Period, Central Charge and Effective Action on Ricci-Flat Manifolds with Special Holonomy

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Contents

CHAPTER 1

Introduction and Overview

In the thesis we primarily concern the study of modern mathematical physics research, especially motivated by (topological) string theory and mirror symmetry. The thesis is divided into two parts. In the first part, we introduce a notion of *numerical vectors* (Chapter 3), which are sort of group homomorphisms preserving Grothendieck-Riemann-Roch (GRR) formula from the Grothendieck group to the cohomology group of some smooth varieties over $\mathbb{C}$, such as Chern character and Mukai vector, and them apply those properties to relevant subjects. Using stability data and t-stability, we introduce notions of *numerical t-stabilities* and *numerical slope functions* on triangulated categories. The study of the derived categories of surfaces and Calabi-Yau threefolds leads us to a conjecture which gives a relation between numerical t-stability and Bridgelands stability on smooth varieties. And when there exists generalized twisted Mukai vectors, we also obtain the results regarding the cohomological Fourier-Mukai (FM) transforms associated to the FM ones on the level of derived categories. In some cases, these cohomological FM transforms agree with the ones on the derived categories of twisted sheaves.

In the second part, we discuss geometric and topological properties of G_2 manifolds, which is a special kind of seven-dimensional space constructed by Dominic Joyce, and these G_2 manifolds are still poorly understood mathematically. In recent years, the situation has improved due to the Kovalev's twisted connected sum constrction, which has been generalized. In the Kovalev limit the Ricci-flat metrics on $X_{L/R}$ approximate the Ricci-flat G_2-metrics and we identify the universal modulus, called the Kovalevton, that parametrizes this limit. Moreover, the low energy effective theory exhibits gauge theory sectors with extended supersymmetry in this limit . The universal (semi-classical) Kähler potential of the effective $\mathcal{N} = 1$ supergravity action is a function of the Kovalevton and the volume modulus of the G_2-manifold. We describe geometric degenerations in $X_{L/R}$, which lead to non-Abelian gauge symmetries enhancements with various matter content. Studying the resulting gauge theory branches, we argue that they lead to transitions compatible with the gluing construction and provide many new explicit examples of G_2-manifolds.

Physics, knowledge of nature, is the scientific study of matter and energy, the effect that they have on each other, and their motion through space and time. Mathematical physics is the field of the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories[1]. In the following, We start with giving a brief overview of the principles of fundamental physics which provide a clever and beautiful picture of universe within a mathematically rigorous framework. Indeed, we still need new physics and mathematical framework which are able to explain open and conceptual questions arising from the unification of quantum theory and general relativity.

[1] Definition in The Journal of Mathematical Physics.

1.1 Principles of fundamental physics

1.1.1 Classical mechanics

Newtonian mechanics, and its abstract, rigorous reformulations: Lagrangian mechanics and Hamiltonian mechanics form foundations of classical physics. Many mathematical concepts and methods are used in classical mechanics, and many modern mathematical theories arose from physical problems in mechanics and later acquired the abstract axiomatic formalization. These ideas and approaches have been extended to other area of physics as relativity, classical and quantum field theory, etc. Furthermore, they also provided basic ideas and examples in differential geometry (Lagrangian mechanics, see § 1.1.3 and symplectic geometry (Hamiltonian mechanics).

Let's imagine we're creating a world. If there is no object, morphisms (principle, rule and law) in it, then nothing would evolve and nothing would be terminated. What do we have is a trivial, boring and stable universe, which is obviously not the universe we're living in. Essence precedes existence for our universe. In the following[2] we list a series of experimental facts (postulates), the basic principle of relativity, and Newton's principle of determinacy which form the basics of mechanics. All these experimental facts are only approximately true and can be refuted by more accurate experiments.

Space and Time: Our space is euclidean, three-dimensional, and time is one-dimensional;

Galileo's Principle of Relativity: there exist coordinate systems, called inertial systems, having properties that all the laws of nature at all moments of times are the same in all inertial systems, and all coordinate systems uniform rectilinear motion with respect to an inertial system are themselves inertial;

Newton's Principle of Determinacy: The initial state of a mechanic system (positions and velocities of its points at some moment of time) uniquely determine all its motions.

Let $\mathbb{A}^n$ be an affine n-dimensional space, i.e. just $\mathbb{R}^n$ without the fixed origin 0. Indeed, there exist a group action $\mathbb{R}^n$ acting on $\mathbb{A}^n$ as the group of parallel transport: for all $x, y \in \mathbb{A}^n$, there exists a unique $v_{x,y} \in \mathbb{R}^n$ such that $v_{x,y} = y - x$, and a distance function (metric) defined as $\rho(x, y) = \|x - y\|$. The postulate of geometric structure of space-time is described as Galilean space-time structure.

The Universe: The universe is a four dimensional affine space $\mathbb{A}^4$, and the points in $\mathbb{A}^4$ are called world events;

Time: Let $t : \mathbb{R}^4 \to \mathbb{R}$ be a linear mapping. The time interval between events $a \in \mathbb{A}^4$ and $b \in \mathbb{A}^4$ is the number $t(b - a)$. If $t(b - a) = 0$, then events a and b are called simultaneous;

Metric: The space of simultaneous events with a given event, i.e. the kernel of t, form a three dimensional affine subspace $\mathbb{A}^3$ in $\mathbb{A}^4$, and the distance function between simultaneous events is $\rho(a, b) = \|a-b\|$ for all $a, b \in \mathbb{A}^3$.

An affine space $\mathbb{A}^4$ equipped with a Galilean space-time structure is called a *Galilean space*. The *Galilean group* is the group of all transformations of a Galilean space which preserve its structure. Galilean transformations are affine transformations of $\mathbb{A}^4$ which preserving time intervals and distance between simultaneous events. The Galilean group of a Galilean coordinate space $\mathbb{R} \times \mathbb{R}^3$ is generated by a uniform motion, a translation and a rotation, and thus its dimension is 10.

Since all motions of a n-points mechanical system are uniquely determined by their initial states at the moment $t_0 \in \mathbb{R}$ (positions $\mathbf{x}(t_0) \in \mathbb{R}^{3n}$ and velocities $\dot{\mathbf{x}}(t_0) \in \mathbb{R}^{3n}$). In particular, a motion is defined by a

[2] We follow the argument in the excellent book [Arn89].

smooth mapping from a interval $I \subset \mathbb{R}$ to $\mathbb{R}^3$, and there exists a function $\mathbf{F} : \mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \mathbb{R} \to \mathbb{R}^{3n}$ such that

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{1.1}$$

called *Newton's equation*. It is the *Newton's second law*. By existence and uniqueness of solutions to ordinary differential equations, the function $\mathbf{F}$ and the initial states $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ uniquely determine a motion. By the Galileo's principle of relativity, Newton's equation (1.1) must be invariant with respect to the Galilean group.

Under the assumption without any previous knowledge of physics, we can deduce the *Newton's first law*: given a mechanical system consists of only one point, its acceleration in an inertial system is equal to zero. Moreover, for a system consists of two points with zero initial velocities in some inertial coordinate systems, the points will stay on the line which connected them at the initial moment (weak version of law of conservation of momentum). In the case of a three points system with their initial velocities are equal to zero in some inertial system, we also can show that the points always remain in the plane which contained them at the initial moment (weak version of law of conservation of angular momentum).

In terms of category theories, the theory of classical mechanics could be considered as a functor (CM functor) from the category of universe to the category of classical mechanics, see Fig. 1.1. Here, the category of universe consists of a set of spacetime as of objects and a set of principles as of morphisms, and the category of classical mechanics consists of a Galilean spacetime and Galilean group.

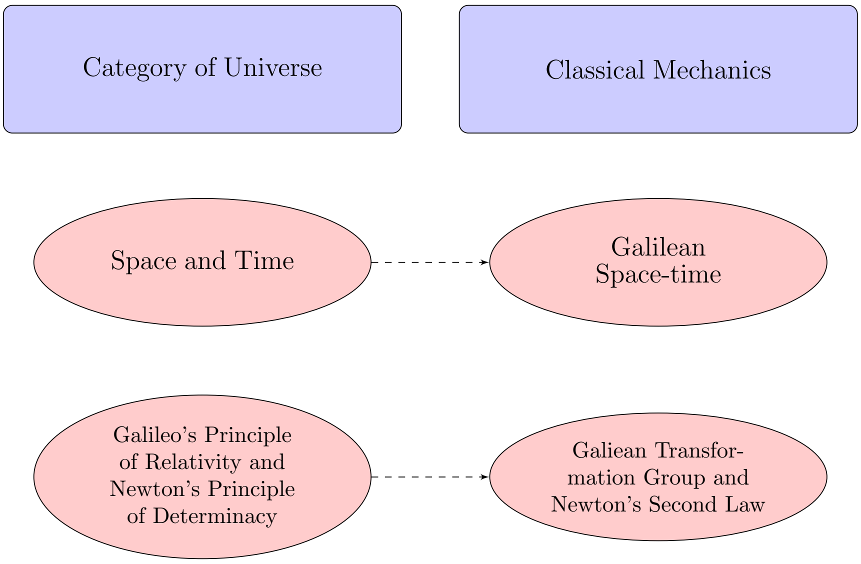

Figure 1.1: CM functor from the category of universe to of classical mechanics.

1.1.2 General relativity

Euclidean geometry had its origins as the description of space-time in physical world, and these physical postulates could be alternatively viewed as mathematical axioms. Indeed, Euclidean geometry gives the local structure of space and time, see § 1.1.1. The mathematical deductions made from the global properties of Euclidean geometry seems to be not validated by any experiments and are not general true. The description should be considered infinitesimally rather then globally. In this subsection[3], we briefly give basic experiment fact and analysis of space-time which lead almost to Lorentzian geometry and (pseudo-)Riemannian geometry.

The fundamental Michelson-Morley experiment (1887) indicates that the velocity of light has an absolute value c, and the findings imply that the set of all possible light rays forms a further invariant of nature. More precisely, given a event o at the moment that the flash of light is emitted and the space of simultaneous events with respect to o, $E_o = \{x \in \mathbb{A}^4 \mid t(x-o) = 0\}$. In the corresponding Galilean coordinate space $\mathbb{R} \times \mathbb{R}^3$, each vector v can be uniquely decomposed as $v = t(v)\tau + \mathbf{v}$, where $t(\tau) = 1$ and $t(\mathbf{x}) = 0$. A light ray sent out at any point x in E_o with the spatial velocity $\mathbf{c}$ is the curve $x + \mathbb{R}(\tau + \mathbf{c})$ in $\mathbb{A}^4$. The corresponding future light cone at o is given as

$$C_o^+ = \left\{ y \in \mathbb{A}^4 \mid \|c\|^2 (t(y-o))^2 = \|\mathbf{y} - \mathbf{o}\|^2, t(y-o) \geq 0 \right\}, \tag{1.2}$$

and $C_x^+ = C_o^+ + (x - o)$. The future light cones at all events are not invariant with respect to Galilean transformations, i.e. uniform motions with some velocities $\mathbf{v}$'s. Thus the experiment is in contradiction to Galileo's principle of relativity which leads to Einstein's special theory of relativity. In the following, we choose units such that $c = 1$ and obtain a new principle.

Invariance of Future Light Cones: For a (local) spacetime identified with $\mathbb{A}^4$, the future light cones C_x^+ at all events $x \in \mathbb{A}^4$ are invariant.

To fulfill the postulate, we have to determine all transformations which leave future light cones structure invariant. In the coordinate system $\mathbb{R} \times \mathbb{R}^3$, we define the metric $\eta(u, v) = -t(u)t(v) + \|\mathbf{u} - \mathbf{v}\|$ for all $u, v \in \mathbb{R} \times \mathbb{R}^3$, and the light cone at a event $x \in \mathbb{A}^4$ is $C_x = \{y \in \mathbb{A}^4 \mid \eta(y - x, y - x) = 0\}$ such that

$$C_x^+ = \{y \in C_x \mid t(y-x) \geq 0\}, \; C_x^- = \{y \in C_x \mid t(y-x) \leq 0\}. \tag{1.3}$$

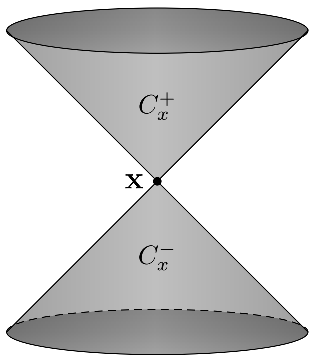

Figure 1.2: C_x^+ is the future light cone and C_x^- the past light cone.

Here the metric η is called a Minkowski metric with signature $(-, +, +, +)$, and the space $(\mathbb{A}^4, \eta)$ is called *Minkowski spacetime*. The group of transformations leaving light cone structures invariant consisting of linear maps $L : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\eta(u, v) = \eta(Lu, Lv)$, for all $u, v \in \mathbb{R}^4$, is called *Lorentz group* denoted by $O(3, 1)$. The isometric group of Minkowski spacetime is called *Poincaré group* consisting of transformations ϕ such that $\phi(x) = L(x - o) + v$ for some $v \in \mathbb{R}^4, o \in \mathbb{A}^4, L \in O(3, 1)$.

[3] We mainly follow the book [Kri99].

Then we can find the invariance group $\mathcal{P}'$ of the light cone structure as $\mathcal{P}' = \{x \mapsto \alpha L(x - o) + v \mid \alpha \in \mathbb{R} \setminus \{0\}, v \in \mathbb{R}^4, o \in \mathbb{A}^4, L \in O(3, 1)\}$. Note that transformations which leave the future light cones C_x^+ invariant must also leave.the light cones $C_x = C_x^+ \cup C_x^-$ invariant. Therefore, we obtain the functor of Einstein's special relativity (SR) as the following figure 1.3.

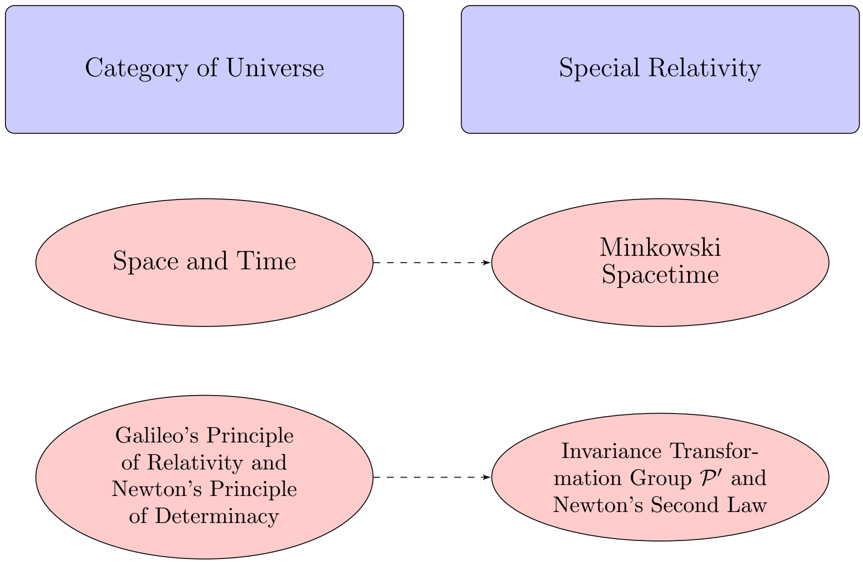

Figure 1.3: SR functor from the category of universe to of special relativity.

Remark 1.1. Until now, we only discuss the local theory of space-time. From global point of view, we have to replace Minkowski spacetime $(\mathbb{A}^4, \eta)$ by a general Lorentzian manifold (M, g). Moreover, we also can infer the existence of a conformal structure C_η on $\mathbb{A}^4$, where $C_\eta = \{\Omega^2\eta \mid \Omega \in C^\infty(\mathbb{A}^4, \mathbb{R}^+ \setminus \{0\})\}$. For further discussion, we refer to [Kri99].

In Einstein's general relativity, one of the most important insights is that gravity and the geometry of spacetime are closed related by the following principle.

Principle of Equivalence: A coordinate system at rest in a gravitational field can be locally identical to a linearly accelerated system relative to an inertial system in special relativity.

Equivalence principle implies that gravitation is an acceleration, rather than a force, and therefore a geometric object. In other word, we should have a equation of the form $\mathcal{D}g = T$, where (M, g) is a Lorentzian manifold, $\mathcal{D}$ is an operator acting on the metric g, and T is a tensor field containing the information of the matter distribution. Indeed, gravity is governed by Einstein's equation as defined below.

Definition 1.2. Einstein's equation is given by

$$Ric - \frac{1}{2}Rg + \Lambda g = \frac{8\pi G}{c^4}T, \tag{1.4}$$

where Λ is the cosmological constant, *Ric* is the Ricci curvature tensor, R is the scalar curvature, G is the gravitational constant, and T is the energy momentum tensor.

If Λ is not zero, then $|\Lambda|$ is very small by astronomical observation. The Newton theory of gravitation arises as a limit $c \to \infty$ if and only if $\Lambda = 0$. Then the functor of general relativity (GR) can be considered as below.

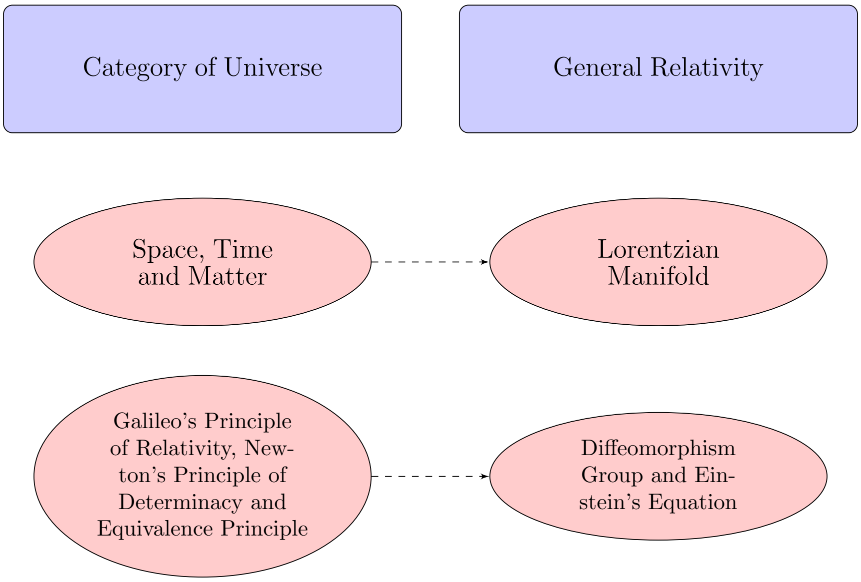

Figure 1.4: GR functor from the category of universe to of general relativity.

Remark 1.3. One way which leads to Einstein's equation (1.4) applies the *Lagrangian formulation* or classical field theory, which is sketched in § 1.1.3.

1.1.3 Classical field theory

There are two particularly important principles of the classical field theory.

Action Principle: We can associate a action function on the manifold of space of physical states to systems in physics, such that these states are the critical points of the action.

Covariance Principle: The groups considered to represent the fundamental symmetries of a physical theory act on the manifold or space of states in a compatible way. For instance, the Galilean group (§ 1.1.1), the Poincaré group (§ 1.1.2), the diffeomorphism group (§ 1.1.2), etc.

Here a state is a complete description of a physical system.

In the classical field theory, spacetime is given by a manifold M, and the *configuration bundle* is a smooth fiber bundle $F \to M$ which contains the possible physical states of the system. A section of F is called a *field*, and the topological *space of fields*,

$$\mathscr{F} := \Gamma^{\infty}(M, F), \tag{1.5}$$

has the structure of a Fréchet manifold. Recall that a Fréchet space is a topological vector space whose topology is defined by a translation invariant metric and complete.

The fields are usually subject to some *field equations* $f(\psi, m) = 0$, where $f : \mathscr{F} \to V$ is smooth map to a vector space V and $(\psi, m) \in (\mathscr{F}, M)$ The set of solutions to the field equation $\mathscr{F}_{\text{shell}} := f^{-1}(0)$ is a subvariety of $\mathscr{F}$. Note that in general $\mathscr{F}_{\text{shell}}$ is not smooth, and not algebraic. Fields in $\mathscr{F}_{\text{shell}}$ are called *on-shell*, and others in $\mathscr{F} \setminus \mathscr{F}_{\text{shell}}$ *off-shell*.

Given a smooth action $S : \mathscr{F} \to \mathbb{R}$, a field theory satisfies the action principle if the condition that $\psi \in \mathscr{F}$ is a solution of the field equation is equivalent to that ψ is a critical point of the action S. More precisely, we have a map called the *Lagrangian*

$$L : \mathscr{F} \longrightarrow \Omega^{\text{top}}(M), \tag{1.6}$$

such that the action becomes

$$S(\psi) := \int_M L(\psi). \tag{1.7}$$

However, the action principle is practically never rigorously true.

In the case of classical mechanics, $M = \mathbb{R}$ is time and F is a trivial bundle $Q \times \mathbb{R} \to \mathbb{R}$, where Q is the *configuration space* (In § 1.1.1, $Q = \mathbb{R}^{3n}$ for a n-point system). So each field can be identified with a curve $q : \mathbb{R} \to Q$. The Lagrangian is given by

$$L(q) := \mathscr{L}(q(t), \dot{q}(t), t)dt, \tag{1.8}$$

where $\mathscr{L}$ is the *Lagrangian function* on the tangent bundle of the configuration space,

$$\mathscr{L} : TQ \times \mathbb{R} \longrightarrow \mathbb{R}. \tag{1.9}$$

In general relativity, M is a Lorentzian manifold, and F is the bundle of Minkowski metrics over M. Given a metric $g \in \mathscr{F}$, the Hilbert Lagrangian is given by

$$L(g) := R(g)\text{vol}_g, \tag{1.10}$$

where $R(g)$ is the scalar curvature and $\text{vol}_g = *1 \in \Omega^{\text{top}}(M)$ the volume form of (M, g).

Let's consider the simple example of one point mechanical system in a potential $V : Q \to \mathbb{R}$ that does not depend on time, $F = \mathbb{R}^3 \times \mathbb{R}$ and $TQ \cong \mathbb{R}^3 \times \mathbb{R}^3$. Then the Lagrangian function of one point particle system of mass m is given by

$$\mathscr{L}(q^i, \dot{q}^i, t) := \frac{1}{2} m \dot{q}^i \dot{q}^i - V(q^i), \tag{1.11}$$

where $(q^i, \dot{q}^i) \in TQ$. Fix a time interval $I = [a, b]$, we get the action

$$S_I(q^i) := \int_I \mathscr{L}(q^i(t), \dot{q}^i(t), t)dt. \tag{1.12}$$

Note that the action $S_{\mathbb{R}}(q^i)$ generally diverge, thus we have to restrict time to an interval I. Consider a variation of the curve $q^i + \varepsilon^i : \mathbb{R} \to Q$, to first order in ε the action is expanded as

$$S(q + \varepsilon) = S(q) + \int_a^b \left(\frac{\partial \mathscr{L}}{\partial q^i} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{q}^i} \right) \varepsilon^i(t)dt + \int_a^b \frac{d}{dt} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i} \varepsilon^i(t) \right) dt + \mathscr{O}(\varepsilon^2). \tag{1.13}$$

The vanishing first integral implies the *Euler-Lagrangian equation*:

$$\frac{\partial \mathscr{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^i} = 0. \tag{1.14}$$

By requiring $\varepsilon^i(a) = \varepsilon^i(b) = 0$, the second integral would vanish. Hence we almost obtain the action principle. But the solutions to the Euler-Lagrangian equation are not really the critical point of the action S, but points where the derivative of S in the direction ε^i vanish. If M is compact without boundary, then all these problems can be solved as assumed in the calculus of variations. However, it is not a natural requirement from the viewpoint of physics. Interested readers could refer to [Del+99] as an introduction, or [Gia09] as a textbook for further discussion. To sum up, we obtain the functor of classical field theory (CF) as the figure 1.5.

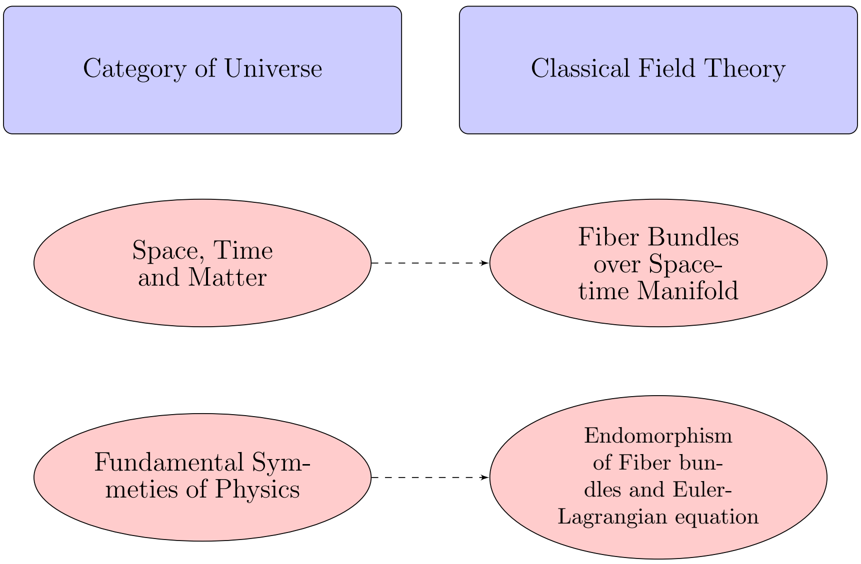

Figure 1.5: CF functor from the category of universe to of classical field theory.

1.1.4 Quantum mechanics

In Hamiltonian mechanics, a classical mechanical problem is characterized by by a *Hamiltonian function* $H(q^i, p^i, t)$, where q^i is a curve in the configuration space, and $p^i := \partial\mathscr{L}/\partial q^i$ with respect to the Lagrangian function $\mathscr{L}(q, \dot{q}, t)$ (see § 1.1.3) is called the *conjugate momentum*. Note that (q^i, p^i) form a system of local coordinates on the cotangent bundle T^*M, which has a canonical symplectic structure described locally by the form $\omega = \sum_i dp^i \wedge dq^i$. In a mechanical system, the Hamiltonian function usually defines the total energy E of the system. The quantum theory developed from 1900 to 1925 by the names of Planck, Einstein, Bohr, etc., yielded that all elementary processes obey the *discontinuous* laws of quanta. Therefore, we must learn as much as possible from the Hamiltonian function H about the quantum mechanical behavior of the system. It turns out that we must determine the possible energy levels, and find out the corresponding *stationary states*. In the following, we would list some important postulates in the quantum mechanics.

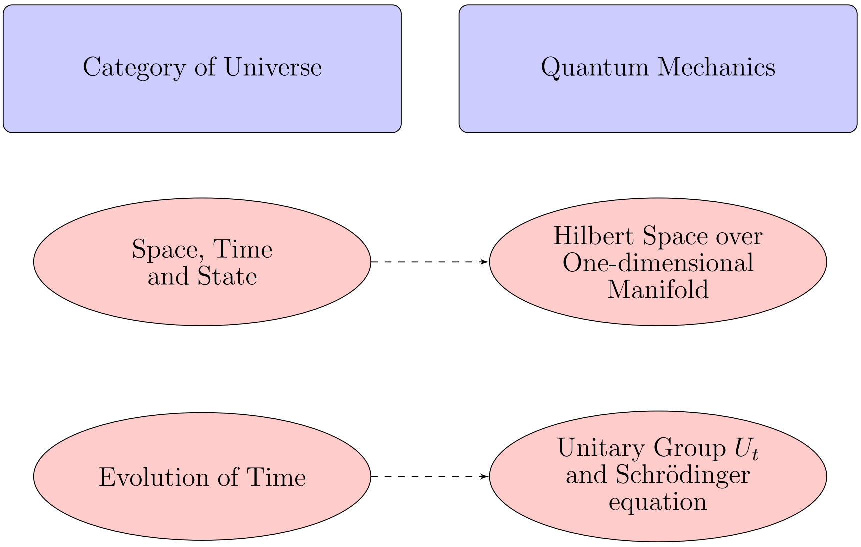

Figure 1.6: QM functor from the category of universe to of quantum mechanics..

Recall that a *state* is a complete description of a physical system, and an *observable* is a property of the system that can be measured in principle.

Postulate 1: In a quantum mechanical system, there is a Hilbert space V such that a state is a line in V. In other word, the set of states is given by $\mathbb{P}(V)$, the projective space of V.

In Dirac notation, $|\psi\rangle$ denotes a vector and $\langle\psi|\phi\rangle$ denotes the inner product in V. A state is represented by a unit-length vector $|\psi\rangle \in L$, i.e. $\langle\psi|\psi\rangle = 1$, where $L \subset V$ is a line containing the origin.

Postulate 2: An *observable* of a quantum mechanical system is a self-adjoint operator in the Hilbert space V. Hence the spectrum of an observable A is real;

Postulate 3: A *measurement* of an observable A picks an eigenstate $|\alpha\rangle$ of A and the observer obtains the corresponding eigenvalue $\alpha \in \mathbb{R}$.

Indeed, given a state $|\psi\rangle$ prior to a measurement, the observer obtains the outcome α with a priori probability $\mathrm{Prob}_{|\psi\rangle}(\alpha) = |\langle\alpha|\psi\rangle|^2$. After the measurement, the system is in the state $|\alpha\rangle$. If the measurement is repeated, the observer obtains α with probability 1. Moreover, if we make many times of the measurement A for the system in initial state $|\psi\rangle$, the expected value of A would approach

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_{\alpha\in\mathrm{Spec}A} \alpha\mathrm{Prob}_{|\psi\rangle}(\alpha). \tag{1.15}$$

Postulate 4: The time evolution of an isolated quantum mechanical system is given by a one-parameter subgroup $\{U_t\}_{t\in\mathbb{R}}$ of the unitary group $U(V)$ of the Hilbert space V.

If the initial state at time $t = 0$ is $|\psi\rangle$, then the time evolved state at time t is $|\psi(t)\rangle := U_t|\psi\rangle$. In particular, the generator of the subgroup $\{U_t\}$ is the self-adjoint operator $\mathscr{H}$, called *Hamiltonian*, the associated operator with the Hamiltonian function, such that

$$\frac{d}{dt}|\psi(t)\rangle = \frac{1}{i\hbar}\mathscr{H}|\psi(t)\rangle, \tag{1.16}$$

which is called *Schrödinger equation*.

Postulate 5: Given two isolated quantum systems with state space V_1 and V_2. We could combine and allow them to interact, and it turns out that the combined system has the product state space $V_1 \otimes V_2$.

There are two types of fundamental particles: bonsons and fermions. Given a system with N identical bosons with state space V, the composite space is the state space $\mathrm{Sym}^N V$. For fermions, the state space is $\bigwedge^N V$. We summarize the quantum mechanics functor as figure 1.6 which can be considered as composition of CM functor and quantization functor. Here, a common quantization procedure is canonical quantization. The relevant classical observables have to be replaced by operators such that the Poisson bracket is preserved in the sense that it is replaced by the commutator of operators in V, i.e.

$$\{A, B\} \longmapsto -\frac{i}{\hbar}\left[\hat{A}, \hat{B}\right]. \tag{1.17}$$

In classical phase space $(q^i, p^j) \in \mathbb{R}^{2n}$, it is natural to require the *Dirac conditions*:

- $\hat{1} = \mathrm{id}_V$;
- $\left[\hat{q}^i, \hat{p}^j\right] = \frac{i}{\hbar}\delta_{ij}$, $\left[\hat{q}^i, \hat{q}^j\right] = \left[\hat{p}^i, \hat{p}^j\right] = 0$.

1.2 Principles of contemporary physics

Mathematical methods and structures of fundamental physics are very well developed (see § 1.1), and we have a very profound understanding of fundamental physics, which have been enormously successful theories of physics in describing known phenomena of our universe on large scales and the sub-atomic world of particles. However, various modern physical theories require rather sophisticated mathematics for their formulation. One of the most difficult problems is to quantize general relativity which one has to generalize quantum field theory in curved spacetime. Indeed, in relativistic quantum field theory, the Standard Model has successfully unify three of the four fundamental interactions, but it is still an open problem in physics and mathematics to develop an mathematical rigorously theory which can unify all four fundamental interactions. Up to now, String theory has been a promising candidate for the unifying theory in physics which provides a framework for incorporating quantum field theory and general relativity Note that even in quantum field theory, the mathematical rigorously structure is still not well known. In this section, we would discuss some basic concepts and axioms in quantum field theory and string theory.

1.2.1 Quantum field theory

Contemporary quantum field theory is mainly developed as quantization of classical field theory (§ 1.1.3). (Although the standard quantization procedure in physics is to use the canonical quantization (eq. 1.17), there still are two important mathematical quantizations: *Geometric quantization* (see [Woo97]) and

Deformation quantization (see [Kon03]).) Indeed, a generating functional of Green functions in perturbative quantum field theory depends on an action functional of classical fields. Let M be a spacetime manifold with a Lorentzian metric, and $\phi \in C^\infty(M, \mathbb{R})$ a scalar field which can describe one possible history of the universe. A typical Lagrangian of interest is

$$\mathscr{L}(\phi) = -\frac{1}{2}\phi(\mathrm{D} + m^2)\phi + \frac{1}{4!}\phi^4, \tag{1.18}$$

where D is the Lorentzian operator analog of the Laplacian, such that the action functional $S(\phi)$ (1.7) would be of the form

$$S(\phi) = \int_{x \in M} \mathscr{L}(\phi)(x). \tag{1.19}$$

Feynman's Sum over Histories: The physical world is in a quantum superposition of all states $\phi \in C^\infty(M, \mathbb{R})$ weighted by $e^{iS(\phi)/\hbar}$.

An *observable* is a function $O : C^\infty(M, \mathbb{R}) \to \mathbb{C}$. Then the correlation function of a set of n observables is defined by the formula

$$\langle O_1, \ldots, O_n \rangle = \int_{\phi \in C^\infty(M)} e^{iS(\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) \mathscr{D}\phi. \tag{1.20}$$

Here $\mathscr{D}\phi$ is the Lebesque measure on the space $C^\infty(M)$. Note that in general the measure $\mathscr{D}\phi$ does not exist, as a Lebesque measure (non-trivial translation invariant) on an infinite dimensional vector space is still unknown. Hence the existence of the measure $\mathscr{D}\phi$ is one of the fundamental problems in quantum field theory.

A more systematic approach to quantum field theory may use axioms. We would follow the argument in the book [Sch08] to present the system of axioms formulated by Arthur Wightman in the early 1950s. Assume that (M, g) is the Minkowski space $\left(\mathbb{R}^{(1,3)}, \eta\right)$. The space of states is the projective space $\mathbb{P}(V)$ of a separable Hilbert space V, and there exists a vacuum vector $\Omega \in V$ of norm 1. We have an unitary representation of the Poincaré group P as $U : P \to \mathrm{U}(V)$, and a collection of field operators $\{\Phi_a\}$, $a \in I$, with a dense subspace $D \in V$ as their common domain such that Ω is in the domain D. Here a field operators is an operator-valued distribution on M, that is $\Phi : \mathscr{S}(M) \to \mathscr{O}(V)$, where $\mathscr{S}$ is the *Schwartz space* of *rapidly decreasing smooth function*, and $\mathscr{O}(V)$ is the set of all densely defined operators in V. Wightman's three axioms are described as below.

Covariance: Ω and D are P-invariant, that is $U(p)\Omega = \Omega$ and $U(p)D \subset D$, for all $p \in P$. Furthermore, D is also invariant in the sense that $\Phi_a(f)D \subset D$, for all $a \in I, f \in \mathscr{S}$, and the actions on V and $\mathscr{S}$ are equivalent, i.e. on D, we have

$$U(p)\Phi(f)U(p)^* = \Phi(pf) \tag{1.21}$$

for all $f \in \mathscr{S}$, and $p \in P$;

Locality: $\Phi_a(f)$ and $\Phi_b(g)$ commute on D, i.e. $[\Phi_a(f), \Phi_b(g)] = \Phi_a(f)\Phi_b(g) - \Phi_b(g)\Phi_a(f) = 0$, if the supports of $f, g \in$ are *space-like separated*, that is $\eta(x, y) < 0$;

Spectrum Condition: The joint spectrum of $\{P_j\}_{j=0\ldots3}$, where P_0 is the Hamiltonian operator $\mathscr{H}$ and P_j the component of the momentum, is contained in the forward cone $C_+ := \{x \in M \mid \eta(x, x) \geq 0, x^1 \geq 0\}$.

To require the vacuum Ω to be unique, we need an additional axiom,

Uniqueness of the Vacuum: The only vectors in V left invariant by the translation are the scalar multiples of the vacuum Ω.

Note that although these axioms seem to be natural, it is always not so easy to find some examples of Wightman quantum field theories, even in the case of free particles theory.

Although Feynman path integral is not rigorously defined, we still can think of quantum field theory as a functor (QFT) from a geometric category, that is a category of manifolds with boundary, to a category of complex vector space. Here objects in the geometric category are closed $(d-1)$-manifolds with metric, and morphsims are d-manifolds with metric providing cobordisms between $(d-1)$-manifolds. In the case of $d = 1$ manifolds, we have the well-known quantum mechanics functor discussed in § 1.1.4. An objects is a finite set of points, zero-dimensional manifold. The complex vector space corresponding to a point is the Hilbert space V, and the simplest morphsim is the interval $[0, t]$ which corresponds the self-adjoint operator U_t on V. We will give a brief review in § 2.1.3.

1.2.2 String theory

In general, a string theory describes the motion of one-dimensional strings, loops (closed string) or segments (open strings), in a Riemannian manifold M, the target space. Precisely, one uses a map from a two-dimensional Riemannian manifold Σ, the world sheet which swept by a string through time, into the target space-time. In case of closed strings, the space of all such configurations is given by the loop space of M which we denote by $\mathscr{L}M$. The Hilbert space of bosonic strings corresponds to the function space on $\mathscr{L}M$, denoted by $\mathscr{H}_{\text{bosonic}} = \Phi(\mathscr{L}M)$ with norm inherited from the metric on M. The Hilbert space of fermionic strings is the space of (semi-infinite) forms on $\mathscr{L}M : \mathscr{H}_{\text{fermionic}} = \bigwedge^{\infty}(\mathscr{L}M)$.

On the large scale, at least larger than the string scale, a string looks like a ordinary particle and the vibrational states of the string determine the mass, charge, and other physical properties. Especially, the *graviton*, the quantum of gravitation, corresponds to one of vibrational state of a bosonic string, and thus string theory is considered as a theory of quantum gravity and a candidate of the unification theory in physics.

In classical bosonic sting theory considered as a classical field theory (§ 1.1.3), a natural action uses the area of the world sheet swept out by the string called *Nambu-Goto action*, that is

$$S_{\text{NG}}(x) := -\frac{1}{\alpha'}\int_{\Sigma} dA = -\frac{1}{\alpha'}\int_{\Sigma} d^2\sigma\sqrt{-\det g}, \tag{1.22}$$

where α', related to the string tension, has the dimension of $[\text{mass}]^{-2}$ such that the action S_{NG} is dimensionless, and g is the Lorentzian metric $(\det g < 0)$ on Σ induced by a embedding $\mathbf{x} : \Sigma \to (M, \eta)$, such that $g := \mathbf{x}^*\eta$, i.e.

$$g_{\mu\nu} = \eta_{ij}\partial_\mu x^i \partial_\nu x^j. \tag{1.23}$$

By the action principle with respect to the embedding $\mathbf{x}$, one can derive the equation of motion (eq. 1.14). However, it is quite difficult to do calculations in terms of the action S_{NG}, thus one introduces another action which also give the same equation of motion called *Polyakov action*, that is

$$S_{\text{P}}(x, h) := -\frac{1}{2\alpha'}\int_{\Sigma} d^2\sigma\sqrt{-\det h}\; h^{\mu\nu} g_{\mu\nu}. \tag{1.24}$$

Here, h is another Lorentzian metric on the world sheet Σ, and the additional variation of $S_{\text{P}}(x, h)$ with respect to h would lead to the former action S_{NG}.

From the covariance principle in the classical field theory, the fundamental symmetries of the bosonic string theory depend on the invariance of group actions on the action function. In the case of the Polyakov action, one can find the invariance action as below.

- Poincaŕe transformation group,
- Reparameterizations of the world sheet Σ,
- Weyl rescalings : $h \mapsto \Omega^2 h,\ \Omega \in \mathbb{R} \setminus \{0\}$.

Obviously, the Nambu-Goto action S_{NG} is only invariant with respect to Poincaŕe transformation group and reparameterizations. Using the action principle and symmetric groups, we obtain the Possion brackets of the classical system, called 2-dimensional *Conformal field theory* (CFT), which are necessary ingredient for the canonical quantization (eq. 1.17). Through the quantization procedure, the algebra of the quantum system would lead to the *Virasoro algebra* as a central extension of the *Witt algebra*, the algebra of the classical system. For good introduction to CFT, one refers to the detailed physics-oriented book [FMS97] and mathematical one [Sch08], in which we can study 2d CFT by some basic concepts and a system of axioms. More advanced mathematical text related to CFT on complex plane called *Vertex algebra* is like [Kac96], and to Vertex algebra on algebraic curves called *Chiral algebra* like [BD04].

In terms of the functorial approach, the string theory is a functor (ST) as (1+1)-dimensional quantum field theory functor, i.e. from a (1+1)d geometric category to a linear category. Since any smooth, connected 1-dimensional manifold is diffeomorphic either to the circle S^1 or to some interval of real numbers, the objects of the (1+1)d geometric category are disjoint unions of circles and oriented intervals with labeled ends. The linear category here is the category of Hilbert spaces and operators. The functor takes disjoint unions to tensor products. In geometry, any oriented surface can be decomposed into a composition of basic surfaces which define the *Frobenius structure*, and a given surface has many different compositions. If the linear category is simply restricted to the category of complex vector spaces and linear maps, there are no further relations on the algebraic structure imposed by consistency of the sewing property. This simple theory is called 2d *Topological field theory* (TFT) and a fundamental algebraic structure of topological string theories, which providing surprising connections to many branches of theoretical physics and mathematics, such as the well-known *Mirror symmetry*, Gromov-Witten invariance, Bridgeland's stability condition, etc. In the chapter 2, we will give a review of topological string theories and relevant mathematical subjects for further studies in the chapter 3.

CHAPTER 2

Topological Strings and D-Branes

In this chapter we would discuss the physics and mathematics background of the chapter 3. In § 2.1 we start with a review of the 2-dimensional supersymmetry quantum field theory in differential approaches, and give a axiomatic functorial definition of the topological field theory and the Frobenius algebra, which is the basic algebraic structure appearing in any 2-dimensional topological field theories. Afterwards, we explain the relations between these approaches, and provide exact constructions of topological field theories from the $\mathcal{N} = (2, 2)$ supersymmetry conformal algebra, and the associated topological sigma models in § 2.2. In particular, we discuss the A-and B-models and the isomorphism between their relevant moduli spaces of the target spaces, which is the origin statement of mirror symmetry.

In § 2.3, we discuss the boundary conditions in the conformal field theory, called a *boundary conformal field theory*, corresponding to the *Dirichlet* and *Neumann* boundary conditions, and then introduce the notion of *D-branes*. In topological string theories, the topological D-branes in the A-model are Lagrangian submanifolds with flat bundles, while in the B-model they are holomorphic submanifolds with holomorphic vector bundles. Homological mirror symmetry conjecture applies to the categories of these topological D-branes. However, there are far too many topological D-branes for all of them to correspond to physical D-branes. We thus study additional stability properties needed for physical D-branes, and Bridgeland's stability condition on triangulated categories discussed in § 2.4.

2.1 2d Supersymmetry quantum field theory

In the section we briefly discuss the basic properties of supersymmetry quantum field theory in 2-dimensional case from three different points of view: the geometric picture (nonlinear sigma model), the algebraic picture (supersymmetry conformal field theory), and the axiomatic approach (topological field theory). However, these three approaches are not equivalent to each other, in other word, given a nonlinear sigma model, it is rather difficult to write down the relevant super conformal algebra, and vice versa. Therefore a trivial property in one picture could become a highly non-trivial problem in physics and mathematics. The most surprising problem is the *Mirror symmetry* which we will give a description from the supersymmetry conformal field theory (trivial) point of view .

2.1.1 Nonlinear sigma model

The nonlinear sigma model is one of the most important geometric realization of the supersymmetric quantum field theory. In the following we would restrict to $\mathcal{N} = (2, 2)$ nonlinear sigma model (For further discussion, see [Wit92], [HKK03], [Asp09]). Let the target space (X, g, B) be a Calabi-Yau manifold, or more general, a Kähler manifold with $c_1 = 0$. Here, g is a metric and B a closed 2-form, called *B-field*. A sigma model is an embedding $\Phi : \Sigma \to X$ from a Riemann surface to the target space as describing the propagation of a string into X. We choose the local coordinate systems $(z, \bar{z})$ on the world sheet Σ and $(\phi^i, \phi^{\bar{i}})$ on X. The Riemann surface Σ is arbitrary which leads to various complicated ways of embedding, allowing a string can be split up into several strings, or to combine several strings to one, i.e. the seesaw property which is the important feature of all string theories.

Recall that a spin^c structure is a pair of holomorphic line bundles (L_1, L_2) such that $L_1 \otimes L_2 \cong K$, the anti-canonical bundle $K \equiv T^*\Sigma$ on Σ. In the case of $L_1 = L_2 = K^{1/2}$, it corresponds to the untwisted sigma model with $\mathcal{N} = (2, 2)$ supersymmetry. Let choose two spin^c structures (L_1, L_2) and (L_3, L_4). Then the fermionic fields ψ are sections of certain bundles on Σ as the following table 2.1.

Fermions	Sections
ψ_+^i	$\Gamma(L_1 \otimes \Phi^* T_X)$
$\psi_+^{\bar{j}}$	$\Gamma(L_2 \otimes \Phi^* \bar{T}_X)$
ψ_-^i	$\Gamma(\bar{L}_3 \otimes \Phi^* T_X)$
$\psi_-^{\bar{j}}$	$\Gamma(\bar{L}_4 \otimes \Phi^* \bar{T}_X)$

Table 2.1: Fermionic fields in the nonlinear sigma model.

Here T_X is the holomorphic tangent bundle on X and $\bar{T}_X$ the antiholomorphic tangent bundle on X. Then the action is the form

$$\begin{aligned} S = \frac{1}{4\pi} \int_\Sigma d^2 z \Bigg\{ & g_{i\bar{j}} \left(\frac{\partial \phi^i}{\partial z} \frac{\partial \phi^{\bar{j}}}{\partial \bar{z}} + \frac{\partial \phi^i}{\partial \bar{z}} \frac{\partial \phi^{\bar{j}}}{\partial z} \right) + B_{i\bar{j}} \left(\frac{\partial \phi^i}{\partial z} \frac{\partial \phi^{\bar{j}}}{\partial \bar{z}} - \frac{\partial \phi^i}{\partial \bar{z}} \frac{\partial \phi^{\bar{j}}}{\partial z} \right) \\ & + i g_{i\bar{j}} \psi_-^{\bar{j}} D \psi_-^i + i g_{i\bar{j}} \psi_+^{\bar{j}} \bar{D} \psi_+^i + R_{i\bar{i}j\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}} \Bigg\}, \end{aligned} \tag{2.1}$$

where R is the curvature tensor of the metric g on X, and D is the covariant derivative deduced from the connection of the metric $\Phi^*(g)$ on Σ as below:

$$D\psi_-^i = \partial \psi_-^i + \partial \phi^j \Gamma^j_{jk} \psi_-^k, \tag{2.2}$$

where ∂ is the usual holomorphic differential.

The supersymmetries are quite complicated and written as the following transformations:

$$\begin{aligned} \delta\phi^i &= i\alpha_- \psi_+^i + i\alpha_+ \psi_-^i \\ \delta\phi^{\bar{i}} &= i\tilde{\alpha}_- \psi_+^{\bar{i}} + i\tilde{\alpha}_+ \psi_-^{\bar{i}} \\ \delta\psi_+^i &= -\tilde{\alpha}_- \partial \phi^i - i\alpha_+ \psi_-^j \Gamma^i_{jk} \psi_+^k \\ \delta\psi_+^{\bar{i}} &= -\alpha_- \partial \phi^{\bar{i}} - i\tilde{\alpha}_+ \psi_-^{\bar{j}} \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \psi_+^{\bar{k}} \\ \delta\psi_-^i &= -\tilde{\alpha}_+ \bar{\partial} \phi^i - i\alpha_- \psi_+^j \Gamma^i_{jk} \psi_-^k \\ \delta\psi_-^{\bar{i}} &= -\alpha_+ \bar{\partial} \phi^{\bar{i}} - i\tilde{\alpha}_- \psi_+^{\bar{j}} \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \psi_-^{\bar{k}} \end{aligned} \tag{2.3}$$

with infinitesimal fermionic parameters α_-, $\tilde{\alpha}_-$, α_+ and $\tilde{\alpha}_+$ which are sections of L_1^{-1}, L_2^{-1}, $\bar{L}_3^{-1}$ and $\bar{L}_4^{-1}$, respectively. The four conserved supercurrents G, generators of the supersymmetry transformations (eq. 2.3) are defined in terms of BRST operator Q as below.

$$\begin{aligned} Q(\alpha) &= \int d\sigma_2 G\alpha \\ \delta W &= -i\,\{Q(\alpha), W\}, \end{aligned} \tag{2.4}$$

for any operator W. We thus denote the four currents by G_+, $\tilde{G}_+$, G_-, $\tilde{G}_-$. Note that a BRST operator Q we are looking for must satisfy the conditions: $Q^2 = 0$, which can be used to define the cohomology, and the stress tensor T can be expressed as $T = \{Q, b\}$ for some local operator b. Here the stress tensor is defined as a variation of the Polyakov action (1.24) with respect to the metric h, i.e. $T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\mathrm{P}}}{\delta h^{\alpha\beta}}$, and the insertion of the stress tensor at some point x in a correlation function generates an infinitesimal metric deformation at the point. There is also an additional $U(1)$ current with holomorphic part J and anti-holomorphic part $\bar{J}$. In the classical theory, these operators can be written as

$$\begin{aligned} T(z) &= -g_{i\bar{j}}\frac{\partial\phi^i}{\partial z}\frac{\partial\phi^{\bar{j}}}{\partial z} + \frac{1}{2}g_{i\bar{j}}\psi_+^i\frac{\partial\psi_+^{\bar{j}}}{\partial z} + \frac{1}{2}g_{i\bar{j}}\psi_+^{\bar{j}}\frac{\partial\psi_+^i}{\partial z} \\ G_+(z) &= \frac{1}{2}g_{i\bar{j}}\psi_+^i\frac{\partial\phi^{\bar{j}}}{\partial z} \\ \tilde{G}_+(z) &= \frac{1}{2}g_{i\bar{j}}\psi_+^{\bar{j}}\frac{\partial\phi^i}{\partial z} \\ J(z) &= \frac{1}{4}g_{i\bar{j}}\psi_+^i\psi_+^{\bar{j}} \end{aligned} \tag{2.5}$$

and the left-handed supercurrents $(T(z), G_+(z), \tilde{G}_+(z), J(z))$ are holomorphic. Thus left-handed and right-handed supercurrents form $\mathcal{N} = (2,2)$ superconformal algebra we will discuss in the following section.

2.1.2 Supersymmetry conformal field theory and mirror symmetry

The $\mathcal{N} = 2$ superconformal algebra (SCA), or super Virasoro algebra, plays an important role in string theory due to its relation to minimal space-time supersymmetry in the compactified theory, although supersymmetry has not been experimentally verified to date (see [BLT12; BP09]). A state in string theory is represented by the superconformal algebra generated by the transformation (2.3). In $\mathcal{N} = (2,2)$ supersymmetry, i.e. $L_j = K^{1/2}$ for $j = 1,2,3,4$, the parameters α_- and $\tilde{\alpha_-}$ are belong to holomorphic sections of $K^{-1/2}$, and α_+ and $\tilde{\alpha}_+$ are belong to anti-holomorphic sections of $\bar{K}^{-1/2}$. The Hilbert space in $\mathcal{N} = 2$ superconformal field theory corresponds to the parameter $e^{2\pi i a}$, $a \in \mathbb{R}$, which labels the isomorphism class of the line bundle $L_1 = K^{1/2}$. The mode expansions of currents thus are

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \\ G(x) &= \sum_{n\in\mathbb{Z}} z^{-n+a-3/2}G_{n-a}, \\ J(x) &= \sum_{n\in\mathbb{Z}} z^{-n-1}J_n. \end{aligned} \tag{2.6}$$

Here we drop the subscript on G_+ and $\tilde{G}_+$. There are two important sectors: *Ramond sector* (R) as $a = 0$ and *Neveu-Schwarz sector* (NS) as $a = 1/2$.

Under the mode expansions, the operator algebra is then equivalent to the $\mathcal{N} = 2$ super Virasora algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{4}(m^3 - m)\delta_{m+n} \\
[J_m, J_n] &= \hat{c}m\delta_{m+n} \\
[L_m, J_n] &= -nJ_{m+n} \\
[L_n, G_{m-a}] &= \left(\frac{n}{2} - m + a\right)G_{m+n-a} \\
[J_n, G_{m-a}] &= G_{m+n-a} \\
[J_n, \tilde{G}_{m-a}] &= -\tilde{G}_{m+n-a} \\
\{G_{n+a}, \tilde{G}_{m-a}\} &= 2L_{m+n} + (n-m+2a)J_{n+m} + \hat{c}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n}.
\end{aligned} \tag{2.7}$$

Here J denotes the generator of U(1) Kac-Moody algebra, L of the Virasoro algebra, and G the anti-commuting generator. m, n are integers. Note that the Cartan subalgebra of the SCA is generated by three generators L_0, $\hat{c}$ and J_0, and the eigenvalue of L_0 is denoted by the weight h called the *Conformal dimension*, and the one of J_0 is denoted by q called the *U(1) charge*. To be precise, a highest weight state is given by

$$L_n|\phi\rangle = 0,\ G_{m-a}|\phi\rangle = 0,\ \tilde{G}_{m-a}|\phi\rangle = 0,\ J_n|\phi\rangle = 0, \tag{2.8}$$

and labeled by the eigenvalues

$$L_0|\phi\rangle = h_\phi|\phi\rangle,\ J_0|\phi\rangle = q_\phi|\phi\rangle. \tag{2.9}$$

A primary field induces a highest weight sate $|\phi\rangle = \phi|0\rangle$. In the sigma model, $\hat{c} = c/3$, where c is the central of the SCA, will equal to the complex dimension of the target space X. A closed string state have both left-handed and right-handed weight and charge denoted by (h_L, q_L) and (h_R, q_R), respectively. Since a_L and a_R are independent, there exist four differential sectors of NS-NS, NS-R, R-NS and R-R.

In an unitary theory, every state in the NS sector has a unique orthogonal decomposition [LVW89] of the form

$$|\phi\rangle = |\phi_0\rangle + G_{-1/2}|\phi_1\rangle + \tilde{G}_{+1/2}|\phi_2\rangle, \tag{2.10}$$

where $|\phi_0\rangle$ is chiral primary, i.e., $L_0|\phi_0\rangle = h|\phi_0\rangle$, $J_0|\phi_0\rangle = q|\phi_0\rangle$ and $\frac{c}{6} \geq h = \frac{q}{2}$. This is the analog of the Hodge decomposition for differential forms. Here we have the correspondence of nilpotent operators

$$(G_{-1/2}, \tilde{G}_{1/2}) \Leftrightarrow (\bar{\partial}, \bar{\partial}^*),\ \left\{G_{-1/2}, \tilde{G}_{1/2}\right\} = 2\left(L_0 - \frac{1}{2}J_0\right) \Leftrightarrow \Delta_{\bar{\partial}} \tag{2.11}$$

and similar relations for complex conjugation, such that

$$\begin{aligned}
\text{chiral fields} &\Longleftrightarrow \text{closed forms} \\
\text{chiral primary fields} &\Longleftrightarrow \text{harmonic forms.}
\end{aligned} \tag{2.12}$$

One important feature of chiral primary fields is that there exists a well-defined product such that chiral primary fields have a ring structure called *Chiral ring*, that is

$$\phi_i \cdot \phi_j = \sum_k C_{ij}^k \phi_k. \tag{2.13}$$

Since $U(1)$ charge is conserved, only fileds with $(h_k = \frac{q_k}{2}, q_k = q_i + q_j)$ appear as chiral primary fields.

Furthermore, in $\mathcal{N} = 2$ super Virasoro algebra there exists a continuous class of automorphisms, or a continuous deformation of generators, called the *Spectral flow*. To be precise, it is defined by

$$
\begin{aligned}
L_n' &= L_n + \eta J_n + \frac{\eta^2}{2}\hat{c}\delta_n, \\
J_n' &= J_n + \hat{c}\eta\delta_n, \\
G_r' &= G_{r+\eta}, \\
\tilde{G}_r' &= \tilde{G}_{r-\eta},
\end{aligned}
\tag{2.14}
$$

where $\eta \in \mathbb{R}$, such that $(L_n', J_n', G_r', \tilde{G}_r')$ still satisfy the algebra (2.7). In particular, for $\eta \in \mathbb{Z} + \frac{1}{2}$ the flow interpolate between the Ramond sector ($a = 0$) and the Neveu-Schwarz sector ($a = \frac{1}{2}$), i.e. a one-to-one mapping between both sectors. In $\mathcal{N} = (2,2)$ superconformal field theory with $(c_L, c_R) = (9,9)$, the chiral primary fields have ($h_L \leq \frac{3}{2}, |q_L| = 2h_L$) and ($h_R \leq \frac{3}{2}, |q_R| = 2h_R$), such that $q_L, q_R \in [-3,3] \subset \mathbb{Z}$. Denote the (c,c) ring for $q_L, q_R > 0$ and the (a,c) ring for $-q_L, q_R > 0$. Then the (c,c) ring is associated with the Dolbeault cohomology group $H^{p,q}(M)$ of a Calabi-Yau 3-fold M and the (a,c) ring with the cohomology group $H^{3-p,q}(M^*)$ of a Calabi-Yau 3-fold M^*. Then under the action of the flow with $(\eta_L, \eta_R) = (-1, 0)$, the automorphism of the SCA therefore interchange the (c,c) ring and (a,c) ring, which induce the relation between the Hodge numbers of M and M^* as

$$
h^{3-p,q}(M^*) = h^{p,q}(M). \tag{2.15}
$$

M^* is called the mirror manifold of M and this relation leads to the well-known *Mirror symmetry* between the mirror pair (M, M^*). Hence the existence of mirror pairs of Calabi-Yau manifolds is trivial from the the $\mathcal{N} = (2,2)$ SCFT point of view, as an automorphism of the super Virasoro algebra, i.e.

$$
SCFT(M, g) \cong SCFT(M^*, g^*) \tag{2.16}
$$

for the mirror pair of (M, M^*).

