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Contents

Introduction

This dissertation consists of three essays in the area of Microeconomic Theory. The three chapters deal with information acquisition or disclosure and asymmetries in private information in various strategic environments. In Chapter 1, in a one-object auction, I analyze how the auction format influences what bidders decide to learn about. In Chapter 2, I analyze the implications of an asymmetry in budget constraints on equilibrium bidding behavior in a first price auction. In Chapter 3, which is joint work with Saskia Fuchs, we analyze to what extent a designer can persuade an informed committee in her favor by strategically disclosing information.

In Chapter 1, I analyze how the choice of an auction formate influences learning about the object for sale. In particular, before the start of the auction bidders can choose about which component or attribute of the object for sale they acquire information. Formally, the valuation of bidders for an object consists of a common value component (which matters to all bidders) and a private value component (which is relevant only to individual bidders). Bidders select about which of these two components they want to acquire noisy information. Learning about a private component yields independent estimates, whereas learning about a common component leads to correlated information between bidders. As all bidders share the common component, only learning about the private component is welfare-enhancing as only the private component matters for the efficient allocation of the object. Acquiring information about the common component reduces efficiency, as it comes at the opportunity costs of not learning about the private component.

I show that in a second price auction, information selection in equilibrium is unique. Bidders only learn about their private component, so an independent private value framework arises endogenously. If this were not the case, a bidder could guarantee himself the same expected gain and a strictly lower payment by decreasing correlation in private information. In an all-pay auction, bidders also prefer information about private components. In a first price auction, increasing correlation strictly elevates the payoff for a bidder under certain conditions.

In Chapter 2, I analyze a first price auction in which bidders have a two-dimensional type: a valuation for the object and a budget constraint. The budgets of the two bidders are private and drawn from different distributions. I propose a new solution technique to pin down a closed-form solution for equilibrium surplus and bidding distributions. This allows me to answer what effect asymmetric budgets have on the equilibrium in contrast to asymmetric valuations, and how bidding aggression reacts to this asymmetry. Furthermore, I analyze whether a seller has any incentives to disclose information about the identity and hence the budget constraints of the bidders before the start of the auction, e.g. by publishing a participants registry. Introducing asymmetry in the budget dimension and not on the valuation dimension and finding a closed-form for the equilibrium is the main contribution to the existing literature (Maskin and Riley, 2000).

My model solves a first price auction for bidders with asymmetrically distributed budget constraints in closed form. I provide a closed-form expression for the set of equilibria in this framework, without imposing any stochastic order on the budget distributions. Expected utility and bidding distributions are unique in equilibrium. If bidders are sufficiently symmetric, the degree of asymmetry in the budgets has no influence on strategies: bidding the entire budget is the unique equilibrium strategy. If asymmetry gets sufficiently severe, mass points in the equilibrium strategies arise. Pure strategy weakly monotonic bidding functions establish existence of such equilibria. I show that a weaker bidder bids more aggressively than his stronger opponent. In contrast to standard results with symmetric budget distributions, a second price auction can yield strictly higher revenue than a first price auction under asymmetric budget constraints.

Chapter 3 asks how to persuade decision makers or protect them against persuasion if they are privately informed. In particular, how can a group of voters be swayed to vote for the preferred outcome of a designer, if the designer can commit to disclosing payoff-relevant information about the quality of the outcome. This paper connects to the recent literature on Bayesian Persuasion (Kamenica and Gentzkow, 2011) and introduces a biased sender into a voting framework in which informed voters vote strategically (Feddersen and Pesendorfer, 1998).

Formally, a biased sender tries to persuade a committee of three members to vote for a proposal by providing public information about its quality. Each voter has some private information about the proposal's quality. We characterize the sender-optimal disclosure policy under unanimity rule when the sender can versus cannot ask voters for a report about their private information. We find that the sender can only profit from asking agents about their private signals when the private information is sufficiently accurate. For all smaller accuracy levels, a sender who cannot elicit the private information is equally well off.

Chapter 1

Knowing What Matters To Others: Information Selection in Auctions

1. Introduction

Bidding preparation for auctions usually involves evaluating multiple characteristics. This paper delves into which characteristics bidders should gather information about and how such decision is influenced by the auction format in cases wherein people cannot take into account all existing information.

These issues are relevant to, for example, corporate takeovers, in which acquiring firms have access to a variety of information about a target company. This information encompasses the R&D activities and the book value. A reasonable assumption is that firms cannot perfectly process or uncover all existing information, and are thus driven to select elements to focus on before bidding takes place. Should an acquiring firm conduct research on aspects that are specific to them, such as their R&D overlap? Or should they focus on factors that also matter to other acquiring firms, such as the book value of a target?

Another example are resource rights auctions for oil fields or timber. Each bidder derives the same monetary value from an unknown volume of oil or timber on a site, and this value stems from the market price. Bidders may incur different costs in extracting the resources from a site because of the use of different drilling or logging technologies and variances in experience levels. I inquire into whether a bidder prefers to perform an exploratory drilling to learn about oil volume (i.e., the common component) or to learn about the costs of extracting the resource through estimations of the drilling costs specific to him (i.e., the private component).

Buying real estate is yet another case that involves evaluating a variety of attributes prior to bidding. These attributes include the costs of maintaining a property, local taxes, mortgage rates, the convenience of travel to work, and personal preference for a property. Do bidders prefer to acquire information on the qualities of a property that are pertinent to all bidders, such as maintenance costs? Or would they rather examine characteristics that are uniquely related to them, such as the convenience of traveling to work from a property.

The contribution of this paper is to investigate the incentives provided by a variety of auction formats regarding information selection. I demonstrate that bidders prefer to learn about their private components in the second price auction (SPA) which is commonly used in the examples[1] described above. I also analyze incentives to selecting information in a first price auction (FPA) and an all-pay auction.[2]

The novelty of this paper lies in its illumination of *which* random variables bidders seek to learn (*information selection*) instead of what *level of accuracy* of information they favor about a given real-valued random variable (*information acquisition*). I isolate incentives for learning about the signal of an opponent: Holding the level of accuracy constant, do bidders prefer their private information to be dependent (information about a common component) or independent (information about a private component)?

The independent private values setting (IPV) and the interdependent values setting (IntV) lead to different theoretical predictions and vary significantly in their implications for auction design and policy.[3] The literature on auctions usually assumes either IPV or IntV setting at the outset of the analysis. In addition, identifying the valuation setting on the basis of data is often challenging if not impossible.[4] By restricting the ability of bidders to learn about more than one attribute, I study which value setting arises endogenously.

For a brief outline of the model, consider two bidders who compete for one indivisible object in a SPA. They share the same common component (e.g., the book value of a firm) and have independent private value components (e.g., match-specific R&D overlap). The valuation of each bidder is the sum of two value components about which they are uninformed. Bidders select between learning about a common or a private component. Information selection is simultaneous and covert. Considering both components is infeasible.[5]

Learning about the common or the private component has equal accuracy. In a single agent problem, an agent would be indifferent between learning about either component, as the two experiments are equally informative about the total value of the object. Yet, in the strategic context of an auction information about the object plays a dual role. Beyond containing information about the object's worth, it is also informative about the signal of the opponent and his bid. Moreover, a rational bidder conditions his estimate of the object not only on his own information, but also on what he learns from the event of winning. Being the highest bidder when the opponent learns about the common component implies that the opponent has a low signal realization. This is bad news for the expected value of the object. In equilibrium, a bidder shades down his bid

[1]See Porter (1995) for a survey of oil and gas lease auctions and Hendricks and Porter (2014) for an analysis of the auction mechanisms in selling resource rights in the U.S. See Gentry and Stroup (2017) for an analysis of auctions and negotiation procedures commonly used in mergers and acquisition, and Chow and Ooi (2014) for real estate land auctions.

[2]As I concentrate on the case of two bidders, my results also hold for the open English auction (equivalent to the SPA) and the Dutch auction (equivalent to the FPA) (Milgrom and Weber, 1982).

[3]In the IPV setup, each bidder's private information matters only to him; in the IntV setup, a bidder's estimate of the object depends on the private information of other bidders.

[4]See Laffont and Vuong (1996) for a general discussion of identifying the value setting in the FPA, and Athey and Levin (2001) for timber auctions.

[5]Learning about a component might involve some actuarial calculations or an experiment, e.g., exploratory drilling. I analyze a scenario where such an experiment is non-divisible, and analyzing both components half-way does not produce meaningful information: drilling half a hole, or calculating only the first half of a cost-benefit analysis is not useful.

due to this so-called winner's curse.

In my model, the extent of the winner's curse and the interdependence between bidders' information are endogenous and depend on which value component bidders learn about. The signals of bidders become more affiliated if they learn about the common component. The winner's curse exacerbates. If other bidders learn only about their private component, their information bears no relevance for other bidders and there is no winner's curse. Two standard valuation settings are nested in my model. An IPV setting arises if both bidders learn only about their private components. A pure common value setting emerges if both bidders learn only about the same common component.

The result for the SPA with two bidders is that in any symmetric equilibrium, information selection is unique: There is only learning about the private component, and an IPV setting arises endogenously. The SPA induces the ex-ante efficient outcome. No resources are wasted by learning about the common component which is irrelevant for efficiency, and the object is allocated to the bidder with the highest estimate of his private component. This result holds in a general class of utility functions.

In the SPA, a bidder could always find a strictly profitable deviation by decreasing interdependence in private signals. The approach is to find a deviation strategy that keeps the expected gain and winning probability constant, while strictly decreasing the expected payment. For a sketch of the argument, consider a candidate equilibrium in which both bidders learn only about the common component and have the highest degree of interdependence in private signals. Then, the following deviation is strictly profitable for a bidder: *Learn about the private component, but bid as if it were a signal about the common component.* This strategy eliminates interdependence in private signals but employs the same bidding function as the candidate equilibrium for tractability.

The expected payment conditional on winning from such a deviation strategy is strictly lower. The higher the interdependence, the higher the distribution of the second order statistic of the opponent's signal and his bid. By decreasing interdependence, the distribution of the second order statistic puts more weight on lower bids, and expected payment strictly decreases.

The expected gain from this deviation is the same as in the candidate equilibrium. For every realization of the total value of the object, the probability of placing the highest bid is the same with the candidate equilibrium and the deviation strategy. However, given a total value for the object, winning probability for different compositions of the two components changes with deviating. In the candidate equilibrium, as both bidders learn about the common component, they win with equal probability for each realization of it. In the deviation strategy, a deviating bidder is more likely to win in states that involve a high private and a low common component, and vice versa. The existence of a deviation strategy that leads to the same expected gain for a strictly lower payment pushes the incentives of bidders in the SPA towards independence, and yields a unique information choice in equilibrium of the SPA.

In a FPA, incentives to select information are opposite to the SPA. As a winning bidder pays his own bid, he does not want to "leave money on the table" by overbidding his opponent by too much. Having a better estimate of the opponent's bid can reduce the expected payment conditional on a win as it reduces the first order statistic of winning bids. Therefore, increasing the dependence of the own signal with the signal of the opponent induces a lower payment if both follow the same bidding function. I show that the ex-ante efficient equilibrium in which bidders learn only about their private component is not robust: After introducing a slight degree of correlation between the common and the private component, bidders prefer more interdependence by learning about the common component.

In addition, I consider information selection in all-pay auctions. For the general case of many bidders, bidders learn only about the private component and an IPV setting arises endogenously. This is because by deviating to the private component, bidders can always guarantee themselves a weakly higher winning probability at every total value of the object, for the same expected payment.

Section 2 describes the related literature. Section 4 introduces the model and the informational framework. The analysis in Section 4 shows the consequences of information selection on the joint signal distributions (Section 4.1) and on the value of the object (Section 4.2). I combine those observations in Section 5 to solve for an equilibrium of the SPA. Then, I show that the results generalize to a broader class of utility functions in Section 6.1, and discuss the effect of more than two bidders in Section 6.2. Finally, I analyze the FPA in Section 7.1, and the all-pay auction in Section 7.2.

2. Related Literature

In the classic literature in Auction Theory, the distribution of private information of bidders is exogenous and does not depend of the choice of the auction format.[6] In their seminal work, Milgrom and Weber (1982) introduce a theory of affiliation in signals, and derive the equilibrium for the SPA, the FPA and English auction. The all-pay auction for affiliated signals has been analyzed by Krishna and Morgan (1997) and recently by Chi et al. (2017).

The literature on information acquisition in auctions[7] endogenizes the private information of bidders, by asking *how much* costly information they seek to acquire.[8] Bidders choose the informativeness of their signal about a single-dimensional payoff relevant variable, usually for a fee that increases in the amount of information it contains.[9]

[6]For an IPV setup, see Vickrey (1961) and Riley and Samuelson (1981). For a common value model, see Wilson (1969) and Milgrom (1981), and Milgrom and Weber (1982) for a general interdependent setup with affiliated signals.

[7]Endogenous information acquisition has been analyzed in other areas of Economics. E.g., see Bergemann and Välimäki (2002), Crémer et al. (2009), Shi (2012) and Bikhchandani and Obara (2017) in optimal and efficient mechanism design, Martinelli (2006) and Gerardi and Yariv (2007) in committees, Crémer and Khalil (1992) and Szalay (2009) in principal-agent-settings, and Rösler and Szentes (2017) in bilateral trade.

[8]In the context of auctions, information acquisition has been modeled in an IPV model (see e.g. Hausch and Li, 1991; Compte and Jehiel, 2007; Gretschko and Wambach, 2014), and in an IntV framework (see e.g. Persico, 2000; Bergemann et al., 2009).

[9]Informativeness criteria include Blackwell sufficiency (Blackwell, 1951), accuracy (Persico, 2000; Lehmann, 1988), dispersion measures (Ganuza and Penalva, 2010), or deciding whether to learn perfectly or nothing about a payoff relevant variable (e.g. Bergemann et al., 2009). Better informativeness usually comes at higher costs.

Persico (2000) considers costly information acquisition in an interdependent value model in the FPA and the SPA. Before bidding, bidders choose the *accuracy* of their signal about a one-dimensional random variable. Accuracy is a statistical order on the informativeness of an experiment by Lehmann (1988).[10] In the model of Persico (2000), learning with higher accuracy has two effects: first, the information about the own valuation becomes more precise; second, bidders obtain a better estimate of the signals of other bidders. Therefore, a higher accuracy inextricably links these two effects. Persico (2000) shows that incentives for information acquisition are stronger in the FPA than in the SPA.

In contrast to Persico (2000), my model fixes the effect of informativeness about the object, and concentrates on choosing more or less correlation with the opponent. In my model, there are two signals available about two payoff-relevant variables. Accuracy of information is fixed and equal in each available signal. In contrast to Persico (2000), bidders in my model have to select the variable about which they prefer to learn. The results in Persico (2000) are of a relative nature: given a level of accuracy acquired in the SPA, the level of accuracy in a FPA is higher.[11] In contrast, my framework provides an absolute prediction: about *which* component do bidders learn.

In Bergemann et al. (2009), the value of an object is a weighted sum of everybody's payoff type. Information acquisition is binary: either learn perfectly about the own payoff-type, or learn nothing. Note that in this formulation, learning cannot introduce any dependence between the signal of bidders, as all payoff types are distributed independently (although they matter to other bidders). With positive interdependencies in payoff types, Bergemann et al. (2009) show that in a generalized Vickrey-Clarke-Groves mechanism[12] bidders acquire more information than would have been socially efficient.

In the IPV setup of Hausch and Li (1991), the SPA and FPA induce equal incentives to acquire information about the one-dimensional value. Stegeman (1996), showing that the incentives to acquire information in an IPV setting coincides in FPA and SPA, and with the incentives of a planner, making information acquisition efficient.

The above literature on information acquisition in auctions considers *covert* information acquisition. That is, bidders do not know how much information their competitors acquire before the auction. Another strand of the literature also analyzes *overt* information acquisition, where bidders observe how much information others acquired before bidding. Hausch and Li (1991) show that the SPA and the FPA induce different incentives to acquire information when information acquisition is overt, and revenue equivalence fails. Compte and Jehiel (2007) show in an IPV setup that an ascending dynamic auction induces more overt information acquisition and higher revenues than a sealed-bid auction. Hernando-Veciana (2009) compares the incentives to overtly acquire information in the English auction and the SPA, when bidders can learn about a common component or about a private component. In his model, it is *exogenous* which component information acquisition is about, while in my model, I endogenize the

[10] The concept of accuracy of of a statistical experiment is established by the name of 'effectiveness' by Lehmann (1988) in the statistical literature.

[11] This holds under appropriate conditions on the marginal costs for increasing accuracy.

[12] See Dasgupta and Maskin (2000) for a generalized Vickrey-Clarke-Groves mechanism in the context of auctions, and Jehiel and Moldovanu (2001) for a general mechanism design setting with externalities in information and allocations.

decision of information selection between the two components.

My paper also relates to the literature on *information choice* in games with strategic complementarities or substitutes, such as Cournot competition, beauty contests and investment games. Hellwig and Veldkamp (2009) ask whether bidders want to coordinate on the same or on different information channels about the same one-dimensional state of the world in a beauty contest game. They show that the choice of information relates to the complementarity of actions in their model: if actions are strategic complements, agents want to know what others know. If actions are strategic substitutes, agents want different information channels.

In a beauty contest game in Myatt and Wallace (2012), agents to choose between multiple information channels about a common state variable. Agents choose how clearly (endogenous noise) to listen to which of many available signals, that vary in accuracy (exogenous noise).

Gendron-Saulnier and Gordon (2017) fix the informativeness of signals, similar to my approach. In their paper, agents have the choice between multiple information channels, that all have the same informativeness: they are all Blackwell sufficient for each other. Information channels vary in the level of dependence they induce between the signals of agents. Actions exhibit strategic complementarities, as in the framework of Hellwig and Veldkamp (2009) and Myatt and Wallace (2012).

There are two major differences between my model and the three papers Hellwig and Veldkamp (2009), Myatt and Wallace (2012) and Gendron-Saulnier and Gordon (2017):[13] bidding functions do not exhibit strategic complementarities in the auction formats in my model (see e.g. Athey, 2002) which leads to a fundamentally different strategic problem. Further, in the above models, all channels contain information about the same single-dimensional payoff-relevant random variable (the one-dimensional state of the world). In contrast, in my model bidders choose about which component of the multidimensional state of the world to learn. Learning about their private component leaves them with an independent signal realization, irrespective of the information acquired by their opponent.

3. Model

3.1 Payoffs

There are two risk-neutral bidders, indexed by $i \in \{1, 2\}$ who compete for one indivisible object. The reservation value of the auctioneer and the outside options of the bidders are zero.

The valuation for the object of bidder i, denoted by $V_i \in \mathbb{R}^+$, depends on two attributes: a common value component S distributed on $[0, 1]$, that is equal for all bidders, and a private value component T_i distributed on $[0, 1]$, the idiosyncratic taste parameter of bidder i.

The the common value component and the private value components $\{S, T_1, T_2\}$

[13] See also Yang (2015) for flexible information acquisition in investment games with strategic complementarities and Denti (2017) for an unrestricted information acquisition technology in potential games.

are drawn mutually independent and identically, each with distribution function $H(\cdot)$, which admits a density function $h(\cdot)$.[14] That is, for all $i \in \{1, 2\}$, $m \in [0, 1]$, it holds that $H(m) = \Pr(S \leq m) = \Pr(T_i \leq m)$. The prior expected value of the components coincide: $\mathbb{E}[S] = \mathbb{E}[T_i]$.

The utility function for each bidder i is

$$V_i = S + T_i.$$

Note that the private component of the other agent $j \neq i$ has no impact on the valuation of bidder i. In Section 6.1, I extend the class of admissible utility functions.

Fix a total value realization v_i. Any $s_i \in [\max\{v_i - 1, 0\}, \min\{v_i, 1\}]$ and $T_i = v_i - s$ is a feasible[15] combination of the components for this particular v_i. As the joint events $(S = s, V_i = v_i)$ and $(S = s, T_i = v_i - s)$ are the same, the density function of the random variable V_i, the overall valuation of bidder i, is

$$h_V(v_i) := \int_{\max\{v_i-1,0\}}^{\min\{v_i,1\}} h(s)h(v_i - s)ds.$$

3.2 Information Structure

Neither the auctioneer, nor the bidders know the realization of any of the value components. Instead, bidders engage in information gathering about their valuations. The information choice of bidder i is one of *information selection*: about which component should he learn.

Bidders choose one experiment X_i which can be one of two random variables: bidders can learn either a random variable X_i^T that is informative about their private component T_i, or a random variable X_i^S that is informative about the common component S. Each signal $X_i \in \{X_i^T, X_i^S\}$ is uninformative about the other attribute. Both signals X_i^T and X_i^S consist of the same compact support $[0, 1]$ and a marginal probability distribution, conditional on the realization of its attribute $\{S, T_i\}$. The marginal distribution of the random variable X_i^T or X_i^S of bidder i has a cumulative distribution function $F^T(\cdot|r)$ or $F^S(\cdot|r)$ for $r \in [0, 1]$, conditional on the state $T_i = r$ or $S = r$.

Assumption
For $K \in \{S, T\}$, for all $r \in [0, 1]$, the distribution $F^K(x_i|r)$ admits a density $f^K(x_i|r)$, such that:

(A1) $\forall x_i \in [0, 1] : f^S(x_i|r) = f^T(x_i|r) =: f(x_i|r)$.

(A2) $\forall x_i' > x_i$, $\frac{f^K(x_i'|r)}{f^K(x_i|r)}$ *strictly increasing in r.*

Assumption A1 implies that an experiment has the same conditional distribution function whether applied to S or T_i. As all components are distributed identically,

[14]The assumption of full support and existence of a density function $h(\cdot)$ is for clarity of the presentation. Results hold if there are only two realizations in the support.

[15]The interval has to account for the fact that each component is distributed with support $[0, 1]$. For example, if $v_i = 1.3$, the common component needs to be at least $s_i = \max\{v_i - 1, 0\} = 0.3$ for value v_i to realize.

Assumption A1 implies the same informativeness on each available signal.[16] For clarity, I sometimes use the superscripts in the exposition to clarify about which component the signal is drawn.

The signals X_i^S and X_i^T satisfy a strong monotone likelihood ratio property (MLRP) in Assumption A2 which broadly speaking states that higher signal realizations are more indicative of higher states. Moreover, I assume that $f(\cdot|r)$ is continuously differentiable in x_i for all r.

Bidders choose the probability ρ_i of applying the signal on the common variable S. The information selection variable $\rho_i \in [0,1]$ is a mixed strategy:[17] With probability ρ_i, bidder i performs an experiment about S. With probability $1-\rho_i$, bidder i learns about attribute T_i. Let $\boldsymbol{\rho} = \{\rho_1, \rho_2\}$ be the vector of information selection variables.

Due to the following assumptions, the private signals of bidders can only be interdependent via learning about the common variable S:

Assumption (CI)
$X_i^S \perp\!\!\!\perp X_j^S \mid S.$

Assumption (IN)
$X_i^T \perp\!\!\!\perp X_j^T$, *and* $X_i^T \perp\!\!\!\perp X_j^S$.

Assumption CI is a conditional independence assumption of X_i^S and X_j^S on S. Together with Assumption A2 (stating that X_i^S and S are affiliated) this implies that the random variables X_1^S and X_2^S are affiliated.[18] According to Assumption IN, if one bidder learns about his private component by observing X_i^T, his signal is independent from both signal X_j^S and X_j^T of his opponent j.

Let $F^S(x) := \Pr(X_i^S \le x) = \int_0^1 F^S(x|s)h(s)ds$ be the unconditional distribution function of a bidders' private signal when he learns about component S, and let $f^S(x)$ be the corresponding density. Analogously, let $F^T(x) := \Pr(X_i^T \le x) = \int_0^1 F^T(x|t)h(t)dt$ be the distribution when applying the experiment on T_i, and $f^T(x)$ the corresponding density. Note that $F^S(x) = F^T(x)$, due to the symmetry of signals and components.

After bidder i chooses what to learn about, he observes signal X_i with the following unconditional distribution function:

$$F(x) := \Pr(X_i \le x|\rho_i) = (1-\rho_i)F^T(x) + \rho_i F^S(x).$$

The unconditional distribution $F(x)$ is not a function of ρ_i, as applying the signal to both components results in the same distribution of signals due to $F^S(x) = F^T(x)$.

Observation 1. *The unconditional distribution function of a signal about each component coincides for any information selection variable ρ_i: $\forall x \in [0,1]$, $F^T(x) = F^S(x) = F(x)$.*

[16]I abstract away from bidders choosing to learn about a component only because it provides more information. Instead, the focus of this paper is to find what dependence bidders prefer between their signal given the same informativeness.

[17]A bidder always observes which experiment was performed for any randomization.

[18]For a formal definition of affiliation, see Appendix A.1, Definition 6. The affiliation between X_i^S and X_j^S follows from combining CI with MLRP.

The next binary example is useful to provide intuition in the following analysis.

Example 1. *Let* $s \in \{0, 1\}$ *and* $t_i \in \{0, 1\}$*, each with equal probability* $\frac{1}{2}$*. Thus,* $\Pr(v_i = 0) = \Pr(v_i = 2) = \frac{1}{4}$*, and* $\Pr(v_i = 1) = \frac{1}{2}$*.*

For $K \in \{S, T\}$ *and* $x_i \in [0, 1]$*, the signal* X_i^K *has conditional density* $f^K(x_i|0) = 2 - 2x_i$ *and* $f^K(x_i|1) = 2x_i$*. The unconditional signal distribution is* $F(x_i) = x_i$ *and the density is* $f(x_i) = 1$*.*

There are no costs associated with the information selection stage beyond the opportunity costs of not learning about the other value component. The *timing* is as follows.

1. An auction format is announced.
2. Nature draws S, T_1, T_2.
3. Bidders simultaneously and privately select their information $\boldsymbol{\rho} := \{\rho_1, \rho_2\}$.
4. Bidders privately observe their signal X_i^S or X_i^T.
5. The auction takes place.

Information selection is *covert*: bidders do not observe which channel others chose to learn about, but make inference about it in equilibrium. Moreover, bidders select their information *after* the auction format is announced. This enables an analysis of the incentives of various auctions on information selection.

4. Information Selection

4.1 Endogenous Correlation

With probability $(1 - \rho_1\rho_2)$ at least one bidder observes a signal about his private attribute T_i and signals are independent by Assumption IN. With the remaining probability $\rho_1\rho_2$, bidders observe correlated signals about the same realization of the common attribute S. In this case, private signals X_i^S and X_j^S are independent conditional on the common value realization s by Assumption CI.

Bidder i forms a belief about the distribution of his opponent's signal, based on the source of his own signal, X_i^T or X_i^S, and its realization $x_i \in [0, 1]$. Bidder i does not know whether his opponent j observed a signal about S or T_j, but draws inference if he expects his opponent to set $\rho_j > 0$, as the following cumulative distributions show.

Let $G^T(x_j|x_i, \rho_j) := \Pr(X_j \leq x_j|X_i^T = x_i, \rho_j)$ be the conditional cumulative distribution function of the 'source-free' signal realization X_j, from the perspective of bidder i with a signal realization $X_i^T = x_i$. The distribution function G^T does not depend on ρ_i, as it already conditions on bidder i having observed a signal X_i^T about T_i. If a bidder learns X_i^T, his signals contains no information about the other bidder due to Assumption IN. Therefore, using Observation 1, for all $x_i \in [0, 1]$ and any information selection $\rho_j \in [0, 1]$ of the opponent,

$$G^T(x_j|x_i, \rho_j) = F(x_j).$$

If bidder i learns about his common component via X_i^S, his signal realization might bear information about his opponent's signal realization. Let $G^S(x_j|x_i,\rho_j) := \Pr(X_j \leq x_j|X_i^S = x_i, \rho_j)$ be the distribution function of the signal realization of bidder $j \neq i$, conditional on $X_i^S = x_i$.

$$G^S(x_j|x_i,\rho_j) = (1-\rho_j)F(x_j) + \rho_j \int_0^1 \frac{f^S(x_i|s)F^S(x_j|s)}{f(x_i)} h(s)ds.$$

The second summand accounts for the correlation in private information if the opponent j also learns about the common component (with probability ρ_j). Then, signals are independent conditional on S by Assumption CI.

4.2 Endogenous Value Setting

The degree of the winner's curse is endogenous in my model. If the opponent of bidder i only learns about his private component T_j, his information is irrelevant for bidder i. Winning at any bid does not provide any further information for bidder i beyond his private signal realization and there is no winner's curse.

If the other bidder j learns about the common component, the event of winning contains information about S for bidder i. If every bidder follows a symmetric and strictly increasing bidding function, winning indicates that bidder j has a lower signal about S than bidder i. This is bad news for the value of the object, and bidders shade their bid down to account for the effect of the winner's curse, to not overbid in case of a win.

Let bidder i observe a signal X_i^K for $K \in \{S, T\}$. His expected value of the object to bidder i, updated only based on his own signal realization is

$$\mathbb{E}\left[V_i|X_i^K = x_i\right] = \int_{\mathcal{V}} v_i h^K(v_i|x_i)dv_i,$$

where $h^K(v_i|x_i)$ is the following probability density function of the value V_i for bidder i conditional on his signal realization $X_i^K = x_i$ about component $K \in \{S, T_i\}$:

$$h^K(v_i|x_i) = \begin{cases} \frac{1}{f^S(x_i)} \int_0^1 \underbrace{f^S(x_i|s)h(s)h(v_i - s)}_{\substack{\text{joint event} \\ X_i^S = x_i, V_i = v_i, S = s}} ds \text{ if } K = S, \\ \frac{1}{f^T(x_i)} \int_0^1 \underbrace{f^T(x_i|t)h(t)h(v_i - t)}_{\substack{\text{joint event} \\ X_i^T = x_i, V_i = v_i, T_i = t}} dt \text{ if } K = T. \end{cases} \tag{1.1}$$

The following observation shows that any information selection leads to the same expected value of the object, conditional on that signal realization alone. This follows immediately from the symmetry of the distributions of the value components S and T_i, $H(m) = \Pr(T_i \leq m) = \Pr(S \leq m)$ and the signals having the same density $f^T(x_i|r) = f^S(x_i|r)$ for each realization $x_i \in [0, 1]$. That is, $h^S(v_i|x_i) = h^T(v_i|x_i)$.

Observation 2. *The object's expected value conditional on signal realization x_i coincides for both signals X_i^S and X_i^T: $\forall x_i \in [0,1]$, $\mathbb{E}\left[V_i|X_i^S = x_i\right] = \mathbb{E}\left[V_i|X_i^T = x_i\right]$.*

Both available signals X_i^S and X_i^T have equal informativeness about V_i, as they lead to the same posterior distribution over the total value. In equilibrium, bidders update about the value of the object, conditional on their signal, and conditional on the event of winning. Being pivotal bears information about the signal realization of the other bidder. The following expression is the value of bidder i after observing an experiment about component $K \in \{S, T\}$, when the signal realization of the opponent is $X_j = x_j$, and the opponent selects ρ_j. For $K \in \{S, T_i\}$,

$$v^K(x_i, x_j|\rho_j) := \mathbb{E}\left[V_i|X_i^K = x_i, X_j = x_j, \rho_j\right].$$

The above definition is based on a source-free signal realization $X_j = x_j$ of the other bidder, as bidder i cannot observe whether it contains information about the common valuation S or the private component T_j of his opponent. However, it conditions on the information selection strategy of the other bidder, ρ_j. This is to capture that information selection is covert. While the choice of ρ_j is unobservable to bidder i, he draws correct inference about it in equilibrium.

The following two value settings are nested in my model:

1. **Independent private values (IPV).** If $\rho_1 = \rho_2 = 0$, private signals X_1^T and X_2^T are independent. The expected value of bidder i does not depend on the signal of bidder j:

$$v^T(x_i, x_j|\rho_j = 0) = \mathbb{E}\left[V_i|X_i^T = x_i\right] = \mathbb{E}\left[T_i|X_i^T = x_i\right] + \mathbb{E}\left[S\right].$$

2. **Common values/ mineral rights model (CV).** If $\rho_1 = \rho_2 = 1$, expected utility of the bidders is symmetric in the two private signals X_1^S and X_2^S:

$$v^S(x_i, x_j|\rho_j = 1) = v^S(x_j, x_i|\rho_j = 1) = \mathbb{E}\left[T_i\right] + \mathbb{E}\left[S|X_i^S = x_i, X_j^S = x_j\right].$$

For example, fix the information choice of bidder j at $\rho_j = 1$ such that he always learns his signal X_j^S. If bidder i learns signal $X_i^S = x_i$ about the common component, his expected value is as described in above CV setting. If bidder i instead learns about his private component via observing X_i^T, his estimate of the object when his opponent has signal realization $X_j^S = x_j$ is

$$v^T(x_i, x_j|\rho_j = 1) = \mathbb{E}\left[T_i|X_i^T = x_i\right] + \mathbb{E}\left[S|X_j^S = x_j\right].$$

Let bidder j select $\rho_j = 1$ and learn about the common component via X_j^S and consider Example 1. Figure 1.1 depicts the expected value of the object for bidder i, when he expects his opponent to have the same signal realization as himself, $X_j^S = x$. The blue dashed line is the expected value $v_i^S(x_i, x_i|\rho_j = 1)$ for bidder i in the CV framework with signal $X_i^S = x_i$. The green solid line is bidder i's expected value $v^T(x, x|\rho_j = 1)$ if he learns about his private component. Expected value is increasing

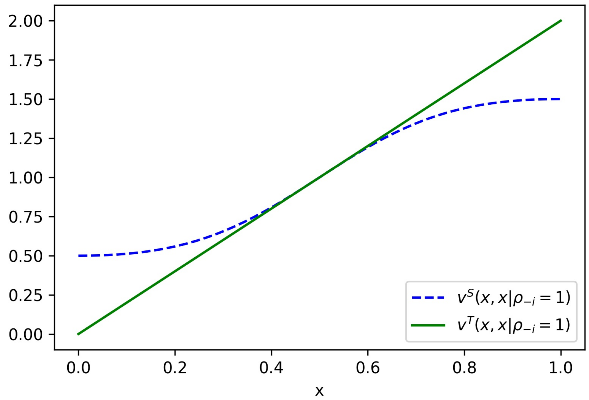


Figure 1.1: Expected valuation of bidder i in Example 1, if he chooses to learn about component $K \in \{S, T_i\}$, the opponent learn about S, and both bidders have the same signal realization x_i.

in the signal realization.[19] The function $v^S(x, x|\rho_j = 1)$ reacts slower to a change in the signal x than $v^T(x, x|\rho_j = 1)$. This is because if a bidder learns about his private component, there is no dependence with his opponent, and therefore, no redundancy in private information. Receiving a low signal is worse news (and receiving a high signal is better news), if it contains information about the private component.

5. Second Price Auction

In this section, two bidders are competing for one indivisible object in a SPA, with no reserve price and an equal tie-breaking rule.[20] If the random vector $\boldsymbol{\rho}$ is exogenous and common knowledge, that is, when there is no information selection stage, the model reduces to Milgrom and Weber (1982). Under endogenous and covert information selection, bidders optimize their own information choice and make inference about the information source of their opponent in equilibrium, as it has an effect on the winning probability, the expected payment and the value of the object conditional on winning.

I consider the following class of equilibria:

Definition 1 (Symmetric Bayes Nash equilibrium)**.** *In a symmetric Bayes Nash equilibrium, bidders*

- *select the same* $\rho_i = \rho^*$*,*
- *after observing* $X_i^S = x$*, bid* $\beta^S(x)$*,*

[19]The expected value $v^K(x_i, x_j|\rho_j = 1)$ of bidder i with own signal x_i and given the signal realization of the opponent x_j is non-decreasing in both arguments. This follows from affiliation of X_i^K with X_j^S (Milgrom and Weber, 1982).

[20]For $N = 2$ bidders, the sealed bid SPA and the open English auction are strategically equivalent (see Milgrom and Weber, 1982). Furthermore, due to the assumption of strictly increasing bidding functions and no atoms in signal distributions, the probability of a tie is zero.

- *after observing $X_i^T = x$, bid $\beta^T(x)$,*

where bidding functions $\beta^S(x)$ and $\beta^T(x)$ are pure and strictly increasing in x, and together with ρ^* constitute mutually best responses.

In the remainder of the paper, the term "equilibrium" refers to an object that satisfies the above definition. Let $CE := \{\rho^*, \beta^S, \beta^T\}$ be a *candidate equilibrium*. The expected utility of bidder i from learning about component $K \in \{S, T_i\}$ and bidding with β_i, who is facing an opponent who plays CE, is denoted by $EU(K, \beta_i|CE)$. It can be separated into his the expected gain $EG(K, \beta_i|CE)$ minus his expected payment $EP(K, \beta_i|CE)$:

$$EU(K, \beta_i|CE) := EG(K, \beta_i|CE) - EP(K, \beta_i|CE). \tag{1.2}$$

First, consider the expected gain of bidder i. In the candidate equilibrium, bidder i expects his opponent to learn X_j^T and bid according to β^T with probability $(1-\rho^*)$; in this case, the expected gain of bidder i is in line 1.3. With the remaining probability ρ^*, his opponent learns a signal X_j^S, bids according to $beta^S$. In this case, the expected gain of bidder i is depicted in line 1.4.

$$EG(K, \beta_i|CE) := (1-\rho^*) \underbrace{\int_{\mathcal{V}} v_i \Pr(i \text{ wins}|v_i, X_i^K, \beta_i, X_j^T, \beta_j^T) h_{\mathcal{V}}(v_i) dv_i}_{\text{Expected gain of bidder i when } j \text{ learns } X_j^T} \tag{1.3}$$

$$+ \rho^* \underbrace{\int_{\mathcal{V}} v_i \Pr(i \text{ wins}|v_i, X_i^K, \beta_i, X_j^S, \beta_j^S) h_{\mathcal{V}}(v_i) dv_i}_{\text{Expected gain of bidder i when } j \text{ learns } X_j^S} . \tag{1.4}$$

Second, consider the expected payment of bidder i. In the SPA, if bidder i wins he pays the bid of his opponent j. Consider the distribution of the signal of the opponent j, conditional on bidder i having a higher signal. This distribution depends on both the information choices of the bidders and their bidding functions.

Let $L \in \{S, T_j\}$ be the component about which bidder j learns signal X_j^L and bids according to β_j^L. Whenever it is well-defined[21], define the cumulative distribution of bidder j's signal realization *conditional* on bidder i winning (when learning X_i^K and bidding β_i^K):

$$H^K(x_j|\beta_i, \beta_j^L, X_j^L) := \Pr(X_j^L \leq x_j | \beta_i(X_i^K) \geq \beta_j^L(X_j^L)). \tag{1.5}$$

Let $h^K(x_j|\beta_i, \beta_j^L, X_j^L)$ be the corresponding density, if it exists. With this information choice K and L, and bidding functions β_i, β_j^L, the overall expected payment of bidder i is:

$$EP(X_i^K, \beta_i|X_j^L, \beta_j^L) := \Pr(\beta_i(X_i^K) \geq \beta_j^L(X_j^L)) \underbrace{\int_0^1 \beta_j^L(x_j) dH^K(x_j|\beta_i, \beta_j^L, X_j^L)}_{\text{payment conditional on winning}} . \tag{1.6}$$

The first factor is the overall probability of bidder i winning. The second factor is the expected bid of j that bidder i has to pay conditional on winning.

[21]That is, if the probability of bidder i winning is non-zero with $\beta_i(X_i^K = 1) > \beta^L(X_j^L = 0)$.

Bidder i does not observe which signal his opponent learns, but expects him to select ρ^* in the candidate equilibrium. Based on this inference, bidder i's expected payment with information choice K and bidding function β_i is

$$EP(X_i^K, \beta_i | CE) = (1 - \rho_j) EP(X_i^K, \beta_i | X_j^T, \beta_j^T) + \rho_j EP(X_i^K, \beta_i | X_j^S, \beta_j^S). \qquad (1.7)$$

The first summand accounts for the possibility of the opponent having observed signal X_j^T times the expected payment in this case. The second summand is the expected payment when facing an opponent with signal X_j^S, weighted with the probability ρ_j of the occurrence of this event.

5.1 Information Selection in Equilibrium

The next theorem establishes the main result for the SPA. It shows that there is no learning about the common component in any equilibrium.

Theorem 1
Information selection is unique in equilibrium, $\rho^* = 0$.
There exists an equilibrium in which $\beta^T(x) = \mathbb{E}\left[V_i | X_i^T = x\right]$.

All proofs are in the appendix, unless stated otherwise. In the remainder of the section, I derive auxiliary results necessary to prove the above theorem.

First, consider any candidate equilibrium in which $\rho^* > 0$. Our goal is to establish that there exists a profitable deviation, as soon as there is positive dependence via learning about the common component. In general, a brute-force maximization approach to find the *best response* to a candidate equilibrium is a fruitless undertaking. This is because simultaneosly varying the information source and bidding function has adverse implications on the winning probability, expected payment and the posterior value of the object conditional on a win, and the overall effect on the payoff becomes intractable. Unless bidders follow the same bidding functions that allow some form of comparability, there is little that can be said about which strategy leads to a higher overall utility.

The trick is to *isolate* the effect on expected gain from the effect on expected payment conditional on a win. I establish existence of a deviation strategy that switches off any change in the expected gain *and* the winning probability. That is, by playing such a deviation strategy a bidder can guarantee himself the same expected gain and the same total winning probability as in in the candidate equilibrium. By picking the deviation strategy accordingly, we can concentrate on the effect on expected payment conditional on a win, as the the other components in Equation 1.2 are held constant. Critically hereby is to employ deviations that involve the same bidding functions between bidders even *after* the a deviation to a different information channel. This ensures that a bidder wins if and only if he has a higher signal than his opponent in certain cases. The following deviation strategy is strictly profitable whenever the candidate equilibrium contains $\rho^* > 0$.

Definition 2. *The deviation strategy (**DS**) for bidder i is the following strategy:*

- *deviate to $\rho_i = 0$,*

- *bid according to $\beta^S(x_i)$ for $X_i^T = x_i$.*

This deviation strategy takes the signal X_i^T about the private component and maps it into a bid with bidding function β^S as if it were the signal about the common component in the candidate equilibrium. While this is not necessarily the optimal bidding behavior learning X_i^T, it is strong enough to establish a profitable deviation over the combination (X_i^S, β^S), which is part of any candidate equilibrium with $\rho^* > 0$.

5.2 Expected Gain

In this section, I compare the expected gain for bidder i from DS to his expected gain from the CE with combination (X_i^S, β^S), if he expects his opponent to play according to CE. I show that the expected probability of winning *conditional on a value realization v_i for bidder i*, is identical in DS and in CE.

Fix a value v_i for bidder i. There are two possibilities that can arise, depending on which information channel bidder j chooses. Bidder i does not know in which possibility he is in, as information selection is covert.

Opponent with signal $\mathbf{X_j^T}$. With probability $(1-\rho^*)$, the opponent of bidder i learns signal X_j^T about his private component, and follows the bidding function β^T. In this situation, a higher signal realization of bidder i does not necessarily imply winning, as this depends on the interplay of the the bidding functions β^S and β^T.

$$DS: \qquad \Pr(i \text{ wins}|v_i, \underbrace{X_i^T, \beta^S}_{DS}, X_j^T, \beta^T) = \Pr(\beta^S(X_i^T) \geq \beta^T(X_j^T)|v_i). \tag{1.8}$$

$$CE: \qquad \Pr(i \text{ wins}|v_i, \underbrace{X_i^S, \beta^S}_{CE}, X_j^T, \beta^T) = \Pr(\beta^S(X_i^S) \geq \beta^T(X_j^T)|v_i). \tag{1.9}$$

Neither playing (X_i^S, β^S) in CE nor DS of bidder i lead to correlation in private information, as by Assumption IN X_j^T is independent from any signal of bidder i. Hence, the probability of a win conditional on any value v_i is the same in Equation 1.8 and Equation 1.9, as the following lemma shows.

Lemma 1
For all v_i, $\Pr(i \textit{ wins}|v_i, \underbrace{X_i^S, \beta^S}_{CE}, X_j^T, \beta^T) = \Pr(i \textit{ wins}|v_i, \underbrace{X_i^T, \beta^S}_{DS}, X_j^T, \beta^T)$.

Note that Lemma 1 does not require bidder i and j to follow the same bidding function. The marginal distribution of both signals X_i^S and X_i^T of bidder i coincide conditional on every value v_i. This follows as both signals have equal marginal distributions. As bidder i follows the same bidding function in CE and DS, also the marginal distribution of *bids* coincides for each value v_i. As the signal of the opponent X_j^T is independent from bidder i for any information choice, the probability of winning is the same in CE and in DS.

Opponent with signal $\mathbf{X_j^S}$. With probability ρ^*, bidder i faces an opponent who learns X_j^S about his common component. In this case, both bidders follow the same bidding function β^S, and bidder i wins if and only if his opponent has a lower signal than him.[22] The winning probabilities for bidder i conditional on v_i in DS and in (X_i^S, β^S) in CE are

$$DS: \qquad \Pr(i \text{ wins}|v_i, \underbrace{X_i^T, \beta^S}_{DS}, X_j^S, \beta^S) = \Pr(X_i^T \geq X_j^S|v_i). \tag{1.10}$$

$$CE: \qquad \Pr(i \text{ wins}|v_i, \underbrace{X_i^S, \beta^S}_{CE}, X_j^S, \beta^S) = \Pr(X_i^S \geq X_j^S|v_i). \tag{1.11}$$

For each total value realization v_i for bidder i, the following theorem pins down the probability of having the highest signal in (X_i^S, β^S) in CE and in DS.

