Relative unitary RZ-spaces and the Arithmetic Fundamental Lemma

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Andreas Johannes Mihatsch
aus
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1. Gutachter: Prof. Dr. Michael Rapoport
2. Gutachter: Prof. Dr. Gerd Faltings

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Zusammenfassung

Die vorliegende Arbeit beweist neue Fälle des *Arithmetischen Fundamentalen Lemmas* (AFL) von Wei Zhang [5]. Dies ist eine vermutete Gleichheit von gewissen Schnittprodukten in unitären Rapoport-Zink-Räumen und gewissen analytischen Ausdrücken. In dieser Zusammenfassung möchten wir die AFL-Vermutung und unser Hauptresultat kurz skizzieren.

Sei $n \geq 1$ und sei E/E_0 eine unverzweigte quadratische Erweiterung lokaler p-adischer Körper[1]. Sei (V, J) ein n-dimensionaler hermitscher E-Vektorraum sodass $\det J \in E_0^\times$ keine Norm ist. Zu diesen Daten konstruieren wir ein formales Schema $\mathcal{N}_{E_0,n}$ als Modulraum von p-divisiblen Gruppen zusammen mit PEL-Zusatzdaten bezüglich der Erweiterung E/E_0. Dies ist ein Beispiel für einen sogenannten *Rapoport-Zink-Raum* [3]. Die unitäre Gruppe $U_{E_0,n} := U(V, J)$ operiert dann in natürlicher Weise auf $\mathcal{N}_{E_0,n}$. Außerdem gibt es für $n \geq 2$ eine natürliche Einbettung

$$\delta : \mathcal{N}_{E_0,n-1} \hookrightarrow \mathcal{N}_{E_0,n},$$

deren Graph wir mit Δ bezeichnen.

Für sogenanntes *regulär halb-einfaches* $g \in U_{E_0,n}$ definieren wir das *Schnittprodukt*

$$\mathrm{Int}(g) := \langle \Delta, (1, g) \cdot \Delta \rangle$$

im Raum $\mathcal{N}_{E_0,n-1} \times \mathcal{N}_{E_0,n}$. Außerdem definieren wir einen gewissen analytischen Ausdruck $\partial O(g)$ als Bahnintegral auf einer p-adischen Lie-Gruppe. Für die genaue Definition verweisen wir auf den Haupttext.

Vermutung 1 (AFL)**.** Für jedes regulär halb-einfache $g \in U_{E_0,n}$ gilt, bis auf ein Vorzeichen, die Gleichheit

$$\partial O(g) = \mathrm{Int}(g) \log(q). \tag{1}$$

Hierbei bezeichne q die Anzahl der Elemente im Restklassenkörper von E_0.

Wei Zhang hat die AFL-Vermutung im Fall $n \leq 3$ verifiziert. In der Arbeit [2] wird das AFL für beliebiges n, aber *minuscules* g bewiesen. (Dies ist eine starke Einschränkung an g.) Das Hauptresultat der vorliegenden Arbeit lässt sich nun wie folgt formulieren.

Sei σ die Galois-Konjugation von E/E_0. Wir nehmen an, dass $\sigma|_{\mathbb{Q}_{p^2}} \neq \mathrm{id}$ und wählen einen Erzeuger ϑ_E der inversen Differente von $E_0/\mathbb{Q}_p$ mit $\mathrm{tr}_{E_0/\mathbb{Q}_p}(\vartheta_E) = 1$. Dann bilden wir die $\mathbb{Q}_{p^2}/\mathbb{Q}_p$-hermitsche Form

$$J_{\mathbb{Q}_{p^2}} := \mathrm{tr}_{E/\mathbb{Q}_{p^2}}(\vartheta_E J).$$

Mit dem Vektorraum V befinden wir uns jetzt gleichzeitig im Kontext der AFL-Vermutung für die Erweiterung E/E_0 und für die Erweiterung $\mathbb{Q}_{p^2}/\mathbb{Q}_p$.
Für jedes $g \in U_{E_0,n}$ bezeichnen wir mit $\mathbb{Z}_{p^2}[g] \subset \mathrm{End}(V)$ die von g erzeugte $\mathbb{Z}_{p^2}$-Algebra. Außerdem bezeichnen wir mit d den Grad $[E_0 : \mathbb{Q}_p]$ und mit i die Inklusion der unitären Gruppen, $U_{E_0,n} \subset U(J_{\mathbb{Q}_{p^2}})$.

[1] Wir nehmen immer $p \neq 2$ an.

Theorem 2. *Sei $g \in U_{E_0,n}$ regulär halb-einfach mit der Eigenschaft, dass $\mathcal{O}_E \subset \mathbb{Z}_{p^2}[g]$ gilt. Dann ist das AFL für g und Grundkörper E_0, also Gleichung* (1)*, äquivalent zum AFL für $i(g)$ und Grundkörper $\mathbb{Q}_p$.*

Insbesondere gilt das AFL für $i(g)$ falls $n \leq 3$.

Der wesentliche Punkt im Beweis des Theorems ist die Konstruktion einer Einbettung

$$\mathcal{N}_{E_0,n} \hookrightarrow \mathcal{N}_{\mathbb{Q}_p,nd}.$$

Diese Konstruktion wird mithilfe der (relativen) Displays von Zink, Lau und Ahsendorf [1] durchgeführt und hat gewisse Ähnlichkeiten mit der Konstruktion in [4].

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Relative unitary RZ-spaces and the Arithmetic Fundamental Lemma

Andreas Mihatsch

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1 Introduction

In [20], Wei Zhang introduces his so-called Arithmetic Fundamental Lemma conjecture (AFL). This is a conjectural identity between certain derivatives of orbital integrals on p-adic[1] symmetric spaces and certain intersection products in unitary Rapoport-Zink spaces. The AFL is proven in the case of dimension $n \leq 3$, see [20]. In the subsequent work [15], Rapoport, Terstiege and Zhang verify the AFL for arbitrary n and so-called minuscule group elements g.

In the present paper, we verify more cases of the AFL for arbitrary n but under restrictive conditions on g. These computations rely on a certain recursion formalism which involves comparison isomorphisms between different Rapoport-Zink spaces. More precisely, we will compare two PEL moduli problems, one for p-divisible groups and one for strict formal $\mathcal{O}$-modules. This comparison relies on the theory of display as developed by Zink [21], Lau [7] and Ahsendorf [2].

There is some resemblance of our comparison isomorphism with the one from Rapoport and Zink in the Drinfeld case, see [17]. However, our moduli problems involve a polarization which adds an additional twist. The reason is that a polarization of a strict formal $\mathcal{O}$-module is not the same as a polarization of the underlying p-divisible group. We treat this problem in the appendix.

Let us briefly mention the following papers around the AFL. First, the AFL is related to an arithmetic Gan-Gross-Prasad conjecture which can be seen as a higher-dimensional generalization of the Gross-Zagier formula, see [4]. We refer to the introduction of [20] for more information on these global aspects.

Second, the AFL from [20] is formulated for an unramified quadratic extension. See [13] and the forthcoming systematic treatment [14] for variants in the ramified situation.

We will now describe our main results in more detail.

Part I: Relative unitary RZ-spaces

We begin by recalling the Vollaard-Wedhorn moduli problem, see [18]. This moduli problem occurs in the formulation of the AFL, see [20, Section 2.2]. Let E/E_0 be an unramified quadratic extension of p-adic local fields with rings of integers $\mathcal{O}_{E_0} \subset \mathcal{O}_E$ and Galois conjugation σ. We denote by $\breve{E}$ the completion of a maximal unramified extension of E with ring of integers $\mathcal{O}_{\breve{E}}$ and residue field $\mathbb{F}$.

Definition 1.1. Let S be a scheme over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.[2] A *hermitian $\mathcal{O}_E$-module* over S is a triple (X, ι, λ) where X/S is a supersingular strict $\mathcal{O}_{E_0}$-module, $\iota : \mathcal{O}_E \longrightarrow \operatorname{End}(X)$ an action and $\lambda : X \xrightarrow{\sim} X^\vee$ a principal polarization that is compatible with the Rosati involution, see Definition 3.1.
The hermitian $\mathcal{O}_E$-module (X, ι, λ) over S is *of signature* (r, s) if, for all $a \in \mathcal{O}_E$,

$$\operatorname{charpol}(\iota(a) \mid \operatorname{Lie}(X))(T) = (T - a)^r (T - \sigma(a))^s \in \mathcal{O}_S[T].$$

Up to quasi-isogeny, there is a unique hermitian $\mathcal{O}_E$-module $(\mathbb{X}_{E_0,(r,s)}, \iota, \lambda)$ of signature (r, s) over $\mathbb{F}$.

[1] Throughout this work, we assume $p \neq 2$.
[2] That is, an $\mathcal{O}_{\breve{E}}$-scheme such that p is locally nilpotent in $\mathcal{O}_S$.

Definition 1.2. For an $\mathcal{O}_{\breve{E}}$-scheme S, we denote by $\overline{S} := S \otimes \mathbb{F}$ its special fiber. Let $\mathcal{N}_{E_0,(r,s)}$ be the following set-valued functor on the category of schemes over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$. To any S, we associate the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$, where (X, ι, λ) is a hermitian $\mathcal{O}_E$-module of signature (r, s) and where

$$\rho : X \times_S \overline{S} \longrightarrow \mathbb{X}_{E_0,(r,s)} \times_{\mathbb{F}} \overline{S}$$

is an E-linear quasi-isogeny such that $\rho^* \lambda = \lambda$.

Proposition 1.3. *The functor $\mathcal{N}_{E_0,(r,s)}$ is representable by a formal scheme which is locally formally of finite type and formally smooth of dimension rs over* $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.

Two remarks are in order. First, if $E_0 = \mathbb{Q}_p$, then this moduli problem is of PEL-type in the sense of Rapoport and Zink, see [16]. By contrast, if $E_0 \neq \mathbb{Q}_p$, then this moduli problem is not covered by their book. This is due to the polarization λ, which is a polarization as a strict $\mathcal{O}_{E_0}$-module. We call $\mathcal{N}_{E_0,(r,s)}$ a *relative Rapoport-Zink space* since the underlying moduli problem is formulated in strict $\mathcal{O}_{E_0}$-modules as opposed to p-divisible groups.

Second, the formal scheme $\mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ uniformizes the supersingular locus in a certain unitary Shimura variety, see [18, Section 5]. Essentially, this follows directly from the moduli description of the Shimura variety in terms of abelian varieties. An analogous result is not known for the formal schemes $\mathcal{N}_{E_0,(1,n-1)}$ a priori.

These two remarks motivate our main result from Part I, which we now state in a rather informal way. See Theorem 4.1 for the precise statement.

Theorem 1.4. *There exists an RZ-space $\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}$ of PEL-type in the sense of [16] together with an isomorphism*

$$\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)} \cong \mathcal{N}_{E_0,(r,s)}.$$

This isomorphism is equivariant with respect to the unitary group acting on both sides.

In particular, the RZ-space $\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}$ is smooth over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$. This is remarkable since we do not impose any conditions on the ramification behavior of $E_0/\mathbb{Q}_p$. Instead, we impose a very specific Kottwitz condition for the moduli problem $\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}$. Namely, the Kottwitz condition has to be induced from the maximal unramified intermediate field $\mathbb{Q}_p \subset E_0^u \subset E_0$ at all but possibly one place $\psi_0 : E_0^u \hookrightarrow \breve{E}$, see Definition 2.8. Our definition bears some similarity with the situation in [17, Equation (2.1)]. But note that the unramified intermediate field does not play a role in loc. cit. Instead, the authors impose the *Eisenstein condition* to get a regular moduli problem. A similar definition is made in [11].

Theorem 1.4 is linked to the AFL in the following way. Let us assume that $E = E_0 \otimes \mathbb{Q}_{p^2}$ and let $d := [E_0 : \mathbb{Q}_p]$. In this case, forgetting the $\mathcal{O}_{E_0}$-action induces an embedding

$$\mathcal{N}_{E_0/\mathbb{Q}_p,(r,n-r)} \hookrightarrow \mathcal{N}_{\mathbb{Q}_p,(r,nd-r)}.$$

If $r = 1$, then its image can be identified with a connected component of a certain cycle which plays a role in the AFL conjecture. This will be explained below, see Theorem 1.10.

Part II: Application to the Arithmetic Fundamental Lemma

We will first give a brief formulation of the AFL conjecture in the *inhomogeneous group version*. In the main text, we will also consider the AFL in the *Lie algebra formulation*. We refer the reader to [13] for the *homogeneous version*.

Let us fix an integer $n \geq 2$ and let W_0 be an $(n-1)$-dimensional E_0-vector space. Set $W := E \otimes W_0$ and $V := W \oplus Eu$. We embed $GL(W)$ into $GL(V)$ as $h \mapsto \mathrm{diag}(h, 1)$. In this way, $GL(W)$ acts by conjugation on $\mathrm{End}(V)$. An element $\gamma \in \mathrm{End}(V)$ is said to be *regular semi-simple*, if its stabilizer for this action is trivial and if its orbit is Zariski-closed.

Let $S(E_0)$ denote the symmetric space

$$S(E_0) := \{\gamma \in \mathrm{End}(V) \mid \gamma\bar{\gamma} = 1\}.$$

It is stable under the action of $GL(W_0)$. We denote its regular semi-simple elements by $S(E_0)_{\mathrm{rs}}$ and form the set-theoretic quotient $[S(E_0)_{\mathrm{rs}}] := GL(W_0)\backslash S(E_0)_{\mathrm{rs}}$.

For a regular semi-simple element $\gamma \in S(E_0)_{\mathrm{rs}}$, for a test function $f \in C_c^\infty(S(E_0))$ and for a complex parameter $s \in \mathbb{C}$, we define the *orbital integral*

$$O_\gamma(f, s) := \int_{GL(W_0)} f(h^{-1}\gamma h)\eta(\det h)|\det h|^s dh,$$

where $\eta : E_0^\times \to \{\pm 1\}$ is the quadratic character associated to E/E_0 by local class field theory and where $|\cdot| := q^{-v(\cdot)}$ is the normalized absolute value. We consider the special value $O_\gamma(f) := O_\gamma(f, 0)$ and the *derived orbital integral*

$$\partial O_\gamma(f) := \frac{d}{ds}\bigg|_{s=0} O_\gamma(f, s).$$

Note that $O_\gamma(f)$ transforms with $\eta\circ\det$ under the action of $GL(W_0)$ on γ. We will define a *transfer factor* $\Omega(\gamma) \in \{\pm 1\}$ which is also η-invariant. Then the product $\Omega(\gamma)O_\gamma(f)$ descends to the quotient $[S(E_0)_{\mathrm{rs}}]$.

Now let $J_0^\flat$ (resp. $J_1^\flat$) be a hermitian form with even discriminant (resp. odd discriminant) on W. For $i = 0, 1$, we extend $J_i^\flat$ to a form J_i on V by defining $J_i(u, u) = 1$ and $u \perp W$. The unitary group $U(J_i^\flat)$ acts by conjugation on the regular semi-simple elements $U(J_i)_{\mathrm{rs}}$ and we form the quotient

$$[U(J_i)_{\mathrm{rs}}] := U(J_i^\flat)\backslash U(J_i)_{\mathrm{rs}}.$$

Definition 1.5. Two elements $\delta \in U(J_i)_{\mathrm{rs}}$ and $\gamma \in S(E_0)_{\mathrm{rs}}$ are said to *match*, if they are conjugate under $GL(W)$ within $\mathrm{End}(V)$.

Lemma 1.6 ([20, Lemma 2.3])**.** *The matching relation induces a bijection*

$$\alpha : [S(E_0)_{\mathrm{rs}}] \cong [U(J_0)_{\mathrm{rs}}] \sqcup [U(J_1)_{\mathrm{rs}}].$$

Now let $\overline{\mathbb{Y}}_{E_0}$ (resp. $\mathbb{X}_{E_0(1,n-2)}$) be a hermitian $\mathcal{O}_E$-module over $\mathbb{F}$ of signature $(0, 1)$ (resp. of signature $(1, n-2)$). We form $\mathbb{X}_{E_0,(1,n-1)} := \mathbb{X}_{E_0,(1,n-2)} \times \overline{\mathbb{Y}}_{E_0}$, which has signature $(1, n-1)$, and consider the associated RZ-spaces $\mathcal{N}_{E_0,(1,n-2)}$ and $\mathcal{N}_{E_0,(1,n-1)}$. Note that there is a unique deformation $\overline{\mathcal{Y}}_{E_0}$ of $\overline{\mathbb{Y}}_{E_0}$ to $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$ by Proposition 1.3. This defines a closed immersion

$$\begin{aligned} \delta : \mathcal{N}_{E_0,(1,n-2)} &\longrightarrow \mathcal{N}_{E_0,(1,n-1)} \\ X &\longmapsto X \times \overline{\mathcal{Y}}_{E_0}. \end{aligned}$$

Its image can be identified with the Kudla-Rapoport divisor $\mathcal{Z}(u)$ associated to the homomorphism $u := 0 \times \mathrm{id} : \overline{\mathbb{Y}}_{E_0} \longrightarrow \mathbb{X}_{E_0,(1,n-2)} \times \overline{\mathbb{Y}}_{E_0}$, see [9].

We can identify the group $\mathrm{Aut}(\mathbb{X}_{E_0,(1,n-2)}, \lambda, \iota)$ with $U(J_1^\flat)$. Then $U(J_1^\flat)$ acts on $\mathcal{N}_{E_0,(1,n-2)}$ by composition in the framing, $g.(X, \lambda, \iota, \rho) = (X, \lambda, \iota, g\rho)$. Similarly, we may identify $\mathrm{Aut}(\mathbb{X}_{E_0,(1,n-1)}, \lambda, \iota)$ with $U(J_1)$ and we can even choose the identification in such a way that δ becomes equivariant with respect to the embedding $U(J_1^\flat) \subset U(J_1)$.

Definition 1.7. (1) For an element $g \in U(J_1)$, we denote by $\mathcal{Z}(g) \subset \mathcal{N}_{E_0,(1,n-1)}$ the locus of (X, ρ) where $\rho^{-1} g \rho \in \mathrm{End}(X)$, see [16, Proposition 2.9].
(2) An element $g \in U(J_1)$ is called *artinian* if the intersection $\mathrm{Im}(\delta) \cap \mathcal{Z}(g)$ is an artinian scheme.
(3) For artinian g, we define the *intersection number*

$$\mathrm{Int}(g) := \mathrm{len}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathrm{Im}(\delta)\cap\mathcal{Z}(g)}.$$

Actually, Wei Zhang defines an intersection product for all regular semi-simple elements $g \in U(J_1)_{\mathrm{rs}}$. Then the schematic intersection $\mathrm{Im}(\delta) \cap \mathcal{Z}(g)$ may be higher-dimensional and higher Tor-terms appear, see Definition 7.2. But note that the results of this paper only apply to the artinian case.

We are now ready to state the AFL conjecture for artinian elements. Let $\Lambda_0 \subset W_0$ be some lattice and set $\Lambda := (\Lambda_0 \otimes \mathcal{O}_E) \oplus \mathcal{O}_E u$. We define $S(\mathcal{O}_{E_0}) := S(E_0) \cap \mathrm{End}(\Lambda)$ and denote its characteristic function by $1_{S(\mathcal{O}_{E_0})}$.

Conjecture 1.8 (AFL, [20, Conjecture 2.9])**.** For every element $\gamma \in S(E_0)_{\mathrm{rs}}$ that matches an artinian element $g \in U(J_1)_{\mathrm{rs}}$, there is an equality

$$\Omega(\gamma)\partial O_\gamma(1_{S(\mathcal{O}_{E_0})}) = -\mathrm{Int}(g)\log(q). \qquad (\mathrm{AFL}_{E_0,(V,J_1),u,g})$$

Here, the indexing quadruple $(E_0, (V, J_1), u, g)$ is chosen in such a way that it allows an unambiguous reconstruction of the terms involved in the equation $(\mathrm{AFL}_{E_0,(V,J_1),u,g})$.

Having stated the AFL, we now formulate our main results for this conjecture. To do this, we assume that $\sigma|_{\mathbb{Q}_{p^2}} \neq \mathrm{id}$. In other words, we assume that $E \cong E_0 \otimes \mathbb{Q}_{p^2}$. Let $d := [E_0 : \mathbb{Q}_p]$ be the degree and let $f := [E_0 : \mathbb{Q}_p]_{\mathrm{inert}}$ be the inertia degree of the field extension $E_0/\mathbb{Q}_p$.

Let us write $V_{\mathbb{Q}_{p^2}}$ for the vector space V, but regarded as $\mathbb{Q}_{p^2}$-vector space. Let ϑ_E be a generator of the inverse different of $E_0/\mathbb{Q}_p$ such that $\mathrm{tr}_{E/\mathbb{Q}_p}(\vartheta_E) = 1$ and let $J_{1,\mathbb{Q}_{p^2}}$ be the following hermitian form on $V_{\mathbb{Q}_{p^2}}$,

$$J_{1,\mathbb{Q}_{p^2}} := \mathrm{tr}_{E/\mathbb{Q}_{p^2}}(\vartheta_E J_1).$$

Note that $J_{1,\mathbb{Q}_{p^2}}(u, u) = 1$ which puts us into the situation of the AFL for the field extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ and the vector space $V_{\mathbb{Q}_{p^2}}$. Let $\overline{\mathbb{Y}}_{\mathbb{Q}_p}$ (resp. $\mathbb{X}_{\mathbb{Q}_p,(1,nd-1)}$) be a hermitian $\mathbb{Z}_{p^2}$-module of signature $(0, 1)$ (resp. of signature $(1, nd-1)$) over $\mathbb{F}$. We can choose an identification

$$\mathrm{End}(V_{\mathbb{Q}_{p^2}}) = \mathrm{End}(\mathbb{X}_{\mathbb{Q}_p,(1,nd-1)}) \qquad (1.1)$$

that is equivariant for the adjoint involution of $J_{1,\mathbb{Q}_{p^2}}$ on the left and the Rosati involution on the right. This induces an identification $U(J_{1,\mathbb{Q}_{p^2}}) \cong \mathrm{Aut}(\mathbb{X}_{\mathbb{Q}_p,(1,nd-1)})$.

Associated to $\mathbb{X}_{\mathbb{Q}_p,(1,nd-1)}$ we have the RZ-space $\mathcal{N}_{\mathbb{Q}_p,(1,nd-1)}$ of hermitian $\mathbb{Z}_{p^2}$-modules of signature $(1, nd-1)$. Via the identification (1.1), we get an action of $\mathcal{O}_E$ on $\mathbb{X}_{\mathbb{Q}_p,(1,nd-1)}$

by quasi-endomorphisms. We denote by $\mathcal{Z}(\mathcal{O}_E) \subset \mathcal{N}_{\mathbb{Q}_p,(1,nd-1)}$ the locus of pairs (X, ρ) such that $\rho^{-1}\mathcal{O}_E\rho \subset \mathrm{End}(X)$. Recall that to any $g \in U(J_{1,\mathbb{Q}_{p^2}})$, we have an associated cycle $\mathcal{Z}(g) \subset \mathcal{N}_{\mathbb{Q}_p,(1,nd-1)}$ by Definition 1.7. This cycle only depends on the $\mathbb{Z}_{p^2}$-algebra spanned by g in $\mathrm{End}(V)$.

Definition 1.9. An element $g \in U(J_1)_{\mathrm{rs}}$ is *of inductive type* if there is an inclusion

$$\mathcal{O}_E \subset \mathbb{Z}_{p^2}[g]$$

where the right hand side denotes the $\mathbb{Z}_{p^2}$-algebra spanned by g in $\mathrm{End}(V)$.

Let us consider the inclusion $i : U(J_1) \hookrightarrow U(J_{1,\mathbb{Q}_{p^2}})$. If $g \in U(J_1)_{\mathrm{rs}}$ is of inductive type, then we get the relation

$$\mathcal{Z}(i(g)) \subset \mathcal{Z}(\mathcal{O}_E).$$

Theorem 1.10. *There is an isomorphism of formal schemes*

$$\mathcal{Z}(\mathcal{O}_E) \cong \coprod_{i=1}^{f} \mathcal{N}_{E_0,(1,n-1)}.$$

This isomorphism is compatible with the formation of $\mathcal{Z}(g)$ in the following sense. If $g \in U(J_1)_{\mathrm{rs}}$ is of inductive type, then

$$\mathcal{Z}(i(g)) \cong \coprod_{i=1}^{f} \mathcal{Z}(g) \subset \mathcal{Z}(\mathcal{O}_E).$$

The proof of this theorem relies on Theorem 1.4. Let us now state our main result on the AFL in the present setting.

Theorem 1.11. *Let $g \in U(J_1)_{\mathrm{rs}}$ be regular semi-simple, artinian and of inductive type. Then there is an equivalence*

$$(\mathrm{AFL}_{E_0,V,u,g}) \quad \Leftrightarrow \quad (\mathrm{AFL}_{\mathbb{Q}_p,V_{\mathbb{Q}_{p^2}},u,i(g)}).$$

Since the AFL has been proven for $n \leq 3$, we get the following corollary.

Corollary 1.12. *Let g be regular semi-simple, artinian and of inductive type and let $n \leq 3$. Then the AFL for g over $\mathbb{Q}_p$,*

$$(\mathrm{AFL}_{\mathbb{Q}_p,V_{\mathbb{Q}_{p^2}},u,i(g)})$$

holds.

In the main text, we prove many variants of this result, see the corollaries in Section 10. These variants also cover the Lie algebra version of the AFL. In the group version, they cover cases beyond the one presented here. Furthermore, we also cover the case of an étale algebra $E_0/\mathbb{Q}_p$. It is worth pointing out, that all these cases are corollaries of the Theorems 10.1 and 10.5. These theorems are formulated in a formalism which encompasses both the AFL in the group and the Lie algebra version for artinian elements, see Section 9.

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Part I

Relative Unitary RZ-spaces

2 Basic Definitions

In this chapter, we formulate a moduli problem of PEL-type as defined by Rapoport and Zink [16]. It generalizes the moduli problem of Vollaard and Wedhorn [18]. In particular, it is also associated to a unitary group for an unramified quadratic extension of p-adic local fields.

2.1 Set-up

Let $p > 2$ be a prime and let E/E_0 be an unramified quadratic extension of p-adic local fields. Let $d := [E_0 : \mathbb{Q}_p]$ with $d = ef$ where e denotes the ramification index and f the inertia degree. We denote the Galois conjugation of E/E_0 by σ and the rings of integers by $\mathcal{O}_{E_0} \subset \mathcal{O}_E$.

Definition 2.1. A *skew-hermitian E-module* $(V, \langle\ ,\ \rangle)$ is an E-vector space together with a perfect alternating $\mathbb{Q}_p$-bilinear pairing $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{Q}_p$ such that $\langle a\ ,\ \rangle = \langle\ , a^\sigma\ \rangle$ for all $a \in E$.

An isomorphism of skew-hermitian E-modules $(V, \langle\ ,\ \rangle)$ and $(V', \langle\ ,\ \rangle)$ is an E-linear isometry $V \cong V'$. We denote by $U(V)$ the group of automorphisms of $(V, \langle\ ,\ \rangle)$.

For every n, there exist two isomorphism classes of skew-hermitian E-modules $(V, \langle\ ,\ \rangle)$ of dimension n. We say that V is *even* if there exists a self-dual $\mathcal{O}_E$-lattice in V. Otherwise we call V *odd*. This distinguishes the two isomorphism classes. Note that $(V, \langle\ ,\ \rangle)$ is even (resp. odd) if and only if the index $[M^\vee : M]$ is even (resp. odd) for every $\mathbb{Z}_p$-lattice $M \subset V$.

The category of skew-hermitian E-modules is endowed with the *adjoint involution* $*$. If V_1 and V_2 are skew-hermitian E-modules, then this is the isomorphism

$$\begin{aligned} * : \mathrm{Hom}_E(V_1, V_2) &\xrightarrow{\cong} \mathrm{Hom}_E(V_2, V_1) \\ f &\longmapsto f^* : V_2 \cong V_2^\vee \xrightarrow{f^\vee} V_1^\vee \cong V_1 \end{aligned}$$

where the identifications $V_1 \cong V_1^\vee$ and $V_2 \cong V_2^\vee$ are induced by the skew-hermitian forms.

2.2 Formal hermitian $\mathcal{O}_E$-modules up to quasi-isogeny

As usual, $\breve{\mathbb{Q}}_p$ denotes the completion of a maximal unramified extension of $\mathbb{Q}_p$. We denote by $E^u \subset E$ the maximal subfield which is unramified over $\mathbb{Q}_p$ and define $\Psi := \mathrm{Hom}_{\mathbb{Q}_p}(E^u, \breve{\mathbb{Q}}_p)$. We choose a decomposition $\Psi = \Psi_0 \sqcup \Psi_1$ such that $\sigma(\Psi_0) = \Psi_1$ and we fix an element $\psi_0 \in \Psi_0$. Finally, we define $\breve{E} := E \otimes_{E^u, \psi_0} \breve{\mathbb{Q}}_p$ which is the completion of a maximal unramified extension of E.

We denote the ring of integers in $\breve{\mathbb{Q}}_p$ (resp. in $\breve{E}$) by $\breve{\mathbb{Z}}_p$ (resp. $\mathcal{O}_{\breve{E}}$). Let $\mathbb{F}$ be their residue field and let $x \mapsto {}^F x$ denote the Frobenius on $\breve{\mathbb{Q}}_p$.

There is a natural identification

$$\mathcal{O}_E \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p = \prod_{\psi \in \Psi} \mathcal{O}_{\breve{E}} \tag{2.1}$$

such that the Frobenius $1 \otimes {}^F$ is homogeneous and acts simply transitive on the indexing set.