2.1.3 2d Topological field theory and Frobenius structure

As the discussion in § 1.2.1, in the functorial approach a topological quantum field theory is a functor from the category of cobordism classes to the category of complex vector space subjects to a collection of axioms due to Atiyah [Ati88]. We now give the axiomatic definition of the topological field theory and follow the book [CK99] (see also [Koc04]).

Definition 2.1. A d-dimensional topological field theory (TFT) is a functor which to each closed oriented $(d-1)$-dimensional manifold Y associates a finite dimensional complex vector space $V(Y)$, and to each oriented d-dimensional manifold X whose boundary ∂X is $(d-1)$-dimensional closed manifolds associates an element $Z_X \in V(\partial X)$, such that $V(Y)$ and Z_X are invariant functorially under isomorphisms of Y and of X, respectively.

A TFT functor satisfies the following axioms:

A1: $V(Y_1 \amalg Y_2) = V(Y_1) \otimes V(Y_2)$.

A2: The empty manifold considered as a closed $(d-1)$-dimensional oriented manifold must be sent to the ground field $\mathbb{C}$, i.e. $V(\emptyset) = \mathbb{C}$.

A3: The empty manifold considered as a closed d-dimensional oriented manifold with empty boundary must be sent to $1 \in \mathbb{C}$, i.e. $Z_\emptyset = 1 \in V(\emptyset) = \mathbb{C}$.

A4: $V(\bar{Y}) \cong V(Y)^*$, where $\bar{Y}$ is the orientation inversed manifold of Y and $Z_{\bar{X}}$ is the adjoint to Z_X. Moreover, If $\partial X = \left(\amalg_{i=1}^k \bar{Y}_i\right) \amalg \left(\amalg_{j=1}^l Y'_j\right)$, then we obtain

$$Z_X \in \mathrm{Hom}_{\mathbb{C}}\left(\otimes_{i=1}^k V(Y_i), \otimes_{j=1}^l V(Y'_j)\right), \tag{2.17}$$

by this axiom and A1.

A5: Let $X = Y \times I$ with $\partial X = \bar{Y} \amalg Y$, where I is an interval in $\mathbb{R}$. We require

$$Z_X = 1_{V(Y)} \in \mathrm{Hom}_{\mathbb{C}}(V(Y), V(Y)). \tag{2.18}$$

A6: Given $\partial X = \bar{Y}_1 \amalg Y_2$ and $\partial X' = \bar{Y}_2 \amalg Y_3$, we can form a new manifold $X \cup_{Y_2} X'$ with boundary $\bar{Y}_1 \amalg Y_3$, by gluing X and X' together along Y_2. By previous axioms, we have $V(\partial X) = \mathrm{Hom}_{\mathbb{C}}(V(Y_1), V(Y_2))$, $V(\partial X') = \mathrm{Hom}_{\mathbb{C}}(V(Y_2), V(Y_3))$, and $V(\partial(X \cup_{Y_2} X')) = \mathrm{Hom}_{\mathbb{C}}(V(Y_1), V(Y_3))$, then we assume $Z_{X \cup_{Y_2} X')} = Z_{X'} \circ Z_X$.

Note that A1 reflects the the postulate 5 of quantum mechanics in § 1.1.4: the state space of two isolated systems is the tensor product of the two state space. These axioms express that the theory is topological, i.e. only depends on the diffeomorphism class of manifolds, not on any geometric data.

We now restrict to the case when $d = 2$ and first define the *Frobenius algebra*.

Definition 2.2. A commutative Frobenius algebra is a commutative, associative algebra $(\mathcal{A}, *)$ with a unit 1 and a non-degenerate inner product $\langle\, , \rangle$ such that

$$\langle a * b, c\rangle = \langle a, b * c\rangle, \tag{2.19}$$

for all $a, b, c \in \mathcal{A}$.

With the Frobenius algebra $(\mathcal{A}, *)$, we can define a three-point correlation function $\langle\, , \, , \rangle : \mathcal{A}^{\otimes 3} \to \mathbb{C}$ by

$$\langle a, b, c\rangle = \langle a * b, c\rangle. \tag{2.20}$$

Similarly the n-point correlation function is defined as

$$\langle a_1, \dots, a_n\rangle = \langle a_1 * \dots * a_{n-1}, a_n\rangle, \tag{2.21}$$

which is totally symmetric in each arguments.

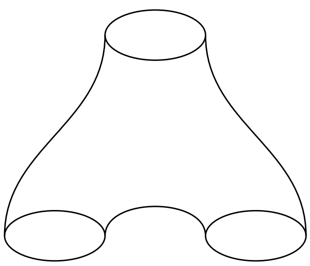

Figure 2.1: The pair of pants.

Let Δ be the standard closed disk and $\partial\Delta = S^1$ be its boundary. Then we denote $\mathcal{H} = V(S^1)$ and $1_0 = Z_\Delta \in V(S^1) = \mathcal{H}$. Since $\bar{S}^1 \cong S^1$ by complex conjugation, it leads to $\mathcal{H} \cong \mathcal{H}^*$ due to the axiom 4. Let $\langle\, , \rangle$ be the natural inner product of the complex vector space $\mathcal{H}$. To construct a product

$* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, we would use the pair of pants Σ, see fig. 2.1. The boundary components of Σ are $\bar{S}^1 \amalg \bar{S}^1 \amalg S^1$, the two copies of $\bar{S}^1$ correspond to the lower boundary circles and the S^1 corresponds to the upper boundary circle. Let $*$ denote the product on $\mathcal{H}$ defined by $Z_\Sigma \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H} \otimes \mathcal{H}, \mathcal{H})$. Thus we have the following proposition.

Proposition 2.3. *The product $*$ is commutative and associative. The element 1_0 is an identity element for $*$, and we have the identification*

$$\langle a * b, c \rangle = \langle a, b * c \rangle, \tag{2.22}$$

for all $a, b, c \in \mathcal{H}$. Hence $(\mathcal{H}, *)$ is a Frobenius algebra.

Proof. Commutativity follows by the twist cobordism corresponding to the twist diffeomorphism which interchanges two components of $\bar{S}^1 \amalg \bar{S}^1$. Now we glue Δ to one of the boundary $\bar{S}^1$ in Σ, and the glued space X is isomorphic to the cylinder $S^1 \times I$, inducing that $Z_X = 1_{\mathcal{H}}$ by the axiom A5. On the other hand, the axiom A6 implies that Z_X is the endomorphism 1_0* on $\mathcal{H}$. Thus we see that 1_0 is the identity for $*$.

To proof the identification, we use the pair of pants Σ with boundary $\partial\Sigma \cong \bar{S}^1 \amalg \bar{S}^1 \amalg \bar{S}^1$, thus $Z_\Sigma \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}, \mathbb{C})$. Using the isomorphism $\bar{S}^1 \cong S^1$ on the third boundary component, we obtain $Z_\Sigma(a, b, c) = \langle a * b, c \rangle$, and on the first component we obtain $Z_\Sigma(a, b, c) = \langle a, b * c \rangle$, Hence the identity follows.

Finally, for the associativity, we consider a 2-sphere Σ' whose boundary components are $\partial\Sigma' \cong \bar{S}^1 \amalg \bar{S}^1 \amalg \bar{S}^1 \amalg S^1$, such that $Z_{\Sigma'} \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}, \mathcal{H})$. By the decomposition of cobordism, we can decompose Σ' into two pair of pants Σ_1 and Σ_2 such as $\Sigma' \cong \Sigma_1 \cup_{S^1} \Sigma_2$ along with the first boundary component S^1 of Σ_2. Thus it turns out that $Z'_\Sigma(a, b, c) = (a * b) * c$ due to the axiom A6. However, we also have another decomposition as $\Sigma' \cong \Sigma_1 \cup_{S^1} \Sigma_2$ together along the second boundary component $S^1 \in \partial\Sigma'$, which induces that $Z'_\Sigma(a, b, c) = a * (b * c)$. Therefore the associativity has been proofed. □

From the above proposition, we see that the finite dimensional Frobenius algebra will uniquely determine a 2d topological field theory. Given the Frobenius algebra of a 2d topological field theory, we can interpret the n-point correlation function in terms of 2d TFT. In the next section, we will give two explicit physical theory known as the *Topological sigma models*.

Remark 2.4. Here we only discuss the cobordisms of closed 1-dimensional manifolds and relevant topological field theory, sometimes called the *Closed topological field theory*. For more general theory including open and closed 1-dimensional manifolds, called the *Open and closed topological field theory*, one can refer to [Asp09; MS06].

2.2 Topological twist and sigma models for closed strings

In order to transform the $\mathcal{N} = 2$ SCA into a topological filed theory, we would impose the algebraic process of *Topological twist* [Wit88] on the SCA. The topological twisted stress tensor is defined by

$$T^{\mathrm{top}} = T + \frac{1}{2}\partial J, \tag{2.23}$$

which is obtained from the original stress tensor by twisting the $U(1)$ current. It induces that the twisted Virasoro generators become

$$L_n^{\mathrm{top}} = L_n - \frac{n+1}{2} J_n. \tag{2.24}$$

In addition we define

$$Q(z) := \frac{1}{\sqrt{2}}G(z),\ G(z) := \frac{1}{\sqrt{2}}\tilde{G}(z), \tag{2.25}$$

one can derive the twisted super Virasora algebra

$$\begin{aligned}
[L^{\mathrm{top}}_m, L^{\mathrm{top}}_n] &= (m-n)L^{\mathrm{top}}_{m+n} \\
[J_m, J_n] &= \hat{c}m\delta_{m+n} \\
[L^{\mathrm{top}}_m, J_n] &= -nJ_{m+n} + \frac{\hat{c}}{2}m(m+1)\delta_{m+n} \\
[L^{\mathrm{top}}_m, G_n] &= (m-n)G_{m+n} \\
[L^{\mathrm{top}}_m, Q_n] &= -nQ_{m+n} \\
[J_m, G_n] &= -G_{m+n} \\
[J_m, Q_n] &= Q_{m+n} \\
\{G_m, Q_n\} &= L^{\mathrm{top}}_{m+n} + nJ_{m+n} + \frac{\hat{c}}{2}m(m+1)\delta_{m+n}.
\end{aligned} \tag{2.26}$$

It turns out that the conformal dimension of any fields is shifted by minus half its $U(1)$ charge as $L_0^{\mathrm{top}} = L_0 - \frac{1}{2}J_0$, and T^{top} becomes a primary field and J is no longer primary due to the term $\frac{\hat{c}}{2}$. Furthermore, the $U(1)$ charge of $Q(z)$ becomes one, which can be used to define a BRST operator (eq. 2.4) as

$$Q = \frac{1}{2\pi i}\oint dzQ(z)\alpha(z), \tag{2.27}$$

where $\alpha(z)$ is a section of L_1^{-1} since $Q(z)$ is a holomorphic 1-form taking value in L_1. In the Ramond sector, L_1 is trivial which leads to a globally defined BRST operator $Q = Q_0$ used to define the BRST cohomology. In this case, $T^{\mathrm{top}}(z)$ and $Q(z)$ are both Q-exact from (eq. 2.26) implying

$$\{Q, G(z)\} = T^{\mathrm{top}}(z),\ \{Q, J(z)\} = -Q(z). \tag{2.28}$$

By the Q-exactness of $T^{\mathrm{top}}(z)$, the correlation functions do not depend on the metric of the world sheet, and hence the twisted theory is a topological theory. Note that there is another topological twist by the spectral flow which replace $J(z) \to -J(z)$ and interchange $G(z)$ and $Q(z)$.

In a $\mathcal{N} = (2,2)$ SCA with left and right sectors, since the left and right SCA's commute with each other, we could define the (twisted) BRST charge using any set of (twisted) BRST operators

$$\begin{aligned}
Q_A &= G_{+,0} + \tilde{G}_{-,0},\ \tilde{Q}_A = \tilde{G}_{+,0} + G_{-,0} \\
Q_B &= \tilde{G}_{+,0} + \tilde{G}_{-,0},\ \tilde{Q}_B = G_{+,0} + G_{-,0},
\end{aligned} \tag{2.29}$$

called the A- and B- topological models respectively. In each case, we define the $U(1)$ charge q to be the eigenvalue with respect to

$$\begin{aligned}
J_A &= J_{L,0} - J_{R,0} \\
J_B &= -J_{L,0} - J_{R,0},
\end{aligned} \tag{2.30}$$

such that

$$[J_i, Q_i] = Q_i \tag{2.31}$$

for $i \in \{A, B\}$.

To define the Frobenius structure in the A- or B-topological field theory (TFT), we need a linear functional used to construct a non-degenerate inner product. Let Ω_L be a local operator with $q_L = \hat{c}$ and $q_R = 0$, which can be identified with the holomorphic $(d, 0)$-form as

$$\Omega_L(z) = \Omega_{i_1 \dots i_d} \psi_+^{i_1} \cdots \psi_+^{i_d}, \tag{2.32}$$

and Ω_R with $q_L = 0$ and $q_R = \hat{c}$ identified with the holomorphic $(0, d)$-form. Now, given $\Omega = \Omega_L \otimes \Omega_R$, the linear functional is defined by

$$\langle O \rangle_{TFT} := \langle \Omega | O | 0 \rangle_{SCFT}, \tag{2.33}$$

where $|\prime\rangle$ is the vacuum and O is a operator. Thus we obtain a pairing

$$\langle O_1 O_2 \rangle_{TFT} = \langle \Omega | O_1 O_2 | 0 \rangle_{SCFT}. \tag{2.34}$$

which gives us the Frobenius structure.

Now we consider $\mathcal{N} = 2$ supersymmetry deformation induced by a local operator O of charge 2. Then the insertion of the operator dO with respect to the world sheet de Rham operator into a correlation would contribute a trivial correlation function, since the location of the operator is unimportant in a topological field theory. It turns out that

$$dO = \{Q, O^{(1)}\}, \tag{2.35}$$

for some operator 1-form $O^{(1)}$ with charge $2 - 1 = 1$, and repeating the process we have

$$dO^{(1)} = \{Q, O^{(2)}\} \tag{2.36}$$

for some operator 2-form $O^{(2)}$ with charge 0. Therefore we find a deformation of the action given by

$$S' = S + t \int_\Sigma O^{(2)} d^2 z. \tag{2.37}$$

Rather than discuss the topological twist in SCFT, the subject of the following sections would be devoted to writing down twisted sigma model action explicitly (see [Wit92] for further discussion).

2.2.1 The A-model

In the A-model, we modify the bundles L_i which the fermionic fields take values in (see table 2.1) as

Fermions	Sections
$\chi^i := \psi_+^i$	$\Gamma(\Phi^* T_X)$
$\chi^{\bar{i}} := \psi_-^{\bar{i}}$	$\Gamma(\Phi^* \bar{T}_X)$
$\psi_z^{\bar{i}} := \psi_+^{\bar{i}}$	$\Gamma(K \otimes \Phi^* \bar{T}_X)$
$\psi_{\bar{z}}^i := \psi_-^i$	$\Gamma(\bar{K} \otimes \Phi^* T_X)$

Table 2.2: Fermionic fields in the A-model.

By setting $\alpha_+ = \tilde{\alpha}_- = 0$ and $\alpha = \alpha_- = \tilde{\alpha}_+$ in eq. (2.3), the symmetry generated by the operator Q fulfill the conditions for the BRST symmetry and satisfies $Q^2 = 0$. The A-twisted action of the sigma model can be written as

$$S_A = i \int_\Sigma \{Q, \mathscr{D}\} - 2\pi i \int_\Sigma \Phi^*(B + i\omega), \tag{2.38}$$

where

$$\mathscr{D} = 2\pi g_{i\bar{j}}(\psi_z^{\bar{j}}\bar{\partial}\phi^i) + \partial\phi^{\bar{i}}\psi_{\bar{z}}^i, \tag{2.39}$$

$B + i\omega \in H^2(X,\mathbb{C})$ is the *complexified Kähler form*, and $\omega = ig_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is the Kähler form. Since the first is Q-exact and $\mathscr{D}$ contains the complex structure, deforming it leads to a Q-exact variation of the action, i.e. trivial in the topological theory. Hence the correlation functions in A-model depend only on the complexified Kähler form $B + i\omega$, and then the moduli space of A-models is the complexified Kähler moduli space of X, the cone in $H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})$ in which ω is a *big* divisor.

The Q-cohomology of the A-model can be represented by local operators that are functions of ϕ and χ only, independent of the derivatives of these fields. Let $W = W_{I_1I_2\cdots I_n}(\phi)d\phi^{I_1}d\phi^{I_2}\cdots d\phi^n$ be an n-form on X, we can define a corresponding local operator

$$O_W(P) = W_{I_1I_2\cdots I_n}\chi^{I_1}\chi^{I_2}\cdots\chi^{I_n}, \tag{2.40}$$

and one can then compute

$$\{Q, O_W\} = -O_{dW}. \tag{2.41}$$

This gives a natural map from de Rham cohomology of X to the Q-cohomology in the A-model and the space of operators is in $H^*(X,\mathbb{C})$.

To evaluate the correlation functions, we have to compute the path integral

$$\langle O_aO_bO_c\cdots\rangle = \int d\phi d\chi d\psi e^{-S_A}O_aO_bO_c\cdots, \tag{2.42}$$

by integrating over all embeddings $\phi : \Sigma \to X$. Note that the second term $\int \Phi^*(B + i\omega)$ in the action (eq. 2.38) depends on the complexified Kähler form and the homotopy class of the map Φ, $\Phi_*(\Sigma) \in H_2(X,\mathbb{Z})$ giving the instanton number. Thus we can rewrite the correlation functions as

$$\langle\cdots\rangle = \sum_{\phi_*(\Sigma)} e^{-2\pi i\int_\Sigma \phi^*(B+i\omega)}\int_{\phi_*\ \text{fixed}} d\phi d\chi d\psi e^{-i\int\{Q,\mathscr{D}\}}\cdots. \tag{2.43}$$

The bosonic part in $\mathscr{D}$ of the action S_A is minimized for the holomorphic maps ϕ, i.e. $\bar{\partial}\phi^i = \partial\phi^{\bar{i}} = 0$, called the *world-sheet instantons*. Thus the infinite-dimensional space of all maps of $\Sigma \to X$ is replaced by the finite-dimensional space of holomorphic maps. The 3-point function then is given (see [HKK03] for the details) by

$$\langle O_aO_bO_c\rangle = \int_X a\wedge b\wedge c + \sum_{\alpha\in I} N_{abc}^\alpha e^{2\pi i\int_\Sigma \phi^*(B+i\omega)}, \tag{2.44}$$

where I is the set of instantons and N_{abc}^α are the intersection numbers on the moduli space of rational curves in X, called the *Gromov-Witten invariants*. Note that if the sum of the degree of forms a, b, c is not $d = \dim(X)$, then the 3-point function vanishes. And we have the 1-point function given by

$$\langle O_a\rangle = \int_X a, \tag{2.45}$$

which is not trivial only if a is the top form on X. Thus these correlation functions induce the Frobenius structure on $H^*(X,\mathbb{C})$.

As discussed above, the algebraic structure of the A-model depends only on the cohomology class of the complexified Kähler forms $B + i\omega$ and not the complex structure of X, nor the Calabi-Yau condition. Thus

X can be any symplectic manifold with a almost complex structure. In this case instantons correspond to pseudo-holomorphic curves. If we neglect the instanton corrections, the algebraic structure is simply given by the wedge product of forms and the deformed ring is called *quantum cohomology ring* (see [CK99; Man96] for a detail account). When X is a Calabi-Yau 3-fold, the deformation we obtain just change the symplectic form ω and the B-field.

2.2.2 The B-model

In the B-model, instead of the form in table 2.1 we replace them with

Fermions	Sections
ψ_+^i	$\Gamma(K \otimes \Phi^* T_X)$
$\psi_-^{\bar{i}}$	$\Gamma(\Phi^* \bar{T}_X)$
$\psi_+^{\bar{i}}$	$\Gamma(\Phi^* \bar{T}_X)$
$\psi_-^{\bar{i}}$	$\Gamma(\bar{K} \otimes \Phi^* T_X)$

Table 2.3: Fermionic fields in the B-model.

We define the world sheet scalars

$$\begin{aligned} \eta^{\bar{j}} &:= \psi_+^{\bar{j}} + \psi_-^{\bar{j}} \\ \theta_j &:= g_{i\bar{k}}(\psi_+^{\bar{k}} - \psi_-^{\bar{k}}), \end{aligned} \tag{2.46}$$

and a 1-form $\rho^j = \rho_z^j + \rho_{\bar{z}}^j$ such that the $(1,0)$-form part is $\rho_z^j = \psi_+^j$ and (0,1)-form part is $\rho_{\bar{z}}^j = \psi_-^j$. For the supersymmetry transformations with the setting $\alpha_\pm = 0$ and $\bar{\alpha}_\pm = \alpha$, this variation induces a BRST charge Q obeying $Q^2 = 0$ modulo the equations of motion.

We can rewrite the action in the form

$$S_B = i \int \{Q, \mathscr{D}\} + W, \tag{2.47}$$

where

$$\begin{aligned} \mathscr{D} &:= g_{j\bar{k}} \left(\rho_z^j \bar{\partial} \phi^{\bar{k}} + \rho_{\bar{z}}^j \partial \phi^{\bar{k}} \right) \\ W &:= \int_\Sigma \left(-\theta_j D\rho^j - \frac{i}{2} R_{j\bar{j}k\bar{k}} \rho^j \wedge \rho^k \eta^{\bar{j}} \theta_l g^{l\bar{k}} \right). \end{aligned} \tag{2.48}$$

Here D is the exterior derivative on Σ acting on forms taking values in $\phi^*(T_X)$ by using the pullback of the Levi-Civita connection of X. To secure the chiral anomaly in defining the phase of the Pfaffian in the path integral, we require an additional condition $c_1(X) = 0$, i.e. X is a Calabi-Yau manifold.

The B-model is also a topological field theory, i.e. independent of the complex structure of Σ and the metric of X. One can see that the variation of the metric on Σ deform the action only by Q-exact forms $\{Q, \dots\}$, and it is less obvious but true that to change the Kähler form ω and the $(1,1)$ component of the B-field also deforms the action by Q-exact forms. Hence the correlation functions of the B-model are all independent of these parameters.

To define the local observables, we consider $(0,q)$ forms on X with values in $\wedge^p T_X$, and an object can be written as

$$V = d\bar{z}^{\bar{i}_1} d\bar{z}^{\bar{i}_2} \cdots d\bar{z}^{\bar{i}_q} V^{j_1 j_2 \cdots j_p}_{\bar{i}_1 \bar{i}_2 \cdots \bar{i}_q} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_p}}, \tag{2.49}$$

we can form a local operator

$$O_V = \eta^{\bar{i}_1} \eta^{\bar{i}_2} \cdots \eta^{\bar{i}_q} V^{j_1 j_2 \cdots j_p}_{\bar{i}_1 \bar{i}_2 \cdots \bar{i}_q} \theta_{j_1} \cdots \theta_{j_p}, \tag{2.50}$$

which is called a $(-p, q)$-form, and

$$\{Q, O_V\} = -O_{\bar{\partial} V}. \tag{2.51}$$

Similarly, this gives a natural map from the Dolbeault cohomolgy on forms with valued in exterior powers of the holomorphic tangent bundle on X to the Q-cohomology of the B-model. Note that the contraction with the holomorphic d-form Ω gives an isomorphism between the space of $(-p, q)$-forms and $(d - p, q)$-forms.

In the B-model, there is no instanton term in the path integral. As $\mathscr{D} = 0$, i.e. $\bar{\partial}\phi^{\bar{k}} = \partial\phi^{\bar{k}} = 0$, we obtain a constant map ϕ of Σ to a point in the target space X. Moreover, there is no quantum correction on the correlation functions, since the action is Q-exact. Under the rescaling of the action by an arbitrary constant, the loop counting parameter $\hbar$, the correlation functions must be invariant, i.e. independent of $\hbar$, and equal to the classical limit $\hbar \to 0$.

The correlation function on the sphere can be express as

$$\langle O_A O_B \cdots \rangle = \int_X \Omega \wedge i_{AB\ldots}\Omega, \tag{2.52}$$

where $AB\ldots$ is a $(0, d))$-form with values in $\wedge^d T_X$ such that $i_{AB\ldots}$, its contraction with the $(d, 0)$-form Ω, is a $(0, d)$-form. When X is a Calabi-Yau 3-fold, the 3-point function is

$$\langle O_A O_B O_C \rangle = \int_X \Omega^{ijk} A_i \wedge B_j \wedge C_k \wedge \Omega. \tag{2.53}$$

Here $A = A^i \frac{\partial}{\partial\phi^i}, B = B^j \frac{\partial}{\partial\phi^j}, C = C^k \frac{\partial}{\partial\phi^k} \in H^1_{\bar{\partial}}(X, T_X)$. The deformation theory corresponding to such $(-1, 1)$-forms are the deformations of the complex structure of X, called the *Kodaira-Spencer theory* (see [Ber+94] for the details). There exist other deformations of the B-model associated to $(-2, 0)$-forms and $(0, 2)$-forms corresponding to the cohomology group $H^0(X, \wedge^2 T_X)$ and $H^2(X, O_X)$, respectively. Such deformations can be understood in terms of generalized complex structure introduced by Hitchin [Hit03; Gua04].

2.2.3 Mirror symmetry for closed strings

To sum up, the A-and B-model actually depend only on half the moduli of the target space X, i.e.

$$\begin{aligned} &\text{A}-\text{model on } X \leftrightarrow \oplus_{p,q} H^q(X, \Omega^p) \leftrightarrow \text{Kähler moduli of } X, \\ &\text{B}-\text{model on } X \leftrightarrow \oplus_{p,q} H^q(X, \wedge^p T_X) \leftrightarrow \text{complex moduli of } X. \end{aligned} \tag{2.54}$$

At the level of $\mathcal{N} = (2, 2)$ SCA, given an A-model on a Kähler manifold Y and a B-model on a Calabi-Yau manifold X, we apply the spectral flow (eq. 2.14) on the $\mathcal{N} = 2$ SCA (eq. 2.7), which acts on the twisted SCA as

$$J \to -J; \quad \bar{J} \to \bar{J}. \tag{2.55}$$

The induced automorphism of the $\mathcal{N} = 2$ SCA gives

$$G_+ \to \tilde{G}_+, \tag{2.56}$$

and preserves the other generators. Hence the spectral flow interchange the A- and B-models from the equations (2.29), and leads to $h^{p,q}(Y) = h^{d-p,q}(X)$ and $\chi(Y) = -\chi(X)$. Thus for an A-model on the target space Y, there is a SCFT isomorphic B-model, and vice versa. Note that it does not mean that there exist a Calabi-Yau manifold X which induce the B-model. Conversely, a B-model on a rigid Calabi-Yau 3-fold, i.e. $h^{2,1}(X) = 0$, gives a counterexample to the reverse claim.

In the geometric approach, given a mirror pair (X, Y), mirror symmetry must map the moduli space of complexified Kähler forms of Y to the moduli space of complex structures of X, and this map is called the *mirror map*. The most-studied example of the mirror pair is to take the Calabi-Yau 3-fold Y to be a quintic hypersurface in $\mathbb{P}^4$, see [Can+91; Mor91; CK99; Voi99] for more details. Thus the moduli space of complexified Kähler forms is one dimenstional as $h^{1,1}(Y) = 1$, and its mirror manifold X should satisfy $h^{2,1}(X) = 1$. X is taken to be a resolution of singularities of a quintic hypersurface Y divided by a $(\mathbb{Z}_5)^3$ action, and defined by the equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0x_1x_2x_3x_4. \tag{2.57}$$

Here the complex structure of X is determined by the single parameter ψ. Therefore, the mirror map should be a map between the complexified Kähler form $B + i\omega$ of Y on the A-model to the complex structure ψ of X on the B-model. By the special geometry structure of moduli spaces ([Str90], [Fre99]), there exists a flat coordinate system on the moduli space. On the A-model, the complexified Kähler forms $B + i\omega$ are the special coordinates. Let e denote the positive generator of $H^2(Y, \mathbb{Z})$, and then we prefer to the complexified Kähler class as $(B + i\omega)e$. However, the complex parameters ψ do not form flat coordinates on the B-model.

In the mirror quintic X, a complex structure of X is uniquely characterized by the class of holomorphic 3-form $\Omega \in \mathbb{P}(H^3(X, \mathbb{C}))$, but not all points in $\mathbb{P}(H^3(X, \mathbb{C}))$. To construct flat coordinates for the complex structure moduli space, we start with choosing a symplectic basis of $H_3(X, \mathbb{Z})$. That is a basis A_k, B^k for $k = 0, \dots, h^{2,1}(X)$ satisfying the intersection rules

$$A_k \cap A_l = 0,\ A_k \cap B^l = \delta_k^l,\ B^k \cap B^l = 0, \tag{2.58}$$

and a relevant Poincaŕe dual basis is denoted by α_k, β^k for all k. Then it turns out that the holomorphic 3-form $\Omega(z) \in H^{3,0}(X_z, \mathbb{C})$ can be expanded in terms of a basis α_k, β^k as

$$\Omega(z) = \varpi^k(z)\alpha_k + \mathcal{F}_k(z)\beta^k, \tag{2.59}$$

where z is the local complex moduli of X, $\varpi^k(z)$ are the A-cycle periods and $\mathcal{F}_k(z)$ are the B-cycle periods defined by

$$\varpi^k(z) = \int_{A_k} \Omega(z), \quad \mathcal{F}_k(z) = \int_{B^k} \Omega(z), \tag{2.60}$$

respectively, and expressed as the period vector $\Pi(z) = (\varpi^i(z), \mathcal{F}_i(z))$. One can pick a primitive element $A_0 \in H_3(X, \mathbb{Z})$, such that $\int_{A_0} \Omega \neq 0$. Therefore $\varpi^0(z) = \int_{A_0} \Omega(z) \neq 0$ in the neighborhood of $z = 0$ and we can define the special A-cycle periods

$$t^k(z) := \frac{\varpi^k(z)}{\varpi^0(z)} = \frac{\int_{A_k} \Omega(z)}{\int_{A_0} \Omega(z)}. \tag{2.61}$$

Then one can proof that the special A-cycle periods form a set of homogeneous special coordinates, and the Hodge-Riemann bilinear relation implies that

$$\frac{\partial}{\partial t^j}\left(\frac{\mathcal{F}_i(z)}{\varpi^0(z)}\right) = \frac{\partial}{\partial t^i}\left(\frac{\mathcal{F}_j(z)}{\varpi^0(z)}\right). \tag{2.62}$$

It turns out that there exist a function $\mathcal{F}(t^1, \ldots, t^n)$, called a *prepotential*, such that

$$\frac{\mathcal{F}_i(z)}{\varpi^0(z)} = \frac{\partial \mathcal{F}}{\partial t^i}. \tag{2.63}$$

In the A-model side, there is the same structure and the complexified Kähler moduli space $H^2(Y, \mathbb{C})$ is embedded into $\mathbb{P}(\oplus_{k=\text{even}} H^k(Y, \mathbb{C}))$. The mirror map thus is a projective linear symplectic map between the two ambient spaces. Under the analysis of the monodromy at the maximally unipotent point, it is natural to expect that the mirror map is given by

$$t^1 = \frac{\int_{A_1} \Omega(z)}{\int_{A_0} \Omega(z)}, \text{ and } q = \exp\left(2\pi i \frac{\int_{A_1} \Omega(z)}{\int_{A_0} \Omega(z)}\right), \tag{2.64}$$

where $q = e^{2\pi i \int_\Sigma \phi^*(B+i\omega)}$ as in eq. (2.44). A physical discussion for the assumption is given in [Ber+94]. For a more detailed mathematical argument, one can refer to [CK99; Voi99] for an more thorough treatment.

2.3 Open strings and D-branes

In the previous section we have discussed supersymmetry conformal field theory defined on compact Riemann surfaces Σ, the world sheet, swept out by the closed strings. On the other hand, strings theory also contains open strings whose world sheet has no trivial boundaries. In the target space the ends of open strings with Dirichlet boundary conditions can be embedded into submanifolds of the target space and such objects are called *D-branes* [Pol96; HIV00; HKK03; Asp04]. Therefore, it is necessary to study the conformal field theory associated to the sectors of open strings, called the *boundary conformal field theory* (BCFT) [Car89; Car04].

2.3.1 Boundary conformal field theory

The boundary conditions of conformal field theories are given by the variation of the Polyakov action S_P (1.24) with respect to the world sheet metric $h_{\mu\nu}$. A natural requirement is that the off-diagonal component of the stress tensor $T_{\mu\nu} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_\text{P}}{\delta h^{\mu\nu}}$ parallel/perpendicular to the boundary should vanish, that is

$$t^\mu n^\nu T_{\mu\nu}|_{\partial\Sigma} = 0, \tag{2.65}$$

where t^μ and n^ν are tangent and normal vectors to the boundary, respectively. This is called the conformal boundary condition, and one can refer to [Wes12] for detail computations. In the sigma model for $B = 0$ defined by a map $\phi : \Sigma \to M$, there are two local boundary conditions

Dirichlet boundary condition: $\phi|_{\partial\Sigma}$ are fixed, or $t^\mu \partial_\mu \phi|_{\partial\Sigma} = 0$;

Neumann boundary condition: $n^\mu \partial_\mu \phi|_{\partial\Sigma} = 0$.

In terms of the left (holomorphic) and right (anti-holomorphic) movers, the $U(1)$ currents become

$$J_L = (t^\mu + n^\mu)\partial_\mu \phi \,, \quad J_R = (t^\mu - n^\mu)\partial_\mu \phi \,, \tag{2.66}$$

such that

$$\begin{aligned} J_L + J_R &= 0 \quad \text{(Dirichlet)}; \\ J_L - J_R &= 0 \quad \text{(Neumann)}. \end{aligned} \tag{2.67}$$

If the stress tensors $T_L = T(z)$ and $T_R = \bar{T}(\bar{z})$ are expanded in terms of the currents $J_L = J(z)$ and $J_R = \bar{J}(\bar{z})$, the boundary conditions (2.67) imply

$$L_n - \bar{L}_n = 0, \tag{2.68}$$

which can be expressed as the condition $T(z) = \bar{T}(\bar{z})$. It turns out that the central charges of left-handed and right-handed conformal field theories have to be the same, that is $c = \bar{c}$. This fact immediately leads the boundary conditions, or D-branes, to be defined only for the Type II string theories, not for the heterotic string theories. However, in an arbitrary conformal field theory there is usually no relevant Lagrangian formulation, and hence no boundary conditions would arise from a variation of the action as above. Thus, we need more general formulations of boundary conditions.

We start with the observation that given a world sheet Σ as a cylinder $\mathbb{R} \times S^1$, by interchanging the coordinates (τ, σ)

$$(\sigma, \tau)_{\text{open string}} \longleftrightarrow (\tau, \sigma)_{\text{closed string}} \,, \tag{2.69}$$

the cylinder partition function of the boundary conformal field theory of open strings can be interpreted as the one of the underlying conformal field theory of closed strings. It turns out that the tree-level amplitude describes the process of the emission of a closed string at one end propagating to the other end where it is absorbed. This duality in string theory is known as the world sheet duality between open and closed strings.

We then can think of the boundary condition as a state $|B\rangle$ in the Hilbert space $\mathcal{H} \otimes \bar{\mathcal{H}}$, which carries two (left and right) commuting Virasoro algebras, i.e.

$$|B\rangle = \sum_{i \in \mathcal{H}, \bar{j} \in \bar{\mathcal{H}}} \alpha_{i\bar{j}} |i, \bar{j}\rangle. \tag{2.70}$$

The conditions (2.67) thus become

$$\begin{aligned} (J_L - J_R)|B_{\mathrm{D}}\rangle &= 0 \quad \text{(Dirichlet)}; \\ (J_L + J_R)|B_{\mathrm{N}}\rangle &= 0 \quad \text{(Neumann)}, \end{aligned} \tag{2.71}$$

so that the condition (2.68) would be

$$(L_n - \bar{L}_n)|B_{\mathrm{N,D}}\rangle = 0. \tag{2.72}$$

Such conditions relating to holomorphic and anti-holomorphic modes on the boundary states are called the *gluing conditions*, and one can show that solutions to the gluing conditions are one-to-one corresponding to primary fields. We thus can view the boundary state $|B\rangle$ as an operator O_B satisfying the relation

$$L_n O_B = O_B L_n, \tag{2.73}$$

and such operator O_B is a sum of projectors on irreducible representations of the Virasoro algebra. The boundary sate related to the projector is called an *Ishibashi state*, and then the physical boundary states are the linear combinations of Ishibashi states. For more details, we refer to the textbook [BP09].

2.3.2 Topological boundary conditions

Topological boundary conditions in topological string theory must be compatible with the topological twist (2.29). For the $\mathcal{N} = (2,2)$ superconformal field theories on a world sheet Σ with the boundaries, it is no longer to preserve the entire $\mathcal{N} = (2,2)$ SCA, since the left and right $U(1)$ currents have to match together at the boundaries. Hence the existence of the D-branes breaks supersymmetry completely. Thus we require the topological boundary conditions must preserve half of the spacetime supersymmetry, in particular to break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. There are two ways to define topological boundary conditions which break half of the supersymmetry, corresponding to the two twistings. These are called A-type and B-type boundary conditions [OOY96] defined by

$$J_L = -J_R, \tag{2.74}$$

$$J_L = J_R, \tag{2.75}$$

respectively. Note that compared with (2.67), these would correspond to Dirichlet and Neumann boundary conditions respectively. Then the conserved charges in the A- and B-models (2.30) are, respectively,

$$\begin{aligned} J_A &= J_L - J_R = \int d\sigma \partial_\tau \phi; \\ J_B &= J_L + J_R = \int d\sigma \partial_\sigma \phi, \end{aligned} \tag{2.76}$$

where the integrals are the realization of the $U(1)$ currents as free bosons. Indeed, this implies that branes in the A-model must preserve A-type N = 2 SCA and branes in the B-model must preserve B-type N = 2 SCA.

Let L be a submanifold of the target space X, $\phi|_{\partial\Sigma} \subset L$ and impose the Dirichlet boundary conditions in the normal directions to L and Neumann conditions in the tangent directions to L. The boundary conditions connect the left and the right moving sectors, and thus can be written [Asp04; Asp09] as

$$\begin{aligned} \frac{\partial \phi^I}{\partial z} &= R^I_J(\phi) \frac{\partial \phi^J}{\partial \bar{z}} \\ \psi^I_+ &= R^I_J(\phi) \psi^J_- \end{aligned} \tag{2.77}$$

where R is an orthogonal matrix with respect to the metric. The eigenvectors with eigenvalue (-1) give Dirichlet boundary conditions and thus span the directions normal to L. The eigenvectors with eigenvalue $(+1)$ of R are associated to directions tangent to L. In the following, we would follow [Asp04; Asp09], and discuss the A-type boundary conditions in the A-model and the B-type boundary conditions in the B-model, called the *A-branes* and *B-branes*, respectively.

2.3.3 The A-branes

In the A-model, the boundary conditions which are consistent with the A-twist are given by

$$R^i_j = R^{\bar{i}}_{\bar{j}} = 0, \tag{2.78}$$

with non-zero off-diagonal terms $R^i_{\bar{j}}$ and $R^{\bar{i}}_j$. Now we choose a vector with eigenvalue $+1$ with respect to R, that is, a tangent vector to the D-brace L. Consider the almost complex structure J with

$$J^m_n = i\delta^m_n\,, \quad J^{\bar{m}}_{\bar{n}} = -i\delta^{\bar{m}}_{\bar{n}}\,. \tag{2.79}$$

One can see that the vector Jv has eigenvalue -1 with respect to R, i.e. normal to L. Again the vector $J^2 v = -v$ is in the tangent direction. Thus J exchanges the directions tangent and normal to the D-brane L, which implies L must be of middle dimension.

Now consider two tangent vectors v and w with eigenvalue $+1$ under R, then w is orthogonal to Jv with respect to the metric g. By definition, the Kähler form on X can be written as $\frac{1}{2}g_{IK}J^K_M d\phi^I d\phi^M$. Then the above arguments induce that L is a Lagrangian submanifold of X. On the other hand, the boundary condition can involve a gauge field, which defines a 1-form A on X, and contribute an additional term into the action, called the boundary action, which can be written as

$$S_b = -\int_{\partial\Sigma} \phi^* A. \tag{2.80}$$

A is a gauge connection and $F = dA$. To preserve BRST symmetry we have to assume $F = 0$, thus A is a flat connection. Moreover, quantum consideration impose additional constrains as the A-branes must preserve the ghost number of the operator product algebra. Let's choose a holomorphic 3-form Ω on X, then the volume form of $L \in X$ may be written as a restriction

$$dV_L = ce^{-i\pi\xi}\Omega|_L, \tag{2.81}$$

where c is a positive real number and ξ is a map from L to a circle, $\xi : L \to S^1$. Thus it induces a map on the fundamental group, called the *Maslov class* of L,

$$\xi_* : \pi_1(L) \to \pi_1(S^1) \cong \mathbb{Z}. \tag{2.82}$$

The condition of cancellations of the ghost number anomaly is related to the vanishing of the Maslov class [HKK03]. The Maslov class is always 0 if the fundamental group of L is trivial, i.e. $\pi_1(L) = 0$. For example, given an one-dimensional complex torus as the target space X, each line of X has the trivial Maslov class, but a contractible loop does not and thus must been excluded from the A-branes.

Now we would study the open string spectrum between a pair of Lagrangian A-branes (L_1, E_1, ∇_1) and (L_2, E_2, ∇_2) [Wit95a]. At first suppose we have a Lagrangian A-brane L with a $U(N)$ vector bundle $E \to L$. Let $L \cong L_1 \cong L_2$. Then the open string states are section of endomorphism of E, i.e. $E^* \otimes E$. As the discussions in § 2.2.1, the Hilbert space of open string states is given by the total de Rham cohomology group

$$\mathcal{H} \cong \bigoplus_k H^k\left(L, \mathrm{Hom}(E_1, E_2)_{\mathbb{C}}\right), \tag{2.83}$$

where the ghost number is given by the degree k. The correlation function would be a sum over holomorphic maps ϕ with compatible boundary conditions. Consider the case of the 3-point function of operators

$$\begin{aligned} A &\in H^1(L, \mathrm{Hom}(E_1, E_2)), \\ B &\in H^1(L, \mathrm{Hom}(E_2, E_3)), \\ C &\in H^1(L, \mathrm{Hom}(E_3, E_1)). \end{aligned} \tag{2.84}$$

We obtain the classical term of the 3-point function which is

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \rangle = \int_L \mathrm{Tr}(A \wedge B \wedge C), \tag{2.85}$$

and the term of the instanton corrections is

$$\sum_{D_j} \pm \exp\left(i \int_{D_j} (B + i\omega) + i \oint_{\partial D_j} A_i \right) N_{abc} \tag{2.86}$$

over all holomorphic disks D_j with $\partial D_j \subset L$. Here A_i is the connection on the bundle E_i, and we have divide the boundary ∂D_j into three arcs label by $i \in \{1, 2, 3\}$ and each A_i is on the ith arc. Moreover, those open string operators stay at the end of the arcs, i.e. $\mathcal{O}_A$ is between arc 1 and arc 2, etc. Note that the ordering of the operators would imply that this open string operator algebra is thus associative but not necessarily commutative. Thus open string chiral operators form a algebra which is not necessarily commutative, but still is associative. On the other hand, elements in H^1 correspond to deformations of Lagrangian L. For an operator $\mathcal{O}$ with $U(1)$ charge 1, there is a unique operator $\mathcal{O}^{(1)}$ with $U(1)$ charge 0 satisfying

$$d\mathcal{O} = \{Q, \mathcal{O}^{(1)}\}, \tag{2.87}$$

and deforming the action by

$$\delta S_b = \int_{\partial\Sigma} \mathcal{O}^{(1)}. \tag{2.88}$$

Thus it preserve the $U(1)$ charge, as its $U(1)$ charge is 0, and conformal invariance linearly.

Furthermore, there is a main difference between the closed and open string deformations. To be precise, the closed string deformation can never be obstructed in the $\mathcal{N} = (2, 2)$ SCFT, but the open string deformation often can be. In order to secure the obstruction problems, the quantum corrections of the open string correlation function as in (2.86) can modify the obstruction theory. The accurate A-branes are those Lagrangian submanifolds satisfying the following quantum obstruction condition:

$$\sum_{D_j} \pm \exp\left(i \int_{D_j} (B + i\omega) + i \oint_{\partial D_j} A_i \right) [\partial D_j] = 0, \tag{2.89}$$

where $[\partial D_j]$ is the homology class of ∂D_j in $H_1(L)$.

To sum up, we conclude the following

Definition 2.5. A *Lagrangian A-brane* (L, E, ∇) is given by an equivalence class of Lagrangian sunmanifolds $L \subset X$ with a flat connection which has trivial Maslov class (2.82) and satisfies the quantum obstruction condition (2.89), modulo Hamiltonian deformations.

Remark 2.6. We only discuss the Lagrangian A-branes with a flat connection. However, in the case of $F \neq 0$, a Calabi-Yau n-fold may have A-branes of real dimension $n + 2p$ for $p \geq 0 \in \mathbb{Z}$. see [OOY96] for more detail arguments.

Let's consider the case of open strings for many A-branes. For simplicity, given a set of A-branes L_i equipped with line bundles E_i. For a pair of A-branes (L_1, L_2), we have a graded Hilbert space of open strings from L_1 to L_2, which is

$$\mathrm{Hom}^*(L_1, L_2) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(L_1, L_2). \tag{2.90}$$

As before, an open string chiral operator corresponds to a state in such a graded Hilbert space of open strings, and the correlation functions of chiral operators form a category of A-branes, an A_∞-category called the *Fukaya category*. This category plays an important role in Kontsevich's homological mirror symmetry [Kon94]. Since the A-model depends on the complexified Kähler forms, the Fukaya category depends on $B + i\omega$ for its objects and composition of morphisms. For a fully comprehensive introduction to the Fukaya category, one can refer to the books [Fuk+09; SS08].

2.3.4 The B-branes

Now we turn to the B-type boundary condition in the B-model. The boundary condition compatible with the B-twist should be

$$R^{\bar{i}}_{j} = R^{i}_{\bar{j}} = 0, \tag{2.91}$$

and the diagonal term R^i_j and $R^{\bar{i}}_{\bar{j}}$ are not zero for the reflection matrix given in (2.77). It turns out that the almost complex structure thus preserves the tangent and normal directions to the D-brane in the B-model. It implies that the D-brane is a complex submanifold of the target space X, and then we conclude that a B-type D-brane wraps holomorphic cycles with even real dimensions in X. Note that there is anther B-brane corresponding to the anti-holomorphic submanifold since we have used the orientation of X.

Similar to the A-model case, the existence of the B-field allows us to introduce a bundle $E \to X$ over the B-brane. By the setting of $B = 0$, the condition of the invariance of the Q-variation of the action from the boundary term would imply that the curvature F of the bundle is a 2-form of type $(1, 1)$ taking values in $\mathrm{End}(E)$ [HIV00; Wit95a]. Thus $E \to X$ is a holomorphic vector bundle, or a locally free coherent sheaf $\mathscr{E}$ on X in algebraic geometry. In general, the $(0, 2)$ part of F is equal to the negative $(0, 2)$ part of B. In a Calabi-Yau 3-fold the $(0, 2)$ part of B is homological trivial and can be zero by a BRST transformation. It means that the B-field makes no contribution to the category of B-branes in a Calabi-Yau 3-fold, but not in a K3 surface or a complex torus.

In the B-model, a local operator given by (2.50) depends on the scalar operators $\eta^{\bar{j}}$ and θ_j (2.46). Then the boundary condition above implies that the fermion θ_j on the boundary can be written as

$$\theta_j = g_{j\bar{k}}(\psi^{\bar{k}}_+ - \psi^{\bar{k}}_-) = F_{j\bar{k}}\eta^{\bar{k}}. \tag{2.92}$$

It turns out that a local operator only depends on ϕ and $\eta^{\bar{k}}$, and a local boundary operator corresponds to a $(0, q)$-form with values in $\mathrm{End}(E)$. As in § 2.2.2, the BRST operator Q is sent to the Dolbeault operator $\bar{\partial}$ in the large volume limit. thus an open string vertex operator related to a string stretching from E_1 to E_2 is given by an element of the Dolbeault cohomology group

$$\bigoplus_q H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E_1, E_2)), \tag{2.93}$$

where the degree q is equal to the ghost number of the operators without ambiguity.

The correction function has no instanton correction in the B-model as before. Then the 3-point function of operators

$$
\begin{aligned}
A &\in H^{0,p}_{\bar{\partial}}(X, \mathrm{Hom}(E_1, E_2)),\\
B &\in H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E_2, E_3)),\\
C &\in H^{0,r}_{\bar{\partial}}(X, \mathrm{Hom}(E_3, E_1)),
\end{aligned}
\tag{2.94}
$$

such that $p + q + r = \dim_{\mathbb{C}}(X)$, is given by the classical term

$$
\langle O_A O_B O_C \rangle = \int_X \mathrm{Tr}(A \wedge B \wedge C) \wedge \Omega. \tag{2.95}
$$

Then the operator product is the ordinary wedge product of Hom-valued forms. The deformation of the Dolbeault operator by $\delta A^{0,1} \in H^1(X, \mathrm{End}(E))$ is

$$
\bar{\partial} + \delta A^{0,1}, \tag{2.96}
$$

i.e. infinitesimal deformation with values in the global Ext group $\mathrm{Ext}^1(\mathscr{E}, \mathscr{E})$, which corresponds to the modification to the boundary action.

2.3.5 The category of B-branes

Now we can define a category of B-branes in the topological string theory following the book [Asp09]. As the discussion above, the category of B-branes includes the holomorphic vector bundles E, or locally free coherent sheaves $\mathscr{E}$ on X as objects, with the morphisms beings the classes of Dolbeault cohomolgy group $H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E, F)) \cong \mathrm{Ext}^q(E, F)$, and the grading of the cohomology group is given by the $U(1)$-charge q, or ghost number given by (2.76). Moreover, Using the free boson realization of the $U(1)$ algebra, the $U(1)$-charge of an open string from a brane E to a brane F is the expectation value of the $U(1)$ current

$$
J_0 = \int_0^{\pi} \partial_\sigma \phi = q + \phi(\pi) - \phi(0), \tag{2.97}
$$

integrated over the open string, where the boundary terms $\phi(\pi)$ and $\phi(0)$ are determined by the boundary conditions E and F, respectively. It means that a boundary condition in the B-model was completely specifies by a holomorphic bundle E with additional quantum number n, which contributes to the $U(1)$ charge. E^n thus is a different boundary condition, so that the $U(1)$ charge of an open string in the Ext group $\mathrm{Ext}^i(E^m, F^n)$ is given by

$$
J = i + n - m. \tag{2.98}
$$

Therefore, boundary conditions in the B-model are graded sums as

$$
E := \bigoplus_{n \in \mathbb{Z}} E^n, \tag{2.99}
$$

where the E^n are different B-type boundary conditions, and the full spectrum of topological open strings then consists of the Q-cohomology classes of maps between pairs of those boundary conditions. Now consider the original B-branes as objects in an additive category $\mathscr{A}$, or $\mathrm{Coh}(X)$, then we have the following proposition

Proposition 2.7. *The full set of B-branes in the B-model corresponds to the homotopy category* $\mathbf{K}(\mathscr{A})$.

Proof. To define the structure of complexes, we first vary the differential by

$$d = \sum_n d_n, \quad d_n \in \mathrm{Ext}^0(E^n, E^{n+1}) = \mathrm{Hom}^0(E^n, E^{n+1}), \tag{2.100}$$

and the corresponding operator $O_d^{(1)}$ obeys

$$\{Q, O_d^{(1)}\} = d_\Sigma d, \tag{2.101}$$

where d_Σ is the de Rham differential on the world sheet. Then the deformation of the action leads to a change in the BRST charge

$$Q' = Q + d. \tag{2.102}$$

To preserve the condition $Q'^2 = 0$, we require that

$$\{Q, d\} + d^2 = 0. \tag{2.103}$$

Since $d_n \in \mathrm{Ext}^0(E^n, E^{n+1})$, we have that $\{Q, d\} = 0$ implying the condition $d^2 = 0$, i.e.

$$d_{n+1} d_n = 0 \tag{2.104}$$

for all n. Hence the condition of nilpotence of the BRST charge is equivalent to the condition the E is a complex

$$\cdots \xrightarrow{d_{n-1}} E^n \xrightarrow{d_n} E^{n+1} \xrightarrow{d_{n+1}} E^{n+2} \xrightarrow{d_{n+2}} \cdots . \tag{2.105}$$

It means that the maps in the complex represent a deformation from original objects. Note that the position of the object in the complex is associated to the $U(1)$ charge of the B-brane.

We now study the spectrum of open strings between the B-branes. Consider open strings from a B-brane $(E^\bullet, d^E)$ to another B-brane $(F^\bullet, d^F)$, then the total BRST charge becomes

$$Q = Q_0 + d^E - d^F, \tag{2.106}$$

acting on a direct sum of morphisms

$$f^{m,n} : E^m \to F^n \tag{2.107}$$

from the complex $E^\bullet$ to the complex $F^\bullet$, and thus the topological open strings correspond to the cohomology classes of the operator. Since the individual morphisms $f^{m,n} \in \mathrm{Hom}(E^m, F^n) = \mathrm{Ext}^0(E^m, F^n)$, thus $Q_0 f^{m,n} = 0$. With a suitable choice of sign, the condition of exactness of morphisms $f^\bullet$ with respect to Q is exactly the condition that $f^\bullet$ is a morphism of complexes. Note that the morphism $f'^\bullet$ differed by a Q-exact morphism, i.e. $f'^\bullet = f^\bullet + Qh^\bullet$, is homotopic to the morphism $f^\bullet$. Furthermore, we can introduce the shift functor $[n]$ which changes the $U(1)$-charge of complexes. □

From the proposition, we see that the category of B-branes contains all of objects and morphisms of the homotopy category $\mathbf{K}(\mathrm{Coh}X)$. On the other hand, the observables in the topological field theory are determined by the spectrum of open strings and their correlation functions, and it implies that a notion of a *physically equivalence*, i.e. the branes with the same behavior in all correlation functions.

Definition 2.8. Two objects $E^\bullet, E'^\bullet \in \mathbf{K}(\mathscr{A})$ are called *physically equivalent* if and only if

$$\begin{aligned} \mathrm{Ext}^p(E^\bullet, F^\bullet) &\cong \mathrm{Ext}^p(E'^\bullet, F^\bullet) \\ \mathrm{Ext}^p(F^\bullet, E^\bullet) &\cong \mathrm{Ext}^p(F^\bullet, E'^\bullet) \end{aligned} \tag{2.108}$$

for all $F^\bullet \in \mathbf{K}(\mathscr{A})$, and preserve the 3-point correlation function $\langle O_A O_B O_C \rangle$ in (2.95).

To be precise, setting $E_2 = E^\bullet$ in (2.94), $E_2' = E'^\bullet$ and

$$\begin{aligned} A' &\in H^1(L, \mathrm{Hom}(E_1, E_2')), \\ B' &\in H^1(L, \mathrm{Hom}(E_2', E_3)), \\ C &\in H^1(L, \mathrm{Hom}(E_3, E_1)), \end{aligned} \tag{2.109}$$

then we obtain

$$\langle O_A O_B O_C \rangle = \langle O_{A'} O_{B'} O_C \rangle, \tag{2.110}$$

for all $E_1, E_3 \in \mathbf{K}(\mathscr{A})$. By the equivalently associativity this condition would imply that all n-point functions are the same, hence all observables do. Thus we can define a quotient category of $\mathbf{K}(\mathscr{A})$ as

Definition 2.9. The category $\mathcal{T}(\mathscr{A})$ is the quotient category of $\mathbf{K}(\mathscr{A})$ by the *physical equivalence*.

Furthermore, the inclusion of $\mathbf{K}(\mathscr{A})$ has the following proposition

Proposition 2.10. *The natural inclusion of* $\mathbf{K}(\mathscr{A})$ *into* $\mathcal{T}(\mathscr{A})$ *maps quasi-isomorphisms into isomorphisms.*

Proof. We consider the cone $C(f^\bullet)$ of a quasi-isomorphism $f^\bullet$. It turns out that $\mathrm{Hom}^*(C(f^\bullet), F^\bullet) \cong \mathrm{Hom}^*(F^\bullet, C(f^\bullet)) \cong 0$ for all $F^\bullet$, as $C(f^\bullet)$ is acyclic. □

On the other hand, we know the derived category $\mathrm{D}(\mathscr{A})$ is the universal category sending quasi-isomorphisms of $\mathbf{K}(\mathscr{A})$ to isomorphisms (see the excellent book [GM02]). Thus the inclusion functor $F : \mathbf{K}(\mathscr{A}) \to \mathcal{T}(\mathscr{A})$ factors uniquely through $\mathrm{D}(\mathscr{A})$, i.e.

$$\begin{tikzcd} \mathbf{K}(\mathscr{A}) \arrow[r, "Q"] \arrow[dr, "F"'] & \mathrm{D}(\mathscr{A}) \arrow[d, "G"] \\ & \mathcal{T}(\mathscr{A}) \end{tikzcd}$$

where Q is the localization functor. Then we have the main theorem of the subsection

Theorem 2.11 ([AL01]). *The quotient category* $\mathcal{T}(\mathscr{A})$ *is equivalent to the derived category* $\mathrm{D}(\mathscr{A})$.

