Zero Partition Cycles

A Recursive Formula for Characteristic Classes of Surface Bundles

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Zusammenfassung

Diese Arbeit beschäftigt sich mit charakteristischen Klassen von Flächenbündeln. Eine vollständige Beschreibung des Rings von stabilen charakteristischen Klassen über den rationalen Zahlen ist nach [16] gegeben durch Polynome in den Klassen κ_i, welche in [17] beschrieben werden.

Diese Arbeit führt weitere charakteristische Klassen ein, welche durch Partitionen von Nullstellen von abelschen Differentialen gegeben sind. Diese Klassen sind nicht neu, sondern stellen eine Übertragung von Strata von abelschen Differentialen (siehe beispielsweise [22]) für reell-differenzierbare Flächenbündel dar.

Das Ziel der Arbeit ist ein Vergleich dieser Partitionsklassen mit den Klassen κ_i. Dies gelingt in umfassender Weise: Durch den in dieser Arbeit bewiesenen Isomorphiesatz 4.4.3 entpuppen sich die Partitionsklassen als alternative Erzeuger des Rings von stabilen charakteristischen Klassen.

Der Beweis des Isomorphiesatzes wird geführt, indem die Partitionsklassen durch ein rekursives Verfahren explizit in Termen der Klassen κ_i dargestellt werden. Dadurch wird das abstrakte Isomorphie-Resultat ergänzt durch ein prinzipiell berechenbares Verfahren für Hin- und Rückrichtungen des Isomorphismus.

Im Zuge des Beweises werden Algebren von Partitionen konstruiert, welche die abstrakten Eigenschaften von Schnitten der Partitionszykeln modellieren. Eine axiomatische Fassung dieser Eigenschaften ist in Definition 2.2.4 gegeben. Die Modelle werden klassifiziert: Sie treten auf in einer Familie, welche von einem komplexen Parameter λ abhängt. Im konkreten Fall der Partitionsklassen ist nur $\lambda = -2$ von Bedeutung.

Zuletzt werden die Partitionsklassen in den erweiterten Fall von speziellen singulären Flächenbündeln übertragen, wie sie in der Kompaktifizierung von Deligne und Mumford [5] auftauchen. Verschiedene Konstruktionen solcher Klassen und deren Eigenschaften sind Gegenstand aktueller Forschung, beispielsweise in [3], [9] und [1]. Die Methoden dieser Arbeit genügen in diesem Fall allerdings nur für ein partielles Resultat. Die Arbeit schließt mit einer konkreten Formulierung des allgemeinen Problems für (verallgemeinerte) Partitionsklassen in der Deligne-Mumford-Kompaktifizierung, und beschreibt die Schwierigkeiten, die eine analoge Behandlung zu Klassen von glatten Flächenbündeln verhindern.

Contents

1 Introduction

Results

This thesis is about rational stable characteristic classes of surface[1] bundles[2]. Analogous to the case of vector bundles, a characteristic class of surface bundles is a rule which assigns to each surface bundle $\pi_E\colon E \to B$ (of a fixed fiber genus g) a cohomology class in $\mathrm{H}^*(B, \mathbb{Q})$, the rule being natural under pullback.

Mumford-Morita-Miller Classes. The Mumford-Morita-Miller classes κ_i, $i \in \{1, 2, \dots\}$, are examples of such classes [17], with the additional feature of being defined for bundles of any fiber genus in a consistent way – they are invariant under a fiberwise connected sum with a trivial torus bundle, a process called stabilization. The determination of all such classes (they are called stable) has been conjectured by Mumford, and proved by Madsen and Weiss in their work [16]: The ring of rational stable characteristic classes $\mathrm{H}^{\mathrm{st}}(\mathbb{Q})$ (with the cup-product) is the polynomial ring in the classes κ_i. This result is recalled in more detail in section 4.2.

Zero Partition Classes. Assume now that the base space is a smooth manifold M, and that the smooth surface bundle is equipped with a fiberwise complex structure J and a fiberwise non-vanishing holomorphic one-form ω_L. Such a structure (J, ω_L) is called an abelian differential, and the connectivity of the space of such structures on a fixed base manifold M increases linearly with the fiber genus g.

In this situation, one can now take the following approach: By a classical theorem it is known that any holomorphic one-form on a Riemann surface of genus g that does not vanish completely, has exactly $2g - 2$ zeroes, counted with multiplicity. As the multiplicities of the zeroes of a holomorphic one-form have to add up to $2g - 2$, they define a decompositition:

$$2g - 2 = n_1 + \ldots + n_k$$

Fix such a decomposition σ. Now the subset of points in M on whose fiber the restriction of ω_L has this fixed zero decomposition is called a **zero partition cycle**. Their definition obviously depends on (J, ω_L).

[1]Assumed to be oriented throughout.

[2]This thesis uses smooth surface bundles. The rational stable characteristic classes of surface bundles are the same as in the topological case, see [17].

But it turns out that (in a way to be defined, see Construction 3.4.4) the zero partition cycles define characteristic (co-)homology classes $h(\sigma)$ which are independent of (J, ω_L), they depend only on the topology of the surface bundle. They moreover define stable characteristic classes. For a surface of fiber genus g the class $h(\sigma)$ is the same as $h(\sigma')$ of its stabilization, where σ' is obtained from σ by adding two simple zeroes.

For the universal example of a surface bundle over the moduli space of Riemann surfaces, the cycles are also known as strata of abelian differentials, and enjoy large interest, for example in [3], [8], [7], and [14], [13], and the related notion for quadratic differentials in [4].

Main Result. The main result of this thesis is the following:

Theorem 1.0.1 (Stable isomorphism)**.** *Let* $\Pi_\Sigma(\infty)$ *be the set of partitions of natural numbers, with two partitions identified under stabilization (addition of* 1*):*

$$(n_1, \dots, n_k) \sim (n_1, \dots, n_k, 1)$$

The map

$$\begin{aligned} \mathbb{Q}[\Pi_\Sigma(\infty)] &\to \mathrm{H}^{\mathrm{st}}(\mathbb{Q}) \\ \sigma &\mapsto h(\sigma), \end{aligned}$$

which sends a partition to its zero partition class, is an isomorphism.

This is the content of the Isomorphism Theorem, Theorem 4.4.3. In simpler terms:

The zero partition classes are a basis for rational stable characteristic classes.

Calculation of Zero Partition Cycles. The proof of the main theorem is carried out via an explicit calculation of the zero partition classes in terms of the Mumford-Morita-Miller classes κ_i.

Due to their importance in the theory of flat surfaces, the calculations include the more general case of projective abelian differentials. These classes are unstable, and their unstability is accounted for by additional terms in the canonical class τ of the line bundle of the projectivization. The nature of this proof improves the result of Theorem 1.0.1:

There is an effective procedure to calculate arbitrary zero partition classes in terms of the classes κ_i *and* τ*, given by Lemma 4.3.1 and Lemma 4.3.2.*

Strategy of the Proof

The machinery developed in this thesis to compute the zero partition classes consists of two mostly independent parts: An algebraic one (section 2) and a differential-topological one (section 3). In section 4, the two parts are put together. Only at the end of that section, the classification of rational stable characteristic classes is used to prove the Isomorphism Theorem. Thus the validity of the calculations, as well as the developed machinery does not depend on this classification, to be possibly applicable in more general settings.

Three types of concepts appear throughout all parts of the proof: The first one is the notion of a primitive partition, which in the algebraic part appears as a generator, in the differential-topological picture as the cycle with only one higher order zero, and in the ring of characteristic classes as (multiples of) the classes κ_i. The second and third one are two kinds of interaction between these objects, transverse and non-transverse ones. In the algebraic picture, these correspond to two types of relations, in the differential-topological picture to transverse and non-transverse intersections of subspaces.

The Algebra Part. This part is an axiomatic approach to algebra structures on the free module on partitions. With minimal assumptions, which account for grading, the symmetry under permutations and restrictions for certain kinds of products (the "transverse" ones), all algebras over the rationals of this type are classified and explicitly constructed.

In general, to construct an algebra with a certain basis, there are several methods. The most common ones include:

(a) Write down a product formula for all products of basis elements and explicitly check associativity.

(b) Construct a faithful action on a vector space (or some module over another algebra), which exhibits the algebra as a subalgebra of the endomorphisms.

(c) Write down generators and relations, and prove that the algebra given by these generators and relations has a basis bijective to the desired one.

In this thesis, method c is used, because generators and relations allow to write down maps from the algebra into another one easily. The tool used to prove the statement of the basis is the Diamond Lemma [2]. This Lemma states that for certain kinds of relations (locally confluent ones), the basis elements of the algebra are given by those words in the generators which cannot be reduced by relations. For commutative algebras (which the algebras in this thesis are), a special case of this is the theory of Gröbner bases. The formulation of these local confluence conditions is the content of Lemma 2.3.5, and they are verified in Lemma 2.3.6.

The Differential Topology Part. For a general abelian differential, a zero partition cycle for a partition σ is not a submanifold. The obstruction to this is twofold: One has to assume that the abelian differential is sufficiently generic for the zero partition cycle to be the image of an immersion, and then one has to pass to its normalization to remove the remaining singularities.

The genericity assumption is made precise by the notion of an admissible abelian differential, which forces the splitting of the zeroes on the fibers to be maximally independent. It is shown that for sufficiently large fiber genus, the space of admissible abelian differentials is an open dense path-connected subspace of the space of all abelian differentials.

In a second step, a normalization of the zero partition cycle is constructed by passing to the branched cover defined by enumerating the single zeroes of the abelian differential.

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2 Algebra of Partitions

This chapter introduces the combinatorial basis of this thesis. The main theme is a correspondence between set partitions and certain subspaces of representations of symmetric groups. The basic construction is the following:

Construction 2.0.1 (Subspace associated with a partition)**.**

- Input:
  (a) A finite set S.
  (b) An equivalence relation $\sim_\pi$ on S.
  (c) A space Z with an action of the symmetric group $\Sigma(S)$ of S.
- Output:
  (d) The subspace $Z(\pi)$ of points fixed by all permutations that interchange only $\sim_\pi$-equivalent points.

Example 2.0.2 (Product Stratification)**.** Let M be a closed complex manifold of real dimension m. The symmetric group $\Sigma(n)$ acts on the n-fold product space $Z = M^n$ permuting the coordinates.

For an equivalence relation $\sim_\pi$, one has:

$$Z(\pi) = \{(x_1, \ldots, x_n) \in Z \mid \forall\, i, j \in \{1, \ldots, n\} \colon i \sim_\pi j \implies x_i = x_j\}$$

These subspaces are closed, naturally oriented submanifolds of Z, and define homology classes $[Z(\pi)] \in \mathrm{H}_{\mathrm{ev}}(Z, \mathbb{Z})$ and Poincare-dual cohomology classes of even degree.

The main interest lies in the (co-)homology classes given by the spaces $Z(\pi)$. Motivated by this, geometric notions are reconstructed algebraically in the partition picture. A basic notion is *dimension*, which gives a grading on the cohomology and has to be interpreted as a grading on partitions.

In a further study of the product structure on cohomology, the notion of *transversality* turns out to be central and is transported to partitions. The products are then distinguished by whether the factors are transverse. The transverse products are easily reconstructed in the partition picture, but the remaining ones do not admit a natural interpretation.

To remedy this fact, a set of axioms is proposed in Definition 2.2.4 to model the non-transverse intersections. Then it is shown that the algebraic models of these axioms naturally form a one-parameter family. These models, called Kraken Algebras, are explicitly constructed.

In the two last subsections, the notions of symmetric and stable partitions are introduced. Related Kraken Algebras are constructed.

2.1 Partitions

Notation 2.1.1 (Symmetric group)**.** Let S be a finite set. The **symmetric group**, that is, the group of permutations of S, is denoted by $\Sigma(S)$. If $S = \{1, \ldots, n\}$, the notation $\Sigma(n)$ is used as well.

Definition 2.1.2 (Partition of a finite set)**.** Let S be a *finite* set.

(a) A **partition** π of S is a decomposition of S into nonempty disjoint subsets. In other words, it is the same as an equivalence relation on S.

(b) The subsets used in the decomposition are called **components** of the partition.

(c) A component containing exactly one element is called **singleton** and synonymously **trivial**.

(d) A partition with exactly one nontrivial component is called **primitive**.

(e) Adhering to the identification of π with an equivalence relation, whenever $i, j \in S$, the notation $i \sim_\pi j$ is used to denote that i and j are equivalent with respect to π.

(f) The notation $\pi = [(i_1, \ldots, i_{k_i}), (j_1, \ldots, j_{k_j}), \ldots]_S$ is used to denote a partition of S into subsets $\{i_1, \ldots, i_{k_i}\}, \{j_1, \ldots, j_{k_j}\}, \ldots$ and possibly additional singletons. In other words, π is denoted by a list of its nontrivial components.

(g) For primitive partitions, the internal round brackets may be omitted.

(h) The set of partitions of S is denoted by $\Pi(S)$.

(i) If $S = \{1, \ldots, n\}$ is the set of the first n natural numbers, the notation $\Pi(n)$ is used concurrently. For partitions of this set, the notation $[\ldots]_n$ is also used.

(j) The group $\Sigma(S)$ acts on $\Pi(S)$. The action of a permutation τ on a partition π is characterized by $i \sim_\pi j \Leftrightarrow \tau(i) \sim_{\tau(\pi)} \tau(j)$.

Example 2.1.3 (Partition of a finite set)**.** Consider the case $S = \{1, \ldots, n\}$, $n \geq 4$. The following are examples of partitions π of S, the first two being primitive.

(a) $\pi = [1,2]_n$. The identification of the first two elements of S.

(b) $\pi = [1,\ldots,n]_n$. The identification of all elements of S. In a partial order among partitions of S, defined in 2.1.11, this will be the unique maximal partition of S.

(c) $\pi_1 = [(1,2),(3,4)]_n$, $\pi_2 = [(1,3),(2,4)]_n$. These are two different non-primitive partitions of S.

(d) $$\Pi(4) = \left\{ \begin{array}{l} [\,]_4, \\ [1,2]_4, [1,3]_4, [1,4]_4, [2,3]_4, [2,4]_4, [3,4]_4, \\ [1,2,3]_4, [1,2,4]_4, [1,3,4]_4, [2,3,4]_4, \\ [(1,2),(3,4)]_4, [(1,3),(2,4)]_4, [(1,4),(2,3)]_4, \\ [1,2,3,4]_4 \end{array} \right\}$$

Definition 2.1.4 (Stabilizers of a partition)**.** Let S be a finite set. Let $\pi \in \Pi(S)$ be a partition.

(a) The **big stabilizer** is defined as:

$$G_0(\pi) = \{\tau \in \Sigma(S) \mid \forall\, i,j \in S \colon i \sim_\pi j \Leftrightarrow \tau(i) \sim_\pi \tau(j)\}$$

It is the stabilizer of π under the action defined in Definition 2.1.2.

(b) The **small stabilizer** is defined as:

$$G_1(\pi) = \{\tau \in \Sigma(S) \mid \forall\, i \in S \colon i \sim_\pi \tau(i)\}$$

As their names suggest, one has $G_1(\pi) \subseteq G_0(\pi)$. One can show that the big stabilizer is the normalizer of the small stabilizer. In particular, the small stabilizer is normal in the big stabilizer.

Example 2.1.5 (Stabilizers of a partition)**.** Consider $\pi = [(1,3),(2,4)]_4$. Then the small stabilizer $G_1(\pi)$ is the symmetry group of a generic rectangle with sides named 1, 2, 3, and 4. The big stabilizer is the symmetry group when the rectangle is a square, and is a non-commutative group with eight elements.

Remark 2.1.6 (Stabilizers are stabilizers)**.** In Example 2.0.2, let M have at least two points. Let $\pi \in \Pi(n)$ be a partition.