Definition 2.2. Let S be a scheme over $\operatorname{Spf} \breve{\mathbb{Z}}_p$. A *(supersingular) hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module over S* is a triple (X, ι, λ) where X/S is a supersingular p-divisible group together with an action $\iota : \mathcal{O}_E \longrightarrow \operatorname{End}(X)$ and a principal polarization $\lambda : X \xrightarrow{\sim} X^\vee$ such that

$$\lambda^{-1}\iota(a^\vee)\lambda = \iota(a^\sigma).$$

An *isomorphism* (resp. *quasi-isogeny*) of two hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules (X, ι, λ) and (X', ι', λ') is an $\mathcal{O}_E$-linear isomorphism (resp. quasi-isogeny) $\mu : X \longrightarrow X'$ of the underlying p-divisible groups such that $\mu^*\lambda' = \lambda$.

We say that the hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module (X, ι, λ) is *of rank n* if the height of X as p-divisible group is $2nd$. This implies $\dim X = nd$.

Definition 2.3. The category of hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules over a scheme S is endowed with the *Rosati involution* $*$. If $(X_1, \iota_1, \lambda_1)$ and $(X_2, \iota_2, \lambda_2)$ are two such modules, then this is the isomorphism

$$\begin{aligned} * : \operatorname{Hom}_{\mathcal{O}_E}(X_1, X_2) &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}_E}(X_2, X_1) \\ f &\longmapsto f^* := \lambda_1^{-1} \circ f^\vee \circ \lambda_2. \end{aligned}$$

By Dieudonné-theory, the category of hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules up to quasi-isogeny over $\operatorname{Spec} \mathbb{F}$ is equivalent to the category of *skew-hermitian E-isocrystals* as we define it now.

Definition 2.4. A *(supersingular) skew-hermitian E-isocrystal* is a tuple $(N, \langle\ ,\ \rangle, F, \iota)$ where N is a finite $\breve{\mathbb{Q}}_p$-vector space, $\langle\ ,\ \rangle : N \times N \longrightarrow \breve{\mathbb{Q}}_p$ is an alternating perfect pairing, $F : N \longrightarrow N$ is an ^{F}-linear isomorphism with all slopes $1/2$ such that $\langle F\ , F\ \rangle = p^F\langle\ ,\ \rangle$ and $\iota : E \longrightarrow \operatorname{End}(N, F)$ is an action of E such that $\langle a\ ,\ \rangle = \langle\ , a^\sigma\ \rangle$ for all $a \in E$.

Proposition 2.5. *There is an equivalence of categories*

$$\begin{aligned} &\{\textit{skew-hermitian } E\textit{-modules } (V, \langle\ ,\ \rangle)\} \\ \cong\ &\{\textit{skew-hermitian } E\textit{-isocrystals } (N, \langle\ ,\ \rangle, F, \iota)\}. \end{aligned}$$

that is compatible with the adjoint involutions on both sides.[3]

In particular for a given rank n, there are precisely two hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules over $\mathbb{F}$ up to quasi-isogeny.

Definition 2.6. A skew-hermitian E-isocrystal is called *even* (resp. *odd*) if it corresponds to an even (resp. odd) skew-hermitian E-module under the above equivalence of categories.

Proof. Given a skew-hermitian E-module $(V, \langle\ ,\ \rangle)$, we define a skew-hermitian E-isocrystal as follows. First, we extend scalars from $\mathbb{Q}_p$ to $\breve{\mathbb{Q}}_p$,

$$N := V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p.$$

[3] The definition of the adjoint involution on skew-hermitian E-isocrystals is just as in the case of skew-hermitian E-modules.

We extend the pairing $\langle\ ,\ \rangle$ in the $\breve{\mathbb{Q}}_p$-bilinear way to N. Note that N is a module over $E \otimes \breve{\mathbb{Q}}_p$ and hence graded according to (2.1),

$$N = \prod_{\psi \in \Psi} N_\psi.$$

The ^{F}-linear operator $\alpha := \mathrm{id}_V \otimes {}^F$ on N is homogeneous in the sense that $\alpha(N_\psi) = N_{{}^F\psi}$. Furthermore, the pairing satisfies

$$\langle\ ,\ \rangle|_{N_\psi \times N_{\psi'}} \equiv 0 \quad \text{if} \;\; \psi' \neq \sigma\psi.$$

We define a supersingular Frobenius $F = \prod F_\psi$ on N via

$$[F_\psi : N_\psi \longrightarrow N_{{}^F\psi}] := \begin{cases} p\alpha_\psi & \text{if } \psi \in \Psi_0 \\ \alpha_\psi & \text{if } \psi \in \Psi_1. \end{cases}$$

Here, $\alpha_\psi : N_\psi \longrightarrow N_{{}^F\psi}$ is the ψ-component of α. It is obvious that (N, F) is supersingular since $F^{2f} = p^f$ on V. In particular, there exists a $\breve{\mathbb{Z}}_p$-lattice $M \subset N$ such that $F^{2f}M = p^f M$.

Furthermore, the previously defined pairing $\langle\ ,\ \rangle$ becomes a polarization on (N, F). To see this, we compute for all $(x, y) \in N_\psi \times N_{\sigma\psi}$ that

$$\langle Fx, Fy\rangle = p\langle \alpha x, \alpha y\rangle = p^F(\langle x, y\rangle).$$

(For the first equality, we used that precisely one out of $\{\psi, \sigma\psi\}$ lies in Ψ_0.)

Finally, we endow N with the E-action on the first factor of $N = V \otimes \breve{\mathbb{Q}}_p$. This action is compatible with $\langle\ ,\ \rangle$ and commutes with F.

Let us give the inverse construction. Given a skew-hermitian E-isocrystal $(N, \langle\ ,\ \rangle, F, \iota)$, we define the ^{F}-linear operator $\alpha = \prod \alpha_\psi$ as

$$[\alpha_\psi : N_\psi \longrightarrow N_{{}^F\psi}] := \begin{cases} p^{-1}F_\psi & \text{if } \psi \in \Psi_0 \\ F_\psi & \text{if } \psi \in \Psi_1. \end{cases} \tag{2.2}$$

We define $V := N^{\alpha=1}$ and restrict the form $\langle\ ,\ \rangle$ to V. Note that α is isoclinic of slope 0 since N is supersingular. Furthermore,

$$\langle \alpha\ , \alpha\ \rangle = p^{-1}\langle F\ , F\ \rangle = {}^F\langle\ ,\ \rangle.$$

So on V, the form $\langle\ ,\ \rangle$ takes values in the ^{F}-invariants of $\breve{\mathbb{Q}}_p$ which agree with $\mathbb{Q}_p$. Finally, the E-action commutes with α and thus E acts on V. This shows that $(V, \langle\ ,\ \rangle)$ defines a skew-hermitian E-module such that $V \otimes \breve{\mathbb{Q}}_p \cong N$.

It is clear that both our constructions are functorial and compatible with the adjoint involutions. $\square$

2.3 Rapoport-Zink spaces

Let $r, s \in \mathbb{Z}_{\geq 0}$ and set $n := r + s$.

Definition 2.7. For $a \in E$, we define the following polynomials.

$$\begin{aligned}
P_{(0,1)}(a;t) &:= \prod_{\psi \in \Psi_1} \psi(\mathrm{charpol}_{E/E^u}(a;t)) && \in E^u[t]. \\
P_{(1,0)}(a;t) &:= P_{(0,1)}(a;t)(t-a)(t-a^\sigma)^{-1} && \in E[t]. \\
P_{(r,s)}(a;t) &:= P_{(1,0)}(a;t)^r P_{(0,1)}(a;t)^s && \in E[t].
\end{aligned}$$

If X is a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module over a $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$-scheme S, then its Lie algebra is Ψ-graded,

$$\mathrm{Lie}(X) = \bigoplus_{\psi \in \Psi} \mathrm{Lie}_\psi(X). \tag{2.3}$$

Here $\mathrm{Lie}_\psi(X)$ is the direct summand on which $\mathcal{O}_{E^u}$ acts via the embedding $\psi : \mathcal{O}_{E^u} \longrightarrow \breve{\mathbb{Z}}_p$. Recall that by definition

$$\mathcal{O}_{\breve{E}} = \mathcal{O}_E \otimes_{\mathcal{O}_{E^u}, \psi_0} \breve{\mathbb{Z}}_p.$$

We consider any $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$-scheme as an $\mathcal{O}_E$-scheme via the first and as a $\breve{\mathbb{Z}}_p$-scheme via the second projection.

Definition 2.8. Let S be a scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$. A hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module of rank n over S is *of signature* (r,s) if the following two conditions hold.

(i) $\mathrm{charpol}(\iota(a) \mid \mathrm{Lie}(X); t) = P_{(r,s)}(a;t) \quad \forall\, a \in \mathcal{O}_E$.

(ii) $(\iota(a) - a)\mid_{\mathrm{Lie}_{\psi_0}(X)} = 0 \quad \forall\, a \in \mathcal{O}_E$.

Here in (i), we view $P_{(r,s)}(a;t)$ as element of $\mathcal{O}_S[t]$ via the structure morphism. Condition (ii) means that $\mathcal{O}_E$ acts on $\mathrm{Lie}_{\psi_0}(X)$ via the structure morphism.

Remark 2.9. (1) In the case $E_0 = \mathbb{Q}_p$, our definition of signature agrees with the one from [18]. More generally in the case of an unramified extension $E/\mathbb{Q}_p$, the condition (ii) is automatically satisfied.

(2) In Definition 2.8, it is enough to demand (i) only for $a \in \mathcal{O}_{E^u}$. Equivalently, one could replace (i) by the following rank condition.

(i') The ranks of the summands in equation (2.3) are as follows:

$$\mathrm{rk}_{\mathcal{O}_S} \mathrm{Lie}_\psi(X) = \begin{cases} 0 & \text{if } \psi \in \Psi_0 \setminus \{\psi_0\} \\ r & \text{if } \psi = \psi_0 \\ ne & \text{if } \psi \in \Psi_1 \setminus \{\sigma\psi_0\} \\ ne - r & \text{if } \psi = \sigma\psi_0. \end{cases}$$

(3) The polynomials $P_{(0,n)}(a;t)$ have coefficients in $E^u \subset \breve{\mathbb{Q}}_p$. Hence in the case of signature $(0,n)$, condition (i) could be formulated for schemes over $\breve{\mathbb{Z}}_p$. We will not need this.

Lemma 2.10. *Consider the quasi-isogeny class of a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module over $\mathbb{F}$ corresponding to a skew-hermitian E-module $(V, \langle\ ,\ \rangle)$ of rank n.*
There exists a formal hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module (X, ι, λ) of signature (r,s) in this class if and only if the parity of V coincides with the parity of r.

We prepare the proof with a simple lemma. Recall the definition of the operator α from equation (2.2).

Lemma 2.11. *Let V be a skew-hermitian E-module and let $N := V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$ be the induced supersingular isocrystal. Then there is a bijection*

$$\{\textit{self-dual } \mathcal{O}_E\textit{-lattices } \Lambda \subset V\} \cong \left\{ \begin{matrix} \textit{self-dual } \mathcal{O}_E\textit{-stable Dieudonné-lattices} \\ M \textit{ in } N \textit{ of signature } (0,n) \end{matrix} \right\}.$$

This bijection is given by $\Lambda \mapsto \Lambda \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ and $M \mapsto M^{\alpha = \mathrm{id}}$.

Proof. This is essentially trivial. Note that Λ has the correct signature by definition of F. Also note that M is stable under α since it has signature $(0,n)$. □

Proof of Lemma 2.10. We first prove the existence of hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules for all signatures. It is enough to consider the cases of signature $(0,1)$ and $(1,0)$. Taking direct products will then settle the general case. The case of signature $(0,1)$ is taken care of by Lemma 2.11 so we are left with the case $(1,0)$.

Let $(V, \langle\ ,\ \rangle)$ be any odd skew-hermitian E-module of rank 1. Let $(N, F, \iota, \langle\ ,\ \rangle)$ be the associated isocrystal. Fix a uniformizer $\pi_E \in E$ and an $\mathcal{O}_E$-lattice $\Lambda \subset V$ such that $\pi_E \Lambda^\vee = \Lambda$.

Let $M := \Lambda \otimes \breve{\mathbb{Z}}_p$ be the associated $\mathcal{O}_E$-stable $\breve{\mathbb{Z}}$-lattice. It decomposes as

$$M = \bigoplus_{\psi \in \Psi} M_\psi.$$

Define $M' := \bigoplus_{\psi \in \Psi} M'_\psi$ as

$$M'_\psi := \begin{cases} M_\psi & \text{if } \psi \in \{{}^{F}\psi_0, {}^{F^2}\psi_0, \ldots, {}^{F^f}\psi_0\} \\ \pi_E^{-1} M_\psi & \text{otherwise}. \end{cases}$$

This lattice is stable under $\mathcal{O}_E$, stable under F and stable under pF^{-1}. Furthermore, it is self-dual of signature $(1,0)$. This finishes the proof of existence.

Now let $(X, \iota, \lambda)/\mathbb{F}$ of signature (r,s) be given. Let $N = \prod_{\psi \in \Psi} N_\psi$ be the isocrystal of X together with the alternating pairing $\langle\ ,\ \rangle$ induced by λ. For any $\mathcal{O}_{\breve{E}}$-lattice $L_{\psi_0} \subset N_{\psi_0}$, we denote by $L_{\psi_0}^\vee \subset N_{\sigma\psi_0}$ its dual with respect to the form $\langle\ ,\ \rangle$.

Let X correspond to the $\mathcal{O}_{\breve{E}}$-lattice $M \subset N$. It is self-dual and thus $M_{\psi_0}^\vee = M_{\sigma\psi_0}$. The signature condition for M implies that

$$[\alpha^f M_{\psi_0} : M_{\psi_0}^\vee] = r.$$

Thus for every $\mathcal{O}_{\breve{E}}$-lattice $L_{\psi_0} \subset N_{\psi_0}$, $[\alpha^f L_{\psi_0} : L_{\psi_0}^\vee] \equiv r \bmod 2$. By Proposition 2.5, there are precisely two quasi-isogeny classes of formal hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules over $\mathbb{F}$. In particular, formal hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules X and X' over $\mathbb{F}$ of signatures (r,s) and (r',s') respectively are quasi-isogeneous if and only $r \equiv r' \bmod 2$. So by Lemma 2.11, r and V have the same parity. □

For any signature (r,s), we fix a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module $\mathbb{X}_{E_0/\mathbb{Q}_p,(r,s)}$ over $\mathbb{F}$ of that signature.[4]

[4]It would be enough to fix any triple $(\mathbb{X}, \iota, \lambda)$ quasi-isogeneous to a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module of signature (r,s).

Definition 2.12. For a given signature (r,s), we consider the following set-valued functor $\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}$ on schemes over $\operatorname{Spf}\mathcal{O}_{\breve{E}}$. It associates to S the set of isomorphism classes of quadruples (X,ι,λ,ρ) where (X,ι,λ) is a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module of signature (r,s) over S and where

$$\rho : X \times_S \overline{S} \longrightarrow \mathbb{X}_{E_0/\mathbb{Q}_p,(r,s)} \times_{\mathbb{F}} \overline{S}$$

is a quasi-isogeny of hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules. Here, $\overline{S}$ denotes the special fiber $\overline{S} = S \times_{\operatorname{Spf}\mathcal{O}_{\breve{E}}} \operatorname{Spec}\mathbb{F}$. The quasi-isogeny ρ is called a *framing* and $\mathbb{X}_{E_0/\mathbb{Q}_p,(r,s)}$ is called the *framing object.*

Lemma 2.13. *The functor $\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}$ is representable by a formal scheme which is locally formally of finite type over $\operatorname{Spf}\mathcal{O}_{\breve{E}}$.*

Proof. This follows from [16]. □

Proposition 2.14. *The formal scheme $\mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}$ is formally smooth over $\operatorname{Spf}\mathcal{O}_{\breve{E}}$ of relative dimension rs.*

The proposition follows from the Grothendieck-Messing Theorem. For a point $X \in \mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}(S)$, we denote by $\mathbb{D}_X$ its covariant crystal on the crystalline site of S. If $S \hookrightarrow S'$ is a pd-thickening, then $\mathbb{D}_X(S')$ is a locally free $\mathcal{O}_{S'}$-module of rank $2nd$. We define its *contraction*

$$\mathcal{D}_X(S') := \mathbb{D}_X(S') \otimes_{\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_{S'}} \mathcal{O}_{S'}. \tag{2.4}$$

Proposition 2.15. *([2, 2.22]) Let X/S be a p-divisible group together with an action by $\mathcal{O}_E$, the maximal order in some étale algebra $E/\mathbb{Q}_p$. Then for all pd-thickenings $S \hookrightarrow S'$, $\mathbb{D}_X(S')$ is locally free over $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_{S'}$.* □

Corollary 2.16. *For any pd-thickening $S \subset S'$, the $\mathcal{O}_{S'}$-module $\mathcal{D}_X(S')$ is locally free of rank n over $\mathcal{O}_{S'}$.*

Proof. The local freeness of $\mathcal{D}_X(S')$ follows from the previous proposition. The rank can be computed at geometric points of S' which is a problem in Dieudonné-theory. □

Proof of Proposition 2.14. We consider the Hodge filtration $\mathcal{F} \subset \mathbb{D}_X(S)$ of a point $X \in \mathcal{N}_{E_0/\mathbb{Q}_p,(r,s)}(S)$. These are $\mathcal{O}_E \otimes \mathcal{O}_S$-modules, so there are decompositions $\mathcal{F} = \prod \mathcal{F}_\psi$ and $\mathbb{D}_X(S) = \prod \mathbb{D}_X(S)_\psi$. From the signature condition it follows that

$$\begin{cases} \mathbb{D}_X(S)_{\psi_0}/\mathcal{F}_{\psi_0} \text{ is projective of rank } r \text{ over } \mathcal{O}_S & \\ \mathcal{F}_{\sigma\psi_0} \subset \mathbb{D}_X(S)_{\sigma\psi_0} \text{ equals } \mathcal{F}_{\psi_0}^{\perp} & \\ \mathcal{F}_\psi = \mathbb{D}_X(S)_\psi & \text{if } \psi \in \Psi_0 \setminus \{\psi_0\} \\ \mathcal{F}_\psi = 0 & \text{if } \psi \in \Psi_1 \setminus \{\sigma\psi_0\}. \end{cases}$$

Furthermore, $\mathcal{O}_E$ acts on the quotient $\operatorname{Lie}_{\psi_0}(X) = \mathbb{D}_X(S)_{\psi_0}/\mathcal{F}_{\psi_0}$ via the structure morphism $\mathcal{O}_E \longrightarrow \mathcal{O}_{\breve{E}} \longrightarrow \mathcal{O}_S$. In other words, $\operatorname{Lie}_{\psi_0}(X)$ is a quotient of $\mathcal{D}_X(S)$.

Now consider a pd-thickening $S \hookrightarrow S'$. Then lifting the Hodge filtration $\mathcal{F}$ to $\mathcal{F}' \subset \mathbb{D}_X(S')$ such that $\mathcal{F}'$ is $\mathcal{O}_E$-stable, isotropic and such that $\mathcal{O}_E$ acts on $\mathcal{F}'_{\psi_0}$ naturally is equivalent to lifting the quotient $\mathcal{D}_X(S) \twoheadrightarrow \operatorname{Lie}_{\psi_0}(X)$.

This deformation problem is formally smooth of relative dimension rs. □

3 Relative variant

Let $K/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_K$. We will now briefly indicate how to generalize the notions of the previous section from p-divisible groups to strict formal $\mathcal{O}_K$-modules. We refer to the Appendix for the definition of strict formal $\mathcal{O}_K$-modules, their polarizations and to the theory of their displays.

Throughout this section, we fix a uniformizer $\pi_K \in \mathcal{O}_K$ and all polarizations of strict formal $\mathcal{O}_K$-modules are taken with respect to this uniformizer, see Remark 11.11. By height (resp. slope) of a relative $\mathcal{O}_K$-module (X, ι), we always mean the relative height (resp. relative slope).

3.1 Relative unitary RZ-spaces

Let $\mathbb{Q}_p \subset K \subset E_0$ be finite extensions and let E/E_0 be an unramified quadratic extension with Galois conjugation σ. Let $E_0^u \subset E^u$ denote the maximal subfields which are unramified over K. As usual, $\breve{K}$ denotes the completion of a maximal unramified extension of K. We choose a decomposition

$$\Psi := \mathrm{Hom}_K(E^u, \breve{K}) = \Psi_0 \sqcup \Psi_1$$

such that $\sigma(\Psi_0) = \Psi_1$.

Definition 3.1. Let S be a scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{K}}$. A *(supersingular) hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module over S* is a triple (X, ι, λ) where X/S is a supersingular[5] strict $\mathcal{O}_K$-module together with an action $\iota : \mathcal{O}_E \longrightarrow \mathrm{End}(X)$ and a principal polarization $\lambda : X \xrightarrow{\sim} X^\vee$ such that

$$\lambda^{-1}\iota(a^\vee)\lambda = \iota(a^\sigma).$$

An *isomorphism* (resp. *quasi-isogeny*) of two hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-modules (X, ι, λ) and (X', ι', λ') is an $\mathcal{O}_E$-linear isomorphism (resp. quasi-isogeny) $\mu : X \longrightarrow X'$ of the underlying strict $\mathcal{O}_K$-modules such that $\mu^*\lambda' = \lambda$.

We say that the hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module (X, ι, λ) is *of rank n* if the height of X as strict $\mathcal{O}_K$-module is $2n[E_0 : K]$. This implies $\dim X = n[E_0 : K]$.

For a $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$-scheme S, the definition of *signature* of a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module over S is completely analogous to Definition 2.8 in the case of p-divisible groups. Again, there is a unique framing object $(\mathbb{X}_{E_0/K,(r,s)}, \iota, \lambda)/\mathbb{F}$ of signature (r, s) up to quasi-isogeny.

Definition 3.2. Let $\mathcal{N}_{E_0/K,(r,s)}$ denote the following functor on the category of schemes over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$. It associates to S the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ where $(X, \iota, \lambda)/S$ is a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module of signature (r, s) and where ρ is an $\mathcal{O}_E$-linear quasi-isogeny

$$\rho : X \times_S \overline{S} \longrightarrow \mathbb{X}_{E_0/K,(r,s)} \times_{\mathbb{F}} \overline{S}$$

that preserves the polarization.

Again it follows from [16] that $\mathcal{N}_{E_0/K,(r,s)}$ is representable by a formal scheme, locally formally of finite type over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$.[6]

[5] This means that all slopes of X (as strict $\mathcal{O}_K$-module) are equal to $1/2$.

[6] The representability result of [16] is easily generalized from p-divisible groups to $\mathcal{O}_K$-modules. Namely both the condition that the $\mathcal{O}_K$-action lifts from the framing object and the condition that the lifted action is strict are closed conditions in the moduli space of [16, Theorem 2.16].

Lemma 3.3. *The formal scheme $\mathcal{N}_{E_0/K,(r,s)}$ is formally smooth of relative dimension rs over* $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.

Proof. The formal smoothness is proved with the same arguments as in the case of p-divisible groups, see Proposition 2.14. $\square$

Notation 3.4. An important special case is that of $K = E_0$. We simplify the notation to

$$\mathcal{N}_{E_0,(r,s)} := \mathcal{N}_{E_0/E_0,(r,s)}$$

and we simply say *hermitian $\mathcal{O}_E$-module* instead of hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module. Note that $\mathcal{N}_{E_0,(r,s)}$ is precisely the problem of Vollaard-Wedhorn [18] but in the relative setting of strict $\mathcal{O}_{E_0}$-modules. It appears in the formulation of the Arithmetic Fundamental Lemma in Wei Zhang [20, Section 2].

4 Comparison with the Vollaard-Wedhorn problem

Let $(V, \langle\ ,\ \rangle)$ be a skew-hermitian E-module as in Definition 2.1. Let K be an intermediate field $\mathbb{Q}_p \subset K \subset E_0$ and choose a generator ϑ_K of the inverse different of $K/\mathbb{Q}_p$. There exists a unique non-degenerate K-bilinear alternating form

$$\langle\ ,\ \rangle_K : V \times V \longrightarrow K$$

such that $\operatorname{tr}_{K/\mathbb{Q}_p}(\vartheta_K \langle\ ,\ \rangle_K) = \langle\ ,\ \rangle$. Furthermore, this form is E-hermitian in the sense that

$$\langle a\ ,\ \rangle_K = \langle\ , a^\sigma\ \rangle_K, \quad a \in E.$$

The groups of E-linear isometries of $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_K$ are then identical. Also note that a lattice $\Lambda \subset V$ is self-dual with respect to the lifted form $\langle\ ,\ \rangle_K$ if and only if it is self-dual for the original form $\langle\ ,\ \rangle$.

For any such intermediate field K, we fix a uniformizer $\pi_K \in \mathcal{O}_K$ in order to talk about polarizations of strict formal $\mathcal{O}_K$-modules, see Remark 11.11. We make $N := (V, \langle\ ,\ \rangle_K) \otimes_K \breve{K}$ into a polarized K-isocrystal as in the proof of Proposition 2.5. The corresponding quasi-isogeny class of hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-modules over $\mathbb{F}$ contains a module of signature (r, s) if and only if r and V have the same parity. This can be proved as in the case of p-divisible groups, see Proposition 2.10.

So if r and V have the same parity, then V gives rise to a whole family of framing objects

$$\{\mathbb{X}_{E_0/K,(r,s)}\}_{\mathbb{Q}_p \subset K \subset E_0}$$

which all come with an action (by quasi-isogenies) of the unitary group $U(V)$. Our main result in this section is that the corresponding RZ-spaces are all isomorphic.

Theorem 4.1. *Let $(V, \langle\ ,\ \rangle)$ be a skew-hermitian E-module and let r have the same parity as V. For any intermediate field $\mathbb{Q}_p \subset K \subset E_0$, there is a $U(V)$-equivariant isomorphism*

$$c : \mathcal{N}_{E_0/K,(r,s)} \cong \mathcal{N}_{E_0,(r,s)}.$$

In particular, the formal scheme $\mathcal{N}_{E_0/K,(r,s)}$ is independent of the choice of the decomposition $\Psi = \Psi_0 \sqcup \Psi_1$.

The proof relies on the following equivalence of categories. We consider the category $\mathit{Sch}/\operatorname{Spf} \mathcal{O}_{\breve{E}}$ of locally noetherian schemes over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$ together with the Zariski topology.

Definition 4.2. We denote by $\mathcal{O}_E$-$\mathcal{O}_K$-*Herm* the stack of hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-modules (X, ι, λ) *that have a signature* over $Sch/\operatorname{Spf}\mathcal{O}_{\breve{E}}$. By the condition, we mean that locally for the Zariski topology, the hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module is of signature (r, s) for some integers $r, s \in \mathbb{Z}_{\geq 0}$. The morphisms in this category are the $\mathcal{O}_E$-linear morphisms of p-divisible groups. In accordance with Notation 3.4, we write $\mathcal{O}_E$-*Herm* for the stack of hermitian $\mathcal{O}_E$-modules.

Theorem 4.3. *There is an isomorphism of stacks on* $Sch/\operatorname{Spf}\mathcal{O}_{\breve{E}}$

$$\mathcal{C} : \mathcal{O}_E\text{-}\mathcal{O}_K\text{-Herm} \xrightarrow{\cong} \mathcal{O}_E\text{-Herm}$$

that satisfies the following properties. It is equivariant for the Rosati involution and it sends objects of signature (r, s) to objects of signature (r, s).

This section is devoted to the proof of these two theorems.

4.1 The unramified case

Proposition 4.4. *Consider an intermediate field $\mathbb{Q}_p \subset K \subset E_0$ and let $E_0^u \subset E_0$ be the maximal subfield which is unramified over K. Then there is an isomorphism of stacks*

$$\mathcal{C} : \mathcal{O}_E\text{-}\mathcal{O}_K\text{-Herm} \xrightarrow{\cong} \mathcal{O}_E\text{-}\mathcal{O}_{E_0^u}\text{-Herm}$$

that is equivariant for the Rosati involutions and sends objects of signature (r, s) to objects of signature (r, s).

Proof. We will construct the functor $\mathcal{C}$ and its quasi-inverse. Let $S = \operatorname{Spec} R$ be an affine scheme over $\operatorname{Spf}\mathcal{O}_{\breve{E}}$ and let (X, ι, λ) be a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module of signature (r, s) over R. We use the notation from Definition 11.7. Let $(P, Q, F, \dot{F})$ be the $\mathcal{O}_K$-display of (X, ι, λ). We denote by $\langle\ ,\ \rangle : P \times P \longrightarrow W_{\mathcal{O}_K}(R)$ the alternating form induced by the polarization λ.

Recall that $\Psi = \operatorname{Hom}_K(E^u, \breve{K})$ and note that there exists a natural morphism $\mathcal{O}_{E^u} \longrightarrow W_{\mathcal{O}_K}(R)$ of $\mathcal{O}_K$-algebras that lifts the morphism $\mathcal{O}_{E^u} \longrightarrow R$, see [3, Lemme 1.2.3]. This morphism induces gradings

$$P = \prod_{\psi\in\Psi} P_\psi, \quad Q = \prod_{\psi\in\Psi} Q_\psi \quad \text{with } Q_\psi = Q \cap P_\psi.$$

For $\psi \notin \{\psi_0, \sigma\psi_0\}$, we define the Frobenius-linear isomorphism

$$[\mathbf{F}_\psi : P_\psi \longrightarrow P_{^F\psi}] := \begin{cases} \dot{F}_\psi & \text{if } \psi \in \Psi_0 \setminus \{\psi_0\} \\ F_\psi & \text{if } \psi \in \Psi_1 \setminus \{\sigma\psi_0\}. \end{cases}$$

Here we used that (X, ι, λ) has a signature, which implies $Q_\psi = P_\psi$ whenever $\psi \in \Psi_0 \setminus \{\psi_0\}$.