Proposition 1
For all total values v_i for bidder i,

$$\Pr(X_i^T \geq X_j^S|v_i) = \Pr(X_i^S \geq X_j^S|v_i) = \frac{1}{2}.$$

Hence, winning probability is equal for *every* value realization v_i in both DS and CE:

$$\Pr(i \text{ wins}|v_i, \underbrace{X_i^S, \beta^S}_{CE}, X_j^S, \beta^S) = \Pr(i \text{ wins}|v_i, \underbrace{X_i^T, \beta^S}, X_j^S, \beta^S).$$

This proposition is more complicated to establish and does not follow from independence as does Lemma 1. This is because if the opponent learns X_j^S about the common component, bidder i has a choice between interdependence in signals (by choosing X_i^S in CE) and independence (by choosing DS and X_i^T). Furthermore, the proposition crucially relies on the fact that there are only two bidders.[23]

It is instructive to consider how winning probability changes in different combinations of S and T_i for bidder i, when deviating to DS from CE. Proposition 1 establishes that winning probability conditional on any value realization v_i is constant. Yet, the particular composition of states S and T_i of components, in which a bidder i wins, changes.

Fix any total value realization v_i, and fix some feasible realization of the common component $s \in [\max\{0, v_i - 1\}, \min\{1, v_i\}]$. Then, $t_i = v_i - s$. If bidder i plays according to CE with (X_j^S, β^S) and faces an X_j^S-type opponent, his probability of winning at this combinations of S and T_i is

$$\Pr(X_i^S \geq X_j^S|S = s, T_i = v_i - s) = \int_0^1 f^S(x|s)F^S(x|s)dx = \frac{1}{2}.$$

If bidder i plays DS instead, his winning probability for this combination v_i and s is

$$\Pr(X_i^T \geq X_j^S|S = s, T_i = v_i - s) = \int_0^1 f^T(x|v_i - s)F^S(x|s)dx.$$

[22]Ties are ignored as they have zero probability.

[23]I extend the proposition in Section 6.2 to more than two bidders.

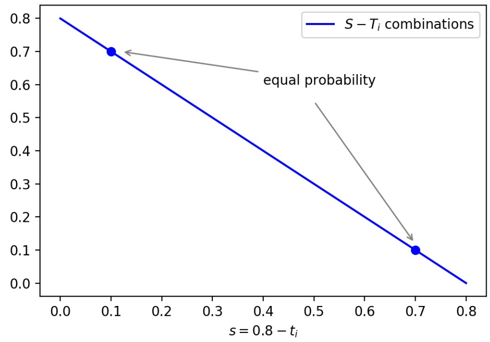


Figure 1.2: Iso-value curve for total values $v_i = 0.8$ of bidder i, showing different combination of feasible value components S and T_i.

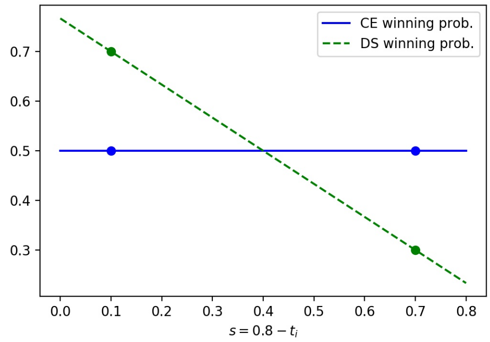


Figure 1.3: Winning probability in DS and (X_i^S, β^S) in CE, for different S-T_i-combinations, given $v_i = 0.8$ and X_j^S. The two green dots in DS sum up 0.5.

The following lemma shows how DS shifts bidder i's winning probability into states with a higher private component realization.

Lemma 2

Fix $v_i \in (0, 2)$ and let bidder j learn X_j^S and bid according to β^S. With DS, bidder i is strictly more (less) likely to win at $S < v_i/2$ ($S > v_i/2$) than with (X_i^S, β^S) in CE. At $S = v_i/2$, winning probability is equal in both strategies.

By deviating to DS, for a given v_i, a bidder is strictly more likely to win in combinations that involve a high T_i, and strictly less likely to win in combination that involve a high S. For example, fix the value $v_i = 0.8$ for bidder i, that stems from any combination of the common component $S \in [0, 0.8]$ and private component $t_i = 0.8 - S$. This is depicted in Figure 1.2, where the diagonal line shows the iso-value curve, that consists of all feasible $S - T_i$ combinations that lead to the same overall value v_i for bidder i.

In CE with (X_i^S, β^S), bidder i wins with equal probability of $\frac{1}{2}$ for *every* realization of $S \in [0.8]$ when facing an opponent j who also learns about the common component. This is because both bidders have access to the same winning technology, and both learn about the same variable S.

With DS, bidder i and bidder j look at different value components. Due to the MLRP, higher signals are more likely for higher realizations of the value components. With DS, bidder i is more likely to win in states with a high *private* component realization, and less likely to win with a high common component realization.

This is depicted in Figure 1.3. The x-axis shows the common component S that is feasible with $v_i = 0.8$. For each feasible s on the x-axis, there exists a unique realization of t_i such that $v_i = 0.8$. The y-axis is the respective winning probability for such a realized pair (S, T_i). The blue solid line at $y = \frac{1}{2}$ shows the winning probability with CE which is constant at one half. The green dashed line sketches the winning probability with DS. The two lines cross exactly at $v_i/2 = 0.4$.

In sum, the overall effect on the winning probability sums up to zero. Winning probability is the same in CE with (X_i^S, β^S) and DS for every v_i. To provide intuition why the effect on overall winning probability given v_i evaporates, fix the following two combinations: $(s = 0.1, t_i = 0.7)$ and $(s = 0.7, t_i = 0.1)$. Those two points are depicted by blue dots on the iso-value curve in Figure 1.2. As S and T_i are distributed identically and independently, both combinations have equal probability of $h(0.1)h(0.7)$.

In CE, winning probability of bidder i is $\frac{1}{2}$ in both those possibilities. This is depicted by the two blue dots on the blue solid line at $y = 1/2$ in Figure 1.3. With DS, if $(s = 0.1, t_i = 0.7)$, the winning probability of bidder i is no longer $\frac{1}{2}$, but higher due to the higher private component in comparison to the low common component. This is depicted by the green dot in the upper left corner of Figure 1.3. However, bidder i loses the exact same winning probability in state $(s = 0.7, t_i = 0.1)$, as there it is his opponent who observes a signal about the higher state s, while bidder i learns about the lower component realization t_i. This is depicted by the green dot in the lower right corner. The overall effect of the change in winning probability in the two combinations balances out to zero. In sum, overall probability of a win conditional on being in one of those two combinations, remains 1/2. This argument works for any two feasible symmetric combinations $(s = a, t_i = v_i - a)$ and $(s = v_i - a, t_i = a)$ for any v_i. Therefore, information selection shuffles the states in which bidder i wins, while keeping the overall probability fixed.

To sum up, given any realization of the total value v_i, DS and (X_i^S, β^S) in CE yield the same probability of winning if his opponent learns about the common component, snd if the opponent learns about his private component. The following corollary shows the impact of DS on the expected gain in Equation 1.3 and Equation 1.4 and on total winning probability in comparison to CE with (X_i^S, β^S). It is an immediate implication of of Lemma 1 and Proposition 1, and the proof is therefore omitted.

Corollary 1. *Expected gain in CE with (S, β^S) and DS coincide. The total winning probability is identical in DS and CE with (S, β^S).*

As winning probability is the same for every v_i, it is also the same overall in CE and DS.

5.3 Expected Payment

The expected payment conditional on winning changes under the deviation strategy. In the following I show, that DS leads to a strictly lower payment by establishing a stochastic dominance order between the payment distributions with and without interdependence in private signals.

Consider the signal distribution of the opponent j, conditional on bidder i winning in Equation 1.5. First, consider bidder i facing a X_j^T-type opponent. If $\beta^S(x_i = 1) \leq \beta^T(x_i = 0)$, bidder i has a zero-probability of winning in DS and in CE with (X_i^S, β^S) against bidder j bidding with β^T. Therefore, deviating to DS does not change the expected payment when facing a X_j^T-type opponent.

If $\beta^S(x_i = 1) > \beta^T(x_i = 0)$, bidder i who employs bidding strategy β^S has a non-zero winning probability when facing a X_j^T-type opponent. The distribution of signals of the loosing bidder j is well-defined. It is $H^T(x_j|\beta^S, \beta^T, X_j^T)$ if bidder i plays DS, and $H^S(x_j|\beta^S, \beta^T, X_j^T)$ if bidder i plays CE with (X_i^S, β^S).

If bidder i faces a X_j^S-type opponent, the distribution of his opponent's signal is $H^T(x_j|\beta^S, \beta^S, X_j^S)$ if bidder i plays DS, and $H^S(x_j|\beta^S, \beta^S, X_j^S)$ if bidder i plays CE with (S, β^S).

Lemma 3 *1. Opponent with signal X_j^T: for all $x_j \in [0,1]$, if $\beta^S(1) > \beta^T(0)$, then*

$$H^S(x_j|\beta^S, \beta^T, X_j^T) = H^T(x_j|\beta^S, \beta^T, X_j^T);$$

2. Opponent with signal X_j^S: $H^S(x_j|\beta^S, \beta^S, X_j^S)$ (strictly) first order stochastically dominates (FOSD) $H^T(x_j|\beta^S, \beta^S, X_j^S)$;

3. Overall expected payment is strictly lower under DS than in CE with (X_i^S, β^S).

The first property says that as long the opponent looks at his private component X_j^T, the expected distribution of payments of bidder i in case of a win does not depend on bidder i's information choice. Note that it does not rely on bidder i and j employing with the same bidding functions, but only bidder i using the same β^S for both his information channels X_i^S and X_j^S. The property holds because if bidder j learns X_j^T, his signal and thus his bid distribution is independent from both signals X_i^S and X_i^T of bidder i. Both these signals of bidder i have the same marginal distribution via Observation 1. The argument is similar to the proof of Lemma 1 and relies on independence in private signals.

The second property establishes that if bidder j learns X_j^S about the common component, and both bidders follow the same bidding function β^S, the event of bidder j having a signal below some x_j *conditional on* bidder i winning is more likely for every x_j. That is, the cumulative distribution of the second order statistic under interdependent signals with X_i^S FOSD the distribution of the second order statistic under independence with X_i^T. By the FOSD, conditional on bidder i winning, the signals and therefore the bids of the opponent are distributed lower in DS than in CE with (X_i^S, β^S).

For a quick sketch[24] of the argument, the following expression is the signal distribution of X_j^S of bidder j, conditional on bidder i playing DS and winning (in this case signals of the two bidders are independent):

$$H^T(x_j|\beta^S, \beta^S, X_j^S) = 2\int_0^{x_j} f(\tilde{x}_j)\,(1 - F(\tilde{x}_j))\,d\tilde{x}_j = 2F(x_j) - F(x_j)^2. \tag{1.12}$$

This is the second order statistic of the two equally distributed independent signals X_i^T and X_j^S, as bidder i pays the second order statistic conditional on winning. Both bidders follow the same bidding function β^S, and bidder i wins if and only if he has a higher signal than his opponent.

If bidder i plays CE with (X_i^S, β^S), conditional on bidder i winning with X_i^S and β^S, the distribution of his opponent's signal X_j^S is the following expression:

$$H^S(x_j|\beta^S, \beta^S, X_j^S) = 2\int_0^{x_j}\int_0^1 f(\tilde{x}_j|s)\,(1 - F(\tilde{x}_j|s))\,h(s)ds d\tilde{x}_j = 2F(x_j) - \int_0^1 F(x_j|s)^2 h(s)ds. \tag{1.13}$$

[24]See the proof of Lemma 3 for a derivation of these cumulative distribution functions.

This is the cumulative distribution function of the second order statistic under correlation between X_i^S and X_j^S via the common component S.

Comparing Equation 1.12 with Equation 1.13 shows that conditionally on a win, less correlation induces a lower distribution of the second order statistic and thus, a lower payment distribution. That is, for all $x_j \in (0,1)$, the Cauchy-Bunyakovsky-Schwarz (strong)[25] inequality establishes

$$F(x_j)^2 = \left(\int_0^1 F(x_j|s)h(s)ds\right)^2 < \underbrace{\int_0^1 h(s)ds}_{=1} \int_0^1 F(x_j|s)^2 h(s)ds.$$

Hence, the probability of paying any bid $\beta^S(x_j)$ or below conditional on winning is lower when playing DS than when playing the candidate equilibrium with X_j^S. Conditional on winning, the lower the distribution of the opponent's signal (i.e. the lower the second order statistic), the lower the expected payment given a fixed bidding strategy β^S of the opponent. Consider the limiting case of almost perfect correlation. Conditional on the event of winning, the bid of the other bidder is close to the own bid. Without correlation, the bid of the opponent conditional on a win is distributed independently. Conditional on winning, a bidder prefers his opponent to bid as low as possible. Positive interdependence raises the expected payment conditional on a win by increasing the distribution of the second order statistic in the sense of FOSD.

To sum up, when facing a X_j^T-type opponent, expected payment is the same in DS and CE with (X_i^S, β^S). Conditional on a win against a X_j^S-type opponent, the payment distribution of bidder i with DS is strictly dominated by the payment distribution with CE and (X_i^S, β^S). Hence, the conditional payment is strictly lower in DS than in CE with (X_i^S, β^S). As the bidding function β^S is strictly increasing in the signal, this follows immediately via strong FOSD in Equation 1.6. By Corollary 1, the winning probability with DS is equal to the winning probability in CE. Hence, the unconditional expected payment is also strictly less with DS with X_j^S of the opponent.

The probability to encounter a X_j^S-type opponent is non-zero in any candidate equilibrium with $\rho^* > 0$. Therefore, the third statement of Lemma 3 follows. Unconditional expected payment from DS is strictly less than in the candidate equilibrium with (X_i^S, β^S).

5.4 Equilibrium and Social Surplus

The advantage of the deviation strategy DS is that it does not modify neither the overall probability of winning for each valuation v_i nor the expected gain, but instead lowers the expected payment in case of a win due to less dependence between the signals of the two bidders. Combined, Corollary 1 and Lemma 3 establish that no $\rho^* > 0$ can be an equilibrium, as DS constitutes a strictly profitable deviation.

For Theorem 1 to hold we need to establish existence of an equilibrium with $\rho^* = 0$. In this case, both bidders learn about their private components, information is only relevant for the bidder who observes it, and bidders are in an IPV setup. Hence, due to Observation 2, a bidder is indifferent between the two signals. The value of information

[25]For the strong Cauchy-Bunyakovsky-Schwarz inequality, see Footnote 33 in the proof of Lemma 3.

from both signals is the same, as both lead to no interdependence with the opponent, and both induce the same best response and posterior about the total value of the object.

Social surplus is maximized if a bidder with the highest expected private component T_i receives the object. All bidders share the same common component S, which therefore plays no role for the social surplus. Ex-ante efficiency requires all bidders to learn only about their private component, to maximize the ex-ante expected social surplus. Information about the common component is not socially valuable, and available only by incurring the opportunity costs of not learning about the private component. Theorem 1 establishes that no equilibrium exists unless $\rho^* = 0$. The *SPA is ex-ante efficient* as it induces $\rho^* = 0$ and allocates efficiently.

6. Generalization

In this section, I analyze the incentives to select information in the SPA for a broader class of utility functions (Section 6.1) and discuss the applicability of my approach for the case of more than two bidders in Section 6.2. In the following, I restrict attention to pure information selection: bidders select information either about their private component via $\rho_i = 0$ or about their common component via $\rho_i = 1$. I show that my approach generalizes to a general class of utility functions, as long as they satisfy a marginal rate of substitution property.

6.1 General Utility Function

In the preceding parts of this paper, a bidder's overall utility function was symmetric, $V_i = S + T_i$. Next, consider a generalized class of utility functions $V_i = u(S, T_i)$ that satisfy the following properties:

1. $u(0,0) \geq 0$, and $u(1,1) < \infty$;

2. $u(\cdot,\cdot)$ is strictly increasing in both arguments;

3. for all $I \in [0,2]$, $u(S, I-S)$ is non-increasing in $S \in [0,1]$.

The first property binds the utility of a bidder above and below such that it is never strictly negative. The second property guarantees that any increase in either of his two components is strictly better for the bidder. The third property is a condition on the marginal rate of substitution between the two components $S \in [0,1]$ and $T_i \in [0,1]$, when their sum is constant at some $I \in [0,2]$. The property states that by substituting T_i with the same amount of S, the bidder is weakly worse off. If the utility function is differentiable in both arguments, the third property simply reduces to a marginal rate of substitution inequality: $\frac{\partial u(\cdot)}{\partial S} \leq \frac{\partial u(\cdot)}{\partial T_i}\Big|_{I=S+T_i}$.

A utility function, that satisfies above assumptions, is for example

$$V_i = \alpha S + (1-\alpha) T_i$$

with $\alpha \in (0, \frac{1}{2}]$. For this particular example, it is straightforward to see that for any sum of the components $I = S + T_i$, we have $\frac{du(S,I-S)}{dS} \leq 0$ whenever $\alpha \leq \frac{1}{2}$. The following proposition extends the result for the SPA for this extended class of utility functions.

Proposition 2
For all utility functions satisfying properties 1.-3., there exists no equilibrium of the SPA in which bidders learn about the common component via $\rho^* = 1$.

The proof is by contradiction, along the lines of the technique developed in Section 5. For a sketch of the argument, consider $\rho^* = 1$ being a candidate equilibrium (CE). That is, in equilibrium both bidders learn only about their common component and expect their opponent to do the same. Then, bidder i can play the following deviation strategy (DS) as in the preceding section and strictly increase his expected utility: *Set $\rho_i = 0$ and observe X_i^T, but bid according to bidding function β^S that bidder i uses in the candidate equilibrium with X_i^S.*

In contrast to the preceding section, the expected gain from DS will be different than in CE. By Proposition 1, with a symmetric utility function $V_i = S + T_i$, it holds for two bidders and any $v_i \in [0, 2]$ realization: $\Pr(X_i^T \geq X_j^S | v_i) = \Pr(X_i^S \geq X_j^S | v_i)$. The theorem conditions on all combinations of S and T_i, that sum up to $I = v_i$, which corresponds to the same utility of v_i for the symmetric utility function $V_i = S + T_i$. Hence, there is no need to differentiate between the sum of the two components and the overall utility for bidder i from this component combination.

Note that for the general utility functions satisfying properties 1.-3., the statement of Proposition 1 holds exactly in the same manner when conditioning on $I = S + T_i$, but no longer on the value realization v_i, which might be different for the same sum I of the two components.[26] That is, we have for all $I \in [0, 2]$

$$\Pr(X_i^T \geq X_j^S | I) = \Pr(X_i^S \geq X_j^S | I).$$

In DS and in CE, bidder i follows β^S, the same bidding function as his opponent in the CE with $\rho^* = 1$. Hence, bidder i wins whenever he has a higher signal realization than his opponent. Thus, the above is the probability of winning conditional on the sum I of the two value components for bidder i.

This establishes that the bidder is equally likely to win, given the sum of the two components. In Section 5.2 I establish that while keeping the overall probability of a win for v_i fixed, DS has an adverse effect on winning in different combinations of S and T_i, as Lemma 2 depicts. Lemma 2 applies exactly in the same way for the case of fixing the sum of the two components I, instead of v_i. Replacing every v_i in the proof by I yields the result.

By deviating to DS, for a given I, a bidder is strictly more likely to win in combinations that involve a high T_i, and strictly less likely to win in combination that involve a high S. By property 3., a bidder prefers those combinations with a higher T_i in which he wins more often over those with a low T_i in which he loses winning probability. Hence, his expected gain from DS is weakly higher than from CE.

[26]The steps of the proof for fixing I instead of v_i are exactly the same as for v_i, and the result follows by simply replacing every v_i in the proof of Proposition 1 by I.

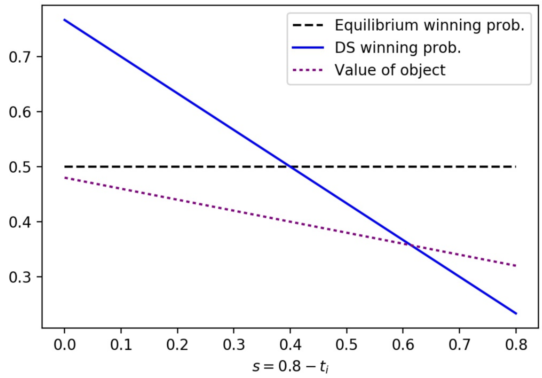


Figure 1.4: Winning probability in DS (blue solid line), the candidate equilibrium (black dashed line) and value of the object (purple dotted) with $\rho^* = 1$, $I = 0.8$, $f(x|r) = (2-2r) + (4r-2)x$ and $V_i = 0.4S + 0.6T_i$.

Example 2. *Let $V_i = 0.4S + 0.6T_i$ and consider the following density of signals for each component realization $r \in [0, 1]$: $f(x|r) = (2-2r) + (4r-2)x$. For this linear example, the winning probability when deviating to DS is $\Pr(X_i^T \geq X_i^S | I, S = s) = \frac{I}{3} - \frac{2s}{3} + \frac{1}{2}$ for all $I \in [0, 2]$ and all feasible $s \in [\max\{0, I-1\}, \min\{1, I\}]$.*

The winning probability at different realizations of S for Example 2 is depicted in Figure 1.4. It shows how the winning probability varies in s for a given sum of the components of bidder i, $I = 0.8$, if the bidder follows DS or plays the candidate equilibrium strategy $\rho^* = 1$ and β^S. For $I = 0.8$, the black dotted line is the winning probability of bidder i with s in the candidate equilibrium, and the blue solid line is his winning probability in s from DS. Note that the s-axis ends at $s = I$: no higher S is compatible with $I = 0.8$. It shows how winning probability under DS is reallocated from states with a high S to states with a lower S (into states that are more desirable for the bidder under property 3. of the utility function), while keeping the overall probability fixed.

The purple dotted line is the object's valuation of bidder i for the specific $I - S$ combination, when his utility function is $V_i = 0.4S + 0.6T_i$ as in Example 2. It sketches that bidder i's expected gain increases from playing DS due to a shift of winning probability from states with low T_i to states with high T_i which the bidder values more.[27]

Note that Lemma 3 for the payment applies without changes: the expected payment is strictly lower under the DS, due to less interdependence between the bids in case of a win. This is because Lemma 3 does not rely on a specific functional form of the bidding function, but holds for any increasing symmetric function via FOSD.

Thus, a bidder is more likely to win when he values the object more. By property 3. of the utility function, he values an increase in T_i more than in S. Instead of winning with probability $\frac{1}{2}$ at every realization of S, via DS a bidder shifts his winning probability to states with higher T_i realization and lower S, and away from states with

[27] In the symmetric setup in Section 5 with the sum of the two components being the utility $V_i = S + T_i = I$, the purple dotted line is constant for every realization of S for fixed I.

a high S realization and lower T_i. As a result, the expected gain from DS is weakly better than in the CE with $\rho^* =$. A proof of this argument that accounts for the prior probability distribution over combinations of S and T_i, is provided in the Appendix in the proof of Proposition 2.

This establishes, that the result that interdependence cannot be sustained in any equilibrium of the SPA is robust to a perturbation of the utility function into a direction, where the private component T_i matters more for the bidder (property 3.).

A natural question is whether a perturbation into the other direction – making S slightly more important for the bidder than T_i – breaks the result. Consider the following utility function: $V_i = \left(\frac{1}{2} + \epsilon\right) S + \left(\frac{1}{2} - \epsilon\right) T_i$, with $\epsilon > 0$, such that $u(S, I - S)$ is strictly increasing in S. Can any equilibrium with learning about the common component via $\rho^* = 1$ be sustained under this utility function? For ϵ sufficiently small, the answer is No. Note that irrespective of the utility function, bidder i can always guarantee himself a strictly lower payment by playing DS. His gain in payment from this deviation is bounded away from zero. Fix some $I = s + t_i$ for bidder i. Under DS, the bidder is more likely to win at states with high T_i and low S, and less likely to win when S is high (which he values more). In Figure 1.4, the purple dotted utility function would be increasing in s, showing that DS shifts his winning probability into unfavorable states combinations. Nevertheless, this loss in expected gain can be made arbitrarily close to zero by choosing ϵ sufficiently small. Therefore, the decrease in payment offsets the loss in gain for a sufficiently small ϵ in the utility function.

Hence, the argument that there cannot be learning about the common component in equilibrium is also robust to making the common component slightly more valuable to the bidder than the private component. Yet, increasing the marginal utility of S further by increasing ϵ eventually breaks the predominance of the gain from lower payment over the lower expected gain from the object. Whether an equilibrium with learning about the common component can be sustained in equilibrium in such a case will depend on the primitives of the model: the utility function $V_i = u(S, T_i)$, the distributions of S and T_i, and the signal distributions $f(x|r)$. The deviation strategy (DS) is no longer suitable for establishing non-existence in such a framework.

6.2 N Bidders

Consider a CE of the SPA with $\rho^* = 1$ and β^S. In the following I show the extension of Proposition 1 to the case of $N > 2$ bidders. Let the utility function be symmetric ($V_i = S + T_i$) as in Section 5.

Consider the same deviation strategy (DS) as for the case $N = 2$, in which bidder i selects $\rho_i = 0$, observes X_i^T and bids according to β^S as if his signal were about the common component in CE.

In both strategies DS and CE, bidder i wins if and only if he has a higher signal realization than all of his opponents, where ties can be ignored. Let

$$Y_i^S = \max_{j \neq i} \{X_1^S, ..., X_{i-1}^S, X_{i+1}^S, ..., X_N^S\}$$

be the highest signal realization of all other bidders but bidder i about the common component.

Due to independence conditional on S, the highest signal Y_i^S of all other bidders has cumulative distribution function

$$G(y) = \int_0^1 F(y|s)^{N-1} h(s) ds.$$

For each total value realization v_i for bidder i the following theorem pins down the probability of winning under DS or CE, depending on whether he observes X_i^T or X_i^S.

Proposition 3
Let all $N-1 \geq 2$ other bidders learn $X_{j \neq i}^S$ about the common component. Then, for all total values $v_i \in [0,2]$ for bidder i:

$$\Pr(X_i^T \geq Y_i^S | v_i) \geq \Pr(X_i^S \geq Y_i^S | v_i) = \frac{1}{N}.$$

The inequality is strict for all $v_i \neq \{0,1\}$.

Let all other bidders learn about the common component S. Fix a total valuation for bidder i, by keeping the sum of the two components equal at $v_i = S + T_i$. The theorem says that, by selecting information about the private component T_i instead of S, bidder i can increase his probability of having the highest signal for all values v_i.

The difference of Proposition 3 to Proposition 1 with $N = 2$, in which winning probability is identical for all v_i, stems from difference in the first order statistic of a bidder's opponents. With $N = 2$, the distribution of the first order statistic of the other bidders is simply the signal distribution of a bidder's single opponent. Moreover, this distribution is the same as a bidder's own signal distribution. With more than one opponent, the first order statistic of the other bidders no longer coincides with the own signal distribution.

This becomes apparent in Figure 1.3 where different winning probabilities are described for the case of two bidders. Consider the same numerical example as before: $v_i = 0.8$ and either $(s = 0.1, t_i = 0.7)$, or $(s = 0.7, t_i = 0.1)$. With two bidders, bidder i gains winning probability in $(s = 0.1, t_i = 0.7)$, but loses the same amount of winning probability in $(s = 0.7, t_i = 0.1)$, as his opponent is symmetric to him and the first order statistic is the same as his own signal distribution.

With more than two bidders, the gain of bidder i from DS in state $(s = 0.1, t_i = 0.7)$ is larger than the winning probability that he loses in state $(s = 0.7, t_i = 0.1)$, when he bids against a higher first order statistic. The next example depicts this intuition for fully revealing signals.

Example 3. *Fix $v_i = 0.8$ and consider two S-T_i-combinations that are compatible with this total value realization for bidder i, $(s = 0.1, t_i = 0.7)$ and $(s = 0.7, t_i = 0.1)$. Both combinations occur with equal probability of $h(0.1)h(0.7)$ as S and T_i are drawn i.i.d.*

Consider fully revealing signals about both value components $K \in \{S, T_i\}$, such that

$$\Pr(X_i^K = x | K = r) = \begin{cases} 1 & \text{if} \quad x = r, \\ 0 & \text{otherwise.} \end{cases}$$

	$(s=0.7, t_i=0.1)$	$(s=0.1, t_i=0.7)$	total winning prob.
CE	$1/N$	$1/N$	$1/N$
DS	0	1	1/2

Table 1.1: Probability of bidder i winning in DS and CE with $\rho^*=1$, conditional on $v_i=1$. Both state combinations have equal probability of $h(0.1)h(0.7)$. Overall winning probability is higher with DS.

If multiple bidders have the same highest signal realization, ties are broken evenly about who wins.[28]

If $(s=0.7, t_i=0.1)$*, all* $N-1$ *other bidders learn a signal* X_j^S *with realization* $x_j=0.7$*. If bidder i learns* X_i^S *as well, he has signal realization* 0.7*, and wins with probability* $\frac{1}{N}$*. If bidder i observes signal* X_i^T *instead about his private component, his signal realization is* 0.1 *and he has zero probability of winning. These probabilities are summarized in the first column of Table 1.1.*

If $(s=0.1, t_i=0.7)$*, all other bidders observe a signal realization* $x_j=0.1$*. If bidder i learns about* S*, he also observes realization* 0.1 *and wins with probability* $\frac{1}{N}$*. If bidder i learns about his private component, his signal realization is* 0.7 *and he wins with probability* 1*. This is summarized in the second column of the Table 1.1.*

Winning probability overall in DS is higher than in CE. In $(s=0.1, t_i=0.7)$*, bidder i has a lot of probability mass of winning to gain by learning about* T_i*. In state* $(s=0.7, t_i=0.1)$*, even if bidder i learns about* S*, his probability of a win is not very high, since the first order statistic of the other bidders is elevated by the high realization of* S*. The gain in probability mass of winning in* $(s=0.1, t_i=0.7)$ *is larger than the loss in* $(s=0.7, t_i=0.1)$*.*

This argument becomes apparent with $N\to\infty$*. As the number of bidders increases and all other bidders learn about the common component, bidder i's probability of winning with CE approaches zero in both* $(s=0.1, t_i=0.7)$ *and* $(s=0.7, t_i=0.1)$*. On the other hand, playing DS always guarantees bidder i a win in state* $(s=0.1, t_i=0.7)$*. It is easy to see that when there are only two bidders, gain and loss in the two states are exactly equal: learning about either component yields the same overall probability* $\frac{1}{2}$ *of having the highest signal for bidder i in above two state realizations. This is evident in the third column of Table 1.1 for* $N=2$*.*

An immediate corollary of Proposition 3 is the following.

Corollary 2. *Let all the opponents of bidder i learn about the common component by observing* $X_{j\neq i}^S$*. For* $N>2$*, the overall probability of winning is strictly higher in DS than in CE.*

As Proposition 3 holds for each realization of v_i, it also holds overall and the proof is therefore omitted.

An overall higher probability of a win at every value realization v_i might seem good news for the overall payoff in DS. Expected gain from DS is clearly strictly higher than the expected gain in CE. Complications arise in expected payment: total winning

[28] In the continuous version of my model, ties have zero probability. In this discrete example, ties occur with strictly positive probability, which requires a tie-breaking rule.

probability in DS is strictly higher than in CE. Hence, the expected payment conditional on a win is multiplied with a higher overall probability of winning in Equation 1.7. The separation approach in the expected utility – keep expected gain and total winning probability constant and focus on the expected payment – is no longer applicable as overall expected payment can strictly increase by switching from CE to DS and needs to be weighted against the gain in expected utility.

7. Alternative Auctions

In this section, I apply the developed technique to two further auction formats, the FPA (Subsection 7.1) and the all-pay auction (Subsection 7.2). As in the preceding section, I restrict attention to pure information selection, $\rho_i \in \{0, 1\}$. For the FPA, I show that $\rho^* = 1$ cannot be ruled out as an equilibrium with the developed approach. Furthermore, $\rho^* = 0$ is not robust in the FPA when introducing a small degree of correlation between the private component and the common component. In the all pay auction with more than two participants, bidders do not want to learn about the common component, and $\rho^* = 0$ is an equilibrium.

7.1 First Price Auction

Two bidders compete in a FPA with no reserve price.[29] Bidders can either learn about the common variable S via observing the random variable X_i^S or learn about the private variable T_i via observing the random variable X_i^T, that is, $\rho_i \in \{0, 1\}$.

In section 5, I derived the necessary toolbox to show why $\rho^* = 1$ cannot arise in any equilibrium of the SPA: a bidder could play a certain deviation strategy that decreases correlation between his signal and the signal of the opponent. Then, bidding as if having observed X_i^S but having truly observed X_i^T yields bidder i the same expected gain (Corollary 1) for a strictly lower payment (Proposition 3).

In the following I show why this argument *cannot* be used for the FPA to rule out $\rho^* = 1$. Let the candidate equilibrium be $\rho^* = 1$, and both bidders bid according to β_f^S. Consider the same deviation strategy as in the SPA for bidder i:

Definition 3 ($\mathbf{DS^f}$). *A deviation strategy (DS^f) for bidder i in the FPA is the following strategy:*

- *deviate to $\rho_i = 0$ and observe X_i^T;*
- *bid according to $\beta_f^S(\cdot)$.*

The expected payoff from this deviation strategy is best evaluated by once again separating the expected gain from the expected payment. In the candidate equilibrium, bidder i is sure that he faces a X_j^S-type opponent. Then, by Proposition 1 and Corollary 1, total winning probability is the same in DS and the candidate equilibrium. That is, $\Pr(X_i^S \geq X_j^S) = \Pr(X_i^T \geq X_j^S) = \frac{1}{2}$. Therefore, expected gain from DS^f is the same as from the equilibrium bidding strategy. This immediately follows from Corollary 1,

[29] As before in the SPA, ties have zero probability and can be ignored.

as the effect of DS on the expected gain coincides in the FPA and the SPA coincide. The difference between SPA and the FPA lies in their payment rule and not in the allocation decision.

Next, I show the effect of DS^f on the expected payment. Similar to the SPA, define the cumulative signal distribution of bidder i, conditional on winning. The definition of this distribution captures his own information choice $K \in \{S, T_i\}$, the information choice of his opponent $L \in \{S, T_j\}$ and both bidding functions β_i^K and β_j^L.

$$H_f^K(x_i|\beta_i^K, \beta_j^L, X_j^K) := \Pr(X_i^K \leq x_i|\beta_i^K(X_i^K) \geq \beta_j^L(X_j^L)).$$

For DS^f, this distribution is the following. The joint event of the common component being $S = s$, bidder i seeing $X_i^T = x_i$ and bidder i winning with β_f^S has density $h(s)f^T(x_i)F^S(x_i|s)$. As the distributions of both components are the same by Assumption A1, I drop the superscripts in the following. Integrating over all common states results yields the distribution:

$$\begin{aligned} H^T(x_i|\beta_f^S, \beta_f^S, X_j^S) =& \frac{\int_0^{x_i}\int_0^1 F(\tilde{x}|s)h(s)dsf(\tilde{x})d\tilde{x}}{\Pr(X_i^S \geq X_j^S)} \\ =& \frac{\int_0^{x_i} F(\tilde{x})f(\tilde{x})d\tilde{x}}{\frac{1}{2}} = F(x_i)^2. \end{aligned}$$

As DS^f involves bidding with the same bidding function β^S as the opponent, the above distribution of signals of bidder i conditional on winning simplifies to the distribution of the first order statistics of two independent signals, X_i^T and X_j^S, each drawn with identical distribution $F(\cdot)$.

Next, consider the cumulative distribution of signals of bidder i who follows the candidate equilibrium strategy. The joint event $S = s$, $X_i^S = x_i$ and bidder i winning with β_f^S has density $h(s)f^S(x_i|s)F^S(x_i|s)$. This results in the following distribution, where I once again drop the superscripts.

$$\begin{aligned} H^S(x_i|\beta_f^S, \beta_f^S, X_j^S) =& \frac{\int_0^{x_i}\int_0^1 f(\tilde{x}|s)F(\tilde{x}|s)h(s)dsd\tilde{x}}{\Pr(X_i^S \geq X_j^S)} \\ =& \frac{\int_0^1\int_0^{x_i} f(\tilde{x}|s)F(\tilde{x}|s)d\tilde{x}h(s)ds}{\frac{1}{2}} \\ =& \int_0^1 F(x_i|s)^2h(s)ds. \end{aligned}$$

Using the strict[30] Cauchy-Bunyakovski-Schwarz inequality we have for all $x \in (0, 1)$,

$$F(x_i)^2 = \left(\int_0^1 F(x_i|s)h(s)ds\right)^2 < \underbrace{\int_0^1 h(s)ds}_{=1.}\int_0^1 F(x_i|s)^2h(s)ds. \tag{1.14}$$

The distribution of the first order statistic is strictly higher under interdependent sig-

[30]See footnote 33 for the strict inequality.

nals, than under independent signals. This establishes FOSD, as $H^T(x_i|\beta_f^S, \beta_f^S, X_j^S) < H^S(x_i|\beta_f^S, \beta_f^S, X_j^S)$ for all $x \in (0,1)$. That is, $H^T(\cdot)$ is FOSD over $H^S(\cdot)$. This immediately translates into a strictly higher payment under DS in case of a win as the bidding function β_f^S is strictly increasing:

$$\int_0^1 \beta_f^S(x_i) dH^S(x_i) < \int_0^1 \beta_f^S(x_i) dH^T(x_i).$$

The expected payment conditional on a win is strictly lower under the original equilibrium strategy than under the constructed deviation DS^f. Not only does decreasing the correlation not help like in the SPA, but it hurts the agent. A bidder still wins with the same probability conditional on any value realization v_i (this is an implication of Proposition 1 and the construction of DS^f using the same bidding strategy as the equilibrium). However, by decreasing correlation with his opponent, a bidder is more likely to win at higher signal realizations which drives up his expected payment. This shows why $\rho^* = 1$ cannot be ruled out as an equilibrium by a deviation strategy of the same kind as in the SPA that decreases interdependence in private information.

Correlation between the components. As in the case with the SPA, an IPV equilibrium with $\rho^* = 0$ always exists. This is because if the opponent of bidder i observes a signal about his *private* component, bidder i is in an IPV setup. Then, bidder i is indifferent between both information channels, as they both contain the same accuracy about the total value v_i and each signal realization leads to the same best response due to Observation 2. Such an equilibrium is a ‘trivial’ equilibrium, as each bidder’s information has neither an effect on interdependence between the signals, nor on total valuations.

Next, I analyze whether the trivial equilibrium with $\rho^* = 0$ is robust to a small degree of interdependence between the bidders. For this purpose, I introduce a slight perturbation into the informational structure. First, the common component S realizes with distribution $H(\cdot)$, as in the Model Section 4. Then, the private components T_1 and T_2 are drawn. In contrast to the analysis before, with probability ϵ the common and private component of bidder i are identical (which is unobserved): $T_i = S$. With probability $1 - \epsilon$, T_i is drawn independently and identically with the same cumulative distribution $H(\cdot)$. Therefore, ϵ captures the correlation between each bidder’s private component and the common component. In the analysis so far, $\epsilon = 0$. Furthermore, with $\epsilon > 0$ the IPV framework is ruled out as the signal of the opponent always contains relevant information about the common component irrespective of its source. That is, learning X_i^T and X_i^S contain information about both components.[31]

The next proposition shows that in the FPA, there cannot exist an equilibrium in which bidders learn signals X_i^T about their private components.

Proposition 4
For $\epsilon > 0$, there exists no symmetric equilibrium of the FPA with $\rho^* = 0$.

[31] An alternative perturbation is the following: both bidders make a small ‘tremble’ when choosing their information source. With probability $1 - \epsilon$ they observe a signal about their preferred value component; with probability ϵ they perform an experiment on the wrong component. This perturbation yields the same results on equilibrium existence as the one introduced in this section.

The proof follows by combining the following two lemmas. Like in the SPA, the proof of the theorem is by contradiction. It relies on providing a deviation strategy and decomposing expected utility into an expected gain (which is the same in $\overline{DS}^f$ and the candidate equilibrium) and an expected payment (which is strictly less under $\overline{DS}^f$). I show that the following deviation strategy $\overline{DS}^f$ is a strictly profitable deviation.

Definition 4. *The deviation strategy ($\overline{DS}^f$) for bidder i in the FPA is the following strategy:*

- *deviate to $\rho_i = 1$,*
- *bid according to $\beta^T_f(x_i)$ for $X^S_i = x_i$.*

The deviation strategy $\overline{DS}^f$ involves changing the information selection strategy from learning X^T_i to learning X^S_i, but not the bidding function. It requires a bidder to learn about the common component, but follow the same bidding function as if the bidder learned X^T_i. It is complementary to the deviation strategies considered before, as its purpose is to increase (not decrease) correlation while following the same bidding function.

The following lemma pins down the effect of $\overline{DS}^f$ on the winning probability for each object value and the expected gain from this deviation.

Lemma 4
For each value v_i, the winning probability of bidder i in $\overline{DS}^f$ equals the winning probability in equilibrium under $\rho^* = 0$. The expected gain from $\overline{DS}^f$ equals the expected gain from the equilibrium with ρ^*.

The proof uses parts of Proposition 1 for the special case of two bidders. As bidders follow the same bidding strategy β^T_f in both the equilibrium and $\overline{DS}^f$, a winning bidder is a bidder with the highest signal realization. Therefore, the proof relies not necessarily on the optimal deviation strategy, but one that uses the same bidding function β^T_f for tractability of the change in winning probability for each v_i. Overall expected gain from the candidate equilibrium and the deviation strategy DS^f is the same.

The next lemma pins down the difference in expected payment between the equilibrium with $\rho^* = 0$ and under the deviation strategy $\overline{DS}^f$.

Lemma 5
Let $\epsilon > 0$. The expected payment with $\overline{DS}^f$ in the FPA is strictly less than in the equilibrium with $\rho^* = 0$.

The argument is similar to the one developed above to show that $\rho^* = 1$ cannot profit from DS^f. Intuitively, achieving a stronger dependence with the opponent reduces the 'money left on the table' in the FPA. A bidder pays his own bid. Conditional on the event of winning, he prefers to outbid his opponent by as little as possible. Whenever the perturbation is inactive as the opponent bidders observed his private component signal X^T_i as selected by $\rho^* = 0$, there is no difference between behaving as in equilibrium and following $\overline{DS}^f$. The deviation strategy comes into play whenever the opponent trembled and observed X_i^S. In this case, a bidder is more likely to observe a signal about the common component and be more correlated with the opponent under the deviation strategy $\overline{DS}^f$ than under the equilibrium strategy. As in both cases, bidders follow the same bidding function, the FOSD argument holds. The expected payment that a bidder has to pay in case of a win is strictly higher with less interdependence, as a bidder is more likely to win when he places a low bid (with higher dependence his opponent is less likely to outbid him in this range). Similarly, with more dependence, a bidder is less likely to win when placing a high bid, as his opponent is more likely to outbid him in the range of higher bids. Expected payment is strictly higher under less correlation, while expected gain is constant. And under the deviation strategy the event of higher correlation (when both bidders observed a signal about S) is more likely to occur.

To sum up Lemma 4 and Lemma 5, the deviation strategy does not change the winning probability or the expected gain from participating, but strictly decreases expected payment. As increasing the dependence in private information with the opponent comes without a loss for expected value, due to the particular construction of $\overline{DS}^f$, it constitutes a strictly profitable deviation. Therefore, $\overline{DS}^f$ is a strictly profitable deviation. The equilibrium $\rho^* = 0$ is not robust to the perturbation of the information structure.

7.2 All-Pay Auction

Consider an all-pay first price auction with N bidders. Bidders submit bids b_i as a function of their signal realization X_i^T or X_i^S. Payment and allocation rule result in the following payoff W_i for bidder i who places bid b_i:

$$W_i = \begin{cases} V_i - b_i \text{ if } b_i > \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i \text{ if } b_i = \max_{j \neq i} b_j \\ -b_i \text{ if } b_i < \max_{j \neq i} b_j \end{cases}$$

Bidders always pay their bid, irrespective of the event of winning. They win if they submitted a higher bid than their opponents. Krishna and Morgan (1997) analyze the all-pay auction in a symmetric interdependent value framework. They show when a symmetric equilibrium in increasing strategies exists.

Denote the bidding function in a candidate equilibrium of the all-pay auction after learning X_i^S by β_a^S, and after learning X_i^T by β_a^T. The next theorem and lemma establish the main result for the all-pay auction about information selection and existence in equilibrium.

Proposition 5
For $N > 2$, there exists no equilibrium of the all-pay auction with $\rho^* = 1$.

Learning about the common component cannot arise in equilibrium. Similar to the proof technique of the SPA, I establish the result by constructing a deviation strategy and decompose it in expected gain and expected payment. It allows an application of Proposition 3 and enables a tractable payoff comparison. By contradiction, consider a candidate equilibrium of the all-pay auction with $\rho^* = 1$ in which all participants bid according to some increasing function $\beta_a^S(x)$.

Definition 5 (DS^a). *A deviation strategy (DS^a) for bidder i in the all-pay auction is:*

- *deviate to $\rho_i = 0$ and observe X_i^T,*
- *bid according to $\beta_a^S(X_i^T)$.*

The deviation strategy DS^a requires a bidder to change the source of his signal (from X_i^S to X_i^T), but follow the same bidding function $\beta_a^S(\cdot)$ as before. Let the utility from the candidate equilibrium (CE) with $\rho^* = 1$ and β_a^S be $EU(S, \beta_a^S|CE)$. Let the expected utility from DS^a be $EU(T_i, \beta_a^S|CE)$. The proof of Proposition 5 shows that $EU(S, \beta_a^S|CE) < EU(T_i, \beta_a^S|CE)$ for $N > 2$.

In the all-pay auction, a bidder pays his own bid. In CE, the expected payment of bidder i is

$$\int_0^1 \beta_a^S(x_i) f(x_i) dx_i.$$

In DS_a^S, expected payment is exactly the same, as both of bidder i's available signals X_i^T and X_i^S induce the same marginal distribution $f(x_i)$. Hence, expected payment in CE and DS^a is the same.