(a) $G_1(\pi)$ is the point stabilizer of $Z(\pi)$.

(b) $G_0(\pi)$ is the set stabilizer of $Z(\pi)$.

Definition 2.1.7 (Codimension grading)**.** Let S be a finite set. Let $\pi \in \Pi(S)$ be a partition of S. Denote the cardinalities of its nontrivial components by

$n = (n_1, \ldots, n_k)$. The **codimension grading** $d(\pi)$ is defined by the following formula:

$$d(\pi) = \sum_{i=1}^{k} (n_i - 1)$$

Lemma 2.1.8 (Codimension is codimension). *In Example 2.0.2, let* $\pi \in \Pi(n)$ *be a partition. The following holds:*

$$\operatorname{codim}_{\mathbb{R}}(Z(\pi), Z) = m \cdot d(\pi)$$

Proof. By the implicit function theorem, one has to show that $Z(\pi)$ is cut out by the correct number of independent regular functions.

Let $x = (x_1, \ldots, x_n) \in Z(\pi)$ be a point, $U \subseteq M$ a (possibly disconnected) chart domain of M containing all points $x_i \in M$, and $(f_1, \ldots, f_m) \colon U \to \mathbb{R}^m$ coordinates on U. Concatenating with the factor projections on U^n, this yields coordinate functions $\{f_{j,k} \mid 1 \leq j \leq m, 1 \leq k \leq n\}$ on U^n, which is a neighborhood of x.

The important point now is that, to cut out $Z(\pi)$, for each component of π of cardinality n_i, we have to require that one coordinate with index in that component equals all the $(n_i - 1)$ other coordinates with index in that component. This means that $Z(\pi)$ is the zero set of the following collection of functions:

$$\{f_{j,k} - f_{j,l} \mid \text{for some component } \pi_i \text{ of } \pi \colon k = \sup \pi_i, l \in \pi_i \setminus \{\sup \pi_i\}\}$$

The derivatives of these functions are easily checked to have full rank, which completes the proof. □

Definition 2.1.9 (Number of nontrivial components). Let S be a finite set. Let $\pi \in \Pi(S)$ be a partition of S. Then $c(\pi)$ denotes its **number of nontrivial components**.

Example 2.1.10 (Partition grading). The partitions $\pi \in \Pi(4)$ have the following codimension gradings and numbers of nontrivial components:

	$c(\pi) = 0$	$c(\pi) = 1$	$c(\pi) = 2$
$d(\pi) = 0$	$[]_4$		
$d(\pi) = 1$		$[1, 2]_4$	
$d(\pi) = 2$		$[1, 2, 3]_4$	$[(1, 2)(3, 4)]_4$
$d(\pi) = 3$		$[1, 2, 3, 4]_4$	

In the table, one representative for each $\Sigma(4)$-orbit is chosen.

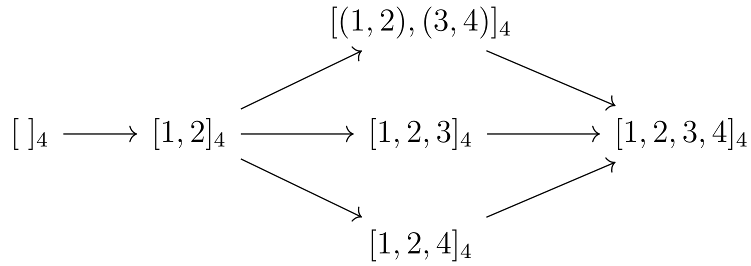
Definition 2.1.11 (Partial order on partitions)**.** Let S be a finite set. Let $\pi_1, \pi_2 \in \Pi(S)$ be two partitions of S.

(a) A partition is said to be **greater than** another partition if it enlarges its components:

$$\pi_1 \geq \pi_2 \Leftrightarrow (\forall i, j \in S\colon i \sim_{\pi_2} j \implies i \sim_{\pi_1} j)$$

(b) With respect to the partial order just defined, the supremum of two partitions π_1, π_2 is called **partition generated by** $\{\pi_1, \pi_2\}$, and denoted as $\pi_1 \vee \pi_2$.

Example 2.1.12 (Partial order on partitions)**.** The diagram below presents some partitions in $\Pi(4)$, where arrows point from smaller to larger elements.



Lemma 2.1.13 (Partition order is inclusion)**.** *In the situation of Example 2.0.2, for two partitions* $\pi_1, \pi_2 \in \Pi(n)$*, the inequality* $\pi_1 \geq \pi_2$ *implies the inclusion* $Z(\pi_1) \subseteq Z(\pi_2)$*. If* M *has at least two points, the converse implication holds.*

Proof. The inequality on the partitions immediately implies the inclusion of the spaces, as the defining equalities are inherited.

For the converse, observe that if $\pi_1 \not\geq \pi_2$, there are indices $i, j \in \{1, \ldots, n\}$ which are π_2-equivalent, but not π_1-equivalent. One can construct from this a point in $Z(\pi_1)$ which has different i- and j-coordinates, hence is not in $Z(\pi_2)$. □

Lemma 2.1.14 (Supremum is intersection)**.** *In the situation of Example 2.0.2, for two partitions* $\pi_1, \pi_2 \in \Pi(n)$*, one has* $Z(\pi_1 \vee \pi_2) = Z(\pi_1) \cap Z(\pi_2)$*.*

Proof. The defining equations of the space associated with the supremum are those which are implied by the intersecting spaces. □

Definition 2.1.15 (Transverse partitions)**.** Let S be a finite set. Two partitions $\pi_1, \pi_2 \in \Pi(S)$ are called **transverse** if $d(\pi_1 \vee \pi_2) = d(\pi_1) + d(\pi_2)$.

Lemma 2.1.16 (Characterisation of transverse partitions)**.** *Let S be a finite set. Let $\pi_1, \pi_2 \in \Pi(S)$ be two partitions.*

(a) $c(\pi_1 \vee \pi_2) \leq c(\pi_1) + c(\pi_2)$

(b) Every partition π_1 can be written as supremum over primitive partitions whose non-trivial component has cardinality 2. The codimension $d(\pi_1)$ is the minimal number of partitions required.

(c) $d(\pi_1 \vee \pi_2) \leq d(\pi_1) + d(\pi_2)$

(d) If π_1, π_2 are primitive, they are transverse if and only if their nontrivial components intersect in at most one element.

Proof.

(a) Every non-trivial component of $\pi_1 \vee \pi_2$ contains a non-trivial component of π_1 or π_2.

(b) Every equivalence relation can be generated by identifications of two elements. The minimal number of identifications needed is, for each component, one less than the cardinality of that component.

(c) Identifications that generate π_1 and π_2 in union also generate their supremum.

(d) If the components intersect in more elements, the supremum needs less identifications. □

Lemma 2.1.17 (Transverse means transverse)**.** *Again in the situation of Example 2.0.2, let M be a nonempty manifold of positive dimension. Then, two partitions $\pi_1, \pi_2 \in \Pi(n)$ are transverse if and only if their associated spaces $Z(\pi_1), Z(\pi_2)$ are transverse.*

With the obvious extension of transversality to more than two partitions, the lemma holds as well.

Proof. The functions in the proof of Lemma 2.1.8 are regular functions to cut out the intersection. □

Example 2.1.18 (Transverse partitions)**.** One has:

$$[(1,2),(3,4)]_4 \vee [1,3]_4 = [1,2,3,4]_4$$
$$[(1,2),(3,4)]_4 \vee [(1,3),(2,4)]_4 = [1,2,3,4]_4$$

Thus, the partitions $[(1,2),(3,4)]_4$ and $[1,3]_4$ are transverse, but $[(1,2),(3,4)]_4$ and $[(1,3),(2,4)]_4$ are not. This shows that the characterization of transverse intersection in Lemma 2.1.16 does not hold for non-primitive partitions.

2.2 The Collision Algebra

Definition 2.2.1 (Collision Algebra)**.** Let n be a natural number. Let R be a commutative unital ring. The **Collision Algebra** $\mathcal{K}_\infty(n,R)$ over R on n elements is the algebra which has as underlying R-module the free R-module on $\Pi(n)$ and whose multiplication is defined as follows:

(a) Transversality: For two transverse partitions $\pi_1, \pi_2 \in \Pi(n)$:

$$\pi_1 \cdot \pi_2 = \pi_1 \vee \pi_2$$

(b) For two non-transverse partitions $\pi_1, \pi_2 \in \Pi(n)$:

$$\pi_1 \cdot \pi_2 = 0$$

This defines an associative and commutative multiplication, because, with a third partition $\pi_3 \in \Pi(n)$, one has independently of the order of multiplication:

$$\pi_1 \cdot \pi_2 \cdot \pi_3 = \begin{cases} \pi_1 \vee \pi_2 \vee \pi_3 & \text{if } d(\pi_1) + d(\pi_2) + d(\pi_3) = d(\pi_1 \vee \pi_2 \vee \pi_3), \\ 0 & \text{else.} \end{cases}$$

By Lemma 2.1.16, the Collision Algebra

- is graded via the codimension grading $d(-)$ and
- is filtered by the free R-submodules generated by $\{\pi \in \Pi(n) \mid c(\pi) \leq k\}_k$.

The symmetric group $\Sigma(n)$ acts on the Collision Algebra by algebra automorphisms permuting the basis elements.

Example 2.2.2 (Collision Algebra)**.** In $\mathcal{K}_\infty(3,\mathbb{Z})$, one has:

$$c([1,2]_3 \cdot [2,3]_3) = c([1,2,3]_3) = 1$$

But $c([1,2]_3) = c([2,3]_3) = 1$, which shows that the number of nontrivial components is not a grading.

Lemma 2.2.3 (Manifold with non-vanishing vector field)**.** *In the situation of Example 2.0.2, assume that M admits a nowhere vanishing real vector field. Then the map*

$$\begin{aligned} \mathcal{K}_\infty(n,R) &\longrightarrow \mathrm{H}^{\mathrm{ev}}(Z,R) \\ \pi &\longmapsto \textit{Poincare-dual of } [Z(\pi)] \end{aligned}$$

is a homomorphism of R-algebras.

Proof. One has to check that for two non-transverse partitions $\pi_1, \pi_2 \in \Pi(n)$, the homological intersection of $Z(\pi_1), Z(\pi_2)$ vanishes.

Because the partitions are non-transverse, there must be some chain of elements $i_1, \ldots, i_k \in \{1, \ldots, n\}$ and a minimal nontrivial chain of equivalences which alternates the equivalence relations:

$$i_1 \sim_{\pi_1} i_2 \sim_{\pi_2} i_3 \sim_{\pi_1} \ldots \sim_{\pi_1} i_k \sim_{\pi_2} i_1$$

Now taking a small flow of the vector field in one of the components appearing in this chain moves the two partitions apart from each other. □

Alternatively, the vanishing of the Euler characteristic number of M can be used to prove the assertion.

The point that enabled this Lemma to work was the fact that the rational vector space generated by the $Z\pi$ was closed under multiplication. In the general case (with non-vanishing Euler characteristic number), this fails, as the self-intersection of the diagonal in the product is a non-zero number of points, which is never given by a class of type $Z\pi$. The easiest counterexample is the space $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, where the self-intersection of the diagonal is not represented by a cycle given by a partition. In general, the homological intersection of two non-transverse cycles $Z(\pi_1)$, $Z(\pi_2)$ is not represented by a cycle associated with a partition.

Now the question how to algebraically model the non-transverse product of the cohomology classes associated with a partition is answered in two extreme cases. There are cases where the product cannot be modeled, and there are cases where it vanishes. In the situation of Example 2.0.2, this covers most cases. But in general, there are more possibilities of multiplicatively closed structures on cohomology classes of partitions, which will be called Deformed Collision Algebras.

The presentation of the Deformed Collision Algebras will proceed as follows. Firstly, a set of axioms is proposed below that clarifies the role of these algebras as deformations of the collision algebra. Secondly, it is shown that in each such algebra, the multiplication of two primitive partitions takes a very specific form (Kraken equation). Thirdly, all models over a field of characteristic zero are constructed.

Definition 2.2.4 (Deformed Collision Algebra)**.** Let n be a natural number. Let R be a commutative unital ring. A **Deformed Collision Algebra** over R on n elements is an associative commutative algebra which has as underlying R-module the free R-module on $\Pi(n)$, is $d(-)$-graded and and whose multiplication fulfills the following relations:

(a) Transversality: For two transverse partitions $\pi_1, \pi_2 \in \Pi(n)$:

$$\pi_1 \cdot \pi_2 = \pi_1 \vee \pi_2$$

(b) For two non-transverse *primitive* partitions, their product is an R-linear combination of primitive partitions.

(c) The product is invariant under the $\Sigma(n)$-action on partitions.

The idea of this definition is to change the multiplication on the Collision Algebra only in lower degrees of its filtration.

Example 2.2.5 (Deformed Collision Algebra)**.** As a simple example, the Deformed Collision Algebras on 3 elements are easily described: The only product of basis elements that is not determined by transversality or degree considerations is the square of a codimension one partition. The only symmetric choice is to fix an $r \in R$ and define:

$$[1,2]_3 \cdot [1,2]_3 = [1,3]_3 \cdot [1,3]_3 = [2,3]_3 \cdot [2,3]_3 = r[1,2,3]_3$$

All choices define a Deformed Collision Algebra, so there is a one-parameter family of them. The parameter λ that occurs later on is in this case $r^{-1} + 1$.

2.3 Kraken Algebras

Definition 2.3.1 (Kraken Algebra)**.** Let n be a natural number. Let R be a commutative unital ring. A **Kraken Algebra** over R on n elements is an associative commutative algebra which has as underlying R-module the free R-module on $\Pi(n)$ and whose multiplication fulfills the following relations:

(a) Transversality: For two transverse partitions $\pi_1, \pi_2 \in \Pi(n)$:

$$\pi_1 \cdot \pi_2 = \pi_1 \vee \pi_2$$

(b) The **Kraken equations**: For natural numbers n_1, n_2, n_3 with

$$n_1 + n_2 \leq n - 1, \quad n_3 \leq n_1, \quad n_3 \leq n_2,$$

there exist **Kraken parameters** $\mu_{n_1,n_2}^{n_3} \in R$, such that:

- (Honest Kraken equations)
For two primitive partitions $\pi_1, \pi_2 \in \Pi(n)$ of codimension $d(\pi_1) = n_1$ and $d(\pi_2) = n_2$ (with $n_1 + n_2 \leq n - 1$), whose nontrivial components intersect in a subset of cardinality n_3+1, the following equation holds:

$$\pi_1 \cdot \pi_2 = \mu_{n_1,n_2}^{n_3} \sum_{\pi \in I} \pi,$$

$$I = \{\pi \in \Pi(n) \mid \pi \text{ primitive}, \pi \geq \pi_1 \vee \pi_2, d(\pi) = d(\pi_1) + d(\pi_2)\}$$

- (Degenerate Kraken equations)
For two primitive partitions $\pi_1, \pi_2 \in \Pi(n)$ with $d(\pi_1) + d(\pi_2) \geq n$,

$$\pi_1 \cdot \pi_2 = 0$$

The symmetric group $\Sigma(n)$ acts on Kraken Algebras by R-module automorphisms permuting the basis elements.

Remark 2.3.2 (Transverse Kraken Equations)**.** It follows from transversality that for $n_1 + n_2 \leq n - 1$, one has $\mu_{n_1,n_2}^{0} = 1$.

The name was chosen by the author with the following picture in mind: Outraged about the non-transverse partitions, a kraken with n_3 tentacles furiously picks additional elements to pad the disappointing $\pi_1 \vee \pi_2$ up to its expected codimension. The unusual kraken was preferred over the more common octopus, because the name of the latter misleadingly refers to the number eight.