We set $P' := P_{\psi_0} \oplus P_{\sigma\psi_0}$ with submodule $Q' := Q_{\psi_0} \oplus Q_{\sigma\psi_0}$ and define the ${}^{F^f}$-linear operators $F', \dot{F}'$ on P' and Q' as

$$F' := \mathbf{F}^{f-1} \circ F, \quad \dot{F}' := \mathbf{F}^{f-1} \circ \dot{F}.$$

Then $\mathcal{P}' := (P', Q', F', \dot{F}')$ defines an f-$\mathcal{O}_K$-display in the sense of Ahsendorf [2]. There is an induced $\mathcal{O}_E$-action and a $W_{\mathcal{O}_K}(R)$-valued alternating pairing $\langle\ ,\ \rangle' := \langle\ ,\ \rangle|_{P'}$ on P'.

Note that f-$\mathcal{O}_K$-displays are the same as windows over the $\mathcal{O}_{E_0^u}$-frame

$$\mathcal{A}_{\mathcal{O}_{E_0^u}/\mathcal{O}_K}(R) := (W_{\mathcal{O}_K}(R), I_{\mathcal{O}_K}(R), {}^{F^f}, {}^{F^{f-1}V^{-1}}).$$

In our case, the pairing $\langle\ ,\ \rangle'$ takes values in $\mathcal{A}_{\mathcal{O}_{E_0^u}/\mathcal{O}_K}(R)$ in the sense that

$$\langle \dot{F}'\ ,\ \dot{F}'\ \rangle' = {}^{F^{f-1}V^{-1}}\langle\ ,\ \rangle'$$

which follows immediately from the identities $\langle F\ , \dot{F}\ \rangle = \langle \dot{F}\ , F\ \rangle = {}^{F}\langle\ ,\ \rangle$ for the pairing $\langle\ ,\ \rangle$ on $\mathcal{P}$. In other words, the pairing defines a principal polarization of the f-$\mathcal{O}_K$-display $\mathcal{P}'$, see Proposition 11.5.

As explained in the appendix, (11.2), base change along the natural strict morphism of frames

$$\mathcal{A}_{\mathcal{O}_{E_0^u}/\mathcal{O}_K}(R) \longrightarrow (W_{\mathcal{O}_{E_0^u}}(R), I_{\mathcal{O}_{E_0^u}}(R), {}^{F'}, {}^{V'^{-1}})$$

defines a principally polarized strict formal $\mathcal{O}_{E_0^u}$-module $\mathcal{C}(X)$ together with a compatible $\mathcal{O}_E$-action. This module is of signature (r,s) and hence an element of $\mathcal{O}_E$-$\mathcal{O}_{E_0^u}$-$\mathit{Herm}(S)$.

Construction of a quasi-inverse of $\mathcal{C}$: Let $\mathcal{P}' := (P', Q', F', \dot{F}')$ be the f-$\mathcal{O}_K$-display associated to a hermitian $\mathcal{O}_E$-$\mathcal{O}_{E_0^u}$-module (X, ι, λ) over S. By functoriality, it comes with an $\mathcal{O}_E$-action and a compatible principal polarization. To construct the associated hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module, we apply a slightly modified version of the construction from the proof of Proposition 13.2.

Note that the $\mathcal{O}_E$-action induces a bigrading, $P' = P_0' \oplus P_1'$, $Q' = Q_0' \oplus Q_1'$. We set

$$\begin{aligned} P_{\psi_0} &:= P_0' \quad & P_{\sigma\psi_0} &:= P_1', \\ Q_{\psi_0} &:= Q_0' \quad & Q_{\sigma\psi_0} &:= Q_1'. \end{aligned}$$

For $i = 1, \ldots, f-1$, we define

$$P_{F^{i+1}\psi_0} := P^{(F)}_{F^i\psi_0}, \quad P_{F^{i+1}\sigma\psi_0} := P^{(F)}_{F^i\sigma\psi_0}.$$

The signature condition forces us to set, for $\psi \notin \{\psi_0, \sigma\psi_0\}$,

$$Q_\psi = \begin{cases} P_\psi & \text{if } \psi \in \Psi_0 \setminus \{\psi_0\} \\ I_{\mathcal{O}_K}(R) P_\psi & \text{if } \psi \in \Psi_1 \setminus \{\sigma\psi_0\}. \end{cases}$$

The display structure is defined by giving a normal decomposition. Let $(P' = L' \oplus T', \phi)$ be a normal decomposition of $\mathcal{P}'$. Then we define a normal decomposition $(P = L \oplus T, \Phi)$ as

$$\begin{aligned} L &= L_{\psi_0} \oplus L_{\sigma\psi_0} \oplus \bigoplus_{\psi \in \Psi_0 \setminus \{\psi_0\}} P_\psi, \\ T &= T_{\psi_0} \oplus T_{\sigma\psi_0} \oplus \bigoplus_{\psi \in \Psi_1 \setminus \{\sigma\psi_0\}} P_\psi \end{aligned}$$

and the ^{F}-linear operator

$$\Phi = \begin{pmatrix} & & & \phi \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \circ {}^F$$

with respect to the decomposition $P = (P_{\psi_0} \oplus P_{\sigma\psi_0}) \oplus (P_{\psi_0} \oplus P_{\sigma\psi_0})^{(F)} \oplus \ldots \oplus (P_{\psi_0} \oplus P_{\sigma\psi_0})^{(F^{f-1})}$. This already defines an f-$\mathcal{O}_K$-display $\mathcal{P} := (P, Q, F, \dot{F})$ equipped with an $\mathcal{O}_E$-action of the correct signature.

We now construct the polarization (as $\mathcal{O}_K$-display) on $\mathcal{P}$. Recall that we have a perfect pairing

$$\langle\ ,\ \rangle_{\psi_0} : P_{\psi_0} \times P_{\sigma\psi_0} \longrightarrow W_{\mathcal{O}_K}(R).$$

There is at most one way to extend this pairing to a pairing $\langle\ ,\ \rangle$ on all of P such that the relations of a polarization are satisfied. Let us explain this for the direct summand $P_{^F\psi_0} \oplus P_{^F\sigma\psi_0}$. If $l + t \in L_{\psi_0} \oplus T_{\psi_0}$ and $l' + t' \in L_{\sigma\psi_0} \oplus T_{\sigma\psi_0}$, then we have to set

$$\begin{aligned} \langle \Phi(l+t), \Phi(l'+t') \rangle &:= \langle \dot{F}(l) + F(t), \dot{F}(l') + F(t') \rangle \\ &= {}^{V^{-1}}\langle l, l' \rangle_{\psi_0} + {}^{F}\langle l, t' \rangle_{\psi_0} + {}^{F}\langle t, l' \rangle_{\psi_0} + \pi {}^{F}\langle t, t' \rangle_{\psi_0}. \end{aligned}$$

Since Φ is a Frobenius-linear isomorphism, this is well-defined and extends in a unique way to all of $P_{^F\psi_0} \oplus P_{^F\sigma\psi_0}$. We apply the same formulas to define $\langle\ ,\ \rangle$ on $P_{^{F^{i+1}}\psi_0} \oplus P_{^{F^{i+1}}\sigma\psi_0}$ for $i = 1, \ldots, f-2$. Note that due to the special form of the normal decomposition at these indices, we get

$$\langle\ ,\ \rangle|_{P_{^{F^{i+1}}\psi_0} \oplus P_{^{F^{i+1}}\sigma\psi_0}} = \langle\ ,\ \rangle|^{(F)}_{P_{^{F^{i}}\psi_0} \oplus P_{^{F^{i}}\sigma\psi_0}}.$$

We leave it to the reader to check that this defines a principal polarization on $\mathcal{P}$ which finishes the proof. $\square$

Remark 4.5. Note that we did not use the assumption of R being noetherian. We will only need this assumption in the ramified situation.

4.2 The totally ramified case

Proposition 4.6. *Let $\mathbb{Q}_p \subset K \subset E_0$ be an intermediate field such that E_0/K is totally ramified. There is an isomorphism of stacks over* $\mathrm{Sch}/\operatorname{Spf} \mathcal{O}_{\breve{E}}$

$$\mathcal{C} : \mathcal{O}_E\text{-}\mathcal{O}_K\text{-}Herm \xrightarrow{\cong} \mathcal{O}_E\text{-}Herm$$

that is equivariant for the Rosati involution and that sends objects of signature (r, s) to objects of signature (r, s).

Proof. The proof consists of three main steps. First, we will construct the functor $\mathcal{C}$. Second, we will prove that $\mathcal{C}$ is an equivalence on reduced $\mathcal{O}_{\breve{E}}$-schemes in characteristic p. Finally, we will prove that $\mathcal{C}$ identifies the deformation theories of X and $\mathcal{C}(X)$. Together with the restriction to noetherian schemes, this will imply the statement.

First step: Construction of the functor $\mathcal{C}$.

Let $S = \operatorname{Spec} R$ be a scheme over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$. Before we begin the construction of $\mathcal{C}$, we spell out the properties of the $\mathcal{O}_K$-display of a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module (X, ι, λ) of signature (r, s) over S. We use the notation from Definition 11.7 and from the beginning of Section 12. We also identify Ψ with $\{0, 1\}$ such that ψ_0 corresponds to 0.

Let $\mathcal{P} := (P, Q, F, \dot{F})$ be the $\mathcal{O}_K$-display of X. The action ι of $\mathcal{O}_E$ on X induces an action of $\mathcal{O}_E$ on the display. This makes P and Q into $\mathcal{O}_E \otimes W_{\mathcal{O}_K}(R)$-modules. Both F and $\dot{F}$ are $\mathcal{O}_E$-linear. The action of the unramified part induces bigradings $P = P_0 \oplus P_1$ and $Q = Q_0 \oplus Q_1$ such that both F and $\dot{F}$ are homogeneous of degree 1.

The signature condition implies that P_0/Q_0 is projective of rank r over R and that P_1/Q_1 is projective of rank $ne-r$ over R. Furthermore, $\mathcal{O}_E$ acts on P_0/Q_0 via the structure morphism. In other words,

$$J_{\mathcal{O}_{E_0}}(R)P_0 \subset Q_0.$$

Recall that by [2, Prop. 2.22], the module P is projective of rank $2n$ over $\mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} W_{\mathcal{O}_K}(R)$.

The polarization induces a perfect alternating form $\langle\ ,\ \rangle : P \times P \longrightarrow W_{\mathcal{O}_K}(R)$ which satisfies $\langle a\ ,\ \rangle = \langle\ , a^\sigma\ \rangle$ for all $a \in \mathcal{O}_E$. In particular, P_0 and P_1 are maximal isotropic subspaces of P which are put into duality by $\langle\ ,\ \rangle$.

Furthermore, $\langle Q, Q\rangle \subset I_{\mathcal{O}_K}(R)$ and the pairing satisfies $\langle \dot{F}\ , \dot{F}\ \rangle = {}^{V^{-1}}\langle\ ,\ \rangle$. In other words, the pairing takes values in the Witt $\mathcal{O}_K$-frame from Definition 11.7,

$$\mathcal{W}_{\mathcal{O}_K}(R) = (W_{\mathcal{O}_K}(R), I_{\mathcal{O}_K}(R), R, {}^F, {}^{V^{-1}}).$$

Construction of $\mathcal{C}(X)$*:* Let $(X, \iota, \lambda)/S$ and $\mathcal{P}$ be as above. As an intermediate step, we construct a polarized window $\mathcal{P}' = (P', Q', F', \dot{F}')$ over a Lubin-Tate frame $\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K}(R)$ together with an $\mathcal{O}_E$-action, see Definition 12.7.

We set $P' := P$ with its given $\mathcal{O}_E$-action. Let ϑ_K be a generator of the inverse different of E_0/K. Consider the $W_{\mathcal{O}_K}(R)$-linear extension of the trace

$$\begin{aligned} \mathbf{tr} : \mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} W_{\mathcal{O}_K}(R) &\longrightarrow W_{\mathcal{O}_K}(R) \\ a \otimes w &\longmapsto \mathrm{tr}_{E_0/K}(\vartheta_K a)w. \end{aligned}$$

Since $\langle\ ,\ \rangle$ is $\mathcal{O}_{E_0}$-equivariant, there is a unique lifting of this form to a perfect $\mathcal{O}_{E_0} \otimes W_{\mathcal{O}_K}(R)$-bilinear alternating form $(\ ,\ ) : P \times P \longrightarrow \mathcal{O}_{E_0} \otimes W_{\mathcal{O}_K}(R)$ such that

$$\langle\ ,\ \rangle = \mathbf{tr} \circ (\ ,\ ).$$

Again, P' is bigraded and we set $Q'_0 := Q_0$. The form $(\ ,\ )$ automatically satisfies $(a\ ,\ ) = (\ , a^\sigma\ )$ and hence Q_0 is totally isotropic. We define

$$Q'_1 := \{p_1 \in P_1 \mid (p_1, Q_0) \subset J_{\mathcal{O}_{E_0}}(R)\}.$$

Note that $(\ ,\ )$ induces a perfect pairing $(P/J_{\mathcal{O}_{E_0}}(R)P) \times (P/J_{\mathcal{O}_{E_0}}(R)P) \longrightarrow R$ by base change. Then Q'_1 is the inverse image of $(Q_0/J_{\mathcal{O}_{E_0}}(R)P_0)^\perp$ under the projection $P_1 \longrightarrow P_1/J_{\mathcal{O}_{E_0}}(R)P_1$. In particular, it is obvious that P_1/Q'_1 is a projective R-module of rank $s = n - r$.

Let $\theta \in \mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} W_{\mathcal{O}_K}(\mathcal{O}_{E_0})$ be an element as in Lemma 12.5.

Lemma 4.7. *There is the relation*

$$\theta Q'_1 \subset Q_1.$$

Proof. By definition, $Q_1 = \{p_1 \in P_1 \mid \langle p_1, Q_0\rangle \subset I_{\mathcal{O}_K}(R)\}$. So given $q'_1 \in Q'_1$, we need to verify that

$$\begin{aligned} \langle \theta q'_1, Q_0\rangle &= \mathbf{tr}\,((\theta q'_1, Q_0)) \\ &= \mathbf{tr}\,(\theta(q'_1, Q_0)) \subset I_{\mathcal{O}_K}(R). \end{aligned}$$

For the second equality, we used that $(\ ,\ )$ is $\mathcal{O}_{E_0} \otimes W_{\mathcal{O}_K}(R)$-bilinear. It is enough to show

$$\mathbf{tr}(\theta J_{\mathcal{O}_{E_0}}(R)) \subset I_{\mathcal{O}_K}(R)$$

which follows from the fact that

$$\theta J_{\mathcal{O}_{E_0}}(R) \subset I_{\mathcal{O}_K}(R).$$

□

In particular, we can define $\dot{F}_1' : Q_1' \longrightarrow P_0$ as $\dot{F}_1'(q_1) := \dot{F}_1(\theta q_1)$. Then $\dot{F}_1'$ is a Frobenius-linear epimorphism, which can be checked at closed points of S.

Let $\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K,\kappa}(R)$ be the Lubin-Tate $\mathcal{O}_{E_0}$-frame from Example 12.9 (2). That is,

$$\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K,\kappa}(R) = (\mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} W_{\mathcal{O}_K}(R), J_{\mathcal{O}_{E_0}}(R), R, \sigma, \dot{\sigma})$$

where $\dot{\sigma}(\xi) = {}^{V^{-1}}(\theta\xi)$. The unit $\kappa \in \mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} W_{\mathcal{O}_K}(R)$ is the element $\dot{\sigma}(\pi_E \otimes 1 - 1 \otimes [\pi_E])$ where $\pi_E \in \mathcal{O}_{E_0}$ is our fixed uniformizer.

We define the Frobenius $F' : P \longrightarrow P$ through the relation

$$F'(x) = \kappa^{-1}\dot{F}'((\pi_E \otimes 1 - 1 \otimes [\pi_E])x).$$

The reader may check that this defines the structure of a $\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K,\kappa}(R)$-window $\mathcal{P}' = (P', Q', F', \dot{F}')$.

Lemma 4.8. *The pairing $(\ ,\ )$ is a principal polarization of the $\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K,\kappa}(R)$-window $\mathcal{P}'$.*

Proof. By definition of Q', the pairing satisfies $(Q', Q') \subset J_{\mathcal{O}_{E_0}}(R)$. We verify for $q_0 \in Q_0', q_1 \in Q_1'$ that

$$\begin{aligned}(\dot{F}'q_0, \dot{F}'q_1) &= (\dot{F}q_0, \dot{F}(\theta q_1)) \\ &= {}^{V^{-1}}(q_0, \theta q_1) \\ &= {}^{V^{-1}}(\theta(q_0, q_1)) = \dot{\sigma}(q_0, q_1).\end{aligned} \tag{4.1}$$

Here, the second equality holds since it does for the pairing $\langle\ ,\ \rangle$. The last equality used the $\mathcal{O}_{E_0}$-bilinearity of the pairing $(\ ,\ )$. □

By Proposition 12.10, there is a strict morphism of $\mathcal{O}_{E_0}$-frames,

$$\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K,\kappa}(R) \longrightarrow \mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_{E_0},\kappa}(R).$$

Base change along this morphism defines a supersingular strict $\mathcal{O}_{E_0}$-module with $\mathcal{O}_E$-action and a principal polarization with values in the $\mathcal{O}_{E_0}$-frame $\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_{E_0},\kappa}(R)$.

The identity on $W_{\mathcal{O}_{E_0}}(R)$ defines a κ/u-isomorphism to the Witt $\mathcal{O}_{E_0}$-frame

$$\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_{E_0},\kappa}(R) \longrightarrow \mathcal{W}_{\mathcal{O}_{E_0}}(R)$$

where u is the unit $u = V_{\pi_E}^{-1}(\pi_E - [\pi_E])$. There exists a unit $\varepsilon \in W_{\mathcal{O}_{E_0}}(\mathcal{O}_{\breve{E}})$ such that

$$\sigma(\varepsilon)\varepsilon^{-1} = \kappa/u.$$

We now apply Lemma 11.2, to get the associated principally polarized $\mathcal{O}_{E_0}$-display $\mathcal{P}'' = (P'', Q'', F'', \dot{F}'')$ with $\mathcal{O}_E$-action. The corresponding strict formal $\mathcal{O}_{E_0}$-module is then the hermitian $\mathcal{O}_E$-module $\mathcal{C}(X)$ we wanted to construct.

It is obvious that $\mathcal{C}$ is functorial in a way compatible with the Rosati involutions.

Second step: $\mathcal{C}$ is an equivalence over reduced schemes in characteristic p.

We will construct a quasi-inverse. Let R be a reduced $\operatorname{Spf} \mathcal{O}_{\breve{E}}$-scheme and let $\mathcal{P}' = (P', Q', F', \dot{F}')$ be the $\mathcal{L}_{\mathcal{O}_{E_0}/\mathcal{O}_K,\kappa}(R)$-frame equipped with an $\mathcal{O}_E$-action ι and a principal polarization λ associated to a hermitian $\mathcal{O}_E$-module over R. We assume that $\mathcal{P}'$ is of signature (r, s). Here, κ is the unit ${}^{V^{-1}}(\theta(\pi_E \otimes 1 - 1 \otimes [\pi_E]))$ from the previous paragraph.

The $\mathcal{O}_E$-action induces gradings $P' = P_0' \oplus P_1'$ and $Q' = Q_0' \oplus Q_1'$. We set $P := P'$ together with its $\mathcal{O}_E$-action.

The polarization λ induces a perfect pairing

$$(\ , \ ) : P_0 \times P_1 \longrightarrow \mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} W_{\mathcal{O}_K}(R)$$

such that

$$Q_1' = \{p_1 \in P_1 \mid (p_1, Q_0') \subset J_{\mathcal{O}_{E_0}}(R)\}.$$

We set $Q_0 := Q_0'$ and

$$Q_1 := \{p_1 \in P_1 \mid (p_1, Q_0) \subset \mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} I_{\mathcal{O}_K}(R)\}.$$

Claim: $Q_1 = \theta Q_1' + I_{\mathcal{O}_K}(R) P_1$.
The relation $\supseteq$ is clear, so we only need to check that $Q_1 \subseteq \theta Q_1'$ mod $I_{\mathcal{O}_K}(R) P_1$. Let us denote by $\overline{P}_0$, $\overline{Q}_0$, etc. the quotients of the various modules by $I_{\mathcal{O}_K}(R)P$. Then the pairing $(\ , \ )$ induces a perfect pairing

$$\overline{P}_0 \times \overline{P}_1 \longrightarrow \mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} R \cong \mathcal{O}_{E_0}/\pi_k \otimes_{\mathcal{O}_K/\pi_k} R.$$

Here, we used that $\pi_K = 0$ in R. We have

$$\begin{aligned} \pi_E \overline{P}_0 \subset \overline{Q}_0 \subset \overline{P}_0 \qquad & \text{and} \\ \pi_E \overline{P}_1 \subset \overline{Q}_1' \subset \overline{P}_1. & \end{aligned}$$

Then by definition, $\overline{Q}_1$ is the orthogonal complement of $\overline{Q}_0$. In particular, it is projective of rank r over R and locally a direct summand of $\overline{P}_1$. Let $\bar{\theta}$ be the image of θ in $\mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi_K \cong \mathcal{O}_{E_0}/\pi_K$. Then $\bar{\theta} \neq 0$, $\pi_E \bar{\theta} = 0$ and multiplication by $\bar{\theta}$ induces an isomorphism

$$\mathcal{O}_{E_0}/\pi_E \otimes R \cong (\pi_E^{e-1})/\pi_K \otimes R.$$

Thus $\bar{\theta}\overline{Q}_1'$ is also of rank r over R and locally a direct summand of $\overline{P}_1$. The claim follows from the relation $\bar{\theta}\overline{Q}_1' \subset \overline{Q}_1$.

Since R is reduced, $W_{\mathcal{O}_K}(R)$ is π_K-torsion free. Also, there exists an inclusion $R \hookrightarrow \prod_{i \in I} k_i$ into a product of perfect fields and hence $\sigma(\theta)$ is not a zero-divisor in $\mathcal{O}_{E_0} \otimes W_{\mathcal{O}_K}(R)$. It follows from the claim that $\dot{F}'|_{Q_1}$ is divisible by $\sigma(\theta)$. Thus we can define a σ-linear epimorphism $\dot{F} : Q_0 \oplus Q_1 \longrightarrow P_0 \oplus P_1$ as

$$\dot{F} = \dot{F}'|_{Q_0} \oplus \sigma(\theta)^{-1} \dot{F}'|_{Q_1}.$$

We leave it to the reader to check that $(P, Q, F, \dot{F})$ is an $\mathcal{O}_K$-display with $\mathcal{O}_E$-action of signature (r, s) and that $\langle \ , \ \rangle := \mathbf{tr} \circ (\ , \ )$ defines a principal polarization, compatible with the $\mathcal{O}_E$-action. This finishes the construction of the quasi-inverse.

Third step: Identifying the deformation theories of X and $\mathcal{C}(X)$.

Let X be a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module over S. Denote by $\mathbb{D}$ (resp. by $\mathbb{D}''$) the $\mathcal{O}_K$-crystal of X (resp. the $\mathcal{O}_{E_0}$-crystal of $\mathcal{C}(X)$) on the category of $\mathcal{O}_K$-pd-thickenings of S (resp. on the category of $\mathcal{O}_{E_0}$-pd-thickenings of S). Note that both crystals are bigraded by the $\mathcal{O}_E$-action and that in the notation of Step 1 above, $\mathbb{D}(S) = P/I_{\mathcal{O}_K}(R)P$ and $\mathbb{D}''(S) = P''/I_{\mathcal{O}_{E_0}}(R)P''$. Furthermore, the Hodge filtration $\mathcal{F} \subset \mathbb{D}(S)$ is given by $Q/I_{\mathcal{O}_K}(R)P$.

We denote by $\mathcal{D}$ the contractions of the crystal $\mathbb{D}$,

$$\mathcal{D}(\tilde{S}) := \mathbb{D}(\tilde{S}) \otimes_{\mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} \mathcal{O}_{\tilde{S}}} \mathcal{O}_{\tilde{S}},$$

for any $\mathcal{O}_K$-pd-thickening $S \longrightarrow \tilde{S}$. Let $\bar{J}$ be the kernel of the projection $\mathcal{O}_{E_0} \otimes_{\mathcal{O}_K} \mathcal{O}_S \longrightarrow \mathcal{O}_S$. Then there are inclusions $\bar{J}\mathbb{D}(S) \subset \mathcal{F} \subset \mathbb{D}(S)$ and we have $\mathcal{D}(S) = \mathbb{D}(S)/\bar{J}\mathbb{D}(S)$. Hence we get a filtration $\mathcal{F}/\bar{J}\mathbb{D}(S) \subset \mathcal{D}(S)$ which we call the Hodge filtration on $\mathcal{D}(S)$.

Lemma 4.9. *Let $S \longrightarrow \tilde{S}$ be a square-zero thickening. There is an $\mathcal{O}_E$-linear identification of the contraction $\mathcal{D}$ of the crystal of X and the 0-component of the crystal of $\mathcal{C}(X)$ evaluated at $\tilde{S}$,*

$$\mathcal{D}(\tilde{S}) \cong \mathcal{D}_0''(\tilde{S}).$$

This identification is functorial in X. In the case $S = \tilde{S}$, the Hodge filtrations on both sides agree.

Corollary 4.10. *Let $S \longrightarrow \tilde{S}$ be a square-zero thickening and let X be a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module over S. Then there is a natural bijection between deformations of X to $\tilde{S}$ and deformations of $\mathcal{C}(X)$ to $\tilde{S}$.*

Furthermore if Y is another $\mathcal{O}_E$-$\mathcal{O}_K$-module over S and if $\tilde{X}, \tilde{Y}$ are deformation of the two modules to $\tilde{S}$, then an $\mathcal{O}_E$-linear homomorphism $f : X \longrightarrow Y$ lifts to $\tilde{X} \longrightarrow \tilde{Y}$ if and only if $\mathcal{C}(f)$ lifts to $\mathcal{C}(\tilde{X}) \longrightarrow \mathcal{C}(\tilde{Y})$.

Proof. The Lemma is clear for $S = \tilde{S}$. Namely let $(P, Q, F, \dot{F})$ be the $\mathcal{O}_K$-display of X and let $(P'', Q'', F'', \dot{F}'')$ be the $\mathcal{O}_{E_0}$-display of $\mathcal{C}(X)$. By construction,

$$\mathcal{D}(S) = (P_0/J_{\mathcal{O}_{E_0}}(R)P_0) = (P_0'/J_{\mathcal{O}_{E_0}}(R)P_0') = P_0''/I_{\mathcal{O}_{E_0}}(R)P_0'' = \mathbb{D}_0''(S).$$

The submodule $Q_0'' \subset P_0''$ is the inverse image of the Hodge filtration

$$(Q_0 + J_{\mathcal{O}_{E_0}}(R)P_0)/I_{\mathcal{O}_K}(R)P_0 \subset P_0/I_{\mathcal{O}_K}(R)P_0$$

and hence the Hodge filtrations on both sides agree.

For a non-trivial square-zero thickening $S \longrightarrow \tilde{S}$, we argue as follows. Locally on S, we can deform X to a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module $\tilde{X}$ of signature (r, s) on $\tilde{S}$. Then $\mathcal{C}(\tilde{X})$ is a deformation of $\mathcal{C}(X)$. The values $\mathbb{D}(\tilde{S})$ and $\mathbb{D}''(\tilde{S})$ can then be computed from the displays of $\tilde{X}$ and $\mathcal{C}(\tilde{X})$ and the above arguments apply.

The corollary is now an immediate application of Grothendieck-Messing deformation theory. Namely as explained in the proof of Proposition 2.14, deformations of X (resp. of $\mathcal{C}(X)$) are in bijection with liftings of the Hodge filtration in $\mathcal{D}(S)$ (resp. of the Hodge filtration in $\mathbb{D}_0''(S)$). A similar result holds for homomorphisms. □

End of Proof: Now let R be noetherian and let $\mathfrak{n} := \ker(R \longrightarrow (R/p)_{\mathrm{red}})$. Then R is $\mathfrak{n}$-adically complete. Applying Corollary 4.10 to the successive quotients $R/\mathfrak{n}^{i+1} \longrightarrow R/\mathfrak{n}^i$, we get both the essential surjectivity and fully faithfulness of $\mathcal{C}$ on R-valued points. This finishes the proof of the proposition and hence of Theorem 4.3. □

4.3 Proof of Theorem 4.1

We now fix a quasi-isogeny on framing objects,

$$\alpha : \mathcal{C}(\mathbb{X}_{E_0/K,(r,s)}) \cong \mathbb{X}_{E_0,(r,s)}. \tag{4.2}$$

Alternatively, we *choose* $\mathcal{C}(\mathbb{X}_{E_0/K,(r,s)})$ as the framing object in the definition of $\mathcal{N}_{E_0,(r,s)}$. Together with $\mathcal{C}$, this induces a morphism

$$c : \mathcal{N}_{E_0/K,(r,s)} \longrightarrow \mathcal{N}_{E_0,(r,s)}$$

which is is an isomorphism by Theorem 4.3. Furthermore, c is equivariant with respect to the isomorphism

$$\begin{aligned} \mathcal{C} : U(\mathbb{X}_{E_0/K,(r,s)}) &\longrightarrow U(\mathcal{C}(\mathbb{X}_{E_0/K,(r,s)})) \\ g &\longmapsto \alpha g \alpha^{-1}. \end{aligned}$$

$\square$

5 Cycles in unitary RZ-spaces

Let $\mathbb{Q}_p \subset K \subset E_0$ be finite extensions and let E/E_0 be an unramified quadratic extension. We choose uniformizers $\pi_K \in \mathcal{O}_K$ and $\pi_E \in \mathcal{O}_{E_0}$ in order to make sense of polarizations of strict $\mathcal{O}_K$-modules (resp. strict $\mathcal{O}_{E_0}$-modules).