Proof. Suppose there exists a pair $E, E' \in \mathrm{D}(\mathscr{A})$ of inequivalent objects which were physical equivalent. Then we obtain a pair of morphisms $\alpha \in \mathrm{Hom}(E, E')$ and $\beta \in \mathrm{Hom}(E', E)$, such that the morphism $\beta\alpha - \mathrm{id}_E \in \mathrm{Hom}(E, E)$ is zero in all correlation functions. However, from Serre duality [Har77] we have

$$\mathrm{Hom}_{D(X)}(\mathscr{E}, \mathscr{F}) \otimes \mathrm{Hom}_{D(X)}(\mathscr{F}, \mathscr{E} \otimes \omega_X[n]) \to H^n(X, \omega_X) \cong \mathbb{C}, \tag{2.111}$$

for all $\mathscr{E}, \mathscr{F} \in \mathrm{D}(X)$, and ω_X is the dualizing sheaf for X. As X is a Calabi-Yau manifold, ω_X is trivial, thus we get a trace map

$$\mathrm{Tr} : \mathrm{Ext}^n(E, E) \to \mathbb{C}, \tag{2.112}$$

which implies that $\beta\alpha - \mathrm{id}_E$ is identically zero, a contradiction. □

2.4 Stability conditions on D-branes

In previous section, we have discussed topological D-branes, i.e. boundary conditions in the A- and B-models, and the spectrum of open strings and the correlation functions in the topological string theories. It turns out that D-branes can naturally be considered as objects in a category, and the morphisms are $\mathbb{Z}$-graded Q-cohomology class. Precisely, the category of A-branes looks like the Fukaya category, and the category of B-branes looks like the derived category of coherent sheaves. Kontsevich's homological mirror symmetry conjecture [Kon94] applies to these categories. However, there are too many objects in the categories for all of them to be associated to physical D-branes. In the case of B-branes, only those sheaves allowing the solutions of the Hermitian Yang-Mill equations are physical D-branes, which corresponds to the μ-stable sheaves. In the A-model, physical A-branes have a relation to *special Lagrangian submanifolds* and a notion of stability [Joy03]. In other words, the physical D-branes depend on more structures, so that their boundary conditions are different, and it may be possible to identify physical D-branes with stable objects in the Fukaya or derived categories in a suitable sense of stability conditions. Hence the mirror symmetry conjecture would lead to

Conjecture 1: The moduli space of stability conditions of a mirror pair are isomorphic.

Conjecture 2: Given a mirror pair X and Y, the category of stable objects in the Fukaya category $\mathrm{Fuk}(X)$ is equivalent to the category of stable objects in $\mathrm{D}(\mathrm{Coh}\,Y)$.

To sum up, we make the table below

	A-branes	B-branes
Geometry structure	Symplectic	Algebraic (Holomorphic)
Category	Fukaya category	Derived category
Topological D-branes	Lagrangian	Complexes of coherent sheaves
Spectrum of open strings	Floer cohomology	Ext's group
Moduli	Complexified Kähler form	Complex structure
Physical D-branes	Special Lagrangians	Π-stable complexes

Table 2.4: A- and B-branes.

In the particle physic theory, the concept of the stability is associated to the *formation* or *decay* processes of particles, that is, the particles combine or split to form other particles. Thus for the D-branes, considerd as particles, we need some suitable definition of the *binding process*. In the remaining section, we would briefly review the geometric stability for A- and B-branes based on the book [Asp09], and make a precise definition of stability conditions on triangulated categories [Bri07].

2.4.1 Stability conditions on A-branes

We start with the definition of special Lagrangian submanifolds given below

Definition 2.12. Given a Calabi-Yau manifold X with a Kähler form ω and a holomorphic volume form Ω, a *special Lagrangian submanifold* (SLAGs) (L, E, ∇) satisfies the following

i) L is a Lagrangian submanifold of X with respect to the Kähler form ω.

ii) The vector bundle E is flat, i.e. $F = 0$.

iii) $\mathrm{Im} e^{-i\pi\xi(L)}\Omega|_L = 0$ for some constant $\xi(L) \in \mathbb{R}$.

Compared with the definition of the Lagrangian A-branes in § 2.3.3, the Maslov class ξ_* in (2.82) is trivial and thus the Maslov condition is always satisfied for a special Lagrangian submanifold. However, the quantum obstruction condition (2.89) is not automatically fulfilled. On the other hand, from (2.81) we have

$$dV_L = ce^{-i\pi\xi(L)}\Omega|_L, \tag{2.113}$$

for some positive real constant c. Now fix the range $0 \le \xi(L) < 2$, the parameter $\xi(L)$ can be written as

$$\xi(L) = \frac{1}{\pi} \arg \frac{\Omega|_L}{dV_L} = \frac{1}{\pi} \arg \int_L \Omega\,, \tag{2.114}$$

which is the argument of the period of the holomorphic volume form associated to the cycle L. Thus $\xi(L)$ only depends on the homology class of L. For physics reasons, we make the following definition

Definition 2.13. The *BPS central charge* of the D-brane L is given by the period

$$Z(L) = \int_L \Omega, \tag{2.115}$$

and the *mass* of the D-brane is the volume of L in the geometric limit, and must be greater or equal to the absolute value of the central charge, i.e.

$$M := \int_L dV_L \ge |Z(L)|, \tag{2.116}$$

and the equality holds if and only if it is a BPS brane.

In the A-model, cohomology classes in $H^{1,1}(X,\mathbb{C})$ correspond to deformations of the complexified Kähler class, and classes in $H^1(L,\mathbb{C})$ correspond to deformation of Lagrangian submanifolds in § 2.3.3. However, most of Lagrangians have no any special Lagrangian equivalent to them by Hamiltonian isotopy, which may be understood as *decay* of special Lagrangians. As mentioned previously, the inverse process of decay is a binding process, i.e. several branes combine into one brane, which is given by Joyce [Joy03] below

Theorem 2.14 ([Joy03]). *Given a family of Calabi-Yau n-folds X_z with complex structures parameterized by $z \in \mathbb{C}$ with $|z|$ small. Suppose X_z contains two special Lagrangians L_1 and L_2 which intersect transversely. Then there exist a special Lagrangian $L_1 \looparrowright L_2 \subset X_z$ which is closed to the connected sum $L_1 \cup L_2$ if and only if $\xi(L_2) \le \xi(L_1)$.*

According to the sign of $\xi(L_2) - \xi(L_1)$, we can separate the moduli space of complex structures into two parts $\mathscr{M}^+$ and $\mathscr{M}^-$ by the wall of marginal stability. So the special Lagrangian $L_1 \looparrowright L_2$ only exists in $\mathscr{M}^+$, and note that $L_1 \looparrowright L_2 \ne L_2 \looparrowright L_1$. In $\mathscr{M}^+$, we have BPS branes L_1, L_2 and $L_1 \looparrowright L_2$, but the mass of $L_1 \looparrowright L_2$ is smaller the sum of the masses of L_1 and L_2. Once we touch the wall, $L_1 \looparrowright L_2$ would become $L_1 \cup L_2$. In $\mathscr{M}^-$ we then only have BPS branes L_1 and L_2, without any smooth special Lagrangian minimizing the volume of $L_1 \cup L_2$, which together break supersymmetry. This is a decay process of a BPS brane $L_1 \looparrowright L_2$ into its factors L_1 and L_2 from the region $\mathscr{M}^+$ into $\mathscr{M}^-$.

$L_1 \cup L_2$ is a special Lagrangian only on the wall due to different values of ξ in the two regions. Since the BPS brane $L_1 \looparrowright L_2$ is homologous to $L_1 \cup L_2$ and minimizes the volume or mass, it turns out that

$$\left|\int_{L_1 \looparrowright L_2} \Omega\right| < \left|\int_{L1} \Omega\right| + \left|\int_{L_2} \Omega\right|, \tag{2.117}$$

and we choose

$$\xi(L_2) < \xi(L_1 \looparrowright L_2) < \xi(L_1). \tag{2.118}$$

We sum up the discussion in the table:

Modui space	Order of ξ	Stability of $L_1 \looparrowright L_2$
$\mathscr{M}^+$	$\xi(L_2) < \xi(L_1 \looparrowright L_2) < \xi(L_1)$	Stable
Wall	$\xi(L_2) = \xi(L_1 \looparrowright L_2) = \xi(L_1) + 1$	Marginally stable
$\mathscr{M}^-$	$\xi(L_2) > \xi(L_1) + 1$	Unstable

Table 2.5: Decay of A-brane.

For relevant physical discussion, one can refer to [SW94; HKK03].

2.4.2 Π-stability on B-branes

In § 2.3.5, it shows that the category of B-branes is just the derived category of coherent sheaves $\mathrm{D}(X)$, a triangulated category (see the textbooks [GM02; KS02]). Given a distinguished triangle in $\mathrm{D}(X)$,

$$\begin{tikzcd} & C \arrow[dl, dashed, "h", "{[1]}"'] & \\ A \arrow[rr, "f"] & & B \arrow[ul, "g"'] \end{tikzcd}$$

for $A, B, C \in \mathrm{D}(X)$, it thus should be understood as the D-branes A and C may potentially be bound to form the bound state B. and morphisms f, g, h correspond to the spectrum of open strings. Therefore, we begin with the translation of axioms in the triangulated category [GM02] into the physical process in the topological string theory.

TR1: a) A can bind with the empty brane 0 to form A.

b) Consider two functorial isomorphic set of objects (A, B, C) and $(A', B', C,)$ on $\mathrm{D}(X)$. If B can potentially decay into A and C, then B' can potentially decay into A' and C'.

c) We can potentially form a bound state of A and B if there exists an open string from A to B.

TR2: If B can potentially decay into A and C, then C can potentially decay into $A[1]$ and B.

TR3: Given two open strings between A and A' and between B and B', there potentially exist open strings between the corresponding bound states.

TR4: If we represent the binding rule by the notion of addition, then we obtain

$$\begin{aligned} C &= A[1] + B \\ &= A[1] + (E + D[-1]) \\ &= (A[1] + E) + D[-1] \\ &= F + D[-1]. \end{aligned} \tag{2.119}$$

On the other hand, since the Fukaya category need not to have a triangulated structure, there exist no any potentially bound state in the Fukaya category. It means that the statement of homological mirror symmetry conjecture have to be modified, by adding extra *potentially stable* A-branes into the Fukaya category to form a triangulated category. The modified statement is given below

Conjecture: Given a mirror pair of Calabi-Yau 3-folds X and Y, then the category $\mathrm{D}(X)$ is equivalent to the category $\mathrm{TwFuk}(Y)$.

Here $\mathrm{TwFuk}(Y)$ is a triangulated category constructed from $\mathrm{Fuk}(Y)$, see the book [SS08] for the detailed construction.

If homological mirror symmetry is true, we could apply the thought of the stability defined on A-branes to the case of B-branes. We now give the definition of the central charge of B-branes.

Definition 2.15. In the large volume limit, the central charge of a B-brane E has the approximate expression[1]

$$Z(E) = \int_X e^{-(B+i\omega)}\mathrm{ch}(E)\sqrt{\mathrm{td}(X)} + O(\alpha'). \tag{2.120}$$

and we define $\xi(E)$ by

$$\xi(E) = \frac{1}{\pi}\arg Z(E) \pmod 2. \tag{2.121}$$

with the property

$$\xi(E[n]) = \xi(E) + n. \tag{2.122}$$

Now compared with the table 2.5, given a distinguished triangle of the form

$$\begin{tikzcd} & C \arrow[dl, dashed, "h"', "{[1]}"] & \\ A \arrow[rr, "f"] & & B, \arrow[ul, "g"'] \end{tikzcd}$$

with A and B are stable, then C is stable with respect to the decay process if and only if $\xi(B) < \xi(A) + 1$. If $\xi(B) = \xi(A) + 1$, then C is marginally stable and $\xi(C) = \xi(B) = \xi(A) + 1$. Thus we can make the following definition of stability.

Definition 2.16 ([Dou01])**.** A B-brane B is called Π-stable if, for all distinguished triangles of the form $\cdots A \to B \to C \to A[1] \cdots$, we have

$$\xi(A) < \xi(B) < \xi(C). \tag{2.123}$$

Otherwise, B is called Π-unstable.

Moreover, by (2.98) and the degree counting, it turns out that if $\xi(A) > \xi(B)$, then $\mathrm{Hom}(A, B) = 0$. We also can consider the case of decays into several number of stable objects, which leads to the following definition.

Definition 2.17. For any object E we define the following set of distinguished triangles, the *Postnikov system*,

$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & E_{n-1} \arrow[rr] \arrow[dl] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & A_{n-1} \arrow[ul, dashed, "{[1]}"] & & A_n \arrow[ul, dashed, "{[1]}"] & \end{tikzcd}$$

[1] The definition here is old-fashioned and incorrect, see chap. 3.

Then E decays into $A_1, A_2, \ldots, A_n$ if

$$\xi(A_1) > \xi(A_2) > \cdots > \xi(A_n). \tag{2.124}$$

It is physically reasonable to expect that the set of n-stable objects A_k satisfying (2.124) for a object E is unique. However, it is still unclear as it is not obvious what we means in CFT by unstable particles. On the other hand, we also can compare the Π-stability and μ-stability in the large volume limit. Indeed, we will recover μ-stability if the condition (2.123) reduces to the condition that the slope satisfies $\mu(E) < \mu(F)$ for any subsheaf F of E if E is μ-stable in the large volume limit [DFR05].

2.4.3 Bridgeland's stability condition

Motivated by Douglas's Π-stability [Dou01] defined on the category of B-branes in the B-model as the discussion in § 2.4.2, Bridgeland made a precise definition of stability conditions on triangulated categories [Bri07]. Moreover, one can show that the space of stability conditions on a reasonable triangulated category is a finite dimensional manifold, providing an geometric invariant of such category.

We begin with Bridgeland's definition:

Definition 2.18. A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category D consists of a group homomorphism $Z : K(\mathrm{D}) \to \mathbb{C}$ called the *central charge*, and full additive subcategories $\mathcal{P}(\phi) \subset \mathrm{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,

(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathrm{D}}(A_1, A_2) = 0$,

(d) for each nonzero object $E \in \mathrm{D}$ there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

Given a stability condition $\sigma = (Z, \mathcal{P})$, each slicing $\mathcal{P}(\phi)$ is an abelian subcategory of D, and the non-zero objects of $\mathcal{P}(\phi)$ are said to be semistable of phase ϕ. The simple objects of $\mathcal{P}(\phi)$ are said to be stable. The category $\mathcal{P}(I)$ is defined as the extension-closed subcategory of D generated by the subcategories $\mathcal{P}(\phi)$ for $\phi \in I \subset \mathbb{R}$. It can be shown that the decompositions of a nonzero object $0 \neq E \in \mathrm{D}$ given by axiom (d) are uniquely defined up to isomorphism, and the objects A_j are called the semistable factors of E with respect to σ. We write $\phi^+_\sigma(E) = \phi_1$ and $\phi^-_\sigma(E) = \phi_n$. The *mass* of E is defined to be the positive real number

$$m_\sigma(E) = \sum_i |Z(A_i)|, \tag{2.125}$$

and by the triangle inequality, we have

$$m_\sigma(E) \geq |Z(E)|. \tag{2.126}$$

The equality holds if and only if E is semistable. This condition is the same as the condition (2.116).

The relation between stability conditions on triangulated categories and stability in abelian category is given by the following proposition.

Proposition 2.19 ([Bri07])**.** *To give a stability condition on* D *is equivalent to giving a bound t-structure on* D *and a stability function on its heart which has the Harder-Narasimhan property.*

Here we recall the definition of a t-structure [GM02; KS02]).

Definition 2.20. A *t-structure* on a triangulated category D is a full subcategory $\mathcal{F} \subset \mathrm{D}$ which is preserved by left-shifts, i.e. $\mathcal{F}[1] \subset \mathcal{F}$, and if we define

$$\mathcal{F}^\perp = \{G \in \mathrm{D} : \mathrm{Hom}_\mathrm{D}(F, G) = 0 \text{ for all F} \in \mathcal{F}\},$$

and such that for every object $E \in \mathrm{D}$ there is a triangle

$$\begin{tikzcd} F \arrow[rr] & & E \arrow[dl] \\ & G \arrow[ul, dashed] & \end{tikzcd}$$

in D with $F \in \mathcal{F}$ and $G \in \mathcal{F}^\perp$.

The *heart* of a t-structure $\mathcal{F} \subset \mathrm{D}$ is the full subcategory

$$\mathcal{A} = \mathcal{F} \cap \mathcal{F}^\perp[1] \subset \mathrm{D}.$$

In [BBD82] it is proved that $\mathcal{A}$ is an abelian category with the short exact sequence $0 \to a_1 \to a_2 \to a_3 \to 0$ in $\mathcal{A}$ being precisely the triangles $a_1 \to a_2 \to a_3 \to a_1[1]$ in D all of whose vertices a_j are objects of $\mathcal{A}$.

Example 2.21. The standard t-structure on the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, given by

$$\mathcal{F} = \{E \in \mathrm{D}(\mathcal{A}) : H^i(E) = \text{for all i} > 0\},$$
$$\mathcal{F}^\perp = \{E \in \mathrm{D}(\mathcal{A}) : H^i(E) = \text{for all i} < 0\}.$$

The heart is the original abelian category $\mathcal{A}$.

A t-structure $\mathcal{F} \subset \mathrm{D}$ is said to be *bounded* if

$$\mathrm{D} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{F}[i] \cap \mathcal{F}^\perp[j].$$

A bounded t-structure $\mathcal{F} \subset \mathrm{D}$ is determined by its heart $\mathcal{A} \subset \mathrm{D}$. In particular, $\mathcal{F}$ is the extension-closed subcategory generated by the subcategories $\mathcal{A}[j]$ for all $j \in \mathbb{Z}_{\geq 0}$. Then we have the following characterization of bounded t-structure.

Lemma 2.22. *Let $\mathcal{A} \subset \mathrm{D}$ is a full additive subcategory of a triangulated category* D*, then $\mathcal{A}$ is the heart of a bounded t-structure on* D *if and only if the two following conditions hold:*

(a) $\mathrm{Hom}_{\mathrm{D}}(A, B[k])$ *for $k < 0$, if A and B are objects of $\mathcal{A}$,*

(b) *for every nonzero object $E \in \mathcal{D}$ there are a finite sequence of integers $k_1 > k_2 > \cdots k_n$ and a collection of triangles*

$$\begin{tikzcd} 0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[rr] & & E_2 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] & \end{tikzcd}$$

with $A_j \in \mathcal{A}[k_j]$ for all j. □

Note that given a slicing $\mathcal{P}$ of a triangulated category D, the subcategory $\mathcal{P}((\phi, \phi + 1]) \subset \mathrm{D}$ is the heart of the t-structure $\mathcal{P}(> \phi)$, see [Bri07, § 3]. To be compared with stability condition on $\mathrm{D}(\mathcal{A})$, a stability function on $\mathcal{A}$ is defined as follow.

Definition 2.23. A *stability function* on an abelian category $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the strict upper half-plane, i.e.

$$Z(E) \in \mathbb{H} = \{r \exp(i\pi\phi) : r \in \mathbb{R}_{>0} \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}.$$

The *phase* of an object $0 \neq E \in \mathcal{A}$ with respect to a stability function $Z : K(\mathcal{A}) \to \mathbb{C}$ is defined to be

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

Thus we can define a notion of stability using the order of phases of objects.

Definition 2.24. Given a stability function on an abelian category $Z : K(\mathcal{A}) \to \mathbb{C}$, an object $0 \neq E \in \mathcal{A}$ is called *semistable* if every object $0 \neq \mathcal{A} \subset E$ satisfies $\phi(A) \leq \phi(E)$, or equivalently every nonzero quotient $E \twoheadrightarrow B$ satisfies $\phi(E) \leq \phi(B)$.

Now once we have a stability, it may be possible to construct a filtration of objects in $\mathcal{A}$.

Definition 2.25. Given an order induced by the phases of objects defined as above in an abelian category $\mathcal{A}$, a *Harder-Narasimhan filtration* of an object $0 \neq E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j / E_{j-1}$ are semitable objects of $\mathcal{A}$ with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

The stability function Z is said to have the *Harder-Narasimhan property* if every nonzero object of $\mathcal{A}$ has a Harder-Narasimhan filtration.

Note that if there exist a morphism $f : E \to F$ between two semistable objects E and F, it turns out that $\phi(E) < \phi(F)$. This simple fact implies the uniqueness of the Harder-Narasimhan filtration, if they exist. The existence of the Harder-Narasimhan filtration is followed a weak condition on $\mathcal{A}$ which forbid to have any infinite ascending or descending chain of this type [Bri07; Rud97].

Now we start to prove the proposition 2.19.

Proof. Let $\sigma = (Z, \mathcal{P})$ be a stability condition on D, and $\mathcal{A} = \mathcal{P}((0, 1]) \subset$ D be the heart of a bounded t-structure $\mathcal{P}(> 0)$ on D. Then the central charge Z defines a stability function on its heart $\mathcal{A}$, and the decomposition of objects of $\mathcal{A}$ given by axiom (d) are the Harder-Narasimhan filtration.

Conversely, suppose $\mathcal{A}$ is the heart of a bounded t-structure on D with a stability function $Z : K(A) \to \mathbb{C}$ on $\mathcal{A}$. We define the full additive subcategory $\mathcal{P}(\phi)$ of D to consist of semistable objects of phase ϕ in $\mathcal{A}$ for $\phi \in (0, 1]$. Then the axioms in Definition 2.18 can be obtain by the Lemma 2.22 and the Harder-Narasimhan filtrations of nonzero objects of $\mathcal{A}$. □

A stability condition is called *locally finite* if there is some $\varepsilon > 0$, such that each quasi-abelian category $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length, i.e. any infinite chain of subobjects or quotients must terminate. Thus every semistable object has a finite Jordan-Hölder filtration into stable objects of the same phase. We can put a topology on the set Stab(D) of locally finite stability conditions on D induced by the metric

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathrm{D}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \in [0, \infty].$$

Then we have the following important theorem which relates a small deformation of the central charge in this metric with a deformation of the stability condition.

Theorem 2.26 ([Bri07])**.** *For each connected component $\Sigma \subset$ Stab(D), there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathrm{D}), \mathbb{C})$ with a well-defined linear topology and a continuous map $\mathcal{Z} : \Sigma \to V(\Sigma)$ which sends a stability condition $(Z, \mathcal{P})$ to its central charge Z.*

It turns out that any deformation of the central charge can be lifted to a unique deformation of the stability condition. If the derived category D(X) is induced by a smooth projective variety X, then the space Stab(X), the set of locally finite stability conditions on D(X) such that the central charge Z factors via the Chern character ch: $K(X) \to H^*(X, \mathbb{Q})$, is a finite dimensional complex manifold. This is the starting point for the next chapter.

CHAPTER 3

Numerical Vectors and Numerical Stability Conditions

This chapter is the main content in the first part of the thesis. We introduce numerical vectors which are the main objects of study in the work. We begin with the discussion of the Gamma class and Gamma conjecture, which serves as the main motivation for the introduction of the numerical vectors. These can be viewed as ring homomorphisms between the Grothendieck group $K(\mathcal{A})$ of an abelian group $\mathcal{A}$ with a bilinear form and a graded vector space over a filed k of characteristic 0 with a quadratic form. We continue with the definition and the basic properties of numerical t-stability conditions, and then give a relation between the numerical t-stability conditions and Bridgeland's stability conditions on smooth projective surfaces in § 3.2. In § 3.3 we construct the cohomolgical Fourier-Mukai transforms induced by the numerical vectors which are compatible with the K-theoretic Fourier-Mukai transforms, and isometric with respect to some quadratic forms on the cohomolgy groups.

3.1 Gamma class and Gamma conjecture

Let us start with a brief discussion of the Gamma conjecture of Sergey Galkin, Vasily Golyshev, and Hiroshi Iritani [GGI16; GI15], which relates the quantum cohomology of a Fano manifold and the Gamma class in terms of differential equations.

3.1.1 Gamma conjecture

For a Fano manifold F, the quantum cohomology algebra defines a quantum (flat) connection over $\mathbb{C}^{\times}$ [Dub96] and its solution is given by a multivalued cohomology-valued function $J_F(t)$, called the *J-function*. The limit of the J-function, under a certain condition, exists and defines the *principle asymptotic class* A_F as :

$$A_F := \lim_{t\to+\infty} \frac{J_F(t)}{\langle[\mathrm{pt}], J_F(t)\rangle} \in H^*(F).$$

The *Gamma conjecture I* claims that the class A_F equals to the Gamma class $\hat{\Gamma}_F = \hat{\Gamma}(TF)$ of the tangent bundle of F, i.e.

$$A_F = \hat{\Gamma}_F.$$

More generally, under semisimplicity assumption of the quantum cohomology of F, one can define *higher* asymptotic classes $A_{F,j}$, $1 < j < N = \dim H^*(F)$ from exponential asymptotics of flat sections of the quantum connection. Then the *Gamma conjecture II* says that the classes $A_{F,i}$ can be written as :

$$A_{F,j} = \hat{\Gamma}_F \cdot \mathrm{Ch}(E_j)$$

for a full exceptional collection $E_1, E_2, \ldots, E_N$ of the derived category $\mathrm{D}(F)$. Here the modified version of the Chern character $\mathrm{Ch}(E)$ is written as $\mathrm{Ch}(E) := (2\pi i)^{\frac{\deg}{2}}\mathrm{ch}(E) = \sum_{p=0}^{\dim F}(2\pi i)^p \mathrm{ch}_p(E)$.

Recently, V. V. Golyshev and D. Zagier presents a proof of the gamma conjecture for Fano 3-folds of Picard rank 1 [GZ16].

3.1.2 Gamma class

Recall that a *multiplicative* characteristic class $\hat{Q}_X$ is a characteristic class satisfies the condition

$$\hat{Q}_{X\times Y} = \hat{Q}_X\hat{Q}_Y.$$

To such a characteristic class, there exists an associated formal power series $Q(z) = \sum_{j=0}^{\infty} b_j z^j$ in the variable z with the constant term $b_0 = 1$, see [HSB95]. Similarly, one can construct an *additive* characteristic class by a formal power series which constant term is zero. In the sequel we consider the product $Q(z_1)Q(z_2)\cdots Q(z_k)$ which is symmetric in the variables $z_1, z_2, \ldots, z_k$, it can be expressed as a formal power series in the symmetric functions $p_1, p_2, \ldots, p_k$ of the variables, that is

$$Q(p_1, p_2, \ldots, p_k) = Q(z_1)Q(z_2)\cdots Q(z_k).$$

Then if we choose $p_i = c_i(X)$ for all i, it would give a characteristic class $\hat{Q}_X$ for smooth projective varieties X.

The *Gamma class* of a smooth projective variety or a complex manifold X is the cohomology class

$$\hat{\Gamma}_X = \prod_{i=1}^{n}\Gamma(1+\alpha_i) \in H^*(X, \mathbb{R})$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the Chern roots of the tangent bundle TX and $\Gamma(x)$ is Euler's Gamma function, which is

$$\log(\Gamma(1+z)) = -\gamma z + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-z)^k$$

so that

$$\Gamma(1+z) = 1 - \gamma z + \left(\zeta(2)+\gamma^2\right)\frac{z^2}{2} - \left(2\zeta(3) + 3\zeta(2)\gamma + \gamma^3\right)\frac{z^3}{6} + \ldots,$$

where γ is Euler's constant and $\zeta(k)$ denotes the Riemann zeta function at k. The Gamma class is explicitly given by the formula,

$$\hat{\Gamma}_X = \exp\left(-\gamma c_1(X) + \sum_{k\geq 2}(-1)^k(k-1)!\zeta(k)\mathrm{ch}_k(TX)\right).$$

The Gamma class can be also regarded as a square root of the Todd class (or $\hat{A}$-class), using the familiar identity

$$\frac{z}{1-e^{-z}} = e^{z/2}\frac{z/2}{\sinh(z/2)} = e^{z/2}\Gamma\left(1+\frac{z}{2\pi i}\right)\Gamma\left(1-\frac{z}{2\pi i}\right).$$

Here the first term induces the Todd class td_X and the middle term induces the $\hat{A}$-class. The first equality thus implies

$$\mathrm{td}_X = e^{c_1(X)/2}\hat{A}(X).$$

As proposed by [Hal+15], one can define an alternative to the square root of the Todd class as below. We write the following equation

$$\sqrt{\frac{z}{1-e^{-z}}}\exp(i\Lambda(z)) = e^{z/4}\Gamma\left(1+\frac{z}{2\pi i}\right)$$

and solve it for $\Lambda(z)$, and z is real. Then

$$\begin{aligned}\Lambda(z) &= \operatorname{Im}\log\Gamma\left(1+\frac{z}{2\pi i}\right)\\ &= \operatorname{Im}\left(-\gamma\frac{z}{2\pi i}+\sum_{n\geq 2}(-1)^n\frac{\zeta(n)}{n}\left(\frac{z}{2\pi i}\right)^2\right)\\ &= \frac{\gamma z}{2\pi}+\sum_{k\geq 1}(-1)^k\frac{\zeta(2k+1)}{2k+1}\left(\frac{z}{2\pi}\right)^{2k+1}.\end{aligned} \tag{3.1}$$

Since the constant term of the formal power series is zero, it can be used to define an additive characteristic class Λ_X, called the *log Gamma class*. In the case of Calabi-Yau X, we obtain

$$\Lambda_X = -\frac{\zeta(3)}{(2\pi)^3}c_3+\frac{\zeta(5)}{(2\pi)^5}(c_5-c_2c_3)-\frac{\zeta(7)}{(2\pi)^7}(c_7-c_3c_4-c_2c_5+c_2^2c_3)+\cdots.$$

So the replacement for the square root of the Todd class td_X is thus a multiplicative characteristic class, called the *complex Gamma class*,

$$\hat{\Gamma}_X^{\mathbb{C}} = \sqrt{\mathrm{td}_X}\exp(i\Lambda_X).$$

Note that if we multiply each cohomology class of $\hat{\Gamma}_X^{\mathbb{C}}$ in $H^k(X)$ by $(2\pi i)^{k/2}$, we would obtain the regular Gamma class $\hat{\Gamma}_X$ defined previously.

3.1.3 Hirzebruch-Riemann-Roch formula and the Mukai pairing

Recall the Chern character [Ful98] defines a ring homomorphism from the Grothendieck group to cohomology

$$\mathrm{ch}: K(X)\to H^{\mathrm{even}}(X,\mathbb{Q}).$$

Note that in general this map is neither injective nor surjective, even tensoring the left side with $\mathbb{Q}$. There is a natural bilinear pairing on the Grothendieck group $K(X)$ given by the Euler characteristic

$$\chi(\mathscr{E},\mathscr{F}) := \sum_k(-1)^k\dim\mathrm{Ext}^k(\mathscr{E},\mathscr{F}),$$

for any class $\mathscr{E},\mathscr{F}\in K(X)$. Using the Hirzebruch-Riemann-Roch formula, the Euler characteristic $\chi(\mathscr{E},\mathscr{F})$ can be expressed as

$$\chi(\mathscr{E},\mathscr{F}) = \chi(X,\mathscr{E}^\vee\otimes\mathscr{F}) = \int_X \mathrm{ch}(\mathscr{E}^\vee).\mathrm{ch}(\mathscr{F}).\mathrm{td}_X.$$

For a smooth projective variety X over $\mathbb{C}$, Hodge theory implies that there is a natural direct sum decomposition

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

with $\overline{H^{p,q}} = H^{q,p}$ and $H^{p,q}(X) \cong H^q(X,\Omega^p)$. The Chern class and all characteristic class are classes of type (p,p).

Definition 3.1. One defines the *Mukai vector* of a class $\mathscr{E} \in K(X)$ as the cohomology class

$$v(\mathscr{E}) := \mathrm{ch}(\mathscr{E}).\sqrt{\mathrm{td}_X}.$$

Note that the existence of the square root of the Todd class can be shown by a formal power series calculation since the constant term of the Todd class td_X is 1 in $H^0(X,\mathbb{Q})$. Hence the induced map combined with Hodge theory

$$v : K(X) \longrightarrow \bigoplus H^{p,p} \cap H^{2p}(X,\mathbb{Q})$$

is additive. To define some duality of $v \in H^*(X,\mathbb{C})$, there is a natural definition :

Definition 3.2. Given a vector $v = \sum_j v_j \in \bigoplus H^j(X,\mathbb{C})$, one defines the dual vector of v as

$$v^\vee := \sum i^j v_j \in H^*(X,\mathbb{C}).$$

One can easily check that this operation is multiplicative, that is, $v^\vee.w^\vee = (v.w)^\vee$. With this notion we have the following lemma

Lemma 3.3. *With this duality it turns out that*

$$\mathrm{td}_X = \mathrm{td}_X^\vee.\exp(c_1(X)).$$

Proof. This can be easily deduced from the splitting principle by taking $\mathrm{td}_X = \prod \frac{\alpha_j}{1-\exp(-\alpha_j)}$, and the right hand side of the equality becomes

$$\mathrm{td}_X^\vee.\exp(c_1(X)) = \prod \frac{(-\alpha_j)}{1-\exp(\alpha_j)}.\prod \exp(\alpha_j),$$

where $\alpha_j, j = 1,2,\ldots$ are the Chern roots of the tangent bundle of X. □

The Mukai pairing on $H^*(X,\mathbb{C})$ for a smooth project variety, introduced by Căldăraru in [Căl05], is the quadratic form

$$\langle v, v'\rangle_X := \int_X \exp(c_1(X)/2).(v^\vee.v).$$

By the construction and the lemma, one can find out that for all $\mathscr{E},\mathscr{F} \in K(X)$, we have

$$\begin{aligned}\chi(\mathscr{E},\mathscr{F}) &= \int_X \mathrm{ch}(\mathscr{E}^\vee).\mathrm{ch}(\mathscr{F}).\mathrm{td}_X \\ &= \int_X \left(\mathrm{ch}(\mathscr{E}^\vee).\sqrt{\mathrm{td}_X}\right).\left(\mathrm{ch}(\mathscr{F}).\sqrt{\mathrm{td}_X}\right) \\ &= \langle v(\mathscr{E}), v(\mathscr{F})\rangle_X.\end{aligned}$$

However, as observed by Iritani [Iri07; Iri09] and Katzarkov-Kontsevich-Pantev [KKP08], to preserve Hirzebruch-Riemann-Roch formula and the Mukai pairing, instead of the Mukai vector we could define a map

$$v_\Lambda(\mathscr{E}) := \mathrm{ch}(\mathscr{E}).\sqrt{\mathrm{td}_X}.\exp(i\Lambda),$$

where Λ satisfies $\Lambda^\vee = -\Lambda$. Therefore, it turns out that

$$\begin{aligned}
\langle v_\Lambda(\mathscr{E}), v_\Lambda(\mathscr{F})\rangle_X &= \int_X \exp(c_1(X)/2).(v_\Lambda(\mathscr{E})^\vee.v_\Lambda(\mathscr{F})) \\
&= \int_X \mathrm{ch}(\mathscr{E})^\vee.\sqrt{\mathrm{td}_X}.\exp(-i\Lambda).\mathrm{ch}(\mathscr{F}).\sqrt{\mathrm{td}_X}.\exp(i\Lambda) \\
&= \int_X \mathrm{ch}(\mathscr{E}^\vee).\mathrm{ch}(\mathscr{F}).\mathrm{td}_X \\
&= \chi(\mathscr{E}, \mathscr{F}).
\end{aligned}$$

We thus call such a map as a *generalized twisted Mukai vector*. Note that in the log Gamma class (3.1) only odd power of z appear in the power series expansion, then $\Lambda_X^\vee = -\Lambda_X$. Hence it can be used to define a generalized twisted Mukai vector.

Now based on the study of Mirror symmetry and relevant phenomena, the Gamma class of a Calabi-Yau 3-fold X appeared in the computation of the mirror periods of hypergeometric functions by Candelas et al. [Can+91]. Libgober showed [Lib99] that the physically relevant n-point correlation functions and their derivatives had asymptotic expansions which are closed related to the Gamma class. Moreover, Kontsevich's homological mirror symmetry suggests that the monodromy of the Picard-Fuchs equation of mirrors should be related to $\mathrm{Aut}(\mathrm{D}(X))$.

In order to fulfill those observations, one of the most important assumption is that the formula of the central charge of the B-branes in the B-model in the large volume limit (Def. 2.15) should be modified by

Definition 3.4. In the large volume limit, the central charge of a B-brane E has the approximate expression

$$Z(E) = \int_X e^{-(B+i\omega)}.\mathrm{ch}(E).\hat{\Gamma}_X + O(\alpha').$$

In other word, we should replace the square root of the Todd class by the Gamma class.

Bridgeland [Bri08] used the formula of the central charge of the B-branes to define the stability function, or slope function, on the derived category of K3 surfaces. In the next subsection, we would apply the ideas given in the section to study numerical stability conditions on triangulated categories.

3.2 Numerical t-stability conditions

In this section we would study the t-stability conditions on triangulated categories introduced by A. L. Gorodentsev, S. A. Kuleshov and A. N. Rudakov in the work [GKR04]. We propose a concept of *numerical vectors* in a Grothendieck group of an abelian category which may be used to construct a *numerical* t-stability condition on the corresponding derived category. In the case of smooth projective surfaces over $\mathbb{C}$, we provide a strategy for constructing Bridgeland's numerical stability conditions [Bri08] from numerical t-stability conditions, and make a conjecture for higher dimensional projective varieties over $\mathbb{C}$.

3.2.1 Numerical vectors

Let $\mathcal{A}$ be an abelian category, and recall that its Grothendieck group, denoted by $K(\mathcal{A})$, is the abelian group presented as having one generator $[A]$ for each object $A \in \mathcal{A}$, with the relation $[A] = [A'] + [A'']$ for each short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$. We have the following useful identities in $K(\mathcal{A})$.

(a) $[0] = 0$.

(b) If $A \cong A'$, then $[A] = [A']$.

(c) $[A' + A''] = [A'] + [A'']$.

It is obvious that if two abelian categories are equivalent, their Grothendieck groups are naturally isomorphic since both of them have the same representation by (b).

An *additive function* from $\mathcal{A}$ to an abelian group Γ is a function f from the objects of $\mathcal{A}$ to Γ so that $f(A) = f(A') + f(A'')$ for each short exact sequence as above in $\mathcal{A}$. The function $[\ ] : A \mapsto [A]$ from $\mathcal{A}$ to $K(\mathcal{A})$ defines an additive function which has the universal property as below.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "{[\bullet]}"] \arrow[dr, "f"'] & K(\mathcal{A}) \arrow[d, "f'"] \\ & \Gamma \end{tikzcd}$$

It means that any additive function f from $\mathcal{A}$ to Γ induces a unique group homomorphism $f' : K(\mathcal{A}) \to \Gamma$ such that $f'([A]) = f(A)$ for each $A \in \mathcal{A}$. For more information on the K-theory, see textbook [Wei13].

Now we introduce a notion of additive vectors in the following.

Definition 3.5. An *additive vector* is an additive function from $\mathcal{A}$ to a finite dimensional graded vector space $\bigoplus_j V^j$ over a field k of characteristic 0, i.e.

$$v : K(\mathcal{A}) \longrightarrow \bigoplus_{j=0}^{n} V^j$$

for some $n \in \mathbb{N}$ and $\dim_k(V^j) < \infty$ for each j.

Suppose there exist a bilinear form on the Grothendieck group $K(\mathcal{A})$

$$(\ ,\ ) : K(\mathcal{A}) \times K(\mathcal{A}) \longrightarrow k$$

and a quadratic form on the graded vector space

$$\langle\ ,\ \rangle : \bigoplus_{j=0}^{n} V^j \otimes \bigoplus_{j=0}^{n} V^j \longrightarrow k.$$

Then we can make the following definition.

Definition 3.6. A *numerical vector* with respect to $(\, , \,)$ is an additive vector v from $(K(\mathcal{A}), (\, , \,))$ to $\left(\bigoplus_{j=0}^{n} V^j, \langle \, , \, \rangle\right)$ satisfying the equality

$$(E, F) = \langle v(E), v(F) \rangle,$$

for all $E, F \in K(\mathcal{A})$.

Example 3.7. In the case of the category of coherent sheaves on smooth projective varieties X over $\mathbb{C}$, there is a natural bilinear pairing on the Grothendieck group $K(\mathcal{A})$ given by the Euler form

$$\chi(\mathscr{E}, \mathscr{F}) := \sum_k (-1)^k \dim \operatorname{Ext}^k(\mathscr{E}, \mathscr{F}),$$

for all $\mathscr{E}, \mathscr{F} \in K(\mathcal{A})$. Now taking the graded vector space to be the cohomology group $H^*(X, \mathbb{Q})$, if we choose the quadratic form to be

$$\langle \alpha, \beta \rangle = \int_X \mathrm{td}_X \cup (\alpha^\vee \cup \beta)$$

for all $\alpha, \beta \in H^*(X, \mathbb{Q})$, then the Chern character actually defines a numerical vector from $K(\mathcal{A})$ to $H^*(X, \mathbb{Q})$ with respect to the Euler characteristic due to Hirzebruch-Riemann-Roch formula

$$\begin{aligned} \chi(\mathscr{E}, \mathscr{F}) &= \int_X \mathrm{ch}(\mathscr{E}^\vee).\mathrm{ch}(\mathscr{F}).\mathrm{td}_X \\ &= \langle \mathrm{ch}(\mathscr{E}), \mathrm{ch}(\mathscr{F}) \rangle. \end{aligned}$$

Example 3.8. The Mukai vector υ and the generalized twisted Mukai vector υ_Λ also defines numerical vectors from $K(\mathcal{A})$ to $H^*(X, \mathbb{Q})$ (or $H^*(X, \mathbb{C})$ for υ_Λ) with respect to the Euler characteristic $\chi(\, , \,)$, if we choose the quadratic form $\langle \, , \, \rangle$ on $H^*(X, \mathbb{Q})$ (or $H^*(X, \mathbb{C})$) to be the Mukai pairing $\langle \, , \, \rangle_X$:

$$\langle \alpha, \beta \rangle_X = \int_X \exp(c_1(X)/2) \cup (\alpha^\vee \cup \beta)$$

for all $\alpha, \beta \in H^*(X, \mathbb{Q})$ or $H^*(X, \mathbb{C})$.

Example 3.9. We also can define a numerical vector with respect to the Euler characteristic $\chi(\, , \,)$ to be

$$\upsilon^\Lambda : K(\mathcal{A}) \to H^*(X, \mathbb{C})$$

for some $\Lambda \in H^*(X, \mathbb{C})$ satisfying the condition $\Lambda^\vee = \Lambda$ such that

$$\upsilon^\Lambda(\mathscr{E}) := \mathrm{ch}(\mathscr{E}).\sqrt{\mathrm{td}_X}.\exp(\Lambda)$$

for each $\mathscr{E} \in K(\mathcal{A})$ with the pairing

$$\langle \alpha, \beta \rangle = \int_X \exp(c_1(X)/2 - 2\Lambda) \cup (\alpha^\vee \cup \beta)$$

for all $\alpha, \beta \in H^*(X, \mathbb{C})$.

From above examples it turns out that in the category of coherent sheaves on smooth projective varieties X, product of a numerical vector and $\exp(\Lambda)$ such that $\Lambda^\vee = -\Lambda$ induces the deformations of the numerical vector within a fixed pairing on $H^*(X, \mathbb{C})$. In the case of $\Lambda^\vee = \Lambda$, it also gives a deformation of the quadratic form on $H^*(X, \mathbb{C})$.

3.2.2 Stability in abelian categories

To define stability in an abelian category $\mathcal{A}$, we need a preorder on the objects of $\mathcal{A}$ such that the *seesaw property* holds. Rudakov made an abstract definition [Rud97] as below.

Definition 3.10 ([Rud97])**.** A *stability structure* on an abelian category $\mathcal{A}$ is a preorder on the objects of $\mathcal{A}$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ for non-zero objects, one of the three conditions is satisfied:

(a) $A \prec B \Leftrightarrow A \prec C \Leftrightarrow B \prec C$;

(b) $A \succ B \Leftrightarrow A \succ C \Leftrightarrow B \succ C$;

(c) $A \asymp B \Leftrightarrow A \asymp C \Leftrightarrow B \asymp C$.

This property is called the *seesaw property*. A non-zero object $A \in \mathcal{A}$ is called *semistable* if $B \preceq A$ for every non-zero subobject $B \subset A$.

Then we have the following consequences derived in [Rud97; Bri07].

Proposition 3.11 ([Rud97; Bri07])**.** *Given a stability structure on an abelian category $\mathcal{A}$.*

(a) *If A, B are semistable and $A \prec B$, then* $\mathrm{Hom}(B, A) = 0$.

(b) *If $\mathcal{A}$ is weakly Artinian and finiteness of chains of factor objects, then for every object $E \in \mathcal{A}$, there exist a unique Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semitable objects of $\mathcal{A}$ with

$$F_n \prec F_{n-1} \prec \cdots \prec F_1.$$

Here weakly Artinian means that there are no infinite chain of subobjects in $\mathcal{A}$

$$\cdots \subset E_3 \subset E_2 \subset E_1$$

with $E_1 \prec E_2 \prec E_3 \prec \cdots$, and finiteness of chains of factor objects means that there are no infinite chain of quotients in $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \cdots$$

with $E_1 \succ E_2 \succ E_3 \succ \cdots$. Note that Rudakov used the condition "weakly Noetherian" rather than finiteness of chains of factor objects. In the thesis it would be sufficient to consider such a bit stronger conditions proposed by Bridgeland.

Proof. (a) Given a non-zero map $f : E \to F$ between semistable objects. Then by the short exact sequences $0 \to \ker f \to E \to \mathrm{im} f \to 0$ and $0 \to \mathrm{im} f \to F \to \mathrm{coker} f \to 0$ with the property of semistability of E, F, we see that $E \prec F$. It follows that Harder-Narasimhan filtrations, if exist, are unique.

(b) First note that every non-zero object $0 \neq E \in \mathcal{A}$ has a semistable subobject $A \subset E$ with $E \prec A$. As E is not semistable there exists a subobect $0 \neq E' \subset E$ with $E \prec E'$. Continuing the process we obtain a chain of subobjects, then the condition of weakly Artinian implies that such a chain must terminate, i.e. we get a semitable subobject. Similarly, every non-zero object of $\mathcal{A}$ has a semistable factor object $E \twoheadrightarrow B$ with $B \prec E$ by the assumption of finiteness of chains of factor objects.

To construct the Harder-Narasimhan filtration, we need the existence of a maximally destabilising quotient (mdq) of non-zero objects. A mdq of a non-zero object $0 \neq E \in \mathcal{A}$ is a non-zero quotient $E \twoheadrightarrow B$ such that (i) if $E \twoheadrightarrow B'$ is a factor object for E then $B \preceq B'$, and (ii) as in (i) but $B \asymp B'$ then the quotient $E \twoheadrightarrow B'$ factors through $E \twoheadrightarrow B \twoheadrightarrow B'$, so B' is a factor object for B. Note that a semistable object is just its mdq.

Choose a non-zero and not semistable object $E \in \mathcal{A}$. Then we have a short exact sequence $0 \to A \to E \to E' \to 0$ with A semistable and $E' \prec E \prec A$. Now suppose $E' \twoheadrightarrow B$ is an mdq for E'. If $E \twoheadrightarrow B'$ with B' semistable and $B' \preceq B$ then $B' \prec A$ which implies the non-existence of non-zero maps between A and B' by (a). Hence the quotient $E \twoheadrightarrow B'$ factors through $E \twoheadrightarrow E' \twoheadrightarrow B'$, which implies that the induced factor object $E \twoheadrightarrow B$ is an mdq for E by condition (i) and (ii). Now repeating the argument for E', and so on, the assumption of finiteness of chains of factor objects implies that the process must terminate. It turns out that every non-zero object of $\mathcal{A}$ has an mdq.

Consider a non-zero object $E \in \mathcal{A}$. If E is semistable then we are done. Otherwise, we have a short exact sequence $0 \to E' \to E \to B \to 0$ with $E \twoheadrightarrow B$ an mdq and $E \prec E'$. Given an mdq $E' \twoheadrightarrow B'$ for E', there are induced short exact sequences $0 \to K \to E' \to B' \to 0$, $0 \to K \to E \to Q \to 0$ and $0 \to B' \to Q \to B \to 0$. Since B is an mdq for E, the second short exact sequence and the condition (ii) imply that $B \prec Q$ and thus $B' \prec B$ by the definition of a stability structure. Repeating the process for E' and so on, we obtain a chain of subobjects of E

$$\cdots \subset E^2 \subset E^1 \subset E^0 = E$$

such that $E^0 \prec E^1 \prec E^2 \prec \cdots$ and with semistable factor objects $F^i = E^i/E^{i+1}$. By the assumption of weakly Artinian, this chain must terminate eventually and gives a Harder-Narasimhan filtration of E. □

To define a suitable ordering we recall the following definition.

Definition 3.12 ([GKR04])**.** Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ be its Grothendieck group. A linear independent system $(x_0, x_1, \ldots, x_r) : K(\mathcal{A}) \to \mathbb{Z}$ of additive functions is called *positive* if for all non-zero $A \in \mathcal{A}$ the conditions below hold:

$$\begin{aligned}
&x_0(A) \geq 0,\\
&x_0(A) = 0 \Rightarrow x_1(A) \geq 0,\\
&x_0(A) = x_1(A) = 0 \Rightarrow x_2(A) \geq 0,\\
&\quad\vdots\\
&x_0(A) = \cdots = x_{r-1}(A) = 0 \Rightarrow x_r(A) > 0.
\end{aligned}$$

If $x_0(A) = \cdots = x_r(A) = 0 \Rightarrow A = 0$ then the positive system is called *exhaustive*. Given a positive system $(x_0, \ldots, x_r)$ on $\mathcal{A}$, one can define the *vector slope* of an object $A \in \mathcal{A}$ with respect to this system is the vector

$$\phi(A) = \left(\underbrace{1, \ldots, 1}_{s}, \theta\left(-\frac{x_{s+1}}{x_s}\right), \theta\left(-\frac{x_{s+2}}{x_s}\right), \ldots, \theta\left(-\frac{x_r}{x_s}\right)\right),$$

where $\theta\left(\frac{a}{b}\right) = \frac{1}{\pi}\operatorname{arccot}\left(\frac{a}{b}\right) \in (0, 1]$, and $s = \min_i\{x_i(A) \neq 0\}$.

The slope ordering on $\mathcal{A}$ is given by the prescription: $A \leq B \Leftrightarrow \phi(A) \leq \phi(B)$ by the lexicographical order. One can check that this slope ordering actually fulfills the seesaw property by comparing the slope components. Note that an inequality between the slope components implies the same inequality between the ratios x_j/x_s, as the mediant $(a+b)/(c+d)$ always lies between a/c and b/d with positive denominators.

In the following we would provide a way to construct a positive system giving a vector slope ordering on an abelian category $\mathcal{A}$. We first make a definition below.

Definition 3.13. A *slope function* $P_t(A) \in \mathbb{Z}[t]$ is an additive polynomial with a finite degree over $\mathbb{Z}$ on an abelian category $\mathcal{A}$ for an indeterminate t such that the non-zero highest coefficient for every non-zero $A \in \mathcal{A}$ is always positive. Precisely,

$$P_t(A) = \sum_{i=0}^{n} a_i t^i$$

with the non-zero highest coefficient $a_{n'} > 0$ for all $A \in \mathcal{A}$. Note that n' and a_i depend on A. Then the exhaustive positive system is given by the vector $(a_n, a_{n-1}, \ldots, a_0)$.

Proposition 3.14. *Given such a slope function P_t defined as above for an abelian category $\mathcal{A}$ ordered by the vector slope induced from P_t, then $\mathcal{A}$ is weakly Artinian. Furthermore, if $\mathcal{A}$ is Noetherian, then $\mathcal{A}$ has finiteness of chains of factor objects, thus $\mathcal{A}$ has the Harder-Narasimhan property.*

Proof. Suppose there exists an infinite chain of subobjects in $\mathcal{A}$

$$\cdots \subset E_3 \subset E_2 \subset E_1$$

with $E_1 \prec E_2 \prec E_3 \prec \cdots$. Since P_t is additive, it leads to $\deg(P_t(E_j)) \geq \deg(P_t(E_{j+1}))$ for $E_{j+1} \subset E_j$. Thus for large enough j we have $\deg(P_t(E_j)) = \deg(P_t(E_{j+1})) = \cdots = d$ and $a_d(E_j) \geq a_d(E_{j+1}) \geq \cdots$. As $a_d(E_j) \in \mathbb{N} \setminus \{0\}$ for all j, this chain can not decrease infinitely, i.e.

$$a_d(E_s) = a_d(E_{s+1}) = \cdots = a$$

for some large enough s. Again by the additivity of P_t, we have the following inequality

$$(a, a_{d-1}(E_s), \ldots, a_0(E_s)) \succ (a, a_{d-1}(E_{s+1}), \ldots, a_0(E_{s+1}))$$

which thus implies $E_s \succ E_{s+1}$, a contradiction.

For the second statement, given any infinite chain of factor objects of $E = E_1$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \cdots,$$

each composite maps $E \twoheadrightarrow E_j$ can fit into short exact sequences

$$0 \to K_j \to E \to E_j \to 0.$$

Then we have a chain

$$0 = L_1 \subset L_2 \subset \cdots \subset E$$

and this chain must terminate since $\mathcal{A}$ is Noetherian. $\square$

Let us consider the category of coherent sheaves $\mathcal{A}$ on a smooth projective variety X over $\mathbb{C}$. Since the category of coherent sheaves is Noetherian, what we need is to find a suitable slope function P_t inducing a (numerical) slope ordering.

Definition 3.15 (Numerical)**.** Recall that the Euler characteristic $\chi(\ ,\ )$ defines a bilinear form on the Grothendieck group $K(X)$, and descends to a non-degenerate form $\chi(\ ,\ )$ on the numerical Grothendieck group $K(X)/K(X)^{\perp}$. A slope function P_t is said to be *numerical* if it factors through some numerical vectors, and the induced positive system is called *numerical*. Here a numerical slope function is allowed to be in $\mathbb{R}[t]$ and the numerical positive system in $\mathbb{R}^n$.

Choose an ample line bundle H on X of dimension n, then there is a natural numerical slope function given by the Hilbert polynomial, if we use the twisted Riemann-Roch map as the numerical vector, that is,

$$v(\mathscr{E}) := \mathrm{ch}(\mathscr{E}).\mathrm{td}_X.\exp(Ht)$$

for all $\mathscr{E} \in K(X)$, and thus the slope function can be written as

$$P_t(\mathscr{E}) := \deg(v(\mathscr{E}))_n = \int_X v(\mathscr{E}) = \chi(X, \mathscr{E}(t)).$$

Here the pairing for the twisted Riemann-Roch map is

$$\langle \alpha, \beta \rangle = \int_X \exp(c_1(X)) \cup \mathrm{td}_X^{-1} \cup (\alpha^{\vee} \cup \beta)$$

for all $\alpha, \beta \in H^*(X, \mathbb{Q})$. Note that although this numerical function P_t is in $\mathbb{Q}[t]$, each coefficient has this form $a_i \in r_i \cdot \mathbb{N}$ for some $r_i \in \mathbb{Q}$. Hence it still satisfies the proposition 3.14. Similarly, if we choose the twisted Chern character as our numerical vector

$$v(\mathscr{E}) := \mathrm{ch}(\mathscr{E}).\exp(Ht)$$

for all $\mathscr{E} \in K(X)$, and the slope function can be defined by

$$P_t(\mathscr{E}) := \deg(v(\mathscr{E}))_n = \int_X v(\mathscr{E}).$$

It also fulfills the proposition 3.14, thus both of them lead to numerical exhaustive positive system with the Harder-Narasimhan property on the category of coherent sheaves $\mathcal{A}$.

Motivated by above observations, we make the following definition.

Definition 3.16. Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ its Grothendieck group. Suppose $K(\mathcal{A})$ is a ring and there exists a graded vector space $\bigoplus_{j=0}^{n} V^j$ over a field k with a product such that $v^i \cdot v^j \in V^{i+j}$ for all $v^i \in V^i$ and $v^j \in V^j$. Assume a numerical vector $v : K(\mathcal{A}) \to \bigoplus_{j=0}^{n} V^j$ is a ring homomorphism. If there is a trace map on the highest degree vector subspace

$$\mathrm{Tr} : V^n \to k \oplus \cdots \oplus k = \sum_{j=0}^{m} kt^j,$$

then we define the *numerical function* corresponding to this numerical vector to be

$$P_v(E) := \mathrm{Tr}(v(E))$$

for all $E \in K(\mathcal{A})$.

To sum up, we recall the formal definition of stability data on $\mathcal{A}$.

Definition 3.17 ([GKR04]). Suppose $\mathcal{A}$ be an abelian category, Φ is a linearly ordered set, and an extension closed subcategory $\Pi_\phi \subset \mathcal{A}$ is given for all $\phi \in \Phi$. *Stability data* on $\mathcal{A}$ is a pair $(\Phi, \{\Pi_\phi\})$ satisfies the following conditions

(a) $\mathrm{Hom}_{\mathcal{A}}(\Pi_{\phi'}, \Pi_{\phi''}) = 0$ for all $\phi' > \phi''$;

(b) each non-zero object $E \in \mathcal{A}$ has a Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

with factors $F_j = E_j/E_{j-1} \in \Pi_{\phi_j}$ and $\phi_i < \phi_j$ for all $i > j$.