The expected gain is strictly higher in DS^a, as due to Proposition 3, the probability of a win is strictly larger at almost all total values v_i if there are more than two bidders.

For the case of two bidders, the expected payment in DS^a and CE with $\rho^* = 1$ is the same as a bidder's bid (and thus, payment) distribution is the same. In contrast to the $N > 2$ bidder case, expected gain is also the same in CE and DS^a. That is, the expected overall utility of bidder i from CE and from DS^a is identical. Whether there exists a strictly profitable deviation over a CE with $\rho^* = 1$ will depend on the characteristics of the signals. One might expect that generically, as β^S is constructed as a best response in CE after seeing X_i^S, there is no reason why it should also constitute a best response after seeing X_i^T, and the bidder could strictly increase his payoff by playing a best response to X_i^T.

Lemma 6
For $N \geq 2$, there exists an equilibrium with $\rho^* = 0$.

The proof is by construction: learning only about the private component T_i and bidding according to the usual IPV bidding function for the all-pay auction $\beta_a^T(x) = \int_0^x \mathbb{E}\left[V_i|X_i^T = \tilde{x}\right] f^T(\tilde{x}) d\tilde{x}$ constitutes a best response to this particular information choice when the opponents also select $\rho^* = 0$ and follow the same bidding function.

Moreover, if $\rho^* = 0$, no other bidder knows anything of relevance to other bidders. Signal realizations of other bidders are independent from one's own signal for any information selection. The value of information conditional on one's signal alone is equal no matter which component the signal was applied to. Due to Observation 2, any signal realization results in the same best response. Bith available signals have the same value of information for a bidder, if the opponents play CE. This establishes existence of an equilibrium with $\rho^* = 0$.

8. Conclusion

If bidders cannot consider all possible information, a question of *which* variables to learn about arises. I analyze this question in the context of auctions. In takeover auctions, out of all the multidimensional information available about the target, which characteristics do bidders choose to focus on? Do they want to know what matters to others – a common variable like the book value – which induces interdependence in private information? Or do bidders prefer to focus on a private component like their specific R&D synergies and receive independent private signals? Bidders are equally well-informed about the object's total value whether they select a signal about the common or the private component.

The focus of this paper is on information selection, specifically *which* payoff-relevant variable to learn about. This contrasts with the literature on information acquisition, which usually asks *how much* information about a single payoff relevant variable a bidder acquires.

In the SPA, information selection in equilibrium is unique. Bidders learn only about their private component. Any candidate equilibrium in which bidders learn with non-zero probability about the common component can be ruled out by an appropriate deviation strategy. The deviation strategy uses the same bidding functions as the candidate equilibrium but induces independent private signals by learning only about the private component. By employing such a deviation strategy, a bidder strictly decreases his expected payment but retains his overall gain and winning probability. By decreasing correlation via learning about the private component, a bidder is more likely to win in states with a high *private* component, and less likely to win in states with a high *common* component, while there is no effect on the overall winning probability.

This paper explores the impact of a selling mechanism on the type of information bidders select. Information about the common component simplifies coordination and is informative about other bidder's bids. However, learning about a common component that matters equally for all bidders is socially wasteful, as this information comes at the opportunity cost of not learning socially valuable information about the private components. A designer who wishes to maximize efficiency should take into consideration, that his auction choice might affect about which value components bidders learn. My analysis suggests that, in such a simplified setting, the SPA is a good choice, as it is ex-ante efficient. It induces learning only about the socially relevant variable and allocates the good efficiently. An IPV setup arises endogenously.

A. Appendices

A.1 Affiliation and Accuracy

The following definition introduces the concept of affiliation between random variables. Affiliation is a strong form of positive correlation, and is a widely used model of statistical dependence in Economics at the latest since the contribution of Milgrom and Weber (1982).[32]

Definition 6 (Milgrom and Weber (1982))**.** *Consider real-valued random variables* $Z_1, ..., Z_k$*, and denote a vector of realizations by* $\boldsymbol{z} := \{z_1, ..., z_k\}$*. Let* $f(\boldsymbol{z})$ *be the density of the realization vector* $\boldsymbol{z}$*. Denote by* $\boldsymbol{z} \vee \boldsymbol{z}'$ *the component-wise maximum, and denote by* $\boldsymbol{z} \wedge \boldsymbol{z}'$ *the component-wise minimum of the two vectors* $\boldsymbol{z}$ *and* $\boldsymbol{z}'$*. Then, the random variables* $Z_1, ..., Z_k$ *are said to be* affiliated *if*

$$\text{for all } \boldsymbol{z}, \boldsymbol{z}' : \quad f(\boldsymbol{z} \vee \boldsymbol{z}')f(\boldsymbol{z} \wedge \boldsymbol{z}') \geq f(\boldsymbol{z})f(\boldsymbol{z}').$$

Observation 3. X_1^S *and* X_2^S *are affiliated.*

This follows from Milgrom and Weber (1982). By Theorem 1, part (ii) in their model, the random variables X_1^S, X_2^S and S are affiliated if their density can be expressed as the product of affiliated non-negative functions. We have $f(x_1, x_2, s) = f^S(x_1|s)f^S(x_2|s)h(s)$ with $f^S(\cdot)$ being non-negative and affiliated due to the strong MLRP. By Theorem 4 in Milgrom and Weber (1982), as the triple of variables X_1^S, X_2^S, S are affiliated, so are the two variables X_1^S and X_2^S.

Note that independence is a special case of affiliation, where above inequality in Definition 6 holds with equality for all realizations $\boldsymbol{z}$ and $\boldsymbol{z}'$. This implies that X_i^T and X_j^S are affiliated, and X_i^T and X_j^T are affiliated, as they are independent due to Assumption IN.

A.2 Proofs

(The proof of Theorem 1 follows after the proof of Lemma 3, by combining the auxiliary results in Lemma 1, Proposition 1 and Lemma 3.)

Proof of Lemma 1. The distribution of X_i^S conditional on v_i and X_i^T conditional on v_i coincide. This is because the density of realization x_i conditional on v_i is $\frac{h^S(v_i|x_i)f^S(x_i)}{h_{\mathcal{V}}(v_i)}$, where $h^S(v_i|x_i)$ as defined in Equation 1.1. As $h^S(v_i|x_i) = h^T(v_i|x_i)$, and $f^S(x_i) = f^T(x_i)$ via Observation 1, this establishes that the signals X_i^S and X_i^T of bidder i are equally distributed conditional on v_i. Therefore, also the marginal distributions $\beta^S(X_i^S)$ and $\beta^S(X_i^T)$ coincide, conditional on v_i.

Due to Assumption IN, any signal of bidder i, X_i^S and X_i^T, is independent from X_j^T. As functions of independent random variables are independent themselves, for any

[32] The concept of affiliation is known in the statistical literature as a multivariate total positivity order MTP_2 (Karlin and Rinott, 1980). For a comparison of affiliation with other forms of positive correlation, see de Castro (2009) in a context of auctions, and Shaked and Shanthikumar (2007) for a general account of positive dependence orders.

information choice $K \in \{S, T_i\}$ of bidder i, the random variable $\beta^S(X_i^K)$ is independent from $\beta^T(X_j^T)$. Therefore,

$$\Pr(\beta^S(X_i^T) \geq \beta^T(X_j^T)|v_i) = \Pr(\beta^S(X_i^S) \geq \beta^T(X_j^T)|v_i),$$

which establishes the proposition for the winning probabilities in Equation 1.8 for DS and Equation 1.9 for CE. □

Proof of Proposition 1. For $v_i = 0$, we have $s = 0$ and $t_i = 0$. Any information selection leads to a density $f(x|0)$, as both signals X_i^S and X_i^T have same density. The density of an opponent with signal X_j^S is $F(x|0)$. The probability of having the highest signal is $\int_0^1 f(x|0)F(x|0)dx_i = \frac{1}{2}$. Similarly, for the total value to be $v_i = 2$, the components need to be $s = 1$ and $t_i = 1$. Then, the probability of having the highest signal with any signal is $\int_0^1 f(x|1)F(x|1)dx = \frac{1}{2}$.

Fix a total value for bidder i at $v_i \in (0, 2)$. Define the set the common component S, that is feasible under this v_i realization as $\mathcal{S}(v_i) := \{s \in S : \exists t_i \in [0, 1] : v_i = s + t_i\} = [\max\{0, v_i - 1\}, \min\{1, v_i\}]$. E.g., if $v_i \geq 1$, we have $\mathcal{S}(v_i) = [v_i - 1, 1]$. If $v_i < 1$, we have $\mathcal{S}(v_i) = [0, v_i]$. Define $\hat{s}(v_i)$ that bisects this interval: $\hat{s}(v_i) := \frac{\max\{0,v_i-1\}+\min\{1,v_i\}}{2} = \frac{v_i}{2}$.

Conditional on the value for bidder i being v_i, the density of the common component being equal to s is $h(s|v_i) := \frac{h(s)h(v_i-s)}{h_{\mathcal{V}}(v_i)}$. This is due to the fact that S and T_i are drawn from an identical distribution with density $h(\cdot)$. Note that $\int_{\mathcal{S}(v_i)} h(s|v_i) = 1$, and $h(s|v_i) = h(v_i - s|v_i)$, as $h(s|v_i)$ is symmetric around $\hat{s}(v_i)$.

If bidder i learns X_i^S and his opponent learns X_j^S, the probability of winning is

$$\begin{aligned}
\Pr(X_i^S \geq X_j^S|v_i) &= \int_{\mathcal{S}(v_i)} \int_0^1 f(x|s)F(x|s)h(s|v_i)dxds \\
&= \int_{\mathcal{S}(v_i)} \underbrace{\left[\frac{1}{2}F(x|s)\right]_0^1}_{=1/2} h(s|v_i)ds \\
&= \frac{1}{2}\int_{\mathcal{S}(v_i)} h(s|v_i)ds = \frac{1}{2}.
\end{aligned}$$

If the common component is s, then, conditional on v_i, bidder i observes a signal about his private component $t_i = v_i - s$. If bidder i learns about his private components via observing X_i^T, his probability of a win is the following.

$$\begin{aligned}
\Pr(X_i^T \geq X_j^S|v_i) &= \int_{\mathcal{S}(v_i)} \int_0^1 f(x|v_i - s)F(x|s)h(s|v_i)dxds && (1.15) \\
&= \int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \int_0^1 f(x|v_i - s)F(x|s)h(s|v_i)dxds && (1.16) \\
&+ \int_{\hat{s}(v_i)}^{\min\{v_i,1\}} \int_0^1 f(x|v_i - s)F(x|s)h(s|v_i)dxds && (1.17)
\end{aligned}$$

The last step followed by splitting up the integral in two intervals. Consider the second integral. Using relabeling and integration by parts, we have

$$\begin{aligned}
&\int_{\hat{s}(v_i)}^{\min\{v_i,1\}} \int_0^1 f(x|v_i - s)F(x|s)h(s|v_i)dxds \\
=&\int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \int_0^1 f(x|s)F(x|v_i - s)h(v_i - s|v_i)dxds \\
=&\int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \left(\underbrace{[F(x|s)F(x|v_i - s)]_0^1}_{=1} - \int_0^1 f(x|v_i - s)F(x|s)dx \right) h(v_i - s|v_i)ds \\
=&\int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} h(v_i - s|v_i)ds - \int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \int_0^1 f(x|v_i - s)F(x|s)dxh(v_i - s|v_i)ds \\
=&\frac{1}{2} - \int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \int_0^1 f(x|v_i - s)F(x|s)h(s|v_i)dxds
\end{aligned}$$

where the last step followed by $h(s|v_i) = h(v_i - s)$ and $\int\limits_{\mathcal{S}(v_i)} h(s|v_i)ds = 1$. Plugging this back into Equation 1.17 yields the result, $\Pr(X_i^T \geq X_j^S|v_i) = \frac{1}{2} = \Pr(X_i^S \geq X_j^S|v_i)$.

□

Proof of Lemma 2. As $f^S(x|r) = f^T(x|r)$ and $F^S(x|r) = F^T(x|r)$, I drop the superscript. Fix the value for bidder i, $v_i \in (0, 2)$.

For every feasible s that can arise with v_i, if both bidders learn about s and bid with β^S, the winning probability is $\frac{1}{2}$:

$$\Pr(X_i^S \geq X_j^S|S = s, T_i = v_i - s) = \int_0^1 f(x|s)F(x|s)dx = \frac{1}{2}.$$

Now consider the winning probability of bidder i with DS when facing opponent with signal X_j^S. For $s = \frac{v_i}{2} = I - s$, it is immediate that $\int_0^1 f(x|v_i - s)F(x|s)dx = \int_0^1 f(x|s)F(x|s)dx = \frac{1}{2}$.

Take any $s < \frac{v_i}{2}$. A consequence of the strong MLRP is FOSD. Thus, for every $x \in (0, 1)$: $F(x|s) > F(x|v_i - s)$ for $s < v_i - s$. Hence, winning probability in DS is

$$\int_0^1 f(x|v_i - s)F(x|s)dx > \int_0^1 f(x|v_i - s)F(x|v_i - s)dx = \frac{1}{2}.$$

Therefore, the winning probability is larger when learning X_i^T about the private value component, if the private component realization $t_i = v_i - s$ is larger than the common component realization s.

Finally, take any $s > \frac{v_i}{2}$. Similarly, due to the strong MLRP we have $F(x|s) < F(x|v_i - s)$ for all $x_i \in (0, 1)$. Thus,

$$\int_0^1 f(x|v_i - s)F(x|s)dx < \int_0^1 f(x|v_i - s)F(x|v_i - s)dx = \frac{1}{2}.$$

□

Proof of Lemma 3. Consider statement 1. of the Lemma. For all realizations $x_j \in [0, 1]$, we have:

$$H^K(x_j|\beta^S, \beta^T, X_j^T) = \frac{\Pr(X_j^T \leq x_j, \beta^S(X_i^K) \geq \beta^T(X_j^T))}{\Pr(\beta^S(X_i^K) \geq \beta^T(X_j^T))}. \tag{1.18}$$

The event that the opponent has signal realization $X_j^T = x_j$ and bidder i has a higher signal when learning about $K \in \{S, T_i\}$ has density $\Pr(\beta_i^S(X_i^K) \geq \beta_j^T(x_j))f^T(x_j)$. Note that $\Pr(\beta^S(X_i^T) \geq \beta^T(x_j)) = \Pr(\beta^S(X_i^S) \geq \beta^T(x_j))$ due to Assumption IN and the same marginal distribution of X_i^S and X_i^T in Observation 1. Therefore, the numerator can be rewritten in the following way and does not depend on the information channel of bidder i:

$$\int_0^{x_j} \Pr(\beta^S(X_i^S) \geq \beta^T(\tilde{x}_j))f^T(\tilde{x}_j)d\tilde{x}_j = \int_0^{x_j} \Pr(\beta^S(X_i^T) \geq \beta^T(\tilde{x}_j))f^T(\tilde{x}_j)d\tilde{x}_j.$$

Next, I establish that the the denominator in Equation 1.18 is equal in CE and in DS. By Lemma 1, we have for all v_i, $\Pr(\beta^S(X_i^S) \geq \beta^T(X_j^T)|v_i) = \Pr(\beta^S(X_i^T) \geq \beta^T(X_j^T)|v_i)$. Hence,

$$\begin{aligned}
\Pr(\beta^S(X_i^S) \geq \beta^T(X_j^T)) &= \int_{\mathcal{V}} \Pr(\beta^S(X_i^S) \geq \beta^T(X_j^T)|v_i)h_{\mathcal{V}}(v_i)dv_i \\
&= \int_{\mathcal{V}} \Pr(\beta^S(X_i^T) \geq \beta^T(X_j^T)|v_i)h_{\mathcal{V}}(v_i)dv_i \\
&= \Pr(\beta^S(X_i^T) \geq \beta^T(X_j^T)).
\end{aligned}$$

This establishes statement 1., as learning about both value components leads to the same numerator and denominator in Equation 1.18.

Next, consider statement 2. I show that when bidder i faces a X_j^S-type opponent, for all $x_j \in (0,1)$ we have $H^S(x_j|\beta^S, \beta^S, X_j^S) < H^T(x_j|\beta^S, \beta^S, X_j^S)$. As both bidders follow the same bidding function β^S, the event of a win of bidder i translates into the event of having a higher signal than his opponent. I depict the cumulative distributions of the looser's signal as an integral over s by exploiting conditional independence in Assumption CI.

The joint event of bidder i winning when learning X_i^S and bidder j having a signal realization $X_j^S = x_j$ has density $\int_0^1 f^S(x_j|s)\left[1 - F^S(x_j|s)\right]h(s)ds$. If bidder i instead learns about X_i^T, his signal does not depend on S. Then, the joint event of him winning and his opponent having a signal realization $X_j^S = x_j$ has density $\int_0^1 f^S(x_j|s)\left[1 - F^T(x_j)\right]h(s)ds = \left[1 - F^T(x_j)\right]\int_0^1 f^S(x_j|s)h(s)ds = \left[1 - F^T(x_j)\right]f^s(x_j)$. Due to Assumption A1, I drop the superscripts of the signal distributions in the following. For all $x_j \in (0,1)$, we have

$$H^S(x_j|\beta^S, \beta^S, X_j^S) = \frac{1}{\Pr(X_i^S \geq X_j^S)}\int_0^{x_j}\int_0^1 f(\tilde{x}_j|s)(1 - F(\tilde{x}_j|s))h(s)dsd\tilde{x}_j.$$

$$H^T(x_j|\beta^S, \beta^S, X_j^S) = \frac{1}{\Pr(X_i^T \geq X_j^S)}\int_0^{x_j}\int_0^1 f(\tilde{x}_j|s)(1 - F(\tilde{x}_j))h(s)dsd\tilde{x}_j.$$

Note that by Corollary 1, $\Pr(X_i^S \geq X_j^S) = \Pr(X_i^T \geq X_j^S) = \frac{1}{2}$. Hence,

$$H^S(x_j|\beta^S,\beta^S,X_j^S) - H^T(x_j|\beta^S,\beta^S,X_j^S) = \tag{1.19}$$
$$= 2\int_0^{x_j}\int_0^1 f(\tilde{x}_j|s)(F(\tilde{x}_j) - F(\tilde{x}_j|s))h(s)dsd\tilde{x}_j \tag{1.20}$$
$$= 2\left[\int_0^{x_j} F(\tilde{x}_j)\int_0^1 f(\tilde{x}_j|s)h(s)dsd\tilde{x}_j - \int_0^{x_j}\int_0^1 f(\tilde{x}_j|s)F(\tilde{x}_j|s))h(s)dsd\tilde{x}_j\right] \tag{1.21}$$
$$= 2\left[\int_0^{x_j} F(\tilde{x}_j)f(\tilde{x}_j)d\tilde{x}_j - \int_0^{x_j}\int_0^1 f(\tilde{x}_j|s)F(\tilde{x}_j|s))h(s)dsd\tilde{x}_j\right] \tag{1.22}$$
$$= 2\left(\frac{F(x_j)^2}{2} - \int_0^1\int_0^{x_j} f(\tilde{x}_j|s)F(\tilde{x}_j|s)d\tilde{x}_j h(s)ds\right) \tag{1.23}$$
$$= \left(F(x_j)^2 - \int_0^1 F(\tilde{x}_j|s)^2 h(s)ds\right). \tag{1.24}$$

By definition, it holds that $F(x_j) = \int_0^1 F(x_j|s)h(s)ds$. This and the strict Cauchy-Bunyakovsky-Schwartz inequality yield for all $x_j \in (0,1)$,

$$F(x_j)^2 = \left[\int_s F(x_j|s)h(s)ds\right]^2 < \underbrace{\int_s h(s)ds}_{=1}\int_s F(x_j|s)^2 h(s)ds.$$

For all $x_j \in (0,1)$, the last inequality is strict, as $F(x_j|s)$ is not constant in the variable s due to the strong MLRP.[33] This establishes that Equation 1.24 is negative for all $x_j \in (0,1)$.

Finally, consider statement 3. If $\beta^S(1) \leq \beta^T(0)$, expected payment against a X_j^T-type is trivially zero in DS and in CE with (X_i^S, β^S). If $\beta^S(1) \leq \beta^T(0)$, Lemma 1. establishes that the expected payment in DS and the candidate equilibrium is also the same when facing a X_j^T-type opponent, *conditional* on a win. As in both cases, the winning probability is also the same due to independence, this also holds for the *unconditional* expected payment:

$$EP(X_i^S,\beta^S|X_j^T,\beta^T) = EP(\underbrace{X_i^T,\beta^S}_{DS}|X_j^T,\beta^T).$$

Statement 2. establishes that when facing a X_j^S-type opponent, the expected payment distribution conditional on a win with DS is dominated by the payment distribution of the candidate equilibrium after X_i^S. As by assumption, bidding function β^S is increasing, FOSD implies a higher expected payment in the candidate equilibrium. Finally, as winning probability overall is the same in DS and the candidate equilibrium, this implies that the *unconditional* expected payment in DS is also lower than in the

[33] This is because the Cauchy-Bunyakovsky-Schwartz inequality $\left[\int_a^b c(s)d(s)ds\right]^2 \leq \int_a^b c(s)^2 ds \cdot \int_a^b d(s)^2 ds$ is strict unless $c(s) = \alpha \cdot d(s)$ for some constant α (see Hardy et al., 1934, Chapter VI). In above argument, $c(s) = \sqrt{h(s)}$, and $d(s) = \sqrt{h(s)}F(x|s)$. Note that $F(x|s)$ is not constant in s due to the strong MLRP unless $x \in \{0,1\}$.

candidate equilibrium. Hence, we have

$$EP(X_i^S, \beta^S | X_j^S, \beta^S) > EP(\underbrace{X_i^T, \beta^S}_{DS} | X_j^S, \beta^S).$$

Therefore, overall expected payment in Equation 1.7 is strictly less under DS than after learning X_i^S in the candidate equilibrium with (X_i^S, β^S). □

Proof of Theorem 1. Corollary 1 establishes the same expected gain and total winning probability in DS and CE. Lemma 3 establishes a strictly lower payment under DS than in CE. This rules out any $\rho^* > 0$ in equilibrium, and establishes the unique information selection $\rho^* = 0$ if an equilibrium exists.

The next steps establish existence. With $\rho^* = 0$, bidders are in an IPV setup. For fixed $\rho^* = 0$, it is a well known result that bidding $\beta^T(x) = \mathbb{E}\left[V_i | X_i^T = x\right]$ is an equilibrium in weakly dominant strategies. Whichever profitable deviation exists without information choice, will also exist in this setup with endogenous information selection. Thus, after learning X_i^T and expecting the opponent to learn about T_j, above bidding function is a weakly dominant strategy.

Therefore, the only deviation we need to consider for bidder i is to deviate and learn about common component. After seeing $X_i^S = x$, bidder i is still in an IPV setup. If his opponent also learns about his private component, bidder i has a weakly dominant strategy to bid his posterior valuation $\mathbb{E}\left[V_i | X_i^S = x\right]$. By Observation 2, for all x, $\mathbb{E}\left[V_i | X_i^S = x\right] = \mathbb{E}\left[V_i | X_i^T = x\right] = \beta^T(x)$. Hence, after deviating to the common component, bidder i has the same best response after each signal realization, for any signal source. As X_i^S and X_i^T are distributed with equal marginal distribution $F(x)$ and are both independent from X_j^T (which the opponent always learns in a candiadte equilibrium with $\rho^* = 0$), the deviating to component S is not strictly profitable as it induces the same expected utility as the caniddate equilibrium with $\rho^* = 0$ when bidding optimally.

□

Proof of Proposition 2. Let $\rho^* = 1$ with bidding function β^S be a candidate equilibrium (CE). Fix the sum $I = S + T_i$ of the two components for bidder i.

Consider first the expected payment. Note that the proof of Proposition 1 holds step by step when instead of fixing v_i, the variable I is fixed. That is, for all I,

$$\Pr(X_i^T \geq X_j^S | I) = \Pr(X_i^S \geq X_j^S | I) = \frac{1}{2}.$$

Holding the sum of the two components fixed, the winning probability in CE or in DS is unchanged for the case of two bidders. Therefore, Lemma 3 holds. Expected payment is strictly lower in DS than in the candidate equilibrium. This is because the proof of Lemma 3 does only rely on the bidding function β^S being strictly increasing, not on any specific functional form. Therefore, varying the utility function does not change the observation that expected payment is strictly less under DS than in CE.

Next, consider the expected gain from DS. Given the sum I, a feasible common component realization lies in the interval $s \in [\underline{s}(I), \overline{s}(I)] := [\max\{I-1, 0\}, \min\{I, 1\}]$.

Denote by $h(s|I) := \frac{h(s)h(I-s)}{h_{\mathcal{I}}(I)}$ the density of the common component conditional on I, where the density of the sum of the two components I is $h_{\mathcal{I}}(I) = \int_0^1 h(s)h(I-s)ds$.

The cumulative distribution function of the common component S, conditional on I and on bidder i winning in the CE is for $s \in [\overline{s}(I), \underline{s}(I)]$:

$$\begin{aligned} J^S(s|I) &:= \Pr(S \leq s|X_i^S \geq X_j^S, I) \\ &= \frac{1}{\Pr(X_i^S \geq X_j^S|I)} \int_{\underline{s}(I)}^{s} h(\tilde{s}|I) \underbrace{\int_0^1 f(x|\tilde{s})F(x|\tilde{s})dx}_{=\frac{1}{2}} d\tilde{s} \\ &= 2\int_{\underline{s}(I)}^{s} h(\tilde{s}|I)\frac{1}{2}d\tilde{s}. \end{aligned}$$

$J^S(s|I) = 0$ for all $s \leq \underline{s}(I)$, where there exists no T_i large enough to sum up to I. Furthermore, $J^S(s|I) = 1$ for all $s \geq I$.

Similarly, let the following be the cumulative distribution function of the common component S, conditional on I and on bidder i winning when following DS.

$$\begin{aligned} J^T(s|I) &= \Pr(S \leq s|X_i^T \geq X_j^S, I) \\ &= \frac{1}{\Pr(X_i^T \geq X_j^S|I)} \int_{\underline{s}(I)}^{s} h(\tilde{s}|I) \int_0^1 f(x|I-\tilde{s})F(x|\tilde{s})dxd\tilde{s} \\ &= 2\int_{\underline{s}(I)}^{s} h(\tilde{s}|I) \underbrace{\int_0^1 f(x|I-\tilde{s})F(x|\tilde{s})dx}_{\triangle(s|I)} d\tilde{s}. \end{aligned}$$

As before, $J^T(s|I) = 0$ for all $s < \underline{s}(I)$ and $J^T(s|I) = 1$ for all $s \geq \overline{s}(I)$.

Next, I show that $J^S(s|I)$ is FOSD over $J^T(s|I)$. Take any $s \leq \frac{I}{2}$. Note that the proof of Lemma 2 holds step-by-step, if conditioning on I instead of v_i. By Lemma 2, $\triangle(s|I) \geq \frac{1}{2}$. Therefore, for all $s \leq \frac{I}{2}$, we have $J^T(s|I) \geq J^S(s|I)$. Note that at $s = \overline{s}(I)$, $J^S(\overline{s}(I)|I) = J^T(\overline{s}(I)|I) = 1$, as no higher S can feasibly occur if the sum of the two components is I. Assume by contradiction that there exists a $s' \in (\frac{I}{2}, \overline{s}(I))$ such that $J^T(s'|I) < J^S(s'|I)$. Then, again due to Lemma 2, $\triangle(s|I) < \frac{1}{2}$ for all $s \in (\frac{I}{2}, I]$. Therefore, if $J^T(s'|I) < J^S(s'|I)$, we must have $J^T(s''|I) < J^S(s''|I)$ for all $s'' > s'$. However, this contradicts $J^S(I|I) = J^T(I|I) = 1$. This establishes FOSD of J^S over J^T: for all $s \in [\underline{s}(I), \overline{s}(I)]$, we have $J^T(s|I) \geq J^S(s|I)$.

Conditional on the sum of the two components being I and bidder i winning, his expected gain in the CE is:

$$\int_{\underline{s}(I)}^{\overline{s}(I)} u(s, I-s)dJ^S(s|I).$$

Note that with $V_i = S + T_i$, the above integral reduces to $u(s, I-s) = I$.

Conditional on the sum of the two components being I and bidder i winning, the expected gain from DS is:

$$\int_{\underline{s}(I)}^{\overline{s}(I)} u(s, I-s) dJ^T(s|I).$$

By assumption (property 3. of the generalized utility function), $u(s, I-s)$ is a non-increasing function. Thus, FOSD implies that $\int_0^1 u(s, I-s) j^S(s|I) ds \leq \int_0^1 u(s, I-s) j^T(s|I) ds$. This establishes the result: DS leads to a weakly higher expected gain conditional on a win, the same probability of a win, and a strictly lower payment.

□

Proof of Proposition 3. The cumulative distribution of the highest signal among $N-1$ bidders who all learn about the common component $S=s$ is

$$G(y) := \Pr(Y_i^S \leq y) = \int_0^1 F(y|s)^{N-1} h(s) ds.$$

For $v_i = 0$, we have $s = 0$ and $t_i = 0$, bidder i's signal follows density $f(x|0)$ for any information selection. The probability of winning for bidder i is $\int_0^1 f(x_i|0) F(x|0)^{N-1} dx_i = \frac{1}{N}$. For $v_i = 2$ (i.e., $s = 1$ and $t_i = 1$), winning probability of bidder i with any signal is $\int_0^1 f(x|1) F(x|1)^{N-1} dx = \frac{1}{N}$. This is because in those two extreme examples, the signal is equally distributed in signals X_i^T and X_i^S.

Next, consider a total value for bidder i at $v_i = v_i \in (0, 2)$. Define the feasible set of the common component by $\mathcal{S}(v_i)$, and let $\hat{s}(v_i)$ be the common component disecting this interval, as defined in the proof of Proposition 1. Similarly, let $h(s|v_i) := \frac{h(s)h(v_i-s)}{h_{\mathcal{V}}(v_i)}$. As before, we have$\int_{\mathcal{S}(v_i)} \frac{h(s)h(v_i-s)}{h_{\mathcal{V}}(v_i)} ds = 1$ and $h(s|v_i) = h(v_i - s|v_i)$.

First, consider the probability of bidder i having the highest signal realization, if bidder i observes the outcome of the experiment X_i^S about the common component.

$$\begin{aligned} \Pr(X_i^S \geq Y_i^S | v_i) &= \int_{\mathcal{S}(v_i)} \int_0^1 f(x|s) F(x|s)^{N-1} h(s|v_i) dx ds \\ &= \int_{\mathcal{S}(v_i)} \underbrace{\left[\frac{1}{n} F(x|s)^N \right]_0^1}_{=1/N} h(s|v_i) ds \\ &= \frac{1}{N} \int_{\mathcal{S}(v_i)} h(s|v_i) ds = \frac{1}{N}. \end{aligned}$$

Learning about the common component as all the other bidders yields a probability of $\frac{1}{N}$ of having the highest signal realization, for every realization of v_i in this symmetric setup.

Next, consider the probability of bidder i having the highest signal realization, if he learns X_i^T and is the only bidder learning about his private component.

$$\Pr(X_i^T \geq Y_i^S | v_i) = \int_{\mathcal{S}(v_i)} \int_0^1 f(x|v_i - s) F(x|s)^{N-1} h(s|v_i) dx ds. \tag{1.25}$$

I use the following abbreviation for clarity of presentation: $\lambda(s,x|v_i) \coloneqq h(s|v_i)F(x|s)^{N-2}$. Then, the probability of having the highest signal with X_i^T can be expressed as

$$\begin{aligned}\Pr(X_i^T \geq Y_i^S|v_i) &= \int_{\mathcal{S}(v_i)}\int_0^1 f(x|v_i-s)F(x|s)\lambda(s,x|v_i)dxds \\ &= \int_{\mathcal{S}(v_i)}\int_0^1 \frac{N-1}{N}f(x|v_i-s)F(x|s)\lambda(s,x|v_i)dxds \\ &\quad + \int_{\mathcal{S}(v_i)}\int_0^1 \frac{1}{N}f(x|v_i-s)F(x|s)\lambda(s,x|v_i)dxds.\end{aligned}$$

Integrating the inner integral of the second summand by parts yields

$$\begin{aligned}&\int_{\mathcal{S}(v_i)}\int_0^1 \frac{1}{N}f(x|v_i-s)F(x|s)\lambda(s,x|v_i)dxds \\ =&\int_{\mathcal{S}(v_i)}\frac{1}{N}\int_0^1 f(x|v_i-s)F(x|s)^{N-1}dxh(s|v_i)ds \\ =&\int_{\mathcal{S}(v_i)}\frac{1}{N}\left(\underbrace{\left[F(x|v_i-s)F(x|s)^{N-1}\right]_0^1}_{=1.} - \int_0^1 (N-1)f(x|s)F(x|s)^{N-2}F(x|v_i-s)dx\right)h(s|v_i)ds \\ =&\frac{1}{N}\underbrace{\int_{\mathcal{S}(v_i)}h(s|v_i)ds}_{=1.} - \int_{\mathcal{S}(v_i)}\int_0^1 \frac{N-1}{N}f(x|s)F(x|s)^{N-2}F(x|v_i-s)h(s|v_i)dxds \\ =&\frac{1}{N} - \int_{\mathcal{S}(v_i)}\int_0^1 \frac{N-1}{N}f(x|s)F(x|v_i-s)\lambda(s,x|v_i)dxds.\end{aligned}$$

Plugging this back into equation 1.26 gives the following expression:

$$\Pr(X_i^T \geq Y_i^S|v_i) = \tag{1.26}$$

$$\frac{1}{N} + \int_{\mathcal{S}(v_i)}\int_0^1 \frac{N-1}{N}\left[f(x|v_i-s)F(x|s) - f(x|s)F(x|v_i-s)\right]\lambda(s,x|v_i)dxds. \tag{1.27}$$

I show that the second summand in equation 1.27 is non-negative. For clarity of presentation, define $\mu(s,x|v_i) \coloneqq f(x|v_i-s)F(x|s) - f(x|s)F(x|v_i-s)$. Plugging in this notation and changing the order of integration in equation 1.27 yields

$$\int_{\mathcal{S}(v_i)}\int_0^1 \frac{N-1}{N}\left[f(x|v_i-s)F(x|s) - f(x|s)F(x|v_i-s)\right]\lambda(s,x|v_i)dxds \tag{1.28}$$

$$= \int_0^1\int_{\mathcal{S}(v_i)}\frac{N-1}{N}\mu(s,x|v_i)\lambda(s,x|v_i)dsdx \tag{1.29}$$

$$= \frac{N-1}{N}\int_0^1\left[\int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)}\mu(s,x|v_i)\lambda(s,x|v_i)ds + \int_{\hat{s}(v_i)}^{\min\{v_i,1\}}\mu(s,x|v_i)\lambda(s,x|v_i)ds\right]dx. \tag{1.30}$$

Note that the second summand can be rewritten as

$$\int_{\hat{s}(v_i)}^{\min\{v_i,1\}} \mu(s,x|v_i)\lambda(s,x|v_i)ds = \int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \mu(v_i-s,x|v_i)\lambda(v_i-s,x|v_i)ds$$
$$= -\int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \mu(s,x|v_i)\lambda(v_i-s,x|v_i),$$

where the first step was by changing the label of the integration variable, and the second line followed from $\mu(s,x|v_i) = -\mu(v_i-s,x|v_i)$. Plugging this back into equation 1.30 yields:

$$\frac{N-1}{N}\int_0^1 \int_{\max\{v_i-1,0\}}^{\hat{s}(v_i)} \mu(s,x|v_i)\left[\lambda(s,x|v_i)-\lambda(v_i-s,x|v_i)\right]dsdx.$$

Consider the expression in the square brackets in the inner integral first,

$$\lambda(s,x|v_i)-\lambda(v_i-s,x|v_i) = h(s|v_i)\left(F(x|s)^{N-2}-F(x|v_i-s)^{N-2}\right).$$

For $N=2$, the expression above is zero, as the term in the brackets is zero for any s,x or v_i, which establishes the theorem for two bidders: winning probability in equation 1.27 is $\frac{1}{2}$.

For $N>2$, the strong MLRP and thus, FOSD[34] imply: for all $a<b$ and for all $x\in(0,1)$, we have $F(x|a)>F(x|b)$. As the integral is below $\hat{s}(v_i)$, we have $s<v_i-t$. Therefore, for $x\in(0,1)$:

$$\lambda(s,x|v_i)-\lambda(v_i-s,x|v_i)>0.$$

Furthermore, note that $\mu(s,x|v_i)\geq 0$ is a reverse hazard rate condition $f(x|v_i-s)F(x|s)-f(x|s)F(x|v_i-s)\geq 0$. A well-known implication of the MLRP is that for all $a<b$, we have reverse hazard rate dominance

$$\frac{f(x|a)}{F(x|a)}\leq\frac{f(x|b)}{F(x|b)}.$$

Due to $s<v_i-s$, it immediately follows that $\mu(s,x|v_i)\geq 0$ in the entire domain of integration. This establishes the non-negativity in the second summand of equation 1.27. Thus, for $N>2$ and $x\in(0,1)$ we have $\Pr(X_i^T\geq Y_i^S|v_i)>\frac{1}{N}$.

□

Proof of Lemma 4. As bidders follow the same bidding function β_f^T in the candidate equilibrium and in $\overline{DS}^f$, after any information choice a bidder wins if and only if he has the highest signal realization.

In the candidate equilibrium, there are four possibilities for bidder i:

1. $S=T_i=T_j$ with probability ϵ^2 (denote the observed signals $X_i^{T=S}$ and $X_j^{T=S}$),

[34] For implications of the MLRP, like FOSD and reverse hazard rate dominance, see Milgrom and Weber (1982).

2. $T_i \neq S \neq T_i$ with probability $(1-\epsilon)^2$ (denote the observed signals $X_i^{T\neq S}$ and $X_j^{T\neq S}$),[35]

3. $T_i = S \neq T_j$ with probability $\epsilon(1-\epsilon)$,

4. $T_i \neq S = T_j$ with probability $\epsilon(1-\epsilon)$.

Consider the winning probability of bidder i conditional on v_i in each of those four possibilities. In possibility 1., if $V_i = v_i$, this implies that $S = T_i = v_i/2$.

$$\Pr(X_i^{T=S} \geq X_j^{T=S}|v_i) = \int_0^1 f(x_i|S = v_i/2)F(x_i|S = v_i/2)dx_i = \frac{1}{2}.$$

In possibility 4., the winning probability is:

$$\Pr(X_i^{T\neq S} \geq X_j^{T=S}|v_i) = \int_0^1 \int_0^1 f(x_i|v_i - s)F(x_i|s)dx_i \frac{h(s)h(v_i - s)}{h_{\mathcal{V}}(v_i)}ds = \frac{1}{2}.$$

The last equality follows from the proof of Proposition 1 (it is the same equation as Equation 1.25) for the case of two bidders.

Furthermore, note that winning probabilities conditional on a win in possibility 2. and 3. are the same, as the following shows:

$$\Pr(X_i^{T=S} \geq X_j^{T\neq S}|v_i) = \int_0^1 \int_0^1 f(x_i|s)F(x_i)dx_i \frac{h(s)h(v_i - s)}{h_{\mathcal{V}}(v_i)}ds$$
$$\Pr(X_i^{T\neq S} \geq X_j^{T\neq S}|v_i) = \int_0^1 \int_0^1 f(x_i|t)F(x_i)dx_i \frac{h(t)h(v_i - t)}{h_{\mathcal{V}}(v_i)}dt.$$

Therefore, in the candidate equilibrium, total winning probability conditional on v_i is:

$$\underbrace{\left(\epsilon^2 + \epsilon(1-\epsilon)\right)}_{\text{1. and 4.}}\frac{1}{2} + \underbrace{\left((1-\epsilon)^2 + \epsilon(1-\epsilon)\right)}_{\text{2. and 3.}}\Pr(X_i^{T=S} \geq X_j^{T\neq S}|v_i)$$
$$= \frac{\epsilon}{2} + (1-\epsilon)\Pr(X_i^{T=S} \geq X_j^{T\neq S}|v_i).$$

If the bidder deviates to $\overline{DS}^f$ instead, he always observes a signal X_i^S based on the realization of S. For his opponent, there are two possibilities: either his opponent' private component is $T_j \neq S$ with probability $(1-\epsilon)$, or it is $T_j = S$ with probability ϵ. Winning probabilities in both cases conditional on v_i are:

$$Pr(X_i^S \geq X_j^{T=S}|v_i) = \int_0^1 f(x_i|S = v_i/2)F(x_i|S = v_i/2)dx_i = \frac{1}{2}.$$
$$\Pr(X_i^S \geq X_j^{T\neq S}|v_i) = \int_0^1 \int_0^1 f(x_i|s)F(x_i)dx_i \frac{h(s)h(v_i - s)}{h_{\mathcal{V}}(v_i)}ds = \Pr(X_i^{T=S} \geq X_j^{T\neq S}|v_i).$$

[35] Note that the probability of both components S and T_i being drawn independently but having the same realization has zero probability as the distribution of each component has no mass points.

Therefore, total winning probability of bidder i conditional on $\overline{DS}^f$ is also

$$\frac{\epsilon}{2} + (1-\epsilon)\Pr(X_i^{T=S} \geq X_j^{T\neq S}|v_i).$$

This establishes that winning probability is equal at every v_i in the candidate equilibrium and $\overline{DS}^f$. Total expected gain is:

$$\int_{\mathcal{V}} v_i h_{\mathcal{V}} \Pr(\text{i wins}|v_i) dv_i$$

Therefore, overall expected gain in the candidate equilibrium and in the $\overline{DS}^f$ is the same. □

Proof of Lemma 5. Consider the distribution of signals of bidder i conditional on winning in the candidate equilibrium, i.e. the distribution of the first order statistic. For the same four possibilities as in Lemma 4, we have the following distributions:

1. $M(x_i|T_i = S, T_j = S) \coloneqq \Pr(X_i \leq x_i|X_i^T \geq X_j^T, T_i = S = T_j) = H^S(x_i|\beta_f^S, \beta_f^S, X_j^S)$,
2. $M(x_i|T_i \neq S, T_j \neq S) \coloneqq \Pr(X_i \leq x_i|X_i^T \geq X_j^T, T_i \neq S \neq T_j) = H^T(x_i|\beta_f^S, \beta_f^S, X_j^T)$,
3. $M(x_i|T_i = S, T_j \neq S) \coloneqq \Pr(X_i \leq x_i|X_i^T \geq X_j^T, T_i = S \neq T_j) = H^S(x_i|\beta_f^S, \beta_f^S, X_j^T)$,
4. $M(x_i|T_i \neq S, T_j = S) \coloneqq \Pr(X_i \leq x_i|X_i^T \geq X_j^T, T_i \neq S = T_j) = H^T(x_i|\beta_f^S, \beta_f^S, X_j^S)$.

The last inequalities followed by definition of H^K as defined in the main part of section 7.1. Due to independence between the signals of the bidders, and the same marginal distribution of both signals of i, we have

$$H^T(x_i|\beta_f^S, \beta_f^S, X_j^T) = H^S(x_i|\beta_f^S, \beta_f^S, X_j^T).$$

Furthermore, as established in the main text of section 7.1 in Inequality 1.14, increasing correlation decreases the first order statistic. That is, for all $x_i \in (0,1)$, we have

$$H^T(x_i|\beta_f^S, \beta_f^S, X_j^S) < H^S(x_i|\beta_f^S, \beta_f^S, X_j^S).$$

Hence, the overall distribution of the first oder statistic in the candidate equilibrium is

$$\begin{aligned} M^C(x_i) = &\underbrace{\epsilon^2}_{1.} H^S(x_i|\beta_f^S, \beta_f^S, X_j^S) + \underbrace{\epsilon(1-\epsilon)}_{4.} H^T(x_i|\beta_f^S, \beta_f^S, X_j^S) \\ &+ \underbrace{\Big((1-\epsilon)^2 + \epsilon(1-\epsilon)\Big)}_{2.\text{ and }3.} H^T(x_i|\beta_f^S, \beta_f^S, X_j^T). \end{aligned}$$

If bidder i instead plays $\overline{DS}^f$, he observes always a signal about X_i^S. With probability ϵ, his distribution in case of a win is $H^S(x_i|\beta_f^S, \beta_f^S, X_j^S)$ (if his opponent's private and common component are the same), and with probability $(1-\epsilon)$, his distribution in case of a win is $H^S(x_i|\beta_f^S, \beta_f^S, X_j^T)$. Thus, his overall distribution of his signal conditional on winning is

$$M^{DS}(x_i) = \epsilon H^S(x_i|\beta_f^S, \beta_f^S, X_j^S) + (1-\epsilon)H^S(x_i|\beta_f^S, \beta_f^S, X_j^T).$$

In Inequality 1.14 I establish that for all $x_i \in (0,1)$, we have $H^T(x_i|\beta_f^S, \beta_f^S, X_j^S) < H^S(x_i|\beta_f^S, \beta_f^S, X_j^S)$. This in turn implies that for all $x_i \in (0,1)$, the distribution of winning bids under the candidate equilibrium is FOSD over $\overline{DS}^f$.

$$M^{DS}(x_i) \geq M^C(x_i).$$

Therefore, expected payment is strictly higher under the candidate equilibrium than in $\overline{DS}^f$. □

Proof of Proposition 4. The proof follows by combining the following two results as described in the main text. Lemma 4 shows that winning probability and expected gain from the deviation strategy $\overline{DS}^f$ is in the deviation strategy and in the candidate equilibrium. Lemma 5 establishes that $\overline{DS}^f$ leads to a strictly lower expected payment. Hence, DS is a strictly profitable deviation. □

Proof of Proposition 5. The proof is by contradiction. I show that expected payoff from DS^a is higher than in a CE with $\rho^* = 1$. Assume that in the candidate equilibrium, $\rho^* = 1$ and bidders follow a strictly increasing pure bidding function $\beta_a^S(x)$. Denote by Y_i the highest signal realization of all bidders but bidder i.