Let $\mathbb{X}_{E_0/K,(r,s)}$ over $\mathbb{F}$ be the framing object for $\mathcal{N}_{E_0/K,(r,s)}$. Recall that $\mathrm{End}^0_E(\mathbb{X}_{E_0/K,(r,s)}) \cong M_n(E)$ is isomorphic to a matrix ring over E, see Proposition 2.5.

Definition 5.1. (1) For a quasi-endomorphism $x \in \mathrm{End}^0_E(\mathbb{X}_{E_0/K,(r,s)})$, we denote by $\mathcal{Z}(x) \subset \mathcal{N}_{E_0/K,(r,s)}$ the closed formal subscheme of objects (X, ρ) such that the quasi-endomorphism $\rho^{-1}x\rho$ is an actual endomorphism, see [16, Proposition 2.9].
(2) For a subring $R \subset \mathrm{End}^0_E(\mathbb{X}_{E_0/K,(r,s)})$, we denote by $\mathcal{Z}(R) \subset \mathcal{N}_{E_0/K,(1,n-1)}$ the closed formal subscheme of objects (X, ρ) such that $\rho^{-1}R\rho \subset \mathrm{End}(X)$. In other words,

$$\mathcal{Z}(R) = \bigcap_{x \in R} \mathcal{Z}(x).$$

Remark 5.2. Note that $\mathcal{Z}(x) = \mathcal{Z}(x^*)$ where $*$ denotes the Rosati involution, see Definition 2.3. Also, $\mathcal{Z}(x)$ only depends on the $\mathcal{O}_E$-algebra $\mathcal{O}_E[x]$ spanned by x. In particular, there are equalities $\mathcal{Z}(x) = \mathcal{Z}(\mathcal{O}_E[x, x^*])$ and

$$\mathcal{Z}(R) = \mathcal{Z}(\mathcal{O}_E[R, R^*]).$$

The second kind of cycle we want to consider is defined as follows. Let $\mathbb{Y}_{E_0/K,(0,1)}$ be a hermitian $\mathcal{O}_E$-$\mathcal{O}_K$-module over $\mathbb{F}$ of signature $(0,1)$. Such an object is unique up to isomorphism.[7] Let $\mathcal{Y}_{E_0/K,(0,1)}$ be a deformation of it to $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$. Such a deformation is unique up to isomorphism according to Proposition 2.14. Via base change, $\mathcal{Y}_{E_0/K,(0,1)}$ is defined on any $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$-scheme.

Definition 5.3. For any quasi-homomorphism $j \in \mathrm{Hom}^0_E(\mathbb{Y}_{E_0/K,(0,1)}, \mathbb{X}_{E_0/K,(r,s)})$, we denote by $\mathcal{Z}(j) \subset \mathcal{N}_{E_0/K,(r,s)}$ the closed formal subscheme of pairs (X, ρ) such that the quasi-homomorphism

$$\rho^{-1} j : \mathcal{Y}_{E_0/K,(0,1)} \longrightarrow X$$

is an actual homomorphism.

[7] This can be checked with Dieudonné theory.

Again, the existence of such a closed formal subscheme follows from [16, Proposition 2.9]. As above, $\mathcal{Z}(j) = \mathcal{Z}(j^*)$ where $j^* : \mathbb{X}_{E_0/K,(r,s)} \longrightarrow \mathbb{Y}_{E_0/K,(0,1)}$ is the Rosati adjoint of j and where $\mathcal{Z}(j^*)$ denotes the locus of (X, ρ) such that $j^*\rho : X \longrightarrow \mathcal{Y}_{E_0/K,(r,s)}$ is a homomorphism. Also, $\mathcal{Z}(j)$ only depends on the span $\mathcal{O}_E j$.

The case of interest for the cycles $\mathcal{Z}(j)$ is that of signature $(1, n-1)$. In this case, $\mathcal{Z}(j) \subset \mathcal{N}_{E_0/K,(1,n-1)}$ is a divisor whenever $j \neq 0$. These divisors were first considered by Kudla and Rapoport, see [9].

Remark 5.4. Let $\mathcal{C}$ be the functor from Theorem 4.3 and let us choose a quasi-isogeny (E-linear, preserving the polarization)

$$\alpha : \mathcal{C}(\mathbb{X}_{E_0/K,(r,s)}) \xrightarrow{\cong} \mathbb{X}_{E_0,(r,s)}$$

and an isomorphism (E-linear, preserving the polarization)

$$\beta : \mathcal{C}(\mathcal{Y}_{E_0/K,(0,1)}) \xrightarrow{\cong} \mathcal{Y}_{E_0,(0,1)}.$$

It then follows from Theorem 4.3 and the construction of c in Theorem 4.1 that

$$c : \mathcal{Z}(x) \cong \mathcal{Z}(\alpha c(x) \alpha^{-1})$$

and

$$c : \mathcal{Z}(j) \cong \mathcal{Z}(\alpha c(j) \beta^{-1}).$$

From now on, we will restrict to the case of signature $(1, n-1)$. Moreover, to simplify the exposition, we will always assume $K = \mathbb{Q}_p$. The generalization to strict formal $\mathcal{O}_K$-modules is left to the reader.

5.1 The cycle $\mathcal{Z}(\mathcal{O}_E)$

We assume that d divides n, say $n = dn'$. We also assume that σ is non-trivial on $\mathbb{Q}_{p^2} \subset E$. In other words, the inertia degree f of E_0 over $\mathbb{Q}_p$ is odd. Let $\mathbb{X}_{\mathbb{Q}_p,(1,n-1)}$ be the framing object for $\mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$, see Notation 3.4. We fix an embedding

$$E \hookrightarrow \mathrm{End}^0_{\mathbb{Q}_{p^2}}(\mathbb{X}_{\mathbb{Q}_p,(1,n-1)})$$

such that the Rosati involution preserves E and agrees with σ. Such an embedding exists since $d \mid n$. Associated to the embedding, we have a cycle $\mathcal{Z}(\mathcal{O}_E) \subset \mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$.

We let $\Psi = \mathrm{Hom}_{\mathbb{Q}_p}(E^u, \breve{\mathbb{Q}}_p) = \Psi_0 \sqcup \Psi_1$ be the unique extension of the decomposition $\{0,1\} = \{0\} \sqcup \{1\}$ for $\mathbb{Q}_{p^2}$. Together with the action of $\mathcal{O}_E$, $\mathbb{X}_{\mathbb{Q}_p,(1,n-1)}$ becomes a framing object for $\mathcal{N}_{E_0/\mathbb{Q}_p,(1,n'-1)}$. Recall that in the definition of the signature condition 2.8, we also had to choose an element $\psi_0 \in \Psi_0$. Let us denote the associated RZ-space by $\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)}$. Then forgetting the E-action induces a morphism

$$\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)} \longrightarrow \mathcal{N}_{\mathbb{Q}_p,(1,n-1)} \tag{5.1}$$

which is a closed immersion.

Theorem 5.5. *The above forgetful map induces an isomorphism*

$$\coprod_{\psi_0 \in \Psi_0} \mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)} \xrightarrow{\cong} \mathcal{Z}(\mathcal{O}_E).$$

Proof. It is obvious that the arrow is a monomorphism with image contained in $\mathcal{Z}(\mathcal{O}_E)$. So we only have to show $\mathcal{Z}(\mathcal{O}_E) \subset \coprod \mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)}$. We do this on S-valued points with S connected.

Let $(X, \iota, \lambda, \rho) \in \mathcal{Z}(\mathcal{O}_E)(S)$. Then the Lie algebra decomposes as

$$\mathrm{Lie}(X) = \prod_{\psi \in \Psi} \mathrm{Lie}(X)_\psi$$

and this decomposition extends the bigrading from the $\mathbb{Z}_{p^2}$-action. Let $\psi_0 \in \Psi_0$ be the unique index (amongst the indices in Ψ_0) such that $\mathrm{Lie}(X)_{\psi_0} \neq 0$. This summand is then a line bundle on S. The $\mathcal{O}_E$-action on $\mathrm{Lie}(X)_{\psi_0}$ induces a morphism

$$S \longrightarrow \operatorname{Spf} \mathcal{O}_E \otimes_{\mathcal{O}_{E^u},\psi_0} \mathcal{O}_{\breve{\mathbb{Z}}_{p^2}} = \operatorname{Spf} \mathcal{O}_{\breve{E}}.$$

(This explains why we restricted ourselves to the case of signature $(1, n-1)$.) Then by definition, X is an S-valued point of $\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)}$. □

Remark 5.6. Note that this construction is compatible with the formation of the cycle $\mathcal{Z}(R)$ in the following sense. Assume that $R \subset \mathrm{End}^0_{\mathbb{Q}_{p^2}}(\mathbb{X}_{\mathbb{Q}_p,(1,n-1)})$ is such that $\mathcal{O}_E \subset R$. Then

$$\mathcal{Z}(R) \subset \mathcal{Z}(\mathcal{O}_E)$$

and

$$c : \coprod_{\psi_0 \in \Psi_0} \mathcal{Z}(R)^{\psi_0} \cong \mathcal{Z}(R).$$

Here, the source are the cycles in the RZ-spaces $\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)}$. Furthermore, the $\mathcal{Z}(R)^{\psi_0}$ are all isomorphic. This can be seen by identifying the $\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)}$ with $\mathcal{N}_{E_0,(1,n'-1)}$ and using Remark 5.4.

We fix a generator ϑ_E of the inverse different of $E_0/\mathbb{Q}_p$ which induces an $\mathcal{O}_{E_0}$-linear isomorphism

$$\phi : \mathcal{O}_{E_0} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{E_0}, \mathbb{Z}_p),\ a \mapsto \phi(a)(-) := \mathrm{tr}_{\mathrm{E_0/\mathbb{Q}_p}}(\vartheta_E a-).$$

Definition 5.7. To any hermitian $\mathbb{Z}_{p^2}$-module (Y, ι, λ), we associate a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module $(\mathcal{O}_{E_0} \otimes Y, \iota', \lambda')$ as follows. The p-divisible group $\mathcal{O}_{E_0} \otimes Y$ is given by the Serre tensor construction. The $\mathcal{O}_E = \mathcal{O}_{E_0} \otimes \mathbb{Z}_{p^2}$-action ι' is simply given by the natural $\mathcal{O}_{E_0}$-action on the first factor and λ' is defined as,

$$\lambda'(a \otimes y) := \phi(a) \otimes \lambda(y).$$

Let $j \in \mathrm{Hom}^0_{\mathbb{Q}_{p^2}}(\mathbb{Y}_{\mathbb{Q}_p,(0,1)}, \mathbb{X}_{\mathbb{Q}_p,(1,n-1)})$ be a quasi-homomorphism with associated divisor $\mathcal{Z}(j) \subset \mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$.

Lemma 5.8. *There is an isomorphism of hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules over $\mathbb{F}$,*

$$\mathcal{O}_{E_0} \otimes \mathbb{Y}_{\mathbb{Q}_p,(0,1)} \cong \mathbb{Y}_{E_0/\mathbb{Q}_p,(0,1)}.$$

Similarly for their deformations to $\operatorname{Spf} \mathcal{O}_{\breve{E}}$,

$$\mathcal{O}_{E_0} \otimes \mathcal{Y}_{\mathbb{Q}_p,(0,1)} \cong \mathcal{Y}_{E_0/\mathbb{Q}_p,(0,1)}.$$

Proof. Up to isomorphism, there is a unique $\mathcal{O}_E$-$\mathbb{Z}_p$-module of signature $(0,1)$ over $\mathbb{F}$. So the statement reduces to the fact that $\mathcal{O}_{E_0} \otimes \mathbb{Y}_{\mathbb{Q}_p,(0,1)}$ has signature $(0,1)$ which is clear. $\square$

Lemma 5.9. *Let $\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)} \longrightarrow \mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ be the embedding from (5.1). Then*

$$\mathcal{Z}(j) \cap \mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)} = \mathcal{Z}(\mathrm{id}_{\mathcal{O}_{E_0}} \otimes j)$$

where $\mathrm{id}_{\mathcal{O}_{E_0}} \otimes j$ is the homomorphism

$$\mathbb{Y}_{E_0/\mathbb{Q}_p,(0,1)} \cong \mathcal{O}_{E_0} \otimes \mathbb{Y}_{\mathbb{Q}_p,(0,1)} \xrightarrow{\mathrm{id}\otimes j} \mathbb{X}_{\mathbb{Q}_p,(1,n-1)}.$$

Proof. The universal p-divisible group over $\mathcal{N}^{\psi_0}_{E_0/\mathbb{Q}_p,(1,n'-1)}$ has an $\mathcal{O}_{E_0}$-action, which implies the relation $\subseteq$. Conversely if $\mathrm{id} \otimes j : \mathcal{O}_{E_0} \otimes \mathbb{Y}_{\mathbb{Q}_p,(0,1)} \longrightarrow \mathbb{X}_{\mathbb{Q}_p,(1,n-1)}$ lifts, then so does its composition with

$$\mathbb{Y}_{\mathbb{Q}_p,(0,1)} \xrightarrow{1\otimes\mathrm{id}} \mathcal{O}_{E_0} \otimes \mathbb{Y}_{\mathbb{Q}_p,(0,1)}.$$

$\square$

5.2 Variant for étale algebras

We first define a variant of $\mathcal{N}_{E_0/\mathbb{Q}_p,(0,n)}$ for a split quadratic extension $E = E_0 \times E_0$. We set $\breve{E} := \breve{E}_0$. As in the non-split case, we choose a decomposition $\Psi := \mathrm{Hom}(E^u, \breve{E}) = \Psi_0 \sqcup \Psi_1$ such that $\sigma(\Psi_0) = \Psi_1$. The definition of a (supersingular) hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module (X, ι, λ) of signature $(0,n)$ over an $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$-scheme S is then completely analogous to the Definition 2.2 in the non-split case.

Note that the condition of X being supersingular poses some restrictions. More precisely, such hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-modules exist if and only if f is even and exactly half of the elements of Ψ_0 factor over the first component $E \longrightarrow E_0 \times 0$. From now on, we assume that this condition is satisfied.

Again, we fix a hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module $(\mathbb{X}_{E/E_0/\mathbb{Q}_p}, \iota, \lambda)$ of signature $(0,n)$ over $\mathbb{F}$ and again, such a choice is unique up to quasi-isogeny.

Definition 5.10. The functor $\mathcal{N}_{E/E_0/\mathbb{Q}_p,(0,n)}$ on schemes over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$ associates to S the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ where (X, ι, λ) is a supersingular hermitian $\mathcal{O}_E$-$\mathbb{Z}_p$-module of signature $(0,n)$ and where

$$\rho : X \times_S \overline{S} \longrightarrow \mathbb{X}_{E/E_0/\mathbb{Q}_p,(0,n)} \times_{\mathbb{F}} \overline{S}$$

is an $\mathcal{O}_E$-linear quasi-isogeny which preserves the polarizations.

As in the non-split case, we see that $\mathcal{N}_{E/E_0/\mathbb{Q}_p,(0,n)}$ is a formal scheme which is étale over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$.

Proposition 5.11. *Let E/E_0 be a quadratic étale algebra and let $U := U(\mathbb{X}_{E/E_0/\mathbb{Q}_p,(0,n)})$ (resp. K) denote the group of E-linear quasi-isogenies (resp. automorphisms) of the framing object that preserve the polarization. Then there is an U-equivariant isomorphism of formal schemes*

$$\mathcal{N}_{E/E_0/\mathbb{Q}_p,(0,n)} \cong \coprod_{U/K} \mathrm{Spf}\, \mathcal{O}_{\breve{E}}.$$

Proof. The group U acts on $\mathcal{N}_{E/E_0/\mathbb{Q}_p,(0,n)}$ by composition in the framing,

$$g.(X,\rho) = (X, g\rho).$$

The stabilizer of $(\mathbb{X}_{E/E_0/\mathbb{Q}_p,(0,n)}, \mathrm{id})$ is the subgroup $K \subset U$. The action of U is transitive on $\mathbb{F}$-points, see Lemma 2.11. (This lemma has an analogue for the split quadratic extension $E = E_0 \times E_0$.) Since $\mathcal{N}_{E/E_0/\mathbb{Q}_p,(0,n)}$ is étale over Spf $\mathcal{O}_{\breve{E}}$, the result follows. $\square$

Now let $E_0/\mathbb{Q}_p$ be any finite étale algebra and set $E := E_0 \otimes \mathbb{Q}_{p^2}$ with Galois conjugation $\sigma = \sigma \otimes \mathrm{id}$. Let

$$E_0 := \prod_{i \in I} E_{0,i}$$

be its decomposition into fields and set $E_i := E_{0,i} \otimes \mathbb{Q}_{p^2}$. We fix an embedding

$$E \hookrightarrow \mathrm{End}^0_{\mathbb{Q}_{p^2}}(\mathbb{X}_{\mathbb{Q}_p,(1,n-1)})$$

that is equivariant for σ and the Rosati-involution. This action makes the isocrystal $N := N(\mathbb{X}_{\mathbb{Q}_p,(1,n-1)})$ of $\mathbb{X}_{\mathbb{Q}_p,(1,n-1)}$ into an $E \otimes \breve{\mathbb{Q}}_p$-module. Let

$$N = \prod_{i \in I} N_i$$

be its decomposition into $E_i \otimes \breve{\mathbb{Q}}_p$-modules. This decomposition is preserved by the Frobenius and is orthogonal with respect to the skew-hermitian $\breve{\mathbb{Q}}_p$-valued form on N. Hence each factor N_i is a skew-hermitian $E_i/E_{0,i}$-isocrystal and we set $n_i := \mathrm{rk}_{E_i \otimes \breve{\mathbb{Q}}_p}(N_i)$.

Definition 5.12. An index $i \in I$ is called *odd* if $E_i/E_{0,i}$ is a field extension and if N_i is an odd skew-hermitian E_i-module, see Proposition 2.5. Otherwise, there exists a self-dual and Frobenius-stable $\mathcal{O}_{E_i} \otimes \breve{\mathbb{Z}}_p$-lattice of signature $(0, n_i)$ in N_i and we call i *even.*

An equivalent way to define the parity of an index is as follows. Let α be the operator from (2.2) and let $V := N^{\alpha=1}$. Then V is a skew-hermitian E-module and there is a decomposition $V = \prod_{i \in I} V_i$ such that each V_i is a skew-hermitian E_i-module. Then an index i is called even (resp. odd) if there exists (resp. does not exist) a self-dual $\mathcal{O}_{E_i}$-lattice in V_i.

Since N itself is odd, there is an odd number of odd indices. Recall that $\mathcal{Z}(\mathcal{O}_E) \subset \mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ denotes the locus to which the $\mathcal{O}_E$-action lifts.

Lemma 5.13. *If there is more than one odd index, then* $\mathcal{Z}(\mathcal{O}_E) = \emptyset$.

Proof. Using the idempotents in $\mathcal{O}_E = \prod_{i \in I} \mathcal{O}_{E_i}$, any point $(X, \iota, \lambda) \in \mathcal{Z}(\mathcal{O}_E)(S)$ has an orthogonal decomposition

$$(X, \iota, \lambda) = \prod_{i \in I} (X_i, \iota, \lambda)$$

where each factor (X_i, ι, λ) is a supersingular hermitian $\mathcal{O}_{E_i}$-$\mathbb{Z}_p$-module. Also, assuming that S is connected, each factor has a well-defined signature (r_i, s_i) and these signatures add up to $(1, n-1)$. In particular, there is exactly one index $i_0 \in I$ with $r_{i_0} = 1$ and $r_i = 0$ for all $i \neq i_0$. Hence N_{i_0} is odd and all other indices are even by Lemma 2.10. $\square$

So from now on, we assume that there is a unique odd index i_0. We denote by $U_{E_i}(N_i)$ the group of E_i-linear automorphisms of N_i which preserve the polarization. Similarly, $U_E(N) = \prod_{i\in I} U_{E_i}(N_i)$ denotes the group of E-linear automorphisms of N which preserve the polarization. For even i, we also fix some self-dual $\mathcal{O}_{E_i}$-stable Dieudonné lattice $M_i \subset N_i$ of signature $(0, n_i)$ and denote by $K_i \subset U_{E_i}(N_i)$ its stabilizer.

At the index i_0, we choose the decomposition

$$\Psi := \operatorname{Hom}(E^u_{i_0}, \breve{E}_{i_0}) = \Psi_0 \sqcup \Psi_1$$

which extends the decomposition from $\mathbb{Q}_{p^2}$. Then it is easy to prove the following proposition.

Proposition 5.14. *There is an $U_E(N)$-equivariant isomorphism of formal schemes over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$,*

$$\mathcal{Z}(\mathcal{O}_E) \cong \coprod_{\psi_0\in\Psi_0} \left(\mathcal{N}^{\psi_0}_{E_{0,i_0}/\mathbb{Q}_p,(1,n_{i_0}-1)}\right) \times \prod_{i\neq i_0} U_{E_i}(N_i)/K_i.$$

□

Of course, we can furthermore identify each connected component with $\mathcal{N}_{E_{0,i_0},(1,n_{i_0}-1)}$.

Part II

Application to the Arithmetic Fundamental Lemma

In the following two sections, we recall the Jacquet-Rallis Fundamental Lemma (FL) and the Arithmetic Fundamental Lemma (AFL) in both the group and the Lie algebra version. In Section 8, we recall the description of the various orbital integrals in terms of lattices from [15]. We use these results to reformulate the FL and the AFL. Finally, we will prove our main result, see Theorems 10.1 and 10.5, together with their corollaries.

6 Analytic set-up

6.1 Symmetric space side

Let $p > 2$ be a prime and let E/E_0 be an unramified quadratic extension of p-adic local fields. We denote their rings of integers by $\mathcal{O}_{E_0} \subset \mathcal{O}_E$. Let q be the cardinality of the residue field of $\mathcal{O}_{E_0}$ and let σ or $a \mapsto \bar{a}$ denote the Galois conjugation on E. Let v be the normalized valuation on E_0 and let $|\cdot| = q^{-v(\cdot)}$ be the associated absolute value. Let $\eta : E_0^\times \longrightarrow \{\pm 1\}, a \mapsto (-1)^{v(a)}$ be the quadratic character associated to E/E_0 by local class field theory.

We fix an E_0-vector space W_0 of dimension $n-1$ with $n \geq 2$ and set $W := E \otimes W_0$. We also form $V_0 := W_0 \oplus E_0 u$ and $V := E \otimes V_0$, where u is some additional vector. Via the embedding

$$GL(W) \hookrightarrow GL(V), \quad g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix},$$

$GL(W)$ acts by conjugation on $\mathrm{End}(V)$.

Definition 6.1. An element $\gamma \in \mathrm{End}(V)$ is *regular semi-simple (with respect to the decomposition $V = W \oplus Eu$)* if its stabilizer in $GL(W)$ is trivial and if its orbit is Zariski closed.

For any subset $X \subset \mathrm{End}(V)$, we denote by X_{rs} the regular semi-simple elements in X.

Lemma 6.2 ([20, Lemma 2.1])**.** *The element $\gamma \in \mathrm{End}(V)$ is regular semi-simple if and only if*

$$\{\gamma^i u\}_{i\geq 0} \quad \textit{and} \quad \{u^\vee \gamma^i\}_{i \geq 0}$$

generate V (resp. $V^\vee$). Here, $u^\vee : W \oplus Eu \longrightarrow E$ is the linear form $(w, \lambda u) \mapsto \lambda$. $\square$

Let $S(E_0)$ be the symmetric space

$$S(E_0) := \{\gamma \in GL(V) \mid \gamma\bar{\gamma} = 1\}.$$

It is stable under the action of $GL(W_0)$. We form the set-theoretic quotient for the conjugation action,

$$[S(E_0)_{\mathrm{rs}}] := GL(W_0)\backslash S(E_0)_{\mathrm{rs}}.$$

Let us fix some $\mathcal{O}_{E_0}$-lattice $\Lambda_0 \subset W_0$ and set $\Lambda := (\mathcal{O}_E \otimes \Lambda_0) \oplus \mathcal{O}_E u$. We normalize the Haar measure on $GL(W_0)$ such that the volume of $GL(\Lambda_0)$ is 1.

For a regular semi-simple element $\gamma \in S(E_0)_{\mathrm{rs}}$, for a test function $f \in C_c^\infty(S(E_0))$ and for a complex parameter $s \in \mathbb{C}$, we define the *orbital integral*

$$O_\gamma(f,s) := \int_{GL(W_0)} f(h^{-1}\gamma h)\eta(\det h)|\det h|^s dh,$$

with special value $O_\gamma(f) := O_\gamma(f,0)$ and derivative

$$\partial O_\gamma(f) := \frac{d}{ds}\bigg|_{s=0} O_\gamma(f,s).$$

These integrals converge absolutely. Note that $O_\gamma(f)$ transforms with $\eta \circ \det$ under the action of $GL(W_0)$ on γ.

Definition 6.3. For $\gamma \in \mathrm{End}(V)_{\mathrm{rs}}$, we define $l(\gamma) := [\mathrm{span}\{\gamma^i u\}_{i=0}^{n-1} : \Lambda] \in \mathbb{Z}$ to be the relative index of the two $\mathcal{O}_E$-lattices $\mathrm{span}\{\gamma^i u\}$ and Λ.

The *transfer factor (with respect to Λ_0)* $\Omega : \mathrm{End}(V)_{\mathrm{rs}} \longrightarrow \{\pm 1\}$ is the function

$$\Omega(\gamma) := (-1)^{l(\gamma)}.$$

Note that Ω is also $\eta \circ \det$-invariant and hence the product $\Omega(\gamma)O_\gamma(f)$ descends to the quotient $[S(E_0)_{\mathrm{rs}}]$.

We also introduce a tangent space version of the notions defined so far. Let

$$\mathfrak{s}(E_0) := \{y \in \mathrm{End}(V) \mid y + \bar{y} = 0\}$$

be the tangent space at 1 of $S(E_0)$. Again we form the quotient by the $GL(W_0)$-action,

$$[\mathfrak{s}(E_0)_{\mathrm{rs}}] := GL(W_0)\backslash \mathfrak{s}(E_0)_{\mathrm{rs}}.$$

For $y \in \mathfrak{s}(E_0)_{\mathrm{rs}}$, for a test function $f \in C_c^\infty(\mathfrak{s}(E_0))$ and for a complex parameter $s \in \mathbb{C}$, we define the orbital integrals $O_y(f,s), O_y(f)$ and $\partial O_y(f)$ by the same formulas as above. Again, $y \mapsto \Omega(y)O_y(f)$ descends to the quotient $[\mathfrak{s}(E_0)_{\mathrm{rs}}]$.

6.2 Unitary Side and orbit matching

Let $J_0^\flat$ and $J_1^\flat$ be two hermitian forms on W such that $J_0^\flat$ is even and such that $J_1^\flat$ is odd. By this we mean that there exists a self-dual lattice for $J_0^\flat$, resp. no self-dual lattice for $J_1^\flat$. Equivalently, we assume that $\eta(\det(J_0^\flat)) = 1$ and $\eta(\det(J_1^\flat)) = -1$. For $i = 0, 1$, we extend the form $J_i^\flat$ to a form J_i on V by setting $J_i(u,u) = 1$ and $u \perp W$. Let $U(J_i^\flat)$ (resp. $U(J_i)$) be the associated unitary groups. The $U(J_i)$ are subsets of $\mathrm{End}(V)$ and hence our definition of regular semi-simple applies to them.

We denote the Lie algebra of $U(J_i)$ by $\mathfrak{u}(J_i)$. The group $U(J_i^\flat)$ acts by conjugation on both $U(J_i)$ and $\mathfrak{u}(J_i)$ and we form the quotients

$$[U(J_i)_{\mathrm{rs}}] := U(J_i^\flat)\backslash U(J_i)_{\mathrm{rs}}$$
$$[\mathfrak{u}(J_i)_{\mathrm{rs}}] := U(J_i^\flat)\backslash \mathfrak{u}(J_i)_{\mathrm{rs}}.$$

Definition 6.4. Two elements $\gamma \in S(E_0)_{\mathrm{rs}}$ and $g \in U(J_i)_{\mathrm{rs}}$ are said to *match* if they are conjugate under $GL(W)$ within $\mathrm{End}(V)$.

Similarly two elements $y \in \mathfrak{s}(E_0)_{\mathrm{rs}}$ and $x \in \mathfrak{u}(J_i)_{\mathrm{rs}}$ are said to match if they are conjugate under $GL(W)$.

Lemma 6.5 ([20, Lemma 2.3]). *The matching relation induces bijections*

$$\alpha : [S(E_0)_{\mathrm{rs}}] \cong [U(J_0)_{\mathrm{rs}}] \sqcup [U(J_1)_{\mathrm{rs}}]$$

and

$$\alpha : [\mathfrak{s}(E_0)_{\mathrm{rs}}] \cong [\mathfrak{u}(J_0)_{\mathrm{rs}}] \sqcup [\mathfrak{u}(J_1)_{\mathrm{rs}}].$$

□

To normalize the Haar measure on $U(J_0^\flat)$, we fix a self-dual lattice $L \subset (W, J_0^\flat)$ and give volume 1 to its stabilizer $K_0^\flat \subset U(J_0^\flat)$. The normalization of the Haar measure on $U(J_1^\flat)$ is not important for us.