Lemma 2.3.3 (Deformed Collision Algebras are Kraken)**.** *Let n be a natural number. Let R be a commutative unital ring. Let $\mathcal{A}$ be a Deformed Collision Algebra over R on n elements.*

(a) $\mathcal{A}$ is generated by primitive partitions of codimension one.

(b) If $n \neq 4$, $\mathcal{A}$ is a Kraken Algebra.

In the exceptional case $n = 4$, there are indeed more Deformed Collision Algebras than Kraken Algebras. While the Kraken Algebras form a one-parameter family, there is a larger two-parameter family of Deformed Kraken Algebras, parametrized by r_1, r_2 in the proof below.

Proof.

(a) Every partition can be written as the transverse product of partitions of codimension one, which proves (a).

(b) It suffices to prove the Kraken equations for primitive partitions π_1, π_2. As the cases $n \leq 2$ are trivial, and $n = 3$ has been discussed in Example 2.2.5, from now on $n \geq 5$ is assumed.

PART A: π_2 HAS CODIMENSION 2. First, assume that π_2 has codimension one. Without loss of generality, one can assume

$$\begin{aligned} \pi_1 &= [1, \ldots, k]_n, \quad k \geq 2, \\ \pi_2 &= [1, 2]_n. \end{aligned}$$

Now one can write:

$$\pi_1 \cdot \pi_2 = \sum_{\pi \in I} r_\pi \pi$$

$$I = \{\pi \in \Pi(n) \mid \pi \text{ primitive}, d(\pi) = d(\pi_1) + d(\pi_2)\}$$

For arbitrary distinct $i, j \leq k, i \geq 2$ one has by transversality:

$$\begin{aligned} [1, \ldots, k]_n \cdot [1, 2]_n &= [2, \ldots, k]_n \cdot [1, i]_n \cdot [1, 2]_n \\ &= [1, \ldots, k]_n \cdot [1, i]_n \\ &= [1, \ldots, i-1, i+1, \ldots, k]_n \cdot [i, j]_n \cdot [1, i]_n \\ &= [1, \ldots, k]_n \cdot [i, j]_n \end{aligned}$$

This shows that r_π does only depend on the $\Sigma(k) \times \Sigma(n-k)$-orbit of π, which is determined by the number of elements its nontrivial component has in common with the set $\{1, \ldots, k\}$. Now the cases $k \leq n-4$, $k = n-3$ and $k = n-2$ are treated separately.

CASE $k \leq n - 4$. Assume the product contains a summand whose primitive partition is not greater than $[1, \ldots, k]_n$. There are at least three elements in $\{1, \ldots, n\}$ which are not in the nontrivial component of that partition. By symmetry one can assume that one of those elements is 1. Let i be another of these elements, which is not 2. Consider the product:

$$([1, \ldots, k]_n \cdot [1, 2]_n) \cdot [1, i]_n = [1, \ldots, k]_n \cdot [1, 2, i]_n$$

Now the left hand side would contain a summand with a non-primitive partition, but the right hand side cannot have such.

CASE $k = n-3$, SUBCASE $n \geq 6$. Assume again the product contains a summand whose primitive partition is not greater than $[1,\dots,k]_n$. There are at least two elements in $\{1,\dots,n\}$ which are not in the nontrivial component of that partition. By symmetry one can assume that one of those elements is 1. Let i be another one of these elements, now one cannot exclude 2. Consider the product:

$$([1,\dots,k]_n \cdot [1,2]_n) \cdot [1,i]_n = [1,\dots,k]_n \cdot ([1,2]_n \cdot [1,i]_n)$$

Now the left hand side would contain a summand with a non-primitive partition, but the right hand side cannot have such by the previous case.

CASE $k = n-3$, SUBCASE $n = 5,\ k = 2$. With the same arguments as before one can assume that:

$$[1,2]_5 \cdot [1,2]_5 = r_1([1,2,3]_5 + [1,2,4]_5 + [1,2,5]_5) + r_2[3,4,5]_5$$

It follows:

$$\begin{aligned}[1,2,3]_5 \cdot [1,3]_5 &= [1,2]_5^2 \cdot [1,3]_5 \\ &= r_1([1,2,3]_5 \cdot [1,3]_5 + [1,2,3,4]_5 + [1,2,3,5]_5) + r_2[1,3,4,5]_5\end{aligned}$$

$$\implies (1-r_1)[1,2,3]_5 \cdot [1,3]_5 = r_1([1,2,3,4]_5 + [1,2,3,5]_5) + r_2[1,3,4,5]_5$$

Now by the symmetry of the left hand side in 1 and 2, $r_2 = 0$.

CASE $k = n-2$. In this case one has:

$$\begin{aligned}[1,\dots,n-2]_n \cdot [1,2]_n = \quad & r_1([1,\dots,n-2,n-1]_n + [1,\dots,n-2,n]_n) \\ & + r_2 \sum_{i=1}^{n-2} [1,i-1,i+1,n]_n\end{aligned}$$

By the previous case, one can define $A \in R$ via:

$$\begin{aligned}&[1,\dots,n-3]_n \cdot [1,2]_n \\ &= A \cdot ([1,\dots,n-3,n-2]_n + [1,\dots,n-3,n-1]_n + [1,\dots,n-3,n]_n)\end{aligned}$$

It follows that:

$$\begin{aligned}&([1,\dots,n-3]_n \cdot [1,2]_n) \cdot [1,n-2]_n \\ &= A \cdot \left((1+r_1)([1,\dots,n-2,n-1]_n + [1,\dots,n-2,n]_n) + r_2 \sum_{i=1}^{n-2} [1,\hat{i},n]_n \right)\end{aligned}$$

Which is equal to:

$$\begin{aligned}
&([1,\dots,n-3]_n\cdot[1,n-2]_n)\cdot[1,2]_n\\
&=r_1([1,\dots,n-2,n-1]_n+[1,\dots,n-2,n]_n)+\ r_2\sum_{i=1}^{n-2}[1,\hat{i},n]_n
\end{aligned}$$

Hence $r_2 = 0$.

Part B: π_2 has arbitrary codimension. In this case, one can without loss of generality assume

$$\begin{aligned}
\pi_1 &= [1,\dots,k]_n, \quad k\geq 2,\\
\pi_2 &= [1,\dots,i,k+1,\dots,k+j]_n, \quad i\leq k.
\end{aligned}$$

Then the following holds:

$$\begin{aligned}
\pi_1\cdot\pi_2 &= \pi_1\cdot[i,k+1,\dots,k+j]_n\cdot[1,\dots,i]_n && \text{by transversality}\\
&= [1,\dots,k+j]_n\cdot[1,\dots,i]_n && \text{by transversality}\\
&= [1,\dots,k+j]_n\cdot[1,\dots,i-1]_n\cdot[1,i]_n\\
&= \sum_{\pi\in I}\mu^{i-2}_{k+j-1,i-2}\,\pi\cdot[1,i]_n && \text{inductively}\\
&= \sum_{\pi\in I}\mu^{i-2}_{k+j-1,i-2}\sum_{\pi'\in I'}\mu^{1}_{k+j+i-3,1}\,\pi' && \text{using part A}\\
&= \sum_{\pi\in I''}(i-1)\mu^{i-2}_{k+j-1,i-2}\,\mu^{1}_{k+j+i-3,1}\,\pi
\end{aligned}$$

Here the following sets are used:

$$\begin{aligned}
I &= \{\pi\in\Pi(n)\mid \pi\text{ primitive}, \pi\geq[1,\dots,k+j]_n, d(\pi)=k+j+i-3\}\\
I' &= \{\pi'\in\Pi(n)\mid \pi'\text{ primitive}, \pi'\geq\pi, d(\pi')=k+j+i-2\}\\
I'' &= \{\pi\in\Pi(n)\mid \pi\text{ primitive}, \pi\geq[1,\dots,k+j]_n, d(\pi)=k+j+i-2\}
\end{aligned}$$

This proves the lemma. □

Lemma 2.3.4 (Kraken Algebras are symmetric)**.** *Let n be a natural number. Let R be a commutative unital ring. Let $\mathcal{A}$ be a Kraken Algebra over R on n elements.*

(a) $\mathcal{A}$ is generated by primitive partitions of codimension one.

(b) $\Sigma(n)$ acts on $\mathcal{A}$ via R-algebra automorphisms.

Proof. Every partition can be written as the transverse product of partitions of codimension one, which proves (a). Now every product can be inductively (on the number of nontrivial components) computed via the Kraken equations. Because the Kraken equations are symmetric, (b) follows. □

Lemma 2.3.5 (Kraken consistency equations)**.** *Let n be a natural number. In a Kraken Algebra on n elements, the Kraken parameters are symmetric in the lower indices and fulfill the following equations:*

For all $a_1, a_2, a_3, b_{12}, b_{13}, b_{23} \geq 0, c \geq -1$, such that

- *$c \geq 0$ or $b_{12}, b_{13} \geq 1$, and*
- *$a_1 + a_2 + a_3 + 2b_{12} + 2b_{13} + 2b_{23} + 3c + 1 \leq n$,*

one has:

$$A \sum_{k=0}^{b_{12}+c} \binom{a_3}{k} \binom{b_{12} + b_{13} + b_{23} + 2c}{b_{12} + c - k} B = C \sum_{k=0}^{b_{13}+c} \binom{a_2}{k} \binom{b_{13} + b_{12} + b_{23} + 2c}{b_{13} + c - k} D$$

with

$$\begin{aligned} A &= \mu^{b_{12}+c}_{a_1+b_{12}+b_{13}+c, a_2+b_{12}+b_{23}+c}, \\ B &= \mu^{b_{13}+b_{23}+c+k}_{a_1+a_2+2b_{12}+b_{13}+b_{23}+2c, a_3+b_{13}+b_{23}+c}, \\ C &= \mu^{b_{13}+c}_{a_1+b_{13}+b_{12}+c, a_3+b_{13}+b_{23}+c}, \\ D &= \mu^{b_{12}+b_{23}+c+k}_{a_1+a_3+2b_{13}+b_{12}+b_{23}+2c, a_2+b_{12}+b_{23}+c}. \end{aligned}$$

These equations are called the ***Kraken consistency equations*** *for n elements.*

Proof. The assertion about the symmetry follows from the commutativity of the multiplication.

By the restrictions on the given numbers, one can find three primitive partitions $\pi_1, \pi_2, \pi_3 \in \Pi(n)$ with codimensions

$$\begin{aligned} d(\pi_1) &= a_1 + b_{12} + b_{13} + c, \\ d(\pi_2) &= a_2 + b_{12} + b_{23} + c, \\ d(\pi_3) &= a_3 + b_{13} + b_{23} + c, \end{aligned}$$

whose nontrivial components intersect pairwise in subsets of cardinalities given by $b_{ij} + c + 1$, and whose triple intersection has cardinality $c + 1$, see Figure 1.

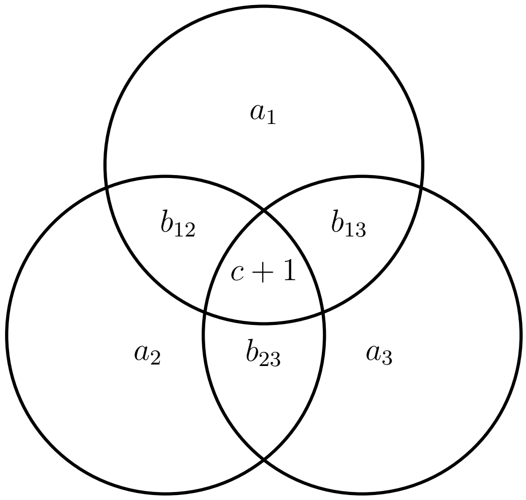


Figure 1: The partitions in the Kraken consistency equation. The circles represent the nontrivial components of the partitions.

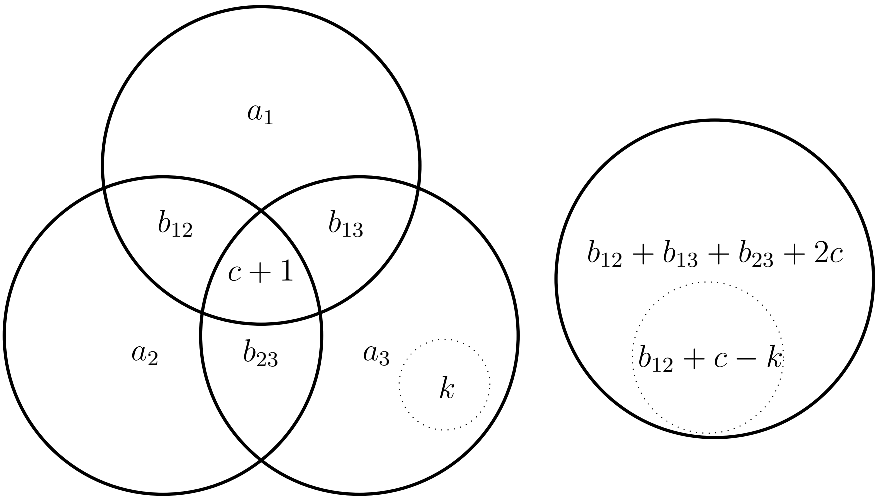


Figure 2: The union of the sets depicted represents a primitive partition that appears in the expansion of the triple product $(\pi_1 \cdot \pi_2) \cdot \pi_3$. In addition to the initial partitions, it contains $b_{12}+b_{13}+b_{23}+2c$ additional elements. The kraken of the first multiplication already picked $b_{12}+c$ new elements, k of them in a_3. The remaining $b_{13}+b_{23}+c$ are picked by the second. The dotted circles are the elements picked by the first kraken.

Then from the definition of Kraken Algebra one has:

$$\pi_1 \cdot \pi_2 = \mu^{b_{12}+c}_{a_1+b_{12}+b_{13}+c, a_2+b_{12}+b_{23}+c} \sum_{\pi \in I_{12}} \pi,$$

$$I_{12} = \{\pi \in \Pi(n) \mid \pi \text{ primitive}, \pi \geq \pi_1 \vee \pi_2, d(\pi) = d(\pi_1) + d(\pi_2)\}$$

To compute the triple product $(\pi_1 \cdot \pi_2) \cdot \pi_3$, one has to consider for all $\pi \in I_{12}$ the products $\pi \cdot \pi_3$. The cardinality of the intersection of their nontrivial components can be written as $b_{13} + b_{23} + c + k + 1$ for some $k \geq 0$, where k is the number of elements that the kraken picked out of the nontrivial component of π_3.

Counting the possibilities (see Figure 2), the triple product takes the following form:

$$A \sum_{k=0}^{b_{12}+c} \binom{a_3}{k} \binom{b_{12} + b_{13} + b_{23} + 2c}{b_{12} + c - k} B \sum_{\pi \in I_{123}} \pi,$$

$$I_{123} = \{\pi \in \Pi(n) \mid \pi \text{ primitive}, \pi \geq \pi_1 \vee \pi_2 \vee \pi_3, d(\pi) = d(\pi_1) + d(\pi_2) + d(\pi_3)\}$$

By the restriction on n, I_{123} is not empty. Thus, the Kraken consistency equation follows from the following equality of products of primitive partitions:

$$(\pi_1 \cdot \pi_2) \cdot \pi_3 = (\pi_1 \cdot \pi_3) \cdot \pi_2$$ □

Lemma 2.3.6 (Solutions of Kraken consistency equations)**.** *Let n be a natural number. Let R be a field of characteristic zero. Let $\{\mu^{n_3}_{n_1,n_2}\}_{n_1,n_2,n_3}$ be a solution to the Kraken consistency equations, indexed over natural numbers n_1, n_2, n_3 with $n_1 + n_2 \leq n - 1$, $n_3 \leq n_1$ and $n_3 \leq n_2$.*

If one of the Kraken parameters is zero, all of them are. The nonzero solutions of the Kraken consistency equations for n elements in R are

$$\mu^{n_3}_{n_1,n_2} = \binom{\lambda - n_1 - n_2 + n_3}{n_3}^{-1}$$

for $\lambda \in R \setminus \{1, \ldots, n-2\}$. Such a λ is called a valid parameter.