The expected payment of bidder i in the CE is:

$$\int_0^1 \beta_a^S(x_i) f^S(x_i) dx_i.$$

The expected payment of bidder i from the DS^a is:

$$\int_0^1 \beta_a^S(x_i) f^T(x_i) dx_i.$$

Due to symmetry, for all $x \in [0,1]$, we have $f^T(x_i) = f^S(x_i)$ according to Observation 1. Hence, the expected payment in the candidate equilibrium is the same as in DS^a.

Next, consider the expected gain from participating in the auction. Fix a value v_i for bidder i, as in the preceding sections. In CE and in DS^a, bidder i wins if he has the highest signal realization, as in both, bidders follow the same bidding function $\beta_a^S(\cdot)$. Formally, the winning probability of bidder i for a fixed value v_i under both regimes is:

$$\begin{aligned} \text{CE:} \qquad & \Pr(\beta_a^S(X_i^S) \geq \beta_a^S(Y_i^S)|v_i) = \underbrace{\Pr(X_i^S \geq Y_i^S|v_i)}_{\star^A}, \\ DS^a: \qquad & \Pr(\beta_a^S(X_i^T) \geq \beta_a^S(Y_i^S)|v_i) = \underbrace{\Pr(X_i^T \geq Y_i^S|v_i)}_{\star^{AA}}. \end{aligned}$$

For $N > 2$, by Proposition 3, the probability of a win is strictly higher with DS^a than with CE for all $v_i \in (0,1)$. Hence, $\star^{AA} > \star^A$ for all $v_i \in (0,2)$. Winning probability in DS^a is strictly higher than in CE for almost all v_i, for the same expected payment. DS^a is a strictly profitable deviation.

□

Proof of Lemma 6. Fix $\rho^* = 0$. This is the standard symmetric IPV setting, with the bidding function in the main text being a best response if all other bidders follow it. That is, given fixed $\rho^* = 0$, this bidding function consitutes a best response for both bidders.

The only part of the proposition left to be shown, is that no bidder has a profitable deviation that involves a different information choice variable ρ_i. Consider bidder i deviating to $\rho_i \neq 0$. Note that the expected utility is a linear combination of the payoff after observing X_i^S with probability ρ_i, and X_i^T with probability $(1 - \rho_i)$. Therefore, it suffices to consider the case $\rho_i = 1$ and showing that it does not lead to a strictly higher payoff than $\rho_i = 0$.

Due to the Independence Assumption IN, neither X_i^T nor X_i^S contain information about the opponent's signal due to Assumption IN. Furthermore, the value of the object conditional on a win does not depend on bidder i's information choice due to Observation 2 and the irrelevance of the opponent's information. Therefore, as the choice of ρ_i impacts neither the joint distribution, nor expected valuation conditional on a win, each bidder is indifferent between each $\rho_i \in [0, 1]$ and plays the same best response after any signal realization (no matter its source). Therefore, the classic equilibrium of the all-pay auction is also an equilibrium of this game that involves information selection. □

Chapter 2

Asymmetric Budget Constraints in a First Price Auction

1. Introduction

Auctions are a widely used method of allocating objects like art or wine, property right or in a procurement setting. For participants in an auction, their willingness to pay might not correspond to their ability to pay. Budget constraints arise e.g. due to credit limits and imperfect capital markets. Budget constraints influence bidding strategies, break the revenue equivalence of standard auctions and lower expected revenue.

Research on standard auctions with budget constrained bidders concentrates on budgets drawn from identical distributions. Yet, in many realistic scenarios, bidders have asymmetric expectations about each others' budgets. The consequences of this asymmetry on bidding behavior and on revenue in the first price auction (FPA) has not been previously studied. Asymmetry in budget constraints can arise in a narrow market, e.g. a telecommunication sector with less than a dozen major players: as there are only a few incumbents, bidders have noisy information about the other bidders and their budgets. Information might stem from previous interaction or from publicly available information, like annual budget reports. Moreover, asymmetric budget distributions might be intended by the auction designer: the auctioneer can reveal the identities of the participants before the start of an auction via a participation register.

The following example illustrates the occurrence of asymmetric budget distributions. Consider the spectrum auction of the U.S. Federal Communications Commission (FCC): 30 bidders registered (Salant, 1997) for the auction in which the rights to provide personal communication services were sold. Assessing the budget constraint of rival bidders was a major part of the preparation before the auction (Salant, 1997). GTE was one of the largest telecommunication firms in the U.S.. It seems reasonable to assume that the expectations of GTE about the budget of a smaller bidder like Poka Lambro, differed from the expectation of the smaller bidder about the resources of GTE.

The contribution of this paper is to identify how asymmetric budget distributions impact bidding and revenue in the FPA. I develop a new solution technique that builds on an indirect utility approach by Che and Gale (1996). This technique allows to characterize the entire set of equilibria[1] via a closed-form solution for expected utility, bidding distributions and revenue. I provide a uniqueness and existence result, and show that asymmetric budget constraints can break the revenue dominance of the first price auction over the second price auction.

Two bidders are competing for one object in a first price auction. Their valuations are common knowledge. Uncertainty stems from the budget dimension. Each bidder has a budget constraint, that is private and drawn independently from different distributions. Budget constraints are hard. That is, there is no credit market, and no bidder can bid more than his budget. The impact of budgets is twofold: budget constraints directly limit the ability to bid. Moreover, budgets have an indirect strategic effect: if the opponent is constraint, the necessary bid to outbid him might be lower than without the constraints. The constrained bidder anticipates this inference of his opponent and shades his bid down even further, and so forth. The extent of these strategic effects varies with the asymmetry in budget distributions.

I solve the FPA, developing a new approach build on Che and Gale (1996). They restrict attention to symmetric equilibria and strictly monotonic bidding functions, for bidders with symmetric budget constraints and the same public valuation for the object. Che and Gale (1996) use a guess and verify approach for the equilibrium utility. The expected utility in their model always equals some exogenous lower bound on utility. This lower bound is the best utility a bidder can achieve if his opponent follows a naive strategy: always bidding his entire budget. Once equilibrium utility is pinned down, symmetric and strictly monotonic bidding strategies in equilibrium can be constructed.

In my model, I allow for asymmetric budget distributions and different valuations. I pin down the relationship between a lower bound utility and the actual equilibrium utility. This relationship is not an equality anymore as in Che and Gale (1996).

If the probability of the opponent having a budget sufficiently close to to one's own budget is high enough (when the reverse hazard rate (RHR) of both bidders is above a certain threshold), the unique equilibrium bidding strategy is to bid the entire budget and the equilibrium utility coincides with the lower bound. The intuition is the following: it is profitable to bid the entire budget, if the gain in winning probability outweighs the higher payment in case of a win. This consideration applies when a bidder is sufficiently weak: when his budget is low relative to the value of the object and the budget distribution of the opponent. As long as this holds, asymmetry in budget distributions does not influence bidding behavior.

If at least one RHR drops below the threshold, mass points in strategies generically occur and at least one bidder achieves a utility strictly above his auxiliary product. When the value of the object and the threat of being poorer than the opponent does not justify bidding the entire budget anymore for at least one bidder, mass points in bidding behavior arise and bidders make each other indifferent by bidding within some interval. Mass points are stable under budget constraints. Those bidders who would like to deviate and bid at the mass point or slightly above, are restrained from deviating by their binding budget constraints.

The uniqueness result pins down a unique equilibrium utility and unique bidding cumulative distribution functions for both bidder. Bidding strategies are uniquely pinned

[1] I do not restrict attention to symmetric equilibria nor to monotonic bidding strategies.

down in the budget domain, where both bidders are bidding their entire budget. Otherwise, mass points are part of the equilibrium and a multiplicity of non-monotonic equilibrium bidding strategies in indifference regions might arise, that all satisfy the unique cumulative bidding distributions and yield the same unique expected utility.

I show existence by providing feasible and optimal weakly monotonic bidding functions. A strictly monotonic equilibrium will usually not exist. Moreover, I show that the symmetric equilibrium utility that Che and Gale (1996) find is the only equilibrium utility their model admits if assuming budget distributions are log-concave; there are no other asymmetric equilibrium utilities. For the special case of RHR dominance in the budget distributions, the weak bidder bids more aggressively than the strong bidder in monotonic strategies. This is in line with the literature on asymmetrically distributed valuations in Maskin and Riley (2000), where the weaker bidder (with regards to the valuation distribution) bids more aggressively.

A revenue-maximizing auctioneer should never disclose the identities of the participants in an auction (e.g. publish a participation register), if bidders are ex-ante symmetric and have identical valuation for the object. The best case scenario for the auctioneer is when utility of the bidders is equal to the lower bound on utility. This is the case in any equilibrium under symmetric budget distributions with equal valuations. Whenever asymmetry becomes sufficiently large, the lower bound may no longer bind for the equilibrium utility.

2. Related Literature

Che and Gale (1996, 1998, 2000) are amongst the first to analyze auctions with budget constrained bidders. In their seminal contributions, they derive the equilibrium for auctions with budget constraints and show that revenue equivalence no longer holds when bidders are symmetrically budget constrained. Research on budget constraints in standard 1-object auctions (see e.g. Che and Gale, 1996, 1998; Kotowski, 2016; Kotowski and Li, 2014) considers symmetric budget distributions. Literature on asymmetrically budget constrained bidders is scarce. Malakhov and Vohra (2008) derive the optimal auction with two bidders, where only one bidder is constrained and his identity is common knowledge. Some literature on multiple object auctions (see e.g. Benoît and Krishna, 2001; Dobzinski et al., 2012) considers asymmetric budgets, however, relies upon common knowledge of budget realizations. I merge the assumption of asymmetric budgets into a framework, that allows for budget realization being private.

The closest paper to my framework is Che and Gale (1996). They consider many bidders with identical commonly known valuation for the object. Budget realizations of the bidders in Che and Gale (1996) are private, identical and independent draws from the same distribution. My model generalizes their model in two directions: first, in my model budgets are drawn from asymmetric distributions. Second, valuations for the object may differ between bidders. This allows me to capture the effect of valuation heterogeneity on bidding strategies. In contrast to Che and Gale (1996), I do not restrict attention to symmetric equilibria, but I impose log-concavity on budget distribution.

The analysis of this paper relates to asymmetric auctions, in which the valuations of bidders are drawn from non-identical distributions, and bidders do not have budget constraints (see the seminal contribution of Maskin and Riley, 2000). Analytical solutions exist for only few particular distributions (see Maskin and Riley (2000) and Kaplan and Zamir (2012) for uniform distributions, Plum (1992) and Cheng (2006) for power distributions). Asymmetric auctions have been approached by perturbation analysis (see e.g. Fibich and Gavious, 2003; Fibich et al., 2004; Lebrun, 2009). Nevertheless, even for two bidders with asymmetrically drawn valuations from the same support, no general closed-form solution is known. Standard auctions no longer yield the same revenue under asymmetric value distributions. Whether the FPA or the SPA yields higher revenue depends on the asymmetry of the value distributions (see e.g. Maskin and Riley (2000), Cantillon (2008), Gavious and Minchuk (2014)).

If bidders are asymmetric not in the valuation, but in the budget dimension, my results apply: in contrast to the literature on asymmetry in valuations, a closed-form solution exists. Revenue can be easily computed. A unique equilibrium utility and bidding distribution exist under mild regularity conditions. This holds for all log-concave budget distributions with same support, without having to impose any stochastic dominance order assumption. Hereby, I do not restrict attention to symmetric or monotonic bidding functions, and I allow for atoms in strategies.

The paper is organized as follows. First, I introduce a simplified version of the model with a strong and a weak bidder in the next section 3. I show how asymmetry influences the equilibrium bidding behavior and revenue of the auctioneer. In the next section, I introduce the general model 4 for different valuations and arbitrary distributions of budgets. In section 5 I introduce the solution technique via lower bound on equilibrium utility. I show how to deduce the unique equilibrium utility via this lower bound. In section 6 I prove existence via pure strategy weakly monotonic bidding strategies and analyze bidding aggression as a function of asymmetry. Section 7 looks at a revenue comparison between standard auctions, restricts the assumption that both bidders have to be budget constrained with positive probability and analyzes information disclosure about budget distributions.

3. Example

Consider a simplified version of the model: two firms compete in a FPA for an object worth v to both. Firm S is the strong bidder in the sense of first order stochastic dominance (FOSD), with a budget distribution governed by the cumulative distribution function (cdf) $F_S(w) = w^2$ on $[0, 1]$. Firm W is the weaker bidder; his budget is distributed with cdf $F_W(w) = w$ on $[0, 1]$. Bids can never strictly exceed the budget, and no ex-post renegotiations are possible.

Let the value of the object be sufficiently high, $v > 2$. Both bidders are always budget constrained under any budget realization. That is, no bidder can ever bid his true value for the object.

Let $b_i(w)$ be the bid of bidder $i \in \{S, W\}$ with budget realization w. In equilibrium, both bidders always bid their entire budget for any realization of the budget[2]: $b_S(w) = b_W(w) = w$, $\forall w \in [0, 1]$. If the opponent sticks to this strategy, it is the best response to also bid the entire budget: the expected payoff from a bid b of a strong bidder,

[2]Always bidding the entire budget is the unique equilibrium, as I show in section 5.

$(v-b)b$, and of a weak bidder, $(v-b)b^2$, are strictly increasing in the bid b over the entire budget domain. The object is so valuable (consider $v \to \infty$), that the increase in winning probability from increasing the bid always offsets the increase in payment in case of a win.

If $v = 1$, bidding the full budget on the entire domain is not an equilibrium anymore.[3] Even if the weak bidder is bidding his entire budget on the full domain, the strong bidder would still never want to bid above 1/2, which is the arg max of $(1-b)b$: the probability mass of the weak bidder being richer than $\frac{1}{2}$ is so low, that the small increase in winning probability does not offset a higher payment in relation to the object value. The following bidding strategies are an equilibrium in the class of weakly monotonic bidding strategies.

$$b_S(w) = \begin{cases} w & \text{if} \quad w < \frac{1}{2}, \\ \frac{1}{2} & \text{if} \quad w \in [\frac{1}{2}, \frac{1}{\sqrt{2}}], \\ 1 - \frac{1}{4w^2} & \text{otherwise}, \end{cases} \qquad \text{and} \qquad b_W(w) = \begin{cases} w & \text{if} \quad w < \frac{1}{2}, \\ 1 - \frac{1}{4w} & \text{otherwise}. \end{cases}$$

Figure 2.1 illustrates the two bidding functions for $v = 1$ as a function of the budget. The blue solid line is the bidding function of the strong bidder, the green dashed line is the bidding function of the weak bidder. In equilibrium, both bidders bid their entire budget for budget realizations in $[0, \frac{1}{2}]$. By the same argument as for $v > 2$, expected payoff from a bid b is strictly increasing in the bid for $b \in [0, 1/2]$. Bidder S places a mass point on $\frac{1}{2}$, and bids below the bidder W for all higher budgets. The highest possible bid $\frac{3}{4}$ is the same for both bidders, and strongly below the value of the object.

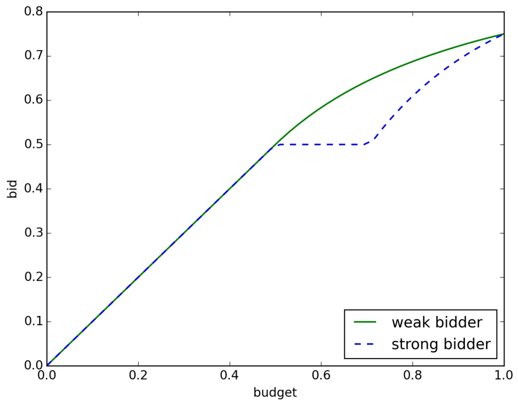


Figure 2.1: Bidding functions for $v = 1$.

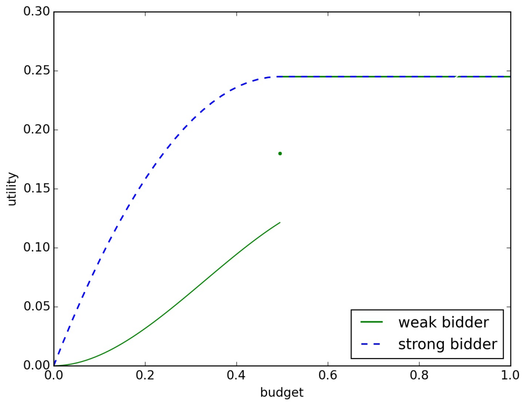


Figure 2.2: Equilibrium utilities for $v = 1$.

Figure 2.2 shows the equilibrium utility associated with these bidding strategies. Both bidders have strictly increasing utility for any budget below $\frac{1}{2}$. Bidder S admits a mass point at $\frac{1}{2}$. He has the same equilibrium utility for any budget above $\frac{1}{2}$; thus,

[3] A similar argument holds for all $v \in (0, 2)$.

he is indifferent between all corresponding bids in $[\frac{1}{2}, \frac{3}{4}]$, including bidding at the mass point. This mass point raises the utility of the weak bidder to the same level as his utility for budgets above $\frac{1}{2}$. A weak bidder with a budget at the mass point (the green dot resulting from an equal tie breaking rule) or slightly below cannot deviate and slightly increase his bid, because his budget binds. A weak bidder with a high budget is indifferent for any bid between $(\frac{1}{2}, \frac{3}{4}]$ and has therefore also no incentive to deviate. Thus, binding budget constraints enable a mass point as part of an equilibrium.

This section contained an example for how bidding strategies and equilibrium utilities look under asymmetrically distributed budgets. Mass points arise, as some profitable deviations are unfeasible due to binding budget constraints. This allows mass points to be part of the equilibrium. In the next section, I show how the unique equilibrium utility can be derived if bidders value the object differently and for arbitrary budget distributions without a FOSD assumption.

4. Model

An auctioneer (she) sells one object without value for her in a first price auction (FPA). She employs an equal tie-breaking rule and no reserve price. There are 2 risk-neutral bidders, indexed by $i \in \{1, 2\}$. Bidder i has valuation v_i for the object. The valuation tuple $\{v_1, v_2\}$ is common knowledge for the bidders.

Each bidder (he) has a private budget w_i, which is drawn independently from an atom-free distribution with a continuous and differentiable cumulative distribution function $F_i(w)$ and probability density function $f_i(w)$. Both distribution functions $\{F_i(w)\}_{i=1,2}$ have common support $[\underline{w}, \overline{w}]$ and are common knowledge. The probability density functions are positive in the interior of the support: $f_i(w) > 0 \ \forall w \in (\underline{w}, \overline{w})$, and $F_i(\underline{w}) = 0 \ \forall i \in \{1, 2\}$. Note that I do not impose any stochastic order between distribution functions $F_1(w)$ and $F_2(w)$, e.g. about stochastic dominance.

Let $\min(v_1, v_2) > \underline{w}$. Both bidders are budget constrained with non-zero probability[4]. Both budget distribution functions satisfy the following assumption:

Assumption 1
$F_1(w)$ and $F_2(w)$ satisfy log-concavity on $(\underline{w}, \overline{w})$.[5]

Equivalently, the reverse hazard rates (RHRs) $\frac{f_i(w)}{F_i(w)}$ are decreasing in w. If a bidder i wins the object by placing a bid b, his quasilinear utility is $v_i - b$.

A bidding strategy of bidder i maps his budget realization w into a probability distribution over feasible bids, $\beta_i : [\underline{w}, \overline{w}] \to \Delta[0, \overline{w}]$. Let $b_i(w)$ be bid realization of bidder i with budget w, to which strategy $\beta_i(w)$ assigns strictly positive probability. Bidders have hard[6] budget constraints: they cannot bid more than they have. A feasible bidding strategy satisfies $b_i(w) \leq w$, for all bids of any budgets w.

[4]If at least one bidder is always unconstrained under any budget realization, this reduces to Bertrand competition, which I analyze in section 7.3.

[5]The assumption of log-concavity is standard in theory to impose some structure, and satisfied by many commonly used distributions. See (Bagnoli and Bergstrom, 2005) for an overview.

[6]An equivalent formulation for my analysis is to impose fines on overbidding and to forbid renegotiation.

Let $G_i(x) = \Pr(b_i \leq x)$ be the cumulative distribution function (cdf) of bidding of bidder i, that is, the probability of bidder i bidding below or equal to x. A *feasibility constraint* holds as a necessity of the hard budget constraints:

$$G_i(x) \geq F_i(x) \quad \forall x \in [0, \overline{w}]. \tag{2.1}$$

Every bidder with a budget below some x has to bid weakly below x. Moreover, a bidder with a budget strictly above x might shade his bid down below x, yielding the weak inequality in the feasibility constraint in equation 2.1. If bidders always bid their entire budget for any budget realization, the above feasibility constraint holds with equality.

Note the non-equivalence of a bidding strategy β_i, and a bidding cdf G_i: multiple bidding strategies might yield the same bidding distribution G_i. A bidding function β_i pins down a unique bidding distribution G_i. Yet, a bidding distribution G_i does not uniquely pin down a bidding strategy of bidder i. [7] In the following, I derive the bidding distributions that arise in equilibrium, and show that they are unique under log-concavity.

5. The First Price Auction

The interim expected utility of bidder i with budget realization w, given bidder j's bidding distribution, is

$$U_i(w) = \max_{0 \leq b_i \leq w} \{(v_i - b_i)[\Pr(b_j < b_i)] + \frac{1}{2}(v_i - b_i)\Pr(b_j = b_i)\}. \tag{2.2}$$

The second summand accounts for the equal sharing rule in case of a tie. In contrast to standard results in Auction Theory with invertible bidding functions, in my model equilibrium strategies may contain mass points. Therefore, the probability of a tie is non-negligible.

Note that $U_i(w)$ is monotonic, and hence contains at most countable discontinuities. Moreover, if the opponent has no mass points in his bidding strategy, the probability of a tie is zero, the second tie-breaking summand drops out, and $U_i(w)$ is continuous on $[\underline{w}, \overline{w}]$. However, due to the individual (potentially slack) budget constraints and the discontinuities in the objective function in equation 2.2, a classic approach with straightforward differentiation and invertible bidding functions is not possible. I solve the problem via an indirect utility approach, using a lower bound on utility.

In the following of section 5, I derive a uniqueness result on equilibrium utilities U_i. This in turn pins down the unique bidding distribution G_i in equilibrium. The existence result follows in section 6, where I show that there always exist weakly monotonic pure

[7] For a sketch of this argument, assume that the bidding function of bidder i consists only of two bids, 0 and 1, which are feasible for any budget realization (i.e. $\underline{w} \geq 1$). Compare the following strategies: In the first bidding strategy, bidder i randomizes with probability 1/2 between those bids, irrespective of any budget realization. In the second bidding strategy, bidder i bids 0 for all budget realizations below the median budget w^m such that $F(w^m) = 1/2$, and bids 1 for all higher budget realizations. Both strategies result in the same bidding distribution $G_i(.)$.

strategy bidding functions, that are feasible and optimal for the bidders, and satisfy the equilibrium bidding cdf G_i.

5.1 The lower bound of equilibrium utilities

I derive the equilibrium utility via an auxiliary expression (referred to as a *lower bound*) for each bidder i for every budget realization w:

$$\underline{U}_i(w) = \max_{b \leq w}(v_i - b)F_j(b) \tag{2.3}$$

This is the maximum expected utility bidder i can achieve given his budget w, conditional on his opponent following a naive strategy: always bidding his entire budget (irrespective of whether it is a best response). This expression is exogenously defined for both bidders, continuous and non-decreasing in the realized budget w.

$\underline{U}_i(w)$ is a *lower bound* on the expected utility $U_i(w)$: if the opponent always bids naively by always bidding his entire budget, winning probability is minimized at every budget in the domain. Any bid wins with a weakly lower probability under the naive strategy than under any other feasible strategy. The expected payoff from any bid under any other strategy of the opponent is weakly greater than under the naive strategy.

Next, I characterize the properties of the lower bound $\underline{U}_i(w)$. Subsequently, I relate those properties to the equilibrium utility.

Let bidder j follow the naive strategy of always bidding his entire budget. Then, a bid b yields a payoff of $(v_i - b)F_j(b)$ for his opponent i. The derivative is $F_j(b)[(v_i - b)\frac{f_j(b)}{F_j(b)} - 1]$. By log-concavity, the reverse hazard rate (RHR) $\frac{f_j(b)}{F_j(b)}$ is monotonically decreasing. Thus, the expression in the squared brackets of the derivative is strictly decreasing in b. This implies that $(v_i - b)F_j(b)$ is quasi-concave and never constant on any open interval. The derivative is positive if the following condition is satisfied:

Definition 7. *The* ***RHR-condition*** *of bidder i holds at b if the following holds:*

$$\frac{f_j(b)}{F_j(b)} > \frac{1}{v_i - b}$$

The expression relates expected benefit and costs of increasing a bid at b. The RHR on the left side corresponds to the probability of the opponent having a budget close to the own bid, conditional on him not having a budget above b. If the opponent has a high RHR, he has a high probability of having a budget close to the own bid, and increasing the bid is profitable if the net utility of a win $v_i - b$ is sufficiently high.

The expression $(v_i - b)F_j(b)$ has a unique global maximum $m_i = argmax_b(v_i - b)F_j(b)$. Note that m_i lies within $\in (\underline{w}, \overline{w}]$ (interior or at the highest possible budget): any bid below or equal to $\underline{w}$ yields payoff 0[8], while any bid $b \in (\underline{w}, v_i)$ yields strictly positive payoff under naive bidding of the opponent. The maximum m_i is increasing in (and always below) v_i: if the value of the object is sufficiently high, bidding the entire budget can be the best response.

[8] A bid $b \leq \underline{w}$ looses with certainty as $F_j(\underline{w}) = 0$ if the other always bids his entire budget.

Observation 4. *Let* $m_i = argmax_b(v_i - b)F_j(b)$*. All* $w \in (\underline{w}, m_i)$ *satisfy the RHR-condition.*

The best response of bidder i, if his opponent follows the naive strategy, is to bid exactly at m_i if his budget allows it. Otherwise, the best response is to bid his entire budget, if he cannot afford m_i; this is because the derivative is strictly positive (negative) for any bid below (above) m_i; no bidder ever bids beyond m_i if his opponent is naive. The resulting lower bound utility is:

$$\underline{U}_i(w) = \begin{cases} (v_i - w)F_j(w) & \text{if } w < m_i \\ (v_i - m_i)F_j(m_i) & \text{if } w \geq m_i \end{cases} \tag{2.4}$$

Figure 2.3 shows the auxiliary lower bound utilities for the example in section 3, with bidder $F_1(w) = w^2$ being the strong bidder, and $F_2(w) = w$ being the weak bidder. $\underline{U}_1(w)$ is strictly increasing up to $m_1 = 1/2$, and constant for higher budget realizations. $\underline{U}_2(w)$ is strictly increasing up to $m_2 = \frac{2}{3}$, and constant thereafter. Intuitively, for small budget realizations in comparison to the object value, the winning probability of any feasible bid is low. The gain in winning probability is worth more than the higher payment due to a higher bid: it is worth bidding the entire budget. For high enough budget realizations, winning probability is high. The focus of the bidder becomes not to bid too much by shading the bid below the budget to m_i.

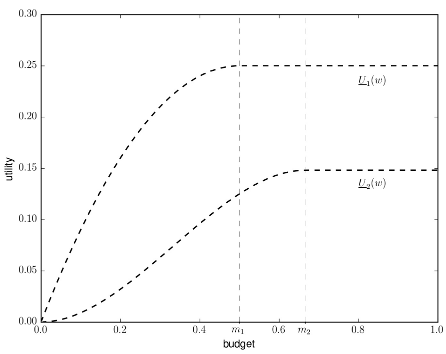


Figure 2.3: Auxiliary utilities with $v_1 = v_2 = 1$,
$F_1(w) = w^2$ and $F_2(w) = w$.

Without loss, relabel m_1 and m_2 such that $m_1 \leq m_2$.

Observation 5. *Both lower bounds* $\underline{U}_i(w)$ *for* $i \in \{1, 2\}$ *are strictly increasing and continuous on* $w \in (\underline{w}, m_1)$*.*

The budget domain can be partitioned into two intervals, on which

1. both $\underline{U}_i(w)$ are strictly increasing (by observation 5 this occurs below m_1);
2. at least one $\underline{U}_i(w)$ is constant (this occurs above m_1).

In the following, I derive the equilibrium utility of the bidders for each of these two scenarios: 1. in section 5.2, and 2. in section 5.3.

Naive bidding cannot be an equilibrium, unless $m_1 = m_2 = \overline{w}$, as strategies in equation 2.4 show. Under the naive assumption on bidder j, bidder i never bids above m_i. If bidder i never bids above m_i, bidder j would never want to bid above m_i either, and the naive assumption on bidder j is unsustainable in equilibrium.

The lower bound utility $\underline{U}_i(w)$ is a generalization of the lower bound expression in Che and Gale (1996), that allows for asymmetric budgets and different valuations. They show that in a symmetric and strictly monotonic equilibrium the lower bound always binds, i.e. $\underline{U}_i(w) = U_i(w)$ for all $w \in [\underline{w}, \overline{w}]$ and for all i. For asymmetric bidders or different valuation, I find that the lower bound does not generically bind and mass points arise.

In the following, I fix candidate equilibrium utilities $U_1(.)$ and $U_2(.)$ and show which shape of them leads to a contradiction, and therefore cannot arise in any equilibrium. In some cases, bidding strategies can be inferred from the shape of the equilibrium utilities, as the next lemma shows.

Lemma 1
Let $U_i(w)$ be strictly increasing on some open interval (w', w''). Then, bidder i with any budget realization within (w', w'') always bids his entire budget.

All proofs are in the appendix. The intuition is that the same feasible bid always yields the same utility to bidder i, no matter what his budget realization is. This is because bidder i always values the object with the same v_i, irrespective of his budget realization[9]. When the utility U_i is strictly increasing in budget realization, bidders i with a budget in this interval have to bid differently, as they get different utility. Moreover, bidders have to bid their entire budget, because that is the only bid, that a bidder i with a lower budget cannot mimic.

The following lemma shows that if a bidder achieves an equilibrium utility strictly above the lower bound, this only occurs because of a mass point in the opponent's strategy. This finding drastically reduces the set of candidate equilibria to consider.

Lemma 2
Let $w' \in (\underline{w}, \overline{w})$. If $U_i(w') > \underline{U}_i(w')$, then

1. *$U_i(.)$ is constant on $(w' - \epsilon, w' + \epsilon)$ for some $\epsilon > 0$, or*

2. *$U_i(.)$ has a discontinuity at w' due to a mass point of opponent at w'.*

The following corollary follows immediately from the contraposition of lemma 2. If equilibrium utility is strictly increasing and continuous, it has to coincide with the lower bound utility.

[9]This is a crucial difference to models where bidder i types have different uncertain valuations for the object, where this kind of inference from equilibrium utility about bidding strategies is not possible: two bidders having different valuations have different expected utilities from bidding the same bid, although winning probability is the same.

Corollary 3. *Let $w' \in (\underline{w}, \overline{w})$. Let $U_i(w)$ be continuous at w' and strictly increasing on $w \in [w', w' + \epsilon)$ or $w \in (w' - \epsilon, w']$ for some $\epsilon > 0$. Then, the lower bound binds at w': $U_i(w') = \underline{U}_i(w')$.*

The proofs of lemma 2 and corollary 3 correspond to the following argument: if the expected utility in equilibrium is strictly increasing in i's budget in some neighborhood, every budget type i bids his full budget in this neighborhood by lemma 1. Let bidder i achieve a utility strictly above his lower bound for some budget. To achieve an expected utility strictly above the lower bound, bids have to have a higher winning probability than under the naive strategy assumption. As bidder i bids his entire budget, this implies for bidder j the same situation for this interval as if bidder i were to follow the naive strategy of always bidding his entire budget: either it is optimal for j to bid his entire budget (if his budget is below the optimal bid m_j), or bidding at m_j yields a higher payoff than bidding in the respective interval above m_j, as shown in equation 2.4. Either way, bidder j does not bid with high enough probability in the respective interval to elevate the utility of bidder i above the lower bound; the lower bound utility binds.

Consider figure 2.2 for an illustration of lemma 2 and corollary 3. Equilibrium utilities of the weak and the strong bidder are strictly increasing in the budget up to $w = \frac{1}{2}$. By corollary 3, equilibrium utilities and lower bounds coincide in this interval: $U_i(w) = \underline{U}_i(w)$ for all $w \in (0, \frac{1}{2})$ for both the strong and the weak bidder. Furthermore, the lower bound utility is a continuous function of the budget. As the equilibrium utility of the weak bidder has a jump discontinuity at $w = \frac{1}{2}$, we have $U_W(\frac{1}{2}) > \underline{U}_W(\frac{1}{2})$. By part 2. of lemma 2, the strong bidder has to admit a mass point in his bidding distribution at the bid $\frac{1}{2}$ in order to raise the equilibrium utility of the weak bidder strictly beyond the lower bound utility.

5.2 Strictly increasing lower bounds

The next theorem shows that as long as both RHR-conditions are satisfied and both lower bound utilities are strictly increasing, bidding the entire budget is the unique equilibrium strategy. This occurs if the probability of the opponent having a budget close to one's own, conditional on not being richer, is sufficiently high, and the object is valuable enough.

Theorem 1
The following holds in any equilibrium for any $i \in \{1, 2\}$.

- *For all $w \in [\underline{w}, m_1)$, the lower bound binds: $U_i(w) = \underline{U}_i(w)$.*
- *Bidders with budget $w \in (\underline{w}, m_1)$ always bid their entire budget.*[10]

Assume the opponent bids his entire budget on $(\underline{w}, m_1)$. Due to the sufficiently high RHRs, any decrease in bidding below the budget looses so much winning probability, that it is not worth the gain from the lower payment in case of a win. This

[10] $b_i(w) = w$ is the only bid to which any equilibrium bidding function assigns strictly positive probability.

guarantees that both bidders bidding their entire budget on this interval is indeed a Nash equilibrium. The proof shows, that this is indeed the unique optimal strategy in any Nash equilibrium. Note that the theorem does not specify the bid of the lowest budget realization $\underline{w}$, while it pins down his equilibrium utility to zero. Yet, as this is a zero-probability event, it does not influence the bidding distributions $G_1(w)$ and $G_2(w)$ and I assume without loss that the lowest bidder always bids his entire budget $b_i(\underline{w}) = \underline{w}$.

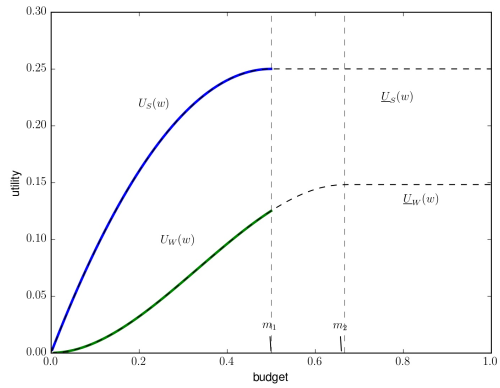


Figure 2.4: Equilibrium utilities below m_1 for
$v = 1$, $F_S(w) = w^2$, $F_W(w) = w$.

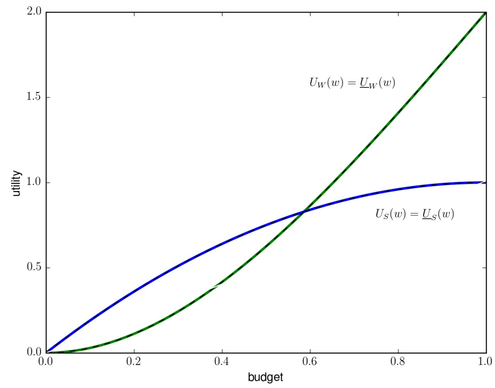


Figure 2.5: Equilibrium utilities below m_1
for $v_S = 2$, $v_W = 3$, $F_S(w) = w^2$, $F_W(w) = w$.

Both players bid their entire budget and achieve their lower bound utility $\underline{U}_i(w) = U_i(w)$ below m_1. Figure 2.4 and 2.5 illustrates this finding for two examples: figure 2.4 shows the equilibrium utility for the example in section 3 with $v_1 = v_2 = 1$. In this case, $m_1 = \frac{1}{2} < m_2 = \frac{2}{3} < \overline{w} = 1$. The upper blue (lower green) line is the equilibrium utility of the strong (weak) bidder. Both coincide with the lower bound utility below m_1.

If the object is valuable enough, and both bidders are likely to be sufficiently rich, the RHR-condition holds on the entire domain: $m_1 = m_2 = \overline{w}$. Figure 2.5 illustrates the equilibrium utility with $v_S = 2$ and $v_W = 3$. For this case, theorem 1 provides a complete characterization equilibrium utilities and bidding strategies. The theorem does not specify the bid and utility at the highest budget $\overline{w}$. Note that any bid below $\overline{w}$ yields strictly weaker payoff than bidding $\overline{w}$. No mass point can arise at $\overline{w}$, as having the highest budget is a zero probability event. Thus, a bidder with budget $\overline{w}$ bids his entire budget and achieves $U_i(\overline{w}) = \underline{U}_i(\overline{w})$.

Observation 6. *The degree of asymmetry is irrelevant for bidding strategies, as long as both RHR-conditions are satisfied.*

Equilibrium strategies are invariant to the degree of asymmetry, as long as both lower bound utilities are strictly increasing. Theorem 2 requires no restrictions on the budget distributions beyond the RHR-conditions. $F_1(w)$ and $F_2(w)$ can be highly asymmetric (e.g. first order stochastic dominance, or RHR dominance), or fully symmetric. The lower bound utilities (the dashed lines in figure 2.4 and 2.5) can be arbitrary distant, or cross multiple times; bidding the entire budget is always the unique equilibrium strategy for interior budgets below m_1.

Note the difference to classical auction theory without budget constraints, where it is optimal to bid the expected value of the second highest bidder, conditional on being the highest. Bidding the expected budget of the second-richest bidder, conditional on having the highest budget, is not a best response in my model: shading the bid below the budget yields such a decrease in winning probability that it does not offset the lower payment.

5.3 Constant lower bounds

If at least one RHR-condition does not hold and at least one $\underline{U}_i$ is constant for w high enough, strategies change abruptly. This occurs with $m_1 < \overline{w}$. Define $b_i^{max} = \sup_{w \in [\underline{w}, \overline{w}]} b_i(w)$ as the highest bid of bidder i in equilibrium under any possible budget realization.

Lemma 3
Let $m_1 < \overline{w}$. Then, the following holds in any equilibrium:

1. *the highest bid of the bidders coincides: $b_{max} = b_1^{max} = b_2^{max}$,*
2. *there exists no mass point at b_{max},*
3. *$m_1 < b_{max} < min(v_1, v_2)$.*

The bid m_1 cannot be the highest bid b_{max}, if $m_1 < \overline{w}$ interior. A positive probability mass of bidders i and j with budget above m_1 never bids below m_1: by theorem 1, any bid below yields a strictly lower payoff than m_1. If m_1 was the highest equilibrium bid, both bidders had a mass point there. Then, both have a profitable deviation to slightly outbid the mass point.[11] Therefore, $b_{max} > m_1$.

Under the naive strategy of the opponent, bidder 1 never bids above m_1: if bidder 2 always bids his entire budget, the winning probability $F_2(b)$ of any bid $b > m_1$ is too low to make such a bid profitable for bidder 1. To induce bidder 1 to bid up to $b_{max} > m_1$, any of his bids $b \in (m_1, b_{max}]$ has to win with a strictly higher probability than under the naive strategy: $G_2(b) > F_2(b)$ for $b \in (m_1, b_{max}]$.

This condition pins down the equilibrium utility of bidder 2: a continuum of bidder 2 types with budget realizations in this interval bids strictly below the budget with some positive probability. This rules out any increase in equilibrium utility $U_2(w)$ for budget realizations within $(m_1, \overline{w}]$, as any increase in equilibrium utility in this interval contradicts the requirement $F_2(b) < G_2(b)$. A similar argument holds for bidder 1:

[11] Two mass points at the same bid cannot occur in the interior of the budget support: both bidders can deviate by bidding slightly above the mass point. This is not the case in the bid cap literature, where two mass points can occur at the highest admissible bid (the bid cap) and therefore the bidding support can have holes: decreasing the bid slightly below the mass point at the cap yields a strictly lower payoff; bidding above is infeasible due to the bid cap. (See e.g. Gavious et al., 2002).

his equilibrium bidding distribution $G_1(w)$ has to induce bidder 2 to bid above m_2 (if $b_{max} > m_2$), which he would never do under the naive strategy of bidder 1. The following lemma formalizes this argument.

Lemma 4
Equilibrium utility is constant at $U_i(w) = v_i - b_{max}$ for budget realizations $w \in (m_j, \overline{w}]$ for $i \in \{1, 2\}$, $i \neq j$. The bidding distributions are $G_i(b) = \frac{v_j - b_{max}}{v_j - b}$ on $b \in (m_i, \overline{w}]$.

Broadly, Lemma 4 states that whenever a bidder's lower bound is constant, his opponent has to have a constant utility for budget realizations within this interval. This pins down the bidding distribution G_i as a function of b_{max} and the opponent's valuation v_j. It is left to elicit the endogenous upper bid limit b_{max}, find the equilibrium bidding distribution G_2 bidder of 2 on (m_1, m_2), and solve what happens at points m_1 and m_2. The following theorem sums up these results; the details are in the appendix.

Theorem 2
The highest bid is

$$b_{max} = \begin{cases} v_1 - (v_1 - m_1)F_2(m_1) & \text{if} \quad \underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2 \\ v_2 - (v_2 - m_2)F_1(m_2) & \text{if} \quad \underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2 \end{cases} \tag{2.5}$$

.

The unique equilibrium bidding distributions are:

$$G_1(b) = \begin{cases} F_1(b) & \forall b \in [\underline{w}, m_1) \\ \frac{v_2 - b_{max}}{v_2 - b} & \forall b \in [m_1, b_{max}] \end{cases} \qquad G_2(b) = \begin{cases} F_2(b) & \forall b \in [\underline{w}, m_1) \\ \frac{(v_1 - m_1)F_2(m_1)}{v_1 - b} & \forall b \in [m_1, m_2) \\ \frac{v_1 - b_{max}}{v_1 - b} & \forall b \in [m_2, b_{max}] \end{cases} \tag{2.6}$$

In lemma 2 I established that any increase in utility beyond the lower bound can only occur due to a mass point. Theorem 2 allows atoms in bidding distributions to exists only at two bids, one per bidder.

Observation 7. *There are at most 2 mass points:*

- *bidder* 1 *admits a mass point at m_1 if $v_2 - b_{max} > \underline{U}_2(m_1)$*
- *bidder* 2 *admits a mass point at m_2 if $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$*

Mass points can occur only at m_1 (in bidder 1's strategy) and m_2 (in bidder 2's strategy). It is always the bidder, whose opponent's RHR drops below the threshold, who places a mass point at that bid. He needs to win with a higher winning probability than under the naive strategy for any higher bid, i.e. he needs to be made at least indifferent between m_i at some higher bids. As this requires his opponent to shade his bid below his budget, his opponent needs to be made indifferent. His indifferent opponent in turn will then employ a bidding strategy that induces the first player to bid above m_i in equilibrium. Moreover, a mass point at b in bidder i's strategy pins down the bidding distribution of the opponent at this point, $G_j(b) = F_j(b)$: poorer bidders j cannot afford to bid at the mass point, richer ones would like to outbid the mass point at least slightly. Only bidder j with budget realization b bids exactly at the mass point. Here, mass points are part of the equilibrium strategy: bidders with a lower budget would like to bid at the mass point of the opponent, but do not have sufficient resources.

The utility for high enough budget realizations is pinned down by theorem 2, as there cannot be a mass point at the highest bid. The highest bid wins with probability one, and yields expected utility $v_i - b_{max}$. Thus, the difference between the equilibrium utilities equals $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$. The difference between the lower bound at the highest budget is $\underline{U}_1(w) - \underline{U}_2(w)$, which is generically not equal to $v_1 - v_2$. The only way to influence the distance between equilibrium utilities is to place one or two mass points at m_1 or m_2. A mass point of bidder 1 at m_1 increases the utility of bidder 2, decreases the difference $U_1(\overline{w}) - U_2(\overline{w})$; a mass point of bidder 2 at m_2 increases the utility of bidder 2 and also increases the difference $U_1(\overline{w}) - U_2(\overline{w})$. As it turns out, there is one unique way to allocate mass points such that the required difference between equilibrium utilities $U_1(\overline{w}) - U_2(\overline{w}) = v_1 - v_2$ is obtained.

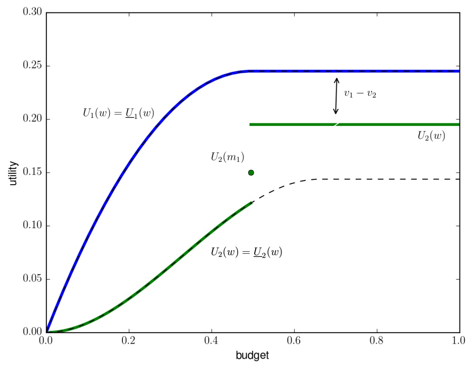


Figure 2.6: Example for $\underline{U}_1(m_1) > \underline{U}_2(m_2)$ and $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$.

