let F be an Easton function for $\mathbb{M}$ and λ a regular M-cardinal. Then $\mathbb{P}_F^{>\lambda}$ is λ^+-closed.*

Proof. Let $\langle p_i \mid i < \lambda \rangle$ be a descending sequence in $\mathbb{P}_F^{>\lambda}$. Let $p = \bigcup_{i<\lambda} p_i$. Then $\mathrm{dom}(p) = \bigcup_{i<\lambda} \mathrm{dom}(p_i) \subseteq \mathrm{Ord} \setminus \lambda^+$. Now if $\kappa \in \mathrm{dom}(p)$, then $\kappa \geq \lambda^+$ and so

$$|\mathrm{dom}(p(\kappa))| = |\bigcup_{i<\lambda} \mathrm{dom}(p_i(\kappa))| < \kappa$$

since κ is regular.

Moreover, if $\kappa \geq \lambda^+$ is a regular cardinal, then

$$|\mathrm{dom}(p) \cap \kappa| = |\bigcup_{i<\lambda} \mathrm{dom}(p_i) \cap \kappa| < \kappa.$$

This shows that $p \in \mathbb{P}_F^{>\lambda}$ and $p \leq_{\mathbb{P}^{>\lambda}} p_i$ for each $i < \lambda$. □

[6]For M-cardinals κ, λ with $\lambda \geq 1$, conditions in $\mathrm{Add}(\kappa, \lambda)$ are partial functions from $\lambda \times \kappa$ to 2 of cardinality $< \kappa$, ordered by reverse inclusion.

Lemma 1.3.16. *Let F be an Easton function for $\mathbb{M}$. For every regular M-cardinal λ, $\mathbb{P}_F^{\leq\lambda}$ satisfies the λ^+-cc*[7].

Proof. For $p \in \mathbb{P}_F^{\leq\lambda}$ let

$$d(p) = \bigcup \{\{\kappa\} \times \mathrm{dom}(p(\kappa)) \mid \kappa \in \mathrm{dom}(p) \cap \lambda^+\}.$$

Then by assumption, $|d(p)| < \lambda$. Now suppose that $A \subseteq \mathbb{P}_F^{\leq\lambda}$ is an antichain of size λ^+. Since $M \models \mathsf{GCH}$, $\lambda^{<\lambda} = \lambda$ and so we can apply the Δ-System Lemma[8]. There is $B \subseteq A$ of size λ^+ such that

$$\{d(p) \mid p \in B\}$$

forms a Δ-system with root r, i.e. for all $p, q \in B$ with $p \neq q$, $d(p) \cap d(q) = r$. So $|r| < \lambda$ and using the GCH we have $2^{|r|} \leq \lambda$. But then there is $C \subseteq B$ of size λ^+ such that for all $p, q \in C$ and for all $\langle \kappa, x \rangle \in r$, $p(\kappa)(x) = q(\kappa)(x)$. But then all elements of C are compatible, a contradiction. □

The idea is that $\mathbb{P}_F$ adds $F(\kappa)$-many Cohen subsets of κ for every $\kappa \in \mathrm{dom}(F)$ and hence in the generic extension we will obtain $2^\kappa = F(\kappa)$. Moreover, we shall see that all axioms of ZFC are preserved. However, at this point we have not yet developed enough tools to prove this in detail. The remaining discussion of Easton forcing is therefore deferred to Section 2.2.

[7] A notion of (class) forcing has the *κ-cc* for some M-cardinal κ, if every antichain has cardinality $< \kappa$.
[8] see [Jec03, Theorem 9.18]

Chapter 2

The forcing theorem

A fundamental result in the context of set forcing is the so-called forcing theorem. It consists of two parts, the first of which states that the forcing relation is definable over the ground model, and the second part postulates that every formula which is true in some generic extension $M[G]$ is forced by some condition in the generic filter G. We will generalize the statement of the forcing theorem to our second-order framework and present both positive and negative results related to the forcing theorem.

2.1 The forcing theorem

In the following, we state the forcing theorem and prove that the forcing theorem for all formulae can be reduced to one atomic instance. Moreover, we introduce an infinitary quantifier-free language and show that the forcing theorem for atomic formulae already implies the forcing theorem for such infinitary formulae.

In this section, we fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^- and a notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$.

Definition 2.1.1. Let $\varphi \equiv \varphi(v_0, \dots, v_{m-1}, V_0, \dots, V_{n-1})$ be an $\mathcal{L}^2_\in$-formula.

(1) We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$, if

$$\{\langle p, \sigma_0, \dots, \sigma_{m-1} \rangle \in P \times (M^{\mathbb{P}})^m \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \dots, \sigma_{m-1}, \Gamma_0, \dots, \Gamma_{n-1})\} \in \mathcal{C}$$

holds for all $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$.

(2) We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$, if for all $\sigma_0, \dots, \sigma_{m-1} \in M^{\mathbb{P}}$, $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with

$$\mathbb{M}[G] \models \varphi(\sigma_0^G, \dots, \sigma_{m-1}^G, \Gamma_0^G, \dots, \Gamma_{n-1}^G),$$

there is a $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \dots, \sigma_{m-1}, \Gamma_0, \dots, \Gamma_{n-1})$.

(3) We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$, if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

Furthermore, we will simply say that $\mathbb{P}$ *satifies the forcing theorem over* $\mathbb{M}$, if $\mathbb{P}$ satisfies the forcing theorem for all $\mathcal{L}_\in$-formulae with class parameters.

In set forcing, the forcing theorem is crucial to prove that separation is preserved; given a formula $\varphi(x,y)$ and names $\sigma,\mu\in M^{\mathbb{P}}$ in some set-sized forcing notion $\mathbb{P}$, we can simply take the name

$$\pi=\{\langle\tau,q\rangle\mid\exists p\geq_{\mathbb{P}} q\,[\langle\tau,p\rangle\in\sigma\wedge q\Vdash^{\mathbb{M}}_{\mathbb{P}}\varphi(\tau,\mu)]\}$$

as a witness for the set $\{x\in\sigma^G\mid\varphi(x,\mu^G)\}$. However, this proof fails in class forcing, since π becomes a class name. Nevertheless, the forcing theorem implies the preservation of the corresponding class axiom, namely first-order class comprehension.

Lemma 2.1.2. *If $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem, then every $\mathbb{P}$-generic extension of $\mathbb{M}$ satisfies first-order class comprehension.*

Proof. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ and let $\varphi(x,\vec{y},\vec{C})$ be a first-order formula with set parameters $\vec{y}$ in $M[G]$ and class parameters $\vec{C}$ in $\mathcal{C}[G]$. Choose names $\vec{\sigma}$ in $M^{\mathbb{P}}$ and $\vec{\Gamma}$ in $\mathcal{C}^{\mathbb{P}}$ for $\vec{C}$. Then

$$\{\langle\tau,p\rangle\mid p\Vdash^{\mathbb{M}}_{\mathbb{P}}\varphi(\tau,\vec{\sigma},\vec{\Gamma})\}\in\mathcal{C}^{\mathbb{P}}$$

is a class name for $\{x\in M[G]\mid\varphi(x,\vec{y},\vec{C})\}$. □

2.1.1 Reducing the forcing theorem to atomic formulae

Our goal is to prove that the definability for one atomic formula already implies the forcing theorem for all $\mathcal{L}_\in$-formulae with class parameters, thus proving Theorem 2.1.5. The first step to achieve this is to show that the definability lemma for one atomic formula already implies the truth lemma. The proofs of Lemma 2.1.3 and Theorem 2.1.5 are essentially the same as the ones given in [HKL+16, Lemma 4.2, Theorem 4.3].

Lemma 2.1.3. *Assume that $\mathbb{P}$ satisfies the definability lemma for "$v_0\in v_1$" or "$v_0=v_1$" over $\mathbb{M}$. Then $\mathbb{P}$ satisfies the forcing theorem for all atomic $\mathcal{L}_\in$-formulae.*

Proof. Suppose first that the definability lemma holds for "$v_0\in v_1$". We denote by $p\Vdash^{\mathbb{M},*}_{\mathbb{P}}\sigma\subseteq\tau$ the statement that for all $\langle\rho,r\rangle\in\sigma$ and for all $q\leq_{\mathbb{P}} p,r$, the class

$$D_{\rho,\tau}=\{s\in\mathbb{P}\mid s\Vdash^{\mathbb{M}}_{\mathbb{P}}\rho\in\tau\}\in\mathcal{C}$$

is dense below q in $\mathbb{P}$. Furthermore, let $p\Vdash^{\mathbb{M},*}_{\mathbb{P}}\sigma=\tau$ denote that $p\Vdash^{\mathbb{M},*}_{\mathbb{P}}\sigma\subseteq\tau$ and $p\Vdash^{\mathbb{M},*}_{\mathbb{P}}\tau\subseteq\sigma$. We show by induction on the lexicographic order on pairs $\langle\operatorname{rank}(\sigma)+\operatorname{rank}(\tau),\operatorname{rank}(\sigma)\rangle$ that the following hold for each $p\in\mathbb{P}$:

(1) $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma\in\tau$ if and only if the class

$$E_{\sigma,\tau}=\{q\in\mathbb{P}\mid\exists\langle\rho,r\rangle\in\tau\,[q\leq_{\mathbb{P}} r\wedge q\Vdash^{\mathbb{M},*}_{\mathbb{P}}\sigma=\rho]\}\in\mathcal{C}$$

is dense below p in $\mathbb{P}$.

(2) $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma\subseteq\tau$ if and only if $p\Vdash^{\mathbb{M},*}_{\mathbb{P}}\sigma\subseteq\tau$. In particular, $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma=\tau$ if and only if $p\Vdash^{\mathbb{M},*}_{\mathbb{P}}\sigma=\tau$.

(3) There is a dense subclass of $\mathbb{P}$ in $\mathcal{C}$ that consists of conditions p in $\mathbb{P}$ such that either $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma\in\tau$ or $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma\notin\tau$.

(4) There is a dense subclass of $\mathbb{P}$ in $\mathcal{C}$ that consists of conditions p in $\mathbb{P}$ such that either $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \subseteq \tau$ or $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \nsubseteq \tau$.

To start the induction, note that if $\operatorname{rank}(\sigma) + \operatorname{rank}(\tau) = 0$, then (1)–(4) trivally hold. Observe that (3) implies that the truth lemma holds for "$v_0 \in v_1$". Furthermore, (4) implies the truth lemma for "$v_0 \subseteq v_1$" and hence also for equality. Suppose now that (1)–(4) are satisfied for all pairs of names $\langle \bar{\sigma}, \bar{\tau} \rangle$ in $M^{\mathbb{P}}$ for which $\langle \operatorname{rank}(\bar{\sigma}) + \operatorname{rank}(\bar{\tau}), \operatorname{rank}(\bar{\sigma}) \rangle$ is lexicographically less than $\langle \operatorname{rank}(\sigma) + \operatorname{rank}(\tau), \operatorname{rank}(\sigma) \rangle$, that is $\operatorname{rank}(\bar{\sigma}) + \operatorname{rank}(\bar{\tau}) \leq \operatorname{rank}(\sigma) + \operatorname{rank}(\tau)$ and in case of equality, we have that $\operatorname{rank}(\bar{\sigma}) < \operatorname{rank}(\sigma)$.

In order to prove (1), pick a condition $p \in \mathbb{P}$ with $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \in \tau$ and $q \leq_{\mathbb{P}} p$. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $q \in G$. Then $\sigma^G \in \tau^G$ by assumption and hence there is $\langle \rho, r \rangle \in \tau$ with $r \in G$ and $\sigma^G = \rho^G$. By our inductive assumption, property (4) yields a condition $s \in G$ with $s \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \rho$ which by (2) is equivalent to $s \Vdash_{\mathbb{P}}^{\mathbb{M},*} \sigma = \rho$. Since G is a filter, there is $t \in G$ with $t \leq_{\mathbb{P}} q, r, s$. In particular, $t \in E_{\sigma,\tau}$. For the other direction, suppose that $E_{\sigma,\tau}$ is dense below p. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$. By density of $E_{\sigma,\tau}$ we can take $q \in G$ and $\langle \rho, r \rangle \in \tau$ such that $q \leq_{\mathbb{P}} r$ and $q \Vdash_{\mathbb{P}}^{\mathbb{M},*} \sigma = \rho$. Then $r \in G$ and so $\rho^G \in \tau^G$. Thus by our inductive assumption, condition (2) implies that $\sigma^G = \rho^G \in \tau^G$.

For (2), suppose first that $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \subseteq \tau$, let $\langle \rho, r \rangle \in \sigma$ and let $q \leq_{\mathbb{P}} p, r$. Take a $\mathbb{P}$-generic filter G with $q \in G$. Then $\rho^G \in \sigma^G \subseteq \tau^G$. By our inductive assumption, we can find $s \in G$ so that $s \Vdash_{\mathbb{P}}^{\mathbb{M}} \rho \in \tau$. Given any $q^* \leq_{\mathbb{P}} q$, by strengthening s if necessary, we can find such $s \leq_{\mathbb{P}} q^*$, as desired. Conversely, assume that $p \Vdash_{\mathbb{P}}^{\mathbb{M},*} \sigma \subseteq \tau$ and let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$. Let $\langle \rho, r \rangle \in \sigma$ with $r \in G$. We have to show that $\rho^G \in \tau^G$. Let $q \in G$ be a common strengthening of p and r. Then by assumption, the class $D_{\rho,\tau}$ is dense below q. By genericity, we can take $s \in D_{\rho,\tau} \cap G$. Using our inductive assumption, this shows that $\rho^G \in \tau^G$, as desired.

For (3), consider the class

$$D = \{p \in \mathbb{P} \mid \forall \langle \rho, r \rangle \in \tau \, \forall q \leq_{\mathbb{P}} p, r \; (q \nVdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \rho)\}.$$

Then our inductive assumptions imply that $D \in \mathcal{C}$. Moreover, condition (1) states that D is nonempty below every $p \in \mathbb{P}$ with $p \nVdash_{\mathbb{P}}^{\mathbb{M}} \sigma \in \tau$. Hence it suffices to show that $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \notin \tau$ for every $p \in D$, since then $D \cup \{p \in \mathbb{P} \mid p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \in \tau\} \in \mathcal{C}$ is a dense class of conditions deciding $\sigma \in \tau$. So take $p \in D$ and suppose that $p \nVdash_{\mathbb{P}}^{\mathbb{M}} \sigma \notin \tau$. Then there is a $\mathbb{P}$-generic filter G containing p such that $\sigma^G \in \tau^G$. Then there must be $\langle \rho, r \rangle \in \tau$ with $r \in G$ and $\sigma^G = \rho^G$. By our inductive assumption, we can find $q \in G$ with $q \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \rho$. By possibly strengthening q using that G is a filter, we may assume that $q \leq_{\mathbb{P}} p, r$. But this contradicts that $p \in D$.

In order to verify (4), we define

$$E = \{p \in \mathbb{P} \mid \exists \langle \rho, r \rangle \in \sigma \; [p \leq_{\mathbb{P}} r \wedge \forall q \leq_{\mathbb{P}} p \; (q \nVdash_{\mathbb{P}}^{\mathbb{M}} \rho \in \tau)]\}.$$

As above, E is in $\mathcal{C}$ inductively, and it is nonempty below every condition which does not force $\sigma \subseteq \tau$. As in the proof of (3) it remains to check that $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \nsubseteq \tau$ for each $p \in E$. Assume, towards a contradiction, that there is $p \in E$ with $p \nVdash_{\mathbb{P}}^{\mathbb{M}} \sigma \nsubseteq \tau$. Then there is a $\mathbb{P}$-generic filter with $p \in G$ and $\sigma^G \subseteq \tau^G$. Let $\langle \rho, r \rangle$ witness that $p \in E$. Then $r \in G$ and so $\rho^G \in \sigma^G \subseteq \tau^G$. Using (3) inductively, we obtain $q \in G$ with $q \Vdash_{\mathbb{P}}^{\mathbb{M}} \rho \in \tau$. But then there is $s \leq_{\mathbb{P}} p, q$, contradicting that $p \in E$.

If the definability lemma holds for "$v_0 = v_1$", we can define the $\mathbb{P}$-forcing relation for "$v_0 \in v_1$" by stipulating (as above) that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau$ if and only if the class

$$\{q \in \mathbb{P} \mid \exists \langle \rho, r \rangle \in \tau \, (q \leq_{\mathbb{P}} r \wedge q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \rho)\} \in \mathcal{C}$$

is dense below p. □

Lemma 2.1.4. *Suppose that $\mathbb{P}$ satisfies the forcing theorem for φ over $\mathbb{M}$. Then the following statements hold:*

(1) *$\mathbb{P}$ satisfies the definability lemma for $\neg\varphi$. Moreover, $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi$ if and only if for all $q \leq_{\mathbb{P}} p$, $q \nVdash^{\mathbb{M}}_{\mathbb{P}} \varphi$.*
(2) *The class $E_\varphi = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \vee p \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi\}$ is dense in $\mathbb{P}$.*
(3) *$\mathbb{P}$ satisfies the truth lemma for $\neg\varphi$.*

Proof. For the first statement, suppose first that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi$. Now if there is $q \leq_{\mathbb{P}} p$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$ then we have $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$ and $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi$, a contradiction. Conversely, suppose that the righthand-side holds and let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$. If $\mathbb{M}[G] \models \varphi$ then by the truth lemma for φ there is $q \in G$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$. Since G is directed, there is $r \in G$ with $r \leq_{\mathbb{P}} p, q$. But then $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$, contradicting our assumption on p.

For (2), suppose that $p \nVdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi$. Then by (1) there is $q \leq_{\mathbb{P}} p$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi$ and so $q \in E_\varphi$.

Finally, we show that $\mathbb{P}$ satisfies the truth lemma for $\neg\varphi$. Suppose that G is $\mathbb{P}$-generic over $\mathbb{M}$ with $\mathbb{M}[G] \models \neg\varphi$. By (1) and (2), the class E_φ is in $\mathcal{C}$ and is dense in $\mathbb{P}$, so there must be $p \in G \cap E_\varphi$. In particular, $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi$. □

Theorem 2.1.5. *If $\mathbb{P}$ satisfies the definability lemma either for "$v_0 \in v_1$" or for "$v_0 = v_1$" over $\mathbb{M}$, then $\mathbb{P}$ satisfies the forcing theorem for every $\mathcal{L}_\in$-formula with class parameters over $\mathbb{M}$.*

Proof. By the previous lemma, we already know that $\mathbb{P}$ satisfies the forcing theorem for "$v_0 \in v_1$" and "$v_0 = v_1$".

Let us next consider the atomic formula "$v_0 \in V_1$" involving one class variable V_1. Let $\sigma \in M^{\mathbb{P}}$ and $\Gamma \in \mathcal{C}^{\mathbb{P}}$. We claim that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma$ if and only if the class

$$D = \{q \in \mathbb{P} \mid \exists \langle \tau, r \rangle \in \Gamma \, (q \leq_{\mathbb{P}} r \wedge q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau)\}$$

is dense below p. Note that $D \in \mathcal{C}$ since the definability lemma holds for equality. First assume that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma$ and let $q \leq_{\mathbb{P}} p$. Let G be a $\mathbb{P}$-generic filter with $q \in G$. Then $\sigma^G \in \Gamma^G$, i.e. there is $\langle \tau, r \rangle \in \Gamma$ such that $r \in G$ and $\sigma^G = \tau^G$. By the truth lemma for "$v_0 = v_1$", there is $s \in G$ such that $s \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$. But then every $t \leq_{\mathbb{P}} q, r, s$ is in D. Conversely, if D is dense in $\mathbb{P}$ and G is $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$, then we find $q \leq_{\mathbb{P}} p$ in $D \cap G$. By definition of D there is $\langle \tau, r \rangle \in \Gamma$ such that $q \leq_{\mathbb{P}} r$ and $q \Vdash_{\mathbb{P}} \sigma = \tau$. Thus using that $q, r \in G$ we get $\sigma^G = \tau^G \in \Gamma^G$. For the truth lemma, suppose that G is a $\mathbb{P}$-generic filter with $\sigma^G \in \Gamma^G$. Then there is $\langle \tau, r \rangle \in \Gamma$ with $r \in G$ such that $\sigma^G = \tau^G$. By the truth lemma for equality, there is $p \in G$ below r with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$.

Now we turn to the formula "$V_0 = V_1$", where both V_0 and V_1 are class variables. As in the case of set names, we define first the forcing relation for "$V_0 \subseteq V_1$". More precisely, we check for $\Sigma, \Gamma \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Sigma \subseteq \Gamma$ if and only if for all $\langle \sigma, r \rangle \in \Sigma$ and for all $q \leq_{\mathbb{P}} p, r$ the class

$$D_{\sigma,\Gamma} = \{s \in \mathbb{P} \mid s \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma\} \in \mathcal{C}$$

is dense below q in $\mathbb{P}$. Given that statement, we can define $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Sigma = \Gamma$ by $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Sigma \subseteq \Gamma$ and $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma \subseteq \Sigma$. To prove the claim, suppose first that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Sigma \subseteq \Gamma$ and let $\langle \sigma, r \rangle \in \Sigma$ and $q \leq_{\mathbb{P}} p, r$. Now if G is $\mathbb{P}$-generic over $\mathbb{M}$ with $q \in G$ then $\sigma^G \in \Sigma^G \subseteq \Gamma^G$, so by the truth lemma for "$v_0 \in V_1$" there is $s \in G$ with $s \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma$. Since G is a filter, there is $t \in G$ with $t \leq_{\mathbb{P}} q, s$. In particular, $t \in D_{\sigma,\Gamma}$. For the converse, let G be a $\mathbb{P}$-generic filter with $p \in G$ and let $\langle \sigma, r \rangle \in \Sigma$ with $r \in G$. Then there is $q \in G$ with $q \leq_{\mathbb{P}} p, r$. By assumption and using the genericity of G, we can choose $s \in G$ with $s \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma$. But then $\sigma^G \in \Gamma^G$. Since σ^G was an arbitrary element of Σ^G, this proves the claim. Similarly, it suffices to show the truth lemma for "$V_0 \subseteq V_1$". Now note that by Lemma 2.1.4 the class

$$D = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Sigma \subseteq \Gamma \vee \exists \langle \sigma, r \rangle \in \Sigma \, (p \leq_{\mathbb{P}} r \wedge p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \notin \Gamma)\}$$

is in $\mathcal{C}$. It suffices to check that D is dense. Let $p \in \mathbb{P}$ with $p \nVdash^{\mathbb{M}}_{\mathbb{P}} \Sigma \subseteq \Gamma$. Then there is a $\mathbb{P}$-generic filter G containing p with $\Sigma^G \nsubseteq \Gamma^G$. In particular, there is $\langle \sigma, r \rangle \in \Sigma$ with $r \in G$ and $\sigma^G \notin \Gamma^G$. Since the truth lemma holds for $\sigma \in \Gamma$, it also holds for $\sigma \notin \Gamma$ by Lemma 2.1.4. In particular, there must be $q \leq_{\mathbb{P}} p, r$ in G with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \notin \Gamma$.

For composite $\mathcal{L}_\in$-formulae with class parameters we can define the forcing relation by the usual recursion:

$$\begin{aligned}
p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \wedge \psi \quad &\Longleftrightarrow \quad p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \text{ and } p \Vdash^{\mathbb{M}}_{\mathbb{P}} \psi \\
p \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi \quad &\Longleftrightarrow \quad \forall q \leq_{\mathbb{P}} p \, (q \nVdash^{\mathbb{M}}_{\mathbb{P}} \varphi) \\
p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall x \varphi(x) \quad &\Longleftrightarrow \quad \forall \sigma \in M^{\mathbb{P}} \, (p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma)).
\end{aligned}$$

We have to verify the above equivalences and that the truth lemma holds for the respective formulae. For conjunctions this is clear and for negations this follows from Lemma 2.1.4.

Let $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall x \varphi(x)$ and $\sigma \in M^{\mathbb{P}}$. Again, if $p \nVdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma)$, then there is a $\mathbb{P}$-generic filter G such that $p \in G$ and $M[G] \models \neg\varphi(\sigma^G)$. Since the truth lemma holds for $\neg\varphi$, there must be some $q \leq_{\mathbb{P}} p$ in G which forces $\neg\varphi(\sigma)$. But then it is impossible that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma)$. The converse is trivial. In order to check the truth lemma, let G be $\mathbb{P}$-generic over $\mathbb{M}$ such that $M[G] \models \forall x \varphi(x)$. We claim that the class

$$E = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall x \varphi(x) \vee \exists \sigma \in M^{\mathbb{P}} \, [p \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi(\sigma)]\} \in \mathcal{C}$$

is dense. Suppose that $p \nVdash^{\mathbb{M}}_{\mathbb{P}} \forall x \varphi(x)$. Then there is a $\mathbb{P}$-generic filter G with $p \in G$ such that $\mathbb{M}[G] \models \exists x \neg\varphi(x)$. In particular, there must be $\sigma \in M^{\mathbb{P}}$ such that $\mathbb{M}[G] \models \neg\varphi(\sigma^G)$. Using the truth lemma for $\neg\varphi(\sigma)$, there is $q \in G$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \neg\varphi(\sigma)$. Since G is directed, there is $r \in G$ with $r \leq_{\mathbb{P}} p, q$. In particular, we have $r \in E$, proving that E is dense. Therefore, there is $p \in G \cap E$. Since $M[G] \models \forall x \varphi$, it follows that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall x \varphi$. □

Corollary 2.1.6. *Suppose that $\mathbb{M} \models \mathsf{KM}$ and $\mathbb{P}$ is a notion of class forcing which satisfies the definability lemma either for "$v_0 \in v_1$" or for "$v_0 = v_1$" over $\mathbb{M}$. Then $\mathbb{P}$ satisfies the forcing theorem for every $\mathcal{L}^2_\in$-formula over $\mathbb{M}$.*

Proof. By the recursion given in the proof of Theorem 2.1.5 remains to check that $\mathbb{P}$ satisfies the forcing theorem for formulae of the form $\forall X\varphi(X)$, where X is a class variable. But as in the case of universal quantifiers restricted to sets, we have

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall X\varphi(X) \iff \forall\Gamma \in \mathcal{C}^{\mathbb{P}} \, [p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\Gamma)].$$

Note that this definition is possible, since KM allows us to perform class recursion. □

Another useful application of Theorem 2.1.5 is the following observation.

Lemma 2.1.7. *Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a dense embedding with the property that for all $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ if and only if $\pi(p) \leq_{\mathbb{Q}} \pi(q)$. If G is $\mathbb{P}$-generic over $\mathbb{M}$ and H is the upwards closure of $\pi''G$ in $\mathbb{Q}$, then $\sigma^G = \pi^*(\sigma)^H$ for every $\sigma \in M^{\mathbb{P}}$. Moreover, if $\mathbb{Q}$ satisfies the forcing theorem then so does $\mathbb{P}$.*

Proof. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ and let H denote the upwards closure of $\pi''G$ in $\mathbb{Q}$. By Lemma 1.2.11, H is $\mathbb{Q}$-generic over $\mathbb{M}$. We show by induction on the name rank that for every $\sigma \in M^{\mathbb{P}}$, $\sigma^G = \pi^*(\sigma)^H$. Suppose that this holds for every $\tau \in M^{\mathbb{P}}$ with $\operatorname{rank}(\tau) < \operatorname{rank}(\sigma)$. Then we have

$$\begin{aligned}
\pi^*(\sigma)^H &= \{\pi^*(\tau)^H \mid \exists p\,[\langle \pi^*(\tau), \pi(p)\rangle \in \pi^*(\sigma) \wedge \pi(p) \in H]\} \\
&= \{\tau^G \mid \exists p\,[\langle \tau, p\rangle \in \sigma \wedge \exists q \in G\,(\pi(q) \leq_{\mathbb{Q}} \pi(p))]\} \\
&= \{\tau^G \mid \exists p\,[\langle \tau, p\rangle \in \sigma \wedge \exists q \in G\,(q \leq_{\mathbb{P}} p)]\} \\
&= \{\tau^G \mid \exists p \in G\,(\langle \tau, p\rangle \in \sigma)\} = \sigma^G.
\end{aligned}$$

For the second claim, note that by Theorem 2.1.5 it is enough to check that for $p \in \mathbb{P}$ and $\sigma, \tau \in M^{\mathbb{P}}$, $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau$ if and only if $\pi(p) \Vdash^{\mathbb{M}}_{\mathbb{Q}} \pi^*(\sigma) \in \pi^*(\tau)$. Suppose first that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau$ and let H be $\mathbb{Q}$-generic with $\pi(p) \in H$ and let G denote $\pi^{-1}[H]$. By Lemma 1.2.11, G is $\mathbb{P}$-generic over $\mathbb{M}$ and $p \in G$. This implies that $\pi^*(\sigma)^H = \sigma^G = \tau^G = \pi^*(\tau)^H$ as desired. The converse is analogous. □

Note that in the second statement of Lemma 2.1.7 the converse may fail, i.e. if $\mathbb{P}$ satisfies the forcing theorem it does not always hold that $\mathbb{Q}$ does so as well. We will prove this in Section 4.2. Furthermore, $\mathbb{Q}$-generic extensions can be strictly larger then their $\mathbb{P}$-generic counterpart. This shall be discussed in 4.5.

2.1.2 The forcing theorem for infinitary formulae

In the following, we introduce an infinitary language and prove that the forcing theorem for atomic $\mathcal{L}_\in$-formulae already implies the forcing theorem for infinitary formulae in the language specified below. This language, that may be of independent interest, is an important tool for our negative results in Section 2.5 as well as to provide an alternative characterization of the forcing theorem in terms of the existence of a Boolean completion in Section 3.2.

Let $\mathcal{L}_{\mathrm{Ord},0}$ denote the infinitary quantifier-free language that allows for set-sized conjunctions and disjunctions. By $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$ we denote the language of such infinitary quantifier-free formulae in the forcing language of $\mathbb{P}$ over M that allows reference to the generic predicate G. More precisely, its constants are all elements of $M^{\mathbb{P}}$, and it has an additional predicate $\dot{G}$. We define $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$ and the class $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$ of Gödel codes of $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formulae by simultaneous recursion:

(1) Atomic $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formulae are of the form $\sigma=\tau, \sigma\in\tau$ or $\check{p}\in\dot{G}$ for $\sigma,\tau\in M^{\mathbb{P}}$ and $p\in\mathbb{P}$, where $\dot{G}=\{\langle\check{p},p\rangle\mid p\in\mathbb{P}\}\in\mathcal{C}^{\mathbb{P}}$ is the canonical class name for the generic filter. Gödel codes of atomic $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formulae are given by

$$\begin{aligned}\ulcorner\check{p}\in\dot{G}\urcorner&=\langle 0,p\rangle\\ \ulcorner\sigma=\tau\urcorner&=\langle 1,\sigma,\tau\rangle\\ \ulcorner\sigma\in\tau\urcorner&=\langle 2,\sigma,\tau\rangle.\end{aligned}$$

(2) If φ is an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formula, then so is $\neg\varphi$, and its Gödel code is given by

$$\ulcorner\neg\varphi\urcorner=\langle 3,\ulcorner\varphi\urcorner\rangle.$$

(3) If $I\in M$ and for every $i\in I$, φ_i is an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formula such that $\langle\ulcorner\varphi_i\urcorner\mid i\in I\rangle\in M$, then so are $\bigvee_{i\in I}\varphi_i$ and $\bigwedge_{i\in I}\varphi_i$ and their Gödel codes are given by

$$\begin{aligned}\ulcorner\bigvee_{i\in I}\varphi_i\urcorner&=\langle 4,I,\{\langle i,\ulcorner\varphi_i\urcorner\rangle\mid i\in I\}\rangle\\ \ulcorner\bigwedge_{i\in I}\varphi_i\urcorner&=\langle 5,I,\{\langle i,\ulcorner\varphi_i\urcorner\rangle\mid i\in I\}\rangle.\end{aligned}$$

Now define $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)\in\mathcal{C}$ to be the class of all Gödel codes of infinitary formulae in the forcing language of $\mathbb{P}$ over M. If G is a $\mathbb{P}$-generic filter over $\mathbb{M}$ and φ is an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formula, then we write φ^G for the formula obtained from φ by replacing each $\mathbb{P}$-name σ occurring in φ by its evaluation σ^G, and by evaluating $\dot{G}$ as G. Note that φ^G is a formula in the infinitary language $\mathcal{L}_{\mathrm{Ord},0}$ with an additional predicate for the generic G. Given an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formula φ and $p\in\mathbb{P}$, we write $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\varphi$ to denote that $\mathbb{M}[G]\models\varphi^G$ whenever G is a $\mathbb{P}$-generic filter over $\mathbb{M}$ with $p\in G$.

Definition 2.1.8. We say that $\mathbb{P}$ *satisfies the uniform forcing theorem for* $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-*formulae* over $\mathbb{M}$, if

$$\{\langle p,\ulcorner\varphi\urcorner\rangle\in P\times\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)\mid p\Vdash^{\mathbb{M}}_{\mathbb{P}}\varphi\}\in\mathcal{C}$$

and $\mathbb{P}$ satisfies the truth lemma for every $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$-formula φ over $\mathbb{M}$, i.e. for every $\mathbb{P}$-generic filter G over $\mathbb{M}$, if $\mathbb{M}[G]\models\varphi^G$ then there is $p\in G$ such that $p\Vdash^{\mathbb{M}}_{\mathbb{P}}\varphi$.

The following lemma will allow us to infer that the uniform forcing theorem for infinitary formulae is equivalent to the forcing theorem for equality.

Lemma 2.1.9. *There is an assignment*

$$\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)\to M^{\mathbb{P}}\times M^{\mathbb{P}},\ \ulcorner\varphi\urcorner\mapsto\langle\nu_{\ulcorner\varphi\urcorner},\mu_{\ulcorner\varphi\urcorner}\rangle$$

such that $\{\langle\ulcorner\varphi\urcorner,\nu_{\ulcorner\varphi\urcorner},\mu_{\ulcorner\varphi\urcorner}\rangle\mid\ulcorner\varphi\urcorner\in\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)\}\in\mathcal{C}$ *and*

$$\mathbb{1}_{\mathbb{P}}\Vdash^{\mathbb{M}}_{\mathbb{P}}\varphi\leftrightarrow\nu_{\ulcorner\varphi\urcorner}=\mu_{\ulcorner\varphi\urcorner}\tag{2.1}$$

for every $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P},M)$*-formula* φ.

Proof. We will argue by induction that, given names ν_ψ and μ_ψ satisfying (2.1) for every proper subformula ψ of φ, we can, uniformly in $\ulcorner\varphi\urcorner$, define $\nu_{\ulcorner\varphi\urcorner}$ and $\mu_{\ulcorner\varphi\urcorner}$ such that (2.1) holds.

Observe that since $\neg\bigwedge_{i\in I}\varphi_i \equiv \bigvee_{i\in I}\neg\varphi_i$ and $\neg\bigvee_{i\in I}\varphi_i \equiv \bigwedge_{i\in I}\neg\varphi_i$ we can assume that all formulae are in negation normal form, i.e. the negation operator is applied to atomic formulae only. Next, due to the equivalences

$$\begin{aligned}
\sigma \neq \tau &\equiv \sigma \not\subseteq \tau \vee \tau \not\subseteq \sigma,\\
\sigma \not\subseteq \tau &\equiv \bigvee_{\langle\pi,p\rangle\in\sigma} (\pi \notin \tau \wedge \check{p} \in \dot{G}),\\
\sigma \notin \tau &\equiv \bigwedge_{\langle\pi,p\rangle\in\tau} (\sigma \neq \pi \vee \check{p} \notin \dot{G}),
\end{aligned}$$

we can further suppose that the only negated formulae are of the form $\check{p} \notin \dot{G}$.

For the atomic cases, let

$$\begin{aligned}
&\nu_{\ulcorner\check{p}\in\dot{G}\urcorner} = \{\langle\check{0},p\rangle\}, && \mu_{\ulcorner\check{p}\in\dot{G}\urcorner} = \check{1},\\
&\nu_{\ulcorner\check{p}\notin\dot{G}\urcorner} = \emptyset, && \mu_{\ulcorner\check{p}\notin\dot{G}\urcorner} = \{\langle\check{0},p\rangle\},\\
&\nu_{\ulcorner\sigma=\tau\urcorner} = \sigma, && \mu_{\ulcorner\sigma=\tau\urcorner} = \tau,\\
&\nu_{\ulcorner\sigma\in\tau\urcorner} = \tau, && \mu_{\ulcorner\sigma\in\tau\urcorner} = \tau \cup \{\langle\sigma,\mathbb{1}_{\mathbb{P}}\rangle\}.
\end{aligned}$$

It is easy to check that (2.1) holds for all atomic formulae.

If φ is a conjunction of the form $\bigwedge_{i\in I}\varphi_i$ and $\nu_{\ulcorner\varphi_i\urcorner}, \mu_{\ulcorner\varphi_i\urcorner}$ have already been defined for $i \in I$, let

$$\begin{aligned}
\nu_{\ulcorner\varphi\urcorner} &= \{\langle\mathsf{op}(\nu_{\ulcorner\varphi_i\urcorner},\check{i}),\mathbb{1}_{\mathbb{P}}\rangle \mid i \in I\} \text{ and}\\
\mu_{\ulcorner\varphi\urcorner} &= \{\langle\mathsf{op}(\mu_{\ulcorner\varphi_i\urcorner},\check{i}),\mathbb{1}_{\mathbb{P}}\rangle \mid i \in I\}.
\end{aligned}$$

If G is $\mathbb{P}$-generic over $\mathbb{M}$ and $\mathbb{M}[G] \models \varphi^G$, then $\mathbb{M}[G] \models \varphi_i^G$ for all $i \in I$. By assumption, this means that $\nu^G_{\ulcorner\varphi_i\urcorner} = \mu^G_{\ulcorner\varphi_i\urcorner}$ for every $i \in I$, thus also $\nu^G_{\ulcorner\varphi\urcorner} = \mu^G_{\ulcorner\varphi\urcorner}$. The converse is similar.

Next suppose that φ is of the form $\bigvee_{i\in I}\varphi_i$. Let $\bar{\nu}_{\ulcorner\varphi_i\urcorner} = \mathsf{op}(\nu_{\ulcorner\varphi_i\urcorner},\check{i})$ and $\bar{\mu}_{\ulcorner\varphi_i\urcorner} = \mathsf{op}(\mu_{\ulcorner\varphi_i\urcorner},\check{i})$ for each $i \in I$. Let

$$\begin{aligned}
\pi_{\ulcorner\varphi\urcorner} &= \{\langle\mathsf{op}(\bar{\nu}_{\ulcorner\varphi_i\urcorner},\bar{\mu}_{\ulcorner\varphi_i\urcorner}),\mathbb{1}_{\mathbb{P}}\rangle \mid i \in I\} \cup \{\langle\mathsf{op}(\bar{\nu}_{\ulcorner\varphi_i\urcorner},\bar{\nu}_{\ulcorner\varphi_i\urcorner}),\mathbb{1}_{\mathbb{P}}\rangle \mid i \in I\},\\
\nu^i_{\ulcorner\varphi\urcorner} &= \pi_{\ulcorner\varphi\urcorner} \setminus \{\langle\mathsf{op}(\bar{\nu}_{\ulcorner\varphi_i\urcorner},\bar{\mu}_{\ulcorner\varphi_i\urcorner}),\mathbb{1}_{\mathbb{P}}\}.
\end{aligned}$$

Now we define

$$\begin{aligned}
\nu_{\ulcorner\varphi\urcorner} &= \{\langle\nu^i_{\ulcorner\varphi\urcorner},\mathbb{1}_{\mathbb{P}}\rangle \mid i \in I\}\\
\mu_{\ulcorner\varphi\urcorner} &= \nu_{\ulcorner\varphi\urcorner} \cup \{\langle\pi_{\ulcorner\varphi\urcorner},\mathbb{1}_{\mathbb{P}}\rangle\}.
\end{aligned}$$

If G is $\mathbb{P}$-generic and $\mathbb{M}[G] \models \varphi^G$ there is some $i \in I$ such that $\mathbb{M}[G] \models \varphi_i^G$. By induction, this implies that $\mathbb{M}[G] \models \nu^G_{\ulcorner\varphi_i\urcorner} = \mu^G_{\ulcorner\varphi_i\urcorner}$. Thus $\pi^G_{\ulcorner\varphi\urcorner} = (\nu^i_{\ulcorner\varphi\urcorner})^G$ and $\nu^G_{\ulcorner\varphi\urcorner} = \mu^G_{\ulcorner\varphi\urcorner}$. For the converse, suppose that there is a generic G such that $\mathbb{M}[G] \models \neg\varphi^G$, hence for every $i \in I$, $\mathbb{M}[G] \models \neg\varphi_i^G$. But then in $M[G]$, for every $i \in I$, we have $\nu^G_{\ulcorner\varphi_i\urcorner} \neq \mu^G_{\ulcorner\varphi_i\urcorner}$. Therefore $\pi^G_{\ulcorner\varphi\urcorner}$ is not of the form $(\nu^i_{\ulcorner\varphi\urcorner})^G$ for any $i \in I$, which shows that $\pi^G_{\ulcorner\varphi\urcorner} \in \mu^G_{\ulcorner\varphi\urcorner} \setminus \nu^G_{\ulcorner\varphi\urcorner}$. □

Corollary 2.1.10. *If $\mathbb{P}$ satisfies the definability lemma either for "$v_0 \in v_1$" or "$v_0 = v_1$" over $\mathbb{M}$, then $\mathbb{P}$ satisfies the uniform forcing theorem for $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae over $\mathbb{M}$.*

Proof. Due to Lemma 2.1.3 we may assume that $\mathbb{P}$ satisfies the forcing theorem for "$v_0 = v_1$" over $\mathbb{M}$. By Lemma 2.1.9 for every $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formula φ there are $\mathbb{P}$-names $\nu_{\ulcorner\varphi\urcorner}$ and $\mu_{\ulcorner\varphi\urcorner}$ such that $\{\langle\ulcorner\varphi\urcorner, \nu_{\ulcorner\varphi\urcorner}, \mu_{\ulcorner\varphi\urcorner}\rangle \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}^{\Vdash}_{\mathrm{Ord}_0}(\mathbb{P}, M)\} \in \mathcal{C}$ and $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \leftrightarrow \nu_{\ulcorner\varphi\urcorner} = \mu_{\ulcorner\varphi\urcorner}$. Therefore, we can define the forcing relation for $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae by stipulating

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \Longleftrightarrow p \Vdash^{\mathbb{M}}_{\mathbb{P}} \mu_{\ulcorner\varphi\urcorner} = \mu_{\ulcorner\varphi\urcorner}.$$

This proves the uniform forcing theorem for $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae over $\mathbb{M}$. □

2.1.3 Products

We prove a class version of the product lemma. This is essentially an adaptation of [Fri00, Lemma 2.27] to our setting. For the sake of completeness, we nevertheless provide a full proof.

Lemma 2.1.11 (Product lemma)**.** *Suppose that $\mathbb{M}$ is a countable transitive model of GB^- and $\mathbb{P}$ and $\mathbb{Q}$ are notions of class forcing for $\mathbb{M}$. Then the following statements hold for $\mathbb{R} = \mathbb{P} \times \mathbb{Q}$.*

(1) *If G is $\mathbb{P}$-generic over $\mathbb{M}$ and H is $\mathbb{Q}$-generic over $\mathbb{M}[G]$, then $G \times H$ is $\mathbb{R}$-generic over $\mathbb{M}$.*
(2) *If K is $\mathbb{R}$-generic over $\mathbb{M}$, then K is of the form $G \times H$, where G is $\mathbb{P}$-generic over $\mathbb{M}$. Moreover, if $\mathbb{P}$ satisfies the forcing theorem over $\mathbb{M}$, then H is $\mathbb{Q}$-generic over $\mathbb{M}[G]$.*

Proof. Suppose first that G is $\mathbb{P}$-generic over $\mathbb{M}$ and H is $\mathbb{Q}$-generic over $\mathbb{M}[G]$. It is easy to check that $G \times H$ is a filter. We claim that $G \times H$ is $\mathbb{R}$-generic over $\mathbb{M}$. Suppose that $D \in \mathcal{C}$ is a dense subclass of $\mathbb{R}$. We claim that

$$D_{\mathbb{Q}} = \{q \in \mathbb{Q} \mid \exists p \in G\,(\langle p, q\rangle \in D)\}$$

is dense in $\mathbb{Q}$. First note that $D_{\mathbb{Q}} = \{\langle\check{q}, p\rangle \mid \langle p, q\rangle \in D\}^G$, so $D_{\mathbb{Q}} \in \mathcal{C}[G]$. In order to prove its density, fix $q \in \mathbb{Q}$ and consider

$$D^q_{\mathbb{P}} = \{p \in \mathbb{P} \mid \exists \bar{q} \leq_{\mathbb{Q}} q\,(\langle p, \bar{q}\rangle \in D)\} \in \mathcal{C}.$$

Then $D^q_{\mathbb{P}}$ is dense by density of D. Choose $p \in G \cap D^q_{\mathbb{P}}$ and $\bar{q} \leq_{\mathbb{Q}} q$ such that $\langle p, \bar{q}\rangle \in D$. Then $\bar{q} \in D_{\mathbb{Q}}$ proving that $D_{\mathbb{Q}}$ is dense. Take now $q \in H \cap D_{\mathbb{Q}}$ and $p \in G$ such that $\langle p, q\rangle \in D$. Then $\langle p, q\rangle \in (G \times H) \cap D$ and so $G \times H$ is $\mathbb{R}$-generic over $\mathbb{M}$.

For the second claim, suppose that K is $\mathbb{R}$-generic over $\mathbb{M}$. We define

$$\begin{aligned} G =& \{p \in \mathbb{P} \mid \exists q\,(\langle p, q\rangle \in K)\} \\ H =& \{q \in \mathbb{Q} \mid \exists p\,(\langle p, q\rangle \in K)\}. \end{aligned}$$

It is obvious that $K \subseteq G \times H$. For the converse, suppose that $\langle p, q\rangle \in G \times H$. Choose $\bar{p} \in \mathbb{P}$ and $\bar{q} \in \mathbb{Q}$ such that $\langle p, \bar{q}\rangle, \langle\bar{p}, q\rangle \in K$. Since K is a filter, there is $\langle\tilde{p}, \tilde{q}\rangle \in K$

with $\langle\tilde{p},\tilde{q}\rangle \leq_{\mathbb{R}} \langle p,\bar{q}\rangle, \langle\bar{p},q\rangle$. In particular, $\langle\tilde{p},\tilde{q}\rangle \leq_{\mathbb{R}} \langle p,q\rangle$ and so $\langle p,q\rangle \in K$. Next, we show that G is $\mathbb{P}$-generic over $\mathbb{M}$. Let $D_{\mathbb{P}} \subseteq \mathbb{P}$ be a dense class in $\mathcal{C}$. Consider $D = \{\langle p,q\rangle \mid p \in D_{\mathbb{P}}\} \in \mathcal{C}$. Clearly, D is dense in $\mathbb{R}$ and so there is $\langle p,q\rangle \in K \cap D$. In particular, $p \in G \cap D_{\mathbb{P}}$. Suppose now that $\mathbb{P}$ satisfies the forcing theorem. We have to verify that H is $\mathbb{Q}$-generic over $\mathbb{M}[G]$. Let $E_{\mathbb{Q}} \subseteq \mathbb{Q}$ be a dense class in $\mathcal{C}[G]$, hence there are a name $\dot{E}_{\mathbb{Q}} \in \mathcal{C}^{\mathbb{P}}$ and a condition $p \in G$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{E}_{\mathbb{Q}}$ is dense". Consider

$$E = \{\langle\bar{p},q\rangle \mid \bar{p} \Vdash^{\mathbb{M}}_{\mathbb{P}} \check{q} \in \dot{E}_{\mathbb{Q}}\} \in \mathcal{C}.$$

Observe that E is a dense subset of $\mathbb{R}$ below $\langle p, \mathbb{1}_{\mathbb{Q}}\rangle$. Let $\langle\bar{p},q\rangle \in K \cap E$. Then $\bar{p} \in G$ and so $q \in H \cap \dot{E}^G_{\mathbb{Q}} = H \cap E_{\mathbb{Q}}$ proving the density of $E_{\mathbb{Q}}$. □

Notice that we do not know whether the product of two forcing notions which satisfy the forcing theorem over some countable transitive model $\mathbb{M} = \langle M,\mathcal{C}\rangle \models \mathsf{GB}^-$ satisfies the forcing theorem over $\mathbb{M}$. The canonical attempt would be to dfine the forcing relation for $\mathbb{P} \times \mathbb{Q}$ by

$$\langle p,q\rangle \Vdash^{\mathbb{M}}_{\mathbb{P}\times\mathbb{Q}} \sigma \in \tau \iff p \Vdash^{\mathbb{M}}_{\mathbb{P}} \text{“}\check{q} \Vdash^{\mathbb{M}[\dot{G}]}_{\mathbb{Q}} \sigma^\star \in \tau^\star\text{”},$$

where for $\sigma \in M^{\mathbb{P}\times\mathbb{Q}}$, $\sigma^\star$ is a $\mathbb{P}$-name for a $\mathbb{Q}$-name defined by the recursion

$$\sigma^\star = \{\langle \mathsf{op}(\tau^\star,\check{q}), p\rangle \mid \langle\tau,\langle p,q\rangle\rangle \in \sigma\}.$$

This is, however, problematic since it is not clear whether $\mathbb{Q}$ still satisfies the forcing theorem over $\mathbb{M}[G]$. For further discussion of this topic, consult Question 4.

2.2 Pretameness

Pretameness was introduced by Sy Friedman in [Fri00] as a property of class-sized forcing notions which not only implies the forcing theorem but also characterizes the preservation of GB^- over models with a hierarchy. Before we are able to state the definition of pretameness, we need the following notation.

Convention. Suppose that $\mathbb{M} = \langle M,\mathcal{C}\rangle$ is a model of GB^-. Given a sequence of the form $\vec{C} = \langle C_i \mid i \in I\rangle$ with $C_i \in \mathcal{C}$ for $i \in I$ and $I \in \mathcal{C}$, we identify $\vec{C}$ with its *code* $\{\langle c,i\rangle \mid i \in I \wedge c \in C_i\}$. In particular, we say that the sequence $\vec{C}$ is an element of $\mathcal{C}$ if its code is in $\mathcal{C}$.

Definition 2.2.1. A notion of forcing $\mathbb{P}$ for $\mathbb{M}$ is *pretame* for $\mathbb{M} = \langle M,\mathcal{C}\rangle \models \mathsf{GB}^-$, if for every $p \in \mathbb{P}$ and for every sequence of dense classes $\langle D_i \mid i \in I\rangle \in \mathcal{C}$ with $I \in M$, there is $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I\rangle \in M$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense in $\mathbb{P}$ below q.

2.2.1 Pretameness implies the forcing theorem

The following theorem is an easy adaptation of [Fri00, Theorem 2.18] to our generalized setting. For the benefit of the reader, we nevertheless include its proof.

Theorem 2.2.2 (Sy Friedman). *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^- with a hierarchy, and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. If $\mathbb{P}$ is pretame over $\mathbb{M}$ then $\mathbb{P}$ satisfies the forcing theorem over $\mathbb{M}$.*

Proof. Suppose that $C = \langle C_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle$ witnesses that $\mathbb{M}$ has a hierarchy. Observe first that by Theorem 2.1.5 it suffices to check the definability of the forcing relation of $\mathbb{P}$ for atomic formulae. To achieve this, we construct a class function

$$F : \mathbb{P} \times M^{\mathbb{P}} \times M^{\mathbb{P}} \times 2 \to M \times 2$$

in $\mathcal{C}$ such that for $p \in \mathbb{P}$ and $\sigma, \tau \in M^{\mathbb{P}}$, $F(p, \sigma, \tau, i) = \langle d, j \rangle$ for some nonempty set $d \subseteq \{q \in \mathbb{P} \mid q \leq_{\mathbb{P}} p\}$ and for all $q \in d$, q decides either $\sigma \in \tau$ (in case $i = 0$) or $\sigma = \tau$ (in case $i = 1$) either positively (if $j = 1$) or negatively (if $j = 0$).[1] Given such F, we can define the $\mathbb{P}$-forcing relation by

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau \iff \forall q \leq_{\mathbb{P}} p \, \exists d \; F(q, \sigma, \tau, 0) = \langle d, 1 \rangle$$

and similarly for $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$.

We are left with defining such a function F by induction on

$$\langle \mathrm{rank}(\sigma) + \mathrm{rank}(\tau), \mathrm{rank}(\sigma) \rangle,$$

ordered lexicographically. If $\mathrm{rank}(\sigma) + \mathrm{rank}(\tau) = 0$, we simply put $F(p, \sigma, \tau, 0) = \langle \{p\}, 0 \rangle$ and $F(p, \sigma, \tau, 1) = \langle \{p\}, 1 \rangle$. Suppose now that $\mathrm{rank}(\sigma) + \mathrm{rank}(\tau) > 0$. We start with defining $F(p, \sigma, \tau, 0)$. By induction, we may assume that for all $\pi \in \mathrm{dom}(\tau)$ and for all $q \in \mathbb{P}$, $F(q, \sigma, \pi, 1)$ has already been defined. There are two cases:

Case 1. There exist $\langle \pi, r \rangle \in \tau$ and $q \leq_{\mathbb{P}} p, r$ such that $F(q, \sigma, \pi, 1) = \langle d, 1 \rangle$ for some $d \in M$. Let $\alpha \in \mathrm{Ord}^M$ be the minimal C-rank of such a set d. Then put $F(p, \sigma, \tau, 0) = \langle e, 1 \rangle$, where

$$e = \bigcup \{d \in C_{\alpha+1} \mid \exists \langle \pi, r \rangle \in \tau \, \exists q \leq_{\mathbb{P}} p, r \; F(q, \sigma, \pi, 1) = \langle d, 1 \rangle\}.$$

Case 2. Suppose we are not in Case 1. For each $\langle \pi, r \rangle \in \tau$, consider

$$D_{\pi,r} = \bigcup \{d \in M \mid \exists q \leq_{\mathbb{P}} p, r \; F(q, \sigma, \pi, 1) = \langle d, 0 \rangle\} \cup \{q \leq_{\mathbb{P}} p \mid q \perp_{\mathbb{P}} r\}.$$

We show that each $D_{\pi,r}$ is dense below p. Let $q \leq_{\mathbb{P}} p$. We want to find $s \leq_{\mathbb{P}} q$ in $D_{\pi,r}$. If $q \perp_{\mathbb{P}} r$ then we are done. Otherwise take $s \leq_{\mathbb{P}} q, r$. Since we are not in Case 1, $F(s, \sigma, \pi, 1) = \langle d, 0 \rangle$ for some $d \in M \setminus \{\emptyset\}$. Since d is nonempty, we may pick some condition $t \in d$. Then $t \in D_{\pi,r}$ and $t \leq_{\mathbb{P}} s \leq_{\mathbb{P}} q$.

By pretameness, there are conditions $q \leq_{\mathbb{P}} p$ and $\langle d_{\pi,r} \mid \langle \pi, r \rangle \in \tau \rangle \in M$ such that each $d_{\pi,r}$ is a subset of $D_{\pi,r}$ which is predense below q. Let $\alpha \in \mathrm{Ord}^M$ be minimal such that there is such q in $C_{\alpha+1}$. Then put $F(p, \sigma, \tau, 0) = \langle e, 0 \rangle$ where

$$e = \{q \in C_{\alpha+1} \cap \mathbb{P} \mid \exists \langle d_{\pi,r} \mid \langle \pi, r \rangle \in \tau \rangle \in M \text{ (each } d_{\pi,r} \subseteq D_{\pi,r} \text{ is predense below } q)\}.$$

[1] If for example $F(p, \sigma, \tau, 0) = \langle d, 1 \rangle$, then for all $q \in d$, $q \Vdash_{\mathbb{P}} \sigma \in \tau$.

Now we define $F(p,\sigma,\tau,1)$. Again, we may inductively assume that for every $\pi \in \mathrm{dom}(\sigma\cup\tau)$ and for every $q\in\mathbb{P}$, $F(q,\pi,\sigma,0)$ and $F(q,\pi,\tau,0)$ have already been defined. As above, we make a case distinction:

Case 1. There exist $\langle\pi,r\rangle\in\sigma\cup\tau$, a condition $q\in\mathbb{P}$ that is stronger than both p and r, $i\in 2$, $d,e\in M$ and $s\in d$ such that $F(q,\pi,\sigma,0)=\langle d,i\rangle$ and $F(s,\pi,\tau,0)=\langle e,1-i\rangle$. Then let $\alpha\in\mathrm{Ord}^M$ be the minimal C-rank of such a set e. Let $F(p,\sigma,\tau,1)=\langle f,0\rangle$, where

$$f=\bigcup\{e\in C_{\alpha+1}\mid \exists\langle\pi,r\rangle\in\sigma\cup\tau\,\exists q\leq_{\mathbb{P}} p,r\,\exists i\in 2\,\exists d\in M\,\exists s\in d\\ (F(q,\pi,\sigma,0)=\langle d,i\rangle\wedge F(s,\pi,\tau,0)=\langle e,1-i\rangle)\}.$$

Case 2. Suppose that we are not in Case 1. For each $\langle\pi,r\rangle\in\sigma\cup\tau$ let

$$D_{\pi,r}=\bigcup\{e\mid \exists q\leq_{\mathbb{P}} r\,\exists i\in 2\,\exists d\,\exists s\in d\,(F(q,\pi,\sigma,0)=\langle d,i\rangle\wedge\\ F(s,\pi,\tau,0)=\langle e,i\rangle)\}\cup\{q\in\mathbb{P}\mid q\perp_{\mathbb{P}} r\}.$$

Since Case 1 fails, each $D_{\pi,r}$ is dense below p. By pretameness there exist $q\leq_{\mathbb{P}} p$ and $\langle d_{\pi,r}\mid\langle\pi,r\rangle\in\sigma\cup\tau\rangle\in M$ such that each $d_{\pi,r}\subseteq D_{\pi,r}$ is predense below q. Let $\alpha\in\mathrm{Ord}^M$ be the least C-rank of such a condition q. Then put $F(p,\sigma,\tau,0)=\langle f,1\rangle$ for

$$f=\{q\in C_{\alpha+1}\cap\mathbb{P}\mid\exists\langle d_{\pi,r}\mid\langle\pi,r\rangle\in\sigma\cup\tau\rangle\in M\,(\text{each } d_{\pi,r}\subseteq D_{\pi,r} \text{ is predense below } q)\}.$$

This finishes the construction of F. It remains to check that F satisfies our desired properties. We proceed by induction. Suppose that $F(p,\sigma,\tau,0)=\langle e,1\rangle$. We have to verify that for every $q\in e$, $q\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma\in\tau$. Take $q\in e$ and a $\mathbb{P}$-generic filter G with $q\in G$. Since we are in Case 1, there is $\langle\pi,r\rangle\in\tau$ and $s\leq_{\mathbb{P}} p,r$ with $F(s,\sigma,\pi,1)=\langle d,1\rangle$ for some d and $q\in d$. Then $q\leq_{\mathbb{P}} s$ and so $s\in G$. But by induction, since $\mathrm{rank}(\pi)<\mathrm{rank}(\tau)$, $q\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma=\pi$ and so $\sigma^G=\pi^G\in\tau^G$.

Secondly, assume that $F(p,\sigma,\tau,0)=\langle e,0\rangle$ and let $q\in e$ and G be $\mathbb{P}$-generic over $\mathbb{M}$ with $q\in G$. Now by Case 2 there is a sequence $\langle d_{\pi,r}\mid\langle\pi,r\rangle\in\tau\rangle$ of sets $d_{\pi,r}\subseteq D_{\pi,r}$ in M which are predense below q. Suppose for a contradiction that $M[G]\models\sigma^G\in\tau^G$. Then there is $\langle\pi,r\rangle\in\tau$ with $r\in G$ and $\sigma^G=\pi^G$. Since $d_{\pi,r}$ is predense below q, there is $s\in d_{\pi,r}\cap G$. Then s is compatible with r and so there are $d\in M$ and $t\leq_{\mathbb{P}} r$ with $F(t,\sigma,\pi,1)=\langle d,0\rangle$ and $s\in d$. By induction, $s\Vdash^{\mathbb{M}}_{\mathbb{P}}\sigma\neq\pi$, contradicting that $\sigma^G=\pi^G$. The proof that $F(p,\sigma,\tau,1)$ is as desired follows in a similar way. □

The next theorem is a version of a theorem of Sy Friedman [Fri00, Lemma 2.19], that we adjusted to our generalized setting.

Theorem 2.2.3. *Let $\mathbb{M}=\langle M,\mathcal{C}\rangle$ be a model of GB^- with a hierarchy witnessed by $\langle C_\alpha\mid\alpha\in\mathrm{Ord}^M\rangle$. If $\mathbb{P}$ is pretame for $\mathbb{M}$ and G is $\mathbb{P}$-generic over $\mathbb{M}$ then $\mathbb{M}[G]$ satisfies GB^- and has a hierarchy. Moreover, if $\mathbb{M}$ satisfies the axiom of choice (resp. global choice), then so does $\mathbb{M}[G]$.*

Proof. Suppose that $\mathbb{P}$ is pretame and that G is $\mathbb{P}$-generic over $\mathbb{M}$. By Lemma 1.2.9, the only non-trivial set axioms are separation and collection. Moreover, collection implies separation. Note that by Theorem 2.2.2, $\mathbb{P}$ satisfies the forcing theorem.

To see that $\mathbb{M}[G]$ satisfies collection, assume that

$$\mathbb{M}[G] \models \forall x \in \sigma^G \, \exists y \; \varphi(x, y, \Gamma^G),$$

where $\sigma \in M^{\mathbb{P}}$, $\Gamma \in \mathcal{C}^{\mathbb{P}}$ and φ is an $\mathcal{L}_\in$-formula with one class parameter. By the truth lemma there is $p \in G$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall x \in \sigma \, \exists y \, \varphi(x, y, \Gamma)$. For each $\langle \pi, r \rangle \in \sigma$, the class

$$D_{\pi,r} = \{s \in \mathbb{P} \mid [s \leq_{\mathbb{P}} p, r \wedge \exists \mu \in M^{\mathbb{P}} \, (s \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\pi, \mu, \Gamma))] \vee s \perp_{\mathbb{P}} r\} \in \mathcal{C}$$

is dense below p in $\mathbb{P}$. By pretameness there is $q \in G$ which strengthens p and there are sets $d_{\pi,r} \subseteq D_{\pi,r}$ for each $\langle \pi, r \rangle \in \sigma$ such that each $d_{\pi,r} \in M$ is predense below q. Using collection in $\mathbb{M}$, there is a set $x \in M$ such that for each $\langle \pi, r \rangle \in \sigma$ and for each $s \in d_{\pi,r}$ there is $\mu \in x$ such that $s \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\pi, \mu, \Gamma)$. Now put

$$\tau = \{\langle \mu, s \rangle \mid \mu \in x \wedge \exists \langle \pi, r \rangle \in \sigma \, [s \in d_{\pi,r} \wedge s \Vdash_{\mathbb{P}} \varphi(\pi, \mu, \Gamma)]\}.$$

By construction, $\mathbb{M}[G] \models \forall x \in \sigma^G \, \exists y \in \tau^G \, \varphi(x, y, \Gamma^G)$.

First-order class comprehension follows from Lemma 2.1.2 using the forcing theorem. The other class axioms of GB^- are trivial. Furthermore, we can define a hierarchy $\langle D_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle$ in $\mathcal{C}[G]$ by

$$D_\alpha = \{x \in \mathbb{M}[G] \mid \exists \sigma \in M^{\mathbb{P}} \cap C_\alpha(\sigma^G = x)\} = \{\langle \sigma, \mathbb{1} \rangle \mid \sigma \in C_\alpha\}^G \in M[G]$$

for every $\alpha \in \mathrm{Ord}^M$. Finally, if $\prec$ is a global well-order of M in $\mathcal{C}$ then

$$x \lhd y \iff \exists \sigma \in M^{\mathbb{P}}[x = \sigma^G \wedge \forall \tau \in M^{\mathbb{P}}(y = \tau^G \to \sigma \prec \tau)]$$

defines a global well-order of $M[G]$ in $\mathcal{C}[G]$. The proof that if the axiom of choice is preserved is analogous. □

Remark 2.2.4. Note that in the proof of Theorem 2.2.3, if we know that $\mathbb{P}$ satisfies the forcing theorem, we do not require $\mathbb{M}$ to have a hierarchy in order to show that GB^- is preserved.

2.2.2 Examples

In this section we present several examples of pretame and non-pretame notions of class forcing.

