Hodge theoretic aspects of Soergel bimodules and representation theory

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Abstract

In the last years, methods coming from Hodge theory have proven to be fruitful in representation theory, most remarkably leading to a new algebraic proof of the Kazhdan-Lusztig conjectures based on the Hodge theory of Soergel bimodules. In this thesis we study several aspects of the connection between Hodge theory and representation theory, following several directions.

We develop Hodge theory for singular Soergel bimodules generalizing the non-singular case, that is we show the hard Lefschetz theorem and Hodge-Riemann bilinear relations for indecomposable singular Soergel bimodules.

Following Looijenga and Lunts, and as a consequence of the aforementioned Hodge theory, we can attach to any Soergel module (or to any Schubert variety) a Lie algebra, called the Néron-Severi Lie algebra. We use this algebra to give an easy Hodge theoretic proof of the Carrell-Peterson criterion for rational smoothness of Schubert varieties. We determine the Néron-Severi Lie algebra for all Schubert varieties in type A and for most Schubert varieties in other types.

In the last part, motivated by modular representation theory, we move to positive characteristic. Here we show that the hard Lefschetz theorem holds for the cohomology with coefficients in a field $\mathbb{K}$ of a flag variety if the characteristic of $\mathbb{K}$ is larger than the number of positive roots.

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Introduction

1 Background

Let Y be a smooth complex projective algebraic variety. A piece of data that we can attach to the cohomology of Y and that distinguishes it from a general manifold is its Hodge structure. Hodge theory was developed in the 50's, and presents a deep tie between algebraic geometry and differential geometry. From Hodge theory we can deduce many consequences about the topology of algebraic varieties: an immediate one is that the cohomology in odd degrees must be even dimensional.

To extend Hodge theory to singular varieties there are two possible directions to follow. The first is to modify the notion of Hodge structure, and this leads to Deligne's definition of mixed Hodge structure. The second is to change the spaces of study, i.e. we replace the usual singular cohomology with its intersection cohomology, introduced in 70's by Goresky and MacPherson [GM80]. It is the latter that plays a role in this thesis.

The hard Lefschetz theorem and the Hodge-Riemann bilinear relations [Sai90] are two direct consequences of Hodge theory that are central throughout this thesis. Assume Y is a projective complex variety. Let $\mathcal{L}$ be a ample line bundle on Y and let λ be its first Chern class. Then for any $k \geq 0$ multiplication by λ on intersection cohomology induces an isomorphism:

$$\lambda^k : IH^{-k}(Y,\mathbb{R}) \to IH^k(Y,\mathbb{R}) \qquad \text{(hard Lefschetz theorem)}$$

Assume further that Y is of Hodge-Tate type, that is in the Hodge decomposition only terms of Hodge type (p,p) appear.[1] Let $P_k = \operatorname{Ker}(\lambda^{k+1} : IH^{-k}(Y,\mathbb{R}) \to IH^{k+2}(Y,\mathbb{R}))$ and let $\langle -,- \rangle$ denote the intersection form on $IH^\bullet(Y,\mathbb{R})$. Then we have:

$$(b,b)_\lambda = \langle b, \lambda^k b\rangle \in (-1)^{(k+\dim Y)/2}\mathbb{R}_{>0} \quad \text{if } 0 \neq b \in P_k \qquad \text{(Hodge-Riemann bil. rel.)}$$

We come now to the connection with representation theory. In 1979 Kazhdan and Lusztig [KL79] conjectured a formula for the characters of highest weight irreducible representations $L(\mu)$ of complex reductive Lie algebras:

$$\operatorname{ch} L(-w\rho - \rho) = \sum_{v \leq w} (-1)^{\ell(v)-\ell(w)} h_{v,w}(1) \operatorname{ch} \Delta(-w\rho-\rho) \qquad \text{(KL conjecture)}$$

Here ρ is half the sum of all positive roots and $\Delta(\mu)$ denotes the Verma module of highest weight μ. The KL polynomials $h_{x,y}$ can be computed using a purely combinatorial algorithm. A few years later KL conjecture was proven by giving a geometric meaning to the KL polynomials $h_{x,y}$ [KL80, BB81, BK81]. In fact, they appear as dimension of the stalks of the intersection cohomology sheaves of Schubert varieties.

[1] For the Hodge-Riemann relations in the general form see for example [dM09a]. We ignore it as all the spaces in which we are interested are of Hodge-Tate type.

In the 90's Soergel [Soe90] proposed a completely algebraic framework to understand the KL conjecture. With the sole input of the action of the Weyl group on the Cartan algebra, he constructed a category of bimodules, today known as Soergel bimodules, which coincide with the equivariant intersection cohomology of Schubert varieties.

Elias and Williamson [EW14] used Soergel bimodules to give a new proof of the KL conjecture avoiding the recourse to geometry. In the setting of Soergel bimodules a crucial point is to show that certain symmetric forms are non-degenerate. These are precisely the forms that the Hodge-Riemann bilinear relations dictate to be positive definite. This is why by proving, now algebraically, Hodge theory for Soergel bimodules Elias and Williamson completed Soergel's program.

The proof of the Hodge theory for Soergel bimodules can be thought as the starting point for this thesis. From here we further investigate the deep relation between representation theory, Soergel bimodules, and Hodge theory. Our investigation follows several largely independent directions.

2 Soergel bimodules

Let (W,S) be a Coxeter system and $\mathfrak{h}$ be a reflection faithful representation of W. The category of Soergel bimodules $\mathbb{S}Bim$ is the full additive subcategory of graded modules over the polynomial ring $R = \mathrm{Sym}(\mathfrak{h}^*)$, generated by direct summands of shifts of Bott-Samelson bimodules

$$BS(\underline{s_1 s_2 \dots s_k}) := R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes \dots \otimes_{R^{s_k}} R$$

where $s_i \in S$ and R^{s_i} denotes the subring of s_i-invariants. Indecomposable self-dual Soergel bimodules are parametrized by elements of W and denoted by B_w.

If W is a Weyl group we have $B_w \cong IH^\bullet_T(X_w,\mathbb{K})$, the torus equivariant intersection cohomology of the Schubert variety X_w. The theory of Soergel bimodules can be developed for any Coxeter group, but in the general case there is no known underlying geometric object. Still, in many aspects these bimodules still behave as if they were the intersection cohomology of some varieties.

The intersection cohomology of Schubert variety $IH^\bullet_T(X_w,\mathbb{K})$ contains a distinguished submodule: the singular cohomology $H^\bullet_T(X_w,\mathbb{K})$. We give a description of this submodule in the diagrammatic language for Soergel bimodules. In this way we can generalize this construction to an analogous bimodule $\widetilde{H}_w \subseteq B_w$ for an arbitrary Coxeter group W. We sketch now this construction.

Libedinsky [Lib08] described a notable basis of homomorphism between Bott-Samelson bimodules $\mathrm{Hom}(BS(\underline{x}), BS(\underline{w}))$ modulo lower terms, called the light leaves basis. By applying these morphism to the lowest degree element $1^\otimes_{\underline{x}} = 1 \otimes 1 \otimes \dots \otimes 1 \in BS(\underline{x})$, and varying $\underline{x}$ over all reduced expression smaller than $\underline{w}$ one obtains a basis of the bimodule $BS(\underline{w})$ itself. Light leaves are parametrized by sequences in $e \in \{0,1\}^k$. Let $\underline{w} = s_1 s_2 \dots s_k$. We say that a light leaf is canonical if for any i we have $s_1^{e_1} s_2^{e_2} \dots s_{i-1}^{e_{i-1}} s_i > s_1^{e_1} s_2^{e_2} \dots s_{i-1}^{e_{i-1}}$. By taking the span of all the non-canonical light leaves we obtain a remarkable submodule $D_{\underline{w}}$ of $BS(\underline{w})$: this submodule does not depend on the choice involved in the light leaves construction and it is fixed by any idempotent of $BS(\underline{w})$. One recovers the cohomology submodule $\widetilde{H}_w$ by taking the orthogonal of $D_{\underline{w}}$ with respect to the intersection form of $BS(\underline{w})$.

One valuable property of the bimodule $\widetilde{H}_w$ is that it comes for free with a distinguished basis: this is the analogue of the Schubert basis, i.e. the basis of the cohomology obtained by considering the fundamental classes of smaller Schubert varieties. As a consequence, the graded rank of $\widetilde{H}_w$ can be readily computed:

$$\operatorname{grrk} \widetilde{H}_w = v^{-\ell(w)} \sum_{x \leq w} v^{2\ell(x)}$$

Fiebig [Fie08] developed a different approach to Soergel bimodules using moment graphs: Soergel bimodules turn out to be equivalent to a certain category of sheaves on the moment graph, and to a certain category of modules over its structure algebra Z. One should think to Z as the equivariant cohomology of the (possibly missing) flag variety. Then the Schubert bases for the bimodules $\widetilde{H}_w$ glue together to a basis $\{\mathcal{P}_x\}_{x \in X}$ of Z. This allows us to prove an isomorphism between Z and Kostant and Kumar's dual nil Hecke ring

Theorem A. *Let Λ the dual nil Hecke ring of W with basis ξ^x, as defined in [KK86a]. Then there exists a W-equivariant isomorphism*

$$\Lambda \cong Z \qquad \xi^x \mapsto \mathcal{P}_x.$$

As the algebra Z is free over R, the quotient $\overline{Z} = \mathbb{K} \otimes_R Z$ also has a Schubert basis. Any Soergel module $\overline{B}_w = \mathbb{K} \otimes_R B_w$ is naturally a module over $\overline{Z}$. We claim that this is the "right" module structure one should equip $\overline{B}_w$ with. In fact, the module $\overline{B}_w$ remains indecomposable over $\overline{Z}$ and we are able to compute the spaces of homomorphisms:

Theorem B (Soergel's hom formula for Soergel modules)**.** *Let B, B' Soergel bimodules. Then*

$$\mathbb{K} \otimes_R \operatorname{Hom}(B, B') \cong \operatorname{Hom}_{\overline{Z}}(\overline{B}, \overline{B'}) \tag{1}$$

and

$$\operatorname{grdim} \operatorname{Hom}_{\overline{Z}}(\overline{B}, \overline{B'}) = (\overline{\operatorname{ch}(B)}, \operatorname{ch}(B')) \tag{2}$$

where $(-,-)$ is the pairing in the Hecke algebra.

We remark that the formulas (1) and (2) do not hold when the obvious R-module structure on $\overline{B}$ and $\overline{B'}$ is considered, at least when W is infinite. In fact, we describe an example, for W of type $\widetilde{A}_2$ in which an indecomposable bimodule B_w gives rise to a module $\overline{B}_w$ which is not indecomposable as a R-module. This answers a question posed by Soergel in [Soe07, Remark 6.8] in the negative.

We go back to Hodge theory: this is another aspect in which Soergel modules behave like the intersection cohomology of Schubert varieties. As already mentioned above, Hodge theory for Soergel modules was shown in [EW14] where the hard Lefschetz theorem and the Hodge-Riemann bilinear relations are established. We examine here the case of singular Soergel modules. For Weyl groups, singular Soergel modules can be realized as intersection cohomology of Schubert varieties in a partial flag variety, hence the Hodge theory in this case can be deduced directly from geometry. Following closely the strategy of Elias and Williamson we can prove it in the generality of arbitrary Coxeter groups.

Let $I \subseteq S$ be a finitary subset, i.e. a subset such that the corresponding parabolic subgroup W_I is finite. If B is a Soergel bimodule then we can consider its restriction B_I to a (R, R^I)-bimodule. The category of singular Soergel bimodule $\mathbb{S}Bim^I$ is the full additive subcategory of (R, R^I)-bimodules generated by direct summands of restrictions of Soergel bimodules B_I. Self-dual indecomposable singular Soergel bimodules are parametrized by cosets $x \in W/W_I$ and denoted by B^I_x. Let $(\mathfrak{h}^*)^I \subseteq \mathfrak{h}^*$ denote the subspace of W_I-invariants.

Theorem C. *Let $\lambda \in (\mathfrak{h}^*)^I$ be such that $\lambda(\alpha_s^\vee) > 0$ for all $s \in S \setminus I$. Then for any $x \in W/W_I$ multiplication by λ on $\overline{B^I_x} = \mathbb{K} \otimes_R B^I_x$ satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations.*

3 Néron-Severi Lie algebra

A remarkable consequence of the hard Lefschetz theorem is that, following Looijenga and Lunts [LL97], we can associate to any complex projective variety a Lie algebra, called the Néron-Severi Lie algebra, acting on its (intersection) cohomology.

For any ρ ample class on Y there exists a Lie algebra $\mathfrak{g}_\rho$, isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, of which ρ is the nil-positive element. The Néron-Severi Lie algebra is the Lie algebra generated by all $\mathfrak{g}_\rho$, with ρ ample class.

The decomposition of $IH(Y) := IH^\bullet(Y, \mathbb{R})$ into irreducible $\mathfrak{g}_\rho$-modules is the primitive decomposition with respect to ρ. The primitive part (i.e. the lowest weight spaces for the $\mathfrak{g}_\rho$-action) inherits a Hodge structure from the Hodge structure of $IH(Y)$ and the Hodge structure of the primitive part determines completely the Hodge structure on $IH(Y)$. However, this decomposition depends on the choice of the ample class ρ. Looijenga and Lunts' initial motivation was to find a "universal" primitive decomposition of $IH(Y)$, not depending on any choice: this is achieved by considering the decomposition of $IH(Y)$ into irreducible $\mathfrak{g}_{NS}(Y)$-modules. This decomposition always exists: in fact one can prove that $\mathfrak{g}_{NS}(Y)$ is semisimple as a direct consequence of the Hodge-Riemann bilinear relations.

As we have discussed above, (singular) Soergel modules possess a Hodge structure, and this means that we can still define a Lie algebra $\mathfrak{g}_{NS}(w)$ for any Soergel module $\overline{B}_w$ in the same way. The semi-simplicity of the Lie algebra $\mathfrak{g}_{NS}(w)$ has an immediate consequence: in fact we can use then the algebra $\mathfrak{g}_{NS}(w)$ to deduce an easy Hodge-theoretic proof of the Carrell-Peterson criterion [Car94]: a Schubert variety X_w is rationally smooth if and only if the Poincaré polynomial of $H(X_w)$ is symmetric. The same proof works for arbitrary Coxeter groups by virtue of the cohomology module $\widetilde{H}_w$ previously discussed.

Looijenga and Lunts went on to compute $\mathfrak{g}_{NS}(X)$ for a flag variety $X = G/B$. They prove that it is "as big as possible," meaning that it is the complete Lie algebra of endomorphisms of $H(X)$ preserving a non-degenerate (either symmetric or antisymmetric depending on the parity of $\dim X$) bilinear form on $H(X)$. In this case we say that $\mathfrak{g}_{NS}(w)$ is maximal.

We explore the case of the Néron-Severi Lie algebra $\mathfrak{g}_{NS}(w)$ of an arbitrary Schubert variety, a question also posed in [LL97]. If $u \in S$ and $wu < w$, the Lie algebra $\mathfrak{g}_{NS}(w)$ contains a Lie algebra isomorphic to $\mathfrak{g}_{NS}(X_w^u) \times \mathfrak{sl}_2$, where X_w^u is the Schubert variety for a minimal parabolic group $\mathbf{P}_u$. Then, using a result of Dynkin on inclusion pairs of irreducible linear groups, we are able to translate the problem: the Lie algebra $\mathfrak{g}_{NS}(X_w)$ is maximal if and only if $IH(X_w)$ does not admit a non-trivial tensor decomposition, that is whenever we write $IH(X_w) = A_1 \otimes_{\mathbb{R}} A_2$, with A_1 (resp. A_2) a R_1 (resp. R_2) module and R_1, R_2 are polynomial algebras with $R = R_1 \otimes_{\mathbb{R}} R_2$, then A_1 or A_2 is one dimensional.

Characterizing for which $w \in W$ there is such a tensor decomposition of $IH(X_w)$ is now a problem of algebraic-combinatorial nature, since we have tools from Schubert calculus at our disposal.

To an element $w \in W$ we associate a directed graph $\mathcal{I}_w$ whose vertices are the simple reflections S, and in which there is an arrow $s \to t$ whenever $ts \leq w$ and $ts \neq st$.

$$1 \rightleftarrows 2 \rightleftarrows 3 \longrightarrow 4 \longleftarrow 5 \longrightarrow 6 \longleftarrow 7 \longleftarrow 8$$

Figure 1: The graph $\mathcal{I}_w$ for the element $w = s_4s_6s_2s_3s_1s_2s_3s_5s_7s_8$ for W of type A_8

The information contained in the graph $\mathcal{I}_w$ allows one to describe $H^4(X_w)$ as a quotient of $\mathrm{Sym}^2(H^2(X_w))$. If the graph $\mathcal{I}_w$ has no sinks we find an obstruction to the existence of non-trivial tensor decompositions.

Theorem D. *If the graph $\mathcal{I}_w$ is connected and $\mathcal{I}_w$ has no sinks, then the Lie algebra $\mathfrak{g}_{NS}(w)$ is maximal.*

It follows for the vast majority of Schubert varieties the Néron-Severi Lie algebra is "as big as possible." In type A we can go further and complete the classification of Néron-Severi Lie algebra.

Theorem E. *Let W a Weyl group of type A_n. For $w \in W$ let $\{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$ the set of sinks in $\mathcal{I}_w$, with $i_1 < i_2 < \ldots < i_k$, so that we can write $w = s_{i_1}s_{i_2}\ldots s_{i_k}v_0v_1\ldots v_k$ with $v_j \in W_{[i_j+1,i_{j+1}-1]}$ (where we set $i_0 = 0$ and $i_{k+1} = n+1$). Then $\mathfrak{g}_{NS}(w)$ is maximal if and only if v_0 and v_k are not the longest element in $W_{[1,i_1-1]}$ and $W_{[i_k+1,n]}$ respectively.*

4 Hard Lefschetz in Positive Characteristic

We now move our focus to the positive characteristic world. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. If we take cohomology or intersection cohomology with coefficients in algebraically closed field $\mathbb{K}$ there is no analogue of Hodge theory: the Hodge-Riemann bilinear relations do not even make sense! Still, asking when the hard Lefschetz theorem holds on the intersection cohomology of a variety remains a valid question.

A first interesting class of examples to consider are the flag varieties. In this case we are able to give a complete answer.

Theorem F. *Let X be a flag variety of a complex reductive group G and let $d = \dim X$. Then if $p > d$ there exists $\lambda \in H^2(X, \mathbb{K})$ such that multiplication by λ has the Lefschetz property, i.e. for any $k \geq 0$ we have an isomorphism*

$$\lambda^k : H^{d-k}(X, \mathbb{K}) \xrightarrow{\sim} H^{d+k}(X, \mathbb{K}).$$

Moreover, if $\mathrm{rk}(G) > 2$ the statement above is a "if and only if".

The motivation for this part also comes from representation theory. In positive characteristic there exists an analogue of the Kazhdan-Lusztig conjecture, known as Lusztig's conjecture.

Let $G^\vee_\mathbb{K}$ be the Langlands dual group of G, defined over $\mathbb{K}$. Lusztig's conjecture [Lus80] predicts a formula for the characters of irreducible $G^\vee_\mathbb{K}$-modules in terms of affine Kazhdan-Lusztig polynomials.

Geometrically, we can approach Lusztig's conjecture by studying Schubert varieties in the affine flag variety of G. Lusztig's conjecture was proven for p very large (with respect to the rank of G) in [AJS94]. In contrast, Williamson [Wil17b] found a family of counterexamples to Lusztig's conjectures for $p = O(c^n)$, with $c \sim 1,101$. It is currently still an open problem to understand more precisely where Lusztig's conjecture holds.

There is a geometric way to understand Lusztig's conjecture. In fact, Fiebig [Fie12] has shown that Lusztig's conjecture is equivalent to the local hard Lefschetz theorem on the stalks of the intersection cohomology sheaves. He used this strategy to prove an upper bound to the exceptional characteristics in Lusztig's conjecture. However, Fiebig's bound seems enormous (roughly $p > n^{n^2}$, for $G = SL_n(\mathbb{K})$) and it is expected that much lower bounds should exist.

This is why we believe that a more precise account on when the (local) hard Lefschetz theorem holds for Schubert variety could be of great importance for applications in modular representation theory.

Structure of the thesis

This thesis consists of six chapters. The first two chapters contain mostly introductory material. In Chapter 1 we review Coxeter group and their Hecke algebras. Here we prove some elementary Lemmas that we are going to need in the following. The main goal of Chapter 2 is to give a geometric motivation for the rest of the thesis: we give two description of the equivariant cohomology of a flag variety, one using the Schubert basis and one in terms of Konstant-Kumar's dual nil-Hecke ring.

Chapters 3 to 6 include the original content of this thesis. The different chapters can for the most part be read independently. However, some results in Chapter 5 for arbitrary Coxeter group are based on Chapters 3 and 4. In Chapter 3 we explain how to define a cohomology submodule and its Schubert basis. Then we use this to show Theorem A and B. Chapter 4 is devoted to the Hodge theory of singular Soergel bimodules. Finally, Chapters 5 and 6 correspond to sections 3 and 4 of the introduction respectively.

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Notation

By graded modules and graded vector spaces we always mean $\mathbb{Z}$-graded. For a graded module M and $i \in \mathbb{Z}$ let $M[i]$ denote the shifted module, i.e. $(M[i])^k = M^{i+k}$. For $p(v) = \sum p_i v^i \in \mathbb{Z}[v, v^{-1}]$ let $M^{\oplus p(v)}$ denote the module $\bigoplus (M[i])^{p_i}$.

Let $\mathbb{K}$ be a field. If V is a graded $\mathbb{K}$ vector space we denote by $\operatorname{grdim} V$ its graded dimension, that is, if $V = \bigoplus_{i\in\mathbb{Z}} V^i$ then

$$\operatorname{grdim} V = \sum (\dim V^i) v^i \in \mathbb{Z}[v, v^{-1}].$$

If M is a finitely generated graded free R-module, we denote by $\operatorname{grrk} M$ the grader rank of M. Usually R will be a polynomial ring over $\mathbb{K}$ with generators in positive degree. We view $\mathbb{K} = R/R_+$ as a R-module, where R_+ stands for the ideal of polynomials without constant term, so we have

$$\operatorname{grrk} M = \operatorname{grdim} \mathbb{K} \otimes_R M \in \mathbb{Z}[v, v^{-1}].$$

If M and N are graded R-module then $\operatorname{Hom}^\bullet(M, N)$ denotes the space of graded homomorphisms of all degrees:

$$\operatorname{Hom}^\bullet(M, N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}(M, N[i]),$$

where Hom denotes the degree-preserving homomorphism (i.e. homogeneous morphisms of degree 0).

If M is a R-algebra, which is graded as a R-module, we say that M is a shifted graded algebra if $M[n]$ is a graded algebra in the usual sense, where n is the degree of the unit of M.

List of recurrent symbols

W, S	Coxeter group and its simple reflections	9
$\mathcal{T}$	reflections in W	9
$\underline{w}$	a (not necessarily reduced) expression	9
ℓ	the length function on W	9
$x \xrightarrow[R]{t} y$	$y = xt$ with $t \in \mathcal{T}$ and $\ell(y) = \ell(x) + 1$	9
$m(\underline{w})$	maximal element smaller then $\underline{w}$	10
def	defect of a 01-sequence	10
Downs	number of Downs of a 01-sequence	10
$\mathfrak{h}$	realization of the Coxeter group	11
$\alpha_t, \alpha_t^\vee$	positive root and coroot corresponding to a reflection $t \in \mathcal{T}$	12
$\Phi, \Phi^\vee$	root and coroot system	12
∂_t	Demazure operator	12
ϖ_s	fundamental weight for $s \in S$	13
p_w	product of all the positive roots sent by w into negative roots	13
$\mathcal{H}$	Hecke algebra of W	13
$\mathbf{H}_x$	standard basis element of $\mathcal{H}$	13
$\underline{\mathbf{H}}_x$	Kazhdan-Lusztig basis element of $\mathcal{H}$	13
$\underline{\mathbf{H}}_{\underline{x}}$	Bott-Samelson basis element of $\mathcal{H}$	13
$h_{y,x}(v)$	Kazhdan-Lusztig polynomial	13
R	symmetric algebra of $\mathfrak{h}^*_{\mathbb{K}}$	14
G, B, T	simply-connected semisimple complex algebraic group, Borel subgroup and maximal torus	15

X	flag variety of G	15
X_w	Schubert variety	16
$P_w, \mathcal{P}_w$	element of the Schubert basis and of the equivariant Schubert basis	16
$\mathbf{P}_I$	parabolic subgroup of G corresponding to I	17
W_I, W^I	parabolic Coxeter group, minimal representatives of W/W_I	17
R^I	W_I-invariants of R	17
Q	field of fractions of R	17
$NH(W)$	nil-Hecke ring of W	18
D_x	basis element of the nil-Hecke ring	18
Λ	dual nil-Hecke ring of W	18
ξ^x	basis element of the dual nil-Hecke ring	18
$e_{x,y}$	equivariant multiplicity	19
$d_{x,y}$	"inverse" equivariant multiplicity	19
$BS(\underline{w})$	Bott-Samelson bimodule	26
$\mathbb{S}Bim$	category of Soergel bimodules	26
$Gr(x)$	twisted graph of x	26
$\Gamma_A B$	sections supported on $Gr(A)$	26
$\Gamma^x B, \Gamma_x B$	"stalk" and "costalk" of a bimodule	27
R_x	standard bimodule	27
B_x	indecomposable Soergel bimodule	27
$\mathcal{F}_\nabla$	category of bimodules with a ∇-flag	28
$\langle -, - \rangle_{BS(\underline{w})}$	intersection form on Bott-Samelson bimodules	29
c_e	string basis element of a Bott-Samelson bimodule	29
$1^{\otimes}_{\underline{w}}$	shifted unit of a Bott-Samelson bimodule	29
$LL_{\underline{w},e}$	light leaf morphism	33
$\Gamma\Gamma_{\underline{w},e}$	flipped light leaf morphism	33
$ll_{\underline{w},e}$	light leaf basis element	34
$D_{\underline{w}}$	bimodule of non-canonical light leaves	36
$\widetilde{H}_{\underline{w}}, \widetilde{H}_w$	cohomology bimodule	38
$\hat{Z}$	structure algebra of the moment graph	39
Z	subring of $\hat{Z}$ of bounded sections	39
τ, σ	left and right R-module structure on Z	39
$\mathcal{P}_{\underline{w},x}$	Schubert basis of the cohomology bimodule	40
$\overline{Z}$	quotient of Z	44
w_I	longest element in W_I	50
$\mathbb{S}Bim^I$	category of I-singular Soergel bimodules	50
B_I	restriction to $\mathbb{S}Bim^I$ of a bimodule $B \in \mathbb{S}Bim$	50
B^I_x	indecomposable singular Soergel bimodule for $x \in W^I$	50
F^I_x	singular Rouquier complex	55
$\mathfrak{g}(V, M)$	Néron-Severi Lie algebra of the V-Lefschetz module M	67
$\mathfrak{aut}(M, \phi)$	Lie algebra of endomorphism of M preserving the form ϕ	67
$\mathfrak{g}_{NS}(w)$	Néron-Severi Lie algebra of the Soergel module $\overline{B_w}$	71
$\mathcal{X}$	W-invariant element of R in degree 4 (aka Killing form)	74
$\mathcal{I}_w$	directed graph associated to w	80
$\mathrm{ht}(\alpha)$	height of the root α	95
$\mathfrak{B}_\Phi, \mathfrak{B}^I_\Phi$	Bruhat graph, parabolic Bruhat graph	97

Chapter 1

Coxeter Groups and Hecke Algebras

1.1 Coxeter groups

The goal of this section is to recall a few basic facts about Coxeter groups and their expressions. A standard reference for Coxeter groups is [Hum90]. We denote by id the identity element of a group.

A *Coxeter group* W is a group which admits a presentation of the form

$$W = \langle s \in S \mid (st)^{m_{st}} = id \text{ for any } s, t \in S \rangle$$

where S is a finite set, $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \dots\} \cup \{\infty\}$ for $s \neq t$ ($m_{st} = \infty$ means that the relation $(st)^{m_{st}} = id$ is missing). The pair (W, S) forms a *Coxeter system* and S is called the set of *simple reflections*. We denote by $\mathcal{T}$ the set of *reflections* in W, that is

$$\mathcal{T} = \bigcup_{w \in W} wSw^{-1}.$$

We call a sequence $\underline{w} = s_1 s_2 \dots s_k$ of elements $s_i \in S$ an *expression*. We say that the *length* of an expression $\underline{w} = s_1 s_2 \dots s_k$ is k. We say that $\underline{w}$ is an expression for $x \in W$ if $s_1 \cdot s_2 \cdot \dots \cdot s_k = x$. It is a *reduced expression* if there exists no expression for w of smaller length. We define the length of $w \in W$ to be the length of a reduced expression for w and we denote it by $\ell(w)$.

The *Bruhat order* is a partial order on W defined as follows: for $v, w \in W$ we say that $v \leq w$ if a subexpression of a reduced expression for w is an expression for v.

If $x, y \in W$ are such that $xt = y$ (resp. $xt = y$), with t reflection, and $\ell(x) + 1 = \ell(y)$ we write $x \xrightarrow[L]{t} y$ (resp. $x \xrightarrow[R]{t} y$). Notice that $x \xrightarrow[L]{t} y$ if and only if $x \xrightarrow[R]{x^{-1}tx} y$. The relations $x \leq y$ with $x \xrightarrow[R]{t} y$ for some $t \in T$ generate the Bruhat order.

The following is a fundamental property of the Bruhat order, and in fact, it completely characterizes it [Deo77, Theorem 1.1].

Proposition 1.1.1 (Property Z). *Let $x, y \in W$ and $s \in S$ such that $xs \geq x$ and $ys \geq y$. Then*

$$x \leq y \iff x \leq ys \iff xs \leq ys.$$

An easy consequence of the Property Z is that for any $x, y \in W$ we have $x \leq max\{y, ys\}$ if and only if $xs \leq max\{y, ys\}$.

Let $\underline{w} = s_1 s_2 \dots s_\ell$ be a (not necessarily reduced) word. We call an element $e \in \{0, 1\}^\ell$ a 01-*sequence* for $\underline{w}$. We denote by $\underline{w}^e$ the element $s_1^{e_1} s_2^{e_2} \dots s_\ell^{e_\ell}$. If $x \in W$ we say $x \leq \underline{w}$ if there exists a 01-sequence e for $\underline{w}$ such that $\underline{w}^e = x$.

For any k such that $0 \leq k \leq \ell$, we further define $\underline{w}_{\leq k} = s_1 s_2 \ldots s_k$ and $\underline{w}^e_{\leq k} = s_1^{e_1} s_2^{e_2} \ldots s_k^{e_k}$. Similarly, we define $\underline{w}_{\geq k}$ and $\underline{w}^e_{\geq k}$.

Lemma 1.1.2. *Let $\underline{w}$ be a word. Then there exists a unique maximal element $m(\underline{w}) \in W$ such that $m(\underline{w}) \leq \underline{w}$.*

Proof. By induction on $\ell(\underline{w})$, we can assume that we have already shown existence and uniqueness of $m(\underline{w})$. Let $\underline{w}' = \underline{w}s$. Then we set $m(\underline{w}') = max\{m(\underline{w}), m(\underline{w})s\}$, i.e.

$$m(\underline{w}') = \begin{cases} m(\underline{w}) & \text{if } m(\underline{w})s < m(\underline{w}), \\ m(\underline{w})s & \text{if } m(\underline{w})s > m(\underline{w}). \end{cases}$$

Clearly, we have $m(\underline{w}') \leq \underline{w}'$. Let $x \leq \underline{w}'$. We can write $x = ys^{\varepsilon}$ with $\varepsilon \in \{0,1\}$ and $y \leq \underline{w}$, hence $y \leq m(\underline{w})$. Now it follows from the Property Z that $x \leq m(\underline{w}')$. □

Notice that $x \leq \underline{w}$ if and only if $x \leq m(\underline{w})$. From a 01-sequence e we can obtain a sequence of elements in $\{U0, U1, D0, D1\}$ as indicated by the following table:

	$e_k = 0$	$e_k = 1$
$\underline{w}^e_{\leq k-1} \cdot s_k > \underline{w}^e_{\leq k-1}$	$U0$	$U1$
$\underline{w}^e_{\leq k-1} \cdot s_k < \underline{w}^e_{\leq k-1}$	$D0$	$D1$

We refer to this sequence as *decoration* of e and to its elements as *bits* of e. Let $\operatorname{def}(e)$ be the defect of e, i.e. the number of $U0$'s minus the number of $D0$'s occurring in the decoration of e. We define $\operatorname{Downs}(e)$ to be the number of D's (both $D1$'s and $D0$'s) of e. We have

$$\operatorname{def}(e) = \ell(\underline{w}) - \ell(\underline{w}^e) - 2\operatorname{Downs}(e). \tag{1.1}$$

Lemma 1.1.3. *Let $\underline{w}$ be a word. For any $x \leq \underline{w}$ there exists a unique 01-sequence e such that $\underline{w}^e = x$ and the decoration of e has only $U0$'s and $U1$'s. Moreover, e is the unique 01-sequence of maximal defect such that $\underline{w}^e = x$, and satisfies $\operatorname{def}(e) = \ell(\underline{w}) - \ell(x)$.*

Proof. We first show the existence. Let $\underline{w} = s_1 \ldots s_\ell$. We start with $x_\ell = x$ and we define recursively, starting with $k = l$ and down to $k = 1$,

$$e_k = \begin{cases} 1 & \text{if } x_k s_k < x_k \\ 0 & \text{if } x_k s_k > x_k \end{cases}, \qquad x_{k-1} = x_k \cdot s_k^{e_k}.$$

It follows that $x_{k-1}s_k > x_{k-1}$ for any k and that $x_{k-1} = min\{x_k, x_k s_k\}$, so at any step we get $x_{k-1} \leq \underline{w}_{\leq k-1}$, as follows by applying Property Z. Hence we have $x_0 = id$ and e is a 01-sequence with $\underline{w}^e = x$ and such that it has only $U1$'s and $U0$'s in its decoration.

Assume now that there are two 01-sequences e and f decorated with only U's and satisfying $\underline{w}^e = \underline{w}^f = x$. If $e_\ell = f_\ell$ we can conclude that $e = f$ by induction on ℓ. Otherwise we can assume $e_\ell = 1$ and $f_\ell = 0$. Now we get $\underline{w}^f_{\leq \ell-1} = x$, and $xs_\ell < x$ because the last bit of e must be a $U1$. But this also means that the last bit of f is a $D0$, hence we get a contradiction.

The last statement follows directly from (1.1). □

Definition 1.1.4. Let $\underline{w}$ be a word and $x \leq \underline{w}$. We call the unique 01-sequence e without D's such that $\underline{w}^e = x$ the *canonical* sequence for x. We denote it by can_x.

1.2 Reflection faithful representations and root systems

Definition 1.2.1. A finite dimensional representation V over a field $\mathbb{K}$ of a Coxeter group W is called *reflection faithful* if it is faithful and, for any $x \in W$, the set of fixed points V^x has codimension 1 in V if and only if $x \in \mathcal{T}$.

Let $\mathbb{K}$ be a field. A *realization* of W is a $\mathbb{K}$-vector space $\mathfrak{h}$ of W over a field $\mathbb{K}$ together with subsets

$$\{\alpha_s\}_{s\in S} \subseteq \mathfrak{h}^* \qquad \text{and} \qquad \{\alpha_s^\vee\}_{s\in S} \subseteq \mathfrak{h}$$

such that $s(v) = v - \alpha_s(v)\alpha_s^\vee$ for all $s \in S$ defines a representation of W on $\mathfrak{h}$. Notice that W acts on $\mathfrak{h}^*$ via the contragredient representation and we have $s(\lambda) = \lambda - \lambda(\alpha_s^\vee)\alpha_s$ for any $s \in S$ and $\lambda \in \mathfrak{h}^*$.

For simplicity here we will consider only three kinds of realizations of W:

Type I) Let $\mathbb{K} = \mathbb{R}$. We fix a finite dimensional real vector space $\mathfrak{h}$ and linearly independent sets $\{\alpha_s\}_{s\in S} \subseteq \mathfrak{h}^*$ and $\{\alpha_s^\vee\}_{s\in S} \subseteq \mathfrak{h}$ such that

$$\alpha_s(\alpha_t^\vee) = -2\cos\left(\frac{\pi}{m_{st}}\right).$$

We further assume that $\mathfrak{h}$ is of minimal dimension amongst vector spaces satisfying these properties.

As shown in [Soe07, Proposition 2.1], the representation $\mathfrak{h}$ is reflection faithful. Notice that if W is finite then $\mathfrak{h}$ is the geometric representation defined in [Hum90, §5.3]. If W is not finite then $\mathfrak{h}$ is not irreducible and contains the geometric representation as a submodule, as follows from the proof of [Soe07, Proposition 2.1].

Type II) Let $\mathbb{K} = \mathbb{R}$. Let $A = (a_{s,t})_{s,t\in S}$ be a generalized symmetrizable Cartan matrix and let $(\mathfrak{h}, \mathfrak{h}^*, \{\alpha_s^\vee\}, \{\alpha_s\})$ a realization of A over $\mathbb{R}$ in the sense of [Kac90] (as in [Kum02, Definition 1.1.2]). We have $\dim \mathfrak{h} = |S| + \operatorname{corank}(A) = 2|S| - \operatorname{rk}(A)$, and the sets $\{\alpha_s^\vee\}_{s\in S} \subseteq \mathfrak{h}^*$ and $\{\alpha_s\}_{s\in S} \subseteq \mathfrak{h}$ are linearly independent and satisfy

$$a_{s,t} = (\alpha_s(\alpha_t^\vee))_{s,t\in S}.$$

Let W the corresponding Coxeter group. Then $\mathfrak{h}^*$ is a representation faithful realization of W [Ric17].

Type III) Let $\mathbb{K}$ be a field such that $\operatorname{char}\mathbb{K} \neq 2$. Let G be a reductive group over $\mathbb{K}$ and let T its maximal torus. Let $\mathfrak{h} = \mathit{Lie}(T)$. There is a natural action of the Weyl group on $\mathfrak{h}$. We assume that the representation so obtained is reflection faithful, which is always the case if $\operatorname{char}\mathbb{K} > 3$ [Lib15, Appendix A].

If $\mathbb{K} = \mathbb{R}$ this coincides with realizations of type II for Cartan matrices of finite type.

If $\mathfrak{h}$ is of Type II or III, then the representation $\mathfrak{h}$ can be obtained by extending scalar to a representation $\mathfrak{h}_{\mathbb{Z}}$ defined over $\mathbb{Z}$. In particular, if $\mathfrak{h}$ is of Type III we have $\mathfrak{h} = \mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ and $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ where

$$\mathfrak{h}_{\mathbb{Z}} = \bigoplus_{s\in S} \mathbb{Z}\alpha^\vee_{s,\mathbb{Z}} \qquad \text{and} \qquad \mathfrak{h}^*_{\mathbb{Z}} = \bigoplus_{s\in S} \mathbb{Z}\alpha_{s,\mathbb{Z}}$$

with $\alpha_s = \alpha_{s,\mathbb{Z}} \otimes 1$ and $\alpha_s^\vee = \alpha^\vee_{s,\mathbb{Z}} \otimes 1$.

Example 1.2.2. Let W be of type G_2 and consider a realization of Type III over a field $\mathbb{K}$ of characteristic p. The simple reflections s and t act on the basis $\{\varpi_s, \varpi_t\}$ of fundamental weights of $\mathfrak{h}^*$ as

$$s = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}.$$

Then $w = stst$ acts as $\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$. It follows that if $p = 3$ this representation is not reflection faithful since $\dim(\mathfrak{h}^*)^{stst} = 1$.

Remark 1.2.3. It would be interesting to consider more general realizations of W, and many statements in this thesis should hold in a larger generality. We require the representation to be reflection faithful to have at our disposal the theory of Soergel bimodules. One could drop this assumption by replacing the category of Soergel bimodules with its diagrammatic counterpart (cf. §3.1.3).

On the other side we will need our realization to have a good notion of positive roots: this is necessary to be able to use the results of Kostant and Kumar for the nil-Hecke ring (see [KK86a, Remark 4.35.b]).

Let

$$\Phi = \{w(\alpha_s) \mid w \in W,\ s \in S\} \subseteq \mathfrak{h}^*$$

be the set of *roots* and

$$\Phi^\vee = \{w(\alpha_s^\vee) \mid w \in W,\ s \in S\} \subseteq \mathfrak{h}$$

be the set of *coroots*.

Assume $\mathfrak{h}$ is a realization of Type I or Type II, thus $\mathbb{K} = \mathbb{R}$. Every root $\alpha \in \Phi$ can be written as $\alpha = \sum_{s \in S} c_s \alpha_s$ with $c_s \in \mathbb{R}$. We say that a root is positive if $c_s > 0$ for all s and negative if $c_s < 0$ for all s.

Let Φ^+ be the set of positive roots and Φ^- be the set of negative roots. We have $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \sqcup \Phi^-$ (cf. [Hum90, §5.4]). Similarly, we have $\Phi^\vee = (\Phi^\vee)^+ \sqcup (\Phi^\vee)^-$.

If $t \in \mathcal{T}$ is a reflection we can write $t = wsw^{-1}$ with $w \in W$, $s \in S$ and $ws > w$. We set $\alpha_t = w(\alpha_s) \in \Phi^+$ and $\alpha_t^\vee = w(\alpha_s^\vee) \in (\Phi^\vee)^+$. We have $t(v) = v - \alpha_t(v)\alpha_t^\vee$. The root α_t and the coroot $\alpha_t^\vee$ are well-defined and the assignments $t \mapsto \alpha_t$, $t \mapsto \alpha_t^\vee$ define bijections $\mathcal{T} \xrightarrow{\sim} \Phi^+$ and $\mathcal{T} \xrightarrow{\sim} (\Phi^\vee)^+$.

Assume now $\mathfrak{h}$ is a realization of Type III. Then we define

$$\Phi_{\mathbb{Z}} = \{w(\alpha_{s,\mathbb{Z}}) \mid w \in W,\ s \in S\} \subseteq \mathfrak{h}^*_{\mathbb{Z}} \quad \text{ and } \quad \Phi^\vee_{\mathbb{Z}} = \{w(\alpha^\vee_{s,\mathbb{Z}}) \mid w \in W,\ s \in S\} \subseteq \mathfrak{h}_{\mathbb{Z}}.$$

Every element $\alpha \in \Phi_{\mathbb{Z}}$ can be written as $\alpha = \sum_{s \in S} c_s \alpha_{s,\mathbb{Z}}$ with $c_s \in \mathbb{Z}$. As before, we define the subsets $\Phi^+_{\mathbb{Z}}$ and $\Phi^-_{\mathbb{Z}}$ and we have $\Phi_{\mathbb{Z}} = \Phi^+_{\mathbb{Z}} \sqcup \Phi^-_{\mathbb{Z}}$. Similarly, we have $\Phi^\vee_{\mathbb{Z}} = (\Phi^\vee_{\mathbb{Z}})^+ \sqcup (\Phi^\vee_{\mathbb{Z}})^-$. For a reflection $t \in \mathcal{T}$ such that wsw^{-1} with $w \in W$, $s \in S$ and $ws > w$ we define $\alpha_{t,\mathbb{Z}} = w(\alpha_{s,\mathbb{Z}}) \in \Phi^+_{\mathbb{Z}}$, $\alpha^\vee_{t,\mathbb{Z}} = w(\alpha^\vee_{s,\mathbb{Z}}) \in (\Phi^\vee_{\mathbb{Z}})^+$, $\alpha_t = w(\alpha_s) = \alpha_{t,\mathbb{Z}} \otimes 1$ and $\alpha_t^\vee = w(\alpha_s^\vee) = \alpha_{t,\mathbb{Z}} \otimes 1$. The assignments $t \mapsto \alpha_{t,\mathbb{Z}}$, $t \mapsto \alpha^\vee_{t,\mathbb{Z}}$ define bijections $\mathcal{T} \xrightarrow{\sim} \Phi^+_{\mathbb{Z}}$ and $\mathcal{T} \xrightarrow{\sim} (\Phi^\vee_{\mathbb{Z}})^+$ (but the map $\mathcal{T} \to \Phi$ defined by $t \mapsto \alpha_t$ need not be injective).

Let R be the ring of regular functions of $\mathfrak{h}$, that is $R = \mathrm{Sym}(\mathfrak{h}^*)$. We regard R as a graded ring, where we set $\deg(\mathfrak{h}^*) = 2$. We denote by R_+ the ideal of R generated by homogeneous polynomials of positive degree. We view $\mathbb{K}$ as a R-module via $\mathbb{K} \cong R/R_+$.

The action of W on $\mathfrak{h}^*$ extends to an action on R. If $t \in \mathcal{T}$ is a reflection we denote by $\partial_t : R \to R$ the so-called *Demazure operator* defined by

$$\partial_t(f) = \frac{f - t(f)}{\alpha_t} \qquad \text{for all } f \in R.$$

If $t = wsw^{-1}$ for $s \in S$ and $w \in W$, with $ws > s$ we have $\partial_t(f) = w\partial_s(w^{-1}(f))$. In particular, if $f \in \mathfrak{h}^*$ we have $\partial_t(f) = \partial_s(w^{-1}(f)) = f(\alpha_t^\vee)$.

For $s \in S$ let $\varpi_s \in \mathfrak{h}^*$ be a fundamental weight for s, that is $\partial_t(\varpi_s) = \delta_{t,s}$ for all $t \in S$. Notice that in general the fundamental weight $\varpi_s \in \mathfrak{h}^*$ is not unique, but it is determined only up to W-invariants.

For an element $w \in W$ we have

$$\ell(w) = \#\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\} = \#\{t \in \mathcal{T} \mid tw < w\}.$$

For later use, we associate to any $w \in W$ a homogeneous polynomial of degree $2\ell(w)$

$$p_w = \prod_{\substack{t \in \mathcal{T} \\ tw < w}} \alpha_t \in R. \tag{1.2}$$

1.3 The Hecke algebra of a Coxeter group

To a Coxeter system (W, S) we associate a $\mathbb{Z}[v, v^{-1}]$-algebra, called the *Hecke algebra* $\mathcal{H}(W, S)$. The algebra $\mathcal{H} := \mathcal{H}(W, S)$ is the unital associative $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\mathbf{H}_s$ for $s \in S$ with relations

$$\mathbf{H}_s^2 = -(v - v^{-1})\mathbf{H}_s + 1, \tag{1.3}$$

$$\underbrace{\mathbf{H}_s\mathbf{H}_t\mathbf{H}_s\dots}_{m_{st}} = \underbrace{\mathbf{H}_t\mathbf{H}_s\mathbf{H}_t\dots}_{m_{st}} \tag{1.4}$$

for all $s, t \in S$. For $x \in W$ we define $\mathbf{H}_x = \mathbf{H}_{s_1}\mathbf{H}_{s_2}\dots\mathbf{H}_{s_l}$ for any reduced expression $x = s_1s_2\dots s_l$. Because of (1.4) this is well-defined.

We denote by $\overline{(-)}$ the involution of $\mathcal{H}$ defined by $\overline{v} = v^{-1}$ and $\overline{\mathbf{H}_s} = \mathbf{H}_s^{-1}$.

Theorem 1.3.1. *[KL79] There exists a unique basis $\{\underline{\mathbf{H}}_x\}_{x \in W}$ of $\mathcal{H}$ as a $\mathbb{Z}[v, v^{-1}]$-module which satisfies for all $x \in W$*

- $\overline{\underline{\mathbf{H}}_x} = \underline{\mathbf{H}}_x$,
- $\underline{\mathbf{H}}_x = \mathbf{H}_x + \sum_{y < x} h_{y,x}(v)\mathbf{H}_y$ *with* $h_{y,x}(v) \in v\mathbb{Z}[v]$.

The basis $\{\underline{\mathbf{H}}_x\}_{x \in W}$ is called the *Kazhdan-Lusztig basis* and the polynomials $h_{y,x}(v)$ are known as *Kazhdan-Lusztig polynomials.*

Warning 1.3.2. In [KL79] a different parametrization of the Kazhdan-Lusztig polynomials is used. Namely, in their notation we have

$$h_{y,x}(v) = v^{\ell(x)-\ell(y)}P_{y,x}(v^{-2}).$$

Notice that we have $\underline{\mathbf{H}}_{id} = \mathbf{H}_{id} = 1$ and $\underline{\mathbf{H}}_s = \mathbf{H}_s + v$. If $\underline{w} = s_1s_2\dots s_k$ is a word we define $\underline{\mathbf{H}}_{\underline{w}} := \underline{\mathbf{H}}_{s_1}\underline{\mathbf{H}}_{s_2}\dots\underline{\mathbf{H}}_{s_k}$.

We also have an anti-involution a of $\mathcal{H}$ defined by $a(v) = v$ and $a(\mathbf{H}_x) = \mathbf{H}_{x^{-1}}$ for $x \in W$. The trace ε is the $\mathbb{Z}[v, v^{-1}]$-linear map defined by $\varepsilon(\mathbf{H}_w) = \delta_{w,id}$. We define a $\mathbb{Z}[v, v^{-1}]$-bilinear pairing

$$(-,-) : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[v, v^{-1}] \tag{1.5}$$

by $(h, h') = \varepsilon(a(h)h')$.

It is easy to check that $\underline{\mathbf{H}}_s$ is biadjoint with respect to this pairing, i.e. $(h\underline{\mathbf{H}}_s, h') = (h, h'\underline{\mathbf{H}}_s)$ and $(\underline{\mathbf{H}}_s h, h') = (h, \underline{\mathbf{H}}_s h')$. Moreover for any $x, y \in W$ we have $(\mathbf{H}_x, \mathbf{H}_y) = \delta_{x,y}$ and from this it follows

$$(\underline{\mathbf{H}}_x, \underline{\mathbf{H}}_y) \in \delta_{x,y} + v\mathbb{Z}[v].$$

Chapter 2

Geometry of Flag Varieties

2.1 Torus equivariant cohomology and Borel-Moore homology

Let T be a complex algebraic torus, i.e. $T \cong (\mathbb{C}^*)^r$ for $r \in \mathbb{N}$. There exists a universal T-bundle $ET \to BT$ such that ET is contractible and the action of T on ET is free. The space ET is unique up to T-homotopy equivalence and BT is unique up to homotopy equivalence. The space BT is called the *classifying space* of T. If $r = 1$, the space ET can be realized as

$$ET = \varinjlim(\mathbb{C}^n \setminus \{0\}) = \mathbb{C}^\infty \setminus \{0\}.$$

The quotient is $BT = ET/T = \varinjlim \mathbb{P}^n = \mathbb{P}^\infty$. In general we realize ET as $(\mathbb{C}^\infty \setminus \{0\})^r$ and $BT \cong (\mathbb{P}^\infty)^r$ (see [Bri00, §1] for more details).

Let $\mathbb{K}$ denote an arbitrary field. If Y is T-space, the equivariant cohomology of Y with coefficients in $\mathbb{K}$ is defined as

$$H^\bullet_T(Y, \mathbb{K}) := H^\bullet(Y \times_T ET, \mathbb{K}).$$

The space $Y \times_T ET$ is the quotient of $Y \times ET$ under the action of T defined by $t \cdot (y, e) = (yt^{-1}, te)$.

Via the pullback, the equivariant cohomology $H^\bullet_T(Y, \mathbb{K})$ is naturally a module over $H^\bullet_T(pt, \mathbb{K}) = H^\bullet(BT, \mathbb{K})$. We can describe $H^\bullet(pt, \mathbb{K})$ as follows. Let

$$\mathbf{X}^*(T) = \{T \to \mathbb{C}^* \mid \text{morphisms of algebraic groups}\}$$

be the group of *characters* of T. We have $\mathbf{X}^*(T) \cong \mathbb{Z}^r$.

To each $\lambda \in \mathbf{X}^*(T)$ we can associate a one-dimensional representation $\mathbb{C}_\lambda$ of T. Let $\mathcal{L}_\lambda$ denote the line bundle $ET \times_T \mathbb{C}_\lambda \to BT$. Then the first Chern class $c_1(\mathcal{L}_\lambda)$ is an element of $H^2(BT, \mathbb{Z})$, thus we obtain a group homomorphism

$$\mathbf{X}^*(T) \to H^2(BT, \mathbb{Z}) = H^2_T(pt, \mathbb{Z}).$$

Let $R = \mathrm{Sym}_{\mathbb{K}}(\mathbf{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{K})$.[1] Then we can extend it to a graded algebra isomorphism

$$R \xrightarrow{\sim} H^\bullet_T(pt, \mathbb{K}),$$

where in R we set $\deg(\mathbf{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{K}) = 2$.

[1] The ring R always implicitly depends on $\mathbb{K}$.

To introduce the equivariant Borel-Moore homology we need to use finite dimensional approximations of ET. Let $ET_m = (\mathbb{C}^{m+1} \setminus \{0\})^r$. Let $H_{BM,\bullet}$ denote the usual (i.e. non-equivariant) Borel-Moore homology (cf. [CG97, §2.6]). The T-equivariant Borel-Moore homology is defined as

$$H^T_{BM,q}(X,\mathbb{K}) := H_{BM,q+2mr}(X \times_T ET_m, \mathbb{K}) \qquad \text{for any } m \gg 0.$$

In fact, for any $m \geq m' \geq \dim(X) - q/2$ the restriction map [CG97, 2.6.21] induces an isomorphism

$$H_{BM,q+2mr}(X \times_T ET_m, \mathbb{K}) \xrightarrow{\sim} H_{BM,q+2m'r}(X \times_T ET_{m'}, \mathbb{K}).$$

Usually the equivariant Borel-Moore homology is non trivial in negative degrees. The cap product

$$H^p_T(X,\mathbb{K}) \times H^T_{BM,q}(X,\mathbb{K}) \to H^T_{BM,q-p}(X,\mathbb{K})$$

equips $H^T_{BM,\bullet}(X,\mathbb{K})$ with a structure of R-module, where $\mathbf{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{K}$ acts with degree -2. We write $H^T_{BM,-\bullet}(X,\mathbb{K})$ for the R-module $H^T_{BM,\bullet}(X,\mathbb{K})$ with the opposite grading, so that $H^T_{BM,-\bullet}(X,\mathbb{K})$ is a graded R-module in the usual sense.

Assume that the Betti numbers of X vanish in odd degree. Then by [Bri00, Lemma 2 and Proposition 1] the graded R-modules $H^\bullet_T(X,\mathbb{K})$ and $H^T_{BM,-\bullet}(X,\mathbb{K})$ are free and we have an isomorphism of graded R-modules

$$H^\bullet_T(X,\mathbb{K}) \cong \operatorname{Hom}^\bullet_R(H^T_{BM,-\bullet}(X,\mathbb{K}), R). \tag{2.1}$$

2.1.1 Equivariant cohomology of the flag variety

Let G be a complex semisimple algebraic group. We further assume that G is connected and simply-connected. Let $B \subseteq G$ be a Borel subgroup and $T \subseteq B$ be a maximal torus. We denote by $\mathfrak{g} \supseteq \mathfrak{b} \supseteq \mathfrak{h}$ the corresponding Lie algebras. The T-action on $\mathfrak{g}$ induces a decomposition into weight spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\Phi \subseteq \mathfrak{h}^*$ is the root system of G. We denote by Φ^+ the set of positive roots, i.e. the set of roots $\alpha \in \Phi$ such that $\mathfrak{g}_\alpha \subseteq \mathfrak{b}$. Let $\Delta \subseteq \Phi^+$ be the corresponding set of simple roots.

We have $\mathbf{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathfrak{h}^*$. Let $\Phi^\vee \subseteq \mathfrak{h}$ denote the *dual root system* or *coroot system*: for any root $\alpha \in \Phi$ we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding coroot. If $(-,-)$ is the Killing form on $\mathfrak{h}^*$, then $\alpha^\vee = \frac{2}{(\alpha,\alpha)}(\alpha,-)$. Because G is simply connected, the character lattice $\mathbf{X}^*(T)$ coincides with the lattice of integral weights $\mathfrak{h}_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha^\vee) \in \mathbb{Z}$ for all $\alpha^\vee \in \Phi^\vee\}$. We set $\mathfrak{h}_{\mathbb{K}} = \mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$.

The *Weyl group* W of G is the group generated by the reflections

$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^* \qquad s_\alpha : \lambda \mapsto \lambda - \lambda(\alpha^\vee)\alpha$$

for $\alpha \in \Phi$. It is a Coxeter group with simple reflections s_α, $\alpha \in \Delta$. We also have $W \cong N_G(T)/T$, where $N_G(T)$ is the normalizer subgroup of T.

We consider the homogeneous space $X := G/B$, called the *flag variety* of G. It is a smooth complex projective variety of dimension equal to $|\Phi^+|$. The Borel subgroup B acts on X and decomposes it in a finite number of orbits, one for each element of W:

$$X = \bigsqcup_{w \in W} B \cdot wB/B.$$

This decomposition is known as the *Bruhat decomposition*. As a variety each B-orbit is isomorphic to an affine space, i.e. we have an isomorphism of algebraic varieties $B\cdot wB/B \cong \mathbb{C}^{\ell(w)}$. This means that the Bruhat decomposition gives a CW-complex structure on X and we can easily use this to compute the homology and cohomology of X.

