

LOWER SEMICONTINUITY, OPTIMIZATION AND
REGULARITY OF VARIATIONAL PROBLEMS UNDER
GENERAL PDE CONSTRAINTS

DISSERTATION

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Abstract

We investigate variational properties of integral functionals defined on spaces of measures satisfying a general PDE constraint. The study of these properties is motivated by the following three problems: existence of solutions, optimality conditions of variational solutions, and regularity of optimal design problems. After the introduction, each chapter of this dissertation corresponds to one of the topics listed above.

The first chapter is introductory, we state the main results of this work and discuss how their different subjects relate to each other. In this chapter we also discuss the historical background in which our work originated.

The second chapter, on the study of existence, focuses in providing sufficient and necessary conditions for the weak* lower semicontinuity of a general class of integral functionals defined for PDE constrained spaces of measures. We provide a characterization based on recent developments on the structure of PDE-constrained measures and their relation to a convexity class (quasiconvexity); our methods rely on blow-up techniques, rigidity arguments, and the study of generalized Young measures.

The second chapter is dedicated to the analysis and derivation of saddle-point conditions of minimizers of convex integral functionals defined on spaces of PDE-constrained measures (even in higher generality than in the first chapter). The analysis is carried out by means of convex analysis and duality methods.

Lastly, the fourth chapter discusses the regularity properties of a general model in optimal design. Our variational model involves a Dirichlet energy term (defined for a general class of elliptic operators) and a perimeter term (often associated to the design). In this work, we use Gamma-convergence techniques and derive a monotonicity formula to show a standard lower bound on the density of the perimeter of optimal designs. The conclusion of the results then follows from standard geometric measure theory arguments.

In memoriam
Adolfo Arroyo Villaseñor (1931–2011)

To my dear family

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1 Introduction

Three of the Hilbert's famous "*Mathematische Probleme*", problems 19th, 20th, and 23rd, discuss the study of *existence*, *uniqueness*, and *regularity properties* of solutions to variational problems.¹ Hilbert's questions cemented the foundations of the modern variational theory of integral functionals which was widely developed throughout the 20th century and continue to raise interest until today. While the better part of the research conducted in this period has been devoted to integrals defined on gradients — this comprises the pioneering work of Morrey [56] on the theory of existence, and the methods of De Giorgi [27] and Nash [64] which constitute a beautiful answer to Hilbert's 20th problem on the regularity of solutions, the evident variety of applications in different areas of physics, mathematics, economy, biology and other engineering-related sciences have provided continual motivation to study differential structures other than the gradient structure. In this general setting, Murat and Tartar [59–61, 73, 74] introduced the theory of *compensated compactness* which develops in the context of *\mathcal{A} -free fields*.

This dissertation focuses in a similar setting, the variational theory of integral functionals defined on functions (or measures) satisfying a general PDE constraint; here, of course, by variational theory we mean existence, conditions of optimality, and regularity of variational solutions.

Due to the amount of material to be presented in this work, we shall postpone precise definitions and complete versions of the results to the next chapters.

1.1 Calculus of Variations in the \mathcal{A} -free setting

Since its inception, a good part of the variational theory of the calculus of variations has focused in the understanding of functionals of the the form

$$u \mapsto \int_{\Omega} f(x, \nabla u(x)) \, dx, \quad \text{where } u \text{ belongs to a class of functions } \mathcal{U}.$$

The systematic study of variational integrals defined on gradients, with a few exceptions, has been successfully developed over the past centuries. Nowadays we have established methods and characterizations — depending on the behavior of integrand f and the class \mathcal{U} — which predict the existence of a minimizers, which frequently possess higher regularity properties than the ones originally prescribed by \mathcal{U} .

In spite of the seemingly well-developed integral theory defined on gradients, we know less when

¹Originally, presented by D. Hilbert in the International Congress of Mathematicians which took place in Paris, 1900. Later translated and published in the english language in [40]

it comes to understanding the integral theory for more general PDE structures. We briefly recall that for a sufficiently regular vector field $v : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$\operatorname{curl} v = 0 \quad \Leftrightarrow \quad v = \nabla u \quad \text{for some } u : \Omega \rightarrow \mathbb{R},$$

where

$$\operatorname{curl} v := \left(\frac{\partial v^j}{\partial x_i} - \frac{\partial v^i}{\partial x_j} \right)_{ij} = 0, \quad 1 \leq i, j \leq d.$$

The need for a well-established variational theory in a more general setting is motivated by the wide variety of physical models arising from more general linear PDE constraints than $\operatorname{curl} v = 0$. This is the case in continuum mechanics, electromagnetism, linear elasticity, linear plate theory models, and various low-volume fraction optimal design problems, just to name a few.

From a variational viewpoint, a sufficiently general and physically relevant problem is the minimization of integral functionals of the form

$$v \mapsto I_f[v] \equiv \int_{\Omega} f(x, v(x)) \, dx, \quad \text{defined in a class of functions } \mathcal{U}, \quad (1.1)$$

whose elements $v : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^N$ satisfy a PDE constraint of the type

$$\mathcal{A}v = \sum_{|\alpha| \leq k} A_{\alpha} \partial^{\alpha} v = 0, \quad \text{in the sense of distributions.} \quad (1.2)$$

Here, we assume that \mathcal{A} satisfies Murat's *constant rank* property — its principal symbol has constant rank as a linear operator when evaluated in \mathbb{S}^{d-1} .

In this thesis we address the questions of existence, optimality and regularity in the setting (1.1)-(1.2) as follows:

Chapter 1. We gather and discuss new developments on the *existence* theory of the minimization of (1.1) under the PDE constraint (1.2) for the unsolved case when $f : \Omega \rightarrow [0, \infty)$ has uniform linear growth. We focus on the lower semicontinuity properties of I_f and provide a characterization of its relaxation on a subspace of measures where (1.1)-(1.2) is a well-posed problem.

Chapter 2. We study the sufficient and necessary *optimality conditions* for minimizers of (1.1)-(1.2) when f has linear-growth and is convex in its second argument. Our techniques involve convex analysis and duality methods.

Chapter 3. We study a general class of optimal design problems — including a perimeter penalization — which are related to the minimization of (1.1)-(1.2) when f is a “double-well energy” with quadratic growth. Our results extend well-known partial *regularity* results for the optimal structures of linear conductivity models to models involving general elliptic systems.

1.2 Theory of existence

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $f : \Omega \rightarrow [0, \infty)$ be a continuous integrand with linear growth at infinity, that is, there exists a positive real number M such that $M^{-1}|A| \leq f(x, A) \leq M(1 + |A|)$ for all $(x, A) \in \Omega \times \mathbb{R}^N$. We focus on the following variational problem:

$$\text{Minimize } I_f \text{ in the space } \ker \mathcal{A} := \{v \in L^1(\Omega; \mathbb{R}^N) : \mathcal{A} v = 0\}.$$

This minimization problem is, in general, not *well-posed* in the sense that minimizers might fail to exist. Concretely, existence by the direct method relies on finding a suitable topology on $\ker \mathcal{A}$ for which minimizing sequences are compact and the functional I_f is lower semicontinuous.² In a nutshell, one aims to find a minimizing sequence $(v_j) \subset \ker \mathcal{A}$ which converges (in some topology τ) to a limit $v_\infty \in \ker \mathcal{A}$, to subsequently apply the lower semicontinuity of I_f (also with respect to τ) from which it follows that v_∞ is a minimizer.

The task of choosing the aforementioned topology can be thought of as a competition between the compactness and continuity properties. The vital point is that, in our setting, $\ker \mathcal{A}$ might fail to be closed for the relevant pre-compact topologies, which in the content of the discussion above means that the candidate minimizer v_∞ might not belong to the admissible class $\ker \mathcal{A}$. To better portray the difficulties arising from the application of the direct method over L^1 spaces, let us take a minimizing sequence (v_j) , i.e., such that

$$I_f[v_j] \rightarrow \inf \left\{ \int_{\Omega} f(x, v(x)) \, dx : v \in \ker \mathcal{A} \right\}.$$

Compactness by relaxation: Under standard coercivity conditions on the integrand (e.g., $f(A) \geq M^{-1}|A|$), it is easy to check that $\sup_j \|v_j\|_{L^1(\Omega)} < \infty$. However, since L^1 spaces are not reflexive, the sequence (v_j) might fail to be pre-compact for the weak L^1 topology — unless, of course, the sequence $(|v_j|)$ is equi-integrable. For this reason we cannot expect that $v_j \rightharpoonup v$ for some $v \in L^1(\Omega; \mathbb{R}^N)$. The usual solution is to extend I_f to a (lower semicontinuous) functional \bar{I}_f defined on a larger class $\bar{\mathcal{U}}$ where minimizing sequences are compact — thus, minimizers can be extracted from minimizing sequences. This procedure is known as *relaxation*. A priori, and in this general setting, there might not be a unique way to relax the problem. In this case it suffices to ignore the differential constraint. We observe that minimizing sequences (v_j) are compact when considering each v_j as a *signed vector-valued measure* via the embedding $L^1(\Omega; \mathbb{R}^N) \xhookrightarrow{c} \mathcal{M}(\Omega; \mathbb{R}^N) \cong (C_0(\Omega; \mathbb{R}^N))^* : v \mapsto v \llcorner \mathcal{L}^d \llcorner \Omega$. Henceforth, the relaxed minimization problem reads

$$\text{Minimize } \bar{I}_f \text{ among Radon measures in } \ker_{\mathcal{M}} \mathcal{A} := \{\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) : \mathcal{A} \mu = 0\}.$$

²While the classical theory concerned mostly the discrimination of (already existing) extremal solutions, the so-called *direct methods* introduced by Hilbert, Lebesgue, and Tonelli provided a new way to study the coveted existence of solutions.

It turns out, as will be motivated in the next subsection, that the extended functional \bar{I}_f takes the form

$$\bar{I}_f[\mu] \equiv \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu}{d|\mu^s|}(x)\right) d|\mu^s|(x), \quad \mu \in \ker \mathcal{M} \mathcal{A},$$

where, here and in what follows,

$$f^{\infty}(x, A) := \lim_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{f(x', tA')}{t}, \quad x \in \bar{\Omega}, A \in \mathbb{R}^N,$$

is the *strong recession function* of f , and where for a Radon measure μ , we write $\mu = \mu^a \mathcal{L}^d + \mu^s$ to denote its Lebesgue–Radon–Nikodým decomposition with respect to \mathcal{L}^d , the d -dimensional Lebesgue measure.

Lower semicontinuity: If \bar{I}_f is lower semicontinuous on weak* convergent \mathcal{A} -free sequences of measures, then $\bar{I}_f[\mu] \leq \lim_{j \rightarrow \infty} \bar{I}_f[v_j d\mathcal{L}^d] = \inf \bar{I}_f$ for every weak* limit μ of a minimizing sequence (v_j) , whence it follows that μ is a *solution* of the relaxed minimization problem. This will, however, fail for general integrands; we shall dedicate the rest of this section to further analysis on the lower semicontinuity properties of \bar{I}_f .

While the lack of weak-compactness on L^1 -bounded sets corresponds to the *concentration* of measure, the lower semicontinuity of \bar{I}_f extends to the scenarios where both concentration and oscillation effects might occur.

Problem 1. *Is there a generic characterization of the integrand $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$, that depends solely on the operator \mathcal{A} , and which is equivalent to the sequential weak* lower semicontinuity of \bar{I}_f when restricted to \mathcal{A} -free sequences of measures? That is, can we characterize those integrands $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ for which*

$$\lim_{j \rightarrow \infty} \bar{I}_f[\mu_j] \geq \bar{I}_f[\mu],$$

for all $\mu_j, \mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ such that $\mu_j \xrightarrow{*} \mu$ and $\mathcal{A} \mu_j = 0$.

1.2.1 The relaxation and the Young measure approach

In several minimization problems it has been observed that optimal designs tend to develop fine oscillations. With the aim of quantifying oscillation effects of weakly convergent sequences in L^p spaces, L. C. Young introduced the so-called Young measures [76–78].³ In this framework one speaks about Young measures *generated* by weakly convergent sequences. Later, the theory of Young measures was extended to the framework of *generalized* Young measures [3, 32], which was introduced to capture both oscillation and concentration effects.

Basically, a (generalized) Young measure generated by a uniformly bounded sequence $(v_j) \subset$

³Young measures were first introduced under the name of *generalized curves*.

$L^1(\Omega; \mathbb{R}^N)$ is a triple $\mathbf{v} = (v_x, \lambda_v, v_x^\infty)_{x \in \bar{\Omega}}$ where for each $x \in \bar{\Omega}$,

$$\begin{aligned} v_x &\in \mathcal{P}(\mathbb{R}^N) \text{ is a probability measure on } \mathbb{R}^N, \\ \lambda_v &\in \mathcal{M}^+(\bar{\Omega}) \text{ is a positive Radon measure on } \bar{\Omega}, \\ v_x^\infty &\in \mathcal{P}(\mathbb{S}^{N-1}) \text{ is a probability measure on } \mathbb{S}^{N-1}, \end{aligned}$$

and for which the limit representation (in the form a pairing)

$$\int_{\Omega} f(x, v_j) \, dx \rightarrow \langle\langle f, \mathbf{v} \rangle\rangle := \int_{\Omega} \underbrace{\left(\int_{\mathbb{R}^N} f(x, A) \, dv_x(A) \right)}_{\text{oscillatory effects}} \, dx + \int_{\bar{\Omega}} \underbrace{\left(\int_{\mathbb{S}^{N-1}} f^\infty(x, A) \, dv_x^\infty(A) \right)}_{\text{concentration of measure}} \, d\lambda_v(x),$$

holds for all continuous $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that the strong recession function f^∞ exists and is also continuous.

Moreover, there is a natural way to identify a Radon measure with an *elementary* Young measure by letting

$$\mu \mapsto \delta[\mu] = \left(\delta_{\mu^a(x)}, |\mu^s|, \delta_{\frac{d\mu}{d|\mu^s|}(x)} \right).$$

Formal derivation of \bar{I}_f . It turns out, as one could already deduce in the form of an ansatz, that the weak* lower semicontinuity of the relaxation of I_f is directly related to the weak* lower semicontinuity properties of the functional

$$\begin{aligned} \bar{I}_f[\mu] &\equiv \langle\langle f, \delta[\mu] \rangle\rangle \\ &= \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) \, dx + \int_{\Omega} f^\infty\left(x, \frac{d\mu}{d|\mu^s|}(x)\right) \, d|\mu^s|(x), \quad \mu \in \ker \mathcal{M} \mathcal{A}. \end{aligned}$$

Let us turn back once again to a weak* convergent sequence $(v_j) \subset \ker \mathcal{A}$, $v_j \mathcal{L}^d \lfloor_{\Omega} \xrightarrow{*} \mu$. By an additional compactness argument on the space of Young measures, we may further assume without loss of generality that (v_j) *generates* a Young measure $\mathbf{v} \in \mathbf{Y}(\Omega; \mathbb{R}^N)$. In particular $I_f[v_j] \rightarrow \langle\langle f, \mathbf{v} \rangle\rangle$, so that Problem 1 reduces to following problem:

Problem 2. *Characterize those continuous integrands $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$, with continuous recession function f^∞ , for which the inequality*

$$\langle\langle f, \mathbf{v} \rangle\rangle \geq \langle\langle f, \delta[\mu] \rangle\rangle$$

holds for all (generalized) Young measures \mathbf{v} satisfying the following properties:

1. *there exists a sequence $(v_j) \subset \ker \mathcal{A}$ which generates the Young measure \mathbf{v} , and*
2. *the barycenter of \mathbf{v} , defined as $[\mathbf{v}] := w^* \text{-lim}_j v_j$, coincides with the measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$.*

Actually, since lower semicontinuity is a local property, it is possible to further split the inequality above into a more precise form by requiring the following *Jensen-type inequalities* to hold:

1. at *regular points*,

$$f\left(\underbrace{\langle \text{id}_{\mathbb{R}^N}, \nu_x \rangle + \langle \text{id}_{\mathbb{R}^N}, \nu_x^\infty \rangle}_{=\frac{d\mu}{d\mathcal{L}^d}(x)} \frac{d\lambda_\nu}{d\mathcal{L}^d}(x)\right) \leq \langle f(x, \cdot), \nu_x \rangle + \langle f^\infty(x, \cdot), \nu_x^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x), \quad (1.3)$$

2. and, at *singular points*,

$$f^\infty\left(\underbrace{\langle \text{id}_{\mathbb{R}^N}, \nu_x^\infty \rangle}_{=\frac{d\mu}{d\lambda_\nu^\infty}(x)}\right) \leq \langle f^\infty(x, \cdot), \nu_x^\infty \rangle. \quad (1.4)$$

for all Young measures ν which are generated by \mathcal{A} -free sequences.

1.2.2 A weak notion of convexity

The formal derivation carried out in the lines above tells us that the lower semicontinuity of integral functionals in the \mathcal{A} -free setting, where both oscillation and concentration of measure is allowed, entails a weak form of *convexity* on the integrand $f(x, \cdot)$.

By Jensen's definition of convexity, which states that a function $h : \mathbb{R}^N \rightarrow \mathbb{R}$ is convex if

$$\int_{\Omega} h(A) \, d\kappa(A) \geq f(A_0)$$

for all probability measures $\kappa \in \mathcal{P}(\Omega; \mathbb{R}^N)$ with center of mass $\int_{\Omega} A \, d\kappa = A_0$, it would seem reasonable to expect \bar{I}_f to be weak* lower semicontinuous (in the sense of measures, on $\ker_{\mathcal{M}} \mathcal{A}$) if and only if

$$f(x, \cdot) \text{ is convex for all } x \in \Omega.$$

However, this first guess is somehow misleading. The subtlety here is the *additional* differential rigidity which \mathcal{A} -free sequences possess. Such questions were first considered by Murat and Tartar [59–61, 73, 74] in their *Compensated compactness* treatise, which, *a grosso modo*, states that oscillation effects may be significantly amortized by the rigidity of a differential constraint. In some sense, one expects $f(x, \cdot)$ to be convex along directions where \mathcal{A} -free sequences may oscillate and/or concentrate, and remain non-convex along all other directions. Therefore, the characterization of the functionals \bar{I}_f which are weak* lower semicontinuous passes through a certain weaker notion of “ \mathcal{A} -quasiconvexity” of $f(x, \cdot)$ and $f^\infty(x, \cdot)$ (compare Jensen's classical definition of convexity with the less restrictive inequalities (1.3)-(1.4)).

In the next lines we briefly discuss the notion \mathcal{A} -quasiconvexity, its origins, and its role as the natural answer to Problems 1 and 2.

The case of gradients

Almost 70 years ago, due to the great success of the direct method, mathematicians dedicated their efforts to investigate certain integrals of the form

$$u \mapsto \int_{\Omega} f(\nabla u) \, dx, \quad \text{where } u : \Omega \rightarrow \mathbb{R}^m \text{ is a Lipschitz function,}$$

and their lower semicontinuity properties under the uniform convergence of Lipschitz functions (weak* convergence in $W^{1,\infty}(\Omega; \mathbb{R}^m)$). The first successful attempt to establish necessary and sufficient conditions for the lower semicontinuity of such functionals was proposed by Morrey [56] through what he defined as a “quasi-convexity” condition on the behavior of f . Specifically, under standard p -growth assumptions, Morrey showed that f is *quasiconvex* if and only if

$$\int_{\Omega} f(\nabla u) \, dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} f(\nabla u_j) \, dx$$

for all weakly convergent sequences

$$u_j \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^m), \quad \text{such that } (|\nabla u_j|^p) \text{ is equi-integrable.}^{45}$$

Here, we say that a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is *quasiconvex* if for every $A \in \mathbb{M}^{m \times d}$,

$$f(A) \leq \int_Q f(A + \nabla \varphi(y)) \, dy \quad \text{for all } \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m),$$

where Q stands for the d -dimensional unit cube.⁶

This characterization covers the theory of existence for integrals defined on gradients under standard p -growth (with $p > 1$); see also [12] for the case of higher-order gradients. However, as we have already discussed, it is far from satisfactory for a number of applications which involve the space $BV(\Omega; \mathbb{R}^m)$ of *functions with bounded variation*.⁷ Understanding the concentration effects of L^1 -bounded sequences of gradients took a considerably longer time. It was not until the early 90’s that Ambrosio & Dal Maso [5], and Fonseca & Müller [38] showed that Morrey’s quasiconvexity condition would remain a necessary and sufficient condition for the lower semicontinuity of the *relaxed*

⁴The sequence $(|\nabla u_j|^p)_j$ is said to be equi-integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_j \left(\int_{\Omega \cap E} |\nabla u_j|^p \, dx \right) \leq \varepsilon, \quad \text{for all } E \subset \Omega \text{ Borel with } \mathcal{L}^d(E) \leq \delta;$$

this is, in turn, a way to prevent concentration of measure in weak* limits of $(|\nabla u_j|^p)$.

⁵Acerbi and Fusco [1] showed that the equi-integrability of $(|\nabla u_j|^p)$ can be dropped from the assumptions.

⁶Kinderlehrer and Pedregal [41] would show, almost 40 years after Morrey’s pioneering work, that the quasiconvexity of $f(x, \cdot)$, in the super-linear case $p > 1$, is equivalent to the Jensen inequality (1.3) on gradient Young measures (Young measures generated by sequences of gradients).

⁷The space of functions with bounded variation $BV(\Omega; \mathbb{R}^m)$ is the space of integrable functions whose distributional derivative is an $\mathbb{M}^{d \times N}$ -valued Radon measure, i.e., $BV(\Omega; \mathbb{R}^m) := \{u \in L^1(\Omega; \mathbb{R}^m) : Du \in \mathcal{M}(\Omega; \mathbb{M}^{d \times m})\}$.

functional

$$u \mapsto \int_{\Omega} f(\nabla u(x)) \, dx + \underbrace{\int_{\Omega} f^{\infty} \left(\frac{dD^s u}{d|D^s u|}(x) \right) d|D^s u|(x)}_{\text{surface energy}},$$

with respect to the weak* convergence in $BV(\Omega; \mathbb{R}^m)$; see also [51] for the case of unsigned integrands.

The \mathcal{A} -free setting

The study of L^p -weak lower semicontinuity of I_f in the \mathcal{A} -free framework (1.1)-(1.2), which corresponds to the absence of concentration effects, is in and of itself a mathematically interesting subject that requires a deeper understanding of the *oscillatory* behavior of L^p -weakly convergent \mathcal{A} -free sequences. It was mostly developed in [39], where the decisive quasiconvexity would be replaced by its natural generalization to \mathcal{A} -free fields, the so-called \mathcal{A} -*quasiconvexity*.

Let us recall from [25, 39] that a Borel function $f: \mathbb{R}^N \rightarrow \mathbb{R}$ is called \mathcal{A} -*quasiconvex* if the Jensen type inequality

$$f(A) \leq \int_Q f(A + w(y)) \, dy \tag{1.5}$$

holds for all $A \in \mathbb{R}^N$ and every Q -periodic $w \in C^\infty(Q; \mathbb{R}^N)$ with

$$\mathcal{A} w = 0 \quad \text{and} \quad \int_Q w(y) \, dy = 0.$$

Specifically, Theorems 3.6 and 3.7 in [39] provide the following characterization:

Theorem 1.1 (Fonseca & Müller '99). *Let $1 \leq p < \infty$ and let $f: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a Carathéodory function. Further assume that f has p -growth at infinity. Then,*

$$\int_{\Omega} f(x, v(x)) \, dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} f(x, v_j(x)) \, dx$$

for every sequence $(v_n) \subset L^p(\Omega; \mathbb{R}^N)$ such that $v_j \rightharpoonup v$ in $L^p(\Omega; \mathbb{R}^N)$ and $\mathcal{A} v_j \rightarrow 0$ in $W^{-k,p}(\Omega; \mathbb{R}^N)$, if and only if $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex for every $x \in \Omega$.⁸

In a similar fashion to the case of gradients, the above characterization renders a complete answer to the existence problem of (1.1)-(1.2) in the case $1 < p < \infty$ (a similar but not identical characterization holds for $p = \infty$). Regarding the case when $p = 1$ (with respect to the weak* topology of measures), substantial advances in the lower semicontinuity and relaxation theory were achieved under the additional assumption that \mathcal{A} is a *first-order* partial differential operator:

Theorem 1.2 (Baía, Chermisi, Matías & Santos '13). *Let \mathcal{A} be a first-order and homogeneous partial differential operator and let $f: \mathbb{R}^N \rightarrow \mathbb{R}$ be an \mathcal{A} -quasiconvex and Lipschitz continuous integrand. Let $(\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ be such that $\mu_j \xrightarrow{*} \mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$, $\mathcal{A} \mu_j \rightarrow 0$ in $W_{\text{loc}}^{-1,q}(\Omega)$ for*

⁸Here, $\mathcal{A}^k := \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha$ is the principal part of \mathcal{A}

some $q \in (1, d/(d-1))$ and $|\mu_j| \stackrel{*}{\rightharpoonup} \Lambda \in \mathcal{M}(\overline{\Omega})$ with $\Lambda(\partial\Omega) = 0$. Then

$$\bar{I}_f[\mu] \leq \liminf_{j \rightarrow \infty} \bar{I}_f[\mu_j].$$

Unfortunately, it is not clear whether similar techniques the ones applied in the proof of the theorem above can be extended to operators of higher order.

New ideas: The case of symmetric gradients

Since the late seventies, there has been a lot of attention paid to linear elasticity models, which involve the minimization of functionals of the form

$$u \mapsto \int_{\Omega} f(x, \mathcal{E}u(x)) \, dx, \quad u \in W^{1,1}(\Omega), \quad (1.6)$$

where

$$\mathcal{E} := \frac{1}{2}(\nabla u + \nabla u^T)$$

is the *symmetrized gradient* (or deformation tensor) of u .

The space $\text{BD}(\Omega)$ of *functions of bounded deformation*, introduced by Pierre-Marie Suquet [70] (see also [53, 71]), is the space containing the integrable \mathbb{R}^d -fields whose distributional *symmetrized derivative*

$$Eu := \frac{1}{2}(Du + Du^T)$$

is a finite Radon measure, that is,

$$\text{BD}(\Omega) = \{ u \in L^1(\Omega; \mathbb{R}^d) : Eu \in \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) \}.$$

Since Eu is a Radon measure, we may split Eu as

$$Eu = \frac{dEu}{d\mathcal{L}^d} + E^s u,$$

corresponding to its Lebesgue–Radon–Nikodým decomposition.

In particular, attention was given to the study of lower semicontinuity and relaxation properties of functionals defined on $\text{BD}(\Omega)$. As opposed to gradients, symmetrized gradients are associated to a *double curl* constraint, that is,

$$\mu \in \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) \quad \text{with} \quad \text{curl curl } \mu = 0 \quad \Leftrightarrow \quad Eu = \mu \quad \text{for some } u \in \text{BD}(\Omega) \quad (\text{locally}),$$

where curl curl is defined as the distributional *second-order* partial differential operator

$$\text{curl curl } \mu := \left(\sum_{i=1}^d \partial_{ik} \mu_{ij} + \partial_{ij} \mu_{ik} - \partial_{jk} \mu_{ii} - \partial_{ii} \mu_{jk} \right)_{jk}, \quad 1 \leq j, k \leq d.$$

Since curl curl is a second-order operator, neither the lower semicontinuity nor the relaxation results for functionals of the form (1.6) could be addressed by means of Theorem 1.2 or similar techniques.

However, through recent developments in the use of rigidity properties and the setting of generalized Young measures, Rindler [67] was able to give the following characterization:

Theorem 1.3 (Rindler '11). *Let $f : \overline{\Omega} \times \mathbb{M}_{\text{sym}}^{d \times d} \rightarrow [0, \infty)$ be a Carathéodory and symmetric quasiconvex integrand.⁹ Further assume that $|f(x, A)| \leq M(1 + |A|)$ for some $M > 0$ and all $x \in \Omega$, $A \in \mathbb{M}_{\text{sym}}^{d \times d}$, and that the strong recession function $f^\infty(x, A)$ exists for all $x \in \overline{\Omega}$, $A \in \mathbb{M}_{\text{sym}}^{d \times d}$ and is (jointly) continuous in $\overline{\Omega} \times \mathbb{M}_{\text{sym}}^{d \times d}$.*

Then, the functional

$$\bar{I}_f[u] := \int_{\Omega} f\left(x, \frac{dEu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^\infty\left(x, \frac{dE^s u}{d|E^s u|}(x)\right) d|E^s u|(x), \quad u \in \text{BD}(\Omega),$$

is sequentially lower semicontinuous with respect to the weak convergence in $\text{BD}(\Omega)$.*

1.2.3 The characterization for operators of arbitrary order

Our results concern PDE constraints $\mathcal{A} \mu = 0$ where \mathcal{A} satisfies Murat's constant rank condition (see [61]), which as seen in the previous discussion, is a long standing assumption in lower semicontinuity results. More precisely, we assume that the *principal symbol* of \mathcal{A} ,

$$\mathbb{A}(\xi) := \sum_{|\alpha|=k} \xi^\alpha A_\alpha,$$

has constant rank as a linear operator in $\text{Lin}(\mathbb{R}^N; \mathbb{R}^n)$, for all $\xi \in \mathbb{S}^{d-1}$. Associated to the principal symbol, we also define the *wave cone* of \mathcal{A} as

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{S}^{d-1}} \ker \mathbb{A}(\xi).$$

With these considerations in mind, we are able to show a lower semicontinuity result and a relaxation result of integral functionals with linear growth assumptions in the \mathcal{A} -free setting:

Theorem 1.4 (A.-R., De Philippis & Rindler '17). *Let $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand. Assume that f has linear growth at infinity and is Lipschitz in its second argument, uniformly in x . Further assume that there exists a modulus of continuity ω such that*

$$|f(x, A) - f(y, A)| \leq \omega(|x - y|)(1 + |A|) \quad \text{for all } x, y \in \Omega, A \in \mathbb{R}^N, \quad (1.7)$$

and that the strong recession function

$$f^\infty(x, A) \quad \text{exists for all } (x, A) \in \Omega \times \text{span } \Lambda_{\mathcal{A}}.$$

⁹In our setting, (curl curl)-quasiconvex.

Then, the functional

$$\bar{I}_f[\mu] := \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x)$$

is sequentially weak* lower semicontinuous for measures on the space $\ker_{\mathcal{M}} \mathcal{A}$ if and only if $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex for every $x \in \Omega$.¹⁰

Moreover, we are able to show the following relaxation result on asymptotically \mathcal{A} -free sequences under the additional assumption that \mathcal{A} is a homogeneous partial differential operator:

Theorem 1.5 (A.-R., De Philippis & Rindler '17). *Let $f: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand. Assume that f has linear growth at infinity, that is uniformly Lipschitz in its second argument, and is such that there exists a modulus of continuity ω as in (1.7). Further we assume that \mathcal{A} is a homogeneous partial differential operator and that the strong recession function*

$$f^{\infty}(x, A) \text{ exists for all } (x, A) \in \Omega \times \text{span } \Lambda_{\mathcal{A}}.$$

Then, for the functional

$$\mathcal{G}[u] := \int_{\Omega} f(x, u(x)) dx, \quad u \in L^1(\Omega; \mathbb{R}^N),$$

the (sequentially) weak* lower semicontinuous envelope

$$\bar{\mathcal{G}}[\mu] := \inf \left\{ \liminf_{j \rightarrow \infty} \mathcal{G}[u_j] : u_j \mathcal{L}^d \xrightarrow{*} \mu \text{ and } \mathcal{A} u_j \rightarrow 0 \text{ in } W^{-k, q} \right\},$$

for some $q \in (1, d/(d-1))$, is given by

$$\bar{\mathcal{G}}[\mu] = \int_{\Omega} Q_{\mathcal{A}} f\left(x, \frac{d\mu}{d|\mu|}(x)\right) dx + \int_{\Omega} (Q_{\mathcal{A}} f)^{\#}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x),$$

where $Q_{\mathcal{A}} f(x, \cdot)$ denotes the \mathcal{A} -quasiconvex envelope of $f(x, \cdot)$ with respect to the second argument and $(Q_{\mathcal{A}} f)^{\#}$ is the upper recession function of $Q_{\mathcal{A}} f$.^{11,12}

¹⁰In spite that f^{∞} may be defined only in the product space $\Omega \times \text{span } \Lambda_{\mathcal{A}}$, the functional \bar{I}_f remains to be well-defined. This owes to a recent development in the structure of \mathcal{A} -free measures by De Philippis & Rindler [29] which states that

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega,$$

whenever $\mathcal{A} \mu = 0$ in Ω ; in the case of gradients ($\mathcal{A} = \text{curl}$) this result was first shown by Alberti [2] and is commonly known as the Rank-one Theorem which essentially states that the singular part of the distributional derivative of a function of bounded variation has rank equal to one.

¹¹For a continuous integrand $h: \mathbb{R}^N \rightarrow \mathbb{R}$, the \mathcal{A} -quasiconvex envelope of h at $A \in \mathbb{R}^N$ is defined as

$$Q_{\mathcal{A}} h(A) := \inf \left\{ \int_Q f(A + w(y)) dy : w \in C_{\text{per}}^{\infty}(Q; \mathbb{R}^N), \mathcal{A} w = 0, \int_Q w dy = 0 \right\};$$

which, for homogeneous operators \mathcal{A} , turns out to be the largest \mathcal{A} -quasiconvex function below h .

¹²For a Borel integrand $g: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ with linear growth at infinity, one may consider a notion of recession function that

Remark 1.6 (Pure constraint). The asymptotically \mathcal{A} -free constraint $\mathcal{A} v_j \rightarrow 0$ appears as the natural convergence associated to the constraint $\mathcal{A}^k \mu = 0$. This follows by observing that the range $q \in (1, d/(d-1))$ corresponds to the embedding $\ker_{\mathcal{A}} \mathcal{A} \xrightarrow{c} W^{-k,q}(\Omega; \mathbb{R}^N)$. However, it is possible reach a similar characterization of the relaxation of I_f with respect to the *pure* constraint

$$\mathcal{A} \mu = 0$$

by requiring Ω to be a *strictly star-shaped* domain (see, e.g., [58], where such a geometrical assumption on the domain was made to address a homogenization problem).

The next table summarizes some of the most substantial advances (some of which have been already discussed) in the study of lower semicontinuity properties of non-convex integrals in the \mathcal{A} -free setting:

Operator	Growth	Author(s)	Characterization
$\mathcal{A} = \text{curl}$ (gradients)	$p > 1$	Morrey '66; Acerbi & Fusco '84	$f(x, \cdot)$ quasiconvex $f(x, \cdot)$ quasiconvex
	$p = 1$	Ambrosio & Dal Maso '92; Fonseca & Müller '93	$f(\cdot)$ quasiconvex $f(x, \cdot)$ quasiconvex
\mathcal{A} homogeneous, of constant rank	$p > 1$	Fonseca & Müller '99	$f(x, \cdot)$ \mathcal{A}^k -quasiconvex
$\mathcal{A} = \text{curl curl}$ (symmetric gradients)	$p = 1$	Barroso, Fonseca & Toader '00 (SBD)	$f(x, \cdot)$ sym. quasiconvex
\mathcal{A} homogeneous, of constant rank	$p = 1$	Fonseca, Leoni & Müller '04 (lower bound on abs. cont. part)	$f(x, \cdot)$ \mathcal{A} -quasiconvex
$\mathcal{A} = \text{curl curl}$ (symmetric gradients)	$p = 1$	Rindler '11 (BD)	$f(x, \cdot)$ sym. quasiconvex
\mathcal{A} of constant rank	$p = 1$	Baía, Cherimisi, Matías & Santos '13 (\mathcal{A} hom. first-order)	$f(\cdot)$ \mathcal{A} -quasiconvex
		Arroyo-Rabasa, De Philippis & Rindler '17 (arbitrary order)	$f(x, \cdot)$ \mathcal{A}^k -quasiconvex

An immediate consequence of the theorem above is the following relaxation in BD which *does not* impose any additional condition on the symmetric-quasiconvex envelope of the integrand (compare with Theorem 1.3):

Corollary 1.7 (BD-relaxation). *Let $f: \Omega \times \mathbb{M}_{\text{sym}}^{d \times d} \rightarrow [0, \infty)$ be a continuous integrand that has linear growth at infinity and is such that there exists a modulus of continuity ω as in (1.7). Further*

is weaker in the sense that it always exists. One such weaker form, the *upper recession function*, is defined by

$$g^\#(x, A) := \limsup_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{g(x', tA')}{t}, \quad (x, A) \in \Omega \times \mathbb{R}^N.$$

assume that the strong recession function

$$f^\infty(x, A) \text{ exists for all } (x, A) \in \Omega \times \mathbb{M}_{\text{sym}}^{d \times d}.$$

Let us consider the functional

$$\mathcal{G}[u] := \int_{\Omega} f\left(x, \frac{dEu}{d\mathcal{L}^d}(x)\right) dx,$$

defined for $u \in \text{LD}(\Omega) := \{u \in \text{BD}(\Omega) : E^s u = 0\}$.

Then, the lower semicontinuous envelope of $\mathcal{G}[u]$ with respect to weak*-convergence in $\text{BD}(\Omega)$, is given by the functional

$$\overline{\mathcal{G}}[u] := \int_{\Omega} SQf\left(x, \frac{d\mathcal{E}u}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} (SQf)^\# \left(x, \frac{dE^s u}{d|E^s u|}(x)\right) d|E^s u|(x), \quad u \in \text{BD}(\Omega),$$

where SQf denotes the symmetric-quasiconvex envelope of f with respect to the second argument.

1.3 Optimality conditions

We continue the analysis of variational properties of PDE constrained integrals with linear growth, in Chapter 3 we focus on the necessary and sufficient conditions for solutions of (1.1)-(1.2) under additional convexity assumptions.

To motivate our discussion, let us briefly recall some well-known facts about the minimization of convex integrals with superlinear growth defined on gradients (we refer the reader to [33] and references therein for an introduction to convex analysis methods).

Let $p > 1$ and let $f \in C^2(\mathbb{M}^{d \times m})$ be a convex integrand with standard p -growth assumptions

$$M^{-1}(1 + |A|^p) \leq |f(A)| \leq M(1 + |A|^p), \quad |Df(A)| \leq M'|A|^{p-1}, \quad \text{for all } A \in \mathbb{M}^{d \times m}.$$

The minimization of the functional

$$u \mapsto \int_{\Omega} f(\nabla u) dx, \quad u \in \mathbf{W}_0^{1,p}(\Omega; \mathbb{R}^m) \tag{1.8}$$

is a well-posed problem in the sense that there exists at least one minimizer $u \in \mathbf{W}_0^{1,p}(\Omega; \mathbb{R}^m)$. Furthermore, due to the growth conditions, it is possible to show that a *necessary and sufficient condition* for \tilde{u} to be a *minimizer* of (1.8) is that \tilde{u} (weakly) solves the correspondent *Euler–Lagrange* equation

$$-\text{div}(Df(\nabla u)) = 0 \text{ in } \Omega,$$

that is,

$$\int_{\Omega} Df(\nabla \tilde{u}) \cdot \nabla \varphi dx = 0 \quad \text{for all } \varphi \in \mathbf{W}_0^{1,p'}(\Omega; \mathbb{R}^m). \tag{1.9}$$

Using standard convex analysis methods and duality arguments one may further derive the so-called

saddle-point condition

$$f(\nabla \tilde{u}) + f^*(\tilde{\tau}) = \langle \tilde{\tau}, \nabla \tilde{u} \rangle_{L^{p'} \times L^p}, \quad (1.10)$$

which holds for every div-free maximizer $\tilde{\tau} \in L^{p'}(\Omega; \mathbb{M}^{d \times N})$ of the dual functional

$$\tau \mapsto - \int_{\Omega} f^*(\tau) \, dx, \quad \operatorname{div} \tau = 0. \quad (1.11)$$

Similarly to (1.9), (1.10) is also a *necessary and sufficient* condition for the extremality of \tilde{u} (and $\tilde{\tau}$). For similar reasons to the ones discussed in earlier sections, the case $p = 1$ presents two main difficulties:

1. In general, the existence of a minimizer $\tilde{u} \in W_0^{1,1}(\Omega; \mathbb{R}^m)$ of (1.8) is not guaranteed.
2. The relaxation in $BV(\Omega; \mathbb{R}^m)$ of (1.8), which is defined by the functional

$$\overline{\mathcal{F}}[u] = \int_{\Omega} f(\nabla u) \, dx + \int_{\Omega} f^{\#} \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u|(x) + \underbrace{\int_{\partial\Omega} f(u \otimes \nu_{\partial\Omega}) \, d\mathcal{H}^{d-1}(x)}_{\text{boundary term}}$$

is a well-posed minimization problem in $BV(\Omega)$. However, the derivation of saddle-point conditions as in (1.10), in this case, is directly linked to the duality pairing $\langle \cdot, \cdot \rangle_{BV^*, BV}$. The lack of reflexivity of BV spaces and the complexity of the dual of $BV(\Omega)$ presents several difficulties in establishing saddle-point conditions.

In spite of these difficulties, Beck and Schmidt [15] were able to characterize the saddle-point conditions in terms of a *generalized duality pairing* $\llbracket \cdot, \cdot \rrbracket_{W^{-1,1}, BV}$ (introduced earlier in [7]). The following theorem is a version of their main result.

Theorem 1.8 (Beck & Schmidt '15). *Let $f : \Omega \times \mathbb{M}^{d \times m} \rightarrow [0, \infty)$ be a continuous integrand. Assume that f has linear growth at infinity and assume that the strong recession function $f^{\infty} : \mathbb{M}^{d \times m} \rightarrow \mathbb{R}$ exists and that $f(x, \cdot) : \mathbb{M}^{d \times m} \rightarrow \mathbb{R}$ is a convex function for all $x \in \Omega$. Then, for $u \in BV(\Omega, \mathbb{R}^m)$ and $\tau \in L_{\operatorname{div}}^{\infty}(\Omega, \mathbb{M}^{d \times N})$ we have the following equivalence: u is a generalized minimizer of (1.8) and τ a solution of (1.11), if and only if the relation*

$$f(x, \nabla u(x)) + f^*(x, \tau(x)) = \tau(x) \cdot \nabla u(x) \quad \text{holds for } \mathcal{L}^d\text{-a.e. } x \in \Omega,$$

and, simultaneously, Du (the distributional derivative of u) satisfies

$$f^{\infty} \left(x, \frac{dD^s u}{d|D^s u|}(x) \right) = \frac{\llbracket \tau, Du \rrbracket}{d|D^s u|}(x) \quad \text{for } |D^s u|\text{-a.e. } x \text{ in } \overline{\Omega},$$

¹³The Fenchel transform of a function $h : \mathbb{R}^N \rightarrow \overline{\mathbb{R}}$ is the lower semicontinuous and convex function $h^* : \mathbb{R}^N \rightarrow \overline{\mathbb{R}}$ defined by the rule

$$h^*(z^*) := \sup_{z \in \mathbb{R}^N} \{z^* \cdot z - h(z)\}.$$

For an integrand $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$, and in a possible abuse of notation, we shall simply write $f^* : \Omega \times \mathbb{R}^N$ to denote its Fenchel transform with respect to the second argument, this is $f^*(x, A) \equiv (f(x, \cdot))^*(A)$.

where $\llbracket \tau, Du \rrbracket$ is the uniquely determined Radon measure on $\overline{\Omega}$ such that

$$\int_{\overline{\Omega}} \varphi \, d\llbracket \tau, Du \rrbracket = - \int_{\Omega} \tau \cdot (u \otimes \nabla \varphi) \, dx, \quad \text{holds for all } \varphi \in C_c^\infty(\mathbb{R}^m).$$

1.3.1 Duality for more general PDE constraints

Motivated by the ansatz that similar saddle-point conditions to the ones established in Theorem 1.8 should hold for minimization problems concerning PDE constraints $\mathcal{A}v = 0$ more general than $\text{curl} v = 0$. We investigate the natural extension of saddle-point conditions to the \mathcal{A} -free setting in a slightly different setting than (1.1)-(1.2).