In the case $\lambda = n - 1$, the formula is supported by the following intuition: The formula then represents the inverse of the number of summands in the Kraken equation, so in this case the padding for the partition is chosen as the mean over an equidistribution.

Proof. In the following, only the nonzero case is considered. With the derived equations below one can show that, if one Kraken parameter is zero, all of them are zero. There are three steps:

STEP 1. Substituting the values

$$a_3 = b_{12} = 1, a_2 = b_{13} = b_{23} = c = 0$$

yields

$$\mu^1_{a_1+1,1}(1 + \mu^1_{a_1+2,1}) = \mu^1_{a_1+2,1},$$

which inductively proves the claim for Kraken parameters of the form $\mu^1_{i,1}$.

STEP 2. Substituting the values

$$a_2 = b_{23} = 0, b_{13} = 1, b_{12} = 2, c = -1$$

yields

$$\mu^1_{a_1+2,1}(1 + a_3\mu^1_{a_1+3,a_3}) = \mu^1_{a_1+a_3+2,1},$$

which proves the claim for Kraken parameters of the form $\mu^1_{i,j}$, except $\mu^1_{2,2}$.

STEP 3. Substituting the values

$$a_3 = 0, b_{13} = b_{23} = 1, c = -1$$

yields

$$b_{12}\mu^{b_{12}-1}_{a_1+b_{12},a_2+b_{12}}\mu^1_{a_1+a_2+2b_{12},1} = \mu^{b_{12}}_{a_1+b_{12}+1,a_2+b_{12}},$$

which inductively proves the claim for all Kraken parameters (it also settles the case $\mu^1_{2,2}$).

To see that the given expressions are indeed solutions of the Kraken consistency equations, it is sufficient to prove the statement that one side of the equation is equal to $\binom{\lambda-a_1-a_2-a_3-b_{12}-b_{13}-b_{23}-c}{b_{12}+b_{13}+b_{23}+2c}^{-1}$, because this expression is invariant under exchanging the indices 2 and 3.

On substituting

$$\begin{aligned} X &= \lambda - a_1 - a_2 - a_3 - b_{12} - b_{13} - b_{23} - c \\ Y &= b_{12} + c \\ Z &= b_{13} + b_{23} + c \end{aligned}$$

the claimed statement becomes equivalent to:

$$\binom{X + a_3}{Y} = \binom{X}{Y + Z} \sum_{k\geq 0} \frac{\binom{a_3}{k}\binom{Y+Z}{Y-k}}{\binom{X-Y+k}{Z+k}}$$

This follows from the identity:

$$\binom{X}{Y+Z}\binom{Y+Z}{Y-k} = \frac{X \cdot \ldots \cdot (X-Y-Z+1)}{(Y-k)!\,(Z+k)!} = \binom{X}{Y-k}\binom{X-Y+k}{Z+k}$$
$$\left[= \binom{X}{Y-k, Z+k, X-Y-Z}\right]$$

□

Theorem 2.3.7 (Classification of Kraken Algebras)**.** *Let n be a natural number. Let R be a field of characteristic zero.*

For $n = 1$ and $n = 2$ there is exactly one Kraken Algebra, namely the Collision Algebra $\mathcal{K}_\infty(n, R)$.

Every Kraken Algebra on $n \geq 3$ elements over R is one of the following algebras:

- *The Collision Algebra $\mathcal{K}_\infty(n, R)$*
- *For a valid parameter $\lambda(\in R \setminus \{0, \ldots, n-2\})$, a Kraken Algebra $\mathcal{K}_\lambda(n, R)$ whose Kraken parameters (indexed by natural numbers n_1, n_2, n_3 with $n_1 + n_2 \leq n - 1$) are given by:*

$$\mu^{n_3}_{n_1,n_2} = \binom{\lambda - n_1 - n_2 + n_3}{n_3}^{-1}$$

These Kraken parameters determine the Kraken Algebra uniquely.

Let λ be a valid parameter. The Kraken Algebra $\mathcal{K}_\lambda(n, R)$ is given as an associative commutative algebra by each of the two following sets of generators and relations:

(a) (Reduction relations)

- *Generators: Primitive partitions on n elements.*
- *Relations: All Kraken equations on n elements, with $\mu^{n_3}_{n_1,n_2}$ given by the formula above.*

(b) (Convenient relations)

- *Generators: Primitive partitions on n elements.*
- *Relations: The Kraken equations for products $\pi_1 \cdot \pi_2$ with $d(\pi_2) = 1$, with $\mu^{n_3}_{n_1,n_2}$ given by the formula above.*

This means that the convenient relations imply all Kraken equations.

Proof. The cases $n = 1$ and $n = 2$ are elementary.

For $n \geq 3$, With the reduction relations, all follows from the Diamond Lemma ([2], Theorem 1.2, with the necessary changes to the commutative case), with reduction system given by the Kraken equations, and partial order given by the number of components. That the ambiguities are resolvable is equivalent to the Kraken consistency equations.

That the convenient relations imply all Kraken equations follows from an inductive calculation on the codimension of π_2. □

2.4 Symmetric Partitions

Definition 2.4.1 (Finite symmetric partition)**.** Let n be a natural number.

(a) A **symmetric partition** σ of n is a presentation of n as a sum of (weakly) descending non-zero natural numbers.

$$n = \sum_{i=1}^{k} j_i, \quad j_1 \geq \ldots \geq j_k$$

(b) The numbers j_i used in the decomposition are called **components** of the symmetric partition.

(c) A component equal to 1 is called **trivial**.

(d) A partition with exactly one nontrivial component is called **primitive**.

(e) The notation $\sigma = \langle j_1, \ldots, j_k \rangle_n$ is used to denote a symmetric partition. All occurrences of 1 can be omitted.

(f) The set of symmetric partitions of n is denoted by $\Pi_\Sigma(n)$.

Symmetric partitions of n are in canonical bijection with the orbits of the canonical action of $\Sigma(n)$ on $\Pi(n)$. For a ring R, this induces a canonical map where each symmetric partition σ is sent to the sum over the element of its orbit $\Sigma(n)\sigma$:

$$\begin{aligned} R[\Pi_\Sigma(n)] &\longrightarrow R[\Pi(n)] \\ \sum_{\sigma \in \Pi_\Sigma(n)} r_\sigma \sigma &\longmapsto \sum_{\pi \in \Pi(n)} r_{\Sigma(n)\pi} \pi \end{aligned}$$

Definition 2.4.2 (Partial order on partitions)**.** Let n be a natural number. Let $\sigma_1, \sigma_2 \in \Pi_\Sigma(n)$ be two symmetric partitions. σ_1 is said to be **greater than** σ_2 if one can replace sets of summands of σ_2 by their sum and thereby obtains σ_1.

This is equivalent to the existence of (non-symmetric) partitions in the orbits such that the same relation holds between them.

Definition 2.4.3. Let n be a natural number. Let R be a field of characteristic zero. Let λ be valid parameter. The **Symmetric Kraken Algebra** $\mathcal{K}^{\Sigma}_{\lambda}(n, R)$ is the algebra of $\Sigma(n)$-invariants of $\mathcal{K}_{\lambda}(n, R)$.

Consider an element in the Kraken Algebra:

$$\sum_{\pi \in \Pi(n)} r_{\pi} \pi \in \mathcal{K}_{\lambda}(n, R)$$

If it is inside the Symmetric Kraken Algebra, the coefficients r_{π} only depend on the $\Sigma(n)$-orbit of π. Therefore, the Symmetric Kraken Algebra can be canonically identified as an R-module with the free R-module on $\Pi_{\Sigma}(n)$.

Example 2.4.4 (Symmetric Kraken Algebra)**.** The algebra $\mathcal{K}^{\Sigma}_{\lambda}(4, R)$ has the basis given by

$$\langle\rangle_4, \langle 2\rangle_4, \langle 2,2\rangle_4, \langle 3\rangle_4, \langle 4\rangle_4$$

and is generated by the elements:

$$\langle 2\rangle_4, \langle 3\rangle_4, \langle 4\rangle_4$$

One computes:

$$\begin{aligned}
\langle 2\rangle_4^2 &= ([1,2]_4 + [1,3]_4 + [1,4]_4 + [2,3]_4 + [2,4]_4 + [3,4]_4)^2 \\
&= 2([(1,2),(3,4)]_4 + [(1,3),(2,4)]_4 + [(1,4),(2,3)]_4) \\
&\quad + (6 + 3\mu^1_{1,1})([1,2,3]_4 + [1,2,4]_4 + [1,3,4]_4 + [2,3,4]_4) \\
&= 2\langle 2,2\rangle_4 + (6 + 3\mu^1_{1,1})\langle 3\rangle_4 \\
\langle 2\rangle_4 \cdot \langle 2,2\rangle_4 &= (12 + 12\mu^1_{1,1})\langle 4\rangle_4 \\
\langle 2\rangle_4 \cdot \langle 3\rangle_4 &= (12 + 12\mu^1_{1,2})\langle 4\rangle_4 \\
\langle 2\rangle_4^3 &= (2(12 + 12\mu^1_{1,1}) + (6 + 3\mu^1_{1,1})(12 + 12\mu^1_{1,2}))\langle 4\rangle_4 \\
&= \left(\frac{24\lambda}{\lambda - 1} + \frac{72\lambda - 36}{\lambda - 2}\right)\langle 4\rangle_4 \\
&= \frac{96\lambda^2 - 156\lambda + 36}{(\lambda-1)(\lambda-2)}\langle 4\rangle_4
\end{aligned}$$

with

$$\mu^1_{1,1} = \frac{1}{\lambda - 1}, \qquad \mu^1_{1,2} = \frac{1}{\lambda - 2}.$$

2.5 Stabilization

Definition 2.5.1 (Stable partition)**.** A **stable partition** is a decomposition of $\mathbb{N}$ into finite subsets, where only finitely many of those contain more than one element.

For any natural number n, there is an injection

$$\begin{aligned} \alpha_n \colon \Pi(n) &\longrightarrow \Pi(n+1) \\ [-]_n &\longmapsto [-]_{n+1} \end{aligned}$$

adding another singleton $\{n+1\}$. The set of stable partitions $\Pi(\infty)$ is a colimit of this system. For a partition $[-]_n$ its image in the colimit is denoted as $[-]$.

Under this identification, all terminology that is defined for partitions of finite sets and that is invariant under this inclusions extends to stable partitions. This applies for example to codimension, non-trivial components, and number of non-trivial components.

Definition 2.5.2 (Stable symmetric partition)**.** A **stable symmetric partition** is a weakly decreasing sequence of natural numbers that is eventually equal to 1.

For any natural number n, there is an injection

$$\begin{aligned} \alpha_n \colon \Pi_\Sigma(n) &\longrightarrow \Pi_\Sigma(n+1) \\ \langle - \rangle_n &\longmapsto \langle - \rangle_{n+1} \end{aligned}$$

adding another 1. The set $\Pi_\Sigma(\infty)$ of stable symmetric partitions is a colimit of this system. It can be canonically identified with the $\Sigma(\infty)$-orbits on $\Pi(\infty)$. For a partition $\langle - \rangle_n$, its image in the colimit is denoted by $\langle - \rangle$.

Definition 2.5.3 (Stable Symmetric Kraken Algebra)**.** Let R be a field of characteristic zero. For any natural number n and parameter λ valid for all natural numbers (a **stable valid parameter**), there is a surjective homomorphism of $d(-)$-graded algebras

$$\begin{aligned} \alpha_n \colon \mathcal{K}^\Sigma_\lambda(n+1, R) &\longrightarrow \mathcal{K}^\Sigma_\lambda(n, R) \\ \sum_{\sigma \in \Pi_\Sigma(n+1)} r_\sigma \sigma &\longmapsto \sum_{\sigma \in \Pi_\Sigma(n)} r_{\alpha_n(\sigma)} \sigma \end{aligned}$$

induced by the injections $\alpha_n \colon \Pi_\Sigma(n) \hookrightarrow \Pi_\Sigma(n+1)$ from Definition 2.5.2.

The **Stable Symmetric Kraken Algebra** $\mathcal{K}^\Sigma_\lambda(\infty, R)$ is the limit over this system (as graded algebras). The underlying R-module can be canonically identified with the free module on the stable partitions $\Pi_\Sigma(\infty)$. Under this identification, a sequence $(\langle - \rangle_n)_n$ is mapped to $\langle - \rangle$.

Theorem 2.5.4 (Stable Symmetric Kraken Algebra is polynomial)**.** *Let R be a field of characteristic zero. Let λ be a valid stable parameter. The algebra $\mathcal{K}^{\Sigma}_{\lambda}(\infty, R)$ is a polynomial algebra over R in the variables $\{\langle i \rangle \mid i \geq 2\}$.*

Proof. In a product $\prod_{i=1}^{k} \langle j_i \rangle$, the unique term with the highest number of nontrivial components is an R-multiple of $\langle j_1, \ldots, j_k \rangle$. Thus, the associated graded algebra to the $c(-)$-filtration is a polynomial algebra, so $\mathcal{K}^{\Sigma}_{\lambda}(\infty, R)$ is one as well. □

3 Admissible Abelian Differentials

This section is concerned with the construction of zero partition cycles in the setting of real differentiable manifolds. The notion has been studied extensively in algebraic and complex geometry, and is there known as strata of abelian differentials.

As mentioned in the introduction, an abelian differential on a surface of genus g has $2g-2$ zeroes counted with multiplicity, hence defines a symmetric partition of $2g-2$. The idea of zero partition cycles is to ask the reverse question: Given a partition of $2g-2$ and a family of abelian differentials on a surface bundle, which abelian differentials have a zero partition greater than or equal to that partition? The **zero partition cycle** is the subset of the base over which these abelian differentials appear.

While complex and algebraic geometry have powerful tools to study singular spaces, the flexibility of real differential geometry allows for very complicated singular phenomena, and it is in general better to restrict to generic cases, where the most complicated behavior does not appear. The kind of genericity suitable for zero partition cycles is given by a notion of independence of the zeroes of abelian differentials, and those abelian differentials which exhibit this trait will be called **admissible abelian differentials**.

The power of admissible abelian differentials comes from two facts: On the one hand, their structure is restrictive enough to guarantee that the zero partition cycle (which might not be a submanifold of the base) yields a well defined homology class. On the other hand, it is indeed a generic condition such that abelian differentials and concordances of abelian differentials[3] are homotopic to admissible ones.

However, even in the case of admissible abelian differentials the zero partition cycle is not a submanifold. Therefore, some work is required to justify that it yields a well defined homology class. This justification will proceed with the construction of an explicit resolution of the appearing singularities, which is given by the **space of ordered zeroes**. This space can be thought of as a branched cover of the base, with branch locus corresponding to the zero partition cycles.

[3] This means abelian differentials on the product of the bundle with $(0,3)$

Notation 3.0.1 (Setup)**.** Throughout this section, the following objects and notations will be frequently used:

BUNDLES.