For a test function $f' \in C_c^\infty(U(J_i))$ (resp. $f' \in C_c^\infty(\mathfrak{u}(J_i))$) and a regular semi-simple element $g \in U(J_i)_{\mathrm{rs}}$ (resp. $x \in \mathfrak{u}(J_i)_{\mathrm{rs}}$), we define the orbital integral

$$O_g(f') := \int_{U(J_i^\flat)} f'(h^{-1}gh)dh, \quad \left(\text{resp. } O_x(f') := \int_{U(J_i^\flat)} f'(h^{-1}xh)dh\right).$$

For fixed f' this function is invariant under the conjugation action of $U(J_i^\flat)$ on g (resp. on x) and hence descends to the quotient $[U(J_i)_{\mathrm{rs}}]$ (resp. $[\mathfrak{u}(J_i)_{\mathrm{rs}}]$).

Definition 6.6. A function $f \in C_c^\infty(S(E_0))$ and a pair of functions (f_0', f_1') in $C_c^\infty(U(J_0)) \times C_c^\infty(U(J_1))$ are said to be *transfers of each other* if, for all $\gamma \in S(E_0)_{\mathrm{rs}}$, there is an equality

$$\Omega(\gamma)O_\gamma(f) = \begin{cases} O_g(f_0') & \text{if } \gamma \text{ matches } g \in U(J_0)_{\mathrm{rs}} \\ O_g(f_1') & \text{if } \gamma \text{ matches } g \in U(J_1)_{\mathrm{rs}}. \end{cases}$$

Similarly, a function $f \in C_c^\infty(\mathfrak{s}(E_0))$ and a pair of functions $(f_0', f_1') \in C_c^\infty(\mathfrak{u}(J_0)) \times C_c^\infty(\mathfrak{u}(J_1))$ are said to be transfers of each other if, for all $y \in \mathfrak{s}(E_0)_{\mathrm{rs}}$, there is an equality

$$\Omega(y)O_y(f) = \begin{cases} O_x(f_0') & \text{if } y \text{ matches } x \in \mathfrak{u}(J_0)_{\mathrm{rs}} \\ O_x(f_1') & \text{if } y \text{ matches } x \in \mathfrak{u}(J_1)_{\mathrm{rs}}. \end{cases}$$

6.3 Jacquet-Rallis Fundamental Lemma

Recall that for the definition of the transfer factor, Definition 6.3, we fixed some $\mathcal{O}_{E_0}$-lattice $\Lambda_0 \subset W_0$ and formed the lattice $\Lambda \subset V$. We define $S(\mathcal{O}_{E_0}) := S(E_0) \cap \mathrm{End}(\Lambda)$ and $\mathfrak{s}(\mathcal{O}_{E_0}) := \mathfrak{s}(E_0) \cap \mathrm{End}(\Lambda)$. Also, let $K_0 \subset U(J_0)$ denote the stabilizer of $L \oplus \mathcal{O}_E u$ where L is some self-dual lattice in $(W, J_0^\flat)$. Similarly, we define $\mathfrak{u}(J_0)(\mathcal{O}_{E_0}) := \mathfrak{u}(J_0) \cap \mathrm{End}(L \oplus \mathcal{O}_E u)$.

Conjecture 6.7 (Jacquet-Rallis Fundamental Lemma). The function $1_{S(\mathcal{O}_{E_0})}$ and the pair $(1_{K_0}, 0)$ are transfers of each other. Equivalently, for all $\gamma \in S(E_0)_{\mathrm{rs}}$,

$$\Omega(\gamma)O_\gamma(1_{S(\mathcal{O}_{E_0})}) = \begin{cases} O_g(1_{K_0})) & \text{if } \gamma \text{ matches } g \in U(J_0) \\ 0 & \text{if } \gamma \text{ matches } g \in U(J_1). \end{cases} \qquad (\mathrm{JR}_{E_0,(V,J_0),u,g})$$

Remark 6.8. (1) Note that the left hand side does not depend on the choice of the lattice Λ_0. Namely if we replace it by $h\Lambda_0$, $h \in GL(W_0)$, then Ω is changed by the sign $(-1)^{v(\det h)}$. But also

$$O_\gamma(1_{hS(\mathcal{O}_{E_0})h^{-1}}) = O_{h^{-1}\gamma h}(1_{S(\mathcal{O}_{E_0})}) = (-1)^{v(\det h)} O_\gamma(1_{S(\mathcal{O}_{E_0})}).$$

(2) The quadruple $(E_0, (V, J_0), u, g)$ is sufficient to formulate the Fundamental Lemma equation. Namely we can define $W := u^\perp$ with form $J_0^\flat := J_0|_W$. The form $J_1^\flat$ can be chosen arbitrarily since it does not play a role in the conjecture. Finally, the left hand side of $(\mathrm{JR}_{E_0,(V,J_0),u,g})$ does not depend on the chosen E_0-structure $W_0 \subset W$, which will follow from Corollary 8.3 below.

The equal characteristic analogue of this conjecture was proven by Zhiwei Yun in [19] in the case $p > n$. Julia Gordon [5] deduced the p-adic case for p sufficiently large. In general, the vanishing part (i.e. the case $g \in U(J_1)$) of the conjecture is known by [15, Corollary 7.3].

Conjecture 6.9 (Jacquet-Rallis Fundamental Lemma, Lie algebra version)**.** The function $1_{\mathfrak{s}(\mathcal{O}_{E_0})}$ and the pair $(1_{\mathfrak{u}(J_0)(\mathcal{O}_{E_0})}, 0)$ are transfers of each other. Equivalently, for all $y \in \mathfrak{s}(E_0)_{\mathrm{rs}}$,

$$\Omega(y) O_y(1_{\mathfrak{s}(\mathcal{O}_{E_0})}) = \begin{cases} O_x(1_{\mathfrak{u}(J_0)(\mathcal{O}_{E_0})}) & \text{if } y \text{ matches } x \in \mathfrak{u}(J_0) \\ 0 & \text{if } y \text{ matches } x \in \mathfrak{u}(J_1). \end{cases} \qquad (\mathfrak{jr}_{E_0,(V,J_0),u,x})$$

As in the group version, the left hand side does not depend on the choice of Λ_0, see Remark 6.8. Similarly, the quadruple $(E_0, (V, J_0), u, x)$ is enough to formulate equation $(\mathfrak{jr}_{E_0,(V,J_0),u,x})$. In the function field setting for $p > n$, the Lie algebra version was also proved by Yun. Gordon deduced the conjecture for p large enough in the p-adic case.

Both orbital integrals appearing in the Jacquet-Rallis Fundamental Lemma can be expressed in terms of lattices. We formulate this now for the unitary side. For the symmetric space side, see Corollaries 8.3 and 8.5. We denote by $\Lambda^\vee$ the J_0-dual of a lattice $\Lambda \subset V$.

Lemma 6.10 ([15, Lemma 7.1])**.** *(1) Let $g \in U(J_0)_{\mathrm{rs}}$ be regular semi-simple and let $L = \mathcal{O}_E[g]u \subset V$ be the g-stable lattice spanned by u. Then*

$$O_g(1_{K_0}) = |\{\Lambda \subset V \mid L \subset \Lambda \subset L^\vee, g\Lambda = \Lambda, \Lambda^\vee = \Lambda\}|.$$

(2) Let $x \in \mathfrak{u}(J_0)_{\mathrm{rs}}$ be regular semi-simple and let $L = \mathcal{O}_E[x]u \subset V$ be the x-stable lattice spanned by u. Then

$$O_x(1_{\mathfrak{u}(J_0)(\mathcal{O}_{E_0})}) = |\{\Lambda \subset V \mid L \subset \Lambda \subset L^\vee, x\Lambda \subseteq \Lambda, \Lambda^\vee = \Lambda\}|.$$

Proof. Part (1) is precisely [15, Lemma 7.1] and part (2) is proved in the same way. □

7 The Arithmetic Fundamental Lemma

7.1 Intersection numbers

Let $n \geq 2$ and let $\mathcal{N}_{E_0,(1,n-2)}$ be the RZ-space from Notation 3.4 with framing object $\mathbb{X}_{E_0,(1,n-2)}$. Let $\overline{\mathbb{Y}}_{E_0} = \mathbb{Y}_{E_0,(0,1)}$ be the hermitian $\mathcal{O}_E$-module of signature $(0,1)$ over $\mathbb{F}$, which is unique up to isomorphism. We choose $\mathbb{X}_{E_0,(1,n-1)} := \mathbb{X}_{E_0,(1,n-2)} \times \overline{\mathbb{Y}}_{E_0}$ as the framing object for $\mathcal{N}_{E_0,(1,n-1)}$.

Let us make $\mathrm{Hom}^0(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-2)})$ into a hermitian E-vector space with form h defined by

$$h(x,y)\mathrm{id}_{\overline{\mathbb{Y}}_{E_0}} = \lambda_{\overline{\mathbb{Y}}_{E_0}}^{-1} \circ x^\vee \circ \lambda_{\mathbb{X}_{E_0,(1,n-2)}} \circ y,$$

see [9, Definition 3.1]. This non-degenerate form is odd by Lemma 2.10, and we fix an isometry

$$(\mathrm{Hom}^0(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-2)}), h) \cong (W, J_1^\flat).$$

We extend this to an isometry

$$\begin{aligned}\mathrm{Hom}^0(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-1)}) = \mathrm{Hom}^0(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-2)}) \times E \cdot \mathrm{id}_{\overline{\mathbb{Y}}_{E_0}} \\ \cong W \oplus Eu = V\end{aligned}$$

by sending $0 \times \mathrm{id}_{\overline{\mathbb{Y}}_{E_0}}$ to u. Via this isomorphism, $\mathrm{End}^0(\mathbb{X}_{E_0,(1,n-1)})$ acts on V which induces an identification

$$\mathrm{End}^0(\mathbb{X}_{E_0,(1,n-1)}) \cong \mathrm{End}(V) \tag{7.1}$$

that is equivariant for the Rosati involution on the left and the adjoint involution of J_1 on the right. The product decompositions $\mathbb{X}_{E_0,(1,n-1)} = \mathbb{X}_{E_0,(1,n-2)} \times \overline{\mathbb{Y}}_{E_0}$ and $V = W \times Eu$ give rise to projection (resp. inclusion) operators to (resp. from) the vector spaces $\mathrm{End}^0(\mathbb{X}_{E_0,(1,n-2)})$, $\mathrm{Hom}(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-2)})$, etc. and $\mathrm{End}(W), \mathrm{Hom}(Eu, W)$, etc. The identification in (7.1) is compatible with all these homomorphisms. Finally, the isomorphism (7.1) identifies the unitary group $U(J_1^\flat)$ (resp. $U(J_1)$) with the group of quasi-isogenies of $\mathbb{X}_{E_0,(1,n-2)}$ (resp. $\mathbb{X}_{E_0,(1,n-1)}$). From now on, we will take the identification from (7.1) as self-evident.

By Proposition 2.14, there is a unique deformation $\overline{\mathcal{Y}}_{E_0}$ of the hermitian $\mathcal{O}_E$-module $\overline{\mathbb{Y}}_{E_0}$ to $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$. We define $\overline{\mathcal{Y}}_{E_0}$ on every $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$-scheme by base change. This induces a closed immersion

$$\delta : \mathcal{N}_{E_0,(1,n-2)} \hookrightarrow \mathcal{N}_{E_0,(1,n-1)}, \quad X \mapsto X \times \overline{\mathcal{Y}}_{E_0} \tag{7.2}$$

which is equivariant with respect to the inclusion $U(J_1^\flat) \hookrightarrow U(J_1)$. Here, the groups act on the RZ-spaces by composition in the framing,

$$g.(X, \rho) := (X, g\rho).$$

Let us consider the graph of δ,

$$\Delta : \mathcal{N}_{E_0,(1,n-2)} \longrightarrow \mathcal{N}_{E_0,(1,n-2)} \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{E}}} \mathcal{N}_{E_0,(1,n-1)}.$$

By abuse of notation, we denote its image also by Δ. Note that the source is regular of dimension $n-1$ while the target is regular of dimension $2(n-1)$. Hence Δ defines a cycle in middle dimension. For $g \in U(J_1)$, we denote by

$$\Delta_g := (1, g)\Delta$$

its translate under g.

Lemma 7.1 ([20, Lemma 2.8])**.** *We assume that $E_0 = \mathbb{Q}_p$. Then for regular semi-simple $g \in U(J_1)_{\mathrm{rs}}$, the schematic intersection $\Delta \cap \Delta_g$ is a projective scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$.* □

Definition 7.2. Let $E_0 = \mathbb{Q}_p$ and let $g \in U(J_1)$ be regular semi-simple. Then we define

$$\mathrm{Int}(g) := \chi(\mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{\Delta_g}).$$

This number is finite by the previous lemma. Moreover, the function $U(J_1)_{\mathrm{rs}} \ni g \mapsto \mathrm{Int}(g)$ descends to the quotient $[U(J_1)_{\mathrm{rs}}]$.

Remark 7.3. Lemma 7.1 is expected to hold for all base fields E_0. In fact, Wei Zhang [20] even states it in this generality. But note that his proof uses p-adic uniformization to reduce the statement to a global results of Kudla and Rapoport, see [18] and [10, Lemma 2.21]. Both the uniformization and the global result are not available for general base fields E_0.

Remark 7.4. All spaces occurring in the definition of $\mathrm{Int}(g)$ are regular. So if the schematic intersection $\Delta \cap \Delta_g$ is 0-dimensional, then there is an equality

$$\mathrm{Int}(g) = \mathrm{len}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\Delta \cap \Delta_g},$$

see [15, Proposition 4.2]. Moreover, the schematic intersection $\Delta \cap \Delta_g$ has the following moduli-theoretic interpretation: The image $\delta(\mathcal{N}_{E_0,(1,n-2)}) \subset \mathcal{N}_{E_0,(1,n-1)}$ can be identified with the KR-divisor $\mathcal{Z}(u)$ from Definition 5.3. Via the second projection, $\Delta \cap \Delta_g$ can then be identified with the formal scheme $\mathcal{Z}(u) \cap \mathcal{Z}(g)$, where $\mathcal{Z}(g)$ is defined in Definition 5.1.

This remark allows us to define $\mathrm{Int}(g)$ also in the case $E_0 \neq \mathbb{Q}_p$, at least for *artinian* g.

Definition 7.5. A quasi-endomorphism $x \in \mathrm{End}^0_E(\mathbb{X}_{E_0,(1,n-1)})$ is called *artinian (with respect to the quasi-homomorphism* $u \in \mathrm{Hom}^0(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-1)})$*)*, if the intersection $\mathcal{Z}(x) \cap \mathcal{Z}(u)$ is an artinian scheme. For artinian x, we define

$$\mathrm{Int}(x) := \mathrm{len}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathcal{Z}(x) \cap \mathcal{Z}(u)}.$$

Remark 7.6. There is no known group-theoretic characterization of the artinian elements in $U(J_1)_{\mathrm{rs}}$.

It is also possible to give a moduli description of $\mathcal{Z}(x) \cap \mathcal{Z}(u)$ in the smaller space $\mathcal{Z}(u) = \delta(\mathcal{N}_{E_0,(1,n-2)})$. If x has the form

$$x = \left(\begin{smallmatrix} x^\flat & v \\ w & d \end{smallmatrix}\right) \in \mathrm{End}^0_E(\mathbb{X}_{E_0,(1,n-2)} \times \overline{\mathbb{Y}}_{E_0}),$$

then

$$\mathcal{Z}(x) \cap \mathcal{Z}(u) = \begin{cases} \emptyset & \text{if } d \notin \mathcal{O}_E \\ \mathcal{Z}(x^\flat) \cap \mathcal{Z}(v) \cap \mathcal{Z}(w^*) & \text{otherwise} \end{cases} \tag{7.3}$$

where $w^* : \overline{\mathbb{Y}}_{E_0} \longrightarrow \mathbb{X}_{E_0,(1,n-2)}$ is the Rosati adjoint of w, see Definition 2.3.

Remark 7.7. If $x \in \mathfrak{u}(J_1) \subset \mathrm{End}^0_E(\mathbb{X}_{(1,n-1)})$, then x has the form

$$x = \left(\begin{smallmatrix} x^\flat & j \\ -j^* & d \end{smallmatrix}\right)$$

with $x^\flat \in \mathfrak{u}(J_1^\flat)$ and $\bar{d} = -d$. So in this case,

$$\Delta \cap \Delta_x \cong \begin{cases} \emptyset & \text{if } d \notin \mathcal{O}_E \\ \mathcal{Z}(x^\flat) \cap \mathcal{Z}(j) & \text{otherwise.} \end{cases}$$

7.2 The conjectures

The Jacquet-Rallis Fundamental Lemma 6.7 gives an expression of the orbital integral function $O_\gamma(1_{S(\mathcal{O}_{E_0})})$ on the unitary side. By contrast, the Arithmetic Fundamental Lemma of Wei Zhang conjecturally expresses the derivative $\partial O_\gamma(1_{S(\mathcal{O}_{E_0})})$ on the unitary side whenever γ matches an element $g \in U(J_1)_{\mathrm{rs}}$. Note that for such γ, the orbital integral $O_\gamma(1_{S(\mathcal{O}_{E_0})})$ vanishes and thus $\partial O_\gamma(1_{S(\mathcal{O}_{E_0})})$ is $\eta \circ \det$-invariant.

Conjecture 7.8 (Arithmetic Fundamental Lemma, Group version, [20])**.** Let $\gamma \in S(E_0)_{\mathrm{rs}}$ be a regular semi-simple element that matches an element $g \in U(J_1)$. Assume that either $E_0 = \mathbb{Q}_p$ or that g is artinian. Then there is an equality

$$\Omega(\gamma)\partial O_\gamma(1_{S(\mathcal{O}_{E_0})}) = -\mathrm{Int}(g)\log(q). \qquad (\mathrm{AFL}_{E_0,(V,J_1),u,g})$$

For the Lie algebra version, we have to restrict to artinian elements since we defined the intersection product only in this case, see Definition 7.5.

Conjecture 7.9 (Arithmetic Fundamental Lemma, Lie algebra version)**.** For any $y \in \mathfrak{s}(E_0)_{\mathrm{rs}}$ matching an artinian element $x \in \mathfrak{u}(J_1)$, there is an equality

$$\Omega(y)\partial O_y(1_{\mathfrak{s}(\mathcal{O}_{E_0})}) = -\mathrm{Int}(x)\log(q). \qquad (\mathfrak{afl}_{E_0,(V,J_1),u,x})$$

Remark 7.10. Just as in the case of the Jacquet-Rallis Fundamental Lemma, see Remark 6.8, the left hand side of the AFL identities does not depend on the chosen lattice Λ_0. Also, it does not depend on the chosen E_0-structure $W_0 \subset W$, which will follow from the Corollaries 8.3 and 8.5 below. Hence the quadruple $(E_0,(V,J_1),u,g)$ (resp. $(E_0,(V,J_1),u,x)$) is enough to formulate the identity $(\mathrm{AFL}_{E_0,V,u,g})$ (resp. $(\mathfrak{afl}_{E_0,V,u,x})$), which justifies our notation.

The group and the Lie algebra version of the AFL are related by the following result.

Proposition 7.11 ([12])**.** *Assume that $q \geq n+2$. Then the AFL for all artinian elements $g \in U(J_1)_{\mathrm{rs}}$ is equivalent to the AFL in the Lie algebra formulation for all artinian $x \in \mathfrak{u}(J_1)_{\mathrm{rs}}$.*[8]

The AFL conjecture has been verified for $n \leq 3$ in [20]. Note that if $n \leq 3$, then any regular semi-simple element $g \in U(J_1)_{\mathrm{rs}}$ is artinian. There exists a slight simplification of this computation in [12] which relies on Proposition 7.11.

More cases of the AFL for any n, but under restrictive conditions on g, have been verified by Rapoport, Terstiege and Zhang in [15]. Note that in these cases, g is also artinian. Up to now, there are no known results for degenerate intersections, i.e. for the case $\dim \Delta \cap \Delta_g \geq 1$.

We have introduced the AFL conjecture in the so-called inhomogeneous formulation. There is also the equivalent homogeneous version which is more systematic from a group-theoretic point of view. We refer the reader to the article [13] of Rapoport, Smithling and Zhang. The three authors also introduce a variant of the AFL for a ramified quadratic extension E/E_0 and verify it for $n \leq 3$.

Before we continue, we would like to modify the AFL for Lie algebras slightly. Namely let $y \in \mathfrak{s}(E_0)_{\mathrm{rs}}$ match $x \in \mathfrak{u}(J_1)$ of the form

$$x = \begin{pmatrix} x^\flat & j \\ -j^* & d \end{pmatrix}.$$

Then d is also the lower right entry of the matrix y. If $d \notin \mathcal{O}_E$, then it is easy to see that both sides of $(\mathfrak{afl}_{E_0,V,u,x})$ vanish.[9] If instead $d \in \mathcal{O}_E$, then we can replace y by $y - d\cdot \mathrm{id}_V$ and x by $x - d\cdot\mathrm{id}_V$ without changing either side of $(\mathfrak{afl}_{E_0,V,u,x})$. Furthermore, $y - d\cdot\mathrm{id}_V$ lies in $\mathfrak{s}(E_0)_{\mathrm{rs}}$ and matches $x - d\cdot\mathrm{id}_V \in \mathfrak{u}(J_1)_{\mathrm{rs}}$.

[8] In [12], it is specified for which x one needs $(\mathrm{AFl}_{E_0,V,u,x})$ to obtain $(\mathrm{AFL}_{E_0,V,u,g})$ and conversely.
[9] In particular, the AFL holds in this case.

Definition 7.12. We define

$$\mathfrak{s}(E_0)^0 := \left\{ \begin{pmatrix} y^\flat & w \\ v & d \end{pmatrix} \in \mathfrak{s}(E_0) \mid d = 0 \right\} \quad \text{and}$$
$$\mathfrak{u}(J)^0 := \left\{ \begin{pmatrix} x^\flat & j \\ -j^* & d \end{pmatrix} \in \mathfrak{u}(J) \mid d = 0 \right\}.$$

Then the matching relation induces a bijection

$$[\mathfrak{s}(E_0)^0_{\mathrm{rs}}] \cong [\mathfrak{u}(J_0)^0_{\mathrm{rs}}] \sqcup [\mathfrak{u}(J_1)^0_{\mathrm{rs}}]$$

and it is enough to consider the Lie algebra formulation of the AFL for $x \in \mathfrak{u}(J_1)^0$.

8 Analytic side of the AFL

In this section, we recall the expression of the orbital integrals $O_\gamma(1_K)$ and $\partial O_\gamma(1_K)$ in terms of lattices from [15, Section 7]. We deduce analogous results for the Lie algebra formulation.

8.1 Orbital integrals: Group version

Let $\gamma \in S(E_0)_{\mathrm{rs}}$ match the element $g \in U(J_0)_{\mathrm{rs}} \sqcup U(J_1)_{\mathrm{rs}}$. From now on, we consider V with the hermitian form $J \in \{J_0, J_1\}$ determined by g.

Recall that $V = W \overset{\perp}{\oplus} Eu$ with $(u,u) = 1$ and note that $u, gu, \ldots, g^{n-1}u$ is a basis of V since g is regular semi-simple. We define a σ-linear involution $\tau : V \longrightarrow V$ by $\tau(g^i u) = g^{-i}u$ for $i = 0, \ldots, n-1$.

Remark 8.1. The involution τ can also be defined as follows. The vector u defines an isomorphism of $E[g]$-modules, $E[g] \cong E[g]u = V$. Under this isomorphism, τ corresponds to the adjoint involution with respect to the hermitian form J on $E[g] \subset \mathrm{End}(V)$.

Let $L := \mathcal{O}_E[g]u$ be the g-stable lattice spanned by u and denote by $L^\vee$ its dual with respect to J. Let

$$M := \{\Lambda \subset V \mid L \subset \Lambda \subset L^\vee,\ g\Lambda \subset \Lambda,\ \Lambda^\tau = \Lambda\}$$

and, for $i \in \mathbb{Z}$,

$$M_i := \{\Lambda \in M \mid \mathrm{len}(\Lambda/L) = i\}.$$

Lemma 8.2 ([15, Proof of Corollary 7.3])**.**

$$\Omega(\gamma) O_\gamma(1_{S(\mathcal{O}_{E_0})}, s) = \sum_{i \in \mathbb{Z}} (-1)^i |M_i| q^{-(i + l(\gamma))s}.$$

□

(Here, $l(\gamma)$ was defined in Definition 6.3.) Taking the value at $s = 0$, resp., taking the derivative at $s = 0$ yields

Corollary 8.3 ([15, Corollary 7.3])**.**

$$\Omega(\gamma) O_\gamma(1_{S(\mathcal{O}_{E_0})}) = \sum_{i \in \mathbb{Z}} (-1)^i |M_i|$$

and, in the case $J = J_1$,

$$\Omega(\gamma) \partial O_\gamma(1_{S(\mathcal{O}_{E_0})}) = -\log(q) \sum_{i \in \mathbb{Z}} (-1)^i i |M_i|. \tag{8.1}$$

□

8.2 Orbital integrals: Lie algebra version

Let $y \in \mathfrak{s}(E_0)^0_{\mathrm{rs}}$ match the element

$$x = \begin{pmatrix} x^\flat & j \\ -j^* & \end{pmatrix} \in \mathfrak{u}(J_0)^0_{\mathrm{rs}} \sqcup \mathfrak{u}(J_1)^0_{\mathrm{rs}}.$$

Let $J \in \{J_0, J_1\}$ be the hermitian form determined by x. Again, $\tau : V \longrightarrow V$ is the adjoint involution on $V = E[x]u \subset \mathrm{End}(V)$, i.e. $\tau x^i u = (-1)^i x^i u$ for $i = 0, \dots, n-1$.

Let $L := \mathcal{O}_E[x]u$ be the x-stable lattice spanned by u and denote by $L^\vee$ its dual. Let

$$M := \{\Lambda \subset V \mid L \subset \Lambda \subset L^\vee,\ x\Lambda \subset \Lambda,\ \Lambda^\tau = \Lambda\}$$

and, for $i \in \mathbb{Z}$,

$$M_i := \{\Lambda \in M \mid \mathrm{len}(\Lambda/L) = i\}.$$

Then the same formula for the orbital integral applies. Its proof is completely analogous to the one for the group version.

Lemma 8.4. *There is an equality*

$$\Omega(y) O_y(1_{\mathfrak{s}(\mathcal{O}_{E_0})}, s) = \sum_{i \in \mathbb{Z}} (-1)^i |M_i| q^{-(i+l(y))s}.$$

□

Corollary 8.5. *Taking the value (resp. the derivative) at $s = 0$ yields the formulas*

$$\Omega(y) O_y(1_{S(\mathcal{O}_{E_0})}) = \sum_{i \in \mathbb{Z}} (-1)^i |M_i|$$

and, in the case $J = J_1$,

$$\Omega(y) \partial O_y(1_{S(\mathcal{O}_{E_0})}) = -\log(q) \sum_{i \in \mathbb{Z}} (-1)^i i |M_i|. \tag{8.2}$$

□

Let us define $L^\flat := \mathcal{O}_E[x^\flat] j \subset W$. We denote the orthogonal projection $V \longrightarrow W$ by pr and denote by pr_u the projection to Eu.

Proposition 8.6. *There is an equality of lattices in $V = W \oplus Eu$,*

$$M = \{\Lambda^\flat \oplus \mathcal{O}_E u \ \mid \ L^\flat \subset \Lambda^\flat \subset L^{\flat,\vee},\ x^\flat \Lambda^\flat \subset \Lambda^\flat,\ (\Lambda^\flat)^\tau = \Lambda^\flat\}. \tag{8.3}$$

We first note the following formula for $v \in V$,

$$xv = x^\flat v + (v, u) j - (j, v) u. \tag{8.4}$$

Lemma 8.7. *Let $\Lambda \subset V$ be any lattice such that $u \in \Lambda$ and $\mathrm{pr}_u(\Lambda) = \mathcal{O}_E u$. Then $\Lambda = (\Lambda \cap W) \oplus \mathcal{O}_E u$ and $\Lambda \cap W = \mathrm{pr}(\Lambda)$. Moreover, $\mathrm{pr}(\Lambda^\vee) = \mathrm{pr}(\Lambda)^\vee$.*

Proof. This is immediate. □

Lemma 8.8. *The inclusion $L \subset L^\vee$ holds if and only if $L^\flat \subset L^{\flat,\vee}$. In this case, $L^\flat = \mathrm{pr}(L)$ and $L^{\flat,\vee} = \mathrm{pr}(L^\vee)$.*

Proof. We first assume that $L \subset L^\vee$. Then $L = \mathrm{pr}(L) \oplus \mathcal{O}_E u$ and $\mathrm{pr}(L)^\vee = \mathrm{pr}(L^\vee)$ by the previous lemma. Hence it is enough to show that $L^\flat = \mathrm{pr}(L)$.

First note that $\mathrm{pr}(L)$ is stable under $\mathrm{pr} \circ x \circ \mathrm{pr} = x^\flat$. Since also $j \in \mathrm{pr}(L)$, we get $L^\flat \subset \mathrm{pr}(L)$. To prove the opposite inclusion, we prove $\mathrm{pr}(x^i u) \subset L^\flat$ by induction.

The case $i = 0$ is clear. Then we use formula (8.4),

$$\mathrm{pr}(x^{i+1}u) = x^\flat x^i u + j(x^i u, u).$$

The first summand lies in $L^\flat$ by induction and by the fact that $L^\flat$ is $x^\flat$-stable. The second summand lies in $L^\flat$ since $j \in L^\flat$ and $(x^i u, u) \in \mathcal{O}_E$ by the assumption $L \subset L^\vee$.