Example 2.2.5. $\mathrm{Col}(\omega, \mathrm{Ord})$ is not pretame. To see this, consider the dense classes

$$D_n = \{p \in \mathrm{Col}(\omega, \mathrm{Ord}^M) \mid n \in \mathrm{dom}(p)\}$$

for all $n \in \omega$. Suppose that there exist a condition $q \in \mathrm{Col}(\omega, \mathrm{Ord}^M)$ and $\langle d_n \mid n \in \omega \rangle \in M$ such that each d_n is a subset of D_n which is predense below q. Now let

$$\alpha = \sup\{\mathrm{ran}(p) \mid p \in \bigcup_{n \in \omega} d_n\}.$$

Let $n \in \omega$ such that $n \notin \mathrm{dom}(q)$ and consider $r = q \cup \{\langle n, \alpha + 1 \rangle\}$. Then r is incompatible with every element of d_n. Nevertheless, it will follow from Lemma 2.4.3 that $\mathrm{Col}(\omega, \mathrm{Ord}^M)$ satisfies the forcing theorem.

Definition 2.2.6. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is said to be *distributive* over $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$, if for every sequence $\langle D_i \mid i \in I\rangle \in \mathcal{C}$ of open dense subclasses of $\mathbb{P}$ with $I \in M$ and for every $p \in \mathbb{P}$ there is $q \leq_{\mathbb{P}} p$ such that $q \in \bigcap_{i\in I} D_i$.

Example 2.2.7. If $\mathbb{P}$ is distributive over some countable transitive model $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ then $\mathbb{P}$ is pretame for $\mathbb{M}$.

Proof. Let $\langle D_i \mid i \in I\rangle$ be a sequence of dense classes in $\mathcal{C}$ and $p \in \mathbb{P}$. For each $i \in I$, we define E_i to be the downwards closure of D_i, i.e. $q \in E_i$ if and only if there is $r \in D_i$ with $q \leq_{\mathbb{P}} r$. Then every E_i is open dense in $\mathbb{P}$. By distributivity of $\mathbb{P}$ there is $q \leq_{\mathbb{P}} p$ with $q \in \bigcap_{i\in I} E_i$. Using collection, we can pick a set $\langle d_i \mid i \in I\rangle$ in M such that each d_i is a subset of D_i with the property that $q \leq_{\mathbb{P}} r$ for all $r \in d_i$. In particular, d_i is predense below q. □

Definition 2.2.8. A notion of class forcing $\mathbb{P}$ for $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is said to be $<$ Ord-*closed*, if it is $< \kappa$-closed for every M-cardinal κ. More precisely, if $\langle p_i \mid i < \kappa\rangle \in M$ is a $\leq_{\mathbb{P}}$-descending sequence of conditions in $\mathbb{P}$ for some M-cardinal κ, then there is $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} p_i$ for every $i < \kappa$.

Example 2.2.9. If $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a model of GBdc^-, then every $<$ Ord-closed forcing notion $\mathbb{P}$ for $\mathbb{M}$ is distributive over $\mathbb{M}$ and hence by Example 2.2.7 also pretame over $\mathbb{M}$. To see this, suppose that $p \in \mathbb{P}$ is an arbitrary condition and $\langle D_i \mid i \in I\rangle \in \mathcal{C}$ is a sequence of open dense subclasses of $\mathbb{P}$. Using choice, we may assume that $I = \kappa$ is an M-cardinal. Using $<$ Ord-closure and dependent choice, we can define a $\leq_{\mathbb{P}}$-decreasing sequence $\langle p_i \mid i < \kappa\rangle$ below p such that $p_i \in D_i$. Then there is $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p_i$ for all $i < \kappa$ and so $q \in \bigcap_{i<\kappa} D_i$.

Example 2.2.10. If M is a model of ZF and $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$, then the class-sized Lévy collapse $\mathbb{P} = \mathrm{Col}(\omega, < \mathrm{Ord})^M$ is pretame for $\mathbb{M}$. To see this, let $I \in M$ and $\langle D_i \mid i \in I\rangle$ be a sequence of dense classes. For each $i \in I$ set $d_i^\alpha = D_i \cap \mathsf{V}_\alpha$ and for $\alpha \in \mathrm{Ord}^M$ we define $F(\alpha)$ to be the minimal $\beta \in \mathrm{Ord}^M$ satisfying

$$\forall p \in \mathrm{Col}(\omega, < \alpha)\, \forall i \in I\, \exists q \in d_i^\beta\, (q \leq_{\mathbb{P}} p).$$

Then F is a continuous total class function on Ord^M. Hence $\delta = \sup\{F^n(0) \mid n < \omega\}$ is a fixed point of F. Consider now $d_i = d_i^\delta$ for each $i \in I$. We claim that each d_i^δ is predense. Let $i \in I$ and $p \in \mathbb{P}$ be an arbitrary condition. Then there is $q \in d_i$ such that $q \leq_{\mathbb{P}} p \upharpoonright (\delta \times \omega)$. In particular, $r = p \cup q$ is a condition witnessing the compatibility of p and q.

Example 2.2.11. Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- and let $\mathbb{W} = \mathbb{W}^M$ denote the forcing notion for adding a global well-order as defined in Section 1.3.2. Then $\mathbb{M} \models \mathsf{GBdc}^-$ if and only if $\mathbb{W}^M$ is pretame over $\mathbb{M}$.

To see this, observe first that if $\mathbb{M} \models \mathsf{GBdc}^-$, then Lemma 1.3.3 and Example 2.2.9 imply that $\mathbb{W}$ is pretame. Conversely, suppose that $\mathbb{W}$ is pretame and let G be $\mathbb{W}$-generic over $\mathbb{M}$. Since $\mathbb{W}$ satisfies the forcing theorem, $\mathbb{M}[G] \models \mathsf{GB}^-$. Note further that $\mathbb{W}$ doesn't add any new sets by Lemma 1.3.3, so $M[G] = M$. Suppose that $\varphi(v_0, v_1)$ is an $\mathcal{L}_\in$-formula with class parameters such that $M \models \forall\vec{x}\exists y\, \varphi(\vec{x}, y)$ and let $x \in M$ and $\alpha \in \mathrm{Ord}^M$. Using the generic global well-order $\prec$ in $\mathcal{C}[G]$, we define a sequence $\langle y_i \mid i < \alpha \rangle$ such that $y_0 = x$ in the following way. Given $\langle y_j \mid j < i \rangle$ for $i < \alpha$, let $y_i \in M$ be the $\prec$-least element of M such that $M \models \varphi(\langle y_j \mid j < i \rangle, y_i)$. Then $\vec{y} = \langle y_i \mid i < \alpha \rangle \in M$ since replacement holds in $\mathbb{M}[G]$. This shows that DC holds. To see that M satisfies the axiom of choice, let $x \in M$ be a set. In $\mathbb{M}[G]$ we can well-order x using $\prec$. Since $\mathbb{W}$ doesn't add any new sets, the well-order already exists in M.

Example 2.2.12. Easton forcing is pretame. Assume that $\mathbb{M}$ is a countable transitive model of $\mathsf{GBc} + \mathsf{GCH}$. Let F be an Easton function for $\mathbb{M}$ and let $\mathbb{P}$ denote $\mathbb{P}_F$ as defined in Section 1.3.5. To see that $\mathbb{P}$ is pretame for $\mathbb{M}$, let $p \in \mathbb{P}$ and let $\langle D_i \mid i < \lambda \rangle$ be a sequence of dense classes below p such that λ is a regular M-cardinal. Let $\langle q_i \mid i < \lambda \rangle$ be an enumeration of $\mathbb{P}^{\leq\lambda}$ and let $h : \lambda \times \lambda \to \lambda$ denote Gödel pairing. We define a $\leq_{\mathbb{P}}$-descending sequence $\langle p_i \mid i < \lambda \rangle$ of conditions in $\mathbb{P}$ such that $p_i^{\leq\lambda} = p^{\leq\lambda}$ for each $i < \lambda$.

- Let $p_0 = p$.
- Given $\langle p_j \mid j < i \rangle$, let $\bar{p}_i = \bigcup_{j<i} p_i$. Then $\bar{p}_i \in \mathbb{P}$ since $\mathbb{P}^{>\lambda}$ is λ^+-closed. Suppose that $i = h(i_0, i_1)$. Then choose $p_i \leq_{\mathbb{P}} \bar{p}_i$ with $p_i^{\leq\lambda} = p^{\leq\lambda}$ such that there is some $r_i \in D_{i_1}$ with $p_i \cup q_{i_0} \leq_{\mathbb{P}} r_i$, if possible. Otherwise, we put $p_i = \bar{p}_i$.

Now let $\bar{p} = \bigcup_{i<\lambda} p_i$ and $d_i = \{r_j \mid r_j \in D_i\}$. Then d_i is predense below $\bar{p}$: Suppose that $r \leq_{\mathbb{P}} \bar{p}$. Let $s \leq_{\mathbb{P}} r$ with $s \in D_i$. Let $j < \lambda$ such that $s^{\leq\lambda} = q_j$. Put $k = h(j, i)$. Then $p_k \cup q_j \leq_{\mathbb{P}} r_k$ and $r_k \in D_i$. In particular, $s \leq_{\mathbb{P}} p_k \cup q_j \leq_{\mathbb{P}} r_k$ and $s \leq_{\mathbb{P}} r$, so r is compatible with $r_k \in d_i$.

The following lemma shows that Easton forcing also preserves the power set axiom.

Lemma 2.2.13. *[Jec03, Lemma 15.19] Let $\mathbb{P}$ and $\mathbb{Q}$ be notions of class forcing which satisfy the forcing theorem over $\mathbb{M}$ such that $\mathbb{P}$ is λ^+-closed and $\mathbb{Q}$ satisfies the λ^+-cc. If $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over $\mathbb{M}$, then every function $f : \lambda \to M$ in $M[G \times H]$ is in $M[H]$. In particular,*

$$\mathcal{P}^{M[G\times H]}(\lambda) = \mathcal{P}^{M[H]}(\lambda).$$

□

Therefore, Easton forcing preserves GBc. Standard forcing arguments imply that in the $\mathbb{P}_F$-generic extension, where F denotes an Easton function, 2^κ becomes $F(\kappa)$ for every regular cardinal κ in the domain of F. This allows us, for example, to obtain a model where $2^\kappa = \kappa^{++}$ for every regular cardinal κ. A detailed proof of the following theorem can be found in any standard textbook on set theory such as [Jec03].

Theorem 2.2.14 (Easton's Theorem, [Eas70])**.** *Let $\mathbb{M} = \langle M\, C \rangle$ be a countable transitive model of GBc such that $M \models \mathsf{GCH}$ and let F be an Easton function for $\mathbb{M}$. If G is $\mathbb{P}_F$-generic over $\mathbb{M}$, then $M[G] \models \mathsf{GBc}$ and for each $\kappa \in \mathrm{dom}(F)$, $M[G] \models 2^\kappa = F(\kappa)$. Moreover, $\mathbb{P}_F$ preserves all cardinals and cofinalities.* □

2.2.3 Two-step iterations

In this section, we consider two-step iterations of class-sized forcing notions. Note that for the construction to work, we require the first forcing notion to be pretame, since otherwise the intermediate model could fail to be a model of GB^-. The following lemma is an adaptation of [Fri00, Lemma 2.30 (a)] to our setting.

Lemma 2.2.15. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- which has a hierarchy, let $\mathbb{P}$ be a pretame notion of class forcing and let $\dot{\mathbb{Q}} \in \mathcal{C}^{\mathbb{P}}$ be a class name for a preorder. Then we define the two-step iteration of $\mathbb{P}$ and $\dot{\mathbb{Q}}$ by*

$$\mathbb{P} * \dot{\mathbb{Q}} = \{\langle p, \dot{q}\rangle \mid p \in \mathbb{P} \wedge p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\} \in \mathcal{C}^{\mathbb{P}}$$

equipped with the ordering given by $\langle p_0, \dot{q}_0\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_1, \dot{q}_1\rangle$ iff $p_0 \leq_{\mathbb{P}} p_1$ and $p_0 \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q}_0 \leq_{\dot{\mathbb{Q}}} \dot{q}_1$. If G is $\mathbb{P}$-generic over $\mathbb{M}$ and H is $\dot{\mathbb{Q}}^G$-generic over $\mathbb{M}[G]$, then

$$G * H = \{\langle p, \dot{q}\rangle \mid p \in G \wedge \dot{q}^G \in H\}$$

is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over $\mathbb{M}$.

Proof. We verify first that $G * H$ is a filter. Let $\langle p_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1\rangle \in \mathbb{P} * \dot{\mathbb{Q}}$. Assume first that $\langle p_0, \dot{q}_0\rangle \in G * H$ and $\langle p_1, \dot{q}_1\rangle \geq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, \dot{q}_0\rangle$. Then $p_1 \in G$ and $p_0 \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q}_0 \leq_{\dot{\mathbb{Q}}} \dot{q}_1$. Since $p_0 \in G$ we have $\dot{q}_0{}^G \leq_{\dot{\mathbb{Q}}^G} \dot{q}_1{}^G$. As H is $\dot{\mathbb{Q}}^G$-generic over $M[G]$ and $\dot{q}_0{}^G \in H$, this implies that $\dot{q}_1{}^G \in H$. Now suppose that $\langle p_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1\rangle \in G * H$. Since G is a filter, there is $r \leq_{\mathbb{P}} p_0, p_1$ in G. Similarly, using that $\dot{q}_0{}^G, \dot{q}_1{}^G \in H$ we can find $\dot{q} \in M^{\mathbb{P}}$ such that $\dot{q}^G \leq_{\dot{\mathbb{Q}}^G} \dot{q}_0{}^G, \dot{q}_1{}^G$. Since $\mathbb{P}$ satisfies the truth lemma, there must be $p \leq_{\mathbb{P}} r$ in G such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q} \leq_{\dot{\mathbb{Q}}} \dot{q}_0, \dot{q}_1$. Hence $\langle p, \dot{q}\rangle \in G * H$ and $\langle p, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1\rangle$.

In order to check genericity, let D be a dense subclass of $\mathbb{P} * \dot{\mathbb{Q}}$ which is in $\mathcal{C}$. Then

$$E = \{\dot{q}^G \mid \exists p \in G\,(\langle p, \dot{q}\rangle \in D)\}$$

is in $\mathcal{C}[G]$. We claim that E is dense in $\dot{\mathbb{Q}}^G$. Let $\dot{q}^G \in \dot{\mathbb{Q}}^G$ and put

$$D(\dot{q}) = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q} \notin \dot{\mathbb{Q}} \vee \exists \dot{r}\,[(p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{r} \leq_{\dot{\mathbb{Q}}} \dot{q}) \wedge \langle p, \dot{r}\rangle \in D]\}.$$

Since $\mathbb{P}$ satisfies the forcing theorem, $D(\dot{q}) \in \mathcal{C}$. We show that it is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$ and suppose that $p \not\Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q} \notin \dot{\mathbb{Q}}$. Using the forcing theorem, we can strengthen p to p_0 such that $p_0 \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$, and therefore we obtain that $\langle p_0, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}}$. Since D is dense, there must be $\langle p_1, \dot{r}\rangle \in D$ with $\langle p_1, \dot{r}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, \dot{q}\rangle$. Thus $p_1 \in D(\dot{q})$ proving that $D(\dot{q})$ is dense. Now let $p \in D(\dot{q}) \cap G$ and take $\dot{r}$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{r} \leq_{\dot{\mathbb{Q}}} \dot{q}$ and $\langle p, \dot{r}\rangle \in D$. Then $\dot{r}^G \leq_{\dot{\mathbb{Q}}^G} \dot{q}^G$ and $\dot{r}^G \in E$. Hence E is dense in $\dot{\mathbb{Q}}^G$. This implies that D meets $G * H$. □

Remark 2.2.16. Notice that if $\mathbb{P}$ is pretame for some countable transitive GB^--model $\mathbb{M}$ with a hierarchy and $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is pretame for $\mathbb{M}[\dot{G}]$", then the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is pretame for $\mathbb{M}$. To see this, observe that by Theorem 2.2.3, $\mathbb{P} * \dot{\mathbb{Q}}$ preserves GB^-. Moreover, it will follow from Lemma 4.3.5 that pretameness is in fact equivalent to the preservation of GB^- over models with a hierarchy. In particular, this implies that $\mathbb{P} * \dot{\mathbb{Q}}$ is pretame.

2.3 The set decision property

In this section, we introduce a simple combinatorial property which implies the forcing theorem. Moreover, we will show that this property exactly characterizes those notions of forcing which do not add any new sets.

Definition 2.3.1. Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Let $\dot{G}$ denote the canonical $\mathbb{P}$-name for the generic filter.

(1) If $p \in \mathbb{P}$ and σ is a $\mathbb{P}$-name, then we define the *p-evaluation* of σ by

$$\sigma^p = \{\tau^p \mid \exists q \in \mathbb{P}\,[\langle \tau, q\rangle \in \sigma \wedge \forall r \leq_{\mathbb{P}} p\,(r \parallel_{\mathbb{P}} q)]\}.$$

(2) Given conditions p and q in $\mathbb{P}$, we write $p \leq^*_{\mathbb{P}} q$ iff $\forall r \leq_{\mathbb{P}} p\,(r \parallel_{\mathbb{P}} q)$ (equivalently, $p \Vdash^{\mathbb{M}}_{\mathbb{P}} q \in \dot{G}$). Note that if $\mathbb{P}$ is separative, then $p \leq^*_{\mathbb{P}} q$ if and only if $p \leq_{\mathbb{P}} q$.
(3) If $A \subseteq \mathbb{P}$ is a set of conditions and $p \in \mathbb{P}$, we write $p \perp_{\mathbb{P}} A$ or $p \leq^*_{\mathbb{P}} A$ if $\forall a \in A\,(p \perp_{\mathbb{P}} a)$ or $\forall a \in A\,(p \leq^*_{\mathbb{P}} a)$ respectively.
(4) If $A \subseteq \mathbb{P}$ is a set of conditions and $p \in \mathbb{P}$, then p *decides* A (we write $p \sim_{\mathbb{P}} A$) if for every $a \in A$, either $p \leq^*_{\mathbb{P}} a$ or $p \perp_{\mathbb{P}} a$.
(5) We say that $\mathbb{P}$ has the *set decision property*, if for every $p \in \mathbb{P}$ and every set $A \subseteq \mathbb{P}$ in M, there is an extension $q \leq_{\mathbb{P}} p$ of p such that q decides A.

Note that if p decides A, then p decides for every condition in A whether it lies in the generic filter or not, i.e. p decides $\dot{G} \cap \check{A}$.

Example 2.3.2. Assuming that $\mathbb{M}$ is a model of GBdc^-, every $<$Ord-closed notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ has the set decision property: Let $p \in \mathbb{P}$ and let $A \subseteq \mathbb{P}$ be a set of conditions. Using the axiom of choice, we can enumerate A as $\{a_i \mid i < \kappa\}$ for some M-cardinal κ. Using dependent choice, we can find a sequence $\langle p_i \mid i < \kappa\rangle$ of conditions such that for every $i < j < \kappa$, $p_j \leq_{\mathbb{P}} a_i$, or $p_j \perp_{\mathbb{P}} a_i$ with the following properties.

- It holds that $p_0 = p$.
- If $p_i \parallel_{\mathbb{P}} a_i$, then p_{i+1} is stronger than both p_i and a_i, if possible, and otherwise, $p_{i+1} = p_i$.
- For a limit ordinal α, p_α is stronger than p_i for all $i < \alpha$. Such p_α exists by $<$ Ord-closure of $\mathbb{P}$.

Now let $q \in \mathbb{P}$ be a condition stronger than every p_i for $i < \kappa$. By construction, q decides A.

Example 2.3.3. The forcing notion $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})^M$ satisfies the set decision property: Suppose that $A \in M$ is a subset of $\mathbb{P}$ and $p \in \mathbb{P}$. Let $\alpha = \sup\{\mathrm{range}(q) \mid q \in A\} + 1$ and $n = \mathrm{dom}(p)$. Then $q = p \cup \{\langle n, \alpha\rangle\}$ decides A: Assume that $a \in A$. If $\mathrm{dom}(a) \leq n$ then either $q \perp_{\mathbb{P}} a$ or $a = q \upharpoonright n$ and hence $q \leq_{\mathbb{P}} a$. Otherwise, $n \in \mathrm{dom}(a)$ and by construction $a(n) < \alpha + 1 = q(n)$ and so $q \perp_{\mathbb{P}} a$.

Lemma 2.3.4. *Let $\mathbb{M}$ be a countable transitive model of GB^-. Then every class forcing $\mathbb{P}$ for $\mathbb{M}$ with the set decision property satisfies the forcing theorem and does not add new sets, that is $M[G] = M$ whenever G is $\mathbb{P}$-generic over $\mathbb{M}$.*

Proof. By Theorem 2.1.5, to verify the forcing theorem it is enough to check that the definability lemma holds for "$v_0 = v_1$". Let $\sigma, \tau \in M^{\mathbb{P}}$. Let $A = \mathbb{P} \cap \mathrm{tc}(\sigma \cup \tau)$ and let $p \in \mathbb{P}$. Then it follows from the set decision property that

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau \iff \forall q \leq_{\mathbb{P}} p\, (q \sim_{\mathbb{P}} A \to q \Vdash_{\mathbb{P}} \sigma = \tau).$$

But if $q \sim_{\mathbb{P}} A$ and $q \in G$ then $\sigma^q = \sigma^G$ (this in particular implies that $\sigma^G \in M$ and hence that $\mathbb{P}$ does not add new sets), thus we obtain $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$ iff $\sigma^q = \tau^q$. Consequently, $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$ can be defined by $\forall q \leq_{\mathbb{P}} p\, (q \sim_{\mathbb{P}} A \to \sigma^q = \tau^q)$. □

Lemma 2.3.5. *Let $\mathbb{M}$ be a countable transitive model of GB^- and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ which adds no new sets. Then $\mathbb{P}$ has the set decision property.*

Proof. Let $A \subseteq \mathbb{P}$ be a set of conditions in M and let $p \in \mathbb{P}$. We have to find $q \leq_{\mathbb{P}} p$ such that $q \sim_{\mathbb{P}} A$. Assume for a contradiction that no such q exists.

Since $\mathbb{M}$ is countable in V, we can enumerate in V all elements of $\mathcal{C}$ that are dense subsets of $\mathbb{P}$ by $\langle D_n \mid n \in \omega \rangle$ and all subsets of A which are elements of M by $\langle x_n \mid n \in \omega \rangle$. Let $\sigma = \{\langle \check{a}, a \rangle \mid a \in A\}$. We will find a $\mathbb{P}$-generic filter G such that $\sigma^G \notin M$, which clearly contradicts our assumption on $\mathbb{P}$. For this we define a $\leq_{\mathbb{P}}$-decreasing sequence of conditions $\langle q_n \mid n \in \omega \rangle$ below p and a sequence $\langle a_n \mid n \in \omega \rangle$ of conditions in A. Let $q_0 = p$. Given q_n, note that by our assumption it cannot be the case that $q_n \leq^*_{\mathbb{P}} x_n$ and $q_n \perp_{\mathbb{P}} (A \setminus x_n)$. Hence there is $a_n \in A$ such that either $a_n \in x_n$ and $q_n \not\leq^*_{\mathbb{P}} a_n$ or $a_n \notin x_n$ and $q_n \parallel_{\mathbb{P}} a_n$. In the first case we pick $r \leq_{\mathbb{P}} q_n$ such that $r \perp_{\mathbb{P}} a_n$. In the second case, we strengthen q_n to $r \leq_{\mathbb{P}} q_n, a_n$. Now take $q_{n+1} \leq_{\mathbb{P}} r$ such that $q_{n+1} \in D_n$. Finally, this means that $G = \{q \in \mathbb{P} \mid \exists n \in \omega\, (q_n \leq_{\mathbb{P}} q)\}$ is a $\mathbb{P}$-generic filter. But since $\mathbb{P}$ doesn't add any new sets and since $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \subseteq \check{A}$, there must be some $n \in \omega$ such that $\sigma^G = x_n$. But we have that either $a_n \in x_n$ and $a_n \perp_{\mathbb{P}} q_{n+1}$, thus $a_n \notin \sigma^G$, or $a_n \notin x_n$ but $q_{n+1} \leq_{\mathbb{P}} a_n$ implying that $a_n \in \sigma^G$. We have thus reached a contradiction. □

Putting together Lemmata 2.3.4 and 2.3.5 we obtain

Corollary 2.3.6. *Every class forcing which does not add new sets satisfies the forcing theorem.* □

2.4 Approachability by projections

In this section, we fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^-. We define a fairly weak combinatorial condition on notions of class forcing that implies the forcing theorem to hold. In particular, this property is satisfied by the forcing notions $\mathrm{Col}(\omega, \mathrm{Ord})^M$, $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ and $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ from Section 1.3.

Definition 2.4.1. We say that a class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ for $\mathbb{M}$ is *approachable by projections*, if it can be written as a continuous, increasing union $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathbb{P}_\alpha$ for a sequence $\langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle \in \mathcal{C}$ of notions of set forcing $\mathbb{P}_\alpha = \langle P_\alpha, \leq_{\mathbb{P}_\alpha} \rangle$, where $\leq_{\mathbb{P}_\alpha}$ is the ordering on P_α induced by $\mathbb{P}$, for which there exists a sequence of maps $\langle \pi_{\alpha+1} \mid \alpha \in \mathrm{Ord}^M \rangle$ so that $\pi_{\alpha+1} \colon \mathbb{P} \to \mathbb{P}_{\alpha+1}$, $\{\langle \alpha, p, \pi_{\alpha+1}(p) \rangle \mid \alpha \in \mathrm{Ord}^M, p \in P\} \in \mathcal{C}$ and for every $\alpha \in \mathrm{Ord}^M$, the following hold:

(1) $\pi_{\alpha+1}(\mathbb{1}_\mathbb{P}) = \mathbb{1}_\mathbb{P}$,
(2) $\forall p, q \in \mathbb{P}\, [p \leq_\mathbb{P} q \rightarrow \pi_{\alpha+1}(p) \leq_\mathbb{P} \pi_{\alpha+1}(q)]$,
(3) $\forall p \in \mathbb{P}\, \forall q \leq_{\mathbb{P}_{\alpha+1}} \pi_{\alpha+1}(p)\, \exists r \leq_\mathbb{P} p\, [\pi_{\alpha+1}(r) \leq_\mathbb{P} q]$,
(4) $\forall p \in \mathbb{P}_\alpha\, \forall q \in \mathbb{P}\, [\pi_{\alpha+1}(q) \leq_\mathbb{P} p \rightarrow q \leq_\mathbb{P} p]$ and
(5) $\pi_{\alpha+1}$ is the identity on $\mathbb{P}_\alpha$.

Note that the first three properties simply state that $\pi_{\alpha+1}$ is a *projection*; this was the motivation for the choice of our terminology. It follows from the definition that each $\pi_{\alpha+1}$ is a dense embedding and thus $\pi''_{\alpha+1}G$ is a $\mathbb{P}_{\alpha+1}$-generic filter whenever G is a $\mathbb{P}$-generic filter.

Lemma 2.4.2. *If $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathbb{P}_\alpha$ is approachable by projections with projections $\pi_{\alpha+1} : \mathbb{P} \rightarrow \mathbb{P}_{\alpha+1}$ and G is $\mathbb{P}$-generic over $\mathbb{M}$, then $M[G] \subseteq \bigcup_{\alpha \in \mathrm{Ord}^M} M[\pi''_{\alpha+1}G]$, and the latter is a union of set-generic extensions of M.*

Proof. If σ is a $\mathbb{P}$-name, then there is $\alpha \in \mathrm{Ord}^M$ such that σ is already a $\mathbb{P}_\alpha$-name. Since $\pi_{\alpha+1}$ is dense, $G_{\alpha+1} = \pi''_{\alpha+1}G$ is $\mathbb{P}_{\alpha+1}$-generic. By property (5) of Definition 2.4.1, $\sigma^G = \sigma^{G_{\alpha+1}} \in M[G_{\alpha+1}]$. □

Lemma 2.4.3. $\mathrm{Col}(\omega, \mathrm{Ord})^M$, $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ *and* $\mathrm{Col}_\geq(\omega, \mathrm{Ord})^M$ *are approachable by projections.*

Proof. Let $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})^M$ and let $\mathbb{P}_\alpha = \mathrm{Col}(\omega, \alpha)$. Furthermore, take $\pi_{\alpha+1}$ to be the map that, for $p \in \mathrm{Col}(\omega, \mathrm{Ord})^M$, replaces the value of $p(n)$ by α whenever $p(n) > \alpha$. Conditions (1), (2) and (5) of Definition 2.4.1 are trivially satisfied. For (3) let $p \in \mathrm{Col}(\omega, \mathrm{Ord})^M$ and $q \in \mathrm{Col}(\omega, \alpha+1)$ such that $q \leq_{\mathbb{P}_{\alpha+1}} \pi_{\alpha+1}(p)$. Then $r = p \cup \{\langle n, q(n)\rangle \mid n \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)\}$ satisfies $r \leq_\mathbb{P} p$ and $\pi_{\alpha+1}(r) \leq_\mathbb{P} q$. For (4), consider $p \in \mathrm{Col}(\omega, \alpha)$ and $q \in \mathrm{Col}(\omega, \mathrm{Ord})^M$ such that $\pi_{\alpha+1}(q) \leq_\mathbb{P} p$. Suppose $n \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. Since $p \in \mathrm{Col}(\omega, \alpha)$, $p(n) < \alpha$ and hence $q(n) = \pi_{\alpha+1}(q)(n) = p(n)$. Thus $q \leq_\mathbb{P} p$. The arguments for $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ and $\mathrm{Col}_\geq(\omega, \mathrm{Ord})^M$ are similar. □

Theorem 2.4.4. *If $\mathbb{P}$ is approachable by projections, then the forcing relation for "$v_0 = v_1$" is definable. Therefore, by Theorem 2.1.5, $\mathbb{P}$ satisfies the forcing theorem for every $\mathcal{L}_\in$-formula with class parameters.*

Proof. Fix any ordinal $\alpha \in M$. We will show by induction on the name rank that the forcing relation for "$v_0 \subseteq v_1$", restricted to names that only mention conditions in $\mathbb{P}_\alpha$, is definable. Then the forcing relation for "$v_0 = v_1$" is definable by $p \Vdash^\mathbb{M}_\mathbb{P} \sigma = \tau$ if and only if $p \Vdash^\mathbb{M}_\mathbb{P} \sigma \subseteq \tau$ and $p \Vdash^\mathbb{M}_\mathbb{P} \tau \subseteq \sigma$. It will be easy to see that this is uniform in α, thus implying the desired statement. For the sake of legibility, we will omit the superscript $\mathbb{M}$ in the notation of the forcing relation.

For $\sigma, \tau \in M^{\mathbb{P}_\alpha}$ and $p \in \mathbb{P}$ define $p \Vdash^*_\mathbb{P} \sigma \subseteq \tau$ if and only if

$$\forall \langle \rho, s\rangle \in \sigma \forall q \leq_\mathbb{P} p \exists r \leq_\mathbb{P} q\, [r \leq_\mathbb{P} s \rightarrow \exists \langle \pi, t\rangle \in \tau (r \leq_\mathbb{P} t \wedge r \Vdash^*_\mathbb{P} \rho = \tau)].$$

Similiarly, let $\pi_{\alpha+1}(p) \Vdash^{*,\alpha+1}_\mathbb{P} \sigma \subseteq \tau$ denote the same formula as above but where p is replaced by $\pi_{\alpha+1}(p)$, $\Vdash^*_\mathbb{P}$ is replaced by $\Vdash^{*,\alpha+1}_\mathbb{P}$ and all quantifiers over $\mathbb{P}$ are restricted to

$\mathbb{P}_{\alpha+1}$. Furthermore, $p \Vdash^*_{\mathbb{P}} \sigma = \tau$ is an abbreviation for $p \Vdash^*_{\mathbb{P}} \sigma \subseteq \tau$ and $p \Vdash^*_{\mathbb{P}} \tau \subseteq \sigma$ and similarly for $\Vdash^{*,\alpha+1}_{\mathbb{P}}$.

We will show by induction on the rank of $\mathbb{P}_\alpha$-names that

$$p \Vdash^*_{\mathbb{P}} \sigma \subseteq \tau \quad \text{if and only if} \quad \pi_{\alpha+1}(p) \Vdash^{*,\alpha+1}_{\mathbb{P}} \sigma \subseteq \tau. \tag{2.2}$$

The right-hand side is clearly definable with parameter α, since $\mathbb{P}_{\alpha+1}$ is a set-sized forcing notion. Moreover, by the usual proof, $p \Vdash^*_{\mathbb{P}} \sigma \subseteq \tau$ if and only if $p \Vdash_{\mathbb{P}} \sigma \subseteq \tau$. This shows that assuming (2.2), the forcing relation for "$v_0 \subseteq v_1$" restricted to $\mathbb{P}_\alpha$-names is definable.

Assume first that $p \Vdash_{\mathbb{P}} \sigma \subseteq \tau$ and let $\langle \rho, s\rangle \in \sigma$ and $\bar{q} \leq_{\mathbb{P}_{\alpha+1}} \pi_{\alpha+1}(p)$. By (3) there is $q \leq_{\mathbb{P}} p$ such that $\pi_{\alpha+1}(q) \leq_{\mathbb{P}} \bar{q}$. By assumption there is $r \leq_{\mathbb{P}} q$ witnessing $p \Vdash^*_{\mathbb{P}} \sigma \subseteq \tau$. Using (2), $\pi_{\alpha+1}(r) \leq_{\mathbb{P}} \bar{q}$. Now if $r \not\leq_{\mathbb{P}} s$, then $\pi_{\alpha+1}(r) \not\leq_{\mathbb{P}} s$ by (4). Otherwise, assume that $\langle \pi, t\rangle \in \tau$ such that $r \leq_{\mathbb{P}} t$ and $r \Vdash^*_{\mathbb{P}} \rho = \pi$. Again by (2) and (5) we have $\pi_{\alpha+1}(r) \leq_{\mathbb{P}} t$ and inductively, $\pi_{\alpha+1}(r) \Vdash^{*,\alpha+1}_{\mathbb{P}} \rho = \pi$.

For the converse, suppose $\pi_{\alpha+1}(p) \Vdash^{*,\alpha+1}_{\mathbb{P}} \sigma \subseteq \tau$ and let $\langle \rho, s\rangle \in \sigma$. Let $q \leq_{\mathbb{P}} p$. By (2) we have $\pi_{\alpha+1}(q) \leq_{\mathbb{P}} \pi_{\alpha+1}(p)$ and thus there is $\bar{r} \leq_{\mathbb{P}_{\alpha+1}} \pi_{\alpha+1}(q)$ witnessing $\pi_{\alpha+1}(p) \Vdash^{*,\alpha+1}_{\mathbb{P}} \sigma \subseteq \tau$. Using (3), choose $r \leq_{\mathbb{P}} q$ such that $\pi_{\alpha+1}(r) \leq_{\mathbb{P}} \bar{r}$. Now if $\bar{r} \perp_{\mathbb{P}_{\alpha+1}} s$, then also $r \perp_{\mathbb{P}} s$ because if $t \leq_{\mathbb{P}} r, s$, then $\pi_{\alpha+1}(t) \leq_{\mathbb{P}} \bar{r}, s$ by (2) and (5). So assume that $\bar{r}$ and s are compatible and take $\bar{u} \in P_{\alpha+1}$ such that $\bar{u} \leq_{\mathbb{P}} \bar{r}, s$. Hence again using that $\pi_{\alpha+1}(p) \Vdash^*_{\mathbb{P}} \sigma \subseteq \tau$ there is $\bar{v} \leq_{\mathbb{P}_{\alpha+1}} \bar{u}$ such that $\bar{v} \leq_{\mathbb{P}} s$ and there is $\langle \pi, t\rangle \in \tau$ such that $\bar{v} \leq_{\mathbb{P}} t$ and $\bar{v} \Vdash^{*,\alpha+1}_{\mathbb{P}} \rho = \pi$. Now since $\bar{v} \leq_{\mathbb{P}_{\alpha+1}} \pi_{\alpha+1}(q)$ there exists $v \leq_{\mathbb{P}} q$ such that $\pi_{\alpha+1}(v) \leq_{\mathbb{P}} \bar{v}$, so $\pi_{\alpha+1}(v) \Vdash^{*,\alpha+1}_{\mathbb{P}} \rho = \tau$. Then by (4) we get that $v \leq_{\mathbb{P}} s, t$ and inductively, $v \Vdash^*_{\mathbb{P}} \rho = \tau$. □

The following lemma shows that the converse does not hold in general.

Lemma 2.4.5. *Suppose that M is a countable transitive model of* ZFC. *There is a tame notion of class forcing for $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$ which is not approachable by projections.*

Proof. Let $\mathbb{P}$ be Jensen coding, as described in [BJW82]. Then $\mathbb{P}$ is a tame notion of forcing, i.e. it preserves ZFC. Without loss of generality, we may assume that $M \models \mathsf{GCH}$; otherwise we can ensure this by previously forcing GCH using a variant of Easton forcing (see [Eas70]). Then the extension of M is $M[G] = \mathsf{L}[x]$ for some real x which is not contained in any set forcing extension of M, so by Lemma 2.4.2, $\mathbb{P}$ cannot be approachable by projections. □

Lemma 2.4.6. *There is a notion of class forcing which is an increasing union of set-sized complete subforcings but not pretame.*

Proof. By Lemma 1.3.2, the forcing notion $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ is an increasing union of set-sized complete subforcings, but it is not pretame since it adds a cofinal function from ω to the ordinals (see Lemma 4.2.1); alternatively one can prove the latter in the same way as for $\mathrm{Col}(\omega, \mathrm{Ord})^M$ (see Example 2.2.5). □

2.5 Failures of the forcing theorem

In set forcing, it is a standard result that every forcing notion satisfies the forcing theorem. This, however, does not carry over to class forcing. In this section we show that both the definability and the truth lemma may fail for class forcing.

2.5.1 A failure of the definability lemma

The main goal of this section is to show that the definability lemma does not hold for every notion of class forcing. More precisely, we will show that if the definability lemma holds for $\mathbb{F}^M$ (as defined in Section 1.3.3) for some transitive countable model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^-, then $\mathcal{C}$ contains a first-order truth predicate for M. In particular, if the ground model is of the form $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$, where M is a countable transitive model of ZF^-, then Tarski's theorem on the undefinability of truth[2] implies that $\mathbb{F}^M$ does not satisfy the definability lemma over $\mathbb{M}$. Furthermore, we will prove that for certain models M of ZFC its forcing relation is not even M-amenable. Whenever it is clear from context which model is referred to, we write $\mathbb{F}$ for $\mathbb{F}^M$. Unless stated otherwise, $\mathbb{M} = \langle M, \mathcal{C}\rangle$ will denote an arbitrary countable transitive model of GB^-.

Following [Dra74, Chapter 3.5], we let $\mathrm{Fml} \subseteq {}^{<\omega}\omega$ denote the set of all codes for $\mathcal{L}_\in$-formulae. Since we work inside some model V of set theory and we use these codes inside countable transitive models that are elements of V together with the corresponding formalized satisfaction relation, we may assume that each element of Fml is the Gödel number $\ulcorner\varphi\urcorner$ of an $\mathcal{L}_\in$-formula φ. For $k \in \omega$, let Fml_k denote the set of all Gödel numbers for formulae with free variables among $\{v_0, \dots, v_{k-1}\}$. For the sake of simplicity, we will assume that every $\mathcal{L}_\in$-formula φ is in the following normal form: Whenever $\exists v_k \psi$ is a subformula of φ, then the free variables of ψ are among $\{v_0, \dots, v_k\}$.

Definition 2.5.1. A relation $T \subseteq \mathrm{Fml}_1 \times M$ is a *first-order truth predicate for M*, if

$$\langle \ulcorner\varphi\urcorner, x\rangle \in T \iff \langle M, \in\rangle \models \varphi(x)$$

holds for every $\ulcorner\varphi\urcorner \in \mathrm{Fml}_1$ and every $x \in M$.

Let G be an $\mathbb{F}$-generic filter over $\mathbb{M}$ and let E and F be defined as in the proof of Lemma 1.3.8. Then

$$T \;=\; \{\langle \ulcorner\varphi\urcorner, x\rangle \in \mathrm{Fml}_1 \times M \mid \langle \omega, E\rangle \models \varphi(F^{-1}(x))\} \subseteq M$$

is a first-order truth predicate for $\mathbb{M}$ and, by Tarski's Undefinability Theorem, T cannot be defined over M by a first-order formula. In the following, we will show that definability of the forcing relation for $\mathbb{F}$ would lead to a first-order definition of T.

Notation. If $\vec{x} = x_0, \dots, x_{k-1}$ is a sequence in M, we say that a sequence $\vec{n} = n_0, \dots, n_{k-1}$ in ω is *appropriate for $\vec{x}$*, if for all $i, j < k$, $x_i = x_j$ if and only if $n_i = n_j$. We inductively define $p^{\vec{x}}_{\vec{n}} \in \mathbb{F}$ as follows, whenever $\vec{n}$ is a sequence of natural numbers which is appropriate for $\vec{x}$.

(1) If $k = 0$, then $p^{\emptyset}_{\emptyset} = \mathbb{1}_{\mathbb{F}}$.
(2) If $\vec{n}, n_k$ is appropriate for $\vec{x}, x_k$, given $p = p^{\vec{x}}_{\vec{n}}$, let $p^{\vec{x},x_k}_{\vec{n},n_k}$ be the condition $q \in \mathbb{F}$ with domain $d_q = d_p \cup \{n_k\}$, $f_q = f_p \cup \{\langle n_k, x_k\rangle\}$ and

$$e_q = e_p \cup \{\langle n_k, n_i\rangle \mid i \in \vec{n} \wedge x_k \in x_i\} \cup \{\langle n_i, n_k\rangle \mid i \in \vec{n} \wedge x_i \in x_k\}.$$

[2] Tarski's Theorem proved first in [Tar36] states that the is truth of formulae in some structure is not first-order definable over that structure; see also [Jec03, Theorem 12.7].

Clearly, we obtain that whenever $\vec{x}$ extends $\vec{y}$, $\vec{n}$ extends $\vec{m}$ and $\vec{n}$ is appropriate for $\vec{x}$, then $p^{\vec{x}}_{\vec{n}} \leq_{\mathbb{F}} p^{\vec{y}}_{\vec{m}}$. Furthermore, we define $p^{\vec{x}}$ to be the condition $p^{\vec{x}}_{\vec{n}}$, where $\vec{n}$ is the lexicographically smallest sequence which is appropriate for $\vec{x}$.

Before we proceed to prove that the definability lemma can fail for $\mathbb{F}$, we need a translation from $\mathcal{L}_{\in}$-formulae to $\mathcal{L}^{\Vdash}_{\omega_1,0}(\mathbb{F}, M)$-formulae so that we can apply Corollary 2.1.10, where $\mathcal{L}^{\Vdash}_{\omega_1,0}(\mathbb{F}, M)$-formulae are $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{F}, M)$-formulae in which all conjunctions and disjunctions are countable.

Notation. Inductively, we assign to every $\mathcal{L}_{\in}$-formula φ whose free variables are among $\{v_0, \ldots, v_{k-1}\}$ and all sequences $\vec{n} = n_0, \ldots, n_{k-1}$ of natural numbers an $\mathcal{L}^{\Vdash}_{\omega_1,0}(\mathbb{F}, M)$-formula $\varphi^*_{\vec{n}}$ as follows:

$$\begin{aligned}
(v_i = v_j)^*_{\vec{n}} &= (\check{n}_i = \check{n}_j)\\
(v_i \in v_j)^*_{\vec{n}} &= (\mathsf{op}(\check{n}_i, \check{n}_j) \in \dot{E})\\
(\neg\varphi)^*_{\vec{n}} &= (\neg\varphi^*_{\vec{n}})\\
(\varphi \vee \psi)^*_{\vec{n}} &= (\varphi^*_{\vec{n}} \vee \psi^*_{\vec{n}})\\
(\exists v_k \varphi)^*_{\vec{n}} &= (\bigvee_{i\in\omega} \varphi^*_{\vec{n},i}).
\end{aligned}$$

If $\vec{n} = 0, \ldots, k-1$, then we simply write φ^* for $\varphi^*_{\vec{n}}$ and if $\vec{x}$ is a sequence in M and $\vec{n}$ is such that $p^{\vec{x}} = p^{\vec{x}}_{\vec{n}}$, then we write $\varphi^*_{\vec{x}}$ for $\varphi^*_{\vec{n}}$. In particular, if v_0 is the only free variable of φ, then φ^*_x is φ^*.

The next lemma is the key ingredient to obtain a first-order truth predicate T for $\mathbb{M}$. We will use the translation of $\mathcal{L}_{\in}$-formulae to $\mathcal{L}^{\Vdash}_{\omega_1,0}(\mathbb{F}, M)$-formulae to define truth by $\langle M, \in\rangle \models \varphi(x)$ if and only if $\langle \omega, E\rangle \models \varphi(n)$, where $n = F(x)$, if and only if $p_x \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_x$.

Lemma 2.5.2. *For every $\mathcal{L}_{\in}$-formula φ with free variables among $\{v_0, \ldots, v_{k-1}\}$ and for all $\vec{x} = x_0, \ldots, x_{k-1} \in M$, the following conditions hold:*

(1) $\mathbb{M} \models \varphi(\vec{x})$ *if and only if* $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{x}}$.
(2) $\mathbb{M} \models \neg\varphi(\vec{x})$ *if and only if* $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg\varphi^*_{\vec{x}}$.

Proof. First, we verify that for every formula φ with free variables in $\{v_0, \ldots, v_{k-1}\}$ and for all $\vec{n} \in \omega^k$ appropriate for $\vec{x}$,

$$p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{x}} \iff p^{\vec{x}}_{\vec{n}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{n}}. \tag{2.3}$$

Let $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{x}}$ and let $\vec{m}$ be such that $p^{\vec{x}} = p^{\vec{x}}_{\vec{m}}$. Consider the automorphism π on $\mathbb{F}$ that for every condition $p = \langle d_p, e_p, f_p\rangle$ replaces every m_i appearing in d_p, e_p and $\mathrm{dom}(f_p)$ by n_i. Clearly, $\pi(p^{\vec{x}}) = p^{\vec{x}}_{\vec{n}}$ and so $p^{\vec{x}}_{\vec{n}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{n}}$. The converse follows in the same way.

Working in V, we now verify (1) and (2) by induction on the complexity of formulae. Observe that it suffices to check only that $\mathbb{M} \models \varphi(\vec{x})$ implies $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{x}}$ and that $\mathbb{M} \models \neg\varphi(\vec{x})$ implies $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg\varphi^*_{\vec{x}}$, since the backwards directions of (1) and (2) immediately follow from the forward directions of (2) and (1) respectively.

For equations this is obvious. Suppose now that $\mathbb{M} \models x \in y$. Let G be generic over $\mathbb{M}$ with $p^{x,y} \in G$. Then by definition of $p^{x,y}$, $\langle 0, 1\rangle \in E$ implying that $\mathbb{M}[G] \models ((v_0 \in v_1)^*)^G$. The converse is similar.

For negations, both (1) and (2) follow directly from the induction hypothesis.

We turn to disjunctions. Assume that $\mathbb{M} \models (\varphi \vee \psi)(\vec{x})$. Without loss of generality, assume that $\mathbb{M} \models \varphi(\vec{x})$. Then inductively, we get that $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{x}}$. But then clearly $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} (\varphi \vee \psi)^*_{\vec{x}}$. Conversely, assume that $\mathbb{M} \models \neg(\varphi \vee \psi)(\vec{x})$. This means that $\mathbb{M} \models \neg\varphi(\vec{x})$ and $\mathbb{M} \models \neg\psi(\vec{x})$. By assumption, this means that $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg\varphi^*_{\vec{x}} \wedge \neg\psi^*_{\vec{x}}$, hence $p^{\vec{x}} \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg(\varphi \vee \psi)^*_{\vec{x}}$.

Suppose now that $\mathbb{M} \models \exists v_k \varphi(\vec{x}, v_k)$. Then there is $y \in M$ such that $\mathbb{M} \models \varphi(\vec{x}, y)$. This means that $p^{\vec{x},y} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{x},y}$. Let $\vec{n}$ be the sequence such that $p^{\vec{x}} = p^{\vec{x}}_{\vec{n}}$. Now observe that by (2.3), we have for every $i \in \omega$ such that $\vec{n}, i$ is appropriate for $\vec{x}, y$ that $p^{\vec{x},y}_{\vec{n},i} \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*_{\vec{n},i}$. Take an $\mathbb{F}$-generic filter G over $\mathbb{M}$ with $p^{\vec{x}} = p^{x}_{\vec{n}} \in G$. By a density argument, there is $i \in \omega$ such that $\vec{n}, i$ is appropriate for $\vec{x}, y$ and $p^{\vec{x},y}_{\vec{n},i} \in G$. By assumption, this implies that $\mathbb{M}[G] \models (\varphi^*_{\vec{n},i})^G$, hence also $\mathbb{M}[G] \models ((\exists v_k \varphi)^*_{\vec{x}})^G$.

Assume now that $\mathbb{M} \models \neg\exists v_k \varphi(\vec{x}, v_k)$. Then for every $y \in M$, $\mathbb{M} \models \neg\varphi(\vec{x}, y)$. Let $\vec{n}$ be the sequence in ω^k with $p^{\vec{x}} = p^{\vec{x}}_{\vec{n}}$. We have to show that $p^{\vec{x}} \Vdash \neg \bigvee_{i\in\omega} \varphi^*_{\vec{n},i}$. Let G be $\mathbb{F}$-generic over $\mathbb{M}$ with $p^{\vec{x}} \in G$ and suppose for a contradiction that $\mathbb{M}[G] \models (\bigvee_{i\in\omega} \varphi^*_{\vec{n},i})^G$. Then there is $i \in \omega$ such that $\mathbb{M}[G] \models (\varphi^*_{\vec{n},i})^G$. Furthermore, there must be some $y \in M$ such that $p^{\vec{x},y}_{\vec{n},i} \in G$. However, since $\mathbb{M} \models \neg\varphi(\vec{x}, y)$, $p^{\vec{x},y} \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg\varphi^*_{\vec{x},y}$ and therefore by (2.3), $p^{\vec{x},y}_{\vec{n},i} \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg\varphi^*_{\vec{n},i}$ which is absurd. □

For the rest of this section, we will assume, without loss of generality, that whenever φ has exactly one free variable v_i, then $i = 0$.

Theorem 2.5.3. *If $\mathbb{F}$ satisfies the definability lemma for "$v_0 \in v_1$" or for "$v_0 = v_1$" over $\mathbb{M}$, then $\mathcal{C}$ contains a first-order truth predicate for M.*

Proof. If the definability lemma holds either for "$v_0 \in v_1$" or for "$v_0 = v_1$", then $\mathbb{F}$ satisfies the uniform forcing theorem for $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{F}, M)$-formulae as a consequence Corollary 2.1.10. But then by Lemma 2.5.2,

$$T = \{\langle \ulcorner\varphi\urcorner, x\rangle \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}_1, x \in M, p^x \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*\} \in \mathcal{C}$$

is a first-order truth predicate for M. □

Theorem 2.5.3 implies that the definability lemma fails whenever $\mathcal{C}$ does not contain a first-order truth predicate for M. An important class of such models is the following.

Corollary 2.5.4. *If M is a countable transitive model of ZF^- and $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$ then $\mathbb{F}^M$ does not satisfy the definability lemma over $\mathbb{M}$.*

Proof. Let M be a countable transitive model of ZF^-. Assume, towards a contradiction, that the class $\{\langle p, \sigma, \tau\rangle \mid p \Vdash^M_{\mathbb{F}} \sigma = \tau\}$ is definable over M. Then $\mathbb{F}^M$ satisfies the definability lemma for atomic formulae over $\mathbb{M}$. By Theorem 2.5.3, there is a first-order truth predicate for M that is first-order definable over M. This contradicts Tarski's theorem on the undefinability of truth. □

We can even do better and find fixed names ν and $\mu \in M^{\mathbb{F}}$ such that the forcing theorem for $\nu = \mu$ implies the existence of a first-order truth predicate.

Lemma 2.5.5. *There exist $\mu, \nu \in M^{\mathbb{F}}$ and $\{\langle \ulcorner\varphi\urcorner, q_{\ulcorner\varphi\urcorner}\rangle \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}_1\} \in M$ such that*

(1) *If* φ *has one free variable and* $x \in M$*, then* $M \models \varphi(x)$ *iff for all* $r \leq_{\mathbb{F}} p^x, q_{\ulcorner\varphi\urcorner}$, $r \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \nu$.
(2) *If* φ *is a sentence, then* $M \models \varphi$ *iff* $q_{\ulcorner\varphi\urcorner} \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \nu$.

Proof. Since the proof of (2) is a simplified version of the proof of (1), we only verify (1). Choose an antichain $\{q^n \mid n \in \omega\} \subseteq \mathbb{F}$ such that for every $n \in \omega$, $0 \notin \mathrm{dom}(q^n)$, e.g. take

$$q^n = \langle \{1, \ldots, n+1\}, \{\langle 1, n+1\rangle\}, \emptyset\rangle.$$

Consider the names $\nu_{\ulcorner\varphi^*\urcorner}, \mu_{\ulcorner\varphi^*\urcorner}$ as defined in Lemma 2.1.9. We will only consider non-atomic formulae, since all atomic formulae with at most one free variable are either tautologically true or false. The proof of Lemma 2.1.9 shows that for $\ulcorner\varphi\urcorner \in \mathrm{Fml}_1$ with φ non-atomic, all elements of $\nu_{\ulcorner\varphi^*\urcorner}, \mu_{\ulcorner\varphi^*\urcorner}$ are of the form $\langle \tau, \mathbb{1}_{\mathbb{F}}\rangle$ for some $\tau \in M^{\mathbb{F}}$. Let $k : \omega \to \mathrm{Fml}_1$ be a bijection and let $j : \omega \to \mathrm{Fml}^{\Vdash}_{\omega_1,0}(\mathbb{F}, M)$ be given by $j(n) = \ulcorner\varphi^*\urcorner$, where $k(n) = \ulcorner\varphi\urcorner$. Now set

$$\nu = \{\langle \tau, q^n\rangle \mid \langle \tau, \mathbb{1}_{\mathbb{F}}\rangle \in \nu_{j(n)}\}$$
$$\mu = \{\langle \tau, q^n\rangle \mid \langle \tau, \mathbb{1}_{\mathbb{F}}\rangle \in \mu_{j(n)}\}.$$

This yields that $q^n \Vdash^{\mathbb{M}}_{\mathbb{F}} \nu = \nu_{j(n)}$ and $q^n \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \mu_{j(n)}$ for each $n \in \omega$. Moreover, since $0 \notin \mathrm{dom}(q^n)$, p^x_0 and q^n are compatible for every $x \in M$ and $n \in \omega$. For $\ulcorner\varphi\urcorner \in \mathrm{Fml}_1$, we put

$$q_{\ulcorner\varphi\urcorner} = q^{k^{-1}(\ulcorner\varphi\urcorner)}.$$

To check (1), suppose first that $M \models \varphi(x)$ for some $\mathcal{L}_{\in}$-formula φ and $x \in M$. Let $r \in \mathbb{F}$ be such that $r \leq_{\mathbb{F}} p^x, q_{\ulcorner\varphi\urcorner}$. Since $r \leq_{\mathbb{F}} q_{\ulcorner\varphi\urcorner}$, $r \Vdash^{\mathbb{M}}_{\mathbb{F}} \nu = \nu_{\ulcorner\varphi^*\urcorner} \wedge \mu = \mu_{\ulcorner\varphi^*\urcorner}$. On the other hand, since $r \leq_{\mathbb{F}} p^x$, Lemma 2.5.2 implies that $r \Vdash^{\mathbb{M}}_{\mathbb{F}} \varphi^*$, i.e. by Lemma 2.1.9, $r \Vdash^{\mathbb{M}}_{\mathbb{F}} \nu = \mu$. Conversely, assume that $M \models \neg\varphi(x)$. By (1) applied to the negation of φ, we have that for all $r \leq_{\mathbb{F}} p^x, q_{\ulcorner\neg\varphi\urcorner}$, $r \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \nu$. Since p^x and $q_{\ulcorner\neg\varphi\urcorner}$ are compatible, such r exists. Now let π be the automorphism on $\mathbb{F}$ which for $p = \langle d_p, e_p, f_p\rangle$ swaps all occurrences of $k^{-1}(\ulcorner\neg\varphi\urcorner)$ and $k^{-1}(\ulcorner\varphi\urcorner)$ in d_p, e_p and $\mathrm{dom}(f_p)$. Then $\pi(q_{\ulcorner\neg\varphi\urcorner}) \leq_{\mathbb{F}} q_{\ulcorner\varphi\urcorner}$ and $\pi(p^x) = p^x$. In particular, $\pi(r) \leq_{\mathbb{F}} p^x, q_{\ulcorner\varphi\urcorner}$ and so $\pi(r) \Vdash^{\mathbb{M}}_{\mathbb{F}} \nu = \nu_{\ulcorner\varphi^*\urcorner} \wedge \mu = \mu_{\ulcorner\varphi^*\urcorner}$. Moreover, since $\pi(r) \leq_{\mathbb{F}} p^x$, by Lemma 2.5.2 we obtain $\pi(r) \Vdash^{\mathbb{M}}_{\mathbb{F}} \neg\varphi^*$. Finally, by Lemma 2.1.9, this proves that $\pi(r) \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu_{\ulcorner\varphi^*\urcorner} \neq \nu_{\ulcorner\varphi^*\urcorner}$ and so $\pi(r) \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu \neq \nu$. □

Corollary 2.5.6. *There exist* $\nu, \mu \in M^{\mathbb{F}}$ *such that if* $\{p \in \mathbb{F} \mid p \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \nu\} \in \mathcal{C}$*, then* $\mathcal{C}$ *contains a first-order truth predicate for* M*. In particular,* $\{p \in \mathbb{F} \mid p \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \nu\}$ *is not first-order definable over* M*.* □

In the remainder of this section, we show that amenability of the forcing relation for the forcing $\mathbb{F}$ can consistently fail.

Definition 2.5.7. Let $\langle M, \in\rangle$ be an $\in$-structure and $C \in \mathcal{P}(M)$. Then we say that C is *M-amenable*, if for every set $x \in M$, $C \cap x \in$ is an element of M.

To prove the non-amenability of the $\mathbb{F}$-forcing relation, we will work with models all of whose ordinals are first-order definable.

Definition 2.5.8. An $\in$-structure $\langle M, \in\rangle$ is said to be

(1) a *Paris model*, if each ordinal in M is definable in M by a $\mathcal{L}_\in$-formula without parameters.
(2) *pointwise definable*, if every element of M is definable by a $\mathcal{L}_\in$-formula without parameters.

Note that the existence of a countable transitive model of ZFC yields the existence of a countable transitive Paris model satisfying the axioms of ZFC – this follows from [HLR13, Theorem 11], where it is shown that every countable transitive model of ZFC has a pointwise definable class forcing extension. However, for our purpose, it is enough to consider Paris models. In Theorem 2.5.12 we provide a simplified argument to verify (the weaker statement) that certain countable transitive models of ZFC have class forcing extensions which are Paris models.

We will work with a countable, transitive Paris model $M \models \mathsf{ZF}^-$. Note that the least α such that $L_\alpha \models \mathsf{ZF}^-$ is such a model.

Lemma 2.5.9. *Let M be a countable transitive Paris model with $M \models \mathsf{ZF}^-$. Then*

$$X = \{q_{\ulcorner\varphi\urcorner} \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}_0, q_{\ulcorner\varphi\urcorner} \Vdash^{\mathbb{M}}_{\mathbb{F}} \mu = \nu\}$$

is not an element of M, where $q_{\ulcorner\varphi\urcorner}, \mu, \nu$ are as in Lemma 2.5.5.

Proof. Suppose for a contradiction that $X \in M$. Observe that for every $\mathcal{L}_\in$-sentence,

$$M \models \varphi \iff q_{\ulcorner\varphi\urcorner} \in X. \tag{2.4}$$

Consider $C = \{\ulcorner\varphi\urcorner \mid q_{\ulcorner\exists! x\in\mathrm{Ord}\,\varphi(x)\urcorner} \in X\}$. Since $X \in M$, so is C. Observe that we can order the elements of C by

$$\ulcorner\varphi\urcorner \prec \ulcorner\psi\urcorner \iff q_{\ulcorner\exists x,y\in\mathrm{Ord}[x<y\wedge\varphi(x)\wedge\psi(y)]\urcorner} \in X.$$

As a consequence of (2.4), we know that $\langle C, \prec\rangle$ has order type Ord^M, a contradiction. □

In particular, this shows that the $\mathbb{F}$-forcing relation need not be amenable over the ground model.

Corollary 2.5.10. *If M is a countable transitive Paris model with $M \models \mathsf{ZF}^-$, then the $\mathbb{F}^M$-forcing relation for "$v_0 = v_1$" is not M-amenable.* □

2.5.2 A failure of the truth lemma

In the following, we show that the truth lemma can consistently fail for class forcing. Note that by Lemma 2.1.9, if we find a notion of class forcing $\mathbb{P}$ and an infinitary formula for which the truth lemma fails, then we automatically obtain that it fails for "$v_0 = v_1$". To see this, suppose that φ is an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formula such that $\mathbb{P}$ does not satisfy the truth lemma for φ. Assume towards a contradiction that the truth lemma for "$v_0 = v_1$" holds and assume that that G is $\mathbb{P}$-generic with $M[G] \models \varphi^G$. Then by Lemma 2.1.9, $M[G] \models \mu^G_{\ulcorner\varphi\urcorner} = \nu^G_{\ulcorner\varphi\urcorner}$ and thus there is $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \mu_{\ulcorner\varphi\urcorner} = \nu_{\ulcorner\varphi\urcorner}$. But then p also forces φ.

If $\mathbb{P}$ is any notion of class forcing that satisfies the forcing theorem, we denote by $\dot{\mathbb{F}}$ the canonical class $\mathbb{P}$-name for $\mathbb{F}^{M[G]}$ in a $\mathbb{P}$-generic extension $M[G]$. As an example of the failure of the truth lemma, we will consider two-step iterations where the second iterand will be of the form $\mathbb{F}^{M[G]}$, where G is generic for the first iterand.

Theorem 2.5.11. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- which has a hierarchy. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ with the following properties:*

(a) *$\mathbb{P}$ is pretame.*
(b) *There is a $\mathbb{P}$-generic filter G such that $M[G]$ is a Paris model.*
(c) *For every $p \in G$ there is a $\mathbb{P}$-generic filter $\bar{G}$ such that $M[\bar{G}]$ is not a Paris model.*

Then the truth lemma fails for $\mathbb{P} * \dot{\mathbb{F}}$.

Proof. We will find an infinitary formula Φ in the language $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P} * \dot{\mathbb{F}}, M)$ such that if $G * H$ is $\mathbb{P} * \dot{\mathbb{F}}$-generic over $\mathbb{M}$, then Φ^{G*H} expresses that $M[G]$ is a Paris model. Using this, we choose G as in (b). Then Φ^{G*H} holds, while by (c), there cannot be a condition in $G * H$ forcing this, implying that the truth lemma fails for $\mathbb{P} * \dot{\mathbb{F}}$.