Throughout Chapter 3 we shall assume that $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is convex in the second argument. We consider the minimization problem (also termed as the *primal problem*):

$$\text{minimize } I_f[u] \text{ among functions in the affine space } u_0 + \ker \mathcal{A}. \quad (\text{P})$$

Instead of $W^{1,1}$, we shall work with the \mathcal{A} -Sobolev space of Ω defined as

$$W^{\mathcal{A},1} := \{ u \in L^1(\Omega; \mathbb{R}^N) : \mathcal{A}u \in L^1(\Omega; \mathbb{R}^n) \}.$$

Since $W^{\mathcal{A},1}(\Omega)$ is a dense subspace of $L^1(\Omega; \mathbb{R}^N)$, we may consider the (possibly unbounded) linear operator $\mathcal{A} : W^{\mathcal{A},1} \subset L^1(\Omega; \mathbb{R}^N) \rightarrow L^1(\Omega; \mathbb{R}^n)$ and its dual $\mathcal{A}^* : D(\mathcal{A}^*) \subset L^\infty(\Omega; \mathbb{R}^n) \rightarrow L^\infty(\Omega; \mathbb{R}^N)$. With these considerations in mind, we also define the *dual problem*:

$$\text{maximize } w \mapsto \int_{\Omega} w^* \cdot \mathcal{A}u_0 \, dx - \int_{\Omega} f^*(x, w^*) \, dx, \quad \text{among fields } w^* \text{ in } D(\mathcal{A}^*). \quad (\text{P}^*)$$

The derivation of the optimality conditions (or saddle-point conditions) of problems (P) and (P*) is based on the introduction of the *set-valued pairing* $\llbracket \cdot, \cdot \rrbracket : \ker_{\mathcal{M}} \mathcal{A} \times D(\mathcal{A}^*) \rightarrow \mathcal{M}(\Omega)$ defined as

$$\begin{aligned} \llbracket \mu, \mathcal{A}^* w^* \rrbracket := & \left\{ \lambda \in \mathcal{M}(\Omega) : (u_n) \subset L^1(\Omega; \mathbb{R}^N) \ker \mathcal{A}, \right. \\ & \left. u_n \rightarrow \mu \text{ area-strictly in } \Omega, \text{ and } (u_n \cdot \mathcal{A}^* w^*) \mathcal{L}^d \xrightarrow{*} \lambda \text{ in } \mathcal{M}(\Omega) \right\}. \end{aligned}$$

Here, we say that a sequence of measures area-strictly converges to a measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ if $\mu_n \xrightarrow{*} \mu$ and $\langle \mu_n \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega)$ where

$$\langle \mu \rangle(\Omega) := \int_{\Omega} \sqrt{1 + \left(\frac{d\mu}{d\mathcal{L}^d} \right)^2} \, dx + |\mu|(\Omega).$$

Remark 1.9 (BV-generalized pairing). For $\mathcal{A} = \text{curl}$, our notion of (set-valued) generalized pairing can be identified with the well-defined Radon measure defined by $\llbracket \cdot, \cdot \rrbracket_{W^{-1,1} \times \text{BV}}$, introduced in [7].

Generalized saddle-point conditions

By means of this generalized pairing we show the intrinsic relation between *generalized minimizers* of (P) and maximizers of (P*) also known as the saddle-point conditions:

Theorem 1.10 (A.-R. '16). *Let $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand with linear growth at infinity such that $f(x, \cdot)$ is convex for all $x \in \Omega$. Further assume that there exists a modulus of continuity ω such that*

$$|f(x, z) - f(y, z)| \leq \omega(|x - y|)(1 + |z|) \quad \text{for all } x, y \in \Omega, z \in \mathbb{R}^N.$$

Then the following conditions are equivalent:

- (i) μ is a generalized solution of problem (P) and w^* is a solution of (P*),
- (ii) The generalized pairing $[[\mu, \mathcal{A}^* w^*]]$ is the singleton containing the measure

$$\lambda := \left(\frac{d\mu}{d\mathcal{L}^d} \cdot \mathcal{A}^* w^* \right) \mathcal{L}^d \llcorner \Omega + f^\infty \left(\cdot, \frac{d\mu}{d|\mu^s|} \right) |\mu^s|,$$

and in particular

$$\frac{d\lambda}{d|\mu^s|}(x) = f^\infty \left(x, \frac{d\mu}{d|\mu^s|} \right) \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Moreover,

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^d}(x) &= \frac{d\mu}{d\mathcal{L}^d}(x) \cdot \mathcal{A}^* w^*(x) \\ &= f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) + f^*(x, \mathcal{A}^* w^*(x)) \end{aligned}$$

for \mathcal{L}^d -a.e. in $x \in \Omega$.

Corollary 1.11 (Interior saddle-point conditions in BD). *Let $f : \Omega \times \mathbb{M}_{\text{sym}}^{d \times d} \rightarrow \mathbb{R}$ be as in the assumptions of Theorem 1.10. Then the (interior) saddle-point conditions associated to the minimization problem*

$$u \mapsto \int_{\Omega} f(x, \mathcal{E}u(x)) \, dx + \int_{\Omega} f^\infty \left(x, \frac{dE^s u}{d|E^s u|}(x) \right) d|E^s u|(x), \quad u \in \text{BD}(\Omega; \mathbb{R}^N),$$

are given by the equations

$$f(x, \mathcal{E}u(x)) + f^*(x, \sigma^*(x)) = \mathcal{E}u(x) \cdot \sigma^*(x) = \frac{d\lambda}{d\mathcal{L}^d}(x), \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega,$$

and

$$f^\infty \left(x, \frac{dE^s u}{d|E^s u|}(x) \right) = \frac{d\lambda}{d|\mu^s|}(x), \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Here, $\sigma^* \in L^\infty(\Omega; \mathbb{M}_{\text{sym}}^{d \times d})$ is a div-free symmetric tensor with $\text{Tr}(\sigma^* \cdot \nu_\Omega) = 0$ that maximizes the

functional

$$w^* \mapsto - \int_{\Omega} f^*(x, w^*) \, dx, \quad w^* \in L_{\text{div}}^{\infty}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}),$$

and λ is the uniquely defined Radon measure in $\mathcal{M}(\Omega)$ such that

$$\int_{\Omega} \varphi \, d\lambda = -\frac{1}{2} \int_{\Omega} \sigma^* \cdot (u \otimes \nabla \varphi + \nabla \varphi \otimes u) \, dx \quad \text{holds for all } \varphi \in C_c^{\infty}(\Omega).$$

Remark 1.12 (Saddle-point conditions in BV). By setting $\mathcal{A} = \text{curl}$, Theorem 1.10 re-proves a variant of Theorem 1.8.

1.4 Regularity: Optimal design problems with a perimeter term

In mathematics and materials science the notion of *optimal design* refers to a subarea of optimal control where the set of *controls* describe the geometries or possible compositions of a body or structure. We focus on the following general setting of the *two-material* optimal design problems for linear models: we look for local saddle-points of the variational problem

$$\min_A \sup_u J(A, u). \quad (\text{odp})$$

Here,

$$J(A, u) := \underbrace{\int_{\Omega} F u \, dx - \int_{\Omega \cap A} \sigma_1 \mathcal{A} u \cdot \mathcal{A} u \, dx - \int_{\Omega \cap A^c} \sigma_2 \mathcal{A} u \cdot \mathcal{A} u \, dx}_{:= D(A, u) \quad \text{“bulk energy”}} + \underbrace{\gamma \mathcal{L}^d(A \cap \Omega)}_{\text{volume term}} + \underbrace{\text{Per}(A; \Omega)}_{\text{surface term}},$$

defined on pairs (A, u) where the *design* $A \subset \mathbb{R}^d$ is prescribed by a Borel set, $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^N$ is the *potential* function, \mathcal{A} is an elliptic operator whose properties will be specified later together with some examples, the *design materials* are represented by symmetric positive definite tensors σ_1, σ_2 , and $F : \Omega \rightarrow \mathbb{R}^N$ is the *source field* associated to the Optimal Design problem.

The *perimeter* term $\text{Per}(A; \Omega)$ — equivalent to $\mathcal{H}^{d-1}(\partial A \cap \Omega)$ on smooth sets $A \subset \mathbb{R}^d$ — prevents highly oscillating pattern formations of designs. To highlight the role of the perimeter let us recall the ideas of Kohn and Strang [45–47] which link the notions of optimal design to the ones of *relaxation*. In the absence of a surface term, one can reformulate (odp) as an integral minimization which absorbs the designs A into a *double-well potential* (see Fig. 1.1)

$$\tau \mapsto \int_{\Omega} W(\tau) \, dx, \quad W(\tau) := \min \{ W_1(\tau) := \sigma_1^{-1} \tau \cdot \tau + \gamma, W_2(\tau) := \sigma_2^{-1} \tau \cdot \tau \},$$

where the candidate fields $\tau : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^N$ satisfy the affine PDE constraint

$$\mathcal{A}^* \tau = F,$$

for some linear PDE operator \mathcal{A}^* — that represents the L^2 adjoint of \mathcal{A} . As was emphasized in earlier sections, minimizers might develop *fine patterns* due to the non-convexity of W which lead to

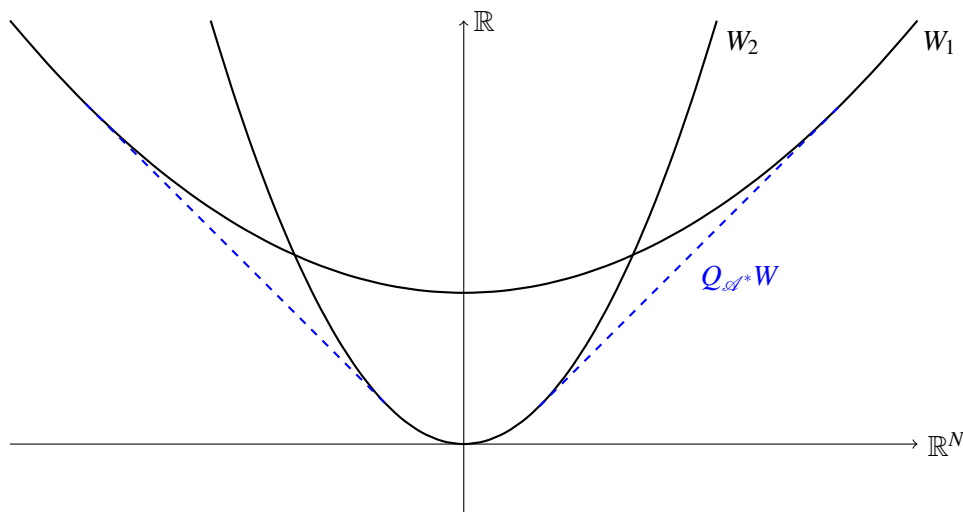


Figure 1.1: The dotted line sketches the graph of the relaxed integrand $Q_{\mathcal{A}^*}W$ in the regions where it does not coincide with the non-convex integrand $W = \min\{W_1, W_2\}$.

the study of the relaxed functional

$$\tau \mapsto \int_{\Omega} Q_{\mathcal{A}^*}W(\tau) \, dx, \quad \mathcal{A}^* \tau = F.$$

However, since the surface term is present, relaxation is unnecessary due to the high energy cost imposed on fine mixtures of the design.

The lower semicontinuity of perimeter functional (see [6, 34]) and the theory of compensated compactness developed by Murat and Tartar (see [62, 63]) provide the necessary compactness and lower semicontinuity properties to show existence of solutions via the direct method. A more interesting and non-trivial problem is to establish the regularity of saddle-points of (odp) to which we will devote our attention:

Problem 3. *Let (A, u) be a saddle-point of (odp). Does the pair (A, u) possess higher regularity properties than the ones prescribed by being an admissible design? Here, we shall understand the regularity of A as the differentiability properties of ∂A when it is seen as a $(d-1)$ -dimensional manifold, and the regularity of u as its integrability and differentiability properties.*

1.4.1 The role of almost perimeter minimizers

The variational properties of a set $A \subset \mathbb{R}^d$, which belongs to a minimizing pair (A, u) of (odp), can be reformulated in a way that resembles those of perimeter minimizers (described below). Indeed, a simple comparison argument and rearrangement of the energy terms yield

$$\text{Per}(A; \Omega) \leq \text{Per}(E; \Omega) + \left(\sup_u D(E, \cdot) - \sup_u D(A, \cdot) \right) + \gamma |\mathcal{L}^d(E \cap \Omega) - \mathcal{L}^d(A \cap \Omega)|,$$

for all measurable $E \subset \mathbb{R}^N$ such that $E \Delta A \subset \subset \Omega$. Thus, the set A *weakly minimizes* the perimeter functional $\text{Per}(\cdot; \Omega)$ in the sense that for every $x \in K \subset \subset \Omega$ there exists a modulus of continuity $\omega_K : [0, \infty) \rightarrow [0, \infty)$ for which

$$\text{Per}(A \cap B_r(x); \Omega) \leq \text{Per}(U \cap B_r(x); \Omega) + \omega_K(r).$$

Moreover, ω_K can be explicitly defined (up to a term of order r^d) as

$$\omega_K(r) := \inf \left\{ \left| \sup_u J(A, \cdot) - \sup_u J(E, \cdot) \right| : (E \Delta A) \subset \subset B_r(x) \text{ and } x \in K \right\}.$$

Having hitherto taken for granted the notion of perimeter, let us now discuss it in more detail along with the attendant regularity properties associated to minimization of perimeter.

The area of an open d -dimensional C^1 -hypersurface $M \subset \mathbb{R}^d$ in Ω is defined as

$$\text{Area}_\Omega(M) := \int_{U \cap \varphi^{-1}(\Omega)} \sqrt{1 + |\varphi|^2} \, dx = \mathcal{H}^{d-1}(M \cap \Omega),$$

where $\varphi : U \subset \mathbb{R}^d \rightarrow M$ is the C^1 -chart that parametrizes it. Stationary “points” of the area functional are called *minimal surfaces*, which in particular are solutions of the area Euler-Lagrange Equation

$$-\text{div} \left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) = 0. \quad (1.12)$$

If M is a minimal surface parametrized by a Lipschitz map φ , it is not hard to see that equation (1.12) is an elliptic PDE to which we can apply standard regularity methods which show that M is an analytic hypersurface.

The topological boundary of a sufficiently regular set $A \subset \mathbb{R}^d$ can be (locally) regarded as an open hypersurface. Hence, by the divergence theorem,

$$\int_{\partial A \cap \Omega} \varphi \cdot \nu_{\partial A} \, d\mathcal{H}^{d-1} = \int_{\Omega \cap A} \text{div} \varphi \, dx = - \int_{\Omega} \varphi \cdot d(\nabla \mathbb{1}_A), \quad \text{for all } \varphi \in C_c^1(\Omega; \mathbb{R}^d),$$

where $\nabla \mathbb{1}_A$ is the distributional derivative of the indicator function $\mathbb{1}_A$. On sufficiently regular sets $A \subset \mathbb{R}^d$, the area functional over the manifold $M = \partial A$ has the alternative representation

$$\text{Area}_\Omega(\partial A) = |\text{D}\mathbb{1}_A|(\Omega) := \sup \left\{ \int_{A \cap \Omega} \text{div} \varphi \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\},$$

which coincides with the norm of $\nabla \mathbb{1}_A$ in $(C_b(\Omega))^*$ — the total variation of the distributional derivative of $\mathbb{1}_A$ in Ω . This motivates the definition of the *perimeter* of a set:

$$\text{Per}(A; \Omega) := |\text{D}\mathbb{1}_A|(\Omega), \quad A \subset \mathbb{R}^d \text{ Borel set.}$$

Of course, every Lipschitz surface M is locally the topological boundary of a set A of finite perimeter and, in this case, $\text{Area}_\Omega(M) = \text{Area}_\Omega(\partial A) = \text{Per}(A; \Omega)$. However, the geometry of a set of finite

perimeter might be rather complicated. For instance, the boundary of a set of finite perimeter might not admit local parametrizations of any kind. In this case, the equality $\text{Area}_\Omega(\partial A) = \text{Per}(A; \Omega)$ is more likely to fail. For this reason De Giorgi introduced the notion of *reduced boundary* of a set, a $(d - 1)$ -dimensional Hausdorff subset $\partial^* A$ of the topological boundary with the property that

$$\text{Per}(A; \Omega) = \mathcal{H}^{d-1}(\Omega \cap \partial^* A). \quad (1.13)$$

In spite of the increase in the complexity of admissible geometries, one can still study certain *partial regularity* properties of sets with *minimal perimeter*, that is, sets $A \subset \mathbb{R}^d$ such that

$$\text{Per}(A; \Omega) \leq \text{Per}(E; \Omega), \quad \text{for all } E \subset \mathbb{R}^d \text{ such that } (E \Delta A) \subset\subset \Omega.$$

De Giorgi showed in [28] that being a perimeter minimizer is a sufficiently rigid property to guarantee $\partial^* A \cap \Omega$ to be an analytic hypersurface. Miranda [55], on the other hand, showed that the difference between topological and reduced boundaries — also known as *singular set* — of perimeter minimizers is small, namely that

$$\mathcal{H}^{d-1}(\Omega \cap (\partial A \setminus \partial^* A)) = 0.$$

Further developments due to Simon and Federer refined this result to the extent that $\mathcal{H}^s(\Omega \cap (\partial A \setminus \partial^* A)) = 0$ for all $s < 8$. Thus, establishing that in low dimensions perimeter minimizers are analytic.

A simple scaling argument dictates that perimeter minimizers are stable under *blow-up* methods. This fact further suggests that the aforementioned regularity results should extend to weaker minimality assumptions, which was later shown by Tamanini [72] in the setting of *almost perimeter minimizers*. A set $A \subset \mathbb{R}^d$ is an almost perimeter minimizer in Ω if there exist $\alpha \in (0, 1/2]$ and a local positive constant c such that

$$\text{Per}(A; \Omega) \leq \text{Per}(E; \Omega) + \underbrace{cr^{d-1+2\alpha}}_{\substack{\text{vanishing term} \\ \text{after blow-up}}},$$

for all $E \subset \mathbb{R}^d$ with $E \Delta A \subset\subset B_r(x) \subset \Omega$. He showed that if $A \subset \mathbb{R}^d$ is an almost perimeter minimizer, then

$$(\Omega \cap \partial^* A) \text{ is a } C^{1,\alpha}\text{-hypersurface} \quad \text{and} \quad \mathcal{H}^s(\Omega \cap (\partial A \setminus \partial^* A)) = 0 \text{ for all } s > 8.$$

This raises the following question:

Problem 4. *If (A, u) is a minimizer pair of (odp) with $A \subset \mathbb{R}^d$, is A an almost perimeter minimizer? Observe that it is enough to show that $\omega_K(r) = o(r^{d-1+2\alpha})$ for some $\alpha \in (0, 1/2]$.*

1.4.2 Regularity of optimal designs: history of the problem

Optimal design problems in linear electrical conductivity models have been considered by Kohn and Strang [45–47], and Murat and Tartar [63]; the success of homogenization and relaxation techniques led to groundbreaking advances in the understanding of pattern formations of optimal structures. However, optimal design problems with a perimeter penalization, like the one we shall consider, were

first developed by Ambrosio and Buttazzo [4] and Lin [52]. An exposition of their model and its regularity properties is provided below.

The linear conductivity equations

An open and bounded set $\Omega \subset \mathbb{R}^d$ represents a container that is occupied by two materials with uniform conductivities $0 < \beta < \alpha < \infty$. The material with conductivity α is distributed along a measurable set $A \subset \Omega$ with a prescribed volume fraction $0 < \lambda < \mathcal{L}^d(\Omega)$; the remaining part $(\Omega \setminus A)$ is occupied by a material with conductivity β . The overall conductivity in the container can be written as

$$\sigma_A(x) = \mathbb{1}_A(x)\alpha + (1 - \mathbb{1}_A(x))\beta$$

The materials are assumed to be linear and *perfectly bonded*, meaning that both the electric potential and the normal electrical current are continuous across the interface. The model is completed by adding a source term $F \in L^\infty(\Omega)$ and assuming (for simplicity) Dirichlet boundary conditions on $\partial\Omega$. The state equation associated to the model reads

$$\begin{aligned} -\operatorname{div}(\sigma_A \nabla u_A) &= F \quad \text{in } \Omega \\ u_A &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the function $u_A : \Omega \rightarrow \mathbb{R}$ models the electrical potential associated to the design A . The energy dissipated in Ω is captured by the functional

$$\int_{\Omega} F u_A \, dx.$$

The optimal design consists of finding designs with minimal combined dissipated and surface energies (among designs with prescribed volume λ). The precise mathematical variational principle being the minimization

$$\inf \left\{ J(A) : A \subset \Omega \text{ is a measurable set and } \mathcal{L}^d(A) = \lambda \right\},$$

where

$$J(A) = \int_{\Omega} f u_A \, dx + \operatorname{Per}(A; \Omega).$$

In order to handle the volume constraint one considers the introduction of a Lagrange multiplier $\gamma \in \mathbb{R}$ giving rise to the following final variant:

$$\inf \left\{ J(A) + \gamma \mathcal{L}^d(A) : A \text{ is a measurable subset of } \Omega \right\}. \quad (1.14)$$

Theorem 1.13 (Ambrosio & Buttazzo '93). *Let $A \subset \Omega$ be an optimal design of the minimization problem (1.14). Then, A is essentially relatively open in the sense that there exists an optimal set $\tilde{A} \subset \Omega$ which is relatively open in Ω , satisfies that $\mathcal{L}^d(A \Delta \tilde{A}) = 0$, and is such that $\partial \tilde{A} = \overline{\partial^* \tilde{A}}$.*

Up to minor considerations regarding the boundary conditions, the regularity is due to Lin [52]:

Theorem 1.14 (Lin '93). *Let $A \subset \Omega$ be an optimal and relatively open profile of the minimization problem (1.14). Then the singular set $\Sigma := (\partial A \setminus \partial^* A) \cap \Omega$ is a relatively closed subset of ∂A with $\mathcal{H}^{d-1}(\Sigma) = 0$. Moreover, there exists $\beta \in (0, 1)$ depending solely on the dimension d such that*

$$\partial^* A \text{ is an open } C^{1,\beta}\text{-hypersurface in } \Omega,$$

and u_A is Lipschitz in $\Omega \setminus \Sigma$.

To see how this model fits in our setting simply set the tensors $\sigma_1 = \alpha \text{id}_{\mathbb{R}^d}$, $\sigma_2 = \beta \text{id}_{\mathbb{R}^d}$, and the operator $\mathcal{A}u = \nabla u$ to be the gradient operator on scalar valued functions (accordingly the adjoint $\mathcal{A}^* \tau = -\text{div } \tau$ is the divergence operator on \mathbb{R}^d -valued fields). Since the dissipated energy $\int_{\Omega} F u_A$ is equivalent to

$$\max \left\{ \int_{\Omega} 2Fu \, dx - \int_{\Omega} \sigma_A \nabla u \cdot \nabla u \, dx : u \in W_0^{1,2}(\Omega) \right\},$$

the minimization of $A \mapsto J(A)$ is indeed a min-max problem by considering over the additional variable $u \in W_0^{1,2}(\Omega)$.

Given a minimizer A of (1.14), Theorem 1.14 provides an answer to Problem 3 by establishing that, up to modifying A in a set of vanishing \mathcal{H}^{d-1} -measure, the reduced boundary $\partial^* A$ is an open $C^{1,\beta}$ -hypersurface in Ω and $\nabla u_A = \tau \in L_{\text{loc}}^{\infty}(\Omega \setminus \Sigma)$.

1.4.3 General elliptic systems

There are similar models following more complicated systems of elliptic equations than the conductivity equations, for example, linear plate theory and linear elasticity models among their respective equivalent formulations. The task of extending the (partial) regularity results from the conductivity setting to more general systems of equations is not trivial. Mainly because the aforementioned results rely on a monotonicity decay property of harmonic maps (essentially, it is possible to reduce the conductivity equations to the Dirichlet equations). This monotonicity, however, fails for general elliptic systems.

Example 1.15 (Linear plate theory). Let $\omega := \Omega \times [-h, h]$ be the reference configuration of a (thin) plate with cross section $\Omega \subset \mathbb{R}^2$ and thickness $2h$. Here, Ω is a C^1 open and bounded set with outer normal $n(x)$. The elastic properties of the plate are described by the two-phase fourth-order tensor

$$\sigma_A(x) := \mathbb{1}_A(x)\sigma_1 + (1 - \mathbb{1}_A(x))\sigma_2, \quad \sigma_1, \sigma_2 \in \mathbb{M}_{\text{sym}}^{2 \times 2 \times 2 \times 2}.$$

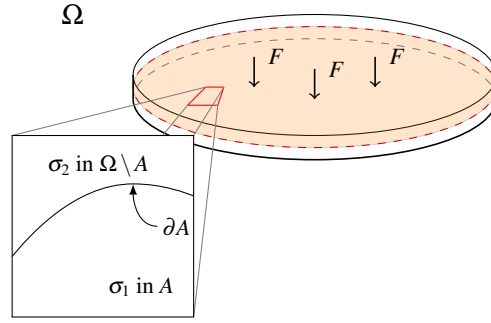


Figure 1.2: The design of a thin plate Ω made up of two materials with elastic moduli σ_1, σ_2 is determined by a design-set A . The vertical deflection u_A is as a consequence of subjecting the plate to the vertical load F .

The equations which describe the vertical displacement $u_A : \Omega \rightarrow \mathbb{R}$ of ω under a load $F \in L^\infty(\Omega)$ are given by the fourth-order system

$$\begin{aligned} \operatorname{div}(\operatorname{div}(\sigma_A \nabla^2 u_A)) &= F \quad \text{in } \Omega \\ u_A = \partial_n u_A &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\nabla^2 u_A$ is the Hessian matrix of u_A .

Example 1.16 (Linear elasticity). Let $\Omega \subset \mathbb{R}^3$ be an elastic body with deformation properties defined by a two-phase design tensor

$$\sigma_A(x) := \mathbb{1}_A(x)\sigma_1 + (1 - \mathbb{1}_A(x))\sigma_2, \quad \sigma_1, \sigma_2 \in \mathbb{M}_{\text{sym}}^{3 \times 3 \times 3}.$$

The linear equations associated to the deformation of Ω by an external force-field $F \in L^\infty(\Omega; \mathbb{R}^3)$ read

$$\begin{aligned} -\operatorname{div}(\sigma_A \mathcal{E} u_A) &= f \quad \text{in } \Omega \\ u_A &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $u_A : \Omega \rightarrow \mathbb{R}^3$ is the resulting deformation potential and $\mathcal{E} u_A := (\nabla u + (\nabla u)^T)/2$ is the *symmetrized gradient* of u_A .

Operators of gradient form

Our results concern the setting of general elliptic systems among which linear plate theory and linear elasticity models are included. The role of the operators $u \mapsto \nabla u, u \mapsto \nabla^k u$, or $u \mapsto \mathcal{E} u$ in different models is reduced to a single model by introducing a class of operators \mathcal{A} , the class of *operators of gradient form*.

The defining properties of this class are the following:

Ellipticity. We say that a k th-order homogeneous operator \mathcal{A} is elliptic if its principal symbol is

injective for all frequencies in Fourier space, i.e., its principal symbol \mathbb{A} has the property that

$$\ker \mathbb{A}(\xi) = \{0\} \subset \mathbb{R}^N, \quad \text{for all } \xi \in \mathbb{S}^{d-1}. \quad (1.15)$$

Compactness. We shall work within a class of operators \mathcal{A} where a Poincaré-type inequality holds.

In other words, we assume that there exists $c_\Omega > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq c_\Omega \|\mathcal{A}u\|_{L^2(\Omega)} \quad \text{for all } u \in \mathbf{W}_0^{k,2}(\Omega; \mathbb{R}^N). \quad (1.16)$$

Exactness. We will further assume that there exists an homogeneous partial differential operator \mathcal{B} such that

$$\mathcal{A}u = v \quad \Leftrightarrow \quad \mathcal{B}v = 0, \quad (1.17)$$

for all $v \in C_c^\infty(\omega; \mathbb{R}^n)$ and every simply connected $\omega \subset \mathbb{R}^d$.

We term the class of operators for which (1.15)-(1.17) hold, *operators of gradient form*.

It turns out that, restricted to square-integrable functions, operators in this class inherit similar properties to those of *gradients*, hence the name. For instance, classical Caccioppoli inequalities can be extended to general Caccioppoli inequalities for elliptic operators. Thus, one might systematically develop a similar regularity theory than the one available for gradients: higher integrability estimates, reverse Hölder estimates, etc.

Partial regularity for models prescribed by operators of gradient form

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded set. Let

$$\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha, \quad A_\alpha \in \text{Lin}(\mathbb{R}^N; \mathbb{R}^n),$$

be a k th-order operator of gradient form. Let $\sigma_1, \sigma_2 \in \text{Sym}(\mathbb{R}^{dN^k} \otimes \mathbb{R}^{dN^k})$ be two (possibly non-ordered) tensors with the property that

$$\frac{1}{M} |P|^2 \leq \sigma_i P \cdot P \leq M |P|^2 \quad \text{for all } P \in \mathbb{R}^{dN^k}; \quad i = 1, 2.$$

We consider the Optimal Design problem of finding the (locally) minimizing configurations $A \subset \mathbb{R}^d$ of the energy

$$J(A) = \int_\Omega F u_A \, dx + \text{Per}(A; \Omega), \quad (1.18)$$

where $u_A \in \mathbf{W}_0^{2,k}(\Omega; \mathbb{R}^N)$ is the unique solution to the elliptic system

$$\mathcal{A}^*(\sigma_A \mathcal{A}u) = F \quad \text{in } \Omega, \quad \text{in the sense of distributions,} \quad (1.19)$$

for some $F \in L^\infty(\Omega; \mathbb{R}^{dN^k})$. In this context we show, under mild assumptions on the regularizing properties of local solutions of the related relaxed problem:

$$\text{minimize } u \mapsto \int_{\Omega} Q_{\mathcal{B}} f(\mathcal{A} u) \, dx, \quad \text{defined on a subclass of } L^2(\Omega; \mathbb{R}^N),$$

that a local minimizer A is, up to a lower dimensional closed set, a C^1 -hypersurface:

Theorem 1.17 (A.-R. '16). *Let A be a local (minimizer) point of (1.18)-(1.19) in Ω . Then there exists a positive constant $\eta \in (0, 1]$ depending only on the dimension d such that, for the singular set $\Sigma = \partial\Omega \setminus \partial^*A$,*

$$\mathcal{H}^{d-1}(\Sigma \cap \Omega) = 0, \quad \text{and} \quad \partial^*A \quad \text{is an open } C^{1,\eta/2}\text{-hypersurface in } \Omega.$$

Moreover if \mathcal{A} is a first-order partial differential operator, then $\mathcal{A} u_A \in C_{\text{loc}}^{0,\eta/8}(\Omega \setminus \Sigma)$; the trace of $\mathcal{A} u_A$ exists on either side of ∂^*A .

2 Lower semicontinuity and relaxation of linear-growth integral functionals

This chapter contains the results obtained in the research paper:

Lower semicontinuity and relaxation of linear-growth integral functionals under PDE constraints

Abstract

We show general lower semicontinuity and relaxation theorems for linear-growth integral functionals defined on vector measures that satisfy linear PDE side constraints (of arbitrary order). These results generalize several known lower semicontinuity and relaxation theorems for BV, BD, and for more general first-order linear PDE side constraints. Our proofs are based on recent progress in the understanding of singularities of measure solutions to linear PDE's and of the generalized convexity notions corresponding to these PDE constraints.

See:

A. Arroyo-Rabasa, G. De Philippis, F. Rindler, *Lower semicontinuity and relaxation of linear-growth integral functionals under PDE constraints*, ArXiv e-prints: 1701.02230 (2017)

2.1 Introduction

The theory of linear-growth integral functionals defined on vector-valued measures satisfying PDE constraints is central to many questions of the calculus of variations. In particular, their relaxation and lower semicontinuity properties have attracted a lot of attention, see for instance [2, 5, 14–16, 20, 28]. In the present work we unify and extend a large number of these results by proving general lower semicontinuity and relaxation theorems for such functionals. Our proofs are based on recent advances in the understanding of the *singularities* that may occur in measures satisfying (under-determined) linear PDEs.

Concretely, let $\Omega \subset \mathbb{R}^d$ be an open and bounded subset with $\mathcal{L}^d(\partial\Omega) = 0$ and consider for a finite vector Radon measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ on Ω with values in \mathbb{R}^N the functional

$$\mathcal{F}^\#[\mu] := \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^\#\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x). \quad (2.1)$$

Here, $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Borel integrand that has *linear growth at infinity*, i.e.,

$$|f(x, A)| \leq M(1 + |A|) \quad \text{for all } (x, A) \in \Omega \times \mathbb{R}^N,$$

whereby the (*generalized*) *recession function*

$$f^\#(x, A) := \limsup_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{f(x', tA')}{t}, \quad (x, A) \in \overline{\Omega} \times \mathbb{R}^N,$$

takes only finite values. Furthermore, on the candidate measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ we impose the k 'th-order linear PDE side constraint

$$\mathcal{A}\mu := \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha \mu = 0 \quad \text{in the sense of distributions.}$$

The coefficient matrices $A_\alpha \in \mathbb{R}^{n \times N}$ are assumed to be constant and we write $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ with $|\alpha| := |\alpha_1| + \dots + |\alpha_d| \leq k$. We call measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ with $\mathcal{A}\mu = 0$ in the sense of distributions *\mathcal{A} -free*.

We will also assume that \mathcal{A} satisfies *Murat's constant rank condition* (see [16, 26]), that is, there exists $r \in \mathbb{N}$ such that

$$\text{rank}(\ker \mathbb{A}^k(\xi)) = r \quad \text{for all } \xi \in \mathbb{S}^{d-1}, \quad (2.2)$$

where

$$\mathbb{A}^k(\xi) := (2\pi i)^k \sum_{|\alpha|=k} \xi^\alpha A_\alpha, \quad \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d},$$

is the *principal symbol* of \mathcal{A} . We also recall the notion of *wave cone* associated to \mathcal{A} , which plays a fundamental role in the study of \mathcal{A} -free fields and first originated in the theory of compensated compactness [12, 24–26, 30, 31].

Definition 2.1. *Let \mathcal{A} be k 'th-order linear PDE operator as above. The wave cone associated to \mathcal{A} is the set*

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi) \subset \mathbb{R}^N.$$

Note that the wave cone contains those amplitudes along which it is possible to construct highly oscillating \mathcal{A} -free fields. More precisely if \mathcal{A} is homogeneous, i.e., $\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$, then $P_0 \in \Lambda_{\mathcal{A}}$ if and only if there exists $\xi \neq 0$ such that

$$\mathcal{A}(P_0 h(x \cdot \xi)) = 0 \quad \text{for all } h \in C^k(\mathbb{R}).$$

Our first main theorem concerns the case when f is \mathcal{A}^k -quasiconvex in its second argument, where

$$\mathcal{A}^k := \sum_{|\alpha|=k} A_\alpha \partial^\alpha$$

is the *principal part* of \mathcal{A} . Recall from [16] that a Borel function $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is called \mathcal{A}^k -*quasiconvex* if

$$h(A) \leq \int_Q h(A + w(y)) \, dy$$

for all $A \in \mathbb{R}^N$ and all Q -periodic $w \in C^\infty(Q; \mathbb{R}^N)$ such that $\mathcal{A}^k w = 0$ and $\int_Q w \, dy = 0$, where $Q := (-1/2, 1/2)^d$ is the unit cube in \mathbb{R}^d .

In order to state our first result, we shall first introduce the notion of *strong recession function* of f , which for $(x, A) \in \overline{\Omega} \times \mathbb{R}^N$ is defined as

$$f^\infty(x, A) := \lim_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{f(x', tA')}{t}, \quad (x, A) \in \overline{\Omega} \times \mathbb{R}^N, \quad (2.3)$$

provided the limit exists.

Theorem 2.2 (lower semicontinuity). *Let $f: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand. Assume that f has linear growth at infinity and is Lipschitz in its second argument and that $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex for all $x \in \Omega$. Further assume that either*

- (i) f^∞ exists in $\Omega \times \mathbb{R}^N$, or
- (ii) f^∞ exists in $\Omega \times \text{span } \Lambda_{\mathcal{A}}$, and there exists a modulus of continuity $\omega: [0, \infty) \rightarrow [0, \infty)$ (increasing, continuous, $\omega(0) = 0$) such that

$$|f(x, A) - f(y, A)| \leq \omega(|x - y|)(1 + |A|) \quad \text{for all } x, y \in \Omega, A \in \mathbb{R}^N. \quad (2.4)$$

Then, the functional

$$\mathcal{F}[\mu] := \int_\Omega f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_\Omega f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x)$$

is sequentially weakly* lower semicontinuous on the space

$$\mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A} := \{\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) : \mathcal{A} \mu = 0\}.$$

Note that according to (2.6) below, $\mathcal{F}[\mu]$ is well defined for $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ since the strong recession function is computed only at amplitudes that belong to $\text{span } \Lambda_{\mathcal{A}}$.

The \mathcal{A}^k -quasiconvexity of $f(x, \cdot)$ is not only a sufficient, but also a *necessary* condition for the sequential weak* lower semicontinuity of \mathcal{F} on $\mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$. In the case of first-order partial differential operator, the proof of the necessity can be found in [16]; the proof in the general case follows by verbatim repeating the same arguments.

Remark 2.3 (asymptotic \mathcal{A} -free sequences). The conclusion of Theorem 2.2 extends to sequences that are only asymptotically \mathcal{A} -free, that is,

$$\mathcal{F}[\mu] \leq \liminf_{j \rightarrow \infty} \mathcal{F}[\mu_j]$$

for all sequences $(\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ such that

$$\mu_j \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N) \quad \text{and} \quad \mathcal{A} \mu_j \rightarrow 0 \text{ in } W^{-k,q}(\Omega; \mathbb{R}^n)$$

for some $1 < q < d/(d-1)$ if $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex for all $x \in \Omega$.

Notice that f^∞ in (2.3) is a limit and, contrary to $f^\#$, it may fail to exist for $A \in (\text{span } \Lambda_{\mathcal{A}}) \setminus \Lambda_{\mathcal{A}}$ (for $A \in \Lambda_{\mathcal{A}}$ the existence of $f^\infty(x, A)$ follows from the \mathcal{A}^k -quasiconvexity, see Corollary 2.31). If we remove the assumption that f^∞ exists for points in the subspace generated by the wave cone $\Lambda_{\mathcal{A}}$, we still have the following partial lower semicontinuity result (cf. [14]).

Theorem 2.4 (partial lower semicontinuity). *Let $f: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand such that $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex for all $x \in \Omega$. Assume that f has linear growth at infinity and is Lipschitz in its second argument, uniformly in x . Further, suppose that there exists a modulus of continuity ω as in (3.2). Then,*

$$\int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx \leq \liminf_{j \rightarrow \infty} \mathcal{F}^\#[\mu_j]$$

for all sequences $\mu_j \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$ such that $\mathcal{A} \mu_j \rightarrow 0$ in $W^{-k,q}(\Omega; \mathbb{R}^N)$. Here,

$$\mathcal{F}^\#[\mu] := \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^\#\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x),$$

and $1 < q < d/(d-1)$.

Remark 2.5. As special cases of Theorem 2.2 we get, among others, the following well-known results:

- (i) For $\mathcal{A} = \text{curl}$, one obtains BV-lower semicontinuity results in the spirit of Ambrosio–Dal Maso [2] and Fonseca–Müller [15].
- (ii) For $\mathcal{A} = \text{curl curl}$, where

$$\text{curl curl } \mu := \left(\sum_{i=1}^d \partial_{ik} \mu_i^j + \partial_{ij} \mu_i^k - \partial_{jk} \mu_i^i - \partial_{ii} \mu_j^k \right)_{j,k=1,\dots,d}$$

is the second order operator expressing the Saint-Venant compatibility conditions (see [16, Example 3.10(e)]), we re-prove the lower semicontinuity and relaxation theorem in the space of functions of bounded deformation (BD) from [28].

- (iii) For first-order operators \mathcal{A} , a similar result was proved in [5].

(iv) Earlier work in this direction is in [14, 16], but there the singular (concentration) part of the functional was not considered.

If we dispense with the assumption of \mathcal{A}^k -quasiconvexity on the integrand, we have the following two relaxation results:

Theorem 2.6 (relaxation). *Let $f: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand that is Lipschitz in its second argument, uniformly in x . Assume also that f has linear growth at infinity (in its second argument) and is such that there exists a modulus of continuity ω as in (3.2). Further, suppose that \mathcal{A} is a homogeneous PDE operator and that the strong recession function*

$$f^\infty(x, A) \text{ exists for all } (x, A) \in \Omega \times \text{span} \Lambda_{\mathcal{A}}.$$

Then, for the functional

$$\mathcal{G}[u] := \int_{\Omega} f(x, u(x)) \, dx, \quad u \in L^1(\Omega; \mathbb{R}^N),$$

the (sequentially) weakly* lower semicontinuous envelope of \mathcal{G} , defined to be

$$\begin{aligned} \overline{\mathcal{G}}[\mu] := \inf \left\{ \liminf_{j \rightarrow \infty} \mathcal{G}[u_j] : (u_j) \subset L^1(\Omega; \mathbb{R}^N), u_j \mathcal{L}^d \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N) \right. \\ \left. \text{and } \mathcal{A} u_j \rightarrow 0 \text{ in } W^{-k, q} \right\}, \end{aligned}$$

where $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ and $1 < q < d/(d-1)$, is given by

$$\overline{\mathcal{G}}[\mu] = \int_{\Omega} Q_{\mathcal{A}} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} (Q_{\mathcal{A}} f)^{\#} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x).$$

Here, $Q_{\mathcal{A}} f(x, \cdot)$ denotes the \mathcal{A} -quasiconvex envelope of $f(x, \cdot)$ with respect to the second argument (see Definition 2.28 below).

If we want to relax in the space $\mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ we need to assume that $L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ is dense in $\mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ with respect to a finer topology than the natural weak* topology (in this context also see [4]).

Theorem 2.7. *Let $f: \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand that is Lipschitz in its second argument, uniformly in x . Assume also that f has linear growth at infinity (in its second argument) and is such that there exists a modulus of continuity ω as in (3.2). Further, suppose that \mathcal{A} is a homogeneous PDE operator, that the strong recession function*

$$f^\infty(x, A) \text{ exists for all } (x, A) \in \Omega \times \text{span} \Lambda_{\mathcal{A}},$$

and that for all $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ there exists a sequence $(u_j) \subset L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ such that

$$u_j \mathcal{L}^d \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N) \text{ and } \langle u_j \mathcal{L}^d \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega), \quad (2.5)$$

where $\langle \cdot \rangle$ is the area functional defined in (2.8). Then, for the functional

$$\mathcal{G}[u] := \int_{\Omega} f(x, u(x)) \, dx, \quad u \in L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A},$$

the weakly* lower semicontinuous envelope of \mathcal{G} , defined to be

$$\overline{\mathcal{G}}[\mu] := \inf \left\{ \liminf_{j \rightarrow \infty} \mathcal{G}[u_j] : (u_j) \subset L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}, u_j \mathcal{L}^d \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N) \right\},$$

is given by

$$\overline{\mathcal{G}}[\mu] = \int_{\Omega} Q_{\mathcal{A}} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) \, dx + \int_{\Omega} (Q_{\mathcal{A}} f)^{\#} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) \, d|\mu^s|(x).$$

Remark 2.8 (density assumptions). Condition (2.5) is automatically fulfilled in the following cases:

(i) For $\mathcal{A} = \text{curl}$, the approximation property (for general domains) is proved in the appendix of [19] (also see Lemma B.1 of [8] for Lipschitz domains). The same argument further shows the area-strict approximation property in the BD-case (also see Lemma 2.2 in [7] for a result which covers the strict convergence).

(ii) If Ω is a *strictly star-shaped* domain, i.e., there exists $x_0 \in \Omega$ such that

$$\overline{(\Omega - x_0)} \subset t(\Omega - x_0) \quad \text{for all } t > 1,$$

then (2.5) holds for every homogeneous operator \mathcal{A} . Indeed, for $t > 1$ we can consider the dilation of μ defined on $t(\Omega - x_0)$ and then mollify it at a sufficiently small scale. We refer for instance to [23] for details.

As a consequence of Theorem 2.7 and of Remark 2.8 we explicitly state the following corollary, which extends the lower semicontinuity result of [28] into a full relaxation result. The only other relaxation result in this direction, albeit for *special* functions of bounded deformation, seems to be in [7], other results in this area are discussed in [28] and the references therein.

Corollary 2.9. *Let $f: \Omega \times \mathbb{R}_{\text{sym}}^{d \times d} \rightarrow [0, \infty)$ be a continuous integrand, uniformly Lipschitz in the second argument, with linear growth at infinity, and such that there exists a modulus of continuity ω as in (3.2). Further, suppose that the strong recession function*

$$f^{\infty}(x, A) \text{ exists for all } (x, A) \in \Omega \times \mathbb{R}_{\text{sym}}^{d \times d}.$$

Consider the functional

$$\mathcal{G}[u] := \int_{\Omega} f(x, \mathcal{E}u(x)) \, dx,$$

defined for $u \in \text{LD}(\Omega) := \{u \in \text{BD}(\Omega) : E^s u = 0\}$, where $Eu := (Du + Du^T)/2 \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{d \times d})$ is

the symmetrized distributional derivative of $u \in \text{BD}(\Omega)$ and where

$$Eu = \mathcal{E}u \mathcal{L}^d \llcorner \Omega + \frac{dE^s u}{d|E^s u|} |E^s u|$$

is its Radon–Nikodým decomposition with respect to \mathcal{L}^d .

Then, the lower semicontinuous envelope of \mathcal{G} with respect to weak*-convergence in $\text{BD}(\Omega)$ is given by the functional

$$\overline{\mathcal{G}}[u] := \int_{\Omega} SQf(x, \mathcal{E}u(x)) \, dx + \int_{\Omega} (SQf)^{\#} \left(x, \frac{dE^s u}{d|E^s u|}(x) \right) d|E^s u|(x),$$

where SQf denotes the symmetric-quasiconvex envelope of f with respect to the second argument (i.e., the curlcurl-quasiconvex envelope of $f(x, \cdot)$ in the sense of Definition 2.28).