Let $X_w = \overline{B\cdot wB/B}$ be the closure of a single orbit. The varieties X_w are in general singular projective varieties and are called *Schubert varieties*. Each Schubert variety X_w is a union of B-orbits and $B\cdot xB/B \subseteq X_w$ if and only if $x \leq w$ in the Bruhat order.

Since all the cells in the Bruhat decomposition have even dimension as real manifolds, we have

$$H_\bullet(X,\mathbb{K}) \cong \bigoplus_{w\in W} \mathbb{K}[X_w],$$

where $[X_w] \in H_{2\ell(w)}(X,\mathbb{K})$ is the fundamental class of X_w.

Similarly, we define $[X_w]_T$ as $[X_w \times_T ET_m] \in H^T_{BM,2\ell(w)}(X,\mathbb{K})$ for any $m \gg 0$. The restriction map $H^T_{BM,\bullet}(X,\mathbb{K}) \to H_\bullet(X,\mathbb{K})$ sends $[X_w]_T$ to $[X_w]$ and induces an isomorphism [Bri00, Proposition 1]:

$$\mathbb{K}\otimes_R H^T_{BM,\bullet}(X,\mathbb{K}) \cong H_\bullet(X,\mathbb{K}).$$

Here $\mathbb{K}$ is regarded as a R-module via the isomorphism $\mathbb{K} \cong R/R_+$ and R_+ stands for the ideal of polynomials without constant term. It follows that $\{[X_w]_T\}_{w\in W}$ is a basis of $H_{BM,-\bullet}(X,\mathbb{K})$ as a R-module.

Because of (2.1) we can define a basis $\{\mathcal{P}_w\}_{w\in W}$ of $H^\bullet_T(X,\mathbb{K})$ dual of $\{[X_w]_T\}_{w\in W}$, that is $\mathcal{P}_w$ is defined by

$$\mathcal{P}_w([X_v]_T) = \delta_{w,v} \qquad \text{for all } v \in W.$$

The basis $\{\mathcal{P}_w\}_{w\in W}$ is known as *Schubert basis*. We have $\deg(\mathcal{P}_w) = 2\ell(w)$.

If $\mathbb{K}$ is a field of characteristic 0, there exists also a second useful description of the equivariant cohomology $H^\bullet_T(X,\mathbb{K})$. Let us denote by $R^W \subseteq R$ the subring of W-invariants. Then we have, as explained in [Bri98, Proposition 1]:

$$H^\bullet_T(X,\mathbb{K}) \cong R \otimes_{R^W} R.$$

Remark 2.1.1. Since $B = TU$ and $U = [B,B]$ is contractible, for any B-space Y we have $H^\bullet_B(Y,\mathbb{K}) \cong H^\bullet_T(Y,\mathbb{K})$. In particular, $H^\bullet_T(X,\mathbb{K}) \cong H^\bullet_B(G/B,\mathbb{K}) \cong H^\bullet_{B\times B}(G,\mathbb{K})$, and this means that $H^\bullet_T(X,\mathbb{K})$ is in a natural way a module over $H^\bullet_{B\times B}(pt,\mathbb{K}) = R\otimes_{\mathbb{K}} R$, that is $H^\bullet_T(X,\mathbb{K})$ is naturally a R-bimodule.

From the equivariant cohomology we can also recover the usual singular cohomology $H^\bullet(X,\mathbb{K})$. In fact, we have

$$H^\bullet(X,\mathbb{K}) \cong \mathbb{K}\otimes_R H^\bullet_T(X,\mathbb{K}) \cong \mathbb{K}\otimes_{R^W} R.$$

In particular, we have $H^\bullet(X,\mathbb{K}) \cong R/R^W_+$ where R^W_+ is the ideal of R generated by homogeneous W-invariant of positive degree. The ring R/R^W_+ is called the *coinvariant ring*.

Let $P_w = 1\otimes \mathcal{P}_w \in H^\bullet(X,\mathbb{K})$. Then $\{P_w\}_{w\in W}$ is a basis $H^\bullet(X,\mathbb{K})$ over $\mathbb{K}$, also called Schubert basis.

Remark 2.1.2. It is false for a general ring $\mathbb{K}$ that $R/R^W_+ \cong H^\bullet(X,\mathbb{K})$. Assume for example $\mathbb{K} = \mathbb{Z}$. Then in general the ring R/R^W_+ is not even free as an abelian group. Using the software Magma [BCP97] we have spotted p-torsion in the coinvariant ring R/R^W_+ as illustrated by Table 2.1. The indicated degree k is the minimum degree in which

Table 2.1:

W	B_5	D_5	F_4	E_6		E_7		E_8	
p	2	2	3	2	3	2	3	2	5
k	22	16	14	16	12	22	12	16	16

such p-torsion appears. From computer computations also follows that there is no p-torsion for G_2, B_n if $n \leq 4$, D_n if $n \leq 4$ and that there is no 2-torsion for F_4. We do not know whether there is 3-torsion for E_8.

However, to have an isomorphism $R/R_+^W \cong H^\bullet(X, \mathbb{K})$ it is sufficient that the primes listed in [Dem73, Proposition 8] are invertible in the ring $\mathbb{K}$. In particular, it follows that there is no p-torsion in type A and C.

For any Schubert variety X_w we have

$$H_T^\bullet(X_w, \mathbb{Z}) = \bigoplus_{x \leq w} R\mathcal{P}_x \quad \text{and} \quad H^\bullet(X_w, \mathbb{Z}) \cong \bigoplus_{x \leq w} \mathbb{Z}P_x.$$

The inclusion map $j_w : X_w \hookrightarrow X$ induces the map $j_w^* : H_T^\bullet(X, \mathbb{Z}) \to H_T^\bullet(X_w, \mathbb{Z})$ given by $j_w^*(\mathcal{P}_x) = \mathcal{P}_x$ if $x \leq w$ and $j_w^*(\mathcal{P}_x) = 0$ otherwise.

For a subset $I \subseteq S$, we denote by W_I the subgroup of W generated by I and by $\mathbf{P}_I \supseteq B$ the parabolic subgroup corresponding to I. The homogeneous space $G/\mathbf{P}_I$ is called *partial flag variety*. We shorten R^{W_I} by R^I. We have

$$H_T^\bullet(G/\mathbf{P}_I, \mathbb{K}) \cong R \otimes_{R^W} R^I \quad \text{and} \quad H^\bullet(G/\mathbf{P}_I, \mathbb{K}) \cong R^I/R_+^W.$$

If $\pi : G/B \to G/\mathbf{P}_I$ is the projection, then $\pi^* : H_T^\bullet(G/\mathbf{P}_I, \mathbb{K}) \to H_T^\bullet(G/B, \mathbb{K})$ is injective, and we can identify an element in $H_T^\bullet(G/\mathbf{P}_I, \mathbb{K})$ with its image under π^* [BGG73]. Let W^I be the set of representatives of minimal length in W/W_I. Then a R-basis for $H_T^\bullet(G/\mathbf{P}_I, \mathbb{K})$ is given by the set

$$\{\mathcal{P}_v \mid v \in W^I\}.$$

Similarly, a $\mathbb{K}$-basis for $H^\bullet(G/\mathbf{P}_I, \mathbb{K})$ is given by the set

$$\{P_v \mid v \in W^I\}.$$

2.2 The nil Hecke ring and its dual

A third, algebraic, description of the equivariant cohomology of the flag variety was given by Konstant and Kumar in [KK86a, KK86b] and Arabia in [Ara86]. It is important to remark that the construction can be generalized to arbitrary Coxeter groups (if we restrict to the realizations of W discussed in §1.2, as pointed out in [KK86a, Remark 4.35(b)], see also [Wil16, §3.4]) In fact, Kostant and Kumar's original motivation was to provide an algebraic description of the (equivariant) cohomology of flag varieties of Kac-Moody groups.

Let Q be the field of fractions of R and let Q_W denote the smash product of Q with W. This means that Q_W is a free left Q-module with basis $\{\delta_w\}_{w \in W}$ and multiplication defined by

$$(f\delta_x)(g\delta_y) = fx(g)\delta_{xy}.$$

In particular, $f\delta_x = \delta_x x^{-1}(f)$. We have an anti-involution $(-)^t$ on Q_W defined by

$$(q\delta_x)^t = x^{-1}(q)\delta_{x^{-1}}.$$

Notice that Q_W is not an algebra over Q since $q\delta_{id}$ is not a central element for $q \in Q$.

For $s \in S$ we define the element

$$D_s = \frac{1}{\alpha_s}(\delta_{id} - \delta_s) = (\delta_{id} + \delta_s)\frac{1}{\alpha_s} \in Q_W.$$

We have $D_s^2 = 0$ and the D_s satisfy the braid relations [KK86a, Proposition 4.2], i.e.

$$\underbrace{D_s D_t D_s \dots}_{m_{st}\text{times}} = \underbrace{D_t D_s D_t \dots}_{m_{st}\text{times}}$$

Hence, for $x \in W$ we can define $D_x = D_{s_1} D_{s_2} \dots D_{s_l}$ where $\underline{x} = s_1 s_2 \dots s_l$ is any reduced expression for x. We have a natural left action of Q_W on Q via $f\delta_x \cdot g = fx(g)$.

Definition 2.2.1. The *nil-Hecke ring* $NH(W)$ is defined to be the ring $\{q \in Q_W \mid q(R) \subseteq R\} \subseteq Q_W$.

Theorem 2.2.2. *[KK86a, Theorem 4.6] The ring $NH(W)$ is a free right R-module with basis $\{D_w\}_{w \in W}$.*

Let $\Omega = \mathrm{Hom}_{-Q}(Q_W, Q)$ be the set of right Q-module morphisms. We can think of Ω as the set of functions $W \to Q$, where to an element $\psi \in \Omega$ corresponds the function $W \to Q$ which sends $x \in W$ to $\psi(\delta_x)$. We regard Ω as a Q-algebra, via point-wise addition, scalar multiplication and multiplication. The algebra Ω has also a structure of left Q_W-module via

$$f \cdot \psi(y) = \psi(f^t \cdot y).$$

Warning 2.2.3. Notice that this defines also a new structure of R-module on Ω via $f \cdot \psi(y) = \psi(f \cdot y)$. However, this does not coincide with the R-action given by point-wise multiplication. To differentiate, we will always write the one induced by the left Q_W action as a left action and the point-wise multiplication as a right action on Ω.

Let us consider the following subspace of Ω:

$$\Lambda := \left\{\psi \in \Omega \mid \psi(NH(W)^t) \subseteq R \text{ and } \psi(D_w^t) \neq 0 \text{ only for a finite number of } w \in W\right\}.$$

Proposition 2.2.4. *[KK86a, Proposition 4.20] The subspace Λ is a R-subalgebra of Ω. Let $\xi^x \in \Omega$ defined by $\xi^x(D_y^t) = \delta_{x,y}$.[2] As a right R-module Λ is free with basis $\{\xi^x\}_{x \in W}$.*

The ring Λ is called the *dual nil-Hecke ring*. It is a graded ring with $\deg(\xi^x) = 2\ell(x)$. It provides a new algebraic description of the equivariant cohomology of flag varieties.

Theorem 2.2.5. *[Ara89] Let A be a generalized Cartan matrix, G the corresponding Kac-Moody group and W its Weyl group. Then there exists an isomorphism*

$$H_T^\bullet(G/B, \mathbb{K}) \cong \Lambda$$

which sends the Schubert basis element $\mathcal{P}_x$ into ξ^x.

Notice that in general the cohomology of the flag variety of a Kac-Moody group is not generated in degree 2, and there is no description available as a "coinvariant ring".

[2] Here $\delta_{x,y}$ denotes the Kronecker delta. It has nothing to do with δ_x defined above!

Warning 2.2.6. In [KK86a] Kostant and Kumar use a different definition of the elements ξ^x and $d_{x,y}$. We denote them by ξ^x_{KK} and $d^{KK}_{x,y}$ to distinguish from the ones used here. We have $\xi^x = (-1)^{\ell(x)}\xi^{x^{-1}}_{KK}$ and $d_{x,y} = (-1)^{\ell(x)} d^{KK}_{x^{-1},y^{-1}}$.

Lemma 2.2.7. *[KK86a, Proposition 4.3(b)] For all $\lambda \in \mathfrak{h}^*$ and $x \in W$ we have*

i) $\lambda \cdot D_x = D_x x^{-1}(\lambda) + \sum_{y \xrightarrow[L]{t} x} D_y \partial_t(\lambda)$;

ii) $\lambda \cdot D^t_x = D^t_x x(\lambda) + \sum_{y \xrightarrow[R]{t} x} D^t_y \partial_t(\lambda)$;

iii) $\lambda \cdot \xi^x = \xi^x x(\lambda) + \sum_{x \xrightarrow[R]{t} y} \xi^y \partial_t(\lambda)$.

Proof. First we consider D_s, for s a simple reflection. We have

$$D_s s(f) = \frac{1}{\alpha_s}(\delta_{id} - \delta_s)s(f) = \frac{s(f)}{\alpha_s}\delta_{id} - \frac{f}{\alpha_s}\delta_s = fD_s - \frac{f - s(f)}{\alpha_s}D_{id} = fD_s - D_{id}\partial_s(f)$$

The general case easily follows by induction using $D_x = D_y D_s$ with $y < x$ and $s \in S$.

The second statement now follows using $\lambda \cdot D^t_x = (D_x \lambda^t)^t = (D_x \lambda)^t$ and that if $y \xrightarrow[L]{t} x$ then $y \xrightarrow[R]{y^{-1}ty} x$ and $\partial_t(x(\lambda)) = -\partial_{y^{-1}ty}(\lambda)$. The third statement follows since $(\lambda \cdot \xi^x)(D^t_y) = \xi^x(\lambda^t \cdot D^t_y) = \xi^x(\lambda \cdot D^t_y)$. ☐

The third statement gives a formula for multiplying a Schubert basis element with a weight. This is often referred to as the *Chevalley formula.*

We can write $D_y = \sum_{x \in W} e_{x,y}\delta_x$, with $e_{x,y} \in Q$. The rational functions $e_{x,y}$ are homogeneous of degree $-2\ell(y)$ and are called *equivariant multiplicities.*

Proposition 2.2.8 ([Wil16, Prop. 3.6])**.** *We have:*

i) $e_{x,y} = 0$ *unless* $x \leq y$;

ii) $e_{y,y} = (-1)^{\ell(y)}(p_y)^{-1}$, *where* p_y *is defined in* (1.2).

We define $d_{x,y} := \xi^x(\delta_{y^{-1}})$. Let $E = (e_{x,y})_{x,y \in W}$ and $D = (d_{x,y})_{x,y \in W}$.

Proposition 2.2.9 ([KK86a, Prop. 4.24])**.** *We have:*

i) $d_{x,y} = 0$ *unless* $x \leq y$

ii) $D = E^{-1}$, *i.e. for any* $x, y \in W$ *we have* $\sum_z d_{x,z}e_{z,y} = \delta_{x,y}$. *In particular, we have* $d_{x,x} = e^{-1}_{x,x} = (-1)^{\ell(x)}p_x$.

iii) For any $x, y \in W$, *the rational fraction* $d_{v,w}$ *belongs to* R *and it is homogeneous of degree* $2\ell(v)$.

iv) For any $\lambda \in \mathfrak{h}^*$, $d_{x,z}(z(\lambda) - x(\lambda)) = \sum_{x \xrightarrow[R]{t} y} \partial_t(\lambda) d_{y,z}$.

Proof. We prove here only iv). This follows from

$$d_{x,z}z(\lambda) = \xi^x(\delta_{z^{-1}}z(\lambda)) = \xi^x(\lambda \cdot \delta_{z^{-1}}) = (\lambda \cdot \xi^x)(\delta_{z^{-1}}) =$$
$$= \xi^x(\delta_{z^{-1}})x(\lambda) + \sum_{x \xrightarrow[R]{t} y} \xi^y(\delta_{z^{-1}})\partial_t(\lambda) = d_{x,z}x(\lambda) + \sum_{x \xrightarrow[R]{t} y} d_{y,z}\partial_t(\lambda). \qquad \square$$

2.3 The affine Grassmannian and the affine flag variety

The *affine Grassmannian* of the simply-connected semisimple group G is defined as

$$\mathcal{G}r := G((t))/G[[t]].^3$$

The affine Grassmannian is a complex ind-projective variety of infinite dimension, i.e. it can be obtained as direct limit of finite dimensional complex projective varieties.

Let $\mathbf{X}_*(T)$ be the *cocharacter lattice*, that is the dual lattice of $\mathbf{X}^*(T)$. Since G is assumed to be simply connected we have $\mathbf{X}_*(T) \cong \mathbb{Z}\Phi^\vee$. Let

$$\mathbf{X}_*(T)_+ = \{\mu \in \mathbf{X}_*(T) \mid \alpha(\mu) \geq 0 \text{ for all } \alpha \in \Phi^+\} \subseteq \mathbf{X}_*(T).$$

Elements of $\mathbf{X}_*(T)$ can also be thought as algebraic morphism $\mathbb{C}^* \to T$, hence as morphisms $\mathbb{C}((t))^* \to T((t))$. Let $t^\mu \in T((t))$ be the image of t under this map. Two important decompositions of $\mathcal{G}r$ are

$$\mathcal{G}r = \bigsqcup_{\mu \in \mathbf{X}_*(T)_+} G[t^{-1}] \cdot t^\mu G[[t]]/G[[t]] \qquad \text{(Birkhoff decomposition [Zhu16, (2.3.1)])}$$

$$\mathcal{G}r = \bigsqcup_{\mu \in \mathbf{X}_*(T)_+} G[[t]] \cdot t^\mu G[[t]]/G[[t]] \qquad \text{(Cartan decomposition [Zhu16, (2.1.2)])}$$

Let $\pi_0 : G[[t]] \to G$ be the map defined by sending $t \mapsto 0$. The group $I = \pi_0^{-1}(B)$ is called the *Iwahori subgroup* of $G((t))$. The quotient

$$\widehat{\mathcal{F}l} = G((t))/I$$

is called the *affine flag variety* of G. Consider now the affine Weyl group $\widetilde{W}$. It is the subgroup of affine transformations of $\mathbf{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the Weyl group W and the coroot lattice $\mathbb{Z}\Phi^\vee$ (which acts by translations). We have $\widetilde{W} \cong \mathbb{Z}\Phi^\vee \rtimes W$, hence $\widetilde{W}/W \cong \mathbb{Z}\Phi^\vee \cong \mathbf{X}_*(T)$. We also have the Bruhat decomposition [Zhu16, 2.1.22]:

$$\mathcal{G}r = \bigsqcup_{\mu \in \mathbf{X}_*(T)} I \cdot t^\mu G[[t]]/G[[t]] \qquad \widehat{\mathcal{F}l} = \bigsqcup_{x \in \widetilde{W}} I \cdot xI/I$$

We mention also another realization of the affine Grassmannian and of the affine flag variety as loop spaces [PS86].

The space $G(\mathbb{C}[t, t^{-1}])$ is the space of algebraic maps $\mathbb{C}^* \to G$. Let K be a maximal compact subgroup of G. Let $L_{pol}K$ be the subspace of $G(\mathbb{C}[t, t^{-1}])$ that sends $S^1 \subseteq \mathbb{C}^*$ into K. We have a subspace $\Omega_{pol}K \subseteq L_{pol}K$ of maps that send $1 \in S^1$ to $1 \in G$. Then the inclusion $\Omega_{pol}K \hookrightarrow G(\mathbb{C}[t, t^{-1}]) \hookrightarrow \mathbb{C}((t))$ induces a homeomorphism $\Omega_{pol}K \cong \mathcal{G}r$ (this is the analogue for loop groups of writing an element in $GL_n(\mathbb{C})$ as a product of a

[3] by $G((t)), G[[t]], G[t]$, etc., we mean the k-rational points of G, with $k = \mathbb{C}((t)), \mathbb{C}[[t]], \mathbb{C}[t]$, etc.

unitary matrix and a upper triangular one). This is proven in [PS86, Theorem 8.6.3] for $G = GL_n(\mathbb{C})$ (see [Zhu16, §1.6] for the general case).

We sketch the proof for $GL_n(\mathbb{C})$: first we identify the affine Grassmannian with the set of $\mathbb{C}[[t]]$-lattices in $\mathbb{C}((t))^n$ as in [Zhu16, §1.1]. Let $H^{(n)} = L^2(S^1, \mathbb{C}^n)$ be the Hilbert space of square-integrable functions from S^1 to $\mathbb{C}^n$. Let $\{e_i\}_{1\leq i\leq n}$ be the standard basis of $\mathbb{C}^n$. Then we can write any element of $f \in H^{(n)}$ as

$$f(z) = \sum_{i=1}^{n} \left(\sum_{k\in\mathbb{Z}} f_{i,k} z^k \right) e_i \qquad \text{with } f_{i,k} \in \mathbb{C}.$$

Let $H_+ = \{f \in H^{(n)} \mid f_{i,k} = 0 \text{ for all } k < 0\} \subseteq H^{(n)}$. Consider the set

$$Gr_{(0)} = \{W \subseteq H^{(n)} \text{ subspace } \mid zW \subseteq W \text{ and } \exists h \geq 0 \text{ such that } z^h H_+ \subseteq W \subseteq z^{-h} H_+\}.$$

For any $\mathbb{C}[[t]]$-lattice $\mathcal{L} \subseteq \mathbb{C}((t))^n$ there exists $h \geq 0$ such that $t^h(\mathbb{C}[[t]])^n \subseteq W \subseteq t^{-h}(\mathbb{C}[[t]])^n$. Let $\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n$ be the standard basis of $\mathbb{C}((t))^n$. If $\widetilde{b}_1, \widetilde{b}_2, \ldots \widetilde{b}_n$ is a basis of $\mathcal{L}$, with

$$\widetilde{b}_j = \sum_{i=1}^{n} \left(\sum_{k\geq -h} c^j_{i,k} t^k \right) \widetilde{e}_i \qquad \text{with } c^j_{i,k} \in \mathbb{C},$$

we associate to $\mathcal{L}$ the subspace $W = W_0 + z^h H_+ \in Gr_{(0)}$, where $W_0 \subseteq H^{(n)}$ is the finite dimensional vector space with basis $b_1, \ldots, b_n$ where

$$b_j = \sum_{i=1}^{n} \left(\sum_{-h\leq k\leq h} c^j_{i,k} z^k \right) e_i.$$

This induces a homeomorphism between $\mathcal{G}r^h := \{\mathcal{L} \in \mathcal{G}r \mid t^h(\mathbb{C}[[t]])^n \subseteq \mathcal{L} \subseteq t^{-h}(\mathbb{C}[[t]])^n\}$ and $Gr^h_{(0)} := \{W \in Gr_{(0)} \mid z^h H_+ \subseteq W \subseteq z^{-h} H_+\}$. They glue together in a homeomorphism between $\mathcal{G}r$ and $Gr_{(0)}$.

For $G = GL_n(\mathbb{C})$ we can take $K = U_n$. The group $L_{pol}K$ acts transitively on $Gr_{(0)}$, and the stabilizer of H_+ is K [PS86, Theorem 8.3.2 and Proposition 8.3.3(a)]. This shows

$$\Omega_{pol} K \cong L_{pol}K/K \cong Gr_{(0)} \cong \mathcal{G}r.$$

Similarly, we have a homeomorphism $\widehat{\mathcal{F}l} \cong L_{pol}K/T_{\mathbb{R}}$, where $T_{\mathbb{R}} = T \cap \mathbb{R}$ (this is the infinite-dimensional analogue of the homeomorphism $K/T_{\mathbb{R}} \cong G/B$). In fact, the affine flag variety can be identified with the set of *full periodic chains of lattices* in $\mathbb{C}((t))^n$ [Gör10, Proposition 2.13], i.e. with the set of chains of $\mathbb{C}[[t]]$-lattices in $\mathbb{C}((t))^n$

$$\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \mathcal{L}_2 \supseteq \ldots \supseteq \mathcal{L}_{n-1} \supseteq t\mathcal{L}_0$$

such that $\dim_{\mathbb{C}}(\mathcal{L}_i/\mathcal{L}_{i+1}) = 1$. Using the same map as above full periodic chains correspond to elements in the set

$$Fl_{(0)} = \left\{ (W_i)_{0\leq i\leq n-1} \in (Gr_{(0)})^n \,\middle|\, \begin{array}{c} W_0 \supseteq W_1 \supseteq \ldots W_{n-1} \supseteq zW_n \\ \text{such that } \dim W_i/W_{i+1} = 1 \text{ for all } i \end{array} \right\}.$$

Again, $L_{pol}K$ acts transitively on $Fl_{(0)}$. In fact, we can find $\gamma \in L_{pol}K$ such that $\gamma(H_+) = W_0$ and the set of chains with $W_0 = H_+$ can be easily identified with the set of flags in $H_+/zH_+ \cong \mathbb{C}^n$. The group K acts on the set of flags of H_+/zH_+ and the action is transitive. Moreover, the stabilizer of the standard flag is $T_{\mathbb{R}}$. We obtain $\widehat{\mathcal{F}l} \cong L_{pol}K/T_{\mathbb{R}}$ (cf. [PS86, Proposition 8.7.6]).

Proposition 2.3.1. *The fiber bundle $p : \widehat{\mathcal{F}l} \to \mathcal{G}r$ is topologically trivial, i.e. $\widehat{\mathcal{F}l} \cong \mathcal{G}r \times G/B$ as topological spaces.*

Proof. We use the identifications $K/T_{\mathbb{R}} \cong G/B$, $\mathcal{G}r \cong \Omega_{pol}K$ and $\widehat{\mathcal{F}l} \cong L_{pol}K/T_{\mathbb{R}}$ as above. It is easy to see that we have a $T_{\mathbb{R}}$-equivariant homeomorphism $\Omega_{pol}K \times K \cong L_{pol}K$. By modding out both sides by $T_{\mathbb{R}}$ we obtain a homeomorphism $\Omega_{pol}K \times K/T_{\mathbb{R}} \xrightarrow{\sim} L_{pol}K/T_{\mathbb{R}}$ defined by $(x, yT_{\mathbb{R}}) \mapsto xyT_{\mathbb{R}}$. This gives an isomorphism of fiber bundles on $\Omega_{pol}K$

$$\begin{tikzcd}
\Omega_{pol}K \times K/T_{\mathbb{R}} \arrow[rr, "\sim"] \arrow[dr] & & L_{pol}K/T_{\mathbb{R}} \arrow[dl, "p"] \\
& \Omega_{pol}K &
\end{tikzcd}$$

It follows that the projection p is a topologically trivial fiber bundle. $\square$

We obtain
$$H^*(\widehat{\mathcal{F}l}, \mathbb{K}) \cong H^*(\mathcal{G}r, \mathbb{K}) \otimes H^*(G/B, \mathbb{K}).$$
(see also [Lee15] for a more detailed description of this isomorphism).

The cohomology of the affine flag variety and of the affine Grassmannian can be described using the nil-Hecke ring. In fact, there exists a Kac-Moody group $\widetilde{G}$ with Weyl group $\widetilde{W}$ (with Borel $\widetilde{B}$ and maximal torus $\widetilde{T}$) such that

$$\widehat{\mathcal{F}l} \cong \widetilde{G}/\widetilde{B} \quad \text{and} \quad \mathcal{G}r \cong \widetilde{G}/\widetilde{\mathbf{P}}$$

where $\widetilde{\mathbf{P}} \subseteq \widetilde{G}$ is the maximal parabolic subgroup corresponding to finite Weyl group $W \subseteq \widetilde{W}$ [Kum02, Chapter XIII]. Let Λ be the dual nil Hecke ring of $\widetilde{W}$ constructed using the realization of type II associated to the affine Cartan matrix of G. We obtain:

$$H^\bullet(\widehat{\mathcal{F}l}, \mathbb{K}) \cong \Lambda \otimes_R \mathbb{K} \quad \text{and} \quad H^\bullet(\mathcal{G}r, \mathbb{K}) \cong \Lambda^W \otimes_R \mathbb{K}.$$ [4]

2.4 Perverse sheaves on the flag variety

We recollect some rudiments about perverse sheaves on the flag variety X to provide a geometric motivation for the introduction of Soergel bimodules in the next chapter. This is also necessary in order to explain the connection with modular representation theory. For a detailed introduction to equivariant sheaves and equivariant perverse sheaves we refer to [BL94].

Let G be a complex semisimple simply-connected algebraic group and let X be its flag variety. Let $\mathcal{D}^b_B(X, \mathbb{K})$ be the bounded B-equivariant derived category of sheaves of $\mathbb{K}$-modules. We have $\mathcal{D}^b_B(X, \mathbb{K}) \cong \mathcal{D}^b_{B\times B}(G, \mathbb{K})$, where $B \times B$ acts on G via $(b, b') \cdot g = bgb'^{-1}$.

Let $\mathcal{P}\mathrm{erv}_B(X, \mathbb{K}) \subseteq \mathcal{D}^b_B(X, \mathbb{K})$ denote the full subcategory of B-equivariant perverse sheaves. The category of perverse sheaves can be obtained as the heart of a t-structure, and so it is an abelian category.

For $w \in W$ let $IC_w^{\mathbb{K}} := IC(X_w, \mathbb{K})$ denote the *intersection cohomology sheaf* of the Schubert variety X_w. The set $\{IC_w^{\mathbb{K}}\}_{w\in W}$ is a complete set of representatives of simple objects in $\mathcal{P}\mathrm{erv}_B(X, \mathbb{K})$ up to isomorphism.

We can equip the category $\mathcal{D}^b_B(X, \mathbb{K})$ with a monoidal structure given by a functor

$$\star : \mathcal{D}^b_B(X, \mathbb{K}) \times \mathcal{D}^b_B(X, \mathbb{K}) \to \mathcal{D}^b_B(X, \mathbb{K})$$

[4] However, $T \subsetneq \widetilde{T}$ so the nil-Hecke ring does not describe directly the T-equivariant cohomology.

as follows. For $\mathcal{F},\mathcal{G} \in \mathcal{D}^b_B(X,\mathbb{K})$. We think $\mathcal{F},\mathcal{G} \in \mathcal{D}^b_{B\times B}(G,\mathbb{K})$, so that $\mathcal{F}\boxtimes\mathcal{G} \in \mathcal{D}^b_{B^4}(G\times G,\mathbb{K})$. Let $\iota : B^3 \to B^4$ the embedding $(b_1,b_2,b_3) \mapsto (b_1,b_2,b_2,b_3)$. Then by restriction we can regard $\mathcal{F}\boxtimes\mathcal{G}$ as a B^3-equivariant sheaf on $G\times G$. There is an equivalence $r : \mathcal{D}^b_{B^3}(G\times G,\mathbb{K}) \xrightarrow{\sim} \mathcal{D}^b_B(G\times_B X,\mathbb{K})$. Finally we have the map $m : G\times_B X \to X$ induced by the multiplication, that is $m(g,x) = gx$. We define

$$\mathcal{F}\star\mathcal{G} := m_*(r(\mathcal{F}\boxtimes\mathcal{G})) \in \mathcal{D}^b_B(X,\mathbb{K}).$$

Let now $\mathbb{K} = \mathbb{Q}$ (or any field of characteristic 0). Let $\mathcal{K}$ the full subcategory of $\mathcal{D}^b_B(X,\mathbb{R})$ whose objects are direct sums of shifts of IC complexes $IC^{\mathbb{Q}}_x := IC_B(X_x,\mathbb{Q})$. It is a consequence of the decomposition theorem [BBD82] applied to the projective map m that the category $\mathcal{K}$ is stable under $\star$. Let $[\mathcal{K}]$ denote the split Grothendieck group of the additive category $\mathcal{K}$. It is a $\mathbb{Z}[v,v^{-1}]$-algebra, where v acts by shifting the degree by one, i.e. $v\cdot\mathcal{F} = \mathcal{F}[1]$.

Theorem 2.4.1. *[KL80, Spr82b] There exists a unique isomorphism of $\mathbb{Z}[v,v^{-1}]$-algebras*

$$\varepsilon : \mathcal{H}(W,S) \xrightarrow{\sim} [\mathcal{K}]$$

such that $\underline{\mathbf{H}}_x \mapsto [IC^{\mathbb{Q}}_x]$ for all $s\in S$.

We can also construct an inverse ch to the isomorphism ε. For $x\in W$, if $\mathcal{F}\in\mathcal{D}^b_B(X,\mathbb{Q})$ we denote by $\mathcal{F}_x$ its stalk in $xB\in X$. Then let

$$h_x(\mathcal{F}) = \sum_{i\in\mathbb{Z}} \dim H^{-i}(\mathcal{F}_x)v^{-\ell(x)+i}$$

and $\mathrm{ch}(\mathcal{F}) = \sum_{x\in W} h_x(\mathcal{F})\mathbf{H}_x$. In particular, we recover the KL polynomials $h_{x,y} = h_x(IC^{\mathbb{Q}}_y)$.

For a complex $\mathcal{F}\in\mathcal{D}^b_{B\times B}(G,\mathbb{Q})$ we can regard its hypercohomology $\mathbb{H}^\bullet(\mathcal{F})$ in a natural way as a graded module over $H^\bullet_{B\times B}(pt,\mathbb{Q}) \cong R\otimes_{\mathbb{Q}} R$, hence as a bimodule over R.

Theorem 2.4.2 (Erweiterungssatz [Soe90, Gin91])**.** *The hypercohomology functor $\mathbb{H}^\bullet$ is fully faithful and monoidal on $\mathcal{K}$, i.e. for any $\mathcal{F},\mathcal{G}\in\mathbb{K}$ we have*

$$\mathrm{Hom}_{\mathcal{D}^b_{B\times B}(G,\mathbb{Q})}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{R\otimes R}(\mathbb{H}^\bullet\mathcal{F},\mathbb{H}^\bullet\mathcal{G})$$

and

$$\mathbb{H}^\bullet(\mathcal{F}\star\mathcal{G}) \cong \mathbb{H}^\bullet\mathcal{F}\otimes_R\mathbb{H}^\bullet\mathcal{G}.$$

In particular, the category $\mathcal{K}$ is equivalent to its essential image under $\mathbb{H}$. The resulting category is called the category of *Soergel bimodules.* In Chapter 3.1 we will give an alternative definition of the category of Soergel bimodules that uses as input only the action of W on $\mathfrak{h}$, so that it can be generalized to any Coxeter group and any reflection faithful realization $\mathfrak{h}$.

Remark 2.4.3. For most of the content in this section we can replace B-equivariant sheaves on X with B-constructible sheaves on X. In this case the hypercohomology will be naturally a R-module, and arguing similarly we obtain the category of *Soergel modules.* For finite Coxeter groups the categories of Soergel modules and Soergel bimodules have very similar behavior. However, we show in §3.6 that this is not necessarily the case for infinite Coxeter groups.

Remark 2.4.4. If $\mathbb{K}$ is a field of positive characteristic the decomposition theorem breaks down, and the category $\mathcal{K}$ is ill-behaved with respect of the monoidal structure $\star$. A natural replacement to perverse sheaves in this case is given by the theory of parity sheaves [JMW16].

2.4.1 Lusztig's conjecture

The reference for this section is [Wil17a]. Let now $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and let $G_{\mathbb{K}}^{\vee}$ be the Langlands dual group of G defined over $\mathbb{K}$ with maximal torus $T_{\mathbb{K}}^{\vee}$. We have $\mathbf{X}^*(T_{\mathbb{K}}^{\vee}) = \mathbf{X}_*(T)$.

We are interested in the category $\operatorname{Rep} G_{\mathbb{K}}$ of finite dimensional algebraic representations of $G_{\mathbb{K}}$ over $\mathbb{K}$. Lusztig's conjecture gives a formula to compute the characters of simple modules in $\operatorname{Rep} G_{\mathbb{K}}^{\vee}$. Before stating it more precisely we need to introduce some terminology.

We choose a Borel subgroup $B_{\mathbb{K}}^{\vee} \subseteq G_{\mathbb{K}}^{\vee}$ corresponding to negative coroots. For any $\lambda \in \mathbf{X}^*(T^{\vee})$ we have the *Weyl module* $\Delta(\lambda) := (\Gamma(G_{\mathbb{K}}^{\vee}/B_{\mathbb{K}}^{\vee}, \mathcal{L}_{\lambda})^*)^{\sigma}$, where σ denotes the Chevalley involution. The character of a Weyl module is "easy" as it can computed using the Weyl character formula.

There is a bijection between

$$\mathbf{X}^*(T^{\vee})_+ = \{\lambda \in X^*(T^{\vee}) \mid \lambda(\alpha^{\vee}) \geq 0 \text{ for all } \alpha \in (\Phi^{\vee})^+\}$$

and simple $G_{\mathbb{K}}$ modules, given by $\lambda \mapsto L(\lambda) = \operatorname{head}(\Delta(\lambda))$. We define the set of *$p$-restricted weights*:

$$\mathbf{X}_1^p := \{\lambda \in \mathbf{X}^*(T^{\vee}) \mid 0 \leq \lambda(\alpha^{\vee}) \leq p \text{ for all } \alpha \in \Delta\}$$

Consider now the affine Weyl group $\widetilde{W}$ of G. The *p-dilated dot action* $\bullet_p$ of the affine Weyl group $\widetilde{W}$ is defined by

$$x \bullet_p \mu = x(\mu + \rho) - \rho \qquad \text{if } x \in W$$

$$\lambda \bullet_p \mu = \mu + p\lambda \qquad \text{if } \lambda \in \mathbb{Z}\Phi$$

Let h be the Coxeter number of W. A proposed version of Lusztig's conjecture [Wil17a, Conjecture 1.20] is the following:

Conjecture 2.4.5. *Assume $p > h$ and $x \bullet_p \mu \in X_p^1$. Then:*

$$\operatorname{ch} L(x \bullet_p \mu) = \sum_{\substack{y \leq x \\ y \bullet_p \mu \in \mathbf{X}^*(T^{\vee})_+}} (-1)^{\ell(y)-\ell(x)} h_{y,x}(1) \operatorname{ch} \Delta(y \bullet_p \mu) \quad \text{(Lusztig's character formula)}$$

where $h_{x,y}$ are the KL polynomials for $\widetilde{W}$.

Using further techniques (Steinberg tensor product theorem, Jantzen translation functors) if Lusztig's character formula holds one can compute the character of any irreducible representation of $G_{\mathbb{K}}^{\vee}$. We know by work of Andersen, Jantzen and Soergel [AJS94] that Lusztig's character formula holds for $p \gg h$, and by work of Williamson [Wil17b] we know that there exists a family of counterexamples to Lusztig's character formula for $p \sim c^h$, with $c \sim 1.10\ldots$. We still do not know precisely where Lusztig's conjecture starts to hold.

There are several deep ties between the representation theory of $G_{\mathbb{K}}^{\vee}$ and the geometry. It is worth to mention the geometric Satake correspondence [MV07], which states that there exists an equivalence of monoidal categories

$$(\mathcal{P}\mathrm{erv}_{G[[t]]}(\mathcal{G}r, \mathbb{K}), \star) \cong (\operatorname{Rep} G_{\mathbb{K}}^{\vee}, \otimes_{\mathbb{K}}).$$

We can also use the geometry of the affine flag variety to control Lusztig's conjecture. For $x \in \widetilde{W}$ let $\widehat{\mathcal{F}l}_x$ be the corresponding Schubert variety, i.e. $\widehat{\mathcal{F}l}_x = \overline{I \cdot xI/I}$.

Theorem 2.4.6. *[FW14, Theorem 9.2] Let $A = \{x \in \widetilde{W} \mid x^{-1} \bullet_p (-2\rho) \in \mathbf{X}_1^p\}$. If there is no p-torsion in the stalk and costalk of $IC(\widehat{\mathcal{F}l}_x, \mathbb{Z})$ for $x \in A$, then Lusztig's conjecture holds.*

Fiebig [Fie08, Theorem 4.6] described an approach to prove the absence of p-torsion. Let $\mathcal{E}(x)$ be the indecomposable parity sheaf defined as in [JMW16, §4.1]. Then it is enough to check for any $x \in A$ that the local hard Lefschetz theorem holds for the parity sheaf $\mathcal{E}(x)$ at any point. In this case every $\mathcal{E}(x)$ can also be obtained as base change of the integral intersection cohomology sheaf $IC(\widehat{\mathcal{F}l}_x, \mathbb{Z})$ [WB12, Proposition 3.11]. We refer to [Wil16] for a precise statement of the local hard Lefschetz theorem in the setting of Soergel bimodules.

Using this approach Fiebig proved an upper bound for Lusztig's conjecture: if p is bigger than a certain number $U(\hat{w}_0)$, defined in [Fie08, §1.3], then Lusztig's conjecture holds.

Chapter 3

Soergel Bimodules, Moment Graphs, and the Hom Formula for Soergel Modules

In this chapter W denotes an arbitrary Coxeter group and $\mathfrak{h}$ is one of the reflection faithful realizations of W over a field $\mathbb{K}$ discussed in §1.2. Recall that $R = \mathrm{Sym}_{\mathbb{K}}(\mathfrak{h}^*)$.

3.1 Soergel bimodules

For $s \in S$ we denote by B_s the graded R-bimodule $R \otimes_{R^s} R[1]$. Let $\underline{w} = s_1 s_2 \dots s_k$ be an expression, not necessarily reduced. The Bott-Samelson bimodule $BS(\underline{w})$ is the graded R-bimodule defined as

$$BS(\underline{w}) = B_{s_1} \otimes_R B_{s_2} \otimes_R \dots \otimes_R B_{s_k} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes \dots \otimes_{R^{s_k}} R[k].$$

Definition 3.1.1. The category of Soergel bimodule $\mathbb{S}Bim$ is the smallest full subcategory of graded R-bimodules that contains all the Bott-Samelson bimodules $BS(\underline{w})$ for any expression $\underline{w}$ and that is closed under grading shifts, finite direct sums and direct summands.

Morphisms in $\mathbb{S}Bim$ are degree-preserving morphisms of R-bimodules, i.e. homogeneous of degree 0. For $B, B' \in \mathbb{S}Bim$ we write

$$\mathrm{Hom}^\bullet(B, B') = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{S}Bim}(B, B'[i]).$$

For $x \in W$ we denote the *twisted graph* of x by $Gr(x)$, that is

$$Gr(x) = \{(x(\lambda), \lambda) \mid \lambda \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{h}$$

For a subset $A \subseteq W$ let $Gr(A) = \bigcup_{x \in A} Gr(x)$. Since R is the ring of regular functions on $\mathfrak{h}$, we can think of any Soergel bimodule B as a quasi-coherent sheaf on $\mathfrak{h} \times \mathfrak{h}$. For any Soergel bimodule B there exists a finite subset $A \subseteq W$ such that B is supported on $Gr(A)$.

For a subset $A \subseteq W$ let

$$\Gamma_A B := \{b \in B \mid \mathrm{supp}\, b \in Gr(A)\}$$

For $x \in W$ we write $\Gamma_{\leq x}B$ for $\Gamma_{\{y|y\leq x\}}B$ and similarly for $\Gamma_{<x}B$, $\Gamma_{\geq x}B$, $\Gamma_{>x}B$, $\Gamma_x B$ and $\Gamma_{\neq x}B$. Let $\Gamma^x B = B/\Gamma_{\neq x}B$.[1] We also write $\Gamma_{\ell\leq i}B$ for $\Gamma_{\{y|\ell(y)\leq i\}}B$ and similarly for $\Gamma_{\ell\geq i}B$.

Notice that $J_x = Ann(Gr(x)) = \{f \in R\otimes R \mid f|_{Gr(x)} = 0\}$ is generated by the elements $x(f)\otimes 1 - 1\otimes f$, for $f \in \mathfrak{h}^*$. If $b \in \Gamma_A B$ then $x(f)b - bf \in \Gamma_{A\setminus\{x\}}B$.

Let R_x be the ring of regular functions of $Gr(x)$, i.e. $R_x = (R \otimes_{\mathbb{K}} R)/J_x$. Then R_x is isomorphic to R as a left graded R-module, and as a right module we have $r \cdot f = x(f)r$, for any $r \in R_x$ and $f \in R$. The bimodule R_x is called *standard bimodule*.

Proposition 3.1.2. *[Soe07, Proposition 6.4 and 6.6] Let B be a Soergel bimodule:*

i) For any $x \in W$ the subspaces $\Gamma_{\leq x}B/\Gamma_{<x}B$, $\Gamma_{\geq x}B/\Gamma_{>x}B$, $\Gamma_x B$ and $\Gamma^x B$ are free graded left R-modules. As graded bimodules they are isomorphic to a direct sum of shifts of the standard bimodule R_x.

ii) The natural maps $\Gamma_x B \to \Gamma_{\leq x}B$ and $\Gamma_{\geq x}B \to \Gamma^x B$ induces isomorphism

$$\Gamma_x B \cong p_x\left(\Gamma_{\leq x}B/\Gamma_{<x}B\right) \qquad \text{and} \qquad \Gamma_{\geq x}B/\Gamma_{>x}B \cong p_x\Gamma^x B$$

where $p_x \in R$ is the polynomial defined in (1.2).

Theorem 3.1.3. *[Soe07, Satz 6.16]*

- *For any $x \in W$ there exists a unique (up to isomorphisms and shifts) indecomposable Soergel bimodule B_x such that B_x is supported on $Gr(\leq x)$ and $\Gamma^x B_x \neq 0$.*
- *Fix a reduced expression $\underline{x}$ for x. For any decomposition of $BS(\underline{x})$ into indecomposable bimodules, B_x is isomorphic to the direct summand containing $1^{\otimes}_{\underline{x}} := 1 \otimes 1 \otimes \ldots \otimes 1 \in BS(\underline{x})$. Moreover, B_x is the unique direct summand of $BS(\underline{x})$ which is not a direct summand of $BS(\underline{y})$ for any expression $\underline{y}$ such that $\ell(\underline{y}) < \ell(\underline{x})$.*
- *Any indecomposable Soergel bimodule is isomorphic to $B_x[k]$, for some $k \in \mathbb{Z}$ and $x \in W$.*

Let $[\mathbb{S}Bim]$ denote the split Grothendieck group the category of Soergel bimodules. We consider $[\mathbb{S}Bim]$ as a $\mathbb{Z}[v, v^{-1}]$ algebra via $v \cdot [B] = [(B[1])]$. The tensor product $\otimes_R$ equips the category $\mathbb{S}Bim$ with a monoidal structure, and this induces a $\mathbb{Z}[v, v^{-1}]$-algebra structure on $[\mathbb{S}Bim]$.

Theorem 3.1.4 (Soergel's Categorification Theorem)**.** *There exists a isomorphism of algebras $\mathcal{E} : \mathcal{H}(W, S) \xrightarrow{\sim} [\mathbb{S}Bim]$ such that $\underline{\mathbf{H}}_s \mapsto [B_s]$*

Using the support filtration we can construct an inverse ch of the isomorphism $\mathcal{E}$ as follows:

$$\mathrm{ch}(B) = \sum_{x\in W} (\mathrm{grrk}\, \Gamma_{\leq x}B/\Gamma_{<x}B) v^{\ell(x)} \mathbf{H}_x.$$

In particular, we have $\mathrm{ch}(B_s) = \underline{\mathbf{H}}_s$ and $\mathrm{ch}(BS(\underline{w})) = \underline{\mathbf{H}}_{\underline{w}}$.

[1] In [Fie08] Fiebig defines $Gr(x)$ to be $Gr(x^{-1})$ in our notation, so the module $\Gamma^x B$ is there denoted by $B^{x^{-1}}$. Yet another notation is used [Wil16]: there $\Gamma^x B$ is denoted by B_x, while $\Gamma_x B$ is denoted by $B^!_x$. Williamson's notation is motivated by the fact $\Gamma^x B$ and $\Gamma_x B$ are the stalks and costalks of the sheaf corresponding to B, when this exists.

Theorem 3.1.5 (Soergel's hom formula [Soe07, Theorem 5.15])**.** *Let B and B' be Soergel bimodules. Then* $\operatorname{Hom}^\bullet(B, B')$ *is a graded free left R-module and*

$$\operatorname{grrk}\operatorname{Hom}^\bullet(B, B') = (\overline{\operatorname{ch}(B)}, \operatorname{ch}(B'))$$

where $(-,-)$ *is the pairing of the Hecke algebra defined in* (1.5)*.*

Theorem 3.1.6 (Soergel's Conjecture / Elias-Williamson theorem [EW14, Theorem 1.1])**.** *Let $\mathbb{K} = \mathbb{R}$. Then* $\operatorname{ch}(B_w) = \underline{\mathbf{H}}_w$*.*

Remark 3.1.7. In [EW14] Elias and Williamson proved Theorem 3.1.6 for realization of Type I, and their proof can be easily adapted to realization of Type II (see [Ric17]). Soergel's conjecture was already known, by geometric means, if $\mathbb{K}$ is of characteristic 0 for realizations of type II or III (see for example [Här99]).

In the following, we will also need to consider a larger category of bimodules, that are well behaved with respect to the support filtration.

Definition 3.1.8. The category of graded *R-bimodules with a ∇-flag* $\mathcal{F}_\nabla$ is the full subcategory of graded R-bimodules B such that

- B is finitely generated both as a left and a right R-module,
- B is supported on $Gr(A)$ for some finite subset $A \subseteq W$,
- for all i the quotients $\Gamma_{\ell\leq i}B/\Gamma_{\ell\leq i-1}B$ are isomorphic to a direct sum of standard bimodules $R_w[k]$, with $\ell(w) = i$.

An important consequences of requiring that our realization is reflection faithful is that there can be a non-trivial extension between the bimodules R_x and R_y if and only if xy^{-1} is a reflection [Soe07, Lemma 5.8]. This allows us to rearrange many terms in the support filtration:

Lemma 3.1.9 (Soergel's hin-und-her Lemma [Soe07, Lemma 6.3])**.** *Let $B \in \mathcal{F}_\nabla$. Fix an enumeration $w_1, w_2, w_3 \dots$ of the elements of W which refines the Bruhat order, i.e. $i \leq j$ if $w_i \leq w_j$. We abbreviate $\Gamma_{\{w_h | h \leq i\}}$ by $\Gamma_{\leq i}B$. Then the inclusion $\Gamma_{\leq w_h}B \hookrightarrow \Gamma_{\leq h}B$ induces an isomorphism*

$$\Gamma_{\leq w_h}B/\Gamma_{<w_h}B \xrightarrow{\sim} \Gamma_{\leq h}B/\Gamma_{\leq h-1}B.$$

3.1.1 Invariant forms and duality of Soergel bimodules

We define the *dual* $\mathbb{D}B$ of a graded R-bimodule B to be $\mathbb{D}B = \operatorname{Hom}^\bullet_{R-}(B, R)$, where $\operatorname{Hom}^\bullet_{R-}(-,-)$ denotes the space of morphisms of left R-modules of all degrees. We can give to $\mathbb{D}B$ a structure of graded R-bimodule via $r_1 f r_2(b) = f(r_1 b r_2)$, for any $f \in \mathbb{D}B$, $b \in B$ and $r_1, r_2 \in R$.

A *left invariant pairing* on two Soergel bimodules B, B' is a homogeneous bilinear form

$$\langle -,- \rangle : B \times B' \to R$$

such that $\langle b, b'f \rangle = \langle bf, b' \rangle$ and $\langle fb, b' \rangle = \langle b, fb' \rangle = f\langle b, b' \rangle$ for all $b \in B$, $b' \in B'$ and $f \in R$.[2]

[2] We use here left invariant form on Soergel bimodules, as in [Wil16]. Notice that in [EW14] the opposite choice is made.

There is a bijection between left invariant pairings on B, B' and R-bimodule morphisms $B \to \mathbb{D}B'$. We say that a pairing is *non-degenerate* if the induced map $B \to \mathbb{D}B'$ is an isomorphism. This is stronger that asking that for any $b \in B$ there exists $b' \in B'$ such that $\langle b, b' \rangle \neq 0$.

Let

$$c_s = \frac{1}{2}\left(\alpha_s \otimes 1 + 1 \otimes \alpha_s\right) \in B_s.$$

The element c_s is, up to scalar, the unique element of B_s of degree 1 such that $fc_s = c_s f$ for all $f \in R$. Thus, the map $R \to B_s$ which sends 1 to c_s is a homomorphism of R-bimodules.

Let $c_{id} = 1 \otimes 1 \in B_s$. The set $\{c_{id}, c_s\}$ is a basis of B_s as a left R-module. By abuse of notation we write $c_s^1 = c_s$ and $c_s^0 = c_{id}$.

Let $\underline{w} = s_1 s_2 \dots s_k$. We call an element of $\{0,1\}^k$ a 01*-sequence*. For a 01-sequence e we define

$$c_e = c_{s_1}^{e_1} c_{s_2}^{e_2} \dots c_{s_k}^{e_k}.$$

The set $\{c_e \mid e \text{ a 01-sequence for } \underline{w}\}$ is a basis of $BS(\underline{w})$ as a free left R-module. We denote $c_{00\dots0}$ by c_{top} and $c_{11\dots1} = 1 \otimes 1 \otimes \dots \otimes 1$ by $1^{\otimes}_{\underline{w}}$.[3] We call it the *string basis* of the Bott-Samelson bimodule. Notice that a Bott-Samelson bimodule is a shifted algebra with respect of component-wise multiplication, and $1^{\otimes}_{\underline{w}}$ is its (shifted) unity. We have

$$\deg(c_e) = -\ell(\underline{w}) + 2 \cdot \#\{k \mid e_k = 0\} = \#\{k \mid e_k = 0\} - \#\{k \mid e_k = 1\}.$$

Let $\mathrm{Tr} : BS(\underline{w}) \to R$ be the left R-linear map which returns the coefficients of c_{top} in the string basis. Let

$$\langle f, g \rangle_{BS(\underline{w})} = Tr(f \cdot g),$$

where $f \cdot g$ stands for the component-wise multiplication in $BS(\underline{w})$. The pairing $\langle -, - \rangle_{BS(\underline{w})}$ is left invariant and it is called the *intersection form*.

The intersection form on Bott-Samelson bimodules is non-degenerate [EW14, Corollary 3.9], hence $BS(\underline{w}) \cong \mathbb{D}BS(\underline{w})$. Since B_x is the unique direct summand of $BS(\underline{x})$, for $\underline{x}$ reduced, such that $\Gamma_x B_x \neq 0$, it follows that $B_x \cong \mathbb{D}B_x$.

Lemma 3.1.10. *The restriction of the intersection form $\langle -, - \rangle_{BS(\underline{x})}$ to the direct summand B_x is non-degenerate.*

Proof. We fix a decomposition of Soergel bimodules $BS(\underline{x}) = B_x \oplus V$. We can view the isomorphism $\Psi : BS(\underline{w}) \xrightarrow{\sim} \mathbb{D}BS(\underline{w})$ induced by the intersection form as a matrix of morphisms:

$$\Psi := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} B_x \\ V \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{D}B_x \\ \mathbb{D}V \end{pmatrix}$$

Let $\mathrm{rad}(\mathbb{S}Bim) \subseteq \mathbb{S}Bim$ denote the radical of the category of Soergel bimodules (see [Kra15, Str95] for the definition of the radical of an additive category, see also [EW14, §6.1]). Let $q : \mathbb{S}Bim \to \mathbb{S}Bim/\mathrm{rad}(\mathbb{S}Bim)$ be the projection functor. We have that a morphism $f : B \to B'$ of Soergel bimodules is an isomorphism if and only if $q(f)$ is also an isomorphism.

Since B_x is not a direct summand of $V \cong \mathbb{D}V$ it follows that $\beta, \gamma \in \mathrm{rad}(\mathbb{S}Bim)$. Since $q(\Psi)$ is an isomorphism then also $q(\alpha)$ and $q(\delta)$ are isomorphisms. It follows that the restrictions of $\langle -, - \rangle_{BS(\underline{x})}$ to both B_x and V is non-degenerate. $\square$

If $\mathbb{K} = \mathbb{R}$, it follows from the Soergel's conjecture and Soergel's hom formula that $\mathrm{End}^0(B_x) \cong \mathbb{R}$ for all $\mathbb{R}$. This means that there exists a unique (up to scalar) non-zero invariant form on B_x and that this invariant form is non-degenerate.

[3] The element $1^{\otimes}_{\underline{w}}$ is often denoted c_{bot}.

3.1.2 Localization of Soergel bimodules

A useful technique to study Soergel bimodules is the *localization* [EW16, §3.6]. Let Q be the field of fractions of R. By tensoring with Q we turn the category of Soergel bimodules into a semisimple category.

The module $Q \otimes_R BS(\underline{w})$ has a natural structure of (ungraded) Q-bimodule. In fact, if $\underline{w} = s_1 s_2 \dots s_k$ we have an isomorphism

$$BS(\underline{w})_Q := Q \otimes_R BS(\underline{w}) \cong Q \otimes_{Q^{s_1}} Q \otimes_{Q^{s_2}} Q \otimes \dots \otimes_{Q^{s_k}} Q$$

Let $Q_x = Q \otimes_R R_x$ be the localization of the standard bimodules. We have

$$\dim_Q \operatorname{Hom}(Q_x, Q_y) = \delta_{x,y}.$$

Let $B_{s,Q} = Q \otimes_R B_s \cong Q \otimes_{Q^s} Q$. It decomposes as $B_{s,Q} = Q_{id} \oplus Q_s$ via the isomorphism $f \otimes g \mapsto (fg, fs(g))$. This induces a decomposition

$$BS(\underline{w})_Q = \bigoplus_{e \in \{0,1\}^l} Q_e \tag{3.1}$$

where $Q_e \cong Q_{\underline{w}^e}$ as a Q-bimodule.