Given a formula φ with exactly one free variable, let $\Psi_{\ulcorner\varphi\urcorner}$ denote the formula

$$\varphi(v_0) \ \wedge\ \forall v_1 \ (\varphi(v_1) \ \rightarrow \ v_1 = v_0)\,,$$

i.e. $\Psi_{\ulcorner\varphi\urcorner}(x)$ states that "*x is unique such that $\varphi(x)$ holds*". Similarly, let $\Omega(x)$ be the formula expressing that x is an ordinal. If G is $\mathbb{P}$-generic over M, then $M[G]$ is a Paris model if and only if for all ordinals $\alpha \in M[G]$ there is φ such that $M[G] \models \Psi_{\ulcorner\varphi\urcorner}(\alpha)$. Now recall that as described in Section 2.5.1, for each φ we can assign to $\Psi_{\ulcorner\varphi\urcorner}$, Ω and $n \in \omega$ infinitary formulae $(\Psi_{\ulcorner\varphi\urcorner})^*_n$ and Ω^*_n in the forcing language of $\mathbb{F}^{M[G]}$ with the properties (as in Lemma 2.5.2)

$$M[G] \models \Psi_{\ulcorner\varphi\urcorner}(x) \iff p^x_n \Vdash^{\mathbb{M}[G]}_{\mathbb{F}^{M[G]}} (\Psi_{\ulcorner\varphi\urcorner})^*_n$$
$$M[G] \models \Omega(x) \iff p^x_n \Vdash^{\mathbb{M}[G]}_{\mathbb{F}^{M[G]}} \Omega^*_n.$$

However, since we will need infinitary formulae in the forcing language of $\mathbb{P} * \dot{\mathbb{F}}$, we have to modify this approach slightly. For a formula ψ and $n \in \omega$, we define ψ^{**}_n in the same way we defined ψ^*_n, but we replace every occurence of some condition $p \in \mathbb{F}^{M[G]}$ by $\langle \mathbb{1}_{\mathbb{P}}, \check{p}\rangle \in \mathbb{P} * \dot{\mathbb{F}}$. Note that this is possible, since for every condition p which appears in ψ^*_n, the function f_p is empty, and so $p \in M$. Let

$$\Phi = \bigwedge_{n\in\omega} \ \bigvee_{\ulcorner\varphi\urcorner\in\mathrm{Fml}_1} [\Omega^{**}_n \ \longrightarrow \ (\Psi_{\ulcorner\varphi\urcorner})^{**}_n]\,.$$

We claim that $M[G]$ is a Paris model if and only if $M[G][H] \models \Phi^{G*H}$ holds for every (or, equivalently, for some) filter H which is $\mathbb{F}^{M[G]}$-generic over $M[G]$. Suppose first that $M[G]$ is a Paris model. Let H be $\mathbb{F}^{M[G]}$-generic over $M[G]$ and $n \in \omega$. By a density argument, there is $x \in M[G]$ such that p^x_n (as defined in $M[G]$) is in H. Since $M[G]$ is a Paris model, either $M[G] \models \neg\Omega(x)$ or there is some formula φ such that $M[G] \models \Psi_{\ulcorner\varphi\urcorner}(x)$. Let $\dot{x} \in M^{\mathbb{P}}$ be a name for x. Since $\mathbb{P}$ is pretame, it satisfies the truth lemma by Theorem 2.2.2 and hence there is $q \in \mathbb{P}$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} [\Omega(\dot{x}) \to \Psi_{\ulcorner\varphi\urcorner}(\dot{x})]$. Let $\dot{p}^x_n$ be a $\mathbb{P}$-name for $p^x_n \in \mathbb{F}^{M[G]}$. Then $\langle q, \dot{p}^x_n\rangle \Vdash^{\mathbb{M}}_{\mathbb{P}*\dot{\mathbb{F}}} [\Omega^{**}_n \to (\Psi_{\ulcorner\varphi\urcorner})^{**}_n]$. To see this, let $\bar{G} * \bar{H}$ be $\mathbb{P} * \dot{\mathbb{F}}$-generic over M with $\langle q, \dot{p}^x_n\rangle \in \bar{G} * \bar{H}$ and put $\bar{x} = \dot{x}^{\bar{G}}$. Then $M[\bar{G}] \models \Psi_{\ulcorner\varphi\urcorner}(\bar{x})$ and $(\dot{p}^x_n)^{\bar{G}} = p^{\bar{x}}_n$. But since $p^{\bar{x}}_n \Vdash^{\mathbb{M}}_{\mathbb{F}^{M[\bar{G}]}} (\Psi_\varphi)^*_n$ by Lemma 2.5.2, it is easy to see that in $M[\bar{G}][\bar{H}]$ also $((\Psi_{\ulcorner\varphi\urcorner})^{**}_n)^{\bar{G}*\bar{H}}$ holds. This proves that $\langle q, \dot{p}^x_n\rangle \Vdash^{\mathbb{M}}_{\mathbb{P}*\dot{\mathbb{F}}} (\Psi_{\ulcorner\varphi\urcorner})^{**}_n$. Now since $\langle q, \dot{p}^x_n\rangle \in G * H$, we have established that $M[G][H] \models \Phi^{G*H}$.

Conversely, suppose that $M[G][H] \models \Phi^{G*H}$. Let $\alpha \in M[G]$ be an ordinal. Let $n \in \omega$ such that $p^\alpha_n \in H$. By assumption, there is $\ulcorner\varphi\urcorner \in \mathrm{Fml}_1$ such that $M[G][H] \models ((\Psi_{\ulcorner\varphi\urcorner})^{**}_n)^{G*H}$. We want to verify that $M[G] \models \Psi_{\ulcorner\varphi\urcorner}(\alpha)$. If not, we can proceed as before and obtain $q \in G$ such that $\langle q, \dot{p}^\alpha_n\rangle \Vdash^{\mathbb{M}}_{\mathbb{P}*\dot{\mathbb{F}}} (\neg\Psi_{\ulcorner\varphi\urcorner})^{**}_n$, which is contradictory. □

By a special case of more general results in [Ena05] and [HLR13] there is a tame[3] notion of class forcing $\mathbb{P}^*$ such that for every countable transitive GBc-model of the form $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$, there is a $\mathbb{P}^*$-generic filter G over $\mathbb{M}$ such that $M[G]$ is pointwise definable.

For the benefit of the reader, we will describe a very simple tame notion of class forcing $\mathbb{P}$ and indicate a proof that there is a $\mathbb{P}$-generic extension which is a Paris model over any countable transitive GBc^--model $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$ such that $\langle M, \in\rangle \models V = L$. The outline of the argument follows the proof of [Ena05, Theorem 2.8] and [HLR13, Theorems 11 and 12].

Theorem 2.5.12. *Suppose that $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$ is a countable transitive model such that $\langle M, \in\rangle \models \mathsf{ZFC} + \mathsf{V} = \mathsf{L}$. Then there is a pretame notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem and a $\mathbb{P}$-generic vilter such that $\langle M[G], \in\rangle$ is a Paris model.*

Proof. $\mathbb{P}$ is a two-step iteration, where the first step is to force with $\mathbb{C} = \langle 2^{<\mathrm{Ord}}, \supseteq\rangle$. Note that $\mathbb{C}$ is $<$ Ord-closed and hence does not add new sets by Example 2.3.2 and Lemma 2.3.4. Following the proof of [HLR13, Theorem 12], we show that there is a $\mathbb{C}$-generic filter U such that all ordinals of M are first-order definable over $\langle M, \in, U\rangle$. Let $\langle \alpha_n \mid n \in \omega\rangle$ be an increasing enumeration (in V) of Ord^M and let $\langle D_n \mid n \in \omega\rangle$ be an enumeration of all dense subsets of $\mathbb{C}$ which are definable over M. Since M satisfies $\mathsf{V} = \mathsf{L}$, each D_n is definable from ordinal parameters. Moroever, if φ_n is the formula defining D_n, we may assume that the parameters occurring in φ_n are among $\{\alpha_i \mid i < n\}$. In order to construct the desired $\mathbb{C}$-generic filter U, we define a $\leq_{\mathbb{C}}$-descending sequence $\langle p_n \mid n \in \omega\rangle$ in the following way. We start with $p_0 = \mathbb{1}_{\mathbb{C}}$.

- Given p_{2n} for some $n \in \omega$, let $p_{2n+1} \leq_{\mathbb{C}} p_{2n}$ be an extension of p_{2n+1} of minimal length such that $p_{2n+1} \in D_n$.
- Given p_{2n+1}, let $p_{2n+2} \leq_{\mathbb{C}} p_{2n+1}$ be the strengthening obtained by concatenating α_n-many 1's and one 0 to the end of p_{2n+1}.

Finally, let $U = \{p \in \mathbb{C} \mid \exists n \in \omega\, (p_n \leq_{\mathbb{C}} p)\}$. By construction, U is $\mathbb{C}$-generic over $\mathbb{M}$. We claim that every ordinal is definable in $\langle M, \in, U\rangle$ without parameters. To see this, we show by induction on $n \in \omega$ that α_n, p_{2n+1} and p_{2n+2} are definable without parameters

[3]Tameness is a strengthening of pretameness defined in [Fri00] which is essentially equivalent to the preservation of GB.

in $\langle M, \in, U\rangle$. Suppose that $n \in \omega$ and $\alpha_i, p_{2i+1}, p_{2i+2}$ have already been defined for all $i < n$. Then p_{2n+1} is defined as the least initial segment q of $\bigcup U$ such that $q \leq_{\mathbb{C}} p_n$ and $M \models \varphi_n(q)$. Since all parameters of φ_n are among $\{\alpha_i \mid i < n\}$, our inductive hypothesis implies that this definition does not require any parameters. Furthermore, p_{2n+2} is defined as the minimal extension of p_{2n+1} obtained by adding a string of 1's and one 0, and α_n is the length of the aforementioned string of 1's.

The second step is to code the generic U into the continuum function, using an Easton iteration. We follow the proof of [HLR13, Theorem 11]. By assumption, we have that $\langle M, \in\rangle \models \mathsf{GCH}$. Note that we may view U as a class $U \subseteq \mathrm{Ord}^M$. We force over $\mathbb{M}[U]$ with the Easton product $\mathbb{Q} = \prod_{\alpha\in U} \mathrm{Add}(\aleph_\alpha, \aleph_\alpha^{++})$, i.e. for every regular M-cardinal κ and for every $q \in \mathbb{Q}$, $|\mathrm{dom}(q)\cap\kappa| < \kappa$. By Example 2.2.12, $\mathbb{Q}$ is pretame and therefore it preserves GB^-. Hence it follows from Remark 2.2.16 that $\mathbb{C} * \dot{\mathbb{Q}}$ is pretame. Furthermore, if H is $\mathbb{Q}$-generic over $\mathbb{M}[U]$ then $M[U][H]$ is a Paris model, since every ordinal is definable in $\langle M, \in, U\rangle$ and U is definable over $M[U][H]$ by $\alpha \in U$ iff $2^{\aleph_\alpha} = \aleph_\alpha^{++}$. □

The forcing notion $\mathbb{P}$ described in Theorem 2.5.12 satisfies (a) and (b) over any countable transitive model of GBc of the form $\langle M, \mathrm{Def}(M)\rangle$. We now present two consistent examples of such models over which $\mathbb{P}$ also satisfies condition (c) in the statement of Theorem 2.5.11.

Example 2.5.13. (1) The simplest possibility is to start with a model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of KM. By forcing over the first-order part $\langle M, \mathrm{Def}(M)\rangle$ of $\mathbb{M}$, we may obtain a $\mathbb{P}$-generic filter G such that $M[G]$ is a Paris model. On the other hand, we can force over the KM-model $\mathbb{M}$ and choose a filter $\bar{G}$ which is $\mathbb{P}$-generic over $\mathbb{M}$. Since $\mathbb{P}$ is tame, by [Ant15, Theorem 23] $\mathbb{M}[\bar{G}] \models \mathsf{KM}$. But this is a contradiction, since no model of KM is a Paris model:

Suppose for a contradiction that $\mathbb{N} = \langle N, \mathcal{D}\rangle$ is a model of KM which is a Paris model. Using that $\mathcal{D}$ contains a first-order truth predicate for formulae with one free variable, it follows that $\mathcal{D}$ contains a surjection from ω to Ord^N, contradicting replacement.

(2) If we want to avoid KM, we can instead start with a countable transitive model $\langle M, \in\rangle$ of ZFC which has cardinality $\aleph_1$ in $\mathsf{L}[M]$ and which is closed under countable sequences in $\mathsf{L}[M]$. Now since the forcing $\mathbb{P}$ is σ-closed[4] in M, for every $p \in \mathbb{P}^M$ there is a $\mathbb{P}^M$-generic filter G_p over M in $\mathsf{L}[M]$ containing p. Since M is uncountable in $\mathsf{L}[M]$, no generic extension of the form $M[G_p]$ is a Paris model. Note that it is easy to obtain such a model M starting in a model of $\mathsf{V} = \mathsf{L}$ with an inaccessible cardinal and then forcing with $\mathrm{Col}(\omega, \omega_1)$.

[4] i.e. $< \omega_1$-closed

Chapter 3

Boolean completions

One of the standard approaches to set forcing is to force with complete Boolean algebras. The crucial observation is that every partial order can be embedded into a complete Boolean algebra, its so-called Boolean completion. Moreover, in set forcing, Boolean completions are always unique up to isomorphisms. In this chapter we investigate to what extent these results carry over to class forcing. In particular, we will show that the existence of a Boolean completion is essentially equivalent to the forcing theorem. Furthermore, we will prove that unions of set-sized complete subforcings always have a Boolean completion, and that in KM every notion of class forcing has a Boolean completion. Finally, we prove that in class forcing Boolean completions need not be unique.

3.1 (Pre-)Boolean algebras and completions

We generalize the concept of a Boolean algebra in order to allow for preorders which are not antisymmetric, and define Boolean completions in the framework of class forcing.

3.1.1 Pre-Boolean algebras

Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a fixed countable transitive model of GB^-. The following generalization of Boolean algebras was already been used by Hamkins and Löwe in the context of modal logic of forcing (see [HL08]).

Definition 3.1.1. A *pre-Boolean algebra* is a structure of the form

$$\mathbb{B} = \langle B, 0_{\mathbb{B}}, \mathbb{1}_{\mathbb{B}}, \neg_{\mathbb{B}}, \vee_{\mathbb{B}}, \wedge_{\mathbb{B}}, \approx_{\mathbb{B}} \rangle^1$$

such that the following statements hold for all $a, b, c \in B$:

(1) $\approx_{\mathbb{B}}$ is an congruence relation on $\langle B, 0_{\mathbb{B}}, \mathbb{1}_{\mathbb{B}}, \neg, \vee, \wedge \rangle$.
(2) $a \vee (b \vee c) \approx_{\mathbb{B}} (a \vee b) \vee c$ and $a \wedge (b \wedge c) \approx_{\mathbb{B}} (a \wedge b) \wedge c$ (*associativity*).
(3) $a \vee b \approx_{\mathbb{B}} b \vee a$ and $a \wedge b \approx_{\mathbb{B}} b \wedge a$ (*commutativity*).
(4) $a \vee (a \wedge b) \approx_{\mathbb{B}} a$ and $a \wedge (a \vee b) \approx_{\mathbb{B}} a$ (*absorption*).
(5) $a \vee 0_{\mathbb{B}} \approx_{\mathbb{B}} a$ and $a \wedge \mathbb{1}_{\mathbb{B}} \approx_{\mathbb{B}} a$ (*identity*).

[1] If the context is clear, we will omit the subscripts of the Boolean operations.

(6) $a \vee (b \wedge c) \approx_{\mathbb{B}} (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) \approx_{\mathbb{B}} (a \wedge b) \vee (a \wedge c)$ (*distributivity*).
(7) $a \vee \neg a \approx_{\mathbb{B}} 1_{\mathbb{B}}$ and $a \wedge \neg a \approx_{\mathbb{B}} 0_{\mathbb{B}}$ (*complements*).

Furthermore, we can define a preorder on the domain B by

$$a \leq_{\mathbb{B}} b \Longleftrightarrow a \vee b \approx_{\mathbb{B}} b.$$

Finally, by $\mathbb{B}^*$ we denote the preorder $\langle B \setminus \{b \in B \mid b \approx_{\mathbb{B}} 0_{\mathbb{B}}\}, \leq_{\mathbb{B}}\rangle$ of all elements of B which are not $\approx_{\mathbb{B}}$-equivalent to $0_{\mathbb{B}}$. Observe that in $\mathbb{B}^*$, $a \perp_{\mathbb{B}^*} b$ if and only if $a \wedge b \approx 0_{\mathbb{B}}$.

It is clear that Boolean algebras constitute a special case of pre-Boolean algebras.

Definition 3.1.2. A pre-Boolean algebra $\mathbb{B}$ is called a *Boolean algebra*, if its equivalence relation $\approx_{\mathbb{B}}$ is given by $=$.

Lemma 3.1.3. *If $\mathbb{B}$ is a pre-Boolean algebra, then $\mathbb{B}^*$ is separative.*

Proof. Suppose that $a, b \in \mathbb{B}^*$ with $a \not\leq_{\mathbb{B}} b$. We show that $a \wedge \neg b \in \mathbb{B}^*$ because this proves the claim, since $a \wedge \neg b \leq_{\mathbb{B}} a$ and $a \wedge \neg b \perp_{\mathbb{B}} b$. Note that

$$\begin{aligned} a &\approx_{\mathbb{B}} a \wedge (a \vee b) \wedge (b \vee \neg b) \approx_{\mathbb{B}} a \wedge (b \vee (a \wedge \neg b)) \\ &\approx_{\mathbb{B}} (a \vee (a \wedge \neg b)) \wedge (b \vee (a \wedge \neg b)) \approx_{\mathbb{B}} (a \wedge b) \vee (a \wedge \neg b). \end{aligned}$$

Hence if $a \wedge \neg b \approx_{\mathbb{B}} 0_{\mathbb{B}}$, then $a \approx_{\mathbb{B}} a \wedge b$ contradicting our assumption. □

We can characterize suprema in terms of the induced preorder which will result very useful for performing class forcing with pre-Boolean algebras.

Lemma 3.1.4. *If $\mathbb{B}$ is a pre-Boolean algebra, then $a \vee b \approx_{\mathbb{B}} c$ if and only if $a, b \leq_{\mathbb{B}} c$ and $\{a, b\}$ is predense in $\mathbb{B}^*$ below c.*

Proof. Suppose first that $a \vee b \approx_{\mathbb{B}} c$. Then $a \wedge c \approx_{\mathbb{B}} a \wedge (a \vee b) \approx_{\mathbb{B}} a$ by the absorption law, so $a \leq_{\mathbb{B}} c$. Similarly, we obtain $b \leq_{\mathbb{B}} c$. Suppose now that $d \in \mathbb{B}^*$ with $d \leq_{\mathbb{B}} c$. We have to check that either $a \wedge d \not\approx_{\mathbb{B}} 0_{\mathbb{B}}$ or $b \wedge d \not\approx_{\mathbb{B}} 0_{\mathbb{B}}$. If this fails, then we obtain

$$0_{\mathbb{B}} \approx_{\mathbb{B}} (a \wedge d) \vee (b \wedge d) \approx_{\mathbb{B}} (a \vee b) \wedge d \approx_{\mathbb{B}} c \wedge d \approx_{\mathbb{B}} d,$$

a contradiction.

Conversely, assume that $a, b \leq_{\mathbb{B}} c$ and $\{a, b\}$ is predense below c. Note first that

$$(a \vee b) \wedge c \approx_{\mathbb{B}} (a \wedge c) \vee (b \wedge c) \approx_{\mathbb{B}} c \vee c \approx_{\mathbb{B}} c,$$

so $c \leq_{\mathbb{B}} a \vee b$. For the other inequality, suppose that $c \not\leq_{\mathbb{B}} a \vee b$. By separativity of $\mathbb{B}^*$ there is $d \leq_{\mathbb{B}} c$ such that $0_{\mathbb{B}} = d \wedge (a \vee b) = (d \wedge a) \vee (d \wedge b)$. But this contradicts our assumption that $\{a, b\}$ is predense below c. □

Lemma 3.1.4 motivates the following definition.

Definition 3.1.5. A pre-Boolean algebra $\mathbb{B}$ is said to be

(1) *M-complete*, if there is a map $\sup_{\mathbb{B}} : \mathcal{P}(B) \cap M \to B$ in $\mathcal{C}$ such that $\sup_{\mathbb{B}}(\{a, b\}) = a \vee_{\mathbb{B}} b$ and the following conditions hold for each $X \in \mathcal{P}(B) \cap M$:
   (a) For each $b \in X$, $b \leq_{\mathbb{B}} \sup_{\mathbb{B}} X$,
   (b) X is predense below $\sup_{\mathbb{B}} X$.
(2) *$\mathcal{C}$-complete*, if there is a map $\sup_{\mathbb{B}} : \mathcal{P}(B) \cap \mathcal{C} \to B$ which is definable over $\mathbb{M}$ and satisfies the properties above for each $X \in \mathcal{P}(B) \cap \mathcal{C}$.

3.1.2 (Pre-)Boolean completions

Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^- and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Unlike in set forcing, in class forcing we distinguish between two variants of Boolean completions, depending on whether there only exist suprema of subsets of $\mathbb{P}$ which are in M, or whether every subset of $\mathbb{P}$ which is an element of $\mathcal{C}$ has a supremum.

Definition 3.1.6. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. A *pre-Boolean M-completion of* $\mathbb{P}$ *in* $\mathbb{M}$ is an M-complete pre-Boolean algebra

$$\mathbb{B} = \langle B, 0_{\mathbb{B}}, 1_{\mathbb{B}}, \neg_{\mathbb{B}}, \vee_{\mathbb{B}}, \wedge_{\mathbb{B}}, \approx_{\mathbb{B}} \rangle$$

with the following properties:

(1) B, all Boolean operations on B and $\approx_{\mathbb{B}}$ are in $\mathcal{C}$.
(2) There is an injective dense embedding $i : \mathbb{P} \to \mathbb{B}^*$ in $\mathcal{C}$.
(3) For all $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ if and only if $i(p) \leq_{\mathbb{B}} i(q)$.

Similarly, we define a *pre-Boolean $\mathcal{C}$-completion of* $\mathbb{P}$ *in* $\mathbb{M}$ to be a pre-Boolean M-completion which is additionally $\mathcal{C}$-complete.

Furthermore, a pre-Boolean M-completion of $\mathbb{P}$ which is a Boolean algebra is called a *Boolean M-completion of* $\mathbb{P}$ *in* $\mathbb{M}$ and a Boolean M-completion which is also $\mathcal{C}$-complete is said to be a *Boolean $\mathcal{C}$-completion of* $\mathbb{P}$ *in* $\mathbb{M}$.

Remark 3.1.7. If $\mathbb{P}$ has a pre-Boolean M-completion $\mathbb{B}$ then $\mathbb{P}$ is separative. This follows from the fact that $\mathbb{B}$ is separative by Lemma 3.1.3 and condition (3) in Definition 3.1.6 above. Conversely, if $\mathbb{P}$ is separative and $i : \mathbb{P} \to \mathbb{Q}$ is any complete embedding, then condition (3) is always satisfied. Therefore, (3) is equivalent to stipulating that $\mathbb{P}$ is separative.

In set forcing, Boolean completions are unique: If $\mathbb{B}_0$ and $\mathbb{B}_1$ are both Boolean completions of $\mathbb{P}$ and $e_0 : \mathbb{P} \to \mathbb{B}_0$ and $e_1 : \mathbb{P} \to \mathbb{B}_1$ are dense embeddings, then one can define an isomorphism from $\mathbb{B}_0$ to $\mathbb{B}_1$ by setting

$$f(b) = \sup_{\mathbb{B}_1} \{e_1(p) \mid p \in \mathbb{P} \wedge e_0(p) \leq_{\mathbb{B}_0} b\}$$

for $b \in \mathbb{B}_0$. Moreover, f fixes $\mathbb{P}$ in the sense that $f(e_0(p)) = e_1(p)$ for every condition $p \in \mathbb{P}$. In class forcing, this proof works only for Boolean $\mathcal{C}$-completions. We will show in Section 3.5 that Boolean M-completions need not be unique in the following sense.

Definition 3.1.8. We say that a notion of class forcing $\mathbb{P}$ *has a unique Boolean M-completion in* $\mathbb{M}$, if $\mathbb{P}$ has a Boolean M-completion $\mathbb{B}_0$ in $\mathbb{M}$ and for every other Boolean M-completion $\mathbb{B}_1$ of $\mathbb{P}$ in $\mathbb{M}$ there is an isomorphism in V between $\mathbb{B}_0$ and $\mathbb{B}_1$ which fixes $\mathbb{P}$. Similarly, by replacing M by $\mathcal{C}$ in the definition above, we define that $\mathbb{P}$ *has a unique Boolean $\mathcal{C}$-completion in* $\mathbb{M}$.

As sketched above, we obtain the following.

Lemma 3.1.9. *Boolean $\mathcal{C}$-completions of notions of class forcing are always unique.* □

3.1.3 The separative quotient

In this section, we fix a ground model $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ which satisfies representatives choice. Since non-separative notions of class forcing do not have (pre)-Boolean completions, we will sometimes have to pass to the so-called *separative quotient* which is separative and forcing equivalent to the original forcing notion. The computations in the present section are adaptations of standard results in set forcing. Accounts on the separative quotient in set forcing can be found in any textbook about set theory such as [Jec03].

Fix a notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$. We define an equivalence relation on P by stipulating

$$p \approx_{\mathbb{P}} q \iff \forall r \in \mathbb{P}\,(r \parallel_{\mathbb{P}} p \leftrightarrow r \parallel_{\mathbb{P}} q).$$

Now using representatives choice, there are $\mathbb{S}(\mathbb{P}) \in \mathcal{C}$ and a surjective map $\pi : \mathbb{P} \to \mathbb{S}(\mathbb{P})$ such that for all $p, q \in \mathbb{P}$, $p \approx_{\mathbb{P}} q$ if and only if $\pi(p) = \pi(q)$. For notational simplicity, we write $[p]$ instead of $\pi(p)$. Furthermore, we define an order on $\mathbb{S}(\mathbb{P})$ by

$$[p] \leq_{\mathbb{S}(\mathbb{P})} [q] \iff \forall r \in \mathbb{P}\,(r \parallel_{\mathbb{P}} p \to r \parallel_{\mathbb{P}} q).$$

Then we have the following properties:

(1) The ordering $\leq_{\mathbb{S}(\mathbb{P})}$ is well-defined.
(2) For all $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ implies $[p] \leq_{\mathbb{S}(\mathbb{P})} [q]$.
(3) Two conditions $p, q \in \mathbb{P}$ are compatible in $\mathbb{P}$ if and only if $[p]$ and $[q]$ are compatible in $\mathbb{S}(\mathbb{P})$.
(4) $\mathbb{S}(\mathbb{P})$ is separative.

The proof of all properties above are standard computations. The following observations can be found in any standard textbook on set theory. For the sake of completeness, we nevertheless give full proofs.

Lemma 3.1.10. *If G is $\mathbb{P}$-generic over $\mathbb{M}$, then $\pi''G$ is $\mathbb{S}(\mathbb{P})$-generic over $\mathbb{M}$. Conversely, if H is $\mathbb{S}(\mathbb{P})$-generic over $\mathbb{M}$, then $\pi^{-1}[H]$ is $\mathbb{P}$-generic over $\mathbb{M}$.*

Proof. Firstly, we assume that G is a $\mathbb{P}$-generic filter and show that $\pi''G$ is $\mathbb{S}(\mathbb{P})$-generic. For upwards closure, let $p \in G$ and $[q] \geq_{\mathbb{S}(\mathbb{P})} [p]$. Clearly, $D = \{r \in \mathbb{P} \mid r \leq_{\mathbb{P}} p, q\}$ is dense below p, hence there must be some $r \in G$ with $r \leq_{\mathbb{P}} p, q$. In particular, $q \in G$ and hence $[q] \in \pi''G$. That all elements of $\pi''G$ are compatible follows from the analogous property for $\mathbb{P}$ and condition (3) above. For genericity, suppose that $\bar{D} \in \mathcal{C}$ is dense open in $\mathbb{S}(\mathbb{P})$. Let $D = \pi^{-1}(\bar{D})$. We show that D is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$. Now by density of $\bar{D}$, there is some $q \in \mathbb{P}$ such that $[q] \leq_{\mathbb{S}(\mathbb{P})} [p]$ and $[q] \in \bar{D}$. In particular, p, q are compatible, so we can take a witness $r \leq_{\mathbb{P}} p, q$. Then by (2), $[r] \leq_{\mathbb{S}(\mathbb{P})} [q]$ and since $\bar{D}$ is open, $[r] \in \bar{D}$. Hence $r \in D$ with $r \leq_{\mathbb{P}} p$. This shows that D is dense. By genericity of $\mathbb{P}$, there is $p \in D \cap G$ and so $[p] \in \pi''G \cap \bar{D}$.

Conversely, let H be $\mathbb{S}(\mathbb{P})$-generic and put $G = \pi^{-1}[H]$. Upwards closure is an immediate consequence of (2) and condition (3) implies that G is a filter. It remains to check that G is generic. Let D be dense in $\mathbb{P}$ and set $\bar{D} = \pi''D$. We show that $\bar{D}$ is dense. Let $p \in \mathbb{P}$ and $q \leq_{\mathbb{P}} p$ such that $q \in D$. Then by (2), $[q] \leq_{\mathbb{S}(\mathbb{P})} [p]$ and $[q] \in \bar{D}$. By genericity of H, there is $[p] \in H \cap \bar{D}$. In particular, $p \in G \cap D$. $\square$

Next we want to check that $\mathbb{P}$ and $\mathbb{S}(\mathbb{P})$ generate the same generic extensions and that the definability lemma for $\mathbb{S}(\mathbb{P})$ implies the definability lemma for $\mathbb{P}$. We show by induction on the rank of σ that for every $\mathbb{P}$-generic filter G and $\sigma \in M^{\mathbb{P}}$, $\pi^*(\sigma)^{\pi''G} = \sigma^G$, where π^* is the map on $M^{\mathbb{P}}$ induced by π. Suppose that this holds for every $\tau \in M^{\mathbb{P}}$ with $\operatorname{rank}(\tau) < \operatorname{rank}(\sigma)$. Let $H = \pi''G$.

$$\begin{aligned}\pi^*(\sigma)^H &= \{\pi^*(\tau)^H \mid \exists [p] \in H\, [\langle \pi^*(\tau), [p]\rangle \in \pi^*(\sigma)]\} \\ &= \{\tau^G \mid \exists p \in G\, (\langle \tau, p\rangle \in \sigma)\} \\ &= \sigma^G.\end{aligned}$$

For the second equality, suppose that $p \in G$ and $\tau \in \operatorname{dom}(\sigma)$ such that $[p] \in H$ and $\langle \pi^*(\tau), [p]\rangle \in \pi^*(\sigma)$. Then there is $q \in \mathbb{P}$ such that $\langle \tau, q\rangle \in \sigma$ and $p \approx_{\mathbb{P}} q$. By density of $\{r \in \mathbb{P} \mid r \leq_{\mathbb{P}} p, q\}$ we obtain that $q \in G$. This shows that whenever G is $\mathbb{P}$-generic, then $M[G] \subseteq M[\pi''G]$. The other inclusion holds as well by surjectivity of π. Now if H is $\mathbb{S}(\mathbb{P})$-generic, then $G = \pi^{-1}[H]$ is $\mathbb{P}$-generic and since $\pi''G = H$, we also have that $M[G] = M[H]$. Thus we have shown the following:

Lemma 3.1.11. *Suppose that $\mathbb{M}$ satisfies representatives choice. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ satisfies the forcing theorem if and only if its separative quotient $\mathbb{S}(\mathbb{P})$ does so. More precisely, if $\sigma, \tau \in M^{\mathbb{P}}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$, then*

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau \iff [p] \Vdash^{\mathbb{M}}_{\mathbb{S}(\mathbb{P})} \pi^*(\sigma) \in \pi^*(\tau)$$

where $\pi : \mathbb{P} \to \mathbb{S}(\mathbb{P}), p \mapsto [p]$ is as defined above. □

3.2 Boolean completions and the forcing theorem

Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a fixed model of GB^-. In this section we investigate the relationship between the forcing theorem and the existence of a (pre)-Boolean M-completion.

3.2.1 Boolean values

Definition 3.2.1 (Boolean values)**.** Let $\mathbb{B} = \langle B, 0_{\mathbb{B}}, \mathbb{1}_{\mathbb{B}}, \neg, \wedge, \vee\rangle$ be an M-complete pre-Boolean algebra. Then we can compute the *Boolean values* of $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{B}, M)$-formulae as in the case of set-sized Boolean algebras by

$$\begin{aligned}&[\![\check{p} \in \dot{G}]\!]_{\mathbb{B}} = p, \\ &[\![\sigma \in \tau]\!]_{\mathbb{B}} = \sup\nolimits_{\mathbb{B}}\{[\![\sigma = \pi]\!]_{\mathbb{B}} \wedge p \mid \langle \pi, q\rangle \in \tau\}, \\ &[\![\sigma = \tau]\!]_{\mathbb{B}} = [\![\sigma \subseteq \tau]\!]_{\mathbb{B}} \wedge [\![\tau \subseteq \sigma]\!]_{\mathbb{B}}, \\ &[\![\sigma \subseteq \tau]\!]_{\mathbb{B}} = \inf\nolimits_{\mathbb{B}}\{\neg p \vee [\![\pi \in \tau]\!]_{\mathbb{B}} \mid \langle \pi, p\rangle \in \sigma\}\end{aligned}$$

for $\sigma, \tau \in M^{\mathbb{B}}$ and $p \in \mathbb{B}$. Furthermore, we can extend this to infinitary formulae in the natural way by stipulating

$$\begin{aligned} [\![\neg\varphi]\!]_{\mathbb{B}} &= \neg[\![\varphi]\!]_{\mathbb{B}}, \\ [\![\bigvee_{i\in I} \varphi_i]\!]_{\mathbb{B}} &= \sup_{\mathbb{B}}\{[\![\varphi_i]\!]_{\mathbb{B}} \mid i \in I\}, \\ [\![\bigwedge_{i\in I} \varphi_i]\!]_{\mathbb{B}} &= \inf_{\mathbb{B}}\{[\![\varphi_i]\!]_{\mathbb{B}} \mid i \in I\}. \end{aligned}$$

Note that this definition does not require class recursion, since we can define it by recursion on the well-founded subformula relation. However, we do not - in general - have Boolean values for all $\mathcal{L}^2_{\in}$-formulae, since this would require us to take suprema of proper classes and hyperclasses of conditions.

Suppose now that $\mathbb{B}$ is a $\mathcal{C}$-complete pre-Boolean algebra. Then we can define Boolean values for all $\mathcal{L}_{\in}$-formulae with class parameters by

$$\begin{aligned} [\![\sigma \in \Gamma]\!]_{\mathbb{B}} &= \sup_{\mathbb{B}}\{[\![\sigma = \tau]\!]_{\mathbb{B}} \wedge p \mid \langle \tau, p\rangle \in \Gamma\}, \\ [\![\Sigma = \Gamma]\!]_{\mathbb{B}} &= [\![\Sigma \subseteq \Gamma]\!]_{\mathbb{B}} \wedge [\![\Gamma \subseteq \Sigma]\!]_{\mathbb{B}}, \\ [\![\Sigma \subseteq \Gamma]\!]_{\mathbb{B}} &= \inf_{\mathbb{B}}\{\neg p \vee [\![\tau \in \Gamma]\!]_{\mathbb{B}} \mid \langle \tau, p\rangle \in \Sigma\}, \\ [\![\forall x \varphi(x)]\!]_{\mathbb{B}} &= \inf_{\mathbb{B}}\{[\![\varphi(\tau)]\!]_{\mathbb{B}} \mid \tau \in M^{\mathbb{P}}\}, \end{aligned}$$

where $\Sigma, \Gamma \in \mathcal{C}^{\mathbb{P}}, \sigma \in M^{\mathbb{P}}$ and φ is a $\mathcal{L}_{\in}$-formula with class parameters. Note that we can not define Boolean values for formulae of the form $\forall X \varphi(X)$, since then we would have to take the supremum of a hyperclass of conditions.

Lemma 3.2.2. *Let $\mathbb{B}$ be an M-complete pre-Boolean algebra and let G be $\mathbb{B}^*$-generic over $\mathbb{M}$. Then we have for every $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{B}, M)$-formula φ,*

$$M[G] \models \varphi^G \iff [\![\varphi]\!]_{\mathbb{B}} \in G. \tag{3.1}$$

Similarly, if $\mathbb{B}$ is a $\mathcal{C}$-complete pre-Boolean algebra and G is $\mathbb{B}^*$-generic over $\mathbb{M}$ then (3.1) *holds for every $\mathcal{L}_{\in}$-formula with class parameters.*

Proof. We first prove (3.1) for atomic formulae. For formulae of the form $\check{p} \in \dot{G}$ this is obvious. For formulae of the form $\sigma \in \tau$ and $\sigma = \tau$ we proceed by induction on the name rank. Suppose now that $M[G] \models \sigma^G \in \tau^G$. Then there is $\langle \pi, p\rangle \in \tau$ such that $p \in G$ and $\sigma^G = \pi^G$. Inductively, we have that $[\![\sigma = \pi]\!]_{\mathbb{B}} \in G$ and hence so is $[\![\sigma = \pi]\!]_{\mathbb{B}} \wedge p$. But then $[\![\sigma \in \tau]\!]_{\mathbb{B}} \in G$. Conversely, assume that $[\![\sigma \in \tau]\!]_{\mathbb{B}} \in G$. Observe that $\{[\![\sigma = \pi]\!]_{\mathbb{B}} \wedge p \mid \langle \pi, p\rangle \in \tau\}$ is predense below $[\![\sigma \in \tau]\!]_{\mathbb{B}}$. By genericity, there is $\langle \pi, p\rangle \in \tau$ such that $[\![\sigma = \pi]\!]_{\mathbb{B}} \wedge p \in G$. Consequently, we have that $\sigma^G = \pi^G \in \tau^G$ as desired.

Next, assume that $\sigma^G \subseteq \tau^G$ and fix $\langle \pi, p\rangle \in \sigma$. If $\neg p \notin G$, then $p \in G$, since $\{p, \neg p\}$ is a maximal antichain. But then $\pi^G \in \sigma^G \subseteq \tau^G$ and by induction, $[\![\pi \in \tau]\!]_{\mathbb{B}} \in G$. Thus $\neg p \vee [\![\pi \in \tau]\!]_{\mathbb{B}} \in G$. Since this holds for every $\langle \pi, p\rangle \in \sigma$, $[\![\sigma \subseteq \tau]\!]_{\mathbb{B}} \in G$. For the converse, let $[\![\sigma \subseteq \tau]\!]_{\mathbb{B}} \in G$ and $\langle \pi, p\rangle \in \sigma$ with $p \in G$. By assumption, $\neg p \vee [\![\pi \in \tau]\!]_{\mathbb{B}} \in G$ and so $[\![\pi \in \tau]\!]_{\mathbb{B}} \in G$. Inductively, we obtain that $\pi^G \in \tau^G$. For formulae of the form $\sigma = \tau$, (3.1) follows directly from the case $\sigma \subseteq \tau$.

If $M[G] \models \neg\varphi^G$, then $M[G] \not\models \varphi^G$ and so $[\![\varphi]\!]_{\mathbb{B}} \notin G$. But this implies that $[\![\neg\varphi]\!]_{\mathbb{B}} = \neg[\![\varphi]\!]_{\mathbb{B}} \in G$. The converse is similar.

Suppose that $M[G] \models \varphi^G$, where φ is a disjunction of the form $\bigvee_{i\in I} \varphi_i$. Then there exists $i_0 \in I$ such that $M[G] \models \varphi_{i_0}^G$. By induction, $[\![\varphi_{i_0}]\!]_{\mathbb{B}} \in G$ and so $[\![\varphi]\!]_{\mathbb{B}} \in G$. The converse follows in the same way using that if $\{[\![\varphi_i]\!]_{\mathbb{B}} \mid i \in I\}$ is predense below $[\![\varphi]\!]_{\mathbb{B}}$. The conjunctive case is easy and shall thus be omitted.

The proofs for the formulae containing class parameters are similar. □

In Section 4.3 we are interested in the preservation of separation. We will show that separation may fail for class forcing. However, all pre-Boolean algebras preserve simple instances of separation, namely those given by rudimentary functions.

Definition 3.2.3. A model X is *rudimentarily closed* if it is closed under the following functions (the *rudimentary functions*):

$$\begin{aligned}
&f_0(x_1,\dots,x_k) = x_i,\\
&f_1(x_1,\dots,x_k) = x_i \setminus x_j,\\
&f_2(x_1,\dots,x_k) = \{x_i, x_j\},\\
&f_3(x_1,\dots,x_k) = h(g_1(x_1,\dots,x_k),\dots g_l(x_1,\dots,x_k)), \quad h, g_i \text{ rudimentary and}\\
&f_4(x_1,\dots,x_k) = \bigcup_{y\in x_1} g(y, x_2,\dots,x_k), \quad g \text{ rudimentary.}
\end{aligned}$$

Lemma 3.2.4. *If $\mathbb{P}$ is an $\mathbb{M}$-complete pre-Boolean algebra and G is $\mathbb{M}$-generic, then $M[G]$ is rudimentarily closed.*

Proof. Closure under projections and compositions of rudimentary functions is obvious. Assume that $\sigma, \tau \in M^{\mathbb{P}}$. Clearly, $\{\langle\rho, [\![\rho \notin \tau]\!]_{\mathbb{B}} \wedge p\rangle \mid \langle\rho, p\rangle \in \sigma\}$ is a name for $\sigma^G \setminus \tau^G$ and $\{\langle\sigma, \mathbb{1}_{\mathbb{P}}\rangle, \langle\tau, \mathbb{1}_{\mathbb{P}}\rangle\}$ is a name for the unordered pair $\{\sigma^G, \tau^G\}$. Next, suppose that $g(v_0, v_1)$ is a rudimentary function. We have to find a name for $\bigcup_{x\in\tau^G} g(x, \sigma^G)$. Since g is rudimentary, for every $\rho \in \operatorname{dom}(\tau)$ there is a $\mathbb{P}$-name $\pi_{\rho,\sigma}$ for $g(\rho^G, \sigma^G)$. Now put

$$\theta = \{\langle\eta, p \wedge q\rangle \mid \exists\rho\,(\langle\rho, p\rangle \in \tau \wedge \langle\eta, q\rangle \in \pi_{\rho,\sigma})\}.$$

Clearly, $\theta^G \subseteq \bigcup_{x\in\tau^G} g(x, \sigma^G)$. For the converse, consider $\langle\rho, p\rangle \in \tau$ with $p \in G$ and $y \in g(\rho^G, \sigma^G)$. Hence there must be $\langle\eta, q\rangle \in \pi_{\rho,\sigma}$ such that $q \in G$ and $y = \eta^G$. Then also $p \wedge q \in G$, so $\eta^G \in \theta^G$. □

As in set forcing, Boolean values are closely related to the forcing theorem.

Lemma 3.2.5. *If $\mathbb{B}$ is an M-complete pre-Boolean algebra, then we can define the forcing relation for $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{B}, M)$-formulae by*

$$p \Vdash^{\mathbb{M}}_{\mathbb{B}^*} \varphi \iff p \leq_{\mathbb{B}} [\![\varphi]\!]_{\mathbb{B}}.$$

Proof. Suppose first that $p \Vdash_{\mathbb{B}} \varphi$ and that $p \not\leq_{\mathbb{B}} [\![\varphi]\!]_{\mathbb{B}}$. By separativity of $\mathbb{B}$, there is $q \leq_{\mathbb{B}} p$ which is incompatible with $[\![\varphi]\!]_{\mathbb{B}}$. Let G be $\mathbb{B}$-generic over $\mathbb{M}$ with $q \in G$. Then $p \in G$ and hence $M[G] \models \varphi^G$. By Lemma 3.2.2 we obtain that $[\![\varphi]\!]_{\mathbb{B}} \in G$ contradicting that $q \perp_{\mathbb{B}} [\![\varphi]\!]_{\mathbb{B}}$. The converse follows directly from the previous lemma. □

Theorem 3.2.6. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^-. Suppose that $\mathbb{P}$ is a separative notion of class forcing for $\mathbb{M}$. Then the following statements are equivalent:*

(1) $\mathbb{P}$ *satisfies the definability lemma for one of the $\mathcal{L}_\in$-formulae "$v_0 \in v_1$" or "$v_0 = v_1$".*
(2) $\mathbb{P}$ *satisfies the forcing theorem for all $\mathcal{L}_\in$-formulae with class parameters.*
(3) $\mathbb{P}$ *satisfies the uniform forcing theorem for all $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formulae.*
(4) $\mathbb{P}$ *has a pre-Boolean M-completion.*

Proof. The equivalence of (1) – (3) follow from Theorem 2.1.5 and Lemma 2.1.9. Assume now that (3) holds. We will construct what could be seen as an analogue of the Lindenbaum algebra. Define a pre-Boolean algebra in the following way: Consider the class $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$ of all Gödel codes of infinitary formulae in the forcing language of $\mathbb{P}$ endowed with the canonical Boolean operations, i.e. suprema and infima are just set-sized disjunctions and conjunctions of formulae and complements are just negations, and with the equivalence relation given by

$$\ulcorner \varphi \urcorner \approx \ulcorner \psi \urcorner \iff \mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \leftrightarrow \psi.$$

It is easy to check that this defines an M-complete pre-Boolean algebra. Moreover, the embedding

$$i : \mathbb{P} \to \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M), p \mapsto \ulcorner \check{p} \in \dot{G} \urcorner$$

is injective and dense.

It remains to prove that (4) implies (1). Assume that $\mathbb{P}$ has a pre-Boolean M-completion $\mathbb{B}(\mathbb{P})$. Without loss of generality, we may assume that the domain of $\mathbb{P}$ is a subclass of the domain of $\mathbb{B}(\mathbb{P})$. Then we have for $\sigma, \tau \in M^{\mathbb{P}}$,

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau \iff p \Vdash^{\mathbb{M}}_{\mathbb{B}(\mathbb{P})^*} \sigma \in \tau.$$

Since by Lemma 3.2.2, $\mathbb{B}(\mathbb{P})^*$ satisfies the definability lemma for $\sigma \in \tau$, by Lemma 2.1.7 so does $\mathbb{P}$. □

Corollary 3.2.7. *Suppose that $\mathbb{M}$ satisfies representatives choice and let $\mathbb{P}$ be a separative and antisymmetric notion of class forcing for $\mathbb{M}$. Then $\mathbb{P}$ satisfies the forcing theorem if and only if it has a Boolean M-completion.*

Moreover, representatives choice is only necessary for the forward direction.

Proof. If $\mathbb{P}$ satisfies the forcing theorem we can proceed as in the proof of Theorem 3.2.6 and consider the pre-Boolean algebra $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$. Let $\approx$ be the equivalence relation given by $\ulcorner \varphi \urcorner \approx \ulcorner \psi \urcorner$ if and only if $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \leftrightarrow \psi$ as above. Since $\mathbb{M}$ satisfies representatives choice, there are $\mathbb{B} \in \mathcal{C}$ and $\pi \in \mathcal{C}$ such that $\pi : \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M) \to \mathbb{B}$ is surjective and such that $\pi(\ulcorner \varphi \urcorner) = \pi(\ulcorner \psi \urcorner)$ if and only if $\ulcorner \varphi \urcorner \approx \ulcorner \psi \urcorner$. Now we can obtain induced Boolean operations on $\mathbb{B}$ in the obvious way and define $0_{\mathbb{B}} = \pi(\ulcorner 0 \neq 0 \urcorner)$ and $\mathbb{1}_{\mathbb{B}} = \pi(\ulcorner 0 = 0 \urcorner)$. Clearly, $\mathbb{B}$ is an M-complete Boolean algebra. We identify $p \in \mathbb{P}$ with the formula $\pi(\ulcorner \check{p} \in \dot{G} \urcorner)$, thus obtaining a dense embedding $i : \mathbb{P} \to \mathbb{B}$ in $\mathcal{C}$. Note that the injectivity of i follows from the antisymmetry and separativity of $\mathbb{P}$. The converse follows from Theorem 3.2.6. □

Corollary 3.2.8. *Suppose that* $\mathbb{M}$ *is a model of* GB^- *which satisfies representatives choice. Then a notion of class forcing* $\mathbb{P}$ *satisfies the forcing theorem if and only if its separative quotient* $\mathbb{S}(\mathbb{P})$ *has a Boolean* M*-completion.*

Proof. This follows directly from Lemma 3.1.11 and Theorem 3.2.7. □

Corollary 3.2.9. *If* $\mathbb{M}$ *is a model of* GB^- *which has a hierarchy then every separative pretame notion of class forcing for* $\mathbb{M}$ *has a pre-Boolean* M*-completion. Moreover, if* $\mathbb{P}$ *is additionally antisymmetric then it has a Boolean* M*-completion.*

Proof. This is an immediate consequence of Theorems 2.2.2 and 3.2.6 and Corollary 3.2.7. □

In Section 4.4 we will show that in some models of GB^- pretameness can in fact be characterized by the existence of a Boolean M-completion.

3.3 Boolean completions in KM

In this section, we prove that in KM every separative notion of class forcing has a Boolean completion. Moreover, we may conclude that in KM, unlike GB^-, every notion of class forcing satisfies the forcing theorem. An alternative proof of the forcing theorem in KM which uses class recursion to prove that the syntactic forcing relation[2] coincides with the semantic one is presented in [Ant15].

Theorem 3.3.1. *If* $\mathbb{M} = \langle M, \mathcal{C}\rangle$ *is a countable transitive model of* KM*, then every separative notion of class forcing* $\mathbb{P}$ *for* $\mathbb{M}$ *has a pre-Boolean* M*-completion. In particular, if* $\mathbb{P}$ *is antisymmetric, it has a Boolean* M*-completion.*

Proof. Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of KM and let $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ be a separative notion of class forcing for $\mathbb{M}$. Making use of a suitable bijection in $\mathcal{C}$, we may assume that $P \cap \mathrm{Ord}^M = \emptyset$. Pick a disjoint partition $\langle A_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ of Ord^M such that $\{\langle \alpha, \beta\rangle \mid \beta \in A_\alpha\} \in \mathcal{C}$. Using class recursion, we define $\subseteq$-increasing sequences $\langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ and $\langle \mathbb{Q}_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ of separative notions of class forcing containing $\mathbb{P}$ such that $\{\langle p_0, p_1, \alpha\rangle \mid p_0 \leq_{\mathbb{P}_\alpha} p_1\} \in \mathcal{C}$, $\{\langle q_0, q_1, \alpha\rangle \mid q_0 \leq_{\mathbb{Q}_\alpha} q_1\} \in \mathcal{C}$ and $\mathbb{P}$ is dense in each $\mathbb{P}^*_\alpha$ and $\mathbb{Q}^*_\alpha$, where $\mathbb{P}^*_\alpha$ and $\mathbb{Q}^*_\alpha$ are the notions of class forcing obtained from $\mathbb{P}_\alpha$ or $\mathbb{Q}_\alpha$ respectively, by removing all conditions p which stengthen every other condition (i.e. which are equivalent to $0_{\mathbb{P}_\alpha}$ or $0_{\mathbb{Q}_\alpha}$ respectively).

Let $P_0 = P$. If α is a limit ordinal, let $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$ and $\leq_{\mathbb{P}_\alpha} = \bigcup_{\beta<\alpha} \leq_{\mathbb{P}_\beta}$ for every $\beta < \alpha$. Suppose that $\mathbb{P}_\alpha$ has been defined. We construct $\mathbb{Q}_\alpha$ by adding suprema for all subsets of P_α in M, and then construct $\mathbb{P}_{\alpha+1}$ by adding negations for all elements of Q_α.

More precisely, let $Q_\alpha = P_\alpha \cup \{\sup A \mid A \in M, A \subseteq P_\alpha\}$, where $\sup A \in A_{2\cdot\alpha}$ is different for each $A \in M$ with $A \subseteq P_\alpha$ and $\{\langle A, \sup A\rangle \mid A \in M, A \subseteq P_\alpha\} \in \mathcal{C}$. Thus

[2]The syntactic forcing relation as defined in [Ant15, Definition 14] for class forcing is an analogue of the one in set forcing (see [Kun80, Ch. VIII, Definition 3.3]).

$Q_\alpha \in \mathcal{C}$ and we define an ordering $\leq_{\mathbb{Q}_\alpha}$ on Q_α with $\leq_{\mathbb{Q}_\alpha} \cap (P_\alpha \times P_\alpha) = \leq_{\mathbb{P}_\alpha}$ and $\leq_{\mathbb{Q}_\alpha} \in \mathcal{C}$ in the following way:

$$\begin{aligned} \sup A \leq_{\mathbb{Q}_\alpha} p &\iff \forall a \in A (a \leq_{\mathbb{P}_\alpha} p), \\ p \leq_{\mathbb{Q}_\alpha} \sup A &\iff A \text{ is predense below } p \text{ in } \mathbb{P}_\alpha, \\ \sup A \leq_{\mathbb{Q}_\alpha} \sup B &\iff \forall a \in A (a \leq_{\mathbb{Q}_\alpha} \sup B) \end{aligned} \tag{3.2}$$

for all $p \in \mathbb{P}_\alpha$, and $A, B \in M$ with $A \subseteq P_\alpha$. Firstly, we check that $\mathbb{P}$ is dense in $\mathbb{Q}^*_\alpha$. If $A \subseteq P_\alpha$ with $A \in M$ such that $\sup A \in \mathbb{Q}^*_\alpha$ then $A \cap \mathbb{P}^*_\alpha \neq \emptyset$. Now if $a \in A$ is in $\mathbb{P}^*_\alpha$ then a strenghtens $\sup A$, and since by assumption $\mathbb{P}$ is dense in $\mathbb{P}^*_\alpha$, there is $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}_\alpha} a$. In particular, $p \leq_{\mathbb{Q}_\alpha} \sup A$. In order to prove that $\mathbb{Q}_\alpha$ is separative, since $\sup\{p\} = p$ for $p \in \mathbb{P}_\alpha$, it suffices to check that whenever $A, B \subseteq \mathbb{P}_\alpha$ are sets with $\sup A \not\leq_{\mathbb{Q}_\alpha} \sup B$, then there is $p \leq_{\mathbb{Q}_\alpha} \sup A$ incompatible with $\sup B$. So suppose that $\sup A \not\leq_{\mathbb{Q}_\alpha} \sup B$. Then there is $a \in A$ with $a \not\leq_{\mathbb{Q}_\alpha} \sup B$, i.e. there exists a strengthening $p \in \mathbb{P}$ of a such that each $q \leq_{\mathbb{P}} p$ is incompatible with every element of B. In particular, p is incompatible with $\sup B$.

Now let $\mathbb{P}_{\alpha+1} = \mathbb{Q}_\alpha \cup \{\neg q \mid q \in \mathbb{Q}_\alpha\}$, where $\neg q \in A_{2\cdot\alpha+1}$ is different for each $q \in \mathbb{Q}_\alpha$, such that $\{\langle q, \neg q\rangle \mid q \in \mathbb{Q}_\alpha\} \in \mathcal{C}$. Thus $\mathbb{P}_{\alpha+1} \in \mathcal{C}$ and we define an ordering $\leq_{\mathbb{P}_{\alpha+1}}$ on $\mathbb{P}_{\alpha+1}$ extending $\leq_{\mathbb{Q}_\alpha}$ such that $\leq_{\mathbb{P}_{\alpha+1}} \in \mathcal{C}$ as follows:

$$\begin{aligned} p \leq_{\mathbb{P}_{\alpha+1}} \neg q &\iff p \perp_{\mathbb{Q}_\alpha} q, \\ \neg p \leq_{\mathbb{P}_{\alpha+1}} q &\iff \forall r \in \mathbb{Q}_\alpha (r \parallel_{\mathbb{Q}_\alpha} p \vee r \parallel_{\mathbb{Q}_\alpha} q), \\ \neg p \leq_{\mathbb{P}_{\alpha+1}} \neg q &\iff q \leq_{\mathbb{Q}_\alpha} p \end{aligned} \tag{3.3}$$

for all $p, q \in \mathbb{Q}_\alpha$. Again, we need to verify that $\mathbb{P}$ is dense in $\mathbb{P}^*_{\alpha+1}$. Let $q \in \mathbb{Q}_\alpha$ be a condition such that $\neg q$ is non-zero, i.e. there is $p \in \mathbb{P}_{\alpha+1}$ such that $\neg q \not\leq_{\mathbb{P}_{\alpha+1}} p$. If $p \in \mathbb{Q}_\alpha$, then this means that there is $r \in \mathbb{Q}_\alpha$ which is incompatible with p and q in $\mathbb{Q}_\alpha$. In particular, $r \leq_{\mathbb{P}_{\alpha+1}} \neg q$ and since $\mathbb{P}$ is dense in $\mathbb{Q}_\alpha$, we can strengthen r to some condition in $\mathbb{P}$. Otherwise, p is of the form $\neg r$ for some $r \in \mathbb{Q}_\alpha$. Therefore, $r \not\leq_{\mathbb{Q}_\alpha} q$ and so by separativity of $\mathbb{Q}_\alpha$ there is $s \in \mathbb{Q}_\alpha$ with $s \leq_{\mathbb{Q}_\alpha} r$ and $s \perp_{\mathbb{Q}_\alpha} q$. But then $s \leq_{\mathbb{P}_{\alpha+1}} \neg q$ and once again we apply the density of $\mathbb{P}$ in $\mathbb{Q}_\alpha$ to obtain the result.

Secondly, we need to check that separativity is preserved. First suppose that $p, q \in \mathbb{Q}_\alpha$ such that $p \not\leq_{\mathbb{P}_{\alpha+1}} \neg q$. Then p and q are compatible, hence there is $r \in \mathbb{Q}_\alpha$ with $r \leq_{\mathbb{Q}_\alpha} p, q$. In particular, r and $\neg q$ are incompatible in $\mathbb{P}_{\alpha+1}$. If $\neg p \not\leq_{\mathbb{P}_{\alpha+1}} q$, then there is $r \in \mathbb{P}$ such that $r \perp_{\mathbb{Q}_\alpha} p$ and $r \perp_{\mathbb{Q}_\alpha} q$. But this means that $r \leq_{\mathbb{P}_{\alpha+1}} \neg p, \neg q$ and so r and q are incompatible in $\mathbb{P}_{\alpha+1}$. Finally, suppose that $\neg p \not\leq_{\mathbb{P}_{\alpha+1}} \neg q$. Then $q \not\leq_{\mathbb{Q}_\alpha} p$, so by separativity of $\mathbb{Q}_\alpha$ there is a strengthening $r \in \mathbb{Q}_\alpha$ of q with $r \perp_{\mathbb{Q}_\alpha} p$. This means that $r \leq_{\mathbb{P}_{\alpha+1}} \neg p$ and clearly r is incompatible with $\neg q$.

Then $\mathbb{B} = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathbb{P}_\alpha = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathbb{Q}_\alpha$ is in $\mathcal{C}$. We define an order on $\mathbb{B}$ by $p \leq_{\mathbb{B}} q$ if and only if there is $\alpha \in \mathrm{Ord}^M$ such that $p, q \in \mathbb{P}_\alpha$ and $p \leq_{\mathbb{P}_\alpha} q$. This defines a preorder. Moreover, we can define the Boolean operations on $\mathbb{B}$ as follows. If $\alpha \in \mathrm{Ord}^M$ is minimal such that $p \in \mathbb{Q}_\alpha$, then let $\neg p$ be as defined in $\mathbb{P}_{\alpha+1}$. If $A \subseteq \mathbb{B}$ and α is the minimal ordinal such that $A \subseteq \mathbb{P}_\alpha$, then we let $\sup A$ be as computed in $\mathbb{Q}_\alpha$. Finally, the equivalence relation on $\mathbb{B}$ is given by

$$p \approx_{\mathbb{B}} q \iff \exists \alpha \in \mathrm{Ord}^M (p, q \in \mathbb{P}_\alpha \wedge p \leq_{\mathbb{P}_\alpha} q \wedge q \leq_{\mathbb{P}_\alpha} p).$$

It is easy to check that this defines an M-complete pre-Boolean completion of $\mathbb{P}$.

If $\mathbb{P}$ is antisymmetric, we can form the quotient of $\mathbb{B}$ modulo $\approx_{\mathbb{B}}$ in order to obtain an M-complete Boolean algebra. For $p \in \bigcup_{\alpha\in\mathrm{Ord}^M} \mathbb{P}_\alpha$, we define

$$[p] = \{q \mid q \approx_{\mathbb{B}} p \wedge \forall r\,[r \approx_{\mathbb{B}} p \rightarrow \mathsf{rnk}(r) \geq \mathsf{rnk}(q)]\} \in M$$

and $\mathbb{B}' = \{[p] \mid \exists\alpha \in \mathrm{Ord}^M(p \in \mathbb{P}_\alpha)\}$. Now we can introduce the ordering, suprema, infima and complements in $\mathbb{B}'$ in the canonical way and let $\mathbb{1}_{\mathbb{B}'} = [\mathbb{1}_{\mathbb{P}}]$ as well as $0_{\mathbb{B}'} = \neg\mathbb{1}_{\mathbb{B}'}$. More precisely, if $A \subseteq \mathbb{B}'$ is a set, then let $\alpha \in \mathrm{Ord}^M$ be least such that every member of A has a representative in $(\mathsf{V}_\alpha)^M$ and $\bar{A} = \{p \in (\mathsf{V}_\alpha)^M \mid \exists\beta \in \mathrm{Ord}^M(p \in \mathbb{P}_\beta \wedge [p] \in A)\}$. Then we can define $\sup A = [\sup_{\mathbb{Q}_\beta} \bar{A}]$, where β is the least ordinal with $\bar{A} \subseteq \mathbb{P}_\beta$. It follows directly from (3.2) that $\sup A$ is well-defined and that it is the supremum of A in $\mathbb{B}'$. Complements are defined in the same way, i.e. for $[p] \in \mathbb{B}'$ with $p \in \mathbb{Q}_\alpha$ put $\neg[p] = [\neg p]$, where $\neg p$ is defined in $\mathbb{P}_{\alpha+1}$. Again, it is a straightforward consequence of (3.3) and the density of $\mathbb{P}$ in $\mathbb{Q}_\alpha^*$ that this is well-defined and that this actually defines the complement of $[p]$ in $\mathbb{B}'$. Furthermore, if $A \in M$ is a subset of $\mathbb{B}'$, then we let $\inf A = \neg(\sup\{\neg a \mid a \in A\})$. Moreover, the embedding $\pi : \mathbb{P} \rightarrow \mathbb{B}' \setminus \{0_{\mathbb{B}'}\}, p \mapsto [p]$ is dense by construction and it follows from $\mathbb{P}$ being separative and antisymmetric that π is injective. □

Note that Owen has erroneously used a similar proof in [Owe08] to show that every class forcing has a Boolean completion. However, Corollary 2.5.4 shows that the definability lemma for atomic formulae can consistently fail, and hence Boolean completions do not always exist.

Corollary 3.3.2. *If $\mathbb{M}$ is a model of* KM*, then every notion of class forcing for $\mathbb{M}$ satisfies the forcing theorem.*

Proof. By Lemma 3.1.11, it is enough to prove that every separative notion of class forcing for $\mathbb{M}$ satisfies the forcing theorem. But this is a direct consequence of Theorems 3.3.1 and 3.2.6. □

In combination with Theorem 2.5.3, the previous theorem can also be used to provide an alternative proof of the following well-known fact.

Corollary 3.3.3. *If $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a model of* KM*, then $\mathcal{C}$ contains a first-order truth predicate for M.*

Proof. By Corollary 3.3.2, $\mathbb{F}$ satisfies the the forcing theorem over $\mathbb{M}$, so by Theorem 2.5.3, $\mathcal{C}$ contains a first-order truth predicate for M. □

3.4 Unions of set-sized complete subforcings

The next theorem shows that approachability by projections is essentially a weakening of being an increasing union of set-sized complete subforcings.

Theorem 3.4.1. *If $\mathbb{P} = \bigcup_{\alpha\in\mathrm{Ord}^M} \mathbb{P}_\alpha$ is an increasing union of set-sized complete subforcings of $\mathbb{P}$ (as witnessed by a class in $\mathcal{C}$), then there is a dense embedding $i : \mathbb{P} \rightarrow \mathbb{B}^*$ in $\mathcal{C}$, where $\mathbb{B}$ is an M-complete Boolean algebra in $\mathcal{C}$ which is approachable by projections. Moreover, $\mathbb{P}$ and $\mathbb{B}^*$ have the same generic extensions, i.e. whenever G is $\mathbb{B}^*$-generic over $\mathbb{M}$, then $M[G] = M[i^{-1}[G]]$.*

Proof. Since every $\mathbb{P}_\alpha$ is a set-sized forcing notion, we can always form its separative quotient $\mathbb{S}(\mathbb{P}_\alpha)$. We can then define an equivalence relation $\approx$ on $\bigcup_{\alpha\in\mathrm{Ord}^M}\mathbb{S}(\mathbb{P}_\alpha)$ by stipulating $[p]_\alpha \approx [q]_\beta$ if and only if $\alpha \leq \beta$ and $[p]_\beta = [q]_\beta$ or $\alpha > \beta$ and $[p]_\alpha = [q]_\alpha$, where $[q]_\alpha$ denotes the equivalence class in $\mathbb{S}(\mathbb{P}_\alpha)$ corresponding to p. If we put

$$[p]_\approx = \{q \in \bigcup_{\alpha\in\mathrm{Ord}^M} \mathbb{S}(\mathbb{P}_\alpha) \mid q \approx p \wedge \exists\alpha \in \mathrm{Ord}^M[q \in \mathbb{S}(\mathbb{P}_\alpha) \wedge \forall\beta < \alpha\forall r \in \mathbb{S}(\mathbb{P}_\beta)(r \not\approx p)]\}$$

then $\mathbb{S} = \{[p]_\approx \mid p \in \bigcup_{\alpha\in\mathrm{Ord}^M}\mathbb{S}(\mathbb{P}_\alpha)\}$ is separative, antisymmetric and it can be written as an increasing union of complete subforcings $\mathbb{S}_\alpha = \{[p]_\approx \mid p \in \mathbb{S}_\alpha\}$. Furthermore, it is easy to check that $\mathbb{P}$ embeds densely into $\mathbb{S}$, and $\mathbb{P}$ and $\mathbb{S}$ have the same generic extensions.