Therefore, one can see that the existence of a Harder-Narasimhan filtration is the weakest property. If the stability data is induced by any numerical slope function on $\mathcal{A}$, it is called *numerical* stability data on $\mathcal{A}$.

3.2.3 t-stability conditions

In this section we would recall some basic properties of t-stabilities we need for the next section. First we give the main definition of t-stability.

Definition 3.18 ([GKR04]). Let $\mathcal{T}$ be a triangulated category, Φ be a linearly ordered set, and a strictly full extension closed subcategory $\Pi_\phi \subset \mathcal{T}$ is given for all $\phi \in \Phi$. A *t-stability* on $\mathcal{T}$ is a pair $(\Phi, \{\Pi_\phi\})$ satisfies the following conditions

(a) $\Pi_\phi[1] = \Pi_{\tau(\phi)}$ with $\tau(\phi) > \phi$ for a bijection $\tau \in \mathrm{Aut}\Phi$;

(b) $\mathrm{Hom}(\Pi_{\phi'}, \Pi_{\phi''}[k]) = 0$ for all $\phi' > \phi''$ and $k \leq 0$;

(c) each non-zero object $E \in \mathcal{T}$ has a Harder-Narasimhan filtration

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \Pi_{\phi_j}$ for all j, and strictly decreasing $\phi_j > \phi_{j+1}$.

The factors A_j are called the semistable factors of E, and the categories Π_{ϕ_j} are called the semistable subcategories of the t-stability.

Lemma 3.19 ([GKR04, Lemma 3.2]). *Let $(\mathcal{F}, \mathcal{F}^\perp)$ be a bounded t-structure on a triangulated category $\mathcal{T}$. Then the extension closed subcategories $\Pi_i := \mathcal{A}[i] = \mathcal{F}[i] \cap \mathcal{F}^\perp[1+i]$ give a t-stability $(\mathbb{Z}, \{\Pi_i\}_{i\in\mathbb{Z}})$ on the category $\mathcal{T}$.*

Proof. By the definition of a t-structure (see Def. 2.20), it is obvious that $\mathrm{Hom}(\Pi_i, \Pi_j[k]) = 0$ for all $i > j$ and $k \le 0$. Here we use the formula $\Pi_i[1] = \Pi_{i+1}$. Now since the t-structure is bounded, for any non-zero object $E \in \mathcal{T}$ there exist two integers $n_+(E) \ge n_-(E)$ such that

$$\begin{aligned} \mathrm{Hom}(E, \mathcal{F}^{\perp}[n]) = 0 \quad &\text{for } n + 1 < n_-(E) \\ \mathrm{Hom}(\mathcal{F}[n], E) = 0 \quad &\text{for } n > n_+(E). \end{aligned}$$

There exists a distinguished triangle

$$\begin{tikzcd} E' \arrow[rr] & & E \arrow[dl] \\ & A_{n_-(E)} \arrow[ul, dashed] & \end{tikzcd}$$

such that $A_{n_-(E)} \in \mathcal{F}^{\perp}[n_-(E) + 1]$ and $E' \in \mathcal{F}[n_-(E) + 1]$. Since $A_{n_-(E)} \in \mathcal{F}[n_-(E)]$, we conclude that $A_{n_-(E)} \in \Pi_{n_-(E)}$. By repeating the process for E' and so on, we then obtain a Harder-Narasimhan filtration. □

We also need the following technical property which allows us to glue some filtrations together.

Proposition 3.20 ([GKR04, Proposition 4.3])**.** *Let $(A_0, A_1, \dots, A_n)$ be factors of a non-zero object E and $(B_{i,0}, B_{i,1}, \dots, B_{i,m_i})$ be factors of the factor A_i. Then E has a filtration with factor objects $(B_{0,0}, \dots, B_{0,m_0}, B_{1,0}, \dots, B_{1,m_1}, \dots, B_{n,0}, \dots, B_{n,m_n})$. The converse statement also holds.*

Proof. (Sketch) Using the composite of morphisms and the octahedron axiom to construct the necessary distinguished triangles. □

By the previous lemma and this technical property, we obtain the following property relating t-stability on a triangulated category $\mathcal{T}$ and stability data on its heart $\mathcal{A}$ of a bounded t-structure.

Proposition 3.21 ([GKR04, Proposition 3.3])**.** *Let $(\Phi, \{\Pi_\phi\})$ be stability data on the heart $\mathcal{A}$ of a bounded t-structure on a triangulated category T. Choose the lexicographical ordering on the set $\mathbb{Z} \times \Phi$, then the pair $(\mathbb{Z} \times \Phi, \Pi_\phi[i])$ gives a t-stability on the triangulated category $\mathcal{T}$.* □

If a t-stability on a triangulated category $\mathcal{T}$ is induced by a *numerical* stability data on its heart $\mathcal{A}$ of a bounded t-structure of $\mathcal{T}$, then such a t-stability is called *numerical*. In Lemma 3.19, it shows that each bounded t-structure on a triangulated category induces a t-stability on it.

Conversely, any t-structure indeed leads to a set of associated t-structures.

Proposition 3.22 ([GKR04, Lemma 5.1])**.** *Let $(\Phi, \{\Pi_\phi\})$ be a t-stability on a triangulated category $\mathcal{T}$. Suppose $\Phi = \Phi_- \sqcup \Phi_+$ is an arbitrary decomposition such $\Phi_\pm \neq \emptyset$ and $\phi_- < \phi_+$ for all $\phi_- \in \Phi_-$ and $\phi_+ \in \Phi_+$. Then the subcategories*

$$\begin{aligned} \mathcal{F} &= \langle \Pi_\phi | \phi \in \Phi_+ \rangle \\ \mathcal{F}^{\perp} &= \langle \Pi_\phi | \phi \in \Phi_- \rangle \end{aligned}$$

give a t-structure on $\mathcal{T}$.

Proof. (Sketch) Directly check the definition of t-structure and use above technical proposition to construct the required distinguished triangles. □

For more details on t-stabilities on triangulated categories, see the original work [GKR04].

3.2.4 Stability conditions on surfaces

This section would contain a construction of Bridgeland's numerical stability conditions from a numerical t-stability on a bounded derived category of a smooth projective surface X. More precisely, to construct a central charge, or slope function in the sense of Bridgeland' stability conditions, by tilting the heart of a bounded t-structure of the derived category $\mathrm{D}(X)$ to reduce the length of a numerical exhaustive positive system induced by a numerical slope function in a numerical t-stability.

First we write down the explicit definition of numerical t-stability condition discussed previously.

Definition 3.23. Let $\mathcal{T}$ be a triangulated category, $P_\nu : K(\mathcal{T})/K(\mathcal{T})^\perp \to \mathbb{R}[t]$ a numerical (slope) function of degree n with respect to a numerical vector and a bilinear form on $K(\mathcal{T})$, and a strictly full extension closed subcategory $\Pi_\phi[i] \subset \mathcal{T}$ given for all $(i, \phi) \in \mathbb{Z} \times \Phi$. A *numerical t-stability* on $\mathcal{T}$ is a pair $(\mathbb{Z} \times \Phi, \{\Pi_\phi[i]\})$ satisfies the following conditions

(a) $\Pi_\phi[1] = \Pi_{\tau(\phi)}$ with $\tau(\phi) > \phi$ for a bijection $\tau \in \mathrm{Aut}\Phi$;

(b) $\mathrm{Hom}(\Pi_{\phi'}, \Pi_{\phi''}[k]) = 0$ for all $\phi' > \phi''$ and $k \leq 0$;

(c) each non-zero object $E \in \mathcal{T}$ has a Harder-Narasimhan filtration

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \Pi_{\phi_j}$ for all j, and strictly decreasing $\phi_j > \phi_{j+1}$. $\Phi^-(E) := \phi_n$ and $\Phi^+(E) := \phi_1$

The factors A_j are called the *numerical* semistable factors of E, and the categories Π_{ϕ_j} are called the *numerical* semistable subcategories of the numerical t-stability.

Here the linear order set Φ is induced by the numerical function P_ν on $K(\mathcal{T})$ defined below.

Definition 3.24. Given a numerical function $P_\nu : K(\mathcal{T})/K(\mathcal{T})^\perp \to \mathbb{R}[t]$ on the Grothendieck group $K(\mathcal{T})$ such that it can be written down as the form

$$P_\nu := \sum_{i=0}^{n} a_i t^i = (a_n, a_{n-1}, \dots, a_0) \in \mathbb{R}^{n+1}.$$

For any $k \in \mathbb{Z}$ the non-zero leading coefficient of $P_\nu([A_{2k}])$ for an object $[A_{2k}] \in K(\mathcal{T})$, $A_{2k} \in \Pi_\phi[2k]$, is positive, i.e. $a_m([A_{2k}]) > 0$ for $m = \max_i\{a_i([A_{2k}]) \neq 0\}$. Then the *vector slope* with respect to P_ν is the vector

$$\phi([A_{2k}]) = \left(\underbrace{1, \dots, 1}_{n-m+1}, \theta\left(-\frac{a_{m-1}}{a_m}\right), \theta\left(-\frac{a_{m-2}}{a_m}\right), \dots, \theta\left(-\frac{a_0}{a_m}\right)\right),$$

where $\theta\left(\frac{x}{y}\right) = \frac{1}{\pi}\mathrm{arccot}\left(\frac{x}{y}\right) \in (0, 1]$. As $a_m([A_{2k+1}]) < 0$, the vector slope is the vector

$$\phi([A_{2k+1}]) = \left(\underbrace{2, \dots, 2}_{n-m+1}, \theta\left(-\frac{a_{m-1}}{a_m}\right), \theta\left(-\frac{a_{m-2}}{a_m}\right), \dots, \theta\left(-\frac{a_0}{a_m}\right)\right),$$

where $\theta\left(\frac{x}{y}\right) = \frac{1}{\pi}\operatorname{arccot}\left(\frac{x}{y}\right) \in (1, 2]$. Then the slope ordering is given by the relation

$$A_k \preceq B_l \Leftrightarrow \phi(A_k) + k \leq \phi(B_l) + l.$$

The bijection $\tau \in \mathrm{Aut}\Phi$ is defined by $\tau\phi := \phi + 1 = (\theta_n + 1, \theta_{n-1} + 1, \ldots, \theta_0 + 1)$.

Compared with the definition of Bridgeland's stability conditions (Def. 2.18) or numerical stability conditions [Bri08], the central charge $Z : K(\mathcal{T}) \to \mathbb{C}$ is a special numerical slope function P_ν of degree 1, that is, $P_\nu = \mathrm{Im}(Z)t - \mathrm{Re}(Z)$. Thus Bridgeland's stability conditions become a special case of t-stability conditions with numerical slope functions of degree 1 on triangulated categories.

Let X be a nonsingular projective variety and $\mathrm{D}(X)$ be its bounded derived category of coherent sheaves, we have the following theorem.

Theorem 3.25. *Given a derived category* $\mathrm{D}(X)$ *of a smooth projective variety* X *over* $\mathbb{C}$*, the numerical slope function*

$$P_\nu(E) := \deg(\nu(E))_n = \int_X \nu(E)$$

corresponding to the numerical vector

$$\nu(E) := \mathrm{ch}(E).\exp(Ht),$$

for all $E \in K(X)$ *and ample classes* H*, determines a numerical t-stability condition on* $\mathrm{D}(X)$.

Proof. Since this slope function induces a exhaustive positive system on the category of coherent sheaves $\mathcal{A}$ by Grothendieck-Riemann-Roch theorem, and $\mathcal{A}$ is Noetherian, this positive system has the Harder-Narasimhan property and thus defines stability data on $\mathcal{A}$. Hence it leads to a numerical t-stability condition on the derived category $\mathrm{D}(X)$. □

In a bounded derived category $\mathrm{D}(X)$ of coherent sheaves of a smooth projective variety X, there is a natural numerical t-stability condition induced by the numerical slope function P_ν with respect to the Chern character and the Euler characteristic. It then is natural expect that there also exists a numerical slope function P_ν of degree 1 inducing the relevant numerical t-stability condition on $\mathrm{D}(X)$.

Conjecture 3.26. *In a bounded derived category* $\mathrm{D}(X)$ *for any smooth projective variety* X*, there always exist a numerical function of degree 1 with respect to the Euler characteristic generating a numerical t-stability or Bridgeland's stability condition on* $\mathrm{D}(X)$.

In the case of algebraic curves, it is automatically fulfilled. See [Mac07; Mac04; Oka04; GKR04] for the full study of stability conditions on curves.

Let us turn to the derived categories of smooth projective surfaces. Suppose X is a smooth projective surface and $K(X)$ is its Grothendieck group. The numerical vector with respect to the category of coherent sheaves is a twisted Chern character defined by $\nu(E) := \mathrm{ch}(E).\exp(Ht)$ for all $E \in K(X)$ and H an ample line bundle on X, then the corresponding numerical function can be expressed as

$$P_\nu(E) = \deg(\nu(E))_2 = \int_X \nu(E) = \left(\frac{1}{2}r(E)H^2, c_1(E).H, \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right)\right).$$

Here $r(E)$ is the rank of E, $c_1(E)$ and $c_2(E)$ are the first and second Chern classes of E, respectively. One can easily check that P_ν defines a numerical exhaustive positive system Φ since the rank of any torsion-free sheaf is positive, $c_1(E).H > 0$ for any torsion sheaf with the support on a curve and $-c_2(E) > 0$ for torsion sheaves in dimension 0.

To construct the heart of a bounded t-structure on $\mathrm{D}(X)$ we recall a very useful method of tilting introduced by D. Happel, I. Reiten and S. Smalø [HRS96].

Definition 3.27. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is called a torsion pair in an abelian category $\mathcal{A}$ if $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$, such that every object $E \in \mathcal{A}$ has a short exact sequence

$$0 \to T \to E \to F \to 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Here the objects of $\mathcal{T}$ and $\mathcal{F}$ are called torsion and torsion-free, respectively.

Proposition 3.28 ([HRS96, Proposition 2.1])**.** *Let $\mathcal{A}$ be the heart of a bounded t-structure on a triangulated category* D*. Given an object $E \in \mathrm{D}$ let $H^i(E) \in \mathcal{A}$ denote the ith cohomology of E with respect to this t-structure. Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. Then the full subcategory*

$$\mathcal{A}^\sharp = \left\{ E \in \mathrm{D} \mid H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T} \right\}$$

is the heart of another bounded t-structure on D. □

First note that the vector $\left(c_1.H, \frac{1}{2}\left(c_1^2 - 2c_2\right)\right)$ provide a positive system on torsion sheaves, so it is natural to consider a tilting of torsion-free sheaves in the category of coherent sheaves $\mathcal{A}$. Recall that given a ample divisor of X, the slope $\mu_H(E)$ of a torsion-free sheaf E on X is defined by the quotient

$$\mu_H(E) = \frac{c_1(E).H}{\mathrm{r}(E)}$$

such that each torsion-free sheaf E on X has a unique Harder-Narasimhan filtration

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E$$

which factors $F_i = E_i/E_{i-1}$ are μ_H-semistable torsion-free sheaves with $\mu_H(F_1) > \mu_H(F_2) > \cdots > \mu_H(F_n)$. A torsion-free sheaf E is said to be μ_H-semisimple if $\mu_H(F) \leq \mu_H(E)$ for any subsheaf $0 \neq F \subset E$ (see [Rud97], or textbooks [Fri98; HL10]). We thus can define a torsion pair as

$$\begin{aligned}\mathcal{T} &= \{\text{torsion sheaves}\} \cup \{E \in \mathcal{A} \mid \mu_H(F_i) > 0 \text{ for all } i\},\\ \mathcal{F} &= \{E \in \mathcal{A} \mid \mu_H(F_i) \leq 0 \text{ for all } i\},\end{aligned}$$

and the tilted category with respect to such a torsion pair would be the abelian category

$$\mathcal{A}^\sharp = \langle \mathcal{F}[1], \mathcal{T} \rangle.$$

We still need the Hodge Index Theorem (see [Har77; GH94]) and Bogomolov-Gieseker Inequality (see [Gie79; Fri98; HL10]) to built necessary slope functions on $\mathcal{A}^\sharp$.

Theorem 3.29 (Hodge Index)**.** *Let H be an ample divisor on the surface X, and suppose $D \not\equiv 0$ is a divisor with $D.H = 0$, then $D.D < 0$.*

Theorem 3.30 (Bogomolov-Gieseker Inequality)**.** *Let X be a n-dimensional smooth projective variety over $\mathbb{C}$ and H be an ample divisor on X. For any torsion-free μ_H-semistable sheaf E, we have the following inequality*

$$H^{n-2}.\left(2r(E)c_2(E) - (r(E) - 1)c_1(E)^2\right) \geq 0.$$

Then we obtain a set of numerical slope functions on $\mathrm{D}(X)$ as below.

Proposition 3.31. *Let $\mathcal{A}^\sharp$ be the tilted subcategory of $\mathrm{D}(X)$ as above. Then we have a set of numerical slope function of degree 1 inducing numerical t-stability or Bridgeland's stability conditions which can be expressed as*

$$P_\nu^\alpha = \left(c_1.H, \frac{1}{2}\left(c_1^2 - 2c_2\right) - r\alpha\right)$$

for any $\alpha > 0 \in \mathbb{R}$ and H is $\mathbb{R}$-ample.

Proof. In the case of any torsion sheaf E supported on a curve, $c_1(E).H > 0$ since $c_1(E)$ is effective and for every torsion-free μ_H-semistable sheaf E with $\mu_H(E) > 0$, $c_1(E).H = r(E)\mu_H(E) > 0$. If E is supported in dimension 0, $-c_2(E) > 0$ by the Grothendieck-Riemann-Roch theorem. Moreover, if E is torsion-free with $\mu_H(E) < 0$, then $P_\nu^\alpha(E[1]) = -P_\nu^\alpha(E)$ so that $c_1(E[1]).H > 0$.

Finally, if $\mu_H(E) = 0$ for a torsion-free μ_H-semistable sheaf E on X, by the Bogomolov-Gieseker Inequality and Hodge Index theorem implying $c_1(E)^2 \leq 0$ we obtain

$$c_1(E)^2 - 2c_2(E) = \frac{-1}{r(E)}\left(2r(E)c_2(E) - (r(E)-1)c_1(E)^2\right) + \frac{1}{r(E)}c_1(E)^2 \leq 0.$$

It turns out that $-\left(c_1(E)^2 - 2c_2(E)\right) + 2r(E)\alpha > 0$ as we require. The only thing we need to check is the existence of the Hader-Narasimhan property which would be discussed in the next lemma.

Lemma 3.32 ([Bri08, Proposition 7.1])**.** *Given a numerical slope function $P_\nu = \sum_{i=0}^{n-1} a_i t^i$ with respect to the Euler characteristic on the tilted subcategory $\mathcal{A}^\sharp \subset \mathrm{D}(X)$ of a n-dimensional non-singular projective variety X as before such that H is $\mathbb{Q}$-ample, $\mathcal{A}^\sharp$ then has the Harder-Narasimhan property.*

Proof. First by Prop. 3.14 $\mathcal{A}^\sharp$ is weakly Artinian so the non-trivial part is the finiteness of chains of factor objects. Suppose we have a chain of epimorphisms

$$E = E_0 \twoheadrightarrow E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots$$

with $E_i \succ E_{i+1}$ for all i and $a_{n-1}(E_i) \geq a_{n-1}(E_{i+1})$. Since the value of a_{n-1} is discrete, there exists a minimal value of $a_{n-1}(E_j)$ for some j. So we can assume that $a_{n-1}(E_j) = a_{n-1}(E_{j+1})$ for all j. Moreover, there are epimorphisms of cohomolgiy sheaves by taking long exact sequences in cohomology

$$H^0(E_0) \twoheadrightarrow H^0(E_1) \twoheadrightarrow H^0(E_2) \twoheadrightarrow \cdots.$$

This chain must terminate as the category of coherent sheaves is Noetherian, so $H^0(E_i) \cong H^0(E_{i+1}) \cong \cdots$ for large enough i. So one can assume $H^0(E) \cong H^0(E_i)$ for all i. Now consider these short exact sequences $0 \to L_i \to E \to E_i \to 0$ which induce a chain

$$0 \subset L_1 \subset L_2 \subset \cdots \subset E$$

with each $a_{n-1}(L_i) = 0$, thus $a_{n-1}(H^0(L_i)) = 0$. If $L_j = L_{j+1} = \cdots$ for large enough j, then we are done. Similarly there are morphisms of sheaves by taking cohomology sheaves

$$0 \subset H^{-1}(L_1) \subset H^{-1}(L_2) \subset \cdots \subset H^{-1}(E).$$

This chain also terminates thus we can assume $H^{-1}(L_i) \cong H^{-1}(L_{i+1})$ for all i. Since $a_{n-1}(H^0(L_i)) = 0$, $H^0(L_i)$ is a torsion sheaf supported in at least codimension 2 and thus $H^0(L_i) \subset H^0(L_{i+1})$. Again by the short exact sequences $0 \to L_i \to E \to E_i \to 0$, they give us the long exact sequences

$$0 \to H^{-1}(L_i) \to H^{-1}(E) \to H^{-1}(E_i) \to H^0(L_i) \to 0.$$

Since $H^{-1}(L_i) \cong H^{-1}(L_{i+1})$ for all i, the images of the middle morphisms in $H^{-1}(E_i)$ for all i are the same denoted by Q. Thus there is a short exact sequence of sheaves

$$0 \to Q \to H^{-1}(E_i) \to H^0(L_i) \to 0$$

for all i. Moreover, Q and $H^{-1}(E_i)$ are torsion-free sheaves, and $H^0(L_i)$ is a torsion sheaf supported in at least codimension 2. It turns out that the chain

$$H^{-1}(E_1) \subset H^{-1}(E_2) \subset \cdots \subset Q^{\vee\vee}$$

also terminates and thus $H^{-1}(E_j) \cong H^{-1}(E_{j+1})$ for large enough j. This proves that $H^0(L_j) \cong H^0(L_{j+1})$ and hence $L_j = L_{j+1}$ for large enough j. □

For the general case of $\mathbb{R}$-ample H on the non-singular projective surface X, the existence of Harder-Narasimhan filtration can be deduced by continuity and the structure of the space of stability conditions. Indeed, the argument in [Bri07, §6 & 7] implies the deformation theory which says that there is a local homeomorphism $\pi : \Sigma \to V(\Sigma)$ which sends a stability condition to its slope function, where Σ is a connected component in the space of stability conditions and $V(\Sigma)$ is the subspace of $\mathrm{Hom}_{\mathbb{Z}}(K(X)/K(X)^{\perp}, \mathbb{C} \simeq \mathbb{R}^2)$ with a well-defined linear topology induced by the generalised norm

$$\|U\|_{(P,\Pi)} = \sup\left\{\frac{|U(E)|}{|Z(E)|} : 0 \neq E \in \Pi_{\phi}\ \forall \phi \in \Phi\right\} \in [0, \infty].$$

In order to make the deformation effective, we need *full* stability conditions introduced in [Bri08], that is $\|U\|_{(P,\Pi)} < \infty$ for any $U \in \mathrm{Hom}_{\mathbb{Z}}(K(X)/K(X)^{\perp}, \mathbb{C})$, which is equivalent to the *support property* introduced in [KS08]. The quadratic form constructed in [MS16, Theorem 6.13] would give the support property we need and this is explained in detail in [MS16, §5 & 6]. □

Moreover, on the nonsingular projective surface X we can consider the numerical vector of the type

$$v(E) := \mathrm{ch}(E).\exp\left((H - \tilde{\beta})t - \frac{(Ht)^2}{2} - \frac{\beta^2}{2H^2}\right),$$

for any $\tilde{\beta} \in H^4(X, \mathbb{R})$ with $\int_X \tilde{\beta} = \beta$, and the corresponding numerical slope function is

$$P_v^{\beta} = \left(c_1.H - r\beta, \frac{1}{2}\left(c_1^2 - 2c_2\right) - r\frac{\beta^2}{2H^2}\right).$$

Instead of Hodge Index Theorem we use the following corollary of Hodge Index theorem on smooth projective surfaces.

Corollary 3.33. *If H is an ample divisor on a smooth projective surface X, and if D is any divisor, then we have the inequality $(D^2)(H^2) \leq (D.H)^2$.*

Proof. Choose a divisor of the form $aD+bH$ with $a = H^2$ and $b = D.H$, which implies $(aD+bH).H = 0$. So $(aD + bH)^2 \leq 0$ by Hodge Index theorem and it turns out that $aD^2 + b^2 - 2(D.H)^2 \leq 0$. Hence we obtain $(D^2)(H^2) - (D.H)^2 \leq 0$. □

Now we choose the torsion pair to be

$$\begin{aligned}\mathcal{T} &= \{\text{torsion sheaves}\} \cup \{E \in \mathcal{A} \mid \mu_H(F_i) > \beta \text{ for all } i\},\\ \mathcal{F} &= \{E \in \mathcal{A} \mid \mu_H(F_i) \leq \beta \text{ for all } i\}.\end{aligned}$$

Then it turns out that for any torsion-free sheaf E with the property $\mu_H(E) = \beta$, we obtain

$$\begin{aligned}c_1(E)^2 - 2c_2(E) - r(E)\frac{\beta^2}{H^2} = \frac{-1}{r(E)}&\left(2r(E)c_2(E) - (r(E)-1)c_1(E)^2\right)\\ &+ \frac{1}{r(E)}\left(c_1(E)^2 - \frac{r(E)^2\beta^2}{H^2}\right) \leq 0.\end{aligned}$$

Here the inequality of the second term in the middle is given as

$$(c_1(E)^2)(H^2) - (r(E)\beta)^2 = (c_1(E)^2)(H^2) - (c_1(E).H)^2 \leq 0$$

by the corollary. Hence the numerical slope function P_ν^β defines a positive system on the tiled category $\mathcal{A}^\sharp$ and induce a numerical t-stability of degree 1 on $\mathrm{D}(X)$. Combined with the proposition 3.31 we can form a more general set of numerical functions on any nonsingular projective surface.

Proposition 3.34. *Let $\mathcal{A}^\sharp$ be the tilted subcategory of $\mathrm{D}(X)$ as above. Then we have a set of numerical slope function of degree 1 inducing numerical t-stability or Bridgeland's stability conditions which can be expressed as*

$$P_\nu^{\alpha\beta} = \left(c_1.H + r\beta, \frac{1}{2}\left(c_1^2 - 2c_2\right) - r\left(\frac{\beta^2}{2H^2} + \alpha\right)\right).$$

for any $\alpha > 0 \in \mathbb{R}$, $\beta \in \mathbb{R}$ and H is $\mathbb{R}$-ample.

Proof. Fist assume β and H are rational classes the value of leading coefficient of $P_\nu^{\alpha\beta}$ is discrete, then we have the Harder-Narasimhan property implying the stability conditions. For general cases, by using Bridgeland's deformation theory we can extend stability conditions from rational to real classes. □

Note that this type of numerical slope functions can be expressed as

$$P_\nu^{\alpha\beta}(E) = \int_X \exp\left((H+\tilde{\beta})t - \frac{(Ht)^2}{2} - \frac{\beta^2}{2H^2} - \tilde{\alpha}\right).\mathrm{ch}(E),$$

where $\tilde{\alpha} \in H^4(X,\mathbb{R})$ with the property $\int_X \tilde{\alpha} = \alpha > 0$ and $\nu = \exp\left((H+\tilde{\beta})t - \frac{(Ht)^2}{2} - \frac{\beta^2}{2H^2} - \tilde{\alpha}\right).\mathrm{ch}(E)$ is the corresponding numerical vector. If we put $t = -i = -\sqrt{-1}$ and time $P_\nu^{\alpha\beta}$ with additional -1, it leads to the familiar form

$$Z(E) := \widetilde{P_\nu^{\alpha\beta}}(E) = -\int_X \exp\left(-i(H+\tilde{\beta}) + \frac{H^2}{2} - \frac{\beta^2}{2H^2} - \tilde{\alpha}\right).\mathrm{ch}(E),$$

which is similar to the stability functions or central charges without the class $-i\tilde{\beta} + \frac{H^2}{2} - \frac{\beta^2}{2H^2} - \tilde{\alpha}$ on smooth projective surfaces constructed in [Bri08; AB13]. Due to the existence of the class in $H^4(X,\mathbb{C})$, we obtain a more general set of stability functions on the bounded derived category of coherent sheaves on any smooth projective surface.

Finally, let us turn to the numerical slope functions corresponding to the central charge of B-branes in the B-model given by the generalized twisted Mukai vector

$$v_{\Lambda_X}(E) := \mathrm{ch}(E).\sqrt{\mathrm{td}_X}.\exp(i\Lambda_X),$$

where Λ_X satisfies $\Lambda_X^\vee = -\Lambda_X$ in (3.1). On any non-singular projective surface X, the square of the Todd class would be written as

$$\sqrt{\mathrm{td}_X} = \left(1, \frac{1}{4}c_1(X), \frac{1}{24}c_2(X) + \frac{1}{96}c_1^2(X)\right)$$

and the log Gamma class can be written as

$$\Lambda_X = \left(0, \frac{\gamma}{2\pi}c_1(X), 0\right)$$

where γ is the Euler number. First consider the numerical vector $\nu = \exp(-\beta - iH).\mathrm{ch}(E).\sqrt{\mathrm{td}_X}$ with $\beta \in H^2(X, \mathbb{R})$ and H is $\mathbb{R}$-ample, then the corresponding central charge can be written as

$$\begin{aligned}
\mathrm{Im}(Z(E)) =& H\left(c_1(E) + \frac{r(E)}{4}c_1(X) - r(E)\beta\right),\\
\mathrm{Re}(Z(E)) =& -\left(\frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{r(E)}{24}\left(c_2(X) + \frac{1}{4}c_1(X)^2\right) + \frac{1}{4}c_1(E)c_1(X)\right.\\
&\left.+\frac{r(E)}{2}(\beta^2 - H^2) - \beta\left(c_1(E) + \frac{r(E)}{4}c_1(X)\right)\right)\\
=& -\frac{1}{2r(E)}\left(\left(c_1(E) + \frac{r(E)}{4}c_1(X) - r(E)\beta\right)^2 - \left(c_1(E)^2 - r(E)\left(c_1(E)^2 - 2c_2(E)\right)\right)\right.\\
&\left.-\frac{r(E)^2}{24}c_1(X)^2 + \frac{r(E)^2}{12}c_2(X) - r(E)^2H^2\right),
\end{aligned}$$

for any $E \in K(X)$. Here the second equality of $\mathrm{Re}Z(E)$ only holds for each torsion-free sheaf. This time we define the torsion pair by

$$\begin{aligned}
\mathcal{T} &= \{\text{torsion sheaves}\} \cup \left\{E \in \mathcal{A} \mid \mu_H(F_i) > H\left(\frac{1}{4}c_1(X) - \beta\right) \text{ for all } i\right\},\\
\mathcal{F} &= \left\{E \in \mathcal{A} \mid \mu_H(F_i) \leq H\left(\frac{1}{4}c_1(X) - \beta\right) \text{ for all } i\right\}.
\end{aligned}$$

As before $\mathrm{Im}Z(E)$ is always positive for any torsion sheaf E supported on curves and -$\mathrm{Re}Z(E)$ do so for any torsion sheaf E supported in dimension 0. For each μ_H-semistable sheaf E with $\mu_H(F_i) = H\left(\frac{1}{4}c_1(X) - \beta\right)$, i.e. $H\left(c_1(E) + \frac{r(E)}{4}c_1(X) - r(E)\beta\right) = 0$, we have the inequalities

$$\begin{aligned}
\left(c_1(E) + \frac{r(E)}{4}c_1(X) - r(E)\beta\right)^2 &\leq 0,\\
-\left(c_1(E)^2 - r(E)\left(c_1(E)^2 - 2c_2(E)\right)\right) &\leq 0
\end{aligned}$$

by Hodge Index Theorem and Bogomolov-Gieseker Inequality. Then Re$Z(E)$ would be positive if it fulfills the following inequality

$$-\frac{1}{24}c_1(X)^2 + \frac{1}{12}c_2(X) - H^2 < 0$$

Interpreting $c_2(X)$ as the topological Euler number χ and $c_1(X)$ the anti-canonical divisor $-K$ yields

$$H^2 > \frac{1}{12}\chi - \frac{1}{24}K^2.$$

In the case of K3 surfaces, $\chi = 24$ and $K \equiv 0$, it turns out that $H^2 > 2$ as proposed by Bridgeland [Bri08].

Corollary 3.35. *The (old) central charges of B-branes in the B-model of the form*

$$Z(E) = -\int_X e^{-(\beta+i\omega)}.\mathrm{ch}(E).\sqrt{td_X}$$

with $\beta \in H^2(X,\mathbb{R})$ and ω is $\mathbb{R}$-ample induce Bridgeland's stability conditions on any smooth projective surfaces X if $\omega^2 > \frac{1}{12}\chi - \frac{1}{24}K^2$.

Replacement of the square root of the Todd class by the Gamma class can be viewed as a deformation of the ample divisor H by the canonical divisor K, that is, $H \mapsto H' := H + (\gamma/2\pi)K$. If H' is also ($\mathbb{R}$-)ample, then the previous corollary implies the Bridgeland's stability on smooth projective surfaces. To prove the ampleness of H', we use the Mori's Cone theorem and Kleiman's Ampleness criterion (see the book [KM98]) :

Theorem 3.36 (Cone Theorem)**.** *Let X be a non-singular projective variety. For any $\epsilon > 0$, and ample divisor H,*

$$\overline{NE}(X) = \overline{NE}(X)_{K+\epsilon H \geq 0} + \sum_{i=1}^{<\infty} \mathbb{R}_{\geq 0}[C_i]$$

where C_i is a rational curve such that $0 < -(C_i.K) \leq \dim X + 1$.

Theorem 3.37 (Kleiman's Ampleness criterion)**.** *Let X be a projective variety and D is an $\mathbb{R}$-divisor. Then D is ample if and only if*

$$D_{>0} \supset \overline{NE}(X) \setminus \{0\}.$$

Thus the only thing we need to check is that $H'.C_i > 0$ for all $i < \infty$, i.e.

$$H.C_i > \frac{\gamma}{2\pi}(-K.C_i).$$

This inequality holds as H is ample, $H.C_i \geq 1$, and $(-K.C_i) \leq 2 + 1$,

$$H.C_i \geq 1 > \frac{3\gamma}{2\pi} \geq \frac{\gamma}{2\pi}(-K.C_i).$$

Then we obtain the following corollary.

Corollary 3.38. *The central charges of B-branes in the B-model of the form*

$$Z(E) = -\int_X e^{-(\beta+i\omega)}.\mathrm{ch}(E).\widehat{\Gamma}_X$$

with $\beta \in H^2(X,\mathbb{R})$ and ω is ample induce Bridgeland's stability conditions on any smooth projective surfaces X if $\omega^2 > \frac{1}{12}\chi - \frac{1}{24}K^2$. Moreover, if ω is $\mathbb{Q}$-ample such that $m\omega$ is ample for some m, then $0 < m < \frac{2\pi}{3\gamma}$ and if ω is $\mathbb{R}$-ample then $\omega.C_i > \frac{3\gamma}{2\pi}$ for all i.

Remark 3.39. For any 3-dimensional non-singular variety, in order to prove the existence of Bridgeland's stability conditions we need some Bogomolov-Gieseker type inequalities involving c_3 conjectured by A. Bayer, E. Macrì and Y. Toda [BMT14]. This conjectural inequality has been proved in some special cases, e.q. $\mathbb{P}^3$ [BMT14], abelian 3-folds [BMS14] and Fano 3-folds of Picard number 1 [Li15].

3.3 Cohomological Fourier-Mukai transforms

Let X be a n-dimensional non-singular variety over $\mathbb{C}$. Mukai vector is a ring homomorphism from the Grothendieck group $K(X)$ to the cohomology group $H^*(X,\mathbb{Q})$ with the Mukai pairing which preserves the Hirzebruch-Riemann-Roch formula, i.e. sending the Euler characteristic to a quadratic form. Furthermore, in general Mukai vector commutes with the Fourier-Mukai transform on K-groups and cohomology groups. However, as discussed in § 3.2.1, Mukai vector can be viewed as a special numerical vector, or generalized twisted Mukai vector. In the following, we would show that generalized twisted Mukai vectors indeed play similar roles as what Mukai vectors do and share similar properties. A much more thorough exposition of Fourier-Mukai transforms is contained in the book [Huy06], or see another [BBH09].

3.3.1 Integral functors and Fourier-Mukai transforms

Let X and Y be non-singular projective varieties over $\mathbb{C}$, and the projections of the Cartesian product $X \times Y$ onto two factors X, Y are denoted by q, p respectively. Precisely,

$$q : X \times Y \longrightarrow X \text{ and } P : X \times Y \longrightarrow Y.$$

Definition 3.40. Let $\mathcal{P}$ be an object in the bounded derived category $\mathrm{D}(X \times Y)$. The induced *integral functor* is the functor

$$\begin{aligned} \Phi_{\mathcal{P}} : \mathrm{D}(X) &\longrightarrow \mathrm{D}(Y) \\ \mathcal{E}^\bullet &\longmapsto p_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P}). \end{aligned}$$

The object $\mathcal{P}$ is called the *kernel* of the integral functor. If $\Phi_{\mathcal{P}}$ is an equivalence, it is called a *Fourier-Mukai transform* and X and Y are called *Fourier-Mukai partners*.

Here, p_*, q^*, and $\otimes$ denote the derived functors between the derived categories. q^* is the regular pull-back if q is flat, and $\otimes\mathcal{P}$ is the regular tensor product as $\mathcal{P}$ is a complex of locally free sheaves. Note that p_*, q^*, and $\otimes$ are exact functors, thus $\Phi_{\mathcal{P}}$ is exact.

Moreover, there is a deep relation between arbitrary functors and Fourier-Mukai type is given by the following theorem of Orlov.

Theorem 3.41 (Orlov)**.** *Let X and Y be two non-singular projective varieties, and a fully faithful exact functor between their bounded derived categories be*

$$F : \mathrm{D}(X) \longrightarrow \mathrm{D}(Y).$$

Suppose F admits left and right adjoint functors, then there exists an object $\mathcal{P} \in \mathrm{D}(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}}$, i.e. $F \simeq \Phi_{\mathcal{P}}$.

Proof. See the original Orlov's work [Orl97] for the highly non-trivial proof of the statement, and a generalization to the case of smooth stacks due to Kawamata [Kaw04]. Note that the proof uses the Postnikov systems which are also used to construct the Harder-Narasimhan filtrations of stability conditions on the derived category discussed in last section. □

Let us give a natural map from the derived category to the Grothendieck group of X already used implicitly in last section. Let $\mathcal{F}^\bullet$ be a bounded complex of coherent sheaves $\mathcal{F}^i$, we define the element

$$[\mathcal{F}^\bullet] := \sum (-1)^i [\mathcal{F}^i]$$

in the Grothendieck group $K(X)$. The ring structure on $K(X)$ is defined by

$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] := [\mathcal{E}_1 \otimes \mathcal{E}_2]$$

for locally free sheaves $\mathcal{E}_i$, since any coherent sheaf on the smooth projective variety X has a finite locally free resolution. Thus we define the map

$$\begin{aligned} [\;] : \mathrm{D}(X) &\longrightarrow K(X), \\ \mathcal{F}^\bullet &\longmapsto [\mathcal{F}^\bullet] = \sum (-1)^i [\mathcal{F}^i], \end{aligned}$$

which satisfies $[\mathcal{F}^\bullet[k]] = (-1)^k [\mathcal{F}^\bullet]$ and $[\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet] = [\mathcal{F}_1^\bullet] + [\mathcal{F}_2^\bullet]$.

For any morphism $f : X \to Y$, the induced pull-back $f^* : K(Y) \to K(X)$ is a ring homomorphism. For any coherent sheaf $\mathcal{F}$ on X, and f is assumed to be projective, we define the generalized direct image

$$f_![\mathcal{F}] := \sum (-1)^i [R^i f_*(\mathcal{F})],$$

which is a group homomorphism from $K(X)$ to $K(Y)$. Note that both maps are compatible with derived pull-back and derived direct image.

Then one can define the *K-theoretic Fourier-Mukai transform* as

$$\begin{aligned} \Phi_E^K : K(X) &\longrightarrow K(Y) \\ F &\longmapsto p_!(q^*(F) \otimes E), \end{aligned}$$

for the kernel $E \in K(X \times Y)$. Moreover the two Fourier-Mukai transforms are compatible, that is, commute in the following digram.

$$\begin{tikzcd} \mathrm{D}(X) \arrow[r, "\Phi_{\mathcal{P}}"] \arrow[d, "{[\;]}"'] & \mathrm{D}(Y) \arrow[d, "{[\;]}"] \\ K(X) \arrow[r, "\Phi^K_{[\mathcal{P}]}"'] & K(Y) \end{tikzcd}$$

Now we consider the complex cohomology group $H^*(X, \mathbb{C})$ which has a natural ring structure. Any morphism $f : X \to Y$ induces a ring homomorphism $f^* : H^*(X, \mathbb{C}) \to H^*(Y, \mathbb{C})$ and if X and Y are compact and connected, one can define the dual map

$$f_* : H^*(X, \mathbb{C}) \longrightarrow H^{*+2\dim(Y)-2\dim(X)}(Y, \mathbb{C})$$

using the Poincaré duality $H^i(X,\mathbb{C}) \simeq H^{2\dim(X)-i}(X,\mathbb{C})^*$. The two maps satisfies the projective formula $f_*(f^*\alpha.\beta)$.

Thus the *cohomological Fourier-Mukai transform* is defined as

$$\begin{aligned}\Phi^H_\alpha : H^*(X,\mathbb{C}) &\longrightarrow H^*(Y,\mathbb{C})\\ \beta &\longmapsto p_*(q^*\beta.\alpha),\end{aligned}$$

for the kernel $\alpha \in H^*(X,\mathbb{C})$. To pass from the Grothendieck group $K(X)$ to the cohomology group $H^*(X,\mathbb{C})$, we would use the numerical vector, generalized twisted Mukai vector defined in § 3.2.1,

$$v_\Lambda : K(X) \longrightarrow H^*(X,\mathbb{C}),$$

by setting

$$v_\Lambda(\mathcal{E}^\bullet) := \mathrm{ch}(\mathcal{E}^\bullet).\sqrt{\mathrm{td}_X}.\exp(\Lambda),$$

where $\Lambda \in H^*(X,\mathbb{C})$ satisfies $\Lambda^\vee = -\Lambda$ and $\mathcal{E}^\bullet := [\mathcal{E}^\bullet]$. To prove the compatibility of two Fourier-Mukai transforms, the main theorem we need is the Grothendieck-Riemann-Roch formula, see [Ful98].

Theorem 3.42 (Grothendieck-Riemann-Roch). *Let $f : X \to Y$ be a smooth projective morphism of nonsingular projective varieties. Then for any $E \in K(X)$ one has*

$$\mathrm{ch}(f_!(E)).\mathrm{td}_Y = f_*(\mathrm{ch}(E).\mathrm{td}_X).$$

3.3.2 Basic properties

We obtain the immediate corollary of compatibility of Fourier-Mukai maps.

Corollary 3.43. *Let E be an object of the Grothendieck group $K(X \times Y)$. Then*

$$\Phi^H_{v_{X^\vee Y}(E)}\left(v_X(F)\right) = v_Y(\Phi^K_E(F))$$

for any $F \in K(X)$. Its means that we have the following commuting diagram

$$\begin{tikzcd} K(X) \arrow[r, "\Phi^K_E"] \arrow[d, "v_X"'] & K(Y) \arrow[d, "v_Y"] \\ H^*(X,\mathbb{C}) \arrow[r, "\Phi^H_{v_{X^\vee Y}(F)}"'] & H^*(Y,\mathbb{C}). \end{tikzcd}$$

Here v_X, v_Y and $v_{X^\vee Y}$ are defined by

$$\begin{aligned} v_X(F) &:= \mathrm{ch}(F).\sqrt{\mathrm{td}_X}.\exp(\Lambda_X),\\ v_Y(F) &:= \mathrm{ch}(F).\sqrt{\mathrm{td}_Y}.\exp(\Lambda_Y),\\ v_{X^\vee Y}(F) &:= \mathrm{ch}(F).\sqrt{\mathrm{td}_{X\times Y}}.q^*\exp(-\Lambda_X).p^*\exp(\Lambda_Y).\end{aligned}$$

Proof. Since any Fourier-Mukai transform is the composition of three exact functors q^*, $\otimes\mathcal{P}$, and p_*, the statement is equivalent to the commutativity of the diagrams

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "v_X"'] & K(X\times Y) \arrow[r, "\cdot E"] \arrow[d, "v_X.(p^*\sqrt{\mathrm{td}_Y})^{-1}"] & K(X\times Y) \arrow[r, "P_!"] \arrow[d, "v_Y.q^*\sqrt{\mathrm{td}_X}"] & K(Y) \arrow[d, "v_Y"] \\
H^*(X) \arrow[r, "q^*"'] & H^*(X\times Y) \arrow[r, ".v_{X^\vee Y}(F)"'] & H^*(X\times Y) \arrow[r, "p_*"'] & H^*(Y).
\end{tikzcd}$$

The commutativity is deduced from the projective formula and the Grothendieck-Riemann-Roch formula. □

Remark 3.44. In general the Chern character does not commute with the Fourier-Mukai transform on K-theory and cohomolgy. On the other hand, for the Chern character and the Riemann-Roch vector v_{RR}, we have the relation

$$\Phi^H_{v_{\mathrm{RR}}(E)}(\mathrm{ch}(F)) = v_{\mathrm{RR}}\left(\Phi^K_E(F)\right)$$

by the commutativity of the diagram below

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "\mathrm{ch}"'] & K(X\times Y) \arrow[r, "\cdot E"] \arrow[d, "\mathrm{ch}"] & K(X\times Y) \arrow[r, "P_!"] \arrow[d, "v_{\mathrm{RR}}"] & K(Y) \arrow[d, "v_{\mathrm{RR}}"] \\
H^*(X) \arrow[r, "q^*"'] & H^*(X\times Y) \arrow[r, ".v_{\mathrm{RR}}(E)"'] & H^*(X\times Y) \arrow[r, "p_*"'] & H^*(Y).
\end{tikzcd}$$

Here the Riemann-Roch vector is defined by $v_{\mathrm{RR}}(F) := \mathrm{ch}(F).\mathrm{td}_X$. Similarly. we also have the following equality

$$\Phi^H_{\mathrm{ch}(E)}(v_{\mathrm{RR}}(F)) = \mathrm{ch}\left(\Phi^K_E(F)\right).$$

by the diagram

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "v_{\mathrm{RR}}"'] & K(X\times Y) \arrow[r, "\cdot E"] \arrow[d, "v_{\mathrm{RR}}.(p^*\mathrm{td}_Y)^{-1}"] & K(X\times Y) \arrow[r, "P_!"] \arrow[d, "v_{\mathrm{RR}}.(p^*\mathrm{td}_Y)^{-1}"] & K(Y) \arrow[d, "\mathrm{ch}"] \\
H^*(X) \arrow[r, "q^*"'] & H^*(X\times Y) \arrow[r, ".\mathrm{ch}(E)"'] & H^*(X\times Y) \arrow[r, "p_*"'] & H^*(Y).
\end{tikzcd}$$

By the same method, we obtain

$$\Phi^H_{\mathrm{ch}(E).q^*\mathrm{td}_X}(\mathrm{ch}(F)) = \mathrm{ch}\left(\Phi^K_E(F)\right).$$

Proposition 3.45. *Let X, Y, and Z be nonsingular projective varieties over $\mathbb{C}$, $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ and $\Phi_{\mathcal{Q}} : \mathrm{D}(Y) \to \mathrm{D}(Z)$ be two Fourier-Mukai transforms. Denote by π_{XY}, π_{YZ} and π_{XZ} the projections from $X\times Y\times Z$ to $X\times Y$, $Y\times Z$ and $X\times Z$, respectively. Suppose $\Phi_{\mathcal{R}} : \mathrm{D}(X) \to \mathrm{D}(Z)$ is their composition, i.e. $\Phi_{\mathcal{R}} = \Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$, where $\mathcal{R} \in \mathrm{D}(X\times Z)$ is defined by*

$$\mathcal{R} := \pi_{XZ*}(\pi^*_{XY}\mathcal{P} \otimes \pi^*_{YZ}\mathcal{Q}).$$

Then

$$\Phi^H_{v_{X^\vee Z}(\mathcal{R})} = \Phi^H_{v_{Y^\vee Z}(\mathcal{Q})} \circ \Phi^H_{v_{X^\vee Y}(\mathcal{P})}.$$

Proof. We denote the projections from $X \times Z$ to X and Z by s and r, from $Y \times Z$ to Y and Z by u and t, respectively. For any cohomology class $\beta \in H^*(X, \mathbb{C})$, it turns out that

$$\begin{aligned}
\Phi^H_{v_{X^\vee Z}(\mathcal{R})}(\beta) &= r_*\left(s^*(\beta).\mathrm{ch}\left(\pi_{XZ*}\left(\pi^*_{XY}\mathcal{P} \otimes \pi^*_{YZ}\mathcal{Q}\right)\right).\sqrt{\mathrm{td}_{X\times Z}}.s^*\exp(-\Lambda_X).r^*\exp(\Lambda_Z)\right) \\
&= r_*\Big(s^*(\beta).\pi_{XZ*}\Big(\pi^*_{XY}\mathrm{ch}(\mathcal{P}).\pi^*_{YZ}\mathrm{ch}(\mathcal{Q}).\pi^*_{XY}p^*\sqrt{\mathrm{td}_Y}.\pi^*_{YZ}u^*\sqrt{\mathrm{td}_Y}. \\
&\quad \pi^*_{XZ}\left(\sqrt{\mathrm{td}_{X\times Z}}.s^*\exp(-\Lambda_X).r^*\exp(\Lambda_Z)\right)\Big)\Big) \\
&= r_*\Big(\pi_{XZ*}\Big(\pi^*_{XZ}s^*(\beta).\pi^*_{XY}\left(\mathrm{ch}(\mathcal{P}).\sqrt{\mathrm{td}_{X\times Y}}.q^*\exp(-\Lambda_X).p^*\exp(\Lambda_Y)\right). \\
&\quad \pi^*_{YZ}\left(\mathrm{ch}(\mathcal{Q}).\sqrt{\mathrm{td}_{Y\times Z}}.u^*\exp(-\Lambda_Y).t^*\exp(\Lambda_Z)\right)\Big)\Big) \\
&= r_*\left(\pi_{XZ*}\left(\pi^*_{XY}\left(q^*(\beta).v_{X^\vee Y}(\mathcal{P})\right).\pi^*_{YZ}\left(v_{Y^\vee Z}(\mathcal{Q})\right)\right)\right) \\
&= t_*\left(\pi_{YZ*}\left(\pi^*_{XY}\left(q^*(\beta).v_{X^\vee Y}(\mathcal{P})\right)\right).v_{Y^\vee Z}(\mathcal{Q})\right) \\
&= t_*\left(u^*\left(p_*\left(q^*(\beta).v_{X^\vee Y}(\mathcal{P})\right)\right).v_{Y^\vee Z}(\mathcal{Q})\right) \qquad (*) \\
&= \Phi^H_{v_{Y^\vee Z}(\mathcal{Q})} \circ \Phi^H_{v_{X^\vee Y}(\mathcal{P})}(\beta).
\end{aligned}$$

Note in the equality $(*)$ we use $\pi_{YZ*} \circ \pi^*_{XY} = u^* \circ p_*$ by flat base change, see [Har77]. □

Proposition 3.46. *Suppose* $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ *is an equivalence of the bounded derived categories, then the induced cohomological Fourier-Mukai transform is a bijection of complex cohomology group, i.e.* $\Phi^H_{v_{X^\vee Y}(\mathcal{P})} : H^*(X, \mathbb{C}) \simeq H^*(Y, \mathbb{C})$.

Proof. Consider the diagonal embedding $\iota : X \simeq \triangle \hookrightarrow X \times X$. It is sufficient to prove that $\Phi^H_{v_{X^\vee X}(\mathcal{O}_\triangle)} = \mathrm{id}$, since $\mathcal{O}_\triangle \in \mathrm{D}(X \times X)$ is the only object inducing the identity Fourier-Mukai transform, and by the previous proposition we obtain $\Phi^H_{v_{Y^\vee X}(\mathcal{P}_R)} \circ \Phi^H_{v_{X^\vee Y}(\mathcal{P})} \simeq \Phi^H_{v_{X^\vee X}(\mathcal{O}_\triangle)} = \mathrm{id}$, and $\Phi^H_{v_{X^\vee Y}(\mathcal{P})} \circ \Phi^H_{v_{Y^\vee X}(\mathcal{P}_R)} \simeq \Phi^H_{v_{Y^\vee Y}(\mathcal{O}_\triangle)} = \mathrm{id}$ for $\mathcal{P}_R := \mathcal{P}^\vee \otimes q^*\omega_X[\dim X]$ with the canonical sheaf ω_X. Then, we have

$$\begin{aligned}
v_{X^\vee X}(\mathcal{O}_\triangle) &= \mathrm{ch}(\mathcal{O}_\triangle).\mathrm{td}_{X\times X}.q^*\exp(-\Lambda_X).p^*\exp(\Lambda_X).\sqrt{\mathrm{td}_{X\times X}}^{-1} \\
&= \iota_*\left(\mathrm{ch}(\mathcal{O}_X).\mathrm{td}_X\right).q^*\exp(-\Lambda_X).p^*\exp(\Lambda_X).\sqrt{\mathrm{td}_{X\times X}}^{-1} \\
&= \iota_*\left(\mathrm{td}_X.\mathrm{td}_X^{-1}\right) \qquad (*) \\
&= \iota_*(1).
\end{aligned}$$

In $(*)$ we use the relation $q \circ \iota \simeq p \circ \iota \simeq \mathrm{id}$. Thus for any $\beta \in H^*(X, \mathbb{C})$,

$$\begin{aligned}
\Phi^H_{v_{X^\vee X}(\mathcal{O}_\triangle)}(\beta) &= p_*\left(q^*(\beta).v_{X^\vee X}(\mathcal{O}_\triangle)\right) = p_*\left(q^*(\beta).\iota_*(1)\right) \\
&= p_*\left(\iota_*\left(\iota^*q^*(\beta)\right)\right) = \beta.
\end{aligned}$$

□

Finally, recall that the Mukai pairing on $H^*(X, \mathbb{C})$ in § 3.2.1 is given by

$$\langle \alpha, \beta \rangle_X = \int_X \exp(c_1(X)/2) \cup (\alpha^\vee \cup \beta)$$

for all $\alpha, \beta \in H^*(X, \mathbb{C})$. and the dual vector of $v \in H^*(X, \mathbb{C})$ is defined by

$$v^\vee := \sum \left(\sqrt{1}\right)^j v_j \in H^*(X, \mathbb{C}).$$

Here $v_j \in H^j(X, \mathbb{C})$. Then we have the following proposition

Proposition 3.47. *Suppose $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ is a Fourier-Mukai transform, i.e. an equivalence of the bounded derived categories. Then the induced cohomological Fourier-Mukai transform $\Phi^H_{\upsilon_{X^\vee Y}(\mathcal{P})}$: $H^*(X, \mathbb{C}) \simeq H^*(Y, \mathbb{C})$ is isometric with respective to the Mukai pairing, that is,*

$$\langle v, v' \rangle_X = \left\langle \Phi^H_{\upsilon_{X^\vee Y}(\mathcal{P})}(v), \Phi^H_{\upsilon_{X^\vee Y}(\mathcal{P})}(v') \right\rangle_X$$

for all $v, v' \in H^*(X, \mathbb{C})$.

Proof. First note that $\left(\Phi^H_{\upsilon_{X^\vee Y}(\mathcal{P})}\right)^{-1} = \Phi^H_{\upsilon_{Y^\vee X}(\mathcal{P}_R)}$ by the previous proposition, where $\mathcal{P}_R = \mathcal{P}^\vee \otimes q^* \omega_X[n]$ with $n := \dim X = \dim Y$. Thus the assertion follows form the equality

$$\left\langle \Phi^H_{\upsilon_{X^\vee Y}(\mathcal{P})}(v), w \right\rangle_Y = \left\langle v, \Phi^H_{\upsilon_{Y^\vee X}(\mathcal{P}_R)}(w) \right\rangle_X$$

for all $v \in H^*(X, \mathbb{C})$ and $w \in H^*(Y, \mathbb{C})$. To prove this equality, we need the easy fact : Given the second projection $p : X \times Y \to Y$, then

$$p_*(v)^\vee = (-1)^{\dim X} p_*(v^\vee)$$

for all $v \in H^*(X \times Y, \mathbb{C})$. Thus we can compute

$$\begin{aligned}
\left\langle \Phi^H_{\upsilon_{X^\vee Y}(\mathcal{P})}(v), w \right\rangle_Y &= \int_Y \exp(c_1(Y)/2).p_*(q^* v.\upsilon_{X^\vee Y}(\mathcal{P}))^\vee.w \\
&= (-1)^n \int_{X \times Y} p^* \exp(c_1(Y)/2).(q^* v.\upsilon_{X^\vee Y}(\mathcal{P}))^\vee.p^* w \\
&= (-1)^n \int_{X \times Y} p^* \exp(c_1(Y)/2).q^* v^\vee.\upsilon_{X^\vee Y}(\mathcal{P})^\vee.p^* w \\
&= (-1)^n \int_{X \times Y} p^* \exp(c_1(Y)/2).q^* v^\vee.\upsilon_{Y^\vee X}(\mathcal{P}^\vee).(\exp(c_1(X \times Y)/2))^{-1}.p^* w \\
&= \int_{X \times Y} q^* v^\vee.\upsilon_{Y^\vee X}(\mathcal{P}_R).q^* \exp(c_1(X)/2).p^* w \\
&= \int_X \exp(c_1(X)/2).v^\vee.q_*(p^* w.\upsilon_{Y^\vee X}(\mathcal{P}_R)) \\
&= \left\langle v, \Phi^H_{\upsilon_{Y^\vee X}(\mathcal{P}_R)}(w) \right\rangle_X .
\end{aligned}$$

□

CHAPTER 4

Kovalev's Construction of Manifolds with Special Holonomy G_2

In this chapter we summarize Kovalev's construction of G_2-manifolds via twisted connected sums [Kov03], which has recently been generalized by A. Corti and M. Haskins and J. Nordström and T. Pacini [Cor+15]. G_2-manifold is a special kind of real seven-dimensional space constructed by Robert Bryant and Salamon [BS89] for non-compact manifolds and by Dominic Joyce [Joy96] for compact ones, and may be viewed as a real version of the Calabi-Yau manifold since the Ricci curvatures of both type of manifolds are trivial. However, these G_2-manifolds are still poorly understood mathematically, compared with the Calabi-Yau manifolds. This situation has been improved by Kovalev's construction, as rich examples of G_2-manifolds can be constructed using the twisted connected sum constructions of Fano, semi-Fano, or weak Fano 3-folds.