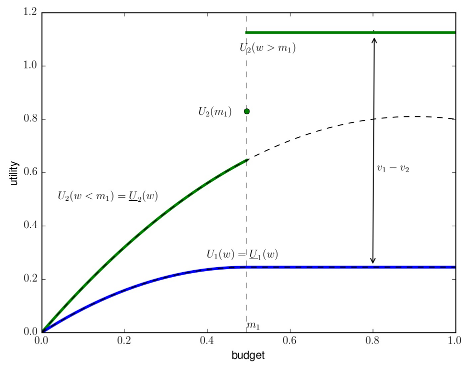
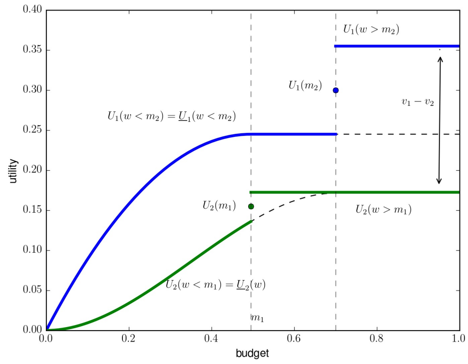


Figure 2.7: Example for $\underline{U}_1(m_1) < \underline{U}_2(m_2)$ and $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$.

By the preceding analysis, equilibrium utility of bidder 1 is constant and equals $v_1 - b_{max}$ for budget realizations in $(m_2, b_{max}]$. Equilibrium utility of bidder 2 equals $v_2 - b_{max}$ for budget realizations in $(m_1, b_{max}]$. The difference between equilibrium utilities in these intervals is $U_1(b_{max}) - U_2(b_{max}) = v_1 - v_2$. The crucial equation is $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$. If this weak inequality holds, the difference between the utilities $U_1(b_{max}) - U_2(b_{max}) = v_1 - v_2$ has to be smaller than the difference between the two auxiliary utilities $\underline{U}_1(m_1) - \underline{U}_2(m_2)$. Figures 2.6 and 2.7 illustrate these cases. Bidder 1 places a mass point at m_1 to decrease the difference in comparison to the lower bound difference. In these cases it is impossible that bidder 2 places a mass point at m_2: the person who places a mass point always achieves at most her lower bound utility at her mass point. This is because his opponent prefers to slightly outbid the mass point and thus, $F_1(m_2) = G_1(m_2)$. Mass point m_2 is the last possibility to change the final equilibrium utilities. If bidder 2 places a mass point at m_2, he achieves a final utility



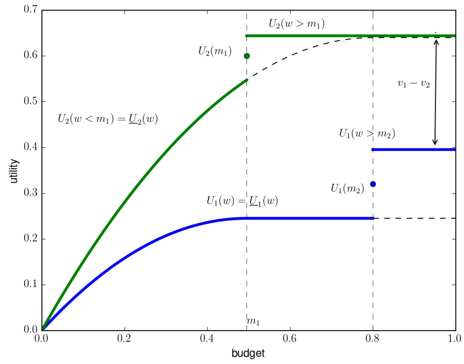


Figure 2.8: Example for $\underline{U}_1(m_1) > \underline{U}_2(m_2)$ and $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$.

Figure 2.9: Example for $\underline{U}_1(m_1) < \underline{U}_2(m_2)$ and $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$.

that equals his lower bound; meanwhile, bidder 1 achieves a utility above his lower bound. These equilibrium utilities would be too far apart, as the distance $v_1 - v_2$ has to be smaller than the difference in lower bounds. Therefore, if $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$, only bidder 1 has a mass point at m_1.

If the weak inequality does not hold ($\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$), the difference between the equilibrium utilities has to be larger then $\underline{U}_1(m_1) - \underline{U}_2(m_2)$. Beyond a mass point from bidder 1 at m_1 to make his opponent indifferent on $(m_1, b_{max}]$, bidder 2 has to place a mass point at bid m_2 to increase the difference in the equilibrium utilities to the required level. These cases are demonstrated in figures 2.8 for $\underline{U}_1(m_1) > \underline{U}_2(m_2)$ and in figure 2.9 for $\underline{U}_1(m_1) < \underline{U}_2(m_2)$.

Figure 2.10 shows the equilibrium utilities for the introductory example with $v_1 = v_2 = 1$ for budget constraints, that satisfy FOSD and with bidder 1 being the strong bidder. In this case, the inequality $\underline{U}_1(m_1) - \underline{U}_2(m_2) > v_1 - v_2 = 0$ is strict. As $v_1 = v_2$, the difference between the final utilities has to be zero: $U_1(b_{max}) - U_2(b_{max}) = v_1 - v_2 = 0$. By the above argument, bidder 1 places a mass point at m_1 such that the equilibrium utilities coincide for high enough budget realizations. From m_1 onwards, both bidders follow the same bidding distributions and therefore, achieve the same utility: the lower bound utility of the relatively stronger bidder. Therefore, the weaker bidder achieves the same utility as his stronger opponent; the lower bound does not necessarily bind for high enough budget realizations in equilibrium.

Figure 2.11 shows the equilibrium utilities for the case of Che and Gale (1996) with symmetric budget distributions and equal valuations for the case where $m_1 < \overline{w}$. As $v_1 - v_2 = 0$, equilibrium utilities of bidding the highest bid have to overlap. Moreover, $m_1 = m_2$ due to symmetry in budget distributions, as the lower bound utilities overlap. Having two mass points at $m_1 = m_2$ is impossible: both bidders would have a deviation. If there exists only one mass point at m_1, there necessarily arises a difference in equilibrium utilities for the highest bid, which cannot occur. Thus, no mass point can occur, and the lower bound utility is binding for the equilibrium utility on the entire budget domain. Che and Gale (1996) arrive at this conclusion

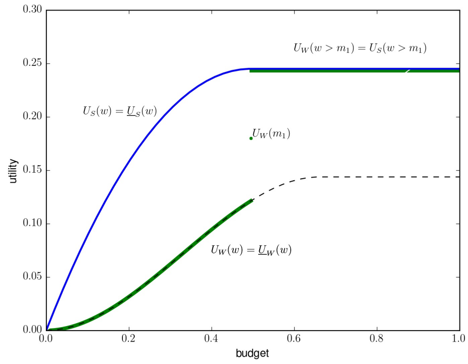


Figure 2.10: Equilibrium utilities for the introductory example with FOSD in budget distributions.

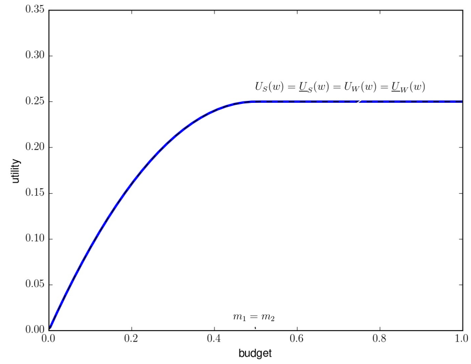


Figure 2.11: Equilibrium utilities for $v_1 = v_2$ and symmetric budgets.

by considering the class of symmetric and strictly monotonic equilibria. Under the additional restriction of log-concavity[12], my findings show that there exists no other asymmetric equilibrium utility; the strictly monotonic bidding strategies in Che and Gale (1996) satisfy the bidding distributions derived in theorem 2. My findings show, that there exist a multiplicity of other equilibria on indifference regions if $m_1 < \overline{w}$, that all satisfy these bidding distributions. I demonstrated that under asymmetric budgets and different valuations mass points arise: a strictly monotonic bidding equilibrium does not exist generically.

In the next section 6 I derive weakly monotonic pure strategy bidding functions, that satisfy above theorem and establish existence of the unique equilibrium shape of this section. The bidding distributions in theorem 2 do not allow to compare the bids of two bidders, which are endowed with the same budget; bidding strategies can be in mixed strategies. Moreover, a direct comparison of bidding aggression here is not possible, as the bidding distributions in theorem 2 combine the bids for each budget with the probability of these budgets being drawn. This cloaks the quantitative shading of the bids below the budgets.

6. Monotonic Equilibrium

Theorem 2 shows uniqueness of bidding distributions $\{G_i\}_{i \in 1,2}$ that any equilibrium necessarily satisfies. The following section establishes existence of such an equilibrium: I derive pure strategy weakly monotonic bidding functions, that are feasible and optimal for the bidders, and satisfy the unique bidding distributions of theorem 2. This allows me to compare bidding aggression for the special case of RHR-dominance in the next section.

[12] Che and Gale (1996) do not require the restriction of log-concavity. I impose log-concavity to restrict the number of cases to consider for the lower bound utilities. Some of my results are applicable without assuming log-concavity, like theorem 1.

6.1 Existence of a monotonic equilibrium

If $m_1 = \overline{w}$, every bidder bids his entire budget for every budget realization. Bidding strategies are unique[13]. If $m_1 < \overline{w}$, bidding strategies are not unique anymore. Both bidders are indifferent between bids in some non-empty interval. In the example in section 3, both bidders are indifferent between any bid in the interval $(\frac{1}{2}, \frac{3}{4}]$. Note that all bidders with a budget in $(\frac{3}{4}, 1]$ are bidding strictly below their budget and are indifferent for any bid in their bidding support. For them, an infinite amount of different equilibrium bidding strategies can be constructed, which all satisfy the same bidding distribution.

The following lemma establishes pure strategy weakly monotonic bidding functions, that satisfy bidding distributions $\{G_i\}_{i\in 1,2}$ from theorem 2.

Proposition 1
Let the budget distributions $\{F_i(w)\}_{i=1,2}$ be log-concave on $(\underline{w}, \overline{w})$. Let v_1, v_2 be common knowledge and $\underline{w} < \min\{v_1, v_2\}$. Then, a pure strategy weakly monotonic equilibrium exists.

Proof. The proof is by construction. I show that the following is a pure strategy equilibrium in weakly monotonic bidding functions.

Case I: let $\underline{U}_1(m_1) - \underline{U}_2(m_2) \geq v_1 - v_2$. Then, $b_{max} = v_1 - (v_1 - m_1)F_2(m_1)$ and the pure strategy monotonic bidding functions are:

$$b_1(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ m_1 & \text{if} \quad w \in [m_1, F_1^{-1}(\frac{v_2 - b_{max}}{v_2 - m_1})], \\ v_2 - \frac{v_2 - b_{max}}{F_1(w)} & \text{otherwise.} \end{cases} \tag{2.7}$$

$$b_2(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ v_1 - \frac{(v_1 - m_1)F_2(m_1)}{F_2(w)} & \text{otherwise.} \end{cases} \tag{2.8}$$

Case II: let $\underline{U}_1(m_1) - \underline{U}_2(m_2) < v_1 - v_2$. Then, $b_{max} = v_2 - (v_2 - m_2)F_1(m_2)$ and pure strategy monotonic bidding functions are:

$$b_1(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ m_1 & \text{if} \quad w \in [m_1, F_1^{-1}(\frac{(v_2 - m_2)F_1(m_2)}{v_2 - m_1})], \\ v_2 - \frac{(v_2 - m_2)F_1(m_2)}{F_1(w)} & \text{otherwise.} \end{cases} \tag{2.9}$$

$$b_2(w) = \begin{cases} w & \text{if} \quad w \in [\underline{w}, m_1), \\ v_1 - \frac{(v_1 - m_1)F_2(m_1)}{F_2(w)} & \text{if} \quad w \in [m_1, F_2^{-1}(\frac{(v_1 - m_1)F_2(m_1)}{v_1 - m_2})), \\ m_2 & \text{if} \quad w \in [F_2^{-1}(\frac{(v_1 - m_1)F_2(m_1)}{v_1 - m_2}), F_2^{-1}(\frac{v_1 - b_{max}}{v_1 - m_2})], \\ v_1 - \frac{v_1 - b_{max}}{F_2(w)} & \text{otherwise.} \end{cases} \tag{2.10}$$

[13] Only the bid of bidders with budget $\underline{w}$ are not uniquely pinned down, but assumed to be equal to the full budget $b(\underline{w}) = \underline{w}$.

First, I show that above bidding functions *satisfy the unique bidding distributions* $\{G_i\}_{i=1,2}$ in theorem 2. For $w < m_1$, $b_i(w) = w$, and $G_i(b) = F_i(b)$ is trivially satisfied. Whenever a bidder with budget realization w is not bidding at a mass point (i.e. whenever bidder i is not bidding m_i), bids are strictly increasing an the bidding function is invertible in this interval. Denote the inverse function as $w_i(b)$, whose output is the budget realization of a bidder i who bids b. Note that $F_i(w_i(b)) = G_i(b)$. Consider for example bidder 2 in equation 2.8 with a budget above m_1. His bidding function $b_2(w)$ can be rewritten using the inverse bidding function $w_2(b)$: $b = v_1 - \frac{(v_1-m_1)F_2(m_1)}{G_2(b)}$. Solving this expression for $G_2(b)$ immediately yields the unique bidding distribution of theorem 2. The same argument applied for the other bidder yields the required unique bidding distributions G_i in all cases.

Next, I show *feasibility* of the bidding functions. This requires to show that $b_i(w) \leq w$ for all w and all i. For any bid equal or below m_1, feasibility is trivially satisfies. It is left to show that 1. $v_i - \frac{(v_i-m_i)F_j(m_i)}{F_j(w)} \leq w$; 2. $v_i - \frac{v_i-b_{max}}{F_j(w)} \leq w$; 3. $F_2^{-1}(\frac{(v_1-m_1)F_2(m_1)}{v_1-m_2}) \geq m_2$, so bidder 2 can afford a bid m_2 in equation 2.10. Rewrite inequality 1. as $(v_i - w)F_j(w) \leq (v_i - m_i)F_j(m_i)$. Note that $(v_i - w)F_j(w) \leq \underline{U}_i(w) \leq \underline{U}_i(m_i) = (v_i - m_i)F_j(m_i)$ establishes feasibility. Rewrite inequality 2. as $(v_i - w)F_j(w) \leq v_i - b_{max}$. Note that $(v_i - w)F_j(w) \leq \underline{U}_i(w) \leq \underline{U}_i(\overline{w}) = v_i - b_{max}$ establishes feasibility. Apply $F_2(.)$ to both sides of equation 3. This yields inequality $(v_1 - m_1)F_2(m_1) \geq (v_1 - m_2)F_2(m_2)$. Note that $(v_1 - m_1)F_2(m_1) = \underline{U}_1(m_1) = \underline{U}_1(m_2) > (v_1 - m_2)F_2(m_2)$ establishes feasibility.

Finally, I show *optimality* of the above bidding functions for the bidders. Any bidder with a budget below m_1 bids his entire budget, as any strictly lower bid yields lower payoff ($\underline{U}_i(w) < \underline{U}_i(w')$ for $w < w' < m_1$); any higher bid above the budget is unfeasible. Consider case I. This corresponds to figures 2.6 and 2.7. Any bid of bidder 1 in the interval $[m_1, b_{max}]$ yields constant utility to bidder 1. It is straightforward to show that any bid above b_{max} or below m_1 yields strictly lower utility - there is no profitable deviation. A similar argument holds for bidder 2: a bidder 2 with budget m_1 has a higher utility from bidding exactly at m_1 due to a mass point of bidder 1 than from any lower bid, and all other higher deviations are unfeasible for him. A bidder with a budget strictly above m_1 is indifferent between any bid on $(m_1, b_{max}]$, and strictly looses from any deviation up or down. Similar straightforward computations yield the optimality of case II.

□

It is straightforward to show, that these bidding functions combined with the budget distributions $F_i(.)$ yield the bidding distribution functions $G_i(.)$ in theorem 2. Optimality is satisfied: bidder i with a budget in $[\underline{w}, m_1)$ would like to bid more, but cannot afford it. Deviating below his budget yields a strictly lower payoff. His downward incentive compatibility (IC) constraints are slack, upward IC constraints do not exist due to feasibility.

6.2 Bidding aggression

The pure strategy monotonic bidding functions in section 6.1 allow a direct comparison in bidding behavior. Which bidder bids more aggressively if both have the same budget?

When comparing bidding aggression in my model, there are two channels of interest. First, how does bidding aggression depend on the budget distribution? Second, how does bidding aggression depend on the valuation for the object?

As theorem 2 shows, bidders with budget realization in $[\underline{w}, m_1)$ always bid their entire budget and are equally aggressive, irrespectively of any order statistic assumption on their budget distributions. If $m_1 = \overline{w}$, under any order statistic, both bidders bid equally aggressive on the entire budget support in my framework.

Definition 8. *F_i dominates F_j in terms of the reverse hazard rate (RHR) if*

$$\frac{f_i(x)}{F_i(x)} \geq \frac{f_j(x)}{F_j(x)}, \quad \forall x \in (\underline{w}, \overline{w}).$$

In the next result, I assume that both bidder have the same valuation v for the object, and one bidder dominates his opponent in terms of reverse hazard rates in the budget distribution. This allows me to elicit the differences in bidding aggression that are only due to differences in the budget distributions, not due to heterogeneity in valuations.

Proposition 2
Let $v_1 = v_2 = v$ and F_i be RHR-dominant over F_j. Then, the dominant bidder bids less aggressively: $b_i(w) \leq b_j(w)$.

Proof. Note that if bidder i RHR-dominates bidder j, it holds that $i = 1$ because $m_1 \leq m_2$. This is because the RHR-condition of the dominant bidder i is stricter than the RHR-condition of the dominated bidder. Therefore, it holds that $\underline{U}_1(m_1) \geq \underline{U}_2(m_2)$, which is case I in proposition 1. Then, the highest bid is $b_{max} = v - (v - m_1)F_2(m_1)$ and using the fact that RHR-dominance implies FOSD, the bidding strategies in the proof of proposition 1 imply that $b_1(w) \leq b_2(w)$ for all $w \in [\underline{2}, \overline{w}]$. □

Maskin and Riley (2000) target a related question for asymmetrically distributed valuations and without budget constraints. They consider a variant of the RHR-dominance on valuation distributions. Maskin and Riley (2000) show that if both bidders have the same valuation, the RHR-dominated bidder bids more aggressively (i.e. higher). This is in line with the findings of this paper on asymmetrically distributed budgets: the weaker bidder in the sense of RHR on budgets bids more aggressively.

The previous findings compared bidding behavior for bidders with equal valuations and distinct bidding distributions. Next, I compare bidding behavior of the diametrical case, bidders with identical budget distributions $F(w) := F_1(w) = F_2(w)$ but different valuations.

Let $v_i > v_j$. Then, the RHR-condition is satisfied for bidder i whenever the RHR-condition is satisfied for bidder j, because $\frac{1}{v_i - w} < \frac{1}{v_j - w}$ for all $w \in (\underline{w}, \min\{v_j, \overline{w}\})$. Therefore, the lower bound of bidder j is the first one to have a kink at $m_j \leq m_i$. Therefore, we have $j = 1$ and $v_2 > v_1$. An example for this case is depicted in figure 2.7. Due to the higher valuation, the lower bound of bidder i is always strictly above the lower bound of bidder j.

Proposition 3
Let $v_i > v_j$ and $F_i(w) = F_j(w)$. Then, $b_i(w) \geq b_j(w)$ for all $w \in [\underline{w}, \overline{w}]$.

Proof. Let $v_2 > v_1$ and $F(w) := F_1(w) = F_2(w)$. Using the fact that $\underline{U}_1(m_1) = \underline{U}_1(m_2) \geq (v_1 - m_2)F(m_2)$, we have:

$$\begin{aligned}\underline{U}_2(m_2) - \underline{U}_1(m_1) &= \underline{U}_2(m_2) - \underline{U}_1(m_2) \\ &\leq (v_2 - m_2)F(m_2) - (v_1 - m_2)F(m_2) \\ &= (v_2 - v_1)F(m_2) \leq v_2 - v_1.\end{aligned}$$

Note that this corresponds to case I of the weakly monotonic bidding strategies in the proof of lemma 1. Using the pure monotonic bidding strategies of case I, it immediately follows that $b_2(w) \geq b_1(w)$ for $w \in [\underline{w}, F^{-1}\left(\frac{v_2 - b_{max}}{v_2 - m_1}\right))$. For $w > F^{-1}\left(\frac{v_2 - b_{max}}{v_2 - m_1}\right)$, we have $b_2(w) \geq b_1(w)$ because the following inequality holds:

$$\begin{aligned} b_1(w) &\leq b_2(w) \\ \Leftrightarrow v_2 - \frac{v_2 - b_{max}}{F(w)} &\leq v_1 \frac{(v_1 - m_1)F(m_1)}{F(w)} \\ \Leftrightarrow v_1(1 - F(w)) &\leq v_2(1 - F(w)) \end{aligned}$$

The last line of the inequality holds by assumption of $v_1 < v_2$.

□

As before, bidders are investing their entire budget up to a certain budget m_1. For any higher budget, the bidder who values the object more, bids more aggressively.

7. Extensions

7.1 Revenue comparison

Revenue equivalence between standard auctions does not hold when bidders are budget constrained, as noted by Che and Gale (1996)[14] and Che and Gale (1998)[15]. Consider a framework where values are identical and public ($v := v_1 = v_2$). If budgets are drawn from a symmetric distribution ($F(w) := F_1(w) = F_2(w)$), Che and Gale (1996) proved, that the FPA dominates the second price auction (SPA) with regards to revenue.

I show that this revenue ranking is not robust under asymmetric budget distributions ($F_1 \neq F_2$): the SPA can perform strictly better than the FPA for sufficiently asymmetric bidders. First, consider bidding strategies in a SPA without reservation values.

Proposition 4
Let $v := v_1 = v_2$ be public information, and budgets distributions governed by cdf $F_1(w)$ and $F_2(w)$. In a SPA without a reservation price, it is a weakly dominant strategy to bid $b_i(w) = \min\{v, w\}$, $\forall i \in \{1, 2\}, \forall w \in [\underline{w}, \overline{w}]$.

[14]Che and Gale (1996) show that the all pay auction dominates the FPA in terms of revenue, with n bidders who all share the same valuation v and have identical budget distributions.

[15]Che and Gale (1998) analyze standard auctions, where budgets and valuations are both private information and symmetrically distributed. They show that the all pay auction dominates the FPA, which itself dominates the second price auction (SPA) in terms of expected revenue for the auctioneer.

Proof. Consider bidder i, who is facing a bid b_j of his opponent and has budget w. Let $v \leq w$. Then, the classical argument of the SPA applies: bidding less ($b_i < v$) potentially looses the auction and forgoes a positive payoff, and changes nothing in case of a win. Bidding higher ($b_i > v$) only changes the outcome if it results in purchasing the object for more than v, and thus, a negative payoff. Let $w < v$. Then, bidding higher than w is infeasible. The only possible deviation is downward, $b_i < w$. However, this is not profitable, because it only changes the outcome if $w > b_j \geq b_i$. In this case the object is lost, while bidding $b_i = w$ would have resulted in strictly positive payoff $v - b_j$. □

Whenever both bidders have a budget above v, the auctioneer gets a payment of the full object value v. Whenever at least one bidder has a budget below the object value, the payoff of the seller is the lowest of the two budgets.

Let $x := \min\{v, \overline{w}\}$ be the highest possible bid under any budget realization. Above bidding strategies in proposition 4 result in the following expected revenue for the designer, where the last line follows by applying integration by parts:

$$\Pi^{SPA} = \int_{\underline{w}}^{x} w\,(f_2(s)(1 - F_1(s) + f_1(s)(1 - F_2(s)))\,dw + x(1 - F_1(x))(1 - F_2(x)) \tag{2.11}$$

$$= \underline{w} + \int_{\underline{w}}^{x} (1 - F_1(s))(1 - F_2(s))ds \tag{2.12}$$

Now consider revenue in a FPA. Because $v := v_1 = v_2$, the auction is always efficient and the total generated surplus is v. The object is always sold, and utilities are linear in the payment. Therefore, the revenue of the seller is the object value v minus the expected utilities of the bidders. That is,

$$\Pi^{FPA} = v - \int_{\underline{w}}^{\overline{w}} U_1(w)dF_1(w) - \int_{\underline{w}}^{\overline{w}} U_2(w)dF_2(w). \tag{2.13}$$

The next proposition shows that the revenue ranking of Che and Gale (1996) does not extend to asymmetric budget distributions.

Proposition 5
Let $v_1 = v_2 = v$ and budgets be drawn with possibly different distributions $F_1(w)$ and $F_2(w)$. Then, the SPA can yield strictly higher revenue than the FPA.

Proof. The proof is by counterexample. Let $w \in [0, 1]$, $F_1(w) = w^9$, $F_2(w) = w^{\frac{1}{9}}$, and both bidders have valuation $v = 0.2$. Note that then, $m_1 = \frac{2}{110}$, and $m_2 = \frac{9}{50}$.

Plugging this into the expected revenue equation 2.12 of the SPA yields approximately $\Pi^{SPA} \approx 0.0494748$.

For the FPA, note that the ex-ante utilities of both bidders are (we are in case I):

$$U_1 = \int_{\underline{w}}^{m_1} (v - w)F_2(w)f_1(w)dw + (v - m_1)F_2(m_1)(1 - F_1(m_1)) \tag{2.14}$$

$$U_2 = \int_{\underline{w}}^{m_1} (v - w)F_2(w)f_1(w)dw + (v - m_1)F_2(m_1)(1 - F_1(m_1)) \tag{2.15}$$

After repeated integration by parts and algebraic manipulations, the expected revenue from the FPA with asymmetric bidders for this numerical example is:

$$\Pi^{FPA} = v - (v - m_1)\left[2F_2(m_1) - F_2(m_1)^2\right] - \int_{\underline{w}}^{m_1} F_1(w)F_2(w)dw \tag{2.16}$$

$$\approx 0.0416594. \tag{2.17}$$

Thus, $\Delta = \Pi^{FPA} - \Pi^{SPA} \approx 0.0416594 - 0.0494748 < 0$. □

In the literature on standard auctions without budget constraints, asymmetrically distributed valuations break revenue equivalence between standard auctions (Maskin and Riley, 2000). A revenue ranking between standard auctions remains an object of investigation, as no general revenue ranking can be established. For some particular distributions, revenue from a FPA is higher than from a SPA (see e.g. Maskin and Riley, 2000). This ranking does not always hold, as Gavious and Minchuk (2014) show that revenue from a SPA can be higher than from a FPA under asymmetry.

With asymmetric budget constraints, but common valuations, I showed that the revenue ranking $\Pi^{FPA} \geq \Pi^{SPA}$ does not hold anymore. It remains an open question under which conditions the FPA perform better than the SPA in a framework with asymmetric budget constraints. Yet, finding a revenue ranking in this framework for particular asymmetric budget constraints might turn out more practical than for asymmetric valuations, because this paper provides a closed form expression for revenue and bidding behavior.

7.2 Information disclosure about ex-ante symmetric bidders

In the following, I allow the auctioneer to endogenize part of the information structure. The auctioneer has the choice whether to disclose the identities of the bidders, e.g. by publishing a participation register. In the following, I again assume that the object is worth the same, v, to every bidder.

Giving up anonymity of the bidders is a relevant strategic decision for the designer. If the auctioneer discloses nothing, bidders are ex-ante symmetric in the sense that their distribution is drawn from the same prior distribution. If the auctioneer publishes a public participation register, bidders can look up annual budget reports and make inference about the budget distributions of the opponents. I show that with ex-ante symmetric bidders, the auctioneer can never gain by disclosing noisy information about the budgets. Note that total surplus in the auction is v. Maximizing the expected revenue of the auctioneer corresponds to minimizing the utilities of both bidders.

Let S be the finite set of budget type distributions, with each $s \in S$ corresponding to a log-concave budget distribution function $F_s(w)$ on equal support $[\underline{w}, \overline{w}]$. The term 'type' in this section refers to the distribution function of budgets, not the budget realization as in the last section. The budget distribution types of bidder 1 and 2 are drawn independently and identically from S, with probability p_s of being type s. Probabilities are non-negative and $\sum_{s \in S} p_s = 1$.

The timing is the following: before the start of the auction, the auctioneer commits whether she wants to publish a participation register. Then, bidders arrive and budget types $F_i(.) \in \{F_s(.)\}_{s\in S}$ are drawn for $i = \{1, 2\}$. Bidders know their own type, but not the type of their opponent. The auctioneer observes both types and publicly announces the types, if she committed to do so. Then, budgets are drawn and observed only by the respective firm. Finally, a FPA takes place.

Lemma 5
Revenue is weakly decreasing, if the auctioneer discloses budget type information about ex-ante symmetric bidders.

Proof.

$$\begin{aligned} U_i(w) = \underline{U}_i(w) &= \max_{b\leq w}(v-b)\sum_{s\in S} p_s F_s(w) & (2.18)\\ &\leq \sum_{s\in S}\max_{b_s\leq w} p_s(v-b_s)F_s(b_s) = \sum_{s\in S} p_s \underline{U}_{i,j=s}(w) & (2.19)\\ &\leq \sum_{s\in S} p_s U_{i,j=s}(w) = E_s(U_{i,j=s}(w)) & (2.20)\end{aligned}$$

□

The total surplus of the auctioneer and the bidders equals to v. Hence, a higher expected utility for bidders corresponds to a lower payoff for the auctioneer. Under disclosure, bidders condition their optimal action upon their opponent's types. With ex-ante symmetric bidders, the lower bound on equilibrium utility always binds (Che and Gale, 1996). The expected lower bound under information disclosure is weakly larger (due to the max-operator) than under no disclosure. Therefore, under information disclosure, even the worst-case scenario for the bidders is weakly better than no disclosure. In section 5 I show, that under sufficient asymmetry, the lower bound is not necessarily binding. Thus, under information disclosure, bidders could be strictly better off than under no information due to an equilibrium above the lower bound, which further decreases the expected revenue for the auctioneer.

In many auction houses, like Sotheby's, bidding is anonymous: bidders take part in an auction, before knowing who their opponents will be. Moreover, during the auction, bidders remain anonymous by placing bids via phone or by raising one's hand. For narrow markets like the telecommunication sector, while usually participants are announced before the start of the auction, this in fact might not constitute a strategic decision of the auction designer but rather a peculiarity of the respective market: anonymity might not be implementable in such a narrow market with few constantly interacting participants.

In this section, I analyzed a very specific information disclosure rule: the auctioneer has the choice whether she wants bidders to remain symmetric, or reveal noisy information about the budget distributions. However, this noisy information is exogenously given and the auctioneer cannot modify its precision or send private and potentially correlated signals. Future research could endogenize the information structure even further by allowing the auctioneer to design the signal precision as in Bergemann and Pesendorfer (2007), in line with the expanding literature on Bayesian persuasion (Kamenica and Gentzkow, 2011). Lemma 5 still holds, if type space S is not given, but designed by the auctioneer; however, any type in S has to satisfy log-concavity and have a strictly positive density on the same support as all other types: $[\underline{w}, \overline{w}]$. Enabling the designer to create types with different support (e.g. by allowing monotone partition of the budget space[16] into a low-budget and a high-budget type interval) will yield further insights about the optimal information disclosure policy.

7.3 At least one bidder with unconstrained liquidity

So far, I assumed that both bidder are constrained with non-zero probability. That is, $\min(v_1, v_2) > \underline{w}$. In this section, I derive an equilibrium when this assumption does not hold: there exists at least one bidder who is not constrained under any budget realization.

Relabel bidder 1 and 2 without loss of generality such that $v_1 \leq v_2$. Let $v_1 \leq \underline{w}$ such that bidder 1 is unconstrained due to his low valuation. First, consider the case $v_1 = v_2 \leq \underline{w}$. This is equivalent to the classic Bertrand competition with unit demand and identical marginal costs. The unique equilibrium is for both bidders to bid at $v_1 = v_2$, and it is in weakly dominated strategies.

Consider $v_1 \neq v_2$.The following lemma describes equilibria in a FPA for two bidders and common knowledge valuations. It follows with slight modification from Blume (2003), who analyze the equilibria in Bertrand competition with heterogeneous marginal costs.

Proposition 6
Let $v_1 < v_2$ and $v_1 \leq \underline{w}$. The following is an equilibrium. Bidder 2 wins the object and places all mass on the same bid b_2^*, where $b_2^* \in [v_1, v_2)$ for $v_2 \leq \underline{w}$, and $b_2^* \in [v_1, \underline{w}]$ for $v_2 > \underline{w}$. Bidder 1 mixes uniformly on $[b_2^* - \eta, b_2^*]$ for $\eta > 0$ small enough.

As soon as one bidder is never budget constrained due to a lower valuation, the financial situation of the other bidder becomes irrelevant for the identity of the winner. Under strict inequality $v_1 < v_2$, a continuum of equilibria with the above structure arises. In each of those equilibria, the bidder with the higher valuation always wins the object with certainty. A FPA is an efficient mechanism in this framework.

Note that bidder 2 bidding exactly at the valuation of his opponent $b_2^* = v_1$ is the only equilibrium in weakly undominated[17] strategies (see Blume, 2003; Kartik, 2011, for this argument in the context of Bertrand competition).

8. Conclusion

I derive equilibrium utilities and bidding distributions for two asymmetrically budget constrained bidders, who compete for an object in a first price auction. Hereby, I allow for any form of asymmetry under the restriction of log-concavity and common full support on budget distributions.

[16] See e.g. Bergemann and Pesendorfer (2007) for disclosing information about valuations, not budgets, in auctions, where monotone partitions arise as part of the optimal disclosure policy.

[17] Kartik (2011) shows for Bertrand competition, that in any Nash equilibrium in weakly undominated strategies, the firm with the lower marginal costs (here: bidder 2) serves the whole market (here: wins the object) at a price equal exactly to the cost of the competitor (here: via bidding v_1).

Che and Gale (1996) show that in a symmetric equilibrium with identically distributed budgets, the equilibrium utilities of the bidders are pinned down by a lower bound on utility. This bound is the highest utility a bidder can achieve, conditional on his opponent following a naive strategy of always bidding his entire budget. I extend the framework of Che and Gale (1996) in two directions: I allow for different valuations for the object, and introduce asymmetric budget constraints. In this framework, the lower bound does not necessarily bind. However, the equilibrium utilities can still be recovered via using the lower bound. I characterize the entire set of equilibria for this class of auctions, without a restriction on symmetric equilibria. My results show that no further asymmetric equilibria exists in the framework of Che and Gale (1996) under the additional assumption of log-concavity.

As long as both RHR are sufficiently high in relation to the value of the object for sale, equilibrium strategies are completely invariant to the degree of asymmetry. The lower bound always binds; bidders bid their entire budget in every Nash equilibrium. If at least one RHR falls below some threshold, lower bounds do not bind anymore. Bidders can achieve a utility strictly above their lower bound. Mass points arise in bidding distribution.

My approach unravels the equilibrium via eliminating any candidate equilibrium shape until only one is left. Mass points in equilibrium strategies are only possible at two points: whenever the RHR-condition holds with equality. Moreover, two bidders cannot have mass points at the same bid. Therefore, there cannot be a mass pint at the highest possible bid. Because the highest bid wins with probability 1, this pins down the utility from the highest bid. In any equilibrium, utility cannot be strictly increasing and be strictly above the lower bound in some interval. I show that there remains only one potential shape for the equilibrium utility left, that satisfies above properties.

I show that the unique remaining equilibrium always exists, as there exist corresponding feasible and optimal weakly monotonic bidding strategies that yield its shape.

A general revenue ranking between the FPA and the SPA with asymmetric budget constraints does not exist. With symmetric budgets, Che and Gale (1996) show that the FPA yields higher revenue than the SPA. I show that with asymmetric budget constraints, the SPA can perform strictly better with regards to revenue than the FPA.

There exists a parallel between my framework and bidders with asymmetrically distributed valuations without budget constraints. If one considers bidders to be asymmetric in the budget distributions and not in the valuation distributions, the problem is solvable under much less restrictive assumptions (log-concavity, same full support), as my results show. I impose no stochastic order or particular distribution on budgets. Under the assumption of RHR-dominance on valuations, Maskin and Riley (2000) show that the weaker bidder bids more aggressively. In my model, under RHR-dominance on budget distributions, the weaker bidder with respect to the budget also bids more aggressively than his stronger opponent.

A. Appendix

Proof of lemma 1. Bidder i has a fixed valuation v_i for the object, irrespective of his budget realization. Thus, for any budget realization, the same feasible bid b always yields the same utility $(v_i - b)\Pr(b_j < b) + \frac{1}{2}(v_i - b)\Pr(b_j = b)$ to bidder i. Therefore, each bidder with a budget realization in (w', w'') has to submit a different bid: per assumption, $U_i(w)$ is strictly increasing. Every budget realization corresponds to a different utility and thus, a different bid.

The following argument establishes that these different bids are uniquely pinned down and equal to the full budget. By contradiction, let bidder i with a budget $\tilde{w} \in (w', w'')$ place a bid below his budget with positive probability: $b_i(\tilde{w}) < \tilde{w}$. The expected utility of this bid is higher than any bid of all types with a lower budget than him, because $U_i(w)$ is strictly increasing. Then, any bidder i with a budget within $[b_i(\tilde{w}), \tilde{w})$ has a profitable deviation: mimicking the $\tilde{w}$-budget type i and bidding $b_i(\tilde{w})$, which is a feasible bid. Therefore, a strictly increasing utility $U_i(w)$ and any bid strictly below the budget on some interval lead to a contradiction: poorer bidders can mimic the behavior of bidders with a strictly higher budget. □

Proof of lemma 2. Assume by contradiction that there exists a $w' \in (\underline{w}, \overline{w})$ such that $U_i(w') > \underline{U}_i(w')$ and $U_i(w)$ is neither discontinuous at w' nor constant in any ϵ-ball around w'. Then, there exists some interval $[w', w' + \epsilon)$ or $(w' - \epsilon, w']$ for some $\epsilon > 0$ sufficiently small where $U_i(.)$ is strictly increasing and continuous. Without loss, assume that this interval is $[w', w' + \epsilon)$.

By lemma 1, $U_i(.)$ strictly increasing implies that all i-types are bidding their entire budget $b_i(w) = w$ on the open interval $(w', w' + \epsilon)$ where expected utility is strictly increasing. The bidding distribution and budget distribution for bidder i coincide on this interval, i.e. $G_i(.) = F_i(.)$. This is because all bidders with a budget lower than in the interval cannot afford to bid in $(w', w' + \epsilon)$, and every bidder i with a higher budget has to bid something else because the utility $U_i(.)$ is non-decreasing[18].

Note that because utility is continuous in $(w', w' + \epsilon)$, this rules out atoms in the bidding distribution of bidder j due to the equal tie-breaking rule. As bidder i achieves a utility strictly above the lower bound, we have for all $w \in (w', w' + \epsilon)$

$$U_i(w) = (v_i - w)G_j(w) > \underline{U}_i(w) \geq (v_i - w)F_j(w).$$

Therefore, we need $G_j(w) > F_j(w)$ for all $w \in (w', w' + \epsilon)$. Some j-types have to bid strictly below their budget in order for a bid w of bidder i to win with a strictly higher probability than under the naive strategy. I now show that this requirement leads to a contradiction.

There exists some open interval $(a, b) \subseteq (w', w' + \epsilon)$, where the expected equilibrium utility of the opponent $U_j(w)$ is either constant or strictly increasing and continuous; this holds because the monotonic function $U_j(w)$ can have only countable discontinuities on any open interval. First, assume there exists an interval (a, b) where $U_j(w)$ is strictly increasing. By the same argument as for bidder i, j-types with a budget within this

[18]This is because bidder i always gets the same expected utility from placing a bid, irrespective of his budget realization.

interval bid their entire budget, $b_j(w) = w$. Hence, bidding and budget distributions coincide: $F_j(w) = G_j(w)$ for all $w \in (a, b)$. This contradicts the requirement $F_j(w) < G_j(w)$ for all $w \in (w', w + \epsilon)$ required by bidder i to achieve a utility strictly above his lower bound.

Second, assume there exists an interval (a, b) where $U_j(w)$ is constant. As $U_j(w)$ constant on (a, b), any bid $w \in (a, b)$ in this interval yields the same constant expected utility $U = U_j(w) = (v_i - w)G_i(w) = (v_i - w)F_i(w)$, due to bidder i's bidding distribution $G_i(w) = F_i(w)$ established above. However, this contradicts the log-concavity assumption on $F_i(w)$, which implies that the function $(v_i - w)F_i(w)$ is never constant on any interval. Thus, a strictly increasing continuous utility strictly above the lower bound is impossible.

I established that whenever equilibrium utility exceeds the lower bound, it is either constant, or discontinuous due to a mass point. It is left show, that in case of a discontinuity at budget w' of bidder i in equilibrium, the mass point of bidder j occurs exactly at the bid w'. Assume by contradiction that the mass point is at $y < w$[19]. As the discontinuity arises at budget w and utility is non-decreasing, no bidder with a budget below w can achieve the same utility; i.e. $U_i(y) < U_i(w)$. However, then all bidder i types with budget within $[y, w)$ have a profitable deviation by mimicking the w-type and bidding y . □

Proof of theorem 1. First, I establish that for the lowest possible budget $\underline{w}$ it always holds that the lower bound binds: $\underline{U}_i(\underline{w}) = U_i(\underline{w})$. Assume that were not true: for some $i \in \{1, 2\}$, we have $\underline{U}_i(\underline{w}) = 0 < U_i(\underline{w})$. Let a bid $b \leq \underline{w}$ be the infimum bid in the entire bidding support of the i-bidder with any budget realization. Because bidder i with any budget realization has a strictly positive utility [20], any bid wins with strictly positive probability. This requires either bidder j to place a mass point at b, or bid with a strictly positive probability below b. The latter is impossible in equilibrium, as it requires some bidder j with a budget above $\underline{w}$ to bid below b. Each such bid of j strictly below b results in an expected utility of zero (it never wins), which is below the lower bound $\underline{U}_j$ and thus, impossible. The former (a mass point at b) cannot be part of an equilibrium, because bidder i would always want to slightly outbid the mass point if feasible to get the discrete jump in winning probability. This implies that $F_i(b) = G_i(b) = 0$. The positive mass of j-bidders who bid at the mass point has utility of zero, again, strictly below the lower bound $\underline{U}_j$ of some, being impossible.

Next, I prove the theorem by contradiction for $w \in (\underline{w}, m_1)$. Assume $U_i(w) > \underline{U}_i(w)$ for some $w \in (\underline{w}, m_1)$. Both $\underline{U}_i(w)$ for $i \in \{1, 2\}$ are strictly increasing on $[\underline{w}, m_1)$ by observation 5. Let $x > w$ be the budget within $(\underline{w}, m_1)$, for which the strictly monotonic lower bound $\underline{U}_i(.)$ catches up and reaches the same value, i.e. $U_i(w) = \underline{U}_i(x)$. If such x does not exists, take m_1. That is, equilibrium utility is strictly above the lower bound, $U_i(.) > \underline{U}_i(.)$, on at least the non-empty interval $[w, x)$. As U_i is a monotonic function, it can have only countable jump discontinuities on (w, x): $U_i(.)$ has to be either i) continuous and strictly increasing, or ii) constant on some subinterval within $[w, x)$. The former i) is ruled out by corollary 3: $U_i(w)$ cannot be both continuously strictly

[19] It cannot be above w as then bidder w could not reach it and no discontinuity could occur.

[20] This is because $U_i(w) \geq U_i(\underline{w}) > 0$ for all w.

increasing and strictly above $\underline{U}_i(w)$. I show in the following, that the latter yields a contradiction as well, if both $\underline{U}_i(w)$ are strictly increasing.

Let $U_i(w) = U > \underline{U}_i(w)$ be constant on some intervals within (w, x). Define $z_i = inf\{w : U_i(w) = U\}$ as the lowest budget, above which bidder i achieves a payoff equal to U. First, let $z_i = \underline{w}$. This is ruled out by the first paragraph of this proof that established that $U_i(\underline{w}) = \underline{U}_i(\underline{w})$. Second, let $z_i > \underline{w}$. Bidder j has a mass point at z_i by lemma 2, as any increase above the lower bound in the interior is due to a mass point. This implies that z_i is indeed the infimum, not the minimum. Bidder i-types with a higher budget than z_i always bid above z_i to extract the additional winning probability from avoiding the sharing rule; therefore, $F_i(z_i) = G_i(z_i)$. However, this yields a contradiction for the utility of bidder j: a mass point of j at z_i implies, that there is a continuum of bidder j with budget above z_i who can at most achieve $(v_j - z_i)F_j(z_i) = \underline{U}_j(z_i)$; otherwise they would have a profitable deviation by bidding above z_i instead of sticking to the mass point. Bidding z_i yields expected payoff of $(v_j - z_i)G_i(z_i) = (v_j - z_i)F_i(z_i)$, which equals the lower bound $\underline{U}_j(z_i)$. Yet, the lower bound is strictly increasing around z_i: if bidder j-types had a constant utility to establish the mass point for higher budget realizations than z_i, their utility would fall strictly below the lower bound on utility, which is impossible.

The previous argument established that $U_i(w) = \underline{U}_i(w)$ below m_1. The second part of the theorem follows from lemma 2: as $\underline{U}_i(w)$ is strictly increasing below m_1, all bidders bid their entire budget on $(\underline{w}, m_1)$.

□

Proof of lemma 3. For the first part of the lemma, assume by contradiction that $b_1^{max} < b_2^{max}$. Any bid of bidder 2, denoted b_2, in the interval $(b_1^{max}, b_2^{max}]$ wins with probability 1 and yields utility of $(v_2 - b_2)$. For any bid in this open interval, there exists a profitable deviation by shading the bid down by $\epsilon > 0$ small enough such that $b_2 - \epsilon > b_1^{max}$. This deviation still wins the object with certainty, however, for a strictly lower payment.

Moreover, $b_{max} < min(v_1, v_2)$. If a bidder can afford to bid her full valuation v_i, this would yield an expected utility of 0 with certainty. However, with $v_i > \underline{w}$ this implies a utility strictly below the lower bound as $0 < \underline{U}_i(w \geq v_i)$, which is impossible in equilibrium: deviating and bidding something in-between $(\underline{w}, min(v_1, v_2))$ yields strictly positive expected utility of at least the lower bound.