By the observation above we can assume without loss of generality that $\mathbb{P}$ is separative and antisymmetric. Since $\mathbb{P}_\alpha$ is set-sized, it has a Boolean completion $\mathbb{B}(\mathbb{P}_\alpha)$ given by the set of all regular open subsets of $\mathbb{P}_\alpha$[3]. Let $e_\alpha : \mathbb{P}_\alpha \to \mathbb{B}(\mathbb{P}_\alpha)^*$ be the canonical dense embedding of $\mathbb{P}_\alpha$ into $\mathbb{B}(\mathbb{P}_\alpha)^*$ for each $\alpha \in \mathrm{Ord}^M$. For ordinals $\alpha \leq \beta$ we define an embedding from $\mathbb{B}(\mathbb{P}_\alpha)$ into $\mathbb{B}(\mathbb{P}_\beta)$ by

$$i_{\alpha\beta} : \mathbb{B}(\mathbb{P}_\alpha) \to \mathbb{B}(\mathbb{P}_\beta), b \mapsto \sup\{e_\beta(p) \mid p \in \mathbb{P}_\alpha \wedge e_\alpha(p) \leq_{\mathbb{B}(\mathbb{P}_\alpha)} b\}.$$

We have the following properties for all ordinals $\alpha \leq \beta \leq \gamma$:

(1) $i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta}$,
(2) $i_{\alpha\beta}$ is injective.

The first statement is a straightforward computation of suprema. For the second claim, assume that $b_0, b_1 \in \mathbb{B}(\mathbb{P}_\alpha)$ are such that $b_0 \not\leq_{\mathbb{B}(\mathbb{P}_\alpha)} b_1$. By separativity of $\mathbb{B}(\mathbb{P}_\alpha)$ and density of $\mathbb{P}_\alpha$, there must be some $p \in \mathbb{P}_\alpha$ such that $e_\alpha(p) \leq_{\mathbb{B}(\mathbb{P}_\alpha)} b_0$ and $e_\alpha(p)\perp_{\mathbb{B}(\mathbb{P}_\alpha)} b_1$. Now consider $q \in \mathbb{P}_\alpha$ with $e_\alpha(q) \leq_{\mathbb{B}(\mathbb{P}_\alpha)} b_1$. Then $e_\alpha(p)\perp_{\mathbb{B}(\mathbb{P}_\alpha)} e_\alpha(q)$, so since e_α and e_β are complete embeddings, we also get $p\perp_{\mathbb{P}_\alpha} q$ and hence $e_\beta(p)\perp_{\mathbb{B}(\mathbb{P}_\beta)} e_\beta(q)$. This shows that $i_{\alpha\beta}(b_0) \neq i_{\alpha\beta}(b_1)$.

Now we define an equivalence relation $\sim$ on $\bigcup_{\alpha\in\mathrm{Ord}}\mathbb{B}(\mathbb{P}_\alpha)$ by setting $b_0 \sim b_1$ iff $b_1 = i_{\alpha\beta}(b_0)$ if $b_0 \in \mathbb{B}(\mathbb{P}_\alpha)$ and $b_1 \in \mathbb{B}(\mathbb{P}_\beta)$ and $\alpha < \beta$ or $b_0 = i_{\alpha\beta}(b_1)$ if $b_0 \in \mathbb{B}(\mathbb{P}_\alpha)$ and $b_1 \in \mathbb{B}(\mathbb{P}_\beta)$ and $\alpha \geq \beta$. Note that the equivalence classes of $\sim$ are, in general, proper classes. We can avoid this problem by considering instead

$$[b] = \{c \in \bigcup_{\alpha\in\mathrm{Ord}^M} \mathbb{B}(\mathbb{P}_\alpha) \mid b \sim c \wedge \exists\alpha \in \mathrm{Ord}[c \in \mathbb{B}(\mathbb{P}_\alpha) \wedge \forall\beta < \alpha\forall d \in \mathbb{B}(\mathbb{P}_\beta)(d \not\sim b)]\}.$$

Now we can define $\mathbb{B}_\alpha = \{[b] \mid b \in \mathbb{B}(\mathbb{P}_\alpha)\}$ equipped with the order given by $[b_0] \leq_{\mathbb{B}_\alpha} [b_1]$ iff $b_0 \leq_{\mathbb{B}(\mathbb{P}_\alpha)} b_1$ for $b_0, b_1 \in \mathbb{B}(\mathbb{P}_\alpha)$. This is well-defined by construction. Finally, let $\mathbb{B} = \bigcup_{\alpha\in\mathrm{Ord}^M}\mathbb{B}_\alpha$, endowed with the ordering given by extending the orderings on the $\mathbb{B}_\alpha$'s. Moreover, we can define the Boolean operations on $\mathbb{B}$ in the canonical way. By construction, $\mathbb{B}$ is an $\mathbb{M}$-complete Boolean algebra. For each $\alpha \in \mathrm{Ord}^M$ and $[b] \in \mathbb{B}^*$ we define the projection

$$\pi_{\alpha+1}([b]) = \sup\{[e_{\alpha+1}(p)] \mid p \in \mathbb{P}_{\alpha+1} \wedge [e_{\alpha+1}(p)] \leq [b]\} = \sup\{[c] \in \mathbb{B}^*_{\alpha+1} \mid [c] \leq [b]\}.$$

[3](see [Jec03, Theorems 7.13, 14.10]

The equality follows from the density of $e_{\alpha+1}$. Straightforward calculations yield those projections to witness that $\mathbb{B}^*$ is approachable by projections. Moreover, $i : \mathbb{P} \to \mathbb{B}^*, i(p) = [e_\alpha(p)]$ for $p \in \mathbb{P}_\alpha$ is a dense embedding.

In order to see that $\mathbb{P}$ and $\mathbb{B}^*$ have the same generic extensions, note that if G is $\mathbb{B}^*$-generic over $\mathbb{M}$, then for every α, $G \cap \mathbb{B}^*_\alpha$ is $\mathbb{B}^*_\alpha$-generic, because $\mathbb{B}^*_\alpha$ is a complete subforcing of $\mathbb{B}^*$. So $M[G] = \bigcup_{\alpha \in \mathrm{Ord}^M} M[G \cap \mathbb{B}^*_\alpha]$. Similiarly, if H is $\mathbb{P}$-generic over $\mathbb{M}$, then $M[H] = \bigcup_{\alpha \in \mathrm{Ord}^M} M[H \cap \mathbb{P}_\alpha]$. Moreover, the set forcings $\mathbb{P}_\alpha, \mathbb{B}(\mathbb{P}_\alpha)^*$ and $\mathbb{B}^*_\alpha$ all have the same generic extensions, i.e. if G is $\mathbb{B}^*$-generic over $\mathbb{M}$, then $M[i^{-1}[G \cap \mathbb{B}^*_\alpha]] = M[G \cap \mathbb{B}^*_\alpha]$. This shows that

$$\begin{aligned} M[i^{-1}[G]] &= \bigcup_{\alpha \in \mathrm{Ord}^M} M[i^{-1}[G] \cap \mathbb{P}_\alpha] = \bigcup_{\alpha \in \mathrm{Ord}^M} M[i^{-1}[G \cap \mathbb{B}^*_\alpha]] \\ &= \bigcup_{\alpha \in \mathrm{Ord}^M} M[G \cap \mathbb{B}^*_\alpha] = M[G] \end{aligned}$$

as desired. □

Corollary 3.4.2. *If $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathbb{P}_\alpha$ is an increasing union of set-sized complete subforcings of $\mathbb{P}$, then $\mathbb{P}$ satisfies the forcing theorem.*

Proof. By passing to the separative quotient of each $\mathbb{P}_\alpha$ as in the proof of Theorem 3.4.1 above, by Lemma 3.1.11 we may assume that $\mathbb{P}$ is separative and antisymmetric. But then the embedding given in the proof of Theorem 3.4.1 can easily be seen to be injective, and so $\mathbb{P}$ has a Boolean M-completion. By Corollary 3.2.7, this shows that $\mathbb{P}$ satisfies the forcing theorem. □

Note that Zarach already proved in [Zar73] that unions of set-sized complete subforcings satisfy the forcing theorem for all $\mathcal{L}_\in$-formulae. A similar result for first-order formulae with class name parameters can be found in the dissertation of Jonas Reitz [Rei06].

Corollary 3.4.3. *If $\mathbb{P}$ is a product or an iteration of length Ord^M of support $< \mathrm{Ord}^M$ of set-sized forcing notions, then $\mathbb{P}$ satisfies the forcing theorem.*

Proof. Since products can always be viewed as iterations, it remains to check the claim for iterations. By Corollary 3.4.2 it suffices to prove that each iteration of length Ord^M is an increasing union of set-sized complete subforcings. Let $\mathbb{P} = \mathbb{P}_{\mathrm{Ord}^M}$ be the final iterand and let $\mathbb{P}_\alpha$ denote the iteration up to $\alpha \in \mathrm{Ord}^M$. Since the support of the iteration is $< \mathrm{Ord}^M$, $\mathbb{P}$ is isomorphic to the union of the $\mathbb{P}_\alpha$, $\alpha \in \mathrm{Ord}^M$. Moreover, it is easy to check that each $\mathbb{P}_\alpha$ is a complete subforcing of $\mathbb{P}$. □

3.5 Non-unique Boolean completions

In this section, we show by means of a concrete example that Boolean M-completions need not be unique. Note that in Section 4.4 we will in fact characterize the existence of a unique Boolean M-completion. Fix a ground model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ which satisfies representatives choice. Note that by Lemma 2.4.3 and Theorem 2.4.4, $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})^M$ satisfies the forcing theorem. Now let $\mathbb{Q}$ be the forcing notion obtained from $\mathbb{P}$ by adding the supremum $e = \sup\{\{\langle 0, \alpha\rangle\} \mid \alpha \text{ even}\}$[4]. It is easy to check that $\mathbb{Q}$ is also approachable by projections and hence it satisfies the forcing theorem. Consequently, by Corollary 3.2.7, both $\mathbb{P}$ and $\mathbb{Q}$ have Boolean M-completions $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$. Since $\mathbb{P}$ embeds densely into $\mathbb{Q}$, $\mathbb{B}(\mathbb{Q})$ is also a Boolean completion of $\mathbb{P}$.

Definition 3.5.1. For $\ulcorner\varphi\urcorner \in \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$, we recursively define the *support* of $\ulcorner\varphi\urcorner$ by

$$\begin{aligned}
\mathrm{supp}(\ulcorner\check{p} \in \dot{G}\urcorner) &= \mathrm{range}(p)\\
\mathrm{supp}(\ulcorner\sigma = \tau\urcorner) = \mathrm{supp}(\ulcorner\sigma \in \tau\urcorner) &= \sup\{\mathrm{range}(p) \mid p \in \mathbb{P} \cap \mathrm{tc}(\sigma \cup \tau)\}\\
\mathrm{supp}(\ulcorner\neg\varphi\urcorner) &= \mathrm{supp}(\ulcorner\varphi\urcorner)\\
\mathrm{supp}(\ulcorner\bigvee_{i\in I} \varphi_i\urcorner) &= \bigcup\{\mathrm{supp}(\ulcorner\varphi_i\urcorner) \mid i \in I\}.
\end{aligned}$$

Recall that the Boolean M-completion $\mathbb{B}(\mathbb{P})$ as constructed in Corollary 3.2.7 is obtained using representatives choice from $\mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$ and the equivalence relation $\ulcorner\varphi\urcorner \approx \ulcorner\psi\urcorner$ if and only if $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \leftrightarrow \psi$, i.e. there is a surjective map $\pi : \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M) \to \mathbb{B}(\mathbb{P})$ such that $\pi(\ulcorner\varphi\urcorner) = \pi(\ulcorner\psi\urcorner)$ if and only if $\ulcorner\varphi\urcorner \approx \ulcorner\psi\urcorner$. We will write $[\ulcorner\varphi\urcorner]$ for $\pi(\ulcorner\varphi\urcorner)$. Next we want to extend the notion of support to elements of $\mathbb{B}(\mathbb{P})$.

Definition 3.5.2. Let $\mathbb{B}(\mathbb{P})$ be the Boolean M-completion of $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})^M$ as described above. If $b \in \mathbb{B}(\mathbb{P})$, then we define the *support* of b by

$$\mathrm{supp}(b) = \min\{\alpha \in \mathrm{Ord}^M \mid \exists\ulcorner\varphi\urcorner \in \mathrm{Fml}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)([\ulcorner\varphi\urcorner] = b \wedge \mathrm{supp}(\ulcorner\varphi\urcorner) = \alpha)\}.$$

Lemma 3.5.3. *For $\alpha \in \mathrm{Ord}^M$ and $n \in \omega$, let $p^n_\alpha = \{\langle n, \alpha\rangle\}$. If $\alpha, \beta \in \mathrm{Ord}^M$ then there is an automorphism $\pi_{\alpha\beta} : \mathbb{B}(\mathbb{P}) \to \mathbb{B}(\mathbb{P})$ such that for every $n \in \omega$, $\pi_{\alpha\beta}(p^n_\alpha) = p^n_\beta$ and for every $b \in \mathbb{B}(\mathbb{P})$ such that $\mathrm{supp}(b) < \min\{\alpha, \beta\}$, $\pi_{\alpha\beta}(b) = b$.*

Proof. We first define an automorphism $\tau_{\alpha\beta}$ on $\mathbb{P}$ which switches α and β in the range of every condition. Then we can extend $\tau_{\alpha\beta}$ to infinitary formulae in the forcing language of $\mathbb{P}$ over M by stipulating

$$\begin{aligned}
\bar{\tau}_{\alpha\beta}(\ulcorner\check{p} \in \dot{G}\urcorner) &= \ulcorner\check{\tau_{\alpha\beta}(p)} \in \dot{G}\urcorner,\\
\bar{\tau}_{\alpha\beta}(\ulcorner\sigma = \tau\urcorner) &= \ulcorner\tau^*_{\alpha\beta}(\sigma) = \tau^*_{\alpha\beta}(\tau)\urcorner,\\
\bar{\tau}_{\alpha\beta}(\ulcorner\sigma \in \tau\urcorner) &= \ulcorner\tau^*_{\alpha\beta}(\sigma) \in \tau^*_{\alpha\beta}(\tau))\urcorner,\\
\bar{\tau}_{\alpha\beta}(\ulcorner\neg\varphi\urcorner) &= \langle 3, \bar{\tau}_{\alpha\beta}(\ulcorner\varphi\urcorner)\rangle,\\
\bar{\tau}_{\alpha\beta}(\ulcorner\bigvee_{i\in I} \varphi_i\urcorner) &= \langle 4, I, \{\langle i, \bar{\tau}_{\alpha\beta}(\ulcorner\varphi_i\urcorner) \mid i \in I\}\rangle,
\end{aligned}$$

where $\tau^*_{\alpha\beta}$ is the map on $M^{\mathbb{P}}$ induced by $\tau_{\alpha\beta}$. Finally, let $\pi_{\alpha\beta} : \mathbb{B}(\mathbb{P}) \to \mathbb{B}(\mathbb{P})$ be the automorphism defined by $\pi_{\alpha\beta}[\ulcorner\varphi\urcorner] = [\bar{\tau}_{\alpha\beta}(\ulcorner\varphi\urcorner)]$. This is clearly well-defined. Suppose that $b \in \mathbb{B}(\mathbb{P})$ with $\gamma = \mathrm{supp}(b) < \min\{\alpha, \beta\}$ and take $\ulcorner\varphi\urcorner$ of support γ such that $b = [\ulcorner\varphi\urcorner]$. Then clearly $\pi_{\alpha\beta}(b) = [\bar{\tau}_{\alpha\beta}(\ulcorner\varphi\urcorner)] = [\ulcorner\varphi\urcorner] = b$. □

[4]i.e. $p \leq_{\mathbb{Q}} e$ iff $0 \in \mathrm{dom}(p)$ and $p(0)$ even. A general method for adding suprema is described in Section 4.1.

Theorem 3.5.4. $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord}^M)$ *does not have a unique Boolean M-completion.*

Proof. Firstly, observe that we can embed $\mathbb{B}(\mathbb{P})$ in $\mathbb{B}(\mathbb{Q})$ by mapping $[\ulcorner\varphi\urcorner]_{\mathbb{P}}$ to $[\ulcorner\varphi\urcorner]_{\mathbb{Q}}$. This embedding fixes $\mathbb{P}$ and it is well-defined since for any two infinitary formulae φ, ψ in the forcing language of $\mathbb{P}$, $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi \leftrightarrow \psi$ iff $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi \leftrightarrow \psi$. This holds since such formulae are quantifier-free and therefore absolute between $\mathbb{P}$- and $\mathbb{Q}$-generic extensions.

We claim that e does not lie in $\mathbb{B}(\mathbb{P})$. Suppose for a contradiction that $e \in \mathbb{B}(\mathbb{P})$ and let $\alpha, \beta > \mathrm{supp}(e)$ such that α is even and β is odd. Clearly, $p^0_\alpha \leq_{\mathbb{B}(\mathbb{P})} e$ since this holds in $\mathbb{Q}$. But then $p^0_\beta = \pi_{\alpha\beta}(p^0_\alpha) \leq_{\mathbb{B}(\mathbb{P})} \pi_{\alpha\beta}(e) = e$ since $\alpha, \beta > \mathrm{supp}(e)$. But this is impossible because β is odd.

We are left with checking that there is no isomorphism between $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$ which fixes $\mathbb{P}$. Inductively, one can easily show that Boolean values $[\![\varphi]\!]_{\mathbb{B}(\mathbb{P})}$ coincide with $[\ulcorner\varphi\urcorner]$, i.e. for every $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{P}, M)$-formula φ, $[\![\varphi]\!]_{\mathbb{B}(\mathbb{P})} = [\ulcorner\varphi\urcorner]$. Again by induction on formula complexity, one can check that each isomorphism which fixes $\mathbb{P}$ already fixes all Boolean values, in particular it fixes $\mathbb{B}(\mathbb{P})$. But $e \in \mathbb{B}(\mathbb{Q}) \setminus \mathbb{B}(\mathbb{P})$, so there is no isomorphism between $\mathbb{B}(\mathbb{P})$ and $\mathbb{B}(\mathbb{Q})$ which fixes $\mathbb{P}$. $\square$

Chapter 4

Characterizations of pretameness and the Ord-cc

In this chapter, we will present various characterizations of pretameness and the Ord-chain condition in terms of several properties which hold for all set-sized forcing notions but may fail for class forcing. Examples are the forcing theorem, the forcing equivalence of forcing notions and their dense subforcings and the existence of nice names for sets of ordinals.

4.1 Preliminaries

Pretameness has already been introduced in Section 2.2, where we prove that over models $\mathbb{M} \models \mathsf{GB}^-$ with a hierarchy, every pretame notion of class forcing satisfies the forcing theorem. In this chapter we will argue that pretameness can in some sense be considered the minimal requirement to avoid pathologies in class forcing. The other – stronger – property that we will characterize is the following.

Definition 4.1.1. We say that a notion of class forcing $\mathbb{P}$ satisfies the Ord-*chain condition* (or simply Ord-*cc*) over $\mathbb{M}$, if every antichain of $\mathbb{P}$ which is in $\mathcal{C}$ is already an element of M.

Note that in the presence of a global well-order the Ord-cc obviously implies pretameness. Hence by Theorem 2.2.2, if $\mathcal{C}$ contains a set-like well-order we obtain that every notion of class forcing with the Ord-cc satisfies the forcing theorem. However, by carefully modifying the proof of Theorem 2.2.2, we can observe that the assumption of a set-like well-order can actually be reduced to the requirement of an arbitrary – not necessarily set-like – global well-order.

Lemma 4.1.2. *If $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a global well-order. Then every notion of class forcing with the* Ord-*cc satisfies the forcing theorem.*

Proof. The proof is exactly the same as the proof of Theorem 2.2.2, but whenever we make reference to the hierarchy, we use instead the global well-order in $\mathcal{C}$. □

However, it is an open question whether in the absence of a global well-order the Ord-cc implies the forcing theorem.

Note that there are several canonical ways to define the Ord-chain condition. Another possibility would be to simply stipulate that all *maximal* antichains are set-sized. However, the following remark shows that this is not equivalent to our definition given in 4.1.1.

Remark 4.1.3. Let $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ such that $\mathbb{M}$ does not satisfy global choice. Then there is a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ such that every maximal antichain of $\mathbb{P}$ is in M but it does not satisfy the Ord-cc over $\mathbb{M}$.

Proof. Let $\mathbb{P}$ be the forcing notion whose conditions are of the form $\mathbb{1}_{\mathbb{P}}$ – the maximal element of $\mathbb{P}$ – or pairs $\langle x, y\rangle$ with $y \in x$, ordered by $\langle x, y\rangle \leq_{\mathbb{P}} \langle x, z\rangle$ for all $x, y, z \in M$ with $y, z \in x$. Clearly, $\mathbb{P}$ has a class-sized antichain given by $\{\langle\{x\}, x\rangle \mid x \in M\}$. However, the only maximal antichain is $\{\mathbb{1}_{\mathbb{P}}\}$, since if A is a non-trivial maximal antichain, then for every $x \in M \setminus \{\emptyset\}$ there is some y such that $\langle x, y\rangle \in A$. But this would define a global choice function on M. □

The next lemma shows that the Ord-cc of some notion of class forcing in fact depends on which classes one considers. In particular, adding a global well-order by forcing can destroy the Ord-chain condition.

Lemma 4.1.4. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- which has a hierarchy and satisfies choice but does not satisfy global choice. Then there is a forcing notion $\mathbb{P}$ for $\mathbb{M}$ which satisfies the* Ord-*cc over $\mathbb{M}$ but whenever $\mathcal{D} \supseteq \mathcal{C}$ is such that $\mathbb{N} = \langle M, \mathcal{D}\rangle \models \mathsf{GB}^-+$ global choice, then $\mathbb{P}$ does not satisfy the* Ord-*cc over* $\mathbb{N}$.

Proof. Let $\langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ witness that $\mathbb{M}$ has a hierarchy. Let

$$X_\alpha = \{f : C_\alpha \to M \mid \forall x \in C_\alpha\, (f(x) \in x)\}$$

denote the set of all choice functions on C_α. Now let $\mathbb{P}$ be the forcing notion whose conditions are in $\bigcup_{\alpha\in\mathrm{Ord}^M} X_\alpha \cup \{\mathbb{1}_{\mathbb{P}}\}$, with the maximal element given by $\mathbb{1}_{\mathbb{P}}$ and the ordering given by $f \leq_{\mathbb{P}} g$ if and only $\mathrm{dom}(f) = \mathrm{dom}(g)$. In particular, if f is a choice function on C_α and g is a choice function on C_β for $\alpha \neq \beta$ then f and g are incompatible. Otherwise, $f \leq_{\mathbb{P}} g$ and $g \leq_{\mathbb{P}} f$. Clearly, $\mathbb{P}$ is not antisymmetric.

We claim that $\mathbb{P}$ satisfies the Ord-cc over $\mathbb{M}$. Suppose that $A \in \mathcal{C}$ is a class-sized antichain. Then for every ordinal $\alpha \in \mathrm{Ord}^M$ there is $\beta \geq \alpha$ such that $A \cap X_\beta \neq \emptyset$. Then we can define a global choice function $F : M \setminus \{\emptyset\} \to M$ as follows. For $x \in M \setminus \{\emptyset\}$ let $\alpha_x \in \mathrm{Ord}^M$ be the least ordinal such that $x \in C_\alpha$ and $A \cap X_\alpha \neq \emptyset$. Then let $F(x) = f(x)$, where $\{f\} = A \cap X_{\alpha_x}$.

On the other hand, suppose that $\mathbb{N} = \langle M, \mathcal{D}\rangle \models \mathsf{GB}^-$ is a model of global choice with $\mathcal{D} \supseteq \mathcal{C}$ and let $F \in \mathcal{D}$ be a global choice function. Then there is a class-sized antichain of $\mathbb{P}$ in $\mathcal{D}$ given by $\{F(X_\alpha) \mid \alpha \in \mathrm{Ord}^M \wedge C_\alpha \neq \emptyset\}$. □

Note that there are models of GB^- which satisfy the conditions of Lemma 4.1.4 by Remark 5.4.4. We can use the forcing notion $\mathbb{P}$ from the previous lemma to prove the following.

Remark 4.1.5. Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$ has a hierarchy but does not satisfy global choice. Then there is a dense embedding $\pi : \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ of notions of class forcing such that $\mathbb{P}$ satisfies the Ord-cc over $\mathbb{M}$ but $\mathbb{Q}$ does not. We can take $\mathbb{P}$ as above and $\mathbb{Q}$ to be the antisymmetric quotient of $\mathbb{P}$, i.e. the forcing notion where all elements of X_α are collapsed to a single element p_α.

In the presence of a global well-order there can be no such counterexample.

Remark 4.1.6. If $\mathcal{C}$ contains a global well-order, and $\pi : \mathbb{P} \to \mathbb{Q}$ is a dense embedding in $\mathcal{C}$ of notions of class forcing such that $\mathbb{P}$ satisfies the Ord-cc over $\mathbb{M}$, then so does $\mathbb{Q}$.

Proof. Suppose that $A \in \mathcal{C}$ is an antichain in $\mathbb{Q}$. Using the global well-order, we choose an antichain $\bar{A}$ for $\mathbb{P}$ which is maximal in $\{p \in \mathbb{P} \mid \exists a \in A\,(\pi(p) \leq_{\mathbb{Q}} a)\}$. In particular, $\bar{A} \in M$. But then we can define a surjection from $\bar{A}$ onto A given by mapping $p \in \bar{A}$ to the unique $a \in A$ such that $\pi(p) \leq_{\mathbb{Q}} a$. Using replacement in M, we obtain that $A \in M$. □

As noted above, if there is a global well-order of M in $\mathcal{C}$, pretameness follows from the Ord-chain condition. The following example shows that the assumption of the global well-order cannot be dropped.

Lemma 4.1.7. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^- which has a hierarchy and satisfies choice but does not satisfy global choice. Then there is a non-pretame forcing notion $\mathbb{P}$ for $\mathbb{M}$ which satisfies the* Ord-*cc.*

Proof. Consider the forcing notion $\mathbb{P}$ defined in Lemma 4.1.4. We modify $\mathbb{P}$ to a notion of class forcing $\mathbb{Q}$ which additionally collapses Ord^M. Conditions in $\mathbb{Q}$ are sequences of the form $p = \langle p_i \mid i < \mathrm{lh}(p) \rangle$, where $\mathrm{lh}(p) \in \omega$ and $p_i \in \bigcup_{\alpha \in \mathrm{Ord}^M} X_\alpha$, where X_α is as in the proof of Lemma 4.1.4. We define $p \leq_{\mathbb{Q}} q$ if and only if $\mathrm{lh}(p) \geq \mathrm{lh}(q)$ and for all $i < \mathrm{lh}(q)$, $p_i \parallel_{\mathbb{P}} q_i$.

Claim 1. $\mathbb{Q}$ *satisfies the* Ord-*cc over* $\mathbb{M}$.

Proof. Suppose that $A \in \mathcal{C}$ is an antichain. We prove that for every condition $p \in \mathbb{Q}$, $A_p = \{q \leq_{\mathbb{Q}} p \mid \exists a \in A\,(a \leq_{\mathbb{Q}} q)\}$ is in M. If this holds, we obtain that $A \in M$, since it is a subclass of $A_{\mathbb{1}_{\mathbb{Q}}} \in M$. By induction on $n \in \omega$, we prove that for every $p \in \mathbb{Q}$,

$$A_p^n = \{q \in A_p \mid \mathrm{lh}(q) - \mathrm{lh}(p) \leq n\}$$

is an element of M. For $n = 0$ this is obvious. Suppose now that for some $n \in \omega$, $A_p^n \in M$ for all $p \in \mathbb{Q}$. Let $p \in \mathbb{Q}$ be a fixed condition. We show that A_p^{n+1} is an elementof M. Note that for every $q \in A_p^{n+1}$, $q \upharpoonright n = \langle q_i \mid i < \mathrm{lh}(q) - 1 \rangle$ is in A_p^n. So there are only set-many possibilities for $q \upharpoonright n$. Hence it suffices to check that for every $r \in A_p^n$, the set of all extensions of r which lie in A_p^{n+1} is in M. But this is obvious, since each such element is of the form $r ^\frown f$, where $f \in \bigcup_{\alpha \in \mathrm{Ord}^M} X_\alpha$ and – as in the previous lemma – class-many options for f would lead to a global choice function in $\mathcal{C}$. □

Claim 2. $\mathbb{Q}$ *is not pretame over* $\mathbb{M}$.

Proof. For each $n \in \omega$, consider the dense class

$$D_n = \{p \in \mathbb{Q} \mid \mathrm{lh}(p) > n\}.$$

Suppose that there are $p \in \mathbb{Q}$ and sets $d_n \subseteq D_n$ which are predense below p. Let $n = \mathrm{lh}(p)$. Now let $\alpha = \sup\{\beta \in \mathrm{Ord}^M \mid \exists q \in d_n\,(q \in X_\beta)\}$. Now we extend p to $\bar{p} = \langle \bar{p}_i \mid i < n+1\rangle$, where $\bar{p}_i = p_i$ for $i < n$ and $\bar{p}_n \in X_{\alpha+1}$. Then $\bar{p}$ is incompatible with every element of d_n, a contradiction. □

This concludes the proof of Lemma 4.1.7. □

Note that we can modify $\mathbb{Q}$ in the proof of Lemma 4.1.7 above to obtain a forcing notion with the same properties as above but which is antisymmetric. This can be achieved by changing the order on $\mathbb{Q}$ as follows.

$$p \leq_{\mathbb{Q}} q \iff p = q \vee [\mathrm{lh}(p) > \mathrm{lh}(q) \wedge \forall i < \mathrm{lh}(q)(\mathrm{dom}(p_i) = \mathrm{dom}(q_i))].$$

However, this forcing notion is not separative.

In set forcing, a standard technique to verify that some forcing notion satifies the κ-cc for some cardinal κ is to apply the Δ-system lemma. In class forcing, the Ord-version of the Δ-system lemma is a useful tool to prove the Ord-cc.

Lemma 4.1.8 (Δ-system lemma, class version)**.** *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^-. Let $\langle x_i \mid i < \mathrm{Ord}^M\rangle$ be a sequence of sets such that $\mathrm{card}(x_i) < \kappa$ for some M-cardinal κ. Then there is a proper class $Z \subseteq \mathrm{Ord}^M$ in $\mathcal{C}$ and a set r such that for all $i, j \in Z$ with $i \neq j$, $x_i \cap x_j = r$.*

Proof. Let $X \subseteq \mathrm{Ord}^M$ be a proper class in $\mathcal{C}$ and $\lambda < \kappa$ such that for each $i \in X$, $\mathrm{card}(x_i) = \lambda$. Let $\mu \leq \lambda$ be maximal such that there is a set r of cardinality μ and a proper class $Y \subseteq X$ such that for each $i \in Y$, $r \subseteq x_i$. Since $\emptyset$ satisfies this property, such μ exists. Note that for each $a \in \bigcup_{i \in Y} x_i$ with $a \notin r$ there is $i(a) \in \mathrm{Ord}^M$ such that for all $i \in Y$ with $i > i(a)$, $a \notin x_i$. This holds because otherwise $r \cup \{a\}$ would be a counterexample to the maximality of r.

Recursively, we define a strictly increasing sequence $\langle i_\xi \mid \xi \in \mathrm{Ord}^M\rangle$ of ordinals in Y. Suppose that the sequence $\langle i_\xi \mid \xi < \eta\rangle$ has already been defined. Now let i_η be the minimal $i \in Y$ such that for all $\xi < \eta$ and for all $a \in x_{i_\xi} \setminus r$, $i > i(a)$. Such an ordinal i exists, since Y is a proper class.

Now put $Z = \{i_\xi \mid \xi \in \mathrm{Ord}^M\}$. We claim that for all ordinals $\xi < \eta$, $x_{i_\xi} \cap x_{i_\eta} = r$. Suppose that $\xi < \eta$. By construction of Y we have that $r \subseteq x_{i_\xi} \cap x_{i_\eta}$. Conversely, let $a \in x_{i_\xi} \cap x_{i_\eta}$ and assume that $a \notin r$. But then $i_\eta > i(a)$ and so $a \notin x_{i_\eta}$, contradicting our assumption. □

Example 4.1.9. The class-sized Lévy collapse $\mathbb{P} = \mathrm{Col}(\omega, < \mathrm{Ord})^M$ satifies the Ord-cc.

Proof. Suppose that $A \subseteq \mathbb{P}$ is a proper class in $\mathcal{C}$. Since $\mathbb{P}$ essentially consists of finite sets of pairs of ordinals, it can be well-ordered and so we can write $A = \{p_i \mid i \in \mathrm{Ord}^M\}$. We apply the Δ-system Lemma 4.1.8 to $\{\mathrm{dom}(p_i) \mid i \in \mathrm{Ord}^M\}$ for $\kappa = \omega$ in order to obtain a proper class $Z \subseteq \mathrm{Ord}^M$ in $\mathcal{C}$ and a finite set r such that for all $i \neq j \in Z$, $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = r$. But since r is finite, there must be $i \neq j$ in Z such that $p_i \upharpoonright r = p_j \upharpoonright r$ and so p_i and p_j are compatible. This proves that A is not an antichain. □

In the following paragraphs, we describe a general method of how to extend a notion of class forcing $\mathbb{P}$ by adding suprema. This technique is a useful tool to characterize both pretameness and the Ord-cc. More precisely, it will frequently be used to show that pretameness (resp. the Ord-cc) of $\mathbb{P}$ is equivalent to $\mathbb{P}$ densely having some property.

Let $S = \langle X_i \mid i \in I\rangle \in \mathcal{C}$ with $I \in M$ be a sequence of subclasses of $\mathbb{P}$. Making use of a suitable bijection in $\mathcal{C}$, we may assume that $P \cap I = \emptyset$. Now let $\mathbb{P}_S$ be the forcing notion with domain $P_S = P \cup I$ ordered by

$$\begin{aligned} &i \leq_{\mathbb{P}_S} p \iff \forall q \in X_i\,(q \leq_{\mathbb{P}} p),\\ &p \leq_{\mathbb{P}_S} i \iff X_i \text{ is predense below } p \text{ in } \mathbb{P},\\ &i \leq_{\mathbb{P}_S} j \iff \forall q \in X_i\,(q \leq_{\mathbb{P}_S} j). \end{aligned}$$

For $i \in I$, we will usually write $\sup X_i$ rather than i. In case that $\sup X_i$ already exists in $\mathbb{P}$, or that $\sup X_i \leq_{\mathbb{P}_S} \sup X_j$ and $\sup X_j \leq_{\mathbb{P}_S} \sup X_i$ for some $i \neq j$, instead of $\mathbb{P}_S$ we need to consider the quotient of $\mathbb{P}_S$ by the equivalence relation $p \sim q$ iff $p \leq_{\mathbb{P}_S} q$ and $q \leq_{\mathbb{P}_S} p$ for $p, q \in P \cup I$, in order to obtain a separative partial order. Since $I \in M$ and $\mathbb{P}$ is separative, all equivalence classes are set-sized, and so this can easily be done and we will identify $\mathbb{P}_S$ with this quotient in this case. We call $\mathbb{P}_S$ the *forcing notion obtained from* $\mathbb{P}$ *by adding* $\sup X_i$ *for all* $i \in I$. Note that by construction, $\mathbb{P}$ is dense in $\mathbb{P}_S$.

Lemma 4.1.10. *Suppose that $\mathbb{P}$ is a notion of class forcing for some countable transitive model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^- which satisfies the forcing theorem. If $S \in \mathcal{C}$ is a finite sequence of subclasses of $\mathbb{P}$, then $\mathbb{P}_S$ satisfies the forcing theorem.*

Proof. By induction, we may assume that S consists of a single class $X \in \mathcal{C}$. Let $\mathbb{Q}$ denote $\mathbb{P}_S$. For $\sigma \in M^{\mathbb{Q}}$, let σ^+ denote the $\mathbb{P}$-name obtained from σ by replacing every occurrence of $\sup A$ in $\operatorname{tc}(\sigma)$ by $\mathbb{1}_{\mathbb{P}}$, and let $\sigma^- \in M^{\mathbb{P}}$ be the name obtained from σ by removing every pair of the form $\langle \mu, \sup A\rangle$ from $\operatorname{tc}(\sigma)$. To be precise about the latter, we inductively define $\sigma^- = \{\langle \tau^-, p\rangle \in \sigma \mid p \neq \sup A\}$. One easily checks that for all $p \in \mathbb{Q}$ and all $\mathbb{Q}$-names σ and τ, we have that $p \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma \in \tau$ if and only if

$$\forall q \in \mathbb{P}\ \left[q \leq_{\mathbb{Q}} p \rightarrow [(q \perp_{\mathbb{P}} A \rightarrow q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma^- \in \tau^-) \wedge (q \leq_{\mathbb{P}} A \rightarrow q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma^+ \in \tau^+)]\right].$$

It is clear that the righthand-side is a consequence of $p \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma \in \tau$. For the converse, suppose that the righthand-side holds and let G be $\mathbb{Q}$-generic with $p \in G$. Observe that $D = \{q \in \mathbb{P} \mid q \perp_{\mathbb{P}} A \vee q \leq_{\mathbb{P}} A\}$ is dense below p, hence there is $q \leq_{\mathbb{P}} p$ with $q \in D \cap G$. If $q \perp_{\mathbb{P}} A$, by assumption $(\sigma^-)^G = (\tau^-)^G$. Moreover, $\sup A \notin G$ and so $\sigma^G = (\sigma^-)^G$ as well as $\tau^G = (\tau^-)^G$. The case that $q \leq_{\mathbb{P}} A$ is similar, since then $\sup A \in G$ and so $\sigma^G = (\sigma^+)^G$ and $\tau^G = (\tau^+)^G$. □

4.2 The forcing theorem

Theorem 2.2.2 shows that over models with a hierarchy, every pretame notion of class forcing satisfies the forcing theorem. By combining this with a generalization of the ideas given in Section 2.5, we will characterize pretameness in terms of the forcing theorem over models without a first-order truth predicate. Furthermore, a strengthening of the forcing theorem is given by the maximality principle which states that if some conditon p forces en existential formula, there is always a witness σ such that p forces σ to satisfy that formula. We will show that the Ord-cc is essentially equivalent to the maximality principle.

4.2.1 Pretameness in terms of the forcing theorem

Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^-. To motivate our result, consider the forcing $\mathbb{P} = \mathrm{Col}_{\mathrm{inj}}(\omega, M)$ whose domain consists of injective functions $p : d_p \to M$ such that d_p is a finite subset of ω. Let now $\mathbb{F} = \mathbb{F}^M$ denote the forcing introduced in Section 1.3.3. Then there is a dense embedding from $\mathbb{P}$ into $\mathbb{F}$ given by $p \mapsto \langle d_p, e_p, p\rangle$, where e_p is the binary relation on d_p defined by

$$\langle i, j\rangle \in e_p \Longleftrightarrow p(i) \in p(j).$$

If $\mathbb{M}$ satisfies for example global choice then it is easy to see that $\mathbb{P}$ satisfies the forcing theorem, since it is then isomorphic to $\mathrm{Col}_{\mathrm{inj}}(\omega, \mathrm{Ord}^M)$, the forcing notion consisting of injective finite partial functions from ω into the ordinals. On the other hand if $\mathcal{C}$ does not contain a first-order truth predicate, then by Theorem 2.5.3, $\mathbb{F}$ does not satisfy the forcing theorem. This already gives a counterexample to the converse of Lemma 2.1.7. We will now generalize this idea and prove that over certain models of GB^-, we can embed every non-pretame notion of class forcing densely into one which does not satisfy the forcing theorem.

The following easy observation will be a key ingredient for our proof.

Lemma 4.2.1. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a set-like well-order of M and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. Then $\mathbb{P}$ is pretame if and only if there exist no M-cardinal κ, $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}^M$ is cofinal".*

Proof. Suppose first that $\mathbb{P}$ is pretame. If $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}^M$ is cofinal", consider

$$D_\alpha = \{q \leq_{\mathbb{P}} p \mid \exists \gamma \in \mathrm{Ord}^M\, (q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma})\} \in \mathcal{C}$$

for $\alpha < \kappa$. By pretameness there are $q \leq_{\mathbb{P}} p$ and sets $d_\alpha \subseteq D_\alpha$ in M which are predense below q. Now let

$$\beta = \sup\{\gamma + 1 \mid \gamma \in \mathrm{Ord}^M \wedge \exists \alpha < \kappa\, \exists r \in d_\alpha\, (r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma})\}.$$

Then $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \mathrm{ran}(\dot{F}) \subseteq \check{\beta}$, a contradiction.

Conversely, suppose that $\langle D_i \mid i \in I\rangle \in \mathcal{C}$ with $I \in M$ is a sequence of dense subclasses of $\mathbb{P}$ and $p \in \mathbb{P}$ is such that there exist no $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I\rangle \in M$ with each $d_i \subseteq D_i$ predense below q. Using the axiom of choice, we may assume that $I = \kappa$ is a cardinal in M. Let $\langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ be a hierarchy on M. Now let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$. In $\mathbb{M}[G]$, let $F : \kappa \to \mathrm{Ord}^M$ be the function defined by

$$F(\alpha) = \min\{\gamma \in \mathrm{Ord}^M \mid C_\gamma \cap D_\alpha \cap G \neq \emptyset\}.$$

Using the forcing theorem and Lemma 2.1.2, we may choose a name $\dot{F} \in \mathcal{C}$ for F and a condition $q \leq_{\mathbb{P}} p$ in G such that the above property of $\dot{F}$ is forced by q. But then q forces that $\dot{F}$ is cofinal in the ordinals – otherwise we could strengthen q to some r which forces the range of $\dot{F}$ to be contained in some ordinal γ and so $d_\alpha = D_\alpha \cap C_\gamma$ would be predense below r for every $\alpha \in I$, contradicting our assumption. □

Note that the forward direction does not require the existence of a set-like well-order in case that $\mathbb{P}$ is known to satisfy the forcing theorem. The next step is to strengthen Lemma 4.2.1.

Lemma 4.2.2. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- and that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. Assume that $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ are such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{F} \colon \check{\kappa} \to \mathrm{Ord}^M$ is cofinal" for some M-cardinal κ. Then there is a class name $\dot{E} \in \mathcal{C}$ and $q \leq_{\mathbb{P}} p$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{E} \colon \check{\kappa} \to \mathrm{Ord}^M$ is surjective."*

Proof. Since $\mathbb{P}$ satisfies the forcing theorem,

$$A = \{\langle r, \alpha, \beta\rangle \mid \exists s \leq_{\mathbb{P}} r\, (s \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta})\} \in \mathcal{C}.$$

Hence there is a sequence $C = \langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle \in \mathcal{C}$ such that each C_α is of the form

$$A_{r,\alpha} = \{\beta \in \mathrm{Ord}^M \mid \exists s \leq_{\mathbb{P}} r\, (s \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta})\}$$

for some $r \in \mathbb{P}$ and $\alpha \in \mathrm{Ord}^M$ such that $A_{r,\alpha}$ is a proper class, and moreover each such class $A_{r,\alpha}$ appears unboundedly often in C.

Claim 3. *There is a class $D = \langle D_\beta \mid \beta \in \mathrm{Ord}^M\rangle$ such that the classes D_β form a partition of Ord^M and $C_\alpha \cap D_\beta$ is a proper class for all $\alpha, \beta \in \mathrm{Ord}^M$.*

Proof. Let $k : \mathrm{Ord}^M \times \mathrm{Ord}^M \to \mathrm{Ord}^M$ be a bijection in $\mathcal{C}$ such that whenever $\bar{\gamma} < \gamma$, $k(\beta, \bar{\gamma}) < k(\beta, \gamma)$. Now we recursively define sets of ordinals $D^\gamma_\beta \in M$ in the following way: We start with $D^0_0 = \emptyset$. Let $\alpha, \beta, \gamma \in \mathrm{Ord}^M$ be such that $\alpha = k(\beta, \gamma)$ and assume that for all $\bar{\beta}, \bar{\gamma}$ with $k(\bar{\beta}, \bar{\gamma}) < \alpha$, $D^{\bar{\gamma}}_{\bar{\beta}}$ has already been defined. Now let $D^\gamma_\beta = \bigcup_{\bar{\gamma}<\gamma} D^{\bar{\gamma}}_\beta \cup \{\delta\}$, where δ is the least ordinal in $C_\gamma \setminus \bigcup\{D^{\bar{\gamma}}_{\bar{\beta}} \mid k(\bar{\beta}, \bar{\gamma}) < \alpha\}$. Finally, put $D_\beta = \bigcup_{\gamma\in\mathrm{Ord}^M} D^\gamma_\beta$ for each $\beta \in \mathrm{Ord}^M$. By construction, if $\beta \neq \bar{\beta}$ then D_β and $D_{\bar{\beta}}$ are disjoint. Moreover, since C_α appears unboundedly often in the enumeration defined above, $C_\alpha \cap D_\beta$ is a proper class for all $\alpha, \beta \in \mathrm{Ord}^M$. □

Suppose that D is a class as in the statement of the previous claim. If G is $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$, let $E \colon \kappa \to \mathrm{Ord}^M$ be the function given by $E(\alpha) = \beta$ if $\dot{F}^G(\alpha) \in D_\beta$. Since $\mathbb{P}$ satisfies the forcing theorem, by Lemma 2.1.2, there is a class name $\dot{E} \in \mathcal{C}$ and a condition $q \in G$ below p which forces that $\dot{E}$ satisfies this definition.

Claim 4. *$q \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{E} \colon \check{\kappa} \to \mathrm{Ord}^M$ is surjective".*

Proof. Suppose the contrary. Since $\mathbb{P}$ satisfies the forcing theorem, there is a condition $r \leq_{\mathbb{P}} q$ and some ordinal β such that $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{E}(\check{\alpha}) \neq \check{\beta}$ for all $\alpha < \kappa$. On the other hand, there is $\alpha < \kappa$ such that $A_{r,\alpha} = \{\beta \in \mathrm{Ord}^M \mid \exists s \leq_{\mathbb{P}} r\, (s \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta})\}$ is a proper class, since otherwise r forces that the range of $\dot{F}$ is bounded in Ord^M, contradicting our assumption on $\dot{F}$. By the previous claim, $A_{r,\alpha} \cap D_\beta$ is nonempty. Choose $\gamma \in A_{r,\alpha} \cap D_\beta$ and $s \leq_{\mathbb{P}} r$ so that $s \Vdash_{\mathbb{P}}^{\mathbb{M}} \dot{F}(\check{\alpha}) = \check{\gamma}$. Then $s \Vdash_{\mathbb{P}}^{\mathbb{M}} \dot{E}(\check{\alpha}) = \check{\beta}$, contradicting our choice of r and of β. □

This completes the proof of Lemma 4.2.2. □

Using the previous lemma, we are now ready to prove the main result of this section.

Theorem 4.2.3. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a set-like well-order of M, but no first-order truth predicate for M.*[1] *If $\mathbb{P}$ is a non-pretame notion of class forcing for $\mathbb{M}$, then there is a notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$ and a dense embedding $\pi\colon \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ such that $\mathbb{Q}$ does not satisfy the forcing theorem.*

Proof. Without loss of generality, we may assume that $\mathbb{P}$ satisfies the forcing theorem. Using Lemmata 4.2.1 and 4.2.2, we can choose $p \in \mathbb{P}$ and a class name $\dot{F}$ such that

$$p \Vdash_{\mathbb{P}}^{\mathbb{M}} \text{“}\dot{F} : \check{\kappa} \to \check{M} \text{ is surjective”}$$

for some M-cardinal κ.

We extend $\mathbb{P}$ to a forcing notion $\mathbb{Q}$ by adding suprema $p_{\alpha,\beta}$ for the classes

$$D_{\alpha,\beta} = \{q \leq_{\mathbb{P}} p \mid q \Vdash_{\mathbb{P}}^{\mathbb{M}} \dot{F}(\check{\alpha}) \in \dot{F}(\check{\beta})\}$$

for all $\alpha, \beta < \kappa$ such that $D_{\alpha,\beta}$ is nonempty. Let $X = \{\langle\alpha,\beta\rangle \in \kappa^2 \mid D_{\alpha,\beta} \neq \emptyset\}$. The following arguments generalize the proof of Theorem 2.5.3. Define

$$\dot{E} = \{\langle \mathsf{op}(\check{\alpha}, \check{\beta}), p_{\alpha,\beta}\rangle \mid \langle\alpha,\beta\rangle \in X\} \in M^{\mathbb{Q}}.$$

Assume for a contradiction that $\mathbb{Q}$ satisfies the forcing theorem. We will use $\dot{E}$ to show that $\mathcal{C}$ contains a first-order truth predicate for M, contradicting our assumptions. The following claim is an analogue of Lemma 1.3.8.

Claim 1. *Let G be $\mathbb{Q}$-generic over $\mathbb{M}$ with $p \in G$ and let $E = \dot{E}^G$ and $F = \dot{F}^G$. Then in $M[G]$ it holds that $\langle\alpha,\beta\rangle \in E$ if and only if $M \models F(\alpha) \in F(\beta)$.*

Proof. Let $\alpha, \beta < \kappa$ such that $\langle\alpha,\beta\rangle \in E$. Then $\langle\alpha,\beta\rangle \in X$ and $p_{\alpha,\beta} \in G$. But by definition of $\leq_{\mathbb{Q}}$, $p_{\alpha,\beta} \Vdash_{\mathbb{Q}} \dot{F}(\check{\alpha}) \in \dot{F}(\check{\beta})$ and therefore $F(\alpha) \in F(\beta)$ as desired. Conversely, suppose that $x \in y$ in M. Since F is surjective, there are $\alpha, \beta < \kappa$ such that $F(\alpha) = x$ and $F(\beta) = y$. Moreover, there must be $q \in G$ which forces that $\dot{F}(\check{\alpha}) \in \dot{F}(\check{\beta})$ and so $q \leq_{\mathbb{Q}} p_{\alpha,\beta}$. In particular, $p_{\alpha\beta} \in G$ and so $\langle\alpha,\beta\rangle \in E$. □

The next step will be to translate $\mathcal{L}_\in$-formulae into infinitary quantifier-free formulae into $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{Q}, M)$-formulae. Inductively, we assign to every $\mathcal{L}_\in$-formula φ with free

[1] Note that by Tarski's theorem on the undefinability of truth, every model of the form $\langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ with a set-like well-order, where $\mathcal{C}$ only consists of the definable subsets of M, satisfies these requirements.

variables in $\{v_0, \ldots, v_{k-1}\}$ and all $\vec{\alpha} = \alpha_0, \ldots, \alpha_{k-1} \in \kappa^k$ an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{Q}, M)$-formula in the following way:

$$\begin{aligned}
(v_i = v_j)^*_{\vec{\alpha}} &= (\check{\alpha}_i = \check{\alpha}_j)\\
(v_i \in v_j)^*_{\vec{\alpha}} &= (\mathsf{op}(\check{\alpha}_i, \check{\alpha}_j) \in \dot{E})\\
(\neg\varphi)^*_{\vec{\alpha}} &= (\neg\varphi^*_{\vec{\alpha}})\\
(\varphi \vee \psi)^*_{\vec{\alpha}} &= (\varphi^*_{\vec{\alpha}} \vee \psi^*_{\vec{\alpha}})\\
(\exists v_k \varphi)^*_{\vec{\alpha}} &= (\bigvee_{\beta<\kappa} \varphi^*_{\vec{\alpha},\beta}).
\end{aligned}$$

Note that by Corollary 2.1.10, if $\mathbb{Q}$ satisfies the definability lemma for "$v_0 \in v_1$" or "$v_0 = v_1$", then it satisfies the forcing theorem for all infinitary formulae in the forcing language of $\mathbb{Q}$. The following claim will allow us to define a first-order truth predicate over M. The next claim is an adaptation of Lemma 2.5.2.

Claim 2. *For every $\mathcal{L}_\in$-formula φ with free variables among $\{v_0, \ldots, v_{k-1}\}$ and for all $\vec{x} = x_0, \ldots, x_{k-1}$ in M, the following statements are equivalent:*

(1) $M \models \varphi(\vec{x})$.
(2) $\forall\vec{\alpha} \in \kappa^k\, \forall q \leq_{\mathbb{P}} p\, [q \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\forall i < k(\dot{F}(\check{\alpha}_i) = \check{x}_i)$" $\rightarrow q \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}]$.
(3) $\exists\vec{\alpha} \in \kappa^k\, \exists q \leq_{\mathbb{P}} p\, [q \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\forall i < k(\dot{F}(\check{\alpha}_i) = \check{x}_i)$" $\wedge\, q \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}]$.

Proof. Observe that since $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to M$ is surjective", (2) always implies (3). Hence it remains to prove that (1) implies (2) and that (3) implies (1). We proceed by induction on the construction of the formula φ.

We start with the atomic formula "$v_i \in v_j$". Without loss of generality, we may assume that $i = 0$ and $j = 1$. Suppose first that $x \in y$ in M. Let $\alpha, \beta < \kappa$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \wedge \dot{F}(\check{\beta}) = \check{y}$. Take a $\mathbb{Q}$-generic filter with $q \in G$. Since $q \leq_{\mathbb{Q}} p_{\alpha,\beta}$, $p_{\alpha,\beta} \in G$. Moreover, $\langle\alpha, \beta\rangle \in \dot{E}^G$, so (2) holds. Assume now that (3) holds, i.e. there are $\alpha, \beta < \kappa$ and $q \leq_{\mathbb{P}} p$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \wedge \dot{F}(\check{\beta}) = \check{y}$ and $q \Vdash^{\mathbb{M}}_{\mathbb{Q}} (v_0 \in v_1)^*_{\alpha,\beta}$. Let G be $\mathbb{Q}$-generic with $q \in G$. Then by assmption $\langle\alpha, \beta\rangle \in \dot{E}^G$ and so $p_{\alpha,\beta} \in G$. In particular, this means that $x = \dot{F}^G(\alpha) \in \dot{F}^G(\beta) = y$. The proof for "$v_i = v_j$" is similar.

Next we turn to negations. Suppose first that $M \models \neg\varphi(\vec{x})$ and let $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall i < k\, (\dot{F}(\check{\alpha}_i) = \check{x}_i)$. Assume, towards a contradiction, that $q \not\Vdash_{\mathbb{Q}} \neg\varphi^*_{\vec{\alpha}}$. Then there is $r \leq_{\mathbb{Q}} q$ with $r \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$. By density, we may assume that $r \in \mathbb{P}$. Then $r \leq_{\mathbb{P}} p$ and so $\vec{\alpha}$ and r witness (3) for φ. By our inductive hypothesis we obtain that $M \models \varphi(\vec{x})$, a contradiction. The implication from (3) to (1) is similar.

Suppose now that $M \models (\varphi\vee\psi)(\vec{x})$. Without loss of generality, assume that $M \models \varphi(\vec{x})$. Now if $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall i < k\, (\dot{F}(\check{\alpha}_i) = \check{x}_i)$, by induction $q \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$. But then in particular $q \Vdash^{\mathbb{M}}_{\mathbb{Q}} (\varphi\vee\psi)^*_{\vec{\alpha}}$. In order to see that (3) implies (1), suppose that $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ witness (3). Then there must be a strengthenig $r \in \mathbb{Q}$ of q which satisfies, without loss of generality, $r \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$. By density of $\mathbb{P}$ in $\mathbb{Q}$, we can assume that $r \in \mathbb{P}$. This means that $\vec{\alpha}$ and r witness that (3) holds for φ, so $M \models \varphi(\vec{x})$.

We are left with the existential case. Assume first that $M \models \exists v_k\varphi(\vec{x})$. Take $y \in M$ such that $M \models \varphi(\vec{x}, y)$ and let $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall i < k\, (\dot{F}(\check{\alpha}_i) = \check{x}_i)$. Let now G be $\mathbb{Q}$-generic with $q \in G$. By an easy density argument there must be $r \leq_{\mathbb{P}} p$

and $\beta < \kappa$ with $r \in G$ and $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\beta}) = \check{y}$. By induction, $r \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\vec{\alpha},\beta}$. In particular, $M[G] \models (\exists v_k \varphi)^*_{\vec{\alpha}}$. The converse follows in a similar way. □

As a consequence of Claim 2, the class

$$T = \{\langle \ulcorner\varphi\urcorner, x\rangle \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}_1 \wedge x \in M \wedge \forall\alpha < \kappa\, \forall q \leq_{\mathbb{P}} p\, (q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \rightarrow q \Vdash^{\mathbb{M}}_{\mathbb{Q}} \varphi^*_{\alpha})\}$$

in $\mathcal{C}$ defines a first-order truth predicate for M, contradicting our assumptions on $\mathbb{M}$. □

Corollary 4.2.4. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of* GB^- *such that $\mathcal{C}$ contains a set-like well-order of M but no first-order truth predicate for M. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it densely satisfies the forcing theorem.* □

Furthermore, the proof of Theorem 4.2.3 yields the following.

Corollary 4.2.5. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of* GB^- *such that $\mathcal{C}$ contains a set-like well-order of M but no first-order truth predicate for M. Then for any M-cardinal κ, there is a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem, such that there is a κ-sequence S in $\mathcal{C}$ of subclasses of $\mathbb{P}$, for which $\mathbb{P}_S$ does not satisfy the forcing theorem.* □

4.2.2 The maximality principle

The definition of the forcing relation in the existential case uses that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \exists x \varphi(x)$ if and only if the class of all $q \leq_{\mathbb{P}} p$ such that there is a $\mathbb{P}$-name σ with $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma)$ is dense below p. The maximality principle states that (in set forcing) it is in fact not necessary to strengthen p in order to obtain a witness for an existential formula. We observe that for notions of class forcing which satisfy the forcing theorem, this principle is equivalent to the Ord-cc over models of GBC.

Definition 4.2.6. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem is said to satisfy the *maximality principle over* $\mathbb{M}$ if whenever $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \exists x \varphi(x, \vec{\sigma}, \vec{\Gamma})$ for some $p \in \mathbb{P}$, some $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_m, \vec{\Gamma})$ with class name parameters $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ and $\vec{\sigma}$ in $(M^{\mathbb{P}})^m$, then there is $\tau \in M^{\mathbb{P}}$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\tau, \vec{\sigma}, \vec{\Gamma})$.

Lemma 4.2.7. *Assume that $\mathbb{M}$ is a model of* GBC *and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. Then $\mathbb{P}$ satisfies the maximality principle if and only if it satisfies the* Ord*-cc over $\mathbb{M}$.*

Proof. Suppose first that $\mathbb{P}$ satisfies the maximality principle and let $A \in \mathcal{C}$ be an antichain in $\mathbb{P}$. Since $\mathcal{C}$ contains a well-ordering of M, we can extend A to a maximal antichain $A' \in \mathcal{C}$. It is enough to show that $A' \in M$. Clearly, $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \exists x\, (x \in \check{A}' \cap \dot{G})$. Using the maximality principle, we obtain $\sigma \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \check{A}' \cap \dot{G}$. But then since $\mathsf{rnk}(\sigma^G) \leq \mathsf{rnk}(\sigma)$ for every $\mathbb{P}$-generic filter G, $A' \subseteq \mathbb{P} \cap (\mathsf{V}_\alpha)^M$ for $\alpha = \mathsf{rnk}(\sigma)$ and so $A' \in M$.

Conversely, assume that $\mathbb{P}$ satisfies the Ord-cc over $\mathbb{M}$ and let $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \exists x \varphi(x, \vec{\sigma}, \vec{\Gamma})$. Using the global well-order in $\mathcal{C}$ we can find an antichain $A \in \mathcal{C}$ which is maximal in

$\{q \leq_{\mathbb{P}} p \mid \exists\sigma \in M^{\mathbb{P}}[q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma,\vec{\sigma},\vec{\Gamma})]\} \in \mathcal{C}$. Note that A is predense below p and that $A \in M$ by assumption. For every $q \in A$, choose a name $\tau_q \in M^{\mathbb{P}}$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\tau_q,\vec{\sigma},\vec{\Gamma})$. Furthermore, for every $\mu \in \operatorname{dom}(\tau_q)$, let A^q_μ be a maximal antichain in $\{r \leq_{\mathbb{P}} q \mid \exists s\,(\langle\mu,s\rangle \in \tau_q \wedge r \leq_{\mathbb{P}} s)\}$. Now put

$$\sigma = \{\langle\mu,r\rangle \mid \exists q \in A\,(\mu \in \operatorname{dom}(\tau_q) \wedge r \in A^q_\mu)\}.$$

By construction, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau_q$ for every $q \in A$ and so $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma)$. □

4.3 Preservation of axioms

As we have seen in Lemma 1.2.9 and Lemma 2.1.2, the only axiom of GB^- which can fail to be preserved by notions of class forcing which satisfy the forcing theorem, is the scheme of collection. In the present section, we investigate failures of separation and, moreover, we will show that the preservation of separation, replacement and collection under class forcing are essentially equivalent and characterize pretameness.

4.3.1 Failures of separation

Let $\mathbb{M} = \langle M,\mathcal{C}\rangle$ be a countable transitive model of GB^-. Recall that $\mathbb{P} = \operatorname{Col}(\omega,\operatorname{Ord})^M$ adds a predicate $F = \bigcup G$ which is a cofinal function from ω to Ord^M. Hence replacement fails in $\mathbb{M}[G]$. It is also easy to see that separation fails in each such extension. We show that there is no name for the set $\{n \in \omega \mid F(n) \text{ is even}\}$. Let $\dot{F}$ be a class name for the generic function F. Assuming that separation holds and using the forcing theorem, there are $p \in \mathbb{P}$ and a $\mathbb{P}$-name σ so that

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \{n \in \check{\omega} \mid \dot{F}(n) \text{ is even}\}.$$

Let $\alpha = \operatorname{rank}(\sigma)$. Now, using an easy density argument, we may extend p to some condition q so that $q(n) = \beta > \alpha$ for some $n \in \omega$. Let π be the automorphism of $\mathbb{P}$ that for any condition r swaps the values β and $\beta+1$ of $r(n)$. Then $\pi^*(\sigma) = \sigma$. Consider $q' = \pi(q)$ and pick a $\mathbb{P}$-generic filter G with $q \in G$, let $G' = \pi''G$ and note that $q' \in G'$ and that $\sigma^G = \sigma^{G'}$. But this equation clearly contradicts that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(n)$ is even if and only if $q' \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(n)$ is odd.

In the counterexample to separation illustrated above, we make reference to the generic filter in the formula for which separation fails. Sometimes, however, we are not interested in the classes that are added by class forcing but only in the sets. This is often the case, for example, when we force over models $\langle M,\operatorname{Def}(M)\rangle$, where M is a model of (some subtheory of) ZFC.

Definition 4.3.1. Let $\mathbb{M}$ be a countable transitive model of GB^-. If $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ and G is $\mathbb{P}$-generic over $\mathbb{M}$, then we call the structure $\langle M[G],\in\rangle$ a $\mathbb{P}$-*generic set extension of* $\mathbb{M}$. We also identify $\langle M[G],\in\rangle$ with $\langle M[G],\operatorname{Def}(M[G])\rangle$.

In the remainder of this section, we consider generic set extensions. We want to show that a failure of separation in such an extension is possible as well. The notion of forcing that will witness this is an adaption of $\mathrm{Col}(\omega, \mathrm{Ord})^M$, which does not only add a predicate that is a cofinal function from ω to Ord^M, but also codes this predicate into the values of the continuum function of $M[G]$.

Theorem 4.3.2. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of* $\mathsf{GBC} + \mathsf{GCH}$. *There is a cofinality-preserving notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ that satisfies the forcing theorem such that separation fails in any $\mathbb{P}$-generic set extension $M[G]$.*

Proof. Let $\mathbb{P}$ be the forcing notion with conditions of the form $p = \langle p(i) \mid i < n(p)\rangle$ for some $n(p) \in \omega$, with each $p(i)$ of the form $p(i) = \langle \alpha_i(p), C_i(p)\rangle$ where $\alpha_i(p)$ is a regular uncountable cardinal and $c_i(p)$ is a condition in $\mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$, the forcing that adds $\alpha_i(p)^{++}$ Cohen subsets of $\alpha_i(p)$, and $\langle \alpha_i(p) \mid i < n(p)\rangle$ is strictly increasing. Given $p \in \mathbb{P}$, we let $\alpha(p) = \langle \alpha_i(p) \mid i < n(p)\rangle$. The ordering on $\mathbb{P}$ is given by stipulating that q is stronger than p iff $\alpha(q)$ end-extends $\alpha(p)$ and for every $i < n(p)$, $c_i(q)$ extends $c_i(p)$ in the usual ordering of $\mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$.