4.1 Geometry and topology of G_2 manifolds

Let us start with a collection of some facts and definition concerning algebra and geometry associated to the Lie groups G_2 and $SU(3)$, which can be found in the article by Bryant [Bry87], or the book by Joyce [Joy00].

4.1.1 The exceptional Lie group G_2

We first give the definition of G_2 groups via the octonion algebra denoted by $\mathbb{O}$ over a field [Bae02].

Definition 4.1. The group G_2 is the automorphism group of the octonion algebra $\mathbb{O}$.

The octonions $\mathbb{O}$ is an 8-dimensional algebra with $\{1, e_1, e_2, \ldots, e_7\}$, and we denote by $\mathrm{Im}\mathbb{O}$ the subspace $\mathrm{Span}\{e_1, e_2, \ldots, e_7\}$. The multiplication is given by the following table 4.1, the result of multiplying the element in the ith row by the element in the jth column. The multiplication rules can be summarized by the relations [Gen+09]

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k,$$

where ε_{ijk} is a completely antisymmetric tensor taking value $+1$ for the indices $ijk = 123$, 145, 176, 246, 257, 347, 365. Note that the definition though is not unique, the others can be obtained by permuting and changing the signs of those basis elements. Pick any triple (e_i, e_j, e_k) with the square of each element equal to -1 such that they anticommute with each other and each product $e_a e_b$ for $a, b \in \{i, j, k\}$, then the

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 4.1: Octonion Multiplication

elements $\{e_i, e_j, e_k\}$ generate all of $\mathbb{O}$. Thus by counting all automorphism of the octonions, we obtain $\dim G_2 = 14$.

An equivalent definition of G_2 below is given by Joyce [Joy00].

Definition 4.2. Let $(x^1, \ldots, x^7)$ be coordinates on $\mathbb{R}^7$ and $dx^{ij\ldots l} = dx^i \wedge dx^j \wedge \cdots dx^l$ on $\mathbb{R}^7$. A 3-form φ_0 is defined on $\mathbb{R}^7$ by

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}.$$

The subgroup of $GL(7, \mathbb{R})$ preserving φ_0 is the exceptional Lie group G_2. It is compact, connected, simply-connected, semi-simple and 14-dimensional, and it also fixes the 4-form

$$*\varphi_0 = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247},$$

the metric $g_0 = d(x^1)^2 + d(x^2)^2 + \cdots + d(x^7)^2$, and the orientation on $\mathbb{R}^7$. Here $*$ is the Hodge star.

Given any $\varphi \in \bigwedge^3(\mathbb{R}^7)$, we define

$$B_\varphi = \mathbb{R}^7 \times \mathbb{R}^7 \to \bigwedge^7 \left(\mathbb{R}^7\right)^*$$

by setting

$$B_\varphi(x, y) = \frac{1}{6}(x \lrcorner \varphi) \wedge (y \lrcorner \varphi) \wedge \varphi,$$

for all $x, y \in \mathbb{R}^7$, which is symmetric bilinear form on $\mathbb{R}^7$ with values in $\bigwedge^7 \left(\mathbb{R}^7\right)^*$. Hence it can be considered as a linear map $Q_\varphi : \mathbb{R}^7 \to \left(\mathbb{R}^7\right)^* \times \bigwedge^7 \left(\mathbb{R}^7\right)^*$, called φ *non-degenerate* if $Q_\varphi \neq 0$. Then as in [Hit00] we can define a volume form vol_φ and a symmetric bilinear form g_φ by

$$\left(\mathrm{vol}_\varphi\right)^9 := \det Q_\varphi,$$
$$g_\varphi \otimes \mathrm{vol}_\varphi := B_\varphi,$$

If we pick $\varphi = \varphi_0$, inducing $g_\varphi = g_0$, then the stabiliser of φ_0 in $GL(7, \mathbb{R})$ which preserves the volume form and the bilinear form g_φ must equal G_2. The set of 3-forms for which there exists an oriented isomorphism to the standard one φ_0 is an open subset of $\bigwedge^3 \left(\mathbb{R}^7\right)^*$. Indeed, since $\dim GL(7, \mathbb{R}) = 49$ and $\dim G_2 = 14$, so the space $GL(7, \mathbb{R})/G_2$ has the dimension equal to $49 - 14 = 35$. However, $\bigwedge^3 \left(\mathbb{R}^7\right)^*$ also has dimension equal to 35, and thus the set of 3-forms isomorphic to φ_0 forms an open subset.

4.1.2 The G_2-structure

Definition 4.3. Let M be an oriented 7-manifold and φ a *positive 3-form*, i.e. $\varphi|_p \simeq \varphi_0$ is an oriented isomorphism for all p. Let Q be the subset of F, the frame bundle of M, consisting of isomorphisms between T_pM and $\mathbb{R}^7$ which identify $\varphi|_p$ and φ_0 for each $p \in M$. Then Q is a principle subbundle of F, with fibre G_2, called a *G_2-structure*. We would refer to the pair $(\varphi, g := g_\varphi)$ as a G_2-structure of M.

Let M be a 7-manifold, (φ, g) a G_2-structure, and ∇ the Levi-Civita connection of g. $\nabla\varphi$ is called the *torsion* of (φ, g), and if $\nabla\varphi = 0$, the G_2-structure (φ, g) is called *torsion-free*.

Definition 4.4. A *G_2-manifold* is a 7-manifold M equipped with a torsion-free G_2-structure (φ, g), or simply denoted by φ. (M, φ) is called a manifold *with holonomy* G_2 if $\mathrm{Hol}(g_\varphi) = G_2$.

The equivalence conditions of the torsion-free G_2-structures are described by the next proposition.

Proposition 4.5 ([Sal89, Lemma 11.5])**.** *Let (M, φ) be a 7-manifold, Then the following are equivalent:*

(i) (φ, g) *is torsion-free,*

(ii) $\mathrm{Hol}(g_\varphi) \subseteq G_2$*,*

(iii) $\nabla\varphi = 0$ *on M,*

(iv) $d\varphi = d^*\varphi = 0$ *on M, and*

(v) $d\varphi = d(*_g\varphi) = 0$ *on M, where $*_g$ is the Hodge star.*

One can see that the condition that (φ, g_φ) is torsion-free is a nonlinear PDE as $*_g$ is a nonlinear map depending on the orientation and metric g, hence on φ, in M.

Moreover, representations of G_2 induce a decomposition of bundles on M into irreducibles of tensor products of the irreducible representations. Thus there is a decomposition of the exterior forms on (M, φ).

Proposition 4.6 ([Joy00, Proposition 10.1.4])**.** *Let M be a 7-manifold with a G_2-structure (φ, g_φ), and Λ^k_l be an irreducible representation of G_2 of dimension l. Then $\bigwedge^k T^*M$ can be decomposed orthogonally into components as follows:*

(i) $\bigwedge^1 T^*M = \Lambda^1_7$,

(ii) $\bigwedge^2 T^*M = \Lambda^2_7 \oplus \Lambda^2_{14}$,

(iii) $\bigwedge^3 T^*M = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$,

(iv) $\bigwedge^4 T^*M = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$,

(v) $\bigwedge^5 T^*M = \Lambda^5_7 \oplus \Lambda^5_{14}$, *and*

(vi) $\bigwedge^6 T^*M = \Lambda^6_7$.

The Hodge star of g_φ gives an isometry between Λ^k_l and Λ^{7-k}_l. Here $\Lambda^3_1 = \langle\varphi\rangle$ and $\Lambda^4_7 = \langle *\varphi\rangle$.

Since $G_2 \subset SO(7)$ is simply-connected, any 7-manifold M with a G_2-structure is a spin manifold. From Salamon [Sal89, Lemma 11.8] and Joyce [Joy00, Proposition 10.1.6], we have

Proposition 4.7. *Let (M, φ) be a 7-manifold with a G_2-structure. Then M is a spin manifold. Moreover if $\mathrm{Hol}(g_\varphi) \subseteq G_2$, or (φ, g_φ) is torsion-free, then g_φ is Ricci-flat and there exists a nontrivial parallel spinor.*

4.1.3 Topology of compact G_2-manifolds

The condition of holonomy G_2 on a 7-manifold M is stronger than the existence of the G_2-structure on M. Joyce then proved the following proposition:

Proposition 4.8 ([Joy00, Proposition 10.2.2])**.** *A compact G_2-manifold M has holonomy G_2 if and only if the fundamental group $\pi_1(M)$ is finite.*

Now from the classification of Riemannian holonomy group, we have the relation of holonomy groups: $SU(2) \subset SU(3) \subset G_2 \subset SO(7)$. We are interested in subholonomy groups $SU(2), SU(3)$ of G_2, as both of them play important roles in Kovalev's construction of compact G_2-manifolds

Let $\{z_1, z_2, \ldots, z^n\}$ be holomorphic coordinates of $\mathbb{C}^n$. Then the standard holomorphic volume form Ω_0 and Kähler form ω_0 are given by

$$\begin{aligned}\Omega_0 &= dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n,\\ \omega_0 &= \frac{i}{2}\left(dz^1 \wedge d\bar{z}^1 + \cdots dz^n \wedge d\bar{z}^n\right).\end{aligned}$$

For any complex n-form Ω which is equivalent to Ω_0 and non-degenerate real 2-form ω equivalent to ω_0, the pair (Ω, ω) satisfies

$$\begin{aligned}\Omega \wedge \omega &= 0,\\ (-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} &= \frac{\omega^n}{n!}.\end{aligned}$$

Suppose $\left\{x^1, \ldots, x^7\right\}$ are orthogonal normalized basis on $\mathbb{R}^7 \simeq \mathbb{R}^1 \times \mathbb{C}^3$, and G is the stabilizer of the basis e_1, mapping the orthogonal complement $e_1^\perp$ to itself. Choosing the holomorphic coordinates $z^1 = x^2 + ix^3, z^2 = x^4 + ix^5, z^3 = x^6 + ix^7$, the action of G on $\mathbb{C}^3$ preserves the following forms

$$\begin{aligned}\omega_0 &= e_1 \lrcorner \varphi_0 = dx^{23} + dx^{45} + dx^{67},\\ \mathrm{Re}\Omega_0 &= \varphi_0|_{e_1^\perp} = dx^{246} - dx^{257} - dx^{347} - dx^{356},\\ \mathrm{Im}\Omega_0 &= e_1 \lrcorner \varphi_0 = -dx^{247} - dx^{256} - dx^{346} + dx^{357},\end{aligned}$$

so that G is contained in $SU(3)$. On the other hand, $SU(3)$ preserves $\varphi_0 = dx^1 \wedge \omega_0 + \mathrm{Re}\Omega_0$, so G is exactly equal to $SU(3)$. In this case of the $SU(3)$-structures on $\mathbb{C}^3$, the pair (Ω, ω) induces a G_2-form and the associated 4-form on $\mathbb{R}^1 \times \mathbb{C}^3$ written as

$$\begin{aligned}\varphi &= dt \wedge \omega + \mathrm{Re}\Omega,\\ *\varphi &= \frac{1}{2}\omega^2 - dt \wedge \mathrm{Im}\Omega.\end{aligned}$$

Thus the stabilizer in G_2 of a non-trivial vector in $\mathbb{R}^7$ is isomorphic to $SU(3)$.

Let us consider the $SU(2)$-structures on $\mathbb{C}^2$ with the holomorphic coordinates $z^1 = x^1 + ix^2$ and $z^2 = x^3 + ix^4$. Let $\omega_0^I := \omega_0$ be the standard Kähler form and $\Omega_0 := \omega_0^J + i\omega_0^K$ be the standard holomorphic volume form on $\mathbb{C}^2$. Here $\{I, J, K\}$ are g_0-orthogonal complex structures on $\mathbb{R}^4$ by the relations $\omega_0^I(x, y) = g_0(Ix, y), \omega_0^j(x, y) = g_0(Jx, y)$, and $\omega_0^K(x, y) = g_0(Kx, y)$. Then the pair (Ω, ω) of the $SU(2)$-structure on $\mathbb{R}^4$ satisfies $\left(\omega^I\right)^2 = \left(\omega^J\right)^2 = \left(\omega^K\right)^2$, and $\omega^I \wedge \omega^J = \omega^j \wedge \omega^K = \omega^K \wedge \omega^I = 0$.

4.1.4 Moduli space of compact G_2-manifolds

We begin with the consideration of deformations of G_2-structures on a 7-manifold M.

Definition 4.9 ([Joy00, Definition 10.3.3])**.** Let (φ, g) be a G_2-structure on a 7-manifold M. Then $\epsilon_1 > 0$ is an universal constant such that if $\tilde{\varphi} \in C^\infty(\bigwedge^3 T^*M)$ and $\|\tilde{\varphi} - \varphi\|_{C^0} < \epsilon_1$, $\tilde{\varphi} \in C^\infty(\mathcal{P}^3 M)$, the set of positive 3-forms on M. In other words, $\tilde{\varphi}$ defines a G_2-structure on the 7-manifold M.

If ϵ_1 is sufficiently small, this condition always holds. Let M be a compact oriented 7-manifold. Let $\mathcal{X}$ be the set of positive 3-forms corresponding to oriented torsion-free G_2-structures, i.e.

$$\mathcal{X} = \left\{\varphi \in C^\infty(\mathcal{P}^3 M) \mid d\varphi = d(*_g\varphi) = 0\right\},$$

and $\mathcal{D}$ be the group of all diffeomorphisms of M isotopic to the identity.

Definition 4.10. The *moduli space of torsion-free G_2-structures* on M is the quotient space $\mathcal{M} = \mathcal{X}/\mathcal{D}$.

Furthermore, $\mathcal{M}$ is a smooth manifold with dimension equal to $b^3(M)$ by the following theorem.

Theorem 4.11 ([Joy00, Theorem 10.4.4])**.** *Let M be a compact 7-manifold, and $\mathcal{M}$ be the moduli space of torsion-free G_2-structures on M. Then $\mathcal{M}$ is a smooth manifold of dimension $b^3(M)$ with the projection $\pi : \mathcal{M} \to H^3(M, \mathbb{R})$ given by $\pi(\varphi\mathcal{D}) = [\varphi]$.*

Note that this theorem is just a local result with little information about the global structure of $\mathcal{M}$. We cannot make sure that $\mathcal{M}$ is not empty, or has only one connected component, or the map π is injective, etc.

4.2 Twisted connected sum constructions

In Kovalev's approach, one can construct a compact real 7-manifold Y via the twisted connected sum of two compatible asymptotically cylindrical Calabi-Yau complex manifolds X_L and X_R along an additional S^1, such that Y is an asymptotically G_2-manifold, i.e. $d(*\varphi) = 0$ in the asymptotically limit.

4.2.1 Asymptotically cylindrical Calabi-Yau 3-folds

We start with a review of the basic definitions and analytic results of asymptotically cylindrical Calabi-Yau 3-folds due to the work by Kovalev [Kov03]. See also [Cor+15].

Definition 4.12. A *Calabi-Yau cylinder X^∞* is a product of a compact K3 surface (S, Ω_S, ω_S) and the algebraic torus $\mathbb{C}^*$. Let $z = \exp(t - i\theta^*)$ be the holomorphic coordinate on $\mathbb{C}^*$, one can define the Käher form and holomorphic volume form by

$$\begin{aligned}\omega^\infty &:= \frac{idz \wedge d\bar{z}}{2z\bar{z}} + \omega_S = dt \wedge d\theta^* + \omega_S \\ \Omega^\infty &:= -\frac{idz}{z}\Omega_S = (d\theta^* - idt) \wedge \Omega_S,\end{aligned}$$

with the metric $g^\infty := dt^2 + d(\theta^*)^2 + g_S$ and the complex structure $I^\infty := I_\mathbb{C} + I_S$ on $X^\infty \simeq \mathbb{C}^* \times S$.

Note that the pair (Ω, ω) with respect to the $SU(2)$-structure $(\Omega^\infty, \omega^\infty)$ also fulfills the relations

$$\Omega \wedge \omega = 0,$$
$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = \frac{\omega^n}{n!}.$$

For any Calabi-Yau cylinder X^∞, with θ denoting the coordinate of S^1 the 3-form

$$\varphi_0 = d\theta \wedge (dt \wedge d\theta^* + \omega_S) + d\theta^* \wedge \mathrm{Re}(\Omega_S) + dt \wedge \mathrm{Im}(\Omega_S)$$

can be used to define a G_2-structure on the product real 7-manifold $X^\infty \times S^1$. Then this G_2-structure is unchanged if the coordinates are exchanged as $(\theta, t, \theta^*, S) \leftrightarrow (\theta^*, -t, \theta, S)$, which is the basic idea of Kovalev's constructions.

Definition 4.13. Let $(X, g, I, \omega, \Omega)$ be a complete (non-compact) Calabi-Yau 3-fold. X is called an *asymptotically cylindrical Calabi-Yau 3-fold* if there exist (i) a compact subset $K \subset X$, (ii) a Calabi-Yau cylinder X^∞ and (iii) a diffeomorphism $\eta : X^\infty \to X \backslash K$ such that for all $k \geq 0$, for some $\lambda > 0$ and as $t \to \infty$,

$$\eta^*\omega - \omega^\infty = d\mu, \quad \text{with } |\nabla^k \mu| = O(e^{-\lambda t})$$
$$\eta^*\Omega - \Omega^\infty = d\nu, \quad \text{with } |\nabla^k \nu| = O(e^{-\lambda t})$$

for some 1-form μ and 2-form ν, where ∇ and $|\cdot|$ are defined by using the metric g^∞ on X^∞. The scale λ is given by $\lambda = \min\{1, \lambda_S\}$, where λ_S is the square root of the smallest positive eigenvalue of the Laplacian of the K3 surface S in the asymptotic Calabi–Yau cylinder X^∞.

Definition 4.14. A *building block* is a nonsingular algebraic 3-fold Z together with a projective morphism $f : Z \to \mathbb{P}^1$ satisfying

(i) the anticanonical class $-K_Z \in H^2(Z)$ is primitive,

(ii) $S = f^*(\infty)$ is a nonsingular K3 surface and $S \sim -K_Z$,

(iii) The inclusion $N \hookrightarrow L$ is primitive, i.e., L/N is torsion-free, with L is the K3 lattice $H^2(S)$ and N the image of $H^2(Z) \to H^2(S)$,

(iv) $H^3(Z)$ is torsion-free and so do $H^4(Z)$.

Note that the fundamental group $\pi_1(Z)$ is alway trivial, as for any disc $\Delta \in \mathbb{P}^1$ containing at most 1 critical value x, $\Delta^\times = \Delta \setminus \{x\}$, $V_\Delta = f^{-1}(\Delta)$ and $V_\Delta^\times = f^{-1}(\Delta^\times)$ the fibers, we have $\pi_1(V_\Delta^\times) = \pi_1(\Delta^\times) = \mathbb{Z}$ by the long exact sequence of homotopy groups in the K3 fibration. The induced morphism $j_* : \pi_1(V_\Delta^\times) \to \pi_1(V_\Delta)$ is surjective, and $f^{-1}(x) = \sum m_i F_i$ with $F_i \subset V = Z \setminus S$ the irreducible components and m_i their multiplicities. Here $\gcd(m_i) = 1$ by the condition (i) of the building block. The image of a loop that loop once around generic point of F_i is trivial, i.e. $m_i j_*(1) = 0$ in $\pi_1(V_\Delta)$. It turns out that $j_*(1) = 0$, as $\gcd(m_i) = 1$, implying that $\pi_1(V_\Delta) = \pi_1(Z \setminus S) = \pi_1(Z) = 0$ by the van Kampen theorem.

A rich class of asymptotically cylindrical Calabi-Yau 3-folds can be constructed using building blocks by the method: Let Z be a closed Kähler 3-fold with a morphism $f : Z \to \mathbb{P}^1$, which has a reduced K3 fiber S that is the anticanonical divisor in Z, and $\omega_S \in H^{1,1}(S)$ is induced from a Kähler class on Z, then $X = Z \setminus S$ is an asymptotically cylindrical Calabi-Yau 3-fold. In the following we introduce the building blocks constructed from the weak Fano 3-folds, called the *building blocks of weak Fano type*.

Proposition 4.15 ([Cor+13, Proposition 5.7]). *Let W be a weak Fano 3-fold, $|S_0, S_\infty| \subset |-K_W|$ a generic pencil with base locus C, Z the blow-up of W along C, $V = Z \setminus S$, and $f : Z \to \mathbb{P}^1$ the K3 fibration induced by the pencil. Then, we have*

(i) *The anticanonical class $-K_Z \in H^2(Z)$ is primitive.*

(ii) *$S = f^{-1}(\infty)$ is a nonsingular K3 surface and $S \sim -K_Z$.*

(iii) *the image N of $H^2(Z) \to H^2(S) = L$ equals that of $H^2(W) \to H^2(S)$ and of $H^2(V) \to H^2(S)$. If W is semi-Fano then $H^2(W) \to H^2(S)$ is injective and $N \hookrightarrow L$ is primitive.*

(iv) *The group $H^3(Z)$ is torsion-free if and only if $H^3(W)$ is.*

Proof. (Sketch) (i) and (ii) follow from the fact that if W is a nonsingular weak Fano 3-fold then a general anticanonical class $S \in |-K_W|$ is a nonsingular K3 surface (see [Cor+13, Theorem 4.7]), and the well-known formula $-K_Z = \pi^*(-K_W) - E$. By the decomposition of the cohomology group of a blow up along a curve (see [GH94, p. 605]), $H^3(Z) \simeq H^3(W) + \mathbb{Z}^{2g(C)}$ implies the condition (iv). To prove (iii) we use the fact, deduced from the relative Lefschetz hyperplane theorem (see [Cor+13, Proposition 3.10]), that if W is a semi-Fano 3-fold and $-K_W \sim S \subset W$ is nonsingular K3 surface then $H^2(W) \to H^2(S)$ is a primitive inclusion. The remaining conditions follow from the results of cohomolgy of building blocks, see § 4.3. □

Finally, the following theorem provides the $SU(3)$-structure $(\Omega^\infty, \omega^\infty)$ on V near the end of infinity.

Theorem 4.16 ([Kov03, Theorem 2.4]). *Let Z be constructed as above with ω' the Kähler fom and $H^1(Z) = 0$, such that a K3 surface $S \subset Z$ is an anti-canonical divisor with trivial self-intersection $S.S = 0$. Then $V = Z \setminus S$ defines an asymptotically cylindrical Calabi-Yau 3-fold.*

Proof. (Sketch) The K3 fibration over $\mathbb{P}^1$ implies that $S.S = 0$ so S has trivial normal bundle in Z. We thus use the triviality of normal bundle U of S to define a local product decomposition $U \simeq \{|z| < 1\} \times S$, $z = \exp(-t - i\theta^*)$. Let $\left(\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0\right)\Big|_S$ is the Calabi-Yau metric in the class $[\omega'|_S]$ on S, where u_0 is a smooth function supported in U. Choose a Kähler form ω_1 on $\mathbb{P}^1$ such that $\omega_1 = (1 + O(|z|^2))idz \wedge d\bar{z}$ for small $|z|$, where $f^{-1}(z = 0) = S$. Rescaling ω_1 by an appropriate positive constant μ, we obtain a positive $(1,1)$-form $\omega_{\text{comp}} = \omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0 + \mu f^*\omega_1$. Moreover, the $(1,1)$-form $\partial\bar{\partial}(\log|s_1|^2)^2$ on $\mathbb{P}^1 \setminus \{z = 0\}$ is positive on a neighborhood of $z = 0$, written as $2(dz/z) \wedge (d\bar{z}/\bar{z})$, where $s_1 \in H^0(\mathbb{P}^1, -K_Z)$. Then we obtain

$$\omega_{\text{cyl}} = \omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0 + \mu f^*\omega_1 + \frac{i}{4}\partial\bar{\partial}\left(\log|s_1 \circ f|^2\right)^2,$$

which has the asymptotic expression

$$\omega_{\text{cyl}}|_{U\setminus S} = dt \wedge d\theta^* + \omega_S + d(e^{-t}\psi_{\text{cyl}}),$$

where ψ_{cyl} is a smooth 1-form bounded with all derivatives on the cylindrical end. Now we set $f_{\text{cyl}} = -\log\frac{\omega_{\text{cyl}}^3}{\omega'^3} - \log|s_1 \circ f|^2$. Due to [Kov03, Proposition 3.9 and 3.16], there exists a smooth function u on V which converges to zero, $|\nabla^k u| < C_k e^{-\lambda t}$ for some $\lambda > 0$ as $t \to \infty$, and is a solution to the equation

$$\omega_g^3 := \left(\omega_{\text{cyl}} + \frac{i}{2\pi}\partial\bar{\partial}u\right)^3 = e^{f_{\text{cyl}}}\omega_{\text{cyl}}^3,$$

such that ω_g is a Ricci-flat Kähler metric on V as we want. Furthermore, if Ω_g is a holomorphic volume form for ω_g, then by straightforward calculation $\Omega_g - \Omega^\infty$ and all derivatives are $O(e^{-t})$, and $\Omega_g - \Omega^\infty = d\Psi$ for some 2-form Ψ in $U \setminus S$. □

4.2.2 Twisted connected sum

We now come to the actual construction of the G_2-manifold Y by first constructing two asymptotically cylindrical Calabi-Yau 3-folds $(X_{L/R}, \Omega_{L/R}, \omega_{L/R})$, then one takes a direct product with S^1 of both of them and glues their asymptotic Calabi-Yau cylinder regions, which are two different copies of the type $Y^\infty_{L/R} = X^\infty_{L/R} \times S^1_{L/R}$ with a twist, which can be visualized as the follwing sketch.

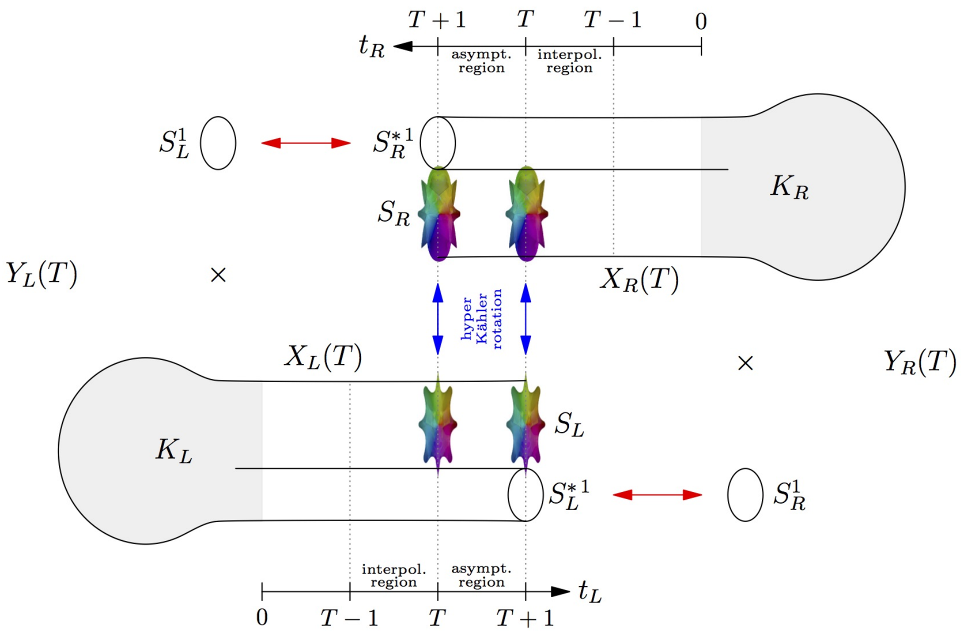

Figure 4.1: Sketch of twist connected sum

Here K_L and K_R indicate their compact regions and $X^\infty_{L/R}$ their asymptotically flat regions. Producted with the left and right circles $S^1_{\theta_{L/R}}$ with indicated radial variables $\theta_{L/R}$ they form the building blocks Y_L and Y_R, which are glued together in the following way. The product G_2 3-form and the associated 4-form on $Y_{L/R} = X_{L/R} \times S^1_{\theta_{L/R}}$ described in § 4.1.3 are expressed by

$$\varphi_{L/R} = d\theta_{L/R} \wedge \omega_{L/R} + \mathrm{Re}\Omega_{L/R},$$
$$*\varphi_{L/R} = \frac{1}{2}\omega^2_{L/R} - d\theta_{L/R} \wedge \mathrm{Im}\Omega_{L/R}.$$

Each asymptotically cylindrical Calabi-Yau $X_{L/R}$ has the circle of the cylinder $S^1_{\theta^*_{L/R}}$, where again $\theta^*_{L/R}$ is the angular variable. On each asymptotic end $X^\infty_{L/R}$ of $X_{L/R}$ which can be identified with $\mathbb{R}^{>0}_{L/R} \times S^1_{\theta^*_{L/R}} \times S_{L/R}$, where $S_{L/R}$ are K3 surfaces of $X_{L/R}$, using the diffeomorphisms $\eta_{L/R}$ we can write

$$\omega_{L/R} = \omega^\infty_{L/R} + d\mu_{L/R},$$
$$\Omega_{L/R} = \Omega^\infty_{L/R} + d\nu_{L/R}.$$

Let $\alpha : \mathbb{R} \to [0,1]$ denote a cut-off function satisfying $\alpha(t) \equiv 0$ for $t \leq 0$ and $\alpha(t) \equiv 1$ for $t \geq 1$. Fixed some $T \to \infty$, the perturbative $SU(3)$-structure on $X^{\infty}_{L/R}$ can be obtained by

$$\begin{aligned}\omega_{T,L/R} &:= \omega_{L/R} - d(\alpha(t-T+1)\mu_{L/R}),\\ \Omega_{T,L/R} &:= \Omega_{L/R} - d(\alpha(t-T+1)\nu_{L/R}).\end{aligned}$$

Thus the perturbed 3-form

$$\varphi_{T,L/R} = d\theta_{L/R} \wedge \omega_{T,L/R} + \mathrm{Re}\Omega_{T,L/R}$$

gives another G_2-structure on $X^{\infty}_{L/R}$ for large enough T since $\mu_{L/R}$ and $\nu_{L/R}$ decay to zero. Note that by construction $\mathrm{d}\varphi_{T,L/R} = 0$. However, the associated 4-form $*_{g_T}\varphi_{T,L/R} \neq \frac{1}{2}\omega^2_{T,L/R} - d\theta_{T,L/R} \wedge \mathrm{Im}\Omega_{T,L/R}$ as the metric is deformed, but their difference and $\mathrm{d}(*_{g_{\varphi_T}}\varphi_T)$ are in the order $O(e^{-\lambda' T})$ for some $\lambda' > 0$ by [Kov03, Lemma 4.25].

The circle $S^1_{\theta_L}$ is glued twisted with respect to a canonical orientation to the circle $S^1_{\theta^*_R}$ and similarly $S^1_{\theta^*_L}$ to $S^1_{\theta_R}$ as indicated by the horizontal black arrows. To preserve the characteristic forms the K3 surface S_L is glued to the K3 surface S_R up to a hyperkähler rotation, as indicated by the blue vertical arrows. The T and its by ± 1 shifted values indicate regions on the cylinder, which are important for quantifying the asymptotics of the metrics involved. The resulting G_2-manifold is called Y_r to indicate its dependence on the hyperkähler rotation r, such that $r^*\omega^I_R = \omega^J_L$, $r^*\omega^J_R = \omega^I_L$ and $r^*\omega^K_R = -\omega^K_L$. Here ω^I, ω^J and ω^K are the three associated Kähler 2-forms of K3 surface S.

Proposition 4.17 ([Kov03, Proposition 4.20])**.** *Suppose that two K3 surfaces $(S, \omega^I, \omega^J, \omega^K)$ and $(S', \omega'^I, \omega'^J, \omega'^K)$ satisfy the matching condition that there exists an isomorphism $h : H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ preserving the cup product and such that $h(\omega'^I) = \omega^J$, $h(\omega'^J) = \omega^I$, $h(\omega'^K) = \omega^K$. Then there is an isomorphism of complex surfaces $r : (S, J) \to (S', I)$, such that $f^* = h$.*

Proof. (Sketch) We have $H^{2,0}(S') = \langle \omega'^J + i\omega'^K \rangle$, $H^{0,2}(S') = \langle \omega'^J - i\omega'^K \rangle$ in (S', I) and $H^{2,0}(S) = \langle \omega^I - i\omega^K \rangle$, $H^{0,2}(S) = \langle \omega^I + i\omega^K \rangle$ in (S, J). Then using the global Torelli theorem for K3 surfaces to obtain the map $f : (S, J) \to (S', I)$. □

The spaces $S^1_{L/R} \times X_{L/R}$ are glued in their asymptotic cylinder region $S^1_{L/R} \times X^{\infty}_{L/R} = S^1_{L/R} \times \mathbb{R}^{>0}_{L/R} \times (S^1)'_{L/R} \times S_{L/R}$ with coordinates $(\varphi_{L/R}, t_{L/R}, \theta_{L/R}, x_{L/R})$ in the annulus region $t_L \in (T, T+1)$ on the $\mathbb{C}^*$ direction by a diffeomorphism F that maps the cylindrical Calabi-Yau region $S^1_R \times X^{\infty}_R$ by the action

$$(\theta_R, t_R, \theta^*_R, x_R) \mapsto (\theta^*_R, 2T + 1 - t_R, \theta_R, r(x_R)) = (\theta_L, t_L, \theta^*_L, x_L)$$

to the cylindrical Calabi-Yau region $S^1_L \times X^{\infty}_L$. Thus the G_2-structures on these region can be written

$$\begin{aligned}\varphi_{T,L/R} = {} & d\theta_{L/R} \wedge dt_{L/R} \wedge d\theta^*_{L/R} + d\theta_{L/R} \wedge \omega^S_{T,L/R}\\ & + d\theta^*_{L/R} \wedge \mathrm{Re}(\Omega^S_{T,L/R}) + dt_{L/R} \wedge \mathrm{Im}(\Omega^S_{T,L/R}),\end{aligned}$$

which is preserved under the action of the diffeomorphism F and thus the compact 7-fold with a well-defined orientation and has a family of G_2-structure φ_T induced from $X_{L/R}$, and $\mathrm{d}\varphi_T = 0$ and $\mathrm{d}(*\varphi_T) = O(e^{-\lambda' T})$ for large enough T. Afterward, Kovalev prove the existence of nearby torsion-free G_2-structures for large enough T [Kov03, Theorem 5.34], thus we have the main theorem

Theorem 4.18. *Let $X_{L/R}$ be two asymptotically cylindrical Calabi-Yau 3-folds and suppose there exists a hyper-Kähler rotation $r : S_L \to S_R$. Define closed G_2-structures $\varphi_r(T)$ on the twisted connected sum*

Y_r as above. Then for sufficiently large T, there is a torsion-free perturbation of $\varphi_r(T)$ by its cohomology class.

By the van Kampen theorem one can show that $\pi_1(Y_r) \simeq \pi_1(X_L) \times \pi_1(X_R)$, which is finite since the later factors are finite, and hence the holonomy group of the metric given by the G_2-structure on Y_r is G_2.

4.2.3 Rescaling

On the 3-dimensional Calabi-Yau cylinder $X^\infty = \mathbb{C}^* \times S$ assuming this normalization for ω_S and Ω_S of the K3 surface S, we can introduce a rescaling factor γ^* of $\frac{\mathrm{d}z}{z}$ on the algebraic torus $\mathbb{C}^*$ such that $\frac{\mathrm{d}z}{z} \mapsto \gamma^* \frac{\mathrm{d}z}{z}$, then the Kähler form and holomorphic volume form on the Calabi-Yau cylinder X^∞ can be written by

$$\begin{aligned} \omega^{\infty,\gamma^*} &= \frac{i\gamma^*\mathrm{d}z \wedge \gamma^*\mathrm{d}\bar{z}}{2z\bar{z}} + \omega_S = \gamma^*\mathrm{d}t \wedge \gamma^*\mathrm{d}\theta^* + \omega_S \\ \Omega^{\infty,\gamma^*} &= -\frac{i\gamma^*\mathrm{d}z}{z}\Omega_S = (\gamma^*\mathrm{d}\theta^* - i\gamma^*\mathrm{d}t) \wedge \Omega_S\,, \end{aligned} \tag{4.1}$$

with the metric $g_{(\Omega,\omega)} = \gamma^{*2}\mathrm{d}t^2 + \gamma^{*2}\mathrm{d}\theta^{*2} + g_{(\Omega_S,\omega_S)}$.

Under the action of the rescaling factor γ^* on the Calabi-Yau cylinder region X^∞ of the asymptotically cylindrical Calabi-Yau 3-fold (X,ω,Ω), for all $k \geq 0$, for some $\lambda > 0$ and as $t \to \infty$, the Kähler form and holomorphic volume form can be written on X^∞ as

$$\begin{aligned} \eta^*\omega - \omega^{\infty,\gamma^*} &= \gamma^{*2}\mathrm{d}\mu, \qquad \text{with} \quad |\nabla^k_{g^\infty}\mu|_\infty = O(e^{-\lambda\gamma^* t}) \\ \eta^*\Omega - \Omega^{\infty,\gamma^*} &= \gamma^*\mathrm{d}\nu, \qquad \text{with} \quad |\nabla^k_{g^\infty}\nu|_\infty = O(e^{-\lambda\gamma^* t})\,. \end{aligned} \tag{4.2}$$

for some smooth 1 form μ and 2 form ν, and the scale λ has inverse length dimension and is determined by the (inverse) length scale of the asymptotic region X^∞,

$$\lambda \;=\; \min\left\{\tfrac{1}{\gamma^*}, \lambda_S\right\}\,,$$

where λ_S is the square root of the smallest positive eigenvalue of the Laplacian of the K3 surface S in the asymptotic Calabi–Yau cylinder X^∞ as before. One can directly construct such Käher form by putting the action of γ^* into the formula of ω_g in the theorem 4.16, then choose suitable coefficients to obtain this Kähler form. Similarly, the holomorphic volume form is constructed in the similar way.

On the other hand, we also can introduce a rescaling factor γ on S^1 in the real seven manifolds $Y = S^1 \times X$, i.e., $\mathrm{d}\theta \mapsto \gamma\mathrm{d}\theta$, then the product G_2 3-form φ^γ and the associated 4-form $*\varphi^\gamma$ are expressed by

$$\varphi^\gamma = \gamma\mathrm{d}\theta \wedge \omega + \mathrm{Re}(\Omega), \qquad *\varphi^\gamma = \frac{1}{2}\omega^2 - \gamma\mathrm{d}\theta \wedge \mathrm{Im}(\Omega), \tag{4.3}$$

As the construction of the G_2 manifolds from twisted connected sum, we must glue two copies of asymptotically cylindrical Calabi-Yau 3 folds times a circle $Y_{L/R} = S^1_{L/R} \times X_{L/R}$ in the annulus region $t_{L/R} \in (T, T+1)$, and thus the G_2 3-forms $\varphi^{\gamma_{L/R},\gamma^*_{L/R}}_{L/R}$ can be written as

$$\begin{aligned} \varphi^{\gamma_{L/R},\gamma^*_{L/R}}_{L/R} &= \gamma_{L/R}\mathrm{d}\theta_{L/R} \wedge \gamma^*_{L/R}\mathrm{d}t_{L/R} \wedge \gamma^*_{L/R}\mathrm{d}\theta^*_{L/R} + \gamma_{L/R}\mathrm{d}\theta_{L/R} \wedge \omega^S_{L/R} \\ &\quad + \gamma^*_{L/R}\mathrm{d}\theta^*_{L/R} \wedge \mathrm{Re}(\Omega^S_{L/R}) + \gamma^*_{L/R}\mathrm{d}t_{L/R} \wedge \mathrm{Im}(\Omega^S_{L/R}). \end{aligned}$$

Under the action of the diffeomorphism F, the matching condition implies that $\gamma := \gamma_L = \gamma_R = \gamma^*_L = \gamma^*_R$, we thus obtain the rescaled G_2 3-forms $\varphi^{\gamma}_{L/R}$ given by

$$\begin{aligned}\varphi^{\gamma}_{L/R} = \gamma \mathrm{d}\theta_{L/R} \wedge \gamma \mathrm{d}t_{L/R} \wedge \gamma \mathrm{d}\theta^{*}_{L/R} + \gamma \mathrm{d}\theta_{L/R} \wedge \omega^{S}_{L/R} \\ + \gamma \mathrm{d}\theta^{*}_{L/R} \wedge \mathrm{Re}(\Omega^{S}_{L/R}) + \gamma \mathrm{d}t_{L/R} \wedge \mathrm{Im}(\Omega^{S}_{L/R})\end{aligned} \tag{4.4}$$

4.3 Topology of Twisted connected sum G_2-manifolds

In this section we would discuss the relations of topology between the asymptotically cylindrical Calabi-Yau 3-folds and the twisted connected sum G_2-manifolds. Indeed, we only restrict to those asymptotically cylindrical Calabi-Yau 3-folds obtained by some building blocks Z, that is, $V = Z \setminus S$ in the definition 4.14.

4.3.1 Cohomology of the building blocks

Let Z be a building block with a K3 fibration $f : Z \to \mathbb{P}^1$ and the K3 surface $S = f^{-1}(\infty) \sim -K_Z$. Since the self-intersection of S is trivial, $S.S = 0$, the normal bundle of S in Z is trivial and thus the inclusion of S into $X = Z \setminus S$ is well-defined up to homotopy. Hence the restriction map $H^m(Z) \to H^m(S) = L$ factors through $H^m(X) = H$, that is, the diagram commute

$$\begin{tikzcd} H^m(Z) \arrow[r] \arrow[dr] & H^m(X) \arrow[d] \\ & H^m(S). \end{tikzcd}$$

We denote by $\rho : H \to L$ the natural restriction map, $K = \ker(\rho)$ the kernel of ρ, and $N = \rho(H) \subset L$ the image of $H^2(Z)$ in $H^2(S)$. N is a primitive sublattice of L, an unimodular K3 lattice , and we define a transcendental lattice T by

$$T = N^{\perp} = \{l \in L \mid \langle l, n\rangle = 0 \quad \text{for all } n \in N\},$$

and thus $L/T \simeq N^* \neq N$. On a polarized K3 surface, N and T stand for the Picard and transcendental lattices, respectively.

Lemma 4.19 ([Cor+13, Lemma 5.3])**.** *Let (Z, S) be a building block, then we have*

(i) $\pi_1(X) = H^1(X) = 0$.

(ii) *there is a split exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{[S]} H^2(Z) \longrightarrow H^2(X) \longrightarrow 0,$$

thus $H^2(Z) \simeq \mathbb{Z}[S] \oplus H^2(X)$, and the image of $H^2(Z) \to L$ is mapped to N.

(iii) *there is a split exact sequence*

$$0 \longrightarrow H^3(Z) \longrightarrow H^3(X) \longrightarrow T \longrightarrow 0,$$

thus $H^3(X) \simeq H^3(Z) \oplus T$.

(iv) *there is a split exact sequence*

$$0 \longrightarrow N^* \longrightarrow H^4(Z) \longrightarrow H^4(X) \longrightarrow 0,$$

thus $H^4(Z) \simeq H^4(X) \oplus N^*$.

(v) $H^5(X) = 0$.

Proof. (Sketch) Let $i : S \to Z$ be the closed immersion and $j : X = Z \setminus S \to Z$ be the open immersion. Then using the distinguished triangle for the constant sheaf $\mathbb{Z}$ [GM02, IV]

$$i_* i^* \mathbb{Z}[-2] \longrightarrow \mathbb{Z} \longrightarrow Rj_* j^* \mathbb{Z} \xrightarrow{+1} \cdots,$$

and taking the long exact sequence, it turns out that

$$0 \longrightarrow H^0(S) \longrightarrow H^2(Z) \longrightarrow H^2(X) \longrightarrow H^1(S) = 0,$$

as $H^1(X) = H^1(Z) = 0$. Since the inclusion $H^0(S) = \mathbb{Z}[S] \sim [-K_Z] \to H^2(Z)$ is primitive, hence this sequence splits.

The higher piece of the long exact sequence gives

$$0 \longrightarrow H^3(Z) \longrightarrow H^3(X) \longrightarrow L \longrightarrow H^4(Z) \longrightarrow H^4(X) \longrightarrow 0.$$

Since the image of $H^2(Z) \to H^2(S)$ is $N \subset L$, the kernel of $H_2(S) \to H_2(Z)$ is $T \subset L^* \simeq L$ which is also the kernel of the map $H^2(S) \to H^4(Z)$, the Poincaré dual of $H_2(S) \to H_2(Z)$. As T is torsion-free, the exact sequence (iii) splits. The inclusion $L/T \simeq N^* \hookrightarrow H^4(Z)$ is primitive, so the sequence (iv) also splits. The condition (v) follows immediately from the last piece of the long exact sequence. □

Corollary 4.20. *The dual statements for the homology group* $H_*(X)$ *are*

(i) $H_1(X) = 0$.

(ii) *there is a split exact sequence*

$$0 \longrightarrow H_2(X) \longrightarrow H_2(Z) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

(iii) *there is a split exact sequence*

$$0 \longrightarrow T^* \longrightarrow H_3(X) \longrightarrow H_3(Z) \longrightarrow 0.$$

(iv) $H_4(X) = K$.

(v) $H_5(X) = 0$.

4.3.2 Cohomology of the G_2-manifolds

For the construction of twisted connected sums in § 4.2.2, the topological data of Y_r can be inferred to a large extent from those asymptotically cylindrical Calabi-Yau 3-folds $(X_{L/R}, S_{L/R})$. Indeed, we have the decomposition $Y_r = Y_L \cup Y_R$ with the common intersection $Y_L \cap Y_R \simeq S \times S^1 \times S^1 \simeq S \times T^2$. To compute the cohomology group and the Betti numbers, we start from the short exact sequence of forms

$$0 \longrightarrow C^n(Y_r) \xrightarrow{r^*} C^n(Y_L) \oplus C^n(Y_R) \xrightarrow{\iota^*} C^n(Y_L \cap Y_R) \longrightarrow 0\ .$$

The maps r^* and ι^* on cochains are induced from the map $r = (r_L, r_R)$ arising from the restrictions $r_{L/R} : Y_{L/R} \to Y_r$ and the map $\iota = (\iota_L, \iota_R)$ given by the inclusions $\iota_{L/R} : Y_L \cap Y_R \hookrightarrow Y_{L/R}$, such that $r^* = r_L^* \oplus -r_R^*$ and $\iota^* = \iota_L^* + \iota_R^*$. Then the induced Mayer–Vietoris sequence reads

$$\cdots \to H^{n-1}(Y_L \cap Y_R) \xrightarrow{\delta} H^n(Y_r) \xrightarrow{r^*} H^n(Y_L) \oplus H^n(Y_R) \xrightarrow{\iota^*} H^n(Y_L \cap Y_R) \to \cdots\ , \tag{4.5}$$

in terms of r^* and ι^* and the coboundary map δ.

From the lemma 4.19, we see that $H^n(Y_{L/R})$ are torsion-free, hence the sequence (4.5) splits in the following sense

$$H^n(Y_r) \simeq \mathrm{Im}(r^{*,n}) \oplus \ker(r^{*,n}) \simeq \ker(\iota^{*,n}) \oplus \mathrm{coker}(\iota^{*,n-1}). \tag{4.6}$$

By the very construction $Y_{L/R} = X_{L/R} \times S^1_{L/R}$ and $Y_L \cap Y_R \simeq S \times T^2$, the Künneth formula implies the decomposition of $H^*(Y_{L/R})$ as

$$H^n(Y_{L/R}) \simeq H^n(X_{L/R}) \oplus H^{n-1}(X_{L/R}),$$
$$H^n(Y_L \cap Y_R) \simeq H^n(S) \oplus H^{n-1}(S) \oplus H^{n-1}(S) \oplus H^{n-2}(S).$$

Due to $\pi(X_{L/R}) = 0$, the van Kampen theorem for the decomposition $Y_r = Y_L \cup Y_R$ with $Y_L \cap Y_R \simeq S \times T^2$ implies $\pi_1(Y_r) = H^1(Y_r) = 0$. At $n = 1$, $H^1(Y_L) \oplus H^1(Y_R) = H^0(X_L) \oplus H^0(X_R)$ and $H^1(Y_L \cap Y_R) = H^0(S) \oplus H^0(S)$, then $\iota^{*,1} : H^0(X_L) \oplus H^0(X_R) \to H^0(S) \oplus H^0(S)$ is a natural isomorphism, and thus $\mathrm{coker}(\iota^{*,1}) = 0$. Therefore, the second cohomology of Y_r is given by

$$H^2(Y_r) = \ker\left(H^2(Y_L) \oplus H^2(Y_R) \xrightarrow{\iota^{*,2}} H^2(Y_L \cap Y_R)\right)\ .$$

Here $H^2(Y_L) \oplus H^2(Y_R) = H^2(X_L) \oplus H^2(X_R)$ and $H^2(Y_L \cap Y_R) = H^2(S) \oplus H^0(S) = L \oplus \mathbb{Z}[S]$. The first part consists of the sum of the individual kernels $K_{L/R}$ of the maps $\iota^{*,2}_{L/R} = \rho_{L/R}$. The second part constitute the cohomology elements of $H^2(Y_L \cap Y_R)$ that are in both images $N_{L/R} = \mathrm{im}\,\rho_{L/R}$. That is to say elements in the kernel $\iota^{*,2}$ arising from the intersection $N_L \cap N_R$. Together we have a split exact sequence

$$0 \longrightarrow K_L \oplus K_R \longrightarrow H^2(Y_r) \longrightarrow N_L \cap N_R \longrightarrow 0,$$

since $N_L \cap N_R$ is torsion-free. It means that we obtain the second cohomology group

$$H^2(Y_r) \ = \ (K_L \oplus K_R) \oplus (N_L \cap N_R)\ . \tag{4.7}$$

Note that the images of $N_{L/R}$ lie in the Picard lattices of the K3 surface S polarized with respect to the complex structures from the Calabi–Yau cylinders $X_{L/R}$. We assume that $N_{L/R}$ are both primitive sublattices of the K3 lattice $L \simeq H^2(S, \mathbb{Z})$.

In a similar fashion we can work out the third cohomology classes with the long exact sequence as

$$\begin{aligned} H^3(Y_r) \;&=\; \text{coker}\left(H^2(Y_L) \oplus H^2(Y_R) \xrightarrow{\iota^{*,2}} H^2(Y_L \cap Y_R)\right) \oplus \\ &\quad \ker\left(H^3(Y_L) \oplus H^3(Y_R) \xrightarrow{\iota^{*,3}} H^3(Y_L \cap Y_R)\right) , \end{aligned}$$

which becomes

$$\begin{aligned} H^3(Y_r) \;&=\; (\mathbb{Z}[S] \oplus L/(N_L + N_R)) \oplus \\ &\quad \left(K_L \oplus K_R \oplus H^3(Z_L) \oplus H^3(Z_R)\right) \oplus (N_L \cap T_R \oplus N_R \cap T_L) \ . \end{aligned} \tag{4.8}$$

The first line are the contributions from the cokernel. They are the induced 2-cocycle generator $\mathbb{Z}[S]$ in $Y_L \cap Y_R \sim T^2 \times S$ and the cokernel elements from $H^2(S)$. The second line furnish the elements in the kernel of $\iota^{*,3}$, which again split into two contributions. Here,

$$\begin{aligned} H^3(Y_L) \oplus H^3(Y_R) \simeq H^3(X_L) \oplus H^2(X_L) \oplus H^3(X_R) \oplus H^2(X_R), \\ H^3(Y_L \cap Y_R) \simeq H^2(S) \oplus H^2(S). \end{aligned}$$

We have those cohomology classes that are in the individual kernels of $\iota^{*,3}_{L/R}$ and those cohomology classes constructed from the intersecting images of $\iota^{*,3}_{L/R}$. The former cohomology classes are identified with $K_{L/R}$ — arising from the product of 2-cocyles in $H^2(X_{L/R})$ and the 1-cocycle generator of $H^1(S^1)$ in $Y_{L/R} \sim S^1 \times X_{L/R}$ — and the 3-cocycle cohomology elements induced from $H^3(Z_{L/R})$ as $H^3(X_{L/R}) \simeq H^3(Z_{L/R}) \oplus T_{L/R}$ in the Lemma 4.19.

However, there are additional 3-cocyle cohomology elements in $H^3(X_{L/R})$, which arise from removing the canonical divisor in $Z_{L/R}$. These cocycles are by construction non-trivial in the asympototic region $Y_L \cap Y_R$ mapping to 2-cocyles in S times the generating 1-cocycle in the asymptotic S^1. These 2-cocyles in S form sublattices $T_{L/R}$. Hence, as before in the long exact sequence argument for globally non-trival 2-cocyle cohomology classes, the intersection $N_L \cap T_R$ and $N_R \cap T_L$ gives rise to additional globally non-trivial three form cohomology elements. Thus we have the following split short exact sequence

$$0 \longrightarrow H^3(Z_{L/R}) \oplus K_{L/R} \longrightarrow \ker\left(\iota^{*,3}_{L/R}\right) \longrightarrow T_{L/R} \cap N_{R/L} \longrightarrow 0,$$

and the decomposition for $H^3(Y_r)$ in (4.8) follows.

To sum up, we have the following theorem

Theorem 4.21. *Let Y_r be a twisted connected sum G_2-manifold constructed from two compatible asymptotically cylindrical Calabi-Yau 3-folds $X_{L/R} = Z_{L/R} \setminus S_{L/R}$. Then*

(i) $\pi_1(Y_r) = H^1(Y_r) = 0$.

(ii) $H^2(Y_r) \simeq (K_L \oplus K_R) \oplus (N_L \cap N_R)$.

(iii) $H^3(Y_r) \simeq \mathbb{Z}[S] \oplus L/(N_L + N_R) \oplus K_L \oplus K_R \oplus H^3(Z_L) \oplus H^3(Z_R) \oplus N_L \cap T_R \oplus N_R \cap T_L$.

For the complete decomposition of the cohomology group of the G_2-manifold Y_r and more detailed discussion about characteristic classes on Y_r, see [Cor+15, §4].

4.4 Explicit examples

In the section, we will review a general and useful method to construct some explicit examples called *orthogonal gluing* in the work [Cor+15] that provides large number of matching asymptotically Calabi-Yau structures which can be used to construct various topologically different G_2-manifolds Y_r with nontrivial $H^2(Y_r)$, and then discuss additional method of *non-generic gluing* such that the resulted Y_r has nontrivial $K_{L/R}$ contributed to $H^2(Y_r)$ and $H^3(Y_r)$.

4.4.1 Orthogonal gluing

Given the pair of primitive embeddings $N_{L/R} \hookrightarrow L$ with signature $(1, r_{L/R} - 1)$ so that N_L and N_R intersect orthogonally in the sense that $N_{L/R}(\mathbb{R}) = \left(N_{L/R}(\mathbb{R}) \cap N_{R/L}(\mathbb{R})\right) \oplus \left(N_{L/R}(\mathbb{R}) \cap T_{R/L}(\mathbb{R})\right)$, where $T_{L/R} = N_{L/R}^{\perp}$ denote the transcendental lattices, and some elements of $N_{L/R}(\mathbb{R}) \cap T_{R/L}(\mathbb{R})$ correnspond to the Kähler classes of some (semi-)Fano 3-folds $Z_{L/R}$.

Definition 4.22 ([Cor+15, Definition 6.17]). Let $\mathcal{Z}$ be a family of of semi-Fano type building blocks and $\mathrm{Amp}_{\mathcal{Z}}$ be an open subcone of the positive cone in $N_{\mathbb{R}}$. A family $\mathcal{Z}$ is called $(N, \mathrm{Amp}_{\mathcal{Z}})$-*generic* if for any $\Pi \in U_{\mathcal{Z}} \subset D_N = \left\{\Pi \in N_{\mathbb{C}} \mid \Pi \wedge \overline{\Pi} > 0\right\}$ and $k \in \mathrm{Amp}_{\mathcal{Z}}$ there is a building block $(Z, S) \in \mathcal{Z}$ and a marking $h : L \to H^2(S; \mathbb{Z})$ such that $h(\Pi) = H^{2,0}(S; \mathbb{Z})$ and $h(k)$ is the image of the restriction of a Kähler class of Z to S.

Proposition 4.23 ([Cor+15, Proposition 6.18]). *Let $\mathcal{Z}_{L/R}$ be $(N_{L/R}, \mathrm{Amp}_{\mathcal{Z}_{L/R}})$-generic families of semi-Fano type building blocks. Suppose that*

(i) *$R = N_L \cap N_R$ is negative definite of rank ρ,*

(ii) *$W = N_L + N_R$ is an orthogonal pushout, i.e., a non-degenerate integral lattice,*

(iii) *$W_{L/R} \cap \mathrm{Amp}_{\mathcal{Z}_{L/R}} \neq \emptyset$, where $W_{L/R} = T_{R/L} \cap N_{L/R}$ are the perpendicular of $N_{R/L}$ in $N_{L/R}$.*

Then one can find a pair of building blocks $(Z_{L/R}, S_{L/R}) \in \mathcal{Z}_{L/R}$ satisfying the matching condition which can solve the matching problem of twisted connected sum construction of G_2 manifolds for semi-Fano 3-folds.