Our proofs are based on new tools to study singularities in PDE-constrained measures. Concretely, we exploit the recent developments on the structure of \mathcal{A} -free measures obtained in [11]. We remark that the study of the singular part – up to now the most complicated argument in the proof – now only requires a fairly straightforward (classical) convexity argument. More precisely, the main theorem of [18] establishes that the restriction of $f^{\#}$ to the linear space spanned by the wave cone is in fact convex at all points of $\Lambda_{\mathcal{A}}$ (in the sense that a supporting hyperplane exists). Moreover, by [11],

$$\frac{d\mu^s}{d|\mu^s|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega. \quad (2.6)$$

Thus, combining these two assertions, we gain classical convexity for $f^{\#}$ at singular points, which can be exploited via the theory of generalized Young measures developed in [1, 13, 19].

Remark 2.10 (different notions of recession function). Note that both in Theorem 2.2 and Theorem 2.6 the existence of the *strong* recession function f^{∞} is assumed, in contrast with the results in [2, 5, 15] where this is not imposed.

The need for this assumption comes from the use of Young measure techniques which seem to be better suited to deal with the singular part of the measure, as we already discussed above. In the aforementioned references a direct blow up approach is instead performed and this allows to deal directly with the functional in (2.1). The blow-up techniques, however, rely strongly on the fact that \mathcal{A} is a homogeneous first-order operator. Indeed, it is not hard to check that for all “elementary” \mathcal{A} -free measures of the form

$$\mu = P_0 \lambda, \quad \text{where } P_0 \in \Lambda_{\mathcal{A}}, \lambda \in \mathcal{M}^+(\mathbb{R}^d),$$

the scalar measure λ is necessarily translation invariant along orthogonal directions to the *characteristic set*

$$\Xi(P_0) := \{ \xi \in \mathbb{R}^d : P_0 \in \ker \mathbb{A}(\xi) \},$$

which turns out to be a subspace of \mathbb{R}^d whenever \mathcal{A} is a first-order operator. The subspace structure

and the aforementioned translation invariance is then used to perform homogenization-type arguments. Due to the lack of linearity of the map

$$\xi \mapsto \mathbb{A}^k(\xi) \quad \text{for } k > 1,$$

the structure of elementary \mathcal{A} -free measures for general operators is more complicated and not yet fully understood (see however [10, 28] for the case $\mathcal{A} = \text{curl curl}$). This prevents, at the moment, the use of a “pure” blow-up techniques and forces us to pass through the combination of the results of [11, 18] with the Young measure approach.

This paper is organized as follows: First, in Section 2.2, we introduce all the necessary notation and prove auxiliary results. Then, in Section 2.3, we establish the central Jensen-type inequalities, which immediately yield the proof of Theorems 2.2 and 2.4 in Section 2.4. The proofs of Theorems 2.6 and 2.7 are given in Section 2.5.

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2.2 Notation and preliminaries

We write $\mathcal{M}(\Omega; \mathbb{R}^N)$ and $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^N)$ to denote the spaces of *bounded Radon measures* and *Radon measures* on $\Omega \subset \mathbb{R}^N$, which are the duals of $C_0(\Omega; \mathbb{R}^N)$ and $C_c(\Omega; \mathbb{R}^N)$ respectively. Here, $C_0(\Omega; \mathbb{R}^N)$ is the completion of $C_c(\Omega; \mathbb{R}^N)$ with respect to the $\|\cdot\|_\infty$ norm, and, in the second case, $C_c(\Omega; \mathbb{R}^N)$ is understood as the nested union of Banach spaces of the form $C_0(K_m)$ where $K_m \nearrow \Omega$ and each K_m is a compact subset of \mathbb{R}^d . The set of *probability measures* over a locally compact space X shall be denoted by

$$\mathcal{M}^1(X) := \left\{ \mu \in \mathcal{M}(X) : \mu \text{ is a positive measure, and } |\mu|(X) = 1 \right\}.$$

We will often make use of the following metrizable principles:

1. Bounded sets of $\mathcal{M}(\Omega; \mathbb{R}^N)$ are metrizable in the sense that there exists a metric d which induces the weak* topology, that is,

$$\sup_{j \in \mathbb{N}} |\mu_j|(\Omega) < \infty \quad \text{and} \quad d(\mu_j, \mu) \rightarrow 0 \quad \Leftrightarrow \quad \mu_j \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^N).$$

2. There exists a complete and separable metric d on $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^N)$. Moreover, convergence with respect to this metric coincides with the weak* convergence of Radon measures (see Remark

14.15 in [21]).

We write the Radon–Nikodým decomposition of a measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ as

$$\mu = \frac{d\mu}{d\mathcal{L}^d} \mathcal{L}^d \llcorner \Omega + \mu^s, \quad (2.7)$$

where $\frac{d\mu}{d\mathcal{L}^d} \in L^1(\Omega; \mathbb{R}^N)$ and $\mu^s \in \mathcal{M}(\Omega; \mathbb{R}^N)$ is singular with respect to \mathcal{L}^d .

2.2.1 Integrands and Young measures

For $f \in C(\Omega \times \mathbb{R}^N)$ define the transformation

$$(Sf)(x, \hat{A}) := (1 - |\hat{A}|)f\left(x, \frac{\hat{A}}{1 - |\hat{A}|}\right), \quad (x, \hat{A}) \in \overline{\Omega} \times \mathbb{B}^N,$$

where \mathbb{B}^N denotes the open unit ball in \mathbb{R}^N . Then, $Sf \in C(\Omega \times \mathbb{B}^N)$. We set

$$\mathbf{E}(\Omega; \mathbb{R}^N) := \{f \in C(\Omega \times \mathbb{R}^N) : Sf \text{ extends to } C(\overline{\Omega \times \mathbb{B}^N})\}.$$

In particular, all $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$ have linear growth at infinity, i.e., there exists a positive constant M such that $|f(x, A)| \leq M(1 + |A|)$ for all $x \in \Omega$ and all $A \in \mathbb{R}^N$. With the norm

$$\|f\|_{\mathbf{E}(\Omega; \mathbb{R}^N)} := \|Sf\|_{\infty}, \quad f \in \mathbf{E}(\Omega; \mathbb{R}^N),$$

the space $\mathbf{E}(\Omega; \mathbb{R}^N)$ turns out to be a Banach space. Also, by definition, for each $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$ the limit

$$f^\infty(x, A) := \lim_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{f(x', tA')}{t}, \quad (x, A) \in \overline{\Omega} \times \mathbb{R}^N,$$

exists and defines a positively 1-homogeneous function called the *strong recession function* of f . Even if one drops the dependence on x , the recession function h^∞ might not exist for $h \in C(\mathbb{R}^N)$. Instead, one can always define the *upper* and *lower recession functions*

$$f^\#(x, A) := \limsup_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{f(x', tA')}{t},$$

$$f_\#(x, A) := \liminf_{\substack{x' \rightarrow x \\ A' \rightarrow A \\ t \rightarrow \infty}} \frac{f(x', tA')}{t},$$

which again turn out to be positively 1-homogeneous. If f is x -uniformly Lipschitz continuous in the A -variable and there exists a modulus of continuity $\omega: [0, \infty) \rightarrow [0, \infty)$ (increasing, continuous, and $\omega(0) = 0$) such that

$$|f(x, A) - f(y, A)| \leq \omega(|x - y|)(1 + |A|), \quad x, y \in \Omega, A \in \mathbb{R}^N,$$

then the definitions of f^∞ , $f^\#$, and $f_\#$ simplify to

$$\begin{aligned} f^\infty(x, A) &:= \lim_{t \rightarrow \infty} \frac{f(x, tA)}{t}, \\ f^\#(x, A) &:= \limsup_{t \rightarrow \infty} \frac{f(x, tA)}{t}, \\ f_\#(x, A) &:= \liminf_{t \rightarrow \infty} \frac{f(x, tA)}{t}. \end{aligned}$$

A natural action of $\mathbf{E}(\Omega; \mathbb{R}^N)$ on the space $\mathcal{M}(\Omega; \mathbb{R}^N)$ is given by

$$\mu \mapsto \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^N}(x)\right) dx + \int_{\Omega} f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x).$$

In particular, for $f(x, A) = \sqrt{1 + |A|^2} \in \mathbf{E}(\Omega; \mathbb{R}^N)$, for which $f^\infty(A) = |A|$, we define the *area functional*

$$\langle \mu \rangle(\Omega) := \int_{\Omega} \sqrt{1 + \left| \frac{d\mu}{d\mathcal{L}^N} \right|^2} dx + |\mu^s|(\Omega), \quad \mu \in \mathcal{M}(\Omega; \mathbb{R}^N). \quad (2.8)$$

In addition to the well-known weak* convergence of measures, we say that a sequence (μ_j) *converges area-strictly* to μ in $\mathcal{M}(\Omega; \mathbb{R}^N)$ if

$$\mu_j \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N) \quad \text{and} \quad \langle \mu_j \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega).$$

This notion of convergence turns out to be stronger than the conventional *strict convergence* of measures, which means that

$$\mu_j \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N) \quad \text{and} \quad |\mu_j|(\Omega) \rightarrow |\mu|(\Omega).$$

Indeed, the area-strict convergence, as opposed to the usual strict convergence, prohibits one-dimensional oscillations. The meaning of area-strict convergence becomes clear when considering the following version of Reshetnyak's continuity theorem, which entails that the topology generated by area-strict convergence is the coarsest topology under which the natural action of $\mathbf{E}(\Omega; \mathbb{R}^N)$ on $\mathcal{M}(\Omega; \mathbb{R}^N)$ is continuous.

Theorem 2.11 (Theorem 5 in [20]). *For every integrand $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$, the functional*

$$\mu \mapsto \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^N}(x)\right) dx + \int_{\Omega} f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x),$$

is area-strictly continuous on $\mathcal{M}(\Omega; \mathbb{R}^N)$.

Remark 2.12. Notice that if $\mu \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^N)$, then $\mu_\varepsilon \rightarrow \mu$ area-strictly, where μ_ε is the mollification of μ with a family of standard convolution kernels, $\mu_\varepsilon := \mu * \rho_\varepsilon$ and $\rho_\varepsilon(x) := \varepsilon^{-d} \rho(x/\varepsilon)$ for $\rho \in C_c^\infty(B_1)$ positive and even function satisfying $\int \rho dx = 1$.

Generalized Young measures form a set of dual objects to the integrands in $\mathbf{E}(\Omega; \mathbb{R}^N)$. We recall

briefly some aspects of this theory, which was introduced by DiPerna and Majda in [13] and later extended in [1, 19].

Definition 2.13 (generalized Young measure). *A generalized Young measure, parameterized by an open set $\Omega \subset \mathbb{R}^d$, and with values in \mathbb{R}^N , is a triple $\nu = (\nu_x, \lambda_\nu, \nu_x^\infty)$, where*

- (i) $(\nu_x)_{x \in \Omega} \subset \mathcal{M}^1(\mathbb{R}^N)$ is a parameterized family of probability measures on \mathbb{R}^N ,
- (ii) $\lambda_\nu \in \mathcal{M}_+(\overline{\Omega})$ is a positive finite Radon measure on $\overline{\Omega}$, and
- (iii) $(\nu_x^\infty)_{x \in \overline{\Omega}} \subset \mathcal{M}^1(\mathbb{S}^{N-1})$ is a parametrized family of probability measures on the unit sphere \mathbb{S}^{N-1} .

Additionally, we require that

- (iv) the map $x \mapsto \nu_x$ is weakly* measurable with respect to \mathcal{L}^d ,
- (v) the map $x \mapsto \nu_x^\infty$ is weakly* measurable with respect to λ_ν , and
- (vi) $x \mapsto \langle |\cdot|, \nu_x \rangle \in L^1(\Omega)$.

The set of all such Young measures is denoted by $\mathbf{Y}(\Omega; \mathbb{R}^N)$.

Similarly we say that $\nu \in \mathbf{Y}_{\text{loc}}(\Omega; \mathbb{R}^N)$ if $\nu \in \mathbf{Y}(E; \mathbb{R}^N)$ for all $E \Subset \Omega$.

Here, weak* measurability means that the functions $x \mapsto \langle f(x, \cdot), \nu_x \rangle$ (respectively $x \mapsto \langle f^\infty(x, \cdot), \nu_x^\infty \rangle$) are Lebesgue measurable (respectively λ_ν -measurable) for all Carathéodory integrands $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ (measurable in their first argument and continuous in their second argument).

For an integrand $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$ and a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$, we define the *duality pairing* between f and ν as follows:

$$\langle\langle f, \nu \rangle\rangle := \int_{\Omega} \langle f(x, \cdot), \nu_x \rangle \, dx + \int_{\overline{\Omega}} \langle f^\infty(x, \cdot), \nu_x^\infty \rangle \, d\lambda_\nu(x).$$

In many cases it will be sufficient to work with functions $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$ that are Lipschitz continuous. The following density lemma can be found in [19, Lemma 3].

Lemma 2.14. *There exists a countable set of functions $\{f_m\} = \{\varphi_m \otimes h_m \in C(\overline{\Omega}) \times C(\mathbb{R}^N) : m \in \mathbb{N}\} \subset \mathbf{E}(\Omega; \mathbb{R}^N)$ such that for two Young measures $\nu_1, \nu_2 \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ the implication*

$$\langle\langle f_m, \nu_1 \rangle\rangle = \langle\langle f_m, \nu_2 \rangle\rangle \quad \forall m \in \mathbb{N} \quad \implies \quad \nu_1 = \nu_2$$

holds. Moreover, all the h_m can be chosen to be Lipschitz continuous.

Since $\mathbf{Y}(\Omega; \mathbb{R}^N)$ is contained in the dual space of $\mathbf{E}(\Omega; \mathbb{R}^N)$ via the duality pairing $\langle\langle \cdot, \cdot \rangle\rangle$, we say that a sequence of Young measures $(\nu_j) \subset \mathbf{Y}(\Omega; \mathbb{R}^N)$ converges weakly* to $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$, in symbols $\nu_j \xrightarrow{*} \nu$, if

$$\langle\langle f, \nu_j \rangle\rangle \rightarrow \langle\langle f, \nu \rangle\rangle \quad \text{for all } f \in \mathbf{E}(\Omega; \mathbb{R}^N).$$

Fundamental for all Young measure theory is the following compactness result, see [19, Section 3.1] for a proof.

Lemma 2.15 (compactness). *Let $(\nu_j) \subset \mathbf{Y}(\Omega; \mathbb{R}^N)$ be a sequence of Young measures satisfying*

- (i) *the functions $x \mapsto \langle |\cdot|, \nu_j \rangle$ are uniformly bounded in $L^1(\Omega)$,*
- (ii) *$\sup_j \lambda_{\nu_j}(\overline{\Omega}) < \infty$.*

Then, there exists a subsequence (not relabeled) and $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ such that $\nu_j \xrightarrow{} \nu$ in $\mathbf{Y}(\Omega; \mathbb{R}^N)$.*

The Radon–Nikodým decomposition (2.7) induces a natural embedding of $\mathcal{M}(\Omega; \mathbb{R}^N)$ into $\mathbf{Y}(\Omega; \mathbb{R}^N)$ via the identification $\mu \mapsto \delta[\mu]$, where

$$(\delta[\mu])_x := \delta_{\frac{d\mu}{d\mathcal{L}^d}}(x), \quad \lambda_{\delta[\mu]} := |\mu^s|, \quad (\delta[\mu])_x^\infty := \delta_{\frac{d\mu^s}{d|\mu^s|}}(x).$$

In this sense, we say that the sequence of measures (μ_j) *generates* the Young measure ν if $\delta[\mu_j] \xrightarrow{*} \nu$ in $\mathbf{Y}(\Omega; \mathbb{R}^N)$; we write

$$\mu_j \xrightarrow{\mathbf{Y}} \nu.$$

The *barycenter* of a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ is defined as the measure

$$[\nu] := \langle \text{id}, \nu_x \rangle \mathcal{L}^d \llcorner \Omega + \langle \text{id}, \nu_x^\infty \rangle \lambda_\nu \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^N).$$

Using the notation above it is clear that for $(\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ it holds that $\mu_j \xrightarrow{*} [\nu]$, as measures on $\overline{\Omega}$, if $\mu_j \xrightarrow{\mathbf{Y}} \nu$.

Remark 2.16. For a sequence $(\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ that area-strictly converges to some limit $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$, it is relatively easy to characterize the (unique) Young measure it generates. Indeed, an immediate consequence of the Separation Lemma 2.14 and Theorem 2.11 is that

$$\mu_j \rightarrow \mu \text{ area-strictly in } \Omega \quad \iff \quad \mu_j \xrightarrow{\mathbf{Y}} \delta[\mu] \in \mathbf{Y}(\Omega; \mathbb{R}^N).$$

Young measures generated by means of periodic homogenization can be easily computed, see Lemma A.1 in [6].

Lemma 2.17 (oscillation measures). *Let $1 \leq p < \infty$ and let $w \in L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$ be a Q -periodic function and let $m \in \mathbb{N}$. Define the (Q/m) -periodic functions $w_m(x) := w(mx)$. Then,*

$$w_m \rightharpoonup \overline{w}(x) := \int_Q w(y) \, dy$$

in $L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$.

In particular, the sequence $(w_m) \subset L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$ generates the homogeneous (local) Young measure $\nu = (\overline{\delta_w}, 0, \bullet) \in \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$ (since λ_ν is the zero measure, the ν_x^∞ component can be occupied

by any parameterized family of probability measures in $\mathcal{M}^1(\mathbb{S}^{N-1})$, where

$$\langle h, \overline{\delta_w} \rangle := \int_Q h(w(y)) \, dy \quad \text{for all } h \in C(\mathbb{R}^d) \text{ with linear growth at infinity.}$$

In some cases it will be necessary to determine the smallest linear space containing the support of a Young measure. With this aim in mind, we state the following version of Theorem 2.5 in [1]:

Lemma 2.18. *Let (u_j) be a sequence in $L^1(\Omega; \mathbb{R}^N)$ generating a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ and let V be a subspace of \mathbb{R}^N such that $u_j(x) \in V$ for \mathcal{L}^d -a.e. $x \in \Omega$. Then,*

- (i) $\text{supp } \nu_x \subset V$ for \mathcal{L}^d -a.e. $x \in \Omega$,
- (ii) $\text{supp } \nu_x^\infty \subset V \cap \mathbb{S}^{N-1}$ for λ_ν -a.e. $x \in \Omega$.

Finally, we have the following approximation lemma, see [1, Lemma 2.3] for a proof.

Lemma 2.19. *Let $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be an upper semicontinuous integrand with linear growth at infinity. Then, there exists a decreasing sequence $(f_m) \subset \mathbf{E}(\Omega; \mathbb{R}^N)$ such that*

$$\inf_{m \in \mathbb{N}} f_m = \lim_{m \rightarrow \infty} f_m = f, \quad \inf_{m \in \mathbb{N}} f_m^\infty = \lim_{m \rightarrow \infty} f_m^\infty = f^\# \quad (\text{pointwise}).$$

Furthermore, the linear growth constants of the f_m 's can be chosen to be bounded by the linear growth constant of f .

By approximation, we thus get:

Corollary 2.20. *Let $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be an upper semicontinuous Borel integrand. Then the functional*

$$\nu \mapsto \int_\Omega \langle f(x, \cdot), \nu_x \rangle \, dx + \int_\Omega \langle f^\#(x, \cdot), \nu_x^\infty \rangle \, d\lambda_\nu(x)$$

is sequentially weakly upper semicontinuous on $\mathbf{Y}(\Omega; \mathbb{R}^N)$.*

Similarly, if $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a lower semicontinuous Borel integrand, then the functional

$$\nu \mapsto \int_\Omega \langle f(x, \cdot), \nu_x \rangle \, dx + \int_\Omega \langle f_\#(x, \cdot), \nu_x^\infty \rangle \, d\lambda_\nu(x)$$

is sequentially weakly lower semicontinuous on $\mathbf{Y}(\Omega; \mathbb{R}^N)$.*

2.2.2 Tangent measures

In this section we recall the notion of tangent measures, as introduced by Preiss [27] (with the exception that we always include the zero measure as a tangent measure).

Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ and consider the map $T^{(x_0, r)}(x) := (x - x_0)/r$, which blows up $B_r(x_0)$, the open ball around $x_0 \in \Omega$ with radius $r > 0$, into the open unit ball B_1 . The *push-forward* of μ under $T^{(x_0, r)}$ is given by the measure

$$T_\#^{(x_0, r)} \mu(B) := \mu(x_0 + rB), \quad B \subset r^{-1}(\Omega - x_0) \text{ a Borel set.}$$

We say that ν is a *tangent measure* to μ at a point $x_0 \in \mathbb{R}^d$ if there exist sequences $r_m > 0$, $c_m > 0$ with $r_m \downarrow 0$ such that

$$c_m T_{\#}^{(x_0, r_m)} \mu \xrightarrow{*} \nu \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N).$$

The set of all such tangent measures is denoted by $\text{Tan}(\mu, x_0)$ and the sequence $c_m T_{\#}^{(x_0, r_m)} \mu$ is called a *blow-up sequence*. Using the canonical zero extension that maps the space $\mathcal{M}(\Omega; \mathbb{R}^N)$ into the space $\mathcal{M}(\mathbb{R}^d; \mathbb{R}^N)$ we may use most of the results contained in the general theory for tangent measures when dealing with tangent measures defined on smaller domains.

Since we will frequently restrict tangent measures to the d -dimensional unit cube $Q := (-1/2, 1/2)^d$, we set

$$\text{Tan}_Q(\mu, x_0) := \{ \sigma \llcorner \bar{Q} : \sigma \in \text{Tan}(\mu, x_0) \}.$$

One can show (see Remark 14.4 in [21]) that for any non-zero $\sigma \in \text{Tan}(\mu, x_0)$ it is always possible to choose the scaling constants $c_m > 0$ in the blow-up sequence to be

$$c_m := c \mu(x_0 + r_m U)^{-1}$$

for any open and bounded set $U \subset \mathbb{R}^d$ containing the origin and with the property that $\sigma(U) > 0$, for some positive constant $c = c(U)$ (this may involve passing to a subsequence).

A special property of tangent measures is that at $|\mu|$ -almost every $x_0 \in \mathbb{R}^d$ it holds that

$$\sigma = \text{w}^* \text{-} \lim_{m \rightarrow \infty} c_m T_{\#}^{(x_0, r_m)} \mu \iff |\sigma| = \text{w}^* \text{-} \lim_{m \rightarrow \infty} c_m T_{\#}^{(x_0, r_m)} |\mu|, \quad (2.9)$$

where the (local*) weak* limits are to be understood in the spaces $\mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$ and $\mathcal{M}_{\text{loc}}^+(\mathbb{R}^d)$, respectively. A proof of this fact can be found in Theorem 2.44 of [3]. In particular, this implies

$$\text{Tan}(\mu, x_0) = \frac{d\mu}{d|\mu|}(x_0) \cdot \text{Tan}(|\mu|, x_0).$$

If $\mu, \lambda \in \mathcal{M}_{\text{loc}}^+(\mathbb{R}^d)$ are two Radon measures with the property that $\mu \ll \lambda$, i.e., that μ is absolutely continuous with respect to λ , then (see Lemma 14.6 of [21])

$$\text{Tan}(\mu, x_0) = \text{Tan}(\lambda, x_0) \quad \text{for } \mu\text{-almost every } x_0 \in \mathbb{R}^d, \quad (2.10)$$

and in particular if $f \in L_{\text{loc}}^1(\mathbb{R}^d, \lambda; \mathbb{R}^N)$, i.e., f is λ -integrable,

$$\text{Tan}(f\lambda, x_0) = f(x_0) \cdot \text{Tan}(\lambda, x_0) \quad \text{for } \lambda\text{-a.e. } x_0 \in \mathbb{R}^d.$$

On the other hand, at every $x_0 \in \text{supp } \mu$ such that

$$\lim_{r \downarrow 0} \frac{\mu(B_r(x_0) \setminus E)}{\mu(B_r(x_0))} = 0$$

for some Borel set $E \subset \mathbb{R}^d$, it holds that

$$\text{Tan}(\mu, x_0) = \text{Tan}(\mu \llcorner E, x_0).$$

A simple consequence of (2.10) is

$$\text{Tan}(|\mu|, x_0) = \text{Tan}(\mathcal{L}^d, x_0) \quad \text{for } \frac{d|\mu|}{d\mathcal{L}^d} \mathcal{L}^d\text{-a.e. } x_0 \in \mathbb{R}^d.$$

This implies

$$\text{Tan}(\mu, x_0) = \left\{ \alpha \frac{d\mu}{d\mathcal{L}^d}(x_0) \mathcal{L}^d : \alpha \in [0, \infty) \right\} \quad \text{for } \mathcal{L}^d\text{-a.e. } x_0 \in \mathbb{R}^d. \quad (2.11)$$

We shall refer to such points as *regular points* of μ (as any blow-up measure is a multiple of the d -dimensional Lebesgue measure). Furthermore, for every regular point x_0 there exists a sequence $r_m \downarrow 0$ and a positive constant c such that

$$cr_m^{-d} (T_{\#}^{(x_0, r_m)} \mu) \xrightarrow{*} \frac{d\mu}{d\mathcal{L}^d}(x_0) \mathcal{L}^d \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N).$$

2.2.3 Rigidity results

As discussed in the introduction, for a linear operator $\mathcal{A} := \sum_{|\alpha| \leq k} A_{\alpha} \partial^{\alpha}$, the wave cone

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \ker \mathbb{A}^k(\xi) \subset \mathbb{R}^N$$

contains those amplitudes along which is possible to have “one-directional” oscillations or concentrations, or equivalently, it contains the amplitudes along which the system loses its ellipticity.

The main result of [11] asserts that the polar vector of the singular part of an \mathcal{A} -free measure μ necessarily has to lie in $\Lambda_{\mathcal{A}}$:

Theorem 2.21. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free Radon measure on Ω with values in \mathbb{R}^N , i.e.,*

$$\mathcal{A} \mu = 0 \quad \text{in the sense of distributions.}$$

Then,

$$\frac{d\mu}{d|\mu|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Remark 2.22. The proof of this result does not require \mathcal{A} to satisfy Murat’s constant rank condition (2.2). However, for the present work, this requirement cannot be dispensed with in the following decomposition by Fonseca and Müller [16, Lemma 2.14], where it is needed for the Fourier projection arguments.

Lemma 2.23 (projection). *Let \mathcal{A} be a homogeneous differential operator satisfying the constant*

rank property (2.2). Then, for every $1 < p < \infty$, there exists a linear projection operator

$$\mathcal{P} : L_{\text{per}}^p(Q; \mathbb{R}^N) \rightarrow L_{\text{per}}^p(Q; \mathbb{R}^N)$$

and a positive constant $c_p > 0$ such that

$$\mathcal{A}(\mathcal{P}u) = 0, \quad \int_Q \mathcal{P}u \, dy = 0, \quad \|u - \mathcal{P}u\|_{L^p(Q)} \leq c_p \|\mathcal{A}u\|_{W_{\text{per}}^{-k,p}(Q)},$$

for every $u \in L_{\text{per}}^p(Q; \mathbb{R}^N)$ with $\int_Q u \, dy = 0$.

Remark 2.24. Here, $W_{\text{per}}^{k,p}(Q)$ ($1 < p < \infty$) denotes the space of $W^{k,p}(Q)$ -maps, which can be Q -periodically extended to a $W_{\text{loc}}^{k,p}(\mathbb{R}^d)$ -map; the space $W_{\text{per}}^{-k,q}(Q)$ with $1/p + 1/q = 1$ is its dual. Note that the dual norm is equivalent to

$$\left\| \mathcal{F}^{-1} \left[\frac{\hat{u}(\xi)}{(1 + |\xi|^2)^{k/2}} \right] \right\|_{L^q(Q)},$$

where $\hat{u}(\xi)$, $\xi \in \mathbb{Z}^d$, denotes the Fourier coefficients on torus and \mathcal{F}^{-1} is the inverse Fourier transform. In the case $\int_Q u \, dx = 0$ (hence $\hat{u}(0) = 0$), this norm is also equivalent to the norm

$$\left\| \mathcal{F}^{-1} \left[\frac{\hat{u}(\xi)}{|\xi|^k} \right] \right\|_{L^q(Q)}$$

since the Fourier multipliers $(1 + |\xi|^2)^{-k/2}$ and $|\xi|^{-k}$ are comparable (by the Mihlin multiplier theorem) for all ξ with $|\xi| \geq 1$.

Proof. The proof given in [16] technically applies only to first-order differential operators. However, the result can be extended to operators of any degree, as long as they are homogeneous. We shortly recall how this is done in the next lines.

By definition,

$$\text{rank } \mathbb{A}^k(\xi) = \text{rank } \mathbb{A}(\xi) = r \quad \text{for all } \xi \in \mathbb{S}^{d-1}. \quad (2.12)$$

For each $\xi \in \mathbb{R}^d$ we write $\mathbb{P}(\xi) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ to denote the *orthogonal projection* onto $\ker \mathbb{A}(\xi)$, and by $\mathbb{Q}(\xi)$ we denote the left inverse of $\mathbb{A}(\xi)$.

It follows from the positive homogeneity of \mathbb{A} that $\mathbb{P} : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$ is 0-homogeneous. Moreover, $(\text{id}_{\mathbb{R}^N} - \mathbb{P}(\xi)) = \mathbb{Q}(\lambda\xi)\mathbb{A}(\lambda\xi) = \lambda^k \mathbb{Q}(\lambda\xi)\mathbb{A}(\xi)$ and hence $\mathbb{Q} : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$ is homogeneous of degree $-k$. In light of (2.12), both maps are smooth (see Proposition 2.7 in [16]).

Since the map $\xi \mapsto \mathbb{P}(\xi)$ is homogeneous of degree 0 and is infinitely differentiable in \mathbb{S}^{N-1} , by Proposition 2.13 in [16], the map defined on $C_{\text{per}}^\infty(Q; \mathbb{R}^N)$ by

$$\mathcal{P}u(w) := \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \mathbb{P}(\xi)_{ij} \hat{u}_j(\xi) e^{2\pi i \xi \cdot w},$$

where $\{\hat{u}(\xi)\}_{\mathbb{Z}^d}$ are the Fourier coefficients of $u \in L^p(Q; \mathbb{R}^N)$, extends to a (p, p) -Fourier multiplier

\mathcal{P} on $L^p(Q; \mathbb{R}^N)$ for all $1 < p < \infty$.

Since $\mathbb{P}(\xi)$ is a projection, so it is \mathcal{P} :

$$\begin{aligned} (\mathcal{P} \circ \mathcal{P})u &= \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} (\mathbb{P}(\xi) \circ \mathbb{P}(\xi)) \hat{u}(\xi) e^{2\pi i \xi \cdot w} \\ &= \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \mathbb{P}(\xi) \hat{u}(\xi) e^{2\pi i \xi \cdot w} = \mathcal{P}u. \end{aligned}$$

Moreover,

$$(\mathcal{A}(\widehat{\mathcal{P}u}))(\xi) = \mathbb{A}(\xi) \widehat{(\mathcal{P}u)}(\xi) = \mathbb{A}(\xi) [\mathbb{P}(\xi) \hat{u}(\xi)] = 0$$

for all $\xi \in \mathbb{Z}^d \setminus \{0\}$. Since $\widehat{(\mathcal{P}u)}(0) = 0$, we get

$$\int_Q \mathcal{P}u \, dy = 0, \quad \text{and} \quad \mathcal{A}(\mathcal{P}u) = 0.$$

Finally, let $u \in C_{\text{per}}^\infty(Q; \mathbb{R}^N)$. We use that \mathbb{A} and \mathbb{Q} are k -homogeneous and $(-k)$ -homogeneous, respectively, to show that

$$\begin{aligned} \hat{u}(\xi) - \widehat{\mathcal{P}u}(\xi) &= (\text{id}_{\mathbb{R}^N} - \mathbb{P}(\xi)) \hat{u}(\xi) \\ &= \mathbb{Q}(\xi) \mathbb{A}(\xi) \hat{u}(\xi) = \mathbb{Q}\left(\frac{\xi}{|\xi|}\right) \frac{1}{|\xi|^k} \mathbb{A}(\xi) \hat{u}(\xi), \end{aligned}$$

for all $\xi \in \mathbb{Z}^d \setminus \{0\}$, and therefore, via the Mihlin multiplier theorem and Remark 2.24, that

$$\|u - \mathcal{P}u\|_{L^p_{\text{per}}(Q)} \leq c_p \|\mathcal{A}u\|_{W_{\text{per}}^{-k,p}(Q)}$$

for all $u \in C_{\text{per}}^\infty(Q; \mathbb{R}^N)$ with $\int_Q u \, dy = 0$; the general case follows by approximation. \square

Lemma 2.23 implies that every Q -periodic $u \in L^p(Q; \mathbb{R}^N)$ with $1 < p < \infty$ and mean value zero can be decomposed as the sum

$$u = v + w, \quad v = \mathcal{P}u,$$

where

$$\mathcal{A}v = 0 \quad \text{and} \quad \|w\|_{L^p(Q)} \leq c_p \|\mathcal{A}u\|_{W_{\text{per}}^{-k,p}(Q)}.$$

A crucial issue in lower semicontinuity problems is the understanding of oscillation and concentration effects in weakly (weakly*) convergent sequences. In our setting, we are interested in sequences of asymptotically \mathcal{A} -free measures generating what we naturally term \mathcal{A} -free Young measures. The study of general \mathcal{A} -free Young measures can be reduced to understanding oscillations in the class of *periodic* \mathcal{A} -free fields. This is expressed in the next lemma, which is a variant of Proposition 3.1 in [14] for higher-order operators (see also Lemma 2.20 in [5]).

Lemma 2.25 (periodic generators). *Let \mathcal{A} be an homogeneous linear partial differential operator*

satisfying the constant rank property (2.2). Let $(u_j), (v_j) \subset L^1(Q; \mathbb{R}^N)$ be sequences such that

$$u_j - v_j \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N) \quad \text{and} \quad |u_j| + |v_j| \xrightarrow{*} \Lambda \quad \text{in } \mathcal{M}^+(\bar{Q})$$

with $\Lambda(\partial Q) = 0$ and

$$\mathcal{A}(u_j - v_j) \rightarrow 0 \quad \text{in } W^{-k,q}(Q; \mathbb{R}^n) \quad \text{for some } 1 < q < d/(d-1).$$

Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a Lipschitz function and assume that the sequence (u_j) generates the Young measure $\mathbf{v} \in \mathbf{Y}(Q; \mathbb{R}^N)$. Then, there exists another sequence $(z_j) \subset C_{\text{per}}^\infty(Q; \mathbb{R}^N)$ such that

$$\mathcal{A} z_j = 0, \quad \int_Q z_j = 0, \quad z_j \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N),$$

and (up to taking a subsequence of the v_j 's) the sequence $(v_j + z_j)$ also generates the Young measure \mathbf{v} , i.e.,

$$(v_j + z_j) \xrightarrow{\mathbf{Y}} \mathbf{v} \quad \text{in } \mathbf{Y}(Q; \mathbb{R}^N).$$

Moreover,

$$\lim_{j \rightarrow \infty} \int_Q f(u_j) \, dx = \lim_{j \rightarrow \infty} \int_Q f(v_j + z_j) \, dx.$$

Note that the sequence (z_j) may depend on the choice of f (since $\mathbb{1}_{\bar{Q}} \otimes f$ is not necessarily in $\mathbf{E}(\Omega; \mathbb{R}^N)$).

Proof. Consider a family of cut-off functions $\psi_m \in C_c^\infty(Q; [0, 1])$ with $\psi_m \equiv 1$ in the set

$$\{y \in Q : \text{dist}(y, \partial Q) > 1/m\}$$

and define

$$w_j^m := (u_j - v_j) \psi_m \in C_c(Q; \mathbb{R}^N).$$

Since $\psi_m \in C_c^\infty(Q)$, it also holds that

$$w_j^m \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N) \quad \text{as } j \rightarrow \infty, \quad \text{for every } m \in \mathbb{N}.$$

Furthermore,

$$\mathcal{A} w_j^m = \mathcal{A}(u_j - v_j) \psi_m + \sum_{\substack{|\alpha|=k, \\ 1 \leq |\beta| \leq k}} c_{\alpha\beta} A_\alpha \partial^{\alpha-\beta} (u_j - v_j) \partial^\beta \psi_m \quad (2.13)$$

where $c_{\alpha\beta} \in \mathbb{N}$. The convergence $u_j - v_j \xrightarrow{*} 0$ and the compact embedding $\mathcal{M}(Q; \mathbb{R}^N) \xrightarrow{c} W^{-1,q}(Q; \mathbb{R}^N)$ entail, via (2.13), the strong convergence

$$\mathcal{A} w_j^m \rightarrow 0 \quad \text{in } W^{-k,q}(Q; \mathbb{R}^n) \quad \text{as } j \rightarrow \infty. \quad (2.14)$$

Let, for $\varepsilon > 0$, $\rho_\varepsilon := \rho(x/\varepsilon)$ where $\rho \in C_c^\infty(B_1)$ is an even mollifier. For every $m \in \mathbb{N}$, let $(\varepsilon(j, m))_j$

be a sequence with $\varepsilon(j, m) \downarrow 0$ as $j \rightarrow \infty$ such that for $\hat{w}_j^m := w_j^m * \rho_{\varepsilon(j, m)}$ it holds that

$$\|w_j^m - \hat{w}_j^m\|_{L^1(Q)} \leq \frac{1}{j}.$$

Fix $\varphi \in W^{k, q}(Q; \mathbb{R}^n) \cap C_c(Q; \mathbb{R}^n)$ and fix $m \in \mathbb{N}$. Then, for $j \in \mathbb{N}$ sufficiently large, it holds that

$$\begin{aligned} |\langle \mathcal{A} \hat{w}_j^m, \varphi \rangle| &= |\langle \mathcal{A} w_j^m, \varphi * \rho_{\varepsilon(j, m)} \rangle| \\ &\leq \|\mathcal{A} w_j^m\|_{W^{-k, q}(Q)} \|\varphi * \rho_{\varepsilon(j, m)}\|_{W^{k, q}(Q)} \\ &\leq \|\mathcal{A} w_j^m\|_{W^{-k, q}(Q)} \|\varphi\|_{W^{k, q}(Q)}. \end{aligned}$$

The case when φ belongs to $W_0^{k, q}(Q; \mathbb{R}^n)$ follows by approximation. Hence, from (2.14) we obtain that

$$\|\mathcal{A} \hat{w}_j^m\|_{W^{-k, q}(Q)} \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad \text{for every } m \in \mathbb{N}. \quad (2.15)$$

The second step consists of applying the projection of Lemma 2.23 to the mollified functions \hat{w}_j^m . Define $\tilde{w}_j^m := \hat{w}_j^m - \int_Q \hat{w}_j^m \, dx$ (by a slight abuse of notation, we also denote by \tilde{w}_j^m its Q -periodic extension to \mathbb{R}^d) and $z_j^m := \mathcal{P} \tilde{w}_j^m$. It follows from Lemma 2.23 that

$$\begin{aligned} \lim_{j \rightarrow \infty} \|\hat{w}_j^m - z_j^m\|_{L^1_{\text{per}}(Q)} &\leq \lim_{j \rightarrow \infty} \|\tilde{w}_j^m - z_j^m\|_{L^q_{\text{per}}(Q)} + \lim_{j \rightarrow \infty} \left| \int_Q \hat{w}_j^m \, dy \right| \\ &\leq c_q \cdot \lim_{j \rightarrow \infty} \|\mathcal{A} \hat{w}_j^m\|_{W^{-k, q}(Q)} + \lim_{j \rightarrow \infty} \left| \int_Q w_j^m \, dy \right| \\ &= 0, \end{aligned} \quad (2.16)$$

where in the first inequality we have exploited that $\mathcal{L}^d(Q) = 1$, and for the last inequality we have used the equality of the norms

$$\|u\|_{W^{-k, p}_{\text{per}}(Q)} = \|u\|_{W^{-k, p}(Q)},$$

which holds for functions $u \in C^\infty_{\text{per}}(Q; \mathbb{R}^d)$ with $u = 0$ on ∂Q and all $1 < p < \infty$, together with (2.15).

Fix $\varphi \otimes g \in C(\bar{Q}) \times W^{1, \infty}(\mathbb{R}^N)$ with $\varphi \otimes g \in \mathbf{E}(Q; \mathbb{R}^N)$. Using the Lipschitz continuity of g , we have that

$$\begin{aligned} \int_Q \varphi g(u_j) \, dy &= \int_Q \varphi g(u_j - v_j + v_j) \, dy \\ &\geq \int_Q \varphi g(\hat{w}_j^m + v_j) \, dy - \|\varphi\|_\infty \cdot \text{Lip}(g) \cdot \int_Q |1 - \psi_m|(|u_j| + |v_j|) \, dy \\ &\quad - \|\varphi\|_\infty \cdot \text{Lip}(g) \cdot \|w_j^m - \hat{w}_j^m\|_{L^1(Q)} \\ &\geq \int_Q \varphi g(z_j^m + v_j) \, dy - \|\varphi\|_\infty \cdot \text{Lip}(g) \cdot \left(\int_Q |1 - \psi_m|(|u_j| + |v_j|) \, dy \right. \\ &\quad \left. + \|w_j^m - \hat{w}_j^m\|_{L^1(Q)} + \|\hat{w}_j^m - z_j^m\|_{L^1(Q)} \right). \end{aligned}$$

By taking the limit as $j \rightarrow \infty$ in the previous inequality we obtain, by (2.16), the lower bound

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_Q \varphi g(u_j) \, dy &\geq \limsup_{j \rightarrow \infty} \int_Q \varphi g(z_j^m + v_j) \, dy \\ &\quad - \|\varphi\|_\infty \cdot \text{Lip}(g) \cdot \Lambda(\bar{Q} \setminus Q_{(m-1)/m}) \quad \text{for all } m \in \mathbb{N}, \end{aligned} \quad (2.17)$$

where $Q_r := rQ$ for $r > 0$. By the same argument one gets

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_Q \varphi g(u_j) \, dy &\leq \liminf_{j \rightarrow \infty} \int_Q \varphi g(z_j^m + v_j) \, dy \\ &\quad + \|\varphi\|_\infty \cdot \text{Lip}(g) \cdot \Lambda(\bar{Q} \setminus Q_{(m-1)/m}) \quad \text{for all } m \in \mathbb{N}. \end{aligned} \quad (2.18)$$

Combining (2.17), (2.18) and using that $\Lambda(\partial Q) = 0$, we first let $j \rightarrow \infty$ and then $m \rightarrow \infty$ to obtain

$$\begin{aligned} \limsup_{m \rightarrow \infty} \limsup_{j \rightarrow \infty} \int_Q \varphi g(z_j^m + v_j) \, dy &\leq \lim_{j \rightarrow \infty} \int_Q \varphi g(u_j) \, dy \\ &\leq \liminf_{m \rightarrow \infty} \liminf_{j \rightarrow \infty} \int_Q \varphi g(z_j^m + v_j) \, dy. \end{aligned}$$

Let $\{g_h\}_{h=0}^\infty$ where $g_0 := \mathbb{1}_Q \otimes f$ and $\{\varphi_h \otimes g_h\}_{h \in \mathbb{N}}$ is the family of integrands appearing in Lemma 2.14. By a diagonalization argument on z_j^m we may find a sequence $(z_j) \subset C_{\text{per}}^\infty(Q; \mathbb{R}^N) \cap \ker \mathcal{A}$ such that

$$\int_Q z_j \, dy = 0 \quad \text{for all } j \in \mathbb{N}, \quad z_j \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N),$$

and, for all $h \in \mathbb{N}_0$,

$$\lim_{j \rightarrow \infty} \int_Q \varphi_h g_h(u_j) \, dy = \lim_{j \rightarrow \infty} \int_Q \varphi_h g_h(z_j + v_j) \, dy. \quad (2.19)$$

Since $(z_j + v_j)$ is uniformly bounded in $L^1(Q; \mathbb{R}^N)$, by Lemma 2.15 we may find a subsequence $(z_{j(i)} + v_{j(i)}) \xrightarrow{\mathbf{Y}} \tilde{\nu} \in \mathbf{Y}(Q; \mathbb{R}^N)$. In particular, since $g_m \in \mathbf{E}(\Omega; \mathbb{R}^N)$ for all $h \in \mathbb{N}$,

$$\lim_{i \rightarrow \infty} \int_Q \varphi_h g_h(z_{j(i)} + v_{j(i)}) = \langle\langle \varphi_h \otimes g_h, \tilde{\nu} \rangle\rangle.$$

By combining with (2.19) we obtain

$$\lim_{j \rightarrow \infty} \int_Q f(u_j) \, dy = \lim_{j \rightarrow \infty} \int_Q f(z_{j(i)} + v_{j(i)}) \, dy,$$

and

$$\langle\langle \varphi_m \otimes g_m, \tilde{\nu} \rangle\rangle = \langle\langle \varphi_m \otimes g_m, \nu \rangle\rangle,$$

where ν is the Young measure generated by u_j . Lemma 2.14 now gives $\nu = \tilde{\nu}$.

□

2.2.4 Scaling properties of \mathcal{A} -free measures

If \mathcal{A} is a homogeneous operator, then

$$\mathcal{A}[T_{\#}^{(x_0, r)} \mu] = 0 \quad \text{on } (x_0 - \Omega)/r,$$

for all \mathcal{A} -free measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$. In general, the re-scaled measure $T_{\#}^{(x_0, r)} \mu$ is a $(T_*^r \mathcal{A})$ -free measure in $(x_0 - \Omega)/r$, where $T_*^r \mathcal{A}$ is the operator defined by

$$T_*^r \mathcal{A} := \sum_{h=0}^k r^{k-h} \mathcal{A}^h,$$

where k is the degree of the operator \mathcal{A} and

$$\mathcal{A}^h := \sum_{|\alpha|=h} A_{\alpha} \partial^{\alpha}, \quad \text{for } h = 0, \dots, k.$$

Notice that, with this convention, $(T_*^r \mathcal{A})^k = \mathcal{A}^k$.