Claim 3. *$\mathbb{P}$ satisfies the forcing theorem over $\mathbb{M}$.*

Proof. We will show that $\mathbb{P}$ is approachable by projections. By Theorem 2.4.4, this implies that $\mathbb{P}$ satisfies the forcing theorem. For this purpose, for each ordinal α let

$$\mathbb{P}_\alpha = \begin{cases} \{p \in \mathbb{P} \mid \alpha_{n(p)-1}(p) < \alpha\}, & \alpha \in \mathrm{Lim}, \\ \{p \in \mathbb{P} \mid \alpha_{n(p)-1}(p) < \alpha \wedge \alpha_{n(p)-1}(p) = \alpha - 1 \rightarrow c_{n(p)-1}(p) = \emptyset\}, & \text{otherwise.} \end{cases}$$

We define projections $\pi_{\alpha+1} : \mathbb{P} \rightarrow \mathbb{P}_{\alpha+1}$ as follows. If $\alpha_{n(p)-1}(p) < \alpha$ then $\pi_{\alpha+1}(p) = p$. Otherwise, let $\pi_{\alpha+1}(p) = \langle p(i) \mid i < k\rangle^\frown\langle \alpha, \emptyset\rangle$, where k is maximal such that $\alpha_{k-1}(p) < \alpha$. It is easy to check that the maps $\pi_{\alpha+1}$ witness that $\mathbb{P}$ is approachable by projections. □

Given a $\mathbb{P}$-generic filter G over $\mathbb{M}$ and a condition $p \in G$, we denote by G_p the $\prod_{i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$-generic filter induced by G.

Claim 4. *For every $p \in \mathbb{P}$ and every $\sigma \in M^{\mathbb{P}}$, there is a condition $q \leq_{\mathbb{P}} p$ and a $\prod_{i<n(q)} \mathrm{Add}(\alpha_i(q), \alpha_i(p)^{++})$-name $\bar{\sigma}$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \bar{\sigma}$.*

Proof. Suppose that $\sigma \in (\mathsf{V}_\gamma)^M$ and $p \in \mathbb{P}$. Now choose $q \leq_{\mathbb{P}} p$ such that $\alpha_{n(q)-1}(q) \geq \gamma$. By recursion on the name rank, we define for $\tau \in M^{\mathbb{P}}$,

$$\tau^q = \{\langle \pi^q, \bar{r}\rangle \mid \langle \pi, r\rangle \in \tau \wedge \alpha(r) \subseteq \alpha(q)\},$$

where for $r \in \mathbb{P}$, $\bar{r} = \langle c_i(r) \mid i < n(q)\rangle \in \prod_{i<n(q)} \mathrm{Add}(\alpha_i(q), \alpha_i(q)^{++})$ with $c_i(r) = \emptyset$ for $n(r) \leq i < n(q)$. Then q and $\bar{\sigma} = \sigma^q$ are as desired, since whenever $\langle \tau, r\rangle \in \sigma$ such that $\alpha(r) \nsubseteq \alpha(q)$ then q and r are incompatible by construction of q. □

Claim 5. *Assume that G is $\mathbb{P}$-generic over $\mathbb{M}$ and $\alpha \in \mathrm{Ord}^M$. Then there is $r \in G$ such that whenever σ is a $\mathbb{P}$-name with $\sigma^G \subseteq \alpha$, then there is a $\prod_{i<n(r)} \mathrm{Add}(\alpha_i(r), \alpha_i(r)^{++})$-name $\tilde{\sigma}$ so that $\tilde{\sigma}^{G_p} = \sigma^G$.*

Proof. We use a reduction argument similar as the one presented in [Jec03, Lemma 15.19]. Let G and $\alpha \in \mathrm{Ord}^M$ be given and choose $r \in G$ such that $\alpha_{n(p)-1}(r) \geq \alpha$. Assume $\sigma \in M^{\mathbb{P}}$ and extend r to p in G to obtain a $\prod_{i<n(q)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$-name $\bar{\sigma}$ such that $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \bar{\sigma}$, using the previous claim. Let $\mathbb{Q} = \prod_{i<n(q)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$ and observe that $\mathbb{Q} \cong \mathbb{Q}_0 \times \mathbb{Q}_1$, where $\mathbb{Q}_0 = \prod_{i<n(r)} \mathrm{Add}(\alpha_i(r), \alpha_i(r)^{++})$ and $\mathbb{Q}_1 = \prod_{n(r)\leq i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$. Moreover, $\mathbb{Q}_0$ satisfies the κ^+-cc and $\mathbb{Q}_1$ is $<\kappa^+$-closed for $\kappa = \alpha_{n(r)-1}(r)$. Let $p \cong \langle p_0, p_1 \rangle$, where $p_0 \in \mathbb{Q}_0$ and $p_1 \in \mathbb{Q}_1$. For each $\beta < \alpha$, we consider the class

$$D_\beta = \{q \leq_{\mathbb{Q}_1} p_1 \mid \exists A \subseteq \mathbb{Q}_0 \, [A \text{ maximal antichain } \wedge \forall a \in A \, (\langle a, q \rangle \text{ decides } \check{\beta} \in \bar{\sigma})]\}.$$

We show that each D_β is open dense below p_1 in $\mathbb{Q}_1$. It is obvious that D_β is open. In order to check density, pick some $q \leq_{\mathbb{Q}_1} p_1$. Inductively, we construct a decreasing sequence $\langle q_i \mid i < \gamma \rangle$ of conditions in $\mathbb{Q}_1$ below q and a sequence $\langle a_i \mid i < \gamma \rangle$ in $\mathbb{Q}_0$ which enumerates an antichain so that each pair $\langle a_i, q_i \rangle$ decides $\check{\beta} \in \bar{\sigma}$, for some $\gamma < \kappa^+$. Suppose that q_i, a_i are given for all $i < \xi$. If $\{a_i \mid i < \xi\}$ is not a maximal antichain, then we can extend both sequences, using that $\mathbb{Q}_1$ is $<\kappa^+$-closed. Since $\mathbb{Q}_0$ satisfies the κ^+-cc, there must be some $\gamma < \kappa^+$ such that $A = \{a_i \mid i < \gamma\}$ is maximal. Invoking the closure of $\mathbb{Q}_1$ once again, we can find $q_\gamma \in \mathbb{Q}_1$ which extends each $q_i, i < \gamma$. Then $q_\gamma \leq_{\mathbb{Q}_1} q$ and $q_\gamma \in D_\beta$ as desired.

Since $\mathbb{Q}_1$ is $<\kappa^+$-closed, $D = \bigcap_{\beta<\alpha} D_\beta$ is also open dense below p_1. Pick $q \in D \cap H$, where H is the $\mathbb{Q}_1$-generic filter induced by G, and for each $\beta < \alpha$, pick a maximal antichain $A_\beta \subseteq \mathbb{Q}_0$ witnessing that $q \in D_\beta$. It follows that

$$\tilde{\sigma} = \{\langle \check{\beta}, a \rangle \mid a \in A_\beta \wedge \langle a, q \rangle \Vdash_{\mathbb{Q}_0 \times \mathbb{Q}_1}^{\mathbb{M}} \check{\beta} \in \bar{\sigma}\} \in M^{\mathbb{Q}_0}$$

is as desired. □

Claim 6. *$\mathbb{P}$ is cofinality-preserving and hence preserves all cardinals.*

Proof. Assume it is not. Let $\sigma \in M^{\mathbb{P}}$ name a witness, i.e. a function f from κ to λ that is cofinal, where $\kappa < \lambda$ are regular cardinals in M. By Claim 5, f has a name in some finite product of Cohen forcings. However, this forcing notion is cofinality-preserving using the GCH, a contradiction. □

Claim 7. *$M[G]$ satisfies the power set axiom, and whenever α is an infinite cardinal of M, $M[G] \models 2^\alpha = \alpha^{++}$ if and only if there are $p \in G$ and $i < n(p)$ such that $\alpha = \alpha_i(p)$.*

Proof. Let α be an infinite M-cardinal. Using Claim 5, choose $p \in G$ such that for every $\sigma \in M^{\mathbb{P}}$ such that $\sigma^G \subseteq \alpha$ there is a $\prod_{i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$-name $\tilde{\sigma}$ with $\tilde{\sigma}^{G_p} = \sigma^G$. Then $\mathcal{P}(\alpha)^{M[G]} = \mathcal{P}(\alpha)^{M[G_p]}$. Since M is a model of ZFC + GCH, together with Claim 6 this proves both statements of the claim. □

Claim 8. *$M[G]$ does not satisfy separation.*

Proof. Suppose the contrary and consider

$$x = \{\langle n, \alpha\rangle \in \omega \times \omega_1 \mid \exists \beta \in \mathrm{Ord}^M (f(n) = \aleph_{\omega_1 \cdot \beta + \alpha})\},$$

where $f(n)$ denotes the n-th cardinal at which the GCH fails. Separation implies that $x \in M[G]$. Since for $\alpha < \omega_1$ the class

$$D_\alpha = \{p \in \mathbb{P} \mid \exists \beta \in \mathrm{Ord}^M \, \exists i < n(p) \, [\alpha_i(p) = \aleph_{\omega_1 \cdot \beta} + \alpha]\}$$

is a dense subclass $\mathbb{P}$ that is definable over M, it follows that x defines a surjection from ω onto ω_1, contradicting that $\mathbb{P}$ is cofinality-preserving. □

This completes the proof of Theorem 4.3.2. □

4.3.2 Separation implies replacement

In this section, we show that over models of GB^- which contain a set-like well-order, the preservation of separation and replacement is equivalent. First we need the following easy characterization of the preservation of separation.

Lemma 4.3.3. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of GB^- and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ that satisfies the forcing theorem. Then the following statements are equivalent:*

(1) *For every $\Gamma \in \mathcal{C}^{\mathbb{P}}$, $\sigma \in M^{\mathbb{P}}$ and $p \in \mathbb{P}$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma \subseteq \sigma$, the class*

$$D_{\Gamma,p} = \{q \leq_{\mathbb{P}} p \mid \exists \tau \in M^{\mathbb{P}} (q \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma = \tau)\} \in \mathcal{C}$$

is dense below p.

(2) *For every $\mathbb{P}$-generic filter G over $\mathbb{M}$, $\mathbb{M}[G]$ satisfies separation.*

Proof. Suppose first that (1) holds and let G be $\mathbb{P}$-generic over $\mathbb{M}$. Let $y \in M[G]$ and φ an $\mathcal{L}_\in$-formula with set paramters $\vec{p}$ in $M[G]$ and class paramters $\vec{C}$ in $\mathcal{C}[G]$. We need to find a set name for $\{x \in y \mid \varphi(x, \vec{z}, \vec{C})\}$. Take names $\sigma \in M^{\mathbb{P}}$ for y and $\vec{\mu}$ and $\vec{\Sigma}$ for $\vec{z}$ and $\vec{C}$. Now consider

$$\Gamma = \{\langle \tau, p\rangle \mid \tau \in \mathrm{dom}(\sigma) \wedge p \Vdash^{\mathbb{M}}_{\mathbb{P}} \tau \in \sigma \wedge \varphi(\tau, \vec{\mu}, \vec{\Sigma})\} \in \mathcal{C}^{\mathbb{P}}.$$

Clearly, $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma \subseteq \sigma$. Using (1), we can take $p \in G \cap D_{\Gamma, \mathbb{1}_{\mathbb{P}}}$ and a name $\tau \in M^{\mathbb{P}}$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma = \tau$. Then τ is as desired.

Suppose now that (2) holds. Assume that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma \subseteq \sigma$ for some $\sigma \in M^{\mathbb{P}}, \Gamma \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ and let $q \leq_{\mathbb{P}} p$. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $q \in G$. Then $\Gamma^G = \{x \in \sigma^G \mid x \in \Gamma^G\} \in M[G]$ since $\mathbb{M}[G]$ satisfies separation. Pick $\tau \in M^{\mathbb{P}}$ such that $\Gamma^G = \tau^G$. Using the forcing theorem, we can find $r \in G$ which strengthens q such that $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma = \tau$. In particular, $r \in D_{\Gamma,p}$. □

Theorem 4.3.4. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a set-like well-order $\prec$. Let $\mathbb{P} \in \mathcal{C}$ be a notion of class forcing which satisfies the forcing theorem and let G be $\mathbb{P}$-generic over $\mathbb{M}$. If $\mathbb{M}[G]$ satisfies separation, then $\mathbb{M}[G]$ satisfies replacement (or equivalently, collection).*

Proof. Assume that replacement fails in $\mathbb{M}[G]$. We want to show that separation fails in $\mathbb{M}[G]$. Using Lemma 4.2.1 and the forcing theorem, we can pick a class name $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in G$ such that

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F} \colon \check{\kappa} \to \mathrm{Ord}^M \text{ is cofinal.}$$

As p plays no role in the proof to come, we may assume that $p = \mathbb{1}_{\mathbb{P}}$. Let $C \subseteq \mathrm{Ord}^M \times \mathrm{Ord}^M$ be an element of $\mathcal{C}$ such that each $C_\gamma = \{\delta \in \mathrm{Ord}^M \mid \langle \gamma, \delta \rangle \in C\}$ for $\gamma \in \mathrm{Ord}^M$ is either of the form

$$A_{p,\alpha} = \{\beta \in \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p \, (q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta})\} \in \mathcal{C}$$

for $p \in \mathbb{P}$ and $\alpha \in \mathrm{Ord}^M$ such that $A_{p,\alpha}$ is a proper class of $\mathbb{M}$, or of the form

$$B_{p,\alpha,\tau} = \{\beta \in \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p \, [q \Vdash^{\mathbb{M}}_{\mathbb{P}} (\dot{F}(\check{\alpha}) = \check{\beta} \wedge \check{\alpha} \in \tau)]\} \in \mathcal{C}$$

for $p \in \mathbb{P}$, $\alpha \in \mathrm{Ord}^M$ and $\tau \in M$ such that $B_{p,\alpha,\tau}$ is a proper class of $\mathbb{M}$, and moreover each such $A_{p,\alpha}$ and $B_{p,\alpha,\tau}$ appears unboundedly often in the enumeration $\langle C_\gamma \mid \gamma \in \mathrm{Ord}^M \rangle$.

Claim 1. *There is $D \in \mathcal{C}$ such that both $C_\gamma \cap D$ and $C_\gamma \setminus D$ are proper classes of $\mathbb{M}$ for all $\gamma \in \mathrm{Ord}^M$.*

Proof. This can be achieved by recursively defining D_γ and D'_γ in M as follows. Let $D_0 = D'_0 = \emptyset$. Suppose that D_γ and D'_γ have already been defined. Let $D_{\gamma+1} \in M$ be the set obtained from D_γ by adding the least ordinal in $C_\gamma \setminus (D_\gamma \cup D'_\gamma)$. Let $D'_{\gamma+1} \in M$ be the set obtained from D'_γ by adding the least ordinal in $C_\gamma \setminus (D_{\gamma+1} \cup D'_\gamma)$. If γ is a limit ordinal, let $D_\gamma = \bigcup_{\delta<\gamma} D_\delta$ and $D'_\gamma = \bigcup_{\delta<\gamma} D'_\delta$. Let $D = \bigcup_{\gamma \in \mathrm{Ord}^M} D_\gamma \in \mathcal{C}$. Since each C_γ appears unboundedly often in the enumeration, both $C_\gamma \cap D$ and $C_\gamma \setminus D$ are proper classes. □

Suppose that $D \in \mathcal{C}$ is as provided by Claim 1. Since $\mathbb{P}$ satisfies the forcing theorem, to show that separation fails in $\mathbb{M}[G]$, by Lemma 4.3.3 it suffices to find $\Gamma \in \mathcal{C}^{\mathbb{P}}$ such that Γ^G is a subset of some $s \in M[G]$ and such that there is no $p \in G$ and no $\tau \in M^{\mathbb{P}}$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma = \tau$. Let $F = \dot{F}^G$. Let $\Gamma \in \mathcal{C}$ be a name for $\{\alpha < \kappa \mid F(\alpha) \in D\} \in \mathcal{C}[G]$. The above set s will be equal to κ. Suppose for a contradiction that $p \in G$ is such that $p \Vdash_{\mathbb{P}} \Gamma = \tau$ for some $\tau \in M^{\mathbb{P}}$. We first claim that there is an $\alpha < \kappa$ such that $A_{p,\alpha}$ is a proper class. Assume that such an α does not exist. Then p forces that the range of $\dot{F}$ is bounded in Ord^M, contradicting our assumption on $\dot{F}$.

The above claim implies that $A_{p,\alpha} \cap D = \{\beta \mid \exists q \leq_{\mathbb{P}} p \, [q \Vdash_{\mathbb{P}} (\dot{F}(\check{\alpha}) = \check{\beta} \wedge \check{\alpha} \in \tau)]\} = B_{p,\alpha,\tau}$ is a proper class. Then $B_{p,\alpha,\tau} \setminus D$ is empty, contradicting the choice of D. □

4.3.3 Pretameness and the preservation of axioms

Recall that in Theorem 2.2.3 we have shown that pretame notions of class forcing preserve GB^- over models with a hierarchy. The following lemma states that there is a converse to this result. This was first observed in [Fri00, Proposition 2.17] in a slightly less general context.

Lemma 4.3.5. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a model of GB^- which has a hierarchy witnessed by $\langle C_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle$. Suppose that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ with the property that for every $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G over $\mathbb{M}$ such that $p \in G$ and replacement holds in $\mathbb{M}[G]$. Then $\mathbb{P}$ is pretame for $\mathbb{M}$.*

Proof. Assume, towards a contradiction, that $\langle D_i \mid i \in I \rangle$ is a sequence of dense classes and p is a condition in $\mathbb{P}$ which witnesses that pretameness fails. By assumption, there is a $\mathbb{P}$-generic filter G containing p such that $\mathbb{M}[G]$ satisfies replacement. Now consider the function

$$F : I \to \mathrm{Ord}^M, F(i) = \min\{\alpha \in \mathrm{Ord}^M \mid G \cap D_i \cap C_\alpha \neq \emptyset\}.$$

Since $\mathbb{M}[G]$ satisfies replacement, $\mathrm{ran}(F) \in M[G]$. Let $\gamma \in \mathrm{Ord}^M$ be such that $\mathrm{ran}(F) \subseteq \gamma$ and

$$D = \{q \leq_{\mathbb{P}} p \mid \exists i \in I \, \forall r \in D_i \cap C_\gamma \, (q \perp_{\mathbb{P}} r)\}.$$

By assumption, D is dense below p. Pick $q \in G \cap D$ and let $i \in I$ such that q is incompatible with all elements of $D_i \cap C_\gamma$. But then $F(i) > \gamma$, a contradiction. □

Corollary 4.3.6. *Suppose that $\mathbb{M}$ is a countable transitive model of GB^- which has a hierarchy. Then a notion of class forcing for $\mathbb{M}$ is pretame if and only if it preserves the axioms of GB^-.*

By combining Corollary 4.3.6 with the results of the previous section, we obtain the following.

Corollary 4.3.7. *Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a set-like well-order of M, and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Then the following statements are equivalent:*

(1) *$\mathbb{P}$ is pretame.*
(2) *$\mathbb{P}$ preserves GB^-.*
(3) *$\mathbb{P}$ preserves collection.*
(4) *$\mathbb{P}$ preserves replacement.*
(5) *$\mathbb{P}$ preserves separation and satisfies the forcing theorem.*

Proof. The implications from (2) to (3), from (3) to (4) and from (4) to (5) are trivial. That (1) implies (2) was shown in Theorem 2.2.3. and that (4) implies (1) follows from Lemma 4.3.5. Finally, the implication from (5) to (4) is shown in Theorem 4.3.4. Note that this is the only time that we are using the fact that $\mathcal{C}$ contains a set-like well-order; for the other implications it suffices to require that $\mathbb{M}$ has a hierarchy. □

4.4 Boolean completions

As has been shown in Section 3.2, the existence of Boolean completions is closely related to the forcing theorem. Namely by Theorem 3.2.6, if $\mathbb{M}$ has a hierarchy and $\mathbb{P}$ is a separative notion of class forcing for $\mathbb{M}$, then $\mathbb{P}$ has a pre-Boolean M-completion iff it satisfies the forcing theorem for all $\mathcal{L}_\in$-formulae. Thus the following is a consequence of Theorem 4.2.4.

Theorem 4.4.1. *Suppose that $\mathcal{C}$ contains a set-like well-order of M, but no first-order truth predicate for M. Then a separative notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame for $\mathbb{M}$ if and only if it densely has a pre-Boolean M-completion.* □

Definition 4.4.2. Suppose that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$. If $A, B \subseteq \mathbb{P}$ with $A, B \in \mathcal{C}$, we say that $\sup_{\mathbb{P}} A = \sup_{\mathbb{P}} B$ if

(1) A is predense below every $b \in B$ and
(2) B is predense below every $a \in A$.

Note that this definition is possible even if the suprema do not exist in $\mathbb{P}$. On the other hand, if $\sup_{\mathbb{P}} A = \sup_{\mathbb{P}} B$ and A has a supremum in $\mathbb{P}$ then so does B and indeed they coincide.

The following observation is a slight strengthening of a lemma which is essentially due to Joel Hamkins.

Lemma 4.4.3. *Suppose that $\mathcal{C}$ contains a set-like well-order of M. If $\mathbb{P}$ does not satisfy the* Ord-*cc, then there is an antichain $A \in \mathcal{C}$ such that for every $B \in M$ with $B \subseteq \mathbb{P}$, $\sup_{\mathbb{P}} B \neq \sup_{\mathbb{P}} A$. In particular, A does not have a supremum in $\mathbb{P}$.*

Proof. Let $A \in \mathcal{C}$ be a class-sized antichain in $\mathbb{P}$. We claim that there is a subclass of A in $\mathcal{C}$ which fulfills the desired properties. Suppose for a contradiction that no such subclass exists. Using the set-like well-order of M, we can assume that the domain of $\mathbb{P}$ is Ord^M. Let $\pi : \mathrm{Ord}^M \to A$ be a bijection in $\mathcal{C}$. Furthermore, there is an injection $\varphi : \mathcal{P}(\mathrm{Ord}^M) \cap M \to \mathrm{Ord}^M$ in $\mathcal{C}$. This gives us a mapping $i : \mathcal{P}(\mathrm{Ord}^M) \cap \mathcal{C} \to \mathrm{Ord}^M$ in V which maps $X \subseteq \mathrm{Ord}^M$ to $\varphi(B)$, where B is the least (with respect to our given global well-order) set $B \subseteq \mathbb{P}$ in M such that $\sup_{\mathbb{P}} \pi''X = \sup_{\mathbb{P}} B$. Since A is an antichain, i is injective. Moreover, whether $i(X) = \alpha$ is definable over $\mathbb{M}$, so

$$C = \{\alpha \in \mathrm{Ord}^M \mid \pi(\alpha) \not\leq_{\mathbb{P}} \alpha \wedge i(X_\alpha) = \alpha\}$$

is in $\mathcal{C}$ for $X_\alpha = \{\beta \in \mathrm{Ord}^M \mid \pi(\beta) \leq_{\mathbb{P}} \alpha\}$.

Claim 1. *For each $\alpha \in \mathrm{Ord}^M$ we have $\alpha \in C$ if and only if there is $X \in \mathcal{P}(\mathrm{Ord}^M) \cap \mathcal{C}$ such that $i(X) = \alpha$ and $\alpha \notin X$.*

Proof. Suppose first that $\alpha \in C$. Then $\alpha \notin X_\alpha$ and so we can choose $X = X_\alpha$. Conversely, suppose that $X \in \mathcal{P}(\mathrm{Ord}^M) \cap \mathcal{C}$ is such that $i(X) = \alpha$ and $\alpha \notin X$. Then $X = X_\alpha$, because $\pi''X$ and $\pi''X_\alpha$ are both subsets of the antichain A and have the same supremum. Hence $\alpha \in C$. □

We will use Claim 1 to derive a contradiction similar to Russell's paradox. Consider $\beta = i(C)$. If $\beta \in C$ then by Claim 1 there is X such that $i(X) = \beta$ but $\beta \notin X$. By injectivity of i, this means that $X = C$, a contradiction. On the other hand, it is also impossible that $\beta \notin C$, since otherwise $X = C$ would witness that $\beta \in C$. □

The following theorem characterizes the Ord-cc in terms of the existence of Boolean completions.

Theorem 4.4.4. *Suppose that* $\mathbb{M} = \langle M, \mathcal{C}\rangle$ *is a countable transitive model of* GB^- *such that* $\mathcal{C}$ *contains a set-like well-order of* M*. Then the following statements are equivalent for every separative partial order* $\mathbb{P}$*:*

(1) $\mathbb{P}$ *satisfies the* Ord-*cc.*
(2) $\mathbb{P}$ *has a unique Boolean* M*-completion.*
(3) $\mathbb{P}$ *has a Boolean* $\mathcal{C}$*-completion.*

Proof. Suppose first that $\mathbb{P}$ satisfies the Ord-cc over $\mathbb{M}$. Since $\mathcal{C}$ contains a global well-order of M, $\mathbb{P}$ is pretame for $\mathbb{M}$ and therefore it has a Boolean M-completion $\mathbb{B}(\mathbb{P}) \in \mathcal{C}$ by Corollary 3.2.9. Let $\mathbb{B}$ be another Boolean M-completion of $\mathbb{P}$. Without loss of generality, we can assume that $\mathbb{P}$ is a subset of $\mathbb{B}$. Then every element $b \in \mathbb{B}$ satisfies $b = \sup_{\mathbb{B}} D_b$, where $D_b = \{p \in \mathbb{P} \mid p \leq_{\mathbb{B}} b\} \in \mathcal{C}$. But using the global well-order of M we obtain that D_b contains an antichain which is maximal in D_b. Moreover, since $\mathbb{P}$ satisfies the Ord-cc, every such antichain lies in M. Furthermore, observe that if A and A' are two antichains which are maximal in D_b, then $\sup_{\mathbb{B}} A = \sup_{\mathbb{B}} A' = b$ and $\sup_{\mathbb{B}(\mathbb{P})} A = \sup_{\mathbb{B}(\mathbb{P})} A'$. But this gives a canonical embedding of $\mathbb{B}$ into $\mathbb{B}(\mathbb{P})$ which fixes $\mathbb{P}$. It is clearly surjective, since the same argument as above can be done within $\mathbb{B}(\mathbb{P})$. This shows that $\mathbb{P}$ has a unique Boolean M-completion.

Suppose now that $\mathbb{B}$ is the unique Boolean M-completion of $\mathbb{P}$ and that $A \subseteq \mathbb{B}$ is a class in $\mathcal{C}$ which does not have a supremum in $\mathbb{B}$. Let $\mathbb{Q}$ be the forcing notion obtained from $\mathbb{P}$ by adding $\sup A$. Note that Corollary 3.2.7, $\mathbb{P}$ satisfies the forcing theorem. Then by Lemma 4.1.10 so does $\mathbb{Q}$ and hence $\mathbb{Q}$ has a Boolean M-completion $\mathbb{B}'$. But by our assumption, $\mathbb{B}$ and $\mathbb{B}'$ are isomorphic and hence $\sup A$ exists in $\mathbb{B}$, a contradiction. This proves that $\mathbb{B}$ is already a Boolean $\mathcal{C}$-completion.

To see that (3) implies (1), suppose that $\mathbb{B}$ is a Boolean $\mathcal{C}$-completion of $\mathbb{P}$. Assume, towards a contradiction, that $\mathbb{P}$ does not satisfy the Ord-cc. Then neither does $\mathbb{B}$. But then by Lemma 4.4.3, $\mathbb{B}$ cannot be $\mathcal{C}$-complete. □

4.5 The extension maximality principle

This section is motivated by the following easy observation which follows from Lemma 1.3.2. The collapse forcing $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$, which consists of functions $n \to \mathrm{Ord}^M$ for $n \in \omega$, is dense in $\mathrm{Col}(\omega, \mathrm{Ord})^M$. However, unlike $\mathrm{Col}(\omega, \mathrm{Ord})^M$ which collapses all M-cardinals, the subforcing $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ does not add any new sets, so $\mathrm{Col}(\omega, \mathrm{Ord})^M$ and $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ do not have the same generic extensions. We will show that, under sufficient conditions on the ground model $\mathbb{M}$, the property of $\mathbb{P}$ of having the same generic extensions as all forcing notions into which $\mathbb{P}$ embeds densely is in fact equivalent to the pretameness of $\mathbb{P}$. Throughout this section, let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be countable transitive model of GB^-.

Definition 4.5.1. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ satisfies the

(1) *extension maximality principle (EMP)* over $\mathbb{M}$, if whenever $\mathbb{Q}$ is a notion of class forcing for $\mathbb{M}$ and $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding in $\mathcal{C}$, then for every $\mathbb{Q}$-generic filter G over $\mathbb{M}$, $M[G] = M[\pi^{-1}(G) \cap \mathbb{P}]$.

(2) *strong extension maximality principle (SEMP)* over $\mathbb{M}$, if whenever $\mathbb{Q}$ is a notion of class forcing for $\mathbb{M}$, $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding in $\mathcal{C}$ and $\sigma \in M^{\mathbb{Q}}$, then there is $\tau \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \pi(\tau)$.

Theorem 4.5.2. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of* GB^- *such that $\mathcal{C}$ contains a set-like well-order of M. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame for $\mathbb{M}$ if and only if it satisfies the forcing theorem and the EMP.*

Proof. Suppose first that $\mathbb{P}$ is pretame. By Theorem 2.2.2, $\mathbb{P}$ satisfies the forcing theorem. Let $\mathbb{Q}$ be a notion of class forcing such that $\mathbb{P}$ embeds densely into $\mathbb{Q}$ and let G be $\mathbb{Q}$-generic over $\mathbb{M}$. Without loss of generality, we may assume that $\mathbb{P}$ is a dense suborder of $\mathbb{Q}$. Fix a $\mathbb{Q}$-name σ. We claim that $\sigma^G \in M[G \cap \mathbb{P}]$. For every $q \in \operatorname{tc}(\sigma) \cap \mathbb{Q}$, let $D_q = \{p \in \mathbb{P} \mid p \leq_{\mathbb{Q}} q \vee p \perp_{\mathbb{Q}} q\}$. Then D_q is a dense subclass of $\mathbb{P}$. By pretameness, there is $p \in G \cap \mathbb{P}$ and there are $d_q \subseteq D_q$ which are predense below p in $\mathbb{P}$. Now we define inductively for every name τ in $\operatorname{tc}(\{\sigma\}) \cap M^{\mathbb{Q}}$,

$$\bar{\tau} = \{\langle \bar{\mu}, r\rangle \mid \exists s\,(\langle \mu, s\rangle \in \tau \wedge r \in d_s \wedge r \leq_{\mathbb{Q}} s)\}.$$

But then $\bar{\sigma} \in M^{\mathbb{P}}$ and $\sigma^G = \bar{\sigma}^{G\cap\mathbb{P}} \in M[G \cap \mathbb{P}]$.

Conversely, assume that $\mathbb{P}$ is not pretame but satisfies the forcing theorem. Then there is a $\mathbb{P}$-generic filter G such that replacement fails in the generic extension $\mathbb{M}[G]$, and by Theorem 4.3.4 so does separation (note that this is where we use the assumption about the set-like well-order). By Lemma 4.3.3 there are $\Gamma \in \mathcal{C}^{\mathbb{P}}$, $\sigma \in M^{\mathbb{P}}$ and $p \in G$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma \subseteq \sigma$ and there are no $q \in G$ and $\tau \in M^{\mathbb{P}}$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma = \tau$. For $\mu \in \operatorname{dom}(\sigma)$ consider

$$A_\mu = \{q \in \mathbb{P} \mid q \Vdash^{\mathbb{M}}_{\mathbb{P}} \mu \in \Gamma\}.$$

Let $\mathbb{Q}$ be the forcing obtained from $\mathbb{P}$ by adding $\sup A_\mu$ for each $\mu \in \operatorname{dom}(\sigma)$ such that A_μ is nonempty below p, as described in Section 4.1. Then $\mathbb{P}$ is a dense subclass of $\mathbb{Q}$. Consider the $\mathbb{Q}$-name

$$\tau = \{\langle \mu, \sup A_\mu\rangle \mid \mu \in \operatorname{dom}(\sigma), A_\mu \text{ is nonempty below } p\} \in M^{\mathbb{Q}}.$$

Let H be the $\mathbb{Q}$-generic induced by G, that is the upwards closure of G in $\mathbb{Q}$. Then τ is a $\mathbb{Q}$-name for Γ^G, so $\tau^H = \Gamma^G \in M[H] \setminus M[G]$, proving the failure of the EMP. □

Note that in the proof of Theorem 4.5.2 we have only used the set-like well-order of M to show that every forcing notion which satisfies the forcing theorem and the EMP is pretame; for the other direction it suffices to assume that $\mathbb{M}$ has a hierarchy.

Example 4.5.3. *Jensen coding* $\mathbb{P}$ (see [BJW82]) is a pretame notion of class forcing which over a model M of ZFC adds a generic real x such that the $\mathbb{P}$-generic extension is of the form $\mathsf{L}[x]$. Moreover, there is a class name Γ for x such that $\mathbb{1}_{\mathbb{P}} \Vdash^{M}_{\mathbb{P}} M[\dot{G}] = \mathsf{L}[\Gamma]$, but there is no set name σ such that $\mathbb{1}_{\mathbb{P}} \Vdash^{M}_{\mathbb{P}} \sigma = \Gamma$. Let $\mathbb{Q}$ be the forcing notion obtained from Jensen coding by adding the suprema $p_n = \sup\{p \in \mathbb{P} \mid p \Vdash^{M}_{\mathbb{P}} \check{n} \in \Gamma\}$. Since $\mathbb{P}$ is pretame and dense in $\mathbb{Q}$, it follows that $\mathbb{Q}$ is also pretame. By Theorem 4.5.2, $\mathbb{P}$ satisfies the EMP and hence $\mathbb{P}$ and $\mathbb{Q}$ produce the same generic extensions. In particular, this means that if G is $\mathbb{Q}$-generic then $M[G \cap \mathbb{P}] = M[G] = \mathsf{L}[\sigma^G]$, where $\sigma = \{\langle \check{n}, p_n\rangle \mid n \in \omega\} \in M^{\mathbb{Q}}$.

Lemma 4.5.4. *Suppose that* $\mathbb{M} = \langle M, \mathcal{C}\rangle$ *is a countable transitive model of* GB^- *and that* $\mathcal{C}$ *contains a set-like well-order of* M*. Then a notion of class forcing* $\mathbb{P}$ *for* $\mathbb{M}$ *satisfies the SEMP if and only if it satisfies the* Ord-*cc over* $\mathbb{M}$*.*

Proof. Suppose first that $\mathbb{P}$ satisfies the Ord-cc. Suppose that there is a dense embedding from $\mathbb{P}$ into some forcing notion $\mathbb{Q}$. Without loss of generality, we may assume that $\mathbb{P}$ is a subclass of $\mathbb{Q}$. We prove by induction on $\operatorname{rank}(\sigma)$ that for every $\sigma \in M^{\mathbb{Q}}$ there is $\bar\sigma \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \bar\sigma$. Assume that this holds for all τ of rank less than $\operatorname{rank}(\sigma)$. Then for every $\tau \in \operatorname{dom}(\sigma)$ there is $\bar\tau \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \tau = \bar\tau$. For each condition $q \in \operatorname{range}(\sigma)$, let $D_q = \{p \in \mathbb{P} \mid p \leq_{\mathbb{Q}} q\}$ and choose an antichain A_q which is maximal in D_q. By assumption, we may do this so that $\langle\langle q, A_q\rangle \mid q \in \operatorname{range}(\sigma)\rangle \in M$. Then put

$$\bar\sigma = \{\langle\bar\tau, p\rangle \mid \exists q \in \mathbb{Q}\,(\langle\tau, q\rangle \in \sigma \wedge p \in A_q)\} \in M^{\mathbb{P}}.$$

By construction, $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \bar\sigma$.

Conversely, suppose that $\mathbb{P}$ does not satisfy the Ord-cc. Then by Lemma 4.4.3 there is an antichain $A \in \mathcal{C}$ such that for no $B \in M$ with $B \subseteq \mathbb{P}$, $\sup_{\mathbb{P}} A = \sup_{\mathbb{P}} B$. Let $\mathbb{Q} = \mathbb{P} \cup \{\sup A\}$ be the extension of $\mathbb{P}$ given by adding the supremum of A. Now consider $\sigma = \{\langle\check{0}, \sup A\rangle\} \in M^{\mathbb{Q}}$. We claim that σ witnesses the failure of the SEMP. Suppose for a contradiction that there is $\tau \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \tau$. Let $\tau = \{\langle\mu_i, p_i\rangle \mid i \in I\}$ for some $I \in M$. But then it is easy to check that $\sup_{\mathbb{P}}\{p_i \mid i \in I\} = \sup_{\mathbb{P}} A$, contradicting our assumption on A. □

4.6 Nice forcing notions

This section is motivated by the observation in Lemma 4.6.2 below, namely that - unlike in the context of set forcing - there are sets of ordinals in class-generic extensions which do not have a nice name. We will characterize both pretameness and the Ord-cc in terms of the existence of nice names for sets of ordinals.

4.6.1 Basic definitions and examples

Throughout this section, $\mathbb{M} = \langle M, \mathcal{C}\rangle$ will denote a countable transitive model of GB^-.

Definition 4.6.1. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. A name $\sigma \in M^{\mathbb{P}}$ for a set of ordinals is a *nice name* if it is of the form $\bigcup_{\alpha<\gamma}\{\check\alpha\} \times A_\alpha$ for some $\gamma \in \operatorname{Ord}^M$, where each $A_\alpha \in M$ is a set-sized antichain of conditions in $\mathbb{P}$.

Lemma 4.6.2. *Let* $\mathbb{P}$ *denote* $\operatorname{Col}(\omega, \operatorname{Ord})^M$*. Then in every* $\mathbb{P}$*-generic extension there is a subset of* ω *which does not have a nice* $\mathbb{P}$*-name.*

Proof. Consider the canonical name $\sigma = \{\langle\check n, \{\langle n, 0\rangle\}\rangle \mid n \in \omega\}$ for the set of natural numbers which are mapped to 0 by the generic function from ω to the ordinals. Let G be $\mathbb{P}$-generic over $\mathbb{M}$. We show that the complement of σ^G is an element of $M[G]$, but does not have a nice $\mathbb{P}$-name in M. Suppose for a contradiction that there are $p \in G$ and a nice $\mathbb{P}$-name

$$\tau = \bigcup_{n\in\omega}\{\check n\} \times A_n \in M,$$

where each $A_n \in M$ is an antichain, such that $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \check{\omega} \setminus \sigma = \tau$. Let $n \in \omega \setminus \mathrm{dom}(p)$ and choose $\alpha > \sup\{r(i) \mid r \in A_n \wedge i \in \mathrm{dom}(r)\}$. Then $q = p \cup \{\langle n, \alpha\rangle\}$ strengthens p and $q \Vdash_{\mathbb{P}}^{\mathbb{M}} \check{n} \in \tau$. Hence there must be some $r \in A_n$ which is compatible with q. But then $n \notin \mathrm{dom}(r)$, so p and $r \cup \{\langle n, 0\rangle\}$ are compatible. Let $s \leq_{\mathbb{P}} p, r \cup \{\langle n, 0\rangle\}$ witness this. Then $s \Vdash_{\mathbb{P}}^{\mathbb{M}} \check{n} \in \sigma \cap \tau$, a contradiction.

That the complement of σ^G has a $\mathbb{P}$-name in M and is thus an element of $M[G]$ follows from a more general result in [HKS16b, Lemma 8.7]. For the benefit of the reader, we provide a shorter proof for the present special case. For each $n \in \omega$, consider the $\mathbb{P}$-name

$$\tau_n = \check{n} \cup \{\langle \check{m}, \{\langle i, 0\rangle \mid n \leq i < m\}\rangle \mid m > n\}.$$

Then each τ_n is a name for the least $k \geq n$ such that $k \notin \sigma^G$. Now put $\tau = \{\langle \tau_n, \mathbb{1}_{\mathbb{P}}\rangle \mid n \in \omega\}$. Since by an easy density argument the complement of σ^G is unbounded in ω, τ is as desired. □

Definition 4.6.3. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is said to be

(1) *nice*, if for every $\gamma \in \mathrm{Ord}^M$, for every $\sigma \in M^{\mathbb{P}}$ and for every $\mathbb{P}$-generic filter G such that $\sigma^G \subseteq \gamma$ there is a nice name $\tau \in M^{\mathbb{P}}$ such that $\sigma^G = \tau^G$.

(2) *very nice*, if for every $\gamma \in \mathrm{Ord}^M$ and for every $\sigma \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \subseteq \check{\gamma}$ there is a nice name $\tau \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \tau$.

Example 4.6.4. (1) By Lemma 4.6.2, $\mathrm{Col}(\omega, \mathrm{Ord})^M$ is not nice.

(2) Suppose that $\mathbb{M}$ has a hierarchy and satisfies the axiom of choice. Then every pretame notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is nice. To see this, let $\gamma \in \mathrm{Ord}^M$ be an ordinal and let $p \in \mathbb{P}$ and $\sigma \in M^{\mathbb{P}}$ be such that $p \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \subseteq \check{\gamma}$.[2] For each $\alpha < \gamma$, consider the class

$$D_\alpha = \{q \leq_{\mathbb{P}} p \mid q \Vdash_{\mathbb{P}}^{\mathbb{M}} \alpha \in \sigma \vee q \Vdash_{\mathbb{P}}^{\mathbb{M}} \alpha \notin \sigma\} \in \mathcal{C},$$

which is dense below p. By pretameness there exist $q \leq_{\mathbb{P}} p$ and for every $\alpha < \gamma$ a set $d_\alpha \subseteq D_\alpha$ in M which is predense below q. Using the axiom of choice, we may choose for every $\alpha < \gamma$ an antichain $a_\alpha \subseteq d_\alpha$ which is maximal in d_α. Finally, let $A_\alpha = \{r \in a_\alpha \mid r \Vdash_{\mathbb{P}}^{\mathbb{M}} \check{\alpha} \in \sigma\}$. Then

$$\tau = \bigcup_{\alpha < \gamma} \{\check{\alpha}\} \times A_\alpha \in M^{\mathbb{P}}$$

is a nice name for a subset of γ with $q \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \tau$.

(3) If $\mathcal{C}$ contains a global well-order of M, then every notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the Ord-cc is very nice: Note first that by Lemma 4.1.2, $\mathbb{P}$ satisfies the forcing theorem. Suppose that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma \subseteq \check{\gamma}$. For every $\alpha < \gamma$, we can choose an antichain A_α which is maximal in $\{q \in \mathbb{P} \mid \exists \langle \mu, p\rangle \in \sigma\, (q \leq_{\mathbb{P}} p \wedge q \Vdash_{\mathbb{P}}^{\mathbb{M}} \mu = \check{\alpha})\}$. Since $\mathbb{P}$ satisfies the Ord-cc, making use of the global well-order we can do this so that $\langle \langle \alpha, A_\alpha\rangle \mid \alpha < \gamma\rangle \in M$. Then

$$\tau = \bigcup_{\alpha < \gamma} \{\check{\alpha}\} \times A_\alpha \in M^{\mathbb{P}}$$

is a nice name and it is easy to check that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}^{\mathbb{M}} \sigma = \tau$.

[2] Note that since $\mathbb{M}$ has a hierarchy, it follows by Theorem 2.2.2 that $\mathbb{P}$ satisfies the forcing theorem.

(4) Every M-complete pre-Boolean algebra $\mathbb{B}$ is very nice, since then we can always define Boolean values $[\![\varphi]\!]_{\mathbb{B}}$ for quantifier-free infinitary formulae φ which mention only set names. More precisely, if $\sigma \in M^{\mathbb{B}}$ such that $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}} \sigma \subseteq \check{\gamma}$ for some ordinal γ, the name

$$\tau = \{\langle \check{\alpha}, [\![\check{\alpha} \in \sigma]\!]_{\mathbb{B}} \rangle \mid \alpha < \gamma\} \in M^{\mathbb{B}}$$

is a nice name with the property that $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}} \sigma = \tau$. In particular, this shows that there are very nice notions of class forcing which are not pretame (for example the Boolean M-completion of $\mathrm{Col}(\omega, \mathrm{Ord})^M$), since every separative notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem has a pre-Boolean M-completion by Theorem 3.2.6.

4.6.2 Pretameness and nice forcing notions

Lemma 4.6.2 suggests that one might try to use the class name $\dot{F}$ for the generic cofinal function $F \colon \kappa \to \mathrm{Ord}^M$ added by a non-pretame notion of class forcing $\mathbb{P}$ (by Lemma 4.2.1) to construct a forcing notion $\mathbb{Q}$ into which $\mathbb{P}$ densely embeds and such that there is a $\mathbb{Q}$-name τ for a subset of κ which has no nice $\mathbb{Q}$-name (the idea would be to obtain $\mathbb{Q}$ by adding the Boolean values of "$\dot{F}(\check{\alpha}) = \check{0}$" for $\alpha < \kappa$, which allow for the construction of a name τ for $\{\alpha < \kappa \mid F(\alpha) \neq 0\}$; in the case of $\mathrm{Col}(\omega, \mathrm{Ord})$, these Boolean values already exist). This approach would indeed work if $\kappa = \omega$, as we can construct such τ by [HKS16b, Lemma 8.7]. Since we however do not know whether names for the complements of (nice) names for subsets of arbitrary ordinals always exist (for more on this topic, consult [HKS16b, Section 8]), we will instead work with a name for an intersection of two nice names. The following lemma shows that such intersections exist in every class forcing extension.

Lemma 4.6.5. *Let $\mathbb{M}$ be a countable transitive model of GB^-. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Let $\alpha \in \mathrm{Ord}^M$ be an ordinal and let $\sigma, \tau \in M^{\mathbb{P}}$ be nice names for subsets of α. If G is $\mathbb{P}$-generic over $\mathbb{M}$, then there is a $\mathbb{P}$-name $\mu \in M^{\mathbb{P}}$ such that $\mu^G = \sigma^G \cap \tau^G$.*

Proof. Since σ and τ are nice names, they are of the form

$$\sigma = \bigcup_{\beta < \alpha} \{\check{\beta}\} \times A_\beta \text{ and } \tau = \bigcup_{\beta < \alpha} \{\check{\beta}\} \times B_\beta,$$

where $\langle A_\beta \mid \beta < \alpha \rangle, \langle B_\beta \mid \beta < \alpha \rangle \in M$. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ and let $\beta_0 \in \alpha$ be minimal such that $\beta_0 \in \sigma^G \cap \tau^G$. Put $\mu = \{\langle \check{\beta}_0, \mathbb{1}_{\mathbb{P}} \rangle\} \cup \{\langle \mu_\beta, p \rangle \mid \beta \in (\beta_0, \alpha), p \in X_\beta\}$, where $\mu_\beta = \check{\beta}_0 \cup \bigcup_{\gamma < \beta} \{\check{\gamma}\} \times Y_\beta$ for every $\beta \in (\beta_0, \alpha) = \{\beta \in \mathrm{Ord}^M \mid \beta_0 < \beta < \alpha\}$. Then

$$(\mu_\beta)^G = \begin{cases} \beta, & \beta \in \tau^G \\ \beta_0, & \text{otherwise.} \end{cases}$$

Clearly, $\mu^G = \sigma^G \cap \tau^G$ is as desired. □

Remark 4.6.6. Note that it is in general not possible to find a $\mathbb{P}$-name μ as in Lemma 4.6.5 such that $\mathbb{1}_{\mathbb{P}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \mu = \sigma \cap \tau$. For example, consider a notion of class forcing $\mathbb{P}$ which contains compatible conditions $p, q \in \mathbb{P}$ such that the class $\{r \in \mathbb{P} \mid r \leq_{\mathbb{P}} p, q\}$ contains no predense subclass that is an element of M. Then there is no such $\mathbb{P}$-name for the intersection of $\sigma = \{\langle \check{0}, p\rangle\}$ and $\tau = \{\langle \check{0}, q\rangle\}$.

Theorem 4.6.7. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of* KM. *Then an antisymmetric separative notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame for $\mathbb{M}$ if and only if it is densely nice.*

Proof. Suppose first that $\mathbb{P}$ is pretame. It is straightforward to check that whenever there is a dense embedding $\pi\colon \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ for some notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$, then $\mathbb{Q}$ is also pretame. Then by Example 4.6.4 (2), every such $\mathbb{Q}$ is nice.

Conversely, suppose that $\mathbb{P}$ is not pretame. By Theorem 3.3.1, $\mathbb{P}$ has a Boolean M-completion. Hence, without loss of generality, we may assume that $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ is an M-complete Boolean algebra. We will extend $\mathbb{P}$ to a notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$ which is not nice and so that $\mathbb{P}$ is a dense subforcing of $\mathbb{Q}$. By Lemma 4.2.1 there are a class name $\dot{F} \in \mathcal{C}^{\mathbb{P}}$, $\kappa \in \mathrm{Ord}$ and $p \in \mathbb{P}$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}$ is cofinal". For the sake of simplicity, suppose that $p = \mathbb{1}_{\mathbb{P}}$.

For every $\alpha, \beta < \kappa$ and $p \in \mathbb{P}$, let

$$X_{p,\alpha,\beta} = \{\langle \gamma, \delta\rangle \in \mathrm{Ord}^M \times \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p\, (q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma} \wedge \dot{F}(\check{\beta}) = \check{\delta})\},$$

and let

$$Y_{p,\alpha} = \{\gamma \in \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p\, (q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma})\}.$$

Claim 1. *For each $p \in \mathbb{P}$ there is $\alpha < \kappa$ such that for all $\beta < \kappa$, $X_{p,\alpha,\beta}$ is a proper class.*

Proof. Suppose the contrary. Then for every $\alpha < \kappa$ there exists $\beta_\alpha < \kappa$ such that $X_{p,\alpha,\beta_\alpha}$ is set-sized. In particular, this implies that for every $\alpha < \kappa$, the class $Y_{p,\alpha}$ is set-sized. But then p forces that the range of $\dot{F}$ is bounded in the ordinals, a contradiction. □

Let $C = \langle C_i \mid i \in \mathrm{Ord}^M\rangle \in \mathcal{C}$ be an enumeration of subclasses of $\mathrm{Ord}^M \times \mathrm{Ord}^M$ such that each C_i is of the form $X_{p,\alpha,\beta}$ for some $p \in \mathbb{P}$ and $\alpha, \beta < \kappa$ such that $X_{p,\alpha,\beta}$ is a proper class, and moreover each $X_{p,\alpha,\beta}$ which is a proper class appears unboundedly often in the enumeration C.

We will next perform a recursive construction to build two classes $D, E \in \mathcal{C}$, in a way that in particular each of $D \cap E$, $D \setminus E$ and $E \setminus D$ has a proper class sized intersection with $Y_{p,\alpha} = \{\gamma \mid \langle \gamma, \gamma\rangle \in X_{p,\alpha,\alpha}\}$ whenever $Y_{p,\alpha}$ is a proper class. The construction of the classes D, E will satisfy further properties which will be used in the proof of Claim 2 below.

Let $D_0 = D'_0 = E_0 = E'_0 = \emptyset$. Suppose that D_i, D'_i, E_i, E'_i have already been defined such that $D_i \cap D'_i = E_i \cap E'_i = D'_i \cap E'_i = \emptyset$ and $D_i \cup D'_i = E_i \cup E'_i$. We define $F_i = D_i \cup D'_i = E_i \cup E'_i$. Let $\langle \gamma_0, \delta_0\rangle, \langle \gamma_1, \delta_1\rangle, \langle \gamma_2, \delta_2\rangle$ be the lexicographically least pairs of ordinals in C_i such that each pair $\langle \gamma_k, \delta_k\rangle$ contains at least one ordinal not in $F_i \cup \{\gamma_j \mid j < k\} \cup \{\delta_j \mid j < k\}$, and γ_0, δ_0 additionally satisfy (if possible)

$$\gamma_0 \notin F_i \wedge \delta_0 \notin D'_i, \tag{4.1}$$

and γ_1, δ_1 satisfy in addition (if such exist)

$$\gamma_1 \notin F_i \cup \{\gamma_0, \delta_0\} \wedge \delta_1 \notin E'_i. \tag{4.2}$$

In the successor step, we will enlarge D_i, D'_i, E_i and E'_i to $D_{i+1}, D'_{i+1}, E_{i+1}$ and E'_{i+1} by putting distinct ordinals, which are not in F_i, into the sets $D_{i+1} \cap E_{i+1}$, $D_{i+1} \cap E'_{i+1}$ and $D'_{i+1} \cap E_{i+1}$. First, we put each ordinal in $\{\gamma_0, \delta_0\}$ which is not in F_i into $D_{i+1} \cap E'_{i+1}$. Next, we put all ordinals amongst $\{\gamma_1, \delta_1\}$ that are not in $F_i \cup \{\gamma_0, \delta_0\}$ into $D'_{i+1} \cap E_{i+1}$. Finally, we put every ordinal in $\{\gamma_2, \delta_2\}$ which is not yet in $F_i \cup \{\gamma_0, \gamma_1, \delta_0, \delta_1\}$ into $D_{i+1} \cap E_{i+1}$. Note that by construction, $D_{i+1} \cap D'_{i+1} = E_{i+1} \cap E'_{i+1} = D'_{i+1} \cap E'_{i+1} = \emptyset$ and $D_{i+1} \cup D'_{i+1} = E_{i+1} \cup E'_{i+1}$.

At limit stages, we take unions, e.g. if j is a limit ordinal, we let $D_j = \bigcup_{i<j} D_i$. Finally, let $D = \bigcup_{i \in \mathrm{Ord}^M} D_i \in \mathcal{C}$ and let $E = \bigcup_{i \in \mathrm{Ord}^M} E_i \in \mathcal{C}$.

Note that at each stage i such that $C_i = X_{p,\alpha,\alpha}$ for some $p \in \mathbb{P}$ and $\alpha \in \mathrm{Ord}^M$, each of the classes $D \cap E$, $D \setminus E$ and $E \setminus D$ obtains a new element from $Y_{p,\alpha}$. Since there are class many such stages, each of $D \cap E$, $D \setminus E$ and $E \setminus D$ has a proper class-sized intersection with $Y_{p,\alpha}$ whenever $Y_{p,\alpha}$ is a proper class.

We define

$$a = \{\alpha < \kappa \mid \exists p \in \mathbb{P}\,(p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \check{D})\} \text{ and}$$
$$b = \{\alpha < \kappa \mid \exists p \in \mathbb{P}\,(p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \check{E})\}.$$

We extend $\mathbb{P}$ to a forcing notion $\mathbb{Q}$ by adding suprema for each of the classes

$$R_\alpha = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \check{D}\} \text{ and}$$
$$S_\beta = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\beta}) \in \check{E}\}$$

for $\alpha \in a$ and $\beta \in b$, as described in Section 4.1. Let $p_\alpha = \sup_{\mathbb{Q}} R_\alpha$ and let $q_\beta = \sup_{\mathbb{Q}} S_\beta$ for $\alpha \in a$ resp. $\beta \in b$. Since $\mathbb{M}$ is a model of KM, $\mathbb{Q}$ satisfies the forcing theorem.

We will show that $\mathbb{Q}$ is not nice. Let $\dot{G}$ denote the canonical class name for the $\mathbb{Q}$-generic filter. Consider the $\mathbb{Q}$-names

$$\sigma = \{\langle \check{\alpha}, p_\alpha \rangle \mid \alpha \in a\} \text{ and } \tau = \{\langle \check{\alpha}, q_\alpha \rangle \mid \alpha \in b\}$$

for $\{\alpha < \kappa \mid \dot{F}^{\dot{G}}(\alpha) \in \check{D}\}$ and $\{\alpha < \kappa \mid \dot{F}^{\dot{G}}(\alpha) \in \check{E}\}$ respectively. It follows from Lemma 4.6.5 that for every $\mathbb{Q}$-generic filter G there is a $\mathbb{Q}$-name μ such that $\mu^G = \sigma^G \cap \tau^G$. We claim that $M^{\mathbb{Q}}$ contains no nice name for $\sigma^G \cap \tau^G$. Suppose for a contradiction that there are $p \in \mathbb{Q}$ and a nice name $\nu \in M^{\mathbb{Q}}$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{Q}} \nu = \sigma \cap \tau$. By density of $\mathbb{P}$ in $\mathbb{Q}$, we may assume that $p \in \mathbb{P}$. Since ν is a nice name, it is of the form

$$\nu = \bigcup_{\alpha<\kappa} \{\check{\alpha}\} \times A_\alpha,$$

where each $A_\alpha \subseteq \mathbb{Q}$ is a set-sized antichain in M.

Let $\alpha < \kappa$ be as in Claim 1. We may assume that A_α only contains conditions which are compatible with p.

Claim 2. *For every $q \in A_\alpha$,*

$$Z_q = \{\gamma \in \mathrm{Ord}^M \mid \exists r \in \mathbb{P}\,(r \leq_{\mathbb{Q}} p, q \text{ and } r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma})\}$$

is a set in M.

Proof. We first consider $q \in A_\alpha \cap \mathbb{P}$. By assumption p and q are compatible, and since $\mathbb{P}$ is a Boolean algebra, $Z_q = Y_{p \wedge q, \alpha}$. Assume for a contradiction that $Y_{p\wedge q,\alpha}$ is a proper class. Then by our construction, $Y_{p\wedge q,\alpha} \setminus D$ is a proper class as well. Take $\gamma \in Y_{p\wedge q,\alpha} \setminus D$ and $r \leq_{\mathbb{P}} p \wedge q$ with $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$. Let G be $\mathbb{Q}$-generic with $r \in G$. Then $p, q \in G$ and so $\alpha \in \nu^G = \sigma^G \cap \tau^G$. On the other hand, since $\dot{F}^G(\alpha) = \gamma \notin D$, we have $p_\alpha \notin G$ and hence $\alpha \notin \sigma^G$. This is a contradiction.

Next, suppose that $q = p_\alpha$ and assume for a contradiction that Z_{p_α} is a proper class. Then $Y_{p,\alpha}$ is a proper class, so $Y_{p,\alpha} \cap (D \setminus E)$ is also a proper class. Now let $r \leq_{\mathbb{P}} p$ and $\gamma \in D \setminus E$ be such that $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$. Then $r \leq_{\mathbb{Q}} p_\alpha$ by the definition of p_α. If G is $\mathbb{Q}$-generic with $r \in G$, then $\alpha \in \nu^G$. Since $\gamma \notin E$, we have $\alpha \notin \tau^G$. This is a contradiction.

Next, suppose that $q = p_\beta \in A_\alpha$ for some $\beta \neq \alpha$. If there is some $\langle\gamma, \delta\rangle \in X_{p,\alpha,\beta}$ such that $\delta \in D$ but $\gamma \notin D \cap E$, then take $r \leq_{\mathbb{P}} p$ such that $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma} \wedge \dot{F}(\check{\beta}) = \check{\delta}$ and a $\mathbb{Q}$-generic filter containig r. Since $\delta \in D$ we have $p_\beta \in G$ and so $\alpha \in \nu^G$. On the other hand, $\dot{F}^G(\alpha) = \gamma \notin D \cap E$, so $\alpha \notin \sigma^G \cap \tau^G$. So there can be no such $\langle\gamma, \delta\rangle \in X_{p,\alpha,\beta}$. Hence for all $\langle\gamma, \delta\rangle \in X_{p,\alpha,\beta}$, if $\delta \in D$ then $\gamma \in D \cap E$. Suppose for a contradiction that Z_{p_β} is a proper class. Consider now the first stage i such that $X_{p,\alpha,\beta} = C_i$. Since $Y_{p,\alpha}$ is a proper class, there is $\langle\gamma, \delta\rangle \in X_{p,\alpha,\beta}$ such that $\gamma \notin F_i$. If there is such a pair which additionally satisfies that $\delta \notin D'_i$, then we are in case (1) in the recursive construction of D and E and so this would imply that γ ends up in $D \setminus E$ and $\delta \in D$. But we have already shown that this is impossible. So for every pair $\langle\gamma, \delta\rangle \in X_{p,\alpha,\beta}$ with $\gamma \notin F_i$ we have $\delta \in D'_i$. In particular, if $\delta \in D$ then $\gamma \in F_i$. But this implies that $Z_{p_\beta} \subseteq F_i$ is not a proper class, which is a contradiction.

The case $q = q_\alpha$ is analogous to the case $q = p_\alpha$. Finally, suppose that $q = q_\beta$ for some $\beta \neq \alpha$. As in the previous case $q = p_\beta$, we can conclude that for all $\langle\gamma, \delta\rangle \in X_{p,\alpha,\beta}$, if $\delta \in E$ then $\gamma \in D \cap E$. As above, we assume that Z_{q_β} is not in M and we let i be the least ordinal such that $C_i = X_{p,\alpha,\beta}$. After choosing γ_0, δ_0 in the recursive construction of D and E, there is still a pair $\langle\gamma_1, \delta_1\rangle$ such that $\gamma_1 \notin F_i^+ = F_i \cup \{\gamma_0, \delta_0\}$, since $Y_{p,\alpha}$ is a proper class. If possible, this pair is chosen such that $\delta_1 \notin E'_i$. But then γ_1 is put into $E \setminus D$ and δ_1 ends up in E. However, we have already argued that this cannot occur. But then for every such pair $\langle\gamma_1, \delta_1\rangle \in X_{p,\alpha,\beta}$ with $\gamma_1 \notin F_i^+$, we have $\delta_1 \in E'_i$, and so Z_{q_β} is contained in the set F_i^+, which is a contradiction. □

By Claim 2 and since $A_\alpha \in M$, we have that

$$B = \bigcup_{q \in A_\alpha} Z_q \in M.$$

Since $Y_{p,\alpha}$ is a proper class, so is $Y_{p,\alpha} \cap D \cap E$ by our construction, and hence there must be some $\gamma \in (Y_{p,\alpha} \cap D \cap E) \setminus B$. Let now $q \leq_{\mathbb{P}} p$ such that $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$ and take a $\mathbb{Q}$-generic filter G with $q \in G$. Then $\dot{F}^G(\alpha) = \gamma \in D \cap E$, so $\alpha \in \sigma^G \cap \tau^G$. Therefore there is some $r \in A_\alpha \cap G$. Take $s \in G$ with $s \leq_{\mathbb{Q}} q, r$. Then $s \Vdash^{\mathbb{M}}_{\mathbb{Q}} \check{\gamma} = \dot{F}(\check{\alpha}) \in \check{B}$, contradicting the choice of γ. □

4.6.3 The Ord-cc and very nice forcing notions

The next theorem shows that the Ord-chain condition can be characterized in terms of very niceness.

Theorem 4.6.8. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a set-like well-order of M. A separative antisymmetric notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem satisfies the* Ord-*cc if and only if it is densely very nice.*

Proof. Suppose first that $\mathbb{P}$ satisfies the Ord-cc and $\mathbb{P}$ embeds densely into $\mathbb{Q}$. It is easy to see that then $\mathbb{Q}$ also satisfies the Ord-cc and so by Example 4.6.4, (3) it is very nice.

Conversely, suppose that $\mathbb{P}$ contains a class-sized antichain. We would like to extend $\mathbb{P}$ via a dense embedding to a partial order which is not very nice. Since $\mathbb{P}$ satisfies the forcing theorem, $\mathbb{P}$ has a Boolean M-completion. As we are only interested in a dense property, we may therefore assume that $\mathbb{P}$ is already an M-complete Boolean algebra.

By the proof of Lemma 4.4.3, we can find three disjoint subclasses of our given class-sized antichain, each of which contains a subclass which does not have a supremum in $\mathbb{P}$. Denote these subclasses without suprema by A, D and E, and let $B = A \cup D$ and $C = A \cup E$.

Claim 1. *At least one of* $\sup B$ *and* $\sup C$ *does not exist in* $\mathbb{P}$.

Proof. We show that if both $\sup B$ and $\sup C$ exist, then so does $\sup A$, contradicting our choice of A. Since $\mathbb{P}$ is an M-complete Boolean algebra, if $\sup B$ and $\sup C$ exist, then so does $p = \sup B \wedge \sup C$. We claim that p is already the supremum of A. It is clear that every element of A is below p. It remains to check that A is predense below p. Let $q \leq_{\mathbb{P}} p$. Since B is predense below q, there are $r \leq_{\mathbb{P}} q$ and $b \in B$ with $r \leq_{\mathbb{P}} b$. Since C is predense below r, there are $s \leq_{\mathbb{P}} r$ and $c \in C$ with $s \leq_{\mathbb{P}} c$. In particular, b and c are compatible. But they both belong to the antichain $B \cup C$, so $b = c \in B \cap C = A$. □

Let $\mathbb{Q}$ be the forcing notion obtained from $\mathbb{P}$ by adding $\sup B$ and $\sup C$. By Lemma 4.1.10, $\mathbb{Q}$ satisfies the forcing theorem. Moreover, it follows from the separativity of $\mathbb{P}$ that $\mathbb{Q}$ is separative. We show that $\mathbb{Q}$ is not very nice. Consider the $\mathbb{Q}$-name

$$\sigma = \{\langle\{\langle\check{0}, \sup B\rangle\}, \sup C\rangle\}.$$

By definition, $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma \subseteq \check{2}$.