To find suitable embeddings $N_{L/R} \hookrightarrow L$ fulfilling orthogonal gluing, we need an additional integral lattice W of N_L and N_R called *orthogonal push-out*. Let $R = N_L \cap N_R$ be a nondegenerate lattice with given primitive inclusions $R \hookrightarrow N_L$, $R \hookrightarrow N_R$. An orthogonal push-out $W = N_L \perp_R N_R$ is a nondegenerate lattice such that $W = N_L + N_R$ and $N_{L/R}^{\perp} \subset N_{R/L}$. However, in general W is not a primitive sublattice of L. If W can be primitively embedded into L, the existence of primitive embedding $N_{L/R} \hookrightarrow L$ could be deduced from results of Nikulin [Nik79]. Once W do exist, it would be unique. In our application L is a $K3$ lattice and $N_{L/R}$ are the polarising lattices of a pair of building blocks $Z_{L/R}$. A sufficient condition for the existence of a primitive embedding $W \hookrightarrow L$ is that

$$\mathrm{rk}N_L + \mathrm{rk}N_R \leq 11\ . \tag{4.9}$$

From the decomposition of $H^3(Y_r)$ in (4.8), it turns out that $\mathrm{Tor}(H^3(Y_r)) \simeq \mathrm{Tor}\left(L/(N_L + N_R)\right) = 0$ if the embedding $W \hookrightarrow L$ is primitive. For more general statement for even nondegenerate lattices, one can refer to [Cor+15, Theorem 6.9].

Furthermore, there is a nice property relating Betti numbers of G_2 manifolds and of building blocks which can be easily deduced from the theorem 4.21.

Corollary 4.24. *For any* G_2*-manifold* Y_r *constructed by the orthogonal gluing of the building blocks* $Z_{L/R}$*, its Betti numbers satisfies the following relation*

$$b^2(Y_r) + b^3(Y_r) = b^3(Z_L) + b^3(Z_R) + 2\mathrm{rk}K_L + 2\mathrm{rk}K_R + 23 \,. \tag{4.10}$$

Note that this formula is not always valid if Y_r is not constructed by the orthogonal guling, and $K_{L/R} = 0$ if these building blocks $Z_{L/R}$ are obtained from some semi-Fano 3-folds (see the proposition 4.15).

Let us start to study some concrete examples of G_2-manifolds by the construction of the orthogonal gluing with nontrivial intersection $R = N_L \cap N_R \neq \emptyset$, and compute the cohomology group and intersection matrix of each building block. We glue orthogonally two building blocks $Z_{L/R}$ obtained from blowing up two Fano 3-folds $W_{L/R}$ in the base locus of a generic anticanonical pencil, both with Picard numbers $\rho = 2$.

Example 4.25. Consider the rank two Fano 3-folds which are No. 2 as W_L and No. 24 as W_R obtained from the Mori-Mikai list [MM81]. W_L is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2,4)$ and the projection of the Cartesian product $\mathbb{P}^1 \times \mathbb{P}^2$ onto two factors $\mathbb{P}^1, \mathbb{P}^2$ are denoted by p, q, respectively. That is

$$p : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1, \quad q : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2.$$

For a double cover over a divisor, $b^2(W_L) = b^2(\mathbb{P}^1 \times \mathbb{P}^2) = 2$. Hence $\mathrm{Pic}(W_L) = \mathbb{Z}\pi^* h_1 + \mathbb{Z}\pi^* h_2$, where $h_1 = p^* c_1(\mathscr{O}_{\mathbb{P}^1}(1)), h_2 = q^* c_1(\mathscr{O}_{\mathbb{P}^2}(1))$ and π is the morphism of double cover $W_L \to \mathbb{P}^1 \times \mathbb{P}^2$. The Euler characteristic $\chi(W_L) = 2\chi(\mathbb{P}^1 \times \mathbb{P}^2) - \chi(D) = 2 \cdot 6 - 46 = -34$, $D = 2h_1 + 4h_2$ is the branch locus. Thus, $b^3(W_L) = 40$. Let $R(\pi) = \pi^* h_1 + 2\pi^* h_2$ be the ramification locus, i.e., $\pi_* R(\pi) = D$, then the anti-canonical divisor $-K_{W_L} = -K_{\mathbb{P}^1 \times \mathbb{P}^2} - R(\pi) = \pi^* h_1 + \pi^* h_2$ and $(-K_{W_L})^3 = 6$. Therefore, the Picard lattice N_L of W_L computed in the basis $\pi^* h_1, \pi^* h_2$ is

$$N_L = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}.$$

We choose a rational basis $\left\{A_L, R = A_L^{\perp}\right\}$, where $A_L = -K_{V_L} = \pi^* h_1 + \pi^* h_2$, $R = 2\pi^* h_1 - \pi^* h_2$, such that $\mathrm{Pic}(V_L) = \mathbb{Z}A_L + \mathbb{Z}R + \frac{1}{3}(A_L + R)\mathbb{Z}$ and Picard lattice $N_{L;\mathbb{Q}}$ thus becomes

$$N_{L;\mathbb{Q}} = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}.$$

W_R is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$, and p and q are the first and second projections of $\mathbb{P}^2 \times \mathbb{P}^2$ to two factors, respectively. Then $\mathrm{Pic}(W_R) = \mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)|_{W_R} = \mathbb{Z}h_{1,W_R} + \mathbb{Z}h_{2,W_R}$, where $h_{1,W_R} = p^* \mathscr{O}_{\mathbb{P}^2}(1)|_{W_R}, h_{2,W_R} = q^* \mathscr{O}_{\mathbb{P}^2}(1)|_{W_R}$, and $W_R = h_{1,W_R} + 2h_{2,W_R}$. The Euler characteristic is $\chi(W_R) = 6$, and the anti-canonical divisor $-K_{W_R} = 2h_{1,W_R} + h_{2,W_R}$. Hence $b^3(W_R) = 0$ and $(-K_{W_R})^3 = 30$. Again, choose a rational basis $\{A_R, R\}$, where $A_R = h_{1,W_R} + h_{2,W_R}$ and $R = h_{1,W_R} - h_{2,W_R}$, such that $\mathrm{Pic}(W_R) = \mathbb{Z}A_R + \mathbb{Z}R + \frac{1}{2}(A + R)\mathbb{Z}$. The Picard lattices $N_R, N_{R;\mathbb{Q}}$ are

$$N_R = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}, \qquad N_{R;\mathbb{Q}} = \begin{pmatrix} 14 & 0 \\ 0 & -6 \end{pmatrix}.$$

We can form a G_2 manifold Y_r with $H^2(Y_r) = \mathbb{Z}R = N_{L;\mathbb{Q}} \cap N_{R;\mathbb{Q}}$ by identifying a sublattice $\mathbb{Z}R$ of the

Picard lattices $N_{L;\mathbb{Q}}$ and $N_{R;\mathbb{Q}}$. Thus the orthogonal pushout $W = N_L + N_R$ exists, given by

$$W = \mathbb{Z}A_L + \mathbb{Z}A_R + \mathbb{Z}R + \frac{1}{3}(A_L + R)\mathbb{Z} + \frac{1}{2}(A_R + R)\mathbb{Z}$$

with the quadratic form

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 14 \end{pmatrix}.$$

One can easily check that $\langle \frac{1}{3}(A_L + R), \frac{1}{2}(A_R + R)\rangle = -1$, hence W is an integral lattice as we desire, and the conditions in the Proposition 4.23 are fulfilled so that we actually obtain a twisted connected sum G_2-manifold Y_r with $H^2(Y_r) \simeq \mathbb{Z}$. Note we also can choose the intersection lattice R to be empty, and obtain another G_2-manifold with $b^2 = 0$.

One important class of toric building blocks can be constructed as follows. Take $P = \mathbb{P}_{\Delta^{(3)}}$, where $\Delta^{(3)}$ is part of a reflexive pair of three dimensional lattice polyhedra $(\Delta^{(3)}, \Delta^{(3)*})$ embedded in the lattice $\Gamma \sim \mathbb{Z}^3$ and its dual lattice Γ^* respectively. By definition

$$\Delta^* = \{x \in \Gamma^*_{\mathbb{R}} \mid \langle x, y\rangle \geq -1, \forall\, y \in \Delta\}$$

and $(\Delta^*)^* = \Delta$. There are 4319 pairs of reflexive polyhedra in 3 dimension[1]. Note that the projective toric variety associated to a reflexive polytope is a Gorenstein Fano variety by [CLS11, Theorem 8.3.4]. For the further construction the asumption is made in [Cor+13] that P is semi Fano with a suitable triangulation, which is for $\mathbb{P}_{\Delta^{(3)}}$ equivalent to the fact that no points lie inside codimension one faces of $\Delta^{(3)}$ [Cor+13]. To be precise, using a triangulation [2] of $\Delta^{(3)}$, P has a toric projective semi-Fano resolution $P' \to P$ resolving the ordinary nodes of P. Then the building block Z is the blow-up along the base locus of a generic anti-canonical pencil.

One of the central points in Batyrevs construction of mirror symmetry is that for $n \leq 4$ there are generically smooth sections of the anti-canonical bunble $|-K_P|$ given in toric coordinates Y_i by

$$W_{\Delta^{(n)}} = \sum_{\nu^{(i)} \in \Delta^{(n)}} a_i \prod_{\nu^{*(k)} \in \Delta^{(n)*}} Y_k^{\langle \nu^{(i)}, \nu^{*(k)}\rangle + 1} = 0\,.$$

In particular for $n = 3$ this section yields a generically smooth $K3$ surface S.

Example 4.26 ([Cor+13, Example 7.10])**.** Let P be a terminal Gorenstein Fano 3-fold associated to the reflexive self-polar polytope with vertices

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix},$$

which is polytope 1942 in Kreuzer–Skarke's database. This polytope and its fan picture can be viewed in Sage and both are the same due to self-polar of the polytope. The anti-canonical divisor of P' with

[1] The list constructed by Kreuzer and Skarke as a training example can be found at http://hep.itp.tuwien.ac.at/~kreuzer/CY/.

[2] Different triangulation leads to different intersections in the Picard lattice.

$H^2(P',\mathbb{Z}) \simeq \mathbb{Z}^{10}$ is the boundary surface of the polytope

$$-K_{P'} = \sum_{i=1}^{9} Q_i + \sum_{j=1}^{4} R_j,$$

where $Q_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ related to a standard parallelogram and $R_j \simeq \mathbb{P}^2$ related to a standard triangle. Moreover, $-K_{P'}|_{Q_i} \simeq \mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathscr{O}_{\mathbb{P}^1}(1)$ and $-K_{P'}|_{R_j} \simeq \mathscr{O}_{\mathbb{P}^2}(1)$, hence the degree of $-K_{P'}$ is $2 \cdot 9 + 4 = 22$ and thus the genus of the base locus curve C is 12. It turns out that $H^3(Z) \simeq H^3(P') \oplus H^1(C) \simeq \mathbb{Z}^{24}$.

Since a generic $S \in |-K_{P'}|$ is an elliptic K3 surface, those curves $S \cap Q_i$ and $S \cap R_j$ are (-2) rational curves, and the dual graph of those (-2) curves also looks like the same polytope above. Compared with Dynkin diagrams one can find an unimodular $E_8(-1)$ lattice and the Picard lattice N of S thus can be decomposed as $N \simeq E_8(-1) \oplus E_8(-1)^{\perp}$. Consider elliptic fibrations on S, we obtain one elliptic fibration with fibers of type $\widetilde{D}_4$ and of type $\widetilde{A}_3$, another with fibers of type $\widetilde{A}_3$ and of type $\widetilde{A}_3$, and the other with fibers of type $\widetilde{A}_2$ and of $\widetilde{A}_3$, which give three relations in $N = \mathrm{Pic}(S)$. Those relations can be used to determine a basis of $E_8(-1)^{\perp}$, and under a small change of coordinates the intersection matrix can be written as

$$N = E_8(-1) \perp \langle -8 \rangle \perp \langle 16 \rangle.$$

We thus can match two copies of $Z_{L/R}$ under perpendicular gluing, i.e., $R = N_L \cap N_R = \emptyset$, by choose primitive embedding of $2 \times \{\langle -8 \rangle \perp \langle 16 \rangle\}$ in $3U \subset L$. Then we have an embedding of $N_L \perp N_R$ in the K3 lattice $L = 2E_8(-1) \perp 3U$ by embedding $\langle 8 \rangle \perp \langle -16 \rangle \perp \langle -8 \rangle \perp \langle 16 \rangle$ in $3U$, $E_8(-1)_L$ in the first copy of $E_8(-1)$ and $E_8(-1)_R$ in the second copy of $E_8(-1)$.

Example 4.27. Consider the terminal Gorenstein Fano 3-fold associated to the reflexive polytope of the fan picture with vertices

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & -1 \end{pmatrix},$$

which is polytope 634 in Kasprzyk's database[3] [Kas06], or 2355 in Kreuzer–Skarke's database. Note that the vertices in Kreuzer–Skarke's database are expressed in PALP form of the dual fan. This polytope is not self-polar and thus in the following we only stay in the fan picture. The anti-canonical divisor is the sum of the vertices

$$-K_{P'} = \sum_{i=1}^{8} R_i + \sum_{j=1}^{6} Q_j,$$

where $R_i \simeq \mathbb{P}^2$ related to the first eight vertices and $Q_j \simeq \mathbb{P}^1 \times \mathbb{P}^1$ related to the remaining ones. As before the degree of $-K_{P'} = 8 + 2 \cdot 6 = 20$ and the genus of the base locus curve C is 11 so that $H^2(Z) \simeq \mathbb{Z}^{22}$.

As the Picard rank of the generic K3 surface $S \in |-K_{P'}|$ is 11, the rational curves $S \cap R_i$ and $S \cap Q_j$ are (-2) curves and the dual graph of (-2) curves is also given by this polytope. It turns out that the sublattice of type $E_8(-1)$ is generated by the curves $R_1, Q_4, R_3, Q_1, Q_5, R_2, Q_3, R_7$ and $N = E_8(-1) \perp E_8(-1)^{\perp}$ with

[3] The graded ring database constructed by A. Kasprzyk can be found at http://www.grdb.co.uk/Index.

rk $\left(E_8(-1)^\perp\right) = 3$. Now the elliptic fibrations on S provide the relations in $\mathrm{Pic}(S) = N$ as below

$$\begin{aligned} R_1 + R_8 + Q_4 + Q_2 &= R_5 + Q_1 + Q_3 + R_4 \\ R_8 + Q_2 + Q_5 + R_2 &= R_6 + Q_1 + R_4 + Q_6 \\ R_8 + Q_5 + R_3 + Q_4 &= R_4 + Q_6 + Q_3 + R_7. \end{aligned}$$

We then have the relations modulo $E_8(-1)$:

$$\begin{aligned} R_8 + Q_2 - R_4 - R_5 &\equiv 0 \\ R_8 + Q_2 - R_4 - R_6 - Q_6 &\equiv 0 \\ R_8 - R_4 - Q_6 &\equiv 0, \end{aligned}$$

and hence R_4, R_6, Q_6 is a basis of N mod $E_8(-1)$. Therefore, the basis of $E_8(-1)^\perp$ is given by the vectors:

$$\begin{aligned} &R_4 + 9R_1 + 18Q_4 + 27R_3 + 14Q_1 + 22Q_5 + 17R_2 + 12Q_3 + 6R_7 \\ &Q_6 + 6R_1 + 11Q_4 + 16R_3 + 8Q_1 + 13Q_5 + 10R_2 + 7Q_3 + 4R_7 \\ &R_6 + 12R_1 + 24Q_4 + 35R_3 + 18Q_1 + 28Q_5 + 21R_2 + 14Q_3 + 7R_7, \end{aligned}$$

and the intersection matrix in the basis is computed to be

$$\begin{pmatrix} 24 & 16 & 32 \\ 16 & 8 & 20 \\ 32 & 20 & 40 \end{pmatrix}.$$

We can choose some building blocks with the Picard number equal to or smaller than 9 to form a compact G_2-manifold under the perpendicular gluing by choosing suitable primitive embeddings of $N_{L/R}$ in $L = 2E_8(-1) \oplus 3U$.

Example 4.28 (Toric semi-Fano 3-fold with Picard rank 2)**.** The corresponding toric terminal Fano 3-fold P is the projective cone in $\mathbb{P}^4$ over a non-singular quadric with vertices

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

which is reflexive polytope 32 in Kasprzyk's database (K32). P is a Gorenstein terminal Fano 3-fold with Picard rank 1, degree 54 and 1 ordinary double point. Then $P' \to P$ is the unique smooth toric semi-Fano 3-fold with Picard rank 2 and the quadratic form of the Picard lattice N is

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

with the discriminant $\Delta = -9$. Note that the anti-canonical divisor is $-K_{P'} = (3, 3)$. Now we choose the ample class $A = (1, 2)$ with $A^2 = 12$ and an orthogonal complement to A in the Picard lattice $e = (-1, 2)$ with $e^2 = -12$.

Given a pair (N_L, N_R) of Picard lattices with Picard rank 2, the condition of the existence of the orthogonal gluing such that the pushout integral lattice W can be primitively embedded into a K3 lattice is equivalent to $e_L^2 = e_R^2$ and $\frac{\Delta_L \Delta_R}{A_L^2 A_R^2}$ is a perfect square [CN14, Lemma 5.8]. Compared to the table 2 in [CN14], we can glue P' and the smooth Fano 3-fold obtained by blowing up of $\mathbb{P}^3$ along an elliptic curve that is the intersection of two quadrics in the Mori-Mukai list ♯25 (MM25). Note that MM5 and MM25 also can be glued together by the orthogonal gluing which is not contained in the article [CN14]. To sum up, we make the table below

No.	$-K^3$	N	Δ	A	e	A^2	e^2	$b^3(Z)$
K32	54	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	-9	(1,2)	(-1,1)	12	-12	56
MM5	12	$\begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix}$	-9	(1,1)	(3,-1)	12	-12	26
MM25	32	$\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$	-16	(1,1)	(2,-1)	12	-12	34

Table 4.2: Rank 2 blocks.

By the condition $\frac{\Delta_L \Delta_R}{A_L^2 A_R^2} = k^2$ for some $k \in \mathbb{Z}$, the allowed matching pairs are (K32, MM25) and (MM5, MM25).

Example 4.29. Consider the Fano 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as Z_L with the Picard lattice N_L of rank 3 generated by $\mathscr{O}(1)_1, \mathscr{O}(1)_2, \mathscr{O}(1)_3$ related to the first, second and third factor, respectively. So the anti-canonical divisor is $-K_{Z_L} = 2\mathscr{O}(1)_1 + 2\mathscr{O}(1)_2 + 2\mathscr{O}(1)_3$ and the intersection matrix on N_L can be computed to be

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

We choose the ample classes $A = m\mathscr{O}(1)_1 + m\mathscr{O}(1)_2 + n\mathscr{O}(1)_3$ for $m, n \geq 1$ and $\gcd(n, m) = 1$, and thus the intersection numbers of A and $\{\mathscr{O}(1)_1, \mathscr{O}(1)_2, \mathscr{O}(1)_3\}$ on $-K_{Z_L}$ are given by $\langle 2m + 2n\rangle \perp \langle 2m + 2n\rangle \perp \langle 4m\rangle$. Thus the basis orthogonal to the ample class A are

$$\begin{aligned} e_1 &:= \mathscr{O}(1)_1 - \mathscr{O}(1)_2, \\ e_2 &:= 2m\mathscr{O}(1)_2 - (n+m)\mathscr{O}(1)_3 \quad \text{for} \quad n + m = 3, 5, \cdots, \\ e_2 &:= m\mathscr{O}(1)_2 - \frac{n+m}{2}\mathscr{O}(1)_3 \quad \text{for} \quad n + m = 2, 4, \cdots, \end{aligned}$$

with the quadratic forms

$$\begin{pmatrix} -4 & 4m \\ 4m & -8m(n+m) \end{pmatrix}_{n+m=3,5,\cdots}, \qquad \begin{pmatrix} -4 & 2m \\ 2m & -2m(n+m) \end{pmatrix}_{n+m=2,4,\cdots}.$$

We can choose the orthogonal basis $e_1 \perp e_2'$ with $e_2' = m\mathscr{O}(1)_1 + m\mathscr{O}(1)_2 - (n+m)\mathscr{O}(1)_3$, so that the self-intersection number is $e_2'.e_2' = -4m(2n+m)$. Note that n and m are co-prime which implies m and $n + m$ are also co-prime. In the basis A, e_1, e_2', the generators of N_L can be expressed as

$$\begin{aligned} 2m(2n+m)\mathscr{O}(1)_1 &= (n+m)A + ne_2' + m(2n+m)e_1, \\ 2m(2n+m)\mathscr{O}(1)_2 &= (n+m)A + ne_2' - m(2n+m)e_1, \\ (2n+m)\mathscr{O}(1)_3 &= A - e_2', \end{aligned}$$

hence

$$N_L = \mathbb{Z}^3 + \frac{1}{2m(2n+m)}\left((n+m)A + ne_2' + m(2n+m)e_1\right)\mathbb{Z}$$
$$+ \frac{1}{2m(2n+m)}\left((n+m)A + ne_2' - m(2n+m)e_1\right)\mathbb{Z} + \frac{1}{2n+m}\left(A - e_2'\right)\mathbb{Z}. \tag{4.11}$$

Now if the building block Z_R is obtained from the Fano 3-folds with Picard lattice N_R of rank 2 of No. 6, 12, 21 and 32 in the Mori-Mukai list such that the Picard lattice is of the form

$$N_R = \mathbb{Z}^2 + \frac{1}{2}(A_R + e)\mathbb{Z} \quad \text{with} \quad e.e = -4. \tag{4.12}$$

Here A_R is the ample class in Z_R (see [Cor+15; CN14]). Then we can form a compact G_2-manifold by identifying e_1 and e, i.e., $R = N_L \cap N_R \simeq \mathbb{Z}e_1$, and the pushout W becomes

$$W = \mathbb{Z}^4 + \frac{1}{2}(A_R + e_1)\mathbb{Z} + \frac{1}{2m(2n+m)}\left((n+m)A + ne_2' + m(2n+m)e_1\right)\mathbb{Z}$$
$$+ \frac{1}{2m(2n+m)}\left((n+m)A + ne_2' - m(2n+m)e_1\right)\mathbb{Z} + \frac{1}{2n+m}\left(A - e_2'\right)\mathbb{Z}, \tag{4.13}$$

with the intersection matrix

$$\begin{pmatrix} 4m(2n+m) & 0 & 0 & 0 \\ 0 & -4m(2n+m) & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & A_R^2 \end{pmatrix}. \tag{4.14}$$

Note that the intersection $e_1^2 = -4$ implies that W is an integral matrix as we need, since $\frac{1}{2}e_1.\frac{1}{2}e_1 = -1$. Similarly we also glue two copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along e_1 to form a compact G_2-manifold Y with $b^2(Y) = 1$. Unlike the case of Picard rank 2, there is no upper bound of A^2 for $H^2(Y) \simeq \mathbb{Z}$ and thus for the given pair of these types we obtain a family of ample classes, satisfying matching conditions, generated by $\mathcal{O}(1)_1$, $\mathcal{O}(1)_2$ and $\mathcal{O}(1)_3$ at the ample class $\mathcal{O}(1)_1 + \mathcal{O}(1)_2 + \mathcal{O}(1)_3$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, but can not deformed simultaneously by $\mathcal{O}(1)_i$ and $\mathcal{O}(1)_j$, for instance, $a\mathcal{O}(1)_1 + b\mathcal{O}(1)_2$.

We make the table 4.3 of some toric (semi-)Fano 3-folds with $e.e = -4$ classes such that each pair satisfies the matching condition. The number in the first column is of the corresponding entry in the Mori-Mukai list of smooth Fanos (MM♯) or in Kasprzyk's database of terminal toric Fanos (K♯). Note that the ample class in the fifth column is not unique and each A in this table has the minimum self-intersection. The table 4.4 include some examples of G_2 manifolds Y by orthogonal gluing of semi-Fano blocks in table 4.3 along the -4 class e such that $b^2(Y) = 1, W = N_+ \perp_e N_-$, or by perpendicular gluing, i.e. $b^2(Y) = 0, W = N_+ \perp N_-$.

Table 4.3: Some rank ≥ 3 (semi-)Fano blocks with $e^2 = -4$.

No.	$-K^3$	rk$N$	N	A	e	A^2	$b^3(Z)$
K62, MM27	48	3	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	(1,1,1)	(1,0,-1)	12	50

Table 4.3: Some rank ≥ 3 (semi-)Fano blocks with $e^2 = -4$.

No.	$-K^3$	rk$N$	N	A	e	A^2	$b^3(Z)$
K68, MM25	44	3	$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{pmatrix}$	(1,2,1)	(-1,1,0)	20	46
K105, MM31	52	3	$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 6 \end{pmatrix}$	(1,1,1)	(0,1,-1)	22	54
K124	48	3	$\begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	(2,2,-1)	(-1,1,0)	32	50
K218, MM12	46	4	$\begin{pmatrix} 2 & 4 & 2 & 0 \\ 4 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & -2 \end{pmatrix}$	(1,2,1,1)	(1,-1,0,0)	46	48
K266, MM10	42	4	$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$	(2,2,1,-1)	(1,-1,0,0)	28	44
K221	38	4	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$	(3,4,-2,-1)	(0,-1,1,1)	32	40
K232	40	4	$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	(2,2,2,-3)	(-1,0,1,0)	30	42
K233	38	4	$\begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -2 \end{pmatrix}$	(-1,2,2,3)	(-1,1,0,0)	24	40
K247	44	4	$\begin{pmatrix} 4 & 3 & 3 & 2 \\ 3 & 0 & 2 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix}$	(-1,2,2,-1)	(0,-1,1,0)	38	46
K257	46	4	$\begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & 1 \\ 3 & 3 & 1 & 6 \end{pmatrix}$	(2,2,-3,2)	(-1,1,0,0)	58	48
K324,MM3	36	5	$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 2 & 1 & 2 & -2 \end{pmatrix}$	(-1,2,1,2,3)	(-1,1,0,0,0)	24	38

Table 4.3: Some rank ≥ 3 (semi-)Fano blocks with $e^2 = -4$.

No.	$-K^3$	rk$N$	N	A	e	A^2	$b^3(Z)$
K369,MM2	36	5	$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}$	(-1,2,3,3,2)	(-1,1,0,0,0)	36	38

Table 4.4: G_2 manifolds Y constructed by orthogonal gluing of (semi-)Fano blocks in table 4.3

Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W	Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W
K62	K62	1	122	$N_+ \perp_e N_-$	K62	K62	0	123	$N_+ \perp N_-$
K62	K68	1	118	$N_+ \perp_e N_-$	K62	K68	0	119	$N_+ \perp N_-$
K62	K105	1	126	$N_+ \perp_e N_-$	K62	K105	0	127	$N_+ \perp N_-$
K62	K124	1	122	$N_+ \perp_e N_-$	K62	K124	0	123	$N_+ \perp N_-$
K62	K218	1	120	$N_+ \perp_e N_-$	K62	K218	0	121	$N_+ \perp N_-$
K62	K266	1	116	$N_+ \perp_e N_-$	K62	K266	0	117	$N_+ \perp N_-$
K62	K221	1	112	$N_+ \perp_e N_-$	K62	K221	0	113	$N_+ \perp N_-$
K62	K232	1	114	$N_+ \perp_e N_-$	K62	K232	0	115	$N_+ \perp N_-$
K62	K233	1	112	$N_+ \perp_e N_-$	K62	K233	0	113	$N_+ \perp N_-$
K62	K247	1	118	$N_+ \perp_e N_-$	K62	K247	0	119	$N_+ \perp N_-$
K62	K257	1	120	$N_+ \perp_e N_-$	K62	K257	0	121	$N_+ \perp N_-$
K62	K324	1	110	$N_+ \perp_e N_-$	K62	K324	0	111	$N_+ \perp N_-$
K62	K369	1	110	$N_+ \perp_e N_-$	K62	K369	0	111	$N_+ \perp N_-$
K68	K68	1	114	$N_+ \perp_e N_-$	K68	K68	0	115	$N_+ \perp N_-$
K68	K105	1	122	$N_+ \perp_e N_-$	K68	K105	0	123	$N_+ \perp N_-$
K68	K124	1	118	$N_+ \perp_e N_-$	K68	K124	0	119	$N_+ \perp N_-$
K68	K218	1	116	$N_+ \perp_e N_-$	K68	K218	0	117	$N_+ \perp N_-$
K68	K266	1	112	$N_+ \perp_e N_-$	K68	K266	0	113	$N_+ \perp N_-$
K68	K221	1	108	$N_+ \perp_e N_-$	K68	K221	0	109	$N_+ \perp N_-$
K68	K232	1	110	$N_+ \perp_e N_-$	K68	K232	0	111	$N_+ \perp N_-$
K68	K233	1	108	$N_+ \perp_e N_-$	K68	K233	0	109	$N_+ \perp N_-$
K68	K247	1	114	$N_+ \perp_e N_-$	K68	K247	0	115	$N_+ \perp N_-$
K68	K257	1	116	$N_+ \perp_e N_-$	K68	K257	0	117	$N_+ \perp N_-$
K68	K324	1	106	$N_+ \perp_e N_-$	K68	K324	0	107	$N_+ \perp N_-$
K68	K369	1	106	$N_+ \perp_e N_-$	K68	K369	0	107	$N_+ \perp N_-$
K105	K105	1	130	$N_+ \perp_e N_-$	K105	K105	0	131	$N_+ \perp N_-$
K105	K124	1	126	$N_+ \perp_e N_-$	K105	K124	0	127	$N_+ \perp N_-$
K105	K218	1	124	$N_+ \perp_e N_-$	K105	K218	0	125	$N_+ \perp N_-$
K105	K266	1	120	$N_+ \perp_e N_-$	K105	K266	0	121	$N_+ \perp N_-$
K105	K221	1	116	$N_+ \perp_e N_-$	K105	K221	0	117	$N_+ \perp N_-$
K105	K232	1	118	$N_+ \perp_e N_-$	K105	K232	0	119	$N_+ \perp N_-$
K105	K233	1	116	$N_+ \perp_e N_-$	K105	K233	0	117	$N_+ \perp N_-$
K105	K247	1	122	$N_+ \perp_e N_-$	K105	K247	0	123	$N_+ \perp N_-$
K105	K257	1	124	$N_+ \perp_e N_-$	K105	K257	0	125	$N_+ \perp N_-$
K105	K324	1	114	$N_+ \perp_e N_-$	K105	K324	0	115	$N_+ \perp N_-$

Table 4.4: G_2 manifolds Y constructed by orthogonal gluing of (semi-)Fano blocks in table 4.3

Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W	Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W
K105	K369	1	114	$N_+\perp_e N_-$	K105	K369	0	115	$N_+\perp N_-$
K124	K124	1	122	$N_+\perp_e N_-$	K124	K124	0	123	$N_+\perp N_-$
K124	K218	1	120	$N_+\perp_e N_-$	K124	K218	0	121	$N_+\perp N_-$
K124	K266	1	116	$N_+\perp_e N_-$	K124	K266	0	117	$N_+\perp N_-$
K124	K221	1	112	$N_+\perp_e N_-$	K124	K221	0	113	$N_+\perp N_-$
K124	K232	1	114	$N_+\perp_e N_-$	K124	K232	0	115	$N_+\perp N_-$
K124	K233	1	112	$N_+\perp_e N_-$	K124	K233	0	113	$N_+\perp N_-$
K124	K247	1	118	$N_+\perp_e N_-$	K124	K247	0	119	$N_+\perp N_-$
K124	K257	1	120	$N_+\perp_e N_-$	K124	K257	0	121	$N_+\perp N_-$
K124	K324	1	110	$N_+\perp_e N_-$	K124	K324	0	111	$N_+\perp N_-$
K124	K369	1	110	$N_+\perp_e N_-$	K124	K369	0	111	$N_+\perp N_-$
K218	K218	1	118	$N_+\perp_e N_-$	K218	K218	0	119	$N_+\perp N_-$
K218	K266	1	114	$N_+\perp_e N_-$	K218	K266	0	115	$N_+\perp N_-$
K218	K221	1	110	$N_+\perp_e N_-$	K218	K221	0	111	$N_+\perp N_-$
K218	K232	1	112	$N_+\perp_e N_-$	K218	K232	0	113	$N_+\perp N_-$
K218	K233	1	110	$N_+\perp_e N_-$	K218	K233	0	111	$N_+\perp N_-$
K218	K247	1	116	$N_+\perp_e N_-$	K218	K247	0	117	$N_+\perp N_-$
K218	K257	1	118	$N_+\perp_e N_-$	K218	K257	0	119	$N_+\perp N_-$
K218	K324	1	108	$N_+\perp_e N_-$	K218	K324	0	109	$N_+\perp N_-$
K218	K369	1	108	$N_+\perp_e N_-$	K218	K369	0	109	$N_+\perp N_-$
K266	K266	1	110	$N_+\perp_e N_-$	K266	K266	0	111	$N_+\perp N_-$
K266	K221	1	106	$N_+\perp_e N_-$	K266	K221	0	107	$N_+\perp N_-$
K266	K232	1	108	$N_+\perp_e N_-$	K266	K232	0	109	$N_+\perp N_-$
K266	K233	1	106	$N_+\perp_e N_-$	K266	K233	0	107	$N_+\perp N_-$
K266	K247	1	112	$N_+\perp_e N_-$	K266	K247	0	113	$N_+\perp N_-$
K266	K257	1	114	$N_+\perp_e N_-$	K266	K257	0	115	$N_+\perp N_-$
K266	K324	1	104	$N_+\perp_e N_-$	K266	K324	0	105	$N_+\perp N_-$
K266	K369	1	104	$N_+\perp_e N_-$	K266	K369	0	105	$N_+\perp N_-$
K221	K221	1	102	$N_+\perp_e N_-$	K221	K221	0	103	$N_+\perp N_-$
K221	K232	1	104	$N_+\perp_e N_-$	K221	K232	0	105	$N_+\perp N_-$
K221	K233	1	102	$N_+\perp_e N_-$	K221	K233	0	103	$N_+\perp N_-$
K221	K247	1	108	$N_+\perp_e N_-$	K221	K247	0	109	$N_+\perp N_-$
K221	K257	1	110	$N_+\perp_e N_-$	K221	K257	0	111	$N_+\perp N_-$
K221	K324	1	100	$N_+\perp_e N_-$	K221	K324	0	101	$N_+\perp N_-$
K221	K369	1	100	$N_+\perp_e N_-$	K221	K369	0	101	$N_+\perp N_-$
K232	K232	1	106	$N_+\perp_e N_-$	K232	K232	0	107	$N_+\perp N_-$
K232	K233	1	104	$N_+\perp_e N_-$	K232	K233	0	105	$N_+\perp N_-$
K232	K247	1	110	$N_+\perp_e N_-$	K232	K247	0	111	$N_+\perp N_-$
K232	K257	1	112	$N_+\perp_e N_-$	K232	K257	0	113	$N_+\perp N_-$
K232	K324	1	102	$N_+\perp_e N_-$	K232	K324	0	103	$N_+\perp N_-$
K232	K369	1	102	$N_+\perp_e N_-$	K232	K369	0	103	$N_+\perp N_-$
K233	K233	1	102	$N_+\perp_e N_-$	K233	K233	0	103	$N_+\perp N_-$
K233	K247	1	108	$N_+\perp_e N_-$	K233	K247	0	109	$N_+\perp N_-$
K233	K257	1	110	$N_+\perp_e N_-$	K233	K257	0	111	$N_+\perp N_-$

Table 4.4: G_2 manifolds Y constructed by orthogonal gluing of (semi-)Fano blocks in table 4.3

Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W	Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W
K233	K324	1	100	$N_+ \perp_e N_-$	K233	K324	0	101	$N_+ \perp N_-$
K233	K369	1	100	$N_+ \perp_e N_-$	K233	K369	0	101	$N_+ \perp N_-$
K247	K247	1	114	$N_+ \perp_e N_-$	K247	K247	0	115	$N_+ \perp N_-$
K247	K257	1	116	$N_+ \perp_e N_-$	K247	K257	0	117	$N_+ \perp N_-$
K247	K324	1	106	$N_+ \perp_e N_-$	K247	K324	0	107	$N_+ \perp N_-$
K247	K369	1	106	$N_+ \perp_e N_-$	K247	K369	0	107	$N_+ \perp N_-$
K257	K257	1	118	$N_+ \perp_e N_-$	K257	K257	0	119	$N_+ \perp N_-$
K257	K324	1	108	$N_+ \perp_e N_-$	K257	K324	0	109	$N_+ \perp N_-$
K257	K369	1	108	$N_+ \perp_e N_-$	K257	K369	0	109	$N_+ \perp N_-$
K324	K324	1	98	$N_+ \perp_e N_-$	K324	K324	0	99	$N_+ \perp N_-$
K324	K369	1	98	$N_+ \perp_e N_-$	K324	K369	0	99	$N_+ \perp N_-$
K369	K369	1	98	$N_+ \perp_e N_-$	K369	K369	0	99	$N_+ \perp N_-$

On the other hand, we can only identify the e_2' vector but not e_1 by orthogonal gluing. Choose $m = 1$ and $n = 2$, then $e_2' = \mathscr{O}(1)_1 + \mathscr{O}(1)_2 - 3\mathscr{O}(1)_3$ and $e_2'.e_2' = -20$. From (4.11), the projection of each generator of the Picard lattice to e_2' is $\pm\frac{1}{5}e_2'$. The smooth Fano 3-fold with Picard rank 2 in the Mori-Mukai list ♯14 has the quadratic form of the Picard lattice

$$\begin{pmatrix} 0 & 5 \\ 5 & 10 \end{pmatrix}$$

with $A = (1, 1)$ and $e = (3, -1)$ such that $A.A = 20$ and $e.e = -20$. Thus the Picard lattice can be expressed as

$$N_R = \mathbb{Z}^2 + \frac{1}{4}(A + e)\mathbb{Z} + \frac{1}{4}(3A - e)\mathbb{Z}. \tag{4.15}$$

Thus W is an integral matrix, since $\frac{1}{5}e_2'.\frac{1}{4}e = -1$ if we identify e_2' and e.

Example 4.30 (Orthogonal gluing along rank two intersection lattice ***R***)**.** We present a particular example with a rank two intersection lattice R with two orthogonal generators e_1 and e_2 both of self-intersection -4, and imposing these two conditions orthogonality and the maximal negative value -4 simplifies the construction of a matching pair. Note that on any smooth $K3$ surface the self-intersection of divisors is even and for each divisor e of self-intersection -2 itself e or $-e$ correspond to a effective divisor (may be reducible), i.e. a curve, with a non-trivial intersection number with any ample class A, which is in violation with the orthogonal gluing assumption.

Our example is based upon gluing a pair of building blocks $(Z_{L/R}, S_{L/R})$ both obtained from the rank five Fano threefold $P_{L/R} = \mathbb{P}^1 \times dP_3$, where dP_3 denotes the del Pezzo surface of degree six, which is the blow-up of $\mathbb{P}^2$ along three non-collinear points p_1, p_2, p_3. This rank five Fano threefold has the Mori–Mukai reference number MM3 and the Kasprzky reference number K324.

First, we collect some basic properties of the del Pezzo surface dP_3. Let E_1, E_2, E_3 be the three exceptional divisors from the blow-ups at the points p_1, p_2, p_3, and let H be the proper transform of the hyperplane class of $\mathbb{P}^2$. These divisors span the Picard lattice of dP_3 and their intersection numbers read

$$E_i.E_j = -\delta_{ij}, \qquad H.H = 1, \qquad H.E_i = 0. \tag{4.16}$$

The ample anti-canonical divisor reads $-K_{dP_3} = 3H - E_1 - E_2 - E_3$. Let us further define the two divisors

$$e_1 = E_1 + E_2 + E_3 - H\,, \qquad e_2 = E_1 - E_2\,, \tag{4.17}$$

which are both differences of rational curves on dP_3. The most important point, however, is that the defined divisors e_1, e_2 of self-intersection -2 are both mutual orthogonal and orthogonal to the class $-K_{dP_3}$ in the Kähler cone $\mathcal{K}(dP_3)$, i.e.,

$$e_1.e_2 = e_1.K_{dP_3} = e_2.K_{dP_3} = 0\,, \quad e_1.e_1 = e_2.e_2 = -2\,. \tag{4.18}$$

Now we return to the rank five Fano threefold $\mathbb{P}^1 \times dP_3$. With the hyperplane divisor h of $\mathbb{P}^1$ and the described divisors of dP_3 the anti-canonical divisor becomes

$$-K_{\mathbb{P}^1 \times dP_3} = 2h - K_{dP_3} = 2h + 3H - E_1 - E_2 - E_3\,, \tag{4.19}$$

Furthermore, the Picard lattice N of the polarized K3 surface S on $\mathbb{P}^1 \times dP_3$ is generated by the divisors h, H, E_1, E_2, E_3 together with the intersection pairing

$$\langle h, h\rangle_N = 0\,, \qquad \langle h, D\rangle_N = -K_{dP_3}.D\,, \qquad \langle D, F\rangle_N = 2D.F\,. \tag{4.20}$$

Here D and F are some divisors on dP_3.

For the orthogonal pushout we generate the rank two lattice R with the two del Pezzo divisors e_1 and e_2 as

$$R = \mathbb{Z}e_1 + \mathbb{Z}e_2\,, \qquad \langle e_i, e_j\rangle_N = -4\delta_{ij}\,, \tag{4.21}$$

where eqs. (4.18) and (4.20) determines the intersection pairing on R. Moreover, the orthogonal complement W of R becomes

$$W = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 \quad \text{with} \quad w_1 = h - K_{dP_3}\,, \quad w_2 = H - E_3\,, \quad w_3 = h\,, \tag{4.22}$$

where in particular the ample generator w_1 is in the Kähler cone $\mathcal{K}(\mathbb{P}^1 \times dP_3)$. As a result for the rank five Picard lattice N of the polarized K3 surface S in $\mathbb{P}^1 \times dP_3$ we arrive with $(w_1, w_2, w_3, e_1, e_2)$ at

$$N = (\mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3) + (\mathbb{Z}e_1 + \mathbb{Z}e_2) + \frac{1}{2}\left(\mathbb{Z}(w_1 + e_1) + \mathbb{Z}(w_1 + w_2 + e_2)\right)\,. \tag{4.23}$$

Now taking the decomposition (4.23) of the Picard lattice for both the left and the right Picard lattice, i.e., $N_L = N_R = N$, we consider the orthogonal pushout $W = N_L \perp_R N_R$, which in the basis $(w_1^L, w_2^L, w_3^L, w_1^R, w_2^R, w_3^R, e_1, e_2)$ takes the form

$$\begin{aligned} W = {} & \left(\mathbb{Z}w_1^L + \mathbb{Z}w_2^L + \mathbb{Z}w_3^L\right) + \left(\mathbb{Z}w_1^R + \mathbb{Z}w_2^R + \mathbb{Z}w_3^R\right) \\ & + (\mathbb{Z}e_1 + \mathbb{Z}e_2) + \frac{1}{2}\left(\mathbb{Z}(w_1^L + e_1) + \mathbb{Z}(w_1^R + e_1)\right) \\ & + \frac{1}{2}\left(\mathbb{Z}(w_1^L + w_2^L + e_2) + \mathbb{Z}(w_1^R + w_2^R + e_2)\right)\,. \end{aligned} \tag{4.24}$$

This orthogonal pushout is well-defined because the potentially non-integral intersections $\langle \frac{1}{2}(w_1^L + e_1), \frac{1}{2}(w_1^R + e_1)\rangle = \langle \frac{1}{2}(w_1^L + w_2^L + e_2), \frac{1}{2}(w_1^R + w_2^R + e_2)\rangle_W = -1$ and $\langle \frac{1}{2}(w_1^L + e_1), \frac{1}{2}(w_1^R + w_2^R + e_2)\rangle_W = \langle \frac{1}{2}(w_1^R + e_1), \frac{1}{2}(w_1^L + w_2^L + e_2)\rangle_W = 0$ are integral. As a result we obtain from this orthogonal pushout along the rank two lattice R the twisted connected G_2-manifold Y with the Betti numbers $b_2(Y) = 2$, $b_3(Y) = 97$. Here we use that $b_3(Z_{L/R}) = \langle K_{\mathbb{P}^1 \times dP_3}, K_{\mathbb{P}^1 \times dP_3} \rangle_N + 2 = 6K_{dP_3}.K_{dP_3} + 2 = 38$ because $b_3(\mathbb{P}^1 \times dP_3) = 0$.

4.4.2 Non-generic orthogonal gluing

In order to give a topology of the gauge group realisations in G_2 compactifications, we are interested in examples of twisted gluing in which there is a contribution to $b^2(Y)$ from the kernels $K_{L/R}$. From the proposition 4.15 each building block Z, obtained by blowing up along the base locus of a generic anticanonical pencil in some semi-Fano 3-folds, has a inclusion map $H^2(X = Z \setminus S) \hookrightarrow H^2(S) = L$. In order to construct some build blocks with non-injective morphism $H^2(X) \to L$, we may use a generic anticanonical pencil in the weak Fano 3-folds. The other way arises by blowing up the base locus of a *non-generic* anticanonical pencil of a semi-Fano 3-fold Z such that $K = \ker \left\{ \rho : H^2(X) \to H^2(S) = L \right\} \neq \emptyset$. Indeed, to construct non-trivial $K_{L/R}$ we can follow the generalization proposed by Kovalev and N.-H. Lee [KL11], or see [Cor+13, Prop. 4.25]. Suppose for a semi-Fano 3-fold P there is a divisor

$$C = C_1 + \cdots + C_n \in |-K_P|_S|,$$

where all C_i are connected smooth curves. Now we can construct the building block Z by the sequence of blow-ups $\pi_{\{C_1,\ldots,C_n\}} : Z \to P$ along the individual curves C_i according to

$$Z \ = \ \mathrm{Bl}_{\{C_1,\ldots,C_n\}} P \ = \ \mathrm{Bl}_{C_n} \mathrm{Bl}_{C_{n-1}} \cdots \mathrm{Bl}_{C_1} P \ .$$

The resulted 3-fold Z is a non-singular building block with non-trivial K:

$$K = m + b^2(P) - \mathrm{rk}\langle C_1, \ldots, C_n, N \rangle - 1, \tag{4.25}$$

where $\langle C_1, \ldots, C_n, N \rangle \subset L \simeq H^2(S, \mathbb{Z})$ is generated by C_i's and N. Thus $K > 0$ if $C_1, \ldots, C_m$ are linear dependent in $H^2(S, \mathbb{R})$. Note that $H^3(Z, \mathbb{Z}) \simeq H^3(P, \mathbb{Z}) \oplus \sum_{i=1}^m H^1(C_i)$. The following example is studied in the articles [Cor+13; Cor+15], or see [HM15], and based on the proposition below.

Proposition 4.31. *Given a non-singular variety Z of dimension 3 with a divisor H such that the corresponding linear system $|H|$ is base point free, and a simple normal crossing divisor X, i.e. $X = X_1 \cup X_2 \cup \cdots \cup X_n$ with irreducible components $X_i, i = 1, \ldots, n$ such that $C_{ij} := X_i \cap X_j$ is a non-singular curve for all $i, j = 1, \ldots, n$. Then general sections s of $|H|$ are non-singular subvarieties of Z such that the curves $C_i = X_i \cap s$ are smooth and mutually intersect transversally.*

Proof. We apply strong Bertini theorem (see, e.g., ref [GH94; Har77]) to Z, $\{X_i\}$ and $\{C_{ij}\}$ which are all non-singular for all i, j. Since the linear system $|H|$ is base point free, generic sections s of $|H|$ with smooth curves $s \cap X_i$ and ordinary points $s \cap C_{ij}$ are non-singular. □

Example 4.32. We begin with the simple Fano 3-fold $W = \mathbb{P}^3$, and consider the non-generic pencil $|S_0, S_\infty| \subset |\mathscr{O}(4)|$ with

$$S_0 = \{x_0 x_1 x_2 x_3 = 0\}$$

the sum of the four coordinate planes, and S_∞ a generic non-singular quartic surface meeting all coordinate planes $\{x_i = 0\}$ transversely. The base locus of the pencil is the union of four non-singular curves $\sum_{i=0}^3 C_i$

and $C_i := \{x_i = 0\} \cap S_\infty$ is a genus 3 curve by the formula

$$2 - 2g = \chi(C_i) = \int_{4H^2} (-H) = -4,$$

where $H = c_1(\mathscr{O}(1))$. Let Z be obtained from $\mathbb{P}^3$ by blowing up the base curve one at a time, and we may assume that by blowing up along C_0, C_1, C_2, C_3 in this order with four associated exceptional divisor E_i such that $b^2(Z) = b^2(\mathbb{P}^3) + 4 = 5$. Then since each E_i is a $\mathbb{P}^1$ fibration over the curve C_i, which is a curve in S_∞ of class $H|_{S_\infty}$, then it turns out that the image of E_i of the restriction map is $\rho(E_i) = H|_{S_\infty}$. Thus we can easily find that the kernel $K = \langle E_1 - E_0, E_2 - E_0, E_3 - E_0\rangle$ is rank three and the Picard lattice $N = \langle \rho(E_0) = H|_{S_\infty}\rangle = \langle 4\rangle$ is rank one.

Let l_0 be the inverse image in Z of a general point of C_0, and l_1 be the inverse image of a general point of C_1. Note that l_0 and l_1 are both projective lines. Then the inverse image of $C_0.C_1$ is a singular curve consists of two lines l_0' and l_1', and we have algebraic equivalence of cycles $l_0 \sim l_0' + l_1'$ and $l_1 \sim l_1'$. Since the rigid holomorphic curve l_0' are contained in E_0 and transverse to E_1, so that $E_1.l_0' = E_1.(l_0 - l_1) = 1$. Hence $E_1.l_1 = -1$ as the generic fiber in E_j does not intersect any exceptional divisor $E_{j\neq i}$, i.e. $E_j.l_i = 0$. By the similar way we obtain $E_2.l_2 = -1$ and $E_3.l_3 = -1$. To compute $E_0.l_0$, we consider the formula $\chi(l_0) = \chi\left(\mathbb{P}^1\right) = 2$ and

$$\chi(l_0) = (-K_{E_0} + l_0).l_0 = -K_{E_0}.l_0.$$

Since $\pi : Z \to \mathbb{P}^3$ is a blow up along the singular curve, the anticanonical divisor is written by

$$K_Z = \pi^* K_{\mathbb{P}^3} + E_0 + E_1 + E_2 + E_3,$$

and the adjunction formula implies

$$K_{E_0} = (K_Z + E_0)\,|_{E_0} = (\pi^* K_{\mathbb{P}^3} + 2E_0 + E_1 + E_2 + E_3)|_{E_0}.$$

Then it turns out that

$$K_{E_0}.l_0 = (\pi^* K_{\mathbb{P}^3} + 2E_0 + E_1 + E_2 + E_3)|_{E_0}.l_0 = 2E_0.l_0,$$

and thus $E_0.l_0 = -1$. Therefore, we conclude that

$$E_i.l_j = -\delta_{ij}.$$

Example 4.33 ([Cor+13, Example 7.11])**.** Choose the semi-Fano P' in the example 4.26 and consider non-generic pencil $|S_0, S_\infty| \subset | - K_{P'}|$ with

$$S_0 = \sum_{i=1}^{9} Q_i + \sum_{j=1}^{4} R_j,$$

and S_∞ a generic non-singular quartic surface meeting all components of S_0 transversely. The base locus of the pencil is the union of 13 non-singular rational curves $C = \sum_{i=1}^{9} C_i + \sum_{j=1}^{4} r_j$. Let Z be obtained from P' by blowing up the base curve one at a time with $H^2(Z) \simeq H^2(P') \oplus \mathbb{Z}^{13} \simeq \mathbb{Z}^{23}$ and $H^3(Z) \simeq 0$. In the same manner as previous example, the image of each exceptional divisor E_i in $H^2(S)$ is just $Q_i|_S$ or $R_j|_S$, so they only contribute to the kernel K of $H^2(X) \to L$ with intersections $E_i.l_j = -\delta_{ij}$ for each l_j the inverse image in Z of a general point of C_j or r_j.

Example 4.34. Again consider the toric Fano 3-fold $P = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as before, and we choose the more interesting pencil as following. Take $S_0 = \sum_{i=1}^{6} D_i$, where all D_i are toric divisors on P, and S is a non-singular element of $|-K_P|$ meeting all the components of S_0 transversely. Then the base curve of the pencil is the sum

$$c = \sum_{i=1}^{6} c_i,$$

where c_i are elliptic curves, since $g(c) = \frac{1}{2}(c^2) + 1$. It turns out that by blowing up along the base locus c, we obtain a non-singular building block Z with $H^2(Z) \simeq H^2(P) \oplus \bigoplus_{i=1}^{6} H^0(c_i) \simeq \mathbb{Z}^9$ and $H^3(Z) \simeq \bigoplus_{i=1}^{6} H^1(c_i) \simeq \mathbb{Z}^{12}$. Note that the image of c_i's in $H^2(S)$ is belong to $N \simeq \rho^*(H^2(P))$, hence the kernel $k = m - 1 = 6 - 1 = 5 > 0$.

Again consider the toric Fano 3-fold $P = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as before, and we choose the more interesting pencil as following. Take $S_0 = \sum_{i=1}^{6} D_i$, where all D_i are toric divisors on P, and S is a non-singular element of $|-K_P|$ meeting all the components of S_0 transversely. Then the base curve of the pencil is the sum

$$c = \sum_{i=1}^{6} c_i, \tag{4.26}$$

where c_i are elliptic curves, since $g(c) = \frac{1}{2}(c^2) + 1$. It turns out that by blowing up along the base locus c, we obtain a non-singular building block Z with $H^2(Z) \simeq H^2(P) \oplus \bigoplus_{i=1}^{6} H^0(c_i) \simeq \mathbb{Z}^9$ and $H^3(Z) \simeq \bigoplus_{i=1}^{6} H^1(c_i) \simeq \mathbb{Z}^{12}$. Note that the image of c_i's in $H^2(S)$ is belong to $N \simeq \rho^*(H^2(P))$, hence the kernel $K = m - 1 = 6 - 1 = 5 > 0$.

By this argument we make the table 4.5 including some semi-Fano building blocks with non-trivial kernel k, and table 4.6 include some examples of G_2 manifolds Y by orthogonal gluing of semi-Fano blocks in table 4.5 along the -4 class e, or by perpendicular gluing.

Table 4.5: Some toric semi-Fano blocks with $K > 0$.