- A fiber bundle $\pi_E\colon E \to B$ over a topological space B with closed oriented surfaces of genus g as fibers. This will be called a **surface bundle**. The negative Euler characteristic number of the fiber is abbreviated as $\chi = 2g - 2$.
- The base space will often be an oriented real manifold M of dimension m. In that case, the bundle is assumed to be smooth and is then called a **surface bundle over a manifold**.
- For a surface bundle over a manifold as above, the **fiber tangent bundle** is the bundle $T_{\pi_E}E = \ker(\mathrm{d}\pi_E\colon TE \to \pi_E^* TB)$ and the **fiber cotangent bundle** is its dual $T^*_{\pi_E}E$.
- For a surface bundle as above, one can consider the **n-fold fiber product** $E^{(n)} = \underbrace{E \times_B \ldots \times_B E}_{n \text{ factors}}$. If the surface bundle was over a manifold, the n-fold fiber product is again a manifold.
- A **complex line bundle** L on B (or M).

STRUCTURES ON BUNDLES.

- For a surface bundle over a manifold, a **fiberwise almost complex structure** is a smooth section $J \in \Gamma(\underline{\mathbf{Hom}}(T_{\pi_E}E, T_{\pi_E}E))$ such that $J^2 = -1$. Notice that on a real two-dimensional manifold, all almost complex structures are integrable. Thus a fiberwise almost complex structure as above induces a complex structure on each fiber.
- For a surface bundle over a manifold with a fiberwise almost complex structure as above, a **fiberwise 1-form** with values in L is a smooth section $\omega_L \in \Gamma(T^*_{\pi_E}E \otimes \pi_E^* L)$ holomorphic with respect to J on each fiber.
- An **abelian differential**[4] is a pair (J, ω_L) as above, such that the fiberwise 1-form ω_L is fiberwise non-zero. That means, for each $x \in M$, the restriction of ω_L to E_x does not vanish completely.

COORDINATES.

- Consider a surface bundle over a manifold with fiberwise complex structure as above. Assume one is given an open subset $U \subseteq M$ of the space and some subset $X \subseteq E|_U$ of the total space over U.

[4]This differs from the usual notion of abelian differential, which just refers to ω_L.

By **local coordinates over U around X**, the following is meant:

(a) An open subset $U_X \subseteq E|_U$ that contains X,

(b) a smooth function (the **fiber coordinate**) $z\colon U_X \to \mathbb{C}$, fiberwise holomorphic with respect to J, and

(c) smooth functions (the **base coordinates**) $\{w_i\colon U_X \to \mathbb{C}\}_i$ that factor over π_E,

such that these functions together are independent on U_X (their derivatives have full rank). It will occur that the base coordinates are considered as functions on the base.

Note that the coordinates are not required to form a complete system of coordinates. This means that the coordinates do not have to define a local diffeomorphism, but just a submersion, in particular they do not have to be injective.

3.1 Zero Partitions

Definition 3.1.1 (Unordered zero partition of a multiset)**.** Let E be a set and $S \subset M$ a finite subset. In this definition, it helps to think of M as some space, probably a surface, and S as set of zeroes of some function or differential form on M.

In the situation described above one usually has more information associated to the elements of S. If S is the a set of zeroes, each element of S has some associated zero order, the order of vanishing of the defining function or differential form. So assume one is given the following data:

- A finite set $S = \{s_1, \ldots, s_k\}$
- For each element $s_i \in S$, a natural number n_i (the zero order)

Then one can enhance S to a multiset, a set where elements can appear with a well defined multiplicity. One denotes the so defined multiset $\mathfrak{m}$ as set of pairs of the form (element, multiplicity):

$$\mathfrak{m} = \{(s_1, n_1), \ldots, (s_k, n_k)\}$$

The cardinality n of $\mathfrak{m}$ is defined to be the sum over all the multiplicities, $n = n_1 + \cdots + n_k$.

Now one can associate with $\mathfrak{m}$ the symmetric partition $\langle \mathfrak{m} \rangle_n \in \Pi_\Sigma(n)$ defined by the sequence of multiplicities. The symmetric partition defined by this is called the **unordered zero partition** $\langle \mathfrak{m} \rangle_n$ of $\mathfrak{m}$.

Definition 3.1.2 (Ordered zero partition of a tuple)**.** In continuation of Definition 3.1.1, label the elements of the zero multiset $\mathfrak{m}$ with the numbers $\{1, \dots, n\}$. This amounts to defining an enumeration function

$$f \colon \{1, \dots, n\} \to S,$$

where, for each $s_i \in S$, the cardinality of the preimage $f^{-1}(\{s_i\})$ is equal to the multiplicity n_i. Such a function can be seen as a tuple $\mathfrak{t}$ of n elements of S, where each element appears according to its multiplicity:

$$\mathfrak{t} = (\mathfrak{t}_1, \dots, \mathfrak{t}_n) = (f(1), \dots, f(n))$$

In this situation, one can define the **ordered zero partition** $[\mathfrak{t}]_n \in \Pi(n)$ of the tuple $\mathfrak{t}$ via:

$$\forall\, i, j \in \{1, \dots, n\} \colon i \sim_{[\mathfrak{t}]_n} j \Leftrightarrow \mathfrak{t}_i = \mathfrak{t}_j$$

The following table summarizes the correspondence of terminology for sets of zeroes and partitions.

	Set of zeroes	Associated partition
ordered	tuple $\mathfrak{t} = (\mathfrak{t}_1, \dots, \mathfrak{t}_n)$	partition $[\mathfrak{t}]_n \in \Pi(n)$
unordered	multiset $\mathfrak{m} = \{(\mathfrak{m}_1, n_1), \dots, (\mathfrak{m}_k, n_k)\}$	symmetric partition $\langle \mathfrak{m} \rangle_n \in \Pi_\Sigma(n)$

3.2 Admissible Abelian Differentials and Deformations

Definition 3.2.1 (Admissible abelian differential)**.** Consider some surface bundle $\pi_E \colon E \to M$ over a manifold. An abelian differential (J, ω_L) is called **admissible** if for each point $x \in M$, there is an open neighborhood U of x in M, such that over U the bundle admits local coordinates $z, \{w_{i,j}\}_{i,j}$ defined around the set of zeroes (indexed by i) of ω_L over U, where $\{w_{i,j}\}_{i,j}$ vanish at x, such that on this neighborhoods ω_L is of the form:

$$\omega_L = \left(\sum_{j=0}^{n_i - 2} w_{i,j} z^j + z^{n_i} + (\text{terms of higher order in } z) \right) \mathrm{d}z$$

The vanishing of the term of order $n_i - 1$ is essential. It can always be arranged by translating z to make this term vanish, but then the independence of the coefficient functions $w_{i,j}$ might be lost. On the other hand, the condition on the coefficient of z^{n_i} to be 1 (instead of some other nonzero number) is not essential, it can be obtained by a rotation. One can additionally assume that all base coordinates $\{w_{i,j}\}_{i,j}$ have a common zero in U.

The intuition behind this definition is that the condition on ω_L forces its zeroes to be as independent as possible.

Remark 3.2.2 (Restriction of admissible abelian differentials)**.** The restriction of an admissible abelian differential to a submanifold of the base does not have to be admissible again.

Definition 3.2.3 (Admissible deformation)**.** Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. An **admissible deformation** is an admissible abelian differential on $E \times (0,3)$ over $M \times (0,3)$, such that the restriction to $E \times \{1\}$ and $E \times \{2\}$ is admissible again.

3.3 Space of Ordered Zeroes

Definition 3.3.1 (Ordered zero partition)**.** Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. For each point $x = (x_1, \dots, x_\chi) \in E^{(\chi)}$ one can define a partition $[[x]]_\chi$ as in Definition 3.1.2 by:

$$\forall\, i, j \in \{1, \dots, \chi\} \colon i \sim_{[[x]]_\chi} j \Leftrightarrow x_i = x_j$$

The function

$$\begin{aligned} [[-]]_\chi \colon E^{(\chi)} &\longrightarrow \Pi(\chi) \\ x &\longmapsto [[x]]_\chi \end{aligned}$$

defined this way is called the **ordered zero partition**.

Remark 3.3.2 (Ordered zero partition is upper continuous)**.** Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. The ordered zero partition $[[-]]_\chi$ is upper continuous. This means, for every $x_1 \in E^{(\chi)}$ there is an open neighborhood $U \subseteq E^{(\chi)}$ such that for all $x_2 \in U$: $[[x_1]]_\chi \geq [[x_2]]_\chi$

Definition 3.3.3 (Space of ordered zeroes)**.** Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. Let (J, ω_L) be an abelian differential. Then the **space of ordered zeroes** Z is defined as follows:

$$\left\{ (x_1, \dots, x_\chi) \in E^{(\chi)} \;\middle|\; \begin{array}{l} \text{The } x_i \text{ are zeroes of } \omega_L, \\ \text{each zero appearing with multiplicity equal} \\ \text{to the zero order of } \omega_L \text{ at that point.} \end{array} \right\}$$

The symmetric group $\Sigma(\chi)$ acts on Z permuting the coordinates.

Lemma 3.3.4 (Transversality of space of ordered zeroes)**.** *Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. Let (J, ω_L) be an admissible abelian differential. Let $\pi \in \Pi(\chi)$ be a partition. Then the subspace Z is a submanifold of $E^{(\chi)}$, and it is transverse to $E^{(\chi)}(\pi)$.*

Proof. Let $\{z_j\}_{1 \leq j \leq \chi}$ be fiber coordinates on the fiber factors. Consider a point $x = (x_1, \ldots, x_\chi)$ in the intersection of Z and $E^{(\chi)}(\pi)$. Let π_i be a component of $[[x]]_\chi$.

By the admissibility of (J, ω_L), around the set of points $x_1, \ldots, x_\chi \in E$ the form ω_L is given by

$$\begin{aligned}\omega_L &= f_i(w, z)\mathrm{d}z \\ &= \left(\sum_{j=0}^{n_i - 2} w_{i,j} z^j + z^{n_i} + (\text{terms of higher order in } z) \right) \mathrm{d}z\end{aligned}$$

with local coordinates $z, \{w_{i,j}\}_{i,j}$ and zero orders n_i.

Then locally at x the space of ordered zeroes is defined by equations of the form $f_i(w, x_j) = 0$. One can write $f_i(-,-)$ as a product

$$f_i(w, z) = \left(\sum_{j=0}^{n_i - 2} \tilde{w}_{i,j} \tilde{z}^j + \tilde{z}^{n_i} \right) \tilde{f}_i(w, z)$$

with new coordinates $\{\tilde{w}_{i,j}\}_{i,j}, \tilde{z}$ and $\tilde{f}_i(-,-)$ nonzero. Hence, from now on one can assume that there are no terms of higher order.

In that case, the space of ordered zeroes is locally defined by the equations

$$\mathrm{e}_k(\{z_i\}_{i \in \pi_j}) = w_{j,k}$$

where $\mathrm{e}_k(-)$ are the elementary symmetric polynomials. This shows that Z is defined as the vanishing locus of independent functions, hence it is a submanifold and a subset of the differentials $\{\mathrm{d}w_{i,j}\}_{i,j}$ forms a local frame of the normal bundle of Z, while $E^{(\chi)}(\pi)$ can be expressed purely in equations on the fiber coordinates. Hence the spaces are transverse. □

Remark 3.3.5 (Invariance under admissible deformation)**.** Let $\pi \in \Pi(\chi)$ be a partition. The fundamental class of $Z(\pi) = Z \cap E^{(\chi)}(\pi)$ in $\mathrm{H}_*(E^{(\chi)}, \mathbb{Z})$ is invariant under admissible deformations of the abelian differential.

3.4 Space of Unordered Zeroes

Definition 3.4.1 (Unordered zero partition)**.** Let $\pi_E\colon E \to M$ be a surface bundle over a manifold with abelian differential (J, ω_L). For each point $x \in M$, the form $\omega_L|_{E_x}$ has χ zeroes, counted with multiplicity. This means, one has a decomposition $\chi = n_1 + \ldots + n_k$ into the orders of the zeroes, in other words, a symmetric partition $\langle \omega_L|_{E_x} \rangle_\chi \in \Pi_\Sigma(\chi)$, as in Definition 3.1.1.

The function

$$\begin{aligned} \langle \omega_L \rangle_\chi \colon M &\longrightarrow \Pi_\Sigma(\chi) \\ x &\longmapsto \langle \omega_L|_{E_x} \rangle_\chi \end{aligned}$$

defined this way is called the **unordered zero partition** of ω_L.

Lemma 3.4.2 (Unordered zero partition is upper continuous)**.** *Let $\pi_E\colon E \to M$ be a surface bundle over a manifold with abelian differential (J, ω_L). The unordered zero partition of ω_L is upper continuous. That means, for every $x_1 \in M$ there is an open neighborhood $U \subseteq M$ such that for all $x_2 \in U$, one has $\langle \omega_L|_{E_{x_1}} \rangle_\chi \geq \langle \omega_L|_{E_{x_2}} \rangle_\chi$, or in other words, locally zeroes only split, but do not collide.*

Proof. For each zero of $\omega_L|_{E_{x_1}}$ over x_1, ω_L can be written in local coordinates around that zero as

$$\omega_L = \left(\sum_{j=0}^{n_i} f_{i,j}(w) z^j + (\text{terms of higher order in } z) \right) \mathrm{d}z$$

where the zero is at $z = 0$. Here n_i is the zero order, and the functions f_{i,n_i} do not vanish at x_1. ☐

Definition 3.4.3 (Space of unordered zeroes)**.** Let $\pi_E\colon E \to M$ be a surface bundle over a manifold. Let (J, ω_L) be an abelian differential. Let $\sigma \in \Pi_\Sigma(\chi)$ be a symmetric partition.

The **space of unordered zeroes** $M[\sigma]$ associated with σ is defined as the preimage of the upper closure of σ under the zero partition function:

$$M[\sigma] = \{x \in M \mid \langle \omega_L|_{E_x} \rangle_\chi \geq \sigma\}$$

By Lemma 3.4.2, this is a closed subset of M.

Construction 3.4.4 (Space of unordered zeroes has well defined fundamental class)**.** Let $\pi_E\colon E \to M$ be a surface bundle over a manifold. Let (J, ω_L) be an admissible abelian differential. Let $\sigma \in \Pi_\Sigma(\chi)$ be a symmetric partition, and write it as $\sigma = \langle n_1, \ldots, n_k \rangle_\chi$. The space of zeroes $M[\sigma]$ is in general not a submanifold of M. Hence the usual notion of fundamental class cannot be applied here. But the problem of $M[\sigma]$ not being a submanifold resides in codimension 2 and higher, hence it still has a well defined fundamental class.

There are different ways to make this statement precise. The most conceptual way is to use stratifolds, a notion introduced in [15] to make the statement *manifold up to codimension* 2 precise. This yields a class in stratifold homology, which has a fundamental class in singular homology.

For the reader who is not familiar with the notion, three methods will be sketched how to obtain a homology class with other methods. These methods yield a rational oriented bordism class, an integral singular homology class and a real homology class, respectively. All classes map to the same class in real singular homology.

Method 1: A Rational Oriented Bordism Class.
Let $\pi \in \Pi(\chi)$ be a partition whose $\Sigma(\chi)$-orbit is σ. The projection $E^{(\chi)} \to M$ induces a map $Z[\pi] \to M[\sigma]$, which is a covering on a dense open set. The group $G_0(\pi)$ acts transitively on $Z[\pi]$ with stabilizer $G_1(\pi)$. Hence, as oriented bordism class in M, one defines:

$$[M[\sigma]] = \frac{|G_0(\pi)|}{|G_1(\pi)|}[Z[\pi]]$$

Method 2: An Integral Singular Homology Class.
The space $M[\sigma]$ is stratified by the spaces $M[\sigma']$ given by symmetric partitions $\sigma' \in \Pi_\Sigma(\chi)$ with $\sigma' \geq \sigma$. Each open stratum is a canonically oriented manifold, and they all have even codimension, because their normal bundle carries a complex structure. One can chose a triangulation of $M[\sigma]$ subordinate to this stratification. The sum of the top simplices of these triangulation defines a homology class

$$[M[\sigma]] \in \mathrm{H}_{m-2d(\sigma)}(M, \mathbb{Z}).$$

Method 3: A Real Homology Class.
This approach uses de Rham cohomology. Let ω be a differential form on M of degree $m - 2d(\sigma)$. Let $U \subseteq M[\sigma]$ be a large subset which is a manifold with boundary ∂U. All singularities of $M[\sigma]$ are contained in the complement of U. Now define (for a reasonable choice of U)

$$\langle \omega, M[\sigma] \rangle = \lim_{U \to M[\sigma]} \int_U \omega$$

and verify that this limit exists and is independent of the cohomology class of ω. This can be done by Stokes' formula and the observation that the measure $|\partial U|$ tends to zero, while ω is bounded in a neighborhood of $M[\sigma]$.