In the sequel it will be often convenient to work with weak* convergent sequences whose elements are $(T_*^r \mathcal{A})$ -free measures; mostly due to a blow-up techniques. The following two results will be useful.

Proposition 2.26 (high-order oscillations I). *Let $r_m \downarrow 0$ be a sequence of positive numbers and let (μ_m) be a sequence of \mathcal{A} -free measures in $\mathcal{M}(\Omega; \mathbb{R}^N)$ with the following property: there are positive constants c_m such that*

$$\gamma_m := c_m T_{\#}^{(x_0, r_m)} \mu_m \xrightarrow{*} \gamma \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N). \quad (2.20)$$

Then,

$$\mathcal{A}^k (c_m T_{\#}^{(x_0, r_m)} \mu_j) \rightarrow 0 \quad \text{in } W^{-k, q}(\Omega) \quad \text{for all } 1 < q < d/(d-1).$$

Proof. Fix $r > 0$. The $(T_*^r \mathcal{A})$ -freeness of each $T_{\#}^{(x_0, r)} \mu_j$ yields

$$\mathcal{A}^k (T_{\#}^{(x_0, r)} \mu_j) = - \sum_{h=0}^{k-1} \mathcal{A}^h (r^{k-h} T_{\#}^{(x_0, r)} \mu_j), \quad (2.21)$$

both sides interpreted in the sense of distributions. This implies that

$$r_m^{k-h} c_m T_{\#}^{(x_0, r_m)} \mu_m \xrightarrow{*} 0 \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N), \quad \text{for every } h = 0, \dots, k-1;$$

in turn, the compact embedding $\mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N) \xrightarrow{c} W_{\text{loc}}^{-1, q}(\mathbb{R}^d; \mathbb{R}^N)$ entails the strong convergence

$$r_m^{k-h} c_m T_{\#}^{(x_0, r_m)} \mu_m \rightarrow 0 \quad \text{in } W_{\text{loc}}^{-1, q}(\mathbb{R}^d; \mathbb{R}^N) \quad \text{for every } h = 0, \dots, k-1.$$

Hence,

$$\mathcal{A}^h (r_m^{k-h} c_m T_{\#}^{(x_0, r)} \mu_m) \rightarrow 0 \quad \text{locally in } W^{-k, q}(\mathbb{R}^d; \mathbb{R}^N) \quad (2.22)$$

for every $h = 0, \dots, k-1$. The assertion then follows from (2.21) and (2.22). \square

Corollary 2.27 (high-order oscillations II). *Let (γ_m) be any blow-up sequence of an \mathcal{A} -free measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$, i.e.,*

$$\gamma_m = c_m T_{\#}^{(x_0, r_m)} \mu \xrightarrow{*} \gamma \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N),$$

for some $x_0 \in \Omega$, $r_m \downarrow 0$, $c_m > 0$, and $\gamma \in \text{Tan}(\mu, x_0)$. Then,

$$\mathcal{A}^k \gamma_m \rightarrow 0 \quad \text{locally in } \mathbf{W}^{-k, q}(\mathbb{R}^d; \mathbb{R}^N).$$

2.2.5 Fourier coefficients of \mathcal{A}^k -free sequences

We shall denote the subspace generated by the wave cone $\Lambda_{\mathcal{A}}$ by

$$V_{\mathcal{A}} := \text{span} \Lambda_{\mathcal{A}} \subset \mathbb{R}^N.$$

Using Fourier series, it is relatively easy to understand the rigidity of \mathcal{A}^k -free periodic fields. To fix ideas, let u be a Q -periodic field in $L_{\text{per}}^2(\mathbb{R}^d; \mathbb{R}^N) \cap \ker \mathcal{A}^k$ with mean value zero (or equivalently $\hat{u}(0) = 0$). Applying the Fourier transform to $\mathcal{A}^k u = 0$, we find that

$$0 = \mathcal{F}(\mathcal{A}^k u)(\xi) = \mathbb{A}^k(\xi) \hat{u}(\xi) \quad \text{for all } \xi \in \mathbb{Z}^d.$$

Hence, $\hat{u}(\xi) \in \ker_{\mathbb{C}} \mathbb{A}^k(\xi)$ for every $\xi \in \mathbb{Z}^d$ (here, $\mathbb{A}^k(\xi)$ is understood as a complex-valued tensor). In particular,

$$\{\hat{u}(\xi) : \xi \in \mathbb{Z}^d\} \subset \mathbb{C} \Lambda_{\mathcal{A}}.$$

Since u is a real vector-valued function, it immediately follows that

$$u \in L_{\text{per}}^2(Q; V_{\mathcal{A}}). \tag{2.23}$$

Using a density argument one can show that, up to a constant term, also functions in $L_{\text{per}}^1(Q; \mathbb{R}^N) \cap \ker \mathcal{A}^k$ take values only in $V_{\mathcal{A}}$. The relevance of this observation will be used later in conjunction with Lemma 2.25 in Lemma 2.38.

2.2.6 \mathcal{A} -quasiconvexity

We state some well-known and some more recent results regarding the properties of \mathcal{A} -quasiconvex integrands. This notion was first introduced by Morrey [22] in the case of curl-free vector fields, where it is known as *quasiconvexity*, and later extended by Dacorogna [9] and Fonseca–Müller [16] to general linear PDE-constraints.

A Borel function $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is called *\mathcal{A} -quasiconvex* if

$$h(A) \leq \int_Q h(A + w(y)) \, dy$$

for all $A \in \mathbb{R}^N$ and all Q -periodic $w \in C^\infty(\mathbb{R}^d; \mathbb{R}^N)$ such that

$$\mathcal{A} w = 0 \quad \text{and} \quad \int_Q w \, dx = 0.$$

For functions h that are not \mathcal{A} -quasiconvex one may define the largest \mathcal{A} -quasiconvex function below h .

Definition 2.28 (\mathcal{A} -quasiconvex envelope). *Given a Borel function $h: \mathbb{R}^N \rightarrow \mathbb{R}$ we define the \mathcal{A} -quasiconvex envelope of h at $A \in \mathbb{R}^N$ as*

$$(Q_{\mathcal{A}}h)(A) := \inf \left\{ \int_Q h(A + w(y)) \, dy : w \in C_{\text{per}}^\infty(Q; \mathbb{R}^N) \cap \ker \mathcal{A}, \int_Q w \, dy = 0 \right\}.$$

For a map $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ we write $Q_{\mathcal{A}}f(x, A)$ for $(Q_{\mathcal{A}}f(x, \cdot))(A)$ by a slight abuse of notation.

We recall from [16] that the \mathcal{A} -quasiconvex envelope of an upper semicontinuous function is \mathcal{A} -quasiconvex and that it is actually the largest \mathcal{A} -quasiconvex function below h .

Lemma 2.29. *If $h: \mathbb{R}^N \rightarrow [0, \infty)$ is upper semicontinuous, then $Q_{\mathcal{A}}h$ is upper semi-continuous and \mathcal{A} -quasiconvex. Furthermore, $Q_{\mathcal{A}}h$ is the largest \mathcal{A} -quasiconvex function below h .*

2.2.7 \mathcal{D} -convexity

Let \mathcal{D} be a balanced cone in \mathbb{R}^N , i.e., we assume that $tA \in \mathcal{D}$ for all $A \in \mathcal{D}$ and every $t \in \mathbb{R}$. A real-valued function $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be \mathcal{D} -convex provided its restrictions to all line segments in \mathbb{R}^N with directions in \mathcal{D} are convex. Here, \mathcal{D} will always be the wave cone $\Lambda_{\mathcal{A}}$ for the linear PDE operator \mathcal{A} .

Lemma 2.30. *Let $h: \mathbb{R}^N \rightarrow [0, \infty)$ be an integrand with linear growth at infinity. Further, suppose that h is \mathcal{A}^k -quasiconvex. Then, h is $\Lambda_{\mathcal{A}}$ -convex.*

Proof. Let $\xi \in \mathbb{S}^{d-1}$ and let $A_1, A_2 \in \mathbb{R}^d$ with $P := A_1 - A_2 \in \ker \mathbb{A}^k(\xi)$. We claim that

$$h(\theta A_1 + (1 - \theta)A_2) \leq \theta h(A_1) + (1 - \theta)h(A_2), \quad \text{for all } \theta \in (0, 1).$$

Fix such a θ and consider the one-dimensional 1-periodic function

$$\chi(s) := (1 - \theta)\mathbb{1}_{[0, \theta)}(s) - \theta\mathbb{1}_{[\theta, 1)}(s), \quad s \in \mathbb{R},$$

which has zero mean value. Fix $\varepsilon \in \min\{\theta/2, (1 - \theta)/2\}$ so that the mollified function $\chi_\varepsilon := \chi * \rho_\varepsilon$ has the following properties:

$$|\{s : \chi_\varepsilon = 1 - \theta\}| \geq \theta - 2\varepsilon, \quad |\{s : \chi_\varepsilon = -\theta\}| \geq (1 - \theta) - 2\varepsilon.$$

Define the sequence of Q -periodic functions

$$u_\varepsilon := P\chi_\varepsilon(y \cdot \xi).$$

By construction, this is a $C_{\text{per}}^\infty(Q; \mathbb{R}^N)$ function, it has zero mean value in Q , and since $P \in \ker \mathbb{A}(\xi)$, it is easy to check that

$$\mathcal{A}^k u_\varepsilon = \frac{d^k \chi_\varepsilon}{ds^k}(y \cdot \xi) \mathbb{A}^k(\xi)P = 0 \quad \text{in the sense of distributions.}$$

Hence, by the definition of \mathcal{A}^k -quasiconvexity and our choice of ε , we have

$$\begin{aligned} h(\theta A_1 + (1 - \theta)A_2) &\leq \int_Q h(\theta A_1 + (1 - \theta)A_2 + u_\varepsilon) \, dy \\ &\leq (\theta - 2\varepsilon)h(A_1) + ((1 - \theta) - 2\varepsilon)h(A_2) \\ &\quad + M(1 + |A_1| + |A_2| + |P|)4\varepsilon \end{aligned}$$

Letting $\varepsilon \downarrow 0$ in the previous inequality yields the claim. \square

The following is an immediate consequence of Lemmas 2.29 and 2.30.

Corollary 2.31. *If $h: \mathbb{R}^N \rightarrow [0, \infty)$ is upper semicontinuous, then $(Q_{\mathcal{A}^k} h)^\#$ is an \mathcal{A}^k -quasiconvex and $\Lambda_{\mathcal{A}}$ -convex function.*

To continue our discussion we define the notion of *convexity at a point*. Let $h: \mathbb{R}^N \rightarrow \mathbb{R}$ be a Borel function. We recall that Jensen's definition of convexity states that h is convex if and only if

$$f\left(\int_{\mathbb{R}^N} A \, d\nu(A)\right) \leq \int_{\mathbb{R}^N} h(A) \, d\nu(A) \quad (2.24)$$

for all probability measures $\nu \in \mathcal{M}^1(\mathbb{R}^N)$.

A Borel function $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be *convex at a point* $A_0 \in \mathbb{R}^N$ if (2.24) holds for all probability measures ν with barycenter A_0 , that is, every $\nu \in \mathcal{M}^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} A \, d\nu = A_0$.

Returning to the convexity properties of \mathcal{A}^k -quasiconvex functions, it was recently shown by Kirchheim and Kristensen [17, 18] that \mathcal{A}^k -quasiconvex and positively 1-homogeneous integrands are actually *convex at points* of $\Lambda_{\mathcal{A}}$ as long as

$$\text{span } \Lambda_{\mathcal{A}} = \mathbb{R}^N. \quad (2.25)$$

In fact, their result is valid in the more general framework of \mathcal{D} -convexity:

Theorem 2.32 (Theorem 1.1 of [18]). *Let \mathcal{D} be a balanced cone of directions in \mathbb{R}^N such that \mathcal{D} spans \mathbb{R}^N . If $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is \mathcal{D} -convex and positively 1-homogeneous, then h is convex at each point of \mathcal{D} .*

Condition (2.25) holds in several applications, for example in the space of gradients ($\mathcal{A} = \text{curl}$) or

the space of divergence-free fields ($\mathcal{A} = \text{div}$). However, it does not necessarily hold in our framework as is evidenced by the operator

$$\mathcal{A} := A_0 \Delta = \sum_{i=1}^d A_0 \partial_{ii},$$

where $A_0 \in \text{Lin}(\mathbb{R}^N; \mathbb{R}^n)$ with $\ker A_0 \neq \mathbb{R}^N$.

Nevertheless, for our purposes it will be sufficient to use the convexity of $f^\#|_{V_{\mathcal{A}}}(x, \cdot)$ in $\Lambda_{\mathcal{A}}$, which is a direct consequence of Theorem 2.32.

Remark 2.33 (automatic convexity). Summing up, in the following we will often make use of the implications from Lemma 2.29, Corollary 2.31 and Theorem 2.32: If $f: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is an integrand with linear growth at infinity, then

$$\begin{aligned} f(x, \cdot) \text{ is } \mathcal{A}^k\text{-quasiconvex and u.s.c.} &\implies \begin{cases} f(x, \cdot) \text{ is } \Lambda_{\mathcal{A}}\text{-convex in } \mathbb{R}^N \text{ and} \\ f^\#|_{V_{\mathcal{A}}}(x, \cdot) \text{ is convex in } \Lambda_{\mathcal{A}} \end{cases}, \\ f \text{ upper semicontinuous} &\implies \begin{cases} \mathcal{Q}_{\mathcal{A}^k} f(x, \cdot) \text{ is } \Lambda_{\mathcal{A}}\text{-convex in } \mathbb{R}^N \text{ and} \\ (\mathcal{Q}_{\mathcal{A}^k} f)^\#|_{V_{\mathcal{A}}}(x, \cdot) \text{ is convex in } \Lambda_{\mathcal{A}} \end{cases}. \end{aligned}$$

2.2.8 Localization principles for Young measures

We state two general localization principles for Young measures, one at *regular* points and another one at *singular* points. These are \mathcal{A} -free versions of the localization principles developed for gradient Young measures and BD-Young measures in [28, 29].

Definition 2.34 (\mathcal{A} -free Young measure). We say that a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^N)$ is an \mathcal{A} -free Young measure in Ω , in symbols $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$, if and only if there exists a sequence $(\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N)$ with $\mathcal{A} \mu_j \rightarrow 0$ in $W^{-k, q}$ for some $1 < q < d/(d-1)$, and such that $\mu_j \xrightarrow{\mathbf{Y}} \nu$ in $\mathbf{Y}(\Omega; \mathbb{R}^N)$.

Proposition 2.35. Let $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free Young measure. Then for \mathcal{L}^d -a.e. $x_0 \in \Omega$ there exists a regular tangent \mathcal{A}^k -free Young measure $\sigma \in \mathbf{Y}_{\mathcal{A}^k}(Q; \mathbb{R}^N)$ to ν at x_0 , that is, σ is generated by a sequence of asymptotically \mathcal{A}^k -free measures and

$$\begin{aligned} [\sigma] &\in \text{Tan}_Q([\nu], x_0), & \sigma_y &= \nu_{x_0} \text{ a.e.}, \\ \lambda_\sigma &= \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0) \mathcal{L}^d \in \text{Tan}_Q(\lambda_\nu, x_0), & \sigma_y^\infty &= \nu_{x_0}^\infty \lambda_\sigma \text{ a.e.} \end{aligned}$$

Moreover, there exists a sequence $(w_j) \subset C_{\text{per}}^\infty(Q; \mathbb{R}^N) \cap \ker \mathcal{A}$ such that $w_j \mathcal{L}^d \xrightarrow{\mathbf{Y}} \sigma$ in $\mathbf{Y}(Q; \mathbb{R}^N)$.

Proposition 2.36. Let $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free Young measure. Then there exists a set $S \subset \Omega$ with $\lambda_\nu^s(\Omega \setminus S) = 0$ such that for all $x_0 \in S$ there exists a non-zero singular tangent \mathcal{A}^k -free Young measure $\sigma \in \mathbf{Y}_{\mathcal{A}^k}(Q; \mathbb{R}^N)$ to ν at x_0 , that is, σ is generated by a sequence of asymptotically \mathcal{A}^k -free

measures and

$$\begin{aligned} [\sigma] &\in \text{Tan}_Q([\nu], x_0), & \sigma_y &= \delta_0 \text{ a.e.}, \\ \lambda_\sigma &\in \text{Tan}_Q(\lambda_\nu^s, x_0), & \lambda_\sigma(Q) &= 1, \quad \lambda_\sigma(\partial Q) = 0, & \sigma_y^\infty &= \nu_{x_0}^\infty \lambda_\sigma \text{-a.e.} \end{aligned}$$

Proof sketches for the last two results can be found in the appendix.

2.3 Jensen's inequalities

In this section we establish generalized Jensen inequalities, which can be understood as a local manifestation of lower semicontinuity. The proof of Theorem 2.2, under Assumption (i), which reads

$$f^\infty(x, A) := \lim_{t \rightarrow \infty} \frac{f(x, tA)}{t} \quad \text{exists for all } (x, A) \in \Omega \times \mathbb{R}^N,$$

will easily follow from Propositions 2.37 and 2.39, by the very same argument used in the proof of (2.32) below.

On the other hand, to prove the Theorem 2.2 under the weaker Assumption (ii),

$$f^\infty(x, A) := \lim_{t \rightarrow \infty} \frac{f(x, tA)}{t} \quad \text{exists for all } (x, A) \in \Omega \times \text{span } \Lambda_{\mathcal{A}},$$

requires to perform a direct blow-up argument for what concerns the regular part of μ and only Proposition 2.39 is used in the proof.

2.3.1 Jensen's inequality at regular points

We first consider regular points.

Proposition 2.37. *Let $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free Young measure. Then, for \mathcal{L}^d -almost every $x_0 \in \Omega$ it holds that*

$$h \left(\langle \text{id}, \nu_{x_0} \rangle + \langle \text{id}, \nu_{x_0}^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0) \right) \leq \langle h, \nu_{x_0} \rangle + \langle h^\#, \nu_{x_0}^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0),$$

for all upper semicontinuous and \mathcal{A}^k -quasiconvex $h: \mathbb{R}^N \rightarrow [0, \infty)$ with linear growth at infinity.

Proof. We make use of Lemma 2.19 to get a collection $\{h_m\} \subset \mathbf{E}(\Omega; \mathbb{R}^N)$ such that $h_m \downarrow h$, $h_m^\infty \downarrow h^\#$ pointwise in Ω and $\bar{\Omega}$ respectively, all h_m are Lipschitz continuous and have uniformly bounded linear growth constants. Fix $x_0 \in \Omega$ such that there exists a regular tangent measure $\sigma \in \mathbf{Y}_{\mathcal{A}^k}(Q; \mathbb{R}^N)$ of ν at x_0 as in Proposition 2.35, which is possible for \mathcal{L}^d -a.e. $x_0 \in \Omega$. The localization principle for regular points tells us that $[\sigma] = A_0 \mathcal{L}^d$ with

$$A_0 := \langle \text{id}, \nu_{x_0} \rangle + \langle \text{id}, \nu_{x_0}^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0) \in \mathbb{R}^N,$$

and that we might find a sequence $z_j \in C_{\text{per}}^\infty(Q; \mathbb{R}^N) \cap \ker \mathcal{A}^k$ with $\int_Q z_j \, dy = 0$ and satisfying

$$(A_0 + z_j) \mathcal{L}^d \xrightarrow{Y} \sigma \quad \text{in } \mathbf{Y}(Q; \mathbb{R}^N). \quad (2.26)$$

Fix $m \in \mathbb{N}$. We use the fact that $\int_Q z_j \, dy = 0$, (2.70) and the \mathcal{A}^k -quasiconvexity of h , to get for every $m \in \mathbb{N}$ that

$$\begin{aligned} \langle h_m, \nu_{x_0} \rangle + \langle h_m^\infty, \nu_{x_0}^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0) &= \frac{1}{|Q_r|} \langle \mathbb{1}_Q \otimes h_m, \sigma \rangle \\ &= \lim_{j \rightarrow \infty} \int_Q h_m(A_0 + z_j(y)) \, dy \\ &\geq \limsup_{j \rightarrow \infty} \int_Q h(A_0 + z_j(y)) \, dy \\ &\geq h(A_0). \end{aligned}$$

The result follows by letting $m \rightarrow \infty$ in the previous inequality and using the monotone convergence theorem. \square

2.3.2 Jensen's inequality at singular points

The strategy for *singular points* differs from the regular case as one cannot simply use the definition of \mathcal{A}^k -quasiconvexity. The latter difficulty arises because the tangent measure at a singular point may not be a multiple of the d -dimensional Lebesgue measure.

In order to circumvent this obstacle, we will first show that the support of the singular part of the Young measures ν^∞ at singular points is contained in the subspace $V_{\mathcal{A}}$ of \mathbb{R}^N (see Lemma 2.38 below). Based on this, we invoke Theorem 2.32, which states that an \mathcal{A}^k -quasiconvex and positively 1-homogeneous function is actually convex at points in $\Lambda_{\mathcal{A}}$ when restricted to $V_{\mathcal{A}}$. Then, the Jensen inequality for \mathcal{A} -free Young measures at singular points follows.

Lemma 2.38. *Let $\sigma \in \mathbf{Y}_{\mathcal{A}^k}(Q; \mathbb{R}^N)$ be an \mathcal{A}^k -free Young measure with $\lambda_\sigma(\partial Q) = 0$. Assume also that*

$$[\sigma] \in \mathcal{M}(Q; V_{\mathcal{A}}).$$

Then,

$$\text{supp } \sigma_x^\infty \subset V_{\mathcal{A}} \cap \mathbb{S}^{N-1} \quad \text{for } \lambda_\sigma\text{-a.e. } x \in Q.$$

Proof. By definition, we may find a sequence $(\mu_j) \subset \mathcal{M}(Q; \mathbb{R}^N)$ with $\mathcal{A} \mu_j \rightarrow 0$ in $W^{-k,q}(Q)$ for some $q \in (1, d/(d-1))$, and such that (μ_j) generates the Young measure σ . Notice that, since \mathcal{A}^k is a homogeneous operator and Q is a strictly star-shaped domain, we may re-scale and mollify each μ_j into some $u_j \in L^2(Q; \mathbb{R}^N)$ with the following property: the sequence (u_j) also generates σ and $\mathcal{A} u_j \rightarrow 0$ in $W^{-k,q}(Q)$. In particular,

$$u_j \mathcal{L}^d \xrightarrow{*} [\sigma] \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N).$$

On the other hand, $\mathcal{A}^k([\sigma]) = 0$ and for every $0 < r < 1$ the measure $T_{\#}^{(0,r)}[\sigma]$ is still an \mathcal{A}^k -free measure on Q . Thus, letting $r \uparrow 1$ and mollifying the measure $T_{\#}^{(0,r)}[\sigma]$ on a sufficiently small scale (with respect to $1 - r$) we might find a sequence $(v_j) \subset L^2(Q; V_{\mathcal{A}}) \cap \ker \mathcal{A}^k$ such that

$$v_j \mathcal{L}^d \xrightarrow{*} [\sigma] \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N).$$

Hence,

$$u_j \mathcal{L}^d - v_j \mathcal{L}^d \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N), \quad |u_j \mathcal{L}^d| + |v_j \mathcal{L}^d| \xrightarrow{*} \Lambda \text{ in } \mathcal{M}^+(\bar{Q})$$

and $\Lambda(\partial Q) = 0$. Here, we have used that $\lambda_{\sigma}(\partial Q) = 0$.

We are now in position to apply Lemma 2.25 to the sequences (u_j) , (v_j) . There exists (possibly passing to a subsequence in the v_j 's) a sequence $z_j \in C_{\text{per}}^{\infty}(Q; \mathbb{R}^N) \cap \ker \mathcal{A}^k$ with $z_j \mathcal{L}^d \xrightarrow{*} 0$ and such that

$$v_j \mathcal{L}^d + z_j \mathcal{L}^d \xrightarrow{Y} \sigma \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N).$$

Recall from observation (2.23) that $z_j \in L^2_{\text{per}}(Q; V_{\mathcal{A}})$ for every $j \in \mathbb{N}$. Therefore,

$$(v_j + z_j) \in L^2(Q; V_{\mathcal{A}}) \quad \text{for all } j \in \mathbb{N}.$$

We conclude with an application of Lemma 2.18 (ii) to the sequence $(v_j + z_j)$, which yields

$$\text{supp } \sigma_x^{\infty} \subset V_{\mathcal{A}} \cap \mathbb{S}^{N-1} \quad \text{for } \lambda_{\sigma}\text{-a.e. } x \in Q.$$

This finishes the proof. □

Proposition 2.39. *Let $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free Young measure. Then for λ_{ν}^s -almost every $x_0 \in \Omega$ it holds that*

$$g(\langle \text{id}, \nu_{x_0}^{\infty} \rangle) \leq \langle g, \nu_{x_0}^{\infty} \rangle$$

for all $\Lambda_{\mathcal{A}}$ -convex and positively 1-homogeneous functions $g : \mathbb{R}^N \rightarrow \mathbb{R}$.

Proof. Step 1: Characterization of the support of \mathcal{A} -free Young measures. Let S be the set given by Proposition 2.36, which has full λ_{ν}^s -measure. Further, also the set

$$S' := \{x \in \Omega : \langle \text{id}, \nu_x^{\infty} \rangle \in \Lambda_{\mathcal{A}}\} \subset \Omega$$

has full λ_{ν}^s -measure: Observe first that

$$[\nu]^s = \langle \text{id}, \nu_x^{\infty} \rangle \lambda_{\nu}^s(dx).$$

Since $[\nu]$ is \mathcal{A} -free, we thus infer from Theorem 2.21 that $\langle \text{id}, \nu_x^{\infty} \rangle \in \Lambda_{\mathcal{A}}$ for $[[\nu]^s]$ -a.e. $x \in \Omega$. On the other hand, $\langle \text{id}, \nu_x^{\infty} \rangle = 0 \in \Lambda_{\mathcal{A}}$ for λ_{ν}^* -a.e. $x \in \Omega$, where λ_{ν}^* is the singular part of λ_{ν}^s with respect to $[[\nu]^s]$. This shows that S' has full λ_{ν}^s -measure.

Fix $x_0 \in S \cap S'$ (which remains of full λ_{ν}^s -measure in Ω). Let $\sigma \in \mathbf{Y}_{\mathcal{A}^k}(Q; \mathbb{R}^N)$ be the non-zero singular tangent Young measure to ν at x_0 given by Proposition 2.36 which according to the same

proposition verifies that $\lambda_\sigma(Q) = 1$ and $\lambda(\partial Q) = 0$. On the one hand, since $x_0 \in S$, it holds that

$$\sigma_y = \delta_0 \quad \mathcal{L}^d\text{-a.e.} \quad \text{and} \quad \sigma_y^\infty = \nu_{x_0}^\infty \quad \lambda_\sigma\text{-a.e.}$$

On the other hand, we use the fact that $x_0 \in S'$ to get

$$\langle \text{id}, \nu_{x_0}^\infty \rangle \in \Lambda_{\mathcal{A}} \quad \text{and} \quad [\sigma] = \langle \text{id}, \nu_{x_0}^\infty \rangle \lambda_\sigma \in \mathcal{M}(Q; V_{\mathcal{A}}). \quad (2.27)$$

Note that, by (2.27), all the hypotheses of Lemma 2.38 are satisfied for σ . Thus,

$$\text{supp } \nu_{x_0}^\infty = \text{supp } \sigma_y^\infty \subset V_{\mathcal{A}} \quad \text{for } \lambda_\sigma\text{-a.e. } y \in Q.$$

This equality and the fact that $\lambda_\sigma(Q) > 0$ (recall that σ is a non-zero singular measure) yield

$$\text{supp } \nu_{x_0}^\infty \subset V_{\mathcal{A}} \quad \text{for } \lambda_\nu^s\text{-a.e. } x_0 \in \Omega. \quad (2.28)$$

Step 2: Convexity of g on $\Lambda_{\mathcal{A}}$. The Kirchheim–Kristensen Theorem 2.32 states that the restriction $g|_{V_{\mathcal{A}}} : V_{\mathcal{A}} \subset \mathbb{R}^N \rightarrow \mathbb{R}$ is a convex function at points $A_0 \in \Lambda_{\mathcal{A}}$. In other words, for every probability measure $\kappa \in \mathcal{P}(\mathbb{R}^N)$ with $\langle \text{id}, \kappa \rangle \in \Lambda_{\mathcal{A}}$ and $\text{supp } \kappa \subset V_{\mathcal{A}}$, the Jensen inequality

$$g\left(\int_{\mathbb{R}^N} A \, d\kappa(A)\right) \leq \int_{\mathbb{R}^N} g(A) \, d\kappa(A)$$

holds. Hence, because of (2.27) and (2.28), it follows that

$$g(\langle \text{id}, \nu_{x_0}^\infty \rangle) \leq \langle g, \nu_{x_0}^\infty \rangle.$$

This proves the assertion. □

The following simple corollary will be important in the proof of Theorem 2.6.

Corollary 2.40. *Let $h : \mathbb{R}^N \rightarrow \mathbb{R}$ be an upper semicontinuous integrand with linear growth at infinity and let $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free Young measure. Then for \mathcal{L}^d -almost every $x_0 \in \Omega$ it holds that*

$$Q_{\mathcal{A}^k} h \left(\langle \text{id}, \nu_{x_0} \rangle + \langle \text{id}, \nu_{x_0}^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0) \right) \leq \langle h, \nu_{x_0} \rangle + \langle h^\#, \nu_{x_0}^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x_0).$$

Moreover, for λ_ν^s -a.e. $x_0 \in \Omega$ it holds that

$$(Q_{\mathcal{A}^k} h)^\#(\langle \text{id}, \nu_{x_0}^\infty \rangle) \leq \langle h^\#, \nu_{x_0}^\infty \rangle$$

Proof. The proof follows by combining Propositions 2.37 and 2.39, Lemma 2.29, Corollary 2.31 and the trivial inequalities $Q_{\mathcal{A}^k} h \leq h$, $(Q_{\mathcal{A}^k} h)^\# \leq h^\#$. □

2.4 Proof of Theorems 2.2 and 2.4

Proof of Theorem 2.2. We will prove Theorem 2.2 in full generality, which means that we consider asymptotically \mathcal{A} -free sequences in the $W^{-k,q}$ -norm for some $q \in (1, d/(d-1))$; see Remark 2.3.

Proof under Assumption (i). Let μ_j be a sequence in $\mathcal{M}(\Omega; \mathbb{R}^N)$ weakly* converging to a limit μ and assume furthermore that $\mathcal{A} \mu_j \rightarrow 0$ in $W^{-k,q}(\Omega; \mathbb{R}^N)$ for some $q \in (1, d/(d-1))$. Up to passing to a subsequence, we might also assume that

$$\liminf_{j \rightarrow \infty} \mathcal{F}[\mu_j] = \lim_{j \rightarrow \infty} \mathcal{F}[\mu_j]$$

and that $\mu_j \xrightarrow{Y} \nu$ for some \mathcal{A} -free Young measure $\nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N)$. Using the continuity of f and representation of Corollary 2.20 we get

$$\mathcal{F}[\mu_j] = \langle \langle f, \delta[\mu_j] \rangle \rangle \rightarrow \langle \langle f, \nu \rangle \rangle \quad \text{as } j \rightarrow \infty.$$

The positivity of f further lets us discard possible concentration of mass on $\partial\Omega$,

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{F}[\mu_j] &= \int_{\Omega} \langle f(x, \cdot), \nu_x \rangle dx + \int_{\Omega} \langle f^{\infty}(x, \cdot), \nu_x^{\infty} \rangle d\lambda_{\nu}(x) \\ &\geq \int_{\Omega} \left(\langle f(x, \cdot), \nu_x \rangle + \langle f^{\infty}(x, \cdot), \nu_x^{\infty} \rangle \frac{d\lambda_{\nu}}{d\mathcal{L}^d}(x) \right) dx \\ &\quad + \int_{\Omega} \langle f^{\infty}(x, \cdot), \nu_x^{\infty} \rangle d\lambda_{\nu}^s(x). \end{aligned} \quad (2.29)$$

By assumption, $f(x, \cdot) \in C(\mathbb{R}^N)$ has linear growth at infinity. Hence we might apply Proposition 2.37 to get

$$f(x, \cdot) \left(\langle \text{id}, \nu_x \rangle + \langle \text{id}, \nu_x^{\infty} \rangle \frac{d\lambda_{\nu}}{d\mathcal{L}^d}(x) \right) \leq \langle f(x, \cdot), \nu_x \rangle + \langle f(x, \cdot)^{\#}, \nu_x^{\infty} \rangle \frac{d\lambda_{\nu}}{d\mathcal{L}^d}(x)$$

for \mathcal{L}^d -a.e. $x \in \Omega$. Likewise, we apply Proposition 2.39 to the functions $f(x, \cdot)^{\#}$ to obtain (recall that under the present assumptions $f^{\infty} = f^{\#}$)

$$f(x, \cdot)^{\infty} (\langle \text{id}, \nu_x^{\infty} \rangle) \leq \langle f(x, \cdot)^{\infty}, \nu_x^{\infty} \rangle$$

at λ_{ν}^s -a.e. $x \in \Omega$. Plugging these two Jensen-type inequalities into (2.29) yields

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{F}[\mu_j] &\geq \int_{\Omega} f(x, \cdot) \left(\langle \text{id}, \nu_x \rangle + \langle \text{id}, \nu_x^{\infty} \rangle \frac{d\lambda_{\nu}}{d\mathcal{L}^d}(x) \right) dx \\ &\quad + \int_{\Omega} f^{\infty}(x, \cdot) (\langle \text{id}_{\mathbb{R}^N}, \nu_x^{\infty} \rangle) d\lambda_{\nu}^s(x). \end{aligned} \quad (2.30)$$

Finally, since $\mu_j \xrightarrow{Y} \nu$, it must hold that

$$\langle \text{id}, \nu_x \rangle + \langle \text{id}, \nu_x^{\infty} \rangle \frac{d\lambda_{\nu}}{d\mathcal{L}^d}(x) = \frac{d\mu}{d\mathcal{L}^d}(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega, \quad \text{and}$$

$$\langle \text{id}_{\mathbb{R}^N}, \mathbf{v}_x^\infty \rangle \lambda_\nu^s = \mu^s \quad \Rightarrow \quad \frac{d\mu^s}{d|\mu^s|}(x) = \frac{\langle \text{id}_{\mathbb{R}^N}, \mathbf{v}_x^\infty \rangle}{|\langle \text{id}_{\mathbb{R}^N}, \mathbf{v}_x^\infty \rangle|} \quad \text{for } \lambda_\nu^s\text{-a.e. } x \in \Omega.$$

We can use this representation and the fact that $f^\infty(x, \cdot)$ is positively 1-homogeneous in the right hand side of (2.30) to conclude

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{F}[\mu_j] &\geq \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx \\ &\quad + \int_{\Omega} f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d(|\langle \text{id}_{\mathbb{R}^N}, \mathbf{v}_x^\infty \rangle| \lambda_\nu^s)(x) \\ &= \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx \\ &\quad + \int_{\Omega} f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x) = \mathcal{F}[\mu]. \end{aligned}$$

This proves the claim under Assumption (i). □

Proof under Assumption (ii). For a measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$, consider the functional

$$\mathcal{F}^\#[\mu; B] := \int_B f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_B f^\# \left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x),$$

defined for any Borel subset $B \subset \Omega$.

Let μ_j be a sequence in $\mathcal{M}(\Omega; \mathbb{R}^N)$ weakly* converging to a limit μ and assume furthermore that $\mathcal{A} \mu_j \rightarrow 0$ in $W^{-k, q}(\Omega; \mathbb{R}^N)$ for some $q \in (1, d/(d-1))$. Define $\lambda_j \in \mathcal{M}^+(\Omega)$ via

$$\lambda_j(B) := \mathcal{F}^\#[\mu_j; B] \quad \text{for every Borel } B \subset \Omega.$$

We may find a (not relabeled) subsequence and positive measures $\lambda, \Lambda \in \mathcal{M}_+(\Omega)$ such that

$$\lambda_j \xrightarrow{*} \lambda, \quad |\mu_j| \xrightarrow{*} \Lambda \quad \text{in } \mathcal{M}^+(\Omega).$$

We claim that

$$\frac{d\lambda}{d\mathcal{L}^d}(x_0) \geq f\left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0)\right) \quad \text{for } \mathcal{L}^d\text{-a.e. } x_0 \in \Omega, \quad (2.31)$$

$$\frac{d\lambda}{d|\mu^s|}(x_0) \geq f^\# \left(x_0, \frac{d\mu^s}{d|\mu^s|}(x_0)\right) \quad \text{for } |\mu^s|\text{-a.e. } x_0 \in \Omega. \quad (2.32)$$

Notice that, if (2.31) and (2.32) hold, then the assertion of the theorem immediately follows. Indeed, there exists a positive Radon measure $\Lambda^* \in \mathcal{M}^+(\Omega)$ (singular to the measure $\mathcal{L}^d + |\mu^s|$) such that

$$\lambda = \frac{d\lambda}{d\mathcal{L}^d} \mathcal{L}^d + \frac{d\lambda}{d|\mu^s|} |\mu^s| + \Lambda^*.$$

Using the Radon-Nikodým theorem, we then obtain that

$$\begin{aligned}
 \liminf_{j \rightarrow \infty} \mathcal{F}^\#[\mu_j] &= \liminf_{j \rightarrow \infty} \lambda_j(\Omega) \\
 &\geq \lambda(\Omega) \\
 &\geq \int_{\Omega} \frac{d\lambda}{d\mathcal{L}^d} dx + \int_{\Omega} \frac{d\lambda}{d|\mu^s|} d|\mu^s| \\
 &\geq \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^\#\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s| \\
 &= \mathcal{F}^\#[\mu].
 \end{aligned} \tag{2.33}$$

With (2.31), (2.32), which are proved below, the result under Assumption (ii) follows. \square

This completes the proof of Theorem 2.2. \square

The following lemma will be used in the proof of (2.31).

Lemma 2.41. *Let $x_0 \in \Omega$ and $R > 0$ be such that $Q_{2R}(x_0) \subset \Omega$. Then, for every $h \in \mathbb{N}$, there exists a sequence $(u_j^h) \subset L^2(\mathbb{R}^d; \mathbb{R}^N)$ such that*

$$\begin{aligned}
 u_j^h &\rightarrow \mu_j \text{ area-strictly in } \mathcal{M}(Q_R(x_0); \mathbb{R}^N) \quad \text{as } h \rightarrow \infty, \quad \text{and} \\
 \|\mathcal{A}^k u_j^h - \mathcal{A}^k \mu_j\|_{W^{-k,q}(Q_R(x_0))} &\rightarrow 0.
 \end{aligned} \tag{2.34}$$

Proof. Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of standard smooth mollifiers. The sequence defined by

$$u_j^h := (\mu_j \llcorner Q_{3R/2}(x_0)) * \rho_{1/h} \in C^\infty(\overline{Q_{2R}(x_0)}; \mathbb{R}^N)$$

satisfies all the conclusion properties as a consequence of the properties of mollification and Remark 2.12 \square

Proof of (2.31). We employ the classical blow-up method to organize the proof. We know from Lebesgue's differentiation theorem and (2.11) that the following properties hold for \mathcal{L}^d -almost every x_0 in Ω :

$$\begin{aligned}
 \frac{d\lambda}{d\mathcal{L}^d}(x_0) &= \lim_{r \downarrow 0} \frac{\lambda(Q_r(x_0))}{r^d} < \infty, \quad \lim_{r \downarrow 0} \frac{|\mu^s|(Q_r(x_0))}{r^d} < \infty, \\
 \lim_{r \downarrow 0} \frac{1}{r^d} \int_{Q_r(x_0)} \left| \frac{d\mu}{d\mathcal{L}^d}(y) - \frac{d\mu}{d\mathcal{L}^d}(x_0) \right| dy &= 0, \\
 \lim_{r \downarrow 0} \frac{1}{r^d} \int_{Q_r} \left| \frac{d\Lambda}{d\mathcal{L}^d}(y) - \frac{d\Lambda}{d\mathcal{L}^d}(x_0) \right| dy &= 0,
 \end{aligned}$$

and

$$\text{Tan}(\mu, x_0) = \left\{ \alpha \cdot \frac{d\mu}{d\mathcal{L}^d}(x_0) \mathcal{L}^d : \alpha \in \mathbb{R}^+ \cup \{0\} \right\}. \tag{2.35}$$

Let $x_0 \in \Omega$ be a point where the properties above are satisfied. Since Ω is an open set, there exists a positive number R such that $Q_{2R(x_0)} \subset \Omega$. From Lemma 2.41, we infer that for almost every $r \in (0, R)$, it holds that

$$\begin{aligned} \mathbf{w}^*\text{-}\lim_{j \rightarrow \infty} \mathbf{w}^*\text{-}\lim_{h \rightarrow \infty} [u_j^h(x_0 + ry) \mathcal{L}_y^d] &= \mathbf{w}^*\text{-}\lim_{j \rightarrow \infty} \mathbf{w}^*\text{-}\lim_{h \rightarrow \infty} r^{-d} T_{\#}^{(x_0, r)} [u_j^h \mathcal{L}^d] \\ &= \mathbf{w}^*\text{-}\lim_{j \rightarrow \infty} r^{-d} T_{\#}^{(x_0, r)} \mu_j \\ &= r^{-d} T_{\#}^{(x_0, r)} \mu, \end{aligned} \quad (2.36)$$

where the weak* convergence is to be understood in $\mathcal{M}(Q; \mathbb{R}^N)$. Thus, choosing a sequence $r \downarrow 0$ with $\lambda_j(\partial Q_r(x_0)) = 0$ and $\Lambda(\partial Q_r(x_0)) = 0$, we get that

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^d}(x_0) &= \lim_{r \rightarrow 0} \lim_{j \rightarrow \infty} \frac{\lambda_j(Q_r(x_0))}{r^d} \\ &= \lim_{r \rightarrow 0} \lim_{j \rightarrow \infty} \frac{\mathcal{F}^{\#}[\mu_j; Q_r(x_0)]}{r^d} \\ &\geq \lim_{r \rightarrow 0} \lim_{j \rightarrow \infty} \limsup_{h \rightarrow \infty} \frac{\mathcal{F}^{\#}[u_j^h \mathcal{L}^d; Q_r(x_0)]}{r^d} \\ &= \lim_{r \rightarrow 0} \lim_{j \rightarrow \infty} \limsup_{h \rightarrow \infty} \int_Q f(x_0 + ry, u_j^h(x_0 + ry)) dy, \end{aligned}$$

where we used Corollary 2.20 and Remark 2.16 for the “ \geq ” estimate.

We may use a suitable diagonalization procedure to find

$$u_r := u_{j(r)}^{h(j(r))} \quad \text{and} \quad \gamma_r := r^{-d} T_{\#}^{(x_0, r)} [u_r \mathcal{L}^d]$$

verifying the following properties:

1. since $y \mapsto u_j^h(x_0 + ry)$ is the density of the measure $r^{-d} T_{\#}^{(x_0, r)} [u_j^h \mathcal{L}^d]$ with respect to \mathcal{L}^d ,

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^d}(x_0) &\geq \lim_{r \rightarrow 0} \int_Q f\left(x_0 + ry, \frac{d\gamma_r}{d\mathcal{L}^d}(y)\right) dy \\ &\geq \lim_{r \rightarrow 0} \int_Q f\left(x_0, \frac{d\gamma_r}{d\mathcal{L}^d}(y)\right) dy \\ &\quad - \lim_{r \rightarrow 0} \left[\omega(r) \left(|\Omega| + \left\| \frac{d\gamma_r}{d\mathcal{L}^d}(y) \right\|_{L^1(\Omega)} \right) \right] \end{aligned} \quad (2.37)$$

2. Through a diagonalization argument we may select $j = j(r)$ and $h = h(j)$ in (2.36) to guarantee that

$$\gamma_r - r^{-d} T_{\#}^{(x_0, r)} \mu \xrightarrow{*} 0 \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N). \quad (2.38)$$

Recall from Section 2.2.2 that (at regular points) there exists a positive constant c such that

$$cr^{-d}T_{\#}^{(x_0,r)}\mu \xrightarrow{*} \frac{d\mu}{d\mathcal{L}^d}(x_0)\mathcal{L}^d \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N),$$

In fact, since

$$\begin{aligned} \left| r^{-d}T_{\#}^{(x_0,r)}\mu(Q) - \frac{d\mu}{d\mathcal{L}^d}(x_0) \right| &= \left| \frac{\mu(Q_r(x_0))}{r^d} - \frac{d\mu}{d\mathcal{L}^d}(x_0) \right| \\ &\leq \int_{Q_r(x_0)} \left| \frac{d\mu}{d\mathcal{L}^d}(x) - \frac{d\mu}{d\mathcal{L}^d}(x_0) \right| dx \\ &\quad + \frac{|\mu^s(Q_r(x_0))|}{r^d} \\ &= o_r(1), \end{aligned}$$

where $o_r(1) \rightarrow 0$ as $r \downarrow 0$. Therefore, the constant c must be equal to 1.