Similarly, every Soergel bimodules B decomposes similarly after localization in a direct sum of standard Q-bimodules. In fact, we have an injection of graded bimodules [Wil16, Equation (6.2)]

$$B \hookrightarrow \bigoplus_{x \in W} \Gamma^x B$$

which becomes an isomorphism after tensoring with Q, thus

$$Q \otimes_R B \cong \bigoplus_{x \in W} Q_x^{\oplus d_x} \qquad \text{where} \qquad d_x = \operatorname{rk}(\Gamma^x B).$$

For $b \in B$ we denote by b_x its projection to $\Gamma^x B$.

Lemma 3.1.11. *Let $A \subseteq W$ be a subset. An element $b \in B$ is in $\Gamma_A B$ if and only if $b_y = 0$ for all $y \in W \setminus A$.*

Proof. As explained in [Soe07, Remark 6.2] if B is a Soergel bimodule then for any $b \in B$ the support of b is union of twisted graphs $Gr(x)$. Hence, we can identify $\Gamma^x B$ with the restriction of B to $Gr(x)$, that is

$$\Gamma^x B \cong B/(Ann(Gr(x))B) = B/J_x B.$$

It follows that $b_x \neq 0$ if and only if $Gr(x) \subseteq \operatorname{supp}(b)$. □

Let B be a Soergel bimodule and let $\phi \in \operatorname{Hom}^\bullet(BS(\underline{w}), B)$ be a morphism. All the direct summands of $BS(\underline{x})_Q$ are isomorphic to Q_y for some $y \leq m(\underline{w})$, where $m(\underline{w})$ is defined in Lemma 1.1.2. It follows immediately from Lemma 3.1.11 that $\operatorname{Im}\phi \subseteq \Gamma_{\leq m(\underline{w})} B$.

3.1.3 Diagrammatic for Soergel bimodules

Let $\mathbb{BS}Bim$ be the category whose objects are the Bott-Samelson bimodules $BS(\underline{w})$ for all expressions $\underline{w}$ and whose morphisms are morphisms of graded bimodules of *all degrees*, i.e. if B, B' are Bott-Samelson bimodules then

$$\mathrm{Hom}_{\mathbb{BS}Bim}(B, B') = \mathrm{Hom}^{\bullet}_{R\otimes R}(B, B') = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{S}Bim}(B, B'[i]),$$

thus $\mathbb{BS}Bim$ is a category enriched in the category of graded R-bimodules.

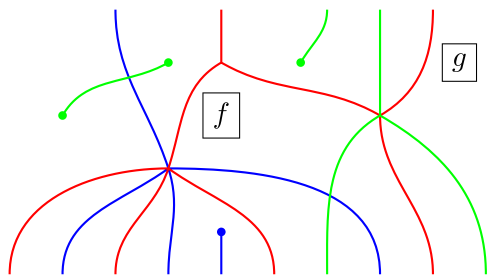
In [EW16] Elias and Williamson define a diagrammatic category $\mathcal{D}$ by generators and relations, using planar diagrams. The category $\mathcal{D}$ is equivalent to the category of Bott-Samelson bimodules $\mathbb{BS}Bim$. We will use this equivalence to depict diagrammatically morphisms in $\mathbb{BS}Bim$ and in $\mathbb{S}Bim$.

To define $\mathcal{D}$, we first assign a different color to each element of S. Then objects in the category $\mathcal{D}$ correspond to sequences of colored dots:

$$\underline{w} = s_1 s_2 \dots s_n \longleftrightarrow \bullet\,\bullet \dots \bullet$$

The morphisms in $\mathcal{D}$ are a linear combination of isotopy classes of some decorated planar diagrams embedded in the strip $\mathbb{R}\times[0,1]$. The edges of this diagram are colored by the elements of S and they may end in a dot of the same color on the boundary of the strip. The connected components of the complement of the diagram can be decorated by elements $f \in R$.

Example 3.1.12. A typical morphism between $ststtsutsu$ and $tsuus$, where $m_{st} = 4$, $m_{su} = 3$ and $m_{tu} = 2$.

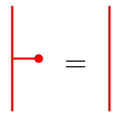


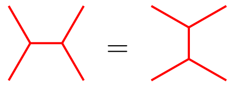
The generating morphisms, i.e. the kinds of vertices allowed in the diagrams, are:

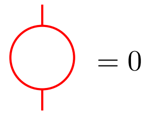
dot trivalent vertex $2m_{s,t}$-valent vertex (here $m_{s,t} = 4$)

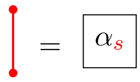
We quotient the so-obtained set of diagrams by the following local relations:

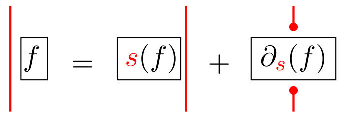
- One color relations:

Frobenius unit: 

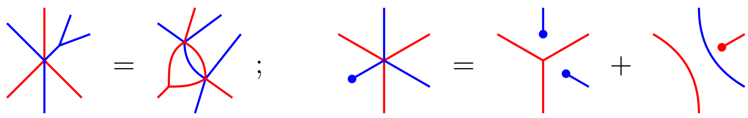
Frobenius associativity: 

Needle relation: 

Barbell relation: 

nil-Hecke relation: 

- Two color relations: Here we illustrate only the case $m_{s,t} = 3$, see [EW16, §5.1] for the general form.



- Three color relations: see (5.8)-(5.13) in [EW16].

We define a functor $\mathcal{D} \to \mathbb{BS}Bim$ by sending $\underline{w}$ to $BS(\underline{w})$ and by specifying the image (and the degrees) of the generating morphisms as in Table 3.1.

We still need to specify the image of the $2m_{st}$-valent vertex. Let $\underline{w}_s = \underbrace{sts\dots}_{m_{st}}$, $\underline{w}_t = \underbrace{tst\dots}_{m_{st}}$ and $w = \underbrace{sts\dots}_{m_{st}} \in W$. Both $BS(\underline{w}_s)$ and $BS(\underline{w}_t)$ have a direct summand isomorphic to B_w. We define the image of the $2m_{st}$-valent vertex to be the composition

$$\phi : BS(\underline{w}_s) \twoheadrightarrow B_w \hookrightarrow BS(\underline{w}_t).$$

This is well defined up to a scalar. In fact, it follows from the Soergel's hom formula and some elementary computation in the Hecke algebra of a dihedral group (see [Lib08, Proposition 4.3]). that

$$\dim \mathrm{Hom}^0(BS(\underline{w}_s), BS(\underline{w}_t)) = 1$$

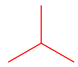
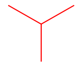
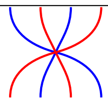
We choose ϕ such that $\phi(1^{\otimes}_{\underline{w}_s}) = 1^{\otimes}_{\underline{w}_t}$.

Under this functor, horizontal juxtaposition corresponds to tensor product of morphisms and vertical juxtaposition corresponds to composition of morphisms.

Theorem 3.1.13. *[EW16, Theorem 6.30] The functor $\mathcal{D} \to \mathbb{BS}Bim$ defined as above is an equivalence of categories.*

Remark 3.1.14. The diagrammatic category $\mathcal{D}$ provides a categorification of the Hecke algebra for much more general realizations than the one discussed in 1.2. For example, one does not need to require faithfulness.

Table 3.1:

	$B_s \to R$ $f \otimes g \mapsto fg$	$\deg = 1$
	$R \to B_s$ $1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$	$\deg = 1$
	$B_sB_s \to B_s$ $f \otimes g \otimes h \mapsto f\partial_s(g) \otimes h$	$\deg = -1$
	$B_s \to B_sB_s$ $f \otimes g \mapsto f \otimes 1 \otimes g$	$\deg = -1$
$\boxed{f}$	$R \to R$ $1 \mapsto f$	$\deg = \deg f$
	$\underbrace{B_sB_tB_s\dots}_{m_{st}} \to \underbrace{B_tB_sB_t\dots}_{m_{st}}$	$\deg = 0$

3.2 An algebraic replacement of the cohomology of Schubert varieties

If X_w is a Schubert variety, then the intersection cohomology $IH^\bullet(X_w, \mathbb{R})$ contains the cohomology $H^\bullet(X_w, \mathbb{R})[\ell(w)]$ as a graded R-submodule.

The goal of this section is to define, for any element $w \in W$, a graded R-sub-bimodule $\widetilde{H}_w$ of the indecomposable Soergel bimodule B_w which works as a replacement for the cohomology ring of a Schubert variety. We will show that $\widetilde{H}_w$ is a R-subbimodule of B_w containg $1^\otimes_{\underline{w}}$ and that

$$\dim(\mathbb{K} \otimes_R \widetilde{H}_w)^k = \#\{v \in W \mid v \leq w \text{ and } 2\ell(v) = k + \ell(w)\}.$$

Remark 3.2.1. For any complex variety Y, there is a natural map $H^\bullet(Y, \mathbb{R})[\dim Y] \to IH^\bullet(Y, \mathbb{R})$, but in general this map need not be injective. In fact, if Y is projective, then the kernel is precisely the non-pure part of $H^\bullet(Y, \mathbb{R})$ [dM09b, Theorem 3.2.1]. Because Schubert varieties have a cell decomposition into complex affine spaces, their cohomology is pure. Hence, we have a natural inclusion $H^\bullet(X_w, \mathbb{R})[\ell(w)] \hookrightarrow IH^\bullet(X_w, \mathbb{R})$ for any $w \in W$.

3.2.1 Light leaves basis of Bott-Samelson bimodules

We use the diagrammatic notation for morphisms between Soergel bimodules from §3.1.3.

In [EW16, Chapter 6] Libedinsky's light leaves are introduced in the diagrammatic setting. We make use of Elias and Williamson's results.

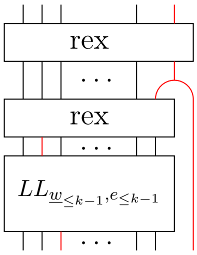
Let $\underline{w}$ an expression and e a 01-sequence with $\underline{w}^e = x$. The Light Leaf $LL_{\underline{w},e}$ is an element in $\mathrm{Hom}^\bullet(BS(\underline{w}), BS(\underline{x}))$, for some choice of a reduced expression $\underline{x}$ of x. For any light leaf $LL_{\underline{w},e}$, let $\Gamma\Gamma_{\underline{w},e} \in \mathrm{Hom}^\bullet(BS(\underline{x}), BS(\underline{w}))$ be the morphism obtained by flipping the diagram of $LL_{\underline{w},e}$ upside down. If $\underline{w}^e = \underline{w}^f$ let $\mathbb{LL}_{\underline{w},e,f} = \Gamma\Gamma_{\underline{w},e} \circ LL_{\underline{w},f}$. We know from [EW16, Theorem 6.11] that the set $\{\mathbb{LL}_{\underline{w},e,f}\}_{\underline{w}^e=\underline{w}^f}$ is a basis of $\mathrm{End}^\bullet(BS(\underline{w}))$ as a left R-module.

If $x = \underline{w}^e$, let $ll_{\underline{w},e} = \Gamma\Gamma_{\underline{w},e}(1^{\otimes}_{\underline{x}})$. We have $\deg(ll_{\underline{w},e}) = -\ell(x) + \mathrm{def}(e)$. In particular, e is a canonical 01-sequence if and only if $\deg(ll_{\underline{w},e}) + 2\ell(x) = \ell(\underline{w})$. From (1.1) we see that if there is at least one D in the decoration of e, then the inequality $\deg(ll_{\underline{w},e}) + 2\ell(x) \leq \ell(\underline{w}) - 2$ holds.

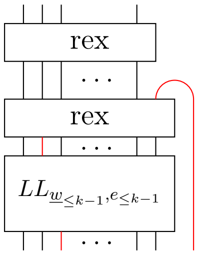
Lemma 3.2.2. *Let $\underline{w}$ be an expression and e be a 01-sequence. Then*

$$LL_{\underline{w},e}(1^{\otimes}_{\underline{w}}) = \begin{cases} 1^{\otimes}_{\underline{x}} & \text{if } e = can_{\underline{w}^e}, \\ 0 & \text{if } e \text{ has (at least) one } D. \end{cases}$$

Proof. The statement easily follows from the definition of light leaves when e has only U's. By induction on $\ell(\underline{w})$ we can assume that the last bit of e is a D and all the others are U's. Then $LL_{\underline{w},e}$ looks like



or



.

The box labeled by "rex" contains only $2m_{st}$-valent vertices. By induction

$$\left(LL_{\underline{w}_{\leq k-1}, e_{\leq k-1}} \otimes \mathrm{Id}_{B_{s_{\ell(\underline{w})}}}\right)(1^{\otimes}_{\underline{w}}) = 1^{\otimes}_{\underline{x}}.$$

Notice that every $2m_{st}$-valent vertex fixes $1 \otimes 1 \otimes \ldots 1$. It follows from Table 3.1 that a trivalent vertex applied to $1 \otimes 1 \otimes 1$ returns 0, thus $LL_{\underline{w},e}(1^{\otimes}_{\underline{w}}) = 0$. □

Every light leaf morphism $\Gamma\Gamma_{\underline{w},e}$ induces a map from the unique summand $Q_x \subseteq BS(\underline{x})_Q$ into $\bigoplus_{f:\underline{w}^f=x} Q_f \subseteq BS(\underline{w})_Q$. For a 01-sequence f with $\underline{w}^f = \underline{w}^e$ let $p^e_f : Q_x \to Q_f$ be the composition with the projection to a single summand Q_f. Since $\mathrm{Hom}_{Q\otimes Q}(Q_x, Q_x) \cong Q$ we can think of p^e_f as an element of Q. The rational function p^e_f may depend on the choices made in the construction of the light leaves basis.

We have a path dominance order on 01-sequences for $\underline{w}$. Namely, we say that $e \geq f$ if $\underline{w}^e_{\leq k} \geq \underline{w}^f_{\leq k}$ for all k. In particular, if $e \geq f$ then $\underline{w}^e \geq \underline{w}^f$.

Lemma 3.2.3. *We have $p^e_f = 0$ unless $f \leq e$ and $p^e_e \in Q$ is invertible.*

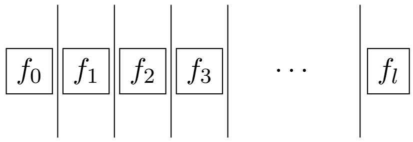
Proof. The proof is completely analogous to the proof of [EW16, Proposition 6.6] where the dual statement is considered. From the inductive construction of light leaves we see that for any k there exists a morphism ϕ such that

$$\Gamma\Gamma_{\underline{w},e} = (\Gamma\Gamma_{\underline{w}_{\leq k}, e_{\leq k}} \otimes Id_{\underline{w}_{\geq k+1}}) \circ \phi.$$

Here $Id_{\underline{w}_{\geq k+1}}$ denotes the identity morphism on $BS(\underline{w}_{\geq k+1})$. Hence the image is contained in all the summands Q_f such that $\underline{w}^f_{\leq k} \leq \underline{w}^e_{\leq k}$, which is exactly the condition for $f \leq e$ in the path dominance order. The same argument as in [EW16, Proposition 6.6] shows that p^e_e is invertible. □

From Lemma 3.2.3 it follows that the elements $\{ll_{\underline{w},e}\}$ are upper-triangular with respect of the decomposition (3.1) if we order summands in the RHS using the path dominance order. We deduce that $\{ll_{\underline{w},e}\}$ is a basis of $BS(\underline{w})_Q$ as a left Q-module. In particular, this means that if $\underline{w}$ is reduced, then $1^{\otimes}_{\underline{w}} = ll_{w,11\ldots 1}$ has a non-trivial component in the unique summand $Q_w \subseteq BS(\underline{w})_Q$.

We claim that $\{ll_{\underline{w},e}\}$ is also a basis of $BS(\underline{w})$ as a left R-module. For this, it remains to show that $\{ll_{\underline{w},e}\}$ generates $BS(\underline{w})$ as a left R-module. We first observe that $\mathrm{span}\{\phi(1^{\otimes}_{\underline{w}}) \mid \phi \in \mathrm{End}^{\bullet}(BS(\underline{w}))\} = BS(\underline{w})$. In fact, if $b = f_0 \otimes f_1 \otimes \ldots \otimes f_l \in BS(\underline{w})$, with $f_0, f_1, \ldots, f_l \in R$, then we have $b = \phi(1^{\otimes}_{\underline{w}})$, where

$\phi :=$  .

Then clearly also the span of all the $\mathbb{LL}_{\underline{w},e,f}(1^{\otimes}_{\underline{w}})$ with $\underline{w}^e = \underline{w}^f$ generates $BS(\underline{w})$. Applying Lemma 3.2.2 we see that $\mathbb{LL}_{\underline{w},e,f}(1^{\otimes}_{\underline{w}}) = ll_{\underline{w},e}$ if f is canonical and 0 otherwise. The claim now follows.

The above discussion, together with Lemma 3.1.11, shows the following proposition:

Proposition 3.2.4. *Let $\underline{w}$ be an expression. The set $\{ll_{\underline{w},e} \mid e \in \{0,1\}^{\ell(\underline{w})}$ and $\underline{w}^e \leq x\}$ is a basis of $\Gamma_{\leq x} BS(\underline{w})$ as a left R-module.*

In particular, the set $\{ll_{\underline{w},e}\}$ with $e \in \{0,1\}^{\ell(\underline{w})}$ is a basis of $BS(\underline{w})$ as a left R-module.

We can use the last Proposition to deduce Deodhar's defect formula [Deo77]:

$$\mathbf{H}_{\underline{w}} = \mathrm{ch}(BS(\underline{w})) = \sum_{e \in \{0,1\}^{\ell(\underline{w})}} v^{\deg(ll_{\underline{w},e})} v^{\ell(\underline{w}^e)} \mathbf{H}_{\underline{w}^e} = \sum_{e \in \{0,1\}^{\ell(\underline{w})}} v^{\operatorname{def} e} \mathbf{H}_{\underline{w}^e}. \tag{3.2}$$

Remark 3.2.5. The result of this section were, in the author's knowledge, published for the first time in the Appendix of [Pat16b]. However, Ben Elias and Geordie Williamson explained canonical subexpression and how to construct the basis $\{ll_{\underline{w},e}\}$ in a master class at the QGM in Aarhus already in 2013. Videos and notes of the lectures are available at `http://qgm.au.dk/video/mc/soergelkl/`.

3.2.2 The cohomology submodule of an indecomposable Soergel bimodule

For any Soergel bimodule B and any $x \in W$ we define a Laurent polynomial $h_x(B) \in \mathbb{Z}[v, v^{-1}]$ by

$$h_x(B)(v) = (\mathrm{grrk}\, \Gamma_{\leq x} B / \Gamma_{< x} B) v^{\ell(x)},$$

so that we have $\mathrm{ch}(B) = \sum_{x \in W} h_x(B) \mathbf{H}_x$.

If $BS(\underline{w})$ is a Bott-Samelson bimodule, from (3.2) we get $h_x(BS(\underline{w})) = \sum_{e:\, \underline{w}^e = x} v^{\operatorname{def}(e)}$. By Lemma 1.1.3 we have $h_x(BS(\underline{w}))(v) = v^{\ell(w)-\ell(x)}$+"terms of lower degree."

If $\mathbb{K} = \mathbb{R}$ it follows form Soergel's conjecture that the polynomials $h_x(B_w) = h_{x,w}$ are the Kazhdan-Lusztig polynomials.

For a general $\mathbb{K}$, the polynomials $h_x(B_w)$ can depend on the realization $\mathfrak{h}^*$: these polynomials are called p-Kazhdan-Lusztig polynomials, where $p = \mathrm{char}(\mathbb{K})$. The p-KL polynomials are discussed in more detail in [JW17].

Lemma 3.2.6. *We have $h_x(B_w) \in \mathbb{N}[v]$ and $h_x(B_w)(v) = v^{\ell(w)-\ell(x)}$+"terms of lower degree," for any $x \leq w$.*

Proof. We show this by induction on $\ell(w)$. If $\underline{w}$ is reduced we have a decomposition:

$$BS(\underline{w}) = B_w \oplus \bigoplus_{y<w} B_y^{\oplus m_y}$$

where $m_y(v) \in \mathbb{Z}[v, v^{-1}]$ is such that $m_y(v) = m_y(v^{-1})$. We have

$$\operatorname{grrk} BS(\underline{w}) = \operatorname{grrk} B_w + \sum_{y<w} m_y \cdot \operatorname{grrk} B_y$$

where grrk is taken with respect to the left R-module structures. Since $1^{\otimes}_{\underline{w}} \in B_w$, for any $y < w$ the bimodule $B_y^{\oplus m_y}$ lies in degree $> -\ell(\underline{w})$ and we have $\deg m_y < \ell(w) - \ell(y)$. The lemma now follows since for all $x \in W$ we have

$$h_x(BS(\underline{w})) = h_x(B_w) + \sum_{y<w} m_y \cdot h_x(B_y).$$

□

Let us consider the following left graded R-submodules of $BS(\underline{w})$:

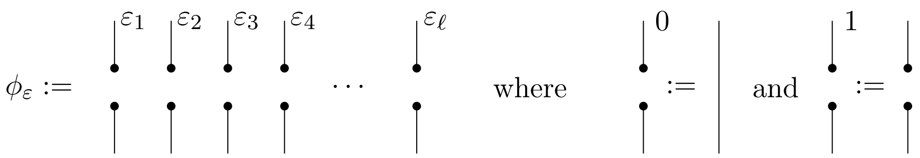
$$C_{\underline{w}} = \sum_{e \text{ canonical}} R \cdot ll_{\underline{w},e} \qquad \text{and} \qquad D_{\underline{w}} = \sum_{e \text{ not canonical}} R \cdot ll_{\underline{w},e}.$$

In general the left module $C_{\underline{w}}$ is not stable under multiplication by R on the right.

Lemma 3.2.7. *Let $D_{\underline{w}}$ as above. Then for any non-canonical* 01*-sequence e we have* $\operatorname{Im}(\Upsilon\Upsilon_{\underline{w},e}) \subseteq D_{\underline{w}}$. *Moreover, $D_{\underline{w}}$ is a graded R-subbimodule of $BS(\underline{w})$.*

Proof. We fix a non-canonical 01-sequence e and let $x = \underline{w}^e$. The light leaf $\Upsilon\Upsilon_{\underline{w},e}$ is a morphism from $BS(\underline{x})$ to $BS(\underline{w})$ for some reduced expression $\underline{x}$ of x. Let $\underline{x} = s_1 s_2 \dots s_l$.

It suffices to show that for any string basis element c_ε we have $\Upsilon\Upsilon_{\underline{w},e}(c_\varepsilon) \in D_{\underline{w}}$. We define the following morphism $\phi_\varepsilon \in \operatorname{End}^\bullet(BS(\underline{x}))$:



(3.3)

Clearly, we have $\phi_\varepsilon(1^{\otimes}_{\underline{x}}) = c_\varepsilon$. Therefore $\Upsilon\Upsilon_{\underline{w},e}(c_\varepsilon) = (\Upsilon\Upsilon_{\underline{w},e} \circ \phi_\varepsilon)(1^{\otimes}_{\underline{x}})$.

Let $\underline{y}$ be the expression obtained from $\underline{x} = s_1 s_2 \dots s_l$ by removing the s_i for i such that $\varepsilon_i = 0$. Then the morphism $\Upsilon\Upsilon_{\underline{w},e} \circ \phi_\varepsilon$ factorizes through $BS(\underline{y})$, hence its image is contained in $\Gamma_{\leq m(\underline{y})} BS(\underline{w})$. By Proposition 3.2.4 we can write:

$$\Upsilon\Upsilon_{\underline{w},e}(c_\varepsilon) = \sum_{f : \underline{w}^f \leq m(\underline{y})} h_f ll_{\underline{w},f}. \tag{3.4}$$

We have

$$\deg \Upsilon\Upsilon_{\underline{w},e}(c_\varepsilon) = \deg ll_{\underline{w},e} + \deg c_\varepsilon = \ell(\underline{w}) - 2 \cdot \operatorname{Downs}(e) - 2\ell(\underline{y}) < \ell(\underline{w}) - 2\ell(m(\underline{y})).$$

Recall that a 01-sequence f is canonical if and only if $\deg ll_{\underline{w},f} = \ell(\underline{w}) - 2\ell(\underline{w}^f)$. Since $\ell(\underline{w}^f) \leq \ell(m(\underline{y}))$, no canonical 01-sequence can appear in the sum in the RHS of (3.4). It follows immediately that $\operatorname{Im}(\Upsilon\Upsilon_{\underline{w},e}) \subseteq D_{\underline{w}}$.

Now the last statement follows since for $f \in R$ we have

$$ll_{\underline{w},e} \cdot f = \Upsilon\Upsilon_{\underline{w},e}(1^{\otimes}_{\underline{x}}) f = \Upsilon\Upsilon_{\underline{w},e}(1^{\otimes}_{\underline{x}} \cdot f) \in D_{\underline{w}}.$$

□

Actually, the same proof shows more generally that if $\phi \in \mathrm{Hom}^\bullet(BS(\underline{x}), BS(\underline{w}))$ is such that $\deg \phi < \ell(\underline{w}) - \ell(\underline{x})$ then $\mathrm{Im}(\phi) \subseteq D_{\underline{w}}$. In this way we deduce that the bimodule $D_{\underline{w}}$ does not depend on the choice of light leaves. Similarly, we also have that if $\phi \in \mathrm{End}^0(BS(\underline{w}))$, then $\phi(D_{\underline{w}}) \subseteq D_{\underline{w}}$.

Let now $\underline{w}$ be a reduced expression. Fix a decomposition of $BS(\underline{w})$ into indecomposable bimodules and let $\mathfrak{e}_w \in \mathrm{End}^0(BS(\underline{w}))$ be the primitive idempotent corresponding to B_w, i.e. $BS(\underline{w}) = \mathrm{Ker}(\mathfrak{e}_{\underline{w}}) \oplus \mathrm{Im}(\mathfrak{e}_{\underline{w}})$ and $\mathrm{Im}(\mathfrak{e}_{\underline{w}}) \cong B_w$. For any x, the map

$$\mathfrak{e}_{\underline{w}}^x : \Gamma_{\leq x} BS(\underline{w})/\Gamma_{<x} BS(\underline{w}) \to \Gamma_{\leq x} B_w / \Gamma_{<x} B_w$$

induced by $\mathfrak{e}_{\underline{w}}$ is split surjective. In particular, we have that

$$\mathrm{grrk}\left(\Gamma_{\leq x} BS(\underline{w})/\Gamma_{<x} BS(\underline{w})\right) = \mathrm{grrk}\left(\Gamma_{\leq x} B_w/\Gamma_{<x} B_w\right) + \mathrm{grrk}\left(\ker \mathfrak{e}_{\underline{w}}^x\right),$$

that is

$$\mathrm{grrk}\left(\ker \mathfrak{e}_{\underline{w}}^x\right) = v^{-\ell(x)} h_x(BS(\underline{w})) - v^{-\ell(x)} h_x(B_w).$$

From Lemma 3.2.6 it follows that $\mathrm{Ker}\, \mathfrak{e}_{\underline{w}}^x$ is generated in degree $< \ell(w) - 2\ell(x)$ as a left R-module.

Lemma 3.2.8. *The kernel of the morphism $\mathfrak{e}_{\underline{w}}$ is contained in $D_{\underline{w}}$.*

Proof. Fix an enumeration $w_1, w_2, w_3 \ldots$ of the elements of W which refines the Bruhat order. Let

$$L = \sum_{i \in I} g_i \cdot ll_{\underline{w}, e_i}$$

be an arbitrary element in $\mathrm{Ker}\, \mathfrak{e}_{\underline{w}} \subseteq BS(\underline{w})$, with $g_i \in R$. Let $x = w_h$ be the element of maximal index in the set $X := \{\underline{w}^{e_i} \mid i \in I\}$. We want to show by induction on h that there are no canonical light leaves appearing in the sum L.

For $y \leq w$ let $F_y := \sum_{e:\, \underline{w}^e = y} R \cdot ll_{\underline{w}, e}$. Then the inclusion $F_y \hookrightarrow BS(\underline{w})$ induces an isomorphism of left R-modules $F_y \cong \Gamma_{\leq y} BS(\underline{w})/\Gamma_{<y} BS(\underline{w})$.

For an integer $k \geq 1$ let us denote by $\Gamma_{\leq k} B$ the submodule of elements supported on $Gr(\{w_1, \ldots, w_k\})$. By Soergel's hin-und-her (Lemma 3.1.9) we have an isomorphism:

$$\Gamma_{\leq x} BS(\underline{w})/\Gamma_{<x} BS(\underline{w}) \cong \Gamma_{\leq h} BS(\underline{w})/\Gamma_{\leq h-1} BS(\underline{w}).$$

Let $L_x = \sum_{i \in I_x} g_i \cdot ll_{\underline{w}, e_i}$, where $I_x = \{i \in I \mid \underline{w}^{e_i} = x\}$. Since x is of maximal index in X, the projection of L and L_x to $\Gamma_{\leq h} BS(\underline{w})/\Gamma_{\leq h-1} BS(\underline{w})$ coincide. Hence $L_x \in \mathrm{Ker}\, \mathfrak{e}_{\underline{w}}^x$ and $L - L_x \in \Gamma_{\leq h-1} BS(\underline{w})$.

The R-module $\mathrm{Ker}\, \mathfrak{e}_{\underline{w}}^x$ is generated in degrees $< \ell(w) - 2\ell(x)$, so we can write

$$L_x = \sum_j h_j r_j,$$

with $h_j \in R$ and $r_j \in F_x$ such that $r_j \in \mathrm{Ker}\, \mathfrak{e}_{\underline{w}}^x$ and $\deg r_j < \ell(w) - 2\ell(x)$. Notice the if f is the canonical 01-sequence for x, then $ll_{\underline{w}, f}$ is a basis element of F_x of degree $\ell(w) - 2\ell(x)$, hence it cannot appear in L_x.

We have also $\deg \mathfrak{e}_{\underline{w}}(r_j) < \ell(w) - 2\ell(x)$ and $\mathfrak{e}_{\underline{w}}(r_j) \in \Gamma_{<x} BS(\underline{w})$. Since all canonical light leaves supported on an element y smaller than x have degree $> \ell(\underline{w}) - 2\ell(x)$, we get $\mathfrak{e}_{\underline{w}}(r_j) \in D_{\underline{w}}$ for all j, and finally $\mathfrak{e}_{\underline{w}}(L_x) \in D_{\underline{w}}$.

Let now $L' = L - L_x + \mathfrak{e}_{\underline{w}}(L_x)$. We have $L' \in \mathrm{Ker}\, \mathfrak{e}_{\underline{w}}$ and $L' \in \Gamma_{\leq h-1} BS(\underline{w})$, so by induction it follows that there are no canonical light leaves appearing when we write L' in the light leaves basis. We have shown that there are no canonical light leaves in L_x and in $\mathfrak{e}_{\underline{w}}(L_x)$, so the statement follows also for L. ☐

It follows that $B_w \cong \mathfrak{e}_{\underline{w}}(C_{\underline{w}}) \oplus \mathfrak{e}_{\underline{w}}(D_{\underline{w}})$ as left R-modules. Moreover, $\mathfrak{e}_{\underline{w}}(D_{\underline{w}})$ is a R-subbimodule of B_w and the restriction of $\mathfrak{e}_{\underline{w}}$ to $C_{\underline{w}}$ is injective.

Definition 3.2.9. We define the *singular cohomology submodule* $\widetilde{H}_{\underline{w}} \subseteq BS(\underline{w})$ to be the orthogonal of $D_{\underline{w}}$ with respect to the intersection form $\langle -,- \rangle_{BS(\underline{w})}$.

Consider the decomposition $BS(\underline{w}) = \operatorname{Im} \mathfrak{e}_{\underline{w}} \oplus \operatorname{Ker} \mathfrak{e}_{\underline{w}}$, with $\operatorname{Im} \mathfrak{e}_{\underline{w}} \cong B_w$. Let $\mathfrak{e}^*_{\underline{w}} \in \operatorname{End}(BS(\underline{w}))$ the adjoint of $\mathfrak{e}^*_{\underline{w}}$ with respect to the intersection form. Since $\deg(\mathfrak{e}^*_{\underline{w}}) = 0$ it preserves $D_{\underline{w}}$. Hence $\mathfrak{e}_{\underline{w}}$ preserves $\widetilde{H}_{\underline{w}}$, thus $\widetilde{H}_{\underline{w}}$ splits as

$$\widetilde{H}_{\underline{w}} = \left(\widetilde{H}_{\underline{w}} \cap \operatorname{Im} \mathfrak{e}_{\underline{w}}\right) \oplus \left(\widetilde{H}_{\underline{w}} \cap \operatorname{Ker} \mathfrak{e}_{\underline{w}}\right).$$

Recall from Lemma 3.1.10 that the restrictions of $\langle -,- \rangle_{BS(\underline{w})}$ to B_w and $\operatorname{Ker} \mathfrak{e}_{\underline{w}}$ is non-degenerate. Since $\operatorname{Ker} \mathfrak{e}_{\underline{w}} \subseteq D_{\underline{w}}$ it follows that $\widetilde{H}_{\underline{w}} \cap \operatorname{Ker} \mathfrak{e}_{\underline{w}} = 0$, hence $\widetilde{H}_{\underline{w}} \subseteq \operatorname{Im} \mathfrak{e}_{\underline{w}}$ and $\mathfrak{e}_{\underline{w}}$ restricts to the identity on $\widetilde{H}_{\underline{w}}$. We also obtain $\widetilde{H}_w := \mathfrak{e}_{\underline{w}}(\widetilde{H}_{\underline{w}}) = \mathfrak{e}_{\underline{w}}(D_{\underline{w}})^{\perp} \subseteq B_w$ where the orthogonal is taken with respect to the restriction of the intersection form to B_w. Finally, we can easily compute the graded rank of $\widetilde{H}_w$:

$$\operatorname{grrk} \widetilde{H}_w = \operatorname{grrk} \widetilde{H}_{\underline{w}} = \overline{\operatorname{grrk} BS(\underline{w}) - \operatorname{grrk} D_{\underline{w}}} = \overline{\operatorname{grrk} C_{\underline{w}}} = \sum_{x \leq w} v^{2\ell(x)-\ell(w)}. \tag{3.5}$$

3.3 Moment graphs of Coxeter groups

There exists a forth description of the equivariant cohomology of the flag variety, obtained by Goresky, Kottwitz and MacPherson [GKM98] using the localization theorem for torus actions.

As pointed out by Fiebig, one can generalize this construction to an arbitrary Coxeter group. Fiebig uses this to obtain a new realization of the category of Soergel bimodules. We show in fact that also in the generality of an arbitrary Coxeter group this construction still returns the dual nil-Hecke ring (even if there is no flag variety of which they are the equivariant cohomology).

We recall the definition of moment graphs and their sheaves from [Fie08].[4]

The moment graph $\mathcal{G} := \mathcal{G}(W, \mathfrak{h})$ is defined as follows: The set of vertices is given by the element $v \in W$. Two vertices v, w are connected by an edge if there exists a reflection $t \in \mathcal{T}$ such that $v = tw$.[5] We label this edge by α_t, where α_t is the positive root corresponding to t.

Definition 3.3.1. A sheaf $\mathcal{M}$ on the moment graph of W is given by

- a graded left R-module $\mathcal{M}^v$ for any $v \in W$;
- for any edge $v - tv$ a graded left R-module $\mathcal{M}^{v-tv}$ such that $\alpha_t \cdot \mathcal{M}^{v-tv} = 0$;
- for any $v \in W$, $t \in \mathcal{T}$ a morphism of graded R-modules $\pi_{v,tv} : \mathcal{M}^v \to \mathcal{M}^{v-tv}$.

We further assume that $\mathcal{M}^v$ is non-zero only for finitely many $v \in W$ and that $\mathcal{M}^v$ is torsion free and finitely generated as a R-module.

[4] Notice that Fiebig uses a different convention: his $Gr(x)$ corresponds to our $Gr(x^{-1})$.
[5] We do not ask here that $\ell(v) = \ell(w) \pm 1$

The *space of global sections* of a sheaf $\mathcal{M}$ is

$$\Gamma(\mathcal{M}) := \left\{ (m_v) \in \prod_{v \in W} \mathcal{M}^v \,\middle|\, \pi_{v,tv}(m_v) = \pi_{tv,v}(m_{tv}) \ \forall v \in W, t \in \mathcal{T} \right\}$$

The *structure algebra* $\hat{Z}$ is the space of global sections of the "constant sheaf" on the moment graph, thus it is defined by

$$\hat{Z} = \left\{ (r_v) \in \prod_{v \in W} R \,\middle|\, r_v \equiv r_{tv} \pmod{\alpha_t} \ \forall v \in W, t \in \mathcal{T} \right\}.$$

For $i \in \mathbb{N}$ let $\hat{Z}_i$ the graded component of $\hat{Z}$, that is $\hat{Z}_i := \{(z_v) \in \hat{Z} \mid \deg z_v = i\}$. We define $Z := \bigoplus_{i \in \mathbb{Z}} \hat{Z}_i$. Then Z is a subring of $\hat{Z}$. We can also describe Z as the subring of section in $\hat{Z}$ with bounded degree, that is $Z = \{(z_v) \in \hat{Z} \mid \exists i : \deg z_v \leq i \text{ for all } v \in W\}$. Notice that for an infinite Coxeter group we have $Z \neq \hat{Z}$.

For any sheaf $\mathcal{M}$, the space of global section $\Gamma(\mathcal{M})$ is in a natural way a graded module over $\hat{Z}$, hence over Z, by point-wise multiplication.

For a subset $\Omega \subseteq W$ we define $\hat{Z}^\Omega$ to be the image of the composition

$$\hat{Z} \hookrightarrow \prod_{v \in W} R \twoheadrightarrow \prod_{v \in \Omega} R.$$

We define Z^Ω similarly. Clearly, for any finite subset Ω we have $Z^\Omega = \hat{Z}^\Omega$. A subset Ω is said to be *upwardly closed* if whenever $v \in \Omega$ and $w \geq v$, then $w \in \Omega$.

Definition 3.3.2. Let Z-mod^f be the full subcategory Z-mod whose objects are Z-modules M which are finitely generated and torsion free over S and such the Z-module structure factors through Z^Ω for some finite $\Omega \subseteq W$.

We define similarly $\hat{Z}$-mod^f. The restriction functor $\hat{Z}$-mod$\to Z$-mod induces an equivalence of categories $\hat{Z}$-mod$^f \xrightarrow{\sim} Z$-mod^f.

Definition 3.3.3. We say that $M \in Z$-mod^f admits a Verma flag if for any upwardly closed subset $\Omega \subseteq W$, the module M^Ω is free as a graded left R-module. We call $\mathcal{V}$ the full subcategory of Z-mod^f of *modules admitting a Verma flag*.

Recall that $\mathcal{F}_\nabla$ is the category of R-bimodules with a ∇-flag.

Theorem 3.3.4. *[Fie08, Theorem 4.3] There is an equivalence of categories $\mathcal{V} \cong \mathcal{F}_\nabla$.*

We sketch now how this equivalence is obtained. We have two morphisms of rings $\tau, \sigma : R \to Z$ defined by

$$(\tau(f))_x = f \text{ and } (\sigma(f))_x = x(f).$$

Hence we have a ring homomorphism $R \otimes_{\mathbb{K}} R \xrightarrow{\tau \otimes \sigma} Z$. By restriction we obtain a functor $F : Z$-mod$\to R \otimes R$-mod which restricts to a functor $F : \mathcal{V} \to \mathcal{F}_\nabla$.

In the other direction, we start with a R-bimodule $B \in \mathcal{F}_\nabla$. To B we associate the sheaf on the moment graph $\mathcal{B}$ such that $\mathcal{B}^v = \Gamma^v B$ and such that

$$\mathcal{B}^{v-tv} = B/(Ann(Gr(v) \cap Gr(tv))B.$$

Since we have $\Gamma^v B = B/Ann(Gr(v))B$ by [Soe07, Remark 6.2], there is a natural projection $\pi_{v,tv} : \mathcal{B}^v \to \mathcal{B}^{v-tv}$ (the bimodules $\mathcal{B}^v$ and $\mathcal{B}^{v-tv}$ are the restrictions of B to $Gr(v)$ and to $Gr(v) \cap Gr(tv)$ respectively). We can therefore define a functor $G : \mathcal{F}_\nabla \to \mathcal{V}$ by $G(B) = \Gamma(\mathcal{B})$.

The two functors F and G are inverse to each other. In particular, we have $B \cong \Gamma(\mathcal{B})$ as an R-bimodule, and since $\mathcal{B}$ is a sheaf on the moment graph we also get a natural structure of Z-module on the bimodule B.

Finally, we describe a W-action on Z. For $z \in \prod_{v\in W} R$ and $x \in W$ we define $x(z)_v = z_{vx}$. This action preserves Z: in fact α_t divides $x(z)_{tv} - x(z)_v = z_{tvx} - z_{vx}$ for all $t \in \mathcal{T}$, $x, v, w \in W$.

Warning 3.3.5. We have two different structures of R-modules on Z, given by τ and σ as defined above. To differentiate between them, we write τ as the left action and σ as the right action. We always think of Z as a R-algebra using the left action τ.

3.4 Schubert basis from Soergel bimodules

We fix a reduced expression $\underline{w} = s_1 s_2 \dots s_l$ throughout this section. Recall from §3.2 that we have a left R-basis of $BS(\underline{w})$ given by light leaves $ll_{\underline{w},e}$. For $x \leq w$ we denote by $\mathcal{C}_{\underline{w},x} = ll_{\underline{w},can_x}$ the canonical 01-sequence can_x of Definition 1.1.4.

Let $D_{\underline{w}}$ be the left R-submodule of $BS(\underline{w})$ spanned by non-canonical light leaves defined in §3.2.2. As shown in Lemma 3.2.7, $D_{\underline{w}}$ is a R-bimodule. Let $\{ll^*_{\underline{w},e}\}$ be the left basis of $BS(\underline{w})$ dual to $\{ll_{\underline{w},e}\}$ with respect to the intersection form. Let $\mathcal{P}_{\underline{w},x} = ll^*_{\underline{w},can_x}$. In other words, $\mathcal{P}_{\underline{w},x}$ is defined by $\langle \mathcal{P}_{\underline{w},x}, D_{\underline{w}}\rangle_{BS(\underline{w})} = 0$ and $\langle \mathcal{P}_{\underline{w},x}, \mathcal{C}_{\underline{w},y}\rangle_{BS(\underline{w})} = \delta_{x,y}$, so $\{\mathcal{P}_{\underline{w},x}\}$ is a basis of $\widetilde{H}_{\underline{w}}$ as a left R-module. It is easy to check that $\mathcal{P}_{\underline{w},id} = 1^{\otimes}_{\underline{w}}$.

Fix $x \leq w$ and $e = can_x$. From $\underline{w}$ and e we obtain a reduced expression $\underline{x} = t_1 t_2 \dots t_k$ for x by removing from $\underline{w}$ all the s_i such that $e_i = 0$.

For $1 \leq i \leq \ell(x)$ let $x_i = t_i t_{i+1} \dots t_k$ and $x_{\ell(x)+1} = id$. We denote by $e(\hat{\imath})$ the 01-sequence obtained by replacing the i-th 1 in e with a 0. Recall the definition of the map $\phi_e \in \mathrm{End}^\bullet(BS(\underline{w}))$ given in (3.3).

Let $\lambda \in \mathfrak{h}^*$. Using repeatedly the nil-Hecke relation (see §3.1.3) on the bottom of the diagram we get

$$\mathcal{C}_{\underline{w},x} \cdot \lambda = \phi_e(1^{\otimes}_{\underline{w}}) \cdot \lambda = \sum_{i=1}^{\ell(x)} \partial_{t_i}(x_{i+1}(\lambda))\phi_{\underline{w},e(\hat{\imath})}(1^{\otimes}_{\underline{w}}) + x(\lambda)\mathcal{C}_{\underline{w},x}$$

If $e(\hat{\imath})$ is canonical, i.e. if it is decorated only with U's, then $\phi_{\underline{w},e(\hat{\imath})}(1^{\otimes}_{\underline{w}}) = c_{e(\hat{\imath})} = \mathcal{C}_{\underline{w},y}$ for some $y < x$. Moreover, $y \xrightarrow[R]{t} x$ where $t = x_{i+1}^{-1} t_i x_{i+1} \in \mathcal{T}$ and $\partial_{t_i}(x_{i+1}(\lambda)) = \partial_t(\lambda)$.

If $e(\hat{\imath})$ is not canonical, then $\phi_{\underline{w},e(\hat{\imath})}(1^{\otimes}_{\underline{w}}) \in \Gamma_{\ell \leq \ell(x)-2} B$. Thus we can write

$$\mathcal{C}_{\underline{w},x} \cdot \lambda = \sum_{y \xrightarrow[R]{t} x} \partial_t(\lambda)\mathcal{C}_{\underline{w},y} + x(\lambda)\mathcal{C}_{\underline{w},x} + \Theta,$$

with $\Theta \in \Gamma_{\ell \leq \ell(x)-2} B$. Furthermore, $\Theta = \sum_j h_j ll_{\underline{w},f_j}$ with $h_j \in R$ and $\ell(\underline{w}^{f_j}) \leq \ell(x) - 2$. The degree of $ll_{\underline{w},f_j}$ is too small for f_j to be canonical, in fact from (1.1) we have

$$\deg ll_{\underline{x},f_j} \leq \deg \mathcal{C}_{\underline{w},x} + 2 = \ell(w) - 2\ell(x) + 2 < \ell(w) - 2\ell(\underline{w}^{f_j}),$$

whence $\Theta \in D_{\underline{w}}$.

We obtain a Chevalley formula for the multiplication in the basis $\{\mathcal{P}_{\underline{w},x}\}$ of $\widetilde{H}_w$:

$$\mathcal{P}_{\underline{w},x} \cdot \lambda = x(\lambda)\mathcal{P}_{\underline{w},x} + \sum_{\substack{x \xrightarrow[R]{t} y \\ y \leq w}} \partial_t(\lambda)\mathcal{P}_{\underline{w},y}. \tag{3.6}$$

Let $\mathcal{BS}(\underline{w})$ be the sheaf on the moment graph obtained from the bimodule $BS(\underline{w})$ as explained below Theorem 3.3.4. Recall that $BS(\underline{w})$ is a R-algebra via componentwise multiplication. For all $v \in W$ and $t \in \mathcal{T}$ the R-modules $\mathcal{BS}(\underline{w})^v = \Gamma^v BS(\underline{w})$ and $\mathcal{BS}(\underline{w})^{v-tv}$ are also naturally R-algebras and the maps π_{v-tv} are morphisms are R-algebras. This shows that the multiplication $BS(\underline{w})$ is compatible with the Z-module structure, that is $BS(\underline{w})$ is naturally a Z-algebra.

Our next goal is to compute explicitly, using the nil-Hecke ring, the element in $\Gamma(\mathcal{BS}(\underline{w}))$ corresponding to $\mathcal{P}_{\underline{w},x}$. The idea is to show that this sections can be obtained as sections of the constant sheaf on the subset $\{x \in W \mid x \leq \underline{w}\}$ of the moment graph, which is a subsheaf of $\mathcal{BS}(\underline{w})$.

For $x \in W$ and $b \in B$ we denote by b_x its image in $\Gamma^x B$. Let $1_x := (\mathcal{P}_{\underline{w},id})_x = (1^{\otimes}_{\underline{w}})_x \in \Gamma^x BS(\underline{w})$.[6] We recall that the right action of R on $\Gamma(\mathcal{BS}(\underline{w}))$ is given by the map σ defined by $\sigma(\lambda)_x = x(\lambda)$. This in fact agrees with the right action on $\Gamma^x B$: the module $\Gamma^x B$ is isomorphic to a direct sum of standard modules R_x, hence for $b \in BS(\underline{w})$ we have $(b \cdot \lambda)_x = b_x \cdot \lambda = x(\lambda)b_x$.

Example 3.4.1. If $s \in S$ with $s \leq w$ if $\lambda = \varpi_s \in \mathfrak{h}^*$ is a fundamental weight corresponding to s (i.e. $\partial_t(\varpi_s) = \delta_{t,s}$ for all $t \in S$) we get

$$\mathcal{P}_{\underline{w},s} = \mathcal{P}_{\underline{w},id} \cdot \varpi_s - \varpi_s \mathcal{P}_{\underline{w},id} = \varpi_s \otimes 1 \otimes \ldots \otimes 1 - 1 \otimes \ldots \otimes 1 \otimes \varpi_s.$$

Hence, for any $x \in W$, we have $(\mathcal{P}_{\underline{w},s})_x = (\varpi_s - x(\varpi_s))1_x$.

Lemma 3.4.2. *For any $x \leq \underline{w}$ we have $\mathcal{P}_{\underline{w},x} \in \Gamma_{\geq x} BS(\underline{w})$, or equivalently $(\mathcal{P}_{\underline{w},x})_y = 0$ unless $x \geq y$.*

Proof. This follows by induction on $\ell(w) - \ell(x)$ using (3.6). The base case follows since we have $\mathcal{P}_{\underline{w},w} \cdot \lambda - w(\lambda)\mathcal{P}_{\underline{w},w} = 0$, which implies $\mathcal{P}_{\underline{w},w} \in \Gamma_w BS(\underline{w})$.

Assume $\mathcal{P}_{\underline{w},y} \in \Gamma_{\geq y} BS(\underline{w})$ for all $y > x$. Then $\mathcal{P}_{\underline{w},x} \cdot \lambda - x(\lambda)\mathcal{P}_{\underline{w},x} \in \Gamma_{>x} BS(\underline{w})$, hence $\mathcal{P}_{\underline{w},x} \in \Gamma_{\geq x} B$. □

Lemma 3.4.3. *For any $x \leq w$ we have $(\mathcal{P}_{\underline{w},x})_x = (-1)^{\ell(x)} p_x 1_x$, where $p_x \in R$ is defined in* (1.2).

Proof. Recall from Proposition 3.1.2 that we have $\Gamma_{\geq x} BS(\underline{w}) / \Gamma_{>x} BS(\underline{w}) \cong p_x \Gamma^x BS(\underline{w})$. The element $\mathcal{P}_{\underline{w},x}$ is homogeneous of degree $-\ell(w) + 2\ell(x)$. Then it maps to an element of minimal degree in $p_x \Gamma^x BS(\underline{w})$, i.e. $(c^*_{\underline{w},x})_x$ must be a scalar multiple of $p_x 1_x$. Let us write $(c^*_{\underline{w},x})_x = \tau p_x 1_x$, with $\tau \in \mathbb{K}$. It remains to show that $\tau = (-1)^{\ell(x)}$.

We make use of the results of [Wil16, §6.7 and §6.8]. We have an embedding

$$BS(\underline{w}) \hookrightarrow \bigoplus_{x \leq w} \Gamma^x BS(\underline{w})$$

which is an isomorphism after tensoring with Q on the left. The form $\langle -, - \rangle_{BS(\underline{w})}$ induces a form $\langle -, - \rangle^x_{BS(\underline{w})}$ on $\Gamma^x BS(\underline{w})$ such that, for any $b, b' \in BS(\underline{w})$, we have

$$\langle b, b' \rangle_{BS(\underline{w})} = \sum_x \langle b_x, b'_x \rangle^x_{BS(\underline{w})}.$$

[6] The element $1_x \in \Gamma^x BS(\underline{w})$ is denoted by $c_{x,\underline{w}}$ in [Wil16, §6.9].

Since $\mathcal{C}_{\underline{w},x} \in \Gamma_{\leq x} BS(\underline{w})$ and $\mathcal{P}_{\underline{w},x} \in \Gamma_{\geq x} BS(\underline{w})$ we have

$$1 = \langle \mathcal{P}_{\underline{w},x}, \mathcal{C}_{\underline{w},x} \rangle_{BS(\underline{w})} = \langle (\mathcal{P}_{\underline{w},x})_x, (\mathcal{C}_{\underline{w},x})_x \rangle^x_{BS(\underline{w})} = \langle \tau p_x 1_x, (\mathcal{C}_{\underline{w},x})_x \rangle^x_{BS(\underline{w})} =$$

$$= \tau p_x \langle 1_x, (\mathcal{C}_{\underline{w},x})_x \rangle^x_{BS(\underline{w})}.$$

It remains to show that $\langle 1_x, (\mathcal{C}_{\underline{w},x})_x \rangle^x_{BS(\underline{w})} = (-1)^{\ell(x)} (p_x)^{-1} = e_{x,x}$. We show this claim by induction on $\underline{w}$.

Let $\underline{w} = \underline{w}' s_l$. Recall the elements $c_{id} = 1 \otimes 1$ and $c_{s_l} = \frac{1}{2}(\alpha_{s_l} \otimes 1 + 1 \otimes \alpha_{s_l})$ of B_{s_l}. Let $e = can_x$. Assume the last bit e is a $U1$, so that $x = x' s_l$, with $x > x'$. Then

$$\mathcal{C}_{\underline{w},x} = \mathcal{C}_{\underline{w}',x'} \otimes \mathcal{C}_{\underline{s},s} = \mathcal{C}_{\underline{w}',x'} \otimes c_{id} \in BS(\underline{w})$$

and $(\mathcal{C}_{\underline{w}',x'})_x = 0$. By [Wil16, Equation (6.8) and Proposition 6.17] we obtain

$$\left\langle 1_x, (\mathcal{C}_{\underline{w},x})_x \right\rangle^x_{BS(\underline{w})} = \frac{1}{x(\alpha_{s_l})} \langle 1_{x'}, \mathcal{C}_{\underline{w}',x'} \rangle^{x'}_{BS(\underline{w}')} = -\frac{1}{x'(\alpha_{s_l})} e_{x',x'} = e_{x,x}$$

Assume now that the last bit of e is a $U0$. Then $\mathcal{C}_{\underline{w},x} = \mathcal{C}_{\underline{w}',x} \otimes \mathcal{C}_{s,id} = \mathcal{C}_{\underline{w}',x} \otimes c_{s_l}$. Therefore

$$\left\langle 1_x, (\mathcal{C}_{\underline{w},x})_x \right\rangle^x_{BS(\underline{w})} = \frac{1}{x(\alpha_{s_l})} \left\langle 1_x, (\mathcal{C}_{\underline{w}',x}) \right\rangle^x_{BS(\underline{x}')} \cdot x(\alpha_{s_l}) = e_{x,x}$$

□

Lemma 3.4.4. *For any $y \leq w$, the map $R \to \Gamma^y(\widetilde{H}_{\underline{w}})$ defined by $1 \mapsto 1_y$ is an isomorphism of R-modules.*

Proof. The module $\Gamma^y \widetilde{H}_{\underline{w}}$ is generated by $(\mathcal{P}_{\underline{w},x})_y$, for $x \in W$. Hence, we have to show that $(\mathcal{P}_{\underline{w},x})_y$ is a multiple of 1_y for all $x \in W$.

We show this by induction on $\ell(y) - \ell(x)$. The case $\ell(x) < \ell(y)$ follows from Lemma 3.4.2 and the case $\ell(x) = \ell(y)$ from Lemma 3.4.3. Recall from (3.6):

$$(y(\lambda) - x(\lambda))(\mathcal{P}_{\underline{w},x})_y = \sum_{\substack{x \xrightarrow[R]{t} z \\ z \leq w}} \partial_t(\lambda)(\mathcal{P}_{\underline{w},z})_y$$

For all the z in the sum we have $\ell(y) - \ell(z) < \ell(y) - \ell(x)$ and by induction we can write $(\mathcal{P}_{\underline{w},z})_y = \gamma_{z,y} 1_y$, for some $\gamma_{z,y} \in R$. Since $x \neq y$, we can choose λ such that $x(\lambda) \neq y(\lambda)$. Recall that $\Gamma^y B$ is a free as a left R-module. Then if $(\mathcal{P}_{\underline{w},x})_y$ were not a multiple of 1_x also $(y(\lambda) - x(\lambda))(\mathcal{P}_{\underline{w},x})_y$ would not be a multiple of 1_x, and we would get a contradiction. □

For any $x, y \in W$, with $x, y \leq w$ the previous Lemma allows us to define $\gamma_{x,y}(\underline{w}) \in R$ such that $(\mathcal{P}_{\underline{w},x})_y = \gamma_{x,y}(\underline{w}) 1_y$. We now show that $\gamma_{x,y}(\underline{w})$ does not depend on $\underline{w}$, as long as x, y are smaller the w. For this, we compare it with the "inverse equivariant multiplicity" $d_{x,y}$ defined in §2.2.

Lemma 3.4.5. *For any $x, y \leq w$ we have $\gamma_{x,y}(\underline{w}) = d_{x,y}$.*

Proof. The statement will follow by induction on $\ell(x) - \ell(y)$. For $\ell(x) \leq \ell(y)$ it follows from Proposition 2.2.9, Lemma 3.4.2 and 3.4.3. In fact, we have $\gamma(\underline{w})_{x,y} = 0$ unless $x \leq y$ and $\gamma(\underline{w})_{x,x} = (-1)^{\ell(x)} p_x = d_{x,x}$ for all $x \leq w$.