No.	K	$b^2(Z)$	$b^3(Z)$
K62, MM27	5	9	12
K68, MM25	5	9	6
K105, MM31	5	9	16
K124	5	9	12
K218, MM12	6	11	12
K266, MM10	6	11	8
K221	6	11	4
K232	6	11	6
K233	6	11	4
K247	6	11	10
K257	6	11	12
K324,MM3	7	13	4
K369,MM2	7	13	4

Table 4.6: G_2 manifolds Y constructed by orthogonal gluing of (semi-)Fano blocks in table 4.5

Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W	Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W
K62	K62	11	56	$N_+ \perp_e N_-$	K62	K62	10	57	$N_+ \perp N_-$
K62	K68	11	50	$N_+ \perp_e N_-$	K62	K68	10	51	$N_+ \perp N_-$
K62	K105	11	60	$N_+ \perp_e N_-$	K62	K105	10	61	$N_+ \perp N_-$
K62	K124	11	56	$N_+ \perp_e N_-$	K62	K124	10	57	$N_+ \perp N_-$
K62	K218	12	57	$N_+ \perp_e N_-$	K62	K218	11	58	$N_+ \perp N_-$
K62	K266	12	53	$N_+ \perp_e N_-$	K62	K266	11	54	$N_+ \perp N_-$
K62	K221	12	49	$N_+ \perp_e N_-$	K62	K221	11	50	$N_+ \perp N_-$
K62	K232	12	51	$N_+ \perp_e N_-$	K62	K232	11	52	$N_+ \perp N_-$
K62	K233	12	49	$N_+ \perp_e N_-$	K62	K233	11	50	$N_+ \perp N_-$
K62	K247	12	55	$N_+ \perp_e N_-$	K62	K247	11	56	$N_+ \perp N_-$
K62	K257	12	57	$N_+ \perp_e N_-$	K62	K257	11	58	$N_+ \perp N_-$
K62	K324	13	50	$N_+ \perp_e N_-$	K62	K324	12	51	$N_+ \perp N_-$
K62	K369	13	50	$N_+ \perp_e N_-$	K62	K369	12	51	$N_+ \perp N_-$
K68	K68	11	44	$N_+ \perp_e N_-$	K68	K68	10	45	$N_+ \perp N_-$
K68	K105	11	54	$N_+ \perp_e N_-$	K68	K105	10	55	$N_+ \perp N_-$
K68	K124	11	50	$N_+ \perp_e N_-$	K68	K124	10	51	$N_+ \perp N_-$
K68	K218	12	51	$N_+ \perp_e N_-$	K68	K218	11	52	$N_+ \perp N_-$
K68	K266	12	47	$N_+ \perp_e N_-$	K68	K266	11	48	$N_+ \perp N_-$
K68	K221	12	43	$N_+ \perp_e N_-$	K68	K221	11	44	$N_+ \perp N_-$
K68	K232	12	45	$N_+ \perp_e N_-$	K68	K232	11	46	$N_+ \perp N_-$
K68	K233	12	43	$N_+ \perp_e N_-$	K68	K233	11	44	$N_+ \perp N_-$
K68	K247	12	49	$N_+ \perp_e N_-$	K68	K247	11	50	$N_+ \perp N_-$
K68	K257	12	51	$N_+ \perp_e N_-$	K68	K257	11	52	$N_+ \perp N_-$
K68	K324	13	44	$N_+ \perp_e N_-$	K68	K324	12	45	$N_+ \perp N_-$
K68	K369	13	44	$N_+ \perp_e N_-$	K68	K369	12	45	$N_+ \perp N_-$
K105	K105	11	64	$N_+ \perp_e N_-$	K105	K105	10	65	$N_+ \perp N_-$
K105	K124	11	60	$N_+ \perp_e N_-$	K105	K124	10	61	$N_+ \perp N_-$
K105	K218	12	61	$N_+ \perp_e N_-$	K105	K218	11	62	$N_+ \perp N_-$
K105	K266	12	57	$N_+ \perp_e N_-$	K105	K266	11	58	$N_+ \perp N_-$
K105	K221	12	53	$N_+ \perp_e N_-$	K105	K221	11	54	$N_+ \perp N_-$
K105	K232	12	55	$N_+ \perp_e N_-$	K105	K232	11	56	$N_+ \perp N_-$
K105	K233	12	53	$N_+ \perp_e N_-$	K105	K233	11	54	$N_+ \perp N_-$
K105	K247	12	59	$N_+ \perp_e N_-$	K105	K247	11	60	$N_+ \perp N_-$
K105	K257	12	61	$N_+ \perp_e N_-$	K105	K257	11	62	$N_+ \perp N_-$
K105	K324	13	54	$N_+ \perp_e N_-$	K105	K324	12	55	$N_+ \perp N_-$
K105	K369	13	54	$N_+ \perp_e N_-$	K105	K369	12	55	$N_+ \perp N_-$
K124	K124	11	56	$N_+ \perp_e N_-$	K124	K124	10	57	$N_+ \perp N_-$
K124	K218	12	57	$N_+ \perp_e N_-$	K124	K218	11	58	$N_+ \perp N_-$
K124	K266	12	53	$N_+ \perp_e N_-$	K124	K266	11	54	$N_+ \perp N_-$
K124	K221	12	49	$N_+ \perp_e N_-$	K124	K221	11	50	$N_+ \perp N_-$
K124	K232	12	51	$N_+ \perp_e N_-$	K124	K232	11	52	$N_+ \perp N_-$
K124	K233	12	49	$N_+ \perp_e N_-$	K124	K233	11	50	$N_+ \perp N_-$
K124	K247	12	55	$N_+ \perp_e N_-$	K124	K247	11	56	$N_+ \perp N_-$
K124	K257	12	57	$N_+ \perp_e N_-$	K124	K257	11	58	$N_+ \perp N_-$

Table 4.6: G_2 manifolds Y constructed by orthogonal gluing of (semi-)Fano blocks in table 4.5

Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W	Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W
K124	K324	13	50	$N_+\perp_e N_-$	K124	K324	12	51	$N_+\perp N_-$
K124	K369	13	50	$N_+\perp_e N_-$	K124	K369	12	51	$N_+\perp N_-$
K218	K218	13	58	$N_+\perp_e N_-$	K218	K218	12	59	$N_+\perp N_-$
K218	K266	13	54	$N_+\perp_e N_-$	K218	K266	12	55	$N_+\perp N_-$
K218	K221	13	50	$N_+\perp_e N_-$	K218	K221	12	51	$N_+\perp N_-$
K218	K232	13	52	$N_+\perp_e N_-$	K218	K232	12	53	$N_+\perp N_-$
K218	K233	13	50	$N_+\perp_e N_-$	K218	K233	12	51	$N_+\perp N_-$
K218	K247	13	56	$N_+\perp_e N_-$	K218	K247	12	57	$N_+\perp N_-$
K218	K257	13	58	$N_+\perp_e N_-$	K218	K257	12	59	$N_+\perp N_-$
K218	K324	14	51	$N_+\perp_e N_-$	K218	K324	13	52	$N_+\perp N_-$
K218	K369	14	51	$N_+\perp_e N_-$	K218	K369	13	52	$N_+\perp N_-$
K266	K266	13	50	$N_+\perp_e N_-$	K266	K266	12	51	$N_+\perp N_-$
K266	K221	13	46	$N_+\perp_e N_-$	K266	K221	12	47	$N_+\perp N_-$
K266	K232	13	48	$N_+\perp_e N_-$	K266	K232	12	49	$N_+\perp N_-$
K266	K233	13	46	$N_+\perp_e N_-$	K266	K233	12	47	$N_+\perp N_-$
K266	K247	13	52	$N_+\perp_e N_-$	K266	K247	12	53	$N_+\perp N_-$
K266	K257	13	54	$N_+\perp_e N_-$	K266	K257	12	55	$N_+\perp N_-$
K266	K324	14	47	$N_+\perp_e N_-$	K266	K324	13	48	$N_+\perp N_-$
K266	K369	14	47	$N_+\perp_e N_-$	K266	K369	13	48	$N_+\perp N_-$
K221	K221	13	42	$N_+\perp_e N_-$	K221	K221	12	43	$N_+\perp N_-$
K221	K232	13	44	$N_+\perp_e N_-$	K221	K232	12	45	$N_+\perp N_-$
K221	K233	13	42	$N_+\perp_e N_-$	K221	K233	12	43	$N_+\perp N_-$
K221	K247	13	48	$N_+\perp_e N_-$	K221	K247	12	49	$N_+\perp N_-$
K221	K257	13	50	$N_+\perp_e N_-$	K221	K257	12	51	$N_+\perp N_-$
K221	K324	14	43	$N_+\perp_e N_-$	K221	K324	13	44	$N_+\perp N_-$
K221	K369	14	43	$N_+\perp_e N_-$	K221	K369	13	44	$N_+\perp N_-$
K232	K232	13	46	$N_+\perp_e N_-$	K232	K232	12	47	$N_+\perp N_-$
K232	K233	13	44	$N_+\perp_e N_-$	K232	K233	12	45	$N_+\perp N_-$
K232	K247	13	50	$N_+\perp_e N_-$	K232	K247	12	51	$N_+\perp N_-$
K232	K257	13	52	$N_+\perp_e N_-$	K232	K257	12	53	$N_+\perp N_-$
K232	K324	14	45	$N_+\perp_e N_-$	K232	K324	13	46	$N_+\perp N_-$
K232	K369	14	45	$N_+\perp_e N_-$	K232	K369	13	46	$N_+\perp N_-$
K233	K233	13	42	$N_+\perp_e N_-$	K233	K233	12	43	$N_+\perp N_-$
K233	K247	13	48	$N_+\perp_e N_-$	K233	K247	12	49	$N_+\perp N_-$
K233	K257	13	50	$N_+\perp_e N_-$	K233	K257	12	51	$N_+\perp N_-$
K233	K324	14	43	$N_+\perp_e N_-$	K233	K324	13	44	$N_+\perp N_-$
K233	K369	14	43	$N_+\perp_e N_-$	K233	K369	13	44	$N_+\perp N_-$
K247	K247	13	54	$N_+\perp_e N_-$	K247	K247	12	55	$N_+\perp N_-$
K247	K257	13	56	$N_+\perp_e N_-$	K247	K257	12	57	$N_+\perp N_-$
K247	K324	14	49	$N_+\perp_e N_-$	K247	K324	13	50	$N_+\perp N_-$
K247	K369	14	49	$N_+\perp_e N_-$	K247	K369	13	50	$N_+\perp N_-$
K257	K257	13	58	$N_+\perp_e N_-$	K257	K257	12	59	$N_+\perp N_-$
K257	K324	14	51	$N_+\perp_e N_-$	K257	K324	13	52	$N_+\perp N_-$
K257	K369	14	51	$N_+\perp_e N_-$	K257	K369	13	52	$N_+\perp N_-$

Table 4.6: G_2 manifolds Y constructed by orthogonal gluing of (semi-)Fano blocks in table 4.5

Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W	Z_+	Z_-	$b^2(Y)$	$b^3(Y)$	W
K324	K324	15	40	$N_+ \perp_e N_-$	K324	K324	14	41	$N_+ \perp N_-$
K324	K369	15	40	$N_+ \perp_e N_-$	K324	K369	14	41	$N_+ \perp N_-$
K369	K369	15	40	$N_+ \perp_e N_-$	K369	K369	14	41	$N_+ \perp N_-$

CHAPTER 5

M-Theory on Twisted Connected Sum G_2-Manifolds

The existence of M-theory, which is a theory that unifies five consistent versions of superstring theory, was first conjectured by Edward Witten in the year of 1995 [Wit95b]. Prior to Witten's work, string theorists found that apparently distinct theories could be identified by mathematical transformations called S-duality and T-duality. However, a complete formulation of M-theory is unknown up-to-date. In one approach to M-theory phenomenology, physicists assume that the seven extra dimensions of M-theory are behaved like a G_2-manifold. Due to the Kovalev's twisted gluing of two products of compatible non-compact asymptotically cylindrical Calabi-Yau 3-folds and an circle, we can study the globally consistent four-dimensional $\mathcal{N} = 1$ supersymmetric M-theory compactifications on G_2-manifolds via the twisted connected sum construction. Since an unique eleven dimensional supergravity action exists, it is supposed to be the effective description of M-theory before the compactification. Therefore we are using much of the properties of the four dimensional theory by the Kaluza Klein reduction of this 11d supergravity to four dimensions.

5.1 M-theory on G_2-manifolds

5.1.1 M-theory

M-theory is an 11-dimensional quantum theory of gravity which includes gravitons, particle-like excitations, and other extended objects known as membranes and five-branes. Although a complete definition of M-theory is not yet known, it seems to unify the three greatest theories of modern theoretic physics:

General relativity: the description of gravity in terms of geometry objects of space-time, i.e. metric and curvature, by the Einstein's equation.
Gauge theory: the description of fundamental forces between elementary particles in terms of connections of some vector bundles.
String theory: a natural generalization of point particles.

Moreover, all five 10-dimensional perturbative superstring theories: type IIA and IIB, type I, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic, have a strong coupling non-perturbative limit whose low energy effective field theory description is the 11-dimensional supergravity theory and which can reduce to the various string theories by Kaluza-Klein compactification, followed by various string dualities. The connections

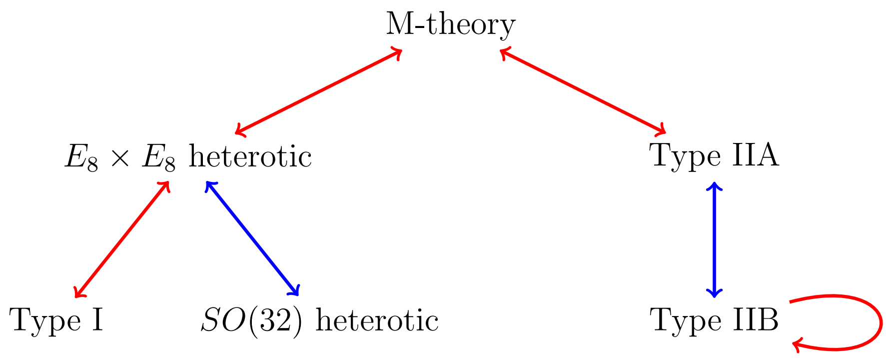

Figure 5.1: A diagram of string theory dualities.

among those theories can be sketched as the diagram 5.1. Here Blue arrows indicate T-duality and Red arrows indicate S-duality.

The M-theory has 11-dimensional supergravity for its low energy limit, and the geometry of the low energy effective action of M-theory is described by an 11-dimensional Lorentz manifold $M^{1,10}$ together with a four-form flux G of an anti-symmetric three-form tensor field C, i.e. $G = \mathrm{d}C$. Since spinors exists in this theory we can assume the Lorentz manifold $M^{1,10}$ is a spin manifold, and thus the first Pontryagin class $p_1(M)$ is divisible by two. We set

$$\lambda(M) = \frac{p_1(M)}{2}, \tag{5.1}$$

and imposes the cohomological flux quantization condition [Wit97]

$$\left[\frac{G}{2\pi}\right] - \frac{\lambda(M)}{2} \in H^4(M;\mathbb{Z}). \tag{5.2}$$

For the 11-dimensional Lorentz manifold $M^{1,10}$ we consider the compactification

$$M^{1,10} = \mathbb{R}^{1,3} \times Y, \tag{5.3}$$

with the 7-dimensional compact smooth manifold Y and the 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ equipped with $\mathcal{N} = 1$ supersymmetry. With the vanishing background fluxes the internal space Y must be a G_2-manifold due to $\mathcal{N} = 1$ supersymmetry vacuum states in the Minkowski space $\mathbb{R}^{1,3}$ [AOS97].

5.1.2 The Kaluza–Klein reduction

Choose the local coordinates $\{x^\mu, y^m\}$ of the 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ with the flat space-time metric $\eta_{\mu\nu}$ and the 7-dimensional compact G_2-manifold Y with the Ricci-flat Riemannian metric g_{mn}, respectively. On the compactification ansatz $M^{1,10} = \mathbb{R}^{1,3} \times Y$, we consider the diagonal metric:

$$\hat{g}(x,y) = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n, \tag{5.4}$$

and the deformation of the background metric, i.e., $\hat{g} \to \hat{g} + \delta\hat{g}$. Then the massless gravitational Kaluza–Klein scalars S^i arise from harmonic 3-forms $\rho_i^{(3)}$, which are solutions to the Einstein equations to linear order in the sense

$$\mathrm{Ric}\left(g + \sum S^i\,\rho_i^{(3)}\right) \;=\; 0 \tag{5.5}$$

and represent a basis for the vector space $H^3(Y)$ of dimension $b_3(Y)$. As discussed in § 4.1.4, the moduli space of torsion-free G_2-structures on Y is $H^3(Y)$, thus such linear order deformations of torsion-free G_2-structures φ are equivalent to the first order metric deformations. The local structure of the moduli space of G_2-manifolds implies that the first order deformations $\rho_i^{(3)}$ to the torsion-free G_2-structure extend order by order to unobstructed finite deformations, which therefore describe locally the moduli space $\mathcal{M}$ of G_2-manifolds.

The massless modes of the 11-dimensional anti-symmetric 3-form tensor field $\hat{C}$ arise from the expansion

$$\hat{C}(x,y) \;=\; \sum_I A^I(x)\wedge\omega_I^{(2)}(y) + \sum_i P^i(x)\wedge\rho_i^{(3)}(y)\,, \tag{5.6}$$

in terms of the harmonic 2-forms $\omega_I^{(2)}$ and 3-forms $\rho_i^{(3)}$ identified with non-trivial cohomology representatives of $H^2(Y)$ and $H^3(Y)$ of dimension $b_2(Y)$ and $b_3(Y)$, respectively. Thus, as there are no dynamical degrees of freedom in 4-dimensional anti-symmetric three-form tensor field and due to the absence of harmonic 1-forms on the internal G_2-manifolds, the 4-dimensional vectors A^I, $I = 1,\ldots,b_2(Y)$, and the 4-dimensional scalars P^i, $i = 1,\ldots,b_3(Y)$, are the only massless modes obtained from the dimensional reduction of the 11-dimensional anti-symmetric 3-form tensor field $\hat{C}$.

Similarly, by the zero-mode analysis for spinors and the irreducible representations of G_2-manifolds in § 4.1.2 we can spell out the massless four-dimensional spectrum in terms of $\mathcal{N} = 1$ supergravity multiplets as obtained from the dimensional reduction of M-theory, or rather of 11-dimensional supergravity, upon a smooth G_2-manifolds Y. It consists of the 4-dimensional supergravity multiplet, $b_3(Y)$ (neutral) chiral multiples Φ^i, and $b_2(Y)$ (Abelian) vector multiplets V^I, as detailed summarized in Table 5.1.

A $\mathcal{N} = 1$ supergravity theory is specified (at second-order derivative) by a Kähler target space for the massless chiral scalars Φ^i with Kähler potential $K(\Phi,\bar{\Phi})$, a holomorphic gauge kinetic coupling matrix $f_{IJ}(\Phi)$, and a holomorphic superpotential $W(\Phi)$. The action for 11-dimensional supergravity theory [CJS78] for the determined spectrum of the massless fields, inserted by the mode expansions for the metric, the anti-symmetric three-form tensor, and the gravitino, can read

$$\begin{aligned} S_{11d} \;=\; \frac{1}{2\kappa_{11}^2}\int\Big(*_{11}\hat{R}_S - \frac{1}{2}d\hat{C}\wedge *_{11}d\hat{C} - *_{11}i\,\bar{\hat{\Psi}}_M\hat{\Gamma}^{MNP}\hat{D}_N\hat{\Psi}_P\Big) \\ - \frac{1}{192\kappa_{11}^2}\int *_{11}\bar{\hat{\Psi}}_M\hat{\Gamma}^{MNPQRS}\hat{\Psi}_N(d\hat{C})_{[PQRS]} \\ - \frac{1}{12\kappa_{11}^2}\int d\hat{C}\wedge d\hat{C}\wedge\hat{C} + \ldots\,, \end{aligned} \tag{5.7}$$

where spinor conjugation is defined by $\bar{\hat{\Psi}} := \hat{\Psi}^\dagger\hat{\Gamma}^0$ in Minkowskian signature. The first line contains the kinetic terms of the 11-dimensional supergravity multiplet, i.e., the Einstein–Hilbert term in terms of the Ricci scalar $\hat{R}_S$, the kinetic term for the anti-symmetric 3-form tensor $\hat{C}$, and the Rarita–Schwinger kinetic term for the gravitino $\hat{\Psi}$. The second line comprises the interaction terms and the third line is the Chern–Simons term of the 11-dimensional supergravity action. There are additional four-fermion interactions denoted by '...' [CJS78]. The coupling constant κ_{11} appearing in the 11-dimensional supergravity action relates to the 11-dimensional Newton constant $\hat{G}_N$, the 11-dimensional Planck length

Multiplicity	Massless 4d component fields		Massless 4d
	bosonic fields	fermionic fields	$\mathcal{N} = 1$ multiplets
1	metric $g_{\mu\nu}$	gravitino Ψ_μ^α	gravity multiplet
$i = 1, \ldots, b_3(Y)$	scalars (S^i, P^i)	spinors χ_α^i	chiral multiplets Φ^i
$I = 1, \ldots, b_2(Y)$	vectors A_μ^I	gauginos λ_α^I	vector multiplets V^I

Table 5.1: The massless 4-dimensional low-energy effective $\mathcal{N} = 1$ supergravity spectrum.

$\hat{\ell}_P$ and Planck mass $\hat{M}_P$ according to

$$\kappa_{11}^2 = 8\pi \hat{G}_N = \frac{(2\pi)^8 \hat{\ell}_P^9}{2} = \frac{(2\pi)^8}{2 \hat{M}_P^9} . \tag{5.8}$$

Let us introduce the moduli-dependent volume $V_Y(S^i)$ of the G_2-manifold Y given by

$$V_Y(S^i) = \int_Y d^7 y \sqrt{\det g(S)_{mn}} . \tag{5.9}$$

Furthermore, we introduce a reference G_2-manifold Y_0 with respect to some background expectation values $S_0^i = \langle S^i \rangle$, upon which we carry out the dimensional reduction. This allows us to introduce the dimensionless (but yet moduli-dependent) volume factor

$$\lambda_0(S^i) = \frac{V_Y(S^i)}{V_{Y_0}} = \frac{1}{7} \int_Y \varphi \wedge *_{g_\varphi} \varphi , \tag{5.10}$$

in terms of the reference volume $V_{Y_0} = V_Y(S_0^i)$. Here the choice of Y_0 fixes via the resulting volume factor V_{Y_0} the normalization of the three-form φ.

To perform the Kaluza–Klein reduction, we have used the Weyl rescaling of the four-dimensional metric according to

$$g_{\mu\nu} \to \frac{g_{\mu\nu}}{\sqrt{\lambda_0(S^i)}} , \tag{5.11}$$

such that the 4-dimensional coupling constant κ_4 — relating to the four-dimensional Newton constant G_N, the 4-dimensional Planck length ℓ_P and the Planck mass M_P — becomes

$$\kappa_4^2 = \frac{\kappa_{11}^2}{V_{Y_0}} , \qquad \kappa_4^2 = 8\pi G_N = 8\pi \ell_P^2 = \frac{8\pi}{M_P^2} . \tag{5.12}$$

The 4-dimensional bosonic action under the dimensional reduction of the Einstein–Hilbert term and the 3-form tensor $\hat{C}$ yields

$$\begin{aligned} S_{4d}^{\text{bos}} = \frac{1}{2\kappa_4^2} \int \Big[*_4 R_S &+ \frac{\kappa_{IJk}}{2V_{Y_0}} \left(S^k F^I \wedge *_4 F^J - P^k F^I \wedge F^J \right) \\ &- \frac{7}{2V_{Y_0}} \int \rho_i^{(3)} \wedge *_7 \rho_j^{(3)} \left(dP^i \wedge *_4 dP^j - dS^i \wedge *_4 dS^j \right) \Big] . \end{aligned} \tag{5.13}$$

in terms of the four-dimensional Hodge star $*_4$, the Ricci scalar R_S with respect to the metric $g_{\mu\nu}$, the reference volume V_{Y_0}, and the seven-dimensional Hodge star $*_7$. Here the couplings κ_{IJk} arise from the topological intersection numbers

$$\kappa_{IJk} \ = \ \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} \tag{5.14}$$

and we have also used the following identification

$$\partial_i \partial_j \log \lambda_0 + \partial_i \log \lambda_0 \partial_j \log \lambda_0 = \frac{2}{7\lambda_0} \int \rho_i^{(3)} \wedge *_7 \rho_j^{(3)} \ . \tag{5.15}$$

We can now bring the (bosonic) action into the conventional form of 4-dimensional $\mathcal{N} = 1$ supergravity [WB92]. To identify the chiral multiplets, that is to say to identify the complex structure of the Kähler target space, we observe that — at least to the leading order — the action of the membrane instantons generating non-perturbative superpotential interactions is given by [HM99]

$$\phi^i = P^i + iS^i \ . \tag{5.16}$$

Hence, due to holomorphy of the $\mathcal{N} = 1$ superpotential, the complex fields ϕ^i furnish complex coordinates of the Kähler target space and thus represent the complex scalar fields in the $\mathcal{N} = 1$ chiral multiplets Φ^i in Table 5.1. This allows us to readily read off from the action the Kähler potential and the gauge kinetic coupling matrix [BW02]

$$K(\phi, \bar{\phi}) \ = \ -3 \log \left(\frac{1}{7} \int_Y \varphi \wedge *_7 \, \varphi \right) , \tag{5.17}$$

$$f_{IJ}(\phi) \ = \ \sum_k 2V_{Y_0} \phi^k \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} \ = \ 2V_{Y_0} \sum_k \kappa_{IJk} \phi^k . \tag{5.18}$$

Note that the holomorphy of the gauge kinetic coupling matrix is in accord with the complex chiral coordinates (5.16). The moduli space metric is then given by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \frac{1}{2V_{Y_0} \lambda_0^2} \int \rho_i^{(3)} \wedge *_7 \rho_j^{(3)} \ . \tag{5.19}$$

Thus we see that in the physical theory the real scalar fields S^i and P^i combine to the complex chiral scalars ϕ^i according to eq. (5.16). These complex scalar fields parametrize locally the (semi-classical) M-theory moduli space $\mathcal{M}_{\mathbb{C}}$ of the G_2 compactification on Y of complex dimension $b_3(Y)$, where the real subspace $\mathrm{Re}(\phi^i) = 0$ of real dimension $b_3(Y)$ is the geometric moduli space $\mathcal{M}$ of G_2 metrics on Y. Note, however, that the derived moduli space $\mathcal{M}_{\mathbb{C}}$ merely arises from the semi-classical dimensional reduction of eleven-dimensional supergravity on the G_2 manifold Y. For the resulting four-dimensional $N = 1$ supersymmetric theory, one expects on general grounds that the flat directions of $\mathcal{M}_{\mathbb{C}}$ are lifted at the quantum level due to non-perturbative effects in M-theory [HM99] — even in the absence of background fluxes.

Finally, let us remark that the presence of non-trivial four-form fluxes G supported on the G_2 manifold Y generates a flux-induced superpotential [BW02]. While the superpotential enters quadratically in the bosonic action, it appears linearly in the fermionic action generating a gravitino mass term M_Ψ [WB92]

$$\mathcal{L}_{4d}^{M_\Psi} = \frac{1}{2} e^{K/2} \left(\bar{W}(\bar{\phi})\, \Psi_\mu^T \gamma^{\mu\nu} \Psi_\nu + W(\phi)\, \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu^* \right) . \tag{5.20}$$

This linear dependence on W allows us to directly derive the superpotential form the dimensional reduction of the gravitino terms. Thus, we obtain the holomorphic superpotential to be

$$W(\phi) = \sum_i \phi^i \int_Y \rho_i^{(3)} \wedge G . \tag{5.21}$$

Our result is in agreement with the derivation of the flux-induced superpotential in refs. [BW02]. Note that — both in the presence and in the absence of background fluxes G — we expect generically additional non-perturbative superpotential contributions arising from membrane instanton effects [HM99].

5.2 Hitchin functional on twisted connected sum G_2-manifolds

First we recall the definition of the Hitchin functional on a real 7-dimensional manifold Y [Hit00; Hit01].

Definition 5.1. Let Y be a real 7-dimensional manifold and $\varphi \in \bigwedge^3 \Gamma(T^*Y)$ be a smooth stable 3-form on Y, i.e. it lies in an open orbit of $GL(Y)$. Thus φ can determine a Riemannian metric g_φ. Then the Hitchin functional Θ is defined on $C^\infty \left(\bigwedge^3 \Gamma(T^*Y) \right)$ by

$$\Theta(\varphi) := \frac{1}{7} \int_Y \varphi \wedge *_{g_\varphi} \varphi.$$

Thus $\Theta(\varphi)$ is the total volume of Y with respect to the metric and orientation determined by φ on Y.

Hitchin proved that a closed stable 3-form φ is a critical point of $\Theta(\varphi)$ in its cohomology class if and only if φ is co-closed, i.e. $\mathrm{d}(*_{g_\varphi} \varphi) = 0$ [Hit01, Theorem 1]. In other word, when restricted to closed G_2-structure φ in a fixed cohomology class, the torsion-free G_2-structures are the critical points of Θ [Hit00, Theorem 19]. On the other hand, from the equation (5.17) we can see that the Hitchin functional determines the Kähler potential in the action of 4-dimensional $\mathcal{N} = 1$ supergravity obtained by the Kaluza–Klein reduction on smooth G_2-manifolds. Due to the Kovalev's construction of compact G_2-manifolds, it allows us to analyze the Hitchin functional on twisted connected sum G_2-manifolds.

Recall that a G_2-manifold Y_r obtained by the twisted connected sum constructions has the decomposition $Y_r = Y_L \cup Y_R$ with the common intersection $Y_L \cap Y_R \simeq S \times T^2$ as discussed in § 4.2. Hence we can decompose the Hitch functional as

$$\begin{aligned} \int_{Y_r = Y_L \cup Y_R} \varphi \wedge *\varphi = & \int_{\left\{K_L \cup (0, \gamma T - 1] \times S_L^{1*} \times S_L\right\} \times S_L^1} \varphi_L \wedge *\varphi_L + \int_{\left\{K_R \cup (0, \gamma T - 1] \times S_R^{1*} \times S_R\right\} \times S_R^1} \varphi_R \wedge *\varphi_R \\ & + \int_{X_L^\infty|_{(\gamma T - 1, \gamma T]} \times S_L^1} \varphi_L \wedge *\varphi_L + \int_{X_R^\infty|_{(\gamma T - 1, \gamma T]} \times S_R^1} \varphi_R \wedge *\varphi_R \\ & + \frac{1}{2} \int_{X_L^\infty|_{(\gamma T, \gamma T + 1]} \times S_L^1} \varphi_L \wedge *\varphi_L + \frac{1}{2} \int_{X_L^\infty|_{(\gamma T, \gamma T + 1]} \times S_R^1} \varphi_R \wedge *\varphi_R , \end{aligned} \tag{5.22}$$

where the first line gives the contribution from the union of the compact subspaces of the asymptotically cylinder CYs $K_{L/R} \subset X_{L/R}$ times an $S^1_{L/R}$ and the CY cylinder region in the interval $(0, \gamma T - 1]$, the second line gives the contribution from the CY cylinder region restricted to the part of the interval $(\gamma T - 1, \gamma T]$, and the last line gives the contribution from the asymptotic ends of the CY cylinders in the interval $(\gamma T, \gamma T + 1]$. The extra factor of $1/2$ in the last line guarantees that we are not overcounting contributions from the left- and right-sides as they are glued together in these region.

In the following, we perform the computation for the left-side only. We discuss the final expression for the Hitchin functional after taking into account the right-side, whose terms can be computed analogously to the terms for the left-side.

5.2.1 Region $K_{L/R} \cup (0, \gamma T - 1] \times S^{1*}_{L/R} \times S_{L/R}$

We start with the Hitchin functional contributed from the product space of the compact subspace K_L and the circle S^1_L which is given by

$$\begin{aligned}\int_{K_L \times S^1_L} \varphi_L \wedge *\varphi_L &= \int_{K_L} \int_0^{2\pi\gamma} \left(\frac{1}{2} \omega_L|^3_{K_L} \wedge d\theta_L + \mathrm{Re}\Omega_L|_{K_L} \wedge \mathrm{Im}\Omega_L|_{K_L} \wedge d\theta_L \right) \\ &= \pi\gamma \int_{K_L} \omega_L|^3_{K_L} + 2\pi\gamma \int_{K_L} \mathrm{Re}\Omega_L|_{K_L} \wedge \mathrm{Im}\Omega_L|_{K_L} \, .\end{aligned} \tag{5.23}$$

Here the product G_2 3-form φ_L and the associated 4-form $*\varphi_L$ in (4.3) are given by

$$\begin{aligned}\varphi_L &= d\theta_L \wedge \omega_L + \mathrm{Re}\Omega_L \, , \\ *\varphi_L &= \frac{1}{2}(\omega_L)^2 - d\theta_L \wedge \mathrm{Im}\Omega_L \, .\end{aligned} \tag{5.24}$$

In the cylinder region $(0, \gamma T - 1]$, the $SU(3)$-structure (ω^T_L, Ω^T_L) in (4.2) are given by

$$\begin{aligned}\omega^T_L &= \omega^\infty_L + d\mu_L := dt \wedge d\theta^*_L + \omega^S_L + d\mu_L \, , \\ \Omega^T_L &= \Omega^\infty_L + d\nu_L := d\theta^*_L \wedge \Omega^S_L - i dt \wedge \Omega^S_L + d\nu_L \, ,\end{aligned} \tag{5.25}$$

and the induced G_2 3-form and 4-form become

$$\begin{aligned}\varphi_L &= d\theta_L \wedge \omega^T_L + \mathrm{Re}\Omega^T_L \, , \\ *\varphi_L &= \frac{1}{2}(\omega^T_L)^2 - d\theta_L \wedge \mathrm{Im}\Omega^T_L \, .\end{aligned} \tag{5.26}$$

The contribution from this CY cylinder region in $(0, \gamma T - 1]$ is then rewritten as

$$\begin{aligned}\int_{(0,T-1] \times S^{1*}_L \times S_L \times S^1_L} \varphi_L \wedge *\varphi_L = &\int_{(0,T-1] \times S^{1*}_L \times S_L \times S^1_L} (d\theta_L \wedge \omega^\infty_L + \mathrm{Re}\Omega^\infty_L) \wedge \left[\frac{1}{2}(\omega^\infty_L)^2 - d\theta_L \wedge \mathrm{Im}\Omega^\infty_L \right] \\ &+ \int_{(0,T-1] \times S^{1*}_L \times S_L \times S^1_L} F(d\mu_L, d\nu_L) \, ,\end{aligned} \tag{5.27}$$

where the correction term $F(d\mu_L, d\nu_L)$ is given by

$$\begin{aligned} F(d\mu_L, d\nu_L) =&(d\theta_L \wedge \omega_L^\infty + \mathrm{Re}\Omega_L^\infty) \wedge \left[\omega_L^\infty \wedge d\mu_L + \frac{1}{2}(d\mu_L)^2 - d\theta_L \wedge \mathrm{Im}d\nu_L\right] \\ &+ (d\theta_L \wedge d\mu_L + \mathrm{Re}d\nu_L) \wedge \left[\frac{1}{2}(\omega_L^\infty + d\mu_L)^2 - d\theta_L \wedge \mathrm{Im}\Omega_L^\infty - d\theta_L \wedge \mathrm{Im}d\nu_L\right] . \end{aligned} \tag{5.28}$$

Inserting the equations (5.25) for $(\omega_L^\infty, \Omega_L^\infty)$ into the first line of above integration implies

$$\begin{aligned} \int\limits_{(0,\gamma T-1]\times S_L^{1*}\times S_L\times S_L^1} & (d\theta_L \wedge \omega_L^\infty + \mathrm{Re}\Omega_L^\infty) \wedge \left[\frac{1}{2}(\omega_L^\infty)^2 - d\theta_L \wedge \mathrm{Im}\Omega_L^\infty\right] \\ &= 4\pi^2\gamma^2(\gamma T - 1) \int\limits_{S_L} \left[\frac{3}{2}(\omega_L^S)^2 + (\mathrm{Re}\Omega_L^S)^2 + (\mathrm{Im}\Omega_L^S)^2\right]. \end{aligned} \tag{5.29}$$

Now the integral of the correction term $F(d\mu_L, d\nu_L)$ can be reduced by degree counting to this form

$$\begin{aligned} \int\limits_{(0,\gamma T-1]\times S_L^{1*}\times S_L\times S_L^1} F(d\mu_L, d\nu_L) = & \int\limits_0^{\gamma T-1} \int\limits_0^{2\pi\gamma} \int\limits_{S_L} \int\limits_0^{2\pi\gamma} \Big[\frac{3}{2}(\omega_L^\infty)^2 \wedge d\mu_L \wedge d\theta_L \\ &+ \mathrm{Re}d\nu_L \wedge \mathrm{Im}\Omega_L^\infty \wedge d\theta_L - \mathrm{Im}d\nu_L \wedge \mathrm{Re}\Omega_L^\infty \wedge d\theta_L \\ &+ \mathrm{Re}d\nu_L \wedge \mathrm{Im}d\nu_L \wedge d\theta_L\Big]. \end{aligned} \tag{5.30}$$

By Stokes's theorem, the first integral above can be evaluated as

$$\begin{aligned} \int\limits_{(0,\gamma T-1]\times S_L^{1*}\times S_L\times S_L^1} & (\omega_L^\infty)^2 \wedge d\mu_L \wedge d\theta_L \\ &= 2\pi\gamma \int\limits_{(0,\gamma T-1]\times S_L^{1*}\times S_L} [2dt \wedge d\theta^* \wedge \omega_L^S \wedge d\mu_L + (\omega_L^S)^2 \wedge d\mu_L] \\ &= 2\pi\gamma \int\limits_{S_L^{1*}\times S_L} \left(\mu_{L|t=\gamma T-1} - \underline{\mu_{L|t=0}}\right) \wedge (\omega_L^S)^2 \end{aligned} \tag{5.31}$$

Due to continuity, the underlined terms must be canceled by part of the contribution from the compact region K_L.

Using the fact that for any smooth complex function $f = f_1 + if_2$, $\mathrm{Re}(df) = d(\mathrm{Re}f)$ and $\mathrm{Im}(df) = d(\mathrm{Im}f)$, we can rewrite

$$\int \mathrm{Re}d\nu_L \wedge \mathrm{Im}\Omega_L^\infty \wedge d\theta_L = \int d(\mathrm{Re}\nu_L) \wedge \mathrm{Im}\Omega_L^\infty \wedge d\theta_L, \tag{5.32}$$

$$\int \mathrm{Im}d\nu_L \wedge \mathrm{Re}\Omega_L^\infty \wedge d\theta_L = \int d(\mathrm{Im}\nu_L) \wedge \mathrm{Re}\Omega_L^\infty \wedge d\theta_L, \tag{5.33}$$

$$\int \mathrm{Re}d\nu_L \wedge \mathrm{Im}d\nu_L \wedge d\theta_L = \int d(\mathrm{Re}\nu_L) \wedge d(\mathrm{Im}\nu_L) \wedge d\theta_L, \tag{5.34}$$

and again applying Stoke's theorem the first integral above is expressed by

$$\begin{aligned}
&\int_0^{\gamma T-1}\int_0^{2\pi\gamma}\int_{S_L}\int_0^{2\pi\gamma} \mathrm{Re}d\nu_L \wedge \mathrm{Im}\Omega_L^{\infty} \wedge d\theta_L = \int_0^{\gamma T-1}\int_0^{2\pi\gamma}\int_{S_L}\int_0^{2\pi\gamma} d(\mathrm{Re}\nu_L) \wedge \mathrm{Im}\Omega_L^{\infty} \wedge d\theta_L \\
&= \int_0^{\gamma T-1}\int_0^{2\pi\gamma}\int_{S_L}\int_0^{2\pi\gamma} \Big[d(\mathrm{Re}\nu_L) \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S \wedge d\theta_L - d(\mathrm{Re}\nu_L) \wedge dt \wedge \mathrm{Re}\Omega_L^S \wedge d\theta_L\Big] \\
&= 2\pi\gamma \int_0^{2\pi\gamma}\int_{S_L} \Big(\mathrm{Re}\nu_{L|t=\gamma T-1} - \underline{\mathrm{Re}\nu_{L|t=0}}\Big) \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S \,,
\end{aligned} \tag{5.35}$$

where the underlined terms also would be canceled by part of the contribution from the boundary of compact region K_L.

In a similar way, we obtain the remaining terms

$$\int_0^{\gamma T-1}\int_0^{2\pi\gamma}\int_{S_L}\int_0^{2\pi\gamma} \mathrm{Im}d\nu_L \wedge \mathrm{Re}\Omega_L^{\infty} \wedge d\theta_L = 2\pi\gamma \int_0^{2\pi\gamma}\int_{S_L} (\mathrm{Im}\nu_{L|t=\gamma T-1} - \underline{\mathrm{Im}\nu_{L|t=0}}) \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S \,, \tag{5.36}$$

$$\int_0^{\gamma T-1}\int_0^{2\pi\gamma}\int_{S_L}\int_0^{2\pi\gamma} \mathrm{Re}d\nu_L \wedge \mathrm{Im}d\nu_L \wedge d\theta_L = 0. \tag{5.37}$$

Then put all together, we have

$$\begin{aligned}
&\int_{\{K_L\cup(S_L^{1*}\times S_L\times(0,\gamma T-1])\}\times S_L^1} \varphi\wedge *\varphi = \pi\gamma \int_{K_L\backslash\partial K_L} \omega_{L|K_L}^3 + 2\pi\gamma \int_{K_L\backslash\partial K_L} \mathrm{Re}\Omega_{L|K_L} \wedge \mathrm{Im}\Omega_{L|K_L} \\
&+ 4\pi^2(\gamma T-1)\gamma^2 \int_{S_L} \left[\frac{3}{2}(\omega_L^S)^2 + (\mathrm{Re}\Omega_L^S)^2 + (\mathrm{Im}\Omega_L^S)^2\right] + 2\pi\gamma \int_{S_L^{1*}\times S_L} \mu_{L|t=\gamma T-1} \wedge (\omega_L^S)^2 \\
&+ 2\pi\gamma \int_0^{2\pi\gamma}\int_{S_L} \mathrm{Re}\nu_{L|t=\gamma T-1} \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S + 2\pi\gamma \int_0^{2\pi\gamma}\int_{S_L} \mathrm{Im}\nu_{L|t=\gamma T-1} \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S \,.
\end{aligned} \tag{5.38}$$

5.2.2 Region $X_{L/R}^{\infty}|_{(\gamma T-1,\gamma T]} \times S_{L/R}^1$

We now turn to the region $X_{L/R}^{\infty}|_{(\gamma T-1,\gamma T]} \times S_{L/R}^1$. Due to the existence of the cut-off function α, we have

$$\begin{aligned}
\omega_L^T &= \omega_L^{\infty} + d\tilde{\mu}_L := dt\wedge d\theta_L^* + \omega_L^S + d[1-\alpha(t-\gamma T+1)]\mu_L \,, \\
\Omega_L^T &= \Omega_L^{\infty} + d\tilde{\nu}_L := d\theta_L^* \wedge \Omega_L^S - i dt\wedge \Omega_L^S + d[1-\alpha(t-\gamma T+1)]\nu_L \,.
\end{aligned} \tag{5.39}$$

The product G_2 3-form φ_L and 4-forms $*\varphi_L$ are given by, respectively,

$$\varphi_L = d\theta_L \wedge \omega_L^T + \mathrm{Re}\Omega_L^T\,, \tag{5.40}$$

$$*\varphi_L = \tfrac{1}{2}(\omega_L^T)^2 - d\theta_L \wedge \mathrm{Im}\Omega_L^T\,. \tag{5.41}$$

Note that this contribution to the Hitchin functional is the same as the one we performed in previous subsection in the region $S_L^{1*} \times S_L \times (0, \gamma T - 1]) \times S_L^1$ for

$$\int\limits_{(S_L^{1*}\times S_L\times(0,\gamma T-1])\times S_L^1} \varphi_L \wedge *\varphi_L\,, \tag{5.42}$$

but now with different correction term $F := F(d\tilde{\mu}_L, d\tilde{\nu}_L)$ and in the region $(\gamma T - 1, \gamma T]$ instead of $(0, \gamma T - 1]$. Hence the Hitchin functional contributed from this region would be

$$\begin{aligned}
\int\limits_{\left\{S_L^{1*}\times S_L\times(\gamma T-1,\gamma T]\right\}\times S_L^1} \varphi_L \wedge *\varphi_L = {} & 4\pi^2\gamma^2 \int\limits_{S_L} \left[\frac{3}{2}(\omega_L^S)^2 + (\mathrm{Re}\Omega_L^S)^2 + (\mathrm{Im}\Omega_L^S)^2\right] \\
& + 2\pi\gamma \int\limits_{S_L^{1*}\times S_L} (\cancelto{0}{\tilde{\mu}_{L|t=\gamma T}} - \underline{\tilde{\mu}_{L|t=\gamma T-1}}) \wedge (\omega_L^S)^2 \\
& + 2\pi\gamma \int_0^{2\pi\gamma}\int\limits_{S_L} (\cancelto{0}{\mathrm{Re}\tilde{\nu}_{L|t=\gamma T}} - \underline{\mathrm{Re}\tilde{\nu}_{L|t=\gamma T-1}}) \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S \\
& + 2\pi\gamma \int_0^{2\pi\gamma}\int\limits_{S_L} (\cancelto{0}{\mathrm{Im}\tilde{\nu}_{L|t=\gamma T}} - \underline{\mathrm{Im}\tilde{\nu}_{L|t=\gamma T-1}}) \wedge d\theta_L^* \wedge \mathrm{Im}\Omega_L^S
\end{aligned} \tag{5.43}$$

Due to continuity, the underlined terms must be canceled by the contribution from the CY cylinder an $t = \gamma T - 1$ we computed in the region $(0, \gamma T - 1]$ because $\tilde{\mu} = \mu$ and $\tilde{\nu} = \nu$ at $t = \gamma T - 1$. The terms that are crossed vanish since $d\tilde{\mu}_L = d\tilde{\nu}_L = 0$ at $t = \gamma T$.

5.2.3 Region $X^\infty_{L/R}|_{(\gamma T, \gamma T+1]} \times S^1_{L/R}$

In this gluing region, the $SU(3)$-structure is expressed by

$$\begin{aligned}
\omega_L^T &= \omega_L^\infty = dt \wedge d\theta_L^* + \omega_L^S\,, \\
\Omega_L^T &= \Omega_L^\infty = d\theta_L^* \wedge \Omega_L^S - i dt \wedge \Omega_L^S\,,
\end{aligned} \tag{5.44}$$

and thus the induced G_2-structure is given by

$$\varphi_L = d\theta_L \wedge \omega_L^T + \mathrm{Re}\Omega_L^T\,, \tag{5.45}$$

$$*\varphi_L = \tfrac{1}{2}(\omega_L^T)^2 - d\theta_L \wedge \mathrm{Im}\Omega_L^T + \tilde{*}\varphi_L^T\,, \tag{5.46}$$

where the extra term $\tilde{*}\varphi_L^T$ in φ_L is due to existence of the cut-off function that modifies the metric, thus the Hodge star $*$, and is of order $O(e^{-(\lambda-\epsilon)\gamma T})$ for any $\lambda > 0$ as introduced in § 4.2.3 with $\epsilon > 0$ (see [Kov03, Lemma 4.25]).

By the same manner as before, we can calculate the Hitchin functional in this gluing region

$$
\begin{aligned}
\int\limits_{X_L^\infty|_{(\gamma T,\gamma T+1]}\times S_L^1} \varphi_L \wedge *\varphi_L = & \int\limits_0^{2\pi\gamma}\int\limits_{S_L}\int\limits_0^{2\pi\gamma}\int\limits_{\gamma T}^{\gamma T+1} \Big[\frac{3}{2} d\theta^* \wedge (\omega_L^S)^2 \wedge d\theta \wedge dt \\
& + d\theta^* \wedge (\mathrm{Re}\Omega_L^S)^2 \wedge d\theta \wedge dt + d\theta^* \wedge (\mathrm{Im}\Omega_L^S)^2 \wedge d\theta \wedge dt\Big] \\
& + \int\limits_0^{2\pi\gamma}\int\limits_{S_L}\int\limits_0^{2\pi\gamma}\int\limits_{\gamma T}^{\gamma T+1} \varphi_L \wedge \tilde{*}\varphi_L \\
= & \, 4\pi^2\gamma^2 \int\limits_{S_L} \Big[\frac{3}{2}(\omega_L^S)^2 + (\mathrm{Re}\Omega_L^S)^2 + (\mathrm{Im}\Omega_L^S)^2\Big] \\
& + O\left(e^{-(\lambda-\epsilon)\gamma T}\right) .
\end{aligned} \tag{5.47}
$$

Note that the last term can be neglected for large enough T.

Now taking account of (5.38), (5.43), and (5.47), the Hitchin functional of a G_2-structure φ on the twisted connected sum G_2-manifold Y_r is given by, for T large enough,

$$
\begin{aligned}
\Theta(\varphi_T) = & \frac{1}{7} \int\limits_{Y_r = Y_L \cup Y_R} \varphi_T \wedge *\varphi_T \\
= & \frac{1}{7}\Bigg(\pi\gamma \int\limits_{K_L} \omega_{L|K_L}^3 + 2\pi\gamma \int\limits_{K_L} \mathrm{Re}\Omega_{L|K_L} \wedge \mathrm{Im}\Omega_{L|K_L} \\
& + 4\pi^2(\gamma T + 1)\gamma^2 \int\limits_{S_L} \Big[\frac{3}{2}(\omega_L^S)^2 + (\mathrm{Re}\Omega_L^S)^2 + (\mathrm{Im}\Omega_L^S)^2\Big] + O\left(e^{-(\lambda-\epsilon)\gamma T}\right)\Bigg) \\
& + (\text{right hand side}).
\end{aligned} \tag{5.48}
$$

Remark 5.2. As T approaches ∞, the G_2-structure φ_T would become torsion-free since $\tilde{*}\varphi_T \to 0$, thus is a critical point of the Hitchin functional. However, due to appearance of the term $4\pi^2(\gamma T + 1)\gamma^2$ it turns out that the Hitchin functional would mainly depend on the volume of the K3 surface glued together and diverge as $T \to \infty$. On the other hand, because of lack of further information about the compact subspaces $K_{L/R} \subset X_{L/R}$, in general we have no method to determine the integration over $K_{L/R}$.

5.3 Hitchin functional and the Kähler potential

Recall that the moduli space $\mathcal{M}$ of torsion-free G_2-structures on a compact 7-manifold Y is a smooth manifold of dimension $b_3(Y)$. The period map $P : \mathcal{M} \to H^3(Y, \mathbb{R})$, $\varphi \mapsto [\varphi]$, is a local diffeomorphism. The Hitchin functional, hence the Kähler potential, is governed by the G_2 3-form φ, which on general grounds in $\mathcal{M}$ enjoys the expansion

$$
\varphi(t) = \sum_{A=1}^{b_3} \varphi^A(t)\Theta_A \,, \tag{5.49}
$$

in terms of a basis Θ_A, $A = 1, \ldots, b_3$, of $H^3(Y,\mathbb{Q})$. In the twisted connected sum construction, we can further decompose the basis elements Θ_A according to the decomposition obtained by (4.8) tensored with rational number $\mathbb{Q}$:

$$H^3(Y,\mathbb{Q}) \;=\; H^3(Z_L) \oplus H^3(Z_R) \oplus K_L \oplus K_R \oplus T_L \cap N_R \oplus T_R \cap N_L \oplus T_{L\cup R} \oplus H^0(S)\,, \tag{5.50}$$

which are respectively generated by the cohomology representatives

$$\langle\!\langle \Theta_A \rangle\!\rangle \;=\; \langle\!\langle \theta^L_l \rangle\!\rangle \oplus \langle\!\langle \theta^R_r \rangle\!\rangle \oplus \langle\!\langle \omega^L_{a_l} \rangle\!\rangle \oplus \langle\!\langle \omega^R_{a_r} \rangle\!\rangle \oplus \langle\!\langle \tau^L_{i_l} \rangle\!\rangle \oplus \langle\!\langle \tau^R_{i_r} \rangle\!\rangle \oplus \langle\!\langle \tau^{L\cup R}_j \rangle\!\rangle \oplus \langle\!\langle V^{K3} \rangle\!\rangle\,. \tag{5.51}$$

It is convenient to think in terms of the Poincaré dual 4-cycles Γ_A in $H_4(Y,\mathbb{Q})$ for the various 3-form contributions, i.e., $\Theta_A = [\Gamma_A]$. Then Poincaré duality of the individual cohomology elements tells us:

- As $H^3(Z_L)$ can be represented by the cohomology group $H^3_c(X_L)$ with compact support up to elements in T_L by the decomposition $H^3_c(X_L) \simeq H^3(Z_L) \oplus T_L$, the cohomology elements θ^L_l are Poincaré dual to 4-cycles $S^1_L \times \gamma^L_l$ in $H_4(Y_L) = H_4(S^1_L \times X_L)$ with a basis of homology 3-cycles γ^L_l in $H_3(X_L)$. Analogously, we find $\theta^R_r = [S^1_R \times \gamma^R_r]$.
- The 3-forms $\omega^L_{a_l}$ arise from 2-forms in $H^2_c(X_L)$. Hence, we find $[\omega^L_{a_l}] = [\Gamma^L_{a_l}]$ with a basis of Poincaré dual homology 4-cycles in $H_4(X_L)$. Analogously, $[\omega^R_{a_r}] = [\Gamma^R_{a_r}]$.
- The 3-forms $\tau^L_{i_l}$ in $Y_L = S^1_{\text{Diag}} \times X_L$ arise form cohomology 2-forms X_L that do not have compact support. Therefore, they are Poincaré dual to relative homology elements $C^L_{i_l}$ in $H_4(X_L,\partial X_L)$ with $0 \neq \partial C^L_{i_l} = S^1_{\text{Diag}} \times \mathcal{T}^L_{i_L}$, where $\mathcal{T}^L_{i_L}$ are transcendental 2-cycles with respect to the K3 surface S_L and Picard 2-cycle with respect to the K3 surface S_R. Analogously, we construct Poincaré dual relative four cycles $C^R_{i_R}$ given rise to transcendental and Picard 2-cycles $\mathcal{T}^R_{i_R}$ in S_R and S_L, respectively.
- Similarly, the 3-forms $\tau^{L\cup R}_j$ yield relative 4-cycles $C^{L\cup R}_j$ that give rise to boundary 3-cycles $\mathcal{T}^{L\cup R}_j$ with $\partial\mathcal{T}^{L\cup R}_j = S^1_{\text{Diag}} \times \mathcal{T}^{L\cup R}_j$, where $\mathcal{T}^{L\cup R}_j$ are transcendental 2-cycles with respect to both K3 surfaces S_R and S_L.
- Finally, the 3-form V^{K3} is Poincaré dual to a relative 4-cycles $\mathcal{V}^{K3}$ with $\partial\mathcal{V}^{K3} = S_L \simeq S_R$.

In a similar fashion we now expand the Poincaré dual 4-form $*_{g(\varphi)}\varphi$ into a basis of cohomology 4-forms $\tilde{\Theta}_A$ according to

$$*_{g(\varphi)}\,\varphi \;=\; \sum_{A=1}^{b_3} \tilde{\varphi}^A(t)\tilde{\Theta}_A\,. \tag{5.52}$$

Note that the moduli dependent coefficient functions $\tilde{\varphi}^A(t)$ are complicated functions of the moduli dependent coefficient functions $\varphi^A(t)$. The idea is now to deduce this dependence in a suitable limit of the twisted connected sum construction of G_2 manifolds.

To achieve this goal we now expand the 4-forms $\tilde{\Theta}_A$ into bases governed by the decomposition dual of (5.50)

$$H^4(Y,\mathbb{Q}) \;=\; H^3(Z_L) \oplus H^3(Z_R) \oplus K^*_L \oplus K^*_R \oplus (T_L \cap N_R)^* \oplus (T_R \cap N_L)^* \oplus T^*_{L\cup R} \oplus H^4(S)\,, \tag{5.53}$$

in terms of the bases elements

$$\langle\!\langle \tilde{\Theta}_A \rangle\!\rangle \;=\; \langle\!\langle \tilde{\theta}^L_l \rangle\!\rangle \oplus \langle\!\langle \tilde{\theta}^R_r \rangle\!\rangle \oplus \langle\!\langle \tilde{\omega}^L_{a_l} \rangle\!\rangle \oplus \langle\!\langle \tilde{\omega}^R_{a_r} \rangle\!\rangle \oplus \langle\!\langle \tilde{\tau}^L_{i_l} \rangle\!\rangle \oplus \langle\!\langle \tilde{\tau}^R_{i_r} \rangle\!\rangle \oplus \langle\!\langle \tilde{\tau}^{L\cup R}_j \rangle\!\rangle \oplus \langle\!\langle V^{T^2} \rangle\!\rangle\,. \tag{5.54}$$

As before, we describe the various forms in terms of Poincaré dual 3-cycles $H_3(Y, \mathbb{Q})$. For the individual contributions we find:

- The 4-forms $\tilde{\theta}^L_l$ have compact support and are Poincaré dual to 3-cycles γ^L_l in $H_3(X_L)$, i.e., $\tilde{\theta}^L_l = [\gamma^L_l]$. Analogously, we find $\tilde{\theta}^R_l = [\gamma^R_l]$.
- The 4-forms $\tilde{\omega}^L_{a_l}$ have compact support and are Poincaré dual to 3-cycles $S^1_L \times \tilde{\Gamma}^L_{a_l}$. Analogously, we have $\tilde{\omega}^R_{a_r} = [S^1_R \times \tilde{\Gamma}^R_{a_r}]$.
- The 4-forms $\tilde{\tau}^L_{i_l}$ are Poincaré dual to relative 3-cycles $\widetilde{C}^L_{i_l}$ such that $\partial\widetilde{C}^L_{i_l} = \mathcal{T}^L_{i_L}$. Analogously, the 4-forms $\tilde{\tau}^R_{i_R}$ are Poincaré dual to $\widetilde{C}^R_{i_r}$ with $\partial\widetilde{C}^R_{i_r} = \mathcal{T}^R_{i_r}$.
- The 4-forms $\tilde{\tau}^{L\cup R}_j$ are Poincaré dual to relative 3-cycles $\widetilde{C}^{L\cup R}_j$ such that $\partial\widetilde{C}^{L\cup R}_j = \mathcal{T}^{L\cup R}_j$.
- Finally, the 4-form V^{T^2} is Poincaré dual to a relative 3-cycles $\mathcal{V}^{T_2}$ with $\partial\mathcal{V}^{T_2} = S^1_L \times S^1_R$.

The non-vanishing intersection pairings are given by[1]

$$
\begin{aligned}
G^L_{l_1 l_2} &= \int_Y \theta^L_{l_1} \wedge \tilde{\theta}^L_{l_2} = \int_{X_L} [\gamma^L_{l_1}] \wedge [\gamma^L_{l_2}]\,, & G^R_{r_1 r_2} &= \int_Y \theta^R_{r_1} \wedge \tilde{\theta}^R_{r_2} = \int_{X_R} [\gamma^R_{r_1}] \wedge [\gamma^R_{r_2}]\,,\\
\mathcal{G}^L_{a_l b_l} &= \int_Y \omega^L_{a_l} \wedge \tilde{\omega}^L_{b_l} = \int_{X_L} [\Gamma^L_{a_l}] \wedge [\tilde{\Gamma}^L_{b_l}]\,, & \mathcal{G}^R_{a_r b_r} &= \int_Y \omega^R_{a_r} \wedge \tilde{\omega}^R_{b_r} = \int_{X_R} [\Gamma^R_{a_r}] \wedge [\tilde{\Gamma}^R_{b_r}]\,,\\
g^L_{i_l j_l} &= \int_Y \tau^L_{i_l} \wedge \tilde{\tau}^L_{j_I} = \int_S [\mathcal{T}^L_{i_l}] \wedge [\mathcal{T}^L_{j_l}]\,, & g^R_{i_r j_r} &= \int_Y \tau^R_{i_r} \wedge \tilde{\tau}^R_{j_r} = \int_S [\mathcal{T}^R_{i_r}] \wedge [\mathcal{T}^R_{j_r}]\,,\\
g^{L\cup R}_{ij} &= \int_Y \tau^{L\cup R}_i \wedge \tilde{\tau}^{L\cup R}_j = \int_S [\mathcal{T}^{L\cup R}_i] \wedge [\mathcal{T}^{L\cup R}_j]\,, & &\\
1 &= \int_Y V^{K3} \wedge V^{T^2}\,. & &
\end{aligned}
\tag{5.55}
$$

in terms of intersection matrices with compact support on X_L and X_R and in terms of the intersection pairing on the K3 surface S.