Therefore, up to taking a further subsequence $r \downarrow 0$, we may assume that $r^{-d}T_{\#}^{(x_0,r)}\mu$ is a blow-up sequence of μ and

$$\text{w}^*_{r \downarrow 0}\text{-}\lim \gamma_r = \text{w}^*_{r \downarrow 0}\text{-}\lim r^{-d}T_{\#}^{(x_0,r)}\mu = \frac{d\mu}{d\mathcal{L}^d}(x_0)\mathcal{L}^d \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N).$$

The next step is to verify that γ_r is asymptotically \mathcal{A}^k -free, or equivalently that $\mathcal{A}^k \gamma_r \rightarrow 0$ in $W^{-k,q}(Q)$. This is a simple consequence of Proposition 2.26 applied to the sequence

$$\gamma_r = c_r T_{\#}^{(x_0,r)}[u_r \mathcal{L}^d] \xrightarrow{*} \frac{d\mu}{d\mathcal{L}^d}(x_0)\mathcal{L}^d \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N),$$

with coefficients $c_r := r^{-d}$.

In particular

$$\begin{aligned} \gamma_r - \frac{d\mu}{d\mathcal{L}^d}(x_0)\mathcal{L}^d &\xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N), \quad \text{and} \\ \mathcal{A}^k \left(\gamma_r - \frac{d\mu}{d\mathcal{L}^d}(x_0)\mathcal{L}^d \right) &\rightarrow 0 \quad \text{in } W^{-k,q}(Q; \mathbb{R}^N). \end{aligned}$$

We are now in a position to apply Lemma 2.25 to the sequence γ_r and the Lipschitz function $f(x_0, \cdot)$, whence there exists a sequence $(z_r) \subset C_{\text{per}}^{\infty}(Q; \mathbb{R}^N)$ such that

$$\mathcal{A} z_r = 0, \quad \int_Q z_r = 0, \quad z_r \xrightarrow{*} 0 \quad \text{in } \mathcal{M}(Q; \mathbb{R}^N),$$

and (up to taking a subsequence)

$$\lim_{r \rightarrow 0} \int_Q f \left(x_0, \frac{d\gamma_r}{d\mathcal{L}^d}(y) \right) dy = \lim_{r \rightarrow 0} \int_Q f \left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0) + z_r(y) \right) dy.$$

Returning to the calculations in (2.37), we use the properties of the sequence (z_r) and the \mathcal{A}^k -

quasiconvexity of $f(x_0, \cdot)$ to obtain the desired lower bound:

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^d}(x_0) &\geq \lim_{r \rightarrow 0} \int_Q f\left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0) + z_r(y)\right) \\ &\quad - \lim_{r \rightarrow 0} \left[\omega(r) \left(|\Omega| + \left\| \frac{d\mu}{d\mathcal{L}^d}(x_0) + z_r \right\|_{L^1(\Omega)} \right) \right] \\ &\geq f\left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0)\right). \end{aligned} \quad (2.39)$$

This proves (2.31). \square

Remark 2.42. If the assumption that $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex is dropped, one can still show that

$$\frac{d\lambda}{d\mathcal{L}^d}(x_0) \geq Q_{\mathcal{A}^k} f\left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0)\right).$$

Indeed, the \mathcal{A}^k -quasiconvexity of $f(x, \cdot)$ has only been used in the last inequality of (2.39) where one can first use the inequality $f(x, \cdot) \geq Q_{\mathcal{A}^k} f(x, \cdot)$ to get

$$\int_Q f\left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0) + z_r(y)\right) \geq \int_Q Q_{\mathcal{A}^k} f\left(x_0, \frac{d\mu}{d\mathcal{L}^d}(x_0) + z_r(y)\right).$$

The assertion then follows by using the \mathcal{A}^k -quasiconvexity of $Q_{\mathcal{A}^k} f(x, \cdot)$.

Proof of (2.32). Passing to a subsequence if necessary, we may assume that

$$\mu_j \xrightarrow{\mathbf{Y}} \nu \quad \text{for some } \nu \in \mathbf{Y}_{\mathcal{A}}(\Omega; \mathbb{R}^N).$$

For each $j \in \mathbb{N}$ set $\nu_j := \delta[\mu_j] \in \mathbf{Y}(\Omega; \mathbb{R}^N)$, the elementary Young measure corresponding to μ_j , so that $\nu_j \xrightarrow{*} \nu$ in $\mathbf{Y}(\Omega; \mathbb{R}^N)$. Define the functional

$$\mathcal{F}_{\#}[\sigma; B] := \int_B \langle f(x, \cdot), \sigma_x \rangle dx + \int_B \langle f_{\#}(x, \cdot), \sigma_x^{\infty} \rangle d\lambda_{\nu}(x), \quad \sigma \in \mathbf{Y}(\Omega; \mathbb{R}^N),$$

where $B \subset \Omega$ is an open set. Observe that, as a functional defined on $\mathbf{Y}(\Omega; \mathbb{R}^N)$, $\mathcal{F}_{\#}$ is sequentially weakly* lower semicontinuous (see Corollary 2.20). We use Assumption (ii), which is equivalent to

$$f^{\#}(x, \cdot) \equiv f_{\#}(x, \cdot) \quad \text{on } V_{\mathcal{A}},$$

and the fact, proved in (2.28), that

$$\text{supp } \nu_x^{\infty} \subset V_{\mathcal{A}} \quad \text{for } \lambda_{\nu}^s\text{-a.e. } x \in \Omega,$$

to get (recall $f \geq 0$)

$$\begin{aligned}
 \liminf_{j \rightarrow \infty} \mathcal{F}^\#[\mu_j; B] &\geq \liminf_{j \rightarrow \infty} \mathcal{F}_\#[\nu_j; B] \\
 &\geq \mathcal{F}_\#[\nu; B] \\
 &\geq \int_B \left(\langle f(x, \cdot), \nu_x \rangle + \langle f_\#(x, \cdot), \nu_x^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x) \right) dx \\
 &\quad + \int_B \langle f_\#(x, \cdot), \nu_x^\infty \rangle d\lambda_\nu^s(x) \\
 &\geq \int_B \langle f^\#(x, \cdot), \nu_x^\infty \rangle d\lambda_\nu^s(x). \tag{2.40}
 \end{aligned}$$

Recall that, for every $x \in \Omega$, the function $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex and hence the function $f^\#(x, \cdot)$ is $\Lambda_{\mathcal{A}}$ -convex and positively 1-homogeneous. An application of the Jensen-type inequality from Proposition 2.39 to the last line yields

$$\liminf_{j \rightarrow \infty} \mathcal{F}[\mu_j; B] \geq \int_B f^\#(x, \langle \text{id}, \nu_x^\infty \rangle) d\lambda_\nu^s(x).$$

Thus, also taking into account $|\mu^s| = |\langle \text{id}, \nu_x^\infty \rangle| \lambda_\nu^s$ and $f^\#(x, \langle \text{id}, \nu_x^\infty \rangle) = f^\#(x, 0) = 0$ for λ_ν^* -a.e. $x \in \Omega$, where λ_ν^* is the singular part of λ_ν^s with respect to $|\mu^s|$, we get

$$\lambda(B) \geq \int_B f^\# \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x),$$

for all open sets $B \subset \Omega$ with $\lambda_\nu^s(\partial B) = 0$. Therefore, by the Besicovitch differentiation theorem and using the modulus of continuity of f in its first argument we get

$$\frac{d\lambda}{d|\mu^s|}(x_0) \geq f^\# \left(x_0, \frac{d\mu^s}{d|\mu^s|}(x_0) \right) \quad \text{for } |\mu^s|\text{-a.e. } x_0 \in \Omega.$$

This proves (2.32). □

Remark 2.43 (recession functions). The only part of the proof where we use the existence of $f^\infty(x, A)$, for $x \in \Omega$ and $A \in V_{\mathcal{A}}$, is in showing that

$$\begin{aligned}
 \mathcal{F}_\#[\nu; B] &\geq \int_B \left(\langle f(x, \cdot), \nu_x \rangle + \langle f_\#(x, \cdot), \nu_x^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x) \right) dx \\
 &\quad + \int_B \langle f^\#(x, \cdot), \nu_x^\infty \rangle d\lambda_\nu^s(x)
 \end{aligned}$$

The need of such an estimate comes from the fact that, in general, we do not know if $f_\#$ is a $\Lambda_{\mathcal{A}}$ -convex function.

Remark 2.44. If we drop the assumption that $f(x, \cdot)$ is \mathcal{A}^k -quasiconvex for every $x \in \Omega$, we can

still show that

$$\begin{aligned} & \int_{\Omega} Q_{\mathcal{A}^k} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx \\ & + \int_{\Omega} (Q_{\mathcal{A}^k} f)^{\#} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x) \leq \liminf_{j \rightarrow \infty} \mathcal{F}[\mu_j] \end{aligned}$$

for every sequence $\mu_j \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$ such that $\mathcal{A} \mu_j \rightarrow 0$ in $W^{-k,q}(\Omega)$. The proof of this fact follows directly from Remark 2.42, the last line of (2.40) together with the modulus of continuity of f in its first argument (for the Besicovitch differentiation arguments), and Corollary 2.40. Observe that one does not require the existence of $(Q_{\mathcal{A}^k} f)^{\infty}$ in $\Omega \times \text{span} \Lambda_{\mathcal{A}}$.

Proof of Theorem 2.4. Note that in the proof of (2.31) we did not use that f^{∞} exists in $\Omega \times \text{span} \Lambda_{\mathcal{A}}$. By the very same argument as in (2.33), is easy to check that Theorem 2.4 is an immediate consequence of (2.31). □

2.5 Proof of Theorems 2.6 and 2.7

We use standard machinery to show the relaxation theorems. Recall that, for Theorems 2.6 and 2.7, we assume that \mathcal{A} is a homogeneous partial differential operator.

2.5.1 Proof of Theorem 2.6

We divide the proof of Theorem 2.6 into three steps. First, we prove that any \mathcal{A} -free measure may be area-strictly approximated by \mathcal{A} -free absolutely continuous measures. Next, we prove the upper bound on absolutely continuous measures, from which the general upper bound follows by approximation. We conclude by observing that the proposed upper bound is weakly* lower semicontinuous as a corollary of Theorem 2.2.

Step 1. The lower bound. The lower bound $\overline{\mathcal{G}} \geq \mathcal{G}_*$, where

$$\mathcal{G}_*[\mu] := \int_{\Omega} Q_{\mathcal{A}} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} (Q_{\mathcal{A}} f)^{\#} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x),$$

is a direct consequence of Remark 2.44 and the fact that \mathcal{A} is a homogeneous partial differential operator ($\mathcal{A} = \mathcal{A}^k$).

Step 2. An area-strictly converging recovery sequence. Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$. We will show that there exists a sequence $(u_j) \subset L^1(\Omega; \mathbb{R}^N)$ for which

$$\begin{aligned} u_j \mathcal{L}^d & \xrightarrow{*} \mu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^N), \quad \langle u_j \mathcal{L}^d \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega), \\ & \text{and } \mathcal{A} u_j \rightarrow 0 \text{ in } W^{-k,q}(\Omega). \end{aligned}$$

Let $\{\varphi_i\}_{i \in \mathbb{N}} \subset C_c^\infty(\Omega)$ be a locally finite partition of unity of Ω . Set

$$\mu_{(i)} := \mu \varphi_i \in \mathcal{M}(\Omega; \mathbb{R}^N),$$

and

$$\mu_{(i)}^a := \mu^a \varphi_i \quad \mu_{(i)}^s := \mu^s \varphi_i.$$

where, as usual,

$$\mu^a = \frac{d\mu}{d\mathcal{L}^d} \mathcal{L}^d \quad \text{and} \quad \mu^s = \mu - \mu^a.$$

Note that, with a slight abuse of notation,

$$\left\| \sum_{i=1}^j \mu_{(i)}^a - \mu^a \right\|_{L^1(\Omega)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Furthermore, for fixed i ,

$$(\mu_{(i)} * \rho_\varepsilon)_{\mathcal{L}^d} \xrightarrow{*} \mu_{(i)}, \quad |\mu_{(i)} * \rho_\varepsilon|(\Omega) \leq |\mu_{(i)}|(\Omega) = \int_{\Omega} \varphi_i d|\mu|, \quad (2.41)$$

and

$$\mu_{(i)}^a * \rho_\varepsilon \rightarrow \mu_{(i)}^a \quad \text{in } L^1(\Omega) \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover,

$$\mathcal{A}(\mu_{(i)} * \rho_\varepsilon) \rightarrow \mathcal{A} \mu_{(i)} \quad \text{in } W^{-k,q}(\Omega) \quad \text{as } \varepsilon \rightarrow 0.$$

Fix $j \in \mathbb{N}$. From (2.41) and the convergence above we might find a sequence $\varepsilon_i(j) \downarrow 0$ such that the measures $\mu_{i,j} := \mu_{(i)} * \rho_{\varepsilon_i(j)}$ and $\mu_{i,j}^a := \mu_{(i)}^a * \rho_{\varepsilon_i(j)}$ verify

$$\begin{aligned} d(\mu_{i,j} \mathcal{L}^d, \mu_{(i)}) &\leq \frac{1}{2^i j}, \\ \|\mu_{i,j}^a - \mu_{(i)}^a\|_{L^1(\Omega)} &\leq \frac{1}{2^i j}, \\ \|\mu_{i,j} - \mu_{(i)}\|_{W^{-k,q}(\Omega)} &\leq \frac{1}{2^i j}, \end{aligned}$$

where d is the metric inducing the weak* convergence on bounded sets of $\mathcal{M}(\Omega; \mathbb{R}^N)$ (the existence of the metric d is a standard result for the duals of separable Banach spaces). Define the integrable functions

$$u_j := \sum_{i=1}^{\infty} \mu_{i,j}, \quad u_j^a := \sum_{i=1}^{\infty} \mu_{i,j}^a.$$

We get

$$d(u_j \mathcal{L}^d, \mu) \leq \sum_{i=1}^{\infty} d(\mu_{i,j} \mathcal{L}^d, \mu_{(i)}) \leq \sum_{i=1}^{\infty} \frac{1}{2^i j} = \frac{1}{j},$$

and in a similar way

$$\begin{aligned}\|\mu_j^a - \mu^a\|_{L^1(\Omega)} &\leq \frac{1}{j}, \\ \|\mathcal{A} \mu_j\|_{W^{-k,q}(\Omega)} &\leq \frac{1}{j},\end{aligned}$$

where we use that μ is \mathcal{A} -free in the second inequality. Observe that (2.41) and that fact that $\{\varphi_i\}_{i \in \mathbb{N}}$ is a partition of unity imply

$$\int_{\Omega} |u_j| \, dx \leq \sum_{i=1}^{\infty} \int_{\Omega} \varphi_i \, d|\mu| \leq |\mu|(\Omega). \quad (2.42)$$

Therefore $\|u_j\|_{L^1(\Omega)}$ is uniformly bounded and hence

$$u_j \mathcal{L}^d \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^N), \quad (2.43)$$

$$\|u_j^a - \mu^a\|_{L^1(\Omega)} \rightarrow 0, \quad (2.44)$$

$$\|\mathcal{A} u_j\|_{W^{-k,q}(\Omega)} \rightarrow 0, \quad (2.45)$$

as $j \rightarrow \infty$. Moreover, the convexity of $z \mapsto |z|$ and (2.42) imply the strict convergence

$$|u_j \mathcal{L}^d|(\Omega) \rightarrow |\mu|(\Omega). \quad (2.46)$$

Thanks to (2.43) and (2.45), to conclude it suffices to show that

$$\lim_{j \rightarrow \infty} \langle u_j \mathcal{L}^d \rangle(\Omega) = \langle \mu \rangle(\Omega). \quad (2.47)$$

Exploiting (2.43), (2.44), (2.46), we get

$$\int_{\Omega} |u_j - u_j^a| \, dx \rightarrow |\mu^s|(\Omega) \quad \text{as } j \rightarrow \infty. \quad (2.48)$$

By the inequality $\sqrt{1+|z|^2} \leq \sqrt{1+|z-w|^2} + |w|$ (for $z, w \in \mathbb{R}^N$), we get

$$\langle u_j \mathcal{L}^d \rangle(\Omega) \leq \langle u_j^a \mathcal{L}^d \rangle(\Omega) + \int_{\Omega} |u_j - u_j^a| \, dx.$$

Hence, again by (2.44) and (2.48)

$$\limsup_{j \rightarrow \infty} \langle u_j \mathcal{L}^d \rangle(\Omega) \leq \langle \mu \rangle(\Omega). \quad (2.49)$$

On the other hand, by the weak* convergence $u_j \mathcal{L}^d \xrightarrow{*} \mu$ and the convexity of $z \mapsto \sqrt{1+|z|^2}$,

$$\liminf_{j \rightarrow \infty} \langle u_j \mathcal{L}^d \rangle(\Omega) \geq \langle \mu \rangle(\Omega).$$

Thus, together with (2.49), (2.47) follows, concluding the proof of the claim.

Step 3.a. Upper bound on absolutely continuous fields. Let us now turn to the derivation of the upper bound for $\overline{\mathcal{G}}[u] = \overline{\mathcal{G}}[u, \mathcal{L}^d]$ where $u \in L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$. For now let us assume additionally the following strengthening of (3.2):

$$f(x, A) - f(y, A) \leq \omega(|x - y|)(1 + f(y, A)) \quad \text{for all } x, y \in \Omega, A \in \mathbb{R}^N. \quad (2.50)$$

It holds that $\mathcal{Q}_{\mathcal{A}} f(x, \cdot)$ is still uniformly Lipschitz in the second variable and

$$\mathcal{Q}_{\mathcal{A}} f(x, A) \leq \mathcal{Q}_{\mathcal{A}} f(y, A) + \omega(|x - y|)(1 + |A|) \quad (2.51)$$

for every $x, y \in \Omega$ and $A \in \mathbb{R}^N$ with a new modulus of continuity (still denoted by ω), which incorporates another multiplicative constant in comparison to the original ω . Indeed, fix $x, y \in \Omega$, $\varepsilon > 0$, and $A \in \mathbb{R}^N$. Let $w \in C_{\text{per}}^\infty(Q; \mathbb{R}^N) \cap \ker \mathcal{A}$ be a function with zero mean in Q such that (recall that $\mathcal{Q}_{\mathcal{A}} f(x, A) := \mathcal{Q}_{\mathcal{A}} f(x, \cdot)(A)$)

$$\int_Q f(y, A + w(z)) \, dz \leq \mathcal{Q}_{\mathcal{A}} f(y, A) + \varepsilon.$$

By assumption, we get

$$\begin{aligned} \int_Q f(x, A + w(z)) \, dz &\leq \int_Q f(y, A + w(z)) \, dz \\ &\quad + \omega(|x - y|) \left(1 + \int_Q f(y, A + w(z)) \, dz \right) \\ &\leq \mathcal{Q}_{\mathcal{A}} f(y, A) + \varepsilon \\ &\quad + \omega(|x - y|)(1 + \mathcal{Q}_{\mathcal{A}} f(y, A) + \varepsilon). \end{aligned}$$

Thus,

$$\mathcal{Q}_{\mathcal{A}} f(x, A) \leq \mathcal{Q}_{\mathcal{A}} f(x, A) + \varepsilon + \omega(|x - y|)(1 + \mathcal{Q}_{\mathcal{A}} f(y, A) + \varepsilon).$$

The linear growth at infinity of f , which is inherited by $\mathcal{Q}_{\mathcal{A}} f$, gives

$$\mathcal{Q}_{\mathcal{A}} f(x, A) \leq \mathcal{Q}_{\mathcal{A}} f(y, A) + \omega(|x - y|)(1 + M(1 + |A|)) + \varepsilon(1 + \omega(|x - y|)).$$

We may now let $\varepsilon \downarrow 0$ in the previous inequality to obtain

$$\mathcal{Q}_{\mathcal{A}} f(x, A) \leq \mathcal{Q}_{\mathcal{A}} f(y, A) + \omega(|x - y|)(M + 1)(1 + |A|).$$

This proves (2.51) provided that (2.50) holds.

Fix $m \in \mathbb{N}$ and consider a partition of \mathbb{R}^d of cubes of side length $1/m$. Let $\{Q_i^m\}_{i=1}^{L(m)}$ be the maximal collection of those cubes with centers $\{x_i^m\}_{i=1}^{L(m)}$ that are compactly contained in Ω . By a version of Besicovitch's Covering Theorem we have

$$\mathcal{L}^d(\Omega) = \sum_{i=1}^{L(m)} \mathcal{L}^d(Q_i^m) + o_m(1),$$

where $o_m(1) \rightarrow 0$ as $m \rightarrow \infty$.

We may approximate u strongly in L^1 by functions $z^m \in L^1(\Omega; \mathbb{R}^N)$ that are piecewise constant on the mesh $\{Q_i^m\}_{i=1}^{L(m)}$ (as $m \rightarrow \infty$). More specifically, we may find functions $z^m \in L^1(\Omega; \mathbb{R}^N)$ such that $z^m = 0$ on $\Omega \setminus \bigcup_i Q_i^m$,

$$z^m = z_i^m \in \mathbb{R}^N \quad \text{on } Q_i^m \quad \text{and} \quad \|u - z^m\|_{L^1(\Omega)} = o_m(1). \quad (2.52)$$

Additionally, for every $m \in \mathbb{N}$, we may find functions $w_i^m \in C_{\text{per}}^\infty(Q; \mathbb{R}^N) \cap \ker \mathcal{A}$ with the following properties

$$\int_Q f(x_i^m, z_i^m + w_i^m(y)) \, dy \leq Q_{\mathcal{A}} f(x_i^m, z_i^m) + \frac{1}{m}, \quad \int_Q w_i^m \, dy = 0. \quad (2.53)$$

Fix $m \in \mathbb{N}$ and let $\varphi_m \in C_c^\infty(Q; [0, 1])$ be a function such that

$$\sum_{i=1}^{L(m)} \|1 - \varphi_m\|_{L^1(Q)} \|w_i^m\|_{L^1(Q)} = \frac{1}{m}, \quad (2.54)$$

We define the functions

$$v_j^m := \sum_{i=1}^{L(m)} \varphi_m(m(x - x_i^m)) \cdot w_i^m(jm(x - x_i^m)) \quad x \in \Omega, \, j \in \mathbb{N}.$$

By Lemma 2.17, the sequence (v_j^m) generates the Young measure

$$v^m = (v_x^m, 0, \cdot) \in \mathbf{Y}(\Omega; \mathbb{R}^N),$$

where for each $x \in \Omega$, v_x^m is the probability measure defined by duality through

$$\langle h, v_x^m \rangle := \sum_{i=1}^{L(m)} \mathbb{1}_{Q_i^m}(x) \int_Q h(\varphi_m(m(x - x_i^m)) \cdot w_i^m(y)) \, dy,$$

on functions $h \in C(\mathbb{R}^N)$ with linear growth.

The central point of this construction is that w_i^m has *zero mean value*, that is, $\int_Q w_i^m \, dy = 0$, whence it follows that

$$v_j^m \mathcal{L}^d \stackrel{*}{\rightharpoonup} \sum_{i=1}^{L(m)} \int_Q \varphi_m(m(x - x_i^m)) w_i^m(y) \, dy = 0 \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^N), \quad (2.55)$$

as $j \rightarrow \infty$. Recall that by construction, $\mathcal{A} w_i^m = 0$ on Q . Hence, using that \mathcal{A} is homogeneous we get

$$\mathcal{A}[w_i^m(jm(\cdot - x_i^m))] = 0 \quad \text{in the sense of distributions on } Q_i^m.$$

Thus, for some coefficients $c_{\alpha,\beta} \in \mathbb{N}$, using the short-hand notation $\psi_m(y) := \varphi_m(my)$ yields

$$\begin{aligned} \mathcal{A} v_j^m &= \sum_{i=1}^{L(m)} \left(\mathcal{A} [w_i^m(jm(\cdot - x_i^m))] \psi_m(\cdot - x_i^m) \right. \\ &\quad \left. + \sum_{\substack{|\alpha|=k, \\ 1 \leq |\beta| \leq k}} c_{\alpha\beta} A_\alpha \partial^{\alpha-\beta} [w_i^m(jm(\cdot - x_i^m))] \partial^\beta [\psi_m(\cdot - x_i^m)] \right) \\ &= \sum_{\substack{|\alpha|=k, \\ 1 \leq |\beta| \leq k}} \left(\sum_{i=1}^{L(m)} c_{\alpha\beta} \partial^{\alpha-\beta} [w_i^m(jm(\cdot - x_i^m))] \partial^\beta [\psi_m(\cdot - x_i^m)] \right), \end{aligned}$$

in the sense of distributions on Ω . Applying Lemma 2.17 to the sequence $(w_i^m(jm(\cdot - x_i^m)))_j$ on each cube Q_i^m we get

$$\begin{aligned} \sum_{i=1}^{L(m)} \mathbb{1}_{Q_i^m} \cdot w_i^m(jm(\cdot - x_i^m)) &\rightharpoonup \sum_{i=1}^{L(m)} \mathbb{1}_{Q_i^m} \cdot \int_{Q_i^m} w_i^m(m(y - x_i^m)) \, dy \\ &= \sum_{i=1}^{L(m)} \mathbb{1}_{Q_i^m} \cdot \int_Q w_i^m(y) \, dy = 0. \end{aligned}$$

Hence, (2.55) and the compact embedding $L^1(\Omega; \mathbb{R}^N) \xhookrightarrow{c} W^{-1,q}(\Omega; \mathbb{R}^N)$ yield

$$\mathcal{A} v_j^m = \sum_{\substack{|\alpha|=k, \\ 1 \leq |\beta| \leq k}} \left(\sum_{i=1}^{L(m)} c_{\alpha\beta} \partial^{\alpha-\beta} [w_i^m(jm(\cdot - x_i^m))] \partial^\beta [\psi_m(\cdot - x_i^m)] \right) \rightarrow 0$$

strongly in $W^{-k,q}(\Omega; \mathbb{R}^N)$, as $j \rightarrow \infty$.

For later use we record:

Remark 2.45. By construction, for every $m, j \in \mathbb{N}$, the function v_j^m is compactly supported in Ω . Up to re-scaling, we may thus assume without loss of generality that $\Omega \subset Q$ and subsequently make use of Lemma 2.25 on the j -indexed sequence (\tilde{v}_j^m) with m fixed, where \tilde{v}_j^m is the zero extension of v_j^m to Q , to find another sequence $(V_j^m) \subset L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ generating the same Young measure ν^m (as $j \rightarrow \infty$).

In the next calculation we use the Lipschitz continuity of $Q_{\mathcal{A}} f(x, \cdot)$ in the second variable, equation (2.52) and the fact that the sequence (v_j^m) generates the Young measure ν^m as j goes to infinity, to get

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{G}[u + v_j^m] &= \lim_{j \rightarrow \infty} \mathcal{G}[z^m + v_j^m] + o_m(1) \\ &= \sum_{i=1}^{L(m)} \int_{Q_i^m} \int_Q f(x, z_i^m + \varphi_m(m(x - x_i^m)) \cdot w_i^m(y)) \, dy \, dx + o_m(1). \end{aligned} \quad (2.56)$$

By a change of variables we can estimate the integrand times $m^d = \mathcal{L}^d(Q_i^m)^{-1}$ on the last line on

each cube of the mesh:

$$\begin{aligned}
 & \int_{Q_i^m} \int_Q f(x, z_i^m + \varphi_m(m(x - x_i^m)) \cdot w_i^m(y)) \, dy \, dx \\
 &= \int_Q \int_Q f(x_i^m + m^{-1}x, z_i^m + \varphi_m(x) \cdot w_i^m(y)) \, dy \, dx \\
 &\leq \int_Q \int_Q f(x_i^m + m^{-1}x, z_i^m + w_i^m(y)) \, dy \, dx + \text{Lip}(f) \|1 - \varphi_m\|_{L^1(Q)} \|w_i^m\|_{L^1(Q)} \\
 &= \int_{Q_i^m} \int_Q f(x, z_i^m + w_i^m(y)) \, dy \, dx + \text{Lip}(f) \|1 - \varphi_m\|_{L^1(Q)} \|w_i^m\|_{L^1(Q)} \\
 &:= I_i^m + II_i^m.
 \end{aligned} \tag{2.57}$$

Using the modulus of continuity of f from (2.50), (2.53) (twice), and $Q_{\mathcal{A}} f \leq f$, we get

$$\begin{aligned}
 I_i^m &\leq \int_{Q_i^m} \int_Q f(x_i^m, z_i^m + w_i^m(y)) \, dy \, dx + \omega(m^{-1}) \left(1 + \int_Q f(x_i^m, z_i^m + w_i^m(y)) \, dy\right) \\
 &\leq Q_{\mathcal{A}} f(x_i^m, z_i^m) + \omega(m^{-1})(1 + f(x_i^m, z_i^m)) + o_m(1).
 \end{aligned} \tag{2.58}$$

Additionally, by (2.54)

$$\sum_{i=1}^{L(m)} \mathcal{L}^d(Q_i^m) II_i^m = o_m(1). \tag{2.59}$$

Returning to (2.56), we can employ (2.51), (2.57), (2.58) and (2.59) to further estimate

$$\begin{aligned}
 & \lim_{j \rightarrow \infty} \mathcal{G}[u + v_j^m] \\
 &\leq \sum_{i=1}^{L(m)} \left\{ \int_{Q_i^m} Q_{\mathcal{A}} f(x_i^m, z_i^m) \, dx + \omega(m^{-1}) \left(\int_{Q_i^m} 1 + f(x_i^m, z_i^m) \, dx \right) \right\} + o_m(1) \\
 &\leq \sum_{i=1}^{L(m)} \left\{ \int_{Q_i^m} Q_{\mathcal{A}} f(x_i^m, z_i^m) \, dx + C \omega(m^{-1}) \left(\int_{Q_i^m} 1 + |z_i^m| \, dx \right) \right\} + o_m(1) \\
 &\leq \sum_{i=1}^{L(m)} \left\{ \int_{Q_i^m} Q_{\mathcal{A}} f(x, z_i^m) \, dx + \tilde{C} \omega(m^{-1}) \left(\int_{Q_i^m} 1 + |z_i^m| \, dx \right) \right\} + o_m(1) \\
 &\leq \int_{\Omega} Q_{\mathcal{A}} f(x, z^m) \, dx + \tilde{C} \omega(m^{-1}) (\|1 + |z^m|\|_{L^1(\Omega)}) + o_m(1) \\
 &= \int_{\Omega} Q_{\mathcal{A}} f(x, u) \, dx + o_m(1),
 \end{aligned}$$

where $o_m(1)$ may change from line to line. Here, we have used the (inherited) Lipschitz continuity of $Q_{\mathcal{A}} f(x, \cdot)$ in the second variable and the fact that $\|u - z^m\|_{L^1(\Omega)} = o_m(1)$ to pass to the last equality. Hence

$$\overline{\mathcal{G}}[u] \leq \inf_{m>0} \lim_{j \rightarrow \infty} \mathcal{G}[u + v_j^m] \leq \int_{\Omega} Q_{\mathcal{A}} f(x, u) \, dx. \tag{2.60}$$

Step 3.b. The upper bound. Fix $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$. By Step 2 we may find a sequence $(u_j) \subset L^1(\Omega; \mathbb{R}^N)$ that area-strictly converges to $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ with $\mathcal{A} u_j \rightarrow 0$ in $W^{-k,q}$. Hence,

by (2.60), Remark 2.16 and Corollary 2.20,

$$\begin{aligned}
 \overline{\mathcal{G}}[\mu] &\leq \liminf_{j \rightarrow \infty} \overline{\mathcal{G}}[u_j] \\
 &\leq \limsup_{j \rightarrow \infty} \langle\langle Q_{\mathcal{A}} f(x, \cdot), \delta[u_j \mathcal{L}^d] \rangle\rangle \\
 &\leq \int_{\Omega} \langle Q_{\mathcal{A}} f(x, \cdot), \delta[\mu]_x \rangle dx + \int_{\Omega} \langle (Q_{\mathcal{A}} f(x, \cdot))^{\#}, \delta[\mu]_x^{\infty} \rangle d\lambda_{\delta[\mu]}(x) \\
 &= \int_{\Omega} Q_{\mathcal{A}} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} (Q_{\mathcal{A}} f)^{\#}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x) \\
 &= \mathcal{G}_*[\mu].
 \end{aligned}$$

Step 4. General continuity condition. It remains to show the upper bound in the case where we only have (3.2) instead of (2.50). As in the previous step, it suffices to show the upper bound on absolutely continuous fields. We let, for fixed $\varepsilon > 0$,

$$f^{\varepsilon}(x, A) := f(x, A) + \varepsilon|A|,$$

which is an integrand satisfying (2.50). Denote the corresponding functionals with f^{ε} in place of f by $\mathcal{G}^{\varepsilon}, \mathcal{G}_*^{\varepsilon}, \overline{\mathcal{G}}^{\varepsilon}$. Then, by the argument in Steps 1–3,

$$\mathcal{G}_*^{\varepsilon} = \overline{\mathcal{G}}^{\varepsilon}.$$

We claim that

$$Q_{\mathcal{A}^k} f^{\varepsilon} \downarrow Q_{\mathcal{A}^k} f \quad \text{pointwise in } \Omega \times \mathbb{R}^N. \quad (2.61)$$

To see this first notice that $\varepsilon \mapsto Q_{\mathcal{A}^k} f^{\varepsilon}(x, A)$ is monotone decreasing for all $x \in \Omega, A \in \mathbb{R}^N$, and

$$Q_{\mathcal{A}^k} f + \varepsilon|\cdot| \leq Q_{\mathcal{A}^k} f^{\varepsilon} \leq f + \varepsilon|\cdot|,$$

which is a simple consequence of Jensen's classical inequality for $|\cdot|$. It follows that the limit

$$g(x, A) := \inf_{\varepsilon > 0} Q_{\mathcal{A}^k} f^{\varepsilon}(x, A) = \lim_{\varepsilon \downarrow 0} Q_{\mathcal{A}^k} f^{\varepsilon}(x, A)$$

defines an upper semicontinuous function $g : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ with bounds

$$Q_{\mathcal{A}^k} f \leq g \leq f.$$

Furthermore, by the monotone convergence theorem, it is easy to check that g is \mathcal{A}^k -quasiconvex, whereby $g = Q_{\mathcal{A}^k} f$ (see Corollary 2.29).

Let us now return to the proof of the upper bound on absolutely continuous fields. By construction,

$$\overline{\mathcal{G}} \leq \overline{\mathcal{G}}^{\varepsilon} = \mathcal{G}_*^{\varepsilon}. \quad (2.62)$$

The monotone convergence theorem and (2.61) yield

$$\overline{\mathcal{G}}[u] \leq \mathcal{G}_*[u, \mathcal{L}^d] \quad \text{for all } u \in L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A},$$

after letting $\varepsilon \downarrow 0$ in (2.62).

The general upper bound then follows in a similar way to the proof under the assumption (2.50). This finishes the proof. \square

2.5.2 Proof of Theorem 2.7

The proof works the same as the proof of Theorem 2.6 with the following additional comments:

Step 1. The lower bound. Since restricting to \mathcal{A} -free sequences is a particular case of the more general convergence $\mathcal{A} u_n \rightarrow 0$ in the space $W^{-k,q}(\Omega; \mathbb{R}^N)$, we can still apply Step 2 in the proof of Theorem 2.6 to prove that $\mathcal{G}_* \leq \overline{\mathcal{G}}$, where for $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$,

$$\mathcal{G}_*[\mu] := \int_{\Omega} Q_{\mathcal{A}} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} (Q_{\mathcal{A}} f)^{\#} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x).$$

Step 2. An \mathcal{A} -free strictly convergent recovery sequence. In this case, this forms part of the assumptions.

Step 3.a. Upper bound on absolutely continuous \mathcal{A} -free fields. An immediate consequence of Remark 2.45 is that one may assume, without loss of generality, that the recovery sequence for the upper bound lies in $\ker \mathcal{A}$. Thus, the upper bound on absolutely continuous fields in the constrained setting also holds.

Step 3.b. The upper bound (assuming (2.50)). The proof is the same as in the proof of Theorem 2.2.

Step 4. General continuity condition. Since assumption (2.50) is a structural property (coercivity) of the integrand and the arguments do not depend on the underlying space of measures, the argument remains the same as in the proof of Theorem 2.6. \square

2.6 Appendix

2.6.1 Proof sketches of the localization principles

Sketch of the proof of Proposition 2.35: In the following we adapt the main steps in proof of the localization principle at regular points which is contained in Proposition 1 of [29]. The statement on the existence of a \mathcal{A} -free and periodic generating sequence is proved in detail.

Let $\mu_j \in \mathcal{M}(\Omega; \mathbb{R}^N)$ be the sequence of asymptotically \mathcal{A} -free measures which generates ν . In the following steps, for an open $\Omega' \subset \mathbb{R}^d$, we will often identify a measure $\mu \in \mathcal{M}(\Omega'; \mathbb{R}^N)$ with its zero extension in $\mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$, and the same for a Young measure $\sigma \in \mathbf{Y}(\Omega'; \mathbb{R}^N)$ and its zero extension in $\mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$.

1. The *first step* consists on showing that, for every $r > 0$, there exists a subsequence of j 's (the

choice of subsequence might depend on r) such that

$$r^{-d} T_{\#}^{(x_0, r)} \mu_j \xrightarrow{\mathbf{Y}} \sigma^{(r)} \quad \text{in } \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N). \quad (2.63)$$

Moreover, for \mathcal{L}^d -a.e. $x_0 \in \Omega$, one can show that a uniform bound

$$\sup_r \langle \mathbb{1}_K \otimes |\cdot|, \sigma^{(r)} \rangle < \infty \quad \text{for every } K \Subset \mathbb{R}^d \quad (2.64)$$

holds; thus, by Lemma 2.15, there exists a sequence of positive numbers $r_m \downarrow 0$ and a Young measure σ for which

$$\sigma^{(r_m)} \xrightarrow{*} \sigma \quad \text{in } \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N).$$

2. The *second step* concerns the quantitative properties of the Young measures $\sigma^{(r_m)}$ with respect to the Young measure ν : for an arbitrary measure $\gamma \in \mathcal{M}(\Omega; \mathbb{R}^N)$, the Radon-Nykodým differentiation theorem yields

$$r^{-d} T_{\#}^{(x_0, r)} \gamma = \frac{d\gamma}{d\mathcal{L}^d}(x_0 + r\cdot) \mathcal{L}^d + \frac{d\gamma}{d|\gamma^s|}(x_0 + r\cdot) r^{-d} T_{\#}^{(x_0, r)} |\gamma^s|.$$

Consider $\sigma^{(r)}$ as an element of $\mathbf{Y}(\bar{Q}; \mathbb{R}^N)$. Fix $\varphi \otimes h \in C(\bar{Q}) \times \mathbf{W}^{1, \infty}(\mathbb{R}^N)$. Using simple change of variables, we get

$$\begin{aligned} \langle \varphi \otimes h, \sigma^{(r)} \rangle &= \lim_{j \rightarrow \infty} \left(\int_Q \varphi(y) \cdot h \left(\frac{d\mu_j}{d\mathcal{L}^d}(x_0 + ry) \right) dy \right. \\ &\quad \left. + \int_{\bar{Q}} \varphi(y) \cdot h^\infty \left(\frac{d\mu_j}{d|\mu_j^s|}(x_0 + ry) \right) d(r^{-d} T_{\#}^{(x_0, r)} |\mu_j^s|)(y) \right) \\ &= r^{-d} \lim_{j \rightarrow \infty} \left(\int_{Q_r(x_0)} \varphi \circ T^{(x_0, r)}(x) \cdot h \left(\frac{d\mu_j}{d\mathcal{L}^d}(x) \right) dx \right. \\ &\quad \left. + \int_{Q_r(x_0)} \varphi \circ T^{(x_0, r)}(x) \cdot h^\infty \left(\frac{d\mu_j}{d|\mu_j^s|}(x) \right) d|\mu_j^s|(x) \right) \\ &= r^{-d} \langle \varphi \circ T^{(x_0, r)} \otimes h, \nu \rangle. \end{aligned} \quad (2.65)$$

3. In the *third step* we let $r = r_m$ in (2.65) and quantify its values as $m \rightarrow \infty$. This will allow us to characterize σ in terms of ν .

Let $\{g_l := \varphi_l \otimes h_l\} \subset C(\bar{Q}) \times \mathbf{W}^{1, \infty}(\mathbb{R}^N)$ be the dense subset of $\mathbf{E}(\bar{Q}; \mathbb{R}^N)$ provided by Lemma 2.14 and further assume that x_0 verifies the following properties: x_0 is a Lebesgue point of the functions

$$x \mapsto \langle h_l, \nu_x \rangle + \langle h_l^\infty, \nu_x^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}^d}(x), \quad \text{for all } l \in \mathbb{N}, \quad (2.66)$$

and x_0 is a regular point of the measure λ_ν , that is,

$$\frac{d\lambda_\nu^s}{d\mathcal{L}^d}(x_0) = \lim_{r \downarrow 0} \frac{\lambda_\nu^s(Q_r(x_0))}{r^d} = 0. \quad (2.67)$$

Consider σ as an element of $\mathbf{Y}(Q; \mathbb{R}^N)$. Setting $r = r_m$ in (2.65) and letting $m \rightarrow \infty$ we get

$$\begin{aligned} \langle\langle g_l, \sigma \rangle\rangle &= \lim_{m \rightarrow \infty} r^{-d} \langle\langle \varphi_l \circ T^{(x_0, r_m)} \otimes h_l, \mathbf{v} \rangle\rangle \\ &= \lim_{m \rightarrow \infty} \left(\int_{Q_{r_m}(x_0)} \varphi_l \left(\frac{x - x_0}{r_m} \right) \left[\langle h_l, \mathbf{v}_x \rangle + \langle h_l^\infty, \mathbf{v}_x^\infty \rangle \frac{d\lambda_{\mathbf{v}}}{d\mathcal{L}^d}(x) \right] dy \right. \\ &\quad \left. + \frac{1}{r^d} \int_{Q_{r_m}(x_0)} \varphi_l \left(\frac{x - x_0}{r_m} \right) \langle h_l^\infty, \mathbf{v}_x^\infty \rangle d\lambda_{\mathbf{v}}^s(x) \right) \\ &= \int_Q \langle g_l(y, \cdot), \mathbf{v}_{x_0} \rangle dy + \int_Q \langle g_l^\infty(y, \cdot), \mathbf{v}_{x_0}^\infty \rangle \frac{d\lambda_{\mathbf{v}}}{d\mathcal{L}^d}(x_0) d(y). \end{aligned}$$

Here, we have used (2.66) and the Dominated Convergence Theorem to pass to the limit in the first summand, and with the help of (2.67), we used that

$$\int_{Q_r(x_0)} \varphi_l \left(\frac{x - x_0}{r} \right) \langle h_l^\infty, \mathbf{v}_x^\infty \rangle d\lambda_{\mathbf{v}}^s(x) \leq \|\varphi\|_\infty \cdot \text{Lip}(h_l) \cdot \lambda_{\mathbf{v}}^s(Q_r(x_0)) = o(r^d)$$

to neglect the second summand in the limiting process.

Since the set $\{g_l\}$ separates $\mathbf{Y}(Q; \mathbb{R}^N)$, Lemma 2.14 tells us that $\sigma_y = \mathbf{v}_{x_0}$, $\sigma_y^\infty = \mathbf{v}_{x_0}^\infty$, $\lambda_\sigma = \frac{d\lambda_{\mathbf{v}}}{d\mathcal{L}^d}(x_0) \mathcal{L}^d$ for \mathcal{L}^d -a.e. $y \in Q$, and that λ_σ^s is the zero measure in $\mathcal{M}(\overline{Q})$; as desired.

4. We use a diagonalization principle (where j is the fast index with respect to m) to find a subsequence $(\mu_{j(m)})$ such that

$$\gamma_m := r_m^{-d} T_{\#}^{(x_0, r_m)} \mu_{j(m)} \xrightarrow{\mathbf{Y}} \sigma \quad \text{in } \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N). \quad (2.68)$$

5. Up to this point, the localization principle presented in Proposition 1 of [29] has been adapted to Young measures without imposing any differential constraint.

Here we additionally require σ to be an \mathcal{A}^k -free Young measure; this is achieved by showing that (γ_m) is asymptotically \mathcal{A}^k -free (on bounded subsets of \mathbb{R}^d): it follows from (2.68) and Theorem 2.11, that for every open $\omega \Subset \mathbb{R}^d$ there exists a positive constant c_ω such that

$$r_m^{-d} T_{\#}^{(x_0, r_m)} |\mu_{j(m)}|(\omega) \leq C_\omega$$

whenever m is sufficiently large. Therefore, the assertion

$$\mathcal{A}^k \gamma_m \rightarrow 0 \quad \text{in } \mathbf{W}_{\text{loc}}^{-k, q},$$

is an immediate consequence of Proposition 2.26 applied to the sequence of measures with elements $\mu_m = \mu_{j(m)}$ and the constants $c_m := r_m^{-d}$.