From (3.6) then we have for any λ.

$$\gamma_{x,y}(\underline{w})(y(\lambda)-x(\lambda)) = \sum_{\substack{x \xrightarrow[R]{t} z \\ z\leq w}} \gamma_{z,y}(\underline{w})\partial_t(\lambda).$$

We also have from Proposition 2.2.9.iv)

$$d_{x,y}(y(\lambda)-x(\lambda)) = \sum_{x \xrightarrow[R]{t} z} d_{z,y}\partial_t(\lambda).$$

Since $x \neq y$ and our realization $\mathfrak{h}$ is faithful, we can choose λ such that $x(\lambda) \neq y(\lambda)$. Since by induction we have $d_{z,y} = \gamma_{z,y}(\underline{w})$ for all z such that $x < z \leq w$ we obtain $\gamma_{x,y}(\underline{w}) = d_{x,y}$. □

This means that for $x \leq w$ we can define a global section of the sheaf $\mathcal{BS}(\underline{w})$ by $(d_{x,y}1_y)_{y\leq w}$.

Let $y \leq w$ and $t \in \mathcal{T}$ be such that $ty < y$. We have $\pi_{y-ty}(d_{x,y}1_y) = \pi_{ty-y}(d_{x,ty}1_{ty})$ in $\Gamma^{y-ty}BS(\underline{w})$. This means that the restriction of $(d_{x,y}-d_{x,ty})1_y$ is zero on the hyperplane $Gr(ty)\cap Gr(y)$ of $Gr(y)$, that is $\alpha_t | d_{x,y}-d_{x,ty}$. Therefore we can also define an element of the structure algebra $\mathcal{P}_x \in Z$ by $(\mathcal{P}_x)_y = d_{x,y}$.

Lemma 3.4.6. *The set $\{\mathcal{P}_x\}_{x\in W}$ is a basis of Z as a R-module.*

Proof. The set $\{\mathcal{P}_x\}_{x\in W} \subseteq Z$ is linearly independent over R since $(\mathcal{P}_x)_y = 0$ for $y \not\geq x$ and $(\mathcal{P}_x)_x \neq 0$.

Let $Z' = \operatorname{span}\langle \mathcal{P}_x \mid x \in W\rangle$. Let $f \in Z$ be homogeneous of degree $2d$ and let $A_d = \{x \in W \mid \ell(x) \leq d\}$. We fix an enumeration $w_1, w_2, w_3 \dots$ of the elements of W which refines the Bruhat order.

Let h be minimal such that $f_{w_h} \neq 0$. Then $p_{w_h} | f_{w_h}$, so

$$f' := f - \frac{f_{w_h}}{p_{w_h}}\mathcal{P}_v \in Z$$

and $f'_{w_i} = 0$ for all $i \leq h$. If we repeat this enough times we end up with $g \in Z'$ of degree $2d$ such that $(f-g)_x = 0$ for all $x \in A_d$.

Assume now that $f \neq g$, so there exists a minimal element $w \in W$ such that $\ell(w) > d$ and $(f-g)_w \neq 0$. But this would imply $p_w|(f-g)_w$, which is impossible since $\deg p_w > 2d$. □

Recall the ring $\Omega = \operatorname{Hom}_{-Q}(Q_W, Q)$ defined in §2.2 and recall its left Q_W-module given by $(f\cdot\psi)(y) = \psi(f^t \cdot y)$ for $\psi \in \Omega$. This allows us to define a W-action on Ω via

$$w(\psi)(y) = (\delta_w \cdot \psi)(y) = \psi(\delta_{w^{-1}}y).$$

The W-action preserves the subalgebra $\Lambda \subseteq \Omega$ [KK86a, Proposition 4.24(g)].

Warning 3.4.7. Remember that we think of Λ as a R-algebra using the right action of R but, on the contrary, we think of Z as a R-algebra via the left action of R.

Theorem 3.4.8. *There exists a W-equivariant isomorphism of graded R-algebras $\Phi : \Lambda \xrightarrow{\sim} Z$ which sends $\xi^x \in \Lambda$ to $\mathcal{P}_x \in Z$.*

Proof. For $\psi \in \Lambda$ we can define $\Phi(\psi)_x = \psi(\delta_{x^{-1}}) \in \prod_{x\in W} R$. The map Φ is a homomorphism of R-algebras from Λ to $\prod_{x\in W} R$.

Then, from Lemma 3.4.5, we get $\Phi(\xi^x) = \mathcal{P}_x$. In particular, $\Phi(\Lambda) \subseteq Z$ and sends a basis as a right R-module in a basis as a left R-module, hence it is an isomorphism.

For any $x, y \in W$ we have

$$\Phi(x \cdot \psi)_y = (x \cdot \psi)(\delta_{y^{-1}}) = \psi(\delta_{x^{-1}y^{-1}}) = \Phi(\psi)_{yx} = (x \cdot \Phi(\psi))_y.$$

It follows that $\Phi(x \cdot \psi) = x \cdot \Phi(\psi)$. □

Notice that the isomorphism Φ also intertwines the left R-action on Λ and the right R-action on Z, that is

$$\Phi(f \cdot \psi) = \Phi(\psi) \cdot f.$$

As a consequence of the above discussion we can think of $\widetilde{H}_{\underline{w}}$ as sections of the constant sheaf on moment graph supported on the set $\{x \mid x \leq w\}$. In this way we can give a natural structure of shifted graded R-algebra on $\widetilde{H}_{\underline{w}}$ (with identity $1^{\otimes}_{\underline{w}}$ lying in degree $-\ell(w)$), so that $\widetilde{H}_{\underline{w}}$ is a subalgebra of $BS(\underline{w})$. The map $p_{\underline{w}} : Z \to \widetilde{H}_{\underline{w}}$ defined by $\mathcal{P}_x \mapsto \mathcal{P}_{\underline{w},x}$ is a surjective R-algebra homomorphism and its kernel $\operatorname{Ker} p_w$ is generated by the elements $\mathcal{P}_y$, $y \not\leq x$.

Let Z_+ be the ideal of Z generated by R_+, that is $Z_+ = \sum_{x\in W} R_+\mathcal{P}_x$. We define

$$\overline{Z} = Z/Z_+ = R/R_+ \otimes_R Z \cong \mathbb{K} \otimes_R Z. \tag{3.7}$$

Let $P_x = 1 \otimes \mathcal{P}_x \in \overline{Z}$. Then $\{P_x\}_{x\in W}$ is a basis of $\overline{Z}$ over $\mathbb{K}$.

Any Soergel bimodule B is in a natural way a module over $\hat{Z}$, hence over Z. We also have by Theorem 3.3.4 that

$$\operatorname{Hom}_{R\otimes R}(B, B') = \operatorname{Hom}_{\hat{Z}}(B, B') = \operatorname{Hom}_Z(B, B').$$

For a Soergel bimodule B we define $\overline{B} = \mathbb{K} \otimes_R B$. This is in a natural way a graded right R-module. All graded right R-modules of this form are called *Soergel modules.* Any Soergel module $\overline{B}$ is in a natural way a module over $\overline{Z}$.

Remark 3.4.9. If W is the Weyl group of a reductive group G, then we have already defined in §2.1.1 the element $\mathcal{P}_x$ as part of the Schubert basis of $H^\bullet_T(X, \mathbb{K})$. However, in this case have an isomorphism $Z \cong H^\bullet_T(X, \mathbb{K})$ and the definition of the basis $\{\mathcal{P}_x\}$ is consistent with the one given above: by Theorem 3.4.8 and Theorem 2.2.5 they both correspond to the basis $\{\xi^x\}$ of the dual nil-Hecke ring.

In fact, we are in the setting of [GKM98]: the moment graph of W can be realized taking as vertices the fixed point on the torus and as edges the 1-dimensional orbits of the torus T on G/B. In particular, the full subgraph of vertices $\leq w$ is the moment graph of the Schubert variety X_w. From Lemma 3.4.4 it follows that we can realize $\widetilde{H}_w$ as cohomology of the constant sheaf (shifted by $-\ell(w) = \deg 1^{\otimes}_{\underline{w}}$) on the moment graph of X_w. Thus we obtain from [GKM98, Theorem 1.2.2] that $\widetilde{H}_w[-\ell(w)]$ is isomorphic to the equivariant cohomology $H_T(X_w)$.

3.4.1 Translation functors on $\overline{Z}$-mod

For $s \in S$ let $Z^s \subseteq Z$ denote the subalgebra of s-invariants.

Lemma 3.4.10. *As a module Z^s is a free R-module, with basis $\{\mathcal{P}_v\}_{vs>v}$.*

Proof. This follows from [KK86a, Lemma 4.34] and Theorem 3.4.8. □

Let $\overline{Z}^s = \mathbb{K} \otimes_R Z^s$. Recall the homomorphism $\sigma : R \to Z$ of §3.3 and consider $\sigma(\alpha_s) \in Z$.

Then Z is a free Z^s-module with basis $\{1, \sigma(\alpha_s)\}$ (cf. [Fie08, Lemma 5.1]). This also implies that $\overline{Z}$ is a free $\overline{Z}^s$-module with basis $\{1, \mu_s\}$, where $\mu_s := 1 \otimes \sigma(\alpha_s) \in \overline{Z}$.

Proposition 3.4.11 (cf. [Fie08, Proposition 5.2] and [Soe07, Proposition 5.10])**.**

- *The two functors $\overline{Z}^s$-mod$\to \overline{Z}$-mod defined by*

$$M \mapsto M \otimes_{\overline{Z}^s} \overline{Z}(2) \quad \text{and} \quad M \mapsto \mathrm{Hom}_{\overline{Z}^s}(\overline{Z}, M)$$

  are equivalent.

- *The functor $\overline{Z}$-mod$\to \overline{Z}$-mod given by*

$$M \mapsto M \otimes_{\overline{Z}^s} \overline{Z}[1]$$

  is self-adjoint.

Proof. Let $\{1^*, \mu_s^*\}$ be the basis of $\mathrm{Hom}_{\overline{Z}^s}(\overline{Z}, \overline{Z}^s)$ dual to $\{1, \mu_s\}$. Since $\deg 1 = \deg 1^* = 0$ and $\deg \mu_s^* = -\deg \mu_s = -2$ we have that the map of $\overline{Z}^s$-modules $\Psi : \overline{Z}(2) \to \mathrm{Hom}_{\overline{Z}^s}(\overline{Z}, \overline{Z}^s)$ defined by $1 \mapsto \mu_s^*$ and $\mu_s \mapsto 1^*$ is an isomorphism. Because $\overline{Z}$ is free as a $\overline{Z}^s$-module, for any $\overline{Z}^s$-module M we have a natural isomorphism of $\overline{Z}$-modules:

$$\begin{array}{ccccc} \mathrm{Hom}_{\overline{Z}^s}(\overline{Z}, M) & \xrightarrow{\sim} & M \otimes_{\overline{Z}^s} \mathrm{Hom}_{\overline{Z}^s}(\overline{Z}, \overline{Z}^s) & \xrightarrow{\Psi^{-1}} & M \otimes_{\overline{Z}^s} \overline{Z}(2) \\ \phi & \longmapsto & \phi(1) \otimes 1^* + \phi(\mu_s) \otimes \mu_s^* & \longmapsto & \phi(1) \otimes \mu_s + \phi(\mu_s) \otimes 1 \end{array}$$

The second statement now follows since the restriction functor $\overline{Z}$-mod$\to \overline{Z}^s$-mod is right adjoint to $- \otimes_{\overline{Z}^s} \overline{Z}$ and left adjoint to $\mathrm{Hom}_{\overline{Z}^s}(\overline{Z}, -)$. □

The following proof is based on unpublished notes by Soergel, in which he considers the case of finite Coxeter groups (Soergel's proof also appears in [Ric17]).

Theorem 3.4.12 (Hom formula for Soergel modules)**.** *Let B, B' Soergel bimodules. Then*

$$\mathbb{K} \otimes_R \mathrm{Hom}_{R \otimes R}(B', B) \cong \mathrm{Hom}_{\overline{Z}}(\mathbb{K} \otimes_R B', \mathbb{K} \otimes_R B).$$

Proof. Let $\Theta : \mathbb{K} \otimes \mathrm{Hom}_{R \otimes R}(B, B') \to \mathrm{Hom}_{\overline{Z}}(\mathbb{K} \otimes_R B, \mathbb{K} \otimes_R B')$ be the map defined by $\Theta(z \otimes \phi)(z' \otimes b) = zz' \otimes \phi(b)$. Since ϕ is a morphism of Soergel bimodules, it is also a morphism of Z-modules, hence the resulting map is a map of $\overline{Z}$-modules.

Because every indecomposable bimodule is a direct summand of a Bott-Samelson bimodule, it is enough to show the theorem for B, B' Bott-Samelson bimodules. Moreover, by adjunction (Proposition 3.4.11 and [Soe07, Proposition 5.10]) we can restrict ourselves to the case $B' = R$, that is to show

$$\mathbb{K} \otimes_R \mathrm{Hom}_{R \otimes R}(R, B) \cong \mathrm{Hom}_{\overline{Z}}(\mathbb{K}, \mathbb{K} \otimes B).$$

By sending $\phi : R \to B$ to $\phi(1)$ we get

$$\mathrm{Hom}_{R \otimes R}(R, B) \cong \{b \in B \mid \lambda b = b\lambda \text{ for all } \lambda \in \mathfrak{h}^*\} = \Gamma_{id} B.$$

On the other hand, similarly, we obtain

$$\mathrm{Hom}_{\overline{Z}}(\mathbb{K}, \mathbb{K}\otimes B) \cong \{b \in \mathbb{K}\otimes_R B \mid P_x \cdot b = 0 \text{ for all } x \in W \setminus \{id\}\} = \bigcap_{x\neq id} Ann\,(P_x)\,.$$

The resulting map $\mathbb{K}\otimes_R \Gamma_{id}B \to \bigcap_{x\neq id} Ann(P_x) \subseteq \mathbb{K}\otimes_R B$ is induced by the inclusion $\Gamma_{id}B \hookrightarrow B$.

For any Soergel bimodule B and any downwardly closed subsets $A \subseteq A' \subseteq W$ the inclusion $\Gamma_A B \hookrightarrow \Gamma_{A'}B$ is a split embedding of left R-modules. This follows because $\mathbb{S}Bim \subseteq \mathcal{F}_\nabla$ (or one can see this more explicitly from Proposition 3.2.4 if B is Bott-Samelson). Therefore we have

$$\mathbb{K}\otimes_R (\Gamma_A B/\Gamma_{A'}B) \cong (\mathbb{K}\otimes_R \Gamma_A B)/(\mathbb{K}\otimes_R \Gamma_{A'}B)$$

In particular, $\mathbb{K}\otimes_R \Gamma_{id}B \subseteq \mathbb{K}\otimes_R B$ and thus Θ is injective.

To show that Θ is also surjective, it is sufficient to show that if $b \in \mathbb{K}\otimes_R B$ and $b \notin \mathbb{K}\otimes_R \Gamma_{id}B$, then there exists $x \in W\setminus\{0\}$ such that $P_x \cdot b \neq 0$.

Fix an enumeration $w_1, w_2, w_3 \ldots$ of the elements of W which refines the Bruhat order. Let $h \in \mathbb{N}$ be such that $b \in \mathbb{K}\otimes_R \Gamma_{\leq h}B$ and $b \notin \mathbb{K}\otimes_R \Gamma_{\leq h-1}B$. Let $x = w_h$.

Multiplication by $\mathcal{P}_x$ induces an isomorphism of R-bimodules

$$\mathcal{P}_x \cdot (-) : \Gamma_{\leq h}B/\Gamma_{\leq h-1}B \xrightarrow{\sim} \Gamma_x B.$$

In fact, on $\Gamma_{\leq h}B/\Gamma_{<h}B \cong \Gamma_{\leq x}B/\Gamma_{<x}B$ multiplying by $\mathcal{P}_x$ is the same as multiplying on the left by $(\mathcal{P}_x)_x = (-1)^{\ell(x)}p_x$, hence its image is $p_x\Gamma_{\leq x}B/\Gamma_{<x}B = \Gamma_x B$ by Proposition 3.1.2.

As a consequence we obtain an isomorphism of right R-modules

$$P_x \cdot (-) : (\mathbb{K}\otimes_R \Gamma_{\leq h}B)/(\mathbb{K}\otimes_R \Gamma_{\leq h-1}B) \xrightarrow{\sim} \mathbb{K}\otimes_R \Gamma_x B$$

Hence to show that $P_x \cdot b \neq 0$ in $\overline{B}$ it is enough to show that for any $x \in W$, we have $\mathbb{K}\otimes_R \Gamma_x B \subseteq \mathbb{K}\otimes_R B$. This is done in the next Lemma. □

Lemma 3.4.13. *For any Soergel bimodule B and for any x the morphism $\Gamma_x B \hookrightarrow B$ is split as left R-modules. In particular, there is an embedding*

$$\mathbb{K}\otimes_R \Gamma_x B \hookrightarrow \mathbb{K}\otimes_R B.$$

Proof. Choose an embedding $B \overset{\oplus}{\subseteq} BS(\underline{w})$. We have already discussed the case $x = id$ in the Theorem above. We have $R_x \in \mathcal{F}_\nabla$, hence by [Soe07, Proposition 5.9(1)] also $BS(\underline{w})\otimes_R R_x \in \mathcal{F}_\nabla$. Since $\mathcal{F}_\nabla$ is closed under taking direct summands, then also $B\otimes R_x \in \mathcal{F}_\nabla$. The map $B \to B\otimes_R R_x$ defined by $b \mapsto b\otimes 1$ is an isomorphism of left R-modules and induces an isomorphism of left R-modules

$$\Gamma_x B \cong \Gamma_{id}(B\otimes_R R_x).$$

Now since the inclusion $\Gamma_{id}(B\otimes_R R_x) \hookrightarrow B\otimes_R R_x$ is split as a map of left R-modules, then also $\Gamma_x B \hookrightarrow B$ is split. □

Corollary 3.4.14. *If B is an indecomposable Soergel bimodule, then $\overline{B}$ is indecomposable as a $\overline{Z}$-module.*

We derive also a formula for the dimension of the space of morphisms between Soergel modules $\mathbb{K} \otimes B$ and $\mathbb{K} \otimes B'$:

$$\operatorname{grdim} \operatorname{Hom}^\bullet_{\overline{Z}}(\mathbb{K} \otimes B, \mathbb{K} \otimes B') = (\overline{\operatorname{ch}(B)}, \operatorname{ch}(B')). \tag{3.8}$$

Remark 3.4.15. Assume $\mathbb{K} = R$. If W is a finite Coxeter group, then the ring Z can be identified with the ring of regular functions of $Gr(W)$ (cf. [Fie08, Theorem 4.3]), which in turn is isomorphic to $R \otimes_{R^W} R$ [Wil11, Lemma 4.3.1]. Hence $\overline{Z} \cong \mathbb{K} \otimes_{R^W} R \cong R/R^W_+$ is the coinvariant ring. In particular, $\overline{Z}$ is generated in degree 2 and the map $R \to \overline{Z}$ is surjective. Clearly, in this case we can replace $\overline{Z}$ by R (acting on the right) in the statement of Theorem 3.4.12 and in (3.8).

3.5 The center of the category of Soergel bimodules

We give a different characterization of the R-subalgebra $\widetilde{H}_{\underline{w}} \subseteq BS(\underline{w})$. We denote the center of a ring A by $\mathcal{Z}(A)$.

Proposition 3.5.1. *Let $\underline{w}$ reduced. Then $\widetilde{H}_{\underline{w}} = \mathcal{Z}(\operatorname{End}^\bullet_{R\otimes R}(BS(\underline{w})))$.*

Proof. Every endomorphism of $BS(\underline{w})$ as an R-bimodule is also an endomorphism as Z-modules, because of Theorem 3.3.4. Hence multiplication defines a map

$$Z \to \mathcal{Z}(\operatorname{End}^\bullet_{R\otimes R}(BS(\underline{w})))$$

whose image is $\widetilde{H}_{\underline{w}}$ (which is seen as a subring of $\operatorname{End}^\bullet(BS(\underline{w}))$ via multiplication). It remains to show $\mathcal{Z}(\operatorname{End}^\bullet(BS(\underline{w}))) \subseteq \widetilde{H}_{\underline{w}}$.

Let $\phi \in \mathcal{Z}(\operatorname{End}^\bullet(BS(\underline{w})))$. Then ϕ commutes with the multiplication by any element of $BS(\underline{w})$. In particular ϕ is an endomorphism of $BS(\underline{w})$ as a module over itself, that is $\phi \in \operatorname{End}^\bullet_{BS(\underline{w})}(BS(\underline{w})) \cong BS(\underline{w})$, so ϕ is the morphism given by multiplication by an element $b \in BS(\underline{w})$.

It remains to show that $\langle b, ll_{\underline{w},e}\rangle_{BS(\underline{w})} = 0$ for any non-canonical 01-sequence e. Fix such a sequence e, and consider $\Psi = \Gamma\Gamma_{\underline{w},e} \circ LL_{\underline{w},can_{\underline{w}}^e} \in \operatorname{End}^\bullet(BS(\underline{w}))$, so that $\Psi(1^\otimes_{\underline{w}}) = ll_{\underline{w},e}$. Since Ψ commutes with ϕ, using Lemma 3.2.7 we get

$$b \cdot ll_{\underline{w},e} = (\phi \circ \Psi)(1^\otimes_{\underline{w}}) = (\Psi \circ \phi)(1^\otimes_{\underline{w}}) = \Psi(b) = \Gamma\Gamma^{\underline{w},e}(LL_{\underline{w},can_{\underline{w}}^e}(b)) \in D_{\underline{w}}.$$

For degree reasons, the restriction of the trace Tr to $D_{\underline{w}}$ is 0. We obtain $\langle \phi, ll_{\underline{w},e}\rangle_{BS(\underline{w})} = 0$. ∎

From this we can easily compute the center of the category of Soergel bimodules. First we recall its definition.

Definition 3.5.2. The center $\mathcal{Z}(\mathcal{A})$ of an additive category $\mathcal{A}$ is the endomorphism ring of the identity functor $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$.

It is well known that if W is a Weyl group, then the center of the category of Soergel bimodules is isomorphic to the equivariant cohomology of the corresponding flag variety (see for example [Str09, §3.1]). We generalize this to any Coxeter group:

Proposition 3.5.3. *The center of the category of Soergel bimodules $\mathcal{Z}(\mathbb{S}Bim)$ is isomorphic to the structure algebra $\hat{Z}$ defined in §3.3.*

Proof. Let $\eta : id_{\mathbb{S}Bim} \to id_{\mathbb{S}Bim}$ be a natural transformation and for any Soergel bimodule B let $\eta_B \in \operatorname{End}^\bullet(B)$ denote the corresponding morphism. We abbreviate $\eta_{\underline{w}}$ for $\eta_{BS(\underline{w})}$. By Proposition 3.5.1 we can think $\eta_{\underline{w}}$ as an element of $\widetilde{H}_w$.

Let $p_x : Z \twoheadrightarrow \widetilde{H}_x[-\ell(x)]$ denote the projection. For $x \leq w$ we have also a natural projection $p_{w,x} : \widetilde{H}_w[-\ell(w)] \twoheadrightarrow \widetilde{H}_x[-\ell(x)]$ defined by $\mathcal{P}_{\underline{w},y} \mapsto \mathcal{P}_{\underline{x},y}$ if $y \leq x$ and $\mathcal{P}_{\underline{w},y} \mapsto 0$ otherwise. The datum $\left(\{\widetilde{H}_x[-\ell(x)]\}_{x\in W}, \{p_{w,x}\}_{x\leq w}\right)$ defines an inverse system on the poset $(W, \leq)$. We have

$$\hat{Z} \cong \varprojlim_{x\in W} \widetilde{H}_x[-\ell(x)].$$

Let now $x \leq w$ and $LL_{\underline{w},can_x} : BS(\underline{w}) \to BS(\underline{x})$ for some reduced expression $\underline{w}, \underline{x}$. Consider the commutative diagram

$$\begin{tikzcd} BS(\underline{w}) \arrow[r, "LL_{\underline{w},can_x}"] \arrow[d, "\eta_{\underline{w}}"] & BS(\underline{x}) \arrow[d, "\eta_{\underline{x}}"] \\ BS(\underline{w}) \arrow[r, "LL_{\underline{w},can_x}"] & BS(\underline{x}) \end{tikzcd}$$

Since $LL_{\underline{w},can_x}$ is a morphism of Z-modules, we obtain

$$p_{w,x}(\eta_{\underline{w}})1^{\otimes}_{\underline{x}} = LL_{\underline{w},can_x}(\eta_{\underline{w}} \cdot 1^{\otimes}_{\underline{w}}) = \eta_{\underline{x}} \cdot LL_{\underline{w},can_x}(1^{\otimes}_{\underline{w}}) = \eta_{\underline{x}} \cdot 1^{\otimes}_{\underline{x}}$$

It follows that $p_{w,x}(\eta_{\underline{w}}) = \eta_{\underline{x}}$, and the tuple $z := (\eta_{\underline{x}})$ is an element of $\varprojlim \widetilde{H}_x[-\ell(x)] \cong \hat{Z}$.

Any Soergel bimodule B can be embedded in a direct sum of shift of bimodules $BS(\underline{x})$, with $\underline{x}$ reduced. It follows that η_B is also multiplication by z.

This shows that the obvious map $\hat{Z} \to \mathcal{Z}(\mathbb{S}Bim)$ is surjective. It is also injective because if $z \in \hat{Z}$ and $z_x \neq 0$, then z acts non-trivially on B_x. □

3.6 Counterexamples

In general, for an infinite Coxeter group it is false that

$$\mathbb{K} \otimes_R \operatorname{Hom}_{R\otimes R}(B, B') \cong \operatorname{Hom}_R(\mathbb{K} \otimes B, \mathbb{K} \otimes B'). \tag{3.9}$$

We discuss now two examples where (3.9) fails. Furthermore, in the first example we show that there exists an indecomposable Soergel bimodule B such that $\mathbb{K} \otimes B$ is not indecomposable as a right R-module.

Example 1: Let $\mathbb{K} = \mathbb{R}$. Let $\widetilde{W}$ be an affine Weyl group and let $\mathfrak{h}$ be a realization for $\widetilde{W}$ of type II. All cohomology and intersection cohomology groups are taken with coefficients in $\mathbb{R}$.

Let $W \subseteq \widetilde{W}$ be the corresponding finite Weyl group and G be the corresponding simply-connected semisimple group associated to W. Recall the definition of the affine Grassmannian $\mathcal{G}r$ and the affine flag variety $\widehat{\mathcal{F}l}$ from §2.3. Let $p : \widehat{\mathcal{F}l} \to \mathcal{G}r$ be the projection. Recall from Proposition 2.3.1 that p is a topologically trivial fiber bundle and that we have an isomorphism of algebras

$$H^\bullet(\widehat{\mathcal{F}l}) \cong H^\bullet(G/B) \otimes_{\mathbb{R}} H^\bullet(\mathcal{G}r). \tag{3.10}$$

In [Här99] Härterich showed that for any $w \in \widetilde{W}$ we have $\mathbb{R} \otimes B_w \cong IH^\bullet(\widehat{\mathcal{F}l}_w)$, where $\widehat{\mathcal{F}l}_w = \overline{I \cdot wI/I} \subseteq \widehat{\mathcal{F}l}$ is the corresponding Schubert variety.

Fix $\overline{w} \in \widetilde{W}/W$ and let $\mathcal{G}r_{\overline{w}} \subseteq \mathcal{G}r$ be the corresponding Schubert variety. Let w be the longest element in the coset $\overline{w}$. Then we have $\widehat{\mathcal{F}l}_w = p^{-1}(\mathcal{G}r_{\overline{w}})$. Since p is a topologically trivial fiber bundle, the same holds for the restriction $p : \widehat{\mathcal{F}l}_w \to \mathcal{G}r_{\overline{w}}$. We have

$$IH^\bullet(\widehat{\mathcal{F}l}_w) \cong H^\bullet(G/B)[d] \otimes_{\mathbb{R}} IH^\bullet(\mathcal{G}r_{\overline{w}}) \tag{3.11}$$

where $d = \dim G/B$. The $H^\bullet(\widehat{\mathcal{F}l})$-module structure on $IH^\bullet(\widehat{\mathcal{F}l}_w)$ is given, in terms of the isomorphism (3.10) and (3.11), by $(f \otimes f')(g \otimes g') = fg \otimes f'g'$. It follows that if $IH^\bullet(\mathcal{G}r_{\overline{w}})$ decomposes as a $Sym(H^2(\mathcal{G}r))$-module, then $H^\bullet(\widehat{\mathcal{F}l}_w)$ decomposes as a R-module.

Now assume further that the group G is simple. It follows that $H^2(\mathcal{G}r)$ is one-dimensional and it is generated by P_u, where u is the unique simple reflection not in W. Therefore $Sym(H^2(\mathcal{G}r))$ is isomorphic to the polynomial ring $\mathbb{R}[x]$, with $\deg(x) = 2$. Note that there are very few Schubert varieties $\mathcal{G}r_{\overline{w}}$ for which we have $\dim IH^i(\mathcal{G}r_{\overline{w}}) \leq 1$ for all i, and that if $\dim IH^i(\mathcal{G}r_{\overline{w}}) \geq 2$ for some i then $IH^\bullet(\mathcal{G}r_{\overline{w}})$ cannot be indecomposable as a $Sym(H^2(\mathcal{G}r))$-module. This describes how to produce many examples of indecomposable Soergel bimodules B_w such that $\overline{B_w}$ is decomposable.

The smallest explicit example is as follows: Let $\widetilde{W}$ be the affine Weyl group of type $\widetilde{A}_2$, that is $\widetilde{W} := \langle s,t,u \rangle$ and $m_{st} = m_{tu} = m_{us} = 3$. Let W be the subgroup generated by s,t and let $w = stutst$. Then $\mathbb{R} \otimes_R B_w = IH^\bullet(\widehat{\mathcal{F}l}_w) = H^\bullet(G/B)[3] \otimes_{\mathbb{R}} IH^\bullet(\mathcal{G}r_{stu})$, where $G = SL_3(\mathbb{C})$. We have $\dim IH^1(\mathcal{G}r_{stu}) \geq \dim H^4(\mathcal{G}r_{stu}) = 2$, since $H^4(\mathcal{G}r_{stu})$ is generated by P_{su} and P_{tu}. Hence the Soergel module $\mathbb{R} \otimes_R B_w$ is not indecomposable as a R-module.

Example 2: The following is another smaller counterexample to (3.9) where we can see in more detail algebraically what happens. Let W be the universal Coxeter group of rank 3, i.e. $W = \langle s,t,u \rangle$ with $m_{st} = m_{tu} = m_{us} = \infty$. Let $w = stustu$ and consider the bimodule $BS(\underline{w})$.

For $e \in \{0,1\}^6$ let c_e be the string basis element defined as in §3.1. Consider the element

$$b := c_{000011} - c_{000101} + c_{000110} - c_{001010} + c_{001100} - c_{010001} - 2c_{010010} +$$

$$+ c_{011000} - c_{010100} + c_{100001} - c_{100010} - c_{101000} + c_{110000} \in BS(\underline{w})^2$$

We have $h_{id}(BS(\underline{w})) = v^6 + 3v^4$, so in particular $\Gamma_{id}BS(\underline{w})$ lies in degree ≥ 4. Then $b \notin \Gamma_{id}(BS(\underline{w}))$ but the projection $\overline{b} \in \overline{BS(\underline{w})}$ belongs to $Ann(R_+)$ (the element v has been found with the help of the software Magma [BCP97], but verifying that $\overline{b} \in Ann(R_+)$ can be easily done by hand). It follows that the map $\mathbb{R} \to \mathbb{R} \otimes_R BS(\underline{w})$ defined by $1 \mapsto \overline{b}$ is a map of right R-bimodules which does not arise from any bimodule map $R \to BS(\underline{w})$.

Remark 3.6.1. These two counterexamples discussed above allow us to answer (negatively) a question posed by Soergel in [Soe07, Remark 6.8]. In general, for infinite Coxeter groups there exists no non-zero function $c_y \in R \otimes R$ homogeneous of degree $2\ell(y)$ such that c_y is supported on $Gr(\leq y)$ and vanishes on $Gr(< y)$. In fact, if such elements $c_y \in R \otimes_{\mathbb{K}} R$ exist, we could use them to play the role of $\mathcal{P}_y$ in the proof of Proposition 3.4.12, and this would imply the isomorphism (3.9).

Chapter 4

Singular Soergel Bimodules and their Hodge Theory

Throughout this chapter we assume $\mathbb{K} = \mathbb{R}$, so that $\mathfrak{h}$ is a realization of type I or II.

4.1 Generalities on one-sided singular Soergel bimodules

For a subset $I \subseteq S$ we denote by W_I the subgroup of W generated by I. We say that I is *finitary* if W_I is finite. If I is finitary we denote by w_I the longest element of W_I.

Recall that R denotes the polynomial ring $\mathrm{Sym}(\mathfrak{h}^*)$. For a finitary I let $R^I := R^{W_I}$ the subring of W_I-invariants. Let $(\mathfrak{h}^*)^I := (\mathfrak{h}^*)^{W_I} \subseteq R^I$. If we regard R as a graded R^I-module, it is free of graded rank $\widetilde{\pi}(W_I)$, the Poincaré polynomial of W_I:

$$\widetilde{\pi}(I) = \sum_{w \in W_I} v^{2\ell(w)}$$

For a finitary subset I we work in the category of graded (R, R^I)-bimodules. For a graded (R, R)-bimodule B we denote by B_I its restriction to a graded (R, R^I)-bimodule.

Definition 4.1.1. The category of I-singular Soergel bimodules $\mathbb{S}Bim^I$ is the full subcategory of graded (R, R^I)-bimodules whose objects are direct summands of B_I for $B \in \mathbb{S}Bim$.

There is a duality functor $\mathbb{D}B = \mathrm{Hom}_{R-}(B, R)$ on $\mathbb{S}Bim^I$. The (R, R^I)-bimodule structure on $\mathbb{D}B$ is given by

$$rfr'(b) = f(rbr') \qquad \text{for any } f \in \mathbb{D}B, b \in B, r \in R, r' \in R^I.$$

Let W^I be the set of minimal representatives for W/W_I. Then self-dual indecomposable I-singular Soergel bimodules B_x^I are parametrized by elements $x \in W^I$. Let $x = s_1 s_2 \dots s_k$ be a reduced expression for $x \in W^I$. Then B_x^I is the unique direct summand of $BS(\underline{s_1 s_2 \dots s_k})_I$ which is not a direct summand of any Bott-Samelson bimodule of smaller length. Equivalently, B_x^I is the unique direct summand of $B_{x,I} := (B_x)_I$ which is not a direct summand of $B_{y,I}$ for any y such that $\ell(y) < \ell(x)$.

Warning 4.1.2. We use a slightly different definition of the duality functor $\mathbb{D}$ respect to [Wil11]. It follows that our self-dual indecomposable bimodules B_x^I coincide with $B_x^I[-\ell(w_I)]$ in Williamson's notation. For us, the bimodules B_x^I are more natural to consider since $\overline{B_x^I} = \mathbb{R} \otimes_R B_x^I$ has symmetric Betti numbers, and in the geometric setting it can be obtained by taking the hypercohomology of an intersection cohomology sheaf.

Recall the Hecke algebra $\mathcal{H}$ of §1.3. For I finitary we define $\mathcal{H}^I = \mathcal{H}\underline{\mathbf{H}}_{w_I}$. This is clearly a left module over the Hecke algebra $\mathcal{H}$. The category $\mathbb{S}Bim^I$ "categorifies" this module. In fact, in analogy with Soergel's categorification Theorem 3.1.4, there is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules

$$\mathrm{ch} : [\mathbb{S}Bim^I] \xrightarrow{\sim} \mathcal{H}^I$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{S}Bim \times \mathbb{S}Bim^I \arrow[r, "- \otimes_R -"] \arrow[d, "\mathrm{ch} \times \mathrm{ch}"'] & \mathbb{S}Bim^I \arrow[d, "\mathrm{ch}"'] \\
\mathcal{H} \times \mathcal{H}^I \arrow[r, "m"] & \mathcal{H}^I
\end{tikzcd}$$

where m is the multiplication in $\mathcal{H}$.

Warning 4.1.3. Since we have a different definition of B^I_x respect to [Wil11], we need to change accordingly the definition of the map ch: we write $\mathrm{ch}(B)$ for $\mathrm{ch}(B[\ell(w_I)])$ in Williamson's notation, so that $\mathrm{ch}(B^I_x)$ in our notation coincides with $\mathrm{ch}(B^I_x)$ in Williamson's notation.

With our convention, we have $\mathrm{ch}(B_I) = \mathrm{ch}(B)\underline{\mathbf{H}}_{w_I}$ (cf. [Wil11, Theorem 6.1.5.(2)]). On the other hand, notice that one need to insert a shift in the statement of [Wil11, Proposition 7.4.3], that is if $x \in W^I$ we have

$$B^I_x \otimes_{R^I} R[\ell(w_I)] \cong B_{xw_I} \in \mathbb{S}Bim.$$

A Kazhdan-Lusztig basis element $\underline{\mathbf{H}}_y$ belongs to $\mathcal{H}^I$ if and only if $y \in W$ is maximal in its coset in W/W_I. For $x \in W^I$ we define $\underline{\mathbf{H}}^I_x = \underline{\mathbf{H}}_{xw_I} \in \mathcal{H}^I$. The set $\{\underline{\mathbf{H}}^I_x\}_{x \in W^I}$ forms a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}^I$, known as the *Kazhdan-Lusztig basis* of $\mathcal{H}^I$.

We have:

Theorem 4.1.4 (Soergel's Hom Formula for Singular Soergel Bimodules [Wil11, Theorem 7.4.1])**.** *Let $B_1, B_2 \in \mathbb{S}Bim^I$. Then $\mathrm{Hom}^\bullet(B_1, B_2)$ is a free graded left R-module and*

$$\mathrm{grrk}\,\mathrm{Hom}^\bullet_{R\otimes R^I}(B_1, B_2) = \frac{1}{\widetilde{\pi}(I)}(\overline{\mathrm{ch}(B_1)}, \mathrm{ch}(B_2)),$$

where $(-,-)$ is the pairing in the Hecke algebra defined in (1.5)*.*

By [Wil11, Theorem 3], Soergel's conjecture for Soergel bimodule (Theorem 3.1.6) implies the corresponding result for singular Soergel bimodules:

Theorem 4.1.5. *For $x \in W^I$ we have $\mathrm{ch}(B^I_x) = \underline{\mathbf{H}}^I_x$.*

It follows that

$$\mathrm{Hom}^i(B^I_x, B^I_y) \cong \begin{cases} 0 & \text{if } i < 0, \text{ or } i = 0 \text{ and } x \neq y \\ \mathbb{R} & \text{if } i = 0 \text{ and } x = y \end{cases} \tag{4.1}$$

We can define a *perverse filtration* τ on any singular Soergel bimodule. In fact, for any $B^I \in \mathbb{S}Bim^I$ we have a (non-canonical) decomposition

$$B^I = \bigoplus (B^I_x[i])^{\oplus m_{x,i}}, \tag{4.2}$$

then we define
$$\tau_{\leq j} B^I = \bigoplus_{i \geq -j} (B^I_x[i])^{\oplus m_{x,i}}.$$

It follows from the vanishing of negative degree homomorphisms (4.1) that the perverse filtration is canonical, i.e. does not depend on the decomposition (4.2).

We say that $B^I \in \mathbb{S}Bim^I$ is *perverse* if we can write $\mathrm{ch}(B^I) = \sum_{x \in W^I} m_x \underline{\mathbf{H}}^I_x$ with $m_x \in \mathbb{Z}_{\geq 0}$. Hence, B^I is perverse if and only if
$$B^I \cong \bigoplus_{x \in W^I} (B^I_x)^{\oplus m_x} \qquad \text{with } m_x \in \mathbb{Z}_{\geq 0}.$$

4.2 Hodge-theoretic statements for singular Soergel modules

If W is a Weyl group of a semisimple algebraic group G, then the category of I-singular Soergel bimodules is equivalent to the additive category generated by shifts of semisimple B-equivariant perverse sheaves on the partial flag variety $G/\mathbf{P}_I$, or alternatively by shifts of semisimple $(B \times \mathbf{P}_I)$-equivariant perverse sheaves on G. The equivalence is given by the hypercohomology, in fact if $\mathcal{F} \in \mathcal{P}\mathrm{erv}_{B \times \mathbf{P}_I}(G, \mathbb{R})$, then $\mathbb{H}^\bullet \mathcal{F}$ is naturally a graded module over $H^\bullet_{B \times \mathbf{P}_I}(pt, \mathbb{R}) \cong R \otimes_{\mathbb{R}} R^I$.

Let For : $\mathcal{F} \in \mathcal{P}\mathrm{erv}_{B \times \mathbf{P}_I}(G, \mathbb{R}) \to \mathcal{P}\mathrm{erv}_{\mathbf{P}_I}(G, \mathbb{R})$ be the functor which "forgets" the B-equivariance. For $w \in W^I$ let $X^I_w = \overline{B \cdot x\mathbf{P}_I/\mathbf{P}_I} \subseteq G/\mathbf{P}_I$ denote the corresponding parabolic Schubert variety. We have
$$IH^\bullet(X^I_w, \mathbb{R}) \cong \mathbb{H}^\bullet(\mathrm{For}(IC_B(X^I_w, \mathbb{R}))) = R/R_+ \otimes_R \mathbb{H}^\bullet(IC_B(X^I_w, \mathbb{R})) = \mathbb{R} \otimes_R IH^\bullet_B(X^I_w, \mathbb{R}).$$
hence $IH^\bullet(X^I_w, \mathbb{R})$ is in a natural way a module over R^I, and we have $IH^\bullet(X^I_w, \mathbb{R}) = \overline{B^I_w}$ (where B^I_w is constructed using the realization of Type II of W).

A line bundle $\mathcal{L}$ on $G/\mathbf{P}_I$ is ample if and only if its first Chern class $\rho = c_1(\mathcal{L}) \in (\mathfrak{h}^*)^I$ is such that we have $\rho(\alpha_t^\vee) > 0$ for all $t \in S \setminus I$. In this geometric setting we can deduce the hard Lefschetz theorem and the Hodge-Riemann bilinear relations with respect to an ample class ρ directly from Hodge theory.

The main goal of this chapter is to establish the hard Lefschetz theorem and the Hodge-Riemann bilinear relations for singular Soergel bimodules for an arbitrary Coxeter group W.

Let $\rho \in (\mathfrak{h}^*)^I \subseteq R^I$. We say that $\rho > 0$ if $\rho(\alpha_s^\vee) > 0$ for any $s \in S \setminus I$. Note that there exists such a ρ with this property since the set $\{\alpha_s^\vee\}_{s \in S}$ is linearly independent in both realizations of type I and II.

We fix now once for all $\rho > 0$ in $(\mathfrak{h}^*)^I$.

Lemma 4.2.1. *Let $w \in W^I$ and $w = s_1 \dots s_l$ a reduced expression for w. Then for any $i \leq l$ we have $(s_{i+1} \dots s_l \rho)(\alpha_{s_i}^\vee) > 0$*

Proof. Since for any i we have $s_i \dots s_l > s_{i+1} \dots s_l$, by [Hum90, Theorem 5.4] it follows that $\beta_i^\vee := s_l \dots s_{i+1}(\alpha_{s_i}^\vee) \in (\Phi^\vee)^+$, therefore
$$(s_{i+1} \dots s_l \rho)(\alpha_{s_i}^\vee) = \rho(\beta_i^\vee) \geq 0.$$

It remains to show that $\rho(\beta_i^\vee) \neq 0$. For this, it is enough to show that $\beta_i^\vee \notin \mathrm{span}\langle \alpha_t^\vee \mid t \in I \rangle$.

Clearly, for any i, $s_i s_{i+1} \dots s_l \in W^I$ and $s_i \dots s_l w_I > s_{i+1} \dots s_l w_I$, hence also
$$w_I s_l \dots s_{i+1}(\alpha_{s_i}^\vee) = w_I(\beta_i^\vee) \in (\Phi^\vee)^+.$$

But w_I is the longest element in W_I, hence it sends every positive coroot in $\mathrm{span}\langle \alpha_t^\vee \mid t \in I\rangle$ into a negative coroot. Since both $\beta_i^\vee$ and $w_I(\beta_i^\vee)$ are positive coroots, it follows that $\beta_i^\vee \notin \mathrm{span}\langle \alpha_t^\vee \mid t \in I\rangle$. $\square$

For $B^I \in \mathbb{S}Bim^I$ let $\overline{B^I} = \mathbb{R} \otimes_R B^I$. This is naturally a right R^I-module.

Warning 4.2.2. To be consistent with the rest of this thesis, our convention for modules is the opposite of [EW14]. For us, Soergel modules are always right R-modules and singular Soergel modules are right R^I-modules.

Theorem 4.2.3 (Hard Lefschetz Theorem for singular Soergel modules)**.** *Let $x \in W^I$. Then multiplication by ρ induces a degree 2 morphism on B_x^I such that for any $i > 0$ the induced map $\rho^i : (\overline{B_x^I})^{-i} \to (\overline{B_x^I})^i$ is an isomorphism.*

For $x \in W^I$ we can choose a reduced expression $\underline{x}$ and an embedding $B_x^I \overset{\oplus}{\subseteq} BS(\underline{x})_I$. Then we define $\langle -,-\rangle_{B_x^I}$ to be the restriction of the intersection form on $BS(\underline{x})$. The form $\langle -,-\rangle_{B_x^I}$ is well defined up to a scalar. We fix the sign by requiring that $\langle 1_{\underline{x}}^{\otimes}, 1_{\underline{x}}^{\otimes} \cdot \rho^{\ell(x)}\rangle_{B_x^I} > 0$ (it follows from the hard Lefschetz theorem that $\langle 1_{\underline{x}}^{\otimes}, 1_{\underline{x}}^{\otimes} \cdot \rho^{\ell(x)}\rangle_{B_x^I} \neq 0$[1]). After fixing in this way the sign we call $\langle -,-\rangle_{B_x^I}$ the *intersection form* of B_x^I. For any $b, b' \in B_x^I$, $f \in R$ and $g \in R^I$ we have

$$\langle fb, b'\rangle_{B_x^I} = \langle b, fb'\rangle_{B_x^I} = f\langle b, b'\rangle_{B_x^I},$$
$$\langle bg, b'\rangle_{B_x^I} = \langle b, b'g\rangle_{B_x^I}.$$

The intersection form induces a real valued symmetric and R^I-invariant form $\langle -,-\rangle_{\overline{B_x^I}}$ on $\overline{B_x^I}$. For $i \geq 0$ we define the Lefschetz form

$$(-,-)_\rho^{-i} = \langle -, - \cdot \rho^i\rangle_{\overline{B_x^I}} : \overline{B_x^I}^{-i} \times \overline{B_x^I}^{-i} \to \mathbb{R}.$$

Theorem 4.2.4 (Hodge-Riemann bilinear Relations for singular Soergel modules)**.** *Let $x \in W^I$. For all $i \geq 0$ the restriction of Lefschetz form $(-,-)_\rho^{-i}$ to $P_\rho^{-i} = \ker(\rho^{i+1}) \subseteq (\overline{B_x^I})^{-i}$ is $(-1)^{(-\ell(x)+i)/2}$-definite.*

The arguments in this chapter will closely follow that of [EW14]. Our focus is on the main modifications that are needed.

There is a major difference with [EW14]. The ultimate goal of Elias and Williamson is in fact to use Hodge theory to show Soergel's conjecture, and thus they need to carry Soergel's conjecture through the induction. We can avoid this, as we can instead deduce Soergel's conjecture for singular Soergel bimodules directly from the non-singular case. This makes several proofs easier.

Remark 4.2.5. In [Wil11] Williamson introduces a category ${}^J\mathbb{S}Bim^I$ of (J,I)-singular Soergel bimodules for any pair of finitary set $J, I \subseteq W$. Aside from one passage in the proof of Lemma 4.5.4 we do not need to consider this generality as the Soergel modules one obtains are the same. In fact, for a double coset $p \in W_J\backslash W/W_I$, if ${}^JB_p^I \in {}^J\mathbb{S}Bim^I$ is the corresponding indecomposable (J,I)-singular Soergel bimodule and $q \in W/W_I$ is the maximal coset contained in p, by [Wil11, Proposition 7.4.3] we have (up to some unspecified shift):

$$R \otimes_{R^J} {}^JB_p^I \cong B_q^I.$$

[1]Without appealing to hard Lefschetz at this stage, one can also notice that the proof of [EW14, Lemma 3.10] works also in our setting because of Lemma 4.2.1.

Hence

$$\overline{{}^JB^I_p} = \mathbb{R} \otimes_{R^J} {}^JB^I_p = \mathbb{R} \otimes_R R \otimes_{R^J} {}^JB^I_p = \mathbb{R} \otimes_R B^I_q = \overline{B^I_q}.$$

Notice also that we give a slightly different definition of the category $\mathbb{S}Bim^I$ compared with [Wil11]. To show that the two definitions are equivalent it is enough to show that for any $x \in W^I$ we can obtain the indecomposable bimodule B^I_x as a direct summand of the restriction B_I of a Soergel bimodule B. Let $\underline{x}$ be a reduced expression for x. We have

$$\mathrm{ch}(BS(\underline{x})_I) = \underline{\mathbf{H}}_{\underline{x}}\underline{\mathbf{H}}_{w_I} = \underline{\mathbf{H}}^I_x + \sum_{W^I \ni y < x} \lambda_y \underline{\mathbf{H}}^I_y,$$

and as in the proof of [Wil11, Theorem 7.4.2] this means that B^I_x is a direct summand of $BS(\underline{x})_I$. However, this simpler definition does not generalize to the (J, I)-case.

4.3 Structure of the proof

Recall that we have fixed $\rho > 0$ in $(\mathfrak{h}^*)^I$. The hard Lefschetz theorem and the Hodge-Riemann relations are considered with respect to the fixed ρ. For $x \in W^I$, $s \in S$ we say:

$$hL(x) := \text{hard Lefschetz holds for } \overline{B^I_x}.$$

$$HR(x) := \text{Hodge-Riemann holds for the Lefschetz form on } B^I_x.$$

$$hL(s,x) := \text{hard Lefschetz holds on } \overline{B_sB^I_x}.$$

$$HR(s,x) := \begin{array}{c}\text{for any reduced expression } \underline{x} \text{ of } x\\ \text{for any embedding } B_sB^I_x \subseteq BS(s\underline{x})_I\\ \text{the restriction of the Lefschetz form of } BS(s\underline{x})\\ \text{satisfies Hodge-Riemann on } \overline{B_sB^I_x}.\end{array}$$

Assume $sxw_I > xw_I$. Then B^I_{sx} is a direct summand of $B_sB^I_x$. Therefore $hL(s,x) \implies hL(sx)$ and $HR(s,x) \implies HR(sx)$.

We will later introduce a deformation L_ζ, for $\zeta \in \mathbb{R}_{\geq 0}$, of the Lefschetz operator, such that L_0 is multiplication by ρ. We say:

$$hL(s,x)_\zeta := \text{hard Lefschetz holds for } L_\zeta \text{ on } \overline{B_sB^I_x}.$$

$$HR(s,x)_\zeta := \begin{array}{c}\text{for any reduced expression } \underline{x} \text{ of } x\\ \text{for any embedding } B_sB^I_x \subseteq BS(s\underline{x})_I\\ \text{the restriction of the Lefschetz form of } BS(s\underline{x})\\ \text{satisfies Hodge-Riemann on } \overline{B_sB^I_x} \text{ with respect to } L_\zeta.\end{array}$$

An elementary argument (Theorem 4.5.1) shows that

$$HR(x) \implies HR(s,x)_\zeta \text{ for } \zeta \gg 0.$$

A crucial observation in [EW14] is that the signature of a family of non-degenerate symmetric forms does not change. This shows

$$hL(s,x)_\zeta \text{ for } \zeta \geq 0 \text{ and } HR(s,x)_\zeta \text{ for } \zeta \gg 0 \implies HR(s,x)_\zeta \text{ for } \zeta \geq 0$$

The critical step is then to show $hL(s,x)_\zeta$ for $\zeta \geq 0$. Note that in the induction step we also need $HR(s,x)_\zeta$ for $sxw_I < xw_I$ and $\zeta > 0$. However, this is the easiest case and it is covered by Theorem 4.5.4. If $sxw_I < xw_I$ we have:

$$hL(x) \implies hL(s,x)_\zeta \text{ for all } \zeta > 0.$$

Thus, together with $HR(s,x)_\zeta$ for $\zeta \gg 0$, $hL(x)$ implies $HR(s,x)_\zeta$ for any $\zeta > 0$.

Assume now $sxw_I > xw_I$. We need to divide into two cases. The case $\zeta > 0$ is done in Theorem 4.5.5. For $\zeta > 0$ we have:

$$\left.\begin{array}{rl} HR(t,z) & \text{for all } t \in S,\ z \in W^I \text{ such that } z < x \text{ and } tzw_I > zw_I \\ HR(s,z)_\zeta & \text{for all } z \in W^I \text{ such that } z < x \\ HR(z) & \text{for all } z \in W^I \text{ such that } z < sx \end{array}\right\} \implies hL(s,x)_\zeta.$$

Finally, the case $\zeta = 0$ is done in Theorem 4.5.6. We have:

$$\left.\begin{array}{rl} HR(x) & \\ HR(t,z) & \text{for all } t \in S,\ z \in W^I \text{ such that } z < x \text{ and } tzw_I > zw_I \\ hL(z) & \text{for all } z \in W^I \text{ such that } z < sx \end{array}\right\} \implies hL(s,x).$$

4.4 Singular Rouquier complexes

Let F be a complex of singular Soergel bimodules. Following the notation of [EW14] we indicate the homological degree on the left, that is:

$$F := [\ldots \to {}^{i-1}F \to {}^{i}F \to {}^{i+1}F \to \ldots].$$

We denote by $\{-\}$ the homological shift, so that ${}^i(F\{1\}) = {}^{i+1}F$.

Let $\mathcal{K}^b(\mathbb{S}Bim^I)$ be the bounded homotopy category of complexes of I-singular Soergel bimodules.

We define ${}^p\mathcal{K}^{\geq 0} := {}^p\mathcal{K}^b(\mathbb{S}Bim^I)^{\geq 0}$ to be the full subcategory of $\mathcal{K}^b(\mathbb{S}Bim^I)$ with objects complexes in $\mathcal{K}^b(\mathbb{S}Bim^I)$ which are isomorphic to a complex F which satisfies $\tau_{\leq -i-1}{}^iF = 0$ for all $i \in \mathbb{Z}$.

Similarly, we define ${}^p\mathcal{K}^{\leq 0} := {}^p\mathcal{K}^b(\mathbb{S}Bim^I)^{\leq 0}$ to be the full subcategory with objects complexes in $\mathcal{K}^b(\mathbb{S}Bim^I)$ which are isomorphic to a complex F which satisfies ${}^iF = \tau_{\leq -i}{}^iF$ for all $i \in \mathbb{Z}$. Let ${}^p\mathcal{K}^0 = {}^p\mathcal{K}^{\geq 0} \cap {}^p\mathcal{K}^{\leq 0}$.

For $s \in S$ let F_s denote the complex

$$F_s = [0 \to \overset{0}{B_s} \xrightarrow{d_s} R[1] \to 0]$$

where $d_s(f \otimes g) = fg$.[2] Then tensoring with F_s on the left induces an equivalence on the category $\mathcal{K}^b(\mathbb{S}Bim^I)$. The inverse is given by tensoring on the left with the complex $E_s = [0 \to R[-1] \xrightarrow{d'_s} \overset{0}{B_s} \to 0]$. Here $d'_s(1) = c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

For any $x \in W^I$ we consider the complex $F_{s_1} \ldots F_{s_k}$ for any reduced expression $s_1 \ldots s_k$ of x. As an object in $\mathcal{K}^b(\mathbb{S}Bim)$ it does not depend on the chosen reduced expression [Rou06, Proposition 9.2]. Hence, also $(F_{s_1} \ldots F_{s_k})_I$ does not depend on the reduced expression as an object in $\mathcal{K}^b(\mathbb{S}Bim^I)$.

We choose $F_x^I \overset{\oplus}{\subseteq} (F_{s_1} \ldots F_{s_k})_I$ to be the corresponding minimal complex (see [EW14, §6.1]), so $F_x^I \cong (F_{s_1} \ldots F_{s_k})_I$ in $\mathcal{K}^b(\mathbb{S}Bim^I)$ and the complex F_x^I does not contain any contractible direct summand. We call F_x^I a *singular Rouquier complex*.

Observe that if F_x is the Rouquier complex for $x \in \mathcal{K}^b(\mathbb{S}Bim)$, i.e. is the minimal complex for $F_{s_1} \ldots F_{s_k}$, then F_x^I can also be obtained as the minimal complex of $F_{x,I} := (F_x)_I$ in $\mathcal{K}^b(\mathbb{S}Bim^I)$.

[2] We use $\overset{0}{-}$ to indicate where the object in homological degree 0 is placed.

Lemma 4.4.1. *Let $x \in W^I$ and $s \in S$.*

i) If $sxw_I > xw_I$ then $F_s B_x^I \in {}^p\mathcal{K}^{\geq 0}$.

ii) If $sxw_I < xw_I$ then $F_s B_x^I \cong B_x^I[-1]$.