Finally, consider the second part of the lemma: if $b_{max} = \overline{w}$, a mass point there is infeasible, as only bidders with budget realization $\overline{w}$ (which is a zero probability event) can afford bidding so high. Assume $b_{max} < \overline{w}$. Let bidder i have a mass point at $b_{max} < min(v_1, v_2)$. Then, any bidder j with budget in $(b_{max}, \overline{w}]$ has a profitable deviation: bidding b_{max} yields expected utility of $(v_j - b_{max})\Pr(b_i < b_{max}) + \frac{1}{2}(v_j - b_{max})\Pr(b_i = b_{max})$. However, if j deviates and bids above the highest bid $b_{max} + \epsilon$ with $\epsilon \to 0$, this yields a higher expected utility $(v_j - b_{max})\Pr(b_i \leq b_{max})$ in the limit. Therefore, this is a profitable deviation. The winning probability discretely increases as the sharing rule no longer applies, for a negligible small increase in payment. In equilibrium, therefore, no bidder has a mass point at the highest equilibrium bid. □

Proof of lemma 4. Assume per contradiction, that bidder i does not have constant utility on $w \in (m_j, \overline{w}]$: there exist two budgets w and w' with $m_j < w < w'$ for which bidder i achieves distinct utilities, i.e. $U_i(w) < U_i(w')$. By lemma 2, any increase beyond the lower bound is due to a mass point. Therefore, the distinct utilities $U_i(w)$ and $U_i(w')$ can be due to 1. a mass point in the bidding cdf of j between w and w' or 2. a strictly increasing $\underline{U}_i(.)$ somewhere between w and w'. Due to log-concavity, the second scenario can only arise on (m_1, m_2) for bidder 2, where $\underline{U}_2$ is still strictly increasing while $\underline{U}_1(w)$ is constant.

For the first scenario, let bidder j have a mass point at $x \in [w, w']$. If bidder i bids x, the sharing rule applies with positive probability. However, slightly bidding above x yields a discrete jump in winning probability (no sharing rule applies) for a negligible higher payment; every bidder with budget above x will bid higher than x. Therefore, we have $F_i(x) = G_i(x)$. However, this implies that the utility of bidder j bidding at the mass point x is below the lower bound, $x > m_j$: $(v_j - x)G_i(x) = (v_j - x)F_i(x) < (v_j - m_j)F_i(m_j) = \underline{U}_i(m_j \leq w \leq \overline{w})$: deviating from the mass point x to m_j yields a strongly higher payoff. Therefore, we cannot have any mass point of j on $(m_j, \overline{w}]$.

For the second scenario, assume we have $\underline{U}_2(w) = U_2(w)$ strictly increasing somewhere on (m_1, m_2). This implies that on this interval, bidder 2 bids his entire budget: $G_2(w) = F_2(w)$. Thus, if bidder 1 bids b_1 anywhere in this interval, his expected utility is below the lower bound: $(v_1 - b_1)G_2(b_1) = (v_1 - b_1)F_2(b_1) < (v_1 - m_1)F_2(m_1) = \underline{U}_1(m_1 \leq w \leq \overline{w})$. But then bidder 1 would never bid in this interval, which contradicts the strictly increasing utility of bidder 2.

There is no mass point at the highest bid b_{max}. Bidding b_{max} wins with certainty and yields utility of $(v_i - b_{max})$. Therefore, any bidder i with budget realization $w \in (m_j, \overline{w}]$ has utility $U_i(m_j < w \leq \overline{w}) = v_i - b_{max}$. Any bid $b \in (m_j, b_{max}]$ of bidder i has the same expected utility $v_i - b_{max} = (v_i - b)G_j(b)$. Rewriting this equation yields the bidding distribution in the lemma. □

Proof of theorem 2. First, I show that bidder 1 has a mass point in his bidding strategy at m_1, because the equilibrium utility of bidder 2 is discontinuous at m_1. By lemma 4, $U_2(w)$ is constant for budget realizations $w \in (m_1, \overline{w}]$. Note that $U_2(w) \geq \underline{U}_2(m_2)$ for $w \in (m_1, \overline{w}]$, with $\underline{U}_2(m_2)$ being the highest value for the lower bound, which equilibrium utility cannot undercut. Moreover, $\underline{U}_2(m_2) > \underline{U}_2(m_1)$ strictly, as the lower bound is strictly increasing below m_2 due to the log-concavity assumption. By theorem 1, the lower bound binds: $\underline{U}_2(w) = U_2(w)$ for $w \in [\underline{w}, m_1)$. Approaching utility of bidder 2 from both sides at m_1, shows the discontinuity and therefore, a mass point of bidder 1: $\lim_{w \nearrow m_1} U_2(w) = \underline{U}_2(m_1) < \underline{U}_2(m_2) \leq \lim_{w \searrow m_1} U_2(w)$. By lemma 2, bidder 1 has to have a mass point at m_1 in his bidding distribution function to enable this jump in expected utility of bidder 2 to achieve a utility of at least the lower bound.

In the next step, I derive the bidding distribution of bidder 2 on $[m_1, m_2)$; this bidding distribution is uniquely pinned down by the equilibrium utility of bidder 1 on $[m_1, m_2)$. As bidder 1 has a mass point at m_1, bidder 2 cannot have a mass point at m_1 as well[21], $F_2(m_1) = G_2(m_1)$. This implies $\underline{U}_1(m_1) = U_1(m_1)$. The lower bound of bidder 1 is constant on (m_1, m_2), so by lemma 2 any increase can only happen due to a mass point of bidder 2 at the respective bid. Let bidder 2 have a mass point at

[21]This is due to the additional winning probability if bidder 2 slightly outbids the mass point: with $lim_{\epsilon \to 0}(v - m_1 + \epsilon)G_1(m_1 + \epsilon) > lim_{\epsilon \to 0}(v - m_1)\Pr(b_1 < m_1) + 1/2(v - m_1)\Pr(b_1 = m_1)$.

$x \in (m_1, m_2)$. This implies $F_1(x) = G_1(x)$ for the bidding distribution of bidder 1: all budget types of bidder 1 who can outbid x, will do so to extract the additional winning probability; the only bidder 1 types who bid x are those with budget equal to x. This yields a profitable deviation for bidder 2 types, who bid at x but have a strictly higher budget: bidding at the mass point yields $(v_2 - x)F_1(x) = \underline{U}_2(x)$, while any higher bid $b > x$ yields a strictly higher payoff of at least $U_2(b) \geq \underline{U}_2(b) > \underline{U}_2(x)$, as the lower bound of bidder 2 is strictly increasing on (m_1, m_2). Hence, bidder 2 has no mass points on $[m_1, m_2)$; equilibrium utility of bidder 1 is therefore continuous and constant on this interval, implying $(v_1 - b)G_2(b) = (v_1 - m_1)F_2(m_1)$, which yields G_2 on the respective interval in the theorem.

In the next step, I pin down b_{max} and discuss, when bidder 2 has a mass point. Let $h, l \in \{1, 2\}$ such that $\underline{U}_h(m_2) \geq \underline{U}_l(m_2)$. For high enough budget realizations, lemma 3 and lemma 4 suggests that utility coincides: $U_1(w > m_2) = U_2(w > m_1) = v - b_{max}$. As equilibrium utilities are always weakly above the lower bound, we have $U_i(w > m_j) = v - b_{max} \geq \underline{U}_h(m_h) = (v - m_h)F_l(m_h)$ for $i = 1, 2$. Otherwise due to equality of the utilities, the lower bound would be undercut somewhere.

Let $h = 1$, i.e. $\underline{U}_1(m_2) \geq \underline{U}_2(m_2)$. I show that the lower bound binds for bidder 1. Assume by contradiction that $U_2(w) = v - b_{max} > \underline{U}_1(m_1)$ on $w \in (m_1, \overline{w}]$. Two paragraphs above I prove, that $U_1(w) = \underline{U}_1(m_1) < U_2(w)$ constant on $[m_1, m_2)$. For $U_1(w > m_2)$ to be equal to $U_2(\overline{w}) = v - b_{max}$, bidder 2 has to have a mass point at m_2: because we rule out every other point for a mass point by lemma 4. However, a mass point of 2 at m_2 would by the same argument as before lead to $F_1(m_2) = G_1(m_2)$. Bidder 2, who bids at the mass point, will get a utility of $(v - m_2)F_1(m_2) < \underline{U}_1(m_1)$ which is a contradiction. Therefore, we have $U_i(w) = v - b_{max} = \underline{U}_1(m_1) = (v - m_1)F_2(m_1)$ in equilibrium and the highest bid is $b_{max} = v - (v - m_1)F_2(m_1)$.

Let $h = 2$ and $\underline{U}_1(m_2) < \underline{U}_2(m_2)$. As the utility is always weakly above the lower bound, we have $U_2(w > m_1) \geq \underline{U}_2(m_2) = (v - m_2)F_1(m_2)$. This is by assumption larger than $\underline{U}_1(m_2)$. For bidder 1 and bidder 2 to have the same expected utility for budget realizations above m_2, we therefore need bidder 2 to have a mass point at m_2 (the paragraph before established that 2 has no mass points on $[m_1, m_2)$; lemma 4 shows no mass point above m_2. Therefore, if a mass point exists in the bidding support of bidder 2, it has to lie at m_2). If bidder 2 has a mass point at m_2, this implies $G_1(m_2) = F_1(m_2)$, as all bidder 1 types who can afford it will bid above the mass point. This in turn pins down the utility of of bidder 2 at the mass point, and thus, for all budgets above it: $U_2(m_2) = (v - m_2)F_1(m_2) = \underline{U}_2(m_2) = v - b_{max}$, and, simultaneously, the upper bid $b_{max} = v - (v - m_2)F_1(m_2)$. □

Chapter 3

Persuading an Informed Committee

1. Introduction

In voting or collective decision making, the persuasion of decision makers through a biased party plays a crucial role. To which extent a biased party can persuade decision makers might depend on how much decision-relevant knowledge they already possess. Consider for example a CEO who tries to convince a board of directors to vote for a new proposal. While the CEO wishes to always implement the proposal to improve short-term firm performance, directors only want to approve the proposal if it increases long-term performance. If directors already have some private knowledge about the long-term effects of the proposal, what is the most promising way to convince them to vote for the proposal? This is the question of this paper.

In our model an information designer requires a unanimous approval of a group of voters to implement a proposal. Depending on the proposal's unknown binary quality, voters either like or dislike the proposal. If the quality was known, all voters would agree on the optimal decision. In contrast, the information designer is biased in that she always wants the proposal to be implemented, irrespective of its quality. Each voter receives a private signal about the proposal having a high or low quality and is, according to his private signal, either optimistic or pessimistic. The information designer chooses a disclosure policy: she sends a public recommendation that is correlated with the quality of the proposal. The term *public* means that the information designer cannot make different recommendations to different voters. After having received the recommendation of the information designer, each voter decides on whether to vote for or against the proposal based on his updated belief. Although voters are aware of the information designer's interest in the proposal, they might nevertheless want to follow her recommendation. This is because the recommendation is based on the true quality of the proposal.

The main contribution of this paper is to unveil the extent to which an information designer can persuade informed voters by choosing the optimal disclosure policy. We characterize when the private information of voters restricts the information designer in her scope for persuasion.

In our benchmark case we consider an *omniscient* information designer who can observe the private signal realizations of all voters. We show that the omniscient information designer recommends to vote for the proposal with probability one when the proposal is of high quality. In the state where the proposal is of low quality, she uses a threshold policy: she recommends the proposal with probability one for any number of optimists above a certain cutoff, and recommends the status quo with certainty below the cutoff. The cutoff is such that a pessimistic voter is indifferent between the proposal and the status quo after the recommendation to vote for the proposal.

Next, we consider an *eliciting* information designer who cannot observe the private information of voters but can ask them for reports about their signal realizations. We show that the eliciting information designer cannot implement the optimal disclosure policy from the omniscient benchmark case and is always worse off compared to the omniscient information designer. This is caused by the optimists having a profitable deviation through misreporting to be pessimists. As a consequence, the eliciting information designer has to give sufficient incentives for truthful reporting by providing voters with more information. This limits the scope of the information designer for persuasion. If the probability of receiving the correct signal is below a lower threshold, the eliciting information designer always recommends to vote for the proposal in the state where voters prefer the proposal. In the state in which voters want to implement the status quo, the probability with which she recommends the proposal is stochastic and decreasing in the accuracy of the private information of voters. This optimal policy of the information designer is equivalent to maximizing the probability of a pessimist to vote for the proposal. In contrast, if the probability of receiving the correct signal is above an upper threshold, the information designer's optimal policy is to maximize the probability of an optimist to vote for the proposal.

Finally, we consider a non-eliciting information designer who can neither observe the signal realizations of voters nor ask voters for reports about their private information. If the probability of receiving the correct signal is below the same lower threshold as in the eliciting case, the optimal disclosure policy of an eliciting and of a non-eliciting information designer are equivalent. Thus, an information designer cannot profit from the ability to ask voters for their private information if the accuracy of voter's private information is not sufficiently high.

We find that voters are better off in the presence of a biased information designer compared to the situation in which they have to decide under unanimity rule only based on their private exogenous information as in Feddersen and Pesendorfer (1998).

2. Related Literature

Our paper belongs to the rapidly growing literature on information design (see Rayo and Segal, 2010; Kamenica and Gentzkow, 2011). While in Kamenica and Gentzkow (2011) there is only one agent that is uninformed, we consider persuasion of a committee of agents that is informed.

Amongst the vast emerging literature on information design, the two strands bearing most resemblance to our paper are first, private information on the receiver's side, and second, persuasion of multiple receivers. Multiple papers extend information design to a setting with *many receivers*. In these papers (Taneva, 2016; Bardhi and Guo, 2018; Alonso and Câmara, 2016; Wang, 2015; Chan et al., 2016; Heese and Lauermann, 2017), agents are aware of the payoff types of each other. There is no uncertainty about the committee constellation, and voters possess no private information about the payoff-relevant state of the world. A crucial difference of these papers to our approach is that in our model, the payoff type of each committee member bears information about the state of the world and is private information. If the committee constellation was known, all voters in our model would agree on the same election outcome.

Taneva (2016) extends the approach of a Bayes correlated equilibrium from Bergemann and Morris (2016a) to a class of Bayesian Persuasion problems with multiple receivers. She fully characterizes a binary-binary[1] model with two receivers and shows that the optimal information structure involves public signals or correlated private signals (not conditionally independent signals). Alonso and Câmara (2016) analyze how a biased sender can influence an uninformed heterogeneous committee of voters with a public signal, as in our model. They elicit the scope for persuasion under different voting rules, and show when agents are worse off. Chan et al. (2016) consider persuading a heterogeneous committee under the restriction to minimal winning coalitions. Wang (2015) compares private persuasion (under the restriction of conditionally independent signals) to public persuasion in collective decision making. She shows, that public persuasion performs weakly better and reveals less information than private persuasion. The closest related to our paper is Bardhi and Guo (2018). They analyze persuasion of a heterogeneous committee, and study a unanimous voting rule. They consider two persuasion regimes: general persuasion (conditional on everybody's payoff type), and individual persuasion (conditional only on own payoff type). Persuasion is private in their model: each agent does not see the messages sent to voters, neither under general nor under individual persuasion. They show that under unanimity, a restriction to a public or private persuasion regime is without loss under some assumptions. Heese and Lauermann (2017) consider persuasion of a heterogeneous committee of voters. They show that the information designer can almost surely guarantee the implementation of her preferred outcome in the limit, as the size of the committee grows sufficiently large.

Amongst the papers considering *private information* on the side of the receiver are Kolotilin et al. (2017), Kolotilin (2018), Bergemann and Morris (2016b) and Bobkova (2017). Kolotilin et al. (2017) study persuasion of one privately informed receiver, who is privately informed about his payoff type. They show that eliciting persuasion is equivalent to non-eliciting persuasion under some conditions[2]. Bergemann et al. (2018) consider a similar environment as Kolotilin et al. (2017) but add monetary transfers, which we do not allow in our framework. Kolotilin (2018) considers an information designer who tries to persuade an informed receiver. Persuasion is non-eliciting: the information designer cannot ask the receiver for his type prior to her information disclosure. Bobkova (2017) considers a stream of short-lived and privately informed buyers, that an information designer (seller) seeks to persuade into buying her product. The seller is restricted in her ability to construct experiments, and has to rely on the private information of previous receivers, that she has to elicit truthfully.

To the best of our knowledge, our framework is the first to introduce private infor-

[1]Binary states and binary actions.

[2]Kolotilin et al. (2017) refer to eliciting and non-eliciting persuasion as public versus private persuasion. See Bergemann and Morris (2018) for a unified terminology, that we follow in this paper.

mation into persuasion of a group. We analyze the problem of an information designer when she first has to squeeze the private information out of multiple agents before she can condition her disclosure policy on it.

The idea of *omniscient* persuasion and *private* persuasion of one privately informed receiver can be found in Bergemann and Morris (2016b). We extend their discussion by providing a comparison of the cases in which the information designer is omniscient and in which the information designer has to first elicit the private signals from multiple agents. A unified perspective of the existing literature on Bayesian Persuasion and information design can be found in Bergemann and Morris (2018).

Finally, our paper relates to the literature on information aggregation in strategic voting, following Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1998). Feddersen and Pesendorfer (1998) show that when voters vote strategically, voting truthfully according to one's own private information is not an equilibrium. Voters condition their strategy on pivotality events, and a unanimous voting rule is a 'uniquely bad' voting rule: it implements the inferior inefficient outcome with a higher probability than any other majority rule. In the model of Feddersen and Pesendorfer (1998), voters have to make their decision based on their private exogenous information only. In our model, we allow the designer to provide further correlated information to all agents, conditional on the true state of the world.

The paper is organized as follows. First, we introduce our model in section 2. In Section 3.1, we first analyze the benchmark case of an omniscient information designer and characterize her optimal disclosure policy. In the subsequent section, we analyze information design with elicitation and show that an eliciting information designer cannot implement the optimal disclosure policy from the omniscient benchmark case. We characterize the optimal disclosure of an eliciting information designer and establish two equivalence results. In section 3.3 we deal with the analysis of a non-eliciting information and prove the equivalence of optimal disclosure policies of an eliciting and a non-eliciting information designer if the accuracy of voters' signals is below some threshold. The last section 3.4 deals with restricted information design. Unlike the unrestricted information designer, the restricted eliciting information designer can achieve the same expected payoff as in the omniscient benchmark case.

3. Model

There are three voters which have to decide whether to vote for a proposal or for a status quo. An information designer tries to influence voters to vote for the proposal. In the following we use information designer and sender synonymously. When a voter i votes for the proposal we write $a_i = 1$, and $a_i = 0$ for the status quo. When the outcome of the ballot is the proposal we write $a = 1$, and $a = 0$ when the status quo is chosen. For example, under unanimity rule the outcome is $a = 1$ if $a_i = 1$ for all i.

Whether a voter likes or dislikes the proposal depends on an uncertain state of the world $\theta \in \{B, G\}$, where $\Pr(\theta = G) = \frac{1}{2}$. Voters have the following utility function:

$$u_i(a, \theta) = \begin{cases} \mathbb{1}_{\{\theta=G\}} - \frac{1}{2} & \text{if } a = 1 \\ 0 & \text{if } a = 0 \end{cases}$$

Hence, all voters agree on the optimal decision if the state was known: when $\theta = G$ all voters want to implement the proposal, while when $\theta = B$, all agree on the status quo. In contrast, the sender always prefers the proposal over the status quo independent of the state of the world. The sender's utility function is given by $u_S(a) = a$.

Each voter receives a private signal $z_i \in \{b, g\}$ that is correlated with the true state of the world in the following way: $\Pr[z_i = g|\theta = G] = \Pr[z_i = b|\theta = B] = p \in (\frac{1}{2}, 1)$. A voter i with a signal $z_i = g$ (referred to as a *good signal*) is more optimistic about the state of the world being $\theta = G$ than a voter with signal $z_i = b$ (referred to as a *bad signal*). Likewise, a voter with bad signal $z_i = b$ considers $\theta = B$ more likely.[3]

Denote the set of signal realizations by $Z = \{g, b\}^3$ with a typical element $z = (z_1, z_2, z_3) \in Z$. Let $k(z)$ be the number of g-signals in a typical signal realization z. By $z_{-i} \in Z_{-i}$ we refer to the signals of all voters except voter i, where Z_{-i} is the set of all signal realizations except voter i's signal. In the following we use the shortcut k to refer to $k(z)$ and k_{-i} to refer to $k(z_{-i})$.

4. Omniscient Information Design

We first analyze the benchmark case in which the sender is omniscient, i.e., observing each voter's private signal. The sender's problem is then to choose a disclosure policy $d : \Theta \times Z \to \triangle(R)$ with public recommendations $r \in R$ to maximize the probability of the event that all voters vote for the proposal. We restrict the analysis to anonymous disclosure policies, which take only the number of good and bad signals into account and not which voter has which signal.

Assumption 1
The sender's disclosure policy is anonymous, i.e., the probability of sending any recommendation is the same for all z, z' with $k(z) = k(z')$.

This assumption allows us to restrict attention to disclosure policies which only condition on the state of the world and on the number of good signal in the population.

The next assumption specifies how a voter behaves under indifference.[4]

Assumption 2
If a voter is indifferent between his actions, he follows the recommendation of the sender.

The following proposition says that we can restrict the sender to only two recommendations $r \in \{\hat{0}, \hat{1}\}$ without loss of generality for optimality. These two recommendations are direct voting recommendations, $\hat{1}$ in favor of the proposal, and $\hat{0}$ in favor of the status quo.

[3]For notational convenience we will sometimes use g_i and b_i respectively as a short cut for voter i having received signal $z_i = g$ and $z_i = b$ respectively.

[4]Note the difference to the sender-preferred equilibrium in Kamenica and Gentzkow (2011): they assumed that if indifferent, their agent votes for the sender-preferred outcome, in our case the proposal. We are interested in partial implementation, and such one-sided tie-breaking rule pro proposal would be with loss of generality in our setting. We will see that in the optimum the sender sometimes wants voters to vote for the status quo if indifferent to achieve the highest outcome. This is driven by pivotality considerations and does not arise in Kamenica and Gentzkow (2011).

Proposition 1
Under unanimity, it is without loss of generality to restrict the message space of the omniscient sender to $R = \{\hat{0}, \hat{1}\}$.

All omitted proofs are in the appendix. The disclosure policy of the sender is:

$$d : \Theta \times \{0, 1, 2, 3\} \rightarrow \triangle\{\hat{0}, \hat{1}\}. \tag{3.1}$$

That is, we are looking for a vector with 8 components $\{d[\hat{1}|\theta, k]\}_{k\in\{0,1,2,3\},\theta\in\{B,G\}} \in [0, 1]^8$. After the sender sends her recommendation, voters have to update their belief about θ and only follow the sender's recommendation if this yields a higher expected utility than disobeying.

Since the decision has to be made under unanimity, after recommendation $\hat{1}$ a voter is pivotal with probability one and after recommendation $\hat{0}$ he is never pivotal. This is because in equilibrium after $r = \hat{1}$ all voters follow the recommendation $\hat{1}$ which is why from the perspective of an individual voter his vote determines the outcome. Similarly, a voter will never be pivotal after $r = \hat{0}$ because all other voters already voted against the proposal which in turn makes one single vote irrelevant under unanimity. As a consequence, a voter will always follow the recommendation $\hat{0}$ and follow the recommendation $\hat{1}$ if his obedience constraint holds:

$$\begin{aligned} & \Pr(\theta = G|\hat{1}, z_i) \geq \frac{1}{2} && (OB^{\hat{1}}_{z_i}) \\ \Leftrightarrow\quad & \Pr(\theta = G|z_i) \sum_{k_{-i}=0}^{2} d[\hat{1}|\theta = G, k_{-i} + k(z_i)] \Pr(k_{-i}|\theta = G) \\ \geq\quad & \Pr(\theta = B|z_i) \sum_{k_{-i}=0}^{2} d[\hat{1}|\theta = B, k_{-i} + k(z_i)] \Pr(k_{-i}|\theta = B) \end{aligned}$$

The omniscient sender's maximization problem is then given by:

$$\max_d \Pr(a = 1) = \max_d \Pr(\hat{1}) = \sum_{\theta\in\{B,G\}} \sum_{k=0}^{3} d[\hat{1}|\theta, k] \Pr(k|\theta) \Pr(\theta)$$

$$s.t.\quad 0 \leq d[r|\theta, k] \leq 1, \quad \forall r \in \{\hat{0}, \hat{1}\},\ \theta \in \{B, G\},\ k \in \{0, 1, 2, 3\} \tag{3.2}$$

$$d[\hat{0}|\theta, k] + d[\hat{1}|\theta, k] = 1, \quad \forall\theta \in \{B, G\},\ k \in \{0, 1, 2, 3\} \tag{3.3}$$

$$\Pr(\theta = G|\hat{1}, z_i) - \frac{1}{2} \geq 0, \qquad \forall z_i \in \{b, g\} \qquad (OB^{\hat{1}}_{z_i})$$

The following lemma states that for the sender it is always optimal to send the recommendation to vote for the proposal when $\theta = G$.

Lemma 1
In any optimal disclosure policy, $d[\hat{1}|\theta = G, k] = 1$ *for all* k.

To give some intuition for Lemma 1, notice that for $\theta = G$ voters agree on the proposal being the more appropriate choice, independent of their private signal. Hence, the sender does not have to convince voters so that she can simply send $\hat{1}$ for $\theta = G$. Assume that the conjecture is false. Then, the sender could increase the probability of implementing the proposal and at the same time relax the voters' obedience constraints by simply increasing $d[\hat{1}|\theta = G, k]$ for those k for which $d[\hat{1}|\theta = G, k] \neq 1$.

The probability of sending a recommendation $\hat{1}$ for the sender is:

$$\Pr(\hat{1}) = 0.5\Pr(\hat{1}|\theta = B) + 0.5\underbrace{\Pr(\hat{1}|\theta = G)}_{=1} \tag{3.4}$$

$$= 0.5\sum_{k=0}^{3} d[\hat{1}|k, \theta = B]\Pr(k|\theta = B) + 0.5 \tag{3.5}$$

Now, consider the obedience constraint for the g-type. It is easy to see that it is always satisfied if Lemma 1 holds.

Lemma 2
The obedience constraint of the g-type is satisfied in any disclosure policy in which $d[\hat{1}|\theta = G, k] = 1$ for all k.

Proof. The obedience constraint of the g-type is:

$$0.5p \geq 0.5(1-p)\sum_{k_{-i}=0}^{2} d[\hat{1}|\theta = B, k_{-i} + 1]\Pr(k_{-i}|\theta = B) \tag{3.6}$$

Note that due to feasibility of the disclosure policy, $d[\hat{1}|\theta, k] \leq 1$ for all θ and all k. Thus,

$$\sum_{k_{-i}=0}^{2} \underbrace{d[\hat{1}|\theta = B, k_{-i} + 1]}_{\leq 1}\Pr(k_{-i}|\theta = B) \tag{3.7}$$

$$\leq \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|\theta = B) = 1$$

Using this in the obedience constraint, we see that it is always satisfied as $0.5p \geq 0.5(1-p) \geq RHS$ always holds. □

The next lemma states the disclosure policy if all voters have a g-signal:

Lemma 3
In any optimal disclosure policy d, it holds that $d[\hat{1}|\theta = B, k = 3] = 1$.

Proof. The probability $d[\hat{1}|\theta = B, k = 3]$ does not show up in the disclosure policy of the b-type, as it only applies when all voters have a g-signal. Therefore, it has no effect on the obedience of the b-type. For the g-type, by Lemma 2 the obedience constraint of the g-type holds in any disclosure policy that sends recommendation $\hat{1}$ whenever the state is $\theta = G$. Therefore, setting $d[\hat{1}|\theta = B, k = 3]$ increases the probability of the proposal being implemented without harming any obedience constraints. □

Using the above findings, the maximization problem of the omniscient unrestricted designer becomes:

$$\max_{d} \quad \frac{1}{2}\sum_{k=0}^{3} d[\hat{1}|\theta = B, k] \Pr(k|\theta = B) + \frac{1}{2}\Pr(k = 3|\theta = B) + \frac{1}{2} \tag{3.8}$$

$$d[\hat{0}|\theta, k] + d[\hat{1}|\theta, k] = 1, \quad \forall\theta \in \{B, G\},\ k \in \{0, 1, 2, 3\} \tag{3.9}$$

$$s.t. \quad (1-p) \geq p \sum_{k_{-i}=0}^{2} d[\hat{1}|\theta = B, k_{-i})] \Pr(k_{-i}|\theta = B) \tag{$OB_b^{\hat{1}}$}$$

By using that $p \Pr(k_{-i}|\theta = B) = \Pr(k|\theta = B)\frac{3-k}{3}$ we can rewrite $OB_b^{\hat{1}}$:

$$(1-p) \geq p \sum_{k_{-i}=0}^{2} d[\hat{1}|\theta = B, k_{-i}] \Pr(k_{-i}|\theta = B)\binom{2}{k_{-i}} \tag{3.10}$$

$$\Leftrightarrow \quad (1-p) \geq \sum_{k=0}^{3} d[\hat{1}|\theta = B, k] \Pr(k|\theta = B)\frac{3-k}{3}. \tag{3.11}$$

Increasing $d[\hat{1}|\theta = B, k]$ for any $k \in \{0, 1, 2\}$ affects differently the obedience constraint and objective function of the designer. While an increase in $d[\hat{1}|\theta = B, k]$ for any particular $k \in \{0, 1, 2\}$ is weighted by the sender with $\Pr(k|\theta = B)$, the b-type weights this increase with $\Pr(k|\theta = B)\frac{3-k}{3}$. In the terminology of the fractional knapsack[5], this means that different k have different value-weight ratios. As a consequence, it will matter for which k the sender will increase the probability of sending recommendation $\hat{1}$ until the constraint binds. The next proposition states that the optimal disclosure policy of the omniscient sender is a monotone threshold policy.

Proposition 2

The unique optimal disclosure policy of the omniscient sender is a monotone cutoff policy with

$$d[\hat{1}|\theta = G, k] = 1 \ \forall k, \quad d[\hat{1}|\theta = B, k] \begin{cases} = 1 & \text{if} \quad k > \tilde{k}, \\ \in [0, 1] & \text{if} \quad k = \tilde{k}, \\ = 0 & \text{if} \quad k < \tilde{k}, \end{cases} \tag{3.12}$$

where $\tilde{k}$ is such that $OB_b^{\hat{1}}$ binds.

Figure 1 shows the optimal policy of the omniscient sender for $p = 0.7$. If $\theta = G$, she sends with the certainty the recommendation $\hat{1}$ irrespective of the number of g-signals. If $\theta = B$, the sender uses a monotone cutoff policy, where she sends $\hat{1}$ with certainty whenever there are at least two voters with a g-signal, mixes whenever there is one voter with a g-signal, and never sends $\hat{1}$ when all have a b-signal.

[5] See appendix for the terminology.

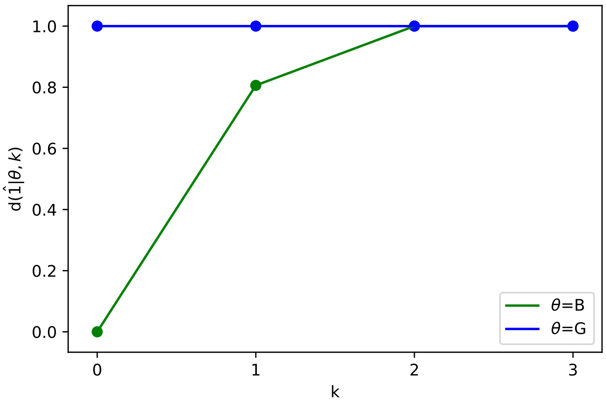


Figure 3.1: optimal disclosure policy for $p = 0.7$.

5. Eliciting Information Design

In the previous section the sender could construct any experiment on the true state of the world and was able to see the private signal realizations of each voter. In this section we assume that the sender cannot see the private information of the voters, but can elicit it in an incentive compatible way. When the sender is eliciting, honesty constraints arise, and we have to check for double deviations: if an agent misreports, does he have a profitable deviation? After misreporting, the agent should not be strictly better of from any possible action after the misreport.

Each voter i sends a message to the sender, a report about his private signal realization: $\hat{z}_i \in \{\hat{g}, \hat{b}\}$. The complete profile of reported signals is then given by $\hat{z} \in \hat{Z}$. We employ the same notation as above, with the restriction, that now the sender conditions not on the number of g-signals in the true signal realizations z, but on the number of $\hat{g}$-reports in the reported signal realization $\hat{z}$. Hence, $k(\hat{z})$ denotes now the number of $\hat{g}$-reports in the reported signal realization $\hat{z}$.

As for the omniscient sender, only two signals suffice for the sender to achieve her highest implementable payoff.

Proposition 3
Under unanimity, it is without loss of generality to restrict the message space of the eliciting sender to $R = \{\hat{0}, \hat{1}\}$.

The sender commits to a disclosure policy $d : k \in \{0, 1, 2, 3\} \to \Delta\{\hat{0}, \hat{1}\}$. Let $U(z_i, \hat{z}_i, a_i(\hat{0}, z_i), a_i(\hat{1}, z_i))$ be the expected utility of a voter with signal z_i, who reports being type $\hat{z}_i$, and votes with probability $a_i(\hat{0}, z_i))$ for the proposal after recommendation $\hat{0}$, and with probability $a_i(\hat{1}, z_i))$ for the proposal after recommendation $\hat{1}$.

We refer to a disclosure policy d of an eliciting sender as *implementable* if and only if it satisfies the obedience and the honesty constraints.

The next observation establishes, that the omniscient sender is strictly better off than the eliciting sender.

Observation 8. *The optimal disclosure policy of the omniscient sender is not implementable when the sender is eliciting.*

It is straightforward that with the optimal disclosure policy in Proposition 2 the g-type will have a profitable deviation from misreporting $\hat{b}$ and following the recommendation. Let $U(z_i, \hat{z}_i', a_i(\hat{0}, z_i), a_i(\hat{1}, z_i))$ be the expected utility of a voter with signal z_i, who reports being type $\hat{z}_i'$, and votes with probability $a_i(\hat{0}, z_i))$ for the proposal after recommendation $\hat{0}$, and with probability $a_i(\hat{1}, z_i))$ for the proposal after recommendation $\hat{1}$. If the g-type is truthful and obedient, his expected utility is

$$\begin{aligned} U(g_i, \hat{g}_i, 0, 1) = &\Pr(\theta = G|g_i)\frac{1}{2} \\ &- \Pr(\theta = B|g_i)\frac{1}{2}\Pr(k_{-i} = 2|\theta = B) \\ &- \Pr(\theta = B|g_i)\frac{1}{2}\sum_{k_{-i}=0}^{1} d[\hat{1}|\theta = B, k_{-i} + 1]\Pr(k_{-i}|\theta = B). \end{aligned}$$

Consider the following deviation: misreport $\hat{b}$ and follow the recommendation. Then, a g-type prevent the event in which all voters have reported a g-signal, the state is $\theta = B$ and the sender sends $\hat{1}$ with probability 1. In all other states, the misreporting does not matter for the disclosure policy of the sender when $\theta = G$ because when $\theta = G$ the sender sends $\hat{1}$ with probability one for all $k \in \{0, 1, 2, 3\}$. When $\theta = B$, the g-type voter profits from misreporting since, $d[\hat{1}|\theta = B, k]$ is decreasing k. Hence, the misreporting g-type will receive recommendation $\hat{1}$ with a (weakly) smaller probability than when being honest in the unfavorable state $\theta = B$. This follows because the disclosure policy is a cutoff policy: misreporting a g-signal gets a more 'favorable' cutoff than when reporting truthfully. The voter's expected payoff when being dishonest is given by

$$\begin{aligned} U(g_i, \hat{b}_i, 0, 1) = &\Pr(\theta = G|g)\frac{1}{2} \\ &- \Pr(\theta = B|g_i)\frac{1}{2}\sum_{k_{-i}=0}^{2} d[\hat{1}|\theta = B, k_{-i}]\Pr(k_{-i}|\theta = B). \end{aligned}$$

Message $\hat{1}$ is sent less frequently if $\theta = B$, hence $U(g_i, \hat{b}_i, 0, 1) > U(g_i, \hat{g}_i, 0, 1)$. The omniscient sender is strictly better of than the eliciting sender under information design. This is in line with the literature. Bergemann and Morris (2016b) show that the implementable set of equilibria is larger for an omniscient than an eliciting sender in a bank run game with one sender and one receiver. Similarly, Bobkova (2017) shows that an omniscient sender has a strictly higher probability of selling a good to a buyer when the sender is omniscient than when she is eliciting.

Since an optimal disclosure policy of the omniscient sender is not implementable for the eliciting sender, we need to solve his maximization problem by taking into account the honesty constraints. The obedience constraint of the g-type after being truthful is

$$U(g_i,\hat{g}_i,0,1)=\Pr(\theta=G|g_i)\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=G,1+k_{-i}]\Pr(k_{-i}|\theta=G) \tag{$OB_g^{\hat{1}}$}$$
$$-\Pr(\theta=B|g_i)\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=B,1+k_{-i}]\Pr(k_{-i}|\theta=B)\geq 0=U(g_i,\hat{g}_i,0,0).$$

The obedience constraint of the b-type after being truthful is

$$U(b_i,\hat{b}_i,0,1)=\Pr(\theta=G|b_i)\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=G,k_{-i}]\Pr(k_{-i}|\theta=G) \tag{$OB_b^{\hat{1}}$}$$
$$-\Pr(\theta=B|b_i)\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=B,k_{-i}]\Pr(k_{-i}|\theta=B)\geq 0=U(b_i,\hat{b}_i,0,0).$$

The honesty constraint of a g-type who is obedient is then given by

$$U(g_i,\hat{g}_i,0,1)=\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=G,1+k_{-i}]\Pr(k_{-i}|\theta=G)p \tag{H_g}$$
$$-d[\hat{1}|\theta=B,1+k_{-i}]\Pr(k_{-i}|\theta=B)(1-p)$$
$$\geq\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=G,k_{-i}]\Pr(k_{-i}|\theta=G)p$$
$$-d[\hat{1}|\theta=B,k_{-i}]\Pr(k_{-i}|\theta=B)(1-p)=U(g_i,\hat{b}_i,0,1).$$

The honesty constraint of a b-type who is obedient is then given by

$$U(b_i,\hat{b}_i,0,1)=\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=G,k_{-i}]\Pr(k_{-i}|\theta=G)(1-p) \tag{H_b}$$
$$-d[\hat{1}|\theta=B,k_{-i}]\Pr(k_{-i}|\theta=B)p$$
$$\geq\sum_{k_{-i}=0}^{2}d[\hat{1}|\theta=G,k_{-i}+1]\Pr(k_{-i}|\theta=G)(1-p)$$
$$-d[\hat{1}|\theta=B,k_{-i}+1]\Pr(k_{-i}|\theta=B)p=U(b_i,\hat{g}_i,0,1).$$

Note that after recommendation $r=\hat{0}$, a voter is never pivotal and hence it does not matter whether he follows or disobeys the recommendation after misreporting. That is, $U(z_i,\hat{z}_i,1,1)=U(z_i,\hat{z}_i,0,1)$. If a voter is not obedient after recommendation $\hat{1}$, i.e., $a_i(\hat{1},z_i))=0$, then his expected utility is simply $U(z_i,\hat{z}_i,0,0)=U(z_i,\hat{z}_i,1,0)=0$. This takes care of all double-deviations, since the obedience constraints guarantee a non-negative payoff.

The maximization problem of the unrestricted eliciting sender is

$$\max_{d} \sum_{\theta\in\{B,G\}} \sum_{k=0}^{3} d[\hat{1}|\theta,k] \Pr(k|\theta)\Pr(\theta) \tag{3.13}$$

$$\text{s.t.} \quad 0 \le d[r|\theta,k] \le 1, \quad \forall r \in \{\hat{0},\hat{1}\}, \theta \in \{B,G\}, k \in \{0,1,2,3\} \tag{3.14}$$

$$d[\hat{1}|\theta,k] + d[\hat{0}|\theta,k] = 1 \quad \forall \theta \in \{B,G\}, k \in \{0,1,2,3\} \tag{3.15}$$

$$U(g_i,\hat{g}_i,0,1) \ge U(g_i,\hat{g}_i,0,0) = 0 \tag{$OB_g^{\hat{1}}$}$$

$$U(b_i,\hat{b}_i,0,1) \ge U(b_i,\hat{b}_i,0,0) = 0 \tag{$OB_b^{\hat{1}}$}$$

$$U(g_i,\hat{g}_i,0,1) \ge U(g_i,\hat{b}_i,0,1) \tag{H_g}$$

$$U(b_i,\hat{b}_i,0,1) \ge U(b_i,\hat{g}_i,0,1). \tag{H_b}$$

Lemma 4
If $OB_b^{\hat{1}}$ and H_g hold, then $OB_g^{\hat{1}}$ is satisfied.

Next, we reforumlte the Primal of the eliciting sender.

$$\max_{\substack{\{d[\hat{1}|\theta,k]\ge 0\} \\ \theta\in\{B,G\} \\ k\in\{0,1,2,3\}}} \sum_{\theta\in\{B,G\}} \sum_{k=0}^{3} d[\hat{1}|\theta,k] \Pr(k|\theta)\Pr(\theta) \tag{3.16}$$

$$\text{s.t.} \quad \sum_{k=0}^{2} \frac{3-k}{3}(d[\hat{1}|\theta=B,k]\Pr(k|\theta=B) - d[\hat{1}|\theta=G,k])\Pr(k|\theta=G) \le 0, \tag{$OB_b^{\hat{1}}$}$$

$$\sum_{k=1}^{3} \frac{k}{3}((d[\hat{1}|\theta=G,k-1] - d[\hat{1}|\theta=G,k])\Pr(k|\theta=G) - (d[\hat{1}|\theta=B,k-1] - d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)) \le 0, \tag{H_g}$$

$$\sum_{k=0}^{2} \frac{3-k}{3}((d[\hat{1}|\theta=G,k+1] - d[\hat{1}|\theta=G,k])\Pr(k|\theta=G) - (d[\hat{1}|\theta=B,k+1] - d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)) \le 0, \tag{H_b}$$

$$d[\hat{1}|\theta,k] - 1 \le 0 \quad \forall \theta \in \{B,G\}, k \in \{0,1,2,3\}. \tag{3.17}$$

Observation 9. *The information designer's maximization problem is equivalent to maximizing $\frac{1}{2}\left(\Pr(\hat{1}|b) + \Pr(\hat{1}|g)\right)$.*

After rewriting $\frac{1}{2}\left(\Pr(\hat{1}|b) + \Pr(\hat{1}|g)\right)$ into

$$\sum_{k=0}^{3} \left(d[\hat{1}|\theta=B,k]\Pr(k|\theta=B)\frac{1}{2} + d[\hat{1}|\theta=G,k]\Pr(k|\theta=G)\frac{1}{2} \right) \underbrace{\left(\frac{k}{3} + \frac{3-k}{3}\right)}_{=1} \tag{3.18}$$

one can directly see that maximizing the ex-ante type is equivalent to the objective function of the eliciting designer.

Proposition 4
The optimal disclosure policy of the eliciting information designer for $p \leq \underline{p}$ is $\forall k$,

$$d[\hat{1}|\theta = G, k] = 1, \quad d[\hat{1}|\theta = B, k] = \frac{(1-p)}{p},$$

where $\underline{p} = \frac{1}{\sqrt{2}}$.

The optimal disclosure policy of the eliciting sender for this interval of accuracy levels does not condition on the information reported by voters. The eliciting sender send $\hat{1}$ in each state of the world with a constant probability, i.e., independent of how many g-signals there were reported. Figure 3.2 shows the optimal disclosure policy of the eliciting sender for $p = 0.6$.

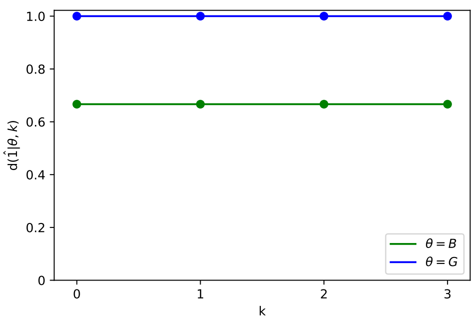


Figure 3.2: optimal policy for $p = 0.6$.

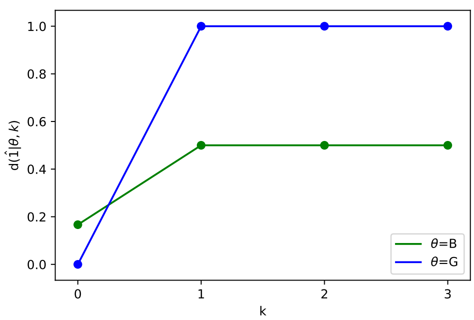


Figure 3.3: optimal policy for $p = 0.75$.

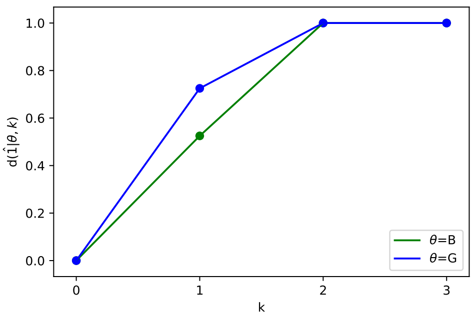


Figure 3.4: optimal policy $p = 0.9$.

Proposition 5
The optimal disclosure policy of the eliciting information designer for $\underline{p} \leq p \leq \overline{p}$ is

$$d[\hat{1}|\theta = B, k] = \begin{cases} \frac{(1-p)}{p}(2p-1) & \text{if} \quad k = 0 \\ 2(1-p) & \text{if} \quad k \neq 0 \end{cases}, \quad d[\hat{1}|\theta = G, k] = \begin{cases} 0 & \text{if} \quad k = 0 \\ 1 & \text{if} \quad k \neq 0, \end{cases} \tag{3.19}$$

where $\underline{p} = \frac{1}{\sqrt{2}}$ *and* $\overline{p} = \frac{1+\sqrt{13}}{6}$.