Claim 2. *There is no nice $\mathbb{Q}$-name τ such that $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \tau$.*

Proof. Suppose for a contradiction that $\tau = \{\check{0}\} \times A_0 \cup \{\check{1}\} \times A_1$, where $A_0, A_1 \in M$ are antichains of $\mathbb{Q}$, and $\mathbb{1}_{\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \tau$. Observe that $A_1 \subseteq \mathbb{P}$, since if for example $\sup B \in A_1$ and G is $\mathbb{Q}$-generic with $\sup B \in G$ and $\sup C \notin G$, then $1 \in \tau^G \setminus \sigma^G$. The same works for $\sup C$. Therefore $\sup A_1$ exists in $\mathbb{P}$. We claim that $\sup A_1$ is the supremum of A.

Firstly, we show that every element of A is below $\sup A_1$. Suppose for a contradiction that there is $a \in A$ with $a \not\leq_{\mathbb{Q}} \sup A_1$. Then by separativity of $\mathbb{Q}$ there is $p \leq_{\mathbb{P}} a$ with $p \perp_{\mathbb{P}} \sup A_1$. In particular, p is incompatible with every element of A_1. Hence if G is a $\mathbb{Q}$-generic filter with $p \in G$ then $1 \in \sigma^G \setminus \tau^G$, contradicting our assumptions on σ and τ.

Secondly, we check that A is predense below $\sup A_1$. Assume, towards a contradiction, that there is $p \leq_{\mathbb{P}} \sup A_1$ with $p \perp_{\mathbb{P}} a$ for each $a \in A$. Now A_1 is predense below p, so there exist $q \leq_{\mathbb{P}} p$ and $r \in A_1$ with $q \leq_{\mathbb{P}} r$. Again, this yields that for any $\mathbb{Q}$-generic filter G with $q \in G$, $1 \in \tau^G$ but $A \cap G = \emptyset$, so it is impossible that $\sup B$ and $\sup C$ are both in G. Hence $1 \notin \sigma^G$, contradicting our assumptions on σ and τ.

We have thus shown that $\sup A$ exists in $\mathbb{P}$, contradicting our choice of A. □

This proves that $\mathbb{Q}$ is not very nice. □

The proof of Theorem 4.6.8 actually shows that every notion of forcing $\mathbb{P}$ which satisfies the forcing theorem but not the Ord-cc, can be densely embedded into a notion of class forcing which satisfies the forcing theorem and is nice but not very nice. To see this, it remains to check that the partial order $\mathbb{Q}$ constructed above is nice. This follows from the following more general result:

Lemma 4.6.9. *Suppose that $\mathbb{P}$ is a notion of class forcing which satisfies the forcing theorem. If $\mathbb{P}$ is nice and $\mathbb{Q}$ is obtained from $\mathbb{P}$ by adding the supremum of some subclass $A \in \mathcal{C}$ of $\mathbb{P}$, then $\mathbb{Q}$ is also nice.*

Proof. Let $\sigma \in M^{\mathbb{Q}}$ and $p \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma \subseteq \check{\gamma}$ for some $p \in \mathbb{Q}$ and $\gamma \in \mathrm{Ord}^M$. Let σ^+ denote the $\mathbb{P}$-name obtained from σ by replacing every ocurrence of $\sup A$ in $\mathrm{tc}(\sigma)$ by $\mathbb{1}_{\mathbb{P}}$, and let σ^- be defined recursively by $\sigma^- = \{\langle \tau^-, p\rangle \in \sigma \mid p \neq \sup A\}$. Let $q \leq_{\mathbb{Q}} p$. Without loss of generality, we can assume that $q \in \mathbb{P}$. If q is incompatible with every element of A, then $q \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \sigma^-$. But then there are $r \leq_{\mathbb{P}} q$ and a nice $\mathbb{P}$-name τ such that $r \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma^- = \tau$ and so $r \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \tau$. If there is some $a \in A$ such that q is compatible with a, let $r \leq_{\mathbb{P}} q, a$. Then $r \Vdash^{\mathbb{M}}_{\mathbb{Q}} \sigma = \sigma^+$ and so as in the previous case we can strengthen r to some s which witnesses that σ^+ has a nice $\mathbb{P}$-name. □

Corollary 4.6.10. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of* GB^- *such that $\mathcal{C}$ contains a set-like well-order of M. Every separative antisymmetric notion of class forcing which satisfies the forcing theorem but not the* Ord-*cc is dense in a notion of class forcing which is nice but not very nice.* □

Chapter 5

Second-order arithmetic, topological regularity and sharps

In this chapter, we will compare the consistency strength of extensions of second-order arithmetic (SOA) with (extensions of ZFC^-. In Section 5.1 we present the folklore result that second-order arithmetic and $\mathsf{ZFC}^- +$ "all sets are countable" are bi-interpretable. In Section 5.2 we will generalize the result mentioned in [KM07] that every model of SOA enhanced with the $\mathbf{\Pi^1_1}$-perfect set property contains inner models of ZFC. More precisely, we will prove that every model of $\mathsf{SOA} + \mathbf{\Pi^1_1}$-determinacy $+\mathbf{\Pi^1_2}$-perfect set property has an inner model of ZFC in which every set of ordinals has a sharp. Section 5.3 is dedicated to the converse of the above mentionend results. This will be achieved using class forcing. Finally, in Section 5.4, we show how to perform class forcing over models of SOA and prove that the $\mathbf{\Pi^1_1}$- PSP is preserved under class forcing for a large class of forcing notions, but is not preserved in general.

5.1 Second-order arithmetic and ZFC^-

In this section, we follow [Sim09]. Firstly, we show how we obtain a model of SOA from a model of ZFC^- in which all sets are countable.

Notation. By $\mathsf{V} = \mathsf{HC}$ we will denote the axiom that states that every set is *hereditarily countable*, i.e. every set is a countable set of hereditarily countable sets. Note that in the presence of countable choice, this is equivalent to stipulating that for every set x, $\mathrm{tc}(\{x\})$ is countable.

Given a model M of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$, we define

$$M^2 = \langle \omega^M, \mathcal{P}(\omega)^M, +^M, \cdot^M, =^M, <^M, \in^M \restriction (\omega^M \times \mathcal{P}(\omega)^M) \rangle.$$

In the case that the context is clear, we omit the superscripts. Furthermore, we will usually identify models of second-order arithmetic with their domain of natural and real numbers, i.e. we would simply write $M^2 = \langle \omega^M, \mathcal{P}(\omega)^M \rangle$.

Lemma 5.1.1. *If M is a model of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$, then M^2 is a model of SOA.*

Proof. It is clear that the axioms of Robinson arithmetic are satisfied. In order to show the induction axiom we proceed by contradiction. Assume that $x \subseteq \omega$ such that $0 \in x$ and for all $n \in x$, $n+1$ is also in x, and let $a \in \omega$ such that $a \notin x$. We define a sequence $\langle a_n \mid n \in \omega \rangle$ of natural numbers which are not in x as follows. Let $a_0 = a$. Given $a_n \notin x$, let $a_{n+1} = a_n - 1$. Note that this is possible, since $a_n \neq 0$. But then $a_{n+1} \in a_n$ for every $n \in \omega$ contradicting the axiom of foundation.

The collection axiom for $\varphi(n)$ is a consequence of Separation applied to the formula $n \in x \wedge \varphi(n)$. It remains to prove choice. For this consider a formula $\varphi(n, x)$ such that $\forall n \in \omega \exists x \subseteq \omega\, \varphi(n, x)$. By collection there is a set y satisfying

$$\forall n \in \omega \exists x \in y\, [x \subseteq \omega \wedge \varphi(n, x)].$$

Using the axiom of choice, we obtain a function f such that

$$\forall n \in \omega [f(n) \subseteq \omega \wedge \varphi(n, f(n))].$$

Separation then allows us to define a set $z = \{k \in \omega \mid \exists m \in f(n)\, [k = (m, n)]\}$ which is as desired. □

The converse is more involved. The idea is to use well-founded trees to code hereditarily countable sets. Let $\mathcal{A} = \langle \mathbb{N}, \mathbb{R} \rangle$ be a fixed model of second-order arithmetic. Using Gödel's β-function one can code finite sets – and in particular finite sequences – of natural numbers in $\mathbb{N}$ as single numbers. We denote by Seq the set of all finite sequences, and for $s \in \mathrm{Seq}$ we denote by $\mathrm{lh}(s)$ the *length* of s. Furthermore, for $i < \mathrm{lh}(s)$ we denote the i-th entry of s by $s(i)$. For sequences $s, t \in \mathrm{Seq}$ we write $s^\frown t$ for the concatenation of s and t and $s \subseteq t$ to state that s is an *initial segment* of t. The empty sequence is given by $\epsilon = \langle\, \rangle$. By a *tree* we denote a set $T \subseteq \mathrm{Seq}$ which is closed under taking initial segments.

Definition 5.1.2. Let $T \subseteq \mathrm{Seq}$ be a tree in $\mathbb{R}$.

(1) A function $f : \mathbb{N} \to \mathbb{N}$ is said to be a *path* through T, if for every $n \in \mathbb{N}$, $\langle F(0), \ldots, F(n-1) \rangle \in T$.
(2) We denote the class of paths through T by $[T]$.
(3) T is said to be *well-founded*, if $[T] = \emptyset$.
(4) Given $s \in T$, we define the *subtree of T at s* by $T_s = \{t \in \mathrm{Seq} \mid s^\frown t \in T\}$. If s is of the form $\langle n \rangle$, then T_s is said to be a *direct subtree* of T.

Furthermore, we will denote the class of well-founded trees by WF.

Remark 5.1.3. Note that a tree T is well-founded if and only if $\langle T, \supseteq \rangle$ is a well-founded relation. To see this, suppose that $\langle s_n \mid n \in \mathbb{N} \rangle$ is an infinite $\supset$-descending sequence in T. Then the sequence $\langle \mathrm{lh}(s_n) \mid n \in \mathbb{N} \rangle$ is strictly increasing and $n < \mathrm{lh}(s_{n+1})$. Then $F(n) = s_{n+1}(n)$ defines a path through T. The converse is obvious, since any path $F \in [T]$ leads to a counterexample of well-foundedness of $\supseteq$ given by $s_n = \langle F(0), \ldots, F(n-1) \rangle$.

Furthermore, in SOA it is possible to use transfinite recursion along a well-founded relation. Simpson [Sim09] gives a precise account on this topic. We apply transfinite recursion in order to use trees as codes for hereditarily countable sets which will then form the domain of the interpretation of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ within a model of SOA.

Each well-founded tree T encodes a hereditarily countable set, namely the set of sets encoded by its direct subtrees. We will define an equivalence relation $=^*$ on WF to identify trees which encode the same set. More precisely, if $S, T \in \mathrm{WF}$ we write $S =^* T$ to denote that

$$\exists X[\langle\epsilon,\epsilon\rangle \in X \wedge \forall n(n \in X \leftrightarrow \exists s \exists t(n = \langle s,t\rangle \wedge \varphi(s,t,S,T,X)))] \tag{5.1}$$

where $\varphi(s,t,S,T,X)$ is the formula

$$\begin{aligned} s \in S \wedge t \in T \wedge &\forall m[s^\frown\langle m\rangle \in S \to \exists n(t^\frown\langle n\rangle \in T \wedge \langle s^\frown\langle m\rangle, t^\frown\langle n\rangle\rangle \in X)] \wedge \\ &\forall n[t^\frown\langle n\rangle \in T \to \exists m(s^\frown\langle m\rangle \in S \wedge \langle s^\frown\langle m\rangle, t^\frown\langle n\rangle\rangle \in X)]. \end{aligned} \tag{5.2}$$

We call a real X witnessing (5.1) an *equivalence code* for S and T.

The intuition behind this definition is that the real X encodes all pairs $\langle s,t\rangle$ such that $S_s =^* T_t$, i.e. X collects the sequences $\langle s,t\rangle$ such that the set encoded by S_s is equal to the set encoded by T_t. The obvious observation that we can make is the following.

Lemma 5.1.4. *The class relation $=^*$ defines an equivalence relation on the class of well-founded trees.*

Proof. Reflexivity is clearly a consequence of arithmetical transfinite recursion applied to the well-founded relation $\langle s,t\rangle \prec \langle s',t'\rangle$ iff $s \supset s' \wedge t \supset t'$ on $T \times T$ for a well-founded tree T. In order to show symmetry, assume that $S =^* T$ is witnessed by the equivalence code X. Then $\{\langle s,t\rangle \mid \langle t,s\rangle \in X\}$ witnesses that $T =^* S$. For transitivity, assume that $S =^* T$ and $T =^* U$ are witnessed by reals X resp. Y. Then

$$Z = \{\langle s,u\rangle \mid \exists t \in T(\langle s,t\rangle \in X \wedge \langle t,u\rangle \in Y)\}$$

witnesses the equivalence of S and U. □

Furthermore, we interpret elementhood as follows.

$$S \in^* T \text{ iff } \exists n\,[\langle n\rangle \in T \wedge S =^* T_{\langle n\rangle}].$$

This means that the elements of a well-founded tree are given by its direct subtrees; or, in terms of hereditarily countable sets, the elements are exactly the sets coded by its direct subtrees. For a tree $T \in \mathrm{WF}$ let

$$[T]_* = \{S \mid S \in \mathcal{T} \wedge S =^* T\}$$

be the equivalence class of T with respect to $=^*$. Now for a model $\mathcal{A}$ of SOA there is a definable $\mathcal{L}_\in$-structure $\mathcal{A}^\in = \langle A^\in, \in\rangle$, with domain given by

$$A^\in = \{[T]_* \mid T \in \mathrm{WF}^{\mathcal{A}}\}$$

and where for any $S, T \in \mathrm{WF}^{\mathcal{A}}$, $[S]_* = [T]_*$ if and only if $S =^* T$ and $[S]_* \in [T]_*$ if and only if $S \in^* T$. Note that $\mathcal{A}^\in$ is definable in $\mathcal{A}$. Now we can finally prove the converse of Lemma 5.1.1.

Theorem 5.1.5. *Let $\mathcal{A} = \langle \mathbb{N}, \mathbb{R} \rangle \models \mathsf{SOA}$. Then $\mathcal{A}^\in \models \mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$.*

Proof. We check all axioms of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ in $\mathcal{A}^\in$. For extensionality, fix $S, T \in \mathrm{WF}^\mathcal{A}$ such that for all $U \in \mathrm{WF}^\mathcal{A}$, $U \in^* S$ iff $U \in^* T$ which is, by transitivity of $=^*$, equivalent to

$$\forall n\, [\langle n \rangle \in S \to \exists m(\langle m \rangle \in T \wedge S_{\langle n \rangle} =^* T_{\langle m \rangle})] \wedge$$
$$\forall m\, [\langle m \rangle \in T \to \exists n(\langle n \rangle \in S \wedge S_{\langle n \rangle} =^* T_{\langle m \rangle})].$$

Using choice, we can choose equivalence codes $X_{n,m}$ such that if $\langle n \rangle \in S, \langle m \rangle \in T$ and $S_{\langle n \rangle} =^* T_{\langle m \rangle}$, then this is witnessed by $X_{n,m}$, and otherwise $X_{n,m} = \emptyset$. Then

$$X = \{\langle \epsilon, \epsilon \rangle\} \cup \{\langle \langle n \rangle ^\frown s, \langle m \rangle ^\frown t \rangle \mid s, t \in X_{n,m}\}$$

is an equivalence code witnessing $S =^* T$.

In order to prove pairing, consider two well-founded trees S, T. Then the tree given by

$$U = \{\epsilon\} \cup \{\langle 0 \rangle ^\frown s \mid s \in S\} \cup \{\langle 1 \rangle ^\frown t \mid t \in T\}$$

has exactly S and T as direct subtrees, so it encodes the pair of the set encoded by S and the set encoded by T.

For the union axiom, let $S \in \mathrm{WF}^\mathcal{A}$. Then the direct subtrees of the tree

$$T = \{\epsilon\} \cup \{\langle (n, m) \rangle ^\frown s \mid \langle n, m \rangle ^\frown s \in S\}.$$

correspond to subtrees of subtrees of S, so T encodes the union of the set coded by S.

In order to prove the axiom of infinity, we construct the natural numbers in $\mathcal{A}^\in$ as follows.

$$0^* = \{\epsilon\},$$
$$(n+1)^* = n^* \cup \{\langle n \rangle ^\frown s \mid s \in n^*\} \text{ for } n > 0.$$

Then the set of natural numbers is interpreted as

$$\mathbb{N}^* = \{\epsilon\} \cup \{\langle n \rangle ^\frown s \mid n \in \mathbb{N} \wedge s \in n^*\}.$$

This tree has clearly infinitely many non-equivalent subtrees.

Next we proof Collection. For the sake of simplicity, we prove the parameter-free version. Let $S \in \mathcal{A}^\in$ and $\varphi(v_0, v_1)$ an $\mathcal{L}_\in$-formula such that

$$\mathcal{A}^\in \models \forall x \in S\, \exists y\, \varphi(x, y).$$

This means that for every n such that $\langle n \rangle \in S$, there is a well-founded non-empty tree T such that $\mathcal{A} \models \varphi^*(S_{\langle n \rangle}, T)$. The axiom of choice in SOA then yields a real T such that for every n such that $\langle n \rangle \in S$, $\mathcal{A} \models \varphi^*(S_{\langle n \rangle}, (T)_n)$. Then putting

$$U = \{\epsilon\} \cup \{\langle n \rangle ^\frown s \mid s \in (T)_n\}$$

we get $\mathcal{A}^\in \models \forall x \in S \exists y \in U \varphi(x, y)$. Note that Separation follows from Collection.

Now we turn to the axiom of foundation. Suppose for a contradiction that $S \in \mathrm{WF}^{\mathcal{A}}$ and $n \in \mathbb{N}$ such that $S \neq \{\epsilon\}$ and

$$\mathcal{A}^{\in} \models \forall x \in S \exists y \, [y \in S \wedge y \in x].$$

Since $S \neq \{\epsilon\}$, there exists $n_0 \in \mathbb{N}$ such that $\langle n_0 \rangle \in S$. Now we construct inductively a sequence $\langle n_k \mid k \in \mathbb{N} \rangle$ such that $\langle n_0, \ldots, n_k \rangle \in S$ and $S_{\langle n_0, \ldots, n_k \rangle} \in^* S$. Having found such a sequence, the function $k \mapsto n_k$ yields a path through S contradicting well-foundedness. Given n_k, define n_{k+1} as follows. Since $S_{\langle n_0, \ldots, n_k \rangle} \in^* S$, there exists $T \in \mathrm{WF}^{\mathcal{A}}$ such that $T \in^* S$ and $T \in^* S_{\langle n_0, \ldots, n_k \rangle}$. But then there is $n_{k+1} \in \mathbb{N}$ such that $\langle n_{k+1} \rangle \in S_{\langle n_0, \ldots, n_k \rangle}$ and $T =^* (S_{\langle n_0, \ldots, n_k \rangle})_{\langle n_{k+1} \rangle} = S_{\langle n_0, \ldots, n_k, n_{k+1} \rangle}$. This shows that $S_{\langle n_0, \ldots, n_k, n_{k+1} \rangle} \in^* S$.

Next we show that every element of $\mathrm{WF}^{\mathcal{A}}$ can be well-ordered. Let $T \in \mathrm{WF}^{\mathcal{A}}$ be an arbitrary well-founded tree. Without loss of generality, we may assume that $T \neq \{\epsilon\}$. For every $S \in^* T$ put

$$n_S = \min\{n \in \mathbb{N} \mid S =^* T_{\langle n \rangle}\}.$$

This allows us to define a well-ordering $\lhd$ of T by stipulating

$$R \lhd S \text{ iff } n_R < n_S$$

for $R, S \in^* T$.

It remains to check that every set in $\mathcal{A}^{\in}$ is hereditarily countable. The transitive closure of a set can be shown to exist without the use of the power set axiom. So it suffices to show that every $T \in \mathrm{WF}^{\mathcal{A}}$ is countable. We need to construct a surjection from $\mathbb{N}^*$ onto T. For this, we can define n_S for $S \in^* T$ just as in the proof of the well-ordering principle. Then by the axioms we have already verified, there exists $P_S \in \mathrm{WF}^{\mathcal{A}}$ encoding the ordered pair $\langle n_S^*, T_{\langle n_S \rangle} \rangle$ for $S \in^* T$. Then the tree

$$F = \{\epsilon\} \cup \{\langle n_S \rangle ^\frown s \mid S \in^* T \wedge s \in P_S\}$$

defines a bijection from a subset of $\mathbb{N}^*$ onto T. □

Now working in $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ we can also introduce the notion of well-founded trees T. Consider the relation of immediate successor, i.e.

$$r_T = \{\langle t ^\frown \langle n \rangle, t \rangle \mid t ^\frown \langle n \rangle \in T\}.$$

Since T is a well-founded tree, we can collapse it recursively in the followsing way.

$$c_T(t) = \{c_T(t ^\frown \langle n \rangle) \mid t ^\frown \langle n \rangle \in T\}$$
$$|T| = c_T(\langle \, \rangle).$$

We can view this as coding the hereditarily countable set $|T|$ by a well-founded tree T. This is especially useful since every hereditarily countable set has such a tree code.

Lemma 5.1.6. *Suppose that $M \models \mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$. Then for any set $a \in M$ there exists a well-founded tree T such that $|T| = a$.*

Proof. Let $a \in M$ be an arbitary set. Then by $\mathsf{V} = \mathsf{HC}$ there is a bijection $f : \omega \to \mathrm{tc}(\{a\})$. Consider the tree given by $t \in T$ if and only if

$$t = \epsilon \vee \exists n \exists s[t = \langle n \rangle ^\frown s \wedge f(n) \in a \wedge \forall i < \mathrm{lh}(s)(f(t(i+1)) \in f(t(i)))].$$

It is easy to check that $|T| = a$ as desired. ☐

Lemma 5.1.7 ($\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$)**.** *If S and T are non-empty well-founded trees, then we have*

(1) *$S =^* T$ if and only if $|S| = |T|$*
(2) *$S \in^* T$ if and only if $|S| \in |T|$.*

Proof. Let S and T be non-empty well-founded trees. For (1), assume first that $S =^* T$ and take an equivalence code X for S and T. Then one can check inductively that for any $\langle s, t \rangle \in X$, $c_S(s) = c_T(t)$. In particular, since $\langle \epsilon, \epsilon \rangle \in X$, this shows that $|S| = c_S(\epsilon) = c_T(\epsilon) = |T|$. Vice versa, suppose that $|S| = |T|$. Then the set

$$\{\langle s, t \rangle \mid s \in S \wedge t \in T \wedge |S_s| = |T_t|\} = \{\langle s, t \rangle \mid s \in S \wedge t \in T \wedge c_S(s) = c_T(t)\}$$

clearly witnesses that $S =^* T$.

We turn to the second claim. Assume that $S \in^* T$. Then there exists $n \in \omega$ such that $\langle n \rangle \in T$ and $S =^* T_{\langle n \rangle}$. By condition (1) this implies that $|S| = |T_{\langle n \rangle}|$. Moreover, one can check by induction that for any $s \in T$ and $t \in T_s$, $c_{T_s}(t) = c_T(s ^\frown t)$. Consequently, we obtain that $|S| = |T_{\langle n \rangle}| = c_{T_{\langle n \rangle}}(\epsilon) = c_T(\langle n \rangle) \in |T|$. The converse is similar. ☐

Now we would like to generalize Lemma 5.1.7. We define an assignment $\varphi \mapsto \varphi^*$ of $\mathcal{L}_\in$-formulae to $\mathcal{L}_2$-formulae where $=$ and $\in$ are translated to $=^*$ and $\in^*$ and quantifiers $(\exists x)$ resp. $(\forall x)$ are interpreted as $(\exists T \in \mathrm{WF})$ resp. $(\forall T \in \mathrm{WF})$.

Lemma 5.1.8 ($\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$)**.** *For any $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ and for any non-empty well-founded trees $T_0, \ldots, T_{n-1}$,*

$$\varphi^*(T_0, \ldots, T_{n-1}) \text{ iff } \varphi(|T_0|, \ldots, |T_{n-1}|).$$

Proof. This is an easy induction on the complexity of φ. If φ is an atomic formula, then we can apply Lemma 5.1.7. For existential resp. universal formulae, it is a consequence of Lemma 5.1.6. ☐

Remark 5.1.9. Lemma 5.1.8 states that if M is a model of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$, then $\varphi^*(T_0, \ldots, T_{n-1})$ formalizes $(M^2)^\in \models \varphi(T_0, \ldots, T_{n_1})$.

For the converse, we code (in SOA) natural numbers and reals as trees in the following way:

$$\begin{aligned} 0^* &= \{\epsilon\}, \\ (n+1)^* &= n^* \cup \{\langle n \rangle ^\frown s \mid s \in n^*\} \text{ for } n \in \mathbb{N}, \\ X^* &= \{\epsilon\} \cup \{\langle n \rangle ^\frown s \mid n \in X \wedge s \in n^*\} \text{ for } X \in \mathbb{R}. \end{aligned}$$

Clearly, every n^* and X^* is a – in the case of n^*, finite – non-empty well-founded tree.

Lemma 5.1.10 ($\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$)**.** *For $n \in \omega$ and $x \subseteq \omega$ we can define n^* resp. x^* as above. Then we have $|n^*| = n$ and $|x^*| = x$. Moreover, this means that $x = y$ if and only if $x^* =^* y^*$ and $n \in x$ if and only if $n^* \in^* x^*$.*

Proof. In order to check $|n^*| = n$ we proceed by induction. If the claim holds for $k < n$ then

$$|n^*| = \{|n^*_{\langle k \rangle}| \mid k < n\} = \{|k^*| \mid k < n\} = n.$$

Secondly, note that

$$|x^*| = \{|x^*_{\langle n \rangle}| \mid n \in x\} = \{|n^*| \mid n \in x\} = x.$$

The last claim is a direct consequence of the above in combination with Lemma 5.1.7. □

Since every $\mathcal{L}_2$-formula φ has a canonical set theoretic interpretation $\bar{\varphi}$ (which we identify with φ), there is an assignment $\varphi \mapsto \varphi^*$ from $\mathcal{L}_2$-formulas to $\mathcal{L}_2$-formulas (where φ^* is identified with $\bar{\varphi}^*$).

Lemma 5.1.11 (SOA)**.** *If $\varphi(v_0, \dots, v_{k-1}, V_0, \dots, V_{l-1})$ is an $\mathcal{L}_2$-formula, then for any $n_0, \dots, n_{k-1}$ and $X_0, \dots, X_{l-1}$,*

$$\varphi(n_0, \dots, n_{k-1}, X_0, \dots, X_{l-1}) \text{ iff } \varphi^*(n^*_0, \dots, n^*_{k-1}, X^*_0, \dots, X^*_{l-1}).$$

Remark 5.1.12. For a model $\mathcal{A}$ of SOA, the formula $\varphi^*(n^*_0, \dots, n^*_{k-1}, X^*_0, \dots, X^*_{l-1})$ is a formalization of the formula $(\mathcal{A}^{\in})^2 \models \varphi(n^*_0, \dots, n^*_{k-1}, X^*_0, \dots, X^*_{l-1})$.

Proof of Lemma 5.1.11. We proceed by induction on the complexity of φ. For atomic formulae, we have to check first that the assignment $\varphi \mapsto \varphi^*$ preserves the non-logical symbols $+, *$ and $<$. Since $+$ etc. are definable in $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$, this will be transferred to the $*$-translation of $\mathcal{L}_{\in}$ into $\mathcal{L}_2$ in the sense that a defining formula φ_+ of $+$ in $\mathcal{L}_{\in}$ will be a defining formula for a function $+^*$. In particular, the resulting function will be canonical, i.e. $n^* +^* m^* = (n+m)^*$ for all $n, m \in \mathbb{N}$. The argument for $*$ and $<$ is similar. The atomic case for formulae of the form $v \in V$ follows directly from the definition.

For formulae of the form $\neg\varphi$ and $\varphi \wedge \psi$ the statement is an easy consequence of the induction hypothesis. For existential formulae observe that $\bar{\varphi}$ relativizes $(\exists n)$ to $(\exists n \in \omega)$ and $(\exists X)$ to $(\exists X \subseteq \omega)$. Now since ω is translated to $\mathbb{N}^*$, all quantifiers appearing in φ^* will again be relativized by $\mathbb{N}^*$. Suppose that φ is of the form $\exists v \psi(v, v_0, \dots, v_{k-1}, V_0, \dots, V_{l-1})$ and assume that $\varphi^*(n^*_0, \dots, n^*_{k-1}, X^*_0, \dots, X^*_{l-1})$ is satisfied. Thus there exists $T \in^* \mathbb{N}^*$ such that $\psi(T, n^*_0, \dots, n^*_{k-1}, X^*_0, \dots, X^*_{l-1})$. But then T is of the form $\mathbb{N}^*_{\langle n \rangle} = n^*$ for some n. Hence by induction hypothesis we get $\psi(n, n_0, \dots, n_{k-1}, X_0, \dots, X_{l-1})$ and thus also $\varphi(n_0, \dots, n_{k-1}, X_0, \dots, X_{l-1})$. In the case of set quantification we check that $T \subseteq^* \mathbb{N}^*$ corresponds to a set of natural numbers, where $\subseteq^*$ is defined in the canonical way. Since $T \subseteq^* \mathbb{N}^*$ stands for $\forall S \in \mathrm{WF}\, [S \in^* T \rightarrow S \in^* \mathbb{N}^*]$ which implies that for every $S \in \mathrm{WF}$ there is n such that $S = \mathbb{N}^*_{\langle n \rangle} = n^*$. This means that $T = X^*$, where

$$X = \{n \mid n^* \in^* T\}.$$

This concludes the proof. □

Finally, we show that SOA and ZFC^- are bi-interpretable und hence prove the same theorems. Moreover, $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ is a conservative extension of SOA.

Theorem 5.1.13. *If $\mathcal{A}$ is a model of* SOA*, then $(\mathcal{A}^\in)^2$ is isomorphic to $\mathcal{A}$. Conversely, for every model $M \models \mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$, the tree model $(M^2)^\in$ is isomorphic to the original model M.*

Proof. The first assertion is a consequence of Lemma 5.1.11. For the converse, note that Lemma 5.1.6 allows us to represent every set $a \in M$ as a tree T with $|T| = a$. Concerning uniqueness, note that we have defined the tree model $(M^2)^\in$ as a class of equivalence classes of trees and Lemma 5.1.7 states that the assignment $a \mapsto [T]_*$ is well-defined. The claim then follows from Lemma 5.1.8. □

Theorem 5.1.14. *The following statements hold.*

(1) *For every $\mathcal{L}_\in$-sentence φ, $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC} \vdash \varphi$ if and only if $\mathsf{SOA} \vdash \varphi^*$.*
(2) *For any $\mathcal{L}_2$-sentence φ, we have $\mathsf{SOA} \vdash \varphi$ if and only if $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC} \vdash \bar{\varphi}$, i.e. by identifying $\bar{\varphi}$ with φ this means that $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ is a conservative extension of* SOA*.*

Proof. For the first part, suppose that φ is an $\mathcal{L}_\in$-sentence such that $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC} \vdash \varphi$ and let $\mathcal{A} \models \mathsf{SOA}$. Then $\mathcal{A}^\in \models \varphi$. But this means exactly that $\mathcal{A} \models \varphi^*$. Conversely, suppose that $\mathsf{SOA} \vdash \varphi^*$ and let M be a model of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$. Therefore we have $M^2 \models \varphi^*$. But then we obtain that $(M^2)^\in \models \varphi$ and since $(M^2)^\in \cong M$ we can conclude that $M \models \varphi$.

We turn to the second statement. The first implication is obvious. For the converse, let φ be an $\mathcal{L}_2$-sentence such that $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC} \vdash \bar{\varphi}$ and let $\mathcal{A} \models \mathsf{SOA}$. Using (1) we obtain that $\mathcal{A} \models \bar{\varphi}^*$ which means by identifying $\bar{\varphi}$ with φ^* that $\mathcal{A} \models \varphi^*$. Then Lemma 5.1.11 implies that $\mathcal{A} \models \varphi$ as desired. □

Theorem 5.1.14 implies that to show that some $\mathcal{L}_2$-sentence is a logical consequence of SOA, we can either argue in SOA or in $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$. From now on, we will therefore switch freely between the two axiom systems.

5.2 Inner models of ZFC and $\mathsf{ZFC}^\#$ in models of SOA

In the following, we will show how to obtain inner models of ZFC and ZFC+"every set of ordinals has a sharp" within models of second-order arithmetic with additional topological regularity and determinacy hypotheses.

5.2.1 Inner models of ZFC

In this section we prove that every model of SOA which satisfies the $\mathbf{\Pi}^1_1$-perfect set property has an inner model of ZFC, notably Gödel's constructible universe. We follow the exposition presented in [KM07] and [War05]. By Theorem 5.1.14, we may work in $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ instead of SOA.

Definition 5.2.1. Let p be a tree in ${}^{<\omega}\omega$, and $s, t \in p$ two nodes.

- s is said to be an *extension* of t, if $t \subseteq s$.
- s and t are said to be *incompatible*, if $s \nsubseteq t$ and $t \nsubseteq s$.

The tree p is called

- *pruned*, if for every $s \in p$ there is a natural number n such that $s ^\frown \langle n \rangle \in p$.
- *perfect*, if every element of p has at least two incompatible extensions in p.

Recall that the topology on the Baire space ${}^\omega\omega$ is defined by taking the closed classes to be of the form $[p]$ for some tree p. Furthermore, a class of reals is said to be *perfect*, if it is of the form $[p]$ for some perfect tree p.

Definition 5.2.2. For a class Γ of $\mathcal{L}_2$-formulae, we define the Γ-*perfect set property* (denoted Γ-PSP) to be the scheme postulating that every uncountable class of reals definable by a formula in Γ has a perfect subclass.

The following standard results carry over to SOA (resp. $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$).

Theorem 5.2.3. *The $\boldsymbol{\Sigma^1_1}$-perfect set property holds in every model of* SOA *resp.* $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$.

Proof. The idea is that trees are thinned out countably many times by removing at each step the isolated points. For a detailed proof, consult [Sim09, Theorem V.4.3]. □

Theorem 5.2.4 (Kondo-Addison)**.** *Let $\varphi(x, \vec{y})$ be a Π^1_1-formula with parameters $\vec{y}$. Then there is a Π^1_1-formula $\psi(x, \vec{y})$ such that the following statements hold.*

(1) $\forall x \forall \vec{y}\, [\psi(x, \vec{y}) \to \varphi(x, \vec{y})]$
(2) $\forall \vec{y}\, [\exists x \varphi(x, \vec{y}) \to \exists x \psi(x, \vec{y})]$
(3) $\forall x \forall x' \forall \vec{y}\, [\psi(x, \vec{y}) \wedge \psi(x', \vec{y}) \to x = x']$.

Proof. The proof be found in [Sim09, Theorem VI.2.6], . □

Lemma 5.2.5. *For every real $x \in {}^\omega\omega$, $\Pi^1_1[x]$-PSP implies $\Sigma^1_2[x]$-PSP.*

Proof. Let $x \in {}^\omega\omega$ and A a $\Sigma^1_2[x]$-class of reals given by the defining formula $\varphi(x, y)$. Since φ is a $\Sigma^1_2[x]$-formula, there is a $\Pi^1_1[x]$-formula $\bar{\varphi}(x, y, z)$ such that $\varphi(x, y)$ is of the form $\exists z \bar{\varphi}(x, y, z)$. Now we can apply Theorem 5.2.4 to $\bar{\varphi}$ in order to find a $\Pi^1_1[x]$-formula $\psi(x, y, z)$ which satisfies properties (1)-(3) as above. Clearly, the first two properties can be summarized as $\forall y\, [\varphi(x, y) \leftrightarrow \exists z \psi(x, y, z)]$. Consider the $\Pi^1_1[x]$-formla $\bar{\psi}(x, y)$ given by $\psi(x, (y)_0, (y)_1)$. Since we assume $\Pi^1_1[x]$-PSP, the class B given by $y \in B$ iff $\bar{\psi}(x, y)$ is either countable or contains a perfect subclass. Clearly, if B is countable then so is A. So suppose that p is a perfect tree such that $[p] \subseteq B$. Let P be the projection of $[p]$ onto the first coordinate. Then P is an uncountable $\boldsymbol{\Sigma^1_1}$-class of reals and consequently, Theorem 5.2.3 implies that it contains a perfect subclass. Since $P \subseteq A$, this concludes the proof of Lemma 5.2.5. □

Now for any real x we can construct the universe $\mathsf{L}[x]$ of constructible sets relative to x in the usual way, since this does not require the power set axiom. It is easy to check that $\mathsf{L}[x]$ is a model of ZFC^-. We will show that using $\Pi^1_1[x]$-PSP one can even prove that the power set axiom holds in $\mathsf{L}[x]$, thus obtaining a model of ZFC.

Definition 5.2.6. Let WO denote the class of well-orders.

(1) Given $x, y \in \mathrm{WO}$, we write $x \leq_{\mathrm{WO}} y$ to denote that there is an order-preserving map from the domain of x to the domain of y. Furthermore, we write $x =_{\mathrm{WO}} y$ if $x \leq_{\mathrm{WO}} y$ and $y \leq_{\mathrm{WO}} x$.
(2) We say that a class $A \subseteq \mathrm{WO}$ is *ordertype bounded*, if there is $x \in \mathrm{WO}$ such that for every $y \in A$, $y \leq_{\mathrm{WO}} x$.

The following classical result carries over to SOA and hence also to $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$.

Theorem 5.2.7. *[Sim09, V.6.2] Every $\boldsymbol{\Sigma}^1_1$-subclass of* WO *is ordertype bounded.* □

Given a real $x \in {}^\omega\omega$, the class of x-constructible reals is a $\Sigma^1_2[x]$-class of reals which has a $\Sigma^1_2[x]$-well-ordering. Now consider the class of reals given by

$$y \in C_x \iff y \in \mathsf{L}[x] \wedge y \in \mathrm{WO} \wedge \forall z\, (z \in \mathrm{WO} \wedge z \prec_x y \to z \neq_{\mathrm{WO}} y).$$

Lemma 5.2.8. *For any real x, C_x is a $\Sigma^1_2[x]$-class of reals.*

Proof. The predicates $y \in \mathsf{L}[x]$ and $y \in \mathrm{WO}$ are $\Sigma^1_2[x]$ resp. Π^1_1, so it suffices to consider the last part which we can restate as

$$\exists z[\langle z, y\rangle \in A \wedge \forall n \in \omega\, [(z)_n \in \mathrm{WO} \to (z)_n \neq_{\mathrm{WO}} y)],$$

where A is the class of $\langle z, y\rangle$ such that $\{(z)_n \mid n \in \omega\} = \{w \in {}^\omega\omega \mid w \prec_x y\}$. By [Add63], it follows that A is $\Sigma^1_2[x]$ and hence so is C_x. □

Note that in $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ we can code ordinals by reals in WO and vice versa as follows.

(1) Given an ordinal α, by $\mathsf{V} = \mathsf{HC}$ we can find a bijection $f_\alpha : \omega \to \alpha$ and define a well-ordering on ω by
$$n <_\alpha m \iff f(n) < f(m).$$
(2) Conversely, consider a real $y \in \mathrm{WO}$. Then define recursively
$$r_y(n) = \sup(\{r_y(m) + 1 \mid m <_y n\})$$
and
$$\|y\| = \sup(\{r_y(n) \mid n \in \omega\}).$$
Clearly, $\|y\|$ is a countable ordinal.

Lemma 5.2.9. *Let $x \in {}^\omega\omega$ be a real. Then the following statements hold.*

(1) *C_x is uncountable in $\mathsf{L}[x]$.*
(2) *C_x contains no perfect subclass.*

Proof. We show first that $\mathrm{WO} \cap \mathsf{L}[x]$ is unbounded in C_x. It suffices to show that for every $\alpha \in \mathrm{Ord}$ which is countable in $\mathsf{L}[x]$ there is $y \in C_x$ such that $\|y\| = \alpha$. As observed above, we can code each $\alpha \in \mathrm{Ord}$ which is countable in $\mathsf{L}[x]$ by a real $y_\alpha \in \mathrm{WO} \cap \mathsf{L}[x]$ such that $\|y_\alpha\| = \alpha$. Moreover, there is $\beta(\alpha) \in \mathrm{Ord}$ such that $y_\alpha \in \mathsf{L}_{\beta(\alpha)}[x]$. By choosing

y_α to be the least such real with respect to the well-ordering of $\mathsf{L}_{\beta(\alpha)}[x]$ we obtain that $y_\alpha \in C_x$ and $\|y_\alpha\| = \alpha$. Now if C_x was countable, then for $\alpha = \sup\{\|y\| \mid y \in C_x\}$, the real y_α would be a $\leq_{\mathrm{WO}}$-upper bound for C_x, contradicting our previous argument. This proves (1).

For the second statement, assume that $P \subseteq C_x$ is a perfect subclass. In particular, P is closed and hence $\mathbf{\Sigma}^1_1$. By Theorem 5.2.7 we know that P is ordertype bounded, so we can find $y \in \mathrm{WO}$ such that for every $z \in P$ we have $z \prec_{\mathrm{WO}} y$. But then P has ordertype at most $\|y\|$ and is therefore countable, a contradiction. □

Theorem 5.2.10. *For every real $x \in {}^\omega\omega$, the following statements are equivalent.*

(1) $\Pi^1_1[x]$-PSP.
(2) $\Sigma^1_2[x]$-PSP.
(3) $\aleph_1^{\mathsf{L}[x]}$ *exists.*
(4) ${}^\omega\omega \cap \mathsf{L}[x]$ *is countable.*

Proof. The implication from (1) to (2) is exactly the statement of Lemma 5.2.5. Now suppose that (2) holds. We prove that $\aleph_1^{\mathsf{L}[x]}$ exists. Due to Lemma 5.2.9 (2) and Lemma 5.2.8, the class C_x must be countable. In particular, there exists $y \in \mathrm{WO}$ such that for all $z \in C_x$, $z \prec_{\mathrm{WO}} y$. However, it follows Lemma 5.2.9 (1) that $\mathrm{WO} \cap \mathsf{L}[x]$ is unbounded in C_x and so $\|y\|$ is uncountable in $\mathsf{L}[x]$. In particular, there must exist a least uncountable ordinal in $\mathsf{L}[x]$ and so $\aleph_1^{\mathsf{L}[x]}$ exists.

To see that (3) implies (4), suppose that $\aleph_1^{\mathsf{L}[x]}$ exists. We prove that $\mathcal{P}(\omega) \cap \mathsf{L}[x] \subseteq \mathsf{L}_{\omega_1^{\mathsf{L}[x]}}[x]$. Since $\aleph_1^{\mathsf{L}[x]}$ is countable, this proves that there are only countably x-constructible reals. Suppose that $\alpha \in \mathrm{Ord}$ and $y \in \mathsf{L}_{\alpha+1}[x] \setminus \mathsf{L}_\alpha[x]$ is a real. We need to prove that α is countable in $\mathsf{L}[x]$. So write $y = \{n \in \omega \mid \varphi(n, x, p)\}$, where p is a parameter in $\mathsf{L}_\alpha[x]$. Now p is definable from a parameter $q \in \mathsf{L}_\beta[x]$ for some $\beta < \alpha$ by a formula ψ. Let n be the maximal number such that φ and ψ are Π_n-formulae. Let $H = H^{\mathsf{L}_\alpha[x]}(\{q, r\}) \prec_{\Sigma_{n+1}} \mathsf{L}_\alpha[x]$ be the Σ_{n+1}-Skolem hull of $\{q, r\}$ in $\mathsf{L}_\alpha[x]$, where r is a parameter appearing in the definition of q. Then by Gödel condensation there is $\bar\alpha \leq \alpha$ such that $H = \mathsf{L}_{\bar\alpha}[x]$. If $\bar\alpha < \alpha$, then $p = h_\psi(q) \in \mathsf{L}_{\bar\alpha}[x]$ and so $y \in \mathsf{L}_{\bar\alpha+1}[x] \subseteq \mathsf{L}_\alpha[x]$ contradicting our assumption. This implies that $\bar\alpha = \alpha$ and hence $\mathsf{L}_\alpha[x]$ is countable. In particular, α is countable and so $\mathcal{P}(\omega)^{\mathsf{L}[x]}$ exists in $\mathsf{L}_{\omega_1}[x]$.

In order to prove that (4) implies (1), the canonical approach would be to apply Mansfield's Theorem[1]. However, its proof relies on the representation of $\Sigma^1_2[x]$-classes of reals as projections of trees on $\omega_1 \times \omega$ and on an iterative process of thinning out such trees, and so it does not carry over to $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$. Instead, one can prove (4) by forcing with $\mathrm{Col}(\omega, \omega_1^{\mathsf{L}[x]})$. For a detailed proof, consult [War05, Theorem 4.11]. □

Theorem 5.2.11. *Assume $\mathbf{\Pi}^1_1$-PSP. Then for every $x \in {}^\omega\omega$, $\mathsf{L}[x] \models \mathsf{ZFC}$.*

Proof. It only remains to show that the power set axiom holds in $\mathsf{L}[x]$. By Theorem 5.2.10, $\mathcal{P}(\omega)^{\mathsf{L}[x]}$ exists in $\mathsf{L}[x]$. It suffices to prove that for every ordinal $\alpha \geq \omega$ the power set of $\mathsf{L}_\alpha[x]$ exists in $\mathsf{L}[x]$. Using $\mathsf{V} = \mathsf{HC}$ we can find a real y such that $\|y\| = \alpha$. In

[1] See [Kan09, Theorem 14.7]

$\mathsf{L}[x,y]$, α is countable and thus so is $\mathsf{L}_\alpha[x]$. Pick a bijection $f:\omega\to(\mathsf{L}_\alpha[x])^{\mathsf{L}_\alpha[x,y]}=\mathsf{L}_\alpha[x]$ in $\mathsf{L}[x,y]$. By the previous argument, $\mathcal{P}(\omega)^{\mathsf{L}[x,y]}$ exists. We define a function

$$g:\mathcal{P}(\omega)^{\mathsf{L}[x,y]}\to\mathsf{L}[x,y], x\mapsto f''x.$$

Put $a=g''(\mathcal{P}(\omega)^{\mathsf{L}[x,y]})$ and $a^{\mathsf{L}[x]}=a\cap\mathsf{L}[x]$. Now for every $s\in a^{\mathsf{L}[x]}$ there exists $\beta(s)\in\mathrm{Ord}$ minimal such that $s\in\mathsf{L}_{\beta(s)}[x]$. Define $\gamma=\sup\{\beta(s)\mid s\in a^{\mathsf{L}[x]}\}$. We claim that $\mathcal{P}(\mathsf{L}_\alpha[x])^{\mathsf{L}[x]}$ is in $\mathsf{L}_\gamma[x]$. Let $s\in\mathsf{L}[x]$ such that $s\subseteq\mathsf{L}_\alpha[x]$. In $\mathsf{L}[x,y]$, $s=g((f^{-1})''s)$, so $s\in a^{\mathsf{L}[x]}$ and thus by construction $a\in\mathsf{L}_{\beta(s)}[x]\subseteq\mathsf{L}_\gamma[x]$. This implies that we can define the power set of $\mathsf{L}_\alpha[x]$ in $\mathsf{L}[x]$ as

$$\mathcal{P}(\mathsf{L}_\alpha[x])^{\mathsf{L}[x]}=\{s\in\mathsf{L}_\gamma[x]\mid s\subseteq\mathsf{L}_\alpha[x]\}$$

concluding the proof of Theorem 5.2.11. □

5.2.2 Inner models of $\mathsf{ZFC}^\#$

In the following, we will generalize the proof from the previous section in order to obtain inner models of ZFC+"every set of ordinals has a sharp" in models of second-order arithmetic plus $\mathbf{\Pi^1_1}$-determinacy and $\mathbf{\Pi^1_2}$-perfect set property.

Firstly, we briefly recall the Ehrenfeucht-Mostowski definition of sharps. We work in $\mathsf{ZFC}^-+\mathsf{V}=\mathsf{HC}$. Let $a\subseteq\mathrm{Ord}$ be a set of ordinals and let $\alpha=\sup(a)$. By $\mathsf{V}=\mathsf{HC}$, α is countable. We expand the language $\mathcal{L}_\in$ by adding constants for a and for every ordinal $\gamma\leq\alpha$. We denote the corresponding language by $\mathcal{L}_a$. Our goal is to obtain a closed unbounded class of indiscernibles for the structure $\langle\mathsf{L}[a],a,\zeta\rangle_{\zeta\leq\alpha}$. As a convention, we will always identify the structure $\langle\mathsf{L}[a],a,\zeta\rangle_{\zeta\leq\alpha}$ with its domain $\mathsf{L}[a]$.

For any given language $\mathcal{L}$ we denote $\mathrm{Fml}_\mathcal{L}$ the set of all Gödel numbers $\ulcorner\varphi\urcorner$ of formulae φ in the language $\mathcal{L}$. If it is clear from the context which language is referred to we may omit the subcript $\mathcal{L}$. Furthermore, for an $\mathcal{L}$-structure $\mathcal{M}$ put

$$\mathrm{Th}(\mathcal{M})=\{\ulcorner\varphi\urcorner\in\mathrm{Fml}_\mathcal{L}\mid\mathrm{free}(\varphi)=\emptyset\wedge\mathcal{M}\models\varphi\}.$$

Definition 5.2.12. Let $\varphi(v_0,v_1,\ldots,v_n)$ be a formula in the language of some structure $\mathcal{M}$ with domain M. Then we define an n-ary function, called *canonical Skolem function*, for φ by

$$h^\mathcal{M}_\varphi(v_1,\ldots,v_n)=\begin{cases}\min\{u\in M\mid\varphi(u,v_1,\ldots,v_n)\}, & \text{if such } u \text{ exists,}\\ \emptyset, & \text{otherwise,}\end{cases}$$

where the minimum is computed with respect to some fixed order $<$ of M (in our case, $<$ is $<_{\mathsf{L}[A]}$). Each set of the form $h^\mathcal{M}_\varphi(\vec{x})$ is said to be a *Skolem term*. For any set $X\subseteq M$ let

$$H^\mathcal{M}(X)=\{h^\mathcal{M}_\varphi(a_0,\ldots,a_{n-1})\mid\ulcorner\varphi(v_0,\ldots,v_{n-1})\urcorner\in\mathrm{Fml}\wedge a_0,\ldots,a_{n-1}\in X\}$$

denote the *Skolem hull* of X. Note that $H^\mathcal{M}(X)$ is an elementary substructure of $\mathcal{M}$. The superscript will be omitted in the case that it is clear which structure it refers to.

Definition 5.2.13. Let $\mathcal{M}$ be a structure with domain M and $\langle I, < \rangle$ be a linearly ordered set such that $I \subseteq M$. Then I is said to be a set of *indiscernibles* for $\mathcal{M}$, if for every formula $\varphi(v_0, \ldots, v_{n-1})$ in the language of $\mathcal{M}$ and for all $x_0 < \cdots < x_{n-1}$ and $y_0 < \cdots < y_{n-1}$ in I,

$$\mathcal{M} \models \varphi(x_0, \ldots, x_{n-1}) \text{ iff } \mathcal{M} \models \varphi(y_0, \ldots, y_{n-1}).$$

From now on, we will stick to the following convention.

Convention. If $\langle I, < \rangle$ is a set of indiscernibles for some structure, then the notation $\vec{x} \in I$ means that $\vec{x}$ is a tuple in I of the form $\langle x_0, \ldots, x_{n-1} \rangle$, where $x_0 < \cdots < x_{n-1}$ for some $n \in \omega$.

Definition 5.2.14. Let $\mathcal{M}$ be a structure with domain M and $\langle I, < \rangle$ a set of indiscernibles for $\mathcal{M}$. We denote by $\mathcal{L}_I$ the language of $\mathcal{M}$ expanded by constants $\langle c_x \mid x \in I \rangle$. Let $\mathcal{M}_I$ be the $\mathcal{L}_I$-structure which extends $\mathcal{M}$ and where each c_x is interpreted by x. For any ordered set $\langle J, \lhd \rangle$ we define $K(\mathcal{M}, I, J)$ to be the theory given by

$$K(\mathcal{M}, I, J) = \{ \ulcorner \varphi(\vec{c_y}) \urcorner \in \mathrm{Fml}_{\mathcal{L}_J} \mid \vec{y} \in J \wedge \ulcorner \varphi(\vec{c_x}) \urcorner \in \mathrm{Th}(\mathcal{M}_I) \}.$$

Note that in the case that $I = J$ we have $K(\mathcal{M}, I, I) = \mathrm{Th}(\mathcal{M}_I)$.

Remark 5.2.15. One can easily check (see [Koe78]) that the theory defined by $K(\mathcal{M}, I, J)$ is complete, consistent, contains $\mathrm{Th}(M)$ and that the $\vec{c_y}$ are indiscernibles, i.e. for any $\ulcorner \varphi \urcorner \in \mathrm{Fml}^{\mathcal{L}_J}$,

$$\ulcorner \varphi(\vec{c_{y_0}}) \leftrightarrow \varphi(\vec{c_{y_1}}) \urcorner \in K(\mathcal{M}, I, J).$$

Lemma 5.2.16. *Assume that $\beta \in \mathrm{Lim}$ is a limit ordinal such that $\beta > \alpha$ and that I is a set of indiscernibles for $\langle \mathsf{L}_\beta[a], a \cap \beta, \zeta \rangle_{\zeta \leq \alpha \cap \beta}$, and let $\gamma \in \mathrm{Lim}$. Then there exists an $\mathcal{L}_a$-model $\mathcal{M}$ and a set of indiscernibles $J \subseteq \mathrm{Ord}$ for $\mathcal{M}$ satisfying*

(1) $\mathrm{Th}(\mathcal{M}, J) = K(\mathsf{L}_\beta[a], I, J)$.
(2) $\mathrm{otp}\,(J) = \gamma$.
(3) $H^{\mathcal{M}}(J) = \mathcal{M}$.

Moreover, $\mathcal{M}$ and J are unique up to isomorphisms. Hence we can define the unique pair $\langle \mathcal{M}, J \rangle$ to be the γ-model *for $\langle \mathsf{L}_\beta[a], I \rangle$.*

Proof. The proof is essentially the same as [Jec03, Lemma 18.7] using the observation that the Compactness Theorem holds in ZFC^-. □

Definition 5.2.17. Let $\langle J, \lhd \rangle$ be an ordered set. Then $K(\mathcal{M}, I, J)$ is said to be

(1) *well-founded,* if for every $\gamma \in \mathrm{Lim}$, the γ-model is well-founded.
(2) *unbounded,* if for every Skolem term $t(v_0, \ldots, v_{n-1})$,

$$\ulcorner t(c_0, \ldots, c_{n-1}) \in \mathrm{Ord} \to t(c_0, \ldots, c_{n-1}) < c_n \urcorner \in K(\mathcal{M}, I, J).$$

(3) *remarkable,* if it is unbounded and for every Skolem term of the form $t(v_0, \ldots, v_{m+n})$,

$$\begin{aligned} &\ulcorner t(c_0, \ldots, c_{m+n}) \in \mathrm{Ord} \wedge t(c_0, \ldots, c_{m+n}) < c_m \\ &\to t(c_0, \ldots, c_{m+n}) = t(c_0, \ldots, c_{m-1}, c_{m+n+1}, \ldots, c_{m+2n+1}) \urcorner \end{aligned}$$

is in $K(\mathcal{M}, I, J)$.

For the rest of this section, we fix $\beta \in \mathrm{Lim}$ and a set I of indiscernibles for $\mathsf{L}_\beta[a]$. Now if for a limit ordinal γ the γ-model $\langle \mathcal{M}, J\rangle$ is well-founded, then its Mostowski collapse is of the form $\mathsf{L}_\delta[a]$ for some $\delta \in \mathrm{Lim}$. In that case we identify the γ-model with $\mathsf{L}_\delta[a]$. We enumerate the indiscernibles for $\mathsf{L}_\delta[a]$ as

$$\langle x_\xi^\gamma \mid \xi < \gamma\rangle.$$

Similarly, we denote the constants c_{x_ξ} by c_ξ for $\xi < \gamma$. In fact, we can do better than this: The next lemma shows that the superscript γ is – under certain conditions – superflous.

Lemma 5.2.18. *[Kan09, Lemma 9.11] Assume that I is a set of indiscernibles for $\mathsf{L}_\beta[a]$ and that $\mathrm{Th}(\mathsf{L}_\beta[a]_I)$ is well-founded and remarkable. Let $\gamma, \delta \in \mathrm{Ord}$ such that $\gamma \leq \delta$ and $\gamma \in \mathrm{Lim}$. Then the Skolem hull of $\{x_\xi^\delta \mid \xi < \gamma\}$ in the δ-model is $\mathsf{L}_{x_\gamma^\delta}[a]$.* □

As a consequence, we obtain that for limit ordinals $\gamma < \delta$, the γ-model is $\mathsf{L}_{x_\gamma^\delta}[a]$ and $\{x_\xi^\delta \mid \xi < \gamma\}$ is the corresponding set of indiscernibles. Hence $x_\xi^\gamma = x_\xi^\delta$ for every $\xi < \gamma$. From now on, we will thus write simply x_ξ for x_ξ^γ. Furthermore, we denote the class of all indiscernibles for $\mathrm{Th}(\mathsf{L}_\beta[a]_I)$ by

$$\bar{I} = \{x_\xi \mid \xi \in \mathrm{Ord}\}$$

and for every $\gamma \geq \omega$ we put

$$I_\gamma = \{x_\xi \mid \xi < \gamma\}.$$

Corollary 5.2.19. *Let $\gamma < \delta$. Then $\mathsf{L}_{x_\gamma}[a]$ is an elementary substructure of $\mathsf{L}_{x_\delta}[a]$.*

Proof. If $\delta \in \mathrm{Lim}$, this follows directly from Lemma 5.2.18. Otherwise, let $\varepsilon \in \mathrm{Lim}$ such that $\mathsf{L}_{x_\gamma}[a], \mathsf{L}_{x_\delta}[a] \in \mathsf{L}_{x_\varepsilon}[a]$. Then for any $\gamma' < \delta'$ such that $\delta' \in \mathrm{Lim}$ and $\gamma', \delta' < \varepsilon$, $\mathsf{L}_{x_\varepsilon}[a] \models$ "$\mathsf{L}_{x_{\gamma'}}[a] \prec \mathsf{L}_{x_{\delta'}}[a]$" which by indescernibility proves the claim. □

Lemma 5.2.20. *The following statements hold.*

(1) *$\bar{I}$ is closed unbounded in* Ord.
(2) *If $y \in \mathsf{L}[a]$, then there exists a Skolem term $t(v_0, \ldots, v_{n-1})$ and $\xi_0 < \cdots < \xi_{n-1}$ such that $y = t^{\mathsf{L}[a]}(x_{\xi_0}, \ldots, x_{\xi_{n-1}})$.*
(3) *For any γ, $\mathsf{L}_{x_\gamma}[a] \prec \mathsf{L}[a]$.*

Proof. In order to show that $\bar{I}$ is unbounded, let $\gamma \in \mathrm{Ord}$. Clearly, we have $\xi \leq x_\xi$ for every ordinal ξ. So in particular this shows that $\gamma < x_{\gamma+1}$. The closure property follows from Lemma 5.2.18.

For the second statement, fix $y \in \mathsf{L}[a]$. Using (1) we can take $\gamma \in \mathrm{Ord}$ such that $y \in \mathsf{L}_{x_\gamma}[a]$. Then there are $\xi_0 < \cdots < \xi_{n-1} < \gamma$ and a Skolem term $t(v_0, \ldots, v_{n-1})$ such that $y = t^{\mathsf{L}_{x_\gamma}[a]}(x_{\xi_0}, \ldots, x_{\xi_{n-1}}) = t^{\mathsf{L}[a]}(x_{\xi_0}, \ldots, x_{\xi_{n-1}})$.

Thirdly, Let $\varphi(v, v_0, \ldots, v_{n-1})$ be a formula such that $\mathsf{L}[a] \models \exists x \varphi(x, y_0, \ldots, y_{n-1})$ for $y_0, \ldots, y_{n-1} \in \mathsf{L}_{x_\gamma}[a]$. Then $\mathsf{L}[a] \models \varphi(h_\varphi(y_0, \ldots, y_{n-1}), a_0, \ldots, a_{n-1})$ and by Lemma 5.2.16 we obtain that $h_\varphi^{\mathsf{L}_{x_\gamma}[a]}(y_0, \ldots, y_{n-1}) \in \mathsf{L}_{x_\gamma}[a]$, so the Tarksi-Vaught criterion completes the proof. □

Lemma 5.2.21. *The class $\bar{I}$ is the unique class of indiscernibles for $\mathsf{L}[a]$ satisfying properties (1) and (2) from Lemma 5.2.20.*

Proof. Assume that $\bar{J}$ is another such class of indiscernibles. Then $\bar{I} \cap \bar{J}$ is closed unbounded in Ord. We claim that

$$\text{Th}(\mathsf{L}[a]_{\bar{J}}) = K(\mathsf{L}[a], \bar{I}, \bar{J}). \tag{5.3}$$

If $\varphi(\vec{v})$ is a formula such that $\ulcorner\varphi(\vec{c_x})\urcorner \in \text{Th}(\mathsf{L}[a]_{\bar{I}})$, then in particular for $\vec{\xi}$ such that $\vec{x_\xi} \in \bar{I} \cap \bar{J}$, we have that $\ulcorner\varphi(\vec{c_{x_\xi}})\urcorner \in \text{Th}(\mathsf{L}[a]_{\bar{I}}) \cap \text{Th}(\mathsf{L}[a]_{\bar{J}})$. So by indiscernibility of $\bar{J}$, $\ulcorner\varphi(\vec{c_y})\urcorner \in \text{Th}(\mathsf{L}[a]_{\bar{J}})$. The other inclusion is shown in a similar way.

Now let $h : \bar{I} \to \bar{J}$ be an order-preserving bijection. We extend h to $\bar{h} : \mathsf{L}[a] \to \mathsf{L}[a]$ by stipulating

$$\bar{h}(t^{\mathsf{L}[a]}(\vec{x_\xi})) = t^{\mathsf{L}[a]}(h(\vec{x_\xi})).$$

By (5.3) $\bar{h}$ is an isomorphism. But then $\bar{h}$ is the identity map, since otherwise there would exist $\xi \in$ Ord such that $\bar{h}(\xi) > \xi$; however, this would mean that $\xi \notin \text{ran}(\bar{h})$ which is a contradiction. In particular, this implies that $\bar{I} = \bar{J}$. □

We now single out the initial segment of $\bar{I}$ of ordertype ω given by

$$I_\omega = \{x_\xi \mid \xi < \omega\}.$$

By Lemma 5.2.18, I_ω is a set of indiscernibles for $\mathsf{L}_{x_\omega}[a]$. Due to the fact that for any $\gamma \in$ Lim that the γ-model with respect to $\langle \mathsf{L}_{x_\omega}[a], I_\omega\rangle$ is equal to the γ-model with respect to the original model $\langle \mathsf{L}_\beta[a], I\rangle$, we obtain that $\text{Th}(\mathsf{L}_{x_\omega}[a]_{I_\omega})$ is well-founded and remarkable. Under the assumption that β and I exist such that $\text{Th}(\mathsf{L}_\beta[a]_I)$ is well-founded and remarkable, I_ω must be unique and I (and hence also $\bar{I}$) can be reconstructed from I_ω.

Theorem 5.2.22. *If there exist an ordinal β and a set $I \subseteq$ Ord of indiscernibles for $\langle \mathsf{L}_\beta[a], a \cap \beta, \zeta\rangle_{\zeta \leq \alpha \cap \beta}$ such that $\text{Th}(\mathsf{L}_\beta[a]_I)$ is remarkable and well-founded, then $I_\omega \subseteq$ Ord is the unique set with the following properties.*

(1) *There exists $\delta \in$ Ord such that I_ω is a set of indiscernibles for $\mathsf{L}_\delta[a]$.*
(2) $\text{otp}\,(I_\omega) = \omega$.
(3) $H^{\mathsf{L}_\delta[a]}(I_\omega) = \mathsf{L}_\delta[a]$.
(4) $\text{Th}(\mathsf{L}_\delta[a]_{I_\omega})$ *is well-founded and remarkable.*

Furthermore, $\bar{I}$ can be reconstructed from I_ω.