Proof. i) From Theorem 4.1.5 we have $\mathrm{ch}(B_s B_x^I) = \underline{\mathbf{H}}_s \underline{\mathbf{H}}_x^I = \underline{\mathbf{H}}_{sx}^I + \sum_{\substack{z \in W^I \\ z < xs}} m_z \underline{\mathbf{H}}_z^I$ with $m_z \in \mathbb{Z}_{\geq 0}$. Hence

$$B_s B_x^I \cong B_{xs}^I \oplus \bigoplus_{\substack{z \in W^I \\ z < xs}} (B_z^I)^{\oplus m_z}.$$

Then the complex

$$F_s B_x^I = [0 \to \overset{0}{B_s B_x^I} \to B_x^I[1] \to 0]$$

is manifestly in ${}^p\mathcal{K}^{\geq 0}$.

ii) We have $\mathrm{ch}(B_s B_x^I) = \underline{\mathbf{H}}_s \underline{\mathbf{H}}_x^I = \underline{\mathbf{H}}_s \underline{\mathbf{H}}_{xw_I} = (v + v^{-1}) \underline{\mathbf{H}}_x^I$. Therefore $B_s B_x^I \cong B_x^I[1] \oplus B_x^I[-1]$ and

$$F_s B_x^I = [0 \to \overset{0}{B_x^I[1] \oplus B_x^I[-1]} \to B_x^I[1] \to 0].$$

Tensoring with F_s induces an equivalence on the category $\mathcal{K}^b(\mathbb{S}Bim^I)$, and since B_x^I is indecomposable also the complex $F_s B_x^I$ must be indecomposable. Therefore the map $\overset{0}{B_x^I[1]} \to B_x^I[1]$ cannot be 0, otherwise $B_x^I[1]$ would be a non-trivial direct summand of $F_s B_x^I$. Since $B_x^I[1] \to B_x^I[1]$ is non zero, it is an isomorphism and $B_x^I[1] \to B_x^I[1]$ is a contractible direct summand that we can remove from the complex. In this way we obtain $F_s B_x^I \cong B_x^I[-1] \in {}^p\mathcal{K}^{\geq 0}$. ☐

Lemma 4.4.2. *Let $F \in {}^p\mathcal{K}^{\geq 0}$. Then $F_s F \in {}^p\mathcal{K}^{\geq 0}$.*

Proof. We denote by $\omega_{\geq k}$ the truncation of complexes, that is

$$\omega_{\geq k} F = [0 \to {}^k F \to {}^{k+1} F \to \dots].$$

We have distinguished triangles

$$\omega_{\geq k+1} F \to \omega_{\geq k} F \to {}^k F\{-k\} \xrightarrow{[1]}$$

$$F_s(\omega_{\geq k+1} F) \to F_s(\omega_{\geq k} F) \to F_s({}^k F\{-k\}) \xrightarrow{[1]}$$

Since $F_s({}^k F\{-k\}) \in {}^p\mathcal{K}^{\geq 0}$ by Lemma 4.4.1, the statement follows by induction on k using the analogue of [EW14, Lemma 6.1]. ☐

Corollary 4.4.3. *For any $x \in W^I$ we have $F_x^I \in {}^p\mathcal{K}^{\geq 0}$.*

Proof. Since $F_x^I \cong F_{s_1} F_{s_2} \dots F_{s_k}(R_I) \in \mathcal{K}^b(\mathbb{S}Bim^I)$ this follows by induction on $\ell(x)$ directly from Lemma 4.4.1 and Lemma 4.4.2. ☐

4.4.1 Singular Rouquier complexes are Δ-split

We can identify $R \otimes_{\mathbb{R}} R^I$ with the ring of regular functions on $\mathfrak{h} \times (\mathfrak{h}/W_I)$. The inclusion $R \otimes_{\mathbb{R}} R^I \hookrightarrow R \otimes_{\mathbb{R}} R$ corresponds to the projection map $\pi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h} \times (\mathfrak{h}/W_I)$.

For a coset $p \in W/W_I$ let $Gr^I(p)$ be the image $Gr(p)$ under π, where $Gr(p) \subseteq \mathfrak{h} \times \mathfrak{h}$ is the twisted graph defined in §3.1. Similarly, if $C \subseteq W/W_I$ let $Gr^I(C) = \bigcup_{p \in C} Gr^I(p)$.

Let $B^I \in \mathbb{S}Bim^I$. For $C \subseteq W/W_I$ let $\Gamma_C B = \{b \in B \mid \operatorname{supp} b \in Gr^I(C)\}$. The functor Γ_C extends to a functor Γ_C from $\mathcal{K}^b(\mathbb{S}Bim^I)$ to the homotopy category of graded (R, R^I)-bimodules, which we denote by $\mathcal{K}^b(R\text{-Mod-}R^I)$.

Let $q : W \to W/W_I$ denote the projection. For $y \in W/W_I$ let us denote by y_- the minimal element in the coset y. The bijection $W^I \cong W/W_I$ induces a Bruhat order on W/W_I, i.e. we say $y \leq z$ if and only if $y_- \leq z_-$. The projection q is a strict morphism of posets:

Lemma 4.4.4. *Let $w \geq v$ in W. Then $q(w) \geq q(v)$.*

Proof. This follows from [Dou90, Lemma 2.2]. □

For any $B \in \mathbb{S}Bim$ and any $C \subseteq W/W_I$ we have by [Wil11, Prop 6.1.6]

$$(\Gamma_{q^{-1}(C)} B)_I = \Gamma_C(B_I). \tag{4.3}$$

Note that $q^{-1}(\geq y) = \{x \in W \mid x \geq y_-\}$. If $x \in W^I$ we write $\Gamma^I_{\geq x}$ for the functor $\Gamma_{\{y \in W^I \mid y \geq x\}}$ on $\mathbb{S}Bim^I$, to differentiate it from the functor $\Gamma_{\geq x}$ on $\mathbb{S}Bim$.

We choose an enumeration $y_1, y_2, y_3, \ldots$ of W/W_I refining the Bruhat order on W/W_I. For any coset $y_i \in W/W_I$ we choose an enumeration $y_{i,1}, y_{i,2}, y_{i,3} \ldots$ of the elements in y_i refining the Bruhat order. Let

$$z_1 = y_{1,1},\ z_2 = y_{1,2},\ \ldots,\ z_{|W_I|} = y_{1,|W_I|},\ z_{|W_I|+1} = y_{2,1},\ z_{|W_I|+2} = y_{2,2} \ldots.$$

By virtue of Lemma 4.4.4, $z_1, z_2, z_3 \ldots$ is also an enumeration of W which refines the Bruhat order.

We denote by $\Gamma^I_{\geq m}$ the functor $\Gamma_{\{y_i : i \geq m\}}$ on $\mathbb{S}Bim^I$ and by $\Gamma_{\geq m}$ the functor $\Gamma_{\{z_i : i \geq m\}}$ on $\mathbb{S}Bim$. For $l \geq k$, let

$$\Gamma^I_{\geq k/\geq l} B := (\Gamma^I_{\geq k} B)/(\Gamma^I_{\geq l} B).$$

The functor $\Gamma^I_{\geq k/\geq l}$ extends to a functor $\Gamma^I_{\geq k/\geq l} : \mathcal{K}^b(\mathbb{S}Bim^I) \to \mathcal{K}^b(R\text{-Mod-}R^I)$. Similarly, we define the functors $\Gamma_{\geq k/\geq l}$, $\Gamma^I_{\geq x/\geq y}$, $\Gamma_{\geq x/\geq y}$. They also extend to functors between the respective homotopy categories.

Fix $y = y_m \in W/W_I$ and $x \in W^I$. We have $(y_m)_- = z_k$ for some k and $(y_{m+1})_- = z_{k+|W_I|}$. Then by the hin-und-her Lemma for singular Soergel bimodules [Wil11, Lemma 6.3.2] we have

$$\Gamma^I_{\geq y/>y}(F^I_x) \cong \Gamma^I_{\geq y/>y}(F_{x,I}) \cong \Gamma^I_{\geq m/\geq m+1}(F_{x,I}) \cong (\Gamma_{\geq k/\geq k+|W_I|} F_x)_I \in \mathcal{K}^b(R\text{-Mod-}R^I)$$

Assume $x \notin y_m$. Then $x = z_j$ with $j < k$ or $j \geq k + |W_I|$. For any i such that $1 \leq i \leq |W_I|$ we have a distinguished triangle in $\mathcal{K}^b(R\text{-Mod-}R)$

$$\Gamma_{\geq k/\geq k+i-1} F_x \to \Gamma_{\geq k/\geq k+i} F_x \to \Gamma_{\geq k+i-1/\geq k+i} F_x \xrightarrow{[1]}$$

and the last term is 0 by [LW14, Prop 3.7]. It follows by induction that $\Gamma_{\geq k/\geq k+|W_I|} F_x \cong 0$, hence

$$\Gamma^I_{\geq y/>y}(F^I_x) \cong \Gamma^I_{\geq y/>y}(F_{x,I}) \cong (\Gamma_{\geq k/\geq k+|W_I|} F_x)_I \cong 0 \in \mathcal{K}^b(R\text{-Mod-}R^I).$$

Assume now $x = y_-$, so that $x = z_k$. Let $R_{x,I} := (R_x)_I$. Since

$$\Gamma_{\geq k/\geq k+1} F_x = R_x[-\ell(x)]$$

the same argument as above shows $\Gamma_{\geq k/\geq k+|W_I|} F_x \cong R_x[-\ell(x)]$, hence

$$\Gamma^I_{\geq x/>x}(F^I_x) \cong \Gamma^I_{\geq x/>x}(F_{x,I}) \cong (\Gamma_{\geq k/\geq k+|W_I|} F_x)_I \cong R_{x,I}[-\ell(x)].$$

We obtain the singular version of [LW14, Prop 3.7]:

Lemma 4.4.5. *Let $x, y \in W^I$. Then*

$$\Gamma^I_{\geq y,>y}(F^I_x) = \begin{cases} 0 & \text{if } y \neq x, \\ R_{x,I}[-\ell(x)] & \text{if } y = x. \end{cases}$$

Remark 4.4.6. If we view F_s as a complex of graded left R-modules it splits, i.e. we have $F_s \cong R[-1]$ in $\mathcal{K}^b(R\text{-mod})$. Similarly $(F_{s_1} F_{s_2} \dots F_{s_k})_I \cong R[-k]$ in $\mathcal{K}^b(R\text{-mod})$. For $x \in W^I$, let $\overline{F^I_x} = \mathbb{R} \otimes_R F^I_x$. It is a complex of graded real vector spaces. It follows that we have:

$$H^i(\overline{F^I_x}) = \begin{cases} \mathbb{R}[-\ell(x)] & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

4.4.2 Singular Rouquier complexes are linear

For us it is important to understand how singular Rouquier complexes look. The idea is to use the first differential in a singular Rouquier complex as a replacement for "weak Lefschetz" in the inductive proof of hard Lefschetz. More precisely, the first differential will have the role of the map ϕ in [EW14, Lemma 2.3]. For this, we first have to show that the first differential is a map of degree 1 between perverse singular Soergel bimodules.

Lemma 4.4.7. *Let $x \in W^I$. For $i > 0$ if ${}^iF^I_x$ contains a direct summand isomorphic to $B^I_z[j]$, then ${}^{i-1}F^I_x$ contains a direct summand isomorphic to $B^I_{z'}[j']$ with $z' > z$ and $j' < j$.*

Proof. The proof is the same of [EW14, Lemma 6.11]. From Theorem 4.1.5 (and the definition of the map ch, cf. [Wil11, §6.3]) we have that for any $y, z \in W^I$ the bimodule $\Gamma^I_{\geq z/>z}(B^I_y)$ is generated in degree $< \ell(z)$ if $y > z$ and $\Gamma^I_{\geq z/>z}(B^I_z) \cong R_{z,I}[-\ell(z)]$.

The image of $B^I_z[j]$ in ${}^{i+1}F^I_x$ is contained in $\tau_{<-j}({}^{i+1}F^I_x)$ because of (4.1): in fact any non-zero homomorphism in degree 0 is an isomorphism and thus yields a contractible direct summand.

Applying $\Gamma^I_{\geq z/>z}$ to F^I_x the direct summand $B^I_z[j]$ induces a summand $R_{z,I}[j - \ell(z)]$. This cannot be a direct summand in $\Gamma^I_{\geq z/>z}(\tau_{<-j}{}^{i+1}F^I_x)$ and cannot survive in the cohomology of the complex because of Lemma 4.4.5. Thus $R_{z,I}[j - \ell(z)]$ must be the image of a direct summand $R_{z,I}[j - \ell(z)]$ in $\Gamma_{\geq z/>z}(\tau_{>-j}({}^{i-1}F_x))$.

This implies that there is a direct summand $B^I_y[k]$ in ${}^{i-1}F_x$ with $y > z$ and $k < j$. $\square$

Theorem 4.4.8. *Let $x \in W^I$ and F^I_x be a singular Rouquier complex. Then:*

i) ${}^0F^I_x = B^I_x$.

ii) For $i \geq 1$, ${}^iF^I_x = \bigoplus (B^I_z[i])^{\oplus m_{z,i}}$ with $z < x$, $z \in W^I$ and $m_{z,i} \in \mathbb{Z}_{\geq 0}$.

In particular, $F^I_x \in {}^p\mathcal{K}^0$.

Proof. We apply the previous lemma. The same proof shows that since ${}^{-1}(F_x^I) = 0$, the only direct summands occurring in ${}^0(F_x^I)$ is B_x^I. By induction we have ${}^iF_x^I = \tau_{\leq -i}F_x^I$ for any $i > 0$. Now ii) follows since we already know $F_x^I \in {}^p\mathcal{K}^{\geq 0}$ from Corollary 4.4.3. $\square$

Remark 4.4.9. We can define the character of a complex $F \in \mathcal{K}^b(\mathbb{S}Bim^I)$ as

$$\mathrm{ch}(F) = \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{ch}({}^iF).$$

If $x \in W^I$ and $\underline{x} = s_1 s_2 \dots s_k$ is a reduced expression we have

$$\mathrm{ch}(F_x^I) = \mathrm{ch}((F_{s_1}F_{s_2}\dots F_{s_k})_I) = \mathbf{H}_x \underline{\mathbf{H}}_{w_I} =: \mathbf{H}_x^I.$$

An immediate consequence of Theorem 4.1.5 is that there is a non trivial morphism of degree i between B_x^I and B_y^I for $x, y \in W^I$ if and only if i and $\ell(x) - \ell(y)$ have the same parity. Because of Theorem 4.4.8 we can write

$$\mathbf{H}_x^I = \sum_{i\geq 0} (-1)^i \,\mathrm{ch}({}^iF_x) = \sum_{y\leq x} g_{y,x} \underline{\mathbf{H}}_y^I$$

with $g_{x,x}(v) = 1$ and $g_{y,x}(v) = \sum_{i>0} m_{y,i}(-v)^i$. The polynomials $g_{y,x}$ are the *parabolic inverse Kazhdan-Lusztig polynomials.* We obtain that for any $y \leq x$ the polynomial $(-1)^{\ell(y)-\ell(x)} g_{y,x}$ has non-negative coefficients.

4.4.3 Singular Rouquier complexes are Hodge-Riemann

The complex F_x^I is a direct summand of $(F_{s_1}\dots F_{s_m})_I$ for a reduced expression $\underline{x} = s_1 \dots s_m$. Hence, for any j, ${}^jF_x^I$ is a direct summand of ${}^j(F_{s_1}\dots F_{s_m})_I$, that is

$${}^jF_x^I \overset{\oplus}{\subseteq} \bigoplus_{\underline{x}'\in\pi(\underline{x},j)} BS(\underline{x}')_I[j]$$

where $\pi(\underline{x}, j)$ is the set of all subexpressions of $\underline{x}$ obtained by omitting j simple reflections.

Fix $\lambda = (\lambda_{\underline{x}'})_{\underline{x}'\in\pi(\underline{x},j)}$ a tuple of strictly positive real numbers. We define a symmetric form $\langle -, - \rangle^\lambda$ on $\bigoplus BS(\underline{x}')_I$ by

$$\langle b, b' \rangle^\lambda = \sum_{\underline{x}'\in\pi(\underline{x},j)} \lambda_{\underline{x}'} \langle b_{\underline{x}'}, b'_{\underline{x}'} \rangle_{BS(\underline{x}')} \text{ for all } b = (b_{\underline{x}'}), b' = (b'_{\underline{x}'}) \in \bigoplus_{\underline{x}'\in\pi(\underline{x},j)} BS(\underline{x}')_I \quad (4.4)$$

where $\langle -, - \rangle_{BS(x')}$ is the intersection form on $BS(\underline{x}')_I = BS(\underline{x}')$ defined in §3.1.1.

We say that F_x^I satisfies the Hodge-Riemann bilinear relations if we can choose an embedding $F_x^I \overset{\oplus}{\subseteq} (F_{s_1}\dots F_{s_m})_I$ such that for all tuples λ as above, multiplication by ρ on the right on ${}^jF_x[-j]$ satisfies the Hodge-Riemann bilinear relations with respect to the form $\langle -, - \rangle^\lambda$.

Proposition 4.4.10. *Assume $HR(s, y)$ for all $s \in S$ and $y \in W^I$ with $y < x$ such that $syw_I > yw_I$. Then F_x^I satisfies the Hodge-Riemann bilinear relations.*

Proof. Fix a reduced expression $\underline{x} = s_1 \dots s_m$ and let $s = s_1$, $\underline{y} = s_2 \dots s_m$. By induction assume F_y^I satisfies Hodge-Riemann so that we can find an appropriate embedding $F_y^I \overset{\oplus}{\subseteq} (F_{s_2}\dots F_{s_m})_I$.

Now F_x^I is a direct summand of $F_sF_y^I$, so we have an embedding

$$ {}^jF_x^I[-j] \overset{\oplus}{\subseteq} {}^j(B_sF_y^I)[-j] \oplus {}^{j-1}F_y^I[-j+1] \overset{\oplus}{\subseteq} \bigoplus_{\underline{y}'\in\pi(\underline{y},j)} B_sBS(\underline{y}')_I \oplus \bigoplus_{\underline{y}''\in\pi(\underline{y},j-1)} BS(\underline{y}'')_I = $$

$$ = \bigoplus_{\underline{x}'\in\pi(\underline{x},j)} BS(\underline{x}'). $$

We fix a tuple $(\mu_{\underline{x}'})_{\underline{x}'\in\pi(\underline{x},j)}$, or equivalently two tuples $(\lambda_{\underline{y}'})_{\underline{y}'\in\pi(\underline{y},j)}$ and $(\sigma_{\underline{y}''})_{\underline{y}''\in\pi(\underline{y},j)}$ of positive real numbers.

From Theorem 4.4.8, we know that ${}^jF_y^I[-j] = \bigoplus (B_z^I)^{\oplus m_z}$ with $m_z \in \mathbb{Z}_{\geq 0}$. Let ${}^jF_y^I[-j] = B^\uparrow \oplus B^\downarrow$ with

$$ B^\uparrow = \bigoplus_{szw_I>zw_I} (B_z^I)^{\oplus m_z} \quad \text{and} \quad B^\downarrow = \bigoplus_{szw_I<zw_I} (B_z^I)^{\oplus m_z}. $$

The decomposition is orthogonal with respect of the restriction of the form $\langle -,-\rangle^\lambda$ since $\mathrm{Hom}^0(B^\uparrow, \mathbb{D}B^\downarrow) = 0$. Then also $B_sB^\uparrow$ and $B_sB^\downarrow$ are orthogonal with respect of the restriction of the form $\langle -,-\rangle^\mu$ on $B_s({}^jF_y^I[-j])$. In fact, for any $b \in B^\uparrow$ and $b' \in B^\downarrow$ we have:

$$ \langle c_{id}b, c_{id}b'\rangle^\mu = \partial_s(\langle b,b'\rangle^\lambda) = 0 \tag{4.5} $$

$$ \langle c_sb, c_{id}b'\rangle^\mu = \langle c_{id}b, c_sb'\rangle^\mu = \langle b,b'\rangle^\lambda = 0 \tag{4.6} $$

$$ \langle c_sb, c_sb'\rangle^\mu = \alpha_s\langle b,b'\rangle^\lambda = 0. \tag{4.7} $$

The bimodule $B_sB^\uparrow$ is perverse while $B_sB^\downarrow = B^\downarrow[-1] \oplus B^\downarrow[1]$. So we have a decomposition

$$ {}^jF_x^I[-j] \overset{\oplus}{\subseteq} B_sB^\downarrow \oplus B_sB^\uparrow \oplus {}^{j-1}F_y^I[-j+1]. \tag{4.8} $$

The inclusion is, by definition, an isometry. This decomposition is orthogonal with respect to the form $\langle -,-\rangle^\mu$. Moreover by (4.1) the image of ${}^jF_x^I[-j] \to B_sB^\downarrow$ is contained in $B^\downarrow[1]$.

Claim 4.4.11. The restriction of the Lefschetz form $(-,-)_\rho^{-k} = \langle -, -\cdot\rho^k\rangle^\mu$ to $(\overline{B^\downarrow[1]})^{-k}$ is zero.

Proof of the claim. Let $B_z^I \overset{\oplus}{\subseteq} B^\downarrow$ and let $z = t_1 \dots t_l$ be a reduced expression. Then $z' := t_2 \dots t_l \in W^I$ and $t_1z'w_I > z'w_I$. Since $B_z^I \overset{\oplus}{\subseteq} B_{t_1}B_{z'}^I$, the hypothesis $HR(t_1,z')$ implies that multiplication by ρ satisfies hard Lefschetz on $\overline{B_z^I}$, hence on $\overline{B^\downarrow}$, i.e.

$$ \rho^i : (\overline{B^\downarrow})^{-i} \xrightarrow{\sim} (\overline{B^\downarrow})^i \text{ for all } i \geq 0. $$

By shifting we get $\rho^i : \overline{B^\downarrow[1]}^{-i-1} \xrightarrow{\sim} \overline{B^\downarrow[1]}^{i-1}$. Let $P_\rho^{-1-i} = \operatorname{Ker}\rho^{i+1} \subseteq \overline{B^\downarrow[1]}^{-i-1}$ so that for any $m \leq 0$ we have

$$ \overline{B^\downarrow[1]}^m = \bigoplus_{j \geq max\{\frac{m+1}{2},0\}} P_\rho^{m-2j} \cdot \rho^j. $$

Let $x \in P_\rho^{m-2j}$, $y \in P_\rho^{m-2k}$ for some $j \geq k \geq 0$, then

$$ (x\rho^j, y\rho^k)_\rho^m = \langle x, y\rho^{j+k-m}\rangle^\mu = 0 $$

because $j+k-m \geq 2k-m$ and $y \in \operatorname{Ker}(\rho^{2k-m})$. $\square$

So the image of $\overline{{}^jF^I_x[-j]}$ in $\overline{B_sB^{\downarrow}}$ does not contribute to the Lefschetz form. We can consider the projection onto the other two factors

$$i : \overline{{}^jF^I_x[-j]} \to \overline{B_sB^{\uparrow}} \oplus \overline{{}^{j-1}F^I_y[-j+1]}$$

which is an isometry.

Furthermore, the map i is injective, in fact ${}^jF^I_x \overset{\oplus}{\subseteq} {}^j(F_sF^I_y)$ is a split inclusion and since when we project to $\mathbb{S}Bim^I / \operatorname{rad}(\mathbb{S}Bim^I)$, the image is contained in $B_sB^{\uparrow} \oplus {}^{j-1}F^I_y[-j+1]$, then also i must be a split injective morphism.

Using the fact that ${}^jF^I_x$ is stable under the Lefschetz operator and that Hodge-Riemann holds by induction for both $\overline{B_sB^{\uparrow}}$ and $\overline{{}^{j-1}F^I_y[-j+1]}$, the thesis follows. □

4.5 Hard Lefschetz for singular Soergel modules

4.5.1 Deforming the Lefschetz operator

Let $B^I \in \mathbb{S}Bim^I$ be a direct summand of $BS(\underline{x})_I$. If the intersection form on B^I is the restriction of the intersection form on $BS(\underline{x})_I$, then we equip B_sB^I with the restriction of the intersection form of $BS(s\underline{x})_I$ (with respect to the embedding $B_sB^I \overset{\oplus}{\subseteq} B_sBS(\underline{x})_I$).

For $\zeta \geq 0$ we define the deformed Lefschetz operator $L_\zeta \in \operatorname{End}^2(B_sB^I)$ as

$$L_\zeta(b_s \otimes b) = b_s \otimes (b \cdot \rho) + \zeta(b_s \cdot \rho) \otimes b \qquad \text{for all } b_s \in B_s \text{ and } b \in B^I.$$

so that for $\zeta = 0$ we recover the Lefschetz operator given by multiplication by ρ on the right.

Theorem 4.5.1. *If $B^I \in \mathbb{S}Bim^I$ is such that $\overline{B^I}$ satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations, than also $\overline{B_sB^I}$ satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations with respect to the Lefschetz operator L_ζ for any $\zeta \gg 0$.*

Proof. The proof is exactly the same as in the non-singular case [EW14, Theorem 5.1]. □

4.5.2 Factoring the Lefschetz operator

Let $\underline{x} = s_1s_2 \ldots s_l$ and $\underline{x}_{\hat{i}} = s_1 \ldots s_{i-1}s_{i+1} \ldots s_l$ for any $1 \leq i \leq l$. Let $\gamma_i = s_l \ldots s_{i+1}(\rho)$ for all $1 \leq i \leq k$. Recall that $\gamma_i > 0$ for all i. We rescale the intersection forms on $BS(\underline{x}_{\hat{i}})_I$ using the tuple $\gamma = (\gamma_i)$ as in (4.4) and we obtain a form $\langle -, - \rangle^\gamma$ on $\bigoplus_i BS(\underline{x}_{\hat{i}})_I$.

Let

$$\phi : \overline{BS(x)_I} \to \bigoplus_{i=1}^{l} \overline{BS(\underline{x}_{\hat{i}})_I}[1]$$

be the map induced by first differential in $(F_{s_1}F_{s_2} \ldots F_{s_l})_I$. We have the following by [EW14, Lemma 6.15]:

Lemma 4.5.2. *We have $\langle b, b'\rho \rangle_{\overline{BS(\underline{x})}} = \langle \phi(b), \phi(b') \rangle^\gamma$ for any $b, b' \in BS(\underline{x})_I$.*

Fix $\zeta > 0$. Let $s = s_1$ so that $\underline{x} = s\underline{x}_{\hat{1}}$. Let now $\mu_i = \gamma_i$ for all $2 \leq i \leq l$ and $\mu_1 = \gamma_1 + \zeta\rho(\alpha_s^\vee)$. Let L_ζ the operator on $B_sBS(\underline{x}_{\hat{1}})_I$ defined in §4.5.1.

Lemma 4.5.3. *We have $\langle b, L_\zeta(b') \rangle_{\overline{BS(\underline{x})}} = \langle \phi(b), \phi(b') \rangle^\mu$ for any $b, b' \in BS(\underline{x})_I$.*

Proof. Assume $b = b_1 \otimes b_2$ and $b' = b'_1 \otimes b'_2$ with $b_1, b'_1 \in B_s$ and $b_2, b'_2 \in BS(\underline{x}_{\hat{1}})_I$. Then

$$\langle b, L_\zeta(b')\rangle_{\overline{BS(\underline{x})}} = \langle b, b'\rho\rangle_{\overline{BS(\underline{x})}} + \zeta\langle b, b'_1\rho \otimes b'_2\rangle_{\overline{BS(\underline{x})}} = \langle \phi(b), \phi(b')\rangle^\gamma + \zeta\langle b, b'_1\rho \otimes b'_2\rangle_{\overline{BS(\underline{x})}}.$$

Now, a straightforward computation as in (4.5)-(4.7) shows that, $\langle b_1 \otimes b_2, b'_1\rho \otimes b'_2\rangle_{\overline{BS(\underline{x})}} = \partial_s(\rho)\langle b_2, b'_2\rangle_{\overline{BS(\underline{x}_{\hat{1}})}}$. The claim follows. $\square$

4.5.3 Proofs of hard Lefschetz

We are now ready to prove hard Lefschetz for the operators L_ζ for $\zeta \geq 0$. As in [EW14, §6.8] we have to divide into three cases.

Theorem 4.5.4 (Hard Lefschetz for $\zeta > 0$, $sxw_I < xw_I$)**.** *Assume $hL(x)$, then $hL(s,x)_\zeta$ holds for any $\zeta > 0$.*

Proof. Let $y \in W_{\{s\}}\backslash W/W_I$ be the double coset containing x. Then there exists an indecomposable $(\{s\}, I)$-singular Soergel bimodule ${}^{\{s\}}B^I \in {}^{\{s\}}\mathbb{S}Bim^I$ such that

$$R \otimes_{R^s} {}^{\{s\}}B^I \cong B^I_x$$

(cf. [Wil11]). Then any decomposition $R \cong R^s \oplus R^s[-2]$ as R^s-modules induces a decomposition

$$B_sB^I_x = R \otimes_{R^s} R \otimes_{R^s} {}^{\{s\}}B^I[1] \cong R \otimes_{R^s} (R^s[1] \oplus R^s[-1]) \otimes_{R^s} {}^{\{s\}}B^I \cong B^I_x[1] \oplus B^I_x[-1]$$

of (R, R^I)-bimodules. We fix a decomposition $R \cong R^s \oplus R^s[-2]$ as in [EW14, Theorem 6.19] and we obtain, by the same computation therein, that the operator L_ζ can be written with respect of the decomposition $\overline{B_sB^I_x} \cong \overline{B^I_x}[1] \oplus \overline{B^I_x}[-1]$ as:

$$L_\zeta = \begin{pmatrix} (-)\cdot\rho & 0 \\ \zeta\rho(\alpha_s^\vee) & (-)\cdot\rho \end{pmatrix} : \begin{pmatrix} \overline{B^I_x}[1] \\ \overline{B^I_x}[-1] \end{pmatrix} \to \begin{pmatrix} \overline{B^I_x}[1] \\ \overline{B^I_x}[-1] \end{pmatrix}.$$

Notice that $\zeta\rho(\alpha_s^\vee) > 0$. We have an isomorphism of graded vector spaces

$$\overline{B_sB^I_x} \cong \mathbb{R}[z]/(z^2)[1] \otimes_\mathbb{R} \overline{B^I_x}$$

and L_ζ acts on $\mathbb{R}[z]/(z^2)[1] \otimes_\mathbb{R} \overline{B^I_x}$ as multiplication by $\zeta\rho(\alpha_s^\vee)z \otimes 1 + 1 \otimes \rho$. Hence L_ζ is the sum of two operators both satisfying hard Lefschetz, hence also L_ζ satisfies it, as it is immediate from the representation theory of $\mathfrak{sl}_2(\mathbb{R})$ (cf. the proof of [EW14, Theorem 6.14]). $\square$

Theorem 4.5.5 (Hard Lefschetz for $\zeta \geq 0$, $sxw_I > xw_I$)**.** *Let $\zeta > 0$ and $sxw_I > xw_I$. Assume*

- *$HR(t,z)$ for all $t \in S$ and $z \in W^I$ such that $z < x$ and $tzw_I > zw_I$,*
- *$HR(s,z)_\zeta$ for all $z \in W^I$ such that $z < x$,*
- *$HR(x)$.*

Then $hL(s,x)_\zeta$ holds.

Proof. Let $x = s_1 \dots s_l$. We define $\gamma_i = (s_{i+1} \dots s_l \rho)(\alpha_{s_i}^\vee)$ for $1 \leq i \leq m$ and $\mu_1 = (x\rho)(\alpha_s^\vee) + \zeta\rho(\alpha_s^\vee)$. From Lemma 4.2.1 we see that all the γ_i's and μ_1 are positive. Let $\mu_i = \gamma_{i+1}$ for $i > 1$. Let ϕ be the first differential in the complex $(F_s F_{s_1} \dots F_{s_l})_I$. Then by Lemma 4.5.3 we have

$$\langle b, b' \cdot \rho \rangle_{\overline{BS(s\underline{x})}} = \langle \phi(b), \phi(b') \rangle^\mu \qquad \text{for all } b, b' \in BS(s\underline{x})_I.$$

The rest of the proof continues as in [EW14, Theorem 6.20] and we only sketch it. By Proposition 4.4.10 we can fix an embedding $F_x^I \overset{\oplus}{\subseteq} (F_{s_1} \dots F_{s_l})_I$ such that F_x^I satisfies Hodge-Riemann with respect of the form $\langle -, - \rangle^\gamma$. The first differential of $F_s F_x^I$ is $B_s B_x^I \xrightarrow{\phi} B_s{}^1 F_x^I \oplus B_x^I[1]$. With respect to this decomposition we write $\phi = (d_1, d_2)$. It is clear that we have $d_1 \circ L_\zeta = L_\zeta \circ d_1$, while

$$d_2(L_\zeta(b)) = d_2(b) \cdot \rho + \zeta\rho \cdot d_2(b).$$

Hence, if we call L the operator on $B_s{}^1 F_x^I \oplus B_x^I[1]$ which is L_ζ on $B_s{}^1 F_x^I$ and $(-) \cdot \rho$ on $B_x^I[1]$, after passing to $\overline{\phi} : \overline{B_s B_x^I} \to \overline{B_s{}^1 F_x^I} \oplus \overline{B_x}[1]$ we have

$$\overline{\phi}(L_\zeta(b)) = L(\overline{\phi}(b)) \qquad \text{for any } b \in B_s B_x^I.$$

Now $\overline{\phi}$ is injective in degree $\leq \ell(x)$ by Remark 4.4.6 and by hypothesis $\overline{B_s{}^1 F_x^I} \oplus \overline{B_x}[1]$ satisfies Hodge-Riemann with respect of L and restriction of $\langle -, - \rangle^\mu$. We can then apply [EW14, Lemma 2.3] to deduce $hL(s,x)_\zeta$. ◻

Theorem 4.5.6 (Hard Lefschetz for $\zeta = 0$, $sxw_I > xw_I$)**.** *Let $sxw_I > xw_I$. Assume*

- *$HR(t,z)$ for all $t \in S$ and $z \in W^I$ such that $z < x$ and $tzw_I > zw_I$,*
- *$HR(x)$,*
- *$hL(z)$ for all $z \in W^I$ such that $z < sx$.*

Then $hL(s,x)$ holds.

Proof. Let $x = s_1 \dots s_l$. We define $\gamma_i = (s_{i+1} \dots s_l \rho)(\alpha_{s_i}^\vee)$ for $1 \leq i \leq m$ and $\mu_1 = (x\rho)(\alpha_s^\vee)$. From Lemma 4.2.1 we see that all the γ_i's and μ_1 are positive. Let $\mu_i = \gamma_{i+1}$ for $i > 1$. Let ϕ the first differential in the complex $(F_s F_{s_1} \dots F_{s_l})_I$. Then by Lemma 4.5.2 we have

$$\langle b, L_\zeta b' \rangle_{\overline{BS(s\underline{x})}} = \langle \phi(b), \phi(b') \rangle^\mu \qquad \text{for all } b, b' \in BS(s\underline{x})_I.$$

The rest of the proof continues as in [EW14, Theorem 6.21] and we only sketch it. By Proposition 4.4.10 we can fix an embedding $F_x^I \overset{\oplus}{\subseteq} (F_{s_1} \dots F_{s_l})_I$ such that F_x^I satisfies Hodge-Riemann with respect of the form $\langle -, - \rangle^\gamma$. Let ${}^1F_x^I[-1] = B^\uparrow \oplus B^\downarrow$ with

$$B^\uparrow = \bigoplus_{szw_I > zw_I} (B_z^I)^{\oplus m_z} \quad \text{and} \quad B^\downarrow = \bigoplus_{szw_I < zw_I} (B_z^I)^{\oplus m_z}.$$

so that $B_s B^\uparrow$ is perverse and $B_s B^\downarrow \cong B^\downarrow[1] \oplus B^\downarrow[-1]$. The first differential of $F_s F_x^I$ is

$$B_s B_x^I \xrightarrow{\phi} B_s B^\uparrow[1] \oplus B_x^I[1] \oplus B^\downarrow \oplus B^\downarrow[2].$$

Because of Proposition 4.4.8 we know that $B^\downarrow[2]$ is contained in a contractible summand of $F_s F_x^I$, hence we can remove it and obtain:

$$B_s B_x^I \xrightarrow{\phi} B_s B^\uparrow[1] \oplus B_x^I[1] \oplus B^\downarrow.$$

With respect to this decomposition we write $\phi = (d_1, d_2, d_3)$. The same argument as in (4.8) shows that the decomposition above of ${}^1(F_s F_x^I)$ is orthogonal with respect to $\langle -, - \rangle^\mu$.

We want to show that if $b \in \overline{B_s B_x^I}^{-k}$ then $b \cdot \rho^k \neq 0$. If $\overline{d_3}(b) \neq 0$ then it follows from hard Lefschetz on $\overline{B^\downarrow}$, which we know by hypothesis since in $B^\downarrow$ only summands B_z^I with $z < sx$ occur.

Assume now $\overline{d_3}(b) = 0$, so that b belongs to $V := \mathrm{Ker}(\overline{d_3}) \subseteq \overline{B_s B_x^I}$. The map $\overline{\phi}$ restricts to a map $V \to \overline{B_s B^\uparrow}[1] \oplus \overline{B_x}[1]$. By hypothesis we have Hodge-Riemann on both $\overline{B_s B^\uparrow}$ and $\overline{B_x}$ for the multiplication by ρ. Now applying [EW14, Lemma 2.3] we obtain that multiplication by ρ satisfies hard Lefschetz on V. This completes the proof. □

4.6 Consequences for non-singular Soergel modules

Let $x \in W$ and $s \in S$ be such that $xs > x$. Let $B_x \in \mathbb{S}Bim$ be the corresponding indecomposable (non-singular) Soergel bimodule. Assume $I = \{s\}$, so that $w_I = s$. Then $(B_x)_I$ is a perverse singular Soergel bimodule, in fact we have:

$$\mathrm{ch}((B_x)_I) = \underline{\mathbf{H}}_x \underline{\mathbf{H}}_s = \underline{\mathbf{H}}_x^I + \sum_{\substack{ys > y \\ y < x}} m_y \underline{\mathbf{H}}_y^I \qquad \text{with } m_y \in \mathbb{Z}_{\geq 0}$$

We obtain the following:

Corollary 4.6.1. *Let $x \in W$ be such that $xs > x$. Then if $\rho > 0$ in $(\mathfrak{h}^*)^s$, i.e. $\rho(\alpha_s^\vee) = 0$ and $\rho(\alpha_t^\vee) > 0$ for all $t \neq s$, multiplication by ρ on $\overline{B_x}$ satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations.*

Proof. Since

$$(B_x)_I \cong B_x^I \oplus \bigoplus_{\substack{ys > y \\ y < x}} (B_y^I)^{\oplus m_y} \tag{4.9}$$

hard Lefschetz for $\overline{B_x}$ follows from $hL(y)$ for all y such that B_y^I is a direct summand in (4.9).

Let ϖ_s be a fundamental weight for s and let $\rho_\zeta = \rho + \zeta \varpi_s$ for $\zeta \geq 0$. Since ρ_ζ satisfies hard Lefschetz on B_x for all $\zeta \geq 0$ and Hodge-Riemann for every $\zeta > 0$ (from the non-singular case), and since the signature of a family of non-degenerate forms does not change, we deduce Hodge-Riemann for $\rho_0 = \rho$. □

Hence, we obtain the results of [EW14] for a slightly larger set of classes ρ.

Remark 4.6.2. Corollary 4.6.1 has a geometric motivation. Assume that W is the Weyl group of a complex semisimple group G. Let $x \in W$ be such that $xs > x$ for $s \in S$ and let $X_x \subseteq G/B$ be the corresponding Schubert variety. Let $\mathbf{P}_s$ be the minimal parabolic subgroup of G containing s. Then the restriction of the projection $G/B \to G/\mathbf{P}_s$ to X_x is semismall. It follows from [dM02, Theorem 2.3.1] that the pull-back of any ample class on $G/\mathbf{P}_s$ satisfies hard Lefschetz and Hodge-Riemann on X_x.

Remark 4.6.3. Assume $w \in W$ such that $ws > w$. Notice that $\mathrm{ch}(B_x) = \underline{\mathbf{H}}_x$ for all $x < ws$ and Corollary 4.6.1 for w imply $\mathrm{ch}(B_{ws}) = \underline{\mathbf{H}}_{ws}$, so if one could prove the previous Corollary by other means, one would obtain an alternative proof of Soergel's conjecture. In fact, let $I = \{s\}$ and fix $\rho > 0$ in $(\mathfrak{h}^*)^I$. Let $x < w \in W$ be such that $xs > x$. Let $P_\rho^{-k} \subseteq (\overline{B_w})^{-k}$ the primitive part, i.e. $P_\rho^{-k} = \ker(\rho^{k+1})$.

We have a symmetric form on $\mathrm{Hom}(B_x^I, (B_w)_I)$ defined by $(f, g) = g^* \circ f \in \mathrm{End}(B_x^I) \cong \mathbb{R}$, where g^* denotes the adjoint with respect of the intersection forms. Then, as in [EW14, Theorem 4.1], the map

$$\mathrm{Hom}(B_x^I, (B_w)_I) \to P_\rho^{-\ell(x)}$$

defined by $f \mapsto f(1^{\otimes}_{\underline{x}})$ is injective and, if we equip $P_\rho^{-\ell(x)}$ with the Lefschetz form, it is an isometry (up to a positive scalar). If $d = \dim \mathrm{Hom}(B_x^I, (B_w)_I)$, it follows that $(B_x^I)^d$ is a direct summand of $(B_w)_I$, hence $(B_{xs})^d$ is a direct summand of $B_w B_s$.

Notice that this proof of Soergel's conjecture is a close translation in the language of Soergel bimodules of the proof of the decomposition theorem for semismall maps given in [dM02].

Example 4.6.4. Let W be the Weyl group of type A_3 with simple reflections labeled s, t, u. Let $I = \{s, t\}$, so that $w_I = sts$. Then $stu \in W^I$ but a simple computation in the Hecke algebra shows that

$$\underline{\mathbf{H}}_{stu}\underline{\mathbf{H}}_{sts} = \underline{\mathbf{H}}^I_{stu} + \underline{\mathbf{H}}^I_u + (v + v^{-1})\underline{\mathbf{H}}^I_{id}.$$

Therefore, the singular Soergel bimodule $(B_{stu})_I$ is not perverse, and no $\rho \in (\mathfrak{h}^*)^I$ satisfies hard Lefschetz on $\overline{B_{stu}}$.

Chapter 5

The Néron-Severi Lie Algebra of Soergel Modules

Let Y be a smooth complex projective variety of dimension n and $\rho \in H^2(Y, \mathbb{R})$ be the Chern class of an ample line bundle on Y. The Hard Lefschetz Theorem states that for any $k \in \mathbb{N}$ cupping with ρ^k yields an isomorphism $\rho^k : H^{n-k}(Y, \mathbb{R}) \to H^{n+k}(Y, \mathbb{R})$. This assures the existence of an adjoint operator $f_\rho \in \mathfrak{gl}(H^\bullet(Y, \mathbb{R}))$ of degree -2 which together with ρ generates a Lie algebra $\mathfrak{g}_\rho$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. In [LL97] Looijenga and Lunts defined the Néron-Severi Lie algebra $\mathfrak{g}_{NS}(Y)$ of Y to be the Lie algebra generated by all the $\mathfrak{g}_\rho$ with ρ an ample class.

In §5.1 we review the definition and properties of Lefschetz modules from [LL97], restricting to the case of Hodge structure of Hodge-Tate type. In §5.2 we explain how to use the Néron-Severi Lie algebra to prove the Carrell-Peterson criterion for rational smoothness of Schubert varieties.

The next sections are devoted to the problem of computing the Néron-Severi Lie algebra of Schubert varieties. In §5.3 we translate this problem: we prove that the Néron-Severi Lie algebra is maximal, i.e. it is the Lie algebra of automorphisms of the (rescaled) intersection form, if the cohomology ring $H^\bullet(X_w, \mathbb{C})$ of a Schubert variety does not admit a tensor decomposition. In §5.4 we introduce a graph $\mathcal{I}_w$ associated to an element $w \in W$. We use the graph $\mathcal{I}_w$ to prove a sufficient condition: if the graph $\mathcal{I}_w$ has no sinks then $H^\bullet(X_w, \mathbb{C})$ is tensor-indecomposable. Finally §5.5, we restricts to the case of Schubert varieties of type A. In this case we have an explicit description of the coinvariant ring and we can exploit it to obtain a complete classification of the Néron-Severi Lie algebras.

5.1 Lefschetz modules

In this section we recall from [LL97] the definition and the main properties of the Néron-Severi Lie algebra.

Let $M = \bigoplus_{k \in \mathbb{Z}} M^k$ be a $\mathbb{Z}$-graded finite dimensional $\mathbb{R}$-vector space. We denote by $h : M \to M$ the map which is multiplication by k on M^k. Let $e : M \to M$ be a linear map of degree 2 (i.e. $e(M^k) \subseteq M^{k+2}$ for any $k \in \mathbb{Z}$). We say that e has the *Lefschetz property* if for any positive integer k, e^k gives an isomorphism between M^{-k} and M^k. The Lefschetz property implies the existence of a unique linear map $f : M \to M$, of degree -2, such that $\{e, h, f\}$ is a $\mathfrak{sl}_2$-triple, i.e. $\{e, h, f\}$ spans a Lie subalgebra of $\mathfrak{gl}(M)$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. We can explicitly construct f as follows: first we decompose $M = \bigoplus_{k \geq 0} \mathbb{R}[e](P_e^{-k})$ where

$P_e^{-k} = \mathrm{Ker}(e^{k+1}|_{M^{-k}})$, then we define, for $p_{-k} \in P_e^{-k}$,

$$f(e^i p_{-k}) = \begin{cases} i(k-i+1)e^{i-1}p_{-k} & \text{if } 0 < i \leq k, \\ 0 & \text{if } i = 0. \end{cases}$$

The uniqueness of f follows from [Bou68, Lemma 11.1.1. (VIII)]:

Lemma 5.1.1. *Let $\{e, h, f\}$ and $\{e, h, f'\}$ be two $\mathfrak{sl}_2$-triples. Then $f = f'$.*

Remark 5.1.2. From the construction of f, we also see that if e and h commute with an endomorphism $\varphi \in \mathfrak{gl}(M)$, then f also commutes with φ.

Lemma 5.1.3. *If h and e belong to a semisimple subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(M)$, then also $f \in \mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is semisimple, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{gl}(M)$ induces a splitting $\mathfrak{g} \oplus \mathfrak{a}$, with $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. If $f = f' + f''$ with $f' \in \mathfrak{g}$ and $f'' \in \mathfrak{a}$, then $\{e, h, f'\}$ is also an $\mathfrak{sl}_2$-triple. The uniqueness of f implies $f = f'$, thus $f \in \mathfrak{g}$. $\square$

Now let V be a finite dimensional $\mathbb{R}$-vector space. We regard it as a graded abelian Lie algebra homogeneous in degree 2 and we consider a graded Lie algebra homomorphism $\mathfrak{e} : V \to \mathfrak{gl}(M)$ (thus the image $\mathfrak{e}(V)$ consists of commuting linear maps of degree 2). We say that M is a *V-Lefschetz module* if there exists $v \in V$ such that $e_v := \mathfrak{e}(v)$ has the Lefschetz property. We denote by $V_{\mathcal{L}} \subseteq V$ the subset of elements satisfying the Lefschetz property. If $\mathfrak{e}$ is injective, and we can always assume so by replacing V with $\mathfrak{e}(V)$, then $V_{\mathcal{L}}$ is Zariski open in V. Thus, if $V_{\mathcal{L}} \neq \emptyset$ there exists a regular map $\mathfrak{f} : V_{\mathcal{L}} \to \mathfrak{gl}(M)$ such that $\{\mathfrak{e}(v), h, \mathfrak{f}(v)\}$ is a $\mathfrak{sl}_2$-triple.

Definition 5.1.4. Let M be a V-Lefschetz module. We define $\mathfrak{g}(V, M)$ to be the Lie subalgebra of $\mathfrak{gl}(M)$ generated by $\mathfrak{e}(V)$ and $\mathfrak{f}(V_{\mathcal{L}})$. We call $\mathfrak{g}(V, M)$ the *Néron-Severi Lie algebra* of the V-Lefschetz module M.

The following simple Lemma is needed in §5.3.2:

Lemma 5.1.5. *Let M be a V-Lefschetz module. Then $M \oplus M$ is also a V-Lefschetz module with respect to the diagonal action of V, and $\mathfrak{g}(V, M) \cong \mathfrak{g}(V, M \oplus M)$.*

Proof. For any $x \in \mathfrak{gl}(M)$ let $x \oplus x \in \mathfrak{gl}(M \oplus M)$ denote the endomorphism defined by $(x \oplus x)(\mu, \mu') = (x(\mu), x(\mu'))$ for all $\mu, \mu' \in M$.

An element $e \in \mathfrak{gl}(M)$ has the Lefschetz property on M if and only if $e \oplus e$ has the Lefschetz property on $M \oplus M$. Moreover if $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(M)$, then clearly $\{e \oplus e, h \oplus h, f \oplus f\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(M \oplus M)$. Therefore the algebra $\mathfrak{g}(V, M \oplus M)$ is generated by the elements $\mathfrak{e}(v) \oplus \mathfrak{e}(v)$, for $v \in V$, and by $\mathfrak{f}(v) \oplus \mathfrak{f}(v)$, for $v \in V_{\mathcal{L}}$. It follows that the map $x \mapsto x \oplus x$ induces an isomorphism $\mathfrak{g}(V, M) \cong \mathfrak{g}(V, M \oplus M)$. $\square$

5.1.1 Polarization of Lefschetz modules

Assume that M is evenly (resp. oddly) graded and let $\phi : M \times M \to \mathbb{R}$ be a non-degenerate symmetric (resp. antisymmetric) form such that $\phi(M^k, M^l) = 0$ unless $k \neq -l$.

We assume for simplicity $V \subseteq \mathfrak{gl}(M)$. We say that V *preserves* ϕ if every $v \in V$ leaves ϕ infinitesimally invariant, that is:

$$\phi(v(x), y) + \phi(x, v(y)) = 0 \quad \forall x, y \in M.$$

Since the Lie algebra $\mathfrak{aut}(M, \phi)$ of endomorphisms preserving ϕ is semisimple, if V preserves ϕ then we can apply Lemma 5.1.3 to deduce that $\mathfrak{g}(V, M) \subseteq \mathfrak{aut}(M, \phi)$.

For any operator $e : M \to M$ of degree 2 preserving ϕ we define a form $\langle -, - \rangle_e$ on M^{-k}, for $k \geq 0$, by $\langle m, m' \rangle_e = \phi(e^k m, m')$. One checks easily that $\langle -, - \rangle_e$ is symmetric.

We say that e is a *polarization* if the symmetric form $\langle -, - \rangle_e$ is definite on the primitive part $P_e^{-k} = \mathrm{Ker}(e^{k+1})|_{M^{-k}}$. If there exists a polarization $e \in V$, then we call (M, ϕ) a *polarized* V-Lefschetz module.

Remark 5.1.6. Each polarization e has the Lefschetz property. The injectivity of $e^k|_{M^{-k}}$ follows easily from the non-degeneracy of $\langle -, - \rangle_e$ on P^{-k}. From the non-degeneracy of ϕ we get $\dim M^{-k} = \dim M^k$ for any $k \geq 0$, hence $e^k|_{M^{-k}}$ is also surjective.

Proposition 5.1.7. *Let (M, ϕ) be a polarized V-Lefschetz module. Then the Lie algebra $\mathfrak{g}(V, M)$ is semisimple.*

Proof. Since $\mathfrak{g}(V, M)$ is generated by commutators, it is sufficient to prove it is reductive. This will be done by proving that the natural representation on M is completely reducible. Let $N \subseteq M$ be a $\mathfrak{g}(V, M)$-submodule. It suffices to show that the restriction of ϕ to N is non-degenerate, so that we can take the ϕ-orthogonal as a complement of N.

Let $e \in V$ be a polarization and let f be such that $\{e, h, f\}$ is a $\mathfrak{sl}_2$-triple. We can decompose N into irreducible $\mathfrak{sl}_2$-modules with respect to this triple. We obtain $N = \bigoplus_{k \geq 0} \mathbb{R}[e] P_{e,N}^{-k}$ where $P_{e,N}^{-k} = \mathrm{Ker}(e^{k+1}|_{N^{-k}})$. This decomposition is ϕ-orthogonal since, if $k > h$, we have
$$\phi(e^a p_{-k}, e^{\frac{k+h}{2} - a} p_{-h}) = (-1)^a \phi(p_{-k}, e^{\frac{k+h}{2}} p_{-h}) = 0$$
for any $p_{-k} \in P_e^{-k}$, $p_{-h} \in P_e^{-h}$ and any integer $a \geq 0$.

We consider now a single summand $\mathbb{R}[e] P_{e,N}^{-k}$. Because the form $\langle -, - \rangle_e$ is definite on $P_{e,N}^{-k} \subseteq P_e^{-k}$, it follows that ϕ is non-degenerate on $P_{e,N}^{-k} + e^k P_{e,N}^{-k}$. Since e preserves ϕ, the restriction of ϕ to $e^a P_{e,N}^{-k} + e^{k-a} P_{e,N}^{-k}$ is also non-degenerate for any $0 \leq a \leq k$. We conclude since the subspaces $e^a P_{e,N}^{-k} + e^{k-a} P_{e,N}^{-k}$ and $e^b P_{e,N}^{-k} + e^{k-b} P_{e,N}^{-k}$ are ϕ-orthogonal for $a \neq b, k - b$. □

Remark 5.1.8. The proof of Proposition 5.1.7 actually shows that the Lie algebra generated by V and $\mathfrak{f}(e)$, where e is a polarization, is semisimple. Therefore, by Lemma 5.1.3, if e is any polarization in V, then V and $\mathfrak{f}(e)$ generate $\mathfrak{g}(V, M)$.

Corollary 5.1.9. *Let (M, ϕ) be a polarized V-Lefschetz module. If $N \subseteq M$ is a graded V-submodule satisfying $\dim N^{-k} = \dim N^k$ for any $k \geq 0$, then there exists a complement $N' \subseteq M$ such that $M = N \oplus N'$ as a $\mathfrak{g}(V, M)$-module.*

Proof. We first show that N is a $\mathfrak{g}(V, M)$-module. For $v \in V_{\mathcal{L}}$ consider the primitive decomposition of M with respect to v:
$$M = \bigoplus_{k \geq 0} \mathbb{R}[v] P_v^{-k}$$
Let $P_{v,N}^{-k} = P_v^{-k} \cap N$ and $\widetilde{N} = \bigoplus_{k \geq 0} \mathbb{R}[v] P_{v,N}^{-k}$. Then $\widetilde{N}$ is a graded vector space contained in N with symmetric Betti numbers and such that v has the Lefschetz property on $\widetilde{N}$. We claim that $\widetilde{N} = N$.

Assume by contradiction $\widetilde{N} \neq N$ and let $-k$ be the minimal degree such that $\widetilde{N}^{-k} \neq N^{-k}$. Consider now $x \in N^{-k} \setminus \widetilde{N}^{-k}$. We have $v^{k+1}(x) \in \widetilde{N}$ because, by symmetry, $\widetilde{N}^{k+2} = N^{k+2}$. It follows that there exists $y \in \widetilde{N}^{-k}$ such that $v^{k+1}(x) = v^{k+1}(y)$, hence $x - y \in P_{v,N}^{-k}$, thus $x \in \widetilde{N}$. It follows that
$$N = \bigoplus_{k \geq 0} \mathbb{R}[v] P_{v,N}^{-k}$$

Now it is clear from the description of the map $\mathfrak{f}(v)$ given at the beginning of §5.1 that $\mathfrak{f}(v)$ preserves N, hence N is a $\mathfrak{g}(V, M)$ module.

Now, as in the proof of Proposition 5.1.7 one can show that the restriction of ϕ to N is non-degenerate, so the ϕ-orthogonal subspace N' is a $\mathfrak{g}(V, M)$-stable complement of N. □

Remark 5.1.10. Let (M, ϕ) be a polarized V-Lefschetz module. The complex vector space $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ acts on $M_{\mathbb{C}} := M \otimes_{\mathbb{R}} \mathbb{C}$. We can therefore define similarly $\mathfrak{g}_{NS}(V_{\mathbb{C}}, M_{\mathbb{C}})$ by taking the complex Lie algebra generated by $V_{\mathbb{C}}$ and $\mathfrak{f}((V_{\mathbb{C}})_{\mathcal{L}})$. Clearly we have

$$\mathfrak{g}_{NS}(V, M) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathfrak{g}_{NS}(V_{\mathbb{C}}, M_{\mathbb{C}}).$$

On the other hand $\mathfrak{g}_{NS}(V, M) \otimes \mathbb{C}$ is a semisimple complex Lie algebra, and since h and $(V_{\mathbb{C}})_{\mathcal{L}}$ lie inside $\mathfrak{g}_{NS}(V, M) \otimes \mathbb{C}$, by Lemma 5.1.3 we have:

$$\mathfrak{g}_{NS}(V, M) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{NS}(V_{\mathbb{C}}, M_{\mathbb{C}}).$$

Remark 5.1.11. The definitions given above arise naturally in the setting of complex projective (or compact Kähler) manifolds. Let Y be a complex projective manifold of complex dimension n and assume that Y is of Hodge-Tate type, i.e. if

$$H^{\bullet}(Y, \mathbb{C}) = \bigoplus_{p,q \geq 0} H^{p,q}$$

is the Hodge decomposition of Y then $H^{p,q} = 0$ for $p \neq q$. In particular, the cohomology of Y vanishes in odd degrees.