In contrast to the previous proposition, for this intermediate interval of accuracy levels, the sender starts to use the information reported by voters. That is, when the private information of voters' is more accurate, the sender conditions her disclosure policy on the number of reported g-signals. Moreover, the eliciting sender's optimal disclosure policy is monotone, i.e., she increases the probability with which she sends the recommendation $\hat{1}$ when the number of reported g-signals increases. Figure 3.3 shows the disclosure policy for $p = 0.75$. If $\theta = G$, with at least one $\hat{g}$-report the sender increases the probability of sending $\hat{1}$ from 0 to 1. Similarly, she sends $\hat{1}$ in $\theta = B$ more often when there is at least one $\hat{g}$-report.

Proposition 6
The optimal disclosure policy of the eliciting information designer for $\overline{p} \leq p < 1$ *is*

$$d[\hat{1}|\theta = B, k] = \begin{cases} 0 \quad \textit{if} \quad k = 0, \\ \frac{(p-\frac{1}{2})(3-p)}{2(2p-1)} \quad \textit{if}\ k = 1, \\ 1 \quad \textit{if} \quad k \in \{2,3\}, \end{cases} \qquad d[\hat{1}|\theta = G, k] = \begin{cases} 0 \quad \textit{if} \quad k = 0, \\ \frac{(p-\frac{1}{2})(p+2)}{2(2p-1)} \quad \textit{if}\ k = 1, \\ 1 \quad \textit{if} \quad k \in \{2,3\}, \end{cases}$$

for $\overline{p} = \frac{1+\sqrt{13}}{6}$.

As for the previous interval of accuracy levels, the sender makes use of the information reported by voters: she changes the probability with which she sends $\hat{1}$ depending on how many g-signals were reported. Moreover, the sender uses a monotone disclosure policy. Figure 3.4 shows the optimal disclosure policy of the eliciting sender for $p = 0.9$. Note the bang-bang structure of the optimal disclosure policy: In both states of the world, the eliciting sender recommends $\hat{1}$ with certainty whenever there are at least two $\hat{g}$-reports, she mixes between $\hat{0}$ and $\hat{1}$ when there is exactly one $\hat{g}$-report, and never recommends $\hat{1}$ when there is no $\hat{g}$-report.

Proposition 7
For $p \leq \underline{p}$*, the information designer's optimal disclosure policy is equivalent to maximizing* $\Pr(\hat{1}|b)$.

Proposition 8
For $\overline{p} \leq p < 1$*, the information designer's optimal disclosure policy is equivalent to maximizing* $\Pr(\hat{1}|g)$.

While Proposition 7 says that the information designer maximizes the probability of a b-type to vote for the proposal if $p \leq \underline{p}$, Proposition 8 says that her optimal disclosure policy is equivalent to maximizing the probability of a g-type to vote for the proposal if $p \geq \overline{p}$. Note that the expected utility of the sender is strictly decreasing in the accuracy of the voters' private signals, that is, the more convinced the voters, the less scope for persuasion. This is depicted in Figure 3.5.

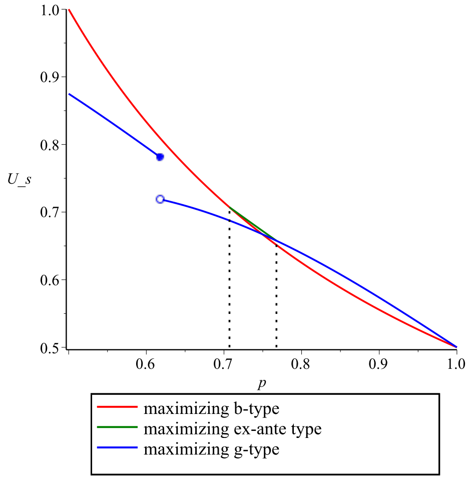


Figure 3.5: expected payoff of the eliciting sender.

6. Non-Eliciting Information Design

In this section the sender cannot ask voters for their private information. In this case the sender is not able to condition her disclosure policy on the private signals or reports of the voters. The following lemma restricts the sender's message set.

Lemma 5
Under unanimity, it is without loss of generality for optimality to restrict the message space of the non-eliciting sender to $R = \{\hat{0}, \widehat{01}, \hat{1}\}$.

As before, obedient voters vote for the proposal after $\hat{1}$ and for the status quo after $\hat{0}$. After $\widehat{01}$, only g-type voters vote for the proposal, and the b-types reject the proposal. To give some intuition for Lemma 3, observe that there exist only only three possible cases that can occur after any recommendation: either both types weakly favor the proposal, only the g-type favors for the proposal, or both types strictly dislike the proposal. A good-type never votes against the proposal while a voter with a bad signal strictly prefers the proposal. This is because a voter with a g-signal is more optimistic about $\theta = G$ than a b-type voter. As a consequence, the above three recommendations are sufficient to capture all the possible cases that can occur.

The disclosure policy of a non-eliciting sender is:

$$d : \Theta \to \Delta\{\hat{0}, \widehat{01}, \hat{1}\}. \tag{3.20}$$

The obedience constraints for each type of voter after $\hat{1}$ are given by:

$$U_i(g_i, a_i(\hat{1}, g_i) = 1) = \sum_{\theta \in \{B,G\}} d[\hat{1}|\theta] \Pr(\theta|g_i)(\mathbb{1}_{\theta=G} - \frac{1}{2}) \geq 0 = U_i(g_i, a_i(\hat{1}, g_i) = 0)$$

$$U_i(b_i, a_i(\hat{1}, b_i) = 1) = \sum_{\theta \in \{B,G\}} d[\hat{1}|\theta] \Pr(\theta|b_i)(\mathbb{1}_{\theta=G} - \frac{1}{2}) \geq 0 = U_i(b_i, a_i(\hat{1}, b_i) = 0)$$

The obedience constraints for each type of voter after $\widehat{01}$ are given by:

$$U_i(g_i, a_i(\widehat{01}, g_i) = 1) = \sum_{\theta \in \{B,G\}} d[\widehat{01}|\theta] \Pr(k = 3|\theta) \Pr(\theta)(\mathbb{1}_{\theta=G} - \frac{1}{2})$$
$$\geq 0 = U_i(g_i, a_i(\hat{1}, g_i) = 0),$$
$$U_i(b_i, a_i(\widehat{01}, b_i) = 1) = \sum_{\theta \in \{B,G\}} d[\widehat{01}|\theta] \Pr(k = 2|\theta) \Pr(\theta)(\mathbb{1}_{\theta=G} - \frac{1}{2})$$
$$\leq 0 = U_i(b_i, a_i(\hat{1}, b_i) = 0).$$

The sender's maximization problem becomes:

$$\max_d \quad \sum_{\theta \in \{B,G\}} (d[\hat{1}|\theta] + d[\widehat{01}|\theta] \Pr(k = 3|\theta)) \Pr(\theta) \tag{3.21}$$
$$\text{s.t.} \quad 0 \leq d[r|\theta] \leq 1, \quad \forall r \in \{\hat{0}, \widehat{01}, \hat{1}\}, \theta \in \{B, G\} \tag{3.22}$$
$$d[\hat{1}|\theta] + d[\widehat{01}|\theta] + d[\hat{0}|\theta] = 1 \quad \forall \theta \in \{B, G\} \tag{3.23}$$
$$U_i(g_i, a_i(\hat{1}, g_i) = 1) \geq U_i(g_i, a_i(\hat{1}, g_i) = 0) = 0 \tag{$OB_g^{\hat{1}}$}$$
$$U_i(b_i, a_i(\hat{1}, b_i) = 1) \geq U_i(b_i, a_i(\hat{1}, b_i) = 0) = 0 \tag{$OB_b^{\hat{1}}$}$$
$$U_i(g_i, a_i(\widehat{01}, g_i) = 1) \geq U_i(g_i, a_i(\widehat{01}, g_i) = 0) \tag{$OB_g^{\widehat{01}}$}$$
$$U_i(b_i, a_i(\widehat{01}, b_i) = 1) \leq U_i(b_i, a_i(\widehat{01}, b_i) = 0) \tag{$OB_b^{\widehat{01}}$}$$

The next result shows the solution to the above problem of a non-eliciting information designer.

Proposition 9
The optimal disclosure policy of the non-eliciting sender for $p \leq \tilde{p}$ is

$$d[\hat{1}|\theta = G] = 1, \qquad d[\hat{1}|\theta = B] = \frac{1-p}{p},$$

$$d[\hat{0}|\theta = G] = 0, \qquad d[\hat{0}|\theta = B] = \frac{2p-1}{p}.$$

The optimal disclosure policy of the non-eliciting sender for $p \geq \tilde{p}$ is

$$d[\hat{1}|\theta = G] = 1 - \frac{(2p-1)(1-p)^3}{(p^4 - (1-p)^4)}, \qquad d[\hat{1}|\theta = B] = 1 - \frac{(2p-1)p^3}{p^4 - (1-p)^4}$$

$$d[\widehat{01}|\theta = G] = \frac{(2p-1)(1-p)^3}{p^4 - (1-p)^4}, \qquad d[\widehat{01}|\theta = B] = \frac{(2p-1)p^3}{p^4 - (1-p)^4},$$

where $\tilde{p} = \sqrt[4]{\frac{1}{2}}$.

Note that for any p, the optimal disclosure policy of the designer never contains more than two messages. When comparing the optimal disclosure policy of an eliciting and non-eliciting information designer, it becomes apparent that they are equivalent for $p \leq \underline{p} = \frac{1}{\sqrt{2}}$. The eliciting information designer has no advantage from asking voters about their private information when the accuracy of signals is sufficiently small.

Corollary 4. *Let $\underline{p} = \frac{1}{\sqrt{2}}$. For $p \leq \underline{p}$, the eliciting sender's optimal disclosure policy is equivalent to the non-eliciting sender's optimal disclosure policy.*

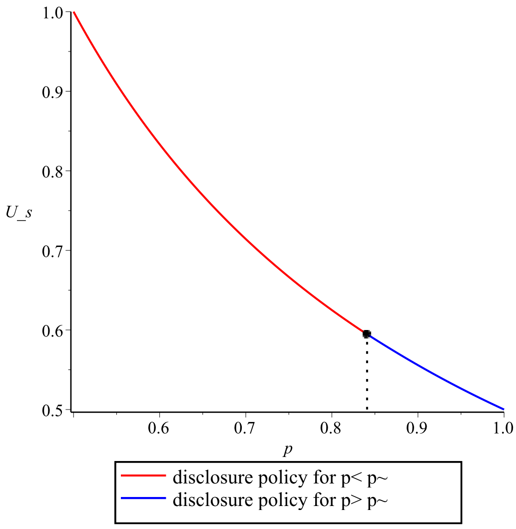


Figure 3.6: non-eliciting sender's expected payoff.

7. Further Remarks

7.1 Eliciting Sender: Comparison to Feddersen and Pesendorfer (1998)

Next, consider the error probabilities of a wrong decision when $\theta = B$ (i.e., $\Pr(a = 1|\theta = B)$) and implementing the status quo when $\theta = G$ (i.e., $\Pr(a = 0|\theta = B)$) under the optimal disclosure policy d of the sender.

We compare the probabilities of making each type of error in our setting to the corresponding probabilities of making each type of error in Feddersen and Pesendorfer (1998), where voters have to act based on their private signals without any coordination device or sender, and the decision rule is unanimity. Applied to our setting, the following is a Nash equilibrium in their model: a voter with a g-signal always votes in favor of the reform, and a b-signal voter in favor of the reform with probability $\frac{\sqrt{p}\left(p+\sqrt{p(1-p)}-1\right)}{p^{\frac{3}{2}}-(1-p)^{\frac{3}{2}}}$.

First, the probability of choosing the proposal when $\theta = B$ in our setting is bigger for all $p \in (\frac{1}{2}, 1)$ (Figure 3.7). Second, the probability of choosing the status quo when $\theta = G$ in our setting is smaller for all $p \in (\frac{1}{2}, 1)$ (Figure 3.8).

Taken together, while in Feddersen and Pesendorfer (1998) the proposal is more often implemented when $\theta = B$, in our setting the proposal is more often implemented

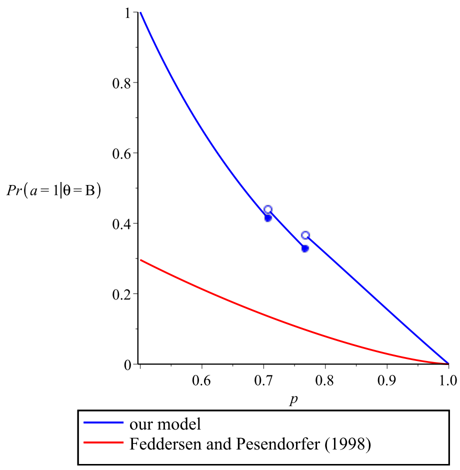


Figure 3.7: error probabilities if $\theta = B$.

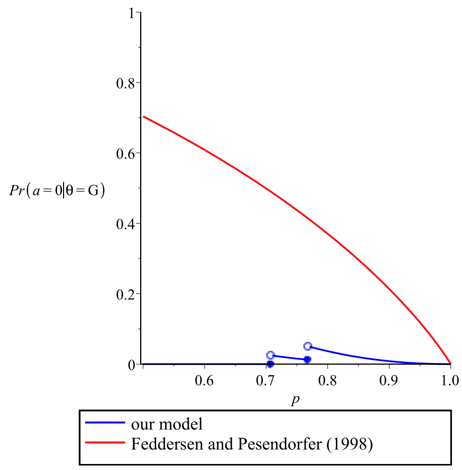


Figure 3.8: error probabilities if $\theta = G$.

when $\theta = G$. A manipulative information designer strictly decreases the probability of rejecting the proposal when the proposal is efficient while increasing the probability of rejecting the status quo when the status quo is efficient.

Overall, the ex-ante type (a voter whose private signal has not yet realized) receives a strictly higher expected payoff in our model than in Feddersen and Pesendorfer (1998).

Proposition 10
Under unanimity, voters have a strictly higher expected utility with an eliciting information designer than in a symmetric equilibrium as in Feddersen and Pesendorfer (1998).

Hence, even with a manipulative sender, voters are better off in expectations compared to the situation in which they have to decide on their own under unanimity.

7.2 Omniscient Sender: One Agent with Multiple Signals

In this section we are comparing our previous results from Section 3.1 to the case where the omniscient sender faces only one voter which receives a signal $s \in \{0, 1, 2, 3\}$. One possible interpretation for this setting is the following: Imagine voters could communicate with each other and exchange their private signals before the information designer sends her public recommendation. In this case every voter has exactly the same information, i.e., knows how many g-signals there are. As a consequence, the sender's problem is equivalent to persuading one representative voter that can possibly have four different signals which correspond to the number of g-signals. Formally, the sender's problem becomes

$$\max_{d} \sum_{\theta \in \{B,G\}} \sum_{k=0}^{3} d[\hat{1}|\theta, k] \Pr(k|\theta) \Pr(\theta) \tag{3.24}$$

$$\Pr(\theta = G|\hat{1}, s) - \frac{1}{2} \geq 0 \quad \forall s \in \{0, 1, 2, 3\}. \tag{$OB^{\hat{1}}$}$$

The obedience constraint of the representative voter can be rewritten into

$$
\begin{aligned}
& \Pr(\theta = G|\hat{1}, s) - \frac{1}{2} \geq 0 && (3.25)\\
\Leftrightarrow \quad & \frac{\Pr(\theta = G, \hat{1}, s)}{\Pr(\hat{1}, s)} \geq \frac{1}{2} && (3.26)\\
\Leftrightarrow \quad & \frac{Pr(\hat{1}|\theta = G, s)\Pr(\theta = G, s)}{\Pr(\hat{1}, s)} \geq \frac{1}{2} && (3.27)\\
\Leftrightarrow \quad & \frac{d[\hat{1}|\theta = G, k]\Pr(k|\theta = G)\Pr(\theta = G)}{\Pr(\hat{1}, k)} \geq \frac{1}{2} && (3.28)\\
\Leftrightarrow \quad & d[\hat{1}|\theta = G, k]\Pr(k|\theta = G) \geq d[\hat{1}|\theta = B, k]\Pr(k|\theta = B) && (3.29)
\end{aligned}
$$

In contrast to before, the omniscient sender who faces only one agent which has one out of four signal realizations, has to take four obedience constraints into account, one for each $s \in \{0, 1, 2, 3\}$. Due to the perfect alignment of interests of the sender and the representative voter in $\theta = G$, the omniscient sender will optimally send $\hat{1}$ in $\theta = G$ with certainty for each $s \in \{0, 1, 2, 3\}$. Simple calculations show that the voter's obedience constraint is slack if the omniscient sender also sends $\hat{1}$ with probability one for $s \in \{2, 3\}$. For $s \in \{0, 1\}$ she will choose $d[\hat{1}|\theta = B, k]$ such that the obedience constraint just binds. That is

$$
\begin{aligned}
& 3p^2(1-p) = d[\hat{1}|\theta = B, k = 1]3p(1-p)^2 && (3.30)\\
\Leftrightarrow \quad & d[\hat{1}|\theta = B, k = 1] = \frac{(1-p)}{p} && (3.31)
\end{aligned}
$$

and

$$
\begin{aligned}
& p^3 = d[\hat{1}|\theta = B, k = 0](1-p)^3 && (3.32)\\
\Leftrightarrow \quad & d[\hat{1}|\theta = B, k = 0] = \frac{(1-p)^3}{p^3}. && (3.33)
\end{aligned}
$$

Hence, when the omniscient sender faces only one representative voter who knows the number of g-signals, she optimally uses the following disclosure policy

$$
d[\hat{1}|\theta = G, k] = 1 \ \forall k, \quad d[\hat{1}|\theta = B, k] = \begin{cases} 1 & \text{if} \quad k \in \{2, 3\} \\ \frac{(1-p)}{p} & \text{if} \quad k = 1 \\ \frac{(1-p)^3}{p^3} & \text{if} \quad k = 0. \end{cases} \qquad (3.34)
$$

The omniscient sender's expected payoff in this case is given by $\frac{1}{2} + \frac{1}{2}(2 - 2p)$. This is always lower than the omniscient sender's expected payoff facing three informed voters who cannot communicate. Since the omniscient sender's optimal disclosure policy in $\theta = G$ is exactly the same in both settings and the total probability with which she sends $\hat{1}$ in $\theta = B$ is smaller when voters know the signals of each other, voters are better off if they know each other's signals.

8. Conclusion

We study a biased sender who tries to persuade three voters to vote for a proposal by sending public recommendations. Voters receive some private signals about the quality of the proposal and only want to approve the proposal if it is of high quality. We characterize the optimal disclosure policy under unanimity rule of a 1. omniscient, 2. eliciting, and 3. non-eliciting sender. We find that the eliciting sender can only profit from her ability to ask voters for their private signals when the accuracy of their private information is sufficiently high. Whenever the accuracy level is below some lower threshold the eliciting sender is just equally well of as the non-eliciting sender who cannot ask voters for reports about their private signals.

We show that depending on the accuracy level of the private signals of voters the optimal disclosure policy of the eliciting sender solves two other related maximization problems: For accuracy levels below some lower threshold, the eliciting sender maximizes the probability that a pessimistic voter votes for the proposal. For accuracy levels above some upper threshold, the eliciting sender maximizes the probability of an optimistic voter approving the proposal. Voters are better off in the presence of a biased informed designer than in a setting where they have to vote under unanimity based on their private exogenous information only as in Feddersen and Pesendorfer (1998).

In this work we consider a sender who knows the true quality of the proposal. Extending the analysis to a restricted sender who is uninformed about the true quality and can only send public recommendations on the basis of the reports made by voters is a potential avenue for future research.

A. Appendix

A.1 Proof of Proposition 1

Proof. We show that given any outcome of disclosure policy d', the sender can implement an outcome equivalent policy d consisting of only two recommendations $R = \{\hat{0}, \hat{1}\}$. Message $\hat{0}$ is the recommendation to vote with probability 1 for the status quo irrespective of the private signal, and $\hat{1}$ is the recommendation to vote with probability 1 for proposal irrespective of the private signal.

Consider any arbitrary disclosure policy of the sender $d' : \Theta \times \{0, \dots, 3\} \to \Delta(R')$, with R' being any arbitrary message set. Denote $r' \in R'$ an element of the message space. Let $a(r', z_i)$ be the probability of a voter i with signal z_i voting for the proposal after seeing recommendation r' under disclosure policy d'.[6]

Now consider a filtering d of the original information disclosure policy d' of the following form.

$$d : R' \times K \to \Delta[\hat{0}, \hat{1}].$$

The new disclosure policy takes the realized message r' in the original disclosure policy and the number of g-signals k of the voters, and maps them into a binary voting recommendation. With slight abuse of notation, denote by $d(\hat{1}|r', k)$ the probability of sending recommendation $\hat{1}$ in favor of the proposal.

Consider the following construction for the new disclosure policy d:

$$d(\hat{1}|r', k) = a(r', g)^k a(r', b)^{3-k}. \tag{3.35}$$

It is immediate that this policy yields the same expected utility to the sender (if implementable), as the probability with which she sends signal $\hat{1}$ corresponds to the probability with which her preferred outcome would have been elected under the original disclosure policy d'.

Furthermore, this disclosure policy is designed to be proportional to the original probability of being pivotal for both voter types. In particular, for $z_i = b$ and $k \in \{0, 1, 2\}$, we have $\Pr(piv|r, k_{-i} = k, b) = a(r', g)^k a(r', b)^{2-k}$ and hence,

$$d(\hat{1}|r', k) = a(r', b) \Pr(piv|r', k_{-i} = k).$$

Similarly, for $z_i = g$ and $k \in \{0, 1, 2\}$, we have $\Pr(piv|r, k_{-i} = k, g) = a(r', g)^{k-1} a(r', b)^{3-k}$ and thus,

$$d(\hat{1}|r', k+1) = a(r', g) \Pr(piv|r', k_{-i} = k).$$

It is left to show that the obedience constraints are also satisfied under the new disclosure policy d'. After public recommendation $\hat{0}$ no voter is ever pivotal. It suffices to show, that both private information types have an incentive to follow the recommendation $\hat{1}$ by voting for the proposal. The obedience constraint of a voter with private signal z_i is:

[6] We assume that voters with the same signal react symmetrically to the same recommendation, $a_i(r', z_i) = a_j(r', z_j)$ if $z_i = z_j$. Hence, we drop index i.

$$\Pr(\theta = G|\hat{1}, z_i, piv) \geq \frac{1}{2} \quad \forall z_i \in \{g_i, b_i\} \tag{3.36}$$

This can be rewritten into

$$\Pr(\theta = G|z_i)\Pr(\hat{1}|\theta = G, z_i)\underbrace{\Pr(piv|\hat{1}, \theta = G, z_i)}_{=1}$$
$$\geq \Pr(\theta = B|z_i)\Pr(\hat{1}|\theta = B, z_i)\underbrace{\Pr(piv|\hat{1}, \theta = B, z_i)}_{=1}.$$

Thus, the two obedience constraints can be expressed as

$$p\Pr(\hat{1}|\theta = G, g_i) \geq (1-p)\Pr(\hat{1}|\theta = B, g_i), \tag{$OB_g^{\hat{1}}$}$$
$$(1-p)\Pr(\hat{1}|\theta = G, b_i) \geq p\Pr(\hat{1}|\theta = B, b_i). \tag{$OB_b^{\hat{1}}$}$$

We can invoke the construction of the new disclosure policy d from d' to rewrite:

$$\Pr(\hat{1}|\theta, g_i) = \sum_{r' \in R'} \Pr(\hat{1}, r'|\theta, g_i) \tag{3.37}$$
$$= \sum_{r' \in R'} \sum_{k_{-i}=0}^{2} \Pr(\hat{1}, r', k_{-i}|\theta, g_i) \tag{3.38}$$
$$= \sum_{r' \in R'} \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|\theta, g_i)\Pr(r'|k_{-i}, \theta, g_i)\Pr(\hat{1}|r', k_{-i}, \theta, g_i) \tag{3.39}$$
$$= \sum_{r' \in R'} \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|\theta, g_i)\Pr(r'|k_{-i}, \theta, g_i)d(\hat{1}|r', k = k_{-i}+1) \tag{3.40}$$

Analogously, for $z_i = b$ we have

$$\Pr(\hat{1}|\theta, b_i) = \sum_{r' \in R'} \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|\theta, b_i)\Pr(r'|k_{-i}, \theta, b_i)d(\hat{1}|r', k = k_{-i}) \tag{3.41}$$

We can use this to rewrite the obedience constraints under the new disclosure policy:

$$\sum_{r' \in R'} \sum_{k_{-i}=0}^{2} p\Pr(k_{-i}|\theta = G, g_i)\Pr(r'|k_{-i}, \theta = G, g_i)d(\hat{1}|r', k = k_{-i}+1)$$
$$\geq \sum_{r' \in R'} \sum_{k_{-i}=0}^{2} (1-p)\Pr(k_{-i}|\theta = B, g_i)\Pr(r'|k_{-i}, \theta = B, g_i)d(\hat{1}|r', k = k_{-i}+1)$$

$$\Leftrightarrow \sum_{r'\in R'} \sum_{k_{-i}=0}^{2} d(\hat{1}|r', k = k_{-i}+1)[p \Pr(k_{-i}|\theta = G, g_i) \Pr(r'|k_{-i}, \theta = G, g_i) \\ -(1-p) \Pr(k_{-i}|\theta = B, g_i) \Pr(r'|k_{-i}, \theta = B, g_i)] \geq 0. \qquad (OB_g^{\hat{1}})$$

Using the construction of the new disclosure policy, we can rewrite

$$\sum_{r'\in R'} a(r', g_i) \sum_{k_{-i}=0}^{2} \Pr(piv|r, k_{-i}, b_i)[p \Pr(k_{-i}|\theta = G, g_i) \Pr(r'|k_{-i}, \theta = G, g_i) \\ -(1-p) \Pr(k_{-i}|\theta = B, g_i) \Pr(r'|k_{-i}, \theta = B, g_i)] \geq 0. \qquad (3.42)$$

Analoguously, for $z_i = b$ we have

$$\sum_{r'\in R'} a(r', b_i) \sum_{k_{-i}=0}^{2} \Pr(piv|r', k_{-i}, b_i)[(1-p) \Pr(k_{-i}|\theta = G, b_i) \Pr(r'|k_{-i}, \theta = G, b_i) \\ -p \Pr(k_{-i}|\theta = B, b_i) \Pr(r'|k_{-i}, \theta = B, b_i)] \geq 0. \qquad (OB_b^{\hat{1}})$$

Note that the original disclosure policy d' is implementable, i.e. the obedience constraint holds for each type $z_i \in \{g, b\}$ and each message $r' \in R'$, when $a_i(r', z_i) > 0$. That is,

$$\sum_{k_{-i}=0}^{2} \Pr(piv|r', k_{-i}, \theta = G)[p \Pr(k_{-i}|\theta = G, g_i) \Pr(r'|k_{-i}, \theta = G, g_i) \\ -(1-p) \Pr(k_{-i}|\theta = B, g_i) \Pr(r'|k_{-i}, \theta = B, g_i)] \geq 0. \qquad (OB_g^{r'})$$

$$\sum_{k_{-i}=0}^{2} \Pr(piv|r', k_{-i}, \theta = G)[(1-p) \Pr(k_{-i}|\theta = G, b_i) \Pr(r'|k_{-i}, \theta = G, b_i) \\ -p \Pr(k_{-i}|\theta = B, b_i) \Pr(r'|k_{-i}, \theta = B, b_i)] \geq 0. \qquad (OB_b^{r'})$$

The inner sums in the obedience constraints under the new disclosure policy d, $OB_b^{\hat{1}}$ and $OB_g^{\hat{1}}$, correspond to the original obedience constraints, $OB_g^{r'}$ and $OB_b^{r'}$, under the former disclosure policy d'. This establishes that the filtering d satisfies both obedience constraints and yields the same payoff to the designer.

□

A.2 Proof of Lemma 1

Proof. We prove this by contradiction. Assume that the disclosure policy d is optimal and that there exists k' such that $d[\hat{1}|\theta = G, k'] \neq 1$. Then, construct a new disclosure policy d' that is equal to the old disclosure policy d for all $k \neq k'$ and θ. For $k = k'$ and $\theta = G$, it sends recommendation $\hat{1}$ with probability 1. That is $d'[\hat{1}|\theta = B, k] = d[\hat{1}|\theta = B, k]\ \forall k'$, $d'[\hat{1}|\theta = G, k] = d[\hat{1}|\theta = G, k]\ \forall k' \neq k$ and $d'[\hat{1}|k'] = 1 \neq d[\hat{1}|k']$. Next we check whether the new disclosure policy d' still fulfills the voters' obedience constraints. Note that if the recommendation $\hat{0}$ was sent, no voter is ever pivotal, which is why this doesn't influence the obedience constraints. Sending the recommendation $\hat{1}$ for any k when $\theta = G$ increases the left hand side of the voters' obedience constraints and thus makes them easier to satisfy. The new disclosure policy relaxed the voters' obedience constraints and sends $\hat{1}$ with a strictly higher probability.

□

A.3 Proof of Proposition 2

Proof. The only part of above proposition that we have not proven is $d[\hat{1}|\theta = B, k \neq 3]$. In the following we will use the greedy algorithm (Dantzig, 1957) to solve this problem. We have a fractional knapsack problem of the following form:

Find $0 \leq x_k = d[\hat{1}|\theta = B, k] \leq 1$ for $k \in \{0, 1, 2\}$ s.t.

1) $\sum_{k=0}^{2} x_k w_k \leq \frac{(1-p)}{p}$ holds and

2) $\sum_{k=0}^{2} x_k v_k$ is maximized,

where $w_k = \frac{3-k}{3} \Pr(k|\theta = B)$ and $v_k = \Pr(k|\theta = B) \cdot 1$. We refer to w_k as the weight and to v_k as the value of k.

Next we calculate the value-per-weight ratio $\rho_k = \frac{v_k}{w_k}$ for $k \in \{0, 1, 2\}$:

$$\rho_k = \frac{\Pr(k|\theta = B)}{\frac{3-k}{3} \Pr(k|\theta = B)} = \frac{3}{3-k} \tag{3.43}$$

Variable ρ_k is increasing in k. Hence, if we sort the k's by decreasing ρ_k, we get the following order $2, 1, 0$. How much probability mass we can place on each k until the obedience constraint of the b-signal voter is binding, will depend on the accuracy level of the voters' private signals p.

□

A.4 Proof of Proposition 3

Proof. Consider the same filtering of the original disclosure policy d' into d as in the proof of Proposition 1:

$$d(\hat{1}|r', k) = a(r', g)^k a(r', b)^{3-k}.$$

It remains to be shown that this disclosure policy d satisfies the honesty constraints for each type. The proof of optimality for the designer, and the validity of the obedience constraints were already established in Proposition 1. We prove that the above filtering satisfies the honesty constraints of the g-type. The argument for the b-type is accordingly, and therefore omitted.

Expected utility in equilibrium with d and d'. First, we show that the old disclosure policy d' and new disclosure policy d yields exactly the same expected utility to the g-type in equilibrium, when reporting truthfully. Let R be the message set of the designer with d'.[7]

$$EU(g_i, \hat{g}_i; d') = \sum_{r\in R}\sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i)\Pr(r|g_i, \hat{g}_i, k_{-i}; d')a_i(r, g)^{k_{-i}+1}a_i(r, b)^{2-k_{-i}}q(r, g_i, \hat{g}_i, k_{-i}), \tag{3.44}$$

where $q(r, g_i, \hat{g}_i, k_{-i}) = E\left[\theta|r, g_i, \hat{g}_i, k_{-i}\right] - \frac{1}{2}$ is the expected net utility from implementing the proposal if a g-type voter reported truthfully, k_{-i} others also have a g-signal and the designer sent recommendation r. The factor $a_i(r, g)^{k_{-i}+1}a_i(r, b)^{2-k_{-i}}$ accounts for the probability of being pivotal $(a_i(r, g)^{k_{-i}}a_i(r, b)^{2-k_{-i}})$ times the probability of the g-type voter i voting for the reform $a_i(r, g)$ in equilibrium.

Next, consider the expected utility under the new disclosure policy d. Whenever the designer sends recommendation $\hat{1}$ (which happens with probability $a_i(r, g)^{k_{-i}+1}a_i(r, b)^{2-k_{-i}}$ if recommendation r would have been sent in d') the reform is implemented.

$$\begin{aligned} EU(g_i, \hat{g}_i; d) &= \sum_{r\in R}\sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i)\Pr(r|g_i, \hat{g}_i, k_{-i}; d')d(\hat{1}|r, k = k_{-i}+1)q(r, g_i, \hat{g}_i, k_{-i}) \\ &= \sum_{r\in R}\sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i)\Pr(r|g_i, \hat{g}_i, k_{-i}; d')a_i(r, g)^{k_{-i}+1}a_i(r, b)^{2-k_{-i}}q(r, g_i, \hat{g}_i, k_{-i}). \end{aligned} \tag{3.45}$$

This coincides with the utility under the original disclosure policy in Equation 3.44, $EU(g_i, \hat{g}_i; d) = EU(g_i, \hat{g}_i; d')$.

Expected utility from misreporting in d'. Next, consider the utility of a g-voter who reports $\hat{b}$ in the original disclosure policy d'. To account for double deviations, we denote by $\tilde{a}_i(r, g_i, \hat{b}_i)$ his action after observing r when reporting $\hat{b}$.

[7] For convenience of notation, we assume that R is finite.

$$EU(g_i, \hat{b}_i; d') = \sum_{r \in R} \max_{\tilde{a}_i(r,g_i,\hat{b}_i) \in [0,1]} \tilde{a}_i(r, g_i, \hat{b}_i) \cdot \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i) \Pr(r|g_i, \hat{b}_i, k_{-i}; d') a_i(r, g)^{k_{-i}} a_i(r, b)^{2-k_{-i}} q(r, g_i, \hat{b}_i, k_{-i}). \tag{3.46}$$

For simplicity of notation, denote by $EU(r|g_i, \hat{b}_i)$ the optimal utility after misreporting and observing recommendation r. Note that $EU(r|g_i, \hat{b}_i) \geq 0$ for all r, as a voter can always derive zero utility by voting against the reform. Hence,

$$EU(g_i, \hat{b}_i; d') = \sum_{r \in R} EU(r|g_i, \hat{b}_i). \tag{3.47}$$

Expected utility from misreporting in d. Finally, consider the expected utility after misreporting in the new disclosure policy d. The voter optimizes over his action $\tilde{a}(\hat{1}, g_i, \hat{b}_i)$ after recommendation $\hat{1}$ after misreporting. For simplicity, we assume that the voter votes for the reform with probability $\tilde{a}(\hat{1}, g_i, \hat{b}_i)$ after both recommendations $\hat{1}$ and $\hat{0}$, as his utility after recommendation $\hat{0}$ yields utility 0 irrespective of his action. The difference to d' is that he might not know which r lead to the recommendation $\hat{1}$.

$$EU(g_i, \hat{b}_i; d) = \max_{\tilde{a}(\hat{1},g_i,\hat{b}_i) \in [0,1]} \tilde{a}(\hat{1}, g_i, \hat{b}_i) \cdot \sum_{r \in R} \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i) \Pr(r|g_i, \hat{b}_i, k_{-i}; d') \underbrace{a_i(r, g)^{k_{-i}} a_i(r, b)^{3-k_{-i}}}_{=\Pr(\hat{1}|r,k=k_{-i})} q(r, g_i, \hat{b}_i, k_{-i}). \tag{3.48}$$

If the voter knew which r of the original disclosure policy d' led to recommendation $\hat{1}$, he would be better off: he could adapt his voting decision $\tilde{a}(\hat{1}, r, g_i, \hat{g}_i)$ to each r (instead of choosing the same $\tilde{a}(\hat{1}, g_i, \hat{g}_i)$ for all r that led to $\hat{1}$). Thus,

$$EU(g_i, \hat{b}_i; d) \leq \sum_{r \in R} \max_{\tilde{a}(\hat{1},r,g_i,\hat{b}_i) \in [0,1]} \tilde{a}(\hat{1}, r, g_i, \hat{b}_i) \cdot \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i) \Pr(r|g_i, \hat{b}_i, k_{-i}; d') \underbrace{a_i(r, g)^{k_{-i}} a_i(r, b)^{3-k_{-i}}}_{=d(\hat{1}|r,k=k_{-i})} q(r, g_i, \hat{b}_i, k_{-i}) \tag{3.49}$$

$$= \sum_{r \in R} a_i(r, b) \max_{\tilde{a}(\hat{1},r,g_i,\hat{b}_i) \in [0,1]} \tilde{a}(\hat{1}, r, g_i, \hat{b}_i) \cdot \sum_{k_{-i}=0}^{2} \Pr(k_{-i}|g_i) \Pr(r|g_i, \hat{b}_i, k_{-i}; d') a_i(r, g)^{k_{-i}} a_i(r, b)^{2-k_{-i}} q(r, g_i, \hat{b}_i, k_{-i}), \tag{3.50}$$

where the last inequality follows by putting the non-negative factor $a_i(r, b)$ outside the maximum. But then note that the maximization problem point-wise after each r is exactly the same as in Equation 3.46 for the original disclosure policy, $EU(r|g_i, \hat{b}_i)$, which is non-negative. Hence, the optimal deviation utility is bounded above by

$$\begin{aligned} EU(g_i, \hat{b}_i; d) &\leq \sum_{r\in R} \underbrace{a_i(r,b)}_{\in[0,1]} \underbrace{EU(r|g_i,\hat{b}_i)}_{\geq 0} && (3.51) \\ &\leq \sum_{r\in R} EU(r|g_i,\hat{b}_i) && (3.52) \\ &= EU(g_i,\hat{b}_i; d'). && (3.53) \end{aligned}$$

The payoff after a misreport and an optimal best response is weakly lower than in the original disclosure policy. Note that the original disclosure policy d' by assumption satisfied all constraints, including the honesty constraint of the g-type. Hence, we established that the honesty constraint of the g-type holds by proving

$$EU(g_i, \hat{g}_i; d) = EU(g_i, \hat{g}_i; d') \geq EU(g_i, \hat{b}_i; d') \geq EU(g_i, \hat{b}_i; d).$$

□

A.5 Proof of Lemma 4

Proof. First, we rewrite the honesty constraint of g-type to sum over k instead of k_{-i}.

$$\begin{aligned} &\sum_{k=1}^{3} \frac{k}{3}((d[\hat{1}|\theta = G, k-1] - d[\hat{1}|\theta = G,k]) \Pr(k|\theta = G) && (H_g) \\ &\qquad - (d[\hat{1}|\theta = B, k-1] - d[\hat{1}|\theta = B,k]) \Pr(k|\theta = B)) \leq 0 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \quad & \underbrace{\sum_{k=1}^{3} \frac{k}{3}(d[\hat{1}|\theta = B,k] \Pr(k|\theta = B) - d[\hat{1}|\theta = G,k]) \Pr(k|\theta = G)}_{OB_g^{\hat{1}}} \\ \leq \quad & \underbrace{\sum_{k=1}^{3} \frac{k}{3}(d[\hat{1}|\theta = B,k-1] \Pr(k|\theta = B) - d[\hat{1}|\theta = G,k-1] \Pr(k|\theta = G)}_{*}. \end{aligned}$$

Observe that we have rewritten H_g such that the LHS of H_g is just $OB_g^{\hat{1}}$. We rewrite $(*)$, i.e., the RHS of H_g, into

$$\sum_{k=0}^{2} \frac{k+1}{3}(d[\hat{1}|\theta = B,k] \Pr(k+1|\theta = B) - d[\hat{1}|\theta = G,k] \Pr(k+1|\theta = G). \qquad (3.54)$$

Next, we subtract $\sum_{k=0}^{2} \frac{3-k}{3}(d[\hat{1}|\theta = B,k] \Pr(k|\theta = B) - d[\hat{1}|\theta = G,k]) \Pr(k|\theta = G) \leq$

0, which is just $OB(b) \leq 0$ rewritten, and get

$$\sum_{k=0}^{2}(d[\hat{1}|\theta = B, k](\frac{3-k}{3}\Pr(k+1|\theta = B) - \frac{k}{3}\Pr(k|\theta = B)) \\ - d[\hat{1}|\theta = G, k](\frac{3-k}{3}\Pr(k+1|\theta = G) - \frac{k}{3}\Pr(k|\theta = G)). \tag{3.55}$$

Rewriting yields

$$\sum_{k=0}^{2}(d[\hat{1}|\theta = B, k]p^{2-k}(1-p)^{k}(1-2p) - d[\hat{1}|\theta = G, k]p^{k}(1-p)^{2-k}(2p-1) \leq 0.$$

Thus, we have that for all $k \leq 2$ the expression above is negative. Note that $(*) - OB_b^{\hat{1}}$ is the sum of these negative terms and is thus also negative. Moreover, $OB_g^{\hat{1}} \leq 0$ implies that $(*) \leq 0$. Since $OB_g^{\hat{1}} \leq (*)$ by equation (3.54), we have that $OB_g^{\hat{1}} \leq 0$, which proves the lemma.

□

A.6 Proof of Proposition 4

Proof. First, we take the Dual of the Primal and get

$$\min_{\substack{\lambda_{OB_b^{\hat{1}}}\geq 0,\ \lambda_{H_g}\geq 0,\ \lambda_{H_b}\geq 0 \\ \{\mu_{\theta,\,k}\geq 0\}_{\theta\in\{B,G\},\,k\in\{0,1,2,3\}}}} \quad \sum_{k=0}^{3}\mu_{\theta=B,\,k} + \sum_{k=0}^{3}\mu_{\theta=G,\,k} \tag{3.56}$$

$$\text{s.t.} \quad p^{3-k}(1-p)^{k}\frac{1}{2}\left(-\binom{3}{k} + (\lambda_{OB_b^{\hat{1}}} + \lambda_{H_b})\binom{2}{k} + \lambda_{H_g}\binom{2}{k-1}\right) \\ - \lambda_{H_g}p^{2-k}(1-p)^{k+1}\binom{2}{k} - \lambda_{H_b}p^{4-k}(1-p)^{k-1}\binom{2}{k-1} \\ + \mu_{\theta=B,\,k} \ \geq\ 0 \quad \forall k \in \{0,1,2,3\} \tag{3.57}$$

$$p^{k}(1-p)^{3-k}\frac{1}{2}\left(-\binom{3}{k} - (\lambda_{OB_b^{\hat{1}}} - \lambda_{H_b})\binom{2}{k} - \lambda_{H_g}\binom{2}{k-1}\right) \\ + \lambda_{H_g}p^{k+1}(1-p)^{2-k}\binom{2}{k} + \lambda_{H_b}p^{k-1}(1-p)^{4-k}\binom{2}{k-1} \\ + \mu_{\theta=G,\,k} \ \geq\ 0 \quad \forall k \in \{0,1,2,3\} \tag{3.58}$$

Let $\{d[\hat{1}|\theta, k] \geq 0\}_{\theta\in\{B,G\},k\in\{0,1,2,3\}}$ be a feasible disclosure policy for the primal, and $\{\vec{\lambda}, \vec{\mu}\}$ feasible vector for the dual. Necessary and sufficient conditions for them to be optimal are

$$\lambda_{OB_b^{\hat{1}}} \cdot \left(\sum_{k=0}^{2} \binom{2}{k} (d[\hat{1}|\theta = B, k] \Pr(k|\theta = B)\frac{1}{2} - d[\hat{1}|\theta = G, k] \Pr(k|\theta = G)\frac{1}{2}) \right) = 0 \tag{3.59}$$

$$\lambda_{H_g} \cdot (\sum_{k=1}^{3} \binom{2}{k-1} ((d[\hat{1}|\theta = G, k-1] - d[\hat{1}|\theta = G, k]) \Pr(k|\theta = G)\frac{1}{2} - (d[\hat{1}|\theta = B, k-1] - d[\hat{1}|\theta = B, k]) \Pr(k|\theta = B)\frac{1}{2})) = 0 \tag{3.60}$$

$$\lambda_{H_b} \cdot (\sum_{k=0}^{2} \binom{2}{k} ((d[\hat{1}|\theta = G, k+1] - d[\hat{1}|\theta = G, k]) \Pr(k|\theta = G)\frac{1}{2} - (d[\hat{1}|\theta = B, k+1] - d[\hat{1}|\theta = B, k]) \Pr(k|\theta = B)\frac{1}{2})) = 0 \tag{3.61}$$

$$\mu_{\theta=B,\,k} \cdot \left(d[\hat{1}|\theta = B, k] - 1\right) = 0 \quad \forall k \in \{0, 1, 2, 3\} \tag{3.62}$$

$$\mu_{\theta=G,\,k} \cdot \left(d[\hat{1}|\theta = G, k] - 1\right) = 0 \quad \forall k \in \{0, 1, 2, 3\} \tag{3.63}$$

$$d[\hat{1}|\theta = B, k] \cdot (p^{3-k}(1-p)^k \left(-\binom{3}{k} + (\lambda_{OB_b^{\hat{1}}} + \lambda_{H_b})\binom{2}{k} + \lambda_{H_g}\binom{2}{k-1} \right) - \lambda_{H_g} p^{2-k}(1-p)^{k+1}\binom{2}{k} - \lambda_{H_b} p^{4-k}(1-p)^{k-1}\binom{2}{k-1} + \mu_{\theta=B,\,k}) = 0 \quad \forall k \in \{0, 1, 2, 3\} \tag{3.64}$$

$$d[\hat{1}|\theta = G, k] \cdot (p^{k}(1-p)^{3-k} \left(-\binom{3}{k} - (\lambda_{OB_b^{\hat{1}}} + \lambda_{H_b})\binom{2}{k} - \lambda_{H_g}\binom{2}{k-1} \right) + \lambda_{H_g} p^{k+1}(1-p)^{2-k}\binom{2}{k} + \lambda_{H_b} p^{k-1}(1-p)^{4-k}\binom{2}{k-1} + \mu_{\theta=G,\,k}) = 0 \quad \forall k \in \{0, 1, 2, 3\}. \tag{3.65}$$

The dual variables

$$\{\vec{\lambda}, \vec{\mu}\} = \begin{pmatrix} \lambda_{OB_b^{\hat{1}}} = \frac{1}{p} \\ \lambda_{H_g} = 1 \\ \lambda_{H_b} = 0 \\ \mu_{\theta=B,k} = 0 \quad \forall k \in \{0, \ldots, 3\} \\ \mu_{\theta=G,\,k} = p^{k-1}(1-p)^{2-k} \cdot \left(p(1-p)\left(\binom{3}{k} + \binom{2}{k-1}\right) + (1-p-p^2)\binom{2}{k} \right) \\ \forall k \in \{0, 1, 2, 3\} \end{pmatrix} \tag{3.66}$$

and the disclosure policy $d[\hat{1}|\theta = G] = 1$, $d[\hat{1}|\theta = B] = \frac{(1-p)}{p}$ fulfill the above complementary slackness conditions for all $p \leq \frac{1}{\sqrt{2}}$.