Proof. Firstly, we verify that I_ω satisfies the desired properties. Lemmata 5.2.18 and 5.2.16 prove (1)-(3) for $\delta = x_\omega$. Remarkability clearly follows from remarkability of $\text{Th}(\mathsf{L}_\beta[a]_I)$. In order to show well-foundedness, we check that for every $\gamma \geq \omega$, the γ-model with respect to $\langle \mathsf{L}_\delta[a], I_\omega\rangle$ is equal to the γ-model with respect to the original structure $\langle \mathsf{L}_\beta[a], I\rangle$. Fix $\gamma \geq \omega$ and let $\langle \mathcal{M}, J\rangle$ be the γ-model obtained from $\langle \mathsf{L}_\delta[a], I_\omega\rangle$ as in Lemma 5.2.16. It suffices to verify that $\text{Th}(\mathcal{M}, J) = K(\mathsf{L}_\beta[a], I, J)$. For this, suppose that $\ulcorner\varphi(\vec{c_y})\urcorner \in \text{Th}(\mathcal{M}, J)$, where $\vec{y} \in J$. Since $\text{Th}(\mathcal{M}, J) = K(\mathsf{L}_\delta[a], I_\omega, J)$, $\ulcorner\varphi(\vec{c_x})\urcorner \in \text{Th}(\mathsf{L}_\delta[a]_{I_\omega})$. But $\text{Th}(\mathsf{L}_\delta[a]_{I_\omega}) = K(\mathsf{L}_\beta[a], I, I_\omega)$ and $I_\omega \subseteq I$, so $\ulcorner\varphi(\vec{c_y})\urcorner \in K(\mathsf{L}_\beta[A], I, J)$.

The other inclusion follows in a similar way. Applying uniqueness in Lemma 5.2.16 yields that $\langle \mathcal{M}, J\rangle$ and $\langle \mathsf{L}_{x_\gamma}[a], I_\gamma\rangle$ are isomorphic and in particular $\langle \mathcal{M}, J\rangle$ is well-founded and remarkable. So $\mathcal{M} = \mathsf{L}_{x_\gamma}[a]$ and using closed unboundedness of I_γ and J in x_γ, similar arguments as in the proof of Lemma 5.2.21 yield that $J = I_\gamma$.

Now we show that I_ω is indeed unique. Suppose that $J \subseteq \mathrm{Ord}$ and $\lambda \in \mathrm{Ord}$ such that J is a set of indiscernibles for $\mathsf{L}_\lambda[a]$ of ordertype ω, $H^{\mathsf{L}_\lambda[a]}(J) = \mathsf{L}_\lambda[a]$ and $\mathrm{Th}(\mathsf{L}_\lambda[a]_J)$ is well-founded and remarkable. Then we can realize the whole construction in a similar way as for $\langle \mathsf{L}_\beta[a], I\rangle$ in order to obtain a closed unbounded class $\bar{J}$ of indiscernibles for $\mathsf{L}[a]$ such that $\mathsf{L}[a]$ is the Skolem hull of $\bar{J}$. But then Lemma 5.2.21 implies that $\bar{I} = \bar{J}$ and by the previous arguments we obtain that $I_\omega = J_\omega = J$ and $\lambda = x_\omega = \delta$. This proves the claim. □

Definition 5.2.23. For a set of ordinals $a \subseteq \mathrm{Ord}$ we denote by

$$a^\# \text{ exists}$$

the statement that there exists a set $I \subseteq \mathrm{Ord}$ such that

(1) $\mathrm{otp}\,(I) = \omega$,
(2) I is a set of indiscernibles for $\mathsf{L}_\delta[a]$ for $\delta = \sup I$,
(3) $H^{\mathsf{L}_\delta[a]}(I) = \mathsf{L}_\delta[a]$,
(4) $\mathrm{Th}(\mathsf{L}_\delta[a]_I)$ is remarkable and well-founded,

and $a^\#$ denotes the unique I satisfying the properties listed above. Observe that for every set $a \subseteq \mathrm{Ord}$, the corresponding *sharp* $a^\#$ is also a set of ordinals. Moreover, we abbreviate the axiom

$$\forall a \subseteq \mathrm{Ord}(a^\# \text{ exists})$$

by $\exists\#$ and for any theory T, we write $\mathsf{T}^\#$ for $\mathsf{T} + \exists\#$.

Remark 5.2.24. The reason why we can take $\delta = \sup I$ in Definition 5.2.23 is that otherwise if $\sup I < \delta$, then $\sup I$ would lie in the Skolem hull of I contradicting the remarkability of $\mathrm{Th}(\mathsf{L}_\delta[a]_I)$.

Remark 5.2.25. If $a^\#$ exists, we can define truth in $\mathsf{L}[a]$ as follows: If $\varphi(v_0, \ldots, v_{n-1})$ is a formula and $y_0, \ldots, y_{n-1} \in \mathsf{L}[a]$, then each y_i can be written as $y_i = t_i^{\mathsf{L}[a]}(\vec{x_i})$ for $\vec{x_i} \in \bar{I}$. If $x_\lambda = \sup\{\sup \vec{x_i} \mid i < n\}$ then every y_i is in $\mathsf{L}_{x_\lambda}[a]$, hence

$$\mathsf{L}[a] \models \varphi(y_0, \ldots, y_{n-1}) \text{ iff } \mathsf{L}_{x_\lambda}[a] \models \varphi(y_0, \ldots, y_{n-1}).$$

Moreover, since there is a formula $\psi_{\varphi,\vec{y}}$ such that the free variables of ψ correspond to the indiscernibles $\vec{x_i}$ appearing in the definition of the y_i as a Skolem term and $\mathsf{L}[a] \models \varphi(\vec{y})$ iff $\mathsf{L}[a] \models \psi(\vec{x})$, where $\vec{x}$ consists of all $\vec{x_i}$. Then

$$\mathsf{L}[a] \models \varphi(y_0, \ldots, y_{n-1}) \text{ iff } \ulcorner \psi(c_0, \ldots, c_k) \urcorner \in \mathrm{Th}(\mathsf{L}_\delta[a]_{a^\#})$$

for k the number of free variables of ψ and $\delta = \sup a^\#$.

Remark 5.2.26. The previous remark motivates an alternative definition of $a^\#$ as $T = \mathrm{Th}(\mathsf{L}_\delta[a]_{a^\#})$, where

(1) T is consistent, complete, all axioms of ZF^- are in T and $\ulcorner\sigma\urcorner \in T$, where σ is the statement with the property that $\langle M, \in\rangle \models \sigma$ if and only if M is of $\mathsf{L}[a]$ or $\mathsf{L}_\delta[a]$ for some $\delta > \omega$.[2],
(2) all c_x for $x \in a^\#$ (since $\operatorname{otp}\left(a^\#\right) = \omega$ we can number them $c_i, i < \omega$) are indiscernibles,
(3) T is remarkable,
(4) T is well-founded.

Note that consistency of T and the existence of the $c_i, i < \omega$, implies that T has a countable model whose interpretation of the c_i is a set of indiscernibles. Then T is well-founded, if and only if every countable model equipped with some set of indescernibles is well-founded.

This means that if such T exists, we can reconstruct $a^\#$ from it as the unique model of T with indiscernibles of order type ω. Moreover, the above characterization allows us to consider $a^\#$ to be a real.

Theorem 5.2.27. *The statement "$a^\#$ exists $\wedge I = a^\#$" is absolute for transitive models M of ZF^- such that $\{a\} \cup \mathrm{Ord} \subseteq M$. Moreover, if $a^\#$ exists in M, then $(a^\#)^M = a^\#$.*

Proof. Let M be a model of ZF^- with $\{a\} \cup \mathrm{Ord} \subseteq M$. Now let $\alpha = \sup a$. Then for any $\delta > \alpha$, we have $(\mathsf{L}_\delta[a])^M = \mathsf{L}_\delta[a]$, since the relation $x = \mathsf{L}_\delta[a]$ is Δ_1 in the parameter a.

Now if $I \subseteq \mathrm{Ord}$ such that $I \in M$, then $\delta = \sup I$ is absolute and so is the Δ_1-statement $\operatorname{otp}(I) = \omega$. Moreover, by absolutenss of the satisfaction relation, we obtain that $(\mathrm{Th}(\mathsf{L}_\delta[a]_I))^M = \mathrm{Th}(\mathsf{L}_\delta[a]_I)$. In particular, this implies absoluteness of the syntactical properties asserting that I is a set of indiscernibles for $\mathsf{L}_\delta[a]$ and that $\mathrm{Th}(\mathsf{L}_\delta[a]_I)$ is remarkable. Now since the well-ordering $<_{\mathsf{L}[a]}$ is absolute, we obtain absoluteness of the Skolem functions, yielding

$$(H^{\mathsf{L}_\delta[a]}(I))^M = H^{\mathsf{L}_\delta[a]}(I).$$

The last property in question is well-foundedness. Note that the relation "$\langle N, e\rangle$ is well-founded" is Δ_1.[3] Now let $\gamma \geq \omega$ be an ordinal. Note that if $\langle N_\gamma, I_\gamma\rangle \in M$ is the corresponding γ-model, then

- if it is well-founded, then it is of the form $\mathsf{L}_\lambda[a]$ for some limit ordinal λ and hence it is also the γ-model in V;
- if it is not well-founded, then by absoluteness of well-foundedness we also obtain a counterexample in V.

Thus we have verified that the properties defining $a^\#$ listed in Definition 5.2.23 are absolute. □

Now assume $(\mathsf{ZFC}^-)^\# + \mathsf{V} = \mathsf{HC}$. We iterate the construction of sharps as follows.

$$^\#\alpha = (\bigcup_{\beta<\alpha} {}^\#\beta)^\# \text{ for } \alpha \in \mathrm{Ord},$$
$$\# = \bigcup_{\alpha\in\mathrm{Ord}} {}^\#\alpha.$$

[2]see [Kan09, Theorem 3.3]
[3]see [Jec03, Lemma 13.11]

Remark 5.2.28. If $\alpha < \beta$, then for all $x \in {}^{\#}\alpha$ and $y \in {}^{\#}\beta$ we have $x < y$. This is clear by definition of ${}^{\#}\beta$, since ${}^{\#}\beta = a^{\#}$ for some $a \subseteq \mathrm{Ord}$ and ${}^{\#}\alpha \subseteq a$. In particular, x is a constant in $\langle \mathsf{L}[a], a, \gamma \rangle_{\gamma \leq \sup a}$.

Then define

$$\mathsf{L}^{\#}_{\alpha} = \mathsf{L}_{\alpha}[\#] \text{ and } \mathsf{L}^{\#} = \mathsf{L}[\#].$$

In particular, this means that $\mathsf{L}^{\#} \models \mathsf{ZFC}^{-}$. We claim that $\mathsf{L}^{\#}$ is closed under sharps. In order to prove this, we require some crucial results concerning sharps which are provable in ZFC. The proofs can all be found in [Jec03] or [Kan09].

Lemma 5.2.29 (ZFC)**.** *If $\Sigma = \mathrm{Th}(\mathsf{L}_{\beta}[a]_I)$ for some $\beta > \sup A$ and a set $I \subseteq \mathrm{Ord}$ of indiscernibles for $\mathsf{L}_{\beta}[a]$, then the following statements are equivalent.*

(1) *For every $\alpha \in \mathrm{Ord}$, the α-model is well-founded.*
(2) *Fore some $\alpha \geq \omega_1$, the α-model is well-founded.*
(3) *For every $\alpha < \omega_1$, the α-model is well-founded.* □

Lemma 5.2.30 (ZFC)**.** *Let $\kappa > \omega$ be a cardinal. If there exists $\beta \in \mathrm{Lim}$ such that $\mathsf{L}_{\beta}[a]$ has a set of ordinal indiscernibles of order type κ, then there is $\gamma \in \mathrm{Lim}$ and a set $I \subseteq \gamma$ of order type κ such that $\mathrm{Th}(\mathsf{L}_{\gamma}[a]_I)$ is remarkable.* □

Theorem 5.2.31 (ZFC)**.** *If $a^{\#}$ exists and $\bar{I}$ is the corresponding closed unbounded class of indiscernibles for $\mathsf{L}[a]$, then every uncountable cardinal $\kappa > \sup a$ lies in $\bar{I}$ and the κ-model is $\mathsf{L}_{\kappa}[a]$.* □

Now we are ready to prove that $\mathsf{L}^{\#}$ is closed under sharps provided that the $\mathbf{\Pi}^1_1$-perfect set property holds.

Theorem 5.2.32. *Assume* $(\mathsf{ZFC}^{-})^{\#} + \mathsf{V} = \mathsf{HC} + \mathbf{\Pi}^1_1\text{-}\mathsf{PSP}$*. Then* $\mathsf{L}^{\#} \models \forall a \subseteq \mathrm{Ord}(a^{\#} \text{ exists})$.

Proof. Let $a \in \mathsf{L}^{\#}$ be a set of ordinals. Let α be a limit ordinal such that $a \in \mathsf{L}^{\#}_{\alpha}$ and $\#$ is cofinal in α. The existence of such α is guaranteed by Remark 5.2.28. Then

$$\# \cap \alpha = \bigcup_{\xi < \gamma} {}^{\#}\xi$$

for some $\gamma \in \mathrm{Ord}$. Consider $b = \# \cap \alpha$. Then $b^{\#} = {}^{\#}\gamma \in \mathsf{L}^{\#}$. Using the observation that $b^{\#}$ can be coded by a real and the $\mathbf{\Pi}^1_1$-PSP we may conclude that $\mathsf{L}[b^{\#}] \models \mathsf{ZFC} + b^{\#}$ exists. Work in $\mathsf{L}[b^{\#}]$. By absoluteness of sharps and since $\mathsf{L}[b^{\#}] \subseteq \mathsf{L}^{\#}$, it is sufficient to prove that $a^{\#}$ exists in $\mathsf{L}[b^{\#}]$. Set $\zeta = \sup b^{\#}$. Then $b^{\#}$ is a set of indiscernibles for $\mathsf{L}_{\zeta}[b]$ of ordertype ω and $\mathrm{Th}(\mathsf{L}_{\zeta}[b]_{b^{\#}})$ is well-founded and remarkable. Now since $\mathsf{L}[b^{\#}] \models \mathsf{ZFC}$, $\aleph_1^{\mathsf{L}[b^{\#}]}$ exists. By well-foundedness of $\mathrm{Th}(\mathsf{L}_{\zeta}[b]_{b^{\#}})$, the ω_1-model is of the form $\mathsf{L}_{\mu}[b]$ with a set I of indiscernibles of order type ω_1 and $H^{\mathsf{L}_{\mu}[b]}(I) = \mathsf{L}_{\mu}[b]$. Observe that since $\mu > \zeta$ we obtain that $a \in \mathsf{L}_{\mu}[b]$, so we can write a as

$$a = t^{\mathsf{L}_{\mu}[b]}(\vec{x})$$

for some Skolem term t and $\vec{x} \in I$. Put $J = I \setminus \sup \vec{x}$ and note that J is a set of indiscernibles of ordertype ω_1 for all $\mathcal{L}_b$-formulae with parameter a. Now since $\mathsf{L}[b] \models \mathsf{ZFC}$

and $\mathsf{L}_\mu[b] \prec \mathsf{L}[b]$ we obtain that $\mathsf{L}_\mu[b] \models \mathsf{ZFC}$. We would like to verify that J is a set of indiscernibles for $\mathsf{L}_\mu[a]$. Let $\varphi(a, \vec{v}, \vec{\gamma})$ be a formula with parameters a, $\vec{\gamma}$ where $\vec{\gamma} \leq \sup a$. We need to check that

$$\mathsf{L}_\mu[a] \models \varphi(a, \vec{y}, \vec{\gamma}) \leftrightarrow \varphi(a, \vec{z}, \vec{\gamma}) \tag{5.4}$$

for all $\vec{y}, \vec{z} \in J$. We work in $\mathsf{L}_\mu[b]$. By indiscernibility we obtain that

$$\mathsf{L}_\mu[b] \models \text{“}\mathsf{L}[a] \models \varphi(a, \vec{y}, \vec{\gamma}) \leftrightarrow \mathsf{L}[a] \models \varphi(a, \vec{z}, \vec{\gamma})\text{”}.$$

By absoluteness of satisfaction this shows that $(\mathsf{L}[a])^{\mathsf{L}_\mu[b]} \models \varphi(a, \vec{y}, \vec{\gamma}) \leftrightarrow \varphi(a, \vec{z}, \vec{\gamma})$. Since $(\mathsf{L}[a])^{\mathsf{L}_\mu[b]} = \mathsf{L}_\mu[a]$, this proves (5.4). Now we can apply Lemma 5.2.30 in order to obtain $\nu \in \mathrm{Ord}$ and a set $K \subseteq \nu$ of indiscernibles of order type ω_1 for $\mathsf{L}_\nu[a]$ such that $\mathrm{Th}(\mathsf{L}_\nu[a]_K)$ is remarkable. Using Lemma 5.2.29 we can conclude that $\mathrm{Th}(\mathsf{L}_\nu[a]_K)$ is well-founded and remarkable. In particular, $a^\#$ exists and $a^\#$ is the initial segment of K of order type ω. □

Theorem 5.2.33. *The classes* ${}^\omega\omega \cap \mathsf{L}^\#$ *and* $<_\# \cap ({}^\omega\omega \times {}^\omega\omega)$ *are* Σ^1_3*, where* $<_\#$ *denotes the canonical well-order of* $\mathsf{L}^\#$*.*

Proof. We follow the detailed proof given in [Koe78]. We outline briefly the idea of the proof. If $x \in \mathsf{L}^\#$ there exists a countable ordinal α such that $x \in \mathsf{L}^\#_\alpha$ and $\# \cap \alpha$ is cofinal in α, and hence there exists $\gamma \in \mathrm{Ord}$ such that

$$\# \cap \alpha = \bigcup_{\xi < \gamma} {}^\#\xi.$$

Since $\mathsf{L}^\#_\alpha$ is countable, it can be represented by a model of the form $\langle \omega, e, s\rangle$, where e is a well-ordering of ω and s is the predicate coding $\# \cap \alpha$. Its Mostowski collapse is given by $\pi : \langle \omega, e, s\rangle \to \langle \mathsf{L}^\#_\alpha, \in, \# \cap \alpha\rangle$ with $\pi''s = \# \cap \alpha$ and $\pi(n) = x$ for some $n \in \omega$. This can be formalized in the following way.

$$\begin{gathered}\exists^1 e \exists^1 s \exists^0 n \, [e \in \mathrm{WO} \wedge \langle \omega, e, s\rangle \models \sigma \wedge s \text{ is cofinal in } \mathrm{Ord}^{\langle \omega, e, s\rangle} \wedge \\ \exists \gamma \in \mathrm{Ord}(\pi''s = \bigcup_{\delta < \gamma} {}^\#\delta) \wedge \pi(n) = x],\end{gathered} \tag{5.5}$$

where σ is the sentence in the language $\mathcal{L}_A$ such that for any transitive class M and any class A, $\langle M, \in, A \cap M\rangle \models \sigma$ iff there exists γ such that $M = \mathsf{L}_\gamma[A]$, and where the quantifiers $\exists^1, \forall^1$ range over reals and $\exists^0, \forall^0$ range over natural numbers.

We need to check that all statements occurring in (5.5) are Σ^1_3 in the parameters e and s. Since WO is Π^1_1, the satisfaction relation is Δ^1_1 and $\{\langle n, m\rangle \mid \pi(n) = m\}$ is arithmetical in e, s, it remains to verify that

$$\exists \gamma \in \mathrm{Ord}\,(\pi''S = \bigcup_{\xi < \gamma} {}^\#\xi)$$

is a $\Sigma^1_3[e, s]$-statement.

To see this, note that γ being a countable ordinal we can code it by some real $y \in \mathrm{WO}$ and we can realize the ${}^{\#}\xi$-construction using a recursive function $f : \omega \to \omega$ such that $s = \bigcup_{m \in \omega} f(m)$ and $\pi(f(n)) = (\pi(\bigcup_{m <_y n} f(m)))^{\#}$. It therefore suffices to show that $\langle \omega, e, s \rangle \models \pi(n) = \pi(m)^{\#}$ for $n, m \in \omega$ is Π^1_2 in the parameters e, s. This can be found in [Kan09, Theorem 14.11]. The only part whose complexity attains $\Pi^1_2[e, s]$ is well-foundedness.

Using this it is straightforward to check that $<_{\#} \cap ({}^{\omega}\omega \times {}^{\omega}\omega)$ is also Σ^1_3:

$$\begin{aligned} x <_{\#} y \leftrightarrow & \exists^1 e \exists^1 s \, [\langle \omega, e, s \rangle \text{ satisfies all properties in (5.5)} \wedge \\ & \exists^0 n \exists^0 m (\pi(n) = x \wedge \pi(m) = y \wedge \langle \omega, e, s \rangle \models \theta(n, m))], \end{aligned}$$

where $\theta(v_0, v_1)$ is a formula in $\mathcal{L}_{\#}$ – the language containing a unary predicate to denote membership in the class $\#$ – such that $x <_{\#} y$ iff there is a limit ordinal δ such that $\delta > \omega$ and $\mathsf{L}^{\#}_{\delta} \models \theta(x, y)$.[4] □

Theorem 5.2.34. *Assume* $(\mathsf{ZFC}^-)^{\#} + \mathsf{V} = \mathsf{HC} + \mathbf{\Sigma^1_3}$-$\mathsf{PSP}$. *Then* $\mathsf{L}^{\#} \models \mathsf{ZFC}^{\#}$.

Proof. It is clear that $\mathsf{L}^{\#} \models \mathsf{ZFC}^-$. Moreover, Theorem 5.2.32 implies that $\mathsf{L}^{\#} \models (\mathsf{ZFC}^-)^{\#}$. This means that it only remains to show that power sets exist in $\mathsf{L}^{\#}$. Consider the class C of reals given by

$$x \in C \iff x \in \mathsf{L}^{\#} \wedge x \in \mathrm{WO} \wedge \forall y \, (y \in \mathrm{WO} \wedge y \prec_{\#} x \to y \neq_{\mathrm{WO}} x).$$

Using the same arguments as in the proof of Lemma 5.2.8, we can show that C is Σ^1_3. Now we proceed exactly as in the proof that $\mathbf{\Pi^1_1}$-PSP implies that $\mathsf{L}[x] \models \mathsf{ZFC}$ for every real x to conclude that $\mathsf{L}^{\#} \models \mathsf{ZFC}$. More explicitly, C is unbounded in $\mathsf{L}^{\#}$ but due to $\mathbf{\Sigma^1_1}$-boundedness, it has no perfect subclass. This shows that C is countable and thus, as in the proof of Theorem 5.2.10, the power set of ω exists in $\mathsf{L}^{\#}$. To prove that for every $\alpha \in \mathrm{Ord}$, $\mathcal{P}(\alpha)^{\mathsf{L}^{\#}}$ exists, take a real x which codes a well-order of ordertype α. It is easy to check that there is a relativized version $\mathsf{L}[x]^{\#}$ of $\mathsf{L}^{\#}$, given by the stipulating

$$\begin{aligned} {}^{\#_x}0 &= x^{\#} \\ {}^{\#_x}\alpha &= (\bigcup_{\beta < \alpha} {}^{\#_x}\beta)^{\#} \text{ for } \alpha > 0, \\ \#_x &= \bigcup_{\alpha \in \mathrm{Ord}} {}^{\#_x}\alpha. \end{aligned}$$

Then $\mathcal{P}(\omega)^{\mathsf{L}[x]^{\#}}$ exists in $\mathsf{L}[x]^{\#} = \mathsf{L}[\#_x]$ by the same argument as above and using $\Sigma^1_3[x]$-PSP. Now we proceed in the same way as in the proof of Theorem 5.2.11 to conclude that the the power set of $\mathsf{L}^{\#}_{\alpha}$ exists in $\mathsf{L}^{\#}$ using the fact that $\mathsf{L}_{\alpha}[x]$ is countable in $\mathsf{L}[x]^{\#}$. □

The next theorem shows that we can further improve the complexity of the perfect set property assumption in Theorem 5.2.34. Its proof is similar to the proof that if $\omega_1^{\mathsf{L}} = \omega_1$, there is a Π^1_1-set of reals without the perfect set property (see [Kan09, Theorem 13.12]).

Theorem 5.2.35. *Assume* $(\mathsf{ZFC}^-)^{\#} + \mathsf{V} = \mathsf{HC}$. *Then* Π^1_2-PSP *fails in* $\mathsf{L}^{\#}$.

[4] see [Kan09, Theorem 3.3 (b)]

Proof. We already know that the class C presented in the proof of Theorem 5.2.34 is a witness to the failure of Σ^1_3-PSP in $\mathsf{L}^\#$, so we have to improve on that attempt in order to obtain a Π^1_2-witness. The idea is to proceed as follows: We want to code ordinals which are countable in $\mathsf{L}^\#$. Again, we will use countable well-orderings to achieve this. Now given some $\alpha \in \mathrm{Ord}$ such that α is countable in $\mathsf{L}^\#$, there is $x_0 \in \mathrm{WO} \cap \mathsf{L}^\#$ such that $\|x_0\| = \alpha$. But $x_0 \in \mathrm{WO} \cap \mathsf{L}^\#$ is Σ^1_3; instead there is a least limit ordinal $\delta > \omega$ such that $\# \cap \alpha$ is cofinal in α and $x_0 \in \mathsf{L}^\#_\delta$. We can code δ and also $\mathsf{L}^\#_\delta$ by a well-ordering x_1 and iterate this process, thus obtaining a real x such that $(x)_i = x_i$ for all $i \in \omega$. We will now do this formally and show that the class of reals x given as above is Π^1_2.

Observe that we can use a real x to code a sequence of reals $\langle (x)_i \mid i \in \omega \rangle$, such that each $(x)_i$ in turn codes a model of the form $M_{(x)_i} = \langle \omega, e_{(x)_i}, s_{(x)_i}, y_{(x)_i} \rangle$, where $e_{(x)_i}$ is a well-order on ω (so that $\langle \omega, e_{(x)_i} \rangle$ will code $\langle \mathsf{L}^\#_\alpha, \in \rangle$ for some α), $s_{(x)_i}$ is a subset of ω (which will be used to code $\# \cap \alpha$) and $y_{(x)_i}$ is a well-order on ω (which we need to code the recursive construction of sharps). Let $A \subseteq {}^\omega\omega$ denote the class of reals defined in the following way:

$$\begin{aligned} x \in A \leftrightarrow x_{(0)} \in \mathrm{WO} \wedge \forall^0 i\, [&M_{(x)_{i+1}} \text{ is well-founded and extensional } \wedge M_{(x)_{i+1}} \models \sigma \\ &\wedge s_{(x)_{i+1}} \text{ is cofinal in } \mathrm{Ord}^{M_{(x)_{i+1}}} \wedge \pi''_{(x)_{i+1}} s_{(x)_{i+1}} = \bigcup_{\xi < \|y\|_{x_{(i+1)}}} {}^\#\xi \\ &\wedge (x)_i \in \mathrm{tr}(M_{(x)_{i+1}}) \wedge \forall^0 k\, (M_{(x)_{i+1}} \restriction k \models \sigma \wedge (x)_i \in \mathrm{tr}(M_{(x)_{i+1}} \restriction k \\ &\to s_{(x)_{i+1}} \restriction k \text{ is not cofinal in } \mathrm{Ord}^{M_{(x)_{i+1}} \restriction k}) \wedge \forall^1 z \forall^0 n \forall^0 m\, (\pi_{(x)_{i+1}}(n) = z \\ &\wedge \pi_{(x)_{i+1}} = (x)_i \wedge M_{(x)_{i+1}} \models \theta(n,m) \to M_z \not\cong M_{(x)_i})]. \end{aligned}$$

Using our computations in the proof of Theorem 5.2.33, it is straightforward to check that A is in fact Π^1_2. It remains to verify that it is uncountable in $\mathsf{L}^\#$ but does not contain a perfect subclass. For the first claim, let x be a real in A. Then there is a sequence $\langle \alpha^x_i \mid i \in \omega \rangle$ of ordinals countable in $\mathsf{L}^\#$ with the following properties.

(a) $M_{(x)_0}$ has ordertype α^x_0 and for each $i \in \omega$, $M_{(x)_{i+1}} \cong \mathsf{L}^\#_{\alpha^x_{i+1}}$ and α^x_{i+1} is the least limit ordinal $> \alpha_i$ such that $(x)_i \in \mathsf{L}^\#_{\alpha^x_{i+1}}$.

(b) For each $i \in \omega$, if $z <_{\mathsf{L}^\#_{\alpha^x_{i+1}}} (x)_i$ then $M_z \not\cong M_{(x)_i}$.

Now observe that for every ordinal α which is countable in $\mathsf{L}^\#$ there is $x \in A$ with $\alpha^x_0 = \alpha$. This shows that A has cardinality $\omega_1^{\mathsf{L}^\#}$.

In order to prove the second claim, suppose that $P \subseteq A$ is a perfect subclass of A. Then P is closed and hence $\boldsymbol{\Sigma^1_1}$. Hence so is the projection $P' = \{(x)_0 \mid x \in P\}$ onto the first coordinate. But then Theorem 5.2.7 implies that P' is ordertype bounded and in particular countable, a contradiction.

□

By combining Theorem 5.2.35 and the arguments in the proof of Theorem 5.2.34 we obtain the following improvement of Theorem 5.2.34.

Corollary 5.2.36. *If* Π^1_2-PSP *holds, then* $\aleph_1^{\mathsf{L}^\#}$ *exists. In particular,* $(\mathsf{ZFC}^-)^\# + \mathsf{V} = \mathsf{HC} + \boldsymbol{\Pi^1_2}$-PSP *implies that* $\mathsf{L}^\# \models \mathsf{ZFC}^\#$. □

Remark 5.2.37. In Lemma 5.2.46 we will prove that $\mathbf{\Pi}^1_2$ is the optimal (boldface) complexity for which we require the perfect set property to hold in order to prove that $\mathsf{L}^\# \models \mathsf{ZFC}^\#$.

5.2.3 $\mathbf{\Pi}^1_1$-determinacy in SOA

We consider infinite games for two players with perfect information. The idea is that in each round one of the players chooses a natural number and this is repeated ω many times such that one obtains a sequence of natural numbers, i.e. a real in the Baire space. For notational convenience, we work in $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$.

Definition 5.2.38. For a class $A \subseteq {}^\omega\omega$ of reals the game $\mathcal{G}(A)$ is defined in the following way. To start with, Player I chooses a natural number $k_0 \in \omega$. Assuming that $k_0, \ldots, k_n$ have been selected, in the case that n is even, $k_{n+1} \in \omega$ is chosen by Player II, in the odd case by Player I. If the resulting $x \in {}^\omega\omega$ given by $x(n) = k_n$ is an element of A (the so-called *payoff class*), Player I wins $\mathcal{G}(A)$, else Player II wins. A *winning strategy* for Player I is a function $\sigma : \bigcup_{n\in\omega} \omega^{2n} \to \omega$ such that Player I wins $\mathcal{G}(A)$, if he plays as follows.

(1) $k_0 = \sigma(\emptyset)$.
(2) Given $k_0, \ldots, k_{2n+1}$, Player I plays $k_{2n+2} = \sigma(\langle k_0, \ldots, k_{2n+1}\rangle)$.

Similarly, a winning strategy for Player II is of the form $\sigma : \bigcup_{n\in\omega} \omega^{2n+1} \to \omega$.
A is said to be *determined*, if either Player I or Player II has a winning strategy.

Remark 5.2.39. If A is a Σ^1_n-class of reals, then the statment "σ is a winning strategy for A" is Π^1_{n+1} in the parameter σ.

Definition 5.2.40. Let Γ be a class of formulas. We define Γ-Det to be the axiom scheme that every class of reals definable by a formula in Γ is determined.

Note that it is equivalent to consider games with payoff classes in the Cantor space ${}^\omega 2$ instead of the Baire space ${}^\omega\omega$. The following classical result was first proved in [Dav64]. For the benefit of the reader, we will provide its proof following [Kan09, Proposition 27.5].

Lemma 5.2.41. *For any class Γ of $\mathcal{L}_2$-formulae,* Γ-Det *implies* Γ-PSP.

Proof. Let $A \subseteq {}^\omega 2$ be a Γ-class of reals. Consider the game

Player I:	s_0		s_2		s_4		$\ldots$
Player II:		k_1		k_3		$\ldots$	

where each $s_i \in {}^{<\omega}2$ and $k_i \in \omega$. Then Player I wins iff $x = s_0 {}^\frown \langle k_1\rangle {}^\frown s_2 {}^\frown \langle k_3\rangle {}^\frown \ldots$ is in A, otherwise Player II wins. We will show the following.

(1) If Player I has a winning strategy, then A has a perfect subclass.
(2) If Player II has a winning strategy, then A is countable.

For (1) consider a winning strategy σ for Player I. Let T be the tree consisting of partial plays of the game $\mathcal{G}(A)$ where Player I plays according to σ, i.e. the closure under initial segments of the set of possible positions when Player I plays according to σ. Then T is obviously perfect and $[T] \subseteq A$.

In order to prove (2), assume that Player II has a winning strategy τ. Consider a position $p = \langle s_0, k_1, \cdots, s_{2n}, k_{2n+1}\rangle$ in $\mathcal{G}(A)$, i.e. $k_1 = \tau(s_0), k_3 = \tau(s_0 {}^\frown \langle k_1\rangle {}^\frown s_2)$ and so on.

Claim 1. *For every $x \in A$ there exists a position p such that $p \subseteq x$ and for every $s \in {}^{<\omega}2$, $p {}^\frown s {}^\frown \tau(p {}^\frown s) \nsubseteq x$.*

Proof. If not, take $x \in A$ witnessing the contrary. We define a strategy σ for Player I as follows: For every position p in the game $\mathcal{G}(A)$ such that $p \subseteq x$ let $\sigma(p) = s$ with the property that $p {}^\frown s {}^\frown \tau(p {}^\frown s) \subseteq x$, otherwise let $\sigma(p) = 0$. Consider the play where Player I plays according to σ and Player II according to τ. Then the resulting real is x, contradicting the assumption that τ is a winning strategy. □

Claim 2. *For every position p there exists at most one $x \in A$ such that $p \subseteq x$ and for every $s \in {}^{<\omega}2$, $p {}^\frown s {}^\frown \tau(p {}^\frown s) \nsubseteq x$.*

Proof. By contradiction, take a position p and $x, y \in A$ witnessing the contrary. Let q be the longest sequence in ${}^{<\omega}2$ such that $x \restriction \mathrm{lh}(q) = y \restriction \mathrm{lh}(q)$. Then obviously $p \subseteq q$ and by assumption $q {}^\frown \tau(q) \nsubseteq x, y$. But then $q {}^\frown \langle 1 - \tau(q)\rangle \subseteq x, y$ which contradicts the maximality of q. □

Claims 1 and 2 imply that there is an injection from A into the set of positions in the game $\mathcal{G}(A)$, and hence A is countable. □

We will now consider the axiom of $\boldsymbol{\Pi^1_1}$-determinacy. In ZFC, this can be characterized in the following way.

Theorem 5.2.42 (ZFC, Martin-Harrington, [Mar70, Har78])**.** *For any real x, $\Pi^1_1[x]$-determinacy holds if and only if $x^\#$ exists.*

Harrington's proof that $\Pi^1_1[x]$-determinacy implies the existence of $x^\#$ applies determinacy to a $\Sigma^1_1[x]$-class $A[x]$, where it is shown that Player II cannot have a winning strategy in the game $\mathcal{G}(A[x])$. Now since $\Pi^1_1[x]$-determinacy is equivalent to $\Sigma^1_1[x]$-determinacy, $\Pi^1_1[x]$-determinacy implies that $A[x]$ is determined. In SOA, we have the following.

Lemma 5.2.43. *If Π^1_1-Det + $\boldsymbol{\Pi^1_1}$-PSP holds, then $0^\#$ exists.*

Proof. Let A denote the Σ^1_1-class of reals considered in Harrington's proof. By assumption, A is determined. Suppose first that there is a winning strategy τ for Player II. Since τ can be coded by a real, the $\boldsymbol{\Pi^1_1}$-PSP implies that $\mathsf{L}[\tau]$ is a model of ZFC which contains a winning strategy for Player II. However, this is impossible by the proof of Harrington's Theorem. It follows that Player I has a winning strategy σ. By assumption, $\mathsf{L}[\sigma] \models$ ZFC. Now since the statement "σ is a winning strategy for Player I" is Π^2_1, Shoenfield absoluteness implies that σ is a winning strategy in $\mathsf{L}[\sigma]$ and so $0^\#$ exists in $\mathsf{L}[\sigma]$. But then $0^\#$ exists by Theorem 5.2.27. □

However, it is an open question whether in second-order arithmetic the existence of $0^{\#}$ follows from Π^1_1-determinacy. Harrington's proof of the existence of $0^{\#}$ is based on the following principle, usually called *Harrington's Principle*.

$$(\mathsf{HP}) \quad \exists x \forall \alpha \in \mathrm{Ord}\,(\alpha \text{ is } X\text{-admissible } \rightarrow \alpha \text{ is a cardinal in } \mathsf{L}).$$

i.e. Harrington showed in [Har78] that analytic determinacy entails (HP) and then applied a theorem of Silver stating that (HP) implies the existence of $0^{\#}$. However, Yong and Schindler [CS15] ruled out the possibility of transferring this proof to SOA by showing that $\mathsf{SOA} + \mathsf{HP}$ is equiconsistent with ZFC. In particular, this implies that a positive answer to the question above would yield a new proof of Harrington's Theorem.

Lemma 5.2.44. *If* $(\mathsf{ZFC}^-)^{\#}$ *holds, then* $\mathsf{L}[x] \models \mathsf{ZFC}$ *for every real* x.

Proof. By the proof of Theorems 5.2.10 and Theorem 5.2.11 it is enough to check that ${}^{\omega}\omega \cap \mathsf{L}[x]$ is countable for every real x. By assumption, $x^{\#}$ exists. Every real $y \in \mathsf{L}[x]$ is of the form $t(x_0, \cdots, x_{n-1})$ for some Skolem term $t(v_0, \cdots, v_{n-1})$ and indiscernibles $x_0, \cdots, x_{n-1} \in x^{\#}$. In particular, in $\mathsf{L}[x^{\#}]$ we can define a real

$$z = \{\langle k, \ulcorner t(c_0, \cdots, c_{n-1}\urcorner\rangle \mid \mathsf{L}[x] \models k \in t(x_0, \cdots, x_{n-1})\}.$$

But then z codes every real in ${}^{\omega}\omega \cap \mathsf{L}[x]$, proving that there are only countably many reals which are constructible from x. □

Theorem 5.2.45. $\mathbf{\Pi^1_1}$*-determinacy holds if and only if* $a^{\#}$ *exists for every set of ordinals* $a \subseteq \mathrm{Ord}$.

Proof. Suppose first that $\mathbf{\Pi^1_1}$-Det holds and let $a \subseteq \mathrm{Ord}$ be a set of ordinals. By $\mathsf{V} = \mathsf{HC}$, a is countable and hence we can enumerate a as $a = \{\alpha_n \mid n \in \omega\}$. Countable ordinals correspond to well-orderings of ω, so we can find a real x such that for every $n \in \omega$, $\|(x)_n\| = \alpha_n$. In particular, $a \in \mathsf{L}[x]$. We show that $x^{\#}$ exists. Consider the game $\mathcal{G}(A[x])$. By $\mathbf{\Pi^1_1}$-Det, $\mathcal{G}(A[x])$ is determined. As in the proof of Lemma 5.2.43, Player II cannot have a winning strategy, so there is a winning strategy σ for Player I. Since $\mathbf{\Pi^1_1}$-Det implies $\mathbf{\Pi^1_1}$-PSP, $\mathsf{L}[x, \sigma] \models \mathsf{ZFC}$ and by Shoenfield Absoluteness, σ is a winning strategy in $\mathcal{G}(A[x])$ in $\mathsf{L}[x, \sigma]$. But this implies that $x^{\#}$ exists in $\mathsf{L}[x, \sigma]$. Work in $\mathsf{L}[x, \sigma]$. There is a closed unbounded class of indiscernibles $\bar{I}$ for $\mathsf{L}[x]$ containing all uncountable cardinals. In particular, by Theorem 5.2.31, there is a cardinal $\kappa > \omega$ such that $a \in \mathsf{L}_\kappa[x]$ and $I_\kappa = \bar{I} \cap \kappa$ is a set of indiscernibles of order type κ for $\mathsf{L}_\kappa[x]$. Now we can write $a = t^{\mathsf{L}_\kappa[x]}(\vec{u})$ for a Skolem term t and indiscernibles $\vec{u} \in I_\kappa$. But then $J = I_\kappa \setminus \sup \vec{u}$ is a set of indiscernibles for $\mathsf{L}_\kappa[a]$ of order type κ. Therefore, Lemma 5.2.30 implies that $a^{\#}$ exists in $\mathsf{L}[x, \sigma]$. Now by Theorem 5.2.27, $a^{\#}$ exists.

Conversely, suppose that every set of ordinals has a sharp. Let φ be a Π^1_1-formula containing the parameter x and let A be the class defined by φ. By Lemma 5.2.44, $\mathsf{L}[x^{\#}] \models \mathsf{ZFC} + x^{\#}$ exists. But by the Martin-Harrington Theorem this implies that $\mathsf{L}[x^{\#}] \models \mathsf{ZFC} + \Pi^1_1[x]$-$\mathsf{Det}$. So in $\mathsf{L}[x^{\#}]$ the game $\mathcal{G}(A)$ is determined. But by Shoenfield absoluteness every winning strategy for $\mathcal{G}(A)$ in $\mathsf{L}[x^{\#}]$ already exists in V, proving that $\mathcal{G}(A)$ is determined. □

It follows from Theoerems 5.2.45 and 5.2.35 that $\mathsf{SOA}+\mathbf{\Pi^1_1}\text{-}\mathsf{Det}+\mathbf{\Pi^1_2}\text{-}\mathsf{PSP}$ implies that $\mathsf{L}^\#$ is a model of $\mathsf{ZFC}^\#$. The next lemma shows that we cannot drop the $\mathbf{\Pi^1_2}$-PSP.

Lemma 5.2.46. $\mathsf{SOA}+\mathbf{\Pi^1_1}\text{-}\mathsf{Det}$ *does not imply that* $\mathsf{L}^\# \models \mathsf{ZFC}^\#$.

Proof. Suppose the contrary. We start with a model M of $\mathsf{ZFC}^\#$. Then $M^2 = \langle \omega^M, \mathcal{P}(\omega)^M \rangle$ is a model of $\mathsf{SOA}+\mathbf{\Pi^1_1}\text{-}\mathsf{Det}$ with $M^2 \in M$. Iteratively, we can construct an $\in$-sequence $\langle N_i \mid i \in \omega \rangle$ in M of models of $\mathsf{SOA}+\mathbf{\Pi^1_1}\text{-}\mathsf{Det}$ as follows.

- Let $N_0 = M^2$.
- Suppose that N_i is given. Then $(\mathsf{L}^\#)^{N_i} \models \mathsf{ZFC}^\#$. We set $N_{i+1} = ((\mathsf{L}^\#)^{N_i})^2$, the analytical part of $(\mathsf{L}^\#)^{N_i}$. Then N_{i+1} is a model of $\mathsf{SOA}+\mathbf{\Pi^1_1}\text{-}\mathsf{Det}$ with $N_{i+1} \in (\mathsf{L}^\#)^{N_i} \subseteq N_i$.

But the existence of such a sequence of models contradicts the axiom of foundation in M. □

5.3 Collapsing the ordinals

Starting with a countable transitive model M of ZFC, we will use the class version of the Lévy collapse (see Section 1.3.4) in order to produce a model of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}+$ full topological regularity, i.e. where all projective classes satisfy the perfect set property, the Baire property and are Lebesgue measurable. If M is in fact a model of $\mathsf{ZFC}^\#$ then the generic extension will additionally satisfy $\mathbf{\Pi^1_1}$-determinacy. Combining this with the results from Sections 5.2 and 5.2 we obtain new equiconsistency results.

We fix a countable transitive model $M \models \mathsf{ZFC}$ and the corresponding GBc-model $\mathbb{M} = \langle M, \mathrm{Def}(M) \rangle$. Let $\mathbb{P}$ denote $\mathrm{Col}(\omega, <\mathrm{Ord})^M$, the class-sized Lévy collapse. We have already observed in Section 1.3.4 that the power set axiom fails in every $\mathbb{P}$-generic extension of $\mathbb{M}$. Since $\mathbb{P}$ is pretame (see Examples 2.2.10 and 4.1.9), every $\mathbb{P}$-generic extension $\mathbb{M}[G]$ satisfies GBc^-. Moreover, by Lemma 1.3.10, every set in $M[G]$ is countable. We show that the proof of the perfect set property in the Solovay model (see [Sol70]) can be transferred to our setting.

Notation. For δ an ordinal in Ord^M, we define the following variants of collapse forcing notions:

$$\mathrm{Col}(\omega, \delta) = \{p \in \mathrm{Col}(\omega, <\mathrm{Ord})^M \mid \mathrm{dom}(p) \subseteq \omega \times \{\delta\}\}$$
$$\mathrm{Col}(\omega, \geq \delta) = \{p \in \mathrm{Col}(\omega, <\mathrm{Ord})^M \mid \mathrm{dom}(p) \subseteq \omega \times \mathrm{Ord}^M \setminus \delta\}.$$

Lemma 5.3.1. *The following properties hold.*

(1) *Let $\alpha \in \mathrm{Ord}^M$ be an ordinal and $\mathbb{P}_0 = \mathrm{Col}(\omega, <\alpha)$ and $\mathbb{P}_1 = \mathrm{Col}(\omega, \geq \alpha)$. Then a filter G is $\mathbb{P}$-generic over M if and only if $G = \{p \cup q \mid p \in G_0 \wedge q \in G_1\}$, where G_0 is $\mathbb{P}_0$-generic over M and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.*

(2) *Suppose that G is $\mathbb{P}$-generic and $f : \gamma \to \mathrm{Ord}^M$ is a function in $M[G]$. Then there is an ordinal δ such that $x \in M[G \cap \mathrm{Col}(\omega, <\delta)]$.*

Proof. (1) This is a direct consequence of the product Lemma 2.1.11, since

$$p \mapsto \langle p \cap \mathrm{Col}(\omega, < \alpha), p \cap \mathrm{Col}(\omega, \geq \alpha)\rangle$$

defines an isomorphism between $\mathbb{P}$ and $\mathbb{P}_0 \times \mathbb{P}_1$.

(2) Let $\dot{f} \in M^{\mathbb{P}}$ be a name for f and $p \in G$ such that $p \Vdash^{\mathbb{M}}_{\mathbb{P}}$ "$\dot{f} : \check{\gamma} \to \mathrm{Ord}^M$". For each $\alpha < \gamma$, let

$$D_\alpha = \{q \leq_{\mathbb{P}} p \mid \exists \xi \in \mathrm{Ord}^M \ (q \Vdash^M_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\xi})\}.$$

Then each D_α is M-definable and dense below p, so by pretameness there are $q \leq_{\mathbb{P}} p$ in G and $\langle d_\alpha \mid \alpha < \gamma\rangle \in M$ such that $d_\alpha \subseteq D_\alpha$ and d_α is predense below q for each $\alpha < \gamma$. Now choose an antichain $a_\alpha \subseteq d_\alpha$ which is maximal in d_α. Let $\delta \in \mathrm{Ord}^M$ be an ordinal such that for every $r \in \bigcup_{\alpha<\gamma} a_\alpha$, $\mathrm{dom}(r) \subseteq \delta \times \omega$. Then we have

$$f(\alpha) = \xi \iff r \Vdash^M_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\xi},$$

where r is the unique element in $a_\alpha \cap G$. In particular, this implies that $f \in G \cap \mathrm{Col}(\omega, < \delta)$.

□

We need the following standard result for set forcing.

Lemma 5.3.2. *[Kan09, Proposition 10.20] Let $\delta \in \mathrm{Ord}^M$ be an ordinal and $\mathbb{Q}$ a set-sized separative partial order such that $|\mathbb{Q}| \leq |\delta|$ and*

$$\mathbb{1}_{\mathbb{Q}} \Vdash^M_{\mathbb{Q}} \text{“}\exists f\,(f : \check{\omega} \to \check{\delta} \textit{ surjective } \wedge f \notin \check{M})\text{”}.$$

Then there is an injective dense embedding of a dense subset of $\mathrm{Col}(\omega, \delta)$ *into* $\mathbb{Q}$. □

The following lemma shows that the Lévy collapse absorbs every forcing notion which adds a single real. The set version of this result was first proved by Solovay in [Sol70]. We follow [Kan09, Proposition 10.21].

Lemma 5.3.3. *If G is $\mathbb{P}$-generic over M, then for every $x \in M[G]$ with $x : \omega \to \mathrm{Ord}^M$ there is a $\mathbb{P}$-generic filter H over $\langle M[x], \mathrm{Def}(M[x])\rangle$ such that $M[G] = M[x][H]$, where $M[x]$ is the smallest model of* ZFC *containing $M \cup \{x\}$.*

Proof. By Lemma 5.3.1 (2) there is an ordinal δ such that $x \in M[G \cap \mathrm{Col}(\omega, < \delta)]$. Consider the following filters.

$$G_0 = G \cap \mathrm{Col}(\omega, < \delta),$$
$$G_1 = G \cap \mathrm{Col}(\omega, \delta),$$
$$G_2 = G \cap \mathrm{Col}(\omega, \geq (\delta + 1)).$$

It follows from 5.3.1 (1) that G_0 is $\mathrm{Col}(\omega, < \delta)$-generic over $\mathbb{M}$. Since $\mathrm{Col}(\omega, < \delta)$ is a set-sized partial order, by [Kan09, Proposition 10.10] there is a set-sized partial order $\mathbb{Q}$ and a $\mathbb{Q}$-generic filter H_0 over $M[x]$ such that $M[x][H_0] = M[G_0]$. Since set-sized forcing notions always have a separative quotient which is forcing equivalent to the original forcing notion, we may additionally assume that $\mathbb{Q}$ is separative. We work in $M[x]$ and set $\mathbb{R} = \mathbb{Q} \times \mathrm{Col}(\omega, \delta)$. By the product lemma for set-sized forcing notions it follows that $M[x][H_0][G_1]$ is an $\mathbb{R}$-generic extension of $M[x]$. Observe that $|\mathbb{R}| \leq |\mathrm{Col}(\omega, <\delta+1)| = |\delta|$ and $\mathbb{R}$ adds a surjective function from ω onto δ, since $\mathrm{Col}(\omega, \delta)$ does so. By Lemma 5.3.2 there is a $\mathrm{Col}(\omega, \delta)$-generic filter H_1 over $M[x]$ such that $M[x][H_1] = M[x][H_0][G_1]$. Moreover, a further application of Lemma 5.3.2 yields that $\mathrm{Col}(\omega, \delta)$ and $\mathrm{Col}(\omega, <\delta+1)$ are forcing equivalent, so there is a $\mathrm{Col}(\omega, <\delta+1)$-generic filter H_2 such that $M[x][H_2] = M[x][H_1]$. Applying Lemma 5.3.1 (1) several times, we obtain

$$M[G] = M[G_0][G_1][G_2] = M[x][H_0][G_1][G_2] = M[x][H_2][G_2]$$

and $H_2 \times G_2$ is $\mathbb{P}$-generic over $\langle M[x], \mathrm{Def}(M[x])\rangle$. □

The following definition generalizes the concept of weak homogeniety to class-sized forcing notions.

Definition 5.3.4. A notion of class forcing $\mathbb{Q}$ is said to be *weakly homogeneous* over a model $\mathbb{N} = \langle N, \mathcal{C}\rangle$ of GB^-, if for all $p, q \in \mathbb{Q}$ there is an automorphism π of $\mathbb{Q}$ in $\mathcal{C}$ such that $\pi(p)$ and q are compatible.

The next lemmata show that, as in the case of set forcing, the maximal element of a weakly homegeneous forcing notion already decides all statements in the forcing language which involve only canonical names and that the class-sized Lévy collapse is weakly homogeneous. The proofs are standard and their set-sized counterparts can be found in [Kan09, Proposition 10.19]

Lemma 5.3.5. *If a notion of class forcing $\mathbb{Q}$ for $\mathbb{N} = \langle N, \mathcal{C}\rangle \models \mathsf{GB}^-$ is weakly homogeneous and satisfies the forcing theorem over $\mathbb{N}$ then for any $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ and for all sets $x_0, \ldots, x_{n-1} \in N$ either $\mathbb{1}_\mathbb{Q} \Vdash^\mathbb{N}_\mathbb{Q} \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ or $\mathbb{1}_\mathbb{Q} \Vdash^\mathbb{M}_\mathbb{Q} \neg\varphi(\check{x}_0, \ldots, \check{x}_{n-1})$.*

Proof. Suppose the contrary. Since $\mathbb{Q}$ satisfies the forcing theorem, there are $p, q \in \mathbb{Q}$ such that $p \Vdash^\mathbb{N}_\mathbb{Q} \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ and $q \Vdash^\mathbb{N}_\mathbb{Q} \neg\varphi(\check{x}_0, \ldots, \check{x}_{n-1})$. Now let $\pi : \mathbb{Q} \to \mathbb{Q}$ be an automorphism with the property that $\pi(p) \parallel_\mathbb{Q} q$. Since $\pi^*(\check{x}_i) = \check{x}_i$ for all $i < n$, we have $\pi(p) \Vdash^\mathbb{N}_\mathbb{Q} \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ contradicting that $\pi(p)$ is compatible with q. □

Lemma 5.3.6. *The Lévy collapse $\mathbb{P} = \mathrm{Col}(\omega, <\mathrm{Ord})^M$ is weakly homogeneous.*

Proof. Given $p, q \in \mathbb{P}$, let $f : \omega \to \omega$ be a bijection such that for all $\langle n, \alpha\rangle \in \mathrm{dom}(p)$, $\langle f(n), \alpha\rangle \notin \mathrm{dom}(q)$. Let π be the automorphism on $\mathbb{P}$ given by

$$\pi(r) = \{\langle\langle f(n), \alpha\rangle, \gamma\rangle \mid \langle\langle n, \alpha\rangle, \gamma\rangle \in r\}.$$

Then $\mathrm{dom}(\pi(p))$ and $\mathrm{dom}(q)$ are disjoint, and so $\pi(p)$ and q are compatible. □

We are now ready to prove that in every $\mathbb{P}$-generic extension, every projective class of reals satisfies the perfect set property. We follow the arguments of [Kan09, Theorem 11.11].

Theorem 5.3.7. *If G is $\mathbb{P}$-generic over $\mathbb{M}$ then $\mathbb{M}[G] \models$ projective* PSP.

Proof. We consider the class of reals to be the Cantor space ${}^{\omega}2$. Let $\mathcal{C}$ be a class of reals definable by $y \in \mathcal{C}$ if and only if $\varphi(x, y)$ for some real parameter $x \in M[G]$. Without loss of generality, we assume that $\mathcal{C}$ is uncountable in $M[G]$. We prove that $\mathcal{C}$ contains a perfect subclass. Since $\mathcal{C}$ is uncountable and ${}^{\omega}2 \cap M[x]$ is countable in $M[G]$, there must be some $y \in \mathcal{C} \setminus M[x]$. By Lemma 5.3.3 there is a $\mathbb{P}$-generic filter H over $M[x]$ such that $M[x][H] = M[G]$. Moreover, by Lemma 5.3.1 (2) there is an ordinal δ such that $y \in M[x][H \cap \mathrm{Col}(\omega, <\delta)]$. Let $\mathbb{Q}$ denote the set-sized partial order $\mathrm{Col}(\omega, <\delta)$. A further application of Lemma 5.3.3 yields a $\mathbb{P}$-generic filter I over $M[x][y]$ such that $M[x][y][I] = M[x][H]$. Moreover, $M[x][y][I] \models \varphi(x, y)$, so using Lemmata 5.3.5 and 5.3.6 we obtain that $M[x][y] \models \tilde{\varphi}(x, y)$, where $\tilde{\varphi}(u, v)$ is the formula defined by $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\check{u}, \check{v})$. Let $\hat{u}$ denote the canonical name of $u \in M[x]$ with respect to $\mathbb{Q}$; and analogously we will use the notation $\hat{C}$ for classes which are definable over $M[x]$. Since $\mathbb{Q}$ satisfies the forcing theorem, there are $p \in \mathbb{Q}$ and a $\mathbb{Q}$-name $\dot{y}$ for y such that

$$p \Vdash^{M}_{\mathbb{Q}} \dot{y} \in {}^{\omega}2 \setminus \widehat{M[x]} \wedge \widehat{M[x]}[\dot{y}] \models \tilde{\varphi}(\hat{x}, \dot{y}).$$

Observe that $\mathcal{P}^{M[x]}(\mathbb{Q})$ is countable in $M[G]$, so we can enumerate the dense subsets of $\mathbb{Q}$ in $M[x]$ by $\langle D_n \mid n \in \omega \rangle$ in $M[G]$. Now the idea is to approximate y in such a way that we obtain a perfect class of different possible reals. We define a condition $p_t \leq_{\mathbb{Q}} p$ for each $t \in {}^{<\omega}2$ in the following way.

- Choose $p_\emptyset \leq_{\mathbb{Q}} p$ such that $p_\emptyset \in D_0$.
- Given p_t such that $\mathrm{lh}(t) = n$, let $k \in \omega$ be minimal such that p_t does not decide $\dot{y}(k)$. Then by density of D_{n+1} we can find $p_{t^\frown\langle 0\rangle}, p_{t^\frown\langle 1\rangle} \leq_{\mathbb{Q}} p_t$ with $p_{t^\frown\langle 0\rangle}, p_{t^\frown\langle 1\rangle} \in D_{n+1}$ which decide distinct values of $\dot{y}(k)$.

Now for every real $z \in {}^{\omega}2$, we obtain a $\mathbb{P}$-generic filter by setting

$$G_z = \{q \in \mathbb{Q} \mid \exists n \in \omega\, (p_{z\restriction n} \leq_{\mathbb{Q}} q)\} \in M[G].$$

Thus by construction we obtain that the reals given by $\dot{x}^{G_z}$, $z \in {}^{\omega}2$, are all distinct. Now consider the subclass of $\mathcal{C}$ given by $\mathcal{P} = \{\dot{y}^{G_z} \mid x \in {}^{\omega}2\}$. Note that $\mathcal{P} = [T]$, where T is the perfect tree consisting of all finite sequences $s \in {}^{<\omega}2$ such that $p_t \Vdash^{M}_{\mathbb{Q}} \dot{y} \restriction \check{n} = \check{s}$ for some $n \in \omega$ and some $t \in {}^{<\omega}2$. By Lemma 5.3.3, we have that $[T] \subseteq \mathcal{C}$. This proves that $\mathcal{C}$ has the perfect set property. □

We denote the property that all projective classes of reals are Lebesgue measurable, satisfy the Baire property and the perfect set property by *projective topological regularity.* Note that we can also prove – as in the Solovay model constructed in [Sol70] (see also [Kan09, Chapter 11]) – that in every $\mathbb{P}$-generic extension projective topological regularity holds. By combining this observation with Theorems 5.2.11 and 5.3.7 we obtain the following equiconsistency result.

Corollary 5.3.8. *The following theories are equiconsistent.*

(1) $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{PSP}$,
(2) $\mathsf{SOA} + \mathbf{\Sigma^1_2}$-$\mathsf{PSP}$,
(3) $\mathsf{SOA}+$ *projective topological regularity,*
(4) ZFC. □

The next step is to show that the Lévy collapse does not affect the existence of sharps. For this we will apply the following characterization which is provable in ZFC. Its proof can be found in [Jec03, Theorem 18.12].

Theorem 5.3.9 (Kunen). *Suppose that M is a model of* ZFC. *For a set $a \subseteq \mathrm{Ord}^M$ the following statements are equivalent.*

(1) *$a^\#$ exists.*
(2) *There is a non-trivial elementary embedding $\langle \mathsf{L}[a], a, \xi\rangle_{\xi \leq \sup a} \to \langle \mathsf{L}[a], a, \xi\rangle_{\xi \leq \sup a}$.* □

Theorem 5.3.10. *Suppose that $M \models \mathsf{ZFC}^\#$ and $\delta \in \mathrm{Ord}^M$. If G is $\mathrm{Col}(\omega, < \delta)$-generic over M, then $M[G] \models \mathsf{ZFC}^\#$.*

Proof. Let $\mathbb{P} = \mathrm{Col}(\omega, < \delta)$ and $a \subseteq \mathrm{Ord}^{M[G]}$ a set of ordinals in $M[G]$. Without loss of generality, we may assume that $\delta \in \mathrm{Ord}^M$ is minimal such that a is in a $\mathrm{Col}(\omega, < \delta)$-generic extension of M (otherwise we show the existence of $a^\#$ in an intermediate model).

Choose a nice $\mathbb{P}$-name $\sigma \in M^{\mathbb{P}}$ for a. Clearly, we can code σ by a set of ordinals. Let $\gamma = \sup \sigma$, where we identify σ with its ordinal code. By minimality of δ we may assume that $\delta \leq \gamma$. Since $M \models \mathsf{ZFC}^\#$, there exists an elementary embedding

$$j : \langle \mathsf{L}[\sigma], \sigma, \xi\rangle_{\xi \leq \gamma} \prec \langle \mathsf{L}[\sigma], \sigma, \xi\rangle_{\xi \leq \gamma}.$$

We want to use this to construct an elementary embedding $\bar{j} : \mathsf{L}[a] \prec \mathsf{L}[a]$. Note that since $\mathbb{P} \subseteq \mathsf{L}$, G is also $\mathbb{P}$-generic over $\mathsf{L}[\sigma]$. We now consider the forcing extension $\mathsf{L}[\sigma] \subseteq \mathsf{L}[a] \subseteq \mathsf{L}[\sigma][G]$. Work in $M[G]$ and define

$$\bar{j} : \langle \mathsf{L}[a], a, \xi\rangle_{\xi \leq \alpha} \to \langle \mathsf{L}[a], a, \xi\rangle_{\xi \leq \alpha},\ \bar{j}(\tau^G) = (j(\tau))^G,$$

where $\alpha = \sup a$. Now to show that $\bar{j}$ is well-defined, note that since $\delta \leq \gamma$, we have $j(p) = p$ for every $p \in \mathbb{P}$. In particular, $\mathbb{P}$-names are always mapped to $\mathbb{P}$-names under j. Secondly, we have to check that if τ, π are $\mathbb{P}$-names in $\mathsf{L}[\sigma]$, then $\tau^G = \pi^G$ in $\mathsf{L}[\sigma][G]$. Using the truth lemma, we can pick $p \in G$ such that in $\mathsf{L}[\sigma]$, $p \Vdash^M_{\mathbb{P}} \tau = \pi$. By elementarity of j, this implies that $\mathsf{L}[\sigma] \models$ "$p \Vdash^M_{\mathbb{P}} j(\tau) = j(\pi)$".

It remains to check that that $\bar{j}$ is an elementary elembedding. Suppose that $\mathsf{L}[a] \models \varphi(x, a, \vec{\xi})$ for some $x \in \mathsf{L}[a]$ and $\vec{\xi}$ a finite sequence of ordinals $\leq \alpha$. Then there exist a $\mathbb{P}$-name $\tau \in \mathsf{L}[\sigma]^{\mathbb{P}}$ and a condition $p \in G$ such that $x = \tau^G$ and

$$p \Vdash^{\mathsf{L}[\sigma]}_{\mathbb{P}} \varphi^{\mathsf{L}[\sigma]}(\tau, \sigma, \vec{\check{\xi}}).$$

By elementarity of j we obtain

$$p = j(p) \Vdash^{\mathsf{L}[\sigma]}_{\mathbb{P}} \varphi^{\mathsf{L}[\sigma]}(j(\tau), \sigma, \vec{\check{\xi}})$$

and hence $\mathsf{L}[\sigma][G] \models \varphi^{\mathsf{L}[a]}(j(\tau)^G, a, \vec{\xi})$. But this means that $\mathsf{L}[a] \models \varphi(j(x), a, \vec{\xi})$ proving elementarity of j. □

Corollary 5.3.11. *Suppose that $M \models \mathsf{ZFC}^\#$ and let G be $\mathrm{Col}(\omega, < \mathrm{Ord})^M$-generic over $\langle M, \mathrm{Def}(M)\rangle$. Then $M[G] \models (\mathsf{ZFC}^-)^\#$.*

Proof. Let $a \in M[G]$ be a set of ordinals. By Lemma 5.3.1 (2) there exists $\delta \in \mathrm{Ord}^M$ such that $a \in M[G \cap \mathrm{Col}(\omega, <\delta)]$. But since $G \cap \mathrm{Col}(\omega, <\delta)$ is $\mathrm{Col}(\omega, <\delta)$-generic over M, we have that $M[G \cap \mathrm{Col}(\omega, <\delta)] \models \mathsf{ZFC}^{\#}$ by Theorem 5.3.10. Hence $a^{\#}$ exists in $M[G \cap \mathrm{Col}(\omega, <\delta)]$. By absoluteness of $a^{\#}$, this implies that $a^{\#}$ exists in $M[G]$. □

Finally, we can state the main result. It is a direct consequence of Theorems 5.2.35, 5.2.45, 5.3.7 and Corollary 5.3.11.