Let $M = H^{\bullet}(Y, \mathbb{R})[n]$ be the cohomology of Y shifted by n and let ϕ be the *rescaled intersection form*:

$$\phi(\alpha, \beta) = (-1)^{\frac{k(k-1)}{2}} \int_Y \alpha \wedge \beta, \qquad \forall \alpha \in H^{n+k}(Y, \mathbb{R}),\ \forall \beta \in H^{n-k}(Y, \mathbb{R}).$$

Notice that ϕ is symmetric (resp. antisymmetric) if n is even (resp. n is odd).

Let $\rho \in H^2(Y, \mathbb{R})$ be the first Chern class of an ample line bundle on Y. Then the Hard Lefschetz theorem and the Hodge-Riemann bilinear relations imply that ρ is a polarization of (M, ϕ). It follows that (M, ϕ) is a polarized Lefschetz module over $H^2(Y, \mathbb{R})$.

We can also replace $H^2(Y, \mathbb{R})$ by the *Néron-Severi group* $NS(Y)$, i.e. the subspace of $H^2(Y, \mathbb{R})$ generated by Chern classes of line bundles on Y. We define the *Néron-Severi Lie algebra* of Y as $\mathfrak{g}_{NS}(Y) = \mathfrak{g}(NS(Y), H^{\bullet}(Y, \mathbb{R})[n])$.

In [LL97] Looijenga and Lunts consider complex manifolds with an arbitrary Hodge structure. To deal with the general case one needs to modify the definition of polarization given here in order to make it compatible with the general form of the Hodge-Riemann bilinear relations.

However, all the Schubert varieties, the case in which we are mostly interested, are of Hodge-Tate type, so for simplicity we can limit ourselves to this case.

5.1.2 Lefschetz modules and weight filtrations

Let V be a finite dimensional $\mathbb{R}$-vector space and (M, ϕ) a polarized V-Lefschetz module. In this section we show how to each element $v \in V$ we can associate a *weight filtration* and to any such filtration we can associate a subalgebra of $\mathfrak{g}(V, M)$. In many situations the knowledge of these subalgebras turns out to be an important tool to study $\mathfrak{g}(V, M)$.

Lemma 5.1.12. *Let e be a nilpotent operator acting on a finite dimensional vector space M such that $e^l \neq 0$ and $e^{l+1} = 0$. Then there exists a unique non-increasing filtration W, called the weight filtration.*

$$\{0\} \subseteq W_l \subseteq W_{l-1} \subseteq \ldots \subseteq W_{-l+1} \subseteq W_{-l} = M$$

such that

- *$e(W_k) \subseteq W_{k+2}$ for all k;*
- *for any $0 \leq k \leq l$, $e^k : \mathrm{Gr}_W^{-k}(M) \to \mathrm{Gr}_W^k(M)$ is an isomorphism, where $\mathrm{Gr}_W^k(M) = W_k/W_{k+1}$.*

Proof. See for example [CEGT14, Proposition A.2.2]. □

Lemma 5.1.13. *Let $e \in V$ (not necessarily a Lefschetz operator). Then there exists a $\mathfrak{sl}_2$-triple $\{e, h', f'\}$ contained in $\mathfrak{g}(V, M)$ such that h' is of degree 0.*

Proof. This is [LL97, Lemma 5.2]. □

Let $\{e, h', f'\}$ be as is Lemma 5.1.13 and $W_\bullet$ be the weight filtration of e. Since h' is semisimple and part of a $\mathfrak{sl}_2$-triple, we have a decomposition in eigenspaces

$$M = \bigoplus_{n \in \mathbb{Z}} (M')^n \quad \text{where} \quad (M')^n = \{x \in M \mid h' \cdot x = nx\}.$$

We can define $\widetilde{W}_k = \bigoplus_{n \geq k} (M')^n$. It is easy to check that $\widetilde{W}_\bullet$ satisfies the defining condition of the weight filtration of e. In particular, $W_\bullet = \widetilde{W}_\bullet$ and h' splits the weight filtration of e, i.e. $W_k = W_{k+1} \oplus (M')^k$ for all k.

Let $h'' = h - h'$. Then (h', h'') is a commuting pair of semisimple elements in $\mathfrak{g}(V, M)$ and it defines a bigrading

$$M^{p,q} = \{m \in M \mid h' \cdot m = pm \text{ and } h'' \cdot m = qm\}$$

on M such that $M^n = \bigoplus_{p+q=n} M^{p,q}$. Furthermore h' and h'' also act via the adjoint representation on $\mathfrak{g}(V, M)$ defining a bigrading $\mathfrak{g}(V, M)^{p,q}$. We have

$$x \in \mathfrak{g}(V, M)^{p,q} \quad \text{if and only if} \quad x(M^{p',q'}) \subseteq M^{p+p',q+q'} \text{ for all } p', q' \in \mathbb{Z}.$$

For $x \in \mathfrak{g}(V, M)$ we denote by $x_{p,q}$ its component in $\mathfrak{g}(V, M)^{p,q}$.

Let $\widetilde{V}$ be a subspace of V containing e and such that, for any $x \in \widetilde{V}$, we have $x(W_k) \subseteq W_{k+2}$ for all k. Consider the graded vector space

$$\mathrm{Gr}_W M = \bigoplus_{k \in \mathbb{Z}} \mathrm{Gr}_W^k M$$

where $\mathrm{Gr}_W^k M$ sits in degree k. Then $\mathrm{Gr}_W M$ is a $\widetilde{V}$-Lefschetz module, so we can define the Lie algebra $\mathfrak{g}(\widetilde{V}, \mathrm{Gr}_W M)$.

Let $x \in \widetilde{V}$. Since $x(W_k) \subseteq W_{k+2}$, then $x((M')^k) \subseteq \bigoplus_{n \geq k+2} (M')^n$. This implies that $x \in \mathfrak{g}(V, M)^{\geq 2, \bullet}$, i.e. $x = x_{2,0} + x_{4,-2} + x_{6,-4} + \ldots$. In particular, if $x, y \in \widetilde{V}$, we have $[x, y] = 0$ and so $[x_{2,0}, y_{2,0}] = [x, y]_{4,0} = 0$.

Let $\widetilde{V}^{2,0} \subseteq \mathfrak{g}(V, M)$ be the span of the degree $(2, 0)$ components of elements of $\widetilde{V}$. The subspace $\widetilde{V}^{2,0}$ is an abelian subalgebra of $\mathfrak{g}(V, M)$. However, notice that in general $\widetilde{V}^{2,0}$ is not a subspace of V. We denote by M' the vector space M with the grading defined by h'. Then M' is a $\widetilde{V}^{2,0}$-Lefschetz module (in fact $e = e_{2,0}$ is a Lefschetz operator on M'), so we can define the algebra $\mathfrak{g}(\widetilde{V}^{2,0}, M')$.

Proposition 5.1.14. *In the setting as above, there exists an isomorphism of Lie algebras* $\mathfrak{g}(\widetilde{V}, \operatorname{Gr}_W M) \cong \mathfrak{g}(\widetilde{V}^{2,0}, M')$. *In particular,* $\mathfrak{g}(V, M)$ *contains a subalgebra isomorphic to* $\mathfrak{g}(\widetilde{V}, \operatorname{Gr}_W M)$.

Proof. Let $\pi_k : W_k \to (M')^k$ be the projection. Then $\bigoplus_k \pi_k : \operatorname{Gr}_W M \to M'$ is an isomorphism of graded vector spaces.

Moreover, the isomorphism $\bigoplus_k \pi_k$ is compatible with the map $\widetilde{V} \to \widetilde{V}^{2,0}$ given by $x \mapsto x_{2,0}$, i.e. for any $x \in \widetilde{V}$ and any $k \in \mathbb{Z}$ the following diagram commutes:

$$\begin{tikzcd} W_{k+2}/W_{k+3} \arrow[r, "\pi_{k+2}"] & (M')^{k+2} \\ W_k/W_{k+1} \arrow[u, "x"] \arrow[r, "\pi_k"] & (M')^k \arrow[u, "x_{2,0}"'] \end{tikzcd}$$

Hence, it follows that $\mathfrak{g}(\widetilde{V}, \operatorname{Gr}_W M) \cong \mathfrak{g}(\widetilde{V}^{2,0}, M')$.

The last statement follows from Lemma 5.1.3, in fact both $\widetilde{V}^{2,0}$ and h' are contained in $\mathfrak{g}(V, M)$, whence $\mathfrak{g}(\widetilde{V}^{2,0}, M') \subseteq \mathfrak{g}(V, M)$. $\square$

5.2 The Carrell-Peterson criterion for rational smoothness

Assume $\mathfrak{h}$ is a realization of Type I or II of a Coxeter group W. A remarkable property of (singular) Soergel modules is that they posses a Hodge theory even when they do not arise from a geometric setting. It follows from Theorem 4.2.3 that if $B^I \in \mathbb{S}Bim^I$ is indecomposable (or more generally, perverse) we can define the Néron-Severi Lie algebra of the singular Soergel module $\overline{B^I}$ as $\mathfrak{g}_{NS}((\mathfrak{h}^*_{\mathbb{R}})^I, \overline{B^I})$.

Let now B^I be indecomposable and $\langle -, - \rangle_{B^I}$ be the intersection form on B^I. For any $k \geq 0$ let

$$\phi(b, b') = (-1)^{\frac{k(k-1)}{2}} \langle b, b' \rangle_{\overline{B^I}} \qquad \text{for any } f \in (\overline{B^I})^k, g \in (\overline{B^I})^{-k}.$$

From Theorem 4.2.4 we see that $\overline{B^I}$ is polarized as a $(\mathfrak{h}^*_{\mathbb{R}})^I$-Lefschetz module with respect to ϕ, hence by Proposition 5.1.7 the Lie algebra $\mathfrak{g}_{NS}((\mathfrak{h}^*_{\mathbb{R}})^I, \overline{B^I})$ is semisimple.

In what follows we will only consider non-singular Soergel modules. For $w \in W$ we define

$$\mathfrak{g}_{NS}(w) := \mathfrak{g}(\mathfrak{h}^*_{\mathbb{R}}, \overline{B_w}).$$

Assume first that W is finite. Recall from Theorem 3.4.12 and Remark 3.4.15 that the modules $\overline{B_w}$ are indecomposable as R-modules. We can now easily apply Corollary 5.1.9 to the polarized $\mathfrak{h}^*_{\mathbb{R}}$-Lefschetz module $\overline{B_w}$.

Corollary 5.2.1. *Let W be a finite Coxeter group, Let N be a non-zero R-submodule of $\overline{B_w}$ such that* $\dim N^{-k} = \dim N^k$ *for any $k \in \mathbb{Z}$. Then $N \cong \overline{B_w}$.*

For a general Coxeter group W we have slightly weaker version of Corollary 5.2.1. Recall the definition of the ring $\overline{Z}$ from (3.7).

Lemma 5.2.2. *Let W be an arbitrary Coxeter group. Let N be a non-zero $\overline{Z}$-submodule of $\overline{B_w}$ such that* $\dim N^{-k} = \dim N^k$ *for any $k \in \mathbb{Z}$. Then $N = \overline{B_w}$.*

Proof. Fix an embedding $B_x \overset{\oplus}{\subseteq} BS(\underline{x})$ and let the intersection form $\langle -,-\rangle_{B_x}$ of B_x be the restriction of the intersection form of $BS(\underline{x})$. Because $\overline{BS(\underline{x})}$ is a commutative $\overline{Z}$-algebra, we have:

$$\langle z\cdot b, b'\rangle_{\overline{B_x}} = \mathrm{Tr}((z\cdot b)\cdot b') = \mathrm{Tr}(b\cdot(z\cdot b')) = \langle b, z\cdot b'\rangle_{\overline{B_x}}.$$

Hence

$$\phi(z\cdot b, b') = (-1)^q\phi(b, z\cdot b') \qquad \text{where } q = \frac{1}{2}\deg(z)(2\deg(b)+\deg(z)-1). \tag{5.1}$$

Because of Corollary 5.1.9 we can find an orthogonal N' of N with respect to ϕ. It follows from (5.1) that also the complement N' is a $\overline{Z}$-submodule of $\overline{B_w}$. □

If $w\in W$ and $s\in S$ such that $ws>w$, then $\overline{B_wB_s} = \overline{B_{ws}} \oplus \bigoplus_{z<ws}(\overline{B_z})^{\oplus m_z}$ for some $m_z\in\mathbb{Z}_{\geq 0}$. In particular, $\overline{B_wB_s}$ is a polarized $\mathfrak{h}^*_{\mathbb{R}}$-Lefschetz module.

Corollary 5.2.3. *Let N be a $\overline{Z}$-submodule of $\overline{B_wB_s}$ such that $\dim N^{-k} = \dim N^k$ for any $k\in\mathbb{Z}$. Then N is a direct summand of $\overline{B_wB_s}$. In particular, if N is indecomposable and $N^{-\ell(ws)}\neq 0$, then $N\cong\overline{B_{ws}}$.*

Recall from (3.5) that we have

$$\operatorname{grrk}\widetilde{H}_w = \sum_{x\leq w} v^{2\ell(x)-\ell(w)}. \tag{5.2}$$

The following result is originally due to Carrell-Peterson [Car94]:

Corollary 5.2.4. *For any $w\in W$ the following are equivalent:*

i) $\widetilde{H}_w = B_w$*;*

ii) $\overline{H_w} = \overline{B_w}$*;*

iii) $\#\{v\in W \mid v\leq w \text{ and } \ell(v)=k\} = \#\{v\in W\mid v\leq w \text{ and } \ell(v)=\ell(w)-k\}$ *for any* $k\in\mathbb{Z}$*;*

iv) All the Kazhdan-Lusztig polynomials $h_{x,w}$ are trivial, i.e. $h_{x,w}(v)=v^{\ell(w)-\ell(x)}$.

Proof. Since both $\widetilde{H}_w$ and B_w are graded free left right module i) and ii) are equivalent.

The cohomology submodule $\overline{H_w}$ is a $\overline{Z}$-submodule of the indecomposable $\overline{Z}$-module $\overline{B_w}$ and

$$\dim\overline{H_w}^k = \#\{v\in W\mid v\leq w \text{ and } 2\ell(v)=\ell(w)+k\}.$$

If $\dim\overline{H_w}^k = \dim\overline{H_w}^{-k}$ for any $0\leq k\leq\ell(w)$, from Corollary 5.2.1 we get that $\overline{H_w}$ and $\overline{B_w}$ must coincide, thus iii) implies ii). Vice versa, ii) implies iii) because $\overline{B_w}$ satisfies $\dim\overline{B_w}^{-k} = \dim\overline{B_w}^k$ for any $k\in\mathbb{Z}$.

We have $\dim\overline{B_w} = \sum_{x\leq w} h_{x,w}(1)$. Since KL polynomials have positive coefficients, because of (5.2) it follows that ii) is equivalent to iv). □

Remark 5.2.5. Let G be a simply-connected complex semisimple algebraic group, X its flag variety and W its Weyl group. Recall the notation from §2.1.1.

The cohomology $H^\bullet(X,\mathbb{R})$ of the flag variety is generated by the algebraic classes P_x, thus in particular in the Hodge decomposition of X only terms of type (p,p) appear.

We have seen that the map $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to H^2(X, \mathbb{R})$ is an isomorphism, so we have $NS(X) = H^2(X) = \mathfrak{h}^*_{\mathbb{R}} = R^2$ (in fact $(R^W_+)^2 = 0$).

Let $w \in W$ and let $IH_w := IH^\bullet(X_w, \mathbb{R})$ be the intersection cohomology of the Schubert variety $X_w = \overline{B \cdot wB} \overset{i}{\hookrightarrow} X$. Assume that $\mathfrak{h}$ is a realization of type II, so that we have $IH_w = \overline{B_w}$ and $H^\bullet(X_w, \mathbb{R})[\ell(w)] = \overline{H_w}$. Therefore

$$\mathfrak{g}_{NS}(X_w) = \mathfrak{g}(\mathfrak{h}^*, IH_w) = \mathfrak{g}(w).$$

If one of the equivalent condition of Corollary 5.2.4 holds for w, then the intersection cohomology sheaf $IC(X_w, \mathbb{R})$ is constant, that is $IC(X_w, \mathbb{R}) \cong \mathbb{R}[\ell(w)]$. In this case the variety X_w is said *rationally smooth.*

5.3 The Néron-Severi Lie algebra of Schubert varieties

In [LL97] Looijenga and Lunts determined the Néron-Severi Lie algebra $\mathfrak{g}_{NS}(X)$ of a flag variety $X = G/B$ of every simple group G: it is the complete algebra of automorphisms of the rescaled intersection form ϕ, i.e. it is a symplectic (resp. orthogonal) algebra if the complex dimension of X is odd (resp. even).

Here we want to extend their results and determine the Lie algebra $\mathfrak{g}_{NS}(w) := \mathfrak{g}_{NS}(X_w)$ for an arbitrary $w \in W$. We restrict to the case of W finite Weyl group. We describe a criterion on the element w for the Lie algebra $\mathfrak{g}_{NS}(w)$ to be "as large as possible". This criterion holds for the majority of the elements w.

5.3.1 Basic properties of the Schubert basis

Let $\{P_v\}_{v\in W}$ be the Schubert basis of $H^\bullet(X, \mathbb{R})$ introduced in Section 2.1.1. The R-module structure of $H^\bullet(X, \mathbb{R})$ can be described in the basis $\{P_v\}_{v\in W}$ by the Chevalley formula (3.6) (or [BGG73, Theorem 3.14]). For any $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ we have:

$$P_w \cdot \lambda = \sum_{w \xrightarrow[R]{t} v} \partial_t(\lambda) P_v \tag{5.3}$$

where the notation $w \xrightarrow[R]{t} v$ means $\ell(v) = \ell(w) + 1$, $v = wt$ and $t \in \mathcal{T}$.

In particular, if $s \in S$ then $P_s \in H^2(X, \mathbb{R}) = \mathfrak{h}^*_{\mathbb{R}}$ can be identified with the fundamental weight corresponding to α_s, i.e. we have $\partial_t(P_s) = \delta_{s,t}$ for any $s, t \in S$. The following Lemma is an easy application of the Chevalley formula (5.3):

Lemma 5.3.1. *In $H^\bullet(X, \mathbb{R})$ we have, for any $s, t \in S$:*

i) $P_s^2 = -\sum_{u \in S \setminus \{s\}} \partial_u(\alpha_s) P_{us}$*;*

ii) $P_s P_t = P_{st}$ *if* $\partial_t(\alpha_s) = 0$*;*

iii) $P_s P_t = P_{st} + P_{ts}$ *if* $\partial_t(\alpha_s) \neq 0$ *(or equivalently* $\partial_s(\alpha_t) \neq 0$*) and* $s \neq t$*.*

Proof. We show i). If $r \in \mathcal{T}$ is a reflection such that $s \xrightarrow[R]{r} sr$, then $\ell(sr) = 2$, thus $\ell(r) = 1$ and $sr > r$ or $\ell(r) = 3$ and $sr < r$. But if $\ell(r) = 1$ then $\partial_r(P_s) = 0$, so we can assume $\ell(r) = 3$. If $\ell(r) = 3$ and $sr < r$ then $r = sus$ with $u \in S$ [Spr82a, Proposition 1]. Now $\partial_r(P_s) = -\partial_u(P_s)$.

The proof of ii) and iii) is similar. ☐

We state here for later reference a preliminary lemma:

Lemma 5.3.2. *If the root system Φ is irreducible (i.e. if the Dynkin diagram of G is connected) then $(R_+^W)^4 \cong \mathbb{R}$ and it is spanned by*

$$\mathcal{X} = \sum_{s,t\in S} c_{st} P_s P_t \qquad \textit{with} \qquad c_{st} = \frac{(\alpha_s, \alpha_t)}{(\alpha_s,\alpha_s)(\alpha_t,\alpha_t)},$$

where $(-,-)$ is the killing form on $\mathfrak{h}_\mathbb{R}^*$.

Proof. A W-invariant element in $\mathfrak{h}_\mathbb{R}^* \otimes \mathfrak{h}_\mathbb{R}^*$ corresponds to a W-equivariant morphism $\mathfrak{h}_\mathbb{R} \to \mathfrak{h}_\mathbb{R}^*$. Since $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$ are irreducible as W-modules, such a morphism is unique up to a scalar. The Killing form $(-,-)$ is W-invariant, hence $\eta \mapsto (\eta, -)$ is the W-equivariant isomorphism $\mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}$. The element in $\mathfrak{h}_\mathbb{R}^* \otimes \mathfrak{h}_\mathbb{R}^*$ the corresponds to the map $(P_s, -) \mapsto P_s$ is

$$2\mathcal{X} := \sum_s P_s \otimes \frac{2}{(\alpha_s,\alpha_s)}\alpha_s = \sum_{s,t} \frac{2\partial_t(\alpha_s)}{(\alpha_s,\alpha_s)} P_s \otimes P_t.$$

Notice that for any $s,t \in S$ we have

$$\frac{\partial_t(\alpha_s)}{(\alpha_s,\alpha_s)} = \frac{(\alpha_s,\alpha_t)}{(\alpha_s,\alpha_s)(\alpha_t,\alpha_t)} = \frac{\partial_s(\alpha_t)}{(\alpha_t,\alpha_t)},$$

hence $\mathcal{X} \in \mathrm{Sym}^2(\mathfrak{h}_\mathbb{R}^*)^W \subseteq (\mathfrak{h}_\mathbb{R}^* \otimes \mathfrak{h}_\mathbb{R}^*)^W$. □

Remark 5.3.3. The element $\mathcal{X}$ is basically (up to a scalar) just the Killing form written in the basis $\{P_s P_t\}_{s,t\in S}$ of $\mathrm{Sym}^2(\mathfrak{h}_\mathbb{R}^*)$. Assume now we have a proper decomposition $\mathfrak{h}_\mathbb{R}^* = \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*$. This induces a decomposition

$$\mathrm{Sym}^2(\mathfrak{h}_\mathbb{R}^*) = \mathrm{Sym}^2(\mathfrak{h}_1^*) \oplus (\mathfrak{h}_1^* \otimes \mathfrak{h}_2^*) \oplus \mathrm{Sym}^2(\mathfrak{h}_2^*).$$

Since the Killing form is non-degenerate on $\mathfrak{h}_\mathbb{R}^*$ we deduce that $\mathcal{X}$ is not contained in $\mathrm{Sym}^2(\mathfrak{h}_1^*)$, otherwise the restriction of $\mathcal{X}$ to $(\mathfrak{h}_1^*)^\perp$ would be 0.

For a simple reflection $u \in S$ let $\mathbf{P}_u := \mathbf{P}_{\{u\}}$ be the minimal parabolic subgroup of G containing u. For any element $w \in W$ such that $wu < w$ we can choose a reduced expression $\underline{w} = st \dots u$. The projection $\pi : G/B \to G/\mathbf{P}_u$ is a $\mathbb{P}^1$-fibration which restricts to a $\mathbb{P}^1$-fibration on X_w since $\overline{BwB} \cdot \mathbf{P}_u = \overline{BwB}$. The image $\pi(X_w) = X_w^u$ is the parabolic Schubert variety of the element w in $G/\mathbf{P}_u$. Let $IH_w^u := IH^\bullet(X_w^u, \mathbb{R})$. Then IH_w^u is a polarized Lefschetz module over $(\mathfrak{h}_\mathbb{R}^*)^u \cong NS(G/\mathbf{P}_u)$, so we can define the Lie algebra $\mathfrak{g}_{NS}(X_w^u) := \mathfrak{g}((\mathfrak{h}_\mathbb{R}^*)^u, IH_w^u)$.

5.3.2 A distinguished subalgebra of $\mathfrak{g}_{NS}(w)$

Let $id \neq w \in W$ and u be a simple reflection such that $wu < w$. Let $\pi : G/B \to G/\mathbf{P}_u$ be the projection as above. We denote by $IC_w := IC(X_w, \mathbb{R})$ (resp. $IC_w^u := IC(X_w^u, \mathbb{R})$) the intersection cohomology sheaf of the variety X_w (resp. X_w^u). Then $R\pi_*(IC_w) \cong IC_w^u[1] \oplus IC_w^u[-1]$ (not canonically) by the Decomposition Theorem (the use of the Decomposition Theorem here can be avoided using an argument of Soergel [Soe00, Lemma 3.3.2]). In particular, as graded vector spaces, we have $IH_w \cong IH_w^u \otimes H^\bullet(\mathbb{P}^1(\mathbb{C}), \mathbb{R})[1]$.

Lemma 5.3.4. *Let $u \in S$ be such that $wu < w$. Then the Lie algebra $\mathfrak{g}_{NS}(w)$ contains a Lie subalgebra isomorphic to $\mathfrak{g}_{NS}(X_w^u)$.*

Proof. Let $\eta \in H^2(X_w^u)$ be the Chern class of an ample line bundle on X_w^u. We can apply Lemma 5.1.13 to find a $\mathfrak{sl}_2$-triple $\{\pi^*\eta, h', f'\}$ inside $\mathfrak{g}_{NS}(w)$ such that h' is of degree 0, i.e. $h'(IH_w^k) \subseteq IH_w^k$ for all k.

Any choice of a decomposition $R\pi_*(IC_w) \cong IC_w^u[1] \oplus IC_w^u[-1]$ induces a splitting $IH_w = IH_w^u[1] \oplus IH_w^u[-1]$ of R^u-modules. One can easily check that weight filtration of the nilpotent element $\pi^*\eta$ is $W_k = (IH_w^u[1])^{k-1} \oplus \bigoplus_{n \geq k} IH_w^n$. Therefore for any $x \in (\mathfrak{h}_{\mathbb{R}}^*)^u$ we have $x(W_k) \subseteq W_{k+2}$.

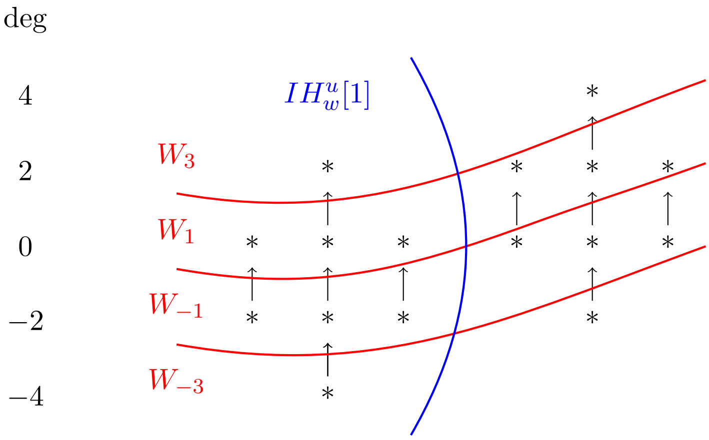
We can now apply Proposition 5.1.14, with $\widetilde{V} = (\mathfrak{h}_{\mathbb{R}}^*)^u$, in order to obtain

$$\mathfrak{g}((\mathfrak{h}_{\mathbb{R}}^*)^u, \mathrm{Gr}_W(IH_w)) \cong \mathfrak{g}(((\mathfrak{h}_{\mathbb{R}}^*)^u)^{2,0}, IH'_w),$$

where IH'_w denotes the vector space IH_w with the grading determined by h'. In particular, $\mathfrak{g}((\mathfrak{h}_{\mathbb{R}}^*)^u, \mathrm{Gr}_W(IH_w))$ is a subalgebra of $\mathfrak{g}_{NS}(w)$.

It is easy to see that $\mathrm{Gr}_W(IH_w) \cong IH_w^u \oplus IH_w^u$ as graded vector spaces, and the isomorphism is compatible with the action of R^u. We conclude using Lemma 5.1.5 that $\mathfrak{g}((\mathfrak{h}_{\mathbb{R}}^*)^u, \mathrm{Gr}_W(IH_w)) \cong \mathfrak{g}_{NS}(X_w^u)$. $\square$

Example 5.3.5. Let $G = SL_4(\mathbb{C})$ so that $W = \mathcal{S}_4$ is the symmetric group on 4 elements, with simple reflections labeled s_1, s_2, s_3. Let $w = s_2 s_1 s_3 s_2$ and $u = s_2$. Let η be an ample Chern class on X_w^u. Then we can draw the action of $\pi^*\eta$ on a basis of IH_w and the weight filtration $W_\bullet$ as follows



We fix η and h' as in Lemma 5.3.4 and let $h'' = h - h'$. Then, as in Section 5.1.2, h' and h'' define a bigrading on IH_w and on $\mathfrak{g}_{NS}(w)$.

Notice that the only eigenvalues of h'' on IH_w are 1 and -1. It follows that $\mathfrak{g}_{NS}(w)$ decomposes as $\mathfrak{g}_{NS}(w) = \mathfrak{g}_{NS}(w)^{\bullet,-2} \oplus \mathfrak{g}_{NS}(w)^{\bullet,0} \oplus \mathfrak{g}_{NS}(w)^{\bullet,2}$. In particular, any element ρ of $\mathfrak{h}_{\mathbb{R}}^*$ can be decomposed as $\rho = \rho_{4,-2} + \rho_{2,0} + \rho_{0,2}$. Moreover, for $\widetilde{\eta} \in (\mathfrak{h}_{\mathbb{R}}^*)^u$ we have $\widetilde{\eta}(W_k) \subseteq W_{k+2}$, hence $\widetilde{\eta} \in \mathfrak{g}_{NS}(w)^{\geq 2,\bullet}$ and $\widetilde{\eta} = \widetilde{\eta}_{4,-2} + \widetilde{\eta}_{2,0}$.

We can now restate and reprove [LL97, Proposition 5.6] in our setting:

Theorem 5.3.6. *If $wu < w$ the Néron-Severi Lie algebra $\mathfrak{g}_{NS}(w)$ contains a Lie subalgebra isomorphic to $\mathfrak{g}_{NS}(X_w^u) \times \mathfrak{sl}_2$.*

Proof. Let ρ be the Chern class of an ample line bundle on X_w. Then by the Relative Hard Lefschetz Theorem [BBD82, Theorem 5.4.10] cupping with ρ induces an isomorphism of R^u-modules:

$$IH_w^u[1] \cong {}^pH^{-1}(R\pi_* IC_w) \xrightarrow{\rho} {}^pH^1(R\pi_* IC_w) \cong IH_w^u[-1].$$

This means that the $(0,2)$-component $\rho_{0,2} \in \mathfrak{g}_{NS}(w)^{0,2}$ of ρ (thus we have $[h', \rho_{0,2}] = 0$ and $[h'', \rho_{0,2}] = 2\rho_{0,2}$) has the Lefschetz property with respect to the grading given by h''. In particular, because of Lemma 5.1.3, we can complete it to an $\mathfrak{sl}_2$-triple $\{\rho_{0,2}, h'', f''_\rho\} \subseteq \mathfrak{g}_{NS}(w)$. The span of $\{\rho_{0,2}, h'', f''_\rho\}$ is a subalgebra of $\mathfrak{g}_{NS}(w)^{0,\bullet}$. In fact, since both $\rho_{0,2}$ and h'' commute with h' so does f''_ρ (see Remark 5.1.2).

Recall from Lemma 5.3.4 that the algebra $\mathfrak{g}_{NS}(X^u_w)$ is isomorphic to $\mathfrak{g}(((\mathfrak{h}^*_\mathbb{R})^u)^{2,0}, IH'_w)$, which in turn is a subalgebra of $\mathfrak{g}_{NS}(w)$. It remains to show that the two subalgebras $\mathfrak{g}(((\mathfrak{h}^*_\mathbb{R})^u)^{2,0}, IH'_w)$ and $\mathrm{span}\{\rho_{0,2}, h'', f''_\rho\} \cong \mathfrak{sl}_2(\mathbb{R})$ intersect trivially and mutually commute.

Since ρ commutes with $\widetilde{\eta}$ for any $\widetilde{\eta} \in (\mathfrak{h}^*_\mathbb{R})^u$, then also $\rho_{0,2}$ commutes with $\widetilde{\eta}_{2,0}$: in fact since $\rho = \rho_{4,-2} + \rho_{2,0} + \rho_{0,2}$ and $\widetilde{\eta} = \widetilde{\eta}_{4,-2} + \widetilde{\eta}_{2,0}$, we have $[\rho_{0,2}, \widetilde{\eta}_{2,0}] = [\rho, \widetilde{\eta}]_{2,2} = 0$. Because $((\mathfrak{h}^*_\mathbb{R})^u)^{2,0}$ and h' commute with $\rho_{0,2}$, so does $\mathfrak{g}(((\mathfrak{h}^*_\mathbb{R})^u)^{2,0}, IH'_w)$. Because $\rho_{0,2}$ and h'' commute with $\mathfrak{g}(((\mathfrak{h}^*_\mathbb{R})^u)^{2,0}, IH'_w)$, so does f''_ρ. We obtain a morphism of Lie algebras

$$\mathfrak{I} : \mathfrak{g}_{NS}(X^u_w) \times \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{g}(((\mathfrak{h}^*_\mathbb{R})^u)^{2,0}, IH'_w) \times \mathrm{span}\{\rho_{0,2}, h'', f''_\rho\} \to \mathfrak{g}_{NS}(w)$$

given by the multiplication. The kernel of $\mathfrak{I}$ is $\mathfrak{g}_{NS}(X^u_w) \cap \mathfrak{sl}_2(\mathbb{R})$ and it is contained in the center of $\mathfrak{sl}_2(\mathbb{R})$, which is trivial. The thesis now follows. □

5.3.3 Irreducibility of the subalgebra and consequences

The goal of the first part of this section is to show the following:

Proposition 5.3.7. *IH^u_w is irreducible as a $\mathfrak{g}_{NS}(X^u_w)$-module.*

We begin with a preparatory lemma:

Lemma 5.3.8. *The cohomology $H^\bullet(G/\mathbf{P}_u, \mathbb{R})$ is generated as an algebra by the first Chern classes, i.e. by $H^2(G/\mathbf{P}_u, \mathbb{R})$.*

Proof. We can identify $H^\bullet(G/\mathbf{P}_u, \mathbb{R})$ with $R^u/(R^W_+)$. The subalgebra R^u is generated by P_s, with $s \in S \setminus \{u\}$, and α_u^2. It is enough to show that $\mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u) \to H^4(G/\mathbf{P}_u, \mathbb{R})$ is surjective, because all the generators of $H^\bullet(G/\mathbf{P}_u, \mathbb{R})$ lie in degrees ≤ 4.

The set $\{P_s\}_{s \in S \setminus \{u\}}$ forms a basis of $H^2(G/\mathbf{P}_u, \mathbb{R}) = NS(G/\mathbf{P}_u) = (\mathfrak{h}^*_\mathbb{R})^u$. We have

$$\dim(R^4)^u = \dim \mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u) + 1.$$

Recall from Lemma 5.3.2 that $(R^W_+)^4 = \mathbb{R}\mathcal{X}$, hence

$$\dim H^4(G/\mathbf{P}_u, \mathbb{R}) = \dim(R^4)^u/(\mathbb{R}\mathcal{X}) = \dim \mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u).$$

So it suffices to show that $\mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u) \to H^4(G/\mathbf{P}_u, \mathbb{R})$ is injective, or in other words that

$$\mathrm{Ker}(\mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u) \to H^4(G/\mathbf{P}_u, \mathbb{R})) = \mathbb{R}\mathcal{X} \cap \mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u) = 0.$$

But since the Killing form is non-degenerate and $(\mathfrak{h}^*_\mathbb{R})^u$ is a proper subspace of $\mathfrak{h}^*_\mathbb{R}$, we have $\mathcal{X} \notin \mathrm{Sym}^2((\mathfrak{h}^*_\mathbb{R})^u)$ (as explained in Remark 5.3.3). □

Proof of Proposition 5.3.7. Since $\mathfrak{g}_{NS}(X^u_w)$ is semisimple, it is enough to show that IH^u_w is an indecomposable $\mathfrak{g}_{NS}(X^u_w)$-module. In particular, it is enough to show that it is indecomposable as a $(\mathfrak{h}^*_\mathbb{R})^u$-module (here regarded as an abelian Lie subalgebra of $\mathfrak{g}_{NS}(X^u_w)$).

The Erweiterungssatz (in the version proved by Ginzburg [Gin91]) states that taking the hypercohomology (as a module over the cohomology of the partial flag variety) is a fully faithful functor on IC complexes of Schubert varieties. In particular, for any $w \in W$, we have:

$$\mathrm{End}_{H^\bullet(G/\mathbf{P}_u,\mathbb{R})\text{-Mod}}(IH_w^u) \cong \mathrm{End}_{D^b(G/\mathbf{P}_u)}(IC(X_w^u,\mathbb{R})).$$

This implies, since $IC(X_w^u,\mathbb{R})$ is a simple perverse sheaf on $G/\mathbf{P}_u$, that IH_w^u is an indecomposable $H^\bullet(G/\mathbf{P}_u,\mathbb{R})$-module. Now Lemma 5.3.8 completes the proof. □

Remark 5.3.9. Proposition 5.3.7 is not true for a general parabolic flag variety. Let $G = SL_4(\mathbb{C})$ so that $W = \mathcal{S}_4$ is the symmetric group on 4 elements, with simple reflections labeled s, t, u. Then $SL_4(\mathbb{C})/\mathbf{P}_{\{s,u\}}$ is isomorphic to $Gr(2,4)$, the Grassmannian of 2-dimensional subspaces in $\mathbb{C}^4$. Since $\dim H^2(Gr(2,4),\mathbb{R}) = 1$ we have $\mathfrak{g}_{NS}(Gr(2,4)) \cong \mathfrak{sl}_2(\mathbb{R})$, but $\dim H^4(Gr(2,4),\mathbb{R}) = 2$ so it cannot be irreducible as a $\mathfrak{g}_{NS}(Gr(2,4))$-module. In fact, $H^\bullet(Gr(2,4),\mathbb{R})$ is not generated by $H^2(Gr(2,4),\mathbb{R})$.

Proposition 5.3.10. *If $\mathfrak{g}_{NS}^{\mathbb{C}}(w) := \mathfrak{g}_{NS}(w) \otimes \mathbb{C}$ is a simple complex Lie algebra, then we have $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w, \phi)$.*

In particular, this implies that the complexification $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ is isomorphic to $\mathfrak{sp}_d(\mathbb{C})$ if $\ell(w)$ is odd, and is isomorphic to $\mathfrak{so}_d(\mathbb{C})$ if $\ell(w)$ is even, with $d = \dim IH_w$.

Proof. Proposition 5.3.7 shows that the Lie algebra $\mathfrak{g}_{NS}(X_w^u) \times \mathfrak{sl}_2(\mathbb{R})$ acts irreducibly on

$$IH_w \cong IH_w^u \otimes_{\mathbb{R}} H^\bullet(\mathbb{P}^1(\mathbb{C}),\mathbb{R})[1].$$

This obviously remains true when one considers, after complexification, the action of $\mathfrak{g}_{NS}^{\mathbb{C}}(X_w^u) \times \mathfrak{sl}_2(\mathbb{C})$ on $IH^\bullet(X_w,\mathbb{C})$.

In [Dyn52, Theorem 2.3], Dynkin classified all the pairs $\mathfrak{g} \subseteq \mathfrak{g}'$ ($\subseteq \mathfrak{gl}(V)$) of complex Lie algebras such that $\mathfrak{g}$ acts irreducibly on a finite dimensional complex vector space V and $\mathfrak{g}'$ is simple. From this classification we see that if $\mathfrak{g} = \widetilde{\mathfrak{g}} \times \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ acts with highest weight 1 then $\mathfrak{g}'$ is one of $\mathfrak{sl}_N$, $\mathfrak{so}_N$ and $\mathfrak{sp}_N$.

We apply now this result to the pair $\mathfrak{g}_{NS}^{\mathbb{C}}(X_w^u) \times \mathfrak{sl}_2(\mathbb{C}) \subseteq \mathfrak{g}_{NS}^{\mathbb{C}}(w)$. Clearly we cannot have $\mathfrak{g}_{NS}^{\mathbb{C}}(w) \cong \mathfrak{sl}(IH^\bullet(X_w,\mathbb{C}))$ since $\mathfrak{g}_{NS}(w) \subseteq \mathfrak{aut}(IH^\bullet(X_w,\mathbb{C}),\phi)$. This implies $\mathfrak{g}_{NS}^{\mathbb{C}}(w) = \mathfrak{aut}(IH^\bullet(X_w,\mathbb{C}),\phi)$, hence $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w,\phi)$. □

Remark 5.3.11. We now discuss which real forms of the symplectic and orthogonal groups occur as $\mathfrak{aut}(IH_w,\phi)$. If $\ell(w)$ is odd there is, up to isomorphism, only one symplectic form on IH_w, hence $\mathfrak{aut}(IH_w,\phi) \cong \mathfrak{sp}_{\dim(IH_w)}(\mathbb{R})$.

If $\ell(w)$ is even we can determine the signature of the symmetric form ϕ on IH_w. If $k > 0$ then ϕ is a non-degenerate pairing between IH_w^k and IH_w^{-k}, hence the signature of $\phi|_{IH_w^k \oplus IH_w^{-k}}$ is $(\dim IH_w^k, \dim IH_w^k)$. The signature of ϕ on IH_w^0 is determined by the Hodge-Riemann bilinear relations: the dimension of the positive part of $\phi|_{IH_w^0}$ is given by

$$\sum_{i=0}^{\lfloor l(w)/4 \rfloor} \dim P^{-\ell(w)+4i} = \sum_{i=0}^{\lfloor l(w)/4 \rfloor} \left(\dim IH_w^{\ell(w)-4i} - \dim IH_w^{\ell(w)-4i+2} \right).$$

5.4 Tensor decomposition of intersection cohomology

We now want to understand for which $w \in W$ the Lie algebra $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ is not simple. The complex Lie algebra $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ acts naturally on $IH^\bullet(X_w,\mathbb{C})$. Recall from Remark 5.1.10 that

$$\mathfrak{g}_{NS}^{\mathbb{C}}(w) \cong \mathfrak{g}(\mathfrak{h}^*, IH^\bullet(X_w,\mathbb{C})).$$

To simplify the notation from now on we will use IH_w to denote $IH^\bullet(X_w,\mathbb{C})$ and H_w to denote $H^\bullet(X_w,\mathbb{C})$[1]. They are both modules over $R = \mathrm{Sym}_{\mathbb{C}}(\mathfrak{h}^*)$.

For any $w \in W$ we have $H_w[\ell(w)] \subseteq IH_w$ (see Remark 3.2.1). In particular, H^2_w acts faithfully on IH_w and we can regard H^2_w as a subspace of $\mathfrak{g}_{NS}(w)$. We recall the following lemma from [LL97, Lemma 1.2]:

Lemma 5.4.1. *Assume there exists a non-trivial decomposition $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$ and consider $\pi_i : \mathfrak{g}^{\mathbb{C}}_{NS}(w) \to \mathfrak{g}_i$ the projections. Then the decomposition is graded and it also induces a decomposition into graded vector spaces $IH_w = IH^{\bullet,0}_w \otimes_{\mathbb{C}} IH^{0,\bullet}_w$ where $IH^{\bullet,0}_w$ (resp. $IH^{0,\bullet}_w$) is an irreducible $\pi_1(H^2_w)$-Lefschetz module (resp. $\pi_2(H^2_w)$-Lefschetz module) with $\mathfrak{g}_1 = \mathfrak{g}(\pi_1(H^2_w), IH^{\bullet,0}_w)$ and $\mathfrak{g}_2 = \mathfrak{g}(\pi_2(H^2_w), IH^{0,\bullet}_w)$.*

For the rest of this chapter we assume that we have a splitting of Lie algebras $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$ and we denote by $\pi_1 : \mathfrak{g}^{\mathbb{C}}_{NS}(w) \to \mathfrak{g}_1$ and $\pi_2 : \mathfrak{g}^{\mathbb{C}}_{NS}(w) \to \mathfrak{g}_2$ the projections. Let $IH_w = IH^{\bullet,0}_w \otimes_{\mathbb{C}} IH^{0,\bullet}_w$ be the induced decomposition.

There exist integers $a, b \geq 0$ such that $IH^{\bullet,0}_w$ (resp. $IH^{0,\bullet}_w$) are not trivial only in degrees between $-a$ and a (resp. between $-b$ and b) with $a, b \geq 0$ and $a + b = \ell(w)$. Moreover both $IH^{-a,0}_w$ and $IH^{0,-b}_w$ are one-dimensional. We define a bigrading on IH_w by $IH^{i,j}_w := IH^{i,0}_w \otimes IH^{0,j}_w$.

5.4.1 Splitting of H^2_w

We can assume from now on $H^2_w = H^2(G/B,\mathbb{C})$. In fact, we can replace G by its Levi subgroup corresponding to the smallest parabolic subgroup of G containing w. This does not change the Schubert variety X_w, the cohomology H_w and the Lie algebra $\mathfrak{g}_{NS}(w)$. In particular, we have $R = \mathrm{Sym}(H^2_w)$.

In general $H_w[\ell(w)] \neq IH_w$, so it is not clear a priori that a tensor decomposition for IH_w descends to one for H_w. Still, this holds in our setting:

Proposition 5.4.2. *Assume we have a decomposition $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$. Then $H^2_w = \pi_1(H^2_w) \oplus \pi_2(H^2_w)$.*

Proof. It is enough to show that $\dim H^2_w \geq \dim \pi_1(H^2_w) + \dim \pi_2(H^2_w)$. We define

$$T := \mathrm{Sym}(\pi_1(H^2_w)) \otimes \mathrm{Sym}(\pi_2(H^2_w)) \cong \mathrm{Sym}(\pi_1(H^2_w) \oplus \pi_2(H^2_w)).$$

We can define a T-module structure on IH_w via $(x \otimes y)(a) = x(a) \otimes y(a)$ for any $x \in \pi_1(H^2_w)$, $y \in \pi_2(H^2_w)$ and $a \in IH_w$.

We have a bigrading $T^{p,q} := \mathrm{Sym}^p(\pi_1(H^2_w)) \otimes \mathrm{Sym}^q(\pi_2(H^2_w))$ on T compatible with the bigrading of IH_w, i.e. $T^{p,q}(IH^{i,j}_w) \subseteq IH^{p+i,q+j}_w$.

The subspace $T^{2,0} \cong \pi_1(H^2_w) \subseteq \mathfrak{g}_1$ acts faithfully on $IH^{\bullet,0}_w$, while $T^{0,2} \cong \pi_2(H^2_w) \subseteq \mathfrak{g}_2$ acts faithfully on $IH^{0,\bullet}_w$. Hence $T^{2,2} \subseteq \mathfrak{g}_1 \otimes \mathfrak{g}_2 \subseteq \mathfrak{gl}(IH^{\bullet,0}_w) \otimes \mathfrak{gl}(IH^{0,\bullet}_w) = \mathfrak{gl}(IH_w)$ acts faithfully on IH_w, i.e. if $t \in T^{2,2}$ acts as 0 on IH_w, then $t = 0$.

Let $\Psi : R \hookrightarrow T$ the inclusion induced by $\Psi(x) = \pi_1(x) + \pi_2(x)$ for any $x \in \mathfrak{h}^*$. We observe that the T-module structure on IH_w extends the R-module structure via Ψ.

We can decompose $P_s = L_s + R_s$ where $L_s = \pi_1(P_s) \in \mathfrak{g}_1$ and $R_s = \pi_2(P_s) \in \mathfrak{g}_2$ for all $s \in S$. Now we consider the element $\mathcal{X} \in (R^4)^W$ defined in Lemma 5.3.2. The R-module structure on IH_w factorizes through $H^\bullet(X,\mathbb{C}) = R/(R^W_+)$, therefore $\Psi(\mathcal{X}) \in T$ acts as 0 on IH_w. In particular, also the component $\Psi(\mathcal{X})^{2,2} \in T^{2,2}$ acts as 0 on IH_w. Since the

[1] The notation here may lead to some confusion: the cohomology H_w is a graded algebra and it is non zero only in non-negative degrees. If $\overline{H_w} = \mathbb{R} \otimes_R \widetilde{H}_w$ we have $H_w = \overline{H_w}[-\ell(w)]$.

action is faithful on $T^{2,2}$ we obtain $\Psi(\mathcal{X})^{2,2} = \sum_{s,t\in S} c_{st}(L_s \otimes R_t + L_t \otimes R_s) = 0 \in T^{2,2}$. Since c_{st} is symmetric we can rewrite it as follows:

$$\sum_{s,t\in S} L_s \otimes c_{st} R_t = 0 \in \pi_1(H^2_w) \otimes \pi_2(H^2_w) \subseteq \mathfrak{g}_1 \otimes \mathfrak{g}_2.$$

Let $S_L \subseteq S$ be such that $\{L_s\}_{s\in S_L}$ is a basis of $\pi_1(H^2_w)$. We can write $L_u = \sum_{s\in S_L} x_{su} L_s$ with $x_{su} \in \mathbb{R}$ for any $u \in S \setminus S_L$. We get

$$\sum_{\substack{s\in S_L\\ t\in S}} L_s \otimes \left(c_{st} + \sum_{u\in S\setminus S_L} x_{su} c_{ut} \right) R_t = 0 \implies \sum_{t\in S} \left(c_{st} + \sum_{u\in S\setminus S_L} x_{su} c_{ut} \right) R_t = 0$$

for any $s \in S_L$. Since $(c_{st})_{s,t\in S}$ is a non-degenerate matrix, it follows that we have $\#(S_L)$ linearly independent equations vanishing on $(R_s)_{s\in S}$, hence $\dim \pi_2(H^2_w) \leq \dim H^2_w - \#(S_L) = \dim H^2_w - \dim \pi_1(H^2_w)$. ☐

In the setting of the previous Proposition, it also follows that $\Psi : R \to T$ is an isomorphism, so we have a bigrading on R compatible with the bigrading on IH_w. Hence $H_w[\ell(w)]$ is also bigraded as a subspace of IH_w, since it is the image of the map of bigraded vector spaces map $R[\ell(w)] \to IH_w$ defined by $x \mapsto x \cdot 1^{\otimes}_w$, where $1^{\otimes}_w$ is any non-zero element in the one dimensional space $IH_w^{-\ell(w)}$.

So we can write

$$H_w^{\bullet,\bullet} = H_w^{\bullet,0} \otimes H_w^{0,\bullet}.$$

We call this a *tensor decomposition* of H_w. It is non trivial if both $H_w^{\bullet,0}$ and $H_w^{0,\bullet}$ are not one dimensional. Note that also the kernel of $R[\ell(w)] \to IH_w$ is bigraded.

Corollary 5.4.3. *If $\mathfrak{g}^{\mathbb{C}}_{NS}(w)$ is not simple, i.e. it admits a non trivial decomposition $\mathfrak{g}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$, then H_w admits a non-trivial decomposition $H_w^{\bullet,\bullet} = H_w^{\bullet,0} \otimes H_w^{0,\bullet}$ as graded algebra.*

Conversely, if H_w does not admit any non-trivial tensor decomposition then $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w, \phi)$.

Proof. The last statement follows from Proposition 5.3.10. ☐

In the next sections we provide a sufficient condition for the Lie algebra $\mathfrak{g}_{NS}(w)$ to be maximal. However, there is a case where the proof is considerably easier and we provide it here for convenience and to motivate the reader.

Recall that for any $w \in W$, the set $\{P_{st}\}_{st\leq w}$ is a basis of H^4_w. In particular, if $st \leq w$ for any $s, t \in S$, we have $H^4_w \cong H^4(G/B, \mathbb{C})$. In this case from Lemma 5.3.2 we have also $\mathrm{Ker}(R^4 \to H^4_w) = (R^W_+)^4 = \mathbb{R}\mathcal{X}$.

Corollary 5.4.4. *Assume that the root system of G is irreducible and suppose that whenever $s_i, s_j \leq w$ then $s_i s_j \leq w$. Then $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w, \phi)$.*

Proof. We assume for contradiction that we have a non-trivial tensor decomposition of H_w, so H^4_w splits as $H^{4,0}_w \oplus H^{2,2}_w \oplus H^{0,4}_w$. This implies that also $K := \mathrm{Ker}(R^4 \to H^4_w)$ splits as $K = K^{4,0} \oplus K^{2,2} \oplus K^{0,4}$ where $K^{i,j} = \mathrm{Ker}(R^{i,j} \to H^{i,j}_w)$. But K is one dimensional and generated by $\mathcal{X}$, thus $\mathcal{X}$ belongs to either $R^{4,0}$, $R^{2,2}$ or $R^{0,4}$, which is impossible since $\mathcal{X}$ is non-degenerate (see Remark 5.3.3).

Now we apply Corollary 5.4.3 to deduce that $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w, \phi)$. ☐

5.4.2 A directed graph associated to an element

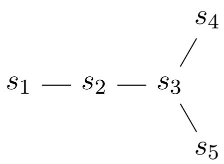
For $w \in W$ we construct an oriented graph $\mathcal{I}_w$ as follows: the vertices are indexed by the set of simple reflections S and we put an arrow $s \to t$ if $ts \leq w$ and $ts \neq st$ (i.e. if $ts \leq w$ and s and t are connected in the Dynkin diagram).

Recall that we assumed, by shrinking to a Levi subgroup, that $s \leq w$ for any $s \in S$. It follows that for any pair $s, t \in S$ we have either $st \leq w$, $ts \leq w$ or both. Hence the graph $\mathcal{I}_w$ is just the Dynkin diagram where each edge $s - t$ is replaced by the arrow $s \leftarrow t$, by the arrow $s \to t$, or by both $s \rightleftarrows t$. In particular, if the Dynkin diagram is connected, then also $\mathcal{I}_w$ is connected. In this case we say that w is *connected*.

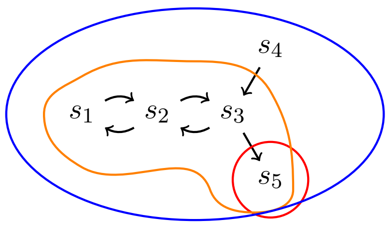
Remark 5.4.5. Since the Dynkin diagram has no loops, then also $\mathcal{I}_w$ has no non-oriented loops (we only consider loops in which for any pair $s, t \in S$ at most one of the arrows $s \to t$ and $t \to s$ occurs).

Definition 5.4.6. We call a subset $C \subseteq S$ *closed* if any arrow in $\mathcal{I}_w$ starting in C ends in C. Union and intersection of closed subsets are still closed. We call a closed singleton in S a *sink*.

Example 5.4.7. Let W be the Coxeter group of D_5. We label the simple reflections as follows:



Consider the element $w = s_1 s_2 s_5 s_3 s_4 s_2 s_1$. Then the graph $\mathcal{I}_w$ associated to w is:



Here the coloured lines describe all the non-empty closed subsets of $\mathcal{I}_w$.

As we show in Lemma 5.4.9, the graph $\mathcal{I}_w$ determines H_w^4 as a quotient of $\mathrm{Sym}^2(H_w^2)$, and we can make use of it to provide obstructions for the algebra $\mathfrak{g}_{NS}(w)$ to admit a non-trivial decomposition, that is to find sufficient conditions for the algebra $\mathfrak{g}_{NS}(w)$ to be simple. Namely, we prove in Theorem 5.4.15 that, if $\mathcal{I}_w$ is connected and has no sinks, then $\mathfrak{g}_{NS}(w)$ is maximal.

5.4.3 Reduction to the connected case

If w is not connected, we can write $w = w_1 w_2$, with $\ell(w) = \ell(w_1) + \ell(w_2)$ such that $s_1 s_2 = s_2 s_1$ for any $s_1 \leq w_1, s_2 \leq w_2$.

Proposition 5.4.8. *If $w = w_1 w_2$ as above, then we have decompositions $IH_w \cong IH_{w_1} \otimes_{\mathbb{C}} IH_{w_2}$ and $\mathfrak{g}_{NS}(w) \cong \mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$.*

Proof. In this case $X_w \cong X_{w_1} \times X_{w_2}$, so $IH_w = IH_{w_1} \otimes IH_{w_2}$. Moreover $H_w^2 = H_{w_1}^2 \oplus H_{w_2}^2$ where H_{w_1} acts on the factor IH_{w_1} while H_{w_2} acts on IH_{w_2}. Since the Lie algebra $\mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$ is semisimple and both h and H_w^2 are contained in $\mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$, from Lemma 5.1.3 we have $\mathfrak{g}_{NS}(w) = \mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$. ☐

5.4.4 The connected case

In view of Proposition 5.4.8 we can restrict ourselves to the case of a connected w.

Lemma 5.4.9. *Let w be connected and let $K = \mathrm{Ker}(\mathrm{Sym}^2(H^2_w) \to H^4_w)$. Then the elements $\mathcal{X}_C := \sum_{s,t\in C} c_{st}P_sP_t$, with C closed, generate K.*

Proof. We know that $\dim K = \#\{(s,t) \in S^2 \mid st \not\leq w\} + 1$ because $\mathrm{Sym}^2(H^2_w) \to H^4_w$ is surjective. Since w is connected, if $st \not\leq w$ then s and t are connected by an edge in the Dynkin diagram and $ts \leq w$.