Remark 3.4.5 (Invariance under admissible deformation)**.** The homology class defined in 3.4.4 is invariant under admissible deformations of the abelian differential.

3.5 Existence of Admissible Abelian Differentials

Lemma 3.5.1 (Existence of abelian differentials)**.** *Let $\pi_E\colon E \to M$ be a surface bundle over a manifold of dimension m. Let L be a complex line bundle over M. Assume $g > m$. Then there exists an abelian differential.*

Proof. The proof naturally decomposes into two steps, the construction of an almost complex structure J, and the construction of a holomorphic one-form ω_L.

Construction of J. By fiberwise orientability, there is a non-vanishing volume form $v \in \Gamma(\Lambda^2(T^*_{\pi_E}E))$. Fix a fiberwise metric $\gamma \in \Gamma(T^*_{\pi_E}E \otimes T^*_{\pi_E}E)$. Now let V, G be the matrices representing v and γ in local coordinates. The product $G{\cdot}V^{-1}$ then defines an section $\gamma v^{-1} \in \Gamma(\underline{\mathbf{Hom}}(T_{\pi_E}E, T_{\pi_E}E))$ which is an endomorphism whose square is a negative scalar function, thus by rescaling gives an almost complex structure.

Construction of ω_L. A holomorphic 1-form can be interpreted as a section over M of the g-dimensional complex vector bundle of holomophic one-forms on the fiber (twisted with L). There is always the zero section. By transversality, it can be made disjoint from the zero section. □

Theorem 3.5.2 (Existence of admissible differentials)**.** *Let $\pi_E\colon E \to M$ be a surface bundle over a manifold of dimension m. Let L be a complex line bundle over M.*

(a) Assume $g > \max(m, 1)$. Then there exists an admissible abelian differential.

(b) Any two admissible differentials are connected via an admissible deformation.

Proof. Denote by $\underline{A}(E, L)$ the bundle over M of abelian differentials on the fibers. The smooth sections $\Gamma^\infty(\underline{A}(E, L))$ are defined to be those sections such that the induced abelian differential on the total space is smooth.

Moreover, denote by $\underline{\mathrm{Diff}}_0(E)$ the bundle of fiberwise diffeomorphisms that are fiberwise isotopic to the identity. The smooth sections $\Gamma^\infty(\underline{\mathrm{Diff}}_0(E))$ are defined to be those sections such that the induced action of such a section on the total space is smooth.

Thirdly, denote by $\underline{\mathcal{T}}(E,L) = \underline{A}(E,L)/\underline{\mathrm{Diff}}_0(E)$ the fiberwise Teichmüller space of abelian differentials. By the theory of abelian differentials (see e.g. [22]), this is a fiber bundle of fiber dimension $8g-6$. By a theorem of Earle-Eells in [6], the bundle $\underline{\mathrm{Diff}}_0(E)$ is fiberwise contractible. Hence, the following induced map on sections is surjective:

$$\Gamma^\infty(\underline{A}(E,L)) \twoheadrightarrow \Gamma^\infty(\underline{\mathcal{T}}(E,L))$$

Assume for a moment that the surface bundle and the line bundle are trivial, $E = F \times M$, $L = \mathbb{C} \times M$.

In this case, $\underline{\mathcal{T}}(E,L) = \mathcal{T}(F) \times M$, where $\mathcal{T}(F)$ denotes the usual Teichmüller space of abelian differentials on F. Fix a point (J, ω_L) in $\mathcal{T}(F)$. The following two steps will construct local parameters of $\mathcal{T}(F)$ around (J, ω_L).

Along the stratum. As usual, let $(x_1, \dots, x_k)$ denote the zeroes of ω_L on F. Fix small open sets U_i around the zeroes x_i, and on those open sets coordinates z_i, such that on U_i, ω_L is given as $z_i^{n_i}\mathrm{d}z_i$.

Coordinates along the stratum are given by the relative homology group with complex coefficients $\mathrm{H}_1(F, \{x_i\}_i, \mathbb{C})$, which has real dimension $4g+2k-2$. Explicitly, let a class in this homology group which is close close to zero be represented by a complex valued one-form ω'_L that vanishes on the U_i. Then the one-form ω_L is replaced by $\omega_L + \omega'_L$, and the smallness of ω'_L guarantees the existence of a unique compatible almost complex structure.

Out of the stratum. In the step before, the one-form ω_L was altered outside of the U_i. To move in a direction transverse to the stratum, one changes ω_L only inside of the U_i. Consider the following variation (χ is a cutoff function):

$$\omega'_L = \left(\chi(|z_i|) \left(\sum_{j=0}^{n_i-2} w_{i,j} z_i^j \right) + z_i^{n_i} \right) \mathrm{d}z_i$$

These variations are parametrized by complex numbers $\{w_{i,j}\}_{i,0\le j\le n_i-2}$. There are a total of $2g-2-k$ of such parameters, so the parameter space has real dimension $4g-4-2k$. Together with the parameters of the last step, this yields the required number of parameters. The parameters $w_{i,j}$ are generators of the symmetric polynomials of the relative periods that appear in the splitting of the multiple zeroes. Thus, this parameters are indeed a local smooth parametrization of $\mathcal{T}(F)$.

Now the theorem follows from Lemma 3.5.1 and the transversality theorem. $\square$

Remark 3.5.3 (Independence of abelian differential)**.** By Theorem 3.5.2, with the Remarks 3.4.5 and 3.3.5, the homology class constructed in 3.4.4 as well as the fundamental class of Z are independent of the abelian differential.

4 Zero Partition Cycles

Notation 4.0.1 (Setup)**.** Throughout this section, the setup defined in 3.0.1 is used. Throughout, $g > \max(m, 1)$ is assumed.

4.1 Definition

Definition 4.1.1 (Unordered zero cycle)**.** Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. Let $\sigma \in \Pi_\Sigma(\chi)$ be a symmetric partition. The **unordered zero cycle** is the the fundamental class of $M[\sigma]$ in $\mathrm{H}_*(M, \mathbb{Q})$ (see 3.4.4). Its Poincare dual is denoted by $h(\sigma) \in \mathrm{H}^{\mathrm{ev}}(M, \mathbb{Q})$.

Definition 4.1.2 (Ordered Zero Cycles)**.** Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. Let $\pi \in \Pi(\chi)$ be a partition. Let Z be the space of ordered zeroes (Definition 3.3.3). The **ordered zero cycle** is the fundamental class of $Z(\pi)$ in $\mathrm{H}_*(Z, \mathbb{Q})$. Its Poincare dual is denoted by $z(\pi) \in \mathrm{H}^{\mathrm{ev}}(Z, \mathbb{Q})$.

Lemma 4.1.3 (Branching of zeroes)**.**

(a) The natural map $\mathrm{H}^{\mathrm{ev}}(\pi_Z, \mathbb{Q}) \colon \mathrm{H}^{\mathrm{ev}}(M, \mathbb{Q}) \to \mathrm{H}^{\mathrm{ev}}(Z, \mathbb{Q})$ *is injective.*

(b) On symmetric partitions, the ordered zero cycles map to cohomology classes on the base space.

$$\begin{array}{ccc} \mathbb{Q}[\Pi(\chi)] & \xrightarrow{z(-)} & \mathrm{H}^{\mathrm{ev}}(Z, \mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathbb{Q}[\Pi_\Sigma(\chi)] & \overset{z(-)}{\dashrightarrow} & \mathrm{H}^{\mathrm{ev}}(M, \mathbb{Q}) \end{array}$$

(c) Let $\sigma \in \Pi_\Sigma(\chi)$ *be a symmetric partition. The unordered zero cycles can be computed in terms of the ordered zero cycles.*

$$h(\sigma) = |G_1(\sigma)|\, z(\sigma)$$

In particular, for a primitive partition $\sigma = \langle k \rangle_\chi$*:*

$$h(\langle k \rangle_\chi) = k!\, z(\langle k \rangle_\chi)$$

Proof.

(a) Both M and Z have the same dimension, π_Z has relative dimension 0, it is a map of degree $\chi!$. For such a map, the postcomposition with the transfer

$\pi_{Z!} \circ \mathrm{H}^{\mathrm{ev}}(\pi_Z, \mathbb{Q})$ is equal to multiplication with the degree: For $\alpha \in \mathrm{H}^k(M, \mathbb{Q})$, $\beta \in \mathrm{H}^{m-k}(M, \mathbb{Q})$, one has

$$\begin{aligned}\langle \alpha \cup \pi_{Z!}\pi_Z^*\beta, M\rangle &= \langle \alpha \cup (\pi_{Z*}(\pi_Z^*\beta)^{PD})^{PD}, M\rangle \\ &= \langle \alpha, \pi_{Z*}(\pi_Z^*\beta)^{PD}\rangle \\ &= \langle \pi_Z^*\alpha, (\pi_Z^*\beta)^{PD}\rangle \\ &= \langle \pi_Z^*\alpha \cup \pi_Z^*\beta, Z\rangle \\ &= \chi! \cdot \langle \alpha \cup \beta, M\rangle\end{aligned}$$

(b) The other composition is given by

$$\mathrm{H}^{\mathrm{ev}}(\pi_Z, \mathbb{Q}) \circ \pi_{Z!} = \sum_{\tau \in \Sigma(\chi)} \tau(-),$$

which is $\chi!$ times a projection onto the image of $\pi_{Z!}$. On the image of symmetric partitions under $z(-)$, it acts as multiplication by $\chi!$, hence they already are in the cohomology of the base.

(c) For a partition $\pi \in \Pi(\chi)$ in the orbit σ one has:

$$h(\sigma) = \sum_{\tau \in \Sigma(\chi)} z(\tau(\pi)) = |G_1(\sigma)|\, z(\sigma) \qquad \square$$

Notation 4.1.4 (Twisting class)**.** Let L be a complex line bundle over a topological space B. The Euler characteristic class of L is denoted by $\tau \in \mathrm{H}^2(B, \mathbb{Q})$.

4.2 Stable Characteristic Classes

Definition 4.2.1 (Characteristic class of surface bundles)**.** Let R be a commutative ring. A **characteristic class of surface bundles of genus** g with coefficients in R is a rule[5] that assigns to every surface bundle $\pi_E \colon E \to B$ a cohomology class $\epsilon(\pi_E) \in \mathrm{H}^*(B, R)$,

$$(\pi_E \colon E \to B) \longmapsto \epsilon(\pi_E) \in \mathrm{H}^*(B, R)$$

such that for every pullback diagram of surface bundles

[5]The choice of this word witnesses the authors feeling of guilt, because set theoretic non-issues are ignored.

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \Big\downarrow{\scriptstyle \pi_{E_1}} & & \Big\downarrow{\scriptstyle \pi_{E_2}} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

one has $\mathrm{H}^*(f, R)(\epsilon(\pi_{E_2})) = \epsilon(\pi_{E_1})$.

Construction 4.2.2 (Bundle stabilization)**.**

- Input:
  (a) A surface bundle $\pi_{E_1} : E_1 \to B$ of genus g.
  (b) A section f_1 of π_{E_1}.
  (c) A trivialization of $T_{\pi_{E_1}} E_1$ along f_1.
- Output:
  (d) A surface bundle $\pi_{E_2} : E_2 \to B$ of genus $g + 1$.
  (e) A section f_2 of π_{E_2}.
  (f) A trivialization of $T_{\pi_{E_2}} E_2$ along f_2.

The construction proceeds with the following steps:

STEP 1. The image of f_1 is removed from E_1.

STEP 2. Using the trivialization, a trivial bundle of tori with two boundary components over B is glued along one boundary component to $E_1 \setminus f_1(B)$.

STEP 3. The other boundary component is collapsed to a point, which yields a new section f_2 together with a trivialization as required.

Definition 4.2.3 (Stable characteristic class)**.** Let R be a commutative unital ring. A **stable characteristic class** with coefficients in R is a sequence of characteristic classes $(\epsilon_g(-))_g$ of surface bundles of genus g, such that whenever a bundle $\pi_{E_2} : E_2 \to B$ is a stabilization of a bundle $\pi_{E_1} : E_1 \to B$ of genus g, one has $\epsilon_g(\pi_{E_1}) = \epsilon_{g+1}(\pi_{E_2})$.

Remark 4.2.4 (Algebraic stabilization)**.** Consider the case $R = \mathbb{Q}$. The process of stabilization can be described algebraically. Let $\mathrm{H}^{\mathrm{ev}}(\mathcal{M}_g, \mathbb{Q})$ denote the ring of even-degree characteristic classes of surface bundles of genus g, and $\mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g,1}, \mathbb{Q})$ the ring of even-degree characteristic classes of surface bundles of genus g with one boundary component in each fiber. Then one has a diagram given by the stabilization process:

$$\begin{array}{ccc} \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g,1},\mathbb{Q}) & \longleftarrow & \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g+1,1},\mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g},\mathbb{Q}) & & \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g+1},\mathbb{Q}) \end{array}$$

It is known ([17], Corollary 3.2) that the vertical morphisms are injective. Thus the ring of stable characteristic classes $\mathrm{H}^{\mathrm{st}}(\mathbb{Q})$ can be viewed as a subring of the graded limit of the upper sequence, and it is actually equal to it. Also, there are no non-trivial odd-degree stable characteristic classes, the emphasis on even degree is made here only to make sure that the the rings under consideration are commutative.

Moreover, in each grade, the sequence stabilizes after some time [11], [19].

Construction 4.2.5 (Mumford-Morita-Miller classes)**.** Early examples for stable characteristic classes were given by Mumford as algebraic cycles and by Morita [17] in the topological setting, the **Mumford-Morita-Miller classes** κ_i.

They are constructed as follows: One considers the fiberwise cotangent bundle of the surface bundle given, which is an oriented plane bundle, takes a power of its Euler class and projects it via a transfer map to the base:

$$\kappa_i = \pi_{E!}(e(T^*_{\pi_E}E)^{i+1})$$

By basic facts on the Euler class, κ_0 is the negative Euler characteristic number of the fiber, which is obviously not stable. For $i \geq 1$ however, the classes are stable. Even though the powers of the Euler class are algebraically related on the total space of the bundle, the transferred classes do not inherit any dependency that way.

The following theorem by Madsen and Weiss resolves a conjecture of Mumford about the structure of rational stable characteristic classes in the positive:

Theorem 4.2.6 (Madsen-Weiss [16], [10], [20], [18])**.** *The canonical map*

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \longrightarrow \mathrm{H}^{\mathrm{st}}(\mathbb{Q})$$

is an isomorphism.

Remark 4.2.7 (Manifold base suffices)**.** In [17], it is shown that two rational characteristic classes are equal if they evaluate the same on surface bundles over closed manifolds. This means that calculations for all even-dimensional base manifolds carry over to the stable characteristic classes in general.