6. So far we have shown that $[\sigma] = A_0 \mathcal{L}^d$ with

$$A_0 := \langle \text{id}, \mathbf{v}_{x_0} \rangle + \langle \text{id}, \mathbf{v}_{x_0}^\infty \rangle \frac{d\lambda_{\mathbf{v}}}{d\mathcal{L}^d}(x_0) \in \mathbb{R}^N,$$

and that σ is generated by a sequence $(\mu_j) \subset \mathcal{M}(Q; \mathbb{R}^N)$ satisfying $\mathcal{A}^k \mu_j \rightarrow 0$. Note that without loss of generality we may assume that the μ_j 's are of the form $u_j \mathcal{L}^d$ where $u_j \in L^1(Q; \mathbb{R}^N)$. Indeed, since

$$\begin{aligned} \gamma_r &:= T_{\#}^{(0,r)} \mu_j \rightarrow \mu_j \quad \text{area strictly in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N), \\ \|\mathcal{A}^k(\gamma_r - \mu_j)\|_{W_{\text{loc}}^{-k,q}(\mathbb{R}^d)} &\rightarrow 0 \quad (\text{as } r \uparrow 1), \end{aligned}$$

and

$$\begin{aligned} \gamma_r * \rho_{\varepsilon} &\rightarrow \gamma_r \quad \text{area strictly in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N), \\ \|\mathcal{A}^k(\gamma_r - \gamma_r * \rho_{\varepsilon})\|_{W_{\text{loc}}^{-k,q}(\mathbb{R}^d)} &\rightarrow 0 \quad (\text{as } \varepsilon \downarrow 0), \end{aligned}$$

we might use a diagonalization argument (relying on the weak*-metrizability of bounded subsets of $\mathbf{E}(Q; \mathbb{R}^N)^*$, and Remarks 2.12 and 2.16), where ε appears as the faster index with respect to r , to find a sequence with elements $u_j := \gamma_{r_j} * \rho_{\varepsilon_{r_j}}$ such that

$$u_j \mathcal{L}^d \xrightarrow{\mathbf{Y}} \sigma \in \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N) \quad \text{and} \quad \mathcal{A}^k u_j \rightarrow 0 \quad \text{in } W_{\text{loc}}^{-k,q}(\mathbb{R}^d). \quad (2.69)$$

Using (2.9), we get

$$|u_j| \mathcal{L}^d \xrightarrow{*} |[\sigma]| = |A_0| \mathcal{L}^d \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^d).$$

Hence, $|u_j| \mathcal{L}^d \xrightarrow{*} \Lambda$ in $\mathcal{M}(\overline{Q})$ with $\Lambda(\partial Q) = 0$. We are in a position to apply Lemma 2.25 to the sequences (u_j) and $(v_j := A_0)$ to find a sequence $z_j \in C_{\text{per}}^{\infty}(Q; \mathbb{R}^N) \cap \ker \mathcal{A}^k$ with $\int_Q z_j \, dy = 0$ and such that (up to taking a subsequence)

$$w_j \mathcal{L}^d := (A_0 + z_j) \mathcal{L}^d \xrightarrow{\mathbf{Y}} \sigma \quad \text{in } \mathbf{Y}(Q; \mathbb{R}^N). \quad (2.70)$$

Since the properties of x_0 that were involved in Steps 1-3 are valid at \mathcal{L}^d -a.e. $x_0 \in \Omega$, the sought localization principle at regular points is proved. \square

Sketch of the proof of Proposition 2.36: The proof of the localization at singular points resembles the one for regular points, with a few exceptions:

1. In this step, we chose $c_r(x_0) := |\mu^s|(Q_r(x_0))^{-1}$ (instead of r^{-d}) so that

$$c_r(x_0) T_{\#}^{(x_0,r)} \mu_j \xrightarrow{\mathbf{Y}} \sigma^{(r)} \quad \text{in } \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N).$$

Moreover, at $\lambda_{\mathbf{v}}^s$ -a.e. $x_0 \in \Omega$, it is possible to show that

$$\sup_{r>0} \langle \mathbb{1}_K \otimes |\cdot|, \sigma^{(r)} \rangle < \infty \quad \text{for every } K \in \mathbb{R}^d. \quad (2.71)$$

By compactness of $\mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N)$, see Lemma 2.15, there exists a sequence of positive numbers

$r_m \downarrow 0$ and a Young measure σ for which

$$\sigma^{(r_m)} \xrightarrow{*} \sigma \quad \text{in } \mathbf{Y}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^N).$$

Moreover, by Preiss' existence result for non-zero tangent measures [27], we may assume that σ and hence λ_σ are non-zero.

2. The calculations of the second step, for the constant $c_r(x_0)$, is

$$\begin{aligned} \langle\langle \varphi \otimes h, \sigma^{(r)} \rangle\rangle &= \lim_{j \rightarrow \infty} \left(\int_Q \varphi(y) \cdot h \left(c_r(x_0) r^d \frac{d\mu_j}{d\mathcal{L}^d}(x_0 + ry) \right) dy \right. \\ &\quad \left. + \int_{\bar{Q}} \varphi(y) \cdot h^\infty \left(c_r(x_0) r^d \frac{d\mu_j}{d|\mu_j^s|}(x_0 + ry) \right) d(r^{-d} T_{\#}^{(x_0, r)} |\mu_j^s|)(y) \right) \\ &= r^{-d} \lim_{j \rightarrow \infty} \left(\int_{Q_r(x_0)} \varphi \circ T^{(x_0, r)}(x) \cdot h \left(c_r(x_0) r^d \frac{d\mu_j}{d\mathcal{L}^d}(x) \right) dx \right. \\ &\quad \left. + \int_{\bar{Q}_r(x_0)} \varphi \circ T^{(x_0, r)}(x) \cdot h^\infty \left(c_r(x_0) r^d \frac{d\mu_j}{d|\mu_j^s|}(x) \right) d|\mu_j^s|(x) \right) \\ &= r^{-d} \langle\langle \varphi \circ T^{(x_0, r)} \otimes h(c_r(x_0) r^d \cdot), \nu \rangle\rangle. \end{aligned} \quad (2.72)$$

3. The assumptions of the third step are substituted by assuming that x_0 is a λ_ν^s -Lebesgue point of the functions

$$x \mapsto \langle |\cdot|, \mathbf{v}_x^\infty \rangle, \quad \left\{ x \mapsto \langle h_l^\infty, \mathbf{v}_x^\infty \rangle \right\} \quad \text{for all } l \in \mathbb{N}. \quad (2.73)$$

We further require that

$$\lim_{r \downarrow 0} \frac{r^d}{\lambda_\nu^s(Q_r(x_0))} = \lim_{r \downarrow 0} c_r(x_0) r^d = 0 \quad (2.74)$$

and we define $S := \{x_0 \in \Omega : (2.73) \text{ and } (2.74) \text{ hold}\}$ which is a set of full λ_ν^s -measure in Ω .

Fix $x_0 \in S$. Setting $r = r_m$ in (2.72) and letting $m \rightarrow \infty$ in this case gives

$$\begin{aligned} \langle\langle \mathbf{1}_Q \otimes |\cdot|, \sigma \rangle\rangle &= \lim_{m \rightarrow \infty} \langle\langle \mathbf{1}_Q \otimes |\cdot|, \sigma^{(r_m)} \rangle\rangle \\ &= \lim_{m \rightarrow \infty} c_m(x_0) \left(\int_{Q_{r_m}(x_0)} \left[\langle |\cdot|, \mathbf{v}_x \rangle + \langle |\cdot|, \mathbf{v}_x^\infty \rangle \frac{d\lambda_\nu}{d\mathcal{L}}(x) \right] dx \right. \\ &\quad \left. + \int_{\bar{Q}_{r_m}(x_0)} \langle |\cdot|, \mathbf{v}_x^\infty \rangle d\lambda_\nu^s(x) \right) \\ &= \langle |\cdot|, \mathbf{v}_{x_0}^\infty \rangle \lim_{m \rightarrow \infty} \left(\int_{\bar{Q}} d(c_m(x_0) T_{\#}^{(x_0, r_m)} \lambda_\nu^s)(y) \right) \\ &= \int_{\bar{Q}} \langle |\cdot|, \mathbf{v}_{x_0}^\infty \rangle d\gamma(y), \quad \text{for some } \gamma \in \text{Tan}(\lambda_\nu^s, x_0), \end{aligned}$$

where, in passing to the third equality we have used that $x_0 \in S$. From the equality above we deduce that $\sigma_y = \delta_0$ for \mathcal{L}^d -a.e. $y \in Q$.

Testing, this time with g_l , we obtain by (2.73) and a similar argument to one above, that

$$\langle\langle g_l, \sigma \rangle\rangle = \int_{\bar{Q}} \varphi_l(y) \langle h_l^\infty, \mathbf{v}_{x_0}^\infty \rangle d\gamma(y),$$

from which we deduce that $\sigma_y^\infty = \mathbf{v}_{x_0}^\infty$ and $\lambda_\sigma \in \text{Tan}(\lambda_v^s, x_0)$.

4. The arguments of Step 4 remain unchanged except that this time one gets

$$\gamma_m := c_m T_{\#}^{(x_0, r_m)} \mu_{j(m)} \xrightarrow{\mathbf{Y}} \sigma \quad \text{in } \mathbf{Y}(Q, \mathbb{R}^N);$$

5. and similarly for Step 5.

(6') Differently from the case at regular points, we want to additionally show $\lambda_\sigma(Q) = 1$ and $\lambda_\sigma(\partial Q) = 0$. There exists $0 < \varepsilon < 1$ such that $\lambda_\sigma(\partial Q_\varepsilon) = 0$. Up to taking $r' = \varepsilon r$ (as well as $r'_m = r_m \varepsilon$ in the arguments of Steps 1-4 above we may assume without loss of generality that $\lambda_\sigma(\partial Q) = 0$ and $\lambda(Q) = 1$.

This proves the localization principle at singular points. □

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3 Relaxation and optimization of convex integrands with linear growth

This chapter contains the results obtained in the research paper:

Relaxation and optimization for linear-growth convex integral functionals under PDE constraints

Abstract

We give necessary and sufficient conditions for the minimality of generalized minimizers of linear-growth integral functionals of the form

$$\mathcal{F}[u] = \int_{\Omega} f(x, u(x)) \, dx, \quad u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^N$$

where u is an integrable function satisfying a general PDE constraint. Our analysis is based on two ideas: a relaxation argument into a subspace of the space of bounded vector-valued Radon measures $\mathcal{M}(\Omega; \mathbb{R}^N)$, and the introduction of a set-valued pairing in $\mathcal{M}(\Omega; \mathbb{R}^N) \times L^\infty(\Omega; \mathbb{R}^N)$. By these means we are able to show an intrinsic relation between minimizers of the relaxed problem and maximizers of its dual formulation also known as the saddle-point conditions. In particular, our results can be applied to relaxation and minimization problems in BV, BD and divergence-free spaces.

See:

A. Arroyo-Rabasa, *Relaxation and optimization for linear-growth convex integral functionals under PDE constraints*, to appear in J. Funct. Anal., 2017 (pre-print: <https://arxiv.org/abs/1603.01310>)

3.1 Introduction

Let Ω be an open and bounded subset of \mathbb{R}^d with $\mathcal{L}^d(\partial\Omega) = 0$. The aim of this work is to establish sufficient and necessary conditions, in the sense of convex duality, for a vector-valued Radon measure

μ to be a *generalized minimizer* of an integral functional of the form

$$\mathcal{F}[u] := \int_{\Omega} f(x, u(x)) \, dx,$$

defined on functions $u : \Omega \rightarrow \mathbb{R}^N$ satisfying a linear PDE constraint of the form

$$\mathcal{A}u = \tau, \quad \text{in the sense of distributions on } \Omega.$$

Here, $\mathcal{A} : \mathcal{M}(\Omega; \mathbb{R}^N) \rightarrow \mathcal{D}'(\Omega; \mathbb{R}^n)$ is a continuous linear partial differential operator defined on the space of bounded vector-valued Radon measures.

As part of our main assumptions, $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ is a continuous and convex integrand, that is, $f(x, \cdot)$ is convex for every $x \in \Omega$. We further assume that f satisfies the following standard linear growth assumptions: there exists a positive constant M such that

$$|f(x, z)| \leq M(1 + |z|), \quad \text{for all } (x, z) \in \Omega \times \mathbb{R}^N. \quad (3.1)$$

Throughout the paper, we shall consider the linear partial differential operator \mathcal{A} as a linear (possibly unbounded) operator $\mathcal{A} : \mathbf{W}^{\mathcal{A},1}(\Omega) \subset L^1(\Omega; \mathbb{R}^N) \rightarrow L^1(\Omega; \mathbb{R}^n)$, where

$$\mathbf{W}^{\mathcal{A},p}(\Omega) := \left\{ u \in L^p(\Omega; \mathbb{R}^N) : \mathcal{A}u \in L^p(\Omega; \mathbb{R}^n) \right\}, \quad 1 \leq p \leq \infty,$$

is the \mathcal{A} -Sobolev space of p -integrable functions on Ω . In this way, \mathcal{A} is densely defined and closed (in the sense of the graph) on $L^1(\Omega; \mathbb{R}^N)$. Whenever we write $\ker \mathcal{A}$ (and $\text{Im } \mathcal{A}$), we will refer to the kernel (and image) of $\mathcal{A} : \mathbf{W}^{\mathcal{A},1}(\Omega) \subset L^1(\Omega; \mathbb{R}^N) \rightarrow L^1(\Omega; \mathbb{R}^n)$. In a possible abuse of notation, we will still denote by $\mu \mapsto \mathcal{A}\mu$ the operator which is originally defined for measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$.

The following examples comprise a general class of linear partial differential operators of the form $\mathcal{A} : \mathcal{M}(\Omega; \mathbb{R}^N) \rightarrow \mathcal{D}'(\Omega; \mathbb{R}^n)$ which are continuous:

Example 3.1 (Operators in divergence form). Let k be a positive integer. Consider the operator in *divergence-form* which assigns, for every $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$, the distribution

$$\mathcal{A}\mu = \sum_{|\alpha| \leq k} \partial^\alpha (A_\alpha \mu), \quad \text{where } A_\alpha \in C(\Omega; \mathbb{M}^{n \times N}).$$

Here, we have defined $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $|\alpha| := |\alpha_1| + \cdots + |\alpha_d|$ for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$. Since the coefficients $A_\alpha(x)$ are continuous in Ω , each term $A_\alpha \mu$ is again a Radon measure and hence the linear operator $\mathcal{A} : \mathcal{M}(\Omega; \mathbb{R}^N) \rightarrow \mathcal{D}'(\Omega; \mathbb{R}^n)$ is well-defined and continuous.

Example 3.2. Alternatively, one might consider operators of the form

$$\mathcal{A}\mu = \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha \mu, \quad A_\alpha \in C^{|\alpha|}(\Omega; \mathbb{R}^n),$$

where each “ $\partial^\alpha \mu$ ” is the α -partial distributional derivative of μ . Observe that, even though this is not an operator in divergence form, the regularity of the coefficients guarantees that each summand $A_\alpha \partial^\alpha \mu$ is again a distribution.

3.1.1 Main results

Let $u_0 \in W^{\mathcal{A},1}(\Omega)$. In this paper we deal with the affine PDE constraint

$$\tau_0 := \mathcal{A} u_0 = \mathcal{A} u.$$

Let us consider the z -variable Fenchel conjugate $f^* : \Omega \times \mathbb{R}^N \rightarrow \overline{\mathbb{R}}$ of f , which is given by the formula

$$f^*(x, z^*) := \sup_{z \in \mathbb{R}^N} \{z^* \cdot z - f(x, z)\}, \quad z^* \in \mathbb{R}^N.^1$$

One way to derive optimality conditions for our constrained problem is to study the relations between the *primal problem*

$$\text{minimize } \left\{ u \mapsto \int_{\Omega} g(x, u) \, dx \right\} \text{ in the affine space } u_0 + \ker \mathcal{A}, \quad (\mathcal{P})$$

and the *dual problem*

$$\text{maximize } \left\{ w^* \mapsto \mathcal{R}[w^*] := \langle w^*, u_0 \rangle - \int_{\Omega} f^*(x, w^*) \, dx \right\} \text{ in } (\ker \mathcal{A})^\perp. \quad (\mathcal{P}^*)$$

Here, $(\ker \mathcal{A})^\perp = \{ w^* \in L^\infty(\Omega; \mathbb{R}^N) : \langle w^*, u \rangle = 0 \text{ for all } u \in \ker \mathcal{A} \}$. Using the duality of \mathcal{A} and \mathcal{A}^* it is elementary to check that

$$\mathcal{F}[u + u_0] \geq \mathcal{R}[w^*], \quad \text{for every } u \in \ker \mathcal{A} \text{ and } w^* \in (\ker \mathcal{A})^\perp,$$

An immediate observation is that the infimum in (\mathcal{P}) is greater or equal than the supremum in (\mathcal{P}^*) . Convex duality is particularly useful when these two extremal quantities agree since it leads to a saddle-point condition between minimizers of the primal problem and maximizers of the dual problem (we refer the reader to [14] for an extensive introduction on this topic). Actually, a simple consequence of the Fenchel–Rockafellar Theorem (see, e.g., [8, Thm. 1.12]) asserts there is in fact no gap between these two problems:

Theorem 3.3. *The problems (\mathcal{P}) and (\mathcal{P}^*) are dual of each other and the infimum in problem (\mathcal{P}) agrees with the supremum in problem (\mathcal{P}^*) , i.e.,*

$$\inf_{\mathcal{A} u = \tau_0} \mathcal{F}[u] = \sup_{w^* \in (\ker \mathcal{A})^\perp} \mathcal{R}[w^*].$$

Moreover, the supremum in the right hand side is in fact a maximum, which is equivalent to problem (\mathcal{P}^*) having at least one solution.

¹ For the sake of simplicity, we depart from the standard notation $(f(x, \bullet))^*$ for the z -variable Fenchel transform

If a *classical* minimizer $u \in L^1(\Omega; \mathbb{R}^N)$ of (\mathcal{P}) exists and w^* is a solution of (\mathcal{P}^*) , then the pairing $\langle w^*, u \rangle$ is a saddle-point of these two variational problems. This constitutive relation between u and w^* can be derived by variational methods and is expressed by the following pointwise characterization:

$$f(x, u(x)) + f^*(x, w^*(x)) = u(x) \cdot w^*(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega.$$

Under standard coercivity assumptions (for example if $M^{-1}(|z| - 1) \leq f(x, z)$ for all $(x, z) \in \Omega \times \mathbb{R}^N$), the infimum of problem (\mathcal{P}) is finite and minimizing sequences are L^1 -uniformly bounded. It is also well-known (see [5, 7, 14, 15, 25]) that if f is sufficiently regular, the convexity of $f(x, \cdot)$ is a sufficient condition to ensure the L^1 -weak sequential lower semicontinuity of \mathcal{F} , i.e.,

$$\liminf_{j \rightarrow \infty} \mathcal{F}[u_j] \geq \mathcal{F}[u], \quad \text{whenever } u_j \rightharpoonup u \text{ in } L^1(\Omega; \mathbb{R}^N).$$

However, due to the lack of weak-compactness of L^1 -bounded sets, we can only hope for compactness in a space of measures, that is,

$$u_j \mathcal{L}^d \xrightarrow{*} \mu \in \mathcal{M}(\Omega; \mathbb{R}^N).$$

This entails the need to relax the functional \mathcal{F} in the space of measures.

We say that $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ is an \mathcal{A} -free measure if $\mathcal{A}\mu = 0$ in the sense of distributions on Ω , the space of \mathcal{A} -free measures will be denoted by $\ker_{\mathcal{M}} \mathcal{A}$.

In order to prove the main relaxation result (see Theorem 3.4 below), we will restrict our analysis to operators for which $\ker \mathcal{A} \subset \ker_{\mathcal{M}} \mathcal{A}$ is densely contained with respect to the area-strict convergence of measures (see Definition 3.9):

Assumption A1. Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ be an \mathcal{A} -free measure. Then, there exists a sequence $(u_j) \subset L^1(\Omega; \mathbb{R}^N) \cap \ker \mathcal{A}$ such that $u_j \mathcal{L}^d$ area-strict converges to μ in Ω .

Theorem 3.4 (Relaxation). Let $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand with linear growth at infinity as in (3.1), and such that $f(x, \cdot)$ is convex for all $x \in \Omega$. Further assume that Assumption A1 holds and that there exists a modulus of continuity ω such that

$$|f(x, z) - f(y, z)| \leq \omega(|x - y|)(1 + |z|) \quad \text{for all } x, y \in \Omega, z \in \mathbb{R}^N. \quad (3.2)$$

Then the weak* lower semicontinuous envelope

$$\overline{\mathcal{F}}[\mu] := \left\{ \liminf_{j \rightarrow \infty} \mathcal{F}[u_j] : u_j \in u_0 + \ker \mathcal{A} \quad \text{and} \quad u_j \mathcal{L}^d \xrightarrow{*} \mu \right\},$$

of the functional

$$\mathcal{F}[u] := \int_{\Omega} f(x, u(x)) \, dx, \quad u \in u_0 + \ker \mathcal{A},$$

is given by the functional

$$\mu \mapsto \int_{\Omega} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} f^{\infty} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x),$$

defined for measures in the affine space $u_0 + \ker_{\mathcal{M}} \mathcal{A}$. Here, $\mu = \frac{d\mu}{d\mathcal{L}^d} \mathcal{L}^d + \mu^s$ is the Radon–Nikodým decomposition of μ with respect to \mathcal{L}^d and

$$f^\infty(x, z) := \lim_{\substack{x' \rightarrow x \\ z' \rightarrow z \\ t \rightarrow \infty}} \frac{f(x', tz')}{t} \quad (x, z) \in \Omega \times \mathbb{R}^N$$

is the recession function of f .

Extending the differential constraint to $\mathcal{M}(\Omega; \mathbb{R}^N)$, the relaxed functional $\overline{\mathcal{F}}$ gives rise to the relaxed problem

$$\text{minimize } \overline{\mathcal{F}} \text{ in the affine space } u_0 + \ker_{\mathcal{M}} \mathcal{A}, \quad (\overline{\mathcal{P}})$$

for which is possible to guarantee the existence of minimizers.

Since a (generalized) minimizer $\bar{\mu}$ may not be absolutely continuous with respect to \mathcal{L}^d , it is not clear in which sense can “ $\bar{\mu} \cdot \bar{w}^*$ ” be considered a saddle-point of $(\overline{\mathcal{P}})$ and (\mathcal{P}^*) . To circumvent the lack of a duality relation in $(\ker_{\mathcal{M}} \mathcal{A}, (\ker \mathcal{A})^\perp)$ we introduce a *set-valued* pairing as follows:

$$\begin{aligned} \llbracket \mu, w^* \rrbracket := & \left\{ \lambda \in \mathcal{M}(\Omega) : (u_j) \subset u_0 + \ker \mathcal{A}, \right. \\ & \left. u_j \rightarrow \mu \text{ area-strictly in } \Omega, \text{ and } (u_j \cdot w^*) \mathcal{L}^d \xrightarrow{*} \lambda \text{ in } \mathcal{M}(\Omega) \right\}. \end{aligned}$$

We stress that, though our notion of *generalized pairing* is that of a set-valued pairing, it reduces to a set containing a single Radon measure if stronger regularity assumptions are posed on its arguments μ or w^* . It should also be noticed that the earlier definitions by Anzellotti [2] for the $(\text{BV}, L^1 \cap \text{div-free})$ duality, and Kohn and Temam [17, 18] in BD with respect to its dual space, both exploit the *potential structure* of gradients and deformation tensors; this structure is in general not available for the constraint $\mu \in \ker_{\mathcal{M}} \mathcal{A}$.

As we will see, it turns out that every $\lambda \in \llbracket \mu, w^* \rrbracket$ is absolutely continuous with respect to $|\mu|$. Even more, its absolutely continuous part with respect to \mathcal{L}^d is fully determined by μ and w^* through the relation

$$\frac{d\lambda}{d\mathcal{L}^d}(x) = \frac{d\mu}{d\mathcal{L}^d}(x) \cdot w^*(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega.$$

This means that, at least formally, elements λ in $\llbracket \mu, w^* \rrbracket$ can be regarded as classical pairings up to a defect singular measure $\lambda^s \perp \mathcal{L}^d$. In fact, λ^s carries the (generalized) saddle-point conditions as illustrated in our main result:

Theorem 3.5 (Conditions for optimality). *Let $f : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ be a continuous integrand with linear growth at infinity as in (3.1) and that $f(x, \cdot)$ is convex for all $x \in \Omega$. Further suppose that Assumption A1 holds and that there exists a modulus of continuity ω such that*

$$|f(x, z) - f(y, z)| \leq \omega(|x - y|)(1 + |z|) \quad \text{for all } x, y \in \Omega, z \in \mathbb{R}^N.$$

Then the following conditions are equivalent:

- (i) μ is a generalized solution of problem (\mathcal{P}) , and w^* is a solution of (\mathcal{P}^*) .
- (ii) The generalized pairing $\llbracket \mu, w^* \rrbracket$ is the singleton containing the measure

$$\lambda := \left(\frac{d\mu}{d\mathcal{L}^d} \cdot w^* \right) \mathcal{L}^d \llcorner \Omega + f^\infty \left(\cdot, \frac{d\mu}{d|\mu^s|} \right) |\mu^s|;$$

in particular

$$\frac{d\lambda}{d|\mu^s|}(x) = f^\infty \left(x, \frac{d\mu}{d|\mu^s|} \right) \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

Moreover, the classical saddle-point conditions

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^d}(x) &= \frac{d\mu}{d\mathcal{L}^d}(x) \cdot w^*(x) \\ &= f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) + f^*(x, w^*(x)) \end{aligned}$$

hold at \mathcal{L}^d -a.e. in $x \in \Omega$.

The paper is organized as follows: Firstly, in Section 3.2 we give a short account of the properties of integral functionals defined on measures and their relation to area-strict convergence. The remainder of the Section recalls some facts of convex duality and the commutativity of the supremum on integral functionals for *PCU-stable* families of measurable functions. In Section 3.3 we rigorously derive the dual variational formulation of (\mathcal{P}) by means of classical convex analysis arguments. Section 3.4 is devoted to the characterization of the relaxed problem $(\overline{\mathcal{P}})$. In Section 3.5, we study the properties of pairing $\llbracket \mu, w^* \rrbracket$, from which the proof of Theorem 3.5 easily follows; applications of our results to BV, BD and other spaces are further discussed throughout the paper. Lastly, in Section 3.6, we apply our results to derive the saddle-point relations of a low-volume fraction model in optimal design.

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3.2 Preliminaries

3.2.1 Notation

We shall work in $\Omega \subset \mathbb{R}^d$, an open and bounded domain.

By $L_\mu^p(\Omega; \mathbb{R}^N)$ we denote the subset of $L_\mu(\Omega; \mathbb{R}^N)$ of μ -measurable functions on Ω with values in \mathbb{R}^N which are p -integrable with respect to a given positive measure μ ; we will simply write $L_\mu^p(\Omega)$ instead of $L_\mu^p(\Omega; \mathbb{R})$, and $L^p(\Omega; \mathbb{R}^N)$ instead of $L_{\mathcal{L}^d}^p(\Omega; \mathbb{R}^N)$, where \mathcal{L}^d stands for the d -dimensional Lebesgue measure.

In the course of this work we confine ourselves to the use of bounded Radon measures, therefore we will use the notation $\mathcal{M}(\Omega; \mathbb{R}^N) \cong (C_0(\Omega; \mathbb{R}^N))^*$ to denote the space of \mathbb{R}^N -valued Radon measures on Ω with finite mass. Similarly to L^p , we will simply write $\mathcal{M}(\Omega)$ instead of $\mathcal{M}(\Omega; \mathbb{R})$. For an arbitrary measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^N)$ we will often write $\frac{d\mu}{d\mathcal{L}^d} \mathcal{L}^d + \mu^s$ to denote its Radon-Nikodým decomposition with respect to \mathcal{L}^d .

We shall write $x \cdot y$ to denote the inner product between two vectors $x, y \in \mathbb{R}^N$. For function and measure spaces, we reserve the notation $\langle \cdot, \cdot \rangle$ to represent the standard pairing between the space and its dual; where no confusion can arise, we shall not emphasize the position of its arguments.

3.2.2 Integrand, lower semicontinuity, and area-strict convergence

We recall some well-known and other recent results concerning integrands and recession functions.

Following [1] and more recently [20], we define $\mathbf{E}(\Omega; \mathbb{R}^N)$ as the class of continuous functions $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that the transformation

$$(Sf)(x, z) := (1 - |z|)f\left(x, \frac{z}{1 - |z|}\right) \quad \text{for } (x, z) \in \Omega \times \mathbb{B}^N,$$

where \mathbb{B}^N is unit open ball in \mathbb{R}^N , can be extended to the space $C(\Omega \times \overline{\mathbb{B}^d})$ by some continuous function \tilde{f} . In particular, for every $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$, there exists a positive constant $M > 0$ such that

$$|f(x, z)| \leq M(1 + |z|) \quad \text{for all } (x, z) \in \Omega \times \mathbb{R}^N,$$

and

$$\tilde{f}(x, z) = \begin{cases} (Sf)(x, z) & \text{if } |z| < 1, \\ f^\infty(x, z) & \text{if } |z| = 1; \end{cases}$$

where the limit

$$f^\infty(x, z) := \lim_{\substack{x' \rightarrow x \\ z' \rightarrow z \\ t \rightarrow \infty}} \frac{f(x', tz)}{t} \quad (x, z) \in \Omega \times \mathbb{R}^N,$$

exists and defines a positively 1-homogeneous function.

Lemma 3.6 (Recession functions I). *If $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous convex integrand with linear growth at infinity with a modulus of continuity ω as in (3.2), then $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$. Moreover, the recession function f^∞ exists, is continuous and has the simplified representation*

$$f^\infty(x, z) = \lim_{t \rightarrow \infty} \frac{f(x, tz)}{t}, \quad \text{for all } (x, z) \in \Omega \times \mathbb{R}^N.$$

Proof. First, we show that $f(x, \cdot)$ is Lipschitz with $\text{Lip}(f(x, \cdot)) \leq M$ (independently of x). Fix $x \in \Omega$, by the convexity assumption we know that $f(x, \cdot) \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^N)$. Therefore,

$$\nabla_z f(x, z) \in \partial_z f(x, z) \quad \text{for } \mathcal{L}^N\text{-almost every } z \in \mathbb{R}^N.$$

Again, by convexity, $p^* \in \partial_z f(x, z)$ if and only if $f^*(p^*) = z \cdot p^* - f(z) \in \mathbb{R}$. Thus, $f^*(\nabla_z f(x, z)) \in \mathbb{R}$

for \mathcal{L}^N -almost every $z \in \mathbb{R}^N$. It is easy to check — using the linear growth assumption on f — that $\{p^* \in \mathbb{R}^N : f^*(p^*) < \infty\} \subset M \cdot \mathbb{B}^N$, whence we deduce that

$$\|\nabla_z f(x, \cdot)\|_{L^\infty} \leq M.$$

The arbitrariness in the choice of x and the continuity of f imply that $f(x, \cdot)$ is x -uniformly Lipschitz. Together with (3.2), this implies that

$$f^\infty(x, z) = \lim_{t \rightarrow \infty} \frac{f(x, tz)}{t} \quad \text{for all } (x, z) \in \Omega \times \mathbb{R}^N,$$

whenever any of the these limits exist. To see that the right hand side above exists in $\Omega \times \mathbb{R}^N$ we simply observe that

$$\frac{f(x, tz)}{t} = \frac{f(x, tz) - f(x, 0)}{t} + \mathcal{O}(t^{-1}) := I_{x,z}(t) + \mathcal{O}(t^{-1}), \quad (3.3)$$

where, by the convexity of f , the functions $I_{x,z}(t) \leq M$ are monotone (in t) for all $(x, z) \in \Omega \times \mathbb{R}^N$.

Finally, to prove that $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$, we are left to show that \tilde{f} is continuous at all $(x, z) \in \Omega \times \partial\mathbb{B}^N$ (this, because $f \in C(\Omega \times \mathbb{R}^N)$). Using the modulus of continuity in (3.2) it is easy to show that f^∞ is continuous on $\Omega \times \partial\mathbb{B}^N$, therefore it suffices to show that

$$\lim_{\substack{x' \rightarrow x \\ |z'| \uparrow 1, z' \rightarrow z}} \tilde{f}(x', z') = f^\infty(x, z) \quad \text{for all } x \in \Omega.$$

Using (3.2) and setting $t(z') := \frac{1}{1-|z'|}$ (which tends to ∞ as $|z'| \uparrow 1$) in the definition of Sf , the argument boils down to the uniqueness of the limit in (3.3) on sequences (t_j) such that $t_j \rightarrow \infty$. \square

We collect some continuity properties of the class $\mathbf{E}(\Omega; \mathbb{R}^N)$ and recession functions in the following lemmas. The first one is a lower semicontinuity result for convex integrands from [1] (see also [15] for the case $f(x, z) = f(z)$). The second is a continuity result, originally proved by Rešetnjak in the case of 1-homogeneous functions [24], but generalized to lower semicontinuous integrands with linear growth.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with $\mathcal{L}^d(\partial\Omega) = 0$. Let $f(x, z) : \Omega \times \mathbb{R}^d \rightarrow (-\infty, \infty]$ be a lower semicontinuous integrand, convex with respect to z , and verifying:*

$$\text{exists } M > 0 \text{ such that } f(x, z) \geq -M(|z| + 1).$$

Then, for every sequence $(u_j) \subset L^1(\Omega; \mathbb{R}^N)$ such that $u_j \mathcal{L}^d \xrightarrow{} \mu$ in $\mathcal{M}(\overline{\Omega}; \mathbb{R}^N)$, one has that*

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_{\Omega} f(x, u_j(x)) \, dx &\geq \\ &\int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) \, dx + \int_{\Omega} f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) \, d|\mu^s|(x). \end{aligned}$$

Before embarking on the proof, let us show by an easy example that the boundary term is necessary

due to possible concentration of measure at $\partial\Omega$.

Example 3.8. Let $d = N = n = 1$, set $\Omega = (0, 1)$ and consider the integrand $f(x, z) = z$ (accordingly $f^\infty(x, z) = z$). Consider the uniformly L^1 -bounded sequence of functions (u_j) where

$$u_j(x) := -j\chi_{(0,1/j)}(x), \quad j \in \mathbb{N}.$$

It is easy to check that $u_j \mathcal{L}^1 \xrightarrow{*} 0$ in $\mathcal{M}(\Omega)$, however, since $\int_\Omega f(u_j) = -1$ for all $j \in \mathbb{N}$ it follows that

$$-1 = \liminf_{j \rightarrow \infty} \int_\Omega f(u_j) < \int_\Omega f(0) \, dx = 0.$$

Hence, the lower semicontinuity fails.

Proof of Lemma 3.7. The proof of this lemma *should*, in practice, follow from the theory developed in [1]. However, due to small imprecisions in their presentation, we have decided to slightly modify the presentation of the proof.

First, let us recall that under the established assumptions, the conclusions of Theorem 5.1 and Remark 5.2 in [1] yield

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_\Omega f(x, u_j(x)) \, dx &\geq \\ &\int_\Omega f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) \, dx + \int_\Omega f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) \, d|\mu^s|(x). \end{aligned} \quad (3.4)$$

Their conclusion is correct as long as there is *no concentration* of measure at the boundary $\partial\Omega$, or, as long as $f \geq 0$ (since then only loss of energy can be accounted on the right hand side limit); see Example 3.8 above. In general, lower semicontinuity might fail for integrands which are unbounded from below (in fact, if f is not x -dependent, the conclusion of Lemma 3.7 holds if and only if $f^\infty \geq 0$, see e.g., Theorem 5.21 in [15]).

In spite of this imprecision, (3.4) holds as long as $(|u_j|)$ does not concentrate on the boundary $\partial\Omega$. Our proof will follow from this argument.

Up to taking a subsequence, we may assume without loss of generality that

$$A_0 := \liminf_{j \rightarrow \infty} \int_\Omega f(x, u_j(x)) \, dx = \lim_{j \rightarrow \infty} \int_\Omega f(x, u_j(x)) \, dx.$$

Let B_R be a ball containing $\overline{\Omega}$. For a measure μ (or function) defined on a smaller domain than B_R , we denote by $\tilde{\mu}$ its natural extension by the zero measure into $\mathcal{M}(B_R; \mathbb{R}^N)$. In this way $\tilde{u}_j \mathcal{L}^d \xrightarrow{*} \tilde{\mu}$ on B_R . Set also $\tilde{f} := \chi_\Omega(x)f(x, z) - \chi_{B_R \setminus \Omega}M(1 + |z|)$. Notice that since Ω is bounded and $\tilde{f}(x, z) \geq -M(1 + |z|)$, the assumptions of the lemma still hold for \tilde{f} and B_R . Also, since $\overline{\Omega} \subset B_R$, the sequence

$|\tilde{u}_j|_{\mathcal{L}^d}$ does not concentrate on ∂B_R . Hence, (3.4) gives

$$\begin{aligned} A_0 &= \liminf_{j \rightarrow \infty} \int_{B_R} \tilde{f}(x, \tilde{u}_j(x)) \, dx \geq \\ &\int_{B_R} \tilde{f}\left(x, \frac{d\tilde{\mu}}{d\mathcal{L}^d}(x)\right) \, dx + \int_{\overline{B_R}} (\tilde{f})^\infty\left(x, \frac{d\tilde{\mu}^s}{d|\tilde{\mu}^s|}(x)\right) \, d|\tilde{\mu}^s|(x) \\ &\geq \int_{\overline{\Omega}} \tilde{f}\left(x, \frac{d\tilde{\mu}}{d\mathcal{L}^d}(x)\right) \, dx + \int_{\overline{\Omega}} f^\infty\left(x, \frac{d\tilde{\mu}^s}{d|\tilde{\mu}^s|}(x)\right) \, d|\tilde{\mu}^s|(x) \\ &\geq \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) \, dx + \int_{\overline{\Omega}} f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) \, d|\mu^s|(x). \end{aligned}$$

Here, we have used that $\tilde{f} \equiv f$ and $\tilde{f}^\infty \equiv f^\infty$, on Ω and $\overline{\Omega}$ respectively (the latter follows directly from the definition of recession function and the fact that $f(x, z) \geq -M(1 + |z|)$). We have also used that $\mathcal{L}^d(\Omega) = 0$ to ensure that

$$\int_{\partial\Omega} \tilde{f}\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) \, dx = 0.$$

□

We introduce the following short notation for the (generalized) area functional

$$\langle \mu \rangle(B) := \int_B \sqrt{1 + \left| \frac{d\mu}{d\mathcal{L}^d}(x) \right|^2} \, dx + |\mu^s|(B), \quad (3.5)$$

defined on Borel sets $B \subset \mathbb{R}^d$.

Definition 3.9 (Area-strict convergence). *We say that a sequence of vector-valued Radon measures μ_j area-strict converges to a measure μ (in Ω) if and only if*

- (i) $\mu_j \xrightarrow{*} \mu$ weak* in $\mathcal{M}(\Omega; \mathbb{R}^N)$, and
- (ii) $\langle \mu_j \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega)$,

for $\langle \cdot \rangle$ the (generalized) area functional defined in (3.5).

Let us recall from [20] that the notion of area-strict convergence is stronger than the *strict convergence* of measures which is obtained by replacing (ii) above with the total-variation continuity

$$(ii') \quad |\mu_j|(\Omega) \rightarrow |\mu|(\Omega).$$

This notion of convergence turns out to be stronger than the usual strict convergence as the latter allows one-dimensional oscillations. The motivation behind the definition of area-strict convergence is that one can formulate the following generalized version of Rešetnjak's Continuity Theorem (see, e.g., [20, Theorem 5]):

Theorem 3.10. *The functional*

$$\mu \mapsto \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x)$$

is area-strict continuous in $\mathcal{M}(\Omega; \mathbb{R}^N)$ for every integrand $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$.

Remark 3.11. It can be easily seen that area-strict convergence is a sharp condition for the continuity of integral functionals defined on measures by taking $f(x, z) := \sqrt{1 + |z|^2} \in \mathbf{E}(\Omega; \mathbb{R}^N)$ and observing that $f^{\infty}(x, z) = |z|$.

The *push-forward* of a measure with respect to a Borel function φ is defined as follows. Let $\varphi : \Omega \rightarrow \Omega'$ be a Borel function, we define the push-forward measure $\varphi_*\mu$ through the assignment $\varphi_*\mu := \mu \circ \varphi^{-1}$. This translates into the following *change of variables* formula: for a map $g : \Omega' \rightarrow \mathbb{R}^N$, it holds that

$$\int_{\Omega'} g d(\varphi_*\mu) = \int_{\Omega} g \circ \varphi d\mu,$$

provided these integrals are well-defined.

Remark 3.12 (Density assumption). There are some operators for which the density Assumption A1 holds:

- (i) The minimization and relaxation on BV-spaces or when $\mathcal{A} = \text{curl}$ (for simply connected domains) is proved in Lemma 1 of [19] where no regularity assumption is imposed on $\partial\Omega$. The same argument further shows the area-strict approximation property in the BD-case (or $\mathcal{A} = \text{curl curl}$); see also Lemma 2.2 in [3] for a result which covers the strict convergence.
- (ii) Let $\mathcal{A} : \mathcal{M}(\Omega; \mathbb{R}^n) \rightarrow \mathcal{D}'(\Omega; \mathbb{R}^n)$ be a k th-order homogeneous partial differential operator with constant coefficients

$$\mathcal{A}\mu = \sum_{|\alpha|=k} A_{\alpha} \partial^{\alpha} \mu, \quad A_{\alpha} \in \mathbb{M}^{n \times N}.$$

Further assume that Ω is a *strictly star-shaped* domain, i.e., there exists $x_0 \in \Omega$ such that

$$\overline{(\Omega - x_0)} \subset t(\Omega - x_0), \quad \forall t > 1.$$

To prove that A1 holds let $\varphi^t : \Omega \rightarrow \{t(\Omega - x_0) + x_0\} : x \mapsto t(x - x_0) + x_0$ and consider the parametrized family of push-forward measures $(\varphi_*^t \mu)_{t>1}$. First, notice that $\overline{\Omega} \subset t(\Omega - x_0) + x_0$. Hence, due to the homogeneity of \mathcal{A} , each $\varphi_*^t \mu$ is an \mathcal{A} -free measure on an open set containing $\overline{\Omega}$. Second, it is relatively easy to check that $\varphi_*^t \mu$ area-strictly converges to μ as $t \downarrow 1$. The last step consists on mollifying each $\varphi_*^t \mu$ by a sufficiently small parameter $\delta(t) \downarrow 0$ with the property that

$$\langle \varphi_*^t \mu * \rho_{\delta} \rangle(\Omega) = \langle \varphi_*^t \mu \rangle(\Omega) + \mathcal{O}(1-t), \quad \text{and} \quad \mathcal{A}(\varphi_*^t \mu * \rho_{\delta}) = 0 \quad \text{on } \Omega.$$

Here, $\rho_{\delta}(x) := \rho(x/\delta)\delta^{-d}$ where $\rho \in C^{\infty}(B_1)$ is a standard mollifier.

The conclusion follows by letting $t \downarrow 1$ in the estimate above.

We refer the reader to [21] where such a geometrical assumption is made to address a homogenization problem in the case $\mathcal{A} = \text{curl}$.

3.2.3 PCU-stability

Next, we recall some facts on the commutativity of the supremum of integral functionals valued on a certain family \mathcal{F} of measurable functions. The definitions and results gathered here can be found in [7, Theorem 1] and [26, Proposition 1.14].

Definition 3.13. Let $L^0(\Omega; \mathbb{R}^N)$ be the space of \mathbb{R}^N -valued measurable functions. A set \mathcal{F} of $L^0(\Omega; \mathbb{R}^N)$ is said to be PCU-stable if for any continuous partition of unity $(\alpha_0, \dots, \alpha_m)$ such that $\alpha_1, \dots, \alpha_m \in C_c(\Omega)$, for every u_1, \dots, u_m in \mathcal{F} , the sum $\sum_{i=1}^m \alpha_i u_i$ belongs to \mathcal{F} .

Theorem 3.14. For any subset \mathcal{F} of $L^0(\Omega; \mathbb{R}^N)$ there exists a smallest closed-valued measurable multifunction Γ such that for all $u \in \mathcal{F}$, $u(x) \in \Gamma(x)$ μ -a.e. (as smallest refers to inclusion). Moreover, there exists a sequence (u_j) in \mathcal{F} such that $\Gamma(x) = \overline{\{u_j(x) : j \in \mathbb{N}\}}$ for μ -a.e. $x \in \Omega$.