Theorem 5.3.12. *The following theories are equiconsistent:*

(1) $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{Det} + \mathbf{\Pi^1_2}$-$\mathsf{PSP}$,
(2) $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{Det} + \mathbf{\Sigma^1_3}$-$\mathsf{PSP}$,
(3) $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{Det}+$ *projective topological regularity,*
(4) $\mathsf{ZFC}^{\#}$. □

5.4 Class forcing over models of SOA

In this section, we outline a simple approach to class forcing over models of second-order arithmetic. Since the natural numbers are always preserved under class forcing, and we are only interested in modifying the reals, we will allow just nice names for real numbers. This approach has the advantage that the forcing theorem is provable for all notions of class forcing. In the case of nice pretame forcing notions, this turns out to be the same as performing class forcing in the classical way. In particular, this applies to many tree forcing notions such as random forcing, Sacks forcing and Laver forcing, if the ground model additionally satisfies dependent choice.

Furthermore, using results from Philipp Schlicht and Fabiana Castiblanco (see [CS16]), we can see that all above-mentioned forcing notions preserve the $\mathbf{\Pi^1_1}$-perfect set property. However, this does not hold in general; for example using reshaping and almost disjoint coding we can force a failure of the $\mathbf{\Pi^1_1}$-PSP.

5.4.1 The basic setup

As we have seen, SOA and ZFC^- are bi-interpretable. Hence class forcing over models of SOA can be done in the same way as over models of ZFC^-. However, there are several obstacles: It is still an open question whether in ZFC^- all pretame class forcings satisfy the forcing theorem. Moreover, since the proof of Collection in generic extensions by pretame class forcings makes use of the forcing theorem, it is unclear whether ZFC^- is preserved under pretame class forcing. One way to solve this problem, is to add a global well-order by a $<$ Ord-closed forcing notion and then code the generic well-order using reshaping and almost disjoint coding. This approach was followed by Carolin Antos in [Ant15]. This is problematic for various reasons. Firstly, this modifies the ground model in an unintended way; instead of performing one forcing, we are in fact performing multiple forcings. Secondly, some properties of forcing notions can get lost if we change the ground model. One example is the Ord-cc (see Lemma 4.1.4). The solution that we propose here is to modify the setting of class forcing by allowing only names for reals, and taking the natural numbers as constants. This has the advantage that it is very easy to prove the forcing theorem. Moreover, this approach is more intuitive, since we do not wish to change the natural numbers of an SOA-model by forcing anyway. Our approach resembles the one proposed by Victoria Gitman in the blog posts [Git13b] and [Git13a] for forcing over models of GBC.

A *notion of class forcing* for a model $\mathcal{A} = \langle \mathbb{N}, \mathbb{R} \rangle \models \mathsf{SOA}$ is a pair $\mathbb{P} = \langle \mathcal{P}, \leq_{\mathbb{P}} \rangle$[5] such that $\mathcal{P}$ is a definable class of reals and $\leq_{\mathbb{P}}$ a definable preorder on $\mathcal{P}$. As in the context of GB^-, we will usually identify $\mathbb{P}$ with its domain.

Let $\mathbb{P} = \langle \mathcal{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$ be a fixed notion of class forcing. A $\mathbb{P}$-*name* is a real Γ such that each $(\Gamma)_n$ is of the form $\langle m, P \rangle$ for some $m \in \mathbb{N}$ and $P \in \mathbb{P}$, where $\langle m, P \rangle$ is a real coding the pair m, P in a canonical way. A definable subclass $\mathcal{D} \subseteq \mathbb{P}$ is said to be *dense*, if for every $P \in \mathbb{P}$ there is $Q \leq_{\mathbb{P}} P$ with $Q \in \mathcal{D}$. A filter $\mathcal{G} \subseteq \mathbb{P}$ is said to be $\mathbb{P}$-generic over $\mathcal{A}$, if it meets every dense subclass of $\mathbb{P}$ which is definable over $\mathcal{A}$. Given such a $\mathbb{P}$-generic filter and a $\mathbb{P}$-name Γ, we define

$$\Gamma^{\mathcal{G}} = \{n \in \mathbb{N} \mid \exists P \in \mathcal{G}\, (\langle n, P \rangle \in \Gamma)\}$$

the *G-evaluation* of Γ and $\mathbb{R}[\mathcal{G}] = \{\Gamma^{\mathcal{G}} \mid \Gamma \text{ a } \mathbb{P}\text{-name}\}$. Then the model $\mathcal{A}[\mathcal{G}] = \langle \mathbb{N}, \mathbb{R}[\mathcal{G}] \rangle$ is called a $\mathbb{P}$-*generic extension* of $\mathcal{A}$. The forcing relation and the forcing theorem are defined as in Chapter 1, where the classes are simply the definable ones. As in our class-theoretic setting, we write

$$P \Vdash^{\mathcal{A}}_{\mathbb{P}} \varphi(n_0, \ldots, n_{k-1}, \Gamma_0, \ldots, \Gamma_{l-1})$$

for $n_0, \ldots, n_{k-1} \in \mathbb{N}$ and $\mathbb{P}$-names $\Gamma_0, \ldots, \Gamma_{l-1}$, if for every $\mathbb{P}$-generic filter $\mathcal{G}$ over $\mathcal{A}$, $\mathcal{A}[\mathcal{G}] \models \varphi(n_0, \ldots, n_{k-1}, \Gamma_0, \ldots, \Gamma_{l-1})$. The forcing theorem is defined as in the context of class forcing over GB^- (see Definition 2.1.1) and, moreover, one can see that for the forcing theorem to hold it suffices to prove the definability lemma for atomic formula of the form $n \in \Gamma$ for $n \in \mathbb{N}$ and $\mathbb{P}$-names Γ.

Lemma 5.4.1. *Let $\mathcal{A} = \langle \mathbb{N}, \mathbb{R} \rangle \models \mathsf{SOA}$ and let $\mathbb{P}$ be a notion of class forcing for $\mathcal{A}$. Then $\mathbb{P}$ satisfies the forcing theorem over $\mathcal{A}$.*

Proof. We claim that for every $P \in \mathbb{P}$, for every $n \in \mathbb{N}$ and for every $\mathbb{P}$-name Γ,

$$P \Vdash^{\mathcal{A}}_{\mathbb{P}} n \in \Gamma \iff \mathcal{D}_{P,\Gamma} = \{Q \leq_{\mathbb{P}} P \mid \exists R\, (\langle n, R \rangle \in \Gamma \wedge Q \leq_{\mathbb{P}} R)\} \text{ is dense below } P.$$

Suppose first that $P \Vdash^{\mathcal{A}}_{\mathbb{P}} n \in \Gamma$ and $Q \leq_{\mathbb{P}} P$. Take a $\mathbb{P}$-generic filter $\mathcal{G}$ with $Q \in \mathcal{G}$. Then $P \in \mathcal{G}$ and so $n \in \Gamma^{\mathcal{G}}$. But this means that there is $\langle n, R \rangle \in \Gamma$ such that $R \in \mathcal{G}$. Since $\mathcal{G}$ is a filter, we have that R and Q are compatible, so there is $S \in \mathcal{G}$ with $S \leq_{\mathbb{P}} Q, R$. In particular, $S \in \mathcal{D}_{P,\Gamma}$. Conversely, assume that the right-hand side holds and let $\mathcal{G}$ be $\mathbb{P}$-generic over $\mathcal{M}$ with $P \in \mathcal{G}$. By assumption, there is $Q \in \mathcal{G} \cap \mathcal{D}_{P,\Gamma}$. Take R such that $\langle n, R \rangle \in \Gamma$ and $Q \leq_{\mathbb{P}} R$. Then $R \in \mathcal{G}$ and so $n \in \Gamma^{\mathcal{G}}$. □

The notion of pretameness is defined in the same way as in Definition 2.2.1 and, as expected, has the property that it implies the preservation of SOA.

Lemma 5.4.2. *Let $\mathcal{A} = \langle \mathbb{N}, \mathbb{R} \rangle \models \mathsf{SOA}$ and let $\mathbb{P}$ be a pretame notion of class forcing for $\mathcal{A}$. If $\mathcal{G}$ is $\mathbb{P}$-generic over $\mathcal{A}$ then $\mathcal{A}[\mathcal{G}] \models \mathsf{SOA}$.*

[5] Usually, $\mathbb{P}$ will have a maximal element denoted by $\mathbb{1}_{\mathbb{P}}$.

Proof. The only problematic axiom is the axiom of comprehension. For this, let φ be an $\mathcal{L}_2$-formula (possibly with parameters). The proof is a simple version of the proof of collection being preserved by pretame notions of class forcing over models of GB^-. Let $\mathcal{G}$ be $\mathbb{P}$-generic over $\mathcal{A}$. For each $n \in \mathbb{N}$, consider

$$\mathcal{D}_n = \{P \in \mathbb{P} \mid P \Vdash^{\mathcal{A}}_{\mathbb{P}} \varphi(n) \vee P \Vdash^{\mathcal{A}}_{\mathbb{P}} \neg\varphi(n)\}.$$

Using pretameness, we can take $P \in G$ and reals D_n which code an enumeration of a dense subset of $\mathcal{D}_n$ of order type ω such that D_n is predense below P. Then

$$\Gamma = \{\langle n, Q\rangle \mid \exists k\, [Q = (D_n)_k \wedge Q \Vdash^{\mathcal{A}}_{\mathbb{P}} \varphi(n)]\}$$

is a $\mathbb{P}$-name for $\{n \in \mathbb{N} \mid \varphi(n)\}$ in $\mathcal{A}[\mathcal{G}]$. □

The designation “class forcing” in the context of second-order arithmetic may seem somewhat misleading: If we conceive a model $\mathcal{M} = \langle \mathbb{N}, \mathbb{R}\rangle$ of SOA as a second-order model, then the “classes” should be the reals, and so a notion of class forcing would simply be a real. However, we are interested in forcing with classes of reals which in this sense then corresponds to so-called *hyperclass forcing*. This has been studied by Antos and Friedman in [AF15] in the setting of KM. Nevertheless, we do not wish to view the reals as classes, but rather the definable classes of reals; this is closer to our intuition of natural numbers and reals. As Section 5.1 suggests, definable class forcing over models of SOA is closely related to definable class forcing over $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$. Given a forcing notion $\mathbb{P}$ over $\mathcal{A}$ defined by $P \in \mathbb{P}$ iff $\varphi(P, X)$ for some parameter $X \in \mathbb{R}$, we can code the elements of $P \in \mathbb{P}$ by well-founded trees P^* and then consider the notion of class forcing $\mathbb{P}^*$ for $\mathcal{A}_\in$ given by $p \in \mathbb{P}^* \iff \varphi^*(p, X^*)$, where φ^* is the $\mathcal{L}_\in$-formula corresponding to φ as defined in Section 5.1. Now if G^* is $\mathbb{P}^*$-generic over $\mathbb{A}_\in = \langle A_\in, \mathrm{Def}(A_\in)\rangle$ then we can consider $\mathcal{G} \subseteq \mathbb{P}$ to be the filter containing all $P \in \mathbb{P}$ such that $P^* \in G^*$. By the results in Section 5.1, it follows that G^* is $\mathbb{P}^*$-generic over $\mathbb{A}_\in$ if and only if $\mathcal{G}$ is $\mathbb{P}$-generic over $\mathcal{A}$. This is essentially the approach that has been pursued in [AF15] to define hyperclass forcing over models of KM.

However, where our version of class forcing over SOA diverges from this approach which uses our previous results about class forcing over models of ZFC^-, is that we restrict ourselves to nice names and can therefore prove the forcing theorem for all $\mathcal{L}_2$-formulae. Our results in Section 2.5 suggest that this is problematic in GB^-. On the other hand, since we want to preserve SOA – and GB^- respectively – we are only interested in those forcing notions which (both over SOA and GB^-) are pretame and satisfy the forcing theorem. It follows from the argument presented in Example 4.6.4 (2) that every pretame forcing notion which satisfies the forcing theorem is nice, so for such forcing notions both the approach to class forcing in SOA presented in the beginning of this section and the approach of forcing over the corresponding model of ZFC^- coincide. Since all the tools that we have developed in the previous chapters about class forcing are set up in the context of (extensions of) GB^-, we will for the rest of this section consider class forcing over models of the form $\mathbb{M} = \langle M, \mathrm{Def}(M)\rangle$, where $M \models \mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$.

5.4.2 Some tree forcings

Over models of second-order arithmetic, some canonical forcing notions to consider are the so-called *arboreal* forcing notions, i.e. partial orders whose conditions are trees. Prominent examples are Cohen forcing, random forcing, Sacks forcing, Laver forcing, Silver forcing, Miller forcing and Mathias forcing. A property that these examples have in common is Axiom A.

For this section, we fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of $\mathsf{GB}^- + \mathsf{V} = \mathsf{HC}$. The example that we are mostly interested in is, of course, when $M \models \mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ and $\mathcal{C} = \mathrm{Def}(M)$, which corresponds to forcing over a model of SOA as discussed above.

Definition 5.4.3. A notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ for $\mathbb{M}$ *satisfies Axiom A over* $\mathbb{M}$, if there is a sequence $\langle \leq_n \mid n \in \omega\rangle$ in $\mathcal{C}$ of partial orders on P with the following properties.

(1) For all $p, q \in \mathbb{P}$, $p \leq_0 q$ implis $p \leq_{\mathbb{P}} q$ and for all $n \in \omega$, $p \leq_{n+1} q$ implies $p \leq_n q$.
(2) Whenever $\langle p_n \mid n \in \omega\rangle$ is a sequence of conditions with $p_{n+1} \leq_n p_n$ for all $n \in \omega$, then there is $q \in \mathbb{P}$ such that $q \leq_n p$ for all $n \in \omega$.
(3) If $A \subseteq \mathbb{P}$ is an antichain in $\mathcal{C}$ and $p \in \mathbb{P}$, then for every $n \in \omega$ there is $q \leq_n p$ such that $\{a \in A \mid q \parallel_{\mathbb{P}} a\}$ is a set.

Sequences $\langle p_n \mid n \in \omega\rangle$ as in (2) are called *fusion sequences*.

It is easy to check that if $\mathcal{C}$ contains a global well-order of M, then every forcing notion $\mathbb{P}$ which satisfies Axiom A over $\mathbb{M}$ is pretame. However, in the absence of a global well-order, Axiom A forcing notions need not be pretame – even if the ground model additionally satisfies dependent choice.

Remark 5.4.4. Let $M \models \mathsf{ZFC}$ such that there is no global choice function for M which is definable over M. If G is $\mathbb{P} = \mathrm{Col}(\omega, < \mathrm{Ord})^M$-generic over M then there is no global choice function for $M[G]$ in $\mathcal{C}[G]$. To see this, suppose otherwise and let $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in G$ such that

$$p \Vdash^M_{\mathbb{P}} \text{“}\dot{F} \text{ is a global choice function”}.$$

Then we can define a global choice function for M as follows. It is easy to see that $\mathbb{P}$ can be well-ordered canonically by some well-order $\prec$, since it essentially consists of finite sets of ordinals. If $x \in M \setminus \{\emptyset\}$ then let $F(x)$ be the set y such that $q \Vdash^M_{\mathbb{P}} \dot{F}(\check{x}) = \check{y}$ for the $\prec$-least $q \leq_{\mathbb{P}} p$ which decides $\dot{F}(\check{x})$. This shows that the Solovay model satisfies $\mathsf{GBc}^- + \mathsf{V} = \mathsf{HC}$, has a hierarchy but global choice fails. In fact, it will follow from Lemma 5.4.7 that it even satisfies GBdc^-. By Lemma 4.1.7 there is a forcing notion which satisfies the Ord-cc but is not pretame for $\langle M, \mathrm{Def}(M)\rangle$. In particular, this forcing notion satisfies Axiom A.

This suggests that we should modify slightly our definition of Axiom A and consider dense classes instead of antichains.

Definition 5.4.5. A notion of class forcing $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}\rangle$ for $\mathbb{M}$ *satisfies Axiom D over* $\mathbb{M}$, if there is a sequence $\langle \leq_n \mid n \in \omega\rangle \in \mathcal{C}$ of partial orders on P with the following properties.

(1) For all $p, q \in \mathbb{P}$, $p \leq_0 q$ implies $p \leq_{\mathbb{P}} q$ and for all $n \in \omega$, $p \leq_{n+1} q$ implies $p \leq_n q$.
(2) Whenever $\langle p_n \mid n \in \omega\rangle$ is a sequence of conditions with $p_{n+1} \leq_n p_n$ for all $n \in \omega$, then there is $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all $n \in \omega$.

(3) If $D \subseteq \mathbb{P}$ is a dense class in $\mathcal{C}$ and $p \in \mathbb{P}$, then for every $n \in \omega$ there is $q \leq_n p$ and a set $d \subseteq D$ in M which is predense below q.

Notice that Axiom D implies Axiom A. All aforementioned tree forcings satisfy Axiom D; however, in some cases, its proof requires dependent choice. The following observation suggests that $\mathsf{GBdc}^- + \mathsf{V} = \mathsf{HC}$ is actually more natural than $\mathsf{GB}^- + \mathsf{V} = \mathsf{HC}$.

Lemma 5.4.6. *Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GBdc^- and let $\mathbb{P}$ be a notion of class forcing which satisfies Axiom D over $\mathbb{M}$. Then $\mathbb{P}$ is pretame for $\mathbb{M}$.*

Proof. Suppose that $\langle D_i \mid i \in I \rangle$ is a sequence of dense classes in $\mathcal{C}$ and $p \in \mathbb{P}$. Since M satisfies the axiom of choice, we may assume that $I = \kappa$ is an M-cardinal. Using DC we can take sequences $\langle p_n \mid n \in \omega \rangle \in M$ of conditions in $\mathbb{P}$ and $\langle d_n \mid n \in \omega \rangle \in M$ of subsets $d_n \subseteq D_n$ with $p_0 = p$ and such that for each $n \in \omega$, $p_{n+1} \leq_n p_n$ and $d_n \subseteq D_n$ is predense below p_{n+1}. Now using property (1) in Definition 5.4.5, there is $q \in \mathbb{P}$ with $q \leq_n p_n$ for all $n \in \omega$. In particular, each d_n is predense below q. $\square$

Moreover, dependent choice holds in the Solovay model.

Lemma 5.4.7. *Let G be $\mathrm{Col}(\omega, < \mathrm{Ord})^M$-generic over M, where M is a model of ZFC. Then $M[G] \models \mathsf{DC}$.*

Proof. Let $\mathbb{P}$ denote $\mathrm{Col}(\omega, < \mathrm{Ord})^M$. Let $x \in M[G]$ be an arbitary set. Without loss of generality, we may assume that $x \in M$ (otherwise we can do the same arguments in some intermediate model). We construct a sequence $\langle \alpha_n \mid n \in \omega \rangle \in M$ as follows. Let $\alpha_0 = \mathsf{rnk}(x)$. Suppose that

$$p \Vdash^M_{\mathbb{P}} \forall \vec{x} \exists y\, \varphi(\vec{x}, y).$$

Hence $p \Vdash^M_{\mathbb{P}} \exists y \varphi(\check{x}, y)$. Since $\mathbb{P}$ satisfies the Ord-cc by Example 4.1.9, it also satisfies the maximality principle by Lemma 4.2.7. Note that we do not require a global well-order, since $\mathbb{P}$ can be canonically well-ordered. Using the maximality principle, we can take $\alpha_1 \geq \alpha_0$ such that there is $\sigma \in M^{\mathbb{P}} \cap \mathsf{V}_{\alpha_1}$ with $p \Vdash^M_{\mathbb{P}} \varphi(\check{x}, \sigma)$. Given $\langle \alpha_i \mid i \leq n \rangle$ let α_{n+1} be the least ordinal $\alpha \geq \alpha_n$ such that there is a sequence $\langle \sigma_i \mid i \leq n+1 \rangle \in \mathsf{V}_\alpha$ of $\mathbb{P}$-names with $\sigma_0 = \check{x}$ and $p \Vdash^M_{\mathbb{P}} \varphi(\langle \sigma_0, \ldots, \sigma_n \rangle, \sigma_{n+1})$. Now let $\alpha = \sup\{\alpha_n \mid n \in \omega\} + 1$. Now using dependent choice in M, there is a sequence of names $\langle \sigma_n \mid n \in \omega \rangle \in \mathsf{V}_\alpha$ such that for all $n \in \omega$, $p \Vdash^M_{\mathbb{P}} \varphi(\langle \sigma_0, \ldots, \sigma_n \rangle, \sigma_{n+1} \rangle)$ with $\sigma_0 = \check{x}$. In particular, in $M[G]$, $\langle \sigma_n^G \mid n \in \omega \rangle$ witness that DC holds. $\square$

Example 5.4.8. Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of $\mathsf{GB}^- + \mathsf{V} = \mathsf{HC}$. Then *Sacks forcing* $\mathbb{S}$ satisfies Axiom D over $\mathbb{M}$. The proof is the same as the proof that in ZFC Sacks forcing satisfies Axiom A. Recall that the conditions of $\mathbb{S}$ are simply perfect trees in the Cantor space, ordered by inclusion. Let $p \in \mathbb{S}$ be a perfect tree. We say that a node $s \in p$ is a *splitting node*, if both $s^\frown 0$ and $s^\frown 1$ are in p. We say that s is an *n-th splitting node*, if there are are exactly n splitting nodes $t \in p$ with $t \subseteq s$. For $n \in \omega$ we define

$$p \leq_n q \iff p \leq_{\mathbb{S}} q \text{ and every } n\text{-th splitting node of } q \text{ is an } n\text{-th splitting node of } p.$$

If $s \in p$ then we denote by $p \upharpoonright s$ the perfect tree given by

$$p \upharpoonright s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}.$$

Moreover, if x is a set of incompatible nodes in p and $q_s \leq_{\mathbb{S}} p \upharpoonright s$ for each $s \in x$ then the *amalgamation of* $\{q_s \mid s \in x\}$ *into* p is the perfect tree given by

$$\{t \in p \mid \forall s \in x\, (t \subseteq s \rightarrow t \in q_s)\}.$$

We check that $\mathbb{S}$ and $\langle \leq_n \mid n \in \omega\rangle$ satisfy the desired properties. The first condition is obvious. For (2), let $\langle p_n \mid n \in \omega\rangle$ be a fusion sequence in $\mathbb{S}$ and let $p = \bigcap_{n\in\omega} p_n$. We check that $p \in \mathbb{S}$. Let $s \in p$ and let n be the number of splitting nodes in p_0 below s. Then the number of splitting nodes below s in p_{n+1} is at most n, so the next splitting node above s in p_{n+1} is also in p.

For condition (3), let $p \in \mathbb{S}$ be a perfect tree, $n \in \omega$ and $D \subseteq \mathbb{S}$ be a dense class in $\mathcal{C}$. Let $\langle s_i \mid i < 2^{n-1}\rangle$ be the set of all n-th splitting nodes in p. For each $i < 2^{n-1}$, let $q_i \leq_{\mathbb{S}} p \upharpoonright s_i$ with $q_i \in D$ and let $q \in \mathbb{S}$ be the amalgamation of $\{q_i \mid i < 2^{n-1}\}$ into p. By construction, $q \leq_n p$ and $d = \{q_i \mid i < 2^{n-1}\}$ is predense below q.

Note that in the above example the witnessing subset d of the dense class D as in condition (2) of Definition 5.4.5 is a finite set, and we do not need to construct a fusion sequence in order to obtain d. In other cases, the situation is more complicated. The following example shows that sometimes we have to make use of fusion sequences to verify the aforementioned property. In particular, for this we require dependent choice.

Example 5.4.9. If $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of $\mathsf{GBdc}^- + \mathsf{V} = \mathsf{HC}$, then Laver forcing satisfies Axiom D over $\mathbb{M}$. Recall that a *Laver tree* is a tree p in ${}^{<\omega}\omega$ with a stem s_p and the property that every node $t \supseteq s_p$ has infinitely many immediate successors. *Laver forcing* $\mathbb{L}$ is the partial order consisting of all laver trees, ordered by inclusion. Note that we can enumerate ${}^{<\omega}\omega$ in a canonical way such that whenever $s \subset t$, then s appears before t and $s^\frown n$ appears before $s^\frown(n+1)$ for each $n \in \omega$. In particular, we can enumerate in the same way all $t \in p$ with $t \supseteq s_p$ as $s^p_0 = s_p, s^p_1, s^p_2, \ldots$. For $n \in \omega$, we define

$$p \leq_n q \Longleftrightarrow p \leq_{\mathbb{L}} q \wedge \forall i \leq n\, (s^p_i = s^q_i).$$

We verify that $\mathbb{L}$ satisfies Axiom D following [Jec03, Lemma 28.16 and Lemma 28.17]. Again, the first property is clear by definition of $\leq_n$. Secondly, let $\langle p_n \mid n \in \omega\rangle$ be a fusion sequence in $\mathbb{L}$ and $p = \bigcap_{n\in\omega} p_n$. Clearly, $s_p = s_{p_0}$. Now let $s \in p$ be an arbitrary node and let $k \in \omega$. We have to find $l \geq k$ such that $s^\frown l \in p$. Take n sufficently large such that there is some $l \geq k$ with $s^\frown l = s^{p_n}_i$ for some $i \leq n$. Then $s^\frown l = s^p_i$ is in p. This proves that p is a condition in $\mathbb{L}$.

For the last property, let $p \in \mathbb{L}$, $n \in \omega$ and $D \subseteq \mathbb{L}$ a dense class in $\mathcal{C}$. For each $i \leq n$, let

$$p_i = \bigcup\{p \upharpoonright s^p_i{}^\frown k \mid s^p_i{}^\frown k \in p \setminus \{s^p_j \mid j \leq n\}\}.$$

We call each p_i an *n-component* of p. Note that $\{p_0, \ldots, p_n\}$ forms a maximal antichain below p. Fix $i \leq n$. Using DC, we define a fusion sequence $\langle q^i_k \mid k \in \omega\rangle$ with $q^i_0 = p_i$. Suppose that the sequence has been defined up to k. Let $r^i_0, \ldots, r^i_k$ be k-components of

q_k^i. For each $l \leq k$, choose $\bar{r}_l^{i,k} \leq_0 r_l^{i,k}$ such that $\bar{r}_l^{i,k} \in D$, otherwise put $\bar{r}_l^{i,k} = r_l^{i,k}$. Let $d_k^i = \{\bar{r}_l^{i,k} \mid l \leq k\} \cap D$. Finally, let $q_{k+1} = \bigcup_{l \leq k} \bar{r}_l$. Now clearly $q^i = \bigcap_{k \in \omega} q_k^i \leq_0 p_i$. Put $q = \bigcup_{i \leq n} q_i$. Note that $q \leq_n p$ and $d = \bigcup\{d_k^i \mid i \leq n, k \in \omega\}$ is predense below q.

Similar arguments work for Miller forcing, Mathias forcing and Silver forcing. We will now show that all of the examples that we have considered so far satisfy the forcing theorem. This follows from the general observation that each of them has a pre-Boolean completion.

Let $\mathcal{B}$ denote the collection of all Borel classes of reals, i.e. the smallest collection of classes of reals containing all open classes which is closed under taking complements and countable unions. As described in [Jec03, pp. 504-507], each Borel class of reals can be coded by a real. Moreover, if $c \in {}^\omega\omega$ is a *Borel code*, then it describes how the corresponding Borel class B_c is constructed from standard open sets by taking complements and countable unions.

Definition 5.4.10. Let $\mathcal{I} \subseteq \mathcal{B}$ be a definable σ-ideal[6] of Borel classes of reals. Then we can define a quasi-complete Boolean algebra $\mathbb{B}_\mathcal{I}$ whose conditions are Borel codes, where for two Borel codes c and d, $c \wedge d$ is the canonical Borel code for $B_c \cap B_d$, $c \vee d$ is the canonical Borel code for $B_c \cup B_d$, $\neg c$ is the Borel code for ${}^\omega\omega \setminus B_c$, $\mathbb{1}_{\mathbb{B}_\mathcal{I}}$ is the Borel code for ${}^\omega\omega$, $0_{\mathbb{B}_\mathcal{I}}$ is the Borel code for the empty set. Moreover, the equivalence relation $\approx_{\mathbb{B}_\mathcal{I}}$ is given by

$$c \approx_{\mathbb{B}_\mathcal{I}} d \iff B_c \triangle B_d \in \mathcal{I}$$

for all $c, d \in \mathbb{B}_\mathcal{I}$.

Example 5.4.11. The following examples are well-known. For further examples, consult [Kho12] and [Zap08].

(1) *Random forcing* is the forcing notion $\mathbb{B}^*_{\mathsf{null}}$, where null is the σ-ideal of classes of reals of Lebesgue measure zero. Moreover, since every Borel class of positive measure contains a closed class of positive measure, $\mathbb{R}$ is forcing equivalent to the subforcing of $\mathbb{B}_{\mathsf{null}}$ consisting of Borel codes for closed classes of reals.

(2) Consider Sacks forcing $\mathbb{S}$. Let ctbl denote the ideal of countable sets of reals. Then $\mathbb{B}_\mathcal{I}$ is a pre-Boolean completion of $\mathbb{S}$. To see this, let $i : \mathbb{S} \to \mathbb{B}^*_{\mathsf{ctbl}}$ be given by mapping a perfect tree p to the canonical Borel code of $[p]$. This map is clearly injective, and it is dense because every Borel class of reals satisfies the perfect set property.

(3) We say that a class of reals $X \subseteq {}^\omega\omega$ is *strongly dominating*, if for every $y \in {}^\omega\omega$ there is $x \in X$ such that $y \leq^* x$, i.e. for all but finitely many $n \in \omega$, $y(n) \leq x(n)$. Furthermore, we define the *Laver ideal* $\mathcal{I}_\mathbb{L}$ to be the ideal of all classes of reals which are not strongly dominating. Then there is an injective dense embedding $\mathbb{L} \to \mathbb{B}^*_{\mathcal{I}_\mathbb{L}}$ by [GRSS95, Lemma 2.3].

Further examples include Mathias forcing, Silver forcing and Miller forcing (see [Kho12, Example 2.15]). In particular, by Theorem 3.2.6, all above mentioned forcing notions satisfy the forcing theorem. Moreover, since they satisfy Axiom D, they are also nice by Lemma 5.4.6 and Example 4.6.4.

[6] Recall that a *σ-ideal* of $\mathcal{B}$ is a subclass of $\mathcal{B}$ which contains $\emptyset$ and is closed under intersections and countable unions.

5.4.3 Preservation of the perfect set property

Let M be a fixed countable transitive model of $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$. We show that all tree forcing notions presented in the previous section preserve the $\boldsymbol{\Pi}^1_1$-PSP. The case of Cohen forcing is very simple, since Cohen forcing is contained in L.

Lemma 5.4.12. *Let $\mathbb{C}$ denote Cohen forcing. If M satisfies $\boldsymbol{\Pi}^1_1$-PSP and G is $\mathbb{C}$-generic over M, then so does $M[G]$. Moreover, if every set of ordinals in M has a sharp, then so does every set of ordinals in $M[G]$.*

Proof. Let $x \in M[G]$ be a real and $\sigma \in M^{\mathbb{C}}$ a name such that $x = \sigma^G$. Clearly, we may assume that σ is a real. Since $M \models \boldsymbol{\Pi}^1_1$-$\mathsf{PSP}$, $\mathsf{L}[\sigma] \models \mathsf{ZFC}$. Moreover, as $\mathbb{C} \subseteq \mathsf{L}$, G is also $\mathbb{C}$-generic over $\mathsf{L}[\sigma]$. In particular, $\mathsf{L}[\sigma][G] \models \mathsf{ZFC}$. But this implies that there are only countably many reals constructible from σ and the generic Cohen real, and hence Theorem 5.2.10, $M[G] \models \Pi^1_1[x]$-PSP.

For the second assertion, suppose that $\sigma \in M^{\mathbb{C}}$ is a name for a set of ordinals in $M[G]$. As before, we may assume that σ is a real. By assumption, $\sigma^{\#}$ exists in M and $\mathsf{L}[\sigma^{\#}] \models \mathsf{ZFC} + \sigma^{\#}$ exists. So in $\mathsf{L}[\sigma^{\#}]$ there is a non-trivial elementary embedding $j : \mathsf{L}[\sigma] \to \mathsf{L}[\sigma]$. But as in the proof of Theorem 5.3.10, we obtain an elementary embedding $\bar{j} : \mathsf{L}[\sigma^G] \to \mathsf{L}[\sigma^G]$ given by mapping $\bar{j}(\tau^G)$ to $(j(\tau))^G$. □

For the other examples, it follows from unpublished work by Fabiana Castiblanco and Philipp Schlicht in [CS16] that Mathias forcing, Sacks forcing, Silver forcing, Laver forcing and Miller forcing preserve the $\boldsymbol{\Pi}^1_1$-PSP and the existence of sharps.

Theorem 5.4.13. *[CS16] The following statements hold.*

(1) *Let $r \in {}^{\omega}\omega$ be a real in M. Then for every Mathias-generic real x over M there exists $y \in M$ such that x is Mathias-generic over $\mathsf{L}[r, y]$.*
(2) *Let $r \in {}^{\omega}\omega$ be a real in M. Then every Sacks real over M is Mathias-generic over $\mathsf{L}[r, y]$ for some real $y \in M$.*
(3) *Let $\dot{y}$ be the canonical name for a Silver real, $x \in {}^{\omega}\omega$ a real in M and p be a condition in Silver forcing $\mathbb{U}$. Then there is $q \leq_{\mathbb{U}} p$ such that*

$$q \Vdash^M_{\mathbb{U}} \text{“}\dot{y} \text{ is } \mathbb{C}\text{-generic over } \mathsf{L}[\check{x}, \check{p}]\text{”}.$$

(4) *Let $\dot{x}$ denote the canonical name for a Laver real, $p \in \mathbb{L}$ and $r \in {}^{\omega}\omega$. Then there is $q \leq_{\mathbb{L}} p$ such that*

$$q \Vdash^M_{\mathbb{L}} \text{“}\dot{x} \text{ is equivalent to a Mathias-generic real over } \mathsf{L}[\check{r}, \check{p}]\text{”}.$$

Moreover, an analogous result holds for Miller forcing. □

We can now apply the above statements to prove that all tree forcings that we have discussed so far preserve the $\boldsymbol{\Pi}^1_1$-PSP. However, it is an open question whether this can be generalized to all forcing notions which satisfy Axiom D.

Corollary 5.4.14. *If M satisfies $\boldsymbol{\Pi}^1_1$-PSP (resp. $M \models \exists\#$), then so does every $\mathbb{P}$-generic extension of M, where $\mathbb{P}$ is either Mathias forcing, Sacks forcing, Silver forcing, Laver forcing or Miller forcing.*

Proof. All proofs are essentially the same as the proof of Lemma 5.4.12 by making use of Theorem 5.4.13. □

Even though we have seen that many canonical forcing notions preserve the perfect set property for $\mathbf{\Pi^1_1}$-classes of reals, this does not hold in general. We give a counterexample by adjoining first an Ord-Cohen subclass A of Ord^M since this yields that the generic extension is of the form $\mathsf{L}[A]$. Using reshaping and almost disjoint coding, we then code A by a real x so that the generic extension is of the form $\mathsf{L}[x]$. Our characterization of the perfect set property presented in Theorem 5.2.10 shows then that the $\mathbf{\Pi^1_1}$-PSP fails in $\mathsf{L}[x]$. This was pointed out to me by Ralf Schindler.

Lemma 5.4.15. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of $\mathsf{GBdc}^- + \mathsf{V} = \mathsf{HC}$. Then there is a notion of class forcing $\mathbb{P}$ for M such that every $\mathbb{P}$-generic extension of M is of the form $\mathsf{L}[A]$ for some class $A \subseteq \mathrm{Ord}^M$ which is definable from G.*

Proof. Let $\mathbb{P}$ denote Ord-Cohen forcing, i.e. conditions are of the form $p : \alpha \to 2$ for some $\alpha \in \mathrm{Ord}^M$, ordered by end-extension. It is clear that $\mathbb{P}$ is $<$ Ord-closed and hence it does not add any new sets and satisfies the forcing theorem. Since $M \models \mathsf{DC}$, Example 2.2.9 implies that $\mathbb{P}$ is pretame. Now if G is $\mathbb{P}$-generic over $\mathbb{M}$ then $F = \bigcup G$ is a function $\mathrm{Ord}^M \to 2$ by standard density arguments. Moreover, $A = \{\alpha \in \mathrm{Ord}^M \mid F(\alpha) = 1\}$ has the desired properties: If $x \in M$ is a set, then using the axiom of choice it can be coded by a set of ordinals $a \subseteq \alpha$ for some $\alpha \in \mathrm{Ord}^M$. Now by genericity there is $\beta \in \mathrm{Ord}^M$ such that for $i < \alpha$, $F(\beta + i) = 1$ if and only if $i \in a$. Thus A codes all sets of ordinals and hence $M = \mathsf{L}[A]$. □

Theorem 5.4.16. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GBdc}^- + \mathsf{V} = \mathsf{HC}$ such that there is $A \in \mathcal{C}$ with $M = \mathsf{L}[A]$. Then there is a pretame notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ such that every $\mathbb{P}$-generic extension is of the form $\mathsf{L}[x]$ for some real x.*

Proof. Our desired forcing notion will be the two-step iteration of a reshaping forcing and almost disjoint coding. Note first that since $\mathbb{M}$ has a hierarchy, by the characterization of pretameness given in Corollary 4.3.6, the two-step iteration of two pretame notions of class forcing is again pretame.

The *reshaping* partial order is defined as follows. Conditions in $\mathbb{P}$ are functions of the form $p : \alpha \to 2$ for some ordinal $\alpha \geq \omega$ with the property that for all $\beta \leq \alpha$, $\mathsf{L}[A \cap \beta, p \restriction \beta] \models |\beta| \leq \omega$, ordered by end-extension.

Claim 1. *For every $\alpha \in \mathrm{Ord}^M$, the class*

$$D_\alpha = \{p \in \mathbb{P} \mid \alpha \leq \mathrm{dom}(p)\} \in \mathcal{C}$$

is dense in $\mathbb{P}$.

Proof. Suppose that $p \in \mathbb{P}$ is an arbitrary condition. If $\mathrm{dom}(p) < \alpha$, then let $x \subseteq \omega$ code a well-order of ordertype α. Now let $q \leq_{\mathbb{P}} p$ be the condition with $\mathrm{dom}(q) = \mathrm{dom}(p) + \omega$ given by $q(\mathrm{dom}(p) + n) = 1$ iff $n \in x$. If $\mathrm{dom}(q) < \alpha$ then let $r \leq_{\mathbb{P}} q$ be the condition with $\mathrm{dom}(r) = \alpha$ with $r(\beta) = 0$ for all $\mathrm{dom}(q) \leq \beta < \alpha$. Since $\mathsf{L}[A \cap \mathrm{dom}(q), q] \models |\alpha| = \omega$, it is clear that r is in fact a condition in D_α. □

Note that in the same way as above, for every $x \in \mathsf{L}[A]$ we can always extend a condition $p \in \mathbb{P}$ to some $q \leq_{\mathbb{P}} p$ which codes x: Since x can be coded by a real, we can code x into the interval $[\mathrm{dom}(p), \mathrm{dom}(p) + \omega)$.

Claim 2. $\mathbb{P}$ *satisfies the set decision property.*

Proof. Let $a \subseteq \mathbb{P}$ be a set in M of conditions in $\mathbb{P}$ and let $p \in \mathbb{P}$. Using $\mathsf{V} = \mathsf{HC}$, we can write $a = \{p_n \mid n \in \omega\}$. Now we define a sequence $\langle q_n \mid n \in \omega\rangle$ in the following way. Let $q_0 \leq_{\mathbb{P}} p$ be a condition which codes the set a. This can be done using the previous claim. Given q_n, let q_{n+1} be the least – with respect to the canonical well-order of $\mathsf{L}[A]$ – condition below q_n which decides p_n. Finally, let $q = \bigcup_{n\in\omega} q_n$. Let $\alpha = \mathrm{dom}(q) = \bigcup_{n\in\omega} \mathrm{dom}(p_n)$. Note that if $\beta < \alpha$, there is $n \in \omega$ such that $\beta \leq \mathrm{dom}(q_n)$. In particular, $\mathsf{L}[A \cap \beta, q \upharpoonright \beta] = \mathsf{L}[A \cap \beta, q_n \upharpoonright \beta] \models |\beta| \leq \omega$, since q_n is a condition in $\mathbb{P}$. Moreover, observe that by our construction above $\langle q_n \mid n \in \omega\rangle$ is definable over $\mathsf{L}_\alpha[A\cap\alpha, q]$ and hence so is $\langle \mathrm{dom}(q_n) \mid n \in \omega\rangle$. But then α is countable in $\mathsf{L}[A \cap \alpha, q]$, since it is the union of ω-many countable sets. This proves that q is indeed a condition in $\mathbb{P}$ which decides a. □

Claim 3. $\mathbb{P}$ *is pretame for* $\mathbb{M}$.

Proof. By Example 2.2.7 suffices to show that $\mathbb{P}$ is distributive over $\mathbb{M}$. We follow the proof of [AF15, Claim 24]. Since the axiom of choice holds in M, it is enough to consider sequences of open dense classes of the form $\langle D_n \mid n \in \omega\rangle \in \mathcal{C}$. Fix a condition $p \in \mathbb{P}$. Suppose that $\langle D_n \mid n \in \omega\rangle$ is Σ_{k+1}-definable over M with parameter $x \in M$. We define a $\leq_{\mathbb{P}}$-descending sequence $\langle p_n \mid n \in \omega\rangle$ of conditions below p such that $p_{n+1} \in D_n$ for each $n \in \omega$ in the following way.

- Since x can be coded by a real and using Claim 1, we can find an extension $p_0 \leq_{\mathbb{P}} p$ such that p_0 codes x.
- Suppose that p_n has already been defined. Now using the global well-order of $\mathsf{L}[A]$, we can choose the least pair $\langle q_n, y_n\rangle$ such that y_n witnesses the Π_k-property that $q_n \in D_n$. Since this property may not be absolute, we further strengthen q_n futher in the following way. Let $\mathsf{L}_{\alpha_n}[A\cap\alpha_n]$ be the transitive collapse of the Σ_k-Skolem hull of $\{q_n, y_n\}$ in $\mathsf{L}[A]$ with respect to the canonical Skolem functions. Now we extend q_n to a condition $p_{n+1} \in \mathbb{P}$ with domain α_n by adding 0's.

Finally, let $q = \bigcup_{n\in\omega} p_n$ and $\beta = \sup\{\alpha_n \mid n \in \omega\}$. Note that the sequence of Skolem hulls and $\langle p_n \mid n \in \omega\rangle$ are elements of $\mathsf{L}_\beta[A]$, since $\mathsf{L}_\beta[A]$ is a Σ_k-elementary substructure of $\mathsf{L}[A]$. Moreover, $\mathrm{dom}(q) = \beta$ and so $\mathsf{L}[A \cap \beta, q] \models |\beta| = \omega$. Hence q is a condition in $\mathbb{P}$ which lies in $\bigcap_{n\in\omega} D_n$. □

Suppose now that G is $\mathbb{P}$-generic over $\mathbb{M}$. Then $\bigcup G$ is a function defined on all ordinals as a consequence of Claim 1. Let $B \subseteq \mathrm{Ord}^M$ be a predicate in $\mathcal{C}[G]$ which codes both A and $\{\alpha \in \mathrm{Ord}^M \mid \exists p \in G\,(\alpha \in \mathrm{dom}(p) \wedge p(\alpha) = 1)\}$. This enables us to choose a sequence $\langle x_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ of reals as follows. Given the sequence $\langle x_\beta \mid \beta < \alpha\rangle$ for some ordinal α, we can choose $x_\alpha \subseteq \omega$ to be the least real $x \in \mathsf{L}[B \cap \alpha]$ with respect to the canonical well-order such that $x \notin \{x_\beta \mid \beta < \alpha\}$. Such a real exists since α is countable in $\mathsf{L}[B \cap \alpha]$. These reals may, however, still not be almost disjoint. For each ordinal α, let $a_\alpha = \{h(x_\alpha \cap n) \mid n \in \omega\}$, where $h : [\omega]^{<\omega} \to \omega$ is a suitable bijection. Then $\langle a_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle$ is a sequence of almost disjoint reals, i.e they have the property that $|a\alpha \cap a_\beta| < \omega$ for $\alpha \neq \beta$.

The second step is to perform *almost disjoint coding* with the help of the sequence $\langle a_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle$. The conditions of $\mathbb{Q}$ are given by pairs $p = \langle s_p, t_p \rangle$ such that $s_p \in [\omega]^{<\omega}$ and $t_p \in [B]^{<\omega}$. The ordering is defined by $p \leq_{\mathbb{P}} q$ if the following conditions hold:

(a) $s_p \supseteq s_q$ and $\min(s_p \setminus s_q) > \max s_q$,
(b) $t_p \supseteq t_q$,
(c) $s_p \setminus s_q \cap \bigcup_{\alpha \in t_q} a_\alpha = \emptyset$.

Clearly, if $p, q \in \mathbb{Q}$ with the same first component $s_p = s_q$ then $r = \langle s_r, t_r \rangle$ with $s_r = s_p$ and $t_r = t_p \cup t_q$ is in $\mathbb{Q}$ and strengthens both p and q. Since there are only countably many possible first components, it follows that $\mathbb{Q}$ satisfies the Ord-cc over $\mathbb{M}$. In particular, $\mathbb{P}$ is pretame.

Now let H be $\mathbb{Q}$-generic over $\mathbb{M}$. Consider the real

$$x = \bigcup \{s_p \mid p \in H\} \subseteq \omega.$$

The next claim shows that x codes B.

Claim 4. *Let α be an ordinal in M. Then $\alpha \in B$ if and only if $x \cap a_\alpha$ is finite.*

Proof. Suppose first that $\alpha \in B$. Note that the class $D_\alpha = \{p \in \mathbb{Q} \mid \alpha \in t_p\} \in \mathcal{C}[G]$ is a dense subclass of $\mathbb{Q}$. If $p \in D_\alpha \cap H$, then $x \cap a_\alpha = s_p \cap a_\alpha$ is finite. Conversely, assume that $\alpha \notin B$ and let $n \in \omega$. Then the class

$$E^n_\alpha = \{p \in \mathbb{Q} \mid \exists m \geq n \, (m \in s_p \cap a_\alpha)\}$$

is dense in $\mathbb{Q}$. To see this, let $p \in \mathbb{Q}$ be an arbitrary condition. Since the a_β's are almost disjoint and t_p is finite, there is $m \geq \max s_p, n$ such that $m \in a_\alpha \setminus \bigcup_{\beta \in t_q} a_\beta$. Then $q = \langle s_p \cup \{m\}, t_p \rangle \leq_{\mathbb{Q}} p$ is in E^n_α. Now let $p \in E^n_\alpha \cap H$. Then there is $m \geq n$ with $m \in s_p \cap a_\alpha \subseteq x \cap a_\alpha$. Since n was arbitrary, this shows that $x \cap a_\alpha$ is infinite. □

Note that since H can be recovered from x as $H = \{p \in \mathbb{Q} \mid s_p \subseteq x\}$ and using Claim 4, it follows that $\mathsf{L}[B][H] = \mathsf{L}[x]$. But this shows that the $\Pi^1_1[x]$-PSP fails, since otherwise by Theorem 5.2.10 there would be only countably many x-constructible reals. □

Chapter 6

Open questions

In the following, we present a selection of open questions related to the topics covered in this thesis. Many of them are mentioned already in our joint papers [HKL+16], [HKS16b] and [HKS16a]. For further interesting open problems, consult the aforementioned papers.

Firstly, there are many interesting open questions related to the forcing theorem. As shown in Theorem 2.2.2, if $\mathbb{M}$ has a hierarchy then every pretame notion of class forcing for $\mathbb{M}$ satisfies the forcing theorem. Moreover, if $\mathcal{C}$ contains a global well-order, then by Lemma 4.1.2, the Ord-cc implies the forcing theorem. Thus we are interested whether these assumptions are necessary.

Question 1. *If $\mathbb{M}$ is a countable transitve model of GB^-, does every pretame notion of class forcing for $\mathbb{M}$ satisfy the forcing theorem? Does every notion of class forcing which satisfies the* Ord*-cc over $\mathbb{M}$ satisfy the forcing theorem?*

In Theorem 4.2.3 we show how to extend a non-pretame notion of class forcing to one which does not satisfy the forcing theorem. However, the proof makes use of the non-existence of a first-order truth predicate in the ground model. Thus the following question arises.

Question 2. *If $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a countable transitive model of GB^- and $\mathbb{N}$ is a model of th form $\langle M, \mathrm{Def}(\mathcal{C} \cup \{T\}) \rangle$, where T is a first-order truth predicate for M, does every notion of class forcing for $\mathbb{M}$ satisfy the forcing theorem over $\mathbb{N}$?*

Our counterexample of the truth lemma given by Theorem 2.5.11 and Example 2.5.13 uses a two-step iteration, where the first iterand is pretame and the second iterand is the forcing notion $\mathbb{F}$ as defined in 1.3.5. However, since we do not know whether the two-step iteration of notions of class forcing satisfying the truth lemma, where the first forcing notion is pretame, again satisfies the truth lemma.

Question 3. *Does the forcing notion $\mathbb{F}$ satisfy the truth lemma over all (some) models of ZF^-?*

Another question related to the forcing theorem is the following.

Question 4. *If $\mathbb{P}$ satisfies the forcing theorem over some countable transitive model $\mathbb{M}$ of GB^-, does $\mathbb{P}$ still satisfy the forcing theorem over some class-generic extension of $\mathbb{M}$? Does the product of two forcing notions which satisfy the forcing theorem still satisfy the forcing theorem?*

Note that a positive answer to the first question above would also yield a postive answer to the second question. Similarly, it is not clear whether pretameness or the preservation of GB^- of some forcing notion holds in class-generic extensions.

Question 5. *Suppose that $\mathbb{P}$ is a notion of class forcing for some countable transitive model $\mathbb{M}$ of GB^- which is pretame for $\mathbb{M}$ (preserves the axioms of GB^-). Is $\mathbb{P}$ still pretame for (does $\mathbb{P}$ still preserve GB^- over) some class-generic extension of $\mathbb{M}$?*

This is particularly interesting, since it would yield that the characterization of pretameness of some forcing notion $\mathbb{P}$ in terms of the preservation of GB^- over models $\mathbb{M} = \langle M, \mathcal{C} \rangle$ with a hierarchy (see Corollary 4.3.6) could be lifted to arbitrary models of GB^- under the assumption that $\mathbb{P}$ satisfies the forcing theorem. This relies on a simple argument which makes use of the product $\mathbb{P} \times \mathbb{W}^M$ and the product lemma.

Question 6. *Let $\mathbb{M}$ be a countable transitive model of GB^- and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Does it hold that $\mathbb{P}$ is pretame for $\mathbb{M}$ if and only if for every $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G over $\mathbb{M}$ with $p \in G$ such that $\mathbb{M}[G] \models \mathsf{GB}^-$?*

The characterization of pretameness in terms of the axiom of GB^- (resp. the preservation of separation) as stated in Corollary 4.3.7 relies heavily on the forcing theorem. In the case of separation, this makes use of Theorem 4.3.4 and hence we additionally assume the existence of a set-like well-order. This motivates the following question.

Question 7. *If $\mathbb{M}$ is a countable transitive model of GB^-, does every notion of class forcing for $\mathbb{M}$ which preserves GB^- satisfy the forcing theorem? Does every notion of class forcing for $\mathbb{M}$ which preserves separation satisfy the forcing theorem?*

Concerning separation, we are also interested in which particular instances can fail in class forcing extensions. It is shown in [HKS16b, Lemma 8.5] that given a nice name $\sigma \in M^{\mathbb{P}}$ for a subset of ω^2, where $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ which is closed under meets[1], that in every $\mathbb{P}$-generic extension the complement $\omega^2 \setminus \sigma^G$ exists. Furthermore, we proved in Lemma 4.6.5 that if $\sigma, \tau \in M^{\mathbb{P}}$ are nice names, then there is a name for the intersection of σ and τ. However, we do not know whether this also holds for names which are not nice.

Question 8. *Suppose that $\mathbb{M}$ is a countable transitive model of GB^- and $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$. If σ, τ are $\mathbb{P}$-names, is there always a $\mathbb{P}$-name for the complement $\sigma \setminus \tau$? Is there a $\mathbb{P}$-name for the intersection of σ and τ? More generally, are class generic extensions always rudimentarily closed?*

[1] i.e. every finite set of conditions in $\mathbb{P}$ has a lower bound in $\mathbb{P}$

Concerning the question above, note that we have proved in Lemma 3.2.4 that class-generic extensions obtained by forcing with an M-complete Boolean algebra are always rudimentarily closed. Furthermore, it is easy to generalize this to M-complete pre-Boolean algebras. However, we do not know whether this holds for all forcing notions.

Note that in the proof of Lemma 1.2.9 we have considered weak union, i.e. the existence of a superset of $\bigcup x$ for every set x, instead of the usual axiom of union. This makes sense, since the axiom of union obviously follows from separation and weak union. Nevertheless, the following open question is still of interest to us.

Question 9. *Does every notion of class forcing preserve the axiom of union?*

Notice that every notion of class forcing $\mathbb{P}$ which has the property that for every $p, q \in \mathbb{P}$ which are compatible there is a set-sized predense subset of conditions strengthening both p and q preserves the axiom of union. If we drop this assumption, we do not know whether this holds.

As we have already mentioned above regarding the forcing theorem, all of our characterizations of pretameness and the Ord-cc require additional assumptions such as the existence of a set-like well-order of the ground model or even class recursion. It would therefore be desirable to investigate whether these assumptions can be dropped.

Question 10. *Can be pretameness be characterized in terms of the existence of nice names for sets of ordinals over models of* GB^- *(or any other theory which does not allow class recursion)? Can we characterize the Ord-cc in terms of very niceness over models of* GB^-*?*

Analogous questions can be posed for all other properties under consideration in Chapter 4. Theorem 4.2.3 shows how pretameness can be characterized in terms of the forcing theorem. The proof shows that if $\mathbb{P}$ is non-pretame over some model of GB^- without a first-order truth predicate then there is a dense embedding from $\mathbb{P}$ into some forcing notion which doesn't satisfy the definability lemma. It is therefore natural to ask whether a similar statement holds for the truth lemma.

Question 11. *Does it hold in (some extension of)* GB^- *that a forcing notion is pretame if and only if it densely satisfies the truth lemma?*

Another topic concerns the equiconsistency results between models of (extensions of) $\mathsf{SOA} + \mathbf{\Pi^1_1}$-$\mathsf{PSP}$ and (extensions of) ZFC. We conjecture that this can be lifted to characterize further large cardinal properties.

Question 12. *Can models of* ZFC *with a measurable cardinal be characterized by models of* SOA *with some additional hypotheses? If M is a model of* $\mathsf{SOA}+$ *projective determinacy, does M have any inner models of* ZFC *with infinitely many Woodin cardinals? Does projective absoluteness hold for all notions of class forcing which satisfy Axiom D?*

As we have seen in Section 5.2.3, one can easily prove in second-order arithmetic that Π^1_1-$\mathsf{Det} + \mathbf{\Pi^1_1}$-$\mathsf{PSP}$ implies the existence of $0^\#$. However, the following is an interesting open question which was formulated in [CS15].

Question 13. *Can one prove in second-order arithmetic that* Π^1_1*-determinacy implies the existence of* $0^\#$*?*

As we have already pointed out before, a positive answer would provide a new prove of the existence of $0^\#$ from Π^1_1-determinacy. Regarding $0^\#$, recall that $0^\#$ is not set-generic over any model M with $0^\# \notin M$. For class forcing, this is, however, still an open question.

Question 14. *Is it consistent that there is a countable transitive model* M *of* ZFC *with* $0^\# \notin M$ *and a class-generic extension* N *of* M *with* $0^\# \in N$*?*

Partial results on this topic have been elaborated by Stanley in [Sta98]. Notably, he shows that $0^\#$ can be invisibly generic over some model of the form L_κ, where invisible genericity is a weaking of class genericity which only requires $\mathsf{L}_\kappa[G]$ to be a model of collection rather than the corresponding second-order model. On the other hand, an easy argument shows that $0^\#$ does not lie in any class-generic extension of L, since otherwise one could use the forcing theorem to define truth over L.

In Section 5.4.3 we have shown examples of class forcing notions over models of $\mathsf{SOA}+\mathsf{DC}$ which preserve the $\mathbf{\Pi^1_1}$-PSP and such which do not.

Question 15. *Does every notion of class forcing which satisfies Axiom D over a model of* $\mathsf{SOA}+\mathsf{DC}$ *preserve the* $\mathbf{\Pi^1_1}$-PSP*? How can one characterize those notions of class forcing which preserve the* $\mathbf{\Pi^1_1}$-PSP *over models of* $\mathsf{SOA}+\mathsf{DC}$*?*

Index

Symbols

A

Bibliography

[Add63] John W Addison, *Some consequences of the axiom of constructibility*, Journal of Symbolic Logic **28** (1963), no. 4, 193–193.

[AF15] Carolin Antos and Sy-David Friedman, *Hyperclass forcing in morse-kelley class theory*, arXiv preprint, arXiv:1510.04082 (2015).

[Ant15] Carolin Antos, *Class-Forcing in Class Theory*, arXiv preprint, arXiv:1503.00116 (2015).

[BJW82] A. Beller, R. Jensen, and P. Welch, *Coding the universe*, London Mathematical Society Lecture Note Series, vol. 47, Cambridge University Press, Cambridge-New York, 1982.

[Coh63] Paul Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148.

[CS15] Yong Cheng and Ralf Schindler, *Harrington's principle in higher order arithmetic*, J. Symb. Log. **80** (2015), no. 2, 477–489.

[CS16] Fabiana Castiblanco and Philipp Schlicht, *Preserving sharps*, In preparation, 2016.

[Dav64] Morton Davis, *Infinite games of perfect information*, Advances in game theory, Princeton Univ. Press, Princeton, N.J., 1964, pp. 85–101.

[Dra74] Frank R Drake, *Set theory: An introduction to large cardinals*, vol. 76, North-Holland, 1974.

[Eas70] William B. Easton, *Powers of regular cardinals*, Ann. Math. Logic **1** (1970), 139–178.

[Ena05] Ali Enayat, *Models of set theory with definable ordinals*, Arch. Math. Logic **44** (2005), no. 3, 363–385.

[Fri00] Sy D. Friedman, *Fine structure and class forcing*, de Gruyter Series in Logic and its Applications, vol. 3, Walter de Gruyter & Co., Berlin, 2000.

[GHJ11] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone, *What is the theory ZFC without power set?*, arXiv preprint (2011).

[Git13a] Victoria Gitman, *Forcing to add proper classes to a model of GBC: An introduction*, Blog Post (2013), http://boolesrings.org/victoriagitman/2013/10/09/forcing-to-add-proper-classes-to-a-model-of-rm-gbc-an-introduction.

[Git13b] ______, *Forcing to add proper classes to a model of GBC: The technicalities*, Blog Post (2013), http://boolesrings.org/victoriagitman/2013/10/12/forcing-to-add-proper-classes-to-a-model-of-rm-gbc-the-technicalities.

[Git14] ______, *Variants of Kelley-Morse set theory*, Blog Post (2014), http://boolesrings.org/victoriagitman/2014/02/24/variants-of-kelley-morse-set-theory.

[GRSS95] Martin Goldstern, Miroslav Repický, Saharon Shelah, and Otmar Spinas, *On tree ideals*, Proc. Amer. Math. Soc. **123** (1995), no. 5, 1573–1581.

[Ham14a] Joel David Hamkins, *Kelley-Morse set theory implies Con(ZFC) and much more*, Blog Post (2014), http://jdh.hamkins.org/km-implies-conzfc.

[Ham14b] ______, *The global choice principle in Gödel-Bernays set theory*, Blog Post (2014), http://jdh.hamkins.org/the-global-choice-principle-in-godel-bernays-set-theory.

[Har77] Leo Harrington, *Long projective wellorderings*, Ann. Math. Logic **12** (1977), no. 1, 1–24.

[Har78] ______, *Analytic determinacy and* $0^\sharp$, J. Symbolic Logic **43** (1978), no. 4, 685–693.

[Hil00] David Hilbert, *Mathematische probleme*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1900** (1900), 253–297.

[HKL$^+$16] Peter Holy, Regula Krapf, Philipp Lücke, Ana Njegomir, and Philipp Schlicht, *Class Forcing, the Forcing Theorem and Boolean Completions*, Accepted for the Journal of Symbolic Logic, 2016.

[HKS16a] Peter Holy, Regula Krapf, and Philipp Schlicht, *Characterizations of Pretameness and the Ord-cc*, Submitted, 2016.

[HKS16b] ______, *Separation in Class Forcing Extensions*, Submitted, 2016.

[HL08] Joel Hamkins and Benedikt Löwe, *The modal logic of forcing*, Transactions of the American Mathematical Society **360** (2008), no. 4, 1793–1817.

[HLR13] Joel David Hamkins, David Linetsky, and Jonas Reitz, *Pointwise definable models of set theory*, J. Symbolic Logic **78** (2013), no. 1, 139–156.

[Jec03] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.

[Kan09] Akihiro Kanamori, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2009.

[Kho12] Yurii Khomskii, *Regularity properties and definability in the real number continuum: idealized forcing, polarized partitions, hausdorff gaps and mad families in the projective hierarchy*, Institute for Logic, Language and Computation, University of Amsterdam, 2012.

[KM07] Peter Koepke and Michael Möllerfeld, *Topological regularity and second order arithmetic*, Unpublished (2007).

[Koe78] Peter Koepke, *Das Modell* $L^{\#}$, Diplomarbeit, Rheinische Friedrich-Wilhelms-Universität Bonn, 1978.

[Kun80] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980, An introduction to independence proofs.

[Mar70] Donald A. Martin, *Measurable cardinals and analytic games*, Fund. Math. **66** (1969/1970), 287–291.

[Owe08] Robert Owen, *Outer Model Theory and the Definability of Forcing*, PhD thesis, 2008.

[Rei06] Jonas Reitz, *The ground axiom*, ProQuest LLC, Ann Arbor, MI, 2006, Thesis (Ph.D.)–City University of New York.

[Rob50] Raphael M. Robinson, *An essentially undecidable axiom system*, Proceedings of the international Congress of Mathematics, vol. 1, 1950, pp. 729–730.

[Sim09] Stephen George Simpson, *Subsystems of second order arithmetic*, vol. 1, Cambridge University Press, 2009.

[Sol70] Robert M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Annals of Mathematics (1970), 1–56.

[Sta98] MC Stanley, *Invisible genericity and* $0^{\#}$, Journal of Symbolic Logic **63** (1998), no. 04, 1297–1318.

[Sta03] M. C. Stanley, *Outer models and genericity*, J. Symbolic Logic **68** (2003), no. 2, 389–418.

[Tar36] Alfred Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen*, Studia Philosophica **1** (1936), 261–405.

[War05] Anna Warzecha, *Zur Stärke topologischer Regularitätsaxiome auf der Basis zweistufiger Arithmetik*, Diplomarbeit, Westfälische Wilhelms-Universität Münster, 2005.

[Zap08] Jindřich Zapletal, *Forcing idealized*, Cambridge Tracts in Mathematics, vol. 174, Cambridge University Press, Cambridge, 2008.

[Zar73] Andrzej Zarach, *Forcing with proper classes*, Fund. Math. **81** (1973), no. 1, 1–27, Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, I.

[Zar82] ———, *Unions of* ZF^-*-models which are themselves* ZF^-*-models*, Logic Colloquium '80 (Prague, 1980), Stud. Logic Foundations Math., vol. 108, North-Holland, Amsterdam-New York, 1982, pp. 315–342.

[Zar96] ———, *Replacement* $\nrightarrow$ *collection*, Gödel'96: Logical foundations of mathematics, computer science and physics—Kurt Gödel's legacy, Brno, Czech Republic, August 1996, proceedings, Association for Symbolic Logic, 1996, pp. 307–322.