4.3 Primitive Cycles and an Intersection Formula

Lemma 4.3.1 (Primitive Cycles)**.** *Let $\pi_E\colon E \to M$ be a surface bundle over a manifold. Let $k \leq \chi$ be a natural number. Consider the primitive symmetric partition $\langle k\rangle_\chi \in \Pi_\Sigma(\chi)$. Then*

$$h(\langle k\rangle_\chi) = \sum_{i=0}^{k-1} e_{k-i}(1,\dots,k)\kappa_{k-i-1}\tau^i \in \mathrm{H}^{2k-2}(M,\mathbb{Q}),$$

where $e_i(-)$ are the elementary symmetric polynomials. In particular if $\tau = 0$:

$$h(\langle k\rangle_\chi) = k!\,\kappa_{k-1}$$

Proof. Let $\mathcal{J}_i(-)$ denote the vector bundle of jets of order i of a vector bundle. The space of k-fold zeroes $E[k] \subseteq E$ is the vanishing locus of the jet prolongation $\mathcal{J}_{k-1}(\omega_L) \in \Gamma(\mathcal{J}_{k-1}(T^*_{\pi_E}E \otimes \pi_E^* L))$. For an admissible abelian differential, this section is transverse to the zero section, hence its zero locus is Poincare dual to the Euler characteristic class of the jet bundle. Using, for $0 \leq i \leq k-1$ the short exact sequence

$$(T^*_{\pi_E}E)^{\otimes i+1} \otimes \pi_E^* L \longrightarrow \mathcal{J}_i(T^*_{\pi_E}E \otimes \pi_E^* L) \longrightarrow \mathcal{J}_{i-1}(T^*_{\pi_E}E \otimes \pi_E^* L),$$

which in local coordinates (with s a section of L) is given by

$$((\mathrm{d}z)^{\otimes i+1} \otimes s) \longmapsto (\mathrm{d}z)^{\otimes i+1} \otimes s,$$

$$\left(\textstyle\sum_{j=0}^{i} f_j \cdot (\mathrm{d}z)^{\otimes j} \otimes s\right) \longmapsto \left(\textstyle\sum_{j=0}^{i-1} f_j \cdot (\mathrm{d}z)^{\otimes j} \otimes s\right),$$

one obtains the formula

$$e(\mathcal{J}_i(T^*_{\pi_E}E \otimes \pi_E^* L)) = \big((i+1)\cdot e(T^*_{\pi_E}E) + \pi_E^*(\tau)\big)\cdot e(\mathcal{J}_{i-1}(T^*_{\pi_E}E \otimes \pi_E^* L)),$$

which inductively proves

$$e(\mathcal{J}_{k-1}(T^*_{\pi_E}E \otimes \pi_E^* L)) = \prod_{i=1}^{k}(i \cdot e(T^*_{\pi_E}E) + \pi_E^*(\tau)).$$

This maps under the transfer map to the claimed expression. □

Lemma 4.3.2 (Intersection formula)**.** *Let $\pi_E \colon E \to M$ be a surface bundle over a manifold. Let $k \leq \chi - 1$ be a natural number. The following holds:*

$$\begin{aligned} &z([1,\dots,k]_\chi) \cup z([1,2]_\chi) \\ &= -\frac{1}{k+1}\left(\sum_{i=k+1}^{\chi} z([1,\dots,k,i]_\chi) - \tau \cup z([1,\dots,k]_\chi)\right) \end{aligned}$$

Proof. Let N be the normal bundle of $Z([1,2]_\chi)$ inside Z. As $Z([1,\dots,k]_\chi)$ is contained in $Z([1,2]_\chi)$, one has:

$$(z([1,\dots,k]_\chi) \cup z([1,2]_\chi)) \cap [Z] = e(N) \cap [Z([1,\dots,k]_\chi)]$$

The bundle N is isomorphic to $T_{\pi_E}E$ pulled back over the double zero. On $Z([1,\dots,k]_\chi)$ the jet prolongation $\mathcal{J}_{k-1}(\omega_L)$ vanishes, so $\mathcal{J}_k(\omega_L)$ can be seen as a section of

$$((T^*_{\pi_E}E)^{\otimes k+1} \otimes \pi_E^* L)|_{Z([1,\dots,k]_\chi)},$$

which for an admissible abelian differential is transverse to the zero section. It vanishes exactly at the zeroes of higher order. Hence:

$$\begin{aligned} \big((k+1)e(T^*_{\pi_E}E) + \tau\big) \cap [Z([1,\dots,k]_\chi)] &= e(((T^*_{\pi_E}E)^{\otimes k+1} \otimes \pi_E^* L)|_{Z([1,\dots,k]_\chi)}) \\ &= \sum_{i=k+1}^{\chi} z([1,\dots,k,i]_\chi) \end{aligned}$$

This proves the lemma. □

This, together with the formula for primitive partitions, suffices to compute arbitrary zero cycles inductively in terms of the classes κ_i and τ.

Example 4.3.3 (Calculation of $h(\langle 2,2\rangle_n)$)**.** Let n be a natural number. This example demonstrates the calculation for the partition $\langle 2,2\rangle_n$. By Lemma 4.3.1 and Lemma 4.1.3:

$$\begin{aligned} h(\langle 2\rangle_n) &= 2\kappa_1 + 3\tau\kappa_0 \\ h(\langle 3\rangle_n) &= 6\kappa_2 + 11\tau\kappa_1 + 6\tau^2\kappa_0 \\ z(\langle 2\rangle_n) &= \frac{1}{2}h(\langle 2\rangle_n) = \kappa_1 + \frac{3}{2}\tau\kappa_0 \\ z(\langle 3\rangle_n) &= \frac{1}{6}h(\langle 2\rangle_n) = \kappa_2 + \frac{11}{6}\tau\kappa_1 + \tau^2\kappa_0 \\ 4z(\langle 2,2\rangle_n) &= h(\langle 2,2\rangle_n) \end{aligned}$$

In the highest filtration, $z(\langle 2,2\rangle_n)$ has to agree with a multiple of $z(\langle 2\rangle_n)^2$:

$$z(\langle 2\rangle_n)^2 = \kappa_1^2 + 3\tau\kappa_0\kappa_1 + \frac{9}{4}\tau^2\kappa_0^2$$

This term can be calculated by the intersection formula:

$$\begin{aligned} z(\langle 2\rangle_n)^2 &= \Bigg(\sum_{1\le i<j\le n} z([i,j]_n)\Bigg)^2 \\ &= \underbrace{2z(\langle 2,2\rangle_n) + 6z(\langle 3\rangle_n)}_{\text{transverse part}} - \underbrace{\frac{1}{3}\left(3z(\langle 3\rangle_n) - \tau z(\langle 2\rangle_n)\right)}_{\text{non-transverse part}} \end{aligned}$$

Together, this yields:

$$\begin{aligned} h(\langle 2,2\rangle_n) &= 2z(\langle 2\rangle_n)^2 - 10z(\langle 3\rangle_n) - \frac{2}{3}\tau z(\langle 2\rangle_n) \\ &= 2\kappa_1^2 - 10\kappa_2 + (6\kappa_0\kappa_1 - 19\kappa_1)\tau + (\frac{9}{2}\kappa_0^2 - 11\kappa_0)\tau^2 \end{aligned}$$

In this example, this already concludes the calculation. Generally, lower order terms will appear that one has to compute recursively.

Lemma 4.3.4 (Map from Kraken Algebra)**.** *Let $\pi_E\colon E \to M$ be a surface bundle over a manifold. If L is trivial, the ordered zero cycles define a map of algebras:*

$$\begin{aligned} \mathcal{K}_{-2}(\chi,\mathbb{Q}) &\longrightarrow \mathrm{H}^{\mathrm{ev}}(Z,\mathbb{Q}) \\ \pi &\longmapsto z(\pi) \end{aligned}$$

This restricts to a map on the Symmetric Kraken Algebra:

$$\mathcal{K}_{-2}^{\Sigma}(\chi,\mathbb{Q}) \longrightarrow \mathrm{H}^{\mathrm{ev}}(M,\mathbb{Q})$$

Proof. This follows from Lemma 4.3.2 and Theorem 2.3.7(b). □

4.4 Stable Isomorphism

Lemma 4.4.1 (Unordered zero cycles yield characteristic classes)**.** *The cohomology classes defined by unordered zero cycles are natural under pullback. Explicitly, for a map $f\colon M_1 \to M$, one can pull back L and E to M_1, and with these bundles on M_1, for any partition $\sigma \in \Pi_\Sigma(\chi)$, the class $h(\sigma)$ on M_1 is the pullback of that class on M.*

In particular, if L is a trivial line bundle, this yields a characteristic class $h(\sigma) \in \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_g,\mathbb{Q})$.

Proof. By Lemma 4.3.1 and Lemma 4.3.2, the classes can be expressed in terms of characteristic classes, so are characteristic classes themselves. □

Lemma 4.4.2 (Map from Stable Symmetric Kraken Algebra)**.** *Let $\pi_E \colon E \to B$ be a surface bundle over a manifold. Let L be the trivial bundle. Then the ordered zero classes of symmetric partitions commute with stabilization, which means that, for all natural numbers g, the following diagram is commutative:*

$$\begin{array}{ccccc}
\mathcal{K}^{\Sigma}_{-2}(2g,\mathbb{Q}) & \xrightarrow{z(-)} & \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g+1},\mathbb{Q}) & \longrightarrow & \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g+1,1},\mathbb{Q}) \\
\downarrow{\scriptstyle \alpha_{2g-1}} & & & & \downarrow \\
\mathcal{K}^{\Sigma}_{-2}(2g-1,\mathbb{Q}) & & & & \\
\downarrow{\scriptstyle \alpha_{2g-2}} & & & & \\
\mathcal{K}^{\Sigma}_{-2}(2g-2,\mathbb{Q}) & \xrightarrow{z(-)} & \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g},\mathbb{Q}) & \longrightarrow & \mathrm{H}^{\mathrm{ev}}(\mathcal{M}_{g,1},\mathbb{Q})
\end{array}$$

Hence it induces a map to the stable characteristic classes:

$$\mathcal{K}^{\Sigma}_{-2}(\infty,\mathbb{Q}) \longrightarrow \mathrm{H}^{\mathrm{st}}(\mathbb{Q})$$

Proof. It suffices to check the commutativity on generators. In all cases, for a natural number k, $z(\langle k\rangle_\chi)$ maps to κ_{k-1}. □

Theorem 4.4.3 (Stable isomorphism)**.** *The map $\mathcal{K}^{\Sigma}_{-2}(\infty,\mathbb{Q}) \to \mathrm{H}^{\mathrm{st}}(\mathbb{Q})$ defined in Lemma 4.4.2. is an isomorphism.*

Proof. It is an algebra homomorphism (Lemma 4.4.2) of polynomial algebras over $\mathbb{Q}$ (Theorem 4.2.6, Theorem 2.5.4) sending generators to generators (Lemma 4.3.1). □

Remark 4.4.4. Besides being dictated by Lemma 4.3.2, the author does not know a meaningful interpretation for the magic parameter $\lambda = -2$.

5 Pinched Surfaces

One might wonder why the previous sections introduced their notions for surface bundles over very specific base manifolds instead of the universal example of a surface bundle, as would be customary in the context of characteristic classes. The reasons are twofold:

- The only good geometric example available as a universal surface bundle is not an honest manifold, but carries the subtle and technically challenging structure of an orbifold or stack. Once one is willing to use these notions, the calculations in principle carry over to this universal example. A common technique to circumvent these notions is the usage of level structures, but the interaction of those with abelian differentials poses more questions than this work is concerned with.
- Another reason is the pivotal usage of Poincare duality throughout the definitions and calculations of zero partition cycles. This makes the usage of a compact base space very appealing, as one has to consider proper non-closed cycles otherwise.

For the next step, however, it seems necessary to use additional notions. Namely, in the world of stacks, there is a way to compactify the universal space for surface bundles, the Deligne-Mumford compactification[5]. It introduces additional points to the moduli space that are represented by pinched surfaces[6].

This compactification was constructed in the setting of algebraic geometry and directly carries over to complex geometry. In differential topology additional difficulties arise, as any good notion of stable bundle does not admit pullbacks over arbitrary differentiable maps. Thus in this case there is a priori a distinction between classes on the Deligne-Mumford space and characteristic classes of stable bundles.

In this section, a more elementary approach is used, that eventually is equivalent to the approach via the Deligne-Mumford compactification. It is shown that the calculation for primitive unordered zero cycles in Lemma 4.3.1 partially extends to these bundles.

Notation 5.0.1 (Setup)**.** Throughout this section, the setup defined in 3.0.1 is used. Throughout, $g > \max(m, 1)$ is assumed.

[6]They are usually called stable surfaces, but this conflicts with the usage of the term stable in this thesis.

5.1 Definition

Definition 5.1.1 (Pinched Surface)**.** Consider the union of the coordinate planes in $\mathbb{C}^2$, $X_0 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$. Without the point $(0, 0)$, this is a smooth submanifold of $\mathbb{C}^2$. For an open subset $U \subseteq X_0$, a smooth function on U is defined to be a function which is the restriction of a smooth function on some open subset of $\mathbb{C}^2$.

A **pinched surface** E is a compact Hausdorff topological space that is locally modeled on X_0, i.e. together with a maximal atlas of charts with codomain in X_0, such that the precomposition with chart transitions sends smooth functions to smooth functions.

If a point $x \in E$ is sent to $(0, 0)$ in some chart, it is sent to $(0, 0)$ in every chart. The subset of these points x of E is called the **singular locus**, and its complement the **smooth locus**. The smooth locus is an ordinary manifold. An **oriented pinched surface** is a pinched surface with an orientation on its smooth locus.

The term *pinched surface* is explained by the fact that the local model X_0 will occur in families of the form $X_w = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = w\}$.

Definition 5.1.2 (Surface with Pairing)**.** A **surface with pairing** is a triple (F, C, τ), where F is a smooth compact surface with a finite subset $C \subset E$ and an involution $\tau \in \Sigma(C)$ without fixed points.

Construction 5.1.3 (Normalization)**.** The following construction shows that the two definitions 5.1.1 and 5.1.2 describe essentially the same kind of object. More precisely, a surface with pairing is the normalization of a pinched surface.

Let F be a pinched surface. Consider the map $f \colon \mathbb{C} \amalg \mathbb{C} \to X_0$, given by the inclusion of the coordinate axes. It is bijective except over the point $(0, 0)$, which has two preimages. Define F^{n} to be the space constructed by gluing the pullbacks of f over local charts. The **normalization** of F is the surface with pairing $(F^{\mathrm{n}}, C, \tau)$. Here C is the set of preimages of $(0, 0)$, and τ interchanges the corresponding preimages.

Now let (F, C, τ) be a surface with pairing. One sets $F^{\mathrm{g}} = F/\tau$. A chart around points $c, \tau(c) \in C$ is given by a disjoint union of a chart around each of the two points composed with f.

The two constructions are inverse to each other, in the sense that for each pinched surface or surface with boundary F there is a canonical bijection $F \to (E^{\mathrm{n}})^{\mathrm{g}}$, or $F \to (E^{\mathrm{g}})^{\mathrm{n}}$, respectively.

Definition 5.1.4 (Pinched abelian differential)**.** Let (F, C, τ) be a surface with pairing. An abelian differential (J, ω_L) on (F, C, τ) is an almost complex structure J on F and a one-form ω_L on $F \setminus C$ holomorphic with respect to J, which has at most simple poles at all points $c \in C$ and such that $\mathrm{Res}|_c\, \omega_L + \mathrm{Res}|_{\tau c}\, \omega_L = 0$.

Definition 5.1.5 (Combinatorial Type)**.** For this definition, a graph Γ consists of a set of vertices $V(\Gamma)$, a set of edges $E(\Gamma)$, and two maps $s, t \colon E(\Gamma) \to V(\Gamma)$ called source and target. A symmetric graph is a graph with an involution ι on $E(\Gamma)$ without fixed points that interchanges source and target. For a vertex $v \in V(\Gamma)$, denote by $E_v(\Gamma)$ the preimage of s.