After inserting $d[\hat{1}|\theta = G] = 1$ and $d[\hat{1}|\theta = B] = \frac{1-p}{p}$ in 3.59-3.61 and 3.62, one can easily see that the terms in brackets are zero. Thus, we get that $\lambda_{OB_b^{\hat{1}}} \geq 0$, $\lambda_{H_g} \geq 0$, $\lambda_{H_b} \geq 0$ and $\mu_{\theta=G,k} \geq 0 \quad \forall\ k \in \{0, \ldots, 3\}$. Since $d[\hat{1}|\theta, k] > 0 \quad \forall\, \theta \in \{B, G\}, k \in \{0, 1, 2, 3\}$, the terms in brackets must be zero. By using that

$$\left(p^3\left(-1 + \lambda_{OB_b^{\hat{1}}} + \lambda_{H_b}\right) - \lambda_{H_g} p^2(1-p) + \mu_{\theta=B,\,k=0}\right) = 0 \quad \text{and} \tag{3.67}$$

$$\left((1-p)^3\left(-1 + \lambda_{H_g}\right) - \lambda_{H_b} p(1-p)^2 + \mu_{\theta=B,\,k=3}\right) = 0 \tag{3.68}$$

we can solve for $\lambda_{OB_b^{\hat{1}}} = \frac{1}{2p}$ and $\lambda_{H_b} = \frac{(1-p)(\lambda_{H_g}-1)}{p}$. For $\lambda_{H_b} \geq 0$ we need that $\lambda_{H_g} \geq \frac{1}{2}$. Choosing $\lambda_{H_g} = \frac{1}{2}$ implies that $\lambda_{H_b} = 0$. Inserting these values for $\lambda_{OB_b^{\hat{1}}}$, λ_{H_g} and λ_{H_b} and solving for $\mu_{\theta=G,\,k=0}$ yields $\mu_{\theta=G,\,k=0} = (1-p)^2\left(\frac{1-2p^2}{p}\right)$, which is ≥ 0 if and only if $0.5 < p \leq \frac{1}{\sqrt{2}}$. For $k \in \{1, 2\}$ we get that

$$\mu_{\theta=G,\,k} = p^{k-1}(1-p)^{2-k} \cdot \left(p(1-p)(\binom{3}{k} + \binom{2}{k-1}) + (1-p-p^2)\binom{2}{k}\right) \geq 0 \tag{3.69}$$

for all $p \leq \frac{1}{\sqrt{2}}$.

□

A.7 Proof of Proposition 5

Proof. The disclosure policy in Proposition 5 and the dual variables

$$\{\vec{\lambda}, \vec{\mu}\} = \begin{pmatrix} \lambda_{OB_b^{\hat{1}}} = \frac{1}{p} \\ \lambda_{H_g} = 1 \\ \lambda_{H_b} = 0 \\ \mu_{\theta=B,k} = 0 \quad \forall k \in \{0, \ldots, 3\} \\ \mu_{\theta=G,k=0} = 0 \\ \mu_{\theta=G,k=1} = 2(1-p)(1+p-3p^2) \\ \mu_{\theta=G,\,k=2} = p(1-p)(5p+1) - p^3 \\ \mu_{\theta=G,\,k=3} = 2p^3 \end{pmatrix} \tag{3.70}$$

fulfill the above complementary slackness conditions for all $p \leq \overline{p}$.

□

A.8 Proof of Proposition 6

Proof. The disclosure policy in Proposition 6 and the dual variables

$$\{\vec{\lambda}, \vec{\mu}\} = \begin{pmatrix} \lambda_{OB_b^{\hat{1}}} = \frac{3(3p^2-3p+1)}{2(2p-1)} \\ \lambda_{H_g} = \frac{3p(1-p)}{2p-1} \\ \lambda_{H_b} = 0 \\ \mu_{\theta=B,k} = 0 \quad \forall k \in \{0,1\} \\ \mu_{\theta=B,k=2} = \frac{3p(1-p)^2(3p(1+p)-1)}{2(2p-1)} \\ \mu_{\theta=B,k=3} = (1-p)^3 \left(\frac{p(3p-1)-1}{2p-1}\right) \\ \mu_{\theta=G,k} = 0 \quad \forall k \in \{0,1\} \\ \mu_{\theta=G,k=2} = \frac{3p^2(1-p)(p(5-3p)-1)}{2(2p-1)} \\ \mu_{\theta=G,k=3} = p^3 \left(\frac{p(5-3p)-1}{2p-1}\right) \end{pmatrix} \tag{3.71}$$

fulfill the above complementary slackness conditions for all $\overline{p} \leq p < 1$.

□

A.9 Proof of Proposition 7

Proof. We prove this proposition by the standard Primal-Dual-technique. The primal of the related problem of maximizing $\Pr(\hat{1}|b)$ is given by:

$$\max_{\substack{\{d[\hat{1}|\theta,k]\geq 0\} \\ \theta\in\{B,G\} \\ k\in\{0,1,2,3\}}} \sum_{k=0}^{3} \left(d[\hat{1}|\theta=B,k] \Pr(k|\theta=B)\frac{1}{2} + d[\hat{1}|\theta=G,k] \Pr(k|\theta=G)\frac{1}{2} \right) \frac{3-k}{3} \tag{3.72}$$

$$\text{s.t.} \quad \sum_{k=0}^{2} \frac{3-k}{3} (d[\hat{1}|\theta=B,k] \Pr(k|\theta=B) - d[\hat{1}|\theta=G,k] \Pr(k|\theta=G)) \leq 0 \tag{$OB_b^{\hat{1}}$}$$

$$\sum_{k=1}^{3} \frac{k}{3} ((d[\hat{1}|\theta=G,k-1] - d[\hat{1}|\theta=G,k]) \Pr(k|\theta=G) - (d[\hat{1}|\theta=B,k-1] - d[\hat{1}|\theta=B,k]) \Pr(k|\theta=B)) \leq 0 \tag{H_g}$$

$$\sum_{k=0}^{2} \frac{3-k}{3} ((d[\hat{1}|\theta=G,k+1] - d[\hat{1}|\theta=G,k]) \Pr(k|\theta=G) - (d[\hat{1}|\theta=B,k+1] - d[\hat{1}|\theta=B,k]) \Pr(k|\theta=B)) \leq 0 \tag{H_b}$$

$$d[\hat{1}|\theta,k] - 1 \leq 0 \quad \forall \theta \in \{B,G\}, k \in \{0,1,2,3\} \tag{3.73}$$

Then, we take the Dual of the Primal and get:

$$\min_{\substack{\lambda_{OB_b^{\hat{1}}}\geq 0,\ \lambda_{H_g}\geq 0,\ \lambda_{H_b}\geq 0 \\ \{\mu_{\theta,\,k}\geq 0\}_{\theta\in\{B,G\},\,k\in\{0,1,2,3\}}}} \quad \sum_{k=0}^{3}\mu_{\theta=B,\,k}+\sum_{k=0}^{3}\mu_{\theta=G,\,k} \tag{3.74}$$

$$\begin{aligned} \text{s.t.}\quad & p^{3-k}(1-p)^k\frac{1}{2}\left(-\frac{3-k}{3}\binom{3}{k}+(\lambda_{OB_b^{\hat{1}}}+\lambda_{H_b})\binom{2}{k}+\lambda_{H_g}\binom{2}{k-1}\right) \\ & -\lambda_{H_g}p^{2-k}(1-p)^{k+1}\binom{2}{k}-\lambda_{H_b}p^{4-k}(1-p)^{k-1}\binom{2}{k-1} \\ & +\mu_{\theta=B,\,k}\ \geq\ 0\quad \forall k\in\{0,1,2,3\} \end{aligned} \tag{3.75}$$

$$\begin{aligned} & p^{k}(1-p)^{3-k}\frac{1}{2}\left(-\frac{3-k}{3}\binom{3}{k}-(\lambda_{OB_b^{\hat{1}}}-\lambda_{H_b})\binom{2}{k}-\lambda_{H_g}\binom{2}{k-1}\right) \\ & +\lambda_{H_g}p^{k+1}(1-p)^{2-k}\binom{2}{k}+\lambda_{H_b}p^{k-1}(1-p)^{4-k}\binom{2}{k-1} \\ & +\mu_{\theta=G,\,k}\ \geq\ 0\quad \forall k\in\{0,1,2,3\}. \end{aligned} \tag{3.76}$$

Let $\{d[\hat{1}|\theta,k]\geq 0\}_{\theta\in\{B,G\},k\in\{0,1,2,3\}}$ be a feasible disclosure policy for the primal, and $\{\vec{\lambda},\vec{\mu}\}$ feasible vector for the dual. Necessary and sufficient conditions for them to be optimal are

$$\lambda_{OB_b^{\hat{1}}}\cdot\left(\sum_{k=0}^{2}\frac{3-k}{3}(d[\hat{1}|\theta=B,k]\Pr(k|\theta=B)\frac{1}{2}-d[\hat{1}|\theta=G,k]\Pr(k|\theta=G)\frac{1}{2})\right)=0 \tag{3.77}$$

$$\begin{aligned} \lambda_{H_g}\cdot(\sum_{k=1}^{3}\frac{k}{3}((d[\hat{1}|\theta=G,k-1]-d[\hat{1}|\theta=G,k])\Pr(k|\theta=G)\frac{1}{2} \\ -(d[\hat{1}|\theta=B,k-1]-d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)\frac{1}{2}))=0 \end{aligned} \tag{3.78}$$

$$\begin{aligned} \lambda_{H_b}\cdot(\sum_{k=0}^{2}\frac{3-k}{3}((d[\hat{1}|\theta=G,k+1]-d[\hat{1}|\theta=G,k])\Pr(k|\theta=G)\frac{1}{2} \\ -(d[\hat{1}|\theta=B,k+1]-d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)\frac{1}{2}))=0 \end{aligned} \tag{3.79}$$

$$\mu_{\theta=B,\,k}\cdot\Big(d[\hat{1}|\theta=B,k]-1\Big)=0\quad\forall k\in\{0,1,2,3\} \tag{3.80}$$

$$\mu_{\theta=G,\,k}\cdot\Big(d[\hat{1}|\theta=G,k]-1\Big)=0\quad\forall k\in\{0,1,2,3\} \tag{3.81}$$

$$d[\hat{1}|\theta=B,k]\cdot(p^{3-k}(1-p)^k\frac{1}{2}\left(-\frac{3-k}{3}\binom{3}{k}+(\lambda_{OB_b^{\hat{1}}}+\lambda_{H_b})\binom{2}{k}+\lambda_{H_g}\binom{2}{k-1}\right) \tag{3.82}$$

$$- \lambda_{H_g} p^{2-k}(1-p)^{k+1}\binom{2}{k} - \lambda_{H_b} p^{4-k}(1-p)^{k-1}\binom{2}{k-1}$$
$$+ \mu_{\theta=B,\,k}\Big) = 0 \quad \forall k \in \{0,1,2,3\}$$

$$d[\hat{1}|\theta = G,k] \cdot \Big(p^k(1-p)^{3-k}\frac{1}{2}\left(-\frac{3-k}{3}\binom{3}{k} - (\lambda_{OB_b^{\hat{1}}} + \lambda_{H_b})\binom{2}{k} - \lambda_{H_g}\binom{2}{k-1}\right)\Big) \tag{3.83}$$
$$+ \lambda_{H_g} p^{k+1}(1-p)^{2-k}\binom{2}{k} + \lambda_{H_b} p^{k-1}(1-p)^{4-k}\binom{2}{k-1}$$
$$+ \mu_{\theta=G,\,k}\Big) = 0 \quad \forall k \in \{0,1,2,3\}$$

The disclosure policy

$$d[\hat{1}|\theta = G] = 1, \qquad d[\hat{1}|\theta = B] = \frac{(1-p)}{p} \tag{3.84}$$

and the dual variables

$$\{\vec{\lambda}, \vec{\mu}\} = \begin{pmatrix} \lambda_{OB_b^{\hat{1}}} = 1 \\ \lambda_{H_g} = 0 \\ \lambda_{H_b} = 0 \\ \mu_{\theta=B,k} = 0 \quad \forall k \in \{0,\dots,3\} \\ \mu_{\theta=G,k=0} = 2(1-p)^3 \\ \mu_{\theta=G,\,k} = 4p^k(1-p)^{3-k} \quad \forall k \in \{1,2\} \\ \mu_{\theta=G,\,k=3} = 0 \end{pmatrix} \tag{3.85}$$

fulfill the above complementary slackness conditions for all $p \in (\frac{1}{2}, 1]$.

□

A.10 Proof of Proposition 8

Proof. The primal of the related problem of maximizing $\Pr(\hat{1}|g)$ is given by:

$$\max_{\substack{\{d[\hat{1}|\theta,k]\geq 0\} \\ \theta\in\{B,G\} \\ k\in\{0,1,2,3\}}} \sum_{k=0}^{3} \left(d[\hat{1}|\theta = B,k]\Pr(k|\theta = B)\frac{1}{2} + d[\hat{1}|\theta = G,k]\Pr(k|\theta = G)\frac{1}{2}\right)\frac{k}{3} \tag{3.86}$$

$$\text{s.t.} \quad \sum_{k=0}^{2} \frac{3-k}{3}(d[\hat{1}|\theta = B,k]\Pr(k|\theta = B)\frac{1}{2} - d[\hat{1}|\theta = G,k])\Pr(k|\theta = G)\frac{1}{2} \leq 0 \quad (OB_b^{\hat{1}})$$

$$\sum_{k=1}^{3}\frac{k}{3}((d[\hat{1}|\theta=G,k-1]-d[\hat{1}|\theta=G,k]\Pr(k|\theta=G)\frac{1}{2}) \qquad (H_g)$$

$$-(d[\hat{1}|\theta=B,k-1]-d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)\frac{1}{2}))\leq 0$$

$$\sum_{k=0}^{2}\frac{3-k}{3}((d[\hat{1}|\theta=G,k+1]-d[\hat{1}|\theta=G,k])\Pr(k|\theta=G)\frac{1}{2} \qquad (H_b)$$

$$-(d[\hat{1}|\theta=B,k+1]-d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)\frac{1}{2})\leq 0$$

$$d[\hat{1}|\theta,k]-1\leq 0 \quad \forall\theta\in\{B,G\},k\in\{0,1,2,3\} \tag{3.87}$$

Then, we take the Dual of the Primal and get

$$\min_{\substack{\lambda_{OB_b^{\hat{1}}}\geq 0,\ \lambda_{H_g}\geq 0,\ \lambda_{H_b}\geq 0\\ \{\mu_{\theta,\,k}\geq 0\}_{\theta\in\{B,G\},\,k\in\{0,1,2,3\}}}} \quad \sum_{k=0}^{3}\mu_{\theta=B,\,k}+\sum_{k=0}^{3}\mu_{\theta=G,\,k} \tag{3.88}$$

$$\text{s.t.}\quad p^{3-k}(1-p)^{k}\frac{1}{2}\left(-\frac{k}{3}\binom{3}{k}+(\lambda_{OB_b^{\hat{1}}}+\lambda_{H_b})\binom{2}{k}+\lambda_{H_g}\binom{2}{k-1}\right) \tag{3.89}$$
$$-\lambda_{H_g}p^{2-k}(1-p)^{k+1}\binom{2}{k}-\lambda_{H_b}p^{4-k}(1-p)^{k-1}\binom{2}{k-1}$$
$$+\mu_{\theta=B,\,k}\ \geq\ 0 \quad \forall k\in\{0,1,2,3\}$$

$$p^{k}(1-p)^{3-k}\frac{1}{2}\left(-\frac{k}{3}\binom{3}{k}-(\lambda_{OB_b^{\hat{1}}}-\lambda_{H_b})\binom{2}{k}-\lambda_{H_g}\binom{2}{k-1}\right) \tag{3.90}$$
$$+\lambda_{H_g}p^{k+1}(1-p)^{2-k}\binom{2}{k}+\lambda_{H_b}p^{k-1}(1-p)^{4-k}\binom{2}{k-1}$$
$$+\mu_{\theta=G,\,k}\ \geq\ 0 \quad \forall k\in\{0,1,2,3\}.$$

Let $\{d[\hat{1}|\theta,k]\geq 0\}_{\theta\in\{B,G\},k\in\{0,1,2,3\}}$ be a feasible disclosure policy for the primal, and $\{\vec{\lambda},\vec{\mu}\}$ feasible vector for the dual. Necessary and sufficient conditions for them to be optimal are

$$\lambda_{OB_b^{\hat{1}}}\cdot\left(\sum_{k=0}^{2}\frac{3-k}{3}(d[\hat{1}|\theta=B,k]\Pr(k|\theta=B)\frac{1}{2}-d[\hat{1}|\theta=G,k]\Pr(k|\theta=G)\frac{1}{2})\right)=0$$

$$\lambda_{H_g}\cdot(\sum_{k=1}^{3}\frac{k}{3}((d[\hat{1}|\theta=G,k-1]-d[\hat{1}|\theta=G,k])\Pr(k|\theta=G)\frac{1}{2}$$
$$-(d[\hat{1}|\theta=B,k-1]-d[\hat{1}|\theta=B,k])\Pr(k|\theta=B)\frac{1}{2}))=0$$

$$\lambda_{H_b} \cdot (\sum_{k=0}^{2} \frac{3-k}{3}((d[\hat{1}|\theta = G, k+1] - d[\hat{1}|\theta = G, k]) \Pr(k|\theta = G)\frac{1}{2} \tag{3.91}$$
$$- (d[\hat{1}|\theta = B, k+1] - d[\hat{1}|\theta = B, k]) \Pr(k|\theta = B)\frac{1}{2})) = 0$$

$$\mu_{\theta=B,\,k} \cdot \Big(d[\hat{1}|\theta = B, k] - 1\Big) = 0 \quad \forall k \in \{0,1,2,3\}$$

$$\mu_{\theta=G,\,k} \cdot \Big(d[\hat{1}|\theta = G, k] - 1\Big) = 0 \quad \forall k \in \{0,1,2,3\}$$

$$d[\hat{1}|\theta = B, k] \cdot (p^{3-k}(1-p)^k \frac{1}{2}\left(-\frac{k}{3}\binom{3}{k} + (\lambda_{OB_b^{\hat{1}}} + \lambda_{H_b})\binom{2}{k} + \lambda_{H_g}\binom{2}{k-1}\right)$$
$$- \lambda_{H_g} p^{2-k}(1-p)^{k+1}\binom{2}{k} - \lambda_{H_b} p^{4-k}(1-p)^{k-1}\binom{2}{k-1}$$
$$+ \mu_{\theta=B,\,k}) = 0 \quad \forall k \in \{0,1,2,3\}$$

$$d[\hat{1}|\theta = G, k] \cdot (p^{k}(1-p)^{3-k} \frac{1}{2}\left(-\frac{k}{3}\binom{3}{k} - (\lambda_{OB_b^{\hat{1}}} + \lambda_{H_b})\binom{2}{k} - \lambda_{H_g}\binom{2}{k-1}\right)$$
$$+ \lambda_{H_g} p^{k+1}(1-p)^{2-k}\binom{2}{k} + \lambda_{H_b} p^{k-1}(1-p)^{4-k}\binom{2}{k-1}$$
$$+ \mu_{\theta=G,\,k}) = 0 \quad \forall k \in \{0,1,2,3\}.$$

The disclosure policy

$$d[\hat{1}|\theta = B, k] = \begin{cases} 0 \quad \text{if} \quad k = 0 \\ \frac{(p-\frac{1}{2})(3-p)}{2(2p-1)} \quad \text{if} \quad k = 1 \\ 1 \quad \text{if} \quad k \in \{2,3\} \end{cases}, \quad d[\hat{1}|\theta = G, k] = \begin{cases} 0 \quad \text{if} \quad k = 0 \\ \frac{(p-\frac{1}{2})(3p^2+5p-2)}{2(6p^2-5p+1)} \quad \text{if} \quad k = 1 \\ 1 \quad \text{if} \quad k \in \{2,3\}. \end{cases} \tag{3.92}$$

and the dual variables

$$\{\vec{\lambda}, \vec{\mu}\} = \begin{pmatrix} \lambda_{OB_b^{\hat{1}}} = \frac{1+3p(p-1)}{2(2p-1)} \\ \lambda_{H_g} = \frac{p(1-p)}{2p-1} \\ \lambda_{H_b} = 0 \\ \mu_{\theta=B,k} = 0 \quad \forall k \in \{0,1\} \\ \mu_{\theta=B,k=2} = \frac{3p(1-p)^2 3(p^2+p-1)}{2(2p-1)} \\ \mu_{\theta=B,k=3} = (1-p)^3\left(\frac{p^2+p-1}{2p-1}\right) \\ \mu_{\theta=G,k} = 0 \quad \forall k \in \{0,1\} \\ \mu_{\theta=G,k=2} = \frac{3p^2(1-p)(3p-p^2-1)}{2(2p-1)} \\ \mu_{\theta=G,k=3} = p^3\left(\frac{3p-p^2-1}{2p-1}\right) \end{pmatrix} \tag{3.93}$$

fulfill the above complementary slackness conditions for all $\overline{p} \leq p < 1$.

□

A.11 Proof of Lemma 5

Proof. Let the designer follow a disclosure policy d', that sends messages $r' \in R'$, and voters responding optimally to this disclosure policy. Denote by $a_i(r', z_i)$ the probability, that voter i with private signal $z_i \in \{g, b\}$ votes $\hat{1}$ after receiving message r'.

The first step is to show, that the g-type is always weakly more optimistic than the b-type for any signal that is sent with strictly positive probability in some state θ. The following formulation makes use of the fact that $\Pr(r'|\theta, z_i) = \Pr(r'|\theta)$ and $\Pr(piv|r', \theta, z_i) = \Pr(piv|r', \theta)$, as the designer cannot use or elicit the private information of the agents.

Lemma 6
Under any non-eliciting disclosure policy, in any equilibrium, the g-type is more optimistic than the b-type: $\Pr(\theta = G|r', g, piv) \geq \Pr(\theta = G|r', b, piv)$.

Proof.

$$\begin{aligned}
&\Pr(\theta = G|r', g, piv) \\
&= \frac{\Pr(\theta = G|g) \Pr(r'|\theta = G) \Pr(piv|r', \theta = G)}{\Pr(\theta = G|g) \Pr(r'|\theta = G) \Pr(piv|r', \theta = G) + \Pr(\theta = B|g) \Pr(r'|\theta = B) \Pr(piv|r', \theta = B)} \\
&= \frac{\frac{1}{2}p \Pr(r'|\theta = G) \Pr(piv|r', \theta = G)}{p \Pr(r'|\theta = G) \Pr(piv|r', \theta = G) + \frac{1}{2}(1-p) \Pr(r'|\theta = B) \Pr(piv|r', \theta = B)} \\
&\geq \frac{\frac{1}{2}(1-p) \Pr(r'|\theta = G) \Pr(piv|r', \theta = G)}{(1-p) \Pr(r'|\theta = G) \Pr(piv|r', \theta = G) + \frac{1}{2}p \Pr(r'|\theta = B) \Pr(piv|r', \theta = B)} \\
&= \Pr(\theta = G|r', b, piv)
\end{aligned}$$

□

Using Lemma 6, the following is a complete partition of the designer's messages:

1. $R'(\hat{0}) := \{r' \in R' : a_i(r', g) = a_i(r', b) = 0\}$
2. $R'(\hat{1}) := \{r' \in R' : a_i(r', g) > 0 \quad \wedge \quad a_i(r', b) > 0\}$
3. $R'(\hat{01}) := \{r' \notin R'(\hat{0}) \cup R'(\hat{1})\}$.

Consider the following alternative policy d, that takes the old disclosure policy d' and maps it into a message space $R = \{\hat{0}, \hat{01}, \hat{1}\}$, for all states $\theta \in \{B, G\}$:

$$d(\hat{0}|\theta, r') = \begin{cases} 1 \quad \text{if} \quad r' \in R'(\hat{0}) \\ 1 - a_i(r', g)^{n-1} \quad \text{if} \quad r' \in R'(\hat{01}) \\ 1 - \Pr(piv|r', \theta) \quad \text{if} \quad r' \in R'(\hat{1}) \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.94}$$

$$d(\hat{01}|\theta, r') = \begin{cases} a_i(r', g)^{n-1} \quad \text{if} \quad r' \in R'(\hat{01}) \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.95}$$

$$d(\hat{1}|\theta, r') = \begin{cases} \Pr(piv|\theta, r') \quad \text{if} \quad r' \in R'(\hat{1}) \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.96}$$

Consider the following equilibrium under the new disclosure policy d:

$$a_i(r, b) = \begin{cases} 0 \quad \text{if} \quad r = \hat{0} \\ 0 \quad \text{if} \quad r = \hat{01} \\ 1 \quad \text{if} \quad r = \hat{1} \end{cases} \qquad \text{and} \qquad a_i(r, g) = \begin{cases} 0 \quad \text{if} \quad r = \hat{0} \\ 1 \quad \text{if} \quad r = \hat{01} \\ 1 \quad \text{if} \quad r = \hat{1} \end{cases} \tag{3.97}$$

First, we establish that the above policy together with the voting behavior in Equation 3.97 is an equilibrium. Then, we show that the designer weakly prefers the disclosure policy d with the restricted message set to the original disclosure policy d'.

Under the new disclosure policy d, after recommendation $\hat{0}$, no agent is ever pivotal; he therefore has no profitable deviation from voting for the status quo.

After realization $\hat{1}$, each agent is pivotal with probability 1. The next lemma is useful in establishing the obedience constraints of voters after realization $\hat{1}$.

Lemma 7
For $r' \in R'(\hat{1})$, we have $\Pr(\theta = G|r', \hat{1}, b) = \Pr^{d'}(\theta = G|r', piv, b)$.

For $r' \in R'(\hat{01})$ and $z_i \in \{b, g\}$, we have $\Pr(\theta = G|r', piv, \hat{01}, z_i) = \Pr^{d'}(\theta = G|r', piv, z_i)$.

Proof. Simple calculation show for $r' \in R'(\hat{1})$:

$$\Pr(\theta = G|r', \hat{1}, b) = \frac{\Pr(\theta = G|r', b) \Pr(\hat{1}|r', \theta = G)}{\Pr(\theta = G|r', b) \Pr(\hat{1}|r', \theta = G) + \Pr(\theta = B|r', b) \Pr(\hat{1}|r', \theta = B)} \tag{3.98}$$

and

$$\overset{d'}{\Pr}(\theta = G|r', piv, b) = \frac{\Pr^{d'}(\theta = G|r', b) \Pr^{d'}(piv|r', \theta = G)}{\Pr^{d'}(\theta = G|r', b) \Pr^{d'}(piv|r', \theta = G) + \Pr^{d'}(\theta = B|r', b) \Pr^{d'}(piv|r', \theta = B)} \tag{3.99}$$

The lemma follows by construction of the new disclosure policy d, where $\Pr(1|r', \hat{\theta} = G) = \Pr(piv|r', \theta = G)$ and $\Pr(1|r', \hat{\theta} = B) = \Pr(piv|r', \theta = B)$.

Analogously, for $r' \in R'(\hat{01})$:

$$\overset{d'}{\Pr}(\theta = G|r', piv, b) = \frac{\Pr^{d'}(\theta = G|r', b) \Pr^{d'}(piv|r', \theta = G)}{\Pr^{d'}(\theta = G|r', b) \Pr^{d'}(piv|r', \theta = G) + \Pr^{d'}(\theta = B|r', b) \Pr^{d'}(piv|r', \theta = B)}$$

and

$$\Pr(\theta = G|r', \hat{1}, b)$$

$$= \frac{\Pr(\theta = G|r', b) \Pr(\hat{01}|r', \theta = G) \Pr(piv|\theta = G, \hat{01})}{\Pr(\theta = G|r', b) \Pr(\hat{1}|r', \theta = G) \Pr(piv|\theta = G, \hat{01}) + \Pr(\theta = B|r', b) \Pr(\hat{1}|r', \theta = B) \Pr(piv|\theta = B, \hat{01})}$$

Under the old disclosure policy, we have $\Pr^{d'}(piv|r', \theta = G) = p^{n-1} a_i(r', g)^{n-1}$. With the new disclosure policy, we have $\underbrace{\Pr(\hat{01}|\theta = G, r')}_{=a_i(r',g)^{n-1}} \underbrace{\Pr(piv|\theta = G, \hat{01})}_{=p^{n-1}}$, which is exactly equal to $\Pr^{d'}(piv|r', \theta = G)$. By the same argument, we have $\Pr^{d'}(piv|r', \theta = B) = \Pr(\hat{01}|\theta = B, r')$, which proves the lemma.

□

Lemma 8
For $r' \in R'(\hat{1})$, we have: $\sum_{r' \in R'(\hat{1})} \Pr(r'|\hat{1}, b) = 1$. For $r' \in R'(\hat{01})$, and $z_i \in \{g, b\}$, we have $\sum_{r' \in R'(\hat{1})} \Pr(r'|\hat{1}, piv, z_i) = 1$.

Proof. First, consider $r' \in R'(\hat{1})$.

$$\begin{aligned}
\sum_{r' \in R'(\hat{1})} \Pr(r'|\hat{1}, b) &= \sum_{r' \in R'(\hat{1})} \frac{\Pr(r', \hat{1}|b)}{\Pr(\hat{1}|b)} \\
&= \frac{\sum_{r' \in R'(\hat{1})} \Pr(r', \hat{1}|b)}{\sum_{r' \in R'(\hat{0}) \vee R'(\hat{01}) \vee R'(\hat{1})} \Pr(\hat{1}, r'|b)} \\
&= \frac{\sum_{r' \in R'(\hat{1})} \Pr(r', \hat{1}|b)}{\sum_{r' \in R'(\hat{0})} \underbrace{\Pr(\hat{1}, r'|b)}_{=0} + \sum_{r' \in R'(\hat{01})} \underbrace{\Pr(\hat{1}, r'|b)}_{=0} + \sum_{r' \in R'(\hat{1})} \Pr(\hat{1}, r'|b)} \\
&= 1
\end{aligned}$$

The last step follows, because $\hat{1}$ is only send with strictly positive probability if $r' \in R'(\hat{1})$.

Next, consider $r' \in R'(\hat{01})$.

$$\sum_{r' \in R'(\hat{01})} \Pr(r'|\hat{01}, piv, z_i) = \sum_{r' \in R'(\hat{01})} \frac{\Pr(r', piv|\hat{01}, z_i)}{\Pr(piv|\hat{01}, z_i)} =$$

$$= \frac{\sum_{r' \in R'(\hat{01})} \Pr(r', piv|\hat{01}, z_i)}{\sum_{r' \in R'(\hat{0})} \underbrace{\Pr(piv, r'|\hat{01}, z_i)}_{=0} + \sum_{r' \in R'(\hat{01})} \Pr(piv, r'|\hat{01}, z_i) + \sum_{r' \in R'(\hat{1})} \underbrace{\Pr(piv, r'|\hat{01}, z_i)}_{=0}}$$

$$= 1.$$

The last step follows because $\hat{01}$ is only send with strictly positive probability if $r' \in R'(\hat{01})$.

□

The belief of each voter after $\hat{1}$ about the state being good, $\Pr(\theta = G|\hat{1}, z_i)$, is a convex combination of the beliefs under the old disclosure policy $\{\Pr(\theta = G|r', piv, z_i)\}_{r' \in R(\hat{1})}$, as the following calculation shows.

$$\begin{aligned}
\Pr(\theta = G|\hat{1}, piv, b) &= \Pr(\theta = G|\hat{1}, b) \\
&= \sum_{r' \in R'(\hat{1})} \Pr(r'|\hat{1}, b) \Pr(\theta = G|r', \hat{1}, b) \\
&=^{\text{Lemma 7}} \sum_{r' \in R'(\hat{1})} \Pr(r'|\hat{1}, b) \underbrace{\overset{d'}{\Pr}(\theta = G|r', piv, b)}_{\geq \frac{1}{2}} \geq \frac{1}{2}.
\end{aligned}$$

We have $\Pr^{d'}(\theta = G|r', piv, b) \geq \frac{1}{2}$, because $a_i(r', b) > 0$ in the original equilibrium for d': the b-type (weakly) prefers the proposal to the status quo.

Because the g-type is more optimistic under any disclosure policy (Lemma 6), we also have $\Pr(\theta = G|\hat{1}, g) \geq \Pr(\theta = G|\hat{1}, b) \geq \frac{1}{2}$. The g-type prefers the proposal to the status quo after observing $\hat{1}$. Both voters have hence no profitable deviation from voting for the proposal.

Finally, consider a signal $\hat{01}$. Using Lemma 6, as the g-type is always more optimistic than the b-type, we have $a_i(r', b) = 0$ and $a_i(r', g) > 0$ for any recommendation $r' \in R'(\hat{01})$. As both voter types are pivotal with non-zero probability (by assumption, r' is sent with strictly positive probability), we have

$$\Pr(\theta = G|\hat{01}, piv, z_i) = \sum_{r' \in R'(\hat{01})} \Pr(r'|\hat{01}, piv, z_i) \Pr(\theta = G|r', \hat{01}, piv, z_i) \tag{3.100}$$

$$= \sum_{r' \in R'(\hat{01})} \Pr(r'|\hat{01}, piv, z_i) \underbrace{\overset{d'}{\Pr}(\theta = G|r', piv, z_i)}_{\geq(\leq)\frac{1}{2} \text{ if } z_i = g(=b)} \tag{3.101}$$

$$\begin{cases} \geq \frac{1}{2} & \text{if} \quad z_i = g \\ \leq \frac{1}{2} & \text{if} \quad z_i = b \end{cases} \tag{3.102}$$

Lemma 8 establishes, that the above is a convex combination; lemma 7 binds each summand below $\frac{1}{2}$ for $z_i = b$, and above $\frac{1}{2}$ for $z_i = g$. Therefore, no voter has a profitable deviation: after $\hat{01}$, the g-type prefers the proposal, and the b-type the status quo.

The last remaining step is to show, that under the alternative constructed policy d', the designer is no worse off than under the disclosure policy d with an arbitrary message space. We prove this by showing that under the new disclosure policy d, the implementation probability of the proposal weakly increases for each r'.

Take $r' \in R'(\hat{0})$. Under both the old and the new disclosure policy, the proposal is implemented with zero probability.

Take $r' \in R'(\hat{01})$. Under the old disclosure policy, the proposal was implemented with probability $p^n a_i(r', g)^n$ if $\theta = G$, and $(1-p)^n a_i(r', g)^n$ if $\theta = B$. Under the new disclosure policy, the proposal is being implemented with probability $p^n a_i(r', g)^{n-1}$ if $\theta = G$, and $(1-p)^n a_i(r', g)^{n-1}$ if $\theta = B$. The probabilities are higher under the new disclosure policy, because $a_i(r', g)^{n-1} \geq a_i(r', g)^n$.

Finally, take $r' \in R'(\hat{1})$. Under the old disclosure policy, the proposal was implemented with probability $\Pr(piv|\theta = G, r')[pa_i(r', g) + (1-p)a_i(r', b)]$ if $\theta = G$, and with probability $\Pr(piv|\theta = B, r')[pa_i(r', b) + (1-p)a_i(r', g)]$ if $\theta = B$. Under the new disclosure policy, the proposal is implemented with probability $\Pr(piv|\theta, r')$, which is weakly higher.

□

A.12 Proof of Proposition 9

Proof. The primal of the sender's problem is

$$\max_{\{d[r|\theta]\geq 0\}_{r\in\{\widehat{01},\hat{1}\},\theta\in\{B,G\}}} \sum_{\theta\in\{B,G\}} (d[\hat{1}|\theta] + d[\widehat{01}|\theta]\Pr(k=3|\theta))\Pr(\theta) \tag{3.103}$$

$$\begin{aligned}
\text{s.t.}\quad & d[\hat{1}|\theta] + d[\widehat{01}|\theta] - 1 \leq 0 \quad \forall\theta \in \{B, G\} && (3.104)\\
& d[\hat{1}|\theta = B](1-p) - d[\hat{1}|\theta = G]p \leq 0 && (OB_g^{\hat{1}})\\
& d[\hat{1}|\theta = B]p - d[\hat{1}|\theta = G](1-p) \leq 0 && (OB_b^{\hat{1}})\\
& d[\widehat{01}|\theta = B](1-p)^3 - d[\widehat{01}|\theta = G]p^3 \leq 0 && (OB_g^{\widehat{01}})\\
& d[\widehat{01}|\theta = B]p(1-p)^2 - d[\widehat{01}|\theta = G]p^2(1-p) \leq 0 && (OB_g^{\widehat{01}})
\end{aligned}$$

Next, we take the dual of the primal and get:

$$\min_{\substack{\lambda_{OBg(\hat{1})}\geq 0,\ \lambda_{OBb(\hat{1})}\geq 0\\ \lambda_{OBg(\widehat{01})}\geq 0,\ \lambda_{OBb(\widehat{01})}\geq 0\\ \mu_{\theta=B}\geq 0,\ \mu_{\theta=G}\geq 0}} \mu_{\theta=B} + \mu_{\theta=G} \tag{3.105}$$

$$\begin{aligned}
\text{s.t.}\quad & -\frac{1}{2}(1 + \lambda_{OBg(\hat{1})}p + \lambda_{OBb(\hat{1})}(1-p)) + \mu_{\theta=G} \geq 0 && (3.106)\\
& -\frac{1}{2}(1 + \lambda_{OBg(\hat{1})}(1-p) - \lambda_{OBb(\hat{1})}p) + \mu_{\theta=B} \geq 0 && (3.107)\\
& -\frac{1}{2}p^2(p + \lambda_{OBg(\widehat{01})}p - \lambda_{OBb(\widehat{01})}(1-p)) + \mu_{\theta=G} \geq 0 && (3.108)\\
& -\frac{1}{2}(1-p)^2((1-p) - \lambda_{OBg(\widehat{01})}(1-p) + \lambda_{OBb(\widehat{01})}p) + \mu_{\theta=B} \geq 0 && (3.109)
\end{aligned}$$

Let $\{d[r|\theta] \geq 0\}_{r\in\{\widehat{01},\hat{1}\}}$ be a feasible disclosure policy for the primal, and $\{\vec{\lambda}, \vec{\mu}\}$

feasible vector for the dual. Necessary and sufficient conditions for the optimal are:

$$\lambda_{OBg(\hat{1})} \cdot \Big(d[\hat{1}|\theta = B](1-p) - d[\hat{1}|\theta = G]p\Big) = 0,$$
$$\lambda_{OBb(\hat{1})} \cdot \Big(d[\hat{1}|\theta = B]p - d[\hat{1}|\theta = G](1-p)\Big) = 0,$$
$$\lambda_{OBg(\widehat{01})} \cdot \Big(d[\widehat{01}|\theta = B](1-p)^3 - d[\widehat{01}|\theta = G]p^3\Big) = 0,$$
$$\lambda_{OBb(\widehat{01})} \cdot \Big(d[\widehat{01}|\theta = B]p(1-p)^2 - d[\widehat{01}|\theta = G]p^2(1-p)\Big) = 0,$$
$$\mu_{\theta=B} \cdot \Big(d[\hat{1}|\theta = B] + d[\widehat{01}|\theta = B] - 1\Big) = 0,$$
$$\mu_{\theta=G} \cdot \Big(d[\hat{1}|\theta = G] + d[\widehat{01}|\theta = G] - 1\Big) = 0,$$
$$d[\hat{1}|\theta = G] \cdot \Big(-\frac{1}{2}(1 + \lambda_{OBg(\hat{1})}p + \lambda_{OBb(\hat{1})}(1-p)) + \mu_{\theta=G}\Big) = 0,$$
$$d[\hat{1}|\theta = B] \cdot \Big(-\frac{1}{2}(1 + \lambda_{OBg(\hat{1})}(1-p) - \lambda_{OBb(\hat{1})}p) + \mu_{\theta=B}\Big) = 0,$$
$$d[\widehat{01}|\theta = G] \cdot \Big(-\frac{1}{2}p^2(p + \lambda_{OBg(\widehat{01})}p - \lambda_{OBb(\widehat{01})}(1-p)) + \mu_{\theta=G}\Big) = 0,$$
$$d[\widehat{01}|\theta = B] \cdot \Big(-\frac{1}{2}(1-p)^2((1-p) - \lambda_{OBg(\widehat{01})}(1-p) + \lambda_{OBb(\widehat{01})}p) + \mu_{\theta=B}\Big) = 0.$$

It can be easily checked by substitution that the disclosure policy

$$\{d[r|\theta] \geq 0\}_{r \in \{\widehat{01}, \hat{1}\}, \theta \in \{B,G\}} = \begin{cases} d[\hat{1}|\theta = G] = 1 \\ d[\hat{1}|\theta = B] = \frac{(1-p)}{p} \\ d[\widehat{01}|\theta = G] = 0 \\ d[\widehat{01}|\theta = B] = 0 \end{cases} \tag{3.110}$$

and the dual variables

$$\{\vec{\lambda}, \vec{\mu}\} = \begin{pmatrix} \lambda_{OBg(\hat{1})} = 0 \\ \lambda_{OBb(\hat{1})} = \frac{1}{p} \\ \lambda_{OBg(\widehat{01})} = 1 + \lambda_{OBb(\widehat{01})}\frac{p}{1-p} \\ \lambda_{OBb(\widehat{01})} = \frac{(2p^4 - \frac{1}{2})(1-p)}{p^3(1-2p)} \\ \mu_{\theta=B} = 0 \\ \mu_{\theta=G} = \frac{1}{2p} \end{pmatrix} \tag{3.111}$$

fulfill the above complementary slackness conditions for all $p \leq \frac{1}{\sqrt[4]{2}} = \tilde{p}$. Analogously, for $p > \tilde{p}$, the disclosure policy from Proposition 10 and the duals

$$\{\vec{\lambda},\vec{\mu}\} = \begin{pmatrix} \lambda_{OBg(\hat{1})} = 0 \\ \lambda_{OBb(\hat{1})} = \frac{1}{p} - \frac{(1-p)^3\left(2p^4-1\right)}{p(p^4-(1-p)^4)} \\ \lambda_{OBg(\widehat{01})} = 1 - \frac{2p^4-1}{p^4-(1-p)^4} \\ \lambda_{OBb(\widehat{01})} = 0 \\ \mu_{\theta=B} = \frac{\frac{1}{2}(1-p)^3\left(2p^4-1\right)}{p^4-(1-p)^4} \\ \mu_{\theta=G} = \frac{1}{2}\left(\frac{1}{p} - \frac{(1-p)^4(2p^4-1)}{p(p^4-(1-p)^4)}\right) \end{pmatrix} \tag{3.112}$$

fulfill the above complementary slackness conditions.

□

A.13 Proof of Proposition 10

Proof. The error probabilities l_G (probabilities of convicting the innocent) and l_B (acquit the guilty) of a wrong decision are found in Feddersen and Pesendorfer (1998). The expected utility of an uninformed voter in Feddersen and Pesendorfer (1998) is

$$\frac{1}{2}\left(1 - l_G - l_B\right) = \frac{1}{4}\frac{(2p-1)^3}{\left(p^{3/2} + p\sqrt{1-p} - \sqrt{1-p}\right)^2}. \tag{3.113}$$

With a manipulative information designer the expected utility of a voter is

$$\frac{1}{2}\Pr(\hat{1}|\theta = G)\frac{1}{2} + \frac{1}{2}\Pr(\hat{1}|\theta = B)\left(-\frac{1}{2}\right). \tag{3.114}$$

Next, we show that the optimal disclosure policy of the designer in each of the three intervals for p yields a strictly higher utility to the voter.

Case 1: $\frac{1}{2} < p \leq \frac{1}{\sqrt{2}}$. Using the optimal disclosure policy in Proposition 4, the utility of the ex ante type is $\frac{1}{4}1 - \frac{1}{4}\frac{1-p}{p} = \frac{1}{4}\frac{2p-1}{p}$. Comparing this with Equation 3.113 shows that the utility in case 1 is strictly higher for all $p \in (\frac{1}{2}, 1)$.

Case 2: $\frac{1}{\sqrt{2}} < p \leq \frac{1+\sqrt{13}}{6}$. With the optimal disclosure policy in Proposition 5, $\Pr(\hat{1}|\theta = G) = 1 - (1-p)^3$ and $\Pr(\hat{1}|\theta = B) = (p-1)(p^2-2)$. A voter's expected utility is $\frac{1}{4}(2p-1)(2-p)$, which can be again easily checked to be larger than Equation 3.113 for all $p \in (\frac{1}{2}, 1)$.

Case 3: $\frac{1+\sqrt{13}}{6} \leq p < 1$. In this scenario, the probabilities of choosing the proposal are $\Pr(\hat{1}|\theta = G) = \frac{1}{4}p\left(3p^3 - 8p^2 + 3p + 6\right)$ and $\Pr(\hat{1}|\theta = B) = \frac{1}{4}(p-1)\left(3p^3 - p^2 - 4p - 4\right)$. This yields an expected utility of $\frac{1}{8}\left[(2p-1)(p+1)(2-p)\right]$. This is higher than the expected utility in Feddersen and Pesendorfer (1998) in Equation 3.113. □

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