We say that Γ is the *essential infimum* of the multifunctions

$$x \mapsto \{u(x) : u \in \mathcal{F}\},$$

in symbols

$$\Gamma(\cdot) = \text{ess sup}\{u(\cdot) : u \in \mathcal{F}\}$$

Theorem 3.15. Let $j : \Omega \times \mathbb{R}^N \rightarrow (-\infty, \infty]$ be a normal convex integrand.² Denote by J the functional

$$u \mapsto \int_{\Omega} j(x, u(x)) \, dx, \quad \text{for all } u \in L^0(\Omega, \mathbb{R}^N).³$$

Let \mathcal{F} be a PCU-stable family in $L^0(\Omega, \mathbb{R}^N)$. Assume furthermore that J is proper within \mathcal{F} , i.e., there exists $u_0 \in \mathcal{F}$ such that $J(u_0) \in \mathbb{R}$. Then,

$$\inf_{u \in \mathcal{F}} J(u) = \int_{\Omega} \inf_{z \in \Gamma(x)} j(x, z) \, dx,$$

and

$$\inf_{z \in \Gamma(x)} j(\cdot, z) = \text{ess sup}\{j(\cdot, u) : u \in \mathcal{F}, J(u) < +\infty\}.$$

3.3 The dual problem

We recall some facts of the theory of convex functions. We follow closely those ideas from [14, Ch. III]. Along this chapter, X and Y will be two topological vector spaces placed in duality with their

²A normal integrand $f : \Omega \times \mathbb{R}^N \rightarrow (-\infty, \infty]$ is a measurable function which is also lower semicontinuous in its second variable

³As usual we define $\int_{\Omega} j(x, u(x)) \, dx = \infty$, as soon as $\int_{\Omega} (j(x, u(x)))^+ \, dx = \infty$.

duals X^* and Y^* by the pairing $\langle \cdot, \cdot \rangle_{X^* \times X}$ (analogously for Y and Y^*). The subscript notation will be dropped as it is understood that the correspondent pairing apply only on their respective domains. For a continuous function $F : X \rightarrow (-\infty, \infty]$, we define a lower semi-continuous, and convex function by letting

$$F^*(u^*) := \sup_X \{ \langle u, u^* \rangle - F(u) \}, \quad u^* \in X^*.$$

This function is known as the conjugate function of F . We will be concerned with the minimization problem

$$\text{minimize } F \text{ in } X, \tag{p}$$

which we term as the *primal problem*.

The dual problem

Let $\Phi^* : X^* \rightarrow \overline{\mathbb{R}}$ be the conjugate of Φ . We define the dual problem of (p) as

$$\text{maximize } \{ p^* \mapsto -\Phi^*(p^*) \} \text{ in } X^*. \tag{p^*}$$

Some of the results of this section are stated under weaker assumptions than the ones previously established in the introduction; however, the results in subsequent sections do require stronger these properties (for a discussion on the sharpness of our assumptions on the integrand f we refer the reader to [1, 7, 15] and references therein).

In this section we study the dual formulation (\mathcal{P}^*) of (\mathcal{P}) in the duality (L^∞, L^1) . Our main goal is to prove Theorem 3.3 which states not only that (\mathcal{P}) and (\mathcal{P}^*) are in duality but that there is no gap between them. The idea is to gather the concepts of the last section to characterize the dual problem (\mathcal{P}^*) as an integral functional in $L^\infty(\Omega; \mathbb{R}^n)$.

For an (measurable) integrand $g : \Omega \times \mathbb{R}^N \rightarrow (-\infty, \infty]$, we will write I_g to denote the functional that assigns

$$u \mapsto \int_\Omega g(x, u(x)) \, dx, \quad u \in L(\Omega; \mathbb{R}^N).$$

Following standard notation we denote, for a Banach space X and a subset $U \subset X$, the U -indicator function $\chi_U : X \rightarrow \overline{\mathbb{R}}$ defined by the functional

$$\chi_U(u) := \begin{cases} 0 & \text{if } u \in U \\ \infty & \text{if } u \in X \setminus U \end{cases},$$

which is lower semicontinuous on $\|\cdot\|_X$ -closed subsets $U \subset X$. If V is a linear subspace of X , the Fenchel transform of the indicator function χ_V is given by another indicator function, namely

$$(\chi_V)^* = \chi_{V^\perp},$$

where $V^\perp := \{ x^* \in X^* : \langle x^*, x \rangle = 0 \, \forall x \in V \}$ is the *orthogonal space* to V .

It will often be convenient to re-write the minimization problem (\mathcal{P}) as

$$\text{minimize } \left\{ u \mapsto I_f(u + u_0) + \chi_{\ker \mathcal{A}}(u) \right\} \text{ in } L^1(\Omega; \mathbb{R}^N). \quad (\mathcal{P})$$

Lemma 3.16. *Let $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous and convex integrand with linear growth at infinity. Then the Fenchel conjugate of the functional $I_f : L^1(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$, is given by the integral functional*

$$u^* \mapsto I_{f^*}(w^*),$$

defined on functions $w^* \in L^\infty(\Omega; \mathbb{R}^N)$. In particular,

$$(I_f(u_0 + \cdot))^*(w^*) = I_{f^*}(w^*) - \langle w^*, u_0 \rangle.$$

Proof. We argue as follows.

Step 1. We point out that $L^1(\Omega; \mathbb{R}^N)$ is a PCU-stable family.

Step 2. Since f has linear growth, $I_f - \langle w^*, \cdot \rangle$ is proper in $L^1(\Omega; \mathbb{R}^N)$.

Step 3. We fix $w^* \in L^\infty(\Omega; \mathbb{R}^N)$ (here, $w^* \in L^\infty(\Omega; \mathbb{R}^N)$ is the representative such that $w^*(x) \in \mathbb{R}^N$ for all $x \in \mathbb{R}^d$) and apply Theorem 3.15 to $\mathcal{F} = L^1(\Omega; \mathbb{R}^N)$ and to

$$j(x, z) = f(x, z) - w^*(x) \cdot z,$$

which remains a convex normal integrand, to find out that

$$(I_f)^*(w^*) = - \inf_{u \in L^1(\Omega; \mathbb{R}^N)} \int_{\Omega} j(x, u(x)) \, dx = - \int_{\Omega} \inf_{z \in \Gamma(x)} j(x, z) \, dx,$$

where $\Gamma(\cdot) = \text{ess sup} \{ u(\cdot) : u \in L^1(\Omega; \mathbb{R}^N) \} = \mathbb{R}^N$. Since $\inf_{z \in \mathbb{R}^N} j(x, z)$ is nothing else than $-f^*(x, w^*(x))$ for a.e. $x \in \Omega$, it follows that

$$(I_f)^*(w^*) = I_{f^*}(w^*).^4$$

The last observation follows from the translation property of the Fenchel transform:

$$(F(x_0 + \cdot))^*(x^*) = F^*(x^*) - \langle x^*, x_0 \rangle.$$

□

Proof of Theorem 3.3. We want to show that if $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous and convex integrand with linear growth at infinity. Then, the dual problem of (\mathcal{P}) reads:

$$\text{maximize } \mathcal{R} \text{ in the space } L^\infty(\Omega; \mathbb{R}^n), \quad (\mathcal{P}^*)$$

⁴Due to the linear-growth assumptions on f , its Fenchel transform is bounded from below. More specifically, $f^*(x, z^*) \geq -f(x, 0) \geq -M$ for all $(x, z^*) \in \Omega \times \mathbb{R}^N$, whence the integral I_{f^*} is well-defined.

where $\mathcal{R} : L^\infty(\Omega; \mathbb{R}^n) \rightarrow \overline{\mathbb{R}}$ is the functional defined as

$$\mathcal{R}[w^*] := -\Phi^*(w^*) = \begin{cases} \langle w^*, u_0 \rangle - I_{f^*}(w^*) & \text{if } w^* \in (\ker \mathcal{A})^\perp \\ -\infty & \text{otherwise} \end{cases}.$$

To show this, we recall the useful well-known duality characterization due to Fenchel and Rockafellar (see, e.g., [8, Theorem 1.12 and Example 4]):

Theorem 3.17 (Fenchel & Rockafellar). *Let X be a Banach space and let $\Phi, \Psi : X \rightarrow (-\infty, \infty]$ be two convex functions. Assume that there is some $u_0 \in \{u \in X : |\Phi(u)|, |\Psi(u)| < \infty\}$ such that Φ is continuous at u_0 . Then*

$$\begin{aligned} \inf_{u \in E} \{\Phi(u) + \Psi(u)\} &= \sup_{w^* \in E^*} \{-\Phi^*(-w^*) - \Psi^*(w^*)\} \\ &= \max_{w^* \in E^*} \{-\Phi^*(-w^*) - \Psi^*(w^*)\}. \end{aligned}$$

The functional $I_f : L^1(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$ is convex and continuous (recall that f is x -uniformly Lipschitz in its second argument). On the other hand, the indicator function $\chi_{\ker \mathcal{A}} : L^1(\Omega; \mathbb{R}^N) \rightarrow (-\infty, \infty]$ is also a convex functional ($\ker \mathcal{A}$ is a closed linear subspace of $L^1(\Omega; \mathbb{R}^N)$). Hence, we may apply the results from the theorem above to $\Phi = I_f(u_0 + \cdot)$ and $\Psi = \chi_{\ker \mathcal{A}}$ to get

$$\inf_{u \in \ker \mathcal{A}} I_f(u) = \max_{w^* \in L^\infty(\Omega; \mathbb{R}^n)} \{-(I_f(u_0 + \cdot))^*(-w^*) - (\chi_{\ker \mathcal{A}})^*(w^*)\}.$$

By Lemma 3.16 we might further use that $-(I_f(u_0 + \cdot))^*(w^*) = \langle w^*, u_0 \rangle - I_{f^*}(w^*)$, whence we obtain the sought equality

$$\inf_{u \in \ker \mathcal{A}} I_f = \max_{w^* \in L^\infty(\Omega; \mathbb{R}^n)} \mathcal{R}[w^*] = \max_{w^* \in (\ker \mathcal{A})^\perp} \langle w^*, u_0 \rangle - I_{f^*}.$$

Notice that the existence of at least one solution of (\mathcal{P}^*) is guaranteed by Theorem 3.17. \square

Corollary 3.18 (Operators with closed range). *Let $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ as in the assumptions of Theorem 3.3. Assume furthermore that $\text{Im } \mathcal{A}$ is closed with respect to the L^1 topology — or equivalently, that $\text{Im } \mathcal{A}^*$ is closed with respect to the L^∞ topology.⁵ Then, the dual problem (\mathcal{P}^*) reads*

$$\text{maximize } \mathcal{R} \text{ in } \text{Im } \mathcal{A}^*.$$

Proof. The proof is an immediate consequence of the identity (see, e. g., [8, Remark 17])

$$(\ker \mathcal{A})^\perp = \text{Im } \mathcal{A}^*.$$

\square

Example 3.19. The next examples (of operators with closed range) are related to low-volume frac-

⁵See, e.g., Remark 17 and Theorem 2.19 in [8]

tion optimal design problems in linear conductivity, linear elasticity, and linear plate theory models.

Let $\Omega \subset \mathbb{R}^d$ be a simply connected Lipschitz domain.

1. Divergence-free fields. Let $\mathcal{A} = \text{div} : \mathcal{M}(\Omega; \mathbb{M}^{d \times m}) \rightarrow \mathcal{D}'(\Omega; \mathbb{R}^d)$ be the divergence operator

$$(\text{div } u)_i := \sum_j \partial_j u_{ij}, \quad 1 \leq i, j \leq d.$$

It is fairly straightforward that the adjoint of $\mathcal{A} : \mathbf{W}^{\text{div},1}(\Omega) \rightarrow \mathbf{L}^1(\Omega; \mathbb{R}^m)$ is the gradient operator $\mathcal{A}^* : \mathbf{W}_0^{1,\infty}(\Omega; \mathbb{R}^m) \rightarrow \mathbf{L}^\infty(\Omega; \mathbb{R}^{d \times m}) : w^* \mapsto \nabla w^*$. Furthermore, due to Poincaré's inequality, $\text{Im } \mathcal{A}^* = \{ \nabla v : v \in \mathbf{W}_0^{1,\infty}(\Omega; \mathbb{R}^m) \}$ is closed with respect to the \mathbf{L}^∞ topology and hence $\text{Im } \mathcal{A}^* = (\text{Im } \mathcal{A})^\perp$.

2. Double divergence-free fields. Consider $\mathcal{A} = \text{div}^2 : \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) \rightarrow \mathcal{D}'(\Omega)$ defined as

$$\text{div}^2 U := \sum_{i,j=1}^d \partial_{ij} U_{ij}.$$

In this case, the adjoint of $\mathcal{A} : \mathbf{W}^{\text{div}^2,1}(\Omega) \rightarrow \mathbf{L}^1(\Omega; \mathbb{R})$ is the operator $\mathcal{A}^* : \mathbf{W}_0^{2,\infty}(\Omega) \rightarrow \mathbf{L}^\infty(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) : U^* \mapsto \nabla^2 U^*$, where $\nabla^2 U^*$ is the Hessian of U^* given by

$$(\nabla^2 U^*)_{ij} := \left(\frac{\partial^2 U^*}{\partial x_i \partial x_j} \right)_{ij}, \quad 1 \leq i, j \leq d.$$

Due to a similar argument as in (1), $\text{Im } \mathcal{A}^* = \{ \nabla^2 v : v \in \mathbf{W}_0^{2,\infty}(\Omega) \}$ is closed with respect to the \mathbf{L}^∞ topology.

3. Symmetric divergence-free fields. Let $\mathcal{A} = \text{div} : \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) \rightarrow \mathcal{D}'(\Omega; \mathbb{R}^d)$. This time the adjoint of $\mathcal{A} : \mathbf{W}^{\text{div},1}(\Omega) \rightarrow \mathbf{L}^1(\Omega; \mathbb{R}^d)$ is the symmetric gradient $\mathcal{A}^* : \mathbf{W}_0^{1,\infty}(\Omega; \mathbb{R}^d) \rightarrow \mathbf{L}^\infty(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) : w^* \mapsto (\nabla w^* + (\nabla w^*)^T)/2$ whose range $\text{Im } \mathcal{A}^*$ is closed with respect to the \mathbf{L}^∞ topology — this follows from Korn's inequality and the classical Sobolev embedding.

Corollary 3.20 (Operators with a potential structure). *Let $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ as in the assumptions of Theorem 3.3. Further assume that $\ker \mathcal{A} = \text{Im } \mathcal{B}$, for some densely defined and closed linear partial differential operator $\mathcal{B} : D(\mathcal{B}^*) \subset \mathbf{L}^1(\Omega; \mathbb{R}^l) \rightarrow \mathbf{L}^1(\Omega; \mathbb{R}^N)$. Then, the dual problem \mathcal{P}^* reads*

$$\text{maximize } \mathcal{H} \text{ in } \ker \mathcal{B}^*.$$

Proof. Since \mathcal{B} is densely defined and closed (in the sense of the graph), it holds that $\ker \mathcal{B}^* = (\text{Im } \mathcal{B})^\perp$ (see, e.g., [8]). Hence, using the exactness of $\text{Im } \mathcal{B} = \ker \mathcal{A}$,

$$\ker \mathcal{B}^* = (\text{Im } \mathcal{B})^\perp = (\ker \mathcal{A})^\perp.$$

The sought assertion then follows from Theorem 3.3. □

Example 3.21 (Gradients). Assume that Ω is a simply connected Lipschitz domain with outer normal vector $\nu_\Omega(y)$ defined at \mathcal{H}^{d-1} -a.e. $y \in \partial\Omega$. The results of Corollary 3.20 apply to the minimization of problems of the form

$$v \mapsto \int_{\Omega} f(x, \nabla v(x)) \, dx, \quad v \in \mathbf{W}^{1,1}(\Omega).$$

Consider the curl and gradient operators defined on measures by

$$\mathcal{A}\mu = \text{curl}\mu = (\partial_k \mu_{ij} - \partial_j \mu_{ik})_{ijk}, \quad 1 \leq j, k \leq d, 1 \leq i \leq m, \quad \mu \in \mathcal{M}(\Omega; \mathbb{R}^{m \times d}),$$

and

$$\mathcal{B}\mu = \text{grad}\mu = (\partial_j \mu^i)_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq d, \quad \mu \in \mathcal{M}(\Omega; \mathbb{R}^m).$$

Since Ω is simply connected, it holds that

$$\begin{aligned} \ker \mathcal{A} &= \{ u \in L^1(\Omega; \mathbb{M}^{m \times d}) : \text{curl} u = 0 \} \\ &= \{ \nabla v : v \in \mathbf{W}^{1,1}(\Omega; \mathbb{R}^m) \} \\ &= \text{Im } \mathcal{B}. \end{aligned}$$

Since Ω is a Lipschitz domain it is easy to show that

$$\mathcal{B}^* w^* = -\text{div} w^* = -\left(\sum_{i=1}^d \partial_j w_{ij}^* \right)_i, \quad 1 \leq i \leq m,$$

in the sense of distributions for any $w^* \in D(\mathcal{B}^*)$, where

$$\begin{aligned} D(\mathcal{B}^*) &= \mathbf{W}_0^{\text{div}, \infty}(\Omega) \\ &:= \{ w^* \in L^\infty(\Omega; \mathbb{M}^{m \times d}) : \text{div} w^* \in L^\infty(\Omega; \mathbb{R}^m), T w^* = 0 \}, \end{aligned}$$

Here, $T : \mathbf{W}^{\text{div}, \infty}(\Omega) \rightarrow L^\infty(\partial\Omega; \mathbb{R}^m)$ is the unique continuous linear map such that

$$T w^* = (w^* \cdot \nu_\Omega)|_{\partial\Omega} \quad \text{for all } w^* \in C^1(\overline{\Omega}; \mathbb{M}^{m \times d}).$$

It follows from Corollary 3.20 that

$$\inf_{v \in \mathbf{W}^{1,1}(\Omega; \mathbb{R}^m)} \mathcal{F}[\nabla v] = \max_{w^* \in \mathbf{W}_0^{\text{div}, \infty}(\Omega)} \mathcal{B}[w^*].$$

See [4] where a generalized pairing in $\text{BV}(\Omega) \times \mathbf{W}^{\text{div}, \infty}(\Omega)$ from [2] is used to derive the correspondent saddle-point conditions.

In a similar fashion one may treat the minimization of integral functionals defined on higher-order gradients, $\nabla^k v = \partial^\alpha v$ with $|\alpha| = k$, by considering a generalized ‘‘curl operator’’ (see, e.g., Example 3.10 (d) in [16]).

Example 3.22 (Linear elasticity). Similarly to the case of gradients, one can deal with the relaxation and optimization in $\text{BD}(\Omega)$ of problems of the form

$$v \mapsto \int_{\Omega} f(x, Ev(x)) \, dx, \quad v \in \text{LD}(\Omega),$$

where, for $v \in L^1(\Omega; \mathbb{R}^d)$, $Ev = (Dv + (Dv)^T)/2$ is the distributional symmetric derivative of v , and

$$\text{LD}(\Omega) := \left\{ v \in L^1(\Omega; \mathbb{R}^d) : Ev \in L^1(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) \right\}.$$

In this case, for $\mu \in \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d})$,

$$\mathcal{A}\mu = \text{curl curl } \mu := \left(\sum_{i=1}^d \partial_{ik} u_{ij} + \partial_{ij} u_{ik} - \partial_{jk} u_{ii} - \partial_{ii} u_{jk} \right)_{j,k=1,\dots,d}, \quad 1 \leq i \leq d,$$

is a second-order partial differential operator expressing the St. Venant compatibility conditions and for $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$,

$$\mathcal{B}\mu = E\mu = \frac{1}{2}(\partial_j \mu_i + \partial_i \mu^j), \quad 1 \leq i, j \leq d.$$

Once again, using that Ω is simply connected,

$$\begin{aligned} \ker \mathcal{A} &= \left\{ u \in L^1(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) : \text{curl curl } u = 0 \right\} \\ &= \left\{ Ev : v \in \text{LD}(\Omega) \right\} \\ &= \text{Im } \mathcal{B}. \end{aligned}$$

As direct consequence of Corollary 3.20 we get

$$\inf_{v \in \text{LD}(\Omega)} \mathcal{F}[Ev] = \max_{w^* \in \text{H}_0^{\text{div}, \infty}(\Omega)} \mathcal{B}[w^*],$$

where

$$\text{H}_0^{\text{div}, \infty}(\Omega) := \left\{ w^* \in L^\infty(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) : \text{div } w^* = 0 \text{ in } \mathcal{D}'(\Omega; \mathbb{R}^d), \text{ and } Tw^* = 0 \right\}.$$

See [17, 18] where saddle-point conditions in BD are established for Hencky plasticity models.

Remark 3.23 (Assumptions I). The results in the present section do not make use of Assumption A1.

3.4 The relaxed problem

So far we have not discussed the optimality conditions for problem (\mathcal{P}) . In part, this owes to the fact that (\mathcal{P}) may not necessarily be well-posed. More precisely, due to the lack of compactness of L^1 -bounded sets one must look into the so-called relaxation of the energy \mathcal{F} . The latter has a meaning by extending the basis space to a subspace of the bounded vector-valued Radon measures

$\mathcal{M}(\Omega; \mathbb{R}^N)$. It is well-known that the largest (below \mathcal{F}) lower semicontinuous functional with respect to the weak*-convergence of measures is given by

$$\overline{\mathcal{F}}[\mu] := \inf \left\{ \liminf \mathcal{F}[u_j] : u_j \xrightarrow{*} \mu, u_j \in u_0 + \ker \mathcal{A} \right\}.$$

Under Assumption A1 it is relatively easy to verify that $\overline{\mathcal{F}}$ is again an integral functional:

Proof of Theorem 3.4. Let $\mu \in u_0 + \ker \mathcal{M} \mathcal{A}$. We divide the proof into three parts:

1. *Lower bound.* Let (u_j) be a sequence in $u_0 + \ker \mathcal{A}$ with the property that

$$u_j \mathcal{L}^d \xrightarrow{*} \mu, \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^N).$$

We want to show that

$$\liminf_{j \rightarrow \infty} \mathcal{F}[u_j] \geq \int_{\Omega} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} f^{\infty} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x)$$

for all sequences $(u_j) \subset \ker \mathcal{A} + u_0$ such that $u_j \mathcal{L}^d \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$.

Up to passing to a subsequence, we may assume that

$$A_0 := \liminf_{j \rightarrow \infty} \mathcal{F}[u_j] = \lim_{j \rightarrow \infty} \mathcal{F}[u_j],$$

and $u_j \xrightarrow{*} \bar{\mu}$ in $\mathcal{M}(\bar{\Omega}; \mathbb{R}^N)$ for some measure $\bar{\mu}$ in $\bar{\Omega}$ with $\bar{\mu} \llcorner \Omega \equiv \mu$.

A simple consequence of Lemma 3.7, applied to (u_j) , is that

$$A_0 \geq \int_{\Omega} f \left(x, \frac{d\bar{\mu}}{d\mathcal{L}^d}(x) \right) dx + \int_{\bar{\Omega}} f^{\infty} \left(x, \frac{d\bar{\mu}^s}{d|\bar{\mu}^s|}(x) \right) d|\bar{\mu}^s|(x).$$

Using that $\bar{\mu} \equiv \mu$ on Ω , we further obtain

$$\begin{aligned} A_0 &\geq \int_{\Omega} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) + \int_{\bar{\Omega}} f^{\infty} \left(x, \frac{d\bar{\mu}^s}{d|\bar{\mu}^s|}(x) \right) d|\bar{\mu}^s|(x) \\ &\geq \int_{\Omega} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} f^{\infty} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x), \end{aligned}$$

where in the last inequality we have used strongly the fact that “ $f \geq 0$ ” to neglect the possible concentration of measure at the boundary $\partial\Omega$.

Thus, taking the infimum over all such sequences $u_j \xrightarrow{*} \mu$ we get

$$\overline{\mathcal{F}}[\mu] \geq \int_{\Omega} f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) dx + \int_{\Omega} f^{\infty} \left(x, \frac{d\mu^s}{d|\mu^s|}(x) \right) d|\mu^s|(x).$$

This proves the lower bound.

2. *Upper bound.* We show that there exists a sequence $(u_j) \subset u_0 + \ker \mathcal{A}$ with $u_j \mathcal{L}^d \xrightarrow{*} \mu$ and such

that

$$\limsup_{j \rightarrow \infty} \mathcal{F}[u_j] \leq \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x).$$

This time we will make use of A1 and Theorem 3.10. Indeed, since $\mu - u_0 \in \ker_{\mathcal{M}} \mathcal{A}$, we may find a sequence $(v_j) \subset \ker \mathcal{A}$ that area-strict converges to $\mu - u_0$. Moreover, since area-strict convergence is stable under translations, the sequence $u_j := u_0 + v_j$ also area-strict converges to μ .

A direct consequence of Theorem 3.10 is that

$$\lim_{j \rightarrow \infty} \mathcal{F}[u_j] = \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x).$$

Therefore, plugging the sequence (u_j) in the definition of $\overline{\mathcal{F}}$ yields

$$\overline{\mathcal{F}}[\mu] \leq \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x).$$

This proves the upper bound.

3. *Conclusion.* A combination of the lower and upper bounds yields that

$$\overline{\mathcal{F}}[\mu] = \int_{\Omega} f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_{\Omega} f^{\infty}\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x),$$

for all $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$. □

Example 3.24 (Relaxation in BV). The space $\text{BV}(\Omega; \mathbb{R}^m)$ of *functions of bounded deformation* is the subspace of $v \in L^1(\Omega; \mathbb{R}^m)$ of functions whose distributional derivative Dv is (can be represented) a finite Radon measure. That is,

$$\text{BV}(\Omega; \mathbb{R}^m) := \left\{ v \in L^1(\Omega; \mathbb{R}^m) : Dv \in \mathcal{M}(\Omega; \mathbb{M}^{m \times d}) \right\}.$$

On simply connected and Lipschitz domains $\Omega \subset \mathbb{R}^d$, we may apply this relaxation result to minimization of problems of the form

$$v \mapsto \mathcal{F}[\nabla v] := \int_{\Omega} f(x, \nabla v(x)) dx, \quad v \in \mathbf{W}^{1,1}(\Omega; \mathbb{R}^m).$$

Indeed, since Ω is simply connected then Remark 3.12 guarantees that

$$\ker_{\mathcal{M}} \mathcal{A} = \left\{ Dv : v \in \text{BV}(\Omega; \mathbb{R}^m) \right\},$$

and therefore Assumption A1 is automatically fulfilled for $\mathcal{A} = \text{curl}$ and hence Theorem 4 yields that the lower semicontinuous envelope of \mathcal{F} with respect to weak* convergence in $\text{BV}(\Omega; \mathbb{R}^m)$ is given

by

$$v \mapsto \overline{\mathcal{F}}[Dv] := \int_{\Omega} f(x, \nabla v(x)) \, dx + \int_{\Omega} f^{\infty} \left(x, \frac{dD^s v}{d|D^s v|}(x) \right) d|D^s v|(x), \quad v \in \mathbf{BV}(\Omega; \mathbb{R}^m).$$

Here,

$$Dv = D^a v + D^s v = \nabla v \mathcal{L}^d + D^s v$$

is the Lebesgue–Radon–Nykodým decomposition of Dv .

Example 3.25 (Relaxation in BD). In the context of linear elasticity and the minimization of linear-growth integral functionals, it is relevant to understand the space $\mathbf{BD}(\Omega)$ of *functions of bounded deformation* which is conformed by functions $v \in L^1(\Omega; \mathbb{R}^d)$ whose distributional *symmetrized derivative*

$$Ev := \frac{1}{2}(Dv + Dv^T)$$

is (or can be represented) a finite Radon measure. In other words,

$$\mathbf{BD}(\Omega) := \left\{ v \in L^1(\Omega; \mathbb{R}^d) : Ev \in \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{d \times d}) \right\};$$

and similarly to the case of gradients, we split

$$Ev = E^a v + E^s v = \mathcal{E} v \mathcal{L}^d + E^s v.$$

On simply connected domains $\Omega \subset \mathbb{R}^d$ it further holds (see Remark 3.12) that

$$\ker_{\mathcal{M}}(\text{curl curl}) = \left\{ Ev : v \in \mathbf{BD}(\Omega) \right\},$$

where curl curl is the second-order operator defined in Example 3.22. Moreover, by Remark 3.12 Assumption A1 is fulfilled for $\mathcal{A} = \text{curl curl}$ and therefore the lower semicontinuous envelope of the functional

$$v \mapsto \int_{\Omega} f(x, \mathcal{E} v(x)) \, dx, \quad v \in \mathbf{LD}(\Omega),$$

with respect to weak* convergence in $\mathbf{BD}(\Omega)$ is given by

$$v \mapsto \int_{\Omega} f(x, \mathcal{E} v(x)) \, dx + \int_{\Omega} f^{\infty} \left(x, \frac{dE^s v}{d|E^s v|}(x) \right) d|E^s v|(x), \quad v \in \mathbf{BD}(\Omega).$$

We conclude this section with a few remarks on the possible concentration of measure at the boundary.

Remark 3.26 (Concentration of measure at the boundary). (i) If one assumes that $\mathcal{L}^d(\partial\Omega) = 0$ then, only concentration of measure at $\partial\Omega$ might undermine the lower semicontinuity. Indeed,

in proving the lower bound we have used the positivity of f to disregard positive concentration of measure at $\partial\Omega$. For an arbitrary (\mathcal{A} -free) sequence (u_j) with $u_j \xrightarrow{*} \mu$ in $\mathcal{M}(\overline{\Omega}; \mathbb{R}^N)$, it might not hold that $|\mu|(\partial\Omega) = 0$. Therefore, using the positivity of f (as in the proof of the lower bound) it may occur that the limes inferior inequality is strict, namely that

$$\liminf_{j \rightarrow \infty} \overline{\mathcal{F}}[u_j] > \overline{\mathcal{F}}[\mu].$$

(ii) However, in terms of the relaxation, Assumption A1 — which can be understood as a density assumption — guarantees that the space of \mathcal{A} -free integrable sequences is sufficiently large to ensure the equality in the limes inferior above is reached for some other sequence. In a way, A1 tells us that for every $\mu \in \ker_{\mathcal{M}} \mathcal{A}$ there exists an \mathcal{A} -free recovery sequence (u_j) which does not concentrate on $\partial\Omega$, this is,

$$|u_j|_{\mathcal{L}^d} \xrightarrow{*} \Lambda, \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad \text{and} \quad \Lambda(\partial\Omega) = 0.$$

Notice also that the proof of the upper bound does not rely on the positivity of f .

(iii) If the positivity of f is dispensed with from in the assumptions, or equivalently if we consider a general signed integrand $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$, there is no hope for lower semicontinuity to hold. The underlying idea is that while f is positive (or bounded from below) only mass can be gained at $\partial\Omega$, which in turn does not affect the lower bound. On the contrary, if f is unbounded from below, the appearance of negative energy at $\partial\Omega$ might not be carried by the limit measure (compare with Example 3.8).

Remark 3.27 (Existence of solutions). Under standard coercivity of the integrand, namely that

$$\frac{1}{M}(|z| - 1) \leq f(x, z) \quad \text{for all } (x, z) \in \Omega \times \mathbb{R}^N.$$

It is relatively easy to check by a diagonal argument that $\overline{\mathcal{F}}$ is actually weak* lower semicontinuous in $u_0 + \ker_{\mathcal{M}} \mathcal{A}$:

Let $\mu_j, \mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ be \mathcal{A} -free measures such that $\mu_j \xrightarrow{*} \mu$. For each $j \in \mathbb{N}$ let $(u_{j_m})_m \subset u_0 + \ker \mathcal{A}$ be a sequence of functions which area-strict converges (as measures) to μ_j so that

$$\infty > M(\mathcal{L}^d(\Omega) + \liminf_{j \rightarrow \infty} |\mu_j|(\Omega)) \geq \liminf_{j \rightarrow \infty} \overline{\mathcal{F}}[\mu_j] = \liminf_{j \rightarrow \infty} \left(\lim_{m \rightarrow \infty} \mathcal{F}[u_{j_m}] \right).$$

It follows from the bound

$$\mathcal{F}[u] \geq \frac{1}{M} (\|u\|_{L^1(\Omega)} - \mathcal{L}^d(\Omega)),$$

that $\sup_{j,m} \|u_{j_m}\|_{L^1(\Omega)} < \infty$. Hence, we might extract a diagonal sequence verifying the following properties:

$$u_{m(j)} := u_{j_m} \xrightarrow{*} \mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A} \quad \text{and} \quad \mathcal{F}[u_{m(j)}] + O(j) = \overline{\mathcal{F}}[\mu_j],$$

where $O(j) \rightarrow 0$ as $j \rightarrow \infty$. The sought lower semicontinuity is then an easy consequence of the lower

bound inequality in the proof of Theorem 3.4,

$$\liminf_{j \rightarrow \infty} \overline{\mathcal{F}}[\mu_j] = \liminf_{j \rightarrow \infty} (\mathcal{F}[u_{m(j)}] + O(j)) \geq \overline{\mathcal{F}}[\mu].$$

Notice that the coercivity assumption on the integrand is crucial for the diagonal argument to work; otherwise, we might not be able to guarantee the weak*-compactness of arbitrary diagonal sequences.

As soon as weak* lower semicontinuity of $\overline{\mathcal{F}}$ is established, we observe (again by coercivity) that minimizing sequences are weak*-bounded (and thus weak* pre-compact). The direct method can be then applied to prove existence of solutions of $(\overline{\mathcal{P}})$.

3.5 The pairing $[[\mu, w^*]]$ and the optimality conditions

The pointwise product $(\mu \cdot v^*)$ of two functions, $\mu \in u_0 + \ker \mathcal{A}$ and $v^* \in (\ker \mathcal{A})^\perp$, may be regarded as the bounded Radon that takes the values

$$B \mapsto \langle \mu, v^* \rangle(B) := \int_{B \cap \Omega} \mu(x) \cdot v^*(x) \, dx, \quad B \subset \mathbb{R}^N \text{ Borel set.}$$

In general, if $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ is only assumed to be vector-valued Radon measure, one cannot simply give a notion to the inner product of μ and v^* (even in the sense of distributions). However, following the interests of our minimization problem, one may define the following generalized pairing by setting

$$[[\mu, v^*]] := \left\{ \lambda \in \mathcal{M}(\Omega) : \exists (u_j) \subset u_0 + \ker \mathcal{A} \text{ such that } \right. \\ \left. (u_j \cdot v^*) \mathcal{L}^d \xrightarrow{*} \lambda \text{ and } (u_j \mathcal{L}^d) \text{ area-strict converges to } \mu \right\}$$

In this way, the set $[[\mu, w^*]]$ contains information on the concentration effects of sequences of the form $(u_j \cdot w^*)$.

The next lines are dedicated to derive the basic properties $[[\mu, w^*]]$.

Theorem 3.28. *Let $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ and let $w^* \in (\ker \mathcal{A})^\perp$. Then*

$$|\lambda|(\omega) \leq |\mu|(\omega) \|w^*\|_{L^\infty(\omega)} \quad \text{for every Borel set } \omega \subset \Omega,$$

for all $\lambda \in [[\mu, w^*]]$.

Proof. Let $\lambda \in [[\mu, w^*]]$. By definition, there exists a sequence of functions $(u_j) \subset L^1(\Omega; \mathbb{R}^N)$ for which the measures $(u_j \mathcal{L}^d)$ area-strict converge to μ and are such that $(u_j \cdot w^*) \mathcal{L}^d \xrightarrow{*} \lambda$. Hence,

$$\liminf_{j \rightarrow \infty} |\langle u_j, w^* \rangle|(\omega) \geq |\lambda|(\omega), \quad \text{for every open set } \omega \subset \Omega. \quad (3.6)$$

On the other hand, by Hölder's inequality, we get the upper bound

$$|\langle u_j, w^* \rangle|(\omega) \leq \|u_j\|(\omega) \|w^*\|_{L^\infty(\omega)}, \quad \text{for every open set } \omega \subset \Omega, \quad (3.7)$$

and every $j \in \mathbb{N}$. Plugging (3.6) into (3.7) and taking the limit as $j \rightarrow \infty$ we get, by Theorem 3.10 (applied to $f(z) = |z|$), that

$$|\lambda|(\omega) \leq |\mu|(\omega) \|w\|_{L^\infty}(\omega), \quad \text{for every open set } \omega \subset \Omega \text{ with } |\mu|(\partial\omega) = 0.$$

The assertion for general Borel sets follows by a density argument. \square

Corollary 3.29. *Let $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ and let $w^* \in (\ker \mathcal{A})^\perp$. If $\lambda \in \llbracket \mu, w^* \rrbracket$, then the Radon measures λ and $|\lambda|$ are absolutely continuous with respect to the measure $|\mu|$ in Ω . Moreover, an application of the Radon-Nikodým differentiation theorem yields*

$$\left\| \frac{d\lambda}{d|\mu|} \right\|_{L^\infty_{|\mu|}} \leq \left\| \frac{d|\lambda|}{d|\mu|} \right\|_{L^\infty_{|\mu|}} \leq \|w^*\|_{L^\infty}.$$

The following proposition plays a crucial role on proving the generalized saddle-point conditions; it characterizes the absolutely continuous part of elements in $\llbracket \mu, w^* \rrbracket$ and gives an upper bound for the density of its singular part.

Theorem 3.30. *Let $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ and $w^* \in (\ker \mathcal{A})^\perp$. If $\lambda \in \llbracket \mu, w^* \rrbracket$ and $\mathcal{R}[w^*] > -\infty$, then*

$$\frac{d\lambda}{d|\mu^s|}(x) \leq f^\infty \left(x, \frac{d\mu}{d|\mu^s|}(x) \right), \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega, \quad (3.8)$$

and

$$\frac{d\lambda}{d\mathcal{L}^d}(x) = \frac{d\mu}{d\mathcal{L}^d}(x) \cdot w^*(x), \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega. \quad (3.9)$$

Proof. Let $\lambda \in \llbracket \mu, w^* \rrbracket$. By definition we may find sequence $(u_j) \subset L^1(\Omega; \mathbb{R}^N)$ that area-strict converges to μ in the sense of Radon measures, i.e., such that

$$u_j \mathcal{L}^d \xrightarrow{*} \mu \in \mathcal{M}(\Omega; \mathbb{R}^N), \quad \langle u_j \mathcal{L}^d \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega),$$

for which

$$(u_j \cdot w^*) \mathcal{L}^d \xrightarrow{*} \lambda, \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^N).$$

Let $x_0 \in (\text{supp } \lambda^s) \cap \Omega$ be a point with the following properties:

$$\frac{d\mu^s}{d|\mu^s|}(x_0) = \frac{d\mu}{d|\mu|}(x_0) = \lim_{r \downarrow 0} \frac{\mu(B_r(x_0))}{|\mu^s|(B_r(x_0))} < \infty, \quad (3.10)$$

$$\lim_{r \downarrow 0} \frac{\int_{B_r(x_0)} \left| \frac{d\mu}{d\mathcal{L}^d}(x) \right| dx}{|\mu^s|(B_r(x_0))} = 0, \quad \lim_{r \downarrow 0} \frac{\int_{B_r(x_0)} \frac{d\lambda}{d\mathcal{L}^d}(x) dx}{|\mu^s|(B_r(x_0))} = 0 \quad (3.11)$$

$$\frac{d\lambda}{d|\mu^s|}(x_0) = \lim_{r \downarrow 0} \frac{\lambda(B_r(x_0))}{|\mu^s|(B_r(x_0))} < \infty. \quad (3.12)$$

Using the principle

$$f^\infty(x, z) \geq \sup \{ z \cdot z^* : z^* \in \mathbb{R}^N, f^*(x, z^*) < +\infty \}$$

and the assumption that $|f^*(x, w^*(x))|$ is essentially bounded by for \mathcal{L}^d a.e. $x \in \Omega$ (here we use that $\mathcal{R}[w^*] > -\infty$), we deduce the simple inequality

$$\int_{B_s(x_0)} f^\infty(x, u_j(x)) dx \geq \int_{B_s(x_0)} u_j \cdot w^* dx, \quad (3.13)$$

for every $s \in (0, \text{dist}(x_0, \partial\Omega))$. We let $j \rightarrow \infty$ on both sides of the inequality to get

$$\lim_{j \rightarrow \infty} \int_{B_s(x_0)} f^\infty(x, u_j(x)) dx \geq \lambda(B_s(x_0)), \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0, \text{dist}(x_0, \partial\Omega)).$$

Recall that $u_j \mathcal{L}^d$ area-strict converges to μ and by construction f^∞ is positively 1-homogeneous in its second argument. Hence, we may apply Theorem 3.10 to the limit in the left hand side of the inequality to obtain

$$\begin{aligned} \frac{1}{|\mu^s|(B_s(x_0))} \int_{B_s(x_0)} f^\infty\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \\ \geq \frac{\lambda(B_s(x_0))}{|\mu^s|(B_s(x_0))}, \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0, \text{dist}(x_0, \partial\Omega)). \end{aligned}$$

Using properties (3.10)-(3.12) we may let $s \downarrow 0$ on the right hand side to deduce that

$$\lim_{s \downarrow 0} \frac{1}{|\mu^s|(B_s(x_0))} \int_{B_s(x_0)} f^\infty\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \geq \frac{d\lambda}{d|\mu^s|}(x_0). \quad (3.14)$$

Moreover, the modulus of continuity of f conveys a similar modulus of continuity for f^∞ , namely that

$$|f^\infty(x, z) - f^\infty(y, z)| \leq \omega(|x - y|)M|z|, \quad \text{for all } x, y \in \Omega \text{ and every } z \in \mathbb{R}^N.$$

Thus, the limit in the left hand side of (3.14) is bounded from above by

$$\lim_{s \downarrow 0} \left(\frac{1}{|\mu^s|(B_s(x_0))} \int_{B_s(x_0)} f^\infty\left(x_0, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) + \omega(s) \cdot M \frac{|\mu|(B_s(x_0))}{|\mu^s|(B_s(x_0))} \right).$$

Using (3.10)-(3.12) and that f^∞ is Lipschitz on its second argument (which follows from the respective Lipschitz continuity of f) we infer from the bound above and (3.14) that

$$f\left(x_0, \frac{d\mu}{d|\mu^s|}(x_0)\right) \geq \frac{d\lambda}{d|\mu^s|}(x_0).$$

The sought statement follows by observing that (3.10)-(3.12) hold simultaneously in Ω for $|\mu^s|$ -a.e. $x_0 \in \Omega$.

For the equality on Lebesgue points, let $x_0 \in \Omega$ be such that

$$\lim_{r \downarrow 0} \frac{|\mu^s|(B_r(x_0))}{r^d} = 0, \quad (3.15)$$

$$\lim_{r \downarrow 0} \frac{1}{r^N} \int_{B_r(x_0)} \left| \frac{d\mu}{d\mathcal{L}^d}(x) - \frac{d\mu}{d\mathcal{L}^d}(x_0) \right| dx = 0, \quad (3.16)$$

and

$$\frac{d(\frac{d\mu}{d\mathcal{L}^d} \cdot w^*)}{d\mathcal{L}^d}(x_0) = \frac{d\mu}{d\mathcal{L}^d}(x_0) \cdot w^*(x_0). \quad (3.17)$$

Set

$$P_0 := \frac{d\mu}{d\mathcal{L}^d}(x_0).$$

Then, by definition, for a.e. $r \in (0, \text{dist}(x_0, \partial\Omega))$ it holds that

$$\begin{aligned} & \left| \lambda(B_r(x_0)) - \int_{B_r(x_0)} P_0 \cdot w^* dx \right| \\ &= \lim_{n \rightarrow \infty} \left| \int_{B_r(x_0)} u_j \cdot w^* dx - \int_{B_r(x_0)} P_0 \cdot w^* dx \right| \\ &\leq \|w^*\|_{L^\infty} \cdot \lim_{j \rightarrow \infty} \int_{B_r(x_0)} |u_j - P_0| dx \\ &\leq \|w^*\|_{L^\infty} \cdot \left(\int_{B_r(x_0)} \left| \frac{d\mu}{d\mathcal{L}^d} - P_0 \right| dx \right. \\ &\quad \left. + |\mu^s|(B_r(x_0)) \right) = o(r^d), \end{aligned}$$

where in the last step we have used that $(u_j - P_0)\mathcal{L}^d$ area-strict converges to $\mu - P_0\mathcal{L}^d$. This follows from Theorem 3.10 and the fact that $(f(\cdot - P_0))^\infty = f^\infty(\cdot)$.

Essentially, this means that computing the Radon-Nikodým derivative of λ at x_0 is equivalent to calculate the correspondent derivative of the measure $(\frac{d\mu}{d\mathcal{L}^d} \cdot w^*)\mathcal{L}^d$ at x_0 . Under this reasoning we use (3.17) to calculate

$$\frac{d\lambda}{d\mathcal{L}^d}(x_0) = \frac{d\mu}{d\mathcal{L}^d}(x_0) \cdot w^*(x_0).$$

Properties (3.15)-(3.17) hold simultaneously for \mathcal{L}^d -a.e. $x_0 \in \Omega$ from where (3.9) follows. \square

Remark 3.31. If w^* is $|\mu^s|$ -measurable, then one can prove (by a similar argument to the one used in the proof of (3.9)) that

$$\frac{d\lambda}{d|\mu^s|}(x_0) = \frac{d\mu}{d|\mu^s|}(x_0) \cdot w^*(x_0), \quad \text{for } |\mu^s|\text{-a.e. } x_0 \in \Omega.$$

For a sequence $(u_j) \subset L(\Omega; \mathbb{R}^N)$ that area-strict converges to some $\mu \in \ker \mathcal{M} \mathcal{A}$ it is automatic to verify, by means of Theorem 3.10, that

$$f(\cdot, u_j) \mathcal{L}^d \xrightarrow{*} f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s| \quad (3.18)$$

in $\mathcal{M}^+(\Omega)$. If one dispenses the assumption that (u_j) area-strict converges μ and only assumes that $u_j \mathcal{L}^d \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$ (or even the stronger strict convergence) the convergence (3.18) may not hold as already observed in Remark 3.11. However, as the next proposition asserts, it does hold for

minimizing sequences if one assumes that the integrand is coercive.