Let (a,b) be any pair of elements of S such that $ba \leq w$ and $ab \not\leq w$, i.e. such that in $\mathcal{I}_w$ there is an arrow $a \to b$ but not an arrow $b \to a$. We can define a proper closed subset C_{ab} by taking the connected component of b in $\mathcal{I}_w$ after erasing the arrow $a \to b$. Since there are no loops in $\mathcal{I}_w$ we have $a \notin C_{ab}$. It is easy to see that $\mathcal{X}_{C_{ab}}$ together with $\mathcal{X} = \mathcal{X}_S$ are linearly independent in $\mathrm{Sym}^2(H^2_w)$: in fact when we write them in the basis $\{P_sP_t\}_{s,t\in S}$ we have $\mathcal{X}_{C_{ab}} \in c_{bb}P_b^2 + \mathcal{R}_{ab}$, where

$$\mathcal{R}_{ab} = \mathrm{span}\langle P_sP_t \mid (s,t) \neq (a,a),(b,b)\rangle,$$

while all the other $\mathcal{X}_{C_{a'b'}}$ are either in $\mathcal{R}_{ab}$ or in $c_{aa}P_a^2 + c_{bb}P_b^2 + \mathcal{R}_{ab}$. Therefore when we quotient to $\mathrm{Sym}^2(H^2_w)/\mathcal{R}_{ab}$, the term $\mathcal{X}_{C_{ab}}$ is the only one which is not proportional to the image of $c_{aa}P_a^2 + c_{bb}P_b^2$.

By the formula for the dimension of K given above, it remains to show that all the $\mathcal{X}_C$, for C closed, lie in K. Let $\overline{y}$ denote the projection of an element $y \in \mathrm{Sym}^2(H^2_w)$ to $H^4(G/B,\mathbb{C})$. Let C be a closed subset and let

$$E := \{a(i) \xrightarrow{i} b(i) \mid a(i) \notin C \text{ and } b(i) \in C\}$$

be the set of arrows starting outside C and ending in C. Applying Lemma 5.3.1, on one hand we obtain:

$$\overline{\mathcal{X}_C} = \sum_{s,t\in C} c_{st}\overline{P_sP_t} \in \mathrm{span}\langle P_{st} \mid s,t\in C\rangle \oplus \mathrm{span}\langle P_{a(i)b(i)} \mid i \in E\rangle \subseteq H^4(G/B,\mathbb{C}). \quad (5.4)$$

On the other hand we have

$$\mathcal{X} - \mathcal{X}_C = \sum_{s,t\notin C} c_{st}P_sP_t + \sum_{i\in E} 2c_{a(i)b(i)}P_{a(i)}P_{b(i)} \in \mathrm{Sym}^2(H^2_w).$$

Since $\overline{\mathcal{X}} = 0$ in $H^4(G/B,\mathbb{C})$, projecting from R^4 to $H^4(G/B,\mathbb{C})$ we obtain

$$\overline{\mathcal{X}_C} \in \mathrm{span}\langle P_{st} \mid s,t \notin C\rangle \oplus \mathrm{span}\langle P_{a(i)b(i)} \mid i\in E\rangle \oplus \mathrm{span}\langle P_{b(i)a(i)} \mid i \in E\rangle. \quad (5.5)$$

Then (5.4) together with (5.5) implies that the projection $\overline{\mathcal{X}_C}$ of $\mathcal{X}_C$ to $H^4(G/B,\mathbb{C})$ lies in $\mathrm{span}\langle P_{a(i)b(i)} \mid i \in E\rangle$. But, for any $i \in E$, $P_{a(i)b(i)}$ projects to 0 in H^4_w since $a(i)b(i) \not\leq w$, whence $\mathcal{X}_C \in K$. ☐

For a closed C let $NS(C) := \mathrm{span}\langle P_s \mid s \in C\rangle \subseteq H^2_w$. The proof of Proposition 5.4.2 applies also to $NS(C)$ if we replace $\mathcal{X}$ by $\mathcal{X}_C = \sum_{s,t\in C} c_{st}P_sP_t$. This means that whenever we have a decomposition $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$, then $NS(C)$ splits compatibly.

Remark 5.4.10. The element $\mathcal{X}_C \in \mathrm{Sym}^2(\mathfrak{h}^*)$ should be thought as the restriction of the Killing form on $\mathrm{span}\langle \alpha_s \mid s \in C\rangle$. This is non denegerate, so it means that $\mathcal{X}_C \notin \mathrm{Sym}^2(V)$ for any proper subspace $V \subsetneq \mathrm{span}\langle\alpha_s \mid s \in C\rangle$ (cf. Remark 5.3.3).

Lemma 5.4.11. *Let $K_C := K \cap \mathrm{Sym}^2(NS(C))$. Then K_C is generated by $\mathcal{X}_D$, with D closed and $D \subseteq C$.*

Proof. Assume that $\sum_i a_i \mathcal{X}_{D_i} \in K \cap \mathrm{Sym}^2(NS(C))$ with D_i closed and $a_i \in \mathbb{C}$. Then it is easy to see that $\sum_i a_i \mathcal{X}_{D_i} = \sum_i a_i \mathcal{X}_{D_i \cap C} \in \mathrm{Sym}^2(NS(C))$. □

For any $s \in S$, let $L_s = \pi_1(P_s) \in \mathfrak{g}_1$ and $R_s = \pi_2(P_s) \in \mathfrak{g}_2$.

Lemma 5.4.12. *Let C be a connected and closed subset of S. Assume that there exists a non-empty closed subset $D \subseteq C$ such that $NS(D) = \pi_1(NS(C))$. Then if D does not contain any sink we have $D = C$.*

Proof. Let $U = C \setminus D$ and $E := \{a(i) \xrightarrow{i} b(i) \mid a(i) \in U \text{ and } b(i) \in D\}$ be the set of arrows starting in U and ending in D. The set $\{P_s\}_{s \in D} = \{L_s\}_{s \in D}$ is a basis of $NS(D) = \pi_1(NS(C))$, therefore the set $\{R_u\}_{u \in U}$ is a basis of $\pi_2(NS(C))$. We assume for contradiction that $U \neq \emptyset$. By writing the $(2,2)$-component of $\mathcal{X}_C - \mathcal{X}_D$ we obtain

$$\sum_{u \in U} \left(\sum_{s \in C} c_{su} L_s \right) \otimes R_u = 0 \in \mathfrak{g}_1 \otimes \mathfrak{g}_2$$

from which we get $\sum_{s \in C} c_{su} L_s = 0$ for any $u \in U$. Let $\widetilde{U}$ be a connected component of U and let

$$\widetilde{E} = \{a(i) \xrightarrow{i} b(i) \mid a(i) \in \widetilde{U} \text{ and } b(i) \in D\} \subseteq E.$$

Since C is connected we have $\widetilde{E} \neq \emptyset$. Since $\widetilde{U}$ is connected and there are no loops in the Dynkin diagram, we have $b(i) \neq b(j)$ for any $i \neq j \in \widetilde{E}$, and moreover there are no arrows between $b(i)$ and $b(j)$. Then for any $u \in \widetilde{U}$ we have

$$0 = \sum_{s \in C} c_{su} L_s = \sum_{s \in \widetilde{U}} c_{su} L_s + \sum_{i \in \widetilde{E}} c_{b(i)u} L_{b(i)}. \tag{5.6}$$

Since the set $\{L_{b(i)}\}_{i \in \widetilde{E}}$ is linearly independent, this can be thought as a non-degenerate system of linear equations in L_s, with $s \in \widetilde{U}$ and it has a unique solution

$$L_s = \sum_{i \in \widetilde{E}} y(s,i) L_{b(i)} = \sum_{i \in \widetilde{E}} y(s,i) P_{b(i)} \quad \text{with } y(s,i) \in \mathbb{R}.$$

Substituting L_s in (5.6) we get

$$\sum_{s \in \widetilde{U}} y(s,i) c_{su} = \begin{cases} 0 & \text{if } u \neq a(i), \\ -c_{a(i)b(i)} & \text{if } u = a(i), \end{cases} \qquad \text{for all } u \in \widetilde{U} \text{ and } i \in \widetilde{E}. \tag{5.7}$$

Claim 5.4.13. We have $y(s,i) > 0$ for any $s \in \widetilde{U}$ and any $i \in \widetilde{E}$.

Proof of the claim. Let $(-,-)$ be the Killing form on $\mathfrak{h}^*$. From Equation (5.7) it is easy to see that

$$\left(\sum_{s \in \widetilde{U}} \frac{y(s,i)}{(\alpha_s, \alpha_s)} \alpha_s, \alpha_u \right) = -\delta_{a(i),u} c_{a(i)b(i)} (\alpha_u, \alpha_u) \qquad \forall u \in \widetilde{U}, \forall i \in \widetilde{E}.$$

Hence $\sum_{s \in \widetilde{U}} \frac{y(s,i)}{(\alpha_s, \alpha_s)} \alpha_s$ is (up to a positive scalar) equal to the fundamental weight of $a(i)$ in the root system generated by the simple roots in $\widetilde{U}$. Now the claim follows from the fact that in any irreducible root system all the dominant weights have only positive coefficients when expressed in the basis of simple roots.

In fact, let $0 \neq \lambda = \sum_{s\in\widetilde{U}} \lambda_s\alpha_s$ and assume $(\lambda, \alpha_s) \geq 0$ for all $s \in \widetilde{U}$. If $\lambda_s < 0$ for some s, then $(\lambda_s, \alpha_s) < 0$. Thus $\lambda_s \geq 0$ for all s. Assume now $\lambda_s = 0$ for some s. Then $(\lambda, \alpha_s) \geq 0$ only if $\lambda_t = 0$ for all $t \in S$ neighboring s in the Dynkin diagram. Since $\widetilde{U}$ is connected we obtain $\lambda_s = 0$ for all s, hence $\lambda = 0$ which is a contradiction. □

For any $s \in \widetilde{U}$ we have $R_s = P_s - \sum_{i\in\widetilde{E}} y(s,i)P_{b(i)} \in \mathfrak{g}_2$. Now consider the element

$$R^{0,4} \ni \sum_{s,t\in\widetilde{U}} c_{st}R_sR_t = \sum_{s,t\in\widetilde{U}} c_{st}\left(P_s - \sum_{i\in\widetilde{E}} y(s,i)P_{b(i)}\right)\left(P_t - \sum_{i\in\widetilde{E}} y(t,i)P_{b(i)}\right) =$$

$$= \left(\sum_{s,t\in\widetilde{U}} c_{st}P_sP_t\right) - 2\sum_{i\in\widetilde{E}}\left(\sum_{s,t\in\widetilde{U}} y(s,i)c_{st}P_t\right)P_{b(i)} + \sum_{i,j\in\widetilde{E}}\left(\sum_{s,t\in\widetilde{U}} y(s,i)y(t,j)c_{st}\right)P_{b(i)}P_{b(j)}$$

$$= \left(\sum_{s,t\in\widetilde{U}} c_{st}P_sP_t\right) + 2\sum_{i\in\widetilde{E}} c_{a(i)b(i)}P_{a(i)}P_{b(i)} - \sum_{i,j\in\widetilde{E}} y(a(j),i)c_{a(j)b(j)}P_{b(i)}P_{b(j)} =$$

$$= \mathcal{X}_{D\cup\widetilde{U}} - \mathcal{X}_D + \Theta, \qquad \text{where } \Theta := -\sum_{i,j\in\widetilde{E}} y(a(j),i)c_{a(j)b(j)}P_{b(i)}P_{b(j)}.$$

Let $p : R^4 \to H^4_w$ denote the projection. The previous equation implies that

$$p\left(\sum_{s,t\in\widetilde{U}} c_{st}R_sR_t\right) = p(\Theta).$$

But $p(\sum_{s,t\in\widetilde{U}} c_{st}R_sR_t) \in H^{0,4}_w$ while $p(\Theta) \in H^{4,0}_w$, because $b(i) \in D$ and $P_{b(i)} \in H^{2,0}_w$ for any $i \in \widetilde{E}$. It follows that $p(\Theta) \in H^{4,0}_w \cap H^{0,4}_w = \{0\}$.

We can write $\Theta = \Theta_1 + \Theta_2$ with

$$\Theta_1 = \sum_{\substack{i,j\in\widetilde{E}\\ i\neq j}} y(a(j),i)c_{a(j)b(j)}P_{b(i)}P_{b(j)} \quad \text{and} \quad \Theta_2 = \sum_{i\in\widetilde{E}} y(a(i),i)c_{a(i)b(i)}P^2_{b(i)}.$$

Since there are no edges between $b(i)$ and $b(j)$, we have that $p(P_{b(i)}P_{b(j)}) = P_{b(i)b(j)}$ for any $i, j \in \widetilde{E}$ such that $i \neq j$. Thus, by Lemma 5.3.1, we have

$$p(\Theta_1) = \sum_{\substack{i,j\in\widetilde{E}\\ i\neq j}} y(a(j),i)c_{a(j)b(j)}P_{b(i)b(j)}$$

$$p(\Theta_2) = -2\sum_{i\in\widetilde{E}} y(a(i),i)c_{a(i)b(i)}\left(\sum_{j\in E_i}\frac{(\alpha_{b(i)}, \alpha_{\beta_i(j)})}{(\alpha_{\beta_i(j)}, \alpha_{\beta_i(j)})}P_{\beta_i(j)b(i)}\right)$$

where $E_i = \{b(i) \xrightarrow{j} \beta_i(j)\}$ is the set of arrows in $\mathcal{I}_w$ starting in $b(i)$. It is easy to see that all the terms in $p(\Theta_1)$ and $p(\Theta_2)$ are linearly independent, whence $p(\Theta_1) + p(\Theta_2) = 0$ if and only if all their terms vanish. Recall that $y(a(i),i)c_{a(i)b(i)} < 0$ for all $i \in \widetilde{E}$. Hence $p(\Theta_1) + p(\Theta_2) = 0$ forces $E_i = \emptyset$ for any $i \in \widetilde{E}$. But this is a contradiction because there are no sinks in D, whence $U = \emptyset$ and $C = D$. □

Lemma 5.4.14. *Let C be a closed and connected subset of S. Assume that there are no sinks in C. Then $NS(C) \subseteq \mathfrak{g}_1$ or $NS(C) \subseteq \mathfrak{g}_2$.*

Proof. We work by induction on the number of vertices in C. There is nothing to prove if $C = \emptyset$.

Let $D \subsetneq C$ be a maximal proper closed subset. The kernel $K_C := K \cap \operatorname{Sym}^2(NS(C))$ is generated by $\mathcal{X}_C$ and $\mathcal{X}_{D'}$ with $D' \subseteq D$. In fact, if $\widetilde{D} \subsetneq C$ is a proper closed subset and $\widetilde{D} \not\subseteq D$, then by maximality $\widetilde{D} \cup D = C$ and $\mathcal{X}_{\widetilde{D}} = \mathcal{X}_C - \mathcal{X}_D + \mathcal{X}_{D \cap \widetilde{D}}$. In particular, we have $\dim K_C = \dim K_D + 1$.

By induction on the number of vertices we can subdivide D into two subsets D_L and D_R, each consisting of the union of connected components of D, such that $NS(D_L) \subseteq \mathfrak{g}_1$ and $NS(D_R) \subseteq \mathfrak{g}_2$.

Since $NS(C)$ splits, then K_C also splits as $K_C^{4,0} \oplus K_C^{2,2} \oplus K_C^{0,4}$ where $K_C^{i,j} = K_C \cap R^{i,j}$. However, $K_C^{2,2} \subseteq K^{2,2} = 0$ since $R^{2,0} \otimes R^{0,2}$ is mapped isomorphically to $H_w^{2,2}$. Using $\dim K_C = \dim K_D + 1$ we get $K_C^{4,0} = K_D^{4,0}$ or $K_C^{0,4} = K_D^{0,4}$. Without loss of generality we can assume $K_C^{4,0} = K_D^{4,0} = K_{D_L}$.

This implies that $\mathcal{X}_C \in K_C^{4,0} \oplus K_C^{0,4} = K_{D_L} \oplus K_C^{0,4}$. It follows that

$$\mathcal{X}_C \in \operatorname{Sym}^2\left(NS(D_L) \oplus \pi_2(NS(C))\right).$$

Since $\mathcal{X}_C$ is non-degenerate on $NS(C)$, we get $NS(D_L) = \pi_1(NS(C))$. Now we can apply Lemma 5.4.12: if $D_L \neq \emptyset$, then $D_L = C$, otherwise $\pi_1(NS(C)) = 0$ and $NS(C) \subseteq \mathfrak{g}_2$. $\square$

Theorem 5.4.15. *For $w \in W$, if the graph $\mathcal{I}_w$ is connected and without sinks, then $\mathfrak{g}_{NS}(w) = \mathfrak{aut}(IH_w, \phi)$.*

Proof. Applying Lemma 5.4.14 to $C = S$ we see that any decomposition of $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ must be trivial, hence by Proposition 5.3.10 we get $\mathfrak{g}_{NS}(w) = \mathfrak{aut}(IH_w, \phi)$. $\square$

Example 5.4.16. It is in general false that $\mathfrak{g}_{NS}(w)$ is simple for any connected w.

Let W be the Weyl group of type A_3 (i.e. $W = S_4$) where $S = \{s, t, u\}$. We consider the element $usts \in W$ whose graph $\mathcal{I}_{usts}$ is

$$\begin{tikzcd} s \arrow[r, bend left] & t \arrow[l, bend left] \arrow[r] & u \end{tikzcd}$$

The closed subsets in $\mathcal{I}_{usts}$ are S, $\{u\}$ and $\emptyset$. Then $\mathfrak{g}_{NS}(usts) \cong \mathfrak{g}_{NS}(u) \times \mathfrak{g}_{NS}(sts) \cong \mathfrak{sp}_2(\mathbb{R}) \times \mathfrak{sp}_6(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sp}_6(\mathbb{R})$. The splitting induced on H_w^2 is

$$H_w^2 = \pi_1(H_w^2) \oplus \pi_2(H_w^2) = \mathbb{C}P_u \oplus \left(\mathbb{C}(P_t - \frac{2}{3}P_u) + \mathbb{C}(P_s - \frac{1}{3}P_u)\right).$$

As we explain in the next section, we have a similar behavior more generally: for any $w \in S_{n+1}$, with $S = \{s_1, \ldots, s_n\}$, such that $w = s_1 w'$ where w' is the longest element in $W_{\{s_2,\ldots,s_n\}}$ the Lie algebra $\mathfrak{g}_{NS}(w)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{g}_{NS}(w')$.

Example 5.4.17. The following example demonstrates that having no sinks in $\mathcal{I}_w$ is not a necessary condition for the algebra $\mathfrak{g}_{NS}(w)$ to be simple.

Let W be the Weyl group of type B_3, where we label the simple reflections as follows:

$$s \;\text{——}\; t \;=\!=\; u$$

Then for $w_1 = usts$ we get again $\mathfrak{g}_{NS}(w_1) \cong \mathfrak{g}_{NS}(u) \times \mathfrak{g}_{NS}(sts) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sp}_6(\mathbb{R})$, but for $w_2 = stut$ the Lie algebra $\mathfrak{g}_{NS}(w_2)$ is simple (hence it is isomorphic to $\mathfrak{so}_{6,6}(\mathbb{R})$). Notice that the graphs $\mathcal{I}_{w_1}$ and $\mathcal{I}_{w_2}$ are isomorphic.

Remark 5.4.18. We have chosen to restrict ourselves to the case of finite Weyl groups in order to be able to state the results using only "classical" Schubert calculus. However, the results given in this section work in the same way for any irreducible finite Coxeter groups using a realization of Type I, i.e. the geometric representation. Note that this includes the groups H_3 and H_4. We briefly explain how.

We replace everywhere the intersection cohomology of Schubert variety IH_w by the indecomposable Soergel modules $\overline{B_w}$ and the Killing form by the positive definite symmetric form B defined in [Hum78, §5.2]. Assume u is a simple reflection such that $wu < u$. Because of Chapter 4 we can define the Néron-Severi Lie algebra $\mathfrak{g}_{NS}(\overline{B^u_{wu}})$ of the singular Soergel module $\overline{B^u_{wu}}$. This Lie algebra is semisimple and its action on $\overline{B^u_{wu}}$ is irreducible.

We need an argument to replace the recourse to the relative hard Lefschetz in the proof of Theorem 5.3.6. We have

$$B^u_{wu} \otimes_{R^u} R[1] \cong B_w,$$

therefore any decomposition $R \cong R^u \oplus R^u[-2]$ as R^u-modules induces a decomposition

$$B_w \cong B^u_{wu}[1] \oplus B^u_{wu}[-1]$$

of (R, R^u) bimodules. We choose this decomposition as in the proof of [EW14, Theorem 6.19] (cf. Theorem 4.5.4). With respect to this decomposition multiplication by ρ induces the map

$$\partial_u(\rho) : B^u_{wu}[1] \to B^u_{wu}[-1]$$

which is clearly an isomorphism if ρ is ample.

The rest of arguments go through using the Schubert basis from Chapter 3. We obtain thus the same criterion: if w is connected and there are not sinks in the graph $\mathcal{I}_w$ then $\mathfrak{g}_{NS}(w)$ is maximal, i.e. it coincides with $\mathfrak{aut}(\overline{B_w}, \phi)$.

For infinite Coxeter groups W our methods do not apply directly. In fact, in general a reflection faithful representation of W is not irreducible, thus Lemma 5.3.2 does not hold and the kernel of the map $R \to \overline{B_w}$ seems harder to compute.

5.5 The complete classification in type A

Theorem 5.4.15 gives a sufficient condition for an element w to have a maximal Néron-Severi Lie algebra $\mathfrak{g}_{NS}(w)$. In the following, we specialize to groups of type A_n. In this case we can go further and explicitly compute the Néron-Severi Lie algebras $\mathfrak{g}_{NS}(w)$ of any element w.

We assume that $W = \mathcal{S}_{n+1}$ is the symmetric group on $n+1$-elements. Let $S = \{s_1, s_2, \dots, s_n\}$. We write P_i for P_{s_i}, and similarly L_i and R_i. We assume that $w \in W$ is not contained in any proper parabolic subgroup $W_I \subseteq W$, so that we can also label the vertices of $\mathcal{I}_w$ by $\{1, 2, \dots n\}$.

We indicate by $[a, b]$ the interval $\{a, a+1, \dots, b\}$. We rescale the Killing form so that $(\alpha, \alpha) = 1$ for any root α, so that we get:

$$c_{ij} = \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{2} & \text{if } |i - j| = 1 \\ 0 & \text{if } |i - j| \geq 2 \end{cases},$$

therefore

$$\mathcal{X} = \sum_{i=1}^{n} P_i^2 - \sum_{i=1}^{n-1} P_i P_{i+1}.$$

Since we are going to apply several times Claim 5.4.13, it is useful to recall that in a root system of type A_n the fundamental weight corresponding to the first simple root is

$$\varpi_1 = \frac{1}{n+1}\left(n\alpha_1 + (n-1)\alpha_2 + \ldots + \alpha_n\right).$$

Recall from Corollary 5.4.3 that it is sufficient to show that H_w does not admit any non-trivial tensor decomposition to show $\mathfrak{g}_{NS}(w) = \mathfrak{aut}(IH_w, \phi)$. So we assume that we have a non-trivial decomposition:

$$H_w^\bullet = H_w^{\bullet,0} \otimes H_w^{0,\bullet}.$$

In view of Theorem 5.4.15 we can assume that $\mathcal{I}_w$ has at least one sink. If s is a sink then $P_s^2 = 0$ and the component of P_s^2 in $H_w^{2,2}$ is $2L_s \otimes R_s$. This can be 0 only if $L_s = 0$ or $R_s = 0$.

The proofs of the next two Lemmas may look rather tedious, and concern some case by case inspections. However, they ultimately rely on the main ideas of §5.4.

Lemma 5.5.1. *Let $C = [a,b]$ be an interval closed in $\mathcal{I}_w$ such that a is the only sink in C. Then one of the following holds:*

1) $NS(C) \subseteq H_w^{2,0}$,

2) $NS(C) \subseteq H_w^{0,2}$,

3) $\pi_1(NS(C)) = \mathbb{C}P_a$ *and* $R_i = P_i - \dfrac{b+1-i}{b-a+1}P_a$ *for all* $i \in [a+1,b]$,

4) $\pi_2(NS(C)) = \mathbb{C}P_a$ *and* $L_i = P_i - \dfrac{b+1-i}{b-a+1}P_a$ *for all* $i \in [a+1,b]$.

Proof. Let $D \subseteq C$ be a maximal closed proper subset containing a. Let $U = C \setminus D$. Since D is maximal, it must consist in one single interval $[a,c]$, with $c < b$, or in two intervals $[a,c] \sqcup [d,b]$, with $c+1 < d$.

As in Lemma 5.4.14 we can assume without loss of generality that $K_C^{4,0} = K_D^{4,0}$, hence

$$\mathcal{X}_C \in \mathrm{Sym}^2(\pi_1(NS(D)) \oplus \pi_2(NS(C))$$

which implies by the non-degeneracy of $\mathcal{X}_C$ that $\pi_1(NS(D)) = \pi_1(NS(C))$. This implies that the R_i's, with $i \in U$ are linearly independent: in fact they must generate the quotient vector space $NS(C)/NS(D) \cong \pi_2(NS(C))/\pi_2(NS(D))$, which has dimension $|U|$. In particular, we have:

$$NS(C) = NS(D) \oplus \mathrm{span}\langle R_i \mid i \in U\rangle.$$

We divide now into the two cases $D = [a,c]$ and $D = [a,c] \sqcup [d,b]$.

Figure 5.1: Two examples of graphs $\mathcal{I}_w$ of some element $w \in \mathcal{S}_7$. In the first example a maximal closed proper subset D is the interval $[1,4]$, in the second $D = [1,3] \cup [5,6]$.

Case 1: $D = [a, c]$.
By writing the $(2, 2)$-component of $\mathcal{X}_C - \mathcal{X}_D$ we have

$$-L_{c+1} \otimes R_c + \sum_{u\in[c+1,b]} (-L_{u-1} + 2L_u - L_{u+1}) \otimes R_u = 0 \in \mathfrak{g}_1 \otimes \mathfrak{g}_2. \tag{5.8}$$

where we write $L_{b+1} = 0$ by abuse of notation. If $R_c \neq 0$ then $R_c \in \pi_2(NS(D))$, thus it is linearly independent from the set $\{R_i\}_{i\in[c+1,b]}$. So we get $L_{c+1} = 0$ and $(-L_{u-1} + 2L_u - L_{u+1}) = 0$ for any $u \in [c+1, b]$. Now, as in the proof of Lemma 5.4.12, we regard this as a linear system in the variable L_c. This system admits a solution if and only $L_c = 0$. But if $L_c = 0$ by induction we see that the only possible case is $NS(D) \subseteq H^{0,2}_w$. Since $\pi_1(NS(C)) = \pi_1(NS(D)) = 0$ we also get $NS(C) \subseteq H^{0,2}_w$.

We assume now $R_c = 0$ and $P_c = L_c$. In this case, solving the system above, by the same argument of Lemma 5.4.12, we find that for any $i \in [c + 1, b]$ we have

$$L_i = \frac{b + 1 - i}{b - c + 1} P_c, \qquad R_i = P_i - \frac{b + 1 - i}{b - c + 1} P_c$$

and that $P_c^2 = 0$. This forces c to be a sink, hence $c = a$.

Case 2: $D = [a, c] \sqcup [d, b]$.
The $(2, 2)$-component of $\mathcal{X}_C - \mathcal{X}_D$ is

$$-L_{c+1} \otimes R_c + \sum_{u\in[c+1,d-1]} (-L_{u-1} + 2L_u - L_{u+1}) \otimes R_u - L_{d-1} \otimes R_d = 0 \in \mathfrak{g}_1 \otimes \mathfrak{g}_2. \tag{5.9}$$

There are no sinks in the closed subset $[d, b]$, hence from Lemma 5.4.14 it follows that either $R_d = 0$ or $R_d = P_d$. The element $P_d \in NS(D)$ is linearly independent from $\{R_i\}_{i\in[c,d-1]}$. Hence, if $R_d = P_d$ we get $L_{d-1} = 0$. Again, the unique solution of the system of equation

$$\begin{cases} L_{d-1} = 0 \\ -L_{u-1} + 2L_u - L_{u+1} = 0 \quad \text{for any } u \in [c + 1, d - 1] \end{cases}$$

is $L_i = 0$ for all $i \in [c, d - 1]$. By induction $NS([a, c]) \subseteq H^{0,2}_w$, and this leads to $NS(C) \subseteq H^{0,2}_w$

We assume now $R_d = 0$. We obtain for all $i \in U$

$$R_i = P_i - \frac{d - i}{d - c} L_c - \frac{i - c}{d - c} P_d.$$

But this, as in Lemma 5.4.12, leads to $P_d^2 = 0$. But d cannot be a sink, so we get a contradiction. □

Lemma 5.5.2. *Let $C = [a, b]$ be an interval closed in $\mathcal{I}_w$ such that a and b are sinks in C. Then either $NS(C) \subseteq H^{2,0}_w$ or $NS(C) \subseteq H^{0,2}_w$.*

Proof. We can assume that a and b are the only sinks in C. In fact, assume for example that there is another sink c with $a < c < b$. Then $NS([a, c]) \subseteq H^{2,0}_w$ implies $P_c \in H^{2,0}_w$, thus also $NS([c, b]) \subseteq H^{2,0}_w$ and $NS(C) \subseteq H^{2,0}_w$.

Let $D \subseteq C$ be a maximal closed proper subset containing a and b and let $U = C \setminus D$. Since D is maximal we have $D = [a, c] \sqcup [d, b]$ for some c and d. Then, arguing as in Lemma 5.5.1, we assume without loss of generality $\pi_1(NS(C)) = \pi_1(NS(D))$. This implies that the R_i's, with $i \in U$ are linearly independent and that

$$NS(C) = NS(D) \oplus \operatorname{span}\langle R_i \mid i \in U\rangle.$$

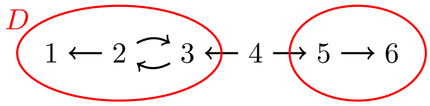


Figure 5.2: An example of a graph $\mathcal{I}_w$ with two extremal sinks.

The $(2,2)$-component of $\mathcal{X}_C - \mathcal{X}_D$ is

$$-L_{c+1} \otimes R_c + \sum_{u \in [c+1, d-1]} (-L_{u-1} + 2L_u - L_{u+1}) \otimes R_u - L_{d-1} \otimes R_d = 0 \in \mathfrak{g}_1 \otimes \mathfrak{g}_2. \quad (5.10)$$

Both R_c and R_d are either 0 or are linearly independent from the set $\{R_i\}_{i \in U}$ (and from each other). If $R_c \neq 0$ the same argument as in the second case of Lemma 5.5.1 shows that $L_i = 0$ for all $i \in [c, d]$. This implies $R_c = P_c$, $R_d = P_d$ and by Lemma 5.5.1 that $NS(D) \subseteq H_w^{0,2}$, whence also $NS(C) \subseteq H_w^{0,2}$. Similarly, if $R_d \neq 0$.

Assume now $R_c = R_d = 0$. Then we obtain for any $i \in U$

$$R_i = P_i - \frac{d-i}{d-c} P_c - \frac{i-c}{d-c} P_d.$$

Now, as in the proof of Lemma 5.4.12 we obtain

$$p\left(\sum_{s,t \in \widetilde{U}} c_{st} R_s R_t \right) = p(\Theta_1) + p(\Theta_2) \in H_w^{0,4} \cap H_w^{4,0} = \{0\},$$

where

$$p(\Theta_1) = p\left(-\frac{2}{d-c} P_c P_d \right) = -\frac{2}{d-c} P_{s_c s_d}$$

$$p(\Theta_2) = p\left(-\frac{d-c-1}{d-c} (P_c^2 + P_d^2) \right) = -2 \frac{d-c-1}{d-c} \left(P_{s_{c-1} s_c} + P_{s_{d+1} s_d} \right)$$

and $p : R^4 \to H_w^4$ is the projection. The element $P_{s_c s_d}$ is a basis element in H_w^4, so we get $0 \neq p(\Theta_1) + p(\Theta_2) = 0$, which is a contradiction. $\square$

5.5.1 The case of an extremal sink

We want now to consider the case of an element w whose graph $\mathcal{I}_w$ has exactly one sink, and this sink is placed in one extreme vertex of the graph $\mathcal{I}_w$. So we can assume that $n \in S$ is the sink. Let $I = \{1, \ldots, n-1\}$. Notice that this implies $w = s_n v$ with $v \in W_I$. Recall that we assumed that w is not contained in any parabolic subgroup W_J of W, hence v is not contained in any parabolic subgroup $W_{J'} \subsetneq W_I$.

We first consider the case $w = s_n w_I$, where w_I is the longest element in W_I. Our first objective is to show that in this case H_w admits a tensor decomposition (cf. Example 5.4.16).

The cohomology of the flag variety X of a group of type A_n can also be described as follows. Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n, x_{n+1}]$ (with $\deg x_i = 2$) and let $W = \mathcal{S}_{n+1}$ act on R by permuting the variables x_i. Then we have an isomorphism

$$H^\bullet(X, \mathbb{C}) \cong R / R_+^W \quad (5.11)$$

$$P_i \mapsto x_1 + x_2 + \ldots + x_i.$$

We consider the graded algebra $A := H^\bullet(X,\mathbb{C})/(P_n^2) = R/(R_+^W, x_{n+1}^2)$. We claim that $A \cong H_w$. Since $P_n^2 = 0$ in H_w the projection $H^\bullet(X,\mathbb{C}) \twoheadrightarrow H_w$ factors through A, so it is enough to show that $\dim A = \dim H_w = 2(n!)$.

Set
$$y_i = x_i + \frac{1}{n}x_{n+1}.$$
Let $J \subseteq R$ be the ideal generated by x_{n+1}^2 and by all the homogeneous symmetric polynomials of positive degree in the variables $y_1, \dots, y_n$. We claim that $J = (R_+^W, x_{n+1}^2)$. In fact, the ideal R_+^W is generated by the polynomials $p_k := x_1^k + x_2^k + \dots x_{n+1}^k$, for $1 \leq k \leq n+1$. We have
$$p_1 = y_1 + y_2 + \dots + y_n$$
and for any $k \geq 2$ $\quad p_k = (y_1^k + y_2^k + \dots y_n^k) - \frac{k}{n}x_n(y_1^{k-1} + y_2^{k-1} + \dots y_n^{k-1}) + x_{n+1}^2 f$

for some polynomial $f \in R$. It follows that
$$A \cong R/(R_+^W, x_{n+1}^2) \cong \mathbb{C}[y_1,\dots,y_n]/\mathbb{C}[y_1,\dots,y_n]_+^{S_n} \otimes_{\mathbb{C}} \mathbb{C}[x_{n+1}]/(x_{n+1}^2). \tag{5.12}$$
Since clearly $\dim A = 2(n!)$ we get $A \cong H_w$. Furthermore, this also shows that $H_w = H^\bullet(X_w,\mathbb{C})$ admits a tensor decomposition. Observe that this tensor decomposition is the same non-trivial decomposition predicted by Lemma 5.5.1(2). The decomposition is clearly defined over $\mathbb{R}$, hence also $H^\bullet(X_w,\mathbb{R})$ admits a tensor decomposition.

The Schubert variety X_w is smooth since the projection $G/B \to G/\mathbf{P}_I$ restricts to a $(\mathbf{P}_I/B)$-bundle $\pi : X_w \to X_w^I = \overline{B \cdot s_n P_I/P_I} \cong \mathbb{P}^1$. Therefore $IH^\bullet(X_w,\mathbb{R}) = H^\bullet(X_w,\mathbb{R})[\ell(w)]$ and, because of Lemma 5.4.1, we obtain a splitting
$$\mathfrak{g}_{NS}(w) = \mathfrak{g}_{NS}(w_I) \times \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{aut}(H^\bullet(G/\mathbf{P}_I,\mathbb{R}),\phi) \times \mathfrak{sl}_2(\mathbb{R}).$$

Consider now an arbitrary element w of the form $s_n v$, with $v \in W_I$ and such that v is not contained in any proper parabolic subgroup of W_I, so the graph $\mathcal{I}_w$ is connected. If $\mathcal{I}_w$ contains more than one sink, it follows from 5.5.2 that H_w does not admit non-trivial tensor-decompositions.

We call R' the first factor of A in (5.12). The ring H_w is a quotient of A. We can assume that n is the only sink in $\mathcal{I}_w$. In this case, it follows by Lemma 5.5.1 that if H_w admits a tensor decomposition then it is induced by the decomposition (5.12). This means that, if we denote by K the kernel of the map $A \twoheadrightarrow H_w$, to show that H_w does not admit any tensor decomposition it is sufficient to show that the ideal K is not generated by elements of R'.

Any Schubert basis element $P_x \in H_{s_n w_I} \cong A$ can be thought as a polynomial in the P_i's or in the x_i's:
$$P_x = g_x(P_1,\dots,P_n) = f_x(x_1,\dots,x_n).$$
Since $x_{n+1}^2 = 0$ in A, it can be easily seen that in A we have:
$$P_x = f_x(y_1,\dots,y_n) - \frac{1}{n}Df_x(y_1,\dots,y_n)x_{n+1} \tag{5.13}$$
where $D : R' \to R'$ is the differential operator
$$D = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \dots + \frac{\partial}{\partial y_n}.$$

Assume that v is not the longest element w_I. Let r be an element of minimal length in the set
$$X = \{y \in W \mid y \leq s_n w_I \text{ and } y \not\leq s_n v\}.$$

From the Property Z (1.1.1), if $y \in X$ then also $s_n y \in X$. It follows that $r \in W_I$.

The kernel K is the ideal generated by all P_x, with $x \in X$, thus P_r is an element of lowest degree in K. By (5.13) we have

$$H_w^{2\ell(w),0} \ni f_r(y_1, y_2, \dots, y_n) = \frac{1}{n} Df_r(y_1, y_2, \dots, y_n) \cdot x_{n+1} \in H_w^{2\ell(w)-2,2},$$

and since $H_w^{2\ell(w),0} \cap H_w^{2\ell(w)-2,2} = \{0\}$ we obtain $Df_r(y_1, y_2, \dots, y_n) \cdot x_{n+1} = 0$, hence $Df_r(y_1, y_2, \dots, y_n) = 0$ in H_w. Since $\deg Df_r = \deg P_r - 2$, the polynomial $Df_r(y_1, \dots, y_n)$ cannot be a non-zero element of K. This implies $Df_r(y_1, y_2, \dots, y_n) = 0$ in R'. Thus, if $Df_r(y_1, y_2, \dots, y_n) \neq 0$ in R' we get a contradiction, which means that H_w is tensor-indecomposable.

We need to recall a few facts about Schubert polynomials in type A. Schubert polynomials for the symmetric group have been intensively studied, both from a geometric and a combinatorial point of view. We refer for example to [Mac91].

Definition 5.5.3. We call *Schubert polynomial* of $w \in \mathcal{S}_{n+1}$ any $f_w \in \mathbb{C}[x_1, x_2, \dots, x_{n+1}]$ such that its projection to $\mathbb{C}[x_1, x_2, \dots, x_{n+1}]/\mathbb{C}[x_1, x_2, \dots, x_{n+1}]_+^{\mathcal{S}_{n+1}}$ coincides with the Schubert basis element P_w (via the isomorphism (5.11)).

In [BJS93] is described a combinatorial formula for Schubert polynomials. We recall briefly their result.

Let $w \in \mathcal{S}_{n+1}$ and let $Rex(w)$ be the set of reduced expression for w. If $w = s_{a_i} s_{a_2} \dots s_{a_l}$ is a reduced expression we denote by $a = (a_1, a_2, \dots, a_l)$ the corresponding element in $Rex(w)$. Let $a \in Rex(w)$. A sequence $(i_1, i_2, \dots, i_l)$ is said *a-compatible* if

$$i_1 \leq i_2 \leq \dots \leq i_l$$

$$i_j \leq a_j \text{ for all } 1 \leq j \leq l$$

$$i_j \geq i_{j+1} \implies a_j > a_{j+1}$$

Let $RK(w) = \{(a, i) \mid a \in Rex(w) \text{ and } i \text{ } a\text{-compatible}\}$. Then [BJS93, Theorem 1.1]

$$f_w(x_1, x_2, \dots, x_n) = \sum_{(a,i) \in RK(w)} x_{i_1} x_{i_2} \dots x_{i_l}. \tag{5.14}$$

The ring $\mathbb{C}[x_1, x_2, \dots, x_{n+1}]/\mathbb{C}[x_1, x_2, \dots, x_{n+1}]_+^{\mathcal{S}_{n+1}}$ admits another useful basis, often referred to as the *Artin basis* [Art59, §II.G]:

$$\{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \mid 0 \leq a_i \leq n + 1 - i\}.$$

Note that all the terms appearing in the sum (5.14) belong to the Artin basis: assume for contradiction that an integer k occurs more than $n + 1 - k$ times in an a-compatible sequence i, so there exists an index b such that

$$i_b = i_{b+1} = \dots = i_{b+n+1-k} = k.$$

This forces

$$a_b > a_{b+1} > \dots > a_{b+n+1-k} \geq i_{b+n+1-k} = k,$$

which is impossible since $a_b \leq n$.

Let $S = \mathbb{C}[z_1, z_2, \dots, z_n]$, so that $S/S_+^{\mathcal{S}_n}$ is the coinvariant ring of W_I. Let $x \in W_I$. It is evident from the formula (5.14) that the Schubert polynomial of x in W coincides with the Schubert polynomial of x in W_I, that is if we denote by $\widetilde{P}_x$ the Schubert basis element of x in $S/S_+^{\mathcal{S}_n}$ then

$$\widetilde{P}_x = f_x(z_1, z_2, \ldots, z_n).$$

It remains to prove to following:

Lemma 5.5.4. *Let $f_x \in \mathbb{C}[z_1, \ldots, z_n]/\mathbb{C}[z_1, \ldots, z_n]_+^{\mathcal{S}_n}$ a Schubert polynomial of $id \neq x \in \mathcal{S}_n$. Let*

$$D = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \ldots + \frac{\partial}{\partial z_n} : \mathbb{C}[z_1, \ldots, z_n]/\mathbb{C}[z_1, \ldots, z_n]_+^{\mathcal{S}_n} \to \mathbb{C}[z_1, \ldots, z_n]/\mathbb{C}[z_1, \ldots, z_n]_+^{\mathcal{S}_n}.$$

Then $Df_x \neq 0$.

Proof. Because of (5.14), any Schubert polynomial f_w can be written in the Artin basis with coefficients in $\mathbb{R}_{\geq 0}$. If the degree of f_w is positive, the differential operator D sends an element of the Artin basis in a positive linear combination of elements of the Artin basis. It follows that Df_x has positive coefficients in the Artin basis, so in particular $Df_x \neq 0$. $\square$

Corollary 5.5.5. *Let $w = s_n v$ with $v \in W_I$ and assume that w is not contained in any proper parabolic subgroup $W_J \subseteq W$. Then $\mathfrak{g}_{NS}(w) = \mathfrak{aut}(IH_w, \phi)$ if and only if v is not the longest element in W_I.*

5.5.2 The general case

We are now ready to determine whether the algebra H_w admits a tensor decomposition for an arbitrary connected element $w \in \mathcal{S}_{n+1}$.

Lemma 5.5.6. *Let a and b be the sinks of w of smallest and largest index respectively. We can write $w = s_a s_b v_1 v_2 v_3$ with $v_1 \in W_{[1,a-1]}$, $v_2 \in W_{[a+1,b-1]}$ and $v_3 \in W_{[b+1,n]}$ (or $w = s_a v_1 v_3$ if $a = b$). Then H_w admits a non-trivial tensor decomposition if and only if $a > 1$ and v_1 is the longest element in $W_{[1,a-1]}$ or $b < n$ and v_3 is the longest element in $W_{[b+1,n]}$.*

Proof. The sets $[1, a]$, $[a, b]$ and $[b, n]$ are closed in $\mathcal{I}_w$. Because of Lemma 5.5.2 we can assume without loss of generality $NS([a, b]) \subseteq H_w^{2,0}$.

Assume that both v_1 and v_3 are not the longest element. Then, as in Corollary 5.5.5 we have that $NS([1, a])$ and $NS([b, n])$ are also contained in $H_w^{2,0}$, hence the only tensor decomposition of H_w is the trivial one.

Assume now that $v_1 = w_{[1,a-1]}$ is the longest element in $W_{[1,a-1]}$, hence $w = s_a v_1 w_2$ with $w_2 \in W_{[a+1,n]}$.

We first assume that w_2 is the longest element in $W_{[a+1,n]}$. Then, similarly to (5.12) one can show that H_w is isomorphic to the algebra

$$H_w \cong A = A_1 \otimes_{\mathbb{C}} A_2 \otimes_{\mathbb{C}} A_3$$

with

$$A_1 = \mathbb{C}[y_1, \ldots, y_a]/\mathbb{C}[y_1, \ldots, y_a]_+^{\mathcal{S}_a}$$
$$A_2 = \mathbb{C}[P_a]/(P_a^2)$$
$$A_3 = \mathbb{C}[y_{a+1}, \ldots, y_{n+1}]/\mathbb{C}[y_{a+1}, \ldots, y_{n+1}]_+^{\mathcal{S}_{n+1-a}}$$

where $P_a = x_1 + \ldots + x_a = -(x_{a+1} + \ldots + x_{n+1})$ and

$$y_i = \begin{cases} x_i + \frac{1}{a}P_a & \text{if } i \leq a, \\ x_i + \frac{1}{n+1-a}P_a & \text{if } i \geq a+1. \end{cases}$$

Assume that w_2 is not the longest element in $W_{[a+1,n]}$. We set $z := s_a w_{[1,a-1]} w_{[a+1,n]}$. Then H_w is the quotient of $A \cong H_z$ by the ideal

$$J = \langle P_x \mid x \leq z \text{ and } x \not\leq w \rangle.$$

All the elements $x \leq z$ are of the form $x = s_a^\varepsilon x_1 x_2$, with $x_1 \in W_{[1,a-1]}$, $x_2 \in W_{[a+1,n]}$ and $\varepsilon \in \{0, 1\}$. We claim that we have $P_x = cP_a^\varepsilon P_{x_1} P_{x_2}$ in H_z, for some $c \in \mathbb{R}$.

For this, we consider the equivariant Schubert basis $\{\mathcal{P}_x\}_{x \leq z}$ of $H_T^\bullet(X_z, \mathbb{C})$ from Chapter 3. We have:

$$\mathcal{P}_a^\varepsilon \mathcal{P}_{x_1} \mathcal{P}_{x_2} = \sum_{y \geq z} c_y \mathcal{P}_y \qquad \text{for } c_y \in R \tag{5.15}$$

where c_y are homogeneous polynomials of degree $2(\varepsilon + \ell(x_1) + \ell(x_2) - \ell(y))$. Since by Lemma 3.4.2 we have $\mathcal{P}_x \in \Gamma_{\geq x} H_T^\bullet(X_z, \mathbb{C})$ for any x, one obtains $y \geq s_a^\varepsilon, x_1, x_2$ for all y appearing in the sum (5.15). Projecting equation (5.15) to $H_z := \mathbb{C} \otimes_R H_T^\bullet(X_z, \mathbb{C})$ means killing all homogenous polynomial of positive degree, hence only the elements y such that $\ell(y) = \varepsilon + \ell(x_1) + \ell(x_2)$ survive. The claim now follows since $x = s_a^\varepsilon x_1 x_2$ is the only element of the required length bigger than s_a^ε, x_1 and x_2 and smaller than z. It follows that

$$J = \langle P_x \mid x \in W_{[a+1,n]} \text{ and } x \not\leq w_2 \rangle.$$

Claim 5.5.7. If $x \in W_{[a+1,n]}$, then $P_x \in H^\bullet(X, \mathbb{C})$ is contained in the subalgebra of $H^\bullet(X, \mathbb{C})$ generated by $P_{a+1}, \ldots, P_n$.

Proof of the claim. It follows immediately from the combinatorial formula (5.14) that if $x \in W_{[1,n-a]}$ then P_w is contained in the algebra generated by $x_1, \ldots, x_{n-a}$, hence in the algebra generated by $P_1, \ldots, P_{n-a}$. Since the map defined $P_i \mapsto P_{n+1-i}$ induced by flipping the Dynkin diagram A_n is an automorphism of $H^\bullet(X, \mathbb{C})$, the claim follows. □

From the claim, it follows that all the generators of J are contained in $A_2 \otimes_{\mathbb{C}} A_3$, hence $J \cong A_1 \otimes_{\mathbb{C}} \widetilde{J}$, where $\widetilde{J} = \langle P_x \mid x \in W_{[a+1,n]} \text{ and } x \not\leq w_2 \rangle$ is an ideal of $A_2 \otimes_{\mathbb{C}} A_3$. We deduce that H_w admits a tensor decomposition of the form

$$A_1 \otimes_{\mathbb{C}} ((A_2 \otimes_{\mathbb{C}} A_3)/\widetilde{J}).$$

The case v_3 longest element in $W_{[b+1,n]}$ is completely symmetric. □

Notice that all the tensor-decompositions of H_w we obtained are defined over $\mathbb{R}$, so we have shown that we have a tensor decomposition of $H^\bullet(X_w, \mathbb{C})$ if and only if we have a tensor decomposition of $H^\bullet(X_w, \mathbb{R})$.

This completes the classification of elements $w \in \mathcal{S}_{n+1}$ such that $H^\bullet(X_w, \mathbb{R})$ admits a non-trivial tensor-decomposition. Recall that we have a decomposition of the Néron-Severi Lie algebra $\mathfrak{g}_{NS}(w)$ if and only if $IH^\bullet(X_w, \mathbb{R})$ admits a non-trivial tensor decomposition (Lemma 5.4.1). Therefore, it remains to show that $IH^\bullet(X_w, \mathbb{R})$ admits a tensor-decomposition compatible with the decomposition of $H^\bullet(X_w, \mathbb{R})$. To show this it is more natural to use the notation coming from Soergel bimodules. Recall that $\overline{B_w} \cong IH^\bullet(X_w, \mathbb{R})$ and that $H^\bullet(X_w, \mathbb{R})[\ell(x)]$ is a R-submodule of $\overline{B_w}$ (here the Soergel bimodules are constructed with respect of the realization of type II of W).

Let $w = s_a w_{[1,a-1]} v = s v w_{[1,a-1]}$ with $v \in W_{[a+1,n]}$. We have just shown that

$$H^\bullet(X_w, \mathbb{R}) = A_1 \otimes_{\mathbb{R}} H^\bullet(X_{s_a v}, \mathbb{R}) \tag{5.16}$$

where A_1 is the subalgebra of $H^\bullet(X_w, \mathbb{R})$ generated by $P_i - \frac{i}{a} P_a$ for $1 \leq i \leq a$. Since $s_a v$ is minimal in its $W/W_{[1,a-1]}$-coset we have

$$\overline{B_w} \cong \overline{B_{s_a v}} \otimes_{R^{[1,a-1]}} R[\ell(w_{[1,a-1]})] \tag{5.17}$$

Since A_1 is a subring of $H^\bullet(X_w, \mathbb{R})$, A_1 acts on $\overline{B_w}$ via multiplication on the right. We regard $\overline{B_{s_a v}}$ as a subspace of $\overline{B_w}$ using (5.17). Therefore we have a map of vector spaces:

$$\Theta : A_1 \otimes_{\mathbb{R}} \overline{B_{s_a v}} \to \overline{B_w}$$

$$a \otimes b \mapsto b \cdot a.$$

We claim that Θ is an isomorphism. The vector spaces $A_1 \otimes_{\mathbb{R}} \overline{B_{s_a v}}$ and $\overline{B_w}$ have the same dimension, so it suffices to show that Θ is surjective. It is clear from (5.17) that $\overline{B_w}$ is generated by $\overline{B_{s_a v}}$ as a R-module, hence as a $H^\bullet(X_w, \mathbb{R})$-module. Because $H^\bullet(X_w, \mathbb{R}) = A_1 \otimes_{\mathbb{R}} H^\bullet(X_{s_a v}, \mathbb{R})$ and $H^\bullet(X_{s_a v}, \mathbb{R})$ preserves $\overline{B_{s_a v}}$, the claim follows.

In this way we obtain a tensor decomposition of $\overline{B_w} = IH^\bullet(X_w, \mathbb{R})$ compatible with the decomposition (5.16) of $H^\bullet(X_w, \mathbb{R})$.

In view of Lemma 5.4.1, we can now give a complete answer to what Néron-Severi Lie algebras look in type A:

Theorem 5.5.8. *Let a and b be the sinks of w of smallest and largest index respectively. If $a < b$ we can write $w = s_a s_b v_1 v_2 v_3$ with $v_1 \in W_{[1,a-1]}$, $v_2 \in W_{[a+1,b-1]}$ and $v_3 \in W_{[b+1,n]}$. Then*

$$\mathfrak{g}_{NS}(w) \cong \begin{cases} \mathfrak{g}_{NS}(v_1) \times \mathfrak{g}_{NS}(s_a s_b v_2) \times \mathfrak{g}_{NS}(v_3) & \text{if } v_1 = w_{[1,a-1]} \text{ and } v_3 = w_{[b+1,n]} \\ \mathfrak{g}_{NS}(v_1) \times \mathfrak{g}_{NS}(s_a s_b v_2 v_3) & \text{if } v_1 = w_{[1,a-1]} \text{ and } v_3 \neq w_{[b+1,n]} \\ \mathfrak{g}_{NS}(v_3) \times \mathfrak{g}_{NS}(s_a s_b v_1 v_2) & \text{if } v_1 \neq w_{[1,a-1]} \text{ and } v_3 = w_{[b+1,n]} \\ \mathfrak{g}_{NS}(s_a s_b v_1 v_2 v_3) & \text{if } v_1 \neq w_{[1,a-1]} \text{ and } v_3 \neq w_{[b+1,n]} \end{cases} \tag{5.18}$$

If $a = b$ we can write $w = s_a v_1 v_3$ with $v_1 \in W_{[1,a-1]}$ and $v_3 \in W_{[a+1,n]}$. Then

$$\mathfrak{g}_{NS}(w) \cong \begin{cases} \mathfrak{g}_{NS}(v_1) \times \mathfrak{g}_{NS}(s_a) \times \mathfrak{g}_{NS}(v_3) & \text{if } v_1 = w_{[1,a-1]} \text{ and } v_3 = w_{[a+1,n]} \\ \mathfrak{g}_{NS}(v_1) \times \mathfrak{g}_{NS}(s_a v_3) & \text{if } v_1 = w_{[1,a-1]} \text{ and } v_3 \neq w_{[a+1,n]} \\ \mathfrak{g}_{NS}(v_3) \times \mathfrak{g}_{NS}(s_a v_1) & \text{if } v_1 \neq w_{[1,a-1]} \text{ and } v_3 = w_{[a+1,n]} \\ \mathfrak{g}_{NS}(s_a v_1 v_3) & \text{if } v_1 \neq w_{[1,a-1]} \text{ and } v_3 \neq w_{[a+1,n]} \end{cases} \tag{5.19}$$

Moreover, all the Lie algebra $\mathfrak{g}_{NS}(x)$ appearing in the RHS of (5.18) *and* (5.19) *are maximal, i.e. we have $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH^\bullet(X_w, \mathbb{R}), \phi)$.*

Proof. If at least one between v_1 and v_3 is not maximal the statement follows from the discussion above. We have also discussed the case v_1, v_3 maximal and $a = b$ in the proof of Lemma 5.5.6. The proof in the remaining case v_1, v_3 maximal and $a < b$ is analogous. □

Chapter 6

The Hard Lefschetz Theorem in Positive Characteristic for Flag Varieties

6.1 Introduction

The hard Lefschetz theorem does not hold over $\mathbb{Z}$: if Y is a complex smooth projective variety of dimension d and $\lambda \in H^2(Y,\mathbb{Z})$ is the first Chern class of an ample line bundle, in general the map

$$\lambda^k : H^{d-k}(Y,\mathbb{Z}) \to H^{d+k}(Y,\mathbb{Z})$$

is not an isomorphism (even if we restricts to varieties with no torsion in the cohomology $H^\bullet(Y,\mathbb{Z})$). In addition, the hard Lefschetz theorem does not even hold when we consider cohomology with coefficient in a field $\mathbb{K}$ of characteristic $p > 0$. We recall the following definition from Chapter 5:

Definition 6.1.1. Let $d \geq 0$ and $V = \bigoplus_{k=0}^{2d} V^k$ be a graded finite dimensional $\mathbb{K}$-vector space. Let $f : V \to V$ be a map of degree 2 (i.e. $f(V^k) \subseteq V^{k+2}$ for any k). We say that f has the *Lefschetz property* on V if for any $0 < k \leq d$ the map $f^k : V^{d-k} \to V^{d+k}$ is an isomorphism.

If V is a graded $\mathbb{K}$-algebra we say that $\eta \in V^2$ has the Lefschetz property on V if the multiplication by η has the Lefschetz property.

Let X be the flag variety of a simply connected group G. The goal of this chapter is to answer the following:

Question 6.1.2. *Let $\mathbb{K}$ be an arbitrary infinite field of characteristic p. For which primes p does there exist $\lambda \in H^2(X,\mathbb{K})$ such that λ has the Lefschetz property on $H^*(X,\mathbb{K})$?*

As explained in the introduction, this is motivated by modular representation theory, and in particular by Lusztig's conjecture. Fiebig's proof of the upper bound on Lusztig's conjecture is based on a rough bound on when local hard Lefschetz holds for Schubert varieties in the affine flag variety. By refining these estimates one could be able to find sharper bounds to Lusztig's conjecture.

We believe that the answer of Question 6.1.2 is a first step in this direction.