Conversely, let us assume that $L^\flat \subset L^{\flat,\vee}$. We need to show that $(x^i u, x^j u) \in \mathcal{O}_E$ for all i, j. Since x is from the unitary Lie algebra, it is enough to prove $(x^i u, u) \in \mathcal{O}_E$ for all i. Again we prove this by induction, the cases $i = 0$ and 1 being clear. We compute

$$-(x^{i+1}u, u) = (x^i u, j) = (x^\flat x^{i-1} u, j) + ((x^{i-1}u, u)j, j) - ((j, x^{i-1}u)u, j).$$

The first summand is integral by assumption on $L^\flat$. In the second summand, the pairing $(x^{i-1}u, u)$ is integral by induction. Hence the second summand is integral by assumption on $L^\flat$. The third summand vanishes. □

Proof of Proposition 8.6. Let $\Lambda \in M$. By Lemma 8.7, it is a direct sum, $\Lambda = \Lambda^\flat \oplus \mathcal{O}_E u$ where $\Lambda^\flat = \mathrm{pr}(\Lambda)$. If $\lambda^\flat \in \Lambda^\flat$, then

$$x^\flat \lambda^\flat = x\lambda^\flat + (j, \lambda^\flat)u \in \Lambda$$

and hence $\Lambda^\flat$ is $x^\flat$-stable. Furthermore, $L^\flat \subset \Lambda^\flat \subset L^{\flat,\vee}$, since this is just the projection of the relation $L \subset \Lambda \subset L^\vee$. Finally, note that τ commutes with the projection pr. Hence $\Lambda^\flat$ has all the properties from (8.3).

Conversely, let us now assume that $\Lambda^\flat$ satisfies all properties from (8.3). We want to show that $\Lambda := \Lambda^\flat \oplus \mathcal{O}_E u \in M$. By Lemma 8.7, $L \subset \Lambda \subset L^\vee$. Furthermore, Λ is τ-stable, since both summands are. It is easy to prove that Λ is also stable under x which concludes the proof. □

9 Reformulation of the FL and the AFL

The results of the previous section allow us to treat the AFL (resp. the FL) for groups and for Lie algebras at the same time. In this section, V is endowed with either of the two hermitian forms, say $J \in \{J_0, J_1\}$. In particular, the adjoint involution $\mathrm{End}(V) \ni x \mapsto x^*$ and the dual lattice $\Lambda \mapsto \Lambda^\vee$ are taken with respect to this form.

Definition 9.1. (1) A pair $(x, j) \in \mathrm{End}_E(V) \times V$ is called *regular semi-simple* if $E[x]j = V$.
(2) The pair (resp. the element x) is called *adjoint-stable* if $\mathcal{O}_E[x] = \mathcal{O}_E[x^*]$.

Remark 9.2. (1) Note that any element $x \in \mathfrak{u}(J)$ is adjoint-stable. An element $g \in U(J_1)$ is adjoint-stable if and only if $g^* = g^{-1} \in \mathcal{O}_E[g]$, which is equivalent to g having integral characteristic polynomial.
(2) Let $x \in \mathrm{End}_E(V)$ be such that $E[x] = E[x^*]$ and let (x, j) be regular semi-simple. Then also $E[x] \cdot (\ , j) = V^\vee$. In particular if $x \in \mathfrak{u}(J)$ or $x \in U(J)$, then x is regular semi-simple in the sense of Lemma 6.2 if and only if (x, u) is a regular semi-simple pair.

Definition 9.3. Let (x,j) be a regular semi-simple and adjoint-stable pair. Then we denote by $L(x,j) := \mathcal{O}_E[x]j$ the x-stable lattice generated by j.

(1) We define the σ-linear involution $\tau(x,j) : V \longrightarrow V$ as follows: The element j induces an isomorphism $\phi : E[x] \cong E[x]j = V$ and we set $\tau(x,j)(v) = \phi(\phi^{-1}(v)^*)$. This is possible since $E[x] = E[x]^*$ by assumption.

(2) We define the sets

$$\begin{aligned} M(x,j) &:= \{\Lambda \subset V \mid L(x,j) \subset \Lambda \subset L(x,j)^\vee, x\Lambda \subset \Lambda, \tau(x,j)\Lambda = \Lambda\}, \\ M(x,j)_i &:= \{\Lambda \in M \mid \mathrm{len}_{\mathcal{O}_E}(\Lambda/L(x,j)) = i\}, \quad i \in \mathbb{Z}. \end{aligned}$$

(3) For $s \in \mathbb{C}$, we define the following numbers.

$$\begin{aligned} O(x,j;s) &:= \sum_{i\in\mathbb{Z}} (-1)^i |M(x,j)_i| q^{-is} \\ O(x,j) &:= O(x,j;0) = \sum_{i\in\mathbb{Z}} (-1)^i |M(x,j)_i| \\ \partial O(x,j) &:= \log(q)^{-1} \left.\frac{d}{ds}\right|_{s=0} O(x,j;s) = -\sum_{i\in\mathbb{Z}} (-1)^i i |M(x,j)_i|. \end{aligned}$$

We will now consider the two possibilities for J separately.

9.1 The Jacquet-Rallis Fundamental Lemma

In this section, $J = J_0$ is the even form.

Definition 9.4. Let (x,j) be a regular semi-simple and adjoint-stable pair. We define

$$I(x,j) := |\{\Lambda \subset V \mid L(x,j) \subset \Lambda \subset L(x,j)^\vee, x\Lambda \subset \Lambda, \Lambda^\vee = \Lambda\}|$$

Remark 9.5. In contrast to the set-up in the previous sections, both this definition and Definition 9.3 make no reference to a codimension 1 subspace $W \subset V$.

Conjecture 9.6 (Fundamental Lemma)**.** Let $(x,j) \in \mathrm{End}_E(V)$ be a regular semi-simple and adjoint-stable pair. Then

$$I(x,j) = O(x,j). \tag{FL(x,j)}$$

In the rest of this subsection, we explain the relation of this conjecture with the Jacquet-Rallis Fundamental Lemma, Conjecture 6.7.

Lemma 9.7. *Let (x,j) be a self-adjoint pair with $x \in \mathfrak{u}(J_0)$. We set $V' := V \oplus Eu'$ with form $J_0 \oplus 1$ and we define*

$$x' := \begin{pmatrix} x & j \\ -j^* & \end{pmatrix} \in \mathfrak{u}(J_0 \oplus 1)^0.$$

Then (x,j) is regular semi-simple if and only if x' is regular semi-simple with respect to $V' = V \oplus Eu'$ in the sense of Definition 6.1. In the regular semi-simple case, (FL(x,j)) *is equivalent to* $(\mathrm{jr}_{E_0,V',u',x'})$.

Proof. This follows from Lemma 6.2, Lemma 6.10 and Corollary 8.5. □

Lemma 9.8. *Let (g,j) be a self-adjoint pair with $g \in U(J_0)$ and $J_0(j,j) \in \mathcal{O}_{E_0}^{\times}$. Then (g,j) is regular semi-simple if and only if g is regular semi-simple with respect to $V = j^{\perp} \oplus Ej$ in the sense of Definition 6.1. In the regular semi-simple case, $(FL(g,j))$ is equivalent to* $(\mathrm{JR}_{E_0,V,j,g})$.

Proof. This follows from Lemma 6.2, Lemma 6.10 and Corollary 8.3. □

Lemma 9.9. *Let (g,j) be a regular semi-simple and self-adjoint pair such that $J_0(j,j) \notin \mathcal{O}_{E_0}$. Then both sides of* $(\mathrm{FL}(g,j))$ *vanish.*

Proof. If $J_0(j,j) \notin \mathcal{O}_{E_0}$, then $L(x,j) \not\subset L(x,j)^{\vee}$. □

Construction 9.10. Let (g,j) be a self-adjoint pair with $g \in U(J_0)$ and $v(J_0(j,j)) \geq 1$. We also assume that[10] $q+1 > n$. Then we define $V' := V \oplus E\tilde{u}$ and $u' := \tilde{u} + j$. We extend J_0 to J_0' on V' by setting $\tilde{u} \perp V$ and $J_0'(u',u') = 1$. We define $W' := (u')^{\perp}$, which is an even hermitian space.

Let $P(t) \in \mathcal{O}_E[t]$ be the characteristic polynomial of g. Note that $q+1$ is the number of residue classes mod π_E of $E^1 := \{a \in E \mid \mathrm{Nm}_{E/E_0}(a) = 1\}$. By assumption this number is lager than $\deg(P)$ and hence there exists $a \in E^1$ such that $P(a) \not\equiv 0 \bmod \pi_E$. We define

$$g' := \begin{pmatrix} g & \\ & a \end{pmatrix} \in U(J_0')$$

where the block matrix decomposition is with respect to $V' = W \oplus E\tilde{u}$.

Lemma 9.11. *Let (g,j) be a self-adjoint pair with $g \in U(J_0)$ and $v(J_0(j,j)) \geq 1$. We also assume that $q+1 > n$. Let V', u' and g' be as above. Then (g,j) is regular semi-simple if and only if g' is regular semi-simple with respect to $V' = W' \oplus Eu'$ in the sense of Definition 6.1. In the regular semi-simple case,* $(\mathrm{FL}(g,j))$ *is equivalent to* $(\mathrm{JR}_{E_0,V',u',g'})$.

Proof. As explained in Lemma 9.8, (g',u') is regular semi-simple if and only if g' is regular semi-simple with respect to $V' = W' \oplus Eu'$ in the sense of Definition 6.1. In this case, $(\mathrm{JR}_{E_0,V',u',g'})$ is equivalent to $(\mathrm{FL}(g',u'))$. So we have to prove that (g',u') is regular semi-simple if and only if (g,j) is and that in this case, $(\mathrm{FL}(g',u'))$ is equivalent to $(\mathrm{FL}(g,j))$.

Due to our special choice of a, there is a decomposition $\mathcal{O}_E[g'] = \mathcal{O}_E[g] \times \mathcal{O}_E$. Its action on $V' = V \oplus E\tilde{u}$ is then a factor-wise action. This already proves the claim about the regular semi-simpleness.

We leave it to the reader to check that there is a bijection

$$M(g,j) \cong M(g',u'), \quad \Lambda \mapsto \Lambda \oplus \mathcal{O}_E\tilde{u}.$$

Of course, $\mathrm{len}(\Lambda/L(g,j)) = \mathrm{len}((\Lambda \oplus \mathcal{O}_E\tilde{u})/L(g',u'))$ and hence $O(g',u') = O(g,j)$. Similarly, one gets that $I(g',u') = I(g,j)$. □

[10] Recall that $n = \dim_E(V)$.

9.2 The Arithmetic Fundamental Lemma

We now assume that $J = J_1$ is the odd hermitian form. In particular, $x \mapsto x^*$, the dual lattice $\Lambda \mapsto \Lambda^\vee$ and M are now defined with respect to this form.

Lemma 9.12. *For all regular semi-simple and self-adjoint pairs (x, j), there is an equality*

$$O(x, j) = 0.$$

Proof. The argument is taken from [15, Corollary 7.3]. Let us assume that $L(x, j) \subset L(x, j)^\vee$ since otherwise $M(x, j) = \emptyset$ and hence $O(x, j) = 0$. Let us also define $l := [L(x, j)^\vee : L(x, j)]$ which is odd since J_1 is the odd form. Then $\Lambda \mapsto \Lambda^\vee$ induces an involution on the set $M(x, j)$ which is fixed point free since it interchanges $M(x, j)_i$ and $M(x, j)_{l-i}$. Thus $|M(x, j)_i| = |M(x, j)_{l-i}|$ and the two summands $(-1)^i |M(x, j)_i|$ and $(-1)^{l-i} |M(x, j)_{l-i}|$ in the definition of $O(x, j)$ cancel. $\square$

Definition 9.13. The pair (x, j) is called *artinian*, if the schematic intersection $\mathcal{Z}(x) \cap \mathcal{Z}(j) \subset \mathcal{N}_{E_0,(1,n-1)}$ is an artinian scheme. In this case, we define

$$\mathrm{Int}(x, j) := \mathrm{len}_{\mathcal{O}_{\breve{E}}}(\mathcal{O}_{\mathcal{Z}(x) \cap \mathcal{Z}(j)}).$$

Conjecture 9.14 (Arithmetic Fundamental Lemma)**.** Let (x, j) be a regular semi-simple, self-adjoint and artinian pair. Then

$$\partial O(x, j) = -\mathrm{Int}(x, j). \qquad (\mathrm{AFL}(x, j))$$

Again, we explain the relation of this conjecture with the AFL from Section 7. This will be very similar to the explanations for the case $J = J_0$.

Lemma 9.15. *Let (x, j) be a self-adjoint pair with $x \in \mathfrak{u}(J_1)$. We set $V' := V \oplus Eu'$ with form $J_1 \oplus 1$ and we define*

$$x' := \begin{pmatrix} x & j \\ -j^* & \end{pmatrix} \in \mathfrak{u}(J_1 \oplus 1)^0.$$

Then (x, j) is regular semi-simple and artinian if and only if x' is regular semi-simple with respect to $V' = V \oplus Eu'$ in the sense of Definition 6.1 and artinian with respect to u' in the sense of Definition 7.5. In the regular semi-simple and artinian case, $(\mathrm{AFL}(x, j))$ *is equivalent to* $(\mathfrak{afl}_{E_0,V',u',x'})$.

Proof. This follows from Lemma 6.2, Remark 7.7 and Corollary 8.5. $\square$

Lemma 9.16. *Let (g, j) be a self-adjoint pair with $g \in U(J_1)$ and $J_1(j, j) \in \mathcal{O}_{E_0}^\times$. Then (g, j) is regular semi-simple and artinian if and only if g is regular semi-simple with respect to $V = j^\perp \oplus Ej$ in the sense of Definition 6.1 and artinian with respect to j in the sense of Definition 7.5. In the regular semi-simple and artinian case,* $(\mathrm{AFL}(g, j))$ *is equivalent to* $(\mathrm{AFL}_{E_0,V,j,g})$.

Proof. This follows from Lemma 6.2, Remark 7.4 and Corollary 8.3. $\square$

Lemma 9.17. *Let (g, j) be a regular semi-simple, self-adjoint and artinian pair such that $J_1(j, j) \notin \mathcal{O}_{E_0}$. Then both sides of* $(\mathrm{AFL}(x, j))$ *vanish.*

Proof. If $J_1(j,j) \notin \mathcal{O}_{E_0}$, then $\mathcal{Z}(u) = \emptyset$. Namely if $X \in \mathcal{N}_{E_0,(1,n-1)}$ is such that $j : \overline{\mathbb{Y}}_{E_0} \longrightarrow \mathbb{X}_{E_0,(1,n-1)}$ lifts to a homomorphism $\overline{\mathcal{Y}}_{E_0} \longrightarrow X$, then also the composition $j^*j \in \mathrm{End}^0_E(\overline{\mathbb{Y}}_{E_0})$ lifts to $\overline{\mathcal{Y}}_{E_0}$. But note that $\mathrm{End}(\overline{\mathbb{Y}}_{E_0}) = E$ and that $j^*j = J_1(j,j)$.

Also, if $J_1(j,j) \notin \mathcal{O}_{E_0}$, then $L(g,j) \not\subset L(g,j)^\vee$ and thus $M = \emptyset$. □

Construction 9.18. Let (g,j) be a self-adjoint pair with $g \in U(J_1)$ and $v(J_1(j,j)) \geq 1$. We also assume that $q+1 > n$. Then we define $V' := V \oplus E\tilde{u}$ and $u' := \tilde{u} + j$. We extend J_1 to J_1' on V' by setting $\tilde{u} \perp V$ and $J_1'(u',u') = 1$. We define $W' := (u')^\perp$, which is an odd hermitian space.

As explained above, there exists $a \in E^1$ such that $P(a) \not\equiv 0 \bmod \pi_E$. We define

$$g' := \begin{pmatrix} g & \\ & a \end{pmatrix} \in U(J_1')$$

where the block matrix decomposition is with respect to $V' = W \oplus E\tilde{u}$.

Lemma 9.19. *Let (g,j) be a self-adjoint pair with $g \in U(J_1)$ and $v(J_1(j,j)) \geq 1$. We also assume that $q+1 > n$. Let V', u' and g' be as above. Then (g,j) is regular semi-simple and artinian if and only if g' is regular semi-simple with respect to $V' = W' \oplus Eu'$ in the sense of Definition 6.1 and artinian with respect to u' in the sense of Definition 7.5. In the regular semi-simple and artinian case, the identity* $(\mathrm{AFL}(g,j))$ *is equivalent to* $(\mathrm{AFL}_{E_0,V',u',g'})$.

Proof. As explained in Lemma 9.16, (g',u') is regular semi-simple and artinian if and only if g' is regular semi-simple with respect to $V' = W' \oplus Eu'$ and artinian with respect to u'. In this case, $(\mathrm{AFL}_{E_0,V',u',g'})$ is equivalent to $(\mathrm{AFL}(g',u'))$. So we have to prove that (g',u') is regular semi-simple and artinian if and only if (g,j) is, in which case the two identities $(\mathrm{AFL}(g',u'))$ and $(\mathrm{AFL}(g,j))$ are equivalent.

Due to our special choice of a, there is a decomposition $\mathcal{O}_E[g'] = \mathcal{O}_E[g] \times \mathcal{O}_E$. Its action on $V' = V \oplus E\tilde{u}$ is then a factor-wise action. This already proves the claim about the regular semi-simpleness.

Note that $J_1'(\tilde{u},\tilde{u}) \in \mathcal{O}_{E_0}^\times$. We leave it to the reader to check that there is a bijection

$$M(g,j) \cong M(g',u'), \quad \Lambda \mapsto \Lambda \oplus \mathcal{O}_E\tilde{u}.$$

Of course, $\mathrm{len}(\Lambda/L(g,j)) = \mathrm{len}((\Lambda \oplus \mathcal{O}_E\tilde{u})/L(g',u'))$ and hence $\partial O(g',u') = \partial O(g,j)$.

We still have to show $\mathrm{Int}(g',u') = \mathrm{Int}(g,j)$. This follows from an identification of formal schemes, $\mathcal{Z}(g) \cap \mathcal{Z}(u) = \mathcal{Z}(g') \cap \mathcal{Z}(u')$.

First note that because of the decomposition $\mathcal{O}_E[g'] = \mathcal{O}_E[g] \times \mathcal{O}_E$, there is an inclusion

$$\mathcal{Z}(g') \subset \mathcal{Z}(\tilde{u}).$$

Even better, the cycle $\mathcal{Z}(g') \subset \mathcal{Z}(\tilde{u}) \cong \mathcal{N}_{E_0,(1,n-1)}$ can be identified with $\mathcal{Z}(g)$. Moreover,

$$\mathcal{Z}(\tilde{u}) \cap \mathcal{Z}(u') = \mathcal{Z}(\tilde{u}) \cap \mathcal{Z}(j)$$

since the intersection only depends on the spanned module

$$\mathcal{O}_E\tilde{u} + \mathcal{O}_E u' \subset \mathrm{Hom}^0(\overline{\mathbb{Y}}_{E_0}, \mathbb{X}_{E_0,(1,n-1)} \times \overline{\mathbb{Y}}_{E_0}).$$

Thus

$$\mathcal{Z}(g') \cap \mathcal{Z}(u') = \mathcal{Z}(g') \cap \mathcal{Z}(\tilde{u}) \cap \mathcal{Z}(u') \cong \mathcal{Z}(g) \cap \mathcal{Z}(j).$$

□

10 Main Results on the AFL

In this section, we specialize to the case $E_0 = \mathbb{Q}_p$. This is only to simplify the exposition. All arguments work in the general case. So V is an n-dimensional $\mathbb{Q}_{p^2}$-vector space with odd hermitian form J_1.

Let $A_0/\mathbb{Q}_p$ be a field extension of degree d such that $A := A_0 \otimes \mathbb{Q}_{p^2}$ is also a field. Then A/A_0 is an unramified quadratic extension and we denote its Galois conjugation also by σ. Let $A \hookrightarrow \mathrm{End}_{\mathbb{Q}_{p^2}}(V)$ be an embedding that is equivariant for the Galois conjugation σ on A and the adjoint involution of J_1 on $\mathrm{End}(V)$. In other words,

$$J_1(a\ ,\ ) = J_1(\ ,\sigma(a)\ ), \quad a \in A.$$

Let ϑ_A be a generator of the inverse different of A_0 and let $J_1^A : V \times V \longrightarrow A$ be the A/A_0-hermitian form characterized by the property that

$$\mathrm{tr}_{A/\mathbb{Q}_{p^2}} \circ \vartheta_A J_1^A = J_1.$$

Note that for any $\mathcal{O}_A$-lattice $\Lambda \subset V$, the dual lattice $\Lambda^\vee$ with respect to the form J_1 is also the dual of Λ with respect to J_1^A. In particular, (V, J_1^A) is an odd hermitian space.[11]

Let $\mathbb{X}_{\mathbb{Q}_p,(1,n-1)}$ be the framing object for $\mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ and set $n' := n/d$. The action of A by quasi-endomorphisms on $\mathbb{X}_{\mathbb{Q}_p,(1,n-1)}$ makes it into a framing object $\mathbb{X}_{A_0/\mathbb{Q}_p,(1,n'-1)}$ for $\mathcal{N}_{A_0/\mathbb{Q}_p,(1,n'-1)}$. We also define $\mathbb{X}_{A_0,(1,n'-1)} := \mathcal{C}(\mathbb{X}_{A_0/\mathbb{Q}_p,(1,n'-1)})$ where $\mathcal{C}$ is the functor from Theorem 4.3.

Similarly, let $\mathbb{Y}_{\mathbb{Q}_p,(0,1)}$ be the hermitian $\mathbb{Z}_{p^2}$-module of signature $(0,1)$ over $\mathbb{F}$, which is unique up to isomorphism. We define $\mathbb{Y}_{A_0/\mathbb{Q}_p,(0,1)} := \mathcal{O}_{A_0} \otimes \mathbb{Y}_{\mathbb{Q}_p,(0,1)}$ with respect to ϑ_A as in Definition 5.7 and set $\mathbb{Y}_{A_0,(0,1)} := \mathcal{C}(\mathbb{Y}_{A_0/\mathbb{Q}_p,(0,1)})$. There is an isomorphism

$$\mathrm{Hom}^0_{\mathbb{Q}_{p^2}}(\mathbb{Y}_{\mathbb{Q}_p,(0,1)}, \mathbb{X}_{\mathbb{Q}_p,(1,n-1)}) \cong \mathrm{Hom}^0_A(\mathbb{Y}_{A_0/\mathbb{Q}_p,(0,1)}, \mathbb{X}_{\mathbb{Q}_p,(1,n'-1)}), \quad j \mapsto \mathrm{id}_{A_0} \otimes_{\mathbb{Q}_p} j$$

which is an isometry with respect to J_1^A on the left and the natural form on the right.[12] Via $\mathcal{C}$, these vector spaces are also isometric to $\mathrm{Hom}^0_A(\mathbb{Y}_{A_0,(0,1)}, \mathbb{X}_{A_0,(1,n'-1)})$.

The point is now that any regular semi-simple, self-adjoint and artinian pair $(x,j) \in \mathrm{End}_{\mathbb{Q}_{p^2}}(V) \times V$ such that x is A-linear gives rise to two AFL identities, one for the base field $\mathbb{Q}_p$ and one for A_0. We denote by $(\mathrm{AFL}(x,j)_{\mathbb{Q}_p})$ the one for $\mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ where the A-action does not play a role. We denote by $(\mathrm{AFL}(x,j)_{A_0}) := (\mathrm{AFL}(\mathcal{C}(x), \mathcal{C}(\mathrm{id}_{A_0} \otimes j)))$ the one for $\mathcal{N}_{A_0,(1,n'-1)}$. Our main result is the following theorem and its corollaries.

Theorem 10.1. *Let $(x,j) \in \mathrm{End}_{\mathbb{Q}_{p^2}}(V) \times V$ be a regular semi-simple, self-adjoint and artinian pair such that $\mathcal{O}_A \subset \mathbb{Z}_{p^2}[x]$. Then (x,j) is also regular semi-simple, self-adjoint and artinian when viewed over A_0 and the two identities $(\mathrm{AFL}(x,j)_{\mathbb{Q}_p})$ and $(\mathrm{AFL}(x,j)_{A_0})$ are equivalent.*

Proof. Let us keep the notation $L(x,j)$, $\tau(x,j)$, $M(x,j)$, $\partial O(x,j), \mathrm{Int}(x,j)$ etc. for the setting over $\mathbb{Q}_p$. We denote by $L(x,j)^A$, $\tau(x,j)^A$, $M(x,j)^A$, $\partial O(x,j)^A, \mathrm{Int}(x,j)^A$ etc. the respective notions for the setting over A_0. It is clear that (x,j) is also regular semi-simple and self-adjoint when viewed over A_0, since $\mathcal{O}_A[x]j = \mathbb{Z}_{p^2}[x]j$ by assumption.

Comparison of the analytic sides of $(\mathrm{AFL}(x,j)_{\mathbb{Q}_p})$ *and* $(\mathrm{AFL}(x,j)_{A_0})$*:*

[11] Recall that this means that there is no self-dual $\mathcal{O}_A$-lattice in V.

[12] We are using here our fixed isometry $V \cong \mathrm{Hom}^0_{\mathbb{Q}_{p^2}}(\mathbb{Y}_{\mathbb{Q}_p,(0,1)}, \mathbb{X}_{\mathbb{Q}_p,(1,n-1)})$.

First note that an x-stable $\mathbb{Z}_{p^2}$-lattice $\Lambda \subset V$ is automatically an $\mathcal{O}_A[x]$-lattice since $\mathbb{Z}_{p^2}[x] = \mathcal{O}_A[x]$. In particular, $L(x,j) = L(x,j)^A$. Similarly, the involutions $\tau(x,j)$ and $\tau(x,j)^A$ agree since they only depend on the (*-stable) ring $\mathbb{Q}_{p^2}[x] = A[x]$ and j. This implies that

$$M(x,j) = M(x,j)^A.$$

Let f be the inertia degree of $A_0/\mathbb{Q}_p$. Then $M(x,j)_i = \emptyset$ if $f \nmid i$ and $M(x,j)_{fi} = M(x,j)_i^A$. Note that f is odd since we assumed that $A_0 \otimes \mathbb{Q}_{p^2}$ is a field. In particular $(-1)^i = (-1)^{fi}$ and hence

$$\partial O(x,j) = \sum_{i\in\mathbb{Z}} (-1)^i i |M(x,j)_i| = f \sum_{i\in\mathbb{Z}} (-1)^i i |M(x,j)_i^A| = f \cdot \partial O(x,j)^A.$$

Comparison of the geometric sides of $(\mathrm{AFL}(x,j)_{\mathbb{Q}_p})$ *and* $(\mathrm{AFL}(x,j)_{A_0})$*:*

By Remark 5.6, the cycle $\mathcal{Z}(x) \subset \mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ can be identified with f copies of $\mathcal{Z}(x)^A$, where $\mathcal{Z}(x)^A := \mathcal{Z}(\mathcal{C}(x))$ is the corresponding cycle in $\mathcal{N}_{A_0,(1,n'-1)}$. By Remark 5.4, this identification is compatible with the formation of KR-divisors and hence

$$\mathcal{Z}(x) \cap \mathcal{Z}(j) = \coprod_{i=1}^{f} \mathcal{Z}(x)^A \cap \mathcal{Z}(j)^A$$

where again $\mathcal{Z}(j)^A = \mathcal{Z}(\mathcal{C}(\mathrm{id}_{A_0} \otimes j))$ is the respective cycle in $\mathcal{N}_{A_0,(1,n'-1)}$. It follows that (x,j) is also artinian when viewed over A_0 and

$$\mathrm{Int}(x,j) = f\mathrm{Int}(x,j)^A.$$

The theorem follows. $\square$

Let us translate this back into statements about the AFL in the original formulation from Sections 6 and 7.

Corollary 10.2. *Let $x \in \mathfrak{u}(J_1)^0_{\mathrm{rs}}$ be regular semi-simple and artinian, of the form*

$$x = \begin{pmatrix} x^\flat & j \\ -j^* & \end{pmatrix}.$$

Let $A_0/\mathbb{Q}_p$ be a field extension such that $A := A_0 \otimes \mathbb{Q}_{p^2}$ is again a field, together with an embedding $\mathcal{O}_A \hookrightarrow \mathbb{Z}_{p^2}[x^\flat]$ that is equivariant for the Galois conjugation σ on $\mathcal{O}_A$ and the adjoint involution * on $\mathbb{Z}_p[x^\flat]$. Let $V^A := W \oplus Au^A$ be the hermitian A-vector space with form $J_1^{\flat,A} \oplus 1$.

(1) Then x can also be viewed as an element of $\mathfrak{u}(J_1^{\flat,A} \oplus 1)^0_{\mathrm{rs}}$ and there is an equivalence

$$(\mathfrak{afl}_{\mathbb{Q}_p,V,u,x}) \;\Leftrightarrow\; (\mathfrak{afl}_{A_0,V^A,u^A,x}).$$

(2) In particular, if $\dim_A(W) \leq 2$, then $(\mathfrak{afl}_{\mathbb{Q}_p,V,u,x})$, holds.

Proof. Part (1) is a combination of Lemma 9.15 and Theorem 10.1. Part (2) follows since the AFL has been proven for $n \leq 3$. $\square$

Similarly in the group-theoretic set-up.