Theorem 3.32 (Uniqueness and improved convergence). *Let $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy the assumptions of Theorem 3.4 and further assume that it is coercive, i.e.,*

$$\frac{1}{M}(1 - |z|) \leq f(x, z) \quad \text{for all } x \in \Omega, z \in \mathbb{R}^N.$$

Let $(u_j) \subset u_0 + \ker \mathcal{A}$ be a minimizing sequence of problem (\mathcal{P}) with $u_j \mathcal{L}^d \xrightarrow{} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$. Then μ is a generalized minimizer of (\mathcal{P}) and the sequence of real-valued radon measures $(f(\cdot, u_j) \mathcal{L}^d \llcorner \Omega)$ weak* converges on Ω , in the sense of Radon measures, to the measure*

$$f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|.$$

Even more, if $f(x, \cdot)$ and $f^\infty(x, \cdot)$ are strictly convex for all $x \in \Omega$, then μ is the unique minimizer of (\mathcal{P}) and $u_j \mathcal{L}^d$ area-strict converges to μ in $\mathcal{M}(\Omega; \mathbb{R}^N)$.

Remark 3.33. Recall that strict convexity for a positively 1-homogeneous function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ — also called strictly convex on norms — is equivalent to the convexity of its unit ball, that is,

$$g(z_1) = g(z_2) = g(z_1 + z_2), \quad \text{for } |z_1| = |z_2| = 1$$

implies

$$z_1 = z_2.$$

In general, strict convexity of a function g does not imply strict convexity of g^∞ (see Remark 5.4 in [1]).

Proof. Set $\Lambda_j \in \mathcal{M}^+(\Omega)$ to be the real-valued Radon measure defined as

$$\Lambda_j(B) := \int_B f(x, u_j(x)) dx, \quad \text{for any open set } B \subset \Omega.$$

Since (u_j) is a minimizing sequence, it is also L^1 -uniformly bounded (see Remark 3.27) and hence

$$\sup_{j \in \mathbb{N}} |\Lambda_j|(\Omega) < +\infty.$$

We may assume, up to taking a subsequence (not re-labeled), that there exist positive Radon measures $\Lambda, \sigma \in \mathcal{M}^+(\Omega)$ for which

$$\Lambda_j \xrightarrow{*} \Lambda, \quad \text{and} \quad |u_j| \mathcal{L}^d \xrightarrow{*} \sigma \quad \text{in } \mathcal{M}^+(\Omega).$$

We do the following observation: the conclusion of Lemma 3.7 also holds any arbitrary open set

$B \subset \Omega$ with $\mathcal{L}^d(\partial B) = 0$. Hence,

$$\Lambda(B) = \lim_{j \rightarrow \infty} \Lambda_j(B) \geq \overline{\mathcal{F}}(\mu, B),$$

for every open subset B of Ω with $\Lambda(\partial B) = \sigma(\partial B) = 0$, and where we have set $\overline{\mathcal{F}}(\mu, \cdot)$ to be the Radon measure that takes the values

$$\overline{\mathcal{F}}(\mu, B) := \int_B f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) dx + \int_B f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) d|\mu^s|(x),$$

on open sets $B \subset \Omega$. Using a density argument of the class of open sets B with $(\mathcal{L}^d + |\sigma|)(\partial B) = 0$ in the Borel σ -algebra $\mathcal{B}(\Omega)$, we conclude that

$$\Lambda \geq \overline{\mathcal{F}}(\mu, \cdot), \quad \text{in the sense of real-valued Radon measures.} \quad (3.19)$$

So far we have not used the fact that (u_j) is a minimizing sequence. Recall that, by definition, this is equivalent to

$$\Lambda_j(\Omega) \rightarrow \Lambda(\Omega) = \overline{\mathcal{F}}(\mu, \Omega) = \inf_{u_0 + \ker \mathcal{A}} \mathcal{F}.$$

The mass convergence above and (3.19) are sufficient conditions for Λ and $\overline{\mathcal{F}}(\mu, \cdot)$ to represent the same Radon measure in $\mathcal{M}(\Omega)$, i.e.,

$$\Lambda = \overline{\mathcal{F}}(\mu, \cdot), \quad \text{in } \mathcal{M}(\Omega).$$

Since the passing to a convergent subsequence was arbitrary, this proves

$$f(\cdot, u_j) \mathcal{L}^d \xrightarrow{*} f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s| \quad \text{in } \mathcal{M}^+(\Omega).$$

To see that for strictly convex integrands μ is the unique minimizer of $(\overline{\mathcal{P}})$, one simply uses the strict convexity of f and f^∞ , and the fact that $\ker_{\mathcal{M}} \mathcal{A}$ is a convex space.

The improvement of convergence to area-strict relies on the theory of generalized Young measures. Its proof is a direct consequence of Theorem 2.5 and Lemma in [1], and Definition 3.9. \square

Remark 3.34. The conclusions of Theorem 3.32 do not rely on Assumption A1. In return, it establishes that coercivity of the integrand is a sufficient condition to ensure the existence of at least one minimizing sequence (u_j) of (\mathcal{P}) and at least one generalized solution μ (generated by (u_j)) of $(\overline{\mathcal{P}})$.

Remark 3.35. The improved convergence for minimizing sequences of strictly convex integrands plays no role in our characterization of the extremality conditions of problems (\mathcal{P}) and (\mathcal{P}^*) . Nevertheless, we have decided to include as it is a standard result for applications in calculus of variations.

We prove our main result:

Proof of Theorem 3.5.

Step 1. $\llbracket \mu, w^* \rrbracket \neq \emptyset$. Let $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ be a generalized solution of problem (\mathcal{P}) . Let (\tilde{u}_j) be the sequence provided by Assumption A1 for which $\tilde{u}_j \mathcal{L}^d$ area-strict converges to $\mu - u_0 \mathcal{L}^d$. Notice that the sequence $(u_j) := (\tilde{u}_j + u_0)$ is a minimizing sequence of (\mathcal{P}) which area-strict converges to μ so that $\llbracket \mu, w^* \rrbracket$ is not the empty set.

Step 2. Necessity. Fix $\lambda \in \llbracket \mu, w^* \rrbracket$. Let $(u_j) \subset u_0 + \ker \mathcal{A}$ be a sequence that area-strict converges to μ and such that $u_j \cdot w^* \mathcal{L}^d$ generates λ . By Theorem 3.10 and the minimality of μ it also holds that (u_j) is a minimizing sequence for problem (\mathcal{P}) .

In return, Theorem 3.32 implies that the sequence of measures $(f(\cdot, u_j) \mathcal{L}^d \llcorner \Omega)$ weak* converges to the Radon measure

$$f\left(x, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s|$$

in $\mathcal{M}^+(\Omega)$. Since f is convex (in its second argument) and lower semicontinuous, it must hold that

$$f^{**}(x, \cdot) = f(x, \cdot), \quad \text{for } \mathcal{L}^d\text{-a.e. every } x \in \Omega.$$

Hence,

$$(f(\cdot, u_j) \mathcal{L}^d)(B) \geq \int_B u_j \cdot w^* dx - \int_B f^*(x, w^*) dx, \quad (3.20)$$

for every Borel subset $B \subset \Omega$. Therefore, by Theorem 3.32 and (3.20) we get (by letting $j \rightarrow \infty$) that

$$\begin{aligned} f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s| \\ \geq \lambda - f^*(\cdot, w^*) \mathcal{L}^d \llcorner \Omega, \end{aligned} \quad (3.21)$$

in the sense of measures. Also, by the equality in Proposition 3.3 we know that $\overline{\mathcal{F}}[\mu] = \mathcal{R}[w^*]$ so that

$$\begin{aligned} & \left(f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s| \right) (\Omega) \\ &= \overline{\mathcal{F}}[\mu] = \mathcal{R}[w^*] \\ &= \langle w^*, u_0 \rangle - (f^*(\cdot, w^*) \mathcal{L}^d)(\Omega) \\ &= (\lambda - f^*(\cdot, w^*) \mathcal{L}^d)(\Omega), \end{aligned}$$

where in the last equality we used that $\lambda(\Omega) = \langle w^*, u_0 \rangle$ for any $\lambda \in \llbracket \mu, w^* \rrbracket$ with $\mu \in u_0 + \ker_{\mathcal{M}} \mathcal{A}$ and $v^* \in (\ker \mathcal{A})^\perp$. The inequality, as measures, in (3.21) and the equality of their total mass in the last formula tells us that the measures in question must be agree as elements of $\mathcal{M}(\Omega)$. In other words,

$$f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s| = \lambda - f^*(\cdot, w^*) \mathcal{L}^d \llcorner \Omega,$$

as measures in $\mathcal{M}(\Omega)$. Finally, we recall the characterization from Theorem 3.30 so that

$$f\left(x, \frac{d\mu}{d\mathcal{L}^d}(x)\right) + f^*(x, w^*(x)) = \frac{d\mu}{d\mathcal{L}^d}(x) \cdot w^*(x),$$

for \mathcal{L}^d -a.e. $x \in \Omega$, whence it follows that

$$\frac{d\lambda}{d|\mu^s|}(x) = f^\infty\left(\frac{d\mu}{d|\mu^s|}(x)\right) \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega.$$

The latter equalities fully characterize $\llbracket \mu, w^* \rrbracket$ by means of Corollary 3.29 and the Radon-Nikodým Decomposition Theorem. In particular, $\llbracket \mu, w^* \rrbracket$ is the singleton

$$\left\{ \left(\frac{d\mu}{d\mathcal{L}^d} \cdot w^* \right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s| \right\}.$$

This proves that (i) implies (ii).

Step 3. Sufficiency. To show that (ii) implies (i) note that we always have $\overline{\mathcal{F}} \geq \sup \mathcal{R}$ (on their respective domains). Hence, it suffices to show that

$$\overline{\mathcal{F}}[\mu] \leq \mathcal{R}[w^*]. \quad (3.22)$$

Indeed, the inequality above implies that μ solves problem (\mathcal{P}) and w^* solves (the relaxation of) problem (\mathcal{P}^*) . To prove (3.22) let $(u_j) \subset u_0 + \ker \mathcal{A}$ be the (area-strict convergent) recovery sequence for μ in the proof Theorem 3.4 so that

$$f(\cdot, u_j) \mathcal{L}^d \llcorner \Omega \xrightarrow{*} f\left(\cdot, \frac{d\mu}{d\mathcal{L}^d}\right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s|.$$

By assumption

$$\lambda_j := (u_j \cdot w^*) \mathcal{L}^d \xrightarrow{*} \lambda := \left(\frac{d\mu}{d\mathcal{L}^d} \cdot w^* \right) \mathcal{L}^d \llcorner \Omega + f^\infty\left(\cdot, \frac{d\mu}{d|\mu^s|}\right) |\mu^s|$$

in $\mathcal{M}(\Omega)$, and therefore using that $\lambda_j(\Omega) = \langle w^*, u_0 \rangle$ for all $j \in \mathbb{N}$, we get that $\lambda(\Omega) = \langle w^*, u_0 \rangle$. The pointwise identities from (ii) then yield

$$\begin{aligned} \mathcal{R}[w^*] &= - \int_{\Omega} f^*(x, w^*) \, dx + \lambda(\Omega) \\ &= - \int_{\Omega} f^*(x, w^*) \, dx + \int_{\Omega} \frac{d\mu}{d\mathcal{L}^d}(x) \cdot w^*(x) \, dx \\ &\quad + \int_{\Omega} f^\infty\left(x, \frac{d\mu}{d|\mu^s|}(x)\right) d|\mu^s|(x) \\ &= \overline{\mathcal{F}}[\mu]. \end{aligned}$$

This proves (3.22). □

Remark 3.36 (Optimality conditions II). In the case that there exists a solution w^* of (\mathcal{P}^*) with

substantially better regularity than the one originally posed by being admissible to its variational problem, say, for example, $w^* \in C(\Omega; \mathbb{R}^N)$ or $w^* \in L^\infty_{|\mu|}(\Omega; \mathbb{R}^n)$. Then, it is easy to verify (cf. Remark 3.31) that

$$f^\infty \left(x, \frac{d\mu^s}{d|\mu^s|}(x_0) \right) = \frac{d\mu}{d|\mu^s|}(x) \cdot w^*(x) \quad \text{for } |\mu^s|\text{-a.e. } x \in \Omega,$$

and

$$f \left(x, \frac{d\mu}{d\mathcal{L}^d}(x) \right) + f^*(x, w^*(x)) = \frac{d\mu}{d\mathcal{L}^d}(x) \cdot w^*(x) \quad \text{for } \mathcal{L}^d\text{-a.e. in } \Omega,$$

are also equivalent to (i) and (ii) in Theorem 3.5.

Corollary 3.37 (BD(Ω)-optimization). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, and simply connected set and let $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy the assumptions of Theorem 3.5. Then the following conditions are equivalent:*

(i) $\tilde{v} \in \text{BD}(\Omega)$ is a minimizer of the functional

$$\begin{aligned} v \mapsto & \int_{\Omega} f(x, \mathcal{E}v(x)) \, dx \\ & + \int_{\Omega} f^\infty \left(x, \frac{dE^s v}{d|E^s v|}(x) \right) \, d|E^s v|(x), \quad v \in \text{BD}(\Omega), \end{aligned}$$

and $\sigma^* \in L^\infty(\Omega; \mathbb{M}_{\text{sym}}^{d \times d})$ is a symmetric div-free tensor with $\text{Tr}(\sigma^* \cdot \nu_\Omega) = 0$ that maximizes the functional

$$w^* \mapsto - \int_{\Omega} f^*(x, w^*(x)) \, dx, \quad w^* \in H_0^{\text{div}, \infty}(\Omega; \mathbb{R}^d).$$

(ii) The measure

$$(\mathcal{E}\tilde{v} \cdot \sigma^*) \mathcal{L}^d \llcorner \Omega + f^\infty \left(\cdot, \frac{dE^s \tilde{v}}{d|E^s \tilde{v}|} \right) |E^s \tilde{v}|$$

coincides with $\lambda \in \mathcal{M}(\Omega)$, the measure uniquely determined by the property

$$\int_{\Omega} \varphi(x) \, d\lambda(x) = -\frac{1}{2} \int_{\Omega} \sigma^*(u \otimes \nabla \varphi + \nabla \varphi \otimes u) \, dx, \quad \text{for all } \varphi \in C_c(\Omega).$$

In particular,

$$\frac{d\lambda}{d|E^s \tilde{v}|}(x) = f^\infty \left(x, \frac{dE^s \tilde{v}}{d|E^s \tilde{v}|} \right) \quad \text{for } |E^s \tilde{v}|\text{-a.e. } x \in \Omega.$$

Moreover,

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^d}(x) &= \mathcal{E}\tilde{v}(x) \cdot \sigma^*(x) \\ &= f(x, \mathcal{E}\tilde{v}(x)) + f^*(x, \sigma^*(x)) \end{aligned}$$

for \mathcal{L}^d -a.e. in $x \in \Omega$.

A similar characterization holds for $\mathcal{A} = \text{curl}$ associated with the minimization on the space BV.

3.6 An application to low-volume fraction optimal design

The model

Consider the physical problem of thermal or electrical conductivity in a given (simply connected) medium $\Omega \subset \mathbb{R}^2$. The conductivity is represented by a positive definite matrix $\sigma_A(x)$ oscillating between two constituent media depending on an indicator set A , in this case with different conductivities α and β with $\alpha > \beta > 0$. More precisely, for a given set $A \subset \mathbb{R}^2$ we let

$$\sigma_A(x) = \chi_A(x)\alpha \text{id}_{\mathbb{R}^2 \times 2} + (1 - \chi_A(x))\beta \text{id}_{\mathbb{R}^2 \times 2}.$$

If we let $\tau \in L^\infty(\Omega)$ be the derivative of the charge of the body, and we impose zero boundary conditions (for simplicity), the model of the conductivity reads

$$\begin{aligned} -\text{div}(\sigma_A \nabla w_A) &= \tau \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where w_A is the electric potential or temperature associated to the conductivity σ_A . The dissipated thermal energy

$$\int_{\Omega} \tau w_A \, dx$$

provides a measure of the global conductivity in Ω . A common problem in optimal design is to find the best conductive material at a low cost in the following sense: production costs or volume constraints are handled by introducing a Lagrange multiplier $\gamma > 0$ on the expensive material α ,

$$J_{\alpha,\beta,\gamma}(A) := \frac{1}{2} \left(\int_{\Omega} \tau w_A \, dx + \gamma \int_{\Omega} \chi_A \, dx \right).$$

How can one mathematically treat the degenerate problem as $\alpha = \infty$ (loss of uniform boundedness)? In other words, what happens as we let $\alpha \rightarrow \infty$ and naturally its cost $\gamma \rightarrow \infty$. This can be considered as an attempt to model perfect isolators. A first step is to understand the meaningful scaling between the parameters α and γ . An easy calculation (see [6]) shows that the only meaningful scaling, up to multiplicative constants, occurs when $\gamma \sim \alpha$. Without loss of generality let us fix $\beta = 1$ and $\gamma = \alpha - 1$. We consider the problem

$$\text{minimize } J_\alpha := J_{\alpha,1,\alpha-1} \text{ among the class of Borel sets } A \subset \Omega. \tag{p_\alpha}$$

In general this problem is not well-posed due to the highly oscillatory behavior of minimizing sequences (and the non-convex nature of the energy). However, according to Murat and Tartar [22, 23], the optimality conditions of the relaxation (p_α) can also be interpreted as the Euler equations corresponding to the minimization problem:

$$\text{minimize } F_\alpha \text{ in the affine space } \left\{ u \in L^2(\Omega; \mathbb{R}^2) : -\text{div } u = \tau \text{ in } \Omega \right\}.$$

Here,

$$F_\alpha[u] := \int_{\Omega} \Psi_\alpha(u(x)) \, dx$$

where, for $\alpha \in [1, \infty]$,

$$\Psi_\alpha(z) = \begin{cases} \frac{|z|^2}{2} & \text{if } |z| \leq 1 \\ |z| - \frac{1}{2} & \text{if } 1 \leq |z| < \alpha \\ \frac{|z|^2}{2\alpha} - \frac{\alpha-1}{2} & \text{if } \alpha \leq |z| \end{cases}.$$

The limit problem and its connection to the elasto-plastic torsion problem

In general the proposed model can be considered for an arbitrary dimension $\Omega \subset \mathbb{R}^d$. The rigorous mathematical tool to understand the governing behavior of the limit problem is Γ -convergence (we refer the reader to [13] for a complete introduction to this topic). The Γ -limit of F_α under the side constraint “ $-\operatorname{div} u = \tau$ ” (with respect to the weak* convergence of measures), as $(\alpha, \gamma) \rightarrow (\infty, \infty)$ with $\alpha \sim \gamma$, is given by $F : \mathcal{M}(\Omega; \mathbb{R}^d) \rightarrow \overline{\mathbb{R}}$ defined as

$$\overline{\mathcal{F}}[\mu] := \begin{cases} \int_{\Omega} \Psi\left(\frac{d\mu}{d\mathcal{L}^d}\right) dx + |\mu^s|(\Omega) & \text{if } -\operatorname{div} \mu = \tau \\ +\infty & \text{otherwise} \end{cases},$$

where $\Psi := \Psi_\infty$. We define the problem

$$\text{minimize } \overline{\mathcal{F}} \text{ in } \mathcal{M}(\Omega; \mathbb{R}^d). \quad (\overline{p})$$

Using the elements of Section 3.3 and Corollary 3.18 one can easily verify that the dual formulation of (\overline{p}) reads:

$$\text{maximize } \mathcal{R} \text{ in } W_0^{1,\infty}(\Omega), \quad (p^*)$$

where

$$\mathcal{R}[w^*] := \begin{cases} \langle \tau, w^* \rangle - \frac{1}{2} \int_{\Omega} |\nabla w^*|^2 dx, & \text{if } \|\nabla w^*\|_{\infty} \leq 1 \\ -\infty & \text{otherwise} \end{cases}.$$

The previous problem, also known as the *elasto-plastic torsion problem*, arises when a long elastic bar with cross section Ω is twisted by an angle proportional to f . For a solution w^* , the set

$$E := \left\{ x \in \Omega : |\nabla w^*(x)| < 1 \right\}$$

is the set of points where the cross section still remains elastic, and the set

$$E' := \left\{ x \in \Omega : |\nabla w^*(x)| = 1 \right\}$$

is the set of points where the material has become plastic due to torsion. We refer to E as the *elastic set* and to E' as the *plastic set*.

Saddle-point optimality conditions

The main reason to compute the Γ -limit for the dual problem lies in the relations given by Theorem 3.5. Indeed, $\Psi \in \mathbf{E}(\Omega; \mathbb{R}^d)$ and $\Psi^\infty = |z|$. Moreover, Theorem 3.3 (see also Example 3.19) states that (\bar{p}) and (p^*) are dual of each other and

$$\inf_{\mathcal{M}(\Omega; \mathbb{R}^d)} \bar{\mathcal{F}} = \max_{W_0^{1,\infty}(\Omega)} \mathcal{R}.$$

It has been shown by Brezis and Stampacchia [9] (see also [10–12]) that for a source term $\tau \in L^p$, there exists a unique solution $w \in W^{2,p}(\Omega)$ of problem (p^*) .

Thence, a solution w^* of (p^*) is such that ∇w^* is $|\mu|$ -measurable for any $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ — indeed, this follows from the Sobolev embedding. In particular, Remark 3.36 and Theorem 3.5 state that every solution μ of (\bar{p}) verifies the following properties:

1. The *classical* saddle-point optimality conditions

$$2\mu^a(x) \cdot \nabla w^*(x) = \begin{cases} |\mu^a(x)|^2 + |\nabla w^*(x)|^2 & \text{if } |\mu^a| < 1 \\ 2|\mu^a(x)| - 1 + |\nabla w^*(x)|^2 & \text{if } |\mu^a| \geq 1 \end{cases},$$

at \mathcal{L}^d -a.e. $x \in \Omega$ (here, we have used the short-hand notation $\mu^a := \frac{d\mu}{d\mathcal{L}^d}$),

2. and, the *singular* optimality conditions

$$1 = \left| \frac{d\mu^s}{d|\mu^s|}(x) \right| = \frac{d\mu^s}{d|\mu^s|}(x) \cdot \nabla w^*(x),$$

which hold at $|\mu^s|$ -a.e. $x \in \Omega$.

These equations are equivalent to the relations

$$\begin{aligned} \mu &= \nabla w^*(x) \mathcal{L}^d \llcorner \Omega \quad \text{on } E, \\ \mu &= \nabla w^*(x) |\mu|, \quad |\mu^a| \geq 1 \quad \text{on } E'. \end{aligned}$$

If we set $\lambda \in \mathcal{M}^+(\Omega)$ to be the positive measure such that $\lambda \llcorner E \equiv 0$ and

$$\lambda \llcorner E' = |\mu| - \mathcal{L}^d,$$

then the characterization of μ and ∇w^* given above and the equation $\operatorname{div} \mu = -\tau$ yield (in the sense of distributions)

$$-\Delta w^* - \operatorname{div} \left(\lambda \frac{\nabla w^*}{|\nabla w^*|} \right) = f. \tag{3.23}$$

Conversely, if we can find a positive measure $\lambda \in \mathcal{M}^+(\Omega)$ which vanishes on E and satisfies (3.23), we may define

$$\bar{\mu} = \nabla w^*(\mathcal{L}^d \llcorner \Omega + \lambda).$$

Clearly $\bar{\mu}$ satisfies the optimality conditions (1) and (2), and by Theorem 3.5 it must be a solution of (\bar{p}) .

Remark 3.38. The analysis of saddle-point conditions for the elasto-plastic torsion problem, and in particular the derivation of (3.23), can also be found in [14, Section 3.4] under the additional assumption that $\mu \in L^2(\Omega; \mathbb{R}^d)$. Notice, however, that (\bar{p}) is not coercive in $L^2(\Omega; \mathbb{R}^d)$ and therefore an square-integrable solution might not exist.

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4 Optimal design problems for elliptic operators

This chapter contains the results obtained in the research paper:

Regularity for free interface variational problems in a general class of gradients

Abstract

We present a way to study a wide class of optimal design problems with a perimeter penalization. More precisely, we address existence and regularity properties of saddle points of energies of the form

$$(u, A) \mapsto \int_{\Omega} 2fu \, dx - \int_{\Omega \cap A} \sigma_1 \mathcal{A} u \cdot \mathcal{A} u \, dx - \int_{\Omega \setminus A} \sigma_2 \mathcal{A} u \cdot \mathcal{A} u \, dx + \text{Per}(A; \bar{\Omega}),$$

where Ω is a bounded Lipschitz domain, $A \subset \mathbb{R}^d$ is a Borel set, $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^m$, \mathcal{A} is an operator of gradient form, and σ_1, σ_2 are two not necessarily well-ordered symmetric tensors. The class of operators of gradient form includes scalar- and vector-valued gradients, symmetrized gradients, and higher order gradients. Therefore, our results may be applied to a wide range of problems in elasticity, conductivity or plasticity models.

In this context and under mild assumptions on f , we show for a solution (w, A) , that the topological boundary of $A \cap \Omega$ is locally a C^1 -hypersurface up to a closed set of zero \mathcal{H}^{d-1} -measure.

See:

A. Arroyo-Rabasa, *Regularity for free interface variational problems in a general class of gradients*, Calc. Var. Partial Differential Equations, vol. 55(6), 154–197, 2017

Disclaimer. *The notation of certain mathematical objects employed in this chapter might not agree with the original notation presented in the published version. This, however, does not represent an alteration of the intellectual presentation of the research paper cited above.*

4.1 Introduction

The problem of finding optimal designs involving two materials goes back to the work of Hashin and Shtrikman. In [20], the authors made the first successful attempt to derive the optimal bounds of a composite material. It was later on, in the series of papers [22–24], that Kohn and Strang described the connection between composite materials, the method of relaxation, and the homogenization theory developed by Murat and Tartar [29, 30]. In the context of homogenization, *better* designs tend to develop finer and finer geometries; a process which results in the creation of non-classical designs. One way to avoid the mathematical abstract of infinitely fine mixtures is to add a cost on the interfacial energy. In this regard, there is a large amount of optimal design problems that involve an interfacial energy and a Dirichlet energy. The study of regularity properties in this setting has been mostly devoted to problems where the Dirichlet energy is related to a scalar elliptic equation; see [6, 14, 18, 21, 25, 26], where partial C^1 -regularity on the interface is shown for an optimization problem oriented to find dielectric materials of maximal conductivity. We shall study regularity properties of similar problems in a rather general framework. Our results extend the aforementioned results to linear elasticity and linear plate theory models.

Before turning to a precise mathematical statement of the problem let us first present the model in linear plate theory that motivated our results. Let $\Omega = \omega \times [-h, h]$ be the reference configuration of a plate of thickness $2h$ and cross section $\omega \subset \mathbb{R}^2$. The linear equations governing a clamped plate Ω as h tends to zero for the Kirchhoff model are

$$\begin{cases} \nabla \cdot \nabla \cdot (\sigma \nabla^2 u) = f & \text{in } \omega, \\ \partial_\nu u = u = 0 & \text{in } \partial\omega, \end{cases} \quad (4.1)$$

where $u : \omega \rightarrow \mathbb{R}$ represents the displacement of the plate with respect to a vertical load $f \in L^\infty(\omega)$, and the design of the plate is described by a symmetric positive definite fourth-order tensor σ (up to a cubic dependence on the constant h). Here, we denote the second gradient by

$$\nabla^2 u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{ij}, \quad i, j = 1, 2.$$

Consider the physical problem of a thin plate Ω made-up of two elastic materials. More precisely, for a given set $A \subset \omega \subset \mathbb{R}^2$ we define the symmetric positive tensor

$$\sigma_A(x) := \mathbb{1}_A \sigma_1 + (1 - \mathbb{1}_A) \sigma_2,$$

where $\sigma_1, \sigma_2 \in \text{Sym}(\mathbb{R}^{2 \times 2}, \mathbb{R}^{2 \times 2})$. In this way, to each Borel subset $A \subset \omega$, there corresponds a displacement $u_A : \omega \rightarrow \mathbb{R}$ solving equation (4.1) with $\sigma = \sigma_A$. One measure of the rigidity of the plate is the so-called compliance, i.e., the work done by the loading. The smaller the compliance, the stiffer the plate is. A reasonable optimal design model consists in finding the most rigid design A

under the aforementioned costs. One seeks to minimize an energy of the form

$$A \mapsto \int_{\omega} \sigma_A \nabla^2 u_A \cdot \nabla^2 u_A \, dx + \text{Per}(A; \omega), \quad \text{among Borel subsets } A \text{ of } \mathbb{R}^2.$$

Optimality conditions for a stiffest plate can be derived by taking local variations on the design. For such analysis to be meaningful, one has to ensure first that the variational equations of optimality have a suitable meaning *in* the interface. Hence, it is natural to ask for the maximal possible regularity of ∂A and $\nabla^2 u_A$.

We will introduce a more general setting where one can replace the second gradient ∇^2 by an operator \mathcal{A} of *gradient type* (see Definition 4.6 and the subsequent examples in the next section for a precise description of this class).

4.1.1 Statement of the problem

Let $d \geq 2$, and let m, k be positive integers. We shall work in $\Omega \subset \mathbb{R}^d$; a nonempty, open, and bounded Lipschitz domain. We also fix a function $f \in L^\infty(\Omega; \mathbb{R}^m)$ and let σ_1 and σ_2 be two positive definite tensors in $\mathbb{M}_{\text{sym}}^{(m \times d^k) \times (m \times d^k)}$ satisfying a strong pointwise Gårding inequality: there exists a positive constant M such that

$$\frac{1}{M} |P|^2 \leq \sigma_i P \cdot P \leq M |P|^2 \quad \text{for all } P \in \mathbb{M}^{m \times d^k}, \quad i \in \{1, 2\}. \quad (4.2)$$

For a fixed Borel set $A \subset \mathbb{R}^d$, define the two-point valued tensor

$$\sigma_A(x) := \mathbb{1}_A \sigma_1 + \mathbb{1}_{(\Omega \setminus A)} \sigma_2. \quad (4.3)$$

We consider an operator $\mathcal{A} : L^2(\Omega; \mathbb{R}^m) \rightarrow W^{-k,2}(\Omega; \mathbb{M}^{m \times d^k})$ of gradient form (see Definition 4.6 in Section 4.2). As a consequence of the definition of operators of gradient form, the following equation

$$\mathcal{A}^*(\sigma_A \mathcal{A} u) = f \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m), \quad u \in W_0^{\mathcal{A}}(\Omega) \subset W_0^{k,2}(\Omega; \mathbb{R}^m), \quad (4.4)$$

has a unique solution (cf. Theorem 4.1). We will refer to equation (4.4) as the *state constraint* and we will denote by w_A its unique solution.

It is a physically relevant question to ask which designs have the least dissipated energy. To this end, consider the energy defined as

$$A \mapsto E(A) := \int_{\Omega} f w_A \, dx + \text{Per}(A; \bar{\Omega}) \quad \text{among Borel subsets } A \text{ of } \mathbb{R}^d.^1$$

We will be interested in the optimal design problem with Dirichlet boundary conditions on sets:

$$\text{minimize} \quad \left\{ E(A) : A \subset \mathbb{R}^d \text{ is a Borel set, } A \cap \Omega^c \equiv A_0 \cap \Omega^c \right\},^2 \quad (4.5)$$

¹Here, $\text{Per}(A; \bar{\Omega}) = |\mu_A|(\bar{\Omega})$, where μ_A is the Gauss-Green measure of A ; see Section 4.2.4.

²Due to the nature of the problem, we cannot replace $\text{Per}(A; \bar{\Omega})$ with $\text{Per}(A; \Omega)$ in $E(A)$ because it is possible that minimizing sequences tend to accumulate perimeter in $\partial\Omega$.

where $A_0 \subset \mathbb{R}^d$ is a set of locally finite perimeter.

Most attention has been drawn to the case where designs are mixtures of two well-ordered materials. The presentation given here places no comparability hypotheses on σ_1 and σ_2 . Instead, we introduce a weaker condition on the decay of generalized minimizers of a double-well problem. Our technique also holds under various constraints other than Dirichlet boundary conditions; in particular, any additional cost that scales as $O(r^{d-1+\varepsilon})$. For example, a constraint on the volume occupied by a particular material (cf. [11, 14, 26]). Lastly, we remark that our technique is robust enough to treat models involving the *maximization* of dissipated energy.

4.1.2 Main results and background of the problem

Existence of a minimizer of (4.5) can be established by standard methods. We are interested in proving that a solution pair (w_A, A) enjoys better *regularity* properties than the ones needed for existence. The notion of regularity for a set A will be understood as the local regularity of ∂A seen as a submanifold of \mathbb{R}^d , whereas the notion of regularity for w_A will refer to its differentiability and integrability properties.

It can be seen from the energy, that the deviation from being a perimeter minimizer for a solution A of problem (4.5) is bounded by the dissipated energy. Therefore, one may not expect better regularity properties for A than the ones for perimeter minimizers; and thus, one may only expect regularity up to singular set (we refer the reader to [5, 13] for classic results, see also [14] for a partial regularity result in a similar setting to ours).

Since a constrained problem may be difficult to treat, we will instead consider an equivalent variational unconstrained problem by introducing a multiplier as follows. Consider the saddle point problem

$$\inf_{A \subset \Omega} \sup_{u \in W_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A), \tag{P}$$

where

$$I_{\Omega}(u, A) := \int_{\Omega} 2fu \, dx - \int_{\Omega} \sigma_A \mathcal{A} u \cdot \mathcal{A} u \, dx + \text{Per}(A; \bar{\Omega}).$$

Our first result shows the equivalence between problem (P) and the minimization problem (4.5) under the state constraint (4.4):

Theorem 4.1 (existence). *There exists a solution (w, A) of problem (P). Furthermore, there is a one to one correspondence*

$$(w, A) \mapsto (w_A, A)$$

between solutions to problem (P) and the minimization problem (4.5) under the state constraint (4.4).

We now turn to the question of regularity. Let us depict an outline of the key steps and results obtained in this regard. The Morrey space $L^{p,\lambda}(\Omega; \mathbb{R}^m)$ is the subspace of $L^p(\Omega; \mathbb{R}^m)$ for which the semi-norm

$$[u]_{L^{p,\lambda}(\Omega)}^p := \sup \left\{ \frac{1}{r^\lambda} \int_{B_r(x)} |u|^p \, dy : B_r(x) \subset \Omega \right\}, \quad 0 < \lambda \leq d,$$

is finite.

The *first step* in proving regularity for solutions (w, A) consists in proving a critical $L^{2, d-1}$ local estimate for $\mathcal{A} w$. This estimate arises naturally since we expect a kind of balance between $\int_{B_r(x)} \sigma_A \mathcal{A} w \cdot \mathcal{A} w \, dy$ and the perimeter part $\text{Per}(A; B_r(x))$ that scales as r^{d-1} in balls of radius r .

To do so, let us recall a related relaxed problem. As part of the assumptions on \mathcal{A} there must exist a constant rank, l th-order differential operator $\mathcal{B} : L^2(\Omega; Z) \rightarrow W^{-l, 2}(\Omega; \mathbb{R}^n)$ with $\text{Ker}(\mathcal{B}) = \mathcal{A}[W^{\mathcal{A}}(\Omega)]$.³ It has been shown by Fonseca and Müller [17], that a necessary and sufficient condition for the lower semi-continuity of integral energies with superlinear growth under a constant rank differential constraint $\mathcal{B}v = 0$ is the \mathcal{B} -quasiconvexity of the integrand. In this context, the \mathcal{B} -free quasiconvex envelope of the double-well $W(P) := \min\{\sigma_1 P \cdot P, \sigma_2 P \cdot P\}$, at a point $P \in Z \subset \mathbb{M}^{m \times d^k}$, is given by

$$Q_{\mathcal{B}}W(P) := \inf \left\{ \int_{[0,1]^d} W(P + v(y)) \, dy : v \in C_{\text{per}}^{\infty}([0,1]^d; Z), \mathcal{B}v = 0 \text{ and } \int_{[0,1]^d} v(y) \, dy = 0 \right\}.$$

The idea is to get an $L^{2, d-1}$ estimate by transferring the regularizing effects from generalized minimizers of the energy $u \mapsto \int_{B_1} W(\mathcal{A} u)$ onto our original problem. In order to achieve this, we use a Γ -convergence argument with respect to a perturbation in the interfacial energy from which the next result follows:

Theorem 4.2 (upper bound). *Let (w, A) be a variational solution of problem (P). Assume that the higher integrability condition*

$$[\mathcal{A} \tilde{u}]_{L^{2, d-\delta}(B_{1/2})}^2 \leq c \|\mathcal{A} \tilde{u}\|_{L^2(B_1)}^2, \quad \text{for some } \delta \in [0, 1) \text{ and some positive constant } c, \quad (\text{Reg})$$

holds for local minimizers of the energy $u \mapsto \int_{B_1} Q_{\mathcal{B}}W(\mathcal{A} u)$, where $u \in W^{\mathcal{A}}(\Omega)$. Then, for every compactly contained set $K \subset\subset \Omega$, there exists a positive constant Λ_K such that

$$\int_{B_r(x)} \sigma_A \mathcal{A} w \cdot \mathcal{A} w \, dy + \text{Per}(A; B_r(x)) \leq \Lambda_K r^{d-1}, \quad (4.6)$$

for all $x \in K$ and every $r \in (0, \text{dist}(K, \partial\Omega))$.

Remark 4.3 (well-ordering assumption). If σ_1, σ_2 are well-ordered, say $\sigma_2 - \sigma_1$ is positive definite, then $Q_{\mathcal{B}}W$ is precisely the quadratic form $\sigma_2 P \cdot P$. Due to standard elliptic regularity results (cf. Lemma 4.11), estimate (Reg) holds for $\delta = 0$; therefore, assuming that the materials are well-ordered is a sufficient condition for the higher integrability assumption (Reg) to hold.

Remark 4.4 (non-comparable materials). In dimensions $d = 2, 3$ and restricted to the setting $\mathcal{A} = \nabla$, $m = 1$, condition (Reg) is strictly weaker than assuming the materials to be well-ordered. Indeed, one can argue by a Moser type iteration as in [12] to lift the regularity of minimizers. For higher-order

³Here, $W^{\mathcal{A}}(\Omega) = \{u \in L^2(\Omega; \mathbb{R}^m) : \mathcal{A} u \in L^2(\Omega; \mathbb{M}^{m \times d^k})\}$ is the \mathcal{A} -Sobolev space of Ω .

gradients or in the case of systems it is not clear to us whether assumption (Reg) is equivalent to the well-ordering of the materials.

The *second step*, consists of proving a *discrete monotonicity* for the excess of the Dirichlet energy on balls under a low perimeter density assumption. More precisely, on the function that assigns

$$r \mapsto \frac{1}{r^{d-1}} \int_{B_r(x)} |\mathcal{A} w|^2 dx, \quad x \in \partial A, r > 0.$$

The discrete monotonicity of the map above, together with the upper bound estimate (4.6), will allow us to prove a local lower bound λ_K on the density of the perimeter:

$$\frac{\text{Per}(A; B_r(x))}{r^{d-1}} \geq \lambda_K \quad \text{for every } x \in (K \cap \partial A), \text{ and every } 0 < r \leq r_K. \quad (\text{LB})$$

As usual, the lower bound on the density of the perimeter is the cornerstone to prove regularity of almost perimeter minimizers. In fact, once the estimate (LB) is proved we simply apply the excess improvement results of [26, Sections 4 and 5] to obtain our main result:

Theorem 4.5 (partial regularity). *Let (w, A) be a saddle point of problem (P) in Ω . Assume that the operator $P_H u = \mathcal{A}^*(\sigma_H \mathcal{A} u)$ is hypoelliptic and regularizing for the half-space problem (see properties (4.60)-(4.61)), and that the higher integrability (Reg) holds. Then there exists a positive constant $\eta \in (0, 1]$ depending only on N such that*

$$\mathcal{H}^{d-1}((\partial A \setminus \partial^* A) \cap \Omega) = 0, \quad \text{and} \quad \partial^* A \quad \text{is an open } C^{1, \eta/2}\text{-hypersurface in } \Omega.$$

Moreover if \mathcal{A} is a first-order partial differential operator, then $\mathcal{A} w \in C_{\text{loc}}^{0, \eta/8}(\Omega \setminus (\partial A \setminus \partial^ A))$; and hence, the trace of $\mathcal{A} w$ exists on either side of $\partial^* A$.*

Let us make a quick account of previous results. To our knowledge, only optimal design problems modeling the maximal dissipation of energy have been treated.

In [6] Ambrosio and Buttazzo considered the case where $\mathcal{A} = \nabla$ is the gradient operator for scalar-valued ($m = 1$) functions and where $\sigma_2 \geq \sigma_1$ in the sense of quadratic forms. The authors proved existence of solutions and showed that, up to choosing a good representative, the topological boundary is the closure of the reduced boundary and $\mathcal{H}^{d-1}(\partial A \setminus \partial^* A) = 0$. Soon after, Lin [26], and Kohn and Lin [21] proved, in the same case, that $\partial^* A$ is an open C^1 -hypersurface. From this point on, there have been several contributions aiming to discuss the optimal regularity of the interface for this particular case. In this regard and in dimension $d = 2$, Larsen [25] proved that connected components of A are C^1 away from the boundary. In arbitrary dimensions, Larsen's argument cannot be further generalized because it relies on the fact that convexity and positive curvature are equivalent in dimension $d = 2$. During the time this project was developed, we have learned that Fusco and Julin [18] found a different proof for the same results as stated in [26]; besides this, De Philippis and Figalli [14] recently obtained an improvement on the dimension of the singular set $(\partial^* A \setminus \partial A)$.

The paper is organized as follows. In the beginning of Section 4.2 we fix notation and discuss some facts of linear operators, Young measures and sets of finite perimeter. We also give the precise defini-

tion of gradient type operators and include a compensated compactness result that will be employed throughout the paper. In Section 4.3 we show the equivalence of the constrained problem (4.4)-(4.5) and the unconstrained problem (P) (Theorem 4.1). In the first part of Section 4.4 we shortly discuss how the higher integrability assumption (Reg) holds for various operators of gradient form. The rest of the section is devoted to the proof of the upper bound (4.6). Section 4.5 is devoted to the proof of the lower bound estimate (LB). Finally, in Section 4.6 we recall the flatness excess improvement [26] from which Theorem 4.5 easily follows.

4.2 Notation and preliminaries

We will write Ω to represent a non-empty, open, bounded subset of \mathbb{R}^d with Lipschitz boundary $\partial\Omega$. The use of capital letters A, B, \dots , will be reserved to denote Borel subsets of \mathbb{R}^d and we will write $\mathfrak{B}(\Omega)$ to denote the Borel σ -algebra in Ω .

The letters x, y will denote points in Ω ; while $z \in \mathbb{R}^m$ and $P \in \mathbb{M}^{m \times d^k}$ will be reserved for \mathbb{R}^m -vectors and $(m \times d^k)$ -matrices in Euclidean space. The Greek letters $\varepsilon, \delta, \rho$ and γ shall be used for general smallness or scaling constants. We follow Lin's convention in [26], bounding constants will be generally denoted by $c_1 \geq c_2 \geq \dots$, while smallness and decay constants will be usually denoted by $\varepsilon_1 \geq \varepsilon_2 \geq \dots$, and $\theta_1 \geq \theta_2 \geq \dots$, respectively. Let us mention that in proving regularity results one may often find it impractical to keep track of numerical constants due to the large amount of parameters; to illustrate better their uses and dependencies we have included a glossary of constants at the end of the paper.

It will often be useful to write a point $x \in \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ as $x = (x', x_d)$, in the same fashion we will also write $\nabla = (\nabla', \partial_d)$ to decompose the gradient operator. The bilinear form $\mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathbb{R} : (x, y) \mapsto x \cdot y$, where q is some positive integer, will stand for the standard inner product between two points while we will use the notation $|x| := \sqrt{x \cdot x}$ to represent the standard q -dimensional Euclidean norm. To denote open balls centered at a point x with radius r we will simply write $B_r(x)$.

We keep the standard notation for L^p and $W^{l,p}$ spaces. We write $C_c^l(\Omega; \mathbb{R}^q)$ to denote the space of functions with values in \mathbb{R}^q and with continuous l th-order derivative, and its subspace of functions compact support respectively. Similar notation stands for $\mathcal{M}(\Omega; \mathbb{R}^q)$, the space of bounded Radon measures in Ω ; and $\mathcal{D}(\Omega; \mathbb{R}^q)$, the space of smooth functions in Ω with compact support. For X and Y Banach spaces, the standard pairing between X and Y will be denoted by $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{R} : (u, v) \mapsto \langle u, v \rangle$.

4.2.1 Operators of gradient form

We introduce an abstract class of linear differential operators $\mathcal{A} : L^2(\Omega; \mathbb{R}^m) \rightarrow W^{-k,2}(\Omega; \mathbb{M}^{m \times d^k})$. This class contains scalar- and vector-valued gradients, higher gradients, and symmetrized gradients among its elements. The motivation behind it is that we may treat different models by employing a general and neat abstract setting. At a first glance this framework may appear too sterile; however, this definition is only meant to capture some of the essential regularity and rigidity properties of gradients.

Let $\mathcal{A} : L^2(\Omega; \mathbb{R}^p) \rightarrow W^{-k,2}(\Omega; \mathbb{R}^q)$ be a k -th order homogeneous partial differential operator of

the form