6.2 Statement of the main result

Let G be complex simply-connected semisimple algebraic group and let $X = G/B$ be its flag variety. Let W the corresponding Weyl group. Recall the relevant notation from Chapter 1 and 2. Let $\{P_w\}_{w\in W}$ be the Schubert basis of $H^\bullet(X,\mathbb{K})$.

The pairing between weights and coroots can be extended to a $\mathbb{K}$-valued pairing between $H^2(X,\mathbb{K})$ and $\mathbb{Z}\Phi^\vee$. We will abuse terminology and refer to the elements of $H^2(X,\mathbb{K})$ as weights.

A first partial answer to the Question 6.1.2 was given by Stembridge. In [Ste02] he computes explicitly the map $\lambda^d : H^0(X,\mathbb{Z}) \to H^{2d}(X,\mathbb{Z})$. We have:

$$\lambda^d \cdot P_e = |\Phi^+|! \prod_{\alpha\in\Phi^+} \frac{\lambda(\alpha^\vee)}{\mathrm{ht}(\alpha)} P_{w_0} \tag{6.1}$$

where $e \in W$ is the identity and $w_0 \in W$ is the longest element of W. The height of a root, here denoted by $\mathrm{ht}(\alpha)$, is the sum of its coordinates when expressed in the basis of simple roots.

From Stembridge's formula (6.1) it follows that if $\mathbb{K}$ is a field of characteristic p and p does not divide $|\Phi^+|!$, i.e. if $p > |\Phi^+|$, then there exists $\lambda \in H^2(X,\mathbb{K})$ such that $\lambda^d : H^0(X,\mathbb{K}) \to H^{2d}(X,\mathbb{K})$ is an isomorphism: we can take, for example, $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$ so that $\rho(\alpha^\vee) = 1$ for every simple root α.

Remark 6.2.1. Let k_i be the number of positive roots of height i. Then we have $k_1 \geq k_2 \geq \ldots$ and $\sum k_i = |\Phi^+|$ (see [Hum90, §3.20]). We can then regard $k_1 \geq k_2 \geq \ldots$ as a partition of $|\Phi^+|$ and consider the dual partition $m_1 \geq m_2 \geq \ldots$, i.e. $m_i = \#\{j \mid k_j \geq i\}$. The integers m_i are the *exponents* of the group W. The values of the exponents (increased by 1) can be found in [Hum90, Table 1, §3.7]. We have

$$\prod_{\alpha\in\Phi^+} \mathrm{ht}(\alpha) = \prod_{j\geq 1} j^{k_j} = \prod_{i\geq 1} m_i!.$$

It follows that the number

$$\frac{|\Phi^+|!}{\prod_{\alpha\in\Phi^+}\mathrm{ht}(\alpha)} = \binom{|\Phi^+|}{m_1, m_2, \ldots}$$

is an integer and it is divided by $\binom{m_{j_1}+m_{j_2}+\ldots m_{j_r}}{m_{j_1}, m_{j_2}, \ldots, m_{j_r}}$, for any finite subset $\{j_1, j_2, \ldots, j_r\}$ of $\mathbb{N}$.

Therefore, from Stembridge's formula, it also follows that there cannot exist λ such that λ^d is an isomorphism if

$$p \Big| \binom{|\Phi^+|}{m_1, m_2, \ldots}. \tag{6.2}$$

Now, using the known explicit values of the exponents, one can easily check that, if p is a prime such that $p \leq |\Phi^+|$ and p is not as in Table 6.1, then (6.2) holds.

It the case listed in Table 6.1 we can compute explicitly, with the help of the software [BCP97], the map $\lambda^k : H^{d-k}(X,\mathbb{K}) \to H^{d+k}(X,\mathbb{K})$ for any $\lambda \in H^2(X,\mathbb{K})$ and any $k \geq 0$. We obtain that there exists $\lambda \in H^2(X,\mathbb{K})$ with the Lefschetz property in the first three cases, namely if $\mathrm{rk}(G) = 2$ (see Example 6.2.3), and that there is not such λ in the last three cases, namely X of type B_3, C_3 or F_4. In fact, for $p = 5$ and X of one of these typesthe map $\lambda^{d-2} : H^2(X,\mathbb{K}) \to H^{2d-2}(X,\mathbb{K})$ is not an isomorphism for all weights $\lambda \in H^2(X,\mathbb{K})$.

Table 6.1:

Φ	p	$\lvert\Phi^+\rvert$	$\exists\lambda$ with Lefschetz property?
A_2	2	3	Yes
B_2	3	4	Yes
G_2	5	6	Yes
B_3	5	9	No
C_3	5	9	No
F_4	5	24	No

Here we give a complete answer to Question 6.1.2. The main result is the following:

Theorem 6.2.2. *Let $\mathbb{K}$ be an infinite field of characteristic $p > 0$. Then there exists $\lambda \in H^2(X, \mathbb{K})$ such that the hard Lefschetz theorem holds for λ on $H^\bullet(X, \mathbb{K})$ if and only if $p > |\Phi^+|$ or Φ and p are as in the first three lines of Table 6.1.*

Example 6.2.3. Let X be of type B_2 and $\mathbb{K}$ be an infinite field of characteristic 3. We label the simple roots in the Dynkin diagram as $\alpha \Rightarrow \beta$. Let $\lambda = a\varpi_\alpha + b\varpi_\beta$ be an arbitrary weight, where $a, b \in \mathbb{K}$ and $\varpi_\alpha, \varpi_\beta$ are the fundamental weights. We can compute explicitly the Lefschetz determinants:

- $D_4(a, b) := \det(\lambda^4 : H^0(X, \mathbb{F}_5) \to H^8(X, \mathbb{F}_5)) = 4ab(a + b)(a + 2b)$;
- $D_2(a, b) := \det(\lambda^2 : H^2(X, \mathbb{F}_5) \to H^6(X, \mathbb{F}_5)) = -(a^2 + 2ab + 2b^2)$.

The polynomials D_2 and D_4 are not identically zero, so there exists λ with the Lefschetz property. For instance, we can choose $\lambda = a\varpi_\alpha + \varpi_\beta$, with $a \in \mathbb{K} \setminus \{0, 1, 2\}$ such that it is not a root of the polynomial $x^2 + 2x + 2$.

Similar elementary computations show that there exists λ with the Lefschetz property on $H^\bullet(X, \mathbb{K})$ if X is of type A_2 (resp. G_2) and $\mathbb{K}$ is a infinite field of characteristic 2 (resp. 5). Thus A_2, B_2 and G_2 are the only types for which there exists $\lambda \in H^2(X, \mathbb{K})$ with the Lefschetz property for a field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) \leq |\Phi^+|$.

Remark 6.2.4. The situation is more subtle if one considers the case of a finite field.

For example, let X be of type B_2 and let $\mathbb{K} = \mathbb{F}_5$. Similarly to Remark 6.2.3, let $\lambda = a\varpi_\alpha + b\varpi_\beta$ with $a, b \in \mathbb{F}_5$. Notice that in this case we have $D_2(a, b) = -(a^2 + 2ab + 2b^2) = -(a+3b)(a+4b)$. It follows that there are no $a, b \in \mathbb{F}_5$ such that λ has the Lefschetz property on $H^\bullet(X, \mathbb{F}_5)$, although $5 > |\Phi^+| = 4$.

6.2.1 Structure of the proof

Using basic Schubert calculus, in §3 we translate the original problem, which is geometric in nature, into a combinatorial one, which is expressed only in terms of the Bruhat graph. In §4 we show how the Bruhat graph can be "degenerated" into a product of simpler graphs (corresponding to maximal parabolic subgroups), and that it is enough to show hard Lefschetz theorem for the latter.

We discuss when the Lefschetz property holds for those simpler graphs (for good choices of the maximal parabolic subgroups) in §5. In §6 we discuss the Lefschetz property for Artinian complete intersection monomial rings, i.e. rings of the form

$$\mathbb{K}[x_1, x_2, \ldots, x_n]/(x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}),$$

opportunely graded: this allows to make use of the knowledge of the Lefschetz property for the single factors to investigate the Lefschetz property for a product of graphs.

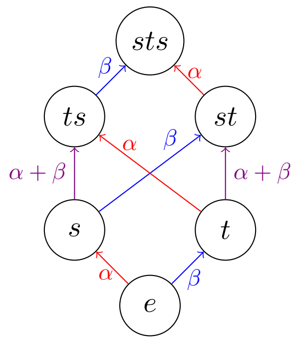
Finally in §7 we put everything together to obtain a proof of Theorem 6.2.2.

6.3 The Bruhat graph of a root system

Definition 6.3.1. We define the *Bruhat graph* $\mathfrak{B}_\Phi$ of Φ. The vertices of the graph are the elements of W. There is an arrow $w \xrightarrow{\gamma^\vee} v$ for $v, w \in W$ if $w \xrightarrow[R]{t_\gamma} v$, i.e. if $\ell(v) = \ell(w) + 1$ and $wt_\gamma = v$, where t_γ is the reflection corresponding to the positive coroot $\gamma^\vee$.

Remark 6.3.2. Our terminology for the Bruhat graph is somewhat non-standard. For example, in [Dye91, Definition 1.1] it is defined to be the graph whose vertices are the elements of W, in which there is an arrow $w \xrightarrow{\gamma^\vee} v$ for $v, w \in W$ whenever $v = wt_\gamma$ and $\ell(v) > \ell(w)$.

Example 6.3.3. If $G = SL_3(\mathbb{C})$, then Φ is the root system of type A_2 and $W \cong \mathcal{S}_3$, the symmetric group on 3 elements. It is generated by the simple transpositions s and t. Let α and β be the two simple *coroots* corresponding to s and t. The Bruhat graph $\mathfrak{B}_\Phi$ is:



We recall Chevalley's formula (3.6) and (5.3). Let $\lambda \in H^2(X, \mathbb{Z})$ be a weight. Then

$$\lambda \cdot P_w = \sum_{w \xrightarrow{\gamma^\vee} v} \lambda(\gamma^\vee) P_v.$$

If $\ell(v) - \ell(w) = k$, let $C_{w,v}(\lambda) \in \mathbb{Z}$ be defined by

$$\lambda^k \cdot P_w = \sum_{\ell(v) = \ell(w)+k} C_{w,v}(\lambda) P_v.$$

Then we have

$$C_{w,v}(\lambda) = \sum \lambda(\gamma_1^\vee)\lambda(\gamma_2^\vee) \dots \lambda(\gamma_k^\vee) = \sum C_{w,w_1}(\lambda) C_{w_1,w_2}(\lambda) \dots C_{w_{k-1},v}(\lambda)$$

where the sum runs over all paths $w \xrightarrow{\gamma_1^\vee} w_1 \xrightarrow{\gamma_2^\vee} w_2 \xrightarrow{\gamma_3^\vee} \dots \xrightarrow{\gamma_k^\vee} v$ in $\mathfrak{B}_\Phi$ connecting w to v.

Let $S \subseteq W$ be the set of simple reflections and $I \subseteq S$ be a subset. Recall that W_I denotes the subgroup generated by the simple reflections in I. We denote by $W^I \subseteq W$ the set of representatives of minimal length in W/W_I. If $J \subseteq I$, then W_I^J is well defined. Let $\Phi(I)$ be the sub-root system of Φ generated by the simple roots in I. Notice that reflections in $\Phi(I)$ correspond to positive roots in $\Phi(I)$.

We fix I. For any $w \in W$ we denote by $w' \in W^I$ and $w'' \in W_I$ the unique elements such that $w = w'w''$. We have $\ell(w) = \ell(w') + \ell(w'')$.

Let $\mathbf{P}_I$ be the parabolic subgroup $B \subseteq \mathbf{P}_I \subseteq G$ corresponding to the subset I. Recall that the projection $G/B \to G/\mathbf{P}_I$ induces an injective map $H^\bullet(G/\mathbf{P}_I, \mathbb{Z}) \to H^\bullet(G/B, \mathbb{Z})$, and that the image is the subspace generated by all the P_w, with $w \in W^I$.

For $s \in S$, let α_s and $\alpha_s^\vee$ denote the corresponding simple root and coroot. Let ϖ_s denote the corresponding fundamental weight, i.e. $\varpi_s(\alpha_t^\vee) = \delta_{s,t}$ for any $t \in S$.

From Chevalley's formula we get that $\varpi_s = P_s$ for any $s \in S$. So the subspace $H^2(G/\mathbf{P}_I, \mathbb{Z}) \subseteq H^2(G/B, \mathbb{Z})$ has as a basis the set $\{\varpi_s\}_{s \in S \setminus I}$.

Definition 6.3.4. Let I be a subset of S. We define the *degeneration map* $\pi_I : \mathbb{Z}\Phi^\vee \to \mathbb{Z}\Phi^\vee$ as follows:

$$\pi_I\left(\sum_{s\in S} c_s\alpha_s^\vee\right) = \begin{cases} \sum_{s\in S} c_s\alpha_s^\vee = \sum_{s\in I} c_s\alpha_s^\vee & \text{if } c_s = 0 \text{ for all } s \in S\setminus I \\ \sum_{s\in S\setminus I} c_s\alpha_s^\vee & \text{otherwise.} \end{cases}$$

For example, if $I = S \setminus \{s\}$, then π_I should be thought as "taking the leading term" of an element in $\mathbb{Z}\Phi^\vee$ after viewing it as a polynomial in the variable $\alpha_s^\vee$.

Definition 6.3.5. Let $I \subseteq S$ be a subset. The *parabolic Bruhat graph* $\mathfrak{B}_\Phi^I$ is a graph whose vertices are the elements in W^I. For any edge $w \xrightarrow{\gamma^\vee} v$ in $\mathfrak{B}_\Phi$, with $w, v \in W^I$, we put an edge $w \xrightarrow{\pi_I(\gamma^\vee)} v$ in $\mathfrak{B}_\Phi^I$, where $\pi_I : \mathbb{Z}\Phi^\vee \to \mathbb{Z}\Phi^\vee$ is the degeneration map.

Notice that if $w, v \in W^I$ with $w \xrightarrow{\gamma^\vee} v$, then $\gamma \notin \Phi(I)$. Hence in Definition 6.3.5 only the second case of the degeneration map π_I is actually used.

We see easily from Chevalley's formula that the graph $\mathfrak{B}_\Phi^I$ describes the multiplication by $\lambda \in H^2(G/\mathbf{P}_I, \mathbb{Z})$ in $H^\bullet(G/\mathbf{P}_I, \mathbb{Z})$ in the Schubert basis $\{P_w\}_{w\in W^I}$, i.e.

$$\lambda \cdot P_w = \sum_{w \xrightarrow{\delta} v \in \mathfrak{B}_\Phi^I} \lambda(\delta) P_v.$$

6.3.1 The degeneration of the Bruhat graph

Fix now $\mathbb{K}$ an arbitrary infinite field and let $\lambda \in H^2(X, \mathbb{K})$ be an arbitrary weight.

We label the elements of $S = \{1, 2, \ldots, n\}$, so that we can express λ as $\sum_{i=1}^n x_i\varpi_i$ with $x_i \in \mathbb{K}$. From now on we will regard the x_i's as indeterminate variables.

After we fix arbitrarily an ordering of the Schubert basis (or, equivalently, of the elements of W) the map $\lambda^k : H^{d-k}(X, \mathbb{K}) \to H^{d+k}(X, \mathbb{K})$ can be thought of as a square matrix with number of columns equal to the number of elements of length $(d-k)/2$ in W. Taking the determinant we obtain a polynomial $D_k(\lambda) = D_k(x_1, \ldots, x_n)$. Since the field $\mathbb{K}$ is infinite, the existence of λ satisfying the Lefschetz property is equivalent to $D_k(x_1, \ldots, x_n) \neq 0$, for all $0 < k \leq n$.

The polynomials $D_k(\lambda)$ appear to be hard to compute explicitly. However, it is sufficient for our purposes to compute a single term in $D_k(\lambda)$: its leading term in the lexicographic order $x_1 > x_2 > \ldots > x_n$.

Definition 6.3.6. Let I be a subset of S. We say that w *I-dominates* v if $w = w'w''$, $v = v'v''$, with $w', v' \in W^I$, $w'', v'' \in W_I$ and $w' \geq v'$, $w'' \geq v''$ ($\geq$ is the usual Bruhat order). We say that an edge $w \xrightarrow{\gamma^\vee} v$ is *I-relevant* if v I-dominates w. A path connecting w to v is I-relevant if all its edges are I-relevant.

The Bruhat order $\leq$ is compatible with the projection $W \to W/W_I = W^I$ (cf. Lemma 4.4.4), that is if $v \geq w$ then $v' \geq w'$. It follows that v I-dominates w if and only if $v \geq w$ and $v'' \geq w''$.

Lemma 6.3.7. *Let $v, w \in W$ such that $v' = w'$. Then $v \geq w$ if and only if $v'' \geq w''$.*

Proof. Let $s \in S$ be such that $sv' < v'$. We have $sv' \in W^I$ by [Deo77, Lemma 3.1], thus $(sv)' = sv'$. Moreover, by the Property Z 1.1.1, we have $v \geq w$ if and only if $sv \geq sw$, so we can easily conclude by induction on $\ell(v')$. $\square$

Lemma 6.3.8. *Let $w \xrightarrow{\gamma^\vee} v$ be an edge in $\mathfrak{B}_\Phi$. Then $w \xrightarrow{\gamma^\vee} v$ is I-relevant if and only if $\ell(v') \leq \ell(w') + 1$.*

Proof. If $w \xrightarrow{\gamma^\vee} v$ is I-relevant, then $\ell(v) = \ell(w) + 1$ and $\ell(v'') \geq \ell(w'')$, so clearly $\ell(v') \leq \ell(w') + 1$.

Conversely, if $\ell(v') = \ell(w')$ then $v' = w'$ because of Lemma 4.4.4. Therefore $v'' > w''$ by Lemma 6.3.7 and $w \xrightarrow{\gamma^\vee} v$ must be I-relevant.

It remains to consider the case $\ell(v') = \ell(w') + 1$, or equivalently $\ell(v'') = \ell(w'')$. We claim that in this case we have $v'' = w''$, whence in particular $w \xrightarrow{\gamma^\vee} v$ is I-relevant. The claim is proven by induction on $\ell(v'') = \ell(w'')$. The case $\ell(v'') = 0$ is clear.

If $s \in I$ then, for any $z \in W$ we have $(zs)' = z'$ and $(zs)'' = z''s$. Let $s \in I$ such that $v''s < v''$. This implies, again by the Property Z, that $w \leq vs$ or $ws \leq vs$.

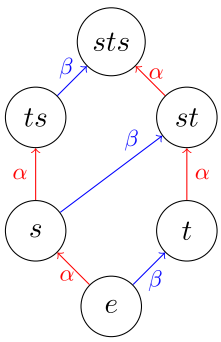
If $w \leq vs < v$, then $w = vs$. Thus we have $w' = (vs)' = v'$, which is a contradiction since $\ell(v') = \ell(w') + 1$. If $ws \leq vs$ then $ws \xrightarrow{s(\gamma)^\vee} vs$ is an edge in $\mathfrak{B}_\Phi$. Since $v' = (vs)'$ and $w' = (ws)'$ we have $\ell((vs)') = \ell((ws)') + 1$ and $\ell((ws)'') = \ell((vs)'') = \ell(v'') - 1$. Hence we can apply the inductive hypothesis to get $w''s = v''s$, thus $w'' = v''$. $\square$

In other words, the proof of Lemma 6.3.8 shows that an edge $w \xrightarrow{\gamma^\vee} v$ in $\mathfrak{B}_\Phi$ is I-relevant if and only if $v' = w'$ or $v'' = w''$.

Definition 6.3.9. The *I-degenerate Bruhat Graph* $\mathfrak{B}_\Phi^{I-\deg}$ is a graph having the same vertices as the Bruhat graph $\mathfrak{B}_\Phi$. The edges in $\mathfrak{B}_\Phi^{I-\deg}$ are the I-relevant edges in $\mathfrak{B}_\Phi$: for any I-relevant edge $w \xrightarrow{\gamma^\vee} v$ in $\mathfrak{B}_\Phi$ we put an edge $w \xrightarrow{\pi_I(\gamma^\vee)} v$ in $\mathfrak{B}_\Phi^{I-\deg}$.

In particular, in the case $I = S \setminus \{s\}$ the edges in $\mathfrak{B}_\Phi^{I-\deg}$ are all labeled by $m\alpha_s$, with $m \in \mathbb{N}_{>0}$, or by a root in $\Phi(I)$.

Example 6.3.10. Let Φ be the root system of type A_2 as in the Example 6.3.3 and let $I = \{t\}$. Then ts does not I-dominate t, although $ts > t$ in the Bruhat order. In fact, $(ts)'' = e \not\geq t = t''$. Thus the edge $t \longrightarrow ts$ is not $\{t\}$-relevant. The degenerate Bruhat graph $\mathfrak{B}_\Phi^{\{t\}-\deg}$ is:



The graph $\mathfrak{B}_\Phi^{I-\deg}$ describes a new action $\overset{I}{\cdot}$ of λ on $H^\bullet(X,\mathbb{K})$. We say

$$\lambda \overset{I}{\cdot} P_w = \sum_{w \xrightarrow{\delta} v \in \mathfrak{B}_\Phi^{I-\deg}} \lambda(\delta) P_v$$

where the sum runs over all edges $w \xrightarrow{\delta} v$ starting in w in $\mathfrak{B}_\Phi^{I-\deg}$ (or equivalently all I-relevant edges starting in w in $\mathfrak{B}_\Phi$). We call it the *I-degenerate action* of λ.

The new graph $\mathfrak{B}_\Phi^{I-\deg}$ can be obtained as product of two smaller graphs. In fact, we have $\mathfrak{B}_\Phi^{I-\deg} \cong \mathfrak{B}_\Phi^I \times \mathfrak{B}_{\Phi(I)}$: at the level of vertices we have a bijection $W = W^I \times W_I$ and, because of Lemma 6.3.8, for any I-relevant edge $w \xrightarrow{\gamma^\vee} v$ we have two cases:

- $w' = v'$ and $w'' t_\gamma = v''$, so $w \xrightarrow{\pi_I(\gamma^\vee)} v$ comes from the edge $w'' \xrightarrow{\gamma^\vee} v''$ in $\mathfrak{B}_{\Phi(I)}$;
- $w'' = v''$ and $w' t_{w''(\gamma)} = v'$, so $w \xrightarrow{\pi_I(\gamma^\vee)} v$ comes from the edge $w' \xrightarrow{\pi_I(w''(\gamma)^\vee)} v'$ in $\mathfrak{B}_\Phi^I$.

Remark 6.3.11. It is not hard to see that the I-degenerate action described by $\mathfrak{B}_\Phi^{I-\deg}$ coincides with the action on $H^\bullet(G/\mathbf{P}_I \times \mathbf{P}_I/B,\mathbb{K}) \cong H^\bullet(G/\mathbf{P}_I,\mathbb{K}) \otimes H^\bullet(\mathbf{P}_I/B,\mathbb{K})$ defined as follows: if $\lambda = \sum_{i\in S} x_i \varpi_i$, $P_1 \in H^\bullet(G/\mathbf{P}_I,\mathbb{K})$ and $P_2 \in H^\bullet(\mathbf{P}_I/B,\mathbb{K})$ then

$$\lambda \overset{I}{\cdot} (P_1 \otimes P_2) = \Big(\sum_{i \in S\setminus I} x_i \varpi_i\Big) \cdot P_1 \otimes P_2 + P_1 \otimes \Big(\sum_{i\in I} x_i\varpi_i\Big) \cdot P_2.$$

For a polynomial $f \in \mathbb{K}[x_1,\dots,x_n]$ we denote by $\deg_i(f)$ its degree in the variable x_i and by $\mathrm{coeff}_{i,a}(f)$ the coefficient of x_i^a in f (thus $\mathrm{coeff}_{i,a}(f)$ is an element of $\mathbb{K}[x_1,\dots x_{i-1},x_{i+1},\dots x_n]$). We set $\deg_i(0) = -1$.

Recall that the elements of S are labeled as $\{1,2,\dots,n\}$ and that $\lambda = \sum_i x_i\varpi_i$ is a formal linear combination of the fundamental weights. We set $I = S \setminus \{1\}$.

We have

$$\deg_1(\lambda(\gamma^\vee)) = \begin{cases} 1 & \text{if } \gamma \in \Phi \setminus \Phi(I) \\ 0 & \text{if } \gamma \in \Phi(I) \end{cases}.$$

Notice that $\gamma \in \Phi(I)$ if and only if $t_\gamma \in W_I$.

Lemma 6.3.12. *Let $w, v \in W$ with $\ell(v) > \ell(w)$. Then:*

i) $\deg_1(C_{w,v}(\lambda)) \leq \ell(v') - \ell(w')$ *and we have equality if and only if there exists an I-relevant path connecting w to v;*

ii) $\mathrm{coeff}_{1,\ell(v')-\ell(w')}(C_{w,v}(\lambda)) \cdot x_1^{\ell(v')-\ell(w')} = \sum_{relevant} \lambda(\pi_I(\gamma_1^\vee))\lambda(\pi_I(\gamma_2^\vee))\dots\lambda(\pi_I(\gamma_k^\vee))$,

where the sum runs over all the I-relevant paths $w \xrightarrow{\gamma_1^\vee} w_1 \xrightarrow{\gamma_2^\vee} w_2 \xrightarrow{\gamma_3^\vee} \dots \xrightarrow{\gamma_k^\vee} v$ connecting w to v in $\mathfrak{B}_\Phi$.

Proof. *i)* We start with the case $\ell(v) = \ell(w) + 1$. If there are no edges connecting w to v in $\mathfrak{B}_\Phi$ then there is nothing to show.

Assume that there is an edge $w \xrightarrow{\gamma^\vee} v$ in $\mathfrak{B}_\Phi$, so that $C_{w,v}(\lambda) = \lambda(\gamma^\vee)$. If $w \xrightarrow{\gamma^\vee} v$ is not I-relevant by Lemma 6.3.8 we have $\ell(v') - \ell(w') \geq 2$, and the statement follows since $\deg_1(C_{w,v}(\lambda)) \leq 1$.

Assume now that $w \xrightarrow{\gamma^\vee} v$ is I-relevant, then $w' = v'$ or $w'' = v''$. Since $w'w''t_\gamma = v'v''$ we see that $w' = v'$ if and only if $t_\gamma \in W_I$, i.e. if and only if $\deg_1(C_{w,v}(\lambda)) = 0$.

The general case $\ell(v) > \ell(w) + 1$ follows since

$$C_{w,v}(\lambda) = \sum C_{w,w_1}(\lambda) C_{w_1,w_2}(\lambda) \dots C_{w_{k-1},v}(\lambda)$$

where the sum runs over all paths $w \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \dots \longrightarrow v$ in $\mathfrak{B}_\Phi$.

ii) We start with the case $\ell(v) = \ell(w) + 1$. If there are no I-relevant edges in $\mathfrak{B}_\Phi$ between w and v then both sides are 0. If there is an I-relevant edge $w \xrightarrow{\gamma^\vee} v$, then $C_{w,v}(\lambda) = \lambda(\gamma^\vee)$ and

$$\mathrm{coeff}_{1,\ell(v')-\ell(w')}(\lambda(\gamma^\vee)) \cdot x_1^{\ell(v')-\ell(w')} = \lambda(\pi_I(\gamma^\vee)).$$

The general case $\ell(v) > \ell(w) + 1$ easily follows. $\square$

We fix now an arbitrary $k \in \{1, 2, \dots, d\}$. Let $D_k^{(1)}(\lambda)$ be the Lefschetz determinant of the I-degenerate action of λ on $H^{d-k}(X, \mathbb{K})$, described by $\mathfrak{B}_\Phi^{I-\deg}$, computed in the same basis used for $D_k(\lambda)$. In other words $D_k^{(1)}(\lambda)$ is the determinant of the map $\lambda^k \overset{I}{\cdot} (-) : H^{d-k}(G/\mathbf{P}_I \times \mathbf{P}_I/B, \mathbb{K}) \to H^{d+k}(G/\mathbf{P}_I \times \mathbf{P}_I/B, \mathbb{K})$ described above.

Lemma 6.3.13. *Let* $M_k = \sum_{\ell(v)=(d+k)/2} l(v') - \sum_{\ell(w)=(d-k)/2} l(w')$. *Then we have:*

i) $\deg_1(D_k(\lambda)) \leq M_k$*;*

ii) The polynomial $D_k^{(1)}(\lambda)$ *is homogeneous of degree* M_k *in* x_1*;*

iii) $\mathrm{coeff}_{1,M_k}(D_k(\lambda)) \cdot x_1^{M_k} = D_k^{(1)}(\lambda)$.

Proof. The determinant polynomial can be expressed as

$$D_k(\lambda) = \sum_\sigma \mathrm{sgn}(\sigma) C_{w_1,\sigma(w_1)}(\lambda) C_{w_2,\sigma(w_2)}(\lambda) \dots C_{w_{n(k)},\sigma(w_{n(k)})}(\lambda)$$

where σ runs over all possible bijections between elements in W of length $(d-k)/2$ and $(d+k)/2$ (and the sign is determined by the chosen order of the Schubert basis). Then i) follows from Lemma 6.3.12.

The terms in the sum which contribute to $\mathrm{coeff}_{1,M_k}(D_k(\lambda))$ are precisely the ones coming from I-relevant paths, i.e. the one which are also in $D_k^{(1)}(\lambda)$, so ii) and iii) also follow. $\square$

We can now reiterate this procedure. Let $S = I_0 \supset I_1 \supset I_2 \supset \dots \supset I_n = \emptyset$ be such that $I_{j-1} \setminus I_j = \{j\}$ for any $1 \leq j \leq n$. We have a length preserving bijection of sets:

$$\Psi : W \cong W^{I_1} \times W_{I_1}^{I_2} \times \dots \times W_{I_{n-1}}.$$

We write $\Psi(w) = \left(w^{(1)}, w^{(2)}, \dots, w^{(n)}\right)$. The degenerated graph $\mathfrak{B}_\Phi^{(1)} := \mathfrak{B}_\Phi^{I_1-\deg}$ is isomorphic to $\mathfrak{B}_\Phi^{I_1} \times \mathfrak{B}_{\Phi(I_1)}$. It can be degenerated again into $\mathfrak{B}_\Phi^{(2)} := \mathfrak{B}_\Phi^{I_1} \times \mathfrak{B}_{\Phi(I_1)}^{I_2\text{-deg}} \cong \mathfrak{B}_\Phi^{I_1} \times \mathfrak{B}_{\Phi(I_1)}^{I_2} \times \mathfrak{B}_{\Phi(I_2)}$, and so on up to

$$\mathfrak{B}_\Phi^{(n-1)} := \mathfrak{B}_\Phi^{I_1} \times \mathfrak{B}_{\Phi(I_1)}^{I_2} \times \dots \times \mathfrak{B}_{\Phi(I_{n-1})}.$$

We set $\mathfrak{B}_\Phi^{(0)} := \mathfrak{B}_\Phi$ and $\mathfrak{B}_\Phi^{(n)} := \mathfrak{B}_\Phi^{(n-1)}$.

Definition 6.3.14. Each of the $\mathfrak{B}_{\Phi}^{(j)}$ describes a new action of λ on $H^{\bullet}(X,\mathbb{K})$, which we call the j^{th}*-degenerate action* and we denote by $\stackrel{j}{\cdot}$. We say that v j*-dominates* w if $v^{(i)} \geq w^{(i)}$ for any $i \leq j$ and $v^{(j+1)} \dots v^{(n)} \geq w^{(j+1)} \dots w^{(n)}$.

We say that an edge $w \xrightarrow{\gamma^\vee} v$ is j*-relevant* if v j-dominates w. A path connecting w to v is j-relevant if all its edges are j-relevant.

For $1 \leq j \leq n$, let $C_{w,v}^{(j)}(\lambda)$ be the coefficient of P_v in $\lambda^h \stackrel{j}{\cdot} P_w$, where $\ell(v) - \ell(w) = h$. Thus Lemma 6.3.12.ii can be restated as:

$$\operatorname{coeff}_{1,\ell(v^{(1)})-\ell(w^{(1)})}(C_{w,v}^{(0)}(\lambda)) \cdot x_1^{\ell(v^{(1)})-\ell(w^{(1)})} = C_{w,v}^{(1)}(\lambda).$$

We also have:

Lemma 6.3.15. *Let $w, v \in W$ with $\ell(v) > \ell(w)$ and $0 \leq j \leq n-1$. Then:*

i) $\deg_{j+1} C_{w,v}^{(j)}(\lambda) \leq \ell(v^{(j+1)}) - \ell(w^{(j+1)})$ *and the equality holds if and only if there is a $(j+1)$-relevant path connecting v and w;*

ii) $\operatorname{coeff}_{j+1,\ell(v^{(j+1)})-\ell(w^{(j+1)})}(C_{w,v}^{(j)}(\lambda)) \cdot x_{j+1}^{\ell(v^{(j+1)})-\ell(w^{(j+1)})} = C_{w,v}^{(j+1)}(\lambda)$;

iii) $C_{w,v}^{(j+1)}(\lambda)$*, regarded as a polynomial in x_i, is homogeneous of degree $\ell(v^{(i)}) - \ell(w^{(i)})$ for $1 \leq i \leq j+1$.*

Proof. The same arguments as in the proof of Lemma 6.3.12 show (i) and (ii). Now (iii) follows by induction on j using (ii). □

For $0 \leq j \leq n$ let $D_k^{(j)}(\lambda)$ be the Lefschetz determinant obtained from the j^{th}-degenerate action of λ, computed in the same bases used for $D_k(\lambda)$. We have

$$D_k^{(j)}(\lambda) = \sum_{\sigma} \operatorname{sgn}(\sigma) C_{w_1,\sigma(w_1)}^{(j)}(\lambda) C_{w_2,\sigma(w_2)}^{(j)}(\lambda) \dots C_{w_{n(k)},\sigma(w_{n(k)})}^{(j)}(\lambda). \tag{6.3}$$

For any $1 \leq j \leq n$ let $M_k^{(j)} = \sum_{\ell(v)=\frac{d+k}{2}} \ell(v^{(j)}) - \sum_{\ell(w)=\frac{d-k}{2}} \ell(w^{(j)})$.

Lemma 6.3.16. *For any $0 \leq j \leq n-1$ we have:*

i) $\deg_{j+1} D_k^{(j)}(\lambda) \leq M_k^{(j+1)}$;

ii) $D_k^{(j)}(\lambda)$ *is homogeneous of degree $M_k^{(i)}$ in x_i for $1 \leq i \leq j$;*

iii) $\operatorname{coeff}_{j+1,M_k^{(j+1)}} D_k^{(j)}(\lambda) \cdot x_{j+1}^{M_k^{(j+1)}} = D_k^{(j+1)}(\lambda)$.

Proof. Using (6.3) and Lemma 6.3.15 this follows arguing just as in Lemma 6.3.13. □

Let $\mu_k = x_1^{M_k^{(1)}} x_2^{M_k^{(2)}} \cdot \ldots \cdot x_n^{M_k^{(n)}}$. We have the following:

Corollary 6.3.17. *All monomials in $D_k(\lambda) = D_k(x_1, \dots, x_n)$ are smaller than μ_k in the lexicographic order.*

The polynomial $D_k^{(n-1)}(\lambda)$ (which is equal to $D_k^{(n)}(\lambda)$) is homogeneous of degree $M_k^{(j)}$ in x_j for any $1 \leq j \leq n$, i.e. $D_k^{(n-1)}(\lambda) = R_k \mu_k$, with $R_k \in \mathbb{K}$, and the coefficient of the monomial μ_k in $D_k(\lambda)$ is R_k.

6.4 Hard Lefschetz for the maximal parabolic flag varieties

To show that the polynomials $D_k(\lambda)$ are not identically zero, it suffices now to show that, for some ordering of the simple reflections, we have $R_k = (\mu_k)^{-1} D_k^{(n-1)}(\lambda) \in \mathbb{K}^*$. This will be done by investigating whether the $(n-1)^{\text{th}}$-degenerate action of a weight λ has the Lefschetz property on $H^\bullet(X, \mathbb{K})$. This coincides with the action on

$$H^\bullet(G/\mathbf{P}_{I_1}, \mathbb{K}) \otimes H^\bullet(\mathbf{P}_{I_1}/\mathbf{P}_{I_2}, \mathbb{K}) \otimes \ldots \otimes H^\bullet(\mathbf{P}_{I_{n-1}}/B, \mathbb{K}),$$

where $\lambda = \sum x_i \varpi_i$ acts as multiplication by

$$x_1\varpi_1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes x_2\varpi_2 \otimes \ldots \otimes 1 + 1 \otimes 1 \otimes \ldots \otimes x_n\varpi_n.$$

Example 6.4.1. Let $W = \mathcal{S}_{n+1}$ be a Weyl group of type A_n. We label the simple reflections as follows:

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } (n-1) \text{ --- } n$$

Then $\mathbf{P}_{I_j}/\mathbf{P}_{I_{j+1}} \cong \mathbb{P}^{n+1-j}(\mathbb{C})$ for all $1 \leq j \leq n$. So the degenerate action of λ can be thought as multiplication by $\sum_{i=1}^n x_i\varpi_i$ on $\mathbb{K}[\varpi_1, \ldots, \varpi_n]/(\varpi_1^{n+1}, \ldots, \varpi_n^2)$.

The aim of this section is to consider the action of the fundamental weight ϖ_j on a single factor $H^\bullet(\mathbf{P}_{I_{j-1}}/\mathbf{P}_{I_j}, \mathbb{K})$. Obviously ϖ_j has the Lefschetz property if and only if $x_j\varpi_j$ has the Lefschetz property for every (or any) $x_j \in \mathbb{K}^*$.

We can assume $j = 1$. Since we can choose arbitrarily the ordering $\{1, 2, \ldots, n\}$ of S, for our goals it is enough for every irreducible root system to check the Lefschetz property on $H^\bullet(G/\mathbf{P}_{S\setminus\{1\}}, \mathbb{K})$ for only one particular choice of $\{1\}$.

Proposition 6.4.2. *Let Φ be an irreducible root system with simple roots S. Then we can always choose $1 \in S$ such that ϖ_1 has the Lefschetz property on $H^\bullet(G/\mathbf{P}_{S\setminus\{1\}}, \mathbb{K})$ for any field of characteristic $p > |\Phi^+|$.*

Proof. We set $I = S \setminus \{1\}$. The proof is divides into cases.

Case A_n: We label the simple reflections as in Example 6.4.1. We can choose $G = SL_{n+1}(\mathbb{C})$. Then the parabolic flag variety $G/\mathbf{P}_I$ is the Grassmannian of lines in $\mathbb{C}^{n+1}$, i.e. it is isomorphic to $\mathbb{P}^n(\mathbb{C})$. Then ϖ_1 has the Lefschetz property in $H^\bullet(G/\mathbf{P}_I, \mathbb{K}) \cong \mathbb{K}[\varpi_1]/(\varpi_1^{n+1})$ for any field $\mathbb{K}$.

Case B_n and C_n: We label the simple reflections as follows

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } (n-1) = n$$

If W is the Weyl group of type B_n (or C_n) is it easy to list all the elements in W^I and to draw the parabolic Bruhat graphs $\mathfrak{B}^I_{B_n}$ and $\mathfrak{B}^I_{C_n}$.

Notice also that $\mathbf{P}_I$ is cominuscule in type B_n and minuscule in type C_n. The parabolic flag varieties $G/\mathbf{P}_I$ are described in detail in these cases in [BL00, §9.3].

$\mathfrak{B}^I_{B_n}$	$\mathfrak{B}^I_{C_n}$
$\begin{tikzcd}[row sep=small] \bullet\ 12\dots(n-1)n(n-1)\dots21 \\ \vdots \arrow[u, "\alpha_1^\vee"] \\ \bullet\ (n-1)n(n-1)\dots21 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ n(n-1)\dots21 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ (n-1)\dots21 \arrow[u, "2\alpha_1^\vee"] \\ \vdots \arrow[u, "\alpha_1^\vee"] \\ \bullet\ 21 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ 1 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ id \arrow[u, "\alpha_1^\vee"] \end{tikzcd}$	$\begin{tikzcd}[row sep=small] \bullet\ 12\dots(n-1)n(n-1)\dots21 \\ \vdots \arrow[u, "\alpha_1^\vee"] \\ \bullet\ (n-1)n(n-1)\dots21 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ n(n-1)\dots21 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ (n-1)\dots21 \arrow[u, "\alpha_1^\vee"] \\ \vdots \arrow[u, "\alpha_1^\vee"] \\ \bullet\ 21 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ 1 \arrow[u, "\alpha_1^\vee"] \\ \bullet\ id \arrow[u, "\alpha_1^\vee"] \end{tikzcd}$

From this it is evident that if Φ is of type C_n then ϖ_1 has the Lefschetz property on $H^\bullet(G/\mathbf{P}_I, \mathbb{K})$ for every field $\mathbb{K}$, while if Φ is of type B_n then ϖ_1 has the Lefschetz property on $H^\bullet(G/\mathbf{P}_I, \mathbb{K})$ if and only if $\mathrm{char}(\mathbb{K}) \neq 2$.

Case D_n: We label the simple reflections as follows:

$$\begin{tikzcd}[column sep=small] & & & & & (n-1) \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & (n-2) \arrow[ur, dash] \arrow[dr, dash] & \\ & & & & & n \end{tikzcd}$$

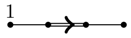
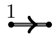
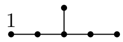
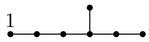
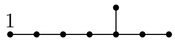
If W is the Weyl group of type D_n is it easy to list all the elements in W^I and to draw parabolic Bruhat graph $\mathfrak{B}^I_{D_n}$. Notice also that $\mathbf{P}_I$ is minuscule and the parabolic flag variety $G/\mathbf{P}_I$ is described in detail in [BL00, §9.3].

The parabolic Bruhat graph $\mathfrak{B}^I_{D_n}$ is:

$$\begin{tikzcd}[row sep=small] & \bullet\ 12\dots(n-2)n(n-1)\dots21 & \\ & \vdots \arrow[u, "\alpha_1^\vee"] & \\ & \bullet\ n(n-1)(n-2)\dots21 \arrow[u, "\alpha_1^\vee"] & \\ n(n-2)\dots21\ \bullet \arrow[ur, "\alpha_1^\vee"] & & \bullet\ (n-1)(n-2)\dots21 \arrow[ul, "\alpha_1^\vee"'] \\ & \bullet\ (n-2)\dots21 \arrow[ul, "\alpha_1^\vee"] \arrow[ur, "\alpha_1^\vee"'] & \\ & \vdots \arrow[u, "\alpha_1^\vee"] & \\ & \bullet\ 21 \arrow[u, "\alpha_1^\vee"] & \\ & \bullet\ 1 \arrow[u, "\alpha_1^\vee"] & \\ & \bullet\ id \arrow[u, "\alpha_1^\vee"] & \end{tikzcd}$$

It follows that ϖ_1 has the Lefschetz property if and only if $\mathrm{char}(\mathbb{K}) \neq 2$.

Exceptional Root Systems: We computed, with the help of the software Magma [BCP97], for each of the exceptional Weyl groups the set of primes p such that ϖ_1 has no Lefschetz property on $H^\bullet(G/\mathbf{P}_{S\setminus\{1\}}, \mathbb{K})$ for an infinite field $\mathbb{K}$ of characteristic p. We indicate in the Dynkin diagram the choice made for the vertex 1.

Root System	$\lvert\Phi^+\rvert$	Dynkin Diagram	Primes with no Lefschetz property for ϖ_1
F_4	24		$2, 3, 13$
G_2	6		2
E_6	36		$2, 3, 13$
E_7	63		$2, 3, 5, 7, 19, 23$
E_8	120		$2, 3, 5, 7, 19, 29, 31, 37, 41, 43, 47, 53$

This completes the proof of Proposition 6.4.2. □

The following Lemma is standard:

Lemma 6.4.3. *Let $\mathbb{K}$ be a field and $V = \bigoplus_{0\le k\le 2d} V^k$ a finite dimensional graded $\mathbb{K}$-vector space. Let $\eta : V \to V$ be a linear map of degree 2 with the Lefschetz property, i.e. $\eta^k : V^{d-k} \to V^{d+k}$ is an isomorphism for any k. Then there exists a decomposition of V, called the* Lefschetz decomposition, *in the form*

$$V = \bigoplus_{\substack{0\le k\le d \\ 1\le i\le r_k}} \mathbb{K}[\eta] p_{k,i}$$

where $\{p_{k,i}\}_{1\le i\le r_k}$ is any basis of $V^{d-k} \cap \mathrm{Ker}(\eta^{k+1})$.

In particular, if $V = \bigoplus_{k,i} \mathbb{K}[\eta]p_{k,i}$ is a primitive decomposition we get a basis $\{\eta^l p_{k,i}\}$ (with $0 \le k \le d$, $1 \le i \le r_k$ and $0 \le l \le k$) of V.

The existence of the Lefschetz decomposition implies that, after changing the basis, the map η can be represented by a graph which is a disjoint union of simple strings.

Example 6.4.4. Let Φ be of type D_4 with the reflections labeled as above. Then, if $\mathrm{char}(\mathbb{K}) \neq 2$, we can choose $\{P_{id}, P_1, P_{21}, P_{321} + P_{421}, P_{321} - P_{421}, 2P_{4321}, 2P_{24321}, 2P_{124321}\}$ as a basis of $H^\bullet(G/\mathbf{P}_{S\setminus\{1\}}, \mathbb{K})$. In this basis multiplication by ϖ_1 is represented by the following graph:

$$P_{id} \xrightarrow{\varpi_1} P_1 \xrightarrow{\varpi_1} P_{21} \xrightarrow{\varpi_1} P_{321} + P_{421} \xrightarrow{\varpi_1} 2P_{4321} \xrightarrow{\varpi_1} 2P_{24321} \xrightarrow{\varpi_1} 2P_{124321}$$

$$P_{321} - P_{421}$$

6.5 Hard Lefschetz for Artinian complete intersection monomial rings

In this section, let $\mathbb{K}$ denote an arbitrary field of characteristic p.

Theorem 6.5.1 ([Pro90]). *Let $A = \mathbb{K}[\varpi_1, \varpi_2, \ldots, \varpi_n]/(\varpi_1^{d_1}, \varpi_2^{d_2}, \ldots, \varpi_n^{d_n})$. We regard A as a graded algebra over $\mathbb{K}$ in which the ϖ_i have degree 2. Let $d = \sum_{i=1}^n (d_i - 1)$. Then if $p > d$ multiplication by $\lambda = \sum x_i \varpi_i$ has the Lefschetz property on A if $x_i \in \mathbb{K}^*$ for all i.*

Let $\lambda = \sum x_i \varpi_i$ with $x_i \in \mathbb{K}^*$. In [Pro90, Corollary 2], Proctor actually gives a closed formula for the determinants $D_k(\lambda)$ of $\lambda^k : A^{d-k} \to A^{d+k}$. From Proctor's formula we can easily check that all the determinants are in $\mathbb{K}^*$ if $p > d$, hence λ has the Lefschetz property on A.

We give here an alternative elementary proof based on the representation theory of $\mathfrak{sl}_2(\mathbb{K})$ (a similar proof for $\mathbb{K}$ of characteristic 0 appears in [Wat87]). If $d \leq 1$ then $A \cong \mathbb{K}$ or $A \cong \mathbb{K}[\varpi_1]/(\varpi_1^2)$. In both cases the statement of Theorem 6.5.1 is clear. We can therefore assume $\operatorname{char} \mathbb{K} = p > 2$. Let

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so that $\{f, h, e\}$ is a basis of $\mathfrak{sl}_2(\mathbb{K})$.

For any integer $0 \leq m \leq p-1$ let $L(m)$ be the irreducible $\mathfrak{sl}_2(\mathbb{K})$-module of highest weight m. These modules can be obtained by reduction from the characteristic 0 case, i.e. $L(m)$ has a basis $\{v_{m-2k}\}_{0 \leq k \leq m}$ such that the action of $\mathfrak{sl}_2(\mathbb{K})$ is described by

$$h \cdot v_i = i v_i, \qquad e \cdot v_i = \frac{m+i+2}{2} v_{i+2}, \qquad f \cdot v_i = \frac{m-i+2}{2} v_{i-2}$$

for any i, where we set $v_{m+2} = v_{-m-2} = 0$.

Let $\mathcal{U} = \mathcal{U}(\mathfrak{sl}_2(\mathbb{K}))$ be the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{K})$. Let M be a $\mathfrak{sl}_2(\mathbb{K})$-module and let $v \in M$ be a highest weight vector of weight a with $0 \leq a \leq p-1$, i.e. $h \cdot v = av$ and $e \cdot v = 0$. Then $\mathcal{U} \cdot v = \operatorname{span}\langle f^k \cdot v \mid k \geq 0 \rangle$ is a submodule of M such that $\dim(\mathcal{U} \cdot v) \geq a+1$. Moreover, $\dim(\mathcal{U} \cdot v) = a+1$ if and only if $\mathcal{U} \cdot v \cong L(a)$.

We consider the Casimir element $C = 2ef + 2fe + h^2 \in \mathcal{U}$. It is easy to check that C lies in the center of $\mathcal{U}$, therefore C acts as a scalar on any highest weight module $\mathcal{U} \cdot v$. If v is of weight m, we get $C \cdot v = (2ef + h^2) \cdot v = (2m + m^2)v$, so C acts as the scalar $2m + m^2$ on $\mathcal{U} \cdot v$.

Proposition 6.5.2. *Let $m_1, m_2, \ldots, m_n$ be non-negative integers such that their sum $d := \sum_{i=1}^n m_i$ is smaller than p. Then $L(m_1) \otimes L(m_2) \otimes \ldots \otimes L(m_n)$ is a semisimple $\mathfrak{sl}_2(\mathbb{K})$-module and it decomposes as $\bigoplus_{a=0}^d L(a)^{\nu_a}$, where ν_a are non-negative integers.*

Proof. By induction it is enough to consider the case $n = 2$. Let $a = m_1$ and $b = m_2$. We can assume $a \geq b$. Let $\{v_{a-2k}\}_{0 \leq k \leq a}$ (resp. $\{w_{b-2k}\}_{0 \leq k \leq b}$) be a basis of $L(a)$ (resp. $L(b)$) as described above.

As in the characteristic 0 case, for any integer k with $0 \leq k \leq b$, there exists a highest weight vector $v_{a-b+2k} \in L(a) \otimes L(b)$ of weight $a - b + 2k$. In fact, e induces a map

$$e : \operatorname{span}\langle v_i \otimes w_j \mid i + j = a - b + 2k \rangle \longrightarrow \operatorname{span}\langle v_i \otimes w_j \mid i + j = a - b + 2k + 2 \rangle$$

which has a non-trivial kernel, as we can easily see by comparing the dimensions.

For any k, we have $(\mathcal{U} \cdot v_{a-b+2k}) \subseteq \operatorname{Ker}(C - 2(a-b+2k) - (a-b+2k)^2)$. Since

$$2(a-b+2k) + (a-b+2k)^2 \not\equiv 2(a-b+2h) - (a-b+2h)^2 \pmod{p}.$$

for any k and h such that $0 \leq k, h \leq b$ and $k \neq h$ we have

$$\bigoplus_{k=0}^{b} \mathcal{U} \cdot v_{a-b+2k} \subseteq \bigoplus_{k=0}^{b} \operatorname{Ker}(C - 2(a-b+2k) - (a-b+2k)^2) \subseteq L(a) \otimes L(b).$$

Now, by comparing the dimensions we must have $\dim(\mathcal{U} \cdot v_{a-b+2k}) = a - b + 2k + 1$, hence $(\mathcal{U} \cdot v_{a-b+2k}) \cong L(a-b+2k)$. Finally we obtain

$$L(a) \otimes L(b) = L(a-b) \oplus L(a-b+2) \oplus \ldots \oplus L(a+b).$$

$\square$

Proof of Theorem 6.5.1. For any $x \in \mathbb{K}^*$, the algebra $\mathbb{K}[\varpi]/(\varpi^a)$ can be seen as a $\mathfrak{sl}_2(\mathbb{K})$-module, where e acts as multiplication by $x\varpi$ and h acts as multiplication by $2k - a + 1$ on ϖ^k. If $a \le p$, then $\mathbb{K}[\varpi]/(\varpi^a) \cong L(a-1)$ as a $\mathfrak{sl}_2(\mathbb{K})$-module.

Therefore, by Proposition 6.5.2, if $d = \sum_{i=1}^n (d_i - 1) < p$ the algebra

$$A \cong \mathbb{K}[\varpi_1]/(\varpi_1^{d_1}) \otimes \mathbb{K}[\varpi_2]/(\varpi_2^{d_2}) \otimes \ldots \otimes \mathbb{K}[\varpi_n]/(\varpi_n^{d_n})$$

is semisimple as a $\mathfrak{sl}_2(\mathbb{K})$-module, where e acts as multiplication by $x_1\varpi_1 + x_2\varpi_2 + \ldots + x_n\varpi_n$ and h acts as multiplication by $(2\sum_{i=1}^n k_i - d)$ on $\varpi_1^{k_1} \otimes \varpi_2^{k_2} \otimes \ldots \otimes \varpi_n^{k_n}$. In particular, we can decompose A as a direct sum of $L(m)$'s, with $m \le p-1$. Now the thesis easily follows since $e \in \mathfrak{sl}_2(\mathbb{K})$ has the Lefschetz property on $L(m)$, for any $0 \le m \le p-1$. $\square$

6.6 Proof of the main theorem

The case $\mathrm{char}(\mathbb{K}) \le |\Phi^+|$ is discussed in Remark 6.2.1 and Example 6.2.3, so we can assume $\mathrm{char}(\mathbb{K}) = p > |\Phi^+|$.

Let $\lambda = \sum_{i=1}^n x_i\varpi_i$ as before. In view of Corollary 6.3.17, it remains to show that the polynomials $D_k^{(n-1)}(\lambda)$, with $1 < k \le n$, are non-zero for some indexing of $S = \{1, 2, \ldots, n\}$. In other words we have to show that the $(n-1)^{\text{th}}$-degenerate action of λ defined by the graph $\mathfrak{B}_\Phi^{(n-1)} := \mathfrak{B}_\Phi^{I_1} \times \mathfrak{B}_{\Phi(I_1)}^{I_2} \times \ldots \times \mathfrak{B}_{\Phi(I_{n-1})}$ satisfies the hard Lefschetz theorem.

Since $\mathrm{char}(\mathbb{K}) = p > |\Phi^+|$, it follows from Proposition 6.4.2 that we can choose an ordering of S such that, for any $1 \le j \le n$ and any $x_j \in \mathbb{K}^*$, $x_j\varpi_j$ has the Lefschetz property on $H^\bullet(\mathbf{P}_{I_{j-1}}/\mathbf{P}_{I_j}, \mathbb{K})$. Therefore, as in Lemma 6.4.3, we have a Lefschetz decomposition

$$H^\bullet(\mathbf{P}_{I_{j-1}}/\mathbf{P}_{I_j}, \mathbb{K}) = \bigoplus_{\substack{0 \le k \le d_j \\ 1 \le i \le r_k}} \mathbb{K}[\varpi_j]p_{k,i}^j$$

where $d_j = \dim(\mathbf{P}_{I_{j-1}}/\mathbf{P}_{I_j})$ and $\{p_{k,i}^j\}_{1 \le i \le r_k}$ is a basis of

$$H^{d_j - k}(\mathbf{P}_{I_{j-1}}/\mathbf{P}_{I_j}, \mathbb{K}) \cap \mathrm{Ker}(\varpi_j^{k+1}).$$

We obtain a decomposition

$$\begin{aligned}
H^\bullet(G/\mathbf{P}_{I_1}, \mathbb{K}) \otimes H^\bullet(\mathbf{P}_{I_1}/\mathbf{P}_{I_2}, \mathbb{K}) \otimes \ldots \otimes H^\bullet(\mathbf{P}_{I_{n-1}}/B, \mathbb{K}) &\cong \\
\cong \bigoplus_{\substack{i_1, i_2, \ldots, i_n \\ k_1, k_2, \ldots, k_n}} \mathbb{K}[\varpi_1]p_{k_1,i_1}^1 \otimes \mathbb{K}[\varpi_2]p_{k_2,i_2}^2 \otimes \ldots \otimes \mathbb{K}[\varpi_n]p_{k_n,i_n}^n &\cong \\
\cong \bigoplus_{\substack{i_1, i_2, \ldots, i_n \\ k_1, k_2, \ldots, k_n}} \mathbb{K}[\varpi_1]/(\varpi_1^{k_1+1}) \otimes \mathbb{K}[\varpi_2]/(\varpi_2^{k_2+1}) \otimes \ldots \otimes \mathbb{K}[\varpi_n]/(\varpi_n^{k_n+1}) &\cong \\
\cong \bigoplus_{\substack{i_1, i_2, \ldots, i_n \\ k_1, k_2, \ldots, k_n}} \mathbb{K}[\varpi_1, \varpi_2, \ldots, \varpi_n]/(\varpi_1^{k_1+1}, \varpi_2^{k_2+1}, \ldots, \varpi_n^{k_n+1})&
\end{aligned}$$

into λ-stable subspaces. Since

$$\sum_{j=1}^n k_j \le \sum_{j=1}^n d_j = \sum_{j=1}^n \dim\left(\mathbf{P}_{I_{j-1}}/\mathbf{P}_{I_j}\right) = \dim(G/B) = |\Phi^+|$$

from Theorem 6.5.1 it follows that λ has the Lefschetz property on every single direct summand of the decomposition. This proves Theorem 6.2.2.

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