Corollary 10.3. *Let $g \in U(J_1)_{\mathrm{rs}}$ be regular semi-simple and artinian with integral characteristic polynomial.*[13] *Let $A_0/\mathbb{Q}_p$ be a field extension such that $A := A_0 \otimes \mathbb{Q}_{p^2}$ is again a field, together with an embedding $\mathcal{O}_A \hookrightarrow \mathbb{Z}_{p^2}[g]$ that is equivariant for the Galois conjugation on A and the adjoint involution on $\mathbb{Z}_{p^2}[g]$. We assume that $J_1^A(u,u) \in \mathcal{O}_{A_0}^\times$, where J_1^A is the lifted hermitian form.*

(1) Then g is also an element of $U(J_1^A)_{\mathrm{rs}}$ and there is an equivalence

$$(AFL_{\mathbb{Q}_p,V,u,g}) \;\Leftrightarrow\; (AFL_{A_0,V,u,g}).$$

(2) In particular, if $\dim_A(V) \leq 3$, then the AFL for g, $(AFL_{\mathbb{Q}_p,V,u,g})$, holds. □

Let us now formulate the variant for $v(J_1^A(u,u)) \geq 1$.

Corollary 10.4. *Assume that $p^f+1 > n$. Let g, A_0, A and the embedding $\mathcal{O}_A \hookrightarrow \mathbb{Z}_{p^2}[g]$ be as in the previous corollary, but let $v(J_1^A(u,u)) \geq 1$. Let $V^A := V \oplus A\tilde{u}$, $u^A := u + \tilde{u}$ and extend J_1^A to a hermitian form $J_1^{A,\sharp}$ on V^A by defining $\tilde{u} \perp V$ and $J_1^{A,\sharp}(u^A, u^A) = 1$.*

Let $P \in A[t]$ be the characteristic polynomial of g as A-linear endomorphism of V and let $a \in A^1$ be such that $P(a) \not\equiv 0$ modulo π_A where π_A is a uniformizer of A. Define $g^A \in U(J_1^{A,\sharp})$ as

$$g^A := \begin{pmatrix} g & \\ & a \end{pmatrix} \in \mathrm{End}(V^A) = \mathrm{End}(V \oplus A\tilde{u}).$$

(1) Then $g^A \in U(J_1^{A,\sharp})_{\mathrm{rs}}$ is regular semi-simple with respect to $V^A = (u^A)^\perp \oplus Au^A$, artinian with respect to u^A and there is an equivalence

$$(\mathrm{AFL}_{\mathbb{Q}_p,V,u,g}) \;\Leftrightarrow\; (\mathrm{AFL}_{A_0,V^A,u^A,g^A}).$$

(2) In particular, if $\dim_A(V) \leq 2$, then the AFL for g, $(\mathrm{AFL}_{\mathbb{Q}_p,V,u,g})$, holds. □

Proof. Part (1) is a combination of Lemma 9.19 and Theorem 10.1. Part (2) follows since the AFL has been proven for $n \leq 3$. □

10.1 Variant for étale algebras

Theorem 10.5. *Let $(x,j) \in \mathrm{End}_{\mathbb{Q}_{p^2}}(V) \times V$ be a regular semi-simple, self-adjoint and artinian pair. Assume that there exists a product decomposition $\mathbb{Z}_{p^2}[x] = R_0 \times R_1$ that is stable under *. Let $V = V_0 \times V_1$ be the corresponding decomposition of V and let (x_0, j_0) resp. (x_1, j_1) denote the components of (x,j) in V_0 resp. V_1. We assume that $J_1|_{V_0}$ is even, which implies that $J_1|_{V_1}$ is odd.*

Then the two identities $(\mathrm{FL}(x_0, j_0))$ and $(\mathrm{AFL}(x_1, j_1))$ imply the identity $(\mathrm{AFL}(x,j))$.

Proof. Computation of the analytic side of $(\mathrm{AFL}(x,j))$:

First note that any x-stable lattice $\Lambda \subset V$ is a product $\Lambda = \Lambda_0 \times \Lambda_1$ where $\Lambda_i \subset V_i$ is an x_i-stable lattice. It is now easy to see that $L(x,j) = L(x_0,j_0) \times L(x_1,j_1)$. Furthermore, $\tau(x,j) = \tau(x_0,j_0) \times \tau(x_1,j_1)$ and hence there is a bijection

$$\begin{aligned} M(x_0,j_0) \times M(x_1,j_1) &\xrightarrow{\cong} M(x,j) \\ (\Lambda_0, \Lambda_1) &\longmapsto \Lambda_0 \times \Lambda_1 \end{aligned}$$

[13] This ensures that $\mathbb{Z}_{p^2}[g]$ is stable under the adjoint involution. Note that both sides of $(\mathrm{AFL}_{\mathbb{Q}_p,V,u,g})$ vanish if the characteristic polynomial of g is not integral. So this is not a serious restriction.

which induces a bijection

$$\coprod_{k+l=m} M(x_0, j_0)_k \times M(x_1, j_1)_l \cong M(x, j)_m.$$

This implies the relation

$$O(x, j; s) = O(x_0, j_0; s) \cdot O(x_1, j_1; s). \tag{10.1}$$

Taking the derivative and using the vanishing part of the FL, Lemma 9.12, we get

$$\partial O(x, j) = O(x_0, j_0) \cdot \partial O(x_1, j_1).$$

Computation of the geometric side of (AFL(x, j))*:* Let $\mathbb{Z}_p \times \mathbb{Z}_p \subset R_0 \times R_1$ be the $\mathbb{Z}_p$-algebra generated by the non-trivial idempotents. Using Proposition 5.14, we get

$$\mathcal{Z}(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}) \cong \left(\coprod_{\{\Lambda_0 \subset V_0 \ \mid\ \Lambda_0^* = \Lambda_0\}} \operatorname{Spf} \breve{\mathbb{Z}}_p \right) \times \mathcal{N}_{\mathbb{Q}_p,(1,n_1-1)}$$

where $n_1 = \dim_{\mathbb{Q}_{p^2}}(V_1)$. By the remarks 5.4 and 5.6, this description is compatible with the formation of $\mathcal{Z}(x)$ and $\mathcal{Z}(j)$ and we get

$$\mathcal{Z}(x) \cap \mathcal{Z}(j) = \left(\coprod_{\{\Lambda_0 \ \mid\ \Lambda_0^* = \Lambda_0, x_0\Lambda_0 \subset \Lambda_0, j_0 \in \Lambda_0\}} \operatorname{Spf} \breve{\mathbb{Z}}_p \right) \times \big(\mathcal{Z}(x_1) \cap \mathcal{Z}(j_1)\big).$$

This implies

$$\operatorname{Int}(x, j) = I(x_0, j_0) \cdot \operatorname{Int}(x_1, j_1)$$

which finishes the proof of the theorem. □

Remark 10.6. Let us assume that the characteristic polynomial of x is integral. Then an inclusion $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^2}[x]$ exists if and only if this polynomial has two different prime factors modulo p. Implicitly, this was already used in [15, Section 8].

We conclude this paper with three corollaries. For this, we take up the notation from Sections 6 and 7.

Corollary 10.7. *Let $x \in \mathfrak{u}(J_1)^0_{\mathrm{rs}}$ be regular semi-simple and artinian, of the form*

$$x = \begin{pmatrix} x^\flat & j \\ -j^* & \end{pmatrix}.$$

Assume that there exists an embedding $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^2}[x^\flat]$ that is equivariant for the factor-wise Galois conjugation on $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and for the adjoint involution of $J_1^\flat$ on $\mathbb{Z}_{p^2}[x^\flat]$. Let $W = W_0 \times W_1$ be the corresponding decomposition of W and assume that $J_1|_{W_0}$ is even. Let $x_0^\flat, x_1^\flat, j_0$ and j_1 be the components of $x^\flat$ and j. For $i = 0, 1$, form the vector space $V_i := W_i \oplus \mathbb{Q}_{p^2} u_i$ where u_i is some additional vector. We extend the form $J_1^\flat|_{W_i}$ to V_i by defining $(u_i, u_i) = 1$ and $u_i \perp W_i$.

Then the element

$$x_i = \begin{pmatrix} x_i^\flat & j_i \\ -j_i^* & \end{pmatrix}$$

lies in $\mathfrak{u}(J_1^\flat|_{W_i} \oplus 1)^0_{\mathrm{rs}}$ and the two identities $(\mathrm{JR}_{\mathbb{Q}_p, V_0, u_0, x_0})$ and $(\mathrm{AFL}_{\mathbb{Q}_p, V_1, u_1, x_1})$ imply the identity $(\mathrm{AFL}_{\mathbb{Q}_p, V, u, x})$.

Proof. This follows from Theorem 10.5, together with Lemmas 9.7 and 9.15. □

Corollary 10.8. *Let $g \in U(J_1)_{\mathrm{rs}}$ be regular semi-simple and artinian. Assume that there exists an embedding $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^2}[g]$ that is equivariant for the Galois conjugation on $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and the adjoint involution of J_1 on $\mathbb{Z}_{p^2}[g]$. Let $V = V_0 \times V_1$ be the corresponding decomposition of V and assume that $J_1|_{V_0}$ is even. Let g_0 and g_1 be the components of g and let u_0 and u_1 be the components of u. Assume that the identity $(\mathrm{FL}(g_0, j_0))$ holds.*

(1) If $J_1(u_1, u_1) \in \mathbb{Z}_p^\times$, then the identity $(\mathrm{AFL}_{\mathbb{Q}_p, V_1, u_1, g_1})$, implies the AFL for g, $(\mathrm{AFL}_{\mathbb{Q}_p, V, u, g})$. In particular, $(\mathrm{AFL}_{\mathbb{Q}_p, V, u, g})$ holds if $\dim V_1 \leq 3$ or if g_1 is minuscule in the sense of [15].

(2) Assume that $p + 1 > n$ and that $v(J_1(u_1, u_1)) \geq 1$. We define $V_1' := V_1 \oplus \mathbb{Q}_{p^2}\tilde{u}_1$ and $u_1' := u_1 + \tilde{u}_1$. We extend $J_1|_{V_1}$ to a hermitian form J_1' on V_1' by defining $\tilde{u}_1 \perp V_1$ and $(u_1', u_1') = 1$. We choose an element $a \in \mathbb{Q}_{p^2}^1$ such that $P(a) \not\equiv 0$ modulo p, where P denotes the characteristic polynomial of g_1 on V_1. We set

$$g_1' := \begin{pmatrix} g_1 & \\ & a \end{pmatrix} \in U(J_1')$$

where the block matrix decomposition is with respect to $V_1' = V \oplus \mathbb{Q}_{p^2}\tilde{u}_1$.

Then $(\mathrm{AFL}_{\mathbb{Q}_p, V_1', u_1', g_1'})$ implies $(\mathrm{AFL}_{\mathbb{Q}_p, V, u, g})$. In particular, $(\mathrm{AFL}_{\mathbb{Q}_p, g})$ holds if $\dim V_1 \leq 2$.

Proof. Part (1) follows from Theorem 10.5 and Lemma 9.16 and the fact that the AFL has been proven in the minuscule case and in the case $n \leq 3$.

Part (2) follows from Theorem 10.5 and Lemma 9.19. □

With an induction argument, we can explicitly formulate the case of more idempotents. Let $A_0/\mathbb{Q}_p$ be a finite étale algebra with decomposition into fields

$$A_0 := \prod_{i \in I} A_{0,i}.$$

We set $A := A_0 \otimes \mathbb{Q}_{p^2}$ and $A_i := A_{0,i} \otimes \mathbb{Q}_{p^2}$, all with Galois conjugation $\sigma := \mathrm{id} \otimes \sigma$. Let $\mathcal{O}_A$ be the ring of integral elements in A.

Any embedding $A \hookrightarrow \mathrm{End}(V)$ that is equivariant for the Galois conjugation of A and the adjoint involution of J_1 on V induces an orthogonal decomposition $V = \prod_{i \in I} V_i$. Just as in Definition 5.12, we call an index i even if there exists a self-dual $\mathcal{O}_{A_i}$-lattice in V_i. Otherwise, we call i odd. Note that since V itself is odd, there is an odd number of odd indices. Also note that if i is odd, then A_i is necessarily a field.

Corollary 10.9. *Let $g \in U(J_1)_{\mathrm{rs}}$ be regular semi-simple and artinian. Assume that there exists an embedding $\mathcal{O}_A \hookrightarrow \mathbb{Z}_{p^2}[g]$ that is equivariant for the Galois conjugation on A and the adjoint involution of J_1 on $\mathbb{Z}_{p^2}[g]$. Let $V = \prod_{i \in I} V_i$ be the corresponding decomposition of V and let $(g_i)_{i \in I}$ and $(u_i)_{i \in I}$ be the components of g and u.*

(1) If there is more than one odd index, then both sides of $(\mathrm{AFL}_{\mathbb{Q}_p, V, u, g})$ vanish.

(2) Otherwise, let $i_0 \in I$ be the unique odd index and let us assume that $(\mathrm{FL}(g_i, u_i))$ holds for $i \neq i_0$. Let us take up the notation from Theorem 10.1 for the factor V_{i_0}. Then

$$(\mathrm{AFL}(g_{i_0}, u_{i_0}))_{A_{i_0}} \quad \Rightarrow \quad (\mathrm{AFL}_{\mathbb{Q}_p, V, u, g}).$$

(3) Under the assumption $J_1^{A_{i_0}}(u_{i_0}, u_{i_0}) \in \mathcal{O}_{A_{0,i_0}}^\times$, we get

$$(\mathrm{AFL}_{A_{0,i_0}, V_{i_0}, u_{i_0}, g_{i_0}}) \quad \Rightarrow \quad (\mathrm{AFL}_{\mathbb{Q}_p, V, u, g}).$$

Proof. First note that $(\mathrm{AFL}_{\mathbb{Q}_p,V,u,g})$ is equivalent to $(\mathrm{AFL}(g,u))$ by Lemma 9.16 and we work with this simpler version.

We first prove Part (1). For the geometric side, note that $\mathcal{Z}(g) \subset \mathcal{Z}(\mathcal{O}_A)$. So if there is more than one odd index, then $\mathcal{Z}(g) \subset \mathcal{N}_{\mathbb{Q}_p,(1,n-1)}$ is empty by Lemma 5.13.

On the analytic side, we use the idempotents $\prod_{i \in I} \mathbb{Z}_{p^2} \subset \mathcal{O}_A \subset \mathbb{Z}_{p^2}[g]$ to get a product decomposition just as in formula (10.1),

$$O(g,u;s) = \prod_{i \in I} O(g_i,u_i;s).$$

Taking the derivative and using the vanishing part of the FL, Lemma 9.12, we get that $\partial O(g,u) = 0$ if there is more than one odd index.

Part (2) follows from Theorem 10.5 by an induction argument and from Theorem 10.1.

Part (3) is then an application of Lemma 9.16. $\square$

Part III

Appendix on strict formal $\mathcal{O}$-modules

Let $\mathcal{O}$ be the ring of integers in a p-adic local field with uniformizer π. Let R be a π-adic $\mathcal{O}$-algebra. In [1], Ahsendorf constructs an equivalence of categories

$$\{\text{strict formal } \mathcal{O}\text{-modules over } R\} \cong \{\text{nilpotent } \mathcal{O}\text{-displays over } R\}\,. \tag{10.2}$$

We refer to [2] for more information. By Lau [8], there is a good notion of duality on the right hand side. This defines good notions of duality and polarization on the left hand side.

By definition, there is also an equivalence

$$\{\text{strict formal } \mathcal{O}\text{-modules over } R\} \cong \{\text{nilpotent displays over } R \text{ with strict } \mathcal{O}\text{-action}\}. \tag{10.3}$$

This equivalence has the advantage that one can forget the $\mathcal{O}$-action on both sides to read off the underlying p-divisible group and its display. This is not possible in (10.2). The aim of this appendix is to identify the correct notion of duality on the right hand side of (10.3).

More precisely, our results are the following. For each finite and totally ramified extension $\mathcal{O} \subset \mathcal{O}'$ of rings of integers in p-adic local fields, we define the *Lubin-Tate $\mathcal{O}'$-frame* $\mathcal{L}_{\mathcal{O}'/\mathcal{O}}(R)$ and prove the equivalence

$$\{\text{strict formal } \mathcal{O}'\text{-modules over } R\} \cong \{\text{nilpotent } \mathcal{L}_{\mathcal{O}'/\mathcal{O}}(R)\text{-windows}\}.$$

Depending on the existence of certain units, this equivalence is compatible with duality, see Lemma 11.2. To prove this compatibility, we reinterpret the construction of Ahsendorf in [2, Definition 2.24] as a base change along a morphism of frames

$$\mathcal{L}_{\mathcal{O}'/\mathcal{O}}(R) \longrightarrow \mathcal{L}_{\mathcal{O}'/\mathcal{O}'}(R).$$

11 Duals and polarizations of windows

We work with the definitions of $\mathcal{O}$-frames and $\mathcal{O}$-windows from [2, Section 3], but keep the terminology of Lau [8] concerning strict and not necessarily strict morphisms of frames. We now recall the definition of the dual $\mathcal{O}$-window.

Let $\mathcal{A} = (S, I, R, \sigma, \dot\sigma)$ be an $\mathcal{O}$-frame and let $\mathcal{P} = (P, Q, F, \dot F)$ be an $\mathcal{A}$-window. Choose a normal decomposition $P = L \oplus T,\ Q = L \oplus IT$ and consider the linearization

$$\mathbf{F} := (\dot F \oplus F)^\sharp : S \otimes_{\sigma, S} (L \oplus T) \longrightarrow P. \tag{11.1}$$

Let $P^\vee := \mathrm{Hom}_S(P, S)$ and $Q^\vee := \{\phi \in P^\vee \mid \phi(Q) \subset I\}$. We define the $\mathcal{A}$-window

$$\mathcal{P}^\vee = (P^\vee, Q^\vee, F^\vee, \dot F^\vee)$$

through the operator $(\mathbf{F}^\vee)^{-1}$ and the normal decomposition $P^\vee = L^\vee \oplus T^\vee, Q^\vee = IL^\vee \oplus T^\vee$, see [2, Lemma 3.6].

Definition 11.1. The $\mathcal{A}$-window $\mathcal{P}^\vee$ is the *dual $\mathcal{A}$-window* of $\mathcal{P}$.

It is clear that dualizing is an anti-equivalence of the category of $\mathcal{A}$-windows and that there is a canonical identification $(\mathcal{P}^\vee)^\vee \cong \mathcal{P}$ coming from the canonical identification $(P^\vee)^\vee \cong P$.

Let

$$\alpha : \mathcal{A} \longrightarrow \mathcal{A}' := (S', I', R', \sigma', \dot\sigma')$$

be a u-morphism of frames for some unit $u \in S'$, i.e. $u\alpha \circ \dot\sigma = \dot\sigma' \circ \alpha$. If $u = 1$, i.e. if α is a strict morphism, and if $\mathcal{P} = (P, Q, F, \dot F)$ is an $\mathcal{A}$-window, then

$$\alpha_*(\mathcal{P}^\vee) \cong (\alpha_* \mathcal{P}^\vee), \tag{11.2}$$

up to the identification $P^\vee \otimes S' \cong (P \otimes S')^\vee$. To treat the case of general u, first recall that the base change along the u-morphism

$$(S, I, R, \sigma, \dot\sigma) \longrightarrow (S, I, R, \sigma, u\dot\sigma)$$

is given by

$$(P, Q, F, \dot F) \mapsto (P, Q, uF, \dot F).$$

Lemma 11.2. *Let $\alpha : \mathcal{A} = (S, I, R, \sigma, \dot\sigma) \longrightarrow (S', I', R', \sigma', \dot\sigma')$ be a u-isomorphism and let $\varepsilon \in S^\times$ be a unit such that $\sigma(\varepsilon)\varepsilon^{-1} = u$. Let $\mathcal{P} = (P, Q, F, \dot F)$ be an $\mathcal{A}$-window. Then multiplication by ε defines an isomorphism*

$$\alpha_*(\mathcal{P}^\vee) \cong (\alpha_* \mathcal{P})^\vee.$$

Proof. Choose a normal decomposition $P = L \oplus T$, $Q = L \oplus IT$ and consider the linearization $\mathbf{F}$ as in (11.1). Then the window $\alpha_*(\mathcal{P}^\vee)$ (resp. $(\alpha_*\mathcal{P})^\vee$) corresponds to the normal decomposition $P^\vee = L^\vee \oplus T^\vee$, $Q^\vee = IL^\vee \oplus T^\vee$ and the operator

$$\left(\begin{smallmatrix} 1 & \\ & u^{-1} \end{smallmatrix}\right) \alpha(\mathbf{F}^\vee)^{-1} \quad \left(\text{resp. } \left(\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right) \alpha(\mathbf{F}^\vee)^{-1}\right).$$

It is clear that multiplication by ε defines an isomorphism. □

Definition 11.3. Let $\mathcal{P}_i = (P_i, Q_i, F_i, \dot F_i)$, for $i = 1, 2, 3$, be three $\mathcal{A}$-windows. We define

$$\mathrm{BiHom}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{P}_3)$$

to be the set of S-bilinear forms $(\ ,\ ) : P_1 \times P_2 \longrightarrow P_3$ such that $(Q_1, Q_2) \subset Q_3$ and such that

$$(\dot F_1 q_1, \dot F_2 q_2) = \dot F_3(q_1, q_2), \quad q_1 \in Q_1, q_2 \in Q_2.$$

Note that $\mathcal{A}$ (or rather just the quadruple $(S, I, \sigma, \dot\sigma)$) is an $\mathcal{A}$-window over itself.

Lemma 11.4. *Let $\mathcal{P} = (P, Q, F, \dot F)$ be an $\mathcal{A}$-window. Then the canonical pairing $\langle\ ,\ \rangle : P \times P^\vee \longrightarrow S$ defines an element in $\mathrm{BiHom}(\mathcal{P} \times \mathcal{P}^\vee, \mathcal{A})$.*

Proof. This can be checked easily after choosing a normal decomposition $P = L \oplus T$, $Q = L \oplus IT$. Let $\mathbf{F}$ be the linearization of $\dot F \oplus F$ as in (11.1). The relation $\langle Q, Q^\vee \rangle \subset I$ is clear. Now for example if $q \in L$ and $\xi q^\vee \in IL^\vee$, then

$$\begin{aligned} \langle \dot F(q), \dot F^\vee(\xi q^\vee) \rangle &= \langle \mathbf{F}(q), \dot\sigma(\xi)(\mathbf{F}^{-1})^\vee(q^\vee) \rangle \\ &= \dot\sigma(\xi)\sigma(\langle q, q^\vee \rangle) \\ &= \dot\sigma(\langle q, \xi q^\vee \rangle). \end{aligned} \tag{11.3}$$

The case $q \in IT$ and $q^\vee \in T^\vee$ is checked analogously. □

Proposition 11.5. *Let $\mathcal{P}_i = (P_i, Q_i, F_i, \dot{F}_i)$, for $i = 1, 2$, be two $\mathcal{A}$-windows. Then pullback of the canonical pairing defines an isomorphism*

$$\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2^\vee) = \mathrm{BiHom}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{A}).$$

This isomorphism is functorial in both $\mathcal{P}_1$ and $\mathcal{P}_2$ and compatible with base change along morphisms of $\mathcal{O}$-frames $\mathcal{A} \longrightarrow \mathcal{A}'$.

Proof. It is clear that the map is injective since it is induced from the analogous isomorphism on underlying S-modules,

$$\mathrm{Hom}_S(P_1, P_2^\vee) \cong \mathrm{BiHom}_S(P_1 \times P_2, S).$$

To prove surjectivity, we consider a homomorphism $f : P_1 \longrightarrow P_2^\vee$ such that the induced bilinear form $(\ ,\ ) : P_1 \times P_2 \longrightarrow S$ lies in $\mathrm{BiHom}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{A})$. We claim that f is a homomorphism of $\mathcal{A}$-windows.

The relation $f(Q_1) \subset Q_2^\vee$ follows immediately from the relation $(Q_1, Q_2) \subset I$. We still have to show $f(\dot{F}_1 q_1) = \dot{F}_2^\vee(f(q_1))$ for all $q_1 \in Q_1$. For this we compute for all $q_2 \in Q_2$,

$$\begin{aligned} \langle f(\dot{F}_1 q_1), \dot{F}_2 q_2 \rangle &= (\dot{F}_1 q_1, \dot{F}_2 q_2) \\ &= \dot{\sigma}(q_1, q_2) \\ &= \dot{\sigma}\langle f(q_1), q_2 \rangle \\ &= \langle \dot{F}_2^\vee f(q_1), \dot{F}_2 f(q_2) \rangle. \end{aligned} \tag{11.4}$$

Now $\dot{F}_2 : Q_2 \longrightarrow P_2$ is a σ-linear epimorphism and $\langle\ ,\ \rangle$ is S-bilinear. This implies

$$\langle f(\dot{F}_1 q_1), p_2 \rangle = \langle \dot{F}_2^\vee(f(q_1)), p_2 \rangle, \quad p_2 \in P_2,$$

which proves $\dot{F}_2^\vee f(q_1) = f(\dot{F}_1 q_1)$. □

In terms of the pairings, the isomorphism from Lemma 11.2 corresponds to scaling the form $\alpha_*(\ ,\ )$ by ε^{-1}.

Definition 11.6. Let $\mathcal{P}$ be an $\mathcal{A}$-window. A *principal polarization* is an isomorphism $\lambda : \mathcal{P} \longrightarrow \mathcal{P}^\vee$ such that $\lambda^\vee : X = (X^\vee)^\vee \longrightarrow X^\vee$ equals $-\lambda$. Equivalently, a polarization is an alternating perfect pairing $\lambda(\ ,\ ) \in \mathrm{BiHom}(\mathcal{P} \times \mathcal{P}, \mathcal{A})$.

11.1 Application to strict formal $\mathcal{O}$-modules

Let us fix a uniformizer $\pi \in \mathcal{O}$. We refer to [3, Section 1.2] for the definition and properties of the relative Witt vectors.

Definition 11.7. For any $\mathcal{O}$-algebra R, we define the *Witt $\mathcal{O}$-frame*[14]

$$\mathcal{W}_\mathcal{O}(R) = (W_\mathcal{O}(R), I_\mathcal{O}(R), {}^F, V)$$

over R as follows. The ring $W_\mathcal{O}(R)$ is the ring of relative $\mathcal{O}$-Witt vectors of R with respect to π. The ideal $I_\mathcal{O}(R)$ is the augmentation ideal

$$I_\mathcal{O}(R) := \ker(W_\mathcal{O}(R) \longrightarrow R)$$

and ^{F} (resp. V) denotes the Frobenius (resp. the Verschiebung). Windows over $\mathcal{W}_\mathcal{O}(R)$ are also called *$\mathcal{O}$-displays over R*, see [2].

[14]There is no need to write the Verschiebung as a superscript in the appendix.

Definition 11.8. Let R be an $\mathcal{O}$-algebra. A *strict $\mathcal{O}$-module* over $S = \operatorname{Spec} R$ is a pair (X, ι) where X/S is a p-divisible group and $\iota : \mathcal{O} \longrightarrow \operatorname{End}(X)$ an action such that $\mathcal{O}$ acts on $\operatorname{Lie}(X)$ via the structure morphism $\mathcal{O} \longrightarrow \mathcal{O}_S$. A strict $\mathcal{O}$-module is called *formal*, if the underlying p-divisible group is formal.

Recall that by Zink [21] and [7] (in the absolute case $\mathcal{O} = \mathbb{Z}_p$) and the extension by Ahsendorf [2] (in the general case) there is an equivalence of categories

$$\{\text{strict formal } \mathcal{O}\text{-modules } /S\} \cong \{\text{nilpotent } \mathcal{O}\text{-displays}/S\}$$

whenever π is nilpotent in R.

Definition 11.9. Let $X = (X, \iota)$ be a strict formal $\mathcal{O}$-module over S with associated $\mathcal{O}$-display $\mathcal{P}$.
i) X is called *biformal* if the dual $\mathcal{O}$-display $\mathcal{P}^\vee$ is also nilpotent. [15]
ii) The dual of a biformal strict $\mathcal{O}$-module X is the strict $\mathcal{O}$-module associated to the dual of its $\mathcal{O}$-display $\mathcal{P}^\vee$.
iii) A *polarization* (resp. *principal polarization*) of the biformal strict $\mathcal{O}$-module X is an isogeny (resp. an isomorphism) $\lambda : X \longrightarrow X^\vee$ such that $\lambda^\vee = -\lambda$.

Remark 11.10. The restriction to biformal strict $\mathcal{O}$-modules is necessary since we only work with $\mathcal{O}$-displays instead of Dieudonné $\mathcal{O}$-displays. See [2, Section 4] for the definition of the dual group in the general case.

Remark 11.11. Note that the definition of the Verschiebung V on $W_\mathcal{O}(R)$ and hence the definition of the dual $\mathcal{O}$-display (resp. the dual strict $\mathcal{O}$-module) depends on the choice of the uniformizer π.

Recall the following results from [2, Section 3]. To any strict formal $\mathcal{O}$-module, there is associated a crystal $\mathbb{D}_X$ on the category of $\mathcal{O}$-pd-thickenings. We denote by $\mathbb{D}_X(S')$ its value at an $\mathcal{O}$-pd-thickening $S \longrightarrow S'$. As in the case of p-divisible groups, there is a Hodge filtration $\mathcal{F} \subset \mathbb{D}_X(S)$ and deformations of X along $\mathcal{O}$-pd-thickenings are in bijection with liftings of the Hodge filtration.