An abelian differential on a pinched surface gives rise to combinatorial data. This data is captured by the following notion of combinatorial type.

A **combinatorial type** is an equivalence class of tuples $[\Gamma, \underline{g}, V_0, \underline{\mu}, \underline{\nu}, \underline{\sigma}]$, where

(a) Γ is a finite symmetric graph,

(b) $\underline{g} \colon V(\Gamma) \to \mathbb{N}$ is a function called **genus**,

(c) $V_0 \subseteq V(\Gamma)$ is a set of vertices called **vanishing vertices**.

(d) $\underline{\mu} \colon V(\Gamma) \setminus V_0 \to \mathbb{N}$ is a function called **vertex order**,

(e) $\underline{\nu} \colon E(\Gamma) \to \mathbb{N} \cup \{\infty\}$ is a function called **edge order**, and

(f) $\underline{\sigma} \colon V(\Gamma) \setminus V_0 \to \Pi_\Sigma(\underline{\mu})$ is a function called **zero partition**,

such that

(a) for all $e \in E(\Gamma)$,
$$\underline{\nu}(e) = \infty \Leftrightarrow s(e) \in V_0,$$

(b) for all $e \in E(\Gamma)$,
$$\underline{\nu}(e) = 0 \Leftrightarrow \underline{\nu}(\iota e) = 0,$$

(c) for all $v \in V(\Gamma) \setminus V_0$,
$$2\underline{g}(v) - 2 = \underline{\mu}(v) + \sum_{e \in E_v(\Gamma)} (\underline{\nu}(e) - 1).$$

Here two tuples are considered to be equivalent if there is a graph isomorphism preserving the additional data.

The genus of a combinatorial type is defined by:
$$g = g(\Gamma) + \sum_{v \in V(\Gamma)} \underline{g}(v)$$

Now let (J, ω_L) be an abelian differential on a surface with pairing (F, C, τ). The combinatorial type of (J, ω_L) is defined as follows:

1. The vertices of Γ are the connected components of F. The edges of Γ are given by the set C. Source and target of $c \in C$ are the connected components containing c and ιc. The involution ι interchanges c with ιc.
2. The genus $\underline{g}$ assigns to each component its genus.
3. The set V_0 is the set of components on which ω_L vanishes completely.
4. The vertex order $\underline{\mu}$ assigns to each component the sum of the zero orders of ω_L on the smooth part of the component.
5. The edge order $\underline{\nu}$ assigns to each $c \in C$ the zero order of ω_L at c.
6. The zero partition $\underline{\sigma}$ assigns to each component the symmetric partition defined by the zero orders of ω_L on the smooth part of the component.

5.2 Compactification of Cycles

Definition 5.2.1 (Pinched surface bundle)**.** Consider a smooth proper surjective map $\pi_E \colon E \to M$ between manifolds. It is called a **pinched surface bundle**, if the following holds:

For each point $x \in E$ there are is an open neighborhood U_1 of x and an open set $U_2 \subseteq M$ with $\pi_E(U_1) \subseteq U_2$, such that there is a pullback diagram

$$\begin{array}{ccccc} U_1 & \longrightarrow & \mathbb{C} \times \mathbb{C} & \qquad & (z_1, z_2) \\ \downarrow & & \downarrow & & \downarrow \\ U_2 & \longrightarrow & \mathbb{C} & & z_1 z_2 \end{array}$$

where the bottom arrow is a submersion.

Construction 5.2.2 (Relative dualizing bundle)**.** Consider complex multiplication as a map $f \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. Over the nonzero numbers, this is a fiber bundle, so has a well defined fiber cotangent bundle, which in this case carries a natural structure of a holomorphic bundle. Over all points in the total space except for $(0, 0)$, there is a well defined fiber cotangent space.

So this is a holomorphic line bundle that is defined outside a submanifold of complex codimension 2, thus it has a unique extension the whole total space, which is called the **relative dualizing bundle**, which will simply be denoted as $T^*_f E$. This can be defined via sections: Holomorphic sections of this bundle on a neighborhood U of $(0,0)$ are holomorphic sections over $U \setminus \{(0,0)\}$.

To be more explicit, denote the two coordinates on $\mathbb{C} \times \mathbb{C}$ by z_1 and z_2. Then the fiber cotangent bundle can be seen as a quotient of the cotangent bundle, where the derivative of the projection vanishes:

$$\mathrm{d}f = z_1 \mathrm{d}z_2 + z_2 \mathrm{d}z_1 = 0$$

This can be used to derive the a priori formal equality

$$\frac{\mathrm{d}z_1}{z_1} = -\frac{\mathrm{d}z_2}{z_2}$$

But the union of the domain of definition of them covers the complement of $(0,0)$. Thus this defines a section of $T^*_f E$, and it can be shown to be a trivialization.

Definition 5.2.3 (Relative dualizing bundle)**.** Let $\pi_E \colon E \to M$ be a pinched surface bundle. The **relative dualizing bundle** $T^*_{\pi_E} E$ is defined

- as the fiber cotangent bundle at the regular points of π_E,
- extended by the local procedure defined above on the others.

With this definition, the notion of abelian differential carries over to pinched surface bundles, where one replaces the fiber cotangent bundle with the relative dualizing bundle.

The discussion above proves the following lemma.

Lemma 5.2.4 (Pinched abelian differential)**.** *Let $\pi_E \colon E \to M$ be a pinched surface bundle with abelian differential (J, ω_L). Then for every $x \in M$, the differential ω_L induces an abelian differential on the normalization $(E_x)^n$ in the sense of* 5.1.4.

Remark 5.2.5 (Local form of pinched abelian differential)**.** Consider a surface with pairing (F, C, τ) with a abelian differential (J, ω_L) on it. Fix a point $c \in C$, and choose coordinates z_1, z_2 around c τc. One can expand ω_L into a power series:

$$\omega_L \sim_c \frac{1}{z_1} \left(\sum_{i=0}^{\infty} w_i^+ z_1^i \right) \mathrm{d}z_1$$

$$\omega_L \sim_{\tau c} \frac{1}{z_2} \left(\sum_{i=0}^{\infty} w_i^- z_2^i \right) \mathrm{d}z_2$$

By the definition of pinched abelian differential, $w_0^+ = w_0^-$.

If ω_L has orders $n^+ - 1, n^- - 1$ at the two points, one can choose coordinates such that:

$$\omega_L = \frac{1}{z_1}\left(\sum_{i=0}^{n^+-1} w_i^+ z_1^i + z_1^{n^+} + (\text{terms of higher order in } z_1)\right) \mathrm{d}z_1$$

$$\omega_L = \frac{1}{z_2}\left(\sum_{i=0}^{n^--1} w_i^- z_2^i + z_2^{n^-} + (\text{terms of higher order in } z_2)\right) \mathrm{d}z_2$$

In this case, one cannot make the terms of order $n^+ - 1$ and $n^- - 1$ vanish, as a translation would not fix c.

Definition 5.2.6 (Admissible pinched abelian differential)**.** Consider some pinched surface bundle $\pi_E \colon E \to M$ over a manifold. A pinched abelian differential (J, ω_L) is called **admissible** if for each point $x \in M$, there is an open neighborhood U of x in M, such that over U the bundle admits local coordinates $z, \{w_{i,j}\}_{i,j}, \{w_{i,j}^{+,-}\}_{i,j}$ with $w_{i,0}^+ = w_{i,0}^-$ defined around the set of zeroes (indexed by i) of ω_L over U, where $\{w_{i,j}\}_{i,j}, \{w_{i,j}^{+,-}\}_{i,j}$ vanish at x, such that on this neighborhoods ω_L is of the form:

(a) For a zero in E_x in the smooth locus:

$$\omega_L = \left(\sum_{j=0}^{n_i-2} w_{i,j} z^j + z^{n_i} + (\text{terms of higher order in } z)\right) \mathrm{d}z$$

(b) For a zero in E_x in the singular locus:

$$\omega_L = \frac{1}{z}\left(\sum_{j=0}^{n_i-1} w_{i,j}^{+,-} z^j + z^{n_i} + (\text{terms of higher order in } z)\right) \mathrm{d}z$$

In contrast with the notion for unpinched surface bundles, this notion is not generic, as the topology of pinched fibers of the bundle may force zeroes of ω_L on the singular points that are not locally generic. Thus, not every pinched surface bundle admits an admissible abelian differential.

Definition 5.2.7 (Pinched zero order)**.** Let $\pi_E \colon E \to M$ be a pinched surface bundle. Let (J, ω_L) be an abelian differential. Consider the bundle locally as a pullback as in the definition of pinched surface bundle, and let z_1 and z_2 be as in the local model. Around a preimage x of the point $(0, 0)$ in the total space, the form ω_L on the fiber has two expansions (on the two local irreducible components)

$$\omega_L(z_1) = f_1(z_1)\frac{\mathrm{d}z_1}{z_1}$$

$$\omega_L(z_2) = f_2(z_2)\frac{\mathrm{d}z_2}{z_2}$$

with $f_1(0) + f_2(0) = 0$. One defines the **zero order** of ω_L at x by:

$$\operatorname{ord}_x \omega_L = \begin{cases} \operatorname{ord}_0 f_1 + \operatorname{ord}_0 f_2 - 1 & \operatorname{ord}_0 f_1, \operatorname{ord}_0 f_2 \geq 1, \\ 0, & \text{else.} \end{cases}$$

Remark 5.2.8 (Definition of zero order is correct)**.** The definition of zero order in Definition 5.2.7 is the same as the dimension of the stalk of local sections of the relative dualizing bundle modulo ω_L. On every fiber, the sum of zero orders is still $2g - 2$.

Definition 5.2.9 (Compactified unordered zero cycles)**.** Let $\pi_E \colon E \to M$ be a pinched surface bundle. Let (J, ω_L) be an admissible abelian differential. Let $\sigma \in \Pi_\Sigma(\chi)$ be a symmetric partition. The **space of unordered zeroes** $M[\sigma]$ is defined as in Definition 3.4.3, with the notion of zero order as in Definition 5.2.7.

The **unordered zero cycle** is again defined as the fundamental class of $M[\sigma]$, and its Poincare dual class is denoted by $h(\sigma)$. Note that this is not necessarily the closure of the space of zeroes on the regular fibers.

Also, the **space of ordered zeroes** Z is defined as before as a subset of $E^{(\chi)}$. Even though this fiber product is not a manifold, the space Z still is one.

Lemma 5.2.10 (Compactified primitive classes)**.** *Let $\pi_E \colon E \to M$ be a pinched surface bundle. Let (J, ω_L) be an admissible abelian differential. In this case, Lemma 4.3.1 still holds.*

Proof. The proof Lemma 4.3.1 does not work directly in this case because it relies on jet bundles. But the following geometric argument (which also proves Lemma 4.3.1) also works in this case:

Let $k \leq \chi$ be a natural number. Consider the subspace $E[k]$ of E of k-fold zeroes. The proof now inductively will show that the fundamental class of $E[k]$ is Poincare dual to:

$$\prod_{i=1}^{k} (i \cdot e(T^*_{\pi_E} E) + e(\pi_E^* L))$$

The start of the induction at $k = 1$ is easily established, as $E[1]$ is the zero locus of a generic section of $T^*_{\pi_E} E \otimes \pi_E^* L$.

For the induction step (k to $k+1$), choose a generic section $s \in (\pi_E^* L)^{-1}$, and denote its zero locus by X. Now the point is that on an k-fold zero, the form $s\omega_L$ specifies $k+1$ directions (the positive real directions), except at the zeroes of higher order, and over X. By the definition of pinched zero order, this holds at these zeroes, too.

Taking $(k+1)$ copies of $E[k+1]$ (locally), one can move each of these copies in one of the specified directions. This makes the space transverse to $E[1]$. This gives:

$$\begin{aligned} [E[k+1]] + [X] \cap [E[k]] &= [E[1]] \cap \big((k+1)[E[k]]\big) \\ \Leftrightarrow [E[k+1]] &= \big((k+1)[E[1]] - [X]\big) \cap [E[k]] \end{aligned}$$

Poincare duality translates this to the statement that $[E[k+1]]$ is Poincare dual to:

$$\big((k+1)e(T^*_{\pi_E} E) + e(\pi_E^* L)\big) \prod_{i=1}^{k} (i \cdot e(T^*_{\pi_E} E) + e(\pi_E^* L)) \qquad \square$$

5.3 Other Cycles in the Compactification

Definition 5.3.1 (Characteristic class of pinched surface bundles)**.** Let R be a commutative unital ring. A **characteristic class of pinched surface bundles of genus** g with coefficients in R is a rule that assigns to every pinched surface bundle over a manifold $\pi_E \colon E \to M$ a cohomology class $\epsilon(\pi_E) \in \mathrm{H}^*(M, R)$,

$$(\pi_E \colon E \to M) \longmapsto \epsilon(\pi_E) \in \mathrm{H}^*(M, R)$$

such that for every pullback diagram of surface bundles

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow{\scriptstyle \pi_{E_1}} & & \downarrow{\scriptstyle \pi_{E_2}} \\ M_1 & \xrightarrow{\ f\ } & M_2 \end{array}$$

such that f and π_{E_2} are transverse, one has $\mathrm{H}^*(f, R)(\epsilon(\pi_{E_2})) = \epsilon(\pi_{E_1})$.

The ring of characteristic classes of pinched surface bundles of genus g with coefficients in R of even degree is denoted as $\mathrm{H}^{\mathrm{ev}}(\overline{\mathcal{M}}_g, R)$.

Conclusion 5.3.2. The formula obtained in 5.2.10 is strongly restricted by the admissibility conditions imposed on the abelian differential. This can be seen in comparison to other result, for example the following (see also [14]):

Proposition 5.3.3 ([3], Proposition 3.2)**.** *In* $\mathrm{Pic}(\mathbb{P}\mathcal{H}) \otimes \mathbb{Q}$*, we have*

$$[\mathbb{P}\overline{\mathcal{H}}(2, 1^{2g-4})] = (6g-6)\psi - 24\lambda + 2\delta_0 + 3\sum_{i=1}^{[g/2]} \delta_i.$$

Here the notation ψ instead of τ is used. This formula agrees with the formula of 5.2.10 only up to boundary classes, which demonstrates that admissible abelian differentials do not see some boundary configurations. To overcome this problem, one had to define a more general notion of admissibility that is truly generic, and deal with the more complicated phenomena that will appear.

From a more general point of view, one can ask for the following generalizations of the results of this thesis:

(a) Can one define a map

$$\mathbb{Q}[\Delta_g] \to \mathrm{H}^{\mathrm{ev}}(\overline{\mathcal{M}}_g, \mathbb{Q})$$

assigning to each combinatorial type the corresponding locus in the base, that is independent of the choice of an abelian differential?

If one replaces the codomain of the map with the coarse moduli space of abelian differentials, such a map exists (and is used for example in the work [3] cited above). Now the map from the fine moduli stack to the coarse moduli space should induce an isomorphism on rational cohomology, and one can hope that the fine moduli stack indeed classifies in a good sense pinched surface bundles (which is not clear in the real differentiable setting).

(b) If the map above is well defined, can one find a formula in terms of the strata for classes given by polynomials in the Mumford-Morita-Miller classes? Can one find a formula that is stable, that means independent of g? How unique is such a presentation?

(c) If one uses a strategy similar to the one pursued in this thesis, it is natural to ask: Is $\mathbb{Q}[\Delta_g]$ an algebra in a natural way?

This question seems to be quite hard, as the map to characteristic classes is unlikely to be injective, and thus, even if the image is closed under multiplication, one does not obtain an induced algebra structure.

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