Then we can spell out explicitly the integral

$$
\begin{aligned}
\int_Y \varphi \wedge *_{g(\varphi)}\varphi = {} & G^L_{l_1 l_2}\varphi^{l_1}(t)\tilde{\varphi}^{l_2}(t) + G^R_{r_1 r_2}\varphi^{r_1}(t)\tilde{\varphi}^{r_2}(t) + \\
& G^L_{l_1 l_2}\varphi^{l_1}(t)\tilde{\varphi}^{l_2}(t) + G^R_{r_1 r_2}\varphi^{r_1}(t)\tilde{\varphi}^{r_2}(t) + \\
& g^L_{i_l j_l}\varphi^{i_l}(t)\tilde{\varphi}^{j_l}(t) + G^R_{i_r j_r}\varphi^{i_r}(t)\tilde{\varphi}^{j_r}(t) + \\
& g^{L\cup R}_{ij}\varphi^i(t)\tilde{\varphi}^j(t) + \varphi^0(t)\tilde{\varphi}^0(t)\,,
\end{aligned}
\tag{5.56}
$$

where $\varphi^0(t)$ and $\tilde{\varphi}^0(t)$ is the coefficient function of V^{K3} and V^{T^2}, respectively. We can further simplify this expression by using the limiting form of the G_2 3-form and the associated 4-form φ and $*\varphi$, which are

$$\varphi \simeq \mathrm{d}\theta \wedge \omega|_K \oplus \mathrm{Re}\Omega|_K \oplus \mathrm{d}\theta \wedge \mathrm{d}t \wedge \mathrm{d}\theta^* \oplus \mathrm{d}\theta \wedge \omega^I_S \oplus \mathrm{d}\theta^* \wedge \omega^J_S \oplus \mathrm{d}t \wedge \omega^K_S\,, \tag{5.57}$$

$$*\varphi \simeq \frac{1}{2}(\omega|^2_K) \oplus \mathrm{d}\theta \wedge \mathrm{Im}\Omega|_K \oplus \frac{1}{2}(\omega^I_S)^2 \oplus \mathrm{d}t \wedge \mathrm{d}\theta^* \wedge \omega^I_S \oplus \mathrm{d}\theta \wedge \mathrm{d}t \wedge \omega^J_S \oplus -\mathrm{d}\theta \wedge \mathrm{d}\theta^* \wedge \omega^K_S\,. \tag{5.58}$$

[1] The spelled out definition of the relative cycles are ambiguous up to cycles without boundary contributions. We choose these ambiguities in such away that only the listed intersection pairings are non-vanishing.

Here we assume each forms are represented by the harmonic forms of the the equivalent cohomology classes. If each form in the G_2 3-form φ is represented by the form induced from the Ricci-flat Kähler metrics on $X_{L/R}$, i.e., $(\omega_{L/R}, \Omega_{L/R})$, then by the direct computation we can recover the result of Hitchin functional in (5.48) without the correction term $O(e^{-(\lambda-\epsilon)\gamma T})$ obtained by analytic method in previous section. Precisely, it is of the form

$$\begin{aligned}\mathrm{Vol}(Y) &\simeq \mathrm{Vol}(S)\cdot\left(4\pi^2 T\gamma^3 + A(\gamma)\gamma^2\right),\\ &\simeq \mathrm{Vol}_{Y_L(\gamma,T)} + \mathrm{Vol}_{Y_R(\gamma,T)} - 4\pi^2 T\gamma^3\mathrm{Vol}(S).\end{aligned} \tag{5.59}$$

Since the sum of the first two terms in the second line counts the volumes twice, we have to subtract the part once.

5.3.1 The Kähler potential

For any G_2-manifold Y there is the universal volume modulus v that is associated to the singlet $H^3_{\mathbf{1}}(Y,\mathbb{Z})$ of the three-form cohomology. It simply rescales the torsion-free G_2-structure φ. In the twisted connected sum we additionally identify the squashing modulus b of the S^3 base in the topological $K3$ fibration of the G_2-manifold $Y \longrightarrow S^3$ as dicussed in [Bra16]. Note that $b \to +\infty$ describes the Kovalev limit. The torsion-free G_2-structure φ depends on these two moduli as

$$\varphi(v,b,\tilde{S}) \;=\; v\left[\left(\rho_0^{\mathrm{ker}} + \sum_{\hat{\imath}}\tilde{S}^{\hat{\imath}}\rho_{\hat{\imath}}^{\mathrm{ker}}\right) + b\left([S] + \sum_{\tilde{\imath}}\tilde{S}^{\tilde{\imath}}\rho_{\tilde{\imath}}^{\mathrm{coker}}\right)\right]\;. \tag{5.60}$$

Here $[S]$ is the harmonic three form that is Poincaré dual to the K3 fiber S. Furthermore, $(\rho_0^{\mathrm{ker}}, \rho_{\hat{\imath}}^{\mathrm{ker}})$ and $\rho_{\tilde{\imath}}^{\mathrm{coker}}$ form a basis of harmonic three-forms arising from the kernel constributions and the cokernel part $L/(N_L + N_R)$, respectively. $\tilde{S}^{\hat{\imath}}$ and $\tilde{S}^{\tilde{\imath}}$ are the respective associated geometric real moduli fields. Thus the two universal $\mathcal{N}=1$ neutral chiral moduli multiplets v and $\varkappa$ are given by

$$\mathrm{Re}(v) \;=\; v\,, \qquad \mathrm{Re}(\varkappa) \;=\; vb\,. \tag{5.61}$$

In particular, we refer to the chiral multiplet $\varkappa$ as the Kovalevton, as it describes in the limit $\mathrm{Re}(\varkappa) \to +\infty$ the Kovalev limit, while keeping $\mathrm{Re}(v)$ constant. The remaining real moduli fields are not universal and relate to the non-universal neutral chiral multiplets as

$$\mathrm{Re}(\phi^{\hat{\imath}}) \;=\; v\tilde{S}^{\hat{\imath}}\,, \qquad \mathrm{Re}(\phi^{\tilde{\imath}}) \;=\; vb\tilde{S}^{\tilde{\imath}}\,. \tag{5.62}$$

They depend on the topological details of the building blocks $(Z_{L/R}, S_{L/R})$ and the choice of gluing diffeomorphism.

While keeping the ratio $\mathrm{Re}(v)/\mathrm{Re}(\varkappa)$ constant, we first establish that the chiral multiplet v directly relates to the (dimensionless) volume modulus $R = \frac{\gamma}{\gamma_0}$, the constant γ_0 has dimension of length, as

$$\mathrm{Re}(v) \;=\; R^3\,. \tag{5.63}$$

This relation comes about because the $\mathrm{Re}(v)$ measures (dimensionless) volumes of three cycles while R measures (dimensionless) length scales in the G_2-manifold Y. Apart from the overall volume dependence, the Kovalevton $\varkappa$ measures the squashed volume of the S^3 base. Therefore, from the expression of the

Hitchin functional we arrive at the relation

$$\mathrm{Re}(\varkappa) = (2\pi)^2 R^3 (2T + \alpha(\tilde{S})) , \tag{5.64}$$

where $\tilde{S}$ denotes collectively the remaining geometric moduli fields $\tilde{S}^{\tilde{\imath}}$ and $\tilde{S}^{\hat{\imath}}$. The moduli dependent function $\alpha(\tilde{S})$ are in principle computable from the (relative) periods and the Kähler forms of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$.

Thus we find that the universal structure of the four-dimensional low energy effective $\mathcal{N} = 1$ supergravity action is governed by the Kähler potential

$$K(\nu, \bar{\nu}, \varkappa, \bar{\varkappa}) = -4 \log(\nu + \bar{\nu}) - 3 \log(\varkappa + \bar{\varkappa}) - 3 \log \left(\frac{1}{4} \mathrm{Vol} S(\tilde{S}) \right) . \tag{5.65}$$

Note that this Kähler potential is only a valid approximation both in the large volume regime and in the Kovalev regime, where quantum corrections and metric corrections of the asymptotically cylindrical Calab–Yau threefolds are suppressed. The semi-classical large volume limit arises for both $\mathrm{Re}(\nu)$ and $\mathrm{Re}(\varkappa)$ to be taken sufficiently large, while the corrections to the G_2-metric in the twisted connected sum are suppressed if in addition $\mathrm{Re}(\varkappa)$ is even (parametrically) larger than $\mathrm{Re}(\nu)$. A detailed analysis of this class of Kähler potential may exhibit interesting phenomenological properties.

We also can study locally the moduli dependence of G_2-manifolds in terms of the periods

$$\Pi_a = \int_{\Gamma_a} \varphi , \tag{5.66}$$

which only depend on the coholomogy class $[\varphi]$ of the G_2-structure, where $\{\Gamma_a\}$ generate the homology group $H_3(Y, \mathbb{Q})$ and $a = b_3(Y)$. Then one can find that it consists of the periods of K3 surface which are well-known and other integration over some 3-cycles supported on the compact subspace K. Since lack of further geometric and topological information of the compact space K, we have no general method to calculate the periods over such 3-cycles.

If the G_2-manifold Y is obtained by the orthogonal gluing, then one part of the topological intersection matrix κ_{IJK} in the gauge kinetic coupling f_{IJ} (5.18) is fully determined by the intersection of Picard lattices $R = N_L \cap N_R$. In the case of the example 4.25 and 4.29, since the kernels K_L and K_R are trivial, the topological intersection matrix K_{IJK}, hence f_{IJ}, only has the contribution from R of rank 1 with self-intersection $\langle -6 \rangle$ or $\langle -4 \rangle$. $H^2(Y)$ is trivial in the example 4.26 and 4.27 due to the perpendicular gluing and the trivialities of K_L and K_R, so that the matrix K_{IJK} is always trivial.

On the other hand, if we choose X_L as the one in the example 4.32 with the nontrivial kernel $K_L = \langle E_1 - E_0, E_2 - E_0, E_3 - E_0 \rangle$. Since the Picard lattice N_L is of rank 1, no matter what X_R is, the intersection R is always trivial by the orthogonal gluing. Hence the K_{IJK} is fully determined by the K_L with the relations $E_i.l_j = -\delta_{ij}$, and K_R. Similarly, the case in the example 4.33 has nontrivial kernel $K_L \simeq \mathbb{Z}^{12}$ with the Picard lattice $N_L = E_8(-1) \perp \langle 8 \rangle \perp \langle -16 \rangle$. If we form a G_2-manifold by perpendicular gluing two copies of this 3-fold, then R is also trivial and thus K_{IJK} is determined by the $K_{L/R}$ with the relations $E_{i,L/R}.l_{j,L/R} = -\delta_{ij}$.

5.4 $\mathcal{N} = 2$ gauge theory sectors

The building blocks $(Z_{L/R}, S_{L/R})$ of the twisted connected G_2-manifolds admit enhancement to $\mathcal{N} = 2$ non-Abelian gauge theory sectors with an interesting branch structure that is geometrically accessible in terms of extremal transitions in the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$. From [AW01] there is a simple hierarchy of real codimension four, six and seven singularities in G_2-manifolds, which respectively lead to non-Abelian gauge groups, non-trivial matter representations, and chirality of the charged $\mathcal{N} = 1$ matter spectrum. However, degenerating the building blocks $(Z_{L/R}, S_{L/R})$ admits non-Abelian gauge groups with non-trivial matter representations, thus we should not expect singularities inducing chirality, as the trivial S^1 fibration in the non-compact seven-manifolds $Y_{L/R}$ prevents the appearance of codimension seven singularities. In our work the non-Abelian gauge theory enhancement arises from singularities along a three cycle $S^1 \times C$, where the curve C of genus g resides in K3-fibers along a circle S^1 in the base Q. Such curve C realizes an ADE singularity in one of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ in the Kovalev limit. In the context of type IIA strings [KM96; KMP96] an ADE singularity along a curve C of genus g yields a four-dimensional $\mathcal{N} = 2$ gauge theory with the associated gauge group G together with g hypermultiplets in the adjoint representation. More general matter representations occurs from points along C where the ADE singularity further enhances, i.e., along real codimension six singularities. For instance, at the intersection point of two curves C and C' of ADE singularities we encounter matter in the bi-fundamental representation of the two associated gauge groups G and G' [KV97]. We find in the following that the described $\mathcal{N} = 2$ gauge theory spectra can indeed be realized within the $\mathcal{N} = 2$ gauge theory sectors of the building blocks $(Z_{L/R}, S_{L/R})$.

5.4.1 Abelian and non-Abelian gauge sectors

In Prop. 4.15 we have to pick two generic global sections s_0 and s_1 of the anti-canonical divisor $-K_P$ on the semi-Fano threefold P to construct the building block $(Z_{L/R}, S_{L/R})$. However, instead of choosing a generic smooth section s_0 as discussed in § 4.4.2, we choose a simple normal crossing section s_0 which factors into a product

$$s_0 \;=\; s_{0,1} \cdots s_{0,n} \,, \tag{5.67}$$

such that $s_{0,i}$ are global sections of line bundles $\mathcal{L}_i$ with $-K_P = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n$. As a consequence the curve $C_{\text{sing}} = \{s_0 = 0\} \cap \{s_1 = 0\}$ becomes reducible and decomposes into

$$C_{\text{sing}} \;=\; \sum_{i=1}^{n} C_i \,, \qquad C_i = \{s_{0,i} = 0\} \cap \{s_1 = 0\} \,, \tag{5.68}$$

where we assume that the individual curves C_i are smooth and reduced, but each C_i is unnecessary to be distinct. By the sequence of blow-ups $\pi_{\{C_1,\ldots,C_n\}} : Z^\sharp \to P$ along the individual smooth curves C_i, we obtain the smooth 3-fold

$$Z^\sharp \;=\; \mathrm{Bl}_{\{C_1,\ldots,C_n\}} P \;=\; \mathrm{Bl}_{C_n} \mathrm{Bl}_{C_{n-1}} \cdots \mathrm{Bl}_{C_1} P \,. \tag{5.69}$$

By blowing up a semi-Fano threefold P the resulting dimension of the kernel K, defined by eq. (4.25) in § 4.4.2

$$\dim K \;=\; n - 1 \,. \tag{5.70}$$

Furthermore, the three-form Betti number $b_3(Z^\sharp)$ of the blown-up threefold $Z^\sharp$ becomes

$$b_3(Z^\sharp) \;=\; b_3(P) + 2 \sum_{i=1}^{n} g(C_i) \,, \tag{5.71}$$

in terms of the three-form Betti number $b_3(P)$ of the semi-Fano threefold P and the genera $g(C_i)$ of the smooth curve components C_i. As all these curves C_i lie in the K3 fiber S, the genus $g(C_i)$ is readily computed by

$$g(C_i) = \frac{1}{2} C_i.C_i + 1 \,, \tag{5.72}$$

with the self-intersections $C_i.C_i$ in S.

To arrive at this gauge theory interpretation, let us consider a semi-Fano threefold P with a curve C_{sing} of the reducible type (5.68) of the factorized global anti-canonical section (5.67). Performing a blow-up along this reducible curve yields the fibration $\pi : Z_{\text{sing}} \to \mathbb{P}^1$ with

$$Z_{\text{sing}} = \text{Bl}_{C_{\text{sing}}} P = \left\{ (x,z) \in P \times \mathbb{P}^1 \,\middle|\, z_0 s_{0,1} \cdots s_{0,n} + z_1 s_1 = 0 \right\} \,. \tag{5.73}$$

In the vicinity of the fiber $\pi^{-1}([1,0])$ the threefold Z_{sing} becomes singular because in the patch of the affine coordinate $t = \frac{z_1}{z_0}$ we get

$$s_{0,1} \cdots s_{0,n} + t s_1 = 0 \,. \tag{5.74}$$

Thus by assumption of transverse intersections among the smooth curves C_i there are conifold singularities at the discrete intersection loci $\mathcal{I}_{ij} = \{t = 0\} \cap \{s_1 = 0\} \cap \{s_{0,i} = 0\} \cap \{s_{0,j} = 0\}$ for $1 \leq i \leq j \leq n$ with $\chi_{ij} = |\mathcal{I}_{ij}|$ intersection points. These numbers are given by

$$\chi_{ij} = C_i.C_j \,, \tag{5.75}$$

in terms of the intersection numbers of the reduced curve C_i and C_j within the K3 surface S. This singularity structure prevails in the asymptotically cylindrical Calabi–Yau threefold $X_{\text{sing}} = Z_{\text{sing}} \setminus S$ as the asymptotic fiber $S = \pi^{-1}([\alpha_0, \alpha_1])$ (for $\alpha_1 \neq 0$) is disjoint from the singular fiber $\pi^{-1}([1,0])$.

Suppose those curves C_i are all distinct, in this IIA picture refs. [Str95; GMS95] establish that the conifold singularities (5.74) yield an Abelian $\mathcal{N} = 2$ gauge theory with charged matter multiplets. Namely, to each curve C_i we assign an Abelian group factor $U(1)_i$ such that the total Abelian gauge group of rank $n-1$ becomes

$$U(1)^{n-1} \simeq \frac{U(1)_1 \times \ldots \times U(1)_n}{U(1)_{\text{Diag}}} \,, \tag{5.76}$$

where $U(1)_{\text{Diag}}$ is the diagonal subgroup of $U(1)_1 \times \ldots \times U(1)_n$. Thus, in the low-energy effective theory we obtain in addition to $(n-1)$ four-dimensional $\mathcal{N} = 2$ $U(1)$ vector multiplets, which decomposes into $(n-1)$ four-dimensional $\mathcal{N} = 1$ $U(1)$ vector multiplets and $n-1$ four-dimensional $\mathcal{N} = 1$ neutral chiral multiplets. Furthermore, to each intersection point in $\mathcal{I}_{ij}$ one assigns a four-dimensioanl $\mathcal{N} = 2$ hypermultiplet of charge $(+1,+1)$ with respect to the $U(1)_i \times U(1)_j$ group factor. Then each of these $\mathcal{N} = 2$ hypermultiplet of charge $(+1,+1)$ decomposes into two four-dimensional $\mathcal{N} = 1$ chiral multiplets of charge $(+1,+1)$ and $(-1,-1)$, respectively. We summarize the resulting spectrum in Table 5.2.

The described four-dimensional $\mathcal{N} = 2$ Abelian gauge theory now predicts a Higgs branch $H^\flat$ and a Coulomb branch $C^\sharp$. On the one hand, generic non-vanishing expectation values of the charged hypermultiplets break the $U(1)^{n-1}$ gauge theory entirely and parametrize the Higgs branch $H^\flat$ of the gauge theory. As a consequence $(n-1)$ charged $\mathcal{N} = 2$ hypermultiplets play the role of $\mathcal{N} = 2$ Goldstone multiplets that combine with the $(n-1)$ short massless $\mathcal{N} = 2$ vector multiplets to $(n-1)$ long massive $\mathcal{N} = 2$ vector multiplets. As a result — according to the spectrum in Table 5.2 — we arrive at the Higgs

Multiplicity	$\mathcal{N} = 2$ multiplets		$\mathcal{N} = 1$ multiplets	
	$U(1)^{n-1}$ charges	multiplet	$U(1)^{n-1}$ charges	multiplet
$n - 1$	$(0, 0, \ldots, 0)$	vector	$(0, \ldots, 0)$	vector
			$(0, \ldots, 0)$	chiral
χ_{ij}	$(0, \ldots, +1_i, \ldots, +1_j, \ldots, 0)$	hyper	$(0, \ldots, +1_i, \ldots, +1_j, \ldots, 0)$	chiral
$1 \leq i < j < n$			$(0, \ldots, -1_i, \ldots, -1_j, \ldots, 0)$	chiral
χ_{in}	$(0, \ldots, +1_i, \ldots, 0)$	hyper	$(0, \ldots, +1_i, \ldots, 0)$	chiral
$1 \leq i < n$			$(0, \ldots, -1_i, \ldots, 0)$	chiral

Table 5.2: The spectrum of the Abelian $\mathcal{N} = 2$ gauge theory sector arising from the conifold singularities in the building block (Z_{sing}, S).

branch $H^\flat$ of complex dimension $h^\flat$ [GMS95]

$$h^\flat = \dim_{\mathbb{C}} H^\flat = 2\left(\sum_{1 \leq i < j \leq n} \chi_{ij}\right) - 2(n-1) . \tag{5.77}$$

Here the factor two takes into account that each hypermultiplet contains two complex scalar fields. This complex dimension readily describes the Higgs branch as parametrized by the expectation values of the corresponding charged $\mathcal{N} = 1$ chiral multiplets. On the other hand the expectation values of the neutral complex scalar fields in the $\mathcal{N} = 2$ vector multiplets furnish the coordinates on the Coulomb branch $C^\sharp$ such that its complex dimension $c^\sharp$ reads

$$c^\sharp = \dim_{\mathbb{C}} C^\sharp = n - 1 . \tag{5.78}$$

In the $\mathcal{N} = 1$ language the Coulomb branch moduli space is parametrized by the expectation value of neutral $\mathcal{N} = 1$ chiral multiplets.

In the geometry, the Higgs branch $H^\flat$ arises from deforming the conifold singularities in X_{sing} to the deformed asymptotically cylindrical Calabi–Yau threefold $X^\flat$ [GMS95]. On the level of the semi-Fano threefold P this amounts to deforming the reducible curve C_{sing} in eq. (5.68) to the smooth reduced curve $C^\flat$ such that building block (Z_{sing}, S) deforms to the building block $(Z^\flat, S^\flat)$, which yields for the kernel $K^\flat$ and the three-form Betti number $b_3(Z^\flat)$ according to eqs. (5.70) and (5.71)

$$\dim K^\flat = 0 , \qquad b_3(Z^\flat) = b_3(P) + C^\flat.C^\flat + 2 . \tag{5.79}$$

Furthermore, the resolution of the conifold singularities in X_{sing} geometrically yields the Coulomb branch $C^\sharp$ of the gauge theory [GMS95], which again on the level of the semi-Fano threefold P realizes the sequential blow-ups (5.69) along the components C_i of C_{sing} to the building block $(Z^\sharp, S^\sharp)$. With eqs. (5.70) and (5.71) the dimension of its kernel $k^\sharp$ and the Betti number $b_3(Z^\sharp)$ becomes

$$\dim K^\sharp = n - 1 , \qquad b_3(Z^\sharp) = b_3(P) + 2n + \sum_{i=1}^{n} C_i.C_i . \tag{5.80}$$

Let us now consider two twisted connected sum G_2-manifolds $Y^\flat$ and $Y^\sharp$ respectively constructed via orthogonal gluing of the left building blocks $(Z^\flat, S^\flat)$ and $(Z^\sharp, S^\sharp)$ with another right building block (Z_R, S_R). Using the equivalence $C^\flat \sim C_1 + \ldots + C_n$ on the semi-Fano threefold P and the definition (5.75) of the multiplicities χ_{ij}, we finally arrive at

$$\begin{aligned} b_2(Y^\flat) &= b_2(Y^\sharp) - (n-1)\,, \\ b_3(Y^\flat) &= b_3(Y^\sharp) + 2\left(\sum_{1\le i<j\le n} \chi_{ij}\right) - 3(n-1)\,. \end{aligned} \tag{5.81}$$

The non-trivial result is now that the derived change in Betti numbers (5.81) between such twisted connected sum G_2-manifolds is in prefect agreement with the phase structure of the proposed $U(1)^{n-1}$ gauge theory. The change in the Betti number b_2 geometrically realizes the difference of massless four-dimensional $\mathcal{N} = 1$ vector multiplets, whereas the difference of four-dimensional $\mathcal{N} = 1$ chiral multiplets reflects the change of the Betti number b_3. This is in agreement with the gauge theory expectation. Passing from the Coulomb branch $C^\sharp$ to the Higgs branch $H^\flat$ via the Higgs mechanism reduces the vector bosons by the rank $(n-1)$ of the gauge group. Furthermore, the difference in the four-dimensional $\mathcal{N} = 1$ chiral multiplets agrees with the change in dimension of the moduli space of these gauge theory phases, i.e.,

$$b_3(Y^\flat) - b_3(Y^\sharp) = b_3^\flat - b_3^\sharp = h^\flat - c^\sharp\,. \tag{5.82}$$

Let us now turn to the enhancement to non-Abelian $\mathcal{N} = 2$ gauge theory sectors in the context of twisted connected G_2-manifolds. We assume that the global section s_0 of $-K_P$ can further degenerate to $s_0 = \tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s}$ with $n = k_1 + \ldots + k_s$ and the singular building block (5.73) reads

$$Z_{\text{sing}} = \left\{(x,z) \in P \times \mathbb{P}^1 \,\middle|\, z_0 \tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s} + z_1 s_1 = 0\right\}\,, \tag{5.83}$$

with the singular equation in the affine coordinate $t = \frac{z_1}{z_0}$ given by

$$\tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s} + t s_1 = 0\,. \tag{5.84}$$

As before we assume that all curves $\tilde{C}_i = \{\tilde{s}_{0,i} = 0\} \cap \{s_1 = 0\}$ are smooth. In the vicinity of the singular fiber $\pi^{-1}([1,0]) \subset Z_{\text{sing}}$ the singular building block (Z_{sing}, S) develops A_{k_i-1}-singularities along those curves $\tilde{C}_i$ with $k_i > 1$.

Again by analyzing the local M-theory geometry on $S^1 \times X_{\text{sing}}$ in terms of its dual type IIA picture on the asymptotically cylindrical Calabi-Yau threefold X_{sing}. Refs. [KMP96; KM96] establish that type IIA string theory on Calabi–Yau threefolds with a genus g curve of A_{k-1} singularities develops a $\mathcal{N} = 2$ $SU(k)$ gauge theory with g four-dimensional $\mathcal{N} = 2$ hypermultiplets in the adjoint representation of $SU(k)$. Furthermore, according to ref. [KV97] each intersection point of two such curves of A_{k_1-1} and A_{k_2-1} singularities contributes a four-dimensional $\mathcal{N} = 2$ hypermultiplet in the bi-fundamental representation $(\mathbf{k_1}, \mathbf{k_2})$ of $SU(k_1) \times SU(k_2)$.

Therefore, we propose for M-theory on the local singular seven space $S^1 \times X_{\text{sing}}$ the following non-Abelian gauge theory description. Firstly, the singularites along the curves $\tilde{C}_i$ determine the gauge

Multiplicity	$\mathcal{N}=2$ multiplets		$\mathcal{N}=1$ multiplets	
	G reps.	multiplet	G reps.	multiplet
$s-1$	$\mathbf{1}$	$U(1)$ vector	$\mathbf{1}$	$U(1)$ vector
			$\mathbf{1}$	chiral
$i=1,\ldots,s$	$\mathbf{adj}_{SU(k_i)}$	$SU(k_i)$ vector	$\mathbf{adj}_{SU(k_i)}$	$SU(k_i)$ vector
			$\mathbf{adj}_{SU(k_i)}$	chiral
$g(\tilde{C}_i)$	$\mathbf{adj}_{SU(k_i)}$	hyper	$\mathbf{adj}_{SU(k_i)}$	chiral
$1\leq i\leq s$			$\mathbf{adj}_{SU(k_i)}$	chiral
$\tilde{\chi}_{ij}$	$(\mathbf{k_i},\mathbf{k_j})_{(+1_i,+1_j)}$	hyper	$(\mathbf{k_i},\mathbf{k_j})_{(+1_i,+1_j)}$	chiral
$1\leq i<j<s$			$(\bar{\mathbf{k}}_\mathbf{i},\bar{\mathbf{k}}_\mathbf{j})_{(-1_i,-1_j)}$	chiral
$\tilde{\chi}_{is}$	$(\mathbf{k_i},\mathbf{k_s})_{(+1_i)}$	hyper	$(\mathbf{k_i},\mathbf{k_s})_{(+1_i)}$	chiral
$1\leq i<s$			$(\bar{\mathbf{k}}_\mathbf{i},\bar{\mathbf{k}}_\mathbf{s})_{(-1_i)}$	chiral

Table 5.3: The spectrum of the $\mathcal{N}=2$ gauge theory sector with gauge group $G=SU(k_1)\times\ldots\times SU(k_s)\times U(1)^{s-1}$ as arising from the non-Abelian building blocks (Z_{sing}, S).

group

$$G \;=\; SU(k_1)\times\ldots\times SU(k_s)\times U(1)^{s-1} \simeq \frac{U(k_1)\times\ldots\times U(k_s)}{U(1)_{\text{Diag}}}\;, \tag{5.85}$$

where any $SU(1)$ factors must be dropped and $U(1)_{\text{Diag}}$ is the diagonal subgroup of $U(k_1)\times\ldots\times U(k_s)$. Secondly, for any i with $k_i>0$ there are $g(\tilde{C}_i)$ four-dimensional $\mathcal{N}=2$ hypermultiplets in the adjoint representation of $SU(k_i)$. Thirdly, we have $\tilde{\chi}_{ij}$ four-dimensional $\mathcal{N}=2$ hypermultiplets in the bi-fundamental representation $(\mathbf{k_i},\mathbf{k_j})_{(+1_i,+1_j)}$ of the gauge group factors $SU(k_1)\times SU(k_1)$, where the subscript indicate the $U(1)$-charges with respect to the diagonal $U(1)_i$ and $U(1)_j$ subgroups of the respective unitary groups $U(k_i)$ and $U(k_j)$ in the relation (5.85). The multiplicities $\tilde{\chi}_{ij}$ are again determined by the intersection numbers of the curves $\tilde{C}_i$ and $\tilde{C}_j$ in the K3 fiber S. The resulting gauge theory spectrum is summarized in Table 5.3.

To analyze the branches of the $\mathcal{N}=2$ gauge theory sectors, we first determine the complex dimension $h^\flat$ of the Higgs branch

$$h^\flat \;=\; \dim_{\mathbb{C}} H^\flat \;=\; 2\left(\sum_{i=1}^{s}(g(\tilde{C}_i)-1)(k_i^2-1)\right)+2\left(\sum_{1\leq i<j\leq s}\tilde{\chi}_{ij}k_ik_j\right)-2(s-1)\;. \tag{5.86}$$

Here, the first term captures the $2(k_i^2-1)$ complex degrees of freedom of the four-dimensional $\mathcal{N}=2$ hypermultiplets in the corresponding adjoint representations of the $SU(k_i)$ gauge group factors, reduced by one adjoint $\mathcal{N}=2$ Goldstone hypermultiplet rendering the four-dimensional $\mathcal{N}=2$ $SU(k_i)$ vector multiplet massive. The second term realizes the complex degrees of freedom of the four-dimensional $\mathcal{N}=2$ matter hypermultiplets in the bi-fundamental representations of the associated special unitary gauge groups and charged with respect to the appropriate $U(1)$ factors. The last term subtracts from the second term the $\mathcal{N}=2$ Goldstone hypermultiplets for higgsing the $(s-1)$ four-dimensional $\mathcal{N}=2$ $U(1)$

vector multiplets.

Next, we derive the complex dimension of the Coulomb branch $C^\flat$, in which the maximal Abelian subgroup $U(1)^{n-1}$ remains unbroken. It is parameterized by the expectation value of all four-dimensional $\mathcal{N} = 2$ hypermultiplet components that are neutral with respect to this unbroken maximal Abelian subgroup. Therefore, the complex dimension $c^\sharp$ of the Coulomb branch becomes

$$c^\sharp = \dim_{\mathbb{C}} C^\sharp = 2\left(\sum_{i=1}^{s} g(\tilde{C}_i)(k_i - 1)\right) + (n-1) . \tag{5.87}$$

The first term counts the traceless neutral diagonal degrees of freedom of the four-dimensional $\mathcal{N} = 2$ matter hypermultiplets in the adjoint representation, while the second term adds the contributions of the complex scalar fields in the four-dimensional unbroken Abelian $\mathcal{N} = 2$ vector multiplets.

To compute the Betti numbers of the twisted connected sum G_2-manifolds as before, using the equivalence relation $C^\flat \sim k_1\tilde{C}_1 + \ldots + k_s\tilde{C}_s$, we calculate the change of Betti numbers

$$\begin{aligned} b_2(Y^\flat) &= b_2(Y^\sharp) - (n-1) , \\ b_3(Y^\flat) &= b_3(Y^\sharp) + \left(\sum_{i=1}^{s} \tilde{\chi}_{ii} k_i(k_i - 1)\right) + 2\left(\sum_{1\le i<j\le s} \tilde{\chi}_{ij} k_i k_j\right) - 3(n-1) . \end{aligned} \tag{5.88}$$

in terms of the intersection numbers $\tilde{\chi}_{ij} = \tilde{C}_i.\tilde{C}_j$ on the K3 surface S. As for the Abelian gauge theory sectors, the change of the two-form Betti number conforms with the difference of the four-dimensional $\mathcal{N} = 1$ vector multiplets in the Higgs and Coulomb branches, given by the rank of the non-Abelian gauge group (5.85). The difference of four-dimensional $\mathcal{N} = 1$ chiral multiplets is accurately predicted by the complex dimensions of the Higgs and Coulomb branches. That is to say with eqs. (5.72), (5.86) and (5.87) we find for the discussed non-Abelian gauge theories

$$b_3(Y^\flat) - b_3(Y^\sharp) = b_3^\flat - b_3^\sharp = \dim_{\mathbb{C}} H^\flat - \dim_{\mathbb{C}} C^\sharp . \tag{5.89}$$

As a result, the topological data of the G_2-manifolds for the Higgs, Coulomb and mixed Higgs–Coulomb phases resulting from a given semi-Fano threefold P are the same for both the discussed Abelian and non-Abelian gauge theory sectors.

5.4.2 Examples with $\mathcal{N} = 2$ gauge theory sectors

We now illustrate the emergence of $\mathcal{N} = 2$ gauge theory sectors in twisted connected sum G_2-manifolds with a few explicit examples:

Example 5.3 (**$SU(4)$** gauge theory from the Fano threefold $\mathbb{P}^3$)**.** Consider the Fano threefold $\mathbb{P}^3$ with the anti-canonical divisor $-K_{\mathbb{P}^3} = 4H$ in terms of the hyperplane class H. Let $\tilde{s}_{0,1}$ and s_1 be a (generic) global section of H and $-K_{\mathbb{P}^3}$, respectively. Then we obtain with eq. (5.83) the resolved building block $Z_{\text{sing}} \subset \mathbb{P}^3 \times \mathbb{P}^1$ as the hypersurface equation

$$\tilde{s}_{0,1}^4 + ts_1 = 0 , \tag{5.90}$$

with the affine coordinate t of the factor $\mathbb{P}^1$. This equation exhibits an A_3 singularity along the curve $\tilde{C}_1 = \{\tilde{s}_{0,1} = 0\} \cap \{s_1 = 0\} \cap \{t = 0\}$, which yields a $\mathcal{N} = 2$ gauge theory sector with gauge group $SU(4)$.

s_0 factors	Gauge Group	$\mathcal{N}=2$ Hypermultiplet spectrum	$h^\flat$	$c^\sharp$	$b_3^\flat$	$b_3^\sharp$	$k^\sharp$
1^4	$SU(4)$	$3\times\mathbf{adj}$	60	21	89	50	3
$1^3\cdot 1$	$SU(3)\times U(1)$	$3\times\mathbf{adj}; 4\times\mathbf{3}_{+1}$	54	15	89	50	3
$1^2\cdot 1^2$	$SU(2)^2\times U(1)$	$3\times(\mathbf{adj},\mathbf{1}); 3\times(\mathbf{1},\mathbf{adj}); 4\times(\mathbf{2},\mathbf{2})_{+1}$	54	15	89	50	3
$1^2\cdot 1\cdot 1$	$SU(2)\times U(1)^2$	$3\times\mathbf{adj}; 4\times\mathbf{2}_{(+1,+1)}; 4\times\mathbf{2}_{(+1,0)}; 4\times\mathbf{2}_{(0,+1)}$	48	9	89	50	3
$1\cdot 1\cdot 1\cdot 1$	$U(1)^3$	$4\times(+1,+1,0); 4\times(+1,0,+1); 4\times(0,+1,+1);$ $4\times(+1,0,0); 4\times(0,+1,0); 4\times(0,0,+1)$	42	3	89	50	3
$2\cdot 1^2$	$SU(2)\times U(1)$	$3\times\mathbf{adj}; 8\times\mathbf{2}_{+1}$	42	8	89	55	2
$2\cdot 1\cdot 1$	$U(1)^2$	$4\times(+1,+1); 8\times(+1,0); 8\times(0,+1)$	36	2	89	55	2
2^2	$SU(2)$	$9\times\mathbf{adj}$	48	19	89	60	1
$2\cdot 2$	$U(1)$	$16\times(+1)$	30	1	89	60	1
$3\cdot 1$	$U(1)$	$12\times(+1)$	22	1	89	68	1

Table 5.4: Depicted are the gauge theory branches of the $SU(4)$ gauge theory of the building blocks associated to the rank one Fano threefold $\mathbb{P}^3$.

We first note that the curves $C^{(k)} = (-K_{\mathbb{P}^3}) \cap (kH)$ have the following intersection numbers on the K3 surface S and — according to eq. (5.72) — genera

$$C^{(k)}.C^{(l)} = 4kl\,, \qquad g(C^{(k)}) = \frac{1}{2}C^{(k)}.C^{(k)} + 1 = 2k^2+1\,. \tag{5.91}$$

Due to the equivalence $\tilde{C}_1 \sim C^{(k)}$ we arrive at $g(\tilde{C}_1)=3$ four-dimensional $\mathcal{N}=2$ hypermultiplets in the adjoint representation of $SU(4)$. This spectrum predicts with eqs. (5.86) and (5.85) the dimensions of the Higgs and Coulomb branches

$$\dim_{\mathbb{C}} H^\flat = 60\,, \qquad \dim_{\mathbb{C}} C^\sharp = 21\,, \qquad \dim_{\mathbb{C}} H^\flat - \dim_{\mathbb{C}} C^\sharp = 39\,. \tag{5.92}$$

By sequentially blowing-up $\mathbb{P}^3$ four times along the curve $\tilde{C}_1$, we arrive at the building block $(Z^\sharp, S^\sharp)$ with

$$\dim k^\sharp = 3\,, \qquad b_3(Z^\sharp) = 4\cdot 2g(\tilde{C}_1) = 24\,. \tag{5.93}$$

Deforming the hypersurface equation (5.90) to $s_0 + ts_1 = 0$ with a generic section of $-K_{\mathbb{P}^3}$, we resolve along the reduced smooth curve $C^\flat \subset \mathbb{P}^3$ with $C^\flat \sim C^{(4)}$ in order to determine the building block $(Z^\flat, S^\flat)$ of the Higgs branch $H^\flat$ with

$$\dim k^\flat = 0\,, \qquad b_3(Z^\flat) = 2g(C^\flat) = 66\,. \tag{5.94}$$

Finally, orthogonally gluing the building blocks $(Z^\flat, S^\flat)$ and $(Z^\sharp, S^\sharp)$ to a suitable right building block (Z_R, S_R), we obtain the twisted connected G_2-manifolds $Y^\flat$ and $Y^\sharp$ with the reduced Betti numbers

$$\begin{aligned} b_2^\flat &= 0\,, & b_3^\flat &= 89\,, \\ b_2^\sharp &= 3\,, & b_3^\sharp &= 50\,, \end{aligned} \tag{5.95}$$

given by

$$b_\ell^\flat = b_\ell(Y^\flat) - \delta_R^{(\ell)} , \qquad b_\ell^\sharp = b_\ell(Y^\sharp) - \delta_R^{(\ell)} , \qquad \ell = 1, 2 . \tag{5.96}$$

Here

$$\delta_R^{(2)} = \dim k_R + \operatorname{rk} R , \qquad \delta_R^{(3)} = b_3(Z_R) + \dim k_R - \operatorname{rk} R . \tag{5.97}$$

We observe that the differences $b_2^\sharp - b_2^\flat = 3$ and $b_3^\flat - b_3^\sharp = 39$ agree with the rank of the gauge group and the change in dimensionality of the Higgs and Coulomb branches, respectively.

By partially deforming the first term $\tilde{s}_{0,1}^4$ in the hypersurface equation (5.90), we can realize hypersurface singularities describing various Abelian and non-Abelian subgroups of $SU(4)$. Such partial deformations geometrically realize mixed Higgs–Coulomb branches of the $SU(4)$ gauge theory. We collect the geometry and phase structure of these mixed Higgs–Coulomb branches in Table 5.4, where the entries of this table are determined with eqs. (5.70), (5.71), (5.86), (5.87), and (5.91). Note that — depending on the breaking pattern of $SU(4)$ arising from partially higgsing — the dimensions of Higgs and Coulumb branches vary because only the charged matter spectrum of the unbroken gauge group plays a role for the Higgs and Coulumb branches in this gauge theory sector. For all entries in Table 5.4 we find that

$$b_3^\flat - b_3^\sharp = h^\flat - c^\sharp , \qquad \dim k^\sharp = \operatorname{rk} G . \tag{5.98}$$

This agreement confirms nicely the correspondence between gauge theory branches and phases of twisted connected G_2-manifolds.

Example 5.4 (Toric Semi-Fano threefolds)**.** In this toric setup the anti-canonical divisor reads

$$- K_{P_\Sigma} = D_1 + \ldots + D_n , \tag{5.99}$$

where the toric divisors D_i correspond to the one-dimensional cones of Σ, that is to say to the rays of the lattice polytope Δ. For smooth toric varieties P_Σ the toric divisors D_i are smooth and intersect transversely [CLS11]. As the anti-canonical divisor $-K_P$ is base point free, by Prop. 4.31 we can find a smooth global section s_1 of the anti-canonical divisor $-K_{P_\Sigma}$ and further generic global sections $s_{0,i}$ of D_i such that the curves $C_i = \{s_{0,i} = 0\} \cap \{s_1 = 0\}$ are smooth and mutually intersect transversely. Hence, the toric semi-Fano threefold P_Σ realizes indeed a $U(1)^{n-1}$ gauge theory sector. The four-dimensional matter spectrum is then given by Table 5.2, where the multiplicities χ_{ij} are the toric triple intersection numbers

$$\chi_{ij} = -K_{P_\Sigma}.D_i.D_j . \tag{5.100}$$

We construct the building blocks $(Z^\sharp, S^\sharp)$ of the Coulomb branch $C^\sharp$ by the sequential blow-ups (5.69) along the curves C_i, while we determine the building block $(Z^\flat, S^\flat)$ of the Higgs branch $H^\flat$ by blowing a smooth curve $C^\flat = \{s_0 = 0\} \cap \{s_1 = 0\}$ obtained by deforming the singular section $s_{0,1} \cdots s_{0,n}$ to generic anti-canonical section s_0. Then we arrive at the twisted connected G_2-manifold $Y^\sharp$ and $Y^\flat$ by orthogonally gluing these gauge theory building blocks with a right building block (Z_R, S_R) in the usual way.

Note that due to linear equivalences among the toric divisors D_i the Abelian gauge theory can enhance to non-Abelian gauge groups as well. Namely, assume that the anti-canonical bundle $-K_{P_\Sigma}$ is linearly equivalent to

$$- K_{P_\Sigma} \sim k_1 \tilde{D}_1 + \ldots + k_s \tilde{D}_s , \tag{5.101}$$

where for some divisors $\tilde{D}_\alpha \sim \sum_i a_{\alpha i} D_i$ with global sections $\tilde{s}_{0,\alpha}$. Furthermore, we require that the curves $\tilde{C}_\alpha$ are smooth and mutually intersect transversely. Then following previous section we arrive at

No.	ρ	Gauge Group	$\mathcal{N}=2$ Hypermultiplet spectrum	$h^\flat$	$c^\sharp$	$b_3^\flat$	$b_3^\sharp$	$k^\sharp$
K24, MM34_2	2	$SU(3)\times SU(2)\times U(1)$	$2\times(\mathbf{adj},\mathbf{1});(\mathbf{1},\mathbf{adj});3\times(\mathbf{3},\mathbf{2})_{+1}$	50	14	79	43	4
K32	2	$SU(3)^2\times U(1)$	$(\mathbf{adj},\mathbf{1});(\mathbf{1},\mathbf{adj});3\times(\mathbf{3},\mathbf{3})_{+1}$	52	13	79	40	5
K35, MM36_2	2	$SU(5)\times SU(2)\times U(1)$	$2\times(\mathbf{adj},\mathbf{1});(\mathbf{5},\mathbf{2})_{+1}$	60	22	87	49	6
K36, MM35_2	2	$SU(4)\times SU(2)\times U(1)$	$2\times(\mathbf{adj},\mathbf{1});2\times(\mathbf{4},\mathbf{2})_{+1}$	54	17	81	44	5
K37, MM33_2	2	$SU(4)\times SU(3)\times U(1)$	$(\mathbf{adj},\mathbf{1});3\times(\mathbf{4},\mathbf{3})_{+1}$	54	12	79	37	6
K62, MM27_3	3	$SU(2)^3\times U(1)^2$	$(\mathbf{adj},\mathbf{1}^2);(\mathbf{1},\mathbf{adj},\mathbf{1});(\mathbf{1}^2,\mathbf{adj});2\times(\mathbf{2}^2,\mathbf{1})_{(1,1)}$ $2\times(\mathbf{2},\mathbf{1},\mathbf{2})_{(1,0)};2\times(\mathbf{1},\mathbf{2}^2)_{(0,1)}$	44	11	73	40	5
K68, MM25_3	3	$SU(3)\times SU(2)\times U(1)^2$	$(\mathbf{adj},\mathbf{1});3\times(\mathbf{3},\mathbf{2})_{(1,1)};2\times(\mathbf{3},\mathbf{1})_{(1,0)};(\mathbf{1},\mathbf{2})_{(0,1)}$	42	9	69	36	6
K105, MM31_3	3	$SU(3)^2\times SU(2)\times U(1)^2$	$(\mathbf{adj},\mathbf{1}^2);(\mathbf{1},\mathbf{adj},\mathbf{1});2\times(\mathbf{3}^2,\mathbf{1})_{(1,1)};(\mathbf{3},\mathbf{1},\mathbf{2})_{(1,0)};$ $(\mathbf{1},\mathbf{3},\mathbf{2})_{(0,1)}$	50	15	77	42	7
K124	3	$SU(4)\times SU(2)^2\times U(1)^2$	$(\mathbf{adj},1,1);2\times(\mathbf{4},\mathbf{2},\mathbf{1})_{(1,1)};2\times(\mathbf{4},\mathbf{1},\mathbf{2})_{(1,0)}$	48	13	73	38	7
K218, MM12_4	4	$SU(4)\times SU(3)\times SU(2)^2\times U(1)^3$	$(\mathbf{adj},\mathbf{1}^3);(\mathbf{4},\mathbf{3},\mathbf{1}^2)_{(1,1,0)};(\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{1})_{(1,0,1)};$ $(\mathbf{4},\mathbf{1}^2,\mathbf{2})_{(1,0,0)};(\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})_{(0,1,1)};(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{2})_{(0,1,0)}$	46	16	71	41	10
K266, MM10_4	4	$SU(3)\times SU(2)^3\times U(1)^3$	$(\mathbf{1},\mathbf{adj},\mathbf{1}^2);(\mathbf{3},\mathbf{2},\mathbf{1}^2)_{(1,1,0)};2\times(\mathbf{3},\mathbf{1},\mathbf{2},\mathbf{1})_{(1,0,1)};$ $2\times(\mathbf{3},\mathbf{1}^2,\mathbf{2})_{(1,0,0)};(\mathbf{1},\mathbf{2}^2,\mathbf{1})_{(0,1,1)};(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_{(0,1,0)}$	42	10	67	35	8
K221	4	$SU(3)\times SU(2)^2\times U(1)^3$	$2\times(\mathbf{3},\mathbf{2},\mathbf{1})_{(1,1,0)};3\times(\mathbf{3},\mathbf{1},\mathbf{2})_{(1,0,1)};(\mathbf{3},\mathbf{1}^2)_{(1,0,0)}$ $2\times(\mathbf{1},\mathbf{2},\mathbf{1})_{(0,1,0)}$	40	7	63	30	7
K232	4	$SU(4)\times SU(2)^3\times U(1)^3$	$2\times(\mathbf{4},\mathbf{2},\mathbf{1}^2)_{(1,1,0)};2\times(\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{1})_{(1,0,1)};$ $2\times(\mathbf{4},\mathbf{1}^2,\mathbf{2})_{(1,0,0)}$	42	9	65	32	9
K233	4	$SU(3)\times SU(2)^2\times U(1)^2$	$3\times(\mathbf{3},\mathbf{2},\mathbf{1})_{(1,1)};3\times(\mathbf{3},\mathbf{1},\mathbf{2})_{(1,0)}$	40	6	63	29	6
K247	4	$SU(4)\times SU(3)^2\times SU(2)\times U(1)^3$	$2\times(\mathbf{4},\mathbf{3},\mathbf{1}^2)_{(1,1,0)};2\times(\mathbf{4},\mathbf{1},\mathbf{3},\mathbf{1})_{(1,0,1)};$ $(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{2})_{(0,1,0)};(\mathbf{1}^2,\mathbf{3},\mathbf{4})_{(0,0,1)}$	46	11	69	34	11
K257	4	$SU(5)\times SU(3)^2\times SU(2)\times U(1)^3$	$2\times(\mathbf{5},\mathbf{3},\mathbf{1}^2)_{(1,1,0)};2\times(\mathbf{5},\mathbf{1},\mathbf{3},\mathbf{1})_{(1,0,1)};$ $(\mathbf{5},\mathbf{1}^2,\mathbf{2})_{(1,0,0)}$	48	12	71	35	12

Table 5.5: The $\mathcal{N}=2$ gauge theory sectors for some smooth toric semi-Fano threefolds P_Σ of Picard rank two and higher.

the $\mathcal{N}=2$ gauge theory sector with gauge group

$$G = SU(k_1)\times\ldots\times SU(k_s)\times U(1)^{s-1} . \tag{5.102}$$

Note that rank of the gauge group $\tilde{n} = k_1 + \ldots + k_s - 1$ is a priori not correlated with the number n of toric divisors. Instead, it depends on the precise nature of the linear equivalences among the toric divisors D_i, $i = 1,\ldots,n$, and the divisors $\tilde{D}_\alpha$, $i = 1,\ldots,s$.

In Table 5.5 we list the gauge theory sectors of a few toric semi-Fano threefolds P_Σ. This table does not display all mixed Higgs–Coulomb branches. Here, we focus on the resulting twisted G_2-manifold $Y^\flat$

and $Y^\sharp$ associated to the Higgs $H^\flat$ and Coulomb branches $C^\sharp$ of the maximally enhanced gauge group of maximal rank, as obtained by the factorization of the anti-canonical bundle $-K_{P_\Sigma}$.

CHAPTER 6

Conclusion and Outlook

In the first part of the thesis, motivated by the Gamma conjecture presented by Iritani [Iri07; Iri09] and Kontsevich [KKP08], we introduce a notion of the numerical vectors, sort of group homomorphisms from the Grothendieck group $K(\mathscr{A})$ of an abelian category $\mathscr{A}$ to a graded finite dimensional vector space $\bigoplus_i V^i$ over a field k which map a bilinear form on $K(\mathscr{A})$ to a quadratic form on $\bigoplus_i V^i$. In any smooth projective variety X over $\mathbb{C}$, the natural numerical vectors are ring homomorphisms from $K(X)$ to $H^*(X, \mathbb{C})$ which preserve the Hirzebruch-Riemann-Roch formula, such as the Chern character and the Mukai vector. Then we construct numerical slope functions induced from the numerical vectors which define stabilities on the abelian category $\mathscr{A}$. These stabilities have the Harder-Narasimhan filtration if $\mathscr{A}$ is Noetherian. Afterward, we introduce a notion of numerical t-stabilities, sort of t-stabilities [GKR04], on a triangulated category $\mathcal{T}$. The numerical slope functions given by t-stabilities defined on $\mathcal{T}$ would reduce to the numerical slope functions on the heart of a bounded t-structure of $\mathcal{T}$ and vice versa. On any smooth projective surface X, we can construct a certain set of numerical slope functions on $\mathrm{D}(X)$ which form a open subspace of the stability manifold of Bridgeland stability conditions on $\mathrm{D}(X)$ [Bri07; Bri08]. We also obtain the corollaries that the central charge of B-branes in B-models really defines the Bridgeland's stability conditions under certain conditions on smooth projective surfaces. Finally, we present the cohomolgical Fourier-Mukai transforms between the numerical vectors on smooth projective varieties over $\mathbb{C}$, which would be isometric with respect to the Mukai pairing between a subspace of numerical vectors.

It is interesting to study numerical t-stabilities and deformations of numerical vectors on nonsingular projective surfaces. In any smooth elliptic curve C_1, any t-stabilities, or Bridgeland's stability conditions, can be refined to some standard finest t-stability which is given by the stability data $(\mathbb{Z} \times \mathcal{M}_1, \{\langle F[i] \rangle\}_{i \in \mathbb{Z}})$ on $\mathrm{Coh}(C_1)$, where $\mathcal{M}_1$ is the set of μ-semistable subcategories $\Pi_{q,F} = \langle F \rangle$ and $F \in \mathscr{M}_q$, the moduli space of μ-stable bundles with $\mu(F) = q$. The finest t-stabilities on $\mathrm{D}(\mathbb{P}^1)$ are exhausted by the standard and exceptional finest t-structures [GKR04]. We may expect that numerical t-stabilities could be refined to the numerical t-stabilities we defined on smooth projective surfaces. Moreover, the stability manifold on any rational curve $\mathbb{P}^1$ is $\mathbb{C}^2$ and on a curve of positive genus is $\mathbb{C} \times \mathbb{H}$ [Mac07; Oka04]. So there should exist a submanifold with boundary of dimension $2b_1 + 2$ in stability manifolds on smooth projective surfaces. Since numerical t-stabilities are induced by numerical vectors, cohomological Fourier-Mukai transforms between those numerical vectors should give morphisms of numerical t-stabilities and thus can be used to analyze stability manifolds. On the other hand, for higher dimensional varieties, to construct the numerical t-stabilities of degree 1, or Bridgeland stability conditions, we need Bogomolov type inequalities including higher Chern classes to produce necessary positive systems for t-stabilities. Note that in general the stability on the tiled category may not be coarser than one on the original category and we do not have general methods which can directly reduce the degree of the numerical t-stability on any triangulated category and generate a new numerical t-stability with lower degree yet. Once we have one then it automatically gives us the necessary Bogomolov type inequalities.

In the second part, we first review geometric and topological properties of compact G_2-manifolds, which are special kind of seven-dimensional space constructed by Dominic Joyce [Joy96; Joy00]. More recently a new construction of G_2-manifolds by twisted gluing of two non-compact asymptotically cylindrical Calabi-Yau 3-folds tensored with an circle has been proposed by Kovalev [Kov03]. The analysis of the gluing [Cor+13; Cor+15] gives a fairly detailed description how the cohomology and homology of the G_2-manifold is concretely constructed from the closed and relative cohomology and homology classes of the building blocks. We use this description to give an account how this information can be used in the Kaluza–Klein reduction to approximate concretely the superpotential, the gauge kinectic terms and the Kähler potential determined by the Hitchin functional for M-theory compactifications on twisted connected sum G_2-manifolds. The Hitchin functional, the corresponding integrals of the volume, on the G_2-manifold can be approximated by integrals on the building blocks and this approximations becomes precisely in the asymptotic limit in which Kovalev argues that the corrections to the harmonicity of the 3-form constructed from the buildings blocks becomes small and thereby proving the existence of the harmonic G_2 3-form φ. From the cohomology of such G_2-manifolds, we established that this class of M-theory compactifications yields two neutral universal $\mathcal{N} = 1$ chiral moduli fields associated to the complexified overall volume modulus ν and the gluing modulus, called the Kovalevton $\varkappa$, respectively. The latter parametrizes the Kovalev limit taken by $\mathrm{Re}(\varkappa) \to \infty$. The expression of the Hitchin functional on twisted connected sum G_2-manifolds obtained by the analytic method coincides with the one obtained by the topological method as we expect.

Moreover, we can identify Abelian and non-Abelian gauge theory enhancements with various matter content from singularities in the asymptotic cylinderical Calabi–Yau threefolds $X_{L/R}$ in codimension four and six that occur in the twisted connected sum Y away from the gluing region. These lead to transitions in the threefolds $X_{L/R}$, whose deformations and resolutions can be described by methods of algebraic geometry familiar in the context of $\mathcal{N} = 2$ theories. The significant point is that these transitions commute with the Kovalev limit and the gluing construction. Namely, they connect G_2 manifolds whose change in the cohomology groups corresponds exactly to the change in the spectrum of $\mathcal{N} = 1$ vector and chiral superfields as predicted by the transitions. Concretely, starting with the equations that describe the blow-up of the anti-canonical divisor in semi-Fano threefolds and analyzing all their possible degenerations leads to a great variety of gauge groups and matter spectra as well as to many novel examples of twisted connected sum G_2-manifolds corresponding to the different branches of these gauge theories.

To construct interesting examples of twisted connected sum G_2-manifolds with nontrivial b_2 associated to the abelian vector multiplets of massless 4d bosonic fields obtained from Kaluza–Klein reduction of 11d supergravity, by orthogonal gluing we need the nontrivial intersection of Picard lattices $R = N_L \cap N_R$ and nontrivial kernel K_L and K_R. One of the matching problems is to find out the ample classes in N_L and N_R restricted from the ample classes on the building blocks and identify the negative definite parts to form the orthogonal pushout $W = N_L + N_R$. Unlike the building block of Fano type, the anticanonical divisor of the building block of semi-Fano type is not a Kähler class, but it is when restricted to a generic anticanonical divisor. Ample cone of building blocks is a proper subcone of the ample cone of the generic K3 surface. Thus it is not easy to choose suitable basis of the Picard lattices of higher rank solving the matching problems, and it is still open question to classify all possibilities and describe a well-defined class of all possible matchings for two given pairs. On the other hand, to completely analyse the periods of G_2-manifolds obtained by the twisted connected sum construction is also a not easy task since we have no further information about the complement of the Calabi-Yau cylinder in the building block. We may think the building block as a K3 fibration over a base which is a fibration of a torus over an interval for which one of the two circles of the torus collapses at each end. Hence the twisted connected sum G_2-manifolds could be described as a K3 fibration over S^3 base which can be compared with M-theory/heterotic duality. It would be interesting to see, if such a speculation could be made precise, namely establishing a duality between M-theory on G_2-manifolds in the Kovalev limit and F-theory on elliptically-fibered Calabi–Yau fourfolds in a certain degeneration limit.

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List of Figures

List of Tables