Now assume that X is biformal. It follows from the definitions that there is a perfect pairing

$$\mathbb{D}_X(S') \times \mathbb{D}_{X^\vee}(S') \longrightarrow \mathcal{O}_{S'}.$$

Furthermore, the Hodge filtration $\mathcal{F} \subset \mathbb{D}_X(S)$ is the orthogonal complement of the Hodge filtration $\mathcal{F}^\vee \subset \mathbb{D}_{X^\vee}(S)$ of the dual $\mathcal{O}$-module. In particular if $\lambda : X \longrightarrow X^\vee$ is a principal polarization, then the induced bilinear form on $\mathbb{D}_X$ is alternating and the Hodge filtration $\mathcal{F} \subset \mathbb{D}_X(S)$ is a Lagrangian subspace. Deformations of (X, λ) along an $\mathcal{O}$-pd-thickening are then in bijection with liftings of the Hodge filtration as a Lagrangian subspace.

12 Lubin-Tate frames

Let $\mathcal{O}'/\mathcal{O}$ be a finite, integrally closed and totally ramified extension of degree e and choose a uniformizer $\pi' \in \mathcal{O}'$. For any $\mathcal{O}'$-algebra R, we consider the ring $\mathcal{O}' \otimes_\mathcal{O} W_\mathcal{O}(R)$. We denote the $\mathcal{O}'$-linear extension of the Frobenius by $\sigma := \mathrm{id}_{\mathcal{O}'} \otimes {}^F$. We also define

$$J_{\mathcal{O}'}(R) := \ker(\mathcal{O}' \otimes_\mathcal{O} W_\mathcal{O}(R) \longrightarrow R).$$

[15] Being biformal is equivalent to the slopes 0 and 1 not occurring in the slope filtration of (the relative $\mathcal{O}$-isocrystal of) X at every geometric point of S.

Our aim is now to define a σ-linear epimorphism $\dot\sigma : J_{\mathcal{O}'}(R) \longrightarrow \mathcal{O}' \otimes W_{\mathcal{O}}(R)$ that makes

$$(\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R), J_{\mathcal{O}}(R), R, \sigma, \dot\sigma)$$

into an $\mathcal{O}'$-frame such that strict formal $\mathcal{O}'$-modules over R are equivalent to windows over that frame.

Definition 12.1. Let R be a π-adic $\mathcal{O}'$-algebra. A *Lubin-Tate $\mathcal{O}$-display over R (for the extension $\mathcal{O}'$)* is an $\mathcal{O}$-display $(P, Q, F, \dot F)$ over R equipped with a strict $\mathcal{O}'$-action such that P is free of rank 1 over $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$.

The strictness implies that $Q = J_{\mathcal{O}'}(R)P$. We will usually choose a generator of P and hence consider $\mathcal{O}$-displays of the form

$$(\mathcal{O}' \otimes W_{\mathcal{O}}(R), J_{\mathcal{O}'}(R), F, \dot F).$$

Here, $\mathcal{O}'$ acts naturally on $\mathcal{O}' \otimes W_{\mathcal{O}}(R)$ and both F and $\dot F$ are σ-linear.

Remark 12.2. (1) The definition could be extended to P being only locally free of rank 1 over $\mathcal{O}' \otimes W_{\mathcal{O}}(R)$. But we will not need this.

(2) Let $u \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ be a unit and let $(P, Q, F, \dot F)$ be a Lubin-Tate $\mathcal{O}$-display over R. Then also $(P, Q, uF, u\dot F)$ is a Lubin-Tate $\mathcal{O}$-display.

(3) Let $\dot F : J_{\mathcal{O}'}(R) \longrightarrow \mathcal{O}' \otimes W_{\mathcal{O}}(R)$ be any σ-linear epimorphism. Then there is at most one way to define a σ-linear endomorphism of $\mathcal{O}' \otimes W_{\mathcal{O}}(R)$ which satisfies the identity

$$\dot F(\xi x) = V^{-1}(\xi) F(x), \quad \xi \in \mathcal{O}' \otimes I_{\mathcal{O}}(R),\ x \in \mathcal{O}' \otimes W_{\mathcal{O}}(R). \tag{12.1}$$

Here, V denotes the $\mathcal{O}'$-linear extension of the π-Verschiebung to $\mathcal{O}' \otimes I_{\mathcal{O}}(R)$. It is given by

$$F(x) = \dot F(V(1)x)$$

and it is now a condition that the so-defined F satisfies the relation (12.1) for all ξ. It is enough to check this for $x = 1$ in which case the condition becomes

$$\dot F(\xi) = V^{-1}(\xi) F(1) = V^{-1}(\xi) \dot F(V(1)), \quad \xi \in \mathcal{O}' \otimes I_{\mathcal{O}}(R). \tag{12.2}$$

Proposition 12.3. *Let R be any π-adic $\mathcal{O}'$-algebra.*

(1) For any Lubin-Tate $\mathcal{O}$-display

$$(\mathcal{O}' \otimes W_{\mathcal{O}}(R), J_{\mathcal{O}'}(R), F, \dot F),$$

the element $\kappa := \dot F(\pi' \otimes 1 - 1 \otimes [\pi'])$ is a unit.

(2) For every unit $\kappa \in \mathcal{O}' \otimes W_{\mathcal{O}}(R)$, there exists a unique Lubin-Tate $\mathcal{O}$-display

$$(\mathcal{O}' \otimes W_{\mathcal{O}}(R), J_{\mathcal{O}'}(R), F, \dot F)$$

such that $\dot F(\pi' \otimes 1 - 1 \otimes [\pi']) = \kappa$.

Zink [21, Proposition 26] proves part (2) in the case $\mathcal{O} = \mathbb{Z}_p$ and π-torsion free R. The proof carries over to the case of general $\mathcal{O}$. Applying this result with $R = \mathcal{O}'$ and using base change, we get the existence of Lubin-Tate $\mathcal{O}$-displays for all π-adic $\mathcal{O}'$-algebras R. Applying (2) of Remark 12.2, we get the existence for all units κ. So we are left with proving (1) and the uniqueness assertion from (2).

Proof of Proposition 12.3, part (1). Let us show that κ is a unit. For this recall the following lemma from [21]. It follows from the fact that $W_{\mathcal{O}}(R)$ is $I_{\mathcal{O}}(R)$-adically complete.

Lemma 12.4. *Let R be a π-adic $\mathcal{O}'$-algebra. Then an element $u \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ is a unit if and only if and only if its image in*

$$(\mathcal{O}'/\pi') \otimes_{\mathcal{O}'} (R/\pi')$$

is. □

In particular, we can check that κ is a unit at geometric points $R/\pi' \longrightarrow k$. But then

$$\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(k) \cong W_{\mathcal{O}'}(k)$$

is a complete DVR with uniformizer π' and residue field k. The element $\pi' \otimes 1 - 1 \otimes [\pi']$ maps to a generator of $J_{\mathcal{O}'}(k) = \pi' W_{\mathcal{O}'}(k)$. In particular, it is sent to a unit by (the base change to k) of $\dot{F}$. This finishes the proof of part (1). □

Lemma 12.5. *There exists an element $\theta \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(\mathcal{O}')$ with the following two properties.*

(i) $\theta J_{\mathcal{O}'}(\mathcal{O}') \subset \mathcal{O}' \otimes I_{\mathcal{O}}(\mathcal{O}')$.

(ii) The image of θ under $\mathcal{O}' \otimes W_{\mathcal{O}}(\mathcal{O}') \longrightarrow \mathcal{O}' \otimes W_{\mathcal{O}}(\mathcal{O}'/\pi') \cong \mathcal{O}'$ has valuation $e-1$.

Proof. First note that for any π-adic $\mathcal{O}'$-algebra R, the ring $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ has the $W_{\mathcal{O}}(R)$-basis

$$1 \otimes 1, (\pi')^i \otimes 1 - 1 \otimes [\pi']^i, \quad i = 1, \dots, e-1.$$

In particular,

$$J_{\mathcal{O}'}(R) = \mathcal{O}' \otimes_{\mathcal{O}} I_{\mathcal{O}}(R) + (\pi' \otimes 1 - 1 \otimes [\pi'])\mathcal{O}' \otimes W_{\mathcal{O}}(R).$$

Thus the first condition is equivalent to $\theta(\pi' \otimes 1 - 1 \otimes [\pi']) \in \mathcal{O}' \otimes I_R$. If $\bar{\theta}$ denotes the image of θ in $\mathcal{O}' \otimes \mathcal{O}'$, then this is equivalent to

$$\bar{\theta}(\pi' \otimes 1 - 1 \otimes \pi') = 0.$$

Informally, we define $\bar{\theta}$ as the fraction

$$\frac{N\pi' \otimes 1 - 1 \otimes N\pi'}{\pi' \otimes 1 - 1 \otimes \pi'} \in \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}'$$

where $N\pi'$ denotes the norm of π' with respect to the ring extension $\mathcal{O}'/\mathcal{O}$. Of course, this does not make sense since the numerator vanishes and the denominator is a zero divisor. The precise definition is as follows. Let

$$(\pi')^e + a_{e-1}(\pi')^{e-1} + \ldots + a_1\pi' + (-1)^e N\pi' = 0$$

be the Eisenstein equation of π'. Then we set

$$(-1)^e\bar{\theta} := -\frac{(\pi')^e \otimes 1 - 1 \otimes (\pi')^e}{\pi' \otimes 1 - 1 \otimes \pi'} - \sum_{i=1}^{e-1} a_i \frac{(\pi')^i \otimes 1 - 1 \otimes (\pi')^i}{\pi' \otimes 1 - 1 \otimes \pi'}$$

where each summand is understood as a geometric series. Let $\theta \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(\mathcal{O}')$ be any lift of $\bar{\theta}$. Then θ satisfies (i) by construction and we are left with verifying (ii).

Consider the quotient

$$\beta : \mathcal{O}' \otimes W_{\mathcal{O}}(\mathcal{O}') \longrightarrow \mathcal{O}' \otimes W_{\mathcal{O}}(\mathcal{O}'/\pi') \longrightarrow \mathcal{O}' \otimes (W_{\mathcal{O}}(\mathcal{O}'/\pi')/\pi).$$

Then θ satisfies (ii) if and only if $\beta(\theta) \neq 0$ and $\pi'\beta(\theta) = 0$. Now note that $\beta(\mathcal{O}' \otimes I_{\mathcal{O}}(\mathcal{O}')) = 0$ and hence β factors through $\mathcal{O}' \otimes \mathcal{O}'$. It is easy to see that the image of $\bar{\theta}$ in $\mathcal{O}' \otimes (W(\mathcal{O}'/\pi')/\pi)$ satisfies these two properties. □

Lemma 12.6. *Let θ be an element as in Lemma 12.5 and let V denote the $\mathcal{O}'$-linear extension of the Verschiebung to $\mathcal{O}' \otimes W_{\mathcal{O}}(R)$. Then*

$$V^{-1}(\theta(\pi' \otimes 1 - 1 \otimes [\pi']))$$

is a unit in $\mathcal{O}' \otimes W_{\mathcal{O}}(\mathcal{O}')$.

Proof. This can be checked in $\mathcal{O}' \otimes W_{\mathcal{O}}(\mathcal{O}'/\pi') \cong \mathcal{O}'$. But here, the Verschiebung V is simply multiplication by π. Using property (ii) from Lemma 12.5, we get the result. □

Proof of Proposition 12.3, part (2). Let us prove the uniqueness of a Lubin-Tate $\mathcal{O}$-display structure

$$(\mathcal{O}' \otimes W_{\mathcal{O}}(R), J_{\mathcal{O}'}(R), F, \dot{F})$$

with $\dot{F}(\pi' \otimes 1 - 1 \otimes [\pi']) = \kappa$.

Note that $\dot{F}$ is determined on $(\pi' \otimes 1 - 1 \otimes [\pi'])\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ by σ-linearity and by κ. Since

$$J_{\mathcal{O}'}(R) = \mathcal{O}' \otimes I_{\mathcal{O}}(R) + (\pi' \otimes 1 - 1 \otimes [\pi'])\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R),$$

we are left with showing that κ determines $\dot{F}$ on $\mathcal{O}' \otimes I_{\mathcal{O}}(R)$. By the relation (12.2), it is enough to show that $F(1)$ is determined by κ.

Now we use the element θ from Lemma 12.5. We set $a := \theta(\pi' \otimes 1 - 1 \otimes [\pi']) \in \mathcal{O}' \otimes I_{\mathcal{O}}(R)$. Then

$$\dot{F}(a) = \dot{F}(\theta(\pi' \otimes 1 - 1 \otimes [\pi'])) = \sigma(\theta)\kappa$$

but also

$$\dot{F}(a) = V^{-1}(a)F(1).$$

By Lemma 12.6, $V^{-1}(a)$ is a unit and hence $F(1)$ is determined by κ. □

Definition 12.7. Let R be a π-adic $\mathcal{O}'$-algebra. A *Lubin-Tate $\mathcal{O}'$-frame over R* is an $\mathcal{O}'$-frame of the form

$$(\mathcal{O}' \otimes W_{\mathcal{O}}(R), J_{\mathcal{O}}(R), R, \sigma, \dot{\sigma})$$

where $\dot{\sigma}$ is a σ-linear epimorphism satisfying the relation analogous to (12.2),

$$\dot{\sigma}(\xi) = V^{-1}(\xi)\dot{\sigma}(V(1)).$$

In other words, $\dot{\sigma}$ is coming from a Lubin-Tate $\mathcal{O}$-display. For a unit $\kappa \in \mathcal{O}' \otimes W_{\mathcal{O}}(R)$, we denote by

$$\mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)$$

the Lubin-Tate $\mathcal{O}'$-frame such that $\dot{\sigma}(\pi' \otimes 1 - 1 \otimes [\pi']) = \kappa$. By Proposition 12.3, such a $\dot{\sigma}$ exists and is unique.

Remark 12.8. By [8, Lemma 2.2], there exists a unique element $s \in \mathcal{O}' \otimes W_{\mathcal{O}}(R)$ such that $\sigma(\xi) = s\dot{\sigma}(\xi)$ for all $\xi \in J_{\mathcal{O}'}(R)$. For the $\mathcal{O}'$-frame $\mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)$, this element is $s = \kappa^{-1}\sigma(\pi' \otimes 1 - 1 \otimes [\pi'])$.

Example 12.9. (1) We consider the case $\mathcal{O}' = \mathcal{O}$. For any π-adic $\mathcal{O}$-algebra R, the Witt $\mathcal{O}$-frame $\mathcal{W}_{\mathcal{O}}(R)$ is an example of a Lubin-Tate $\mathcal{O}$-frame. It agrees with $\mathcal{L}_{\mathcal{O}/\mathcal{O},\varepsilon}(R)$, where $\varepsilon \in W_{\mathcal{O}}(R)$ is the unit

$$\varepsilon = V^{-1}(\pi - [\pi]).$$

(2) We return to the case of an arbitrary totally ramified extension $\mathcal{O}'/\mathcal{O}$. Let θ be an element as in Lemma 12.5. We define $\dot{\sigma} : J_{\mathcal{O}'}(R) \longrightarrow \mathcal{O}' \otimes W_{\mathcal{O}}(R)$ as

$$\dot{\sigma}(x) = V^{-1}(\theta x).$$

Then for $\xi \in \mathcal{O}' \otimes I_{\mathcal{O}}(R)$,

$$\dot{\sigma}(\xi) = V^{-1}(\theta\xi) = \sigma(\theta)V^{-1}(\xi) = V^{-1}(\xi)\dot{\sigma}(V(1))$$

because of the identity $\theta V(1) = V(\sigma(\theta))$. Thus $\dot{\sigma}$ defines the Lubin-Tate $\mathcal{O}'$-frame $\mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)$ where κ is the unit $V^{-1}(\theta(\pi' \otimes 1 - 1 \otimes [\pi']))$ from Lemma 12.6.

We now consider a tower of extensions $\mathcal{O}''/\mathcal{O}'/\mathcal{O}$, all totally ramified. We fix uniformizers π'', π' and π in the respective rings. Recall from [3] that for any $\mathcal{O}'$-algebra R, there is a natural map of $\mathcal{O}$-algebras

$$\alpha : W_{\mathcal{O}}(R) \longrightarrow W_{\mathcal{O}'}(R).$$

This map is Frobenius equivariant and satisfies $\alpha \circ V_\pi = \frac{\pi}{\pi'} V_{\pi'} \circ \alpha$ where V_π and $V_{\pi'}$ denote the respective Verschiebung maps.

Proposition 12.10. *Let R be a π-adic $\mathcal{O}''$-algebra and let $\kappa \in \mathcal{O}'' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ be a unit. Then the natural map of $\mathcal{O}''$-algebras*

$$\alpha : \mathcal{O}'' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) \longrightarrow \mathcal{O}'' \otimes_{\mathcal{O}'} W_{\mathcal{O}'}(R)$$

induces a strict morphism of $\mathcal{O}''$-frames

$$\mathcal{L}_{\mathcal{O}''/\mathcal{O},\kappa}(R) \longrightarrow \mathcal{L}_{\mathcal{O}''/\mathcal{O}',\alpha(\kappa)}(R).$$

In other words, α commutes with the $\dot{\sigma}$-operators.

Proof. Let us consider the Lubin-Tate $\mathcal{O}$-display $(\mathcal{O}'' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R), J_{\mathcal{O}}(R), F, \dot{\sigma})$ underlying the $\mathcal{O}''$-frame $\mathcal{L}_{\mathcal{O}''/\mathcal{O},\kappa}(R)$. By [2, Proposition 2.23], there exists a Lubin-Tate $\mathcal{O}'$-display $(\mathcal{O}'' \otimes_{\mathcal{O}'} W_{\mathcal{O}'}(R), J_{\mathcal{O}'}(R), F', \dot{\sigma}')$ over R such that $\alpha \circ \dot{\sigma} = \dot{\sigma}' \circ \alpha$. The corresponding Lubin-Tate $\mathcal{O}''$-frame then equals $\mathcal{L}_{\mathcal{O}''/\mathcal{O}',\alpha(\kappa)}(R)$ which proves the proposition. $\square$

12.1 Windows over Lubin-Tate frames

Proposition 12.11. *Let $\mathcal{O}'/\mathcal{O}$ be a totally ramified extension of rings of integers in p-adic local fields. Let R be a π-adic $\mathcal{O}'$-algebra and $\kappa \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$ a unit. Then there is an equivalence of categories*

$$\{\textit{strict formal } \mathcal{O}'\textit{-modules over } R\} \cong \{\textit{nilpotent } \mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)\textit{-windows}\}.$$

This equivalence is compatible with base change in R and with base change along the morphisms of $\mathcal{O}'$-frames

$$\mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R) \longrightarrow \mathcal{L}_{\mathcal{O}'/\tilde{\mathcal{O}},\tilde{\kappa}}(R)$$

for intermediate extensions $\mathcal{O} \subset \tilde{\mathcal{O}} \subset \mathcal{O}'$.

Proof. Let X/R be a formal $\mathcal{O}$-module equipped with a strict $\mathcal{O}'$-action $\iota : \mathcal{O}' \longrightarrow \mathrm{End}(X)$. Let $\mathcal{P} := (P, Q, F, \dot{F})$ be its $\mathcal{O}$-display. Then P is naturally an $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R)$-module and $J_{\mathcal{O}'}(R)P \subset Q$. Furthermore, the map $\dot{F}$ is a σ-linear epimorphism $Q \longrightarrow P$. Then there is at most one way to define a σ-linear operator $F' : P \longrightarrow P$ which makes $(P, Q, F', \dot{F})$ into an $\mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)$-window, namely

$$F'(x) := \kappa^{-1}\dot{F}((\pi' \otimes 1 - 1 \otimes [\pi'])x).$$

We need to verify that this F' satisfies

$$\dot{F}(\xi x) = \dot{\sigma}(\xi)F'(x), \quad \xi \in J_{\mathcal{O}'}(R),\ x \in P. \tag{12.3}$$

It is enough to verify this for one single κ since all other choices multiply both sides of the equation by a unit. So we choose

$$\kappa = V^{-1}(\theta(\pi' \otimes 1 - 1 \otimes [\pi']))$$

where θ is an element as in Lemma 12.5. In other words, we work with the Lubin-Tate $\mathcal{O}'$-frame from Example 12.9 (2).

Both sides in equation (12.3) are σ-linear, so it is enough to verify the relation for $\xi = (\pi' \otimes 1 - 1 \otimes [\pi'])$ or $\xi \in I_{\mathcal{O}}(R)$. (These elements generate $J_{\mathcal{O}'}(R)$ as ideal.) The case $\xi = (\pi' \otimes 1 - 1 \otimes [\pi'])$ is the definition of F'. In the case $\xi \in I_{\mathcal{O}}(R)$, we compute

$$\begin{aligned}
\dot{\sigma}(\xi)F'(x) &= V_{\pi}^{-1}(\theta\xi)F'(x) \\
&= \sigma(\theta)V^{-1}(\xi)F'(x) \\
&= \sigma(\theta)V^{-1}(\xi)\kappa^{-1}\dot{F}((\pi' \otimes 1 - 1 \otimes [\pi'])x) \\
&= V^{-1}(\xi)\kappa^{-1}\dot{F}(\theta(\pi' \otimes 1 - 1 \otimes [\pi'])x) \\
&= V^{-1}(\xi)F(x) = \dot{F}(\xi x).
\end{aligned}$$

In the last step, we used that $\mathcal{P}$ is an $\mathcal{O}$-display.

Thus we get a functor

$$\{\text{strict formal } \mathcal{O}'\text{-modules over } R\} \longrightarrow \{\text{nilpotent } \mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)\text{-windows}\}$$

which commutes with base change in R and in the Lubin-Tate $\mathcal{O}'$-frame.

To prove that this functor is an equivalence, we construct its inverse. Again it suffices to do this in the special case of $\kappa = V^{-1}(\theta(\pi' \otimes 1 - 1 \otimes [\pi']))$. Given any $\mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R)$-window $(P, Q, F', \dot{F})$, we define a σ-linear operator $F : P \longrightarrow P$ by the formula

$$F(x) = F'(\theta x)$$

We only need to check that this defines an $\mathcal{O}$-display, i.e. that $\dot{F}(\xi x) = V^{-1}(\xi)F(x)$ for all $\xi \in I_{\mathcal{O}}(R)$. But

$$\dot{F}(\xi x) = V^{-1}(\theta\xi)F'(x) = \sigma(\theta)V^{-1}(\xi)F'(x) = V^{-1}(\xi)F(x).$$

The compatibility with base change along the morphisms of frames $\alpha : \mathcal{L}_{\mathcal{O}'/\mathcal{O},\kappa}(R) \longrightarrow \mathcal{L}_{\mathcal{O}'/\tilde{\mathcal{O}},\tilde{\kappa}}(R)$ is clear. Namely let $\alpha_*\mathcal{P} = (P', Q', F', \dot{F}')$ be the base change of $\mathcal{P}$ and let $\mathcal{P}'' = (P'', Q'', F'', \dot{F}')$ be the $\mathcal{L}_{\mathcal{O}'/\tilde{\mathcal{O}},\tilde{\kappa}}(R)$-window constructed from the $\tilde{\mathcal{O}}$-display of (X, ι). Then

$$P' = (\mathcal{O}' \otimes_{\tilde{\mathcal{O}}} W_{\tilde{\mathcal{O}}}(R)) \otimes P = P''$$

by [2, Definition 2.24] which relates the $\mathcal{O}$-display of X and its $\tilde{\mathcal{O}}$-display. Furthermore, the submodules Q' and Q'' agree under this identification. Now both $\dot{F}'$ and $\dot{F}''$ are determined by the condition that they agree with $\dot{F}$ on the image of Q. Since F' and F'' are determined by $\dot{F}'$ and $\dot{F}''$, the windows $\mathcal{P}'$ and $\mathcal{P}''$ agree. □

Corollary 12.12. *The morphisms of frames from Proposition 12.10*

$$\mathcal{L}_{\mathcal{O}''/\mathcal{O},\kappa}(R) \longrightarrow \mathcal{L}_{\mathcal{O}''/\mathcal{O}',\kappa'}(R)$$

are all crystalline, i.e. they induce equivalences on their categories of windows.

Proof. This is just a reformulation of the fact that the equivalence in the previous proposition commutes with the base change along such morphisms of $\mathcal{O}''$-frames. □

13 The unramified case

For completeness, we also include the case of an unramified extension $\mathcal{O}'/\mathcal{O}$. Let f be the degree of the extension. Again we fix a uniformizer $\pi \in \mathcal{O}$. For a π-adic $\mathcal{O}'$-algebra R, there exists a unique morphism

$$\mathcal{O}' \longrightarrow W_{\mathcal{O}}(R)$$

that lifts the given morphism $\mathcal{O}' \longrightarrow R$. In particular, there is a direct product decomposition

$$\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R) \cong \prod_{\mathbb{Z}/f} W_{\mathcal{O}}(R).$$

Definition 13.1. For a π-adic $\mathcal{O}'$-algebra R, we define the $\mathcal{O}'$-frame

$$\mathcal{A}_{\mathcal{O}'/\mathcal{O}}(R) := (W_{\mathcal{O}}(R), I_{\mathcal{O}}(R), {}^{F^f}, {}^{F^{f-1}}V^{-1}).$$

Windows over $\mathcal{A}_{\mathcal{O}'/\mathcal{O}}(R)$ are also called f-$\mathcal{O}$-displays, see [2].

In his thesis [1], Ahsendorf constructs a functor

$$\gamma : \{\text{strict formal } \mathcal{O}'\text{-modules over} R\} \longrightarrow \{\text{nilpotent } f\text{-}\mathcal{O}\text{-displays over } R\}.$$

Furthermore, the natural morphism

$$W_{\mathcal{O}}(R) \longrightarrow W_{\mathcal{O}'}(R)$$

induces a strict morphism of frames

$$\mathcal{A}_{\mathcal{O}'/\mathcal{O}}(R) \longrightarrow \mathcal{W}_{\mathcal{O}'}(R)$$

and thus gives rise to a functor

$$\delta : \{f\text{-}\mathcal{O}\text{-displays over } R\} \longrightarrow \{\mathcal{O}'\text{-displays over} R\}$$

that is compatible with duality by Lemma 11.2. Ahsendorf proves that the composition of these functors is an equivalence of categories. We slightly strengthen this result as follows.

Proposition 13.2. *Let R be a noetherian π-adic $\mathcal{O}'$-algebra. Then the above functors γ and δ are both equivalences of categories (when restricted to the full subcategories of nilpotent windows).*

Proof. By Ahsendorf, the composition $\delta \circ \gamma$ is an equivalence of categories, at least when restricted to the full subcategories of nilpotent windows. It is hence enough to prove that either of these functors is an equivalence. It would even be enough to just prove the faithfulness of δ. But for later use, we construct a quasi-inverse for γ. For this, we first recall the construction of this functor.

Let $\mathcal{P} = (P, Q, F, \dot{F})$ be the $\mathcal{O}$-display of a strict formal $\mathcal{O}'$-module over R. Then the natural map $\mathcal{O}' \longrightarrow W_{\mathcal{O}}(R)$ induces a $\mathbb{Z}/f$-grading

$$P = \bigoplus_{i \in \mathbb{Z}/f} P_i$$

such that both F and $\dot{F}$ are homogeneous of degree 1. The strictness implies that $Q = Q_0 \oplus P_1 \oplus \ldots \oplus P_{f-1}$. In particular, the restriction $\dot{F}_i := \dot{F}|_{Q_i}$ is an ^{F}-linear isomorphism $P_i \longrightarrow P_{i+1}$ for $i = 1, \ldots, f-1$. The f-$\mathcal{O}$-display is now given by

$$(P_0, Q_0, \dot{F}^{f-1} \circ F|_{P_0}, \dot{F}^f|_{Q_0}).$$

Let us phrase this construction in terms of an $\mathcal{O}'$-stable normal decomposition $P = L \oplus T$. The $\mathcal{O}'$-stability is equivalent to the fact that both L and T are compatible with the grading, hence have the form

$$\begin{aligned} L &= L_0 \oplus P_1 \oplus \ldots \oplus P_{f-1}, \\ T &= T_0 \oplus 0 \oplus \ldots \oplus 0. \end{aligned}$$

Let $\Phi := \dot{F}|_L \oplus F|_T = \oplus_{i \in \mathbb{Z}/f} \Phi_i$ be the ^{F}-linear automorphism of P associated to the normal decomposition. We use Φ_i to identify P_{i+1} with $P_i^{(F)}$. Hence the display $\mathcal{P}$ together with its $\mathcal{O}'$-action can be describes as follows. The modules are of the form

$$\begin{aligned} P &= P_0 \oplus P_0^{(F)} \oplus P_0^{(F^2)} \oplus \ldots \oplus P_0^{(F^{f-1})}, \\ Q &= Q_0 \oplus P_0^{(F)} \oplus P_0^{(F^2)} \oplus \ldots \oplus P_0^{(F^{f-1})} \end{aligned}$$

and the display structure is given by the normal decomposition

$$\begin{aligned} L &= L_0 \oplus P_0^{(F)} \oplus P_0^{(F^2)} \oplus \ldots \oplus P_0^{(F^{f-1})}, \\ T &= T_0 \oplus 0 \oplus \ldots \oplus 0 \end{aligned}$$

and the ^{F}-linear operator

$$\Phi = \begin{pmatrix} & & & & \phi \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}$$

where $\phi = \Phi_{f-1}$.

It is now obvious, how to invert this construction. Given an $\mathcal{O}'$-display $\mathcal{P}' = (P', Q', F', \dot{F}')$, we set $P_0 := P'$ and $Q_0 := Q'$. Then we define P and Q by the above formulas. If $(P' = L' \oplus T', \phi)$ is a normal decomposition of $\mathcal{P}'$, then we set $L_0 := L'$, $T_0 := T'$ and define a normal decomposition and the operator Φ by the above formulas.

Functoriality of this construction is clear. $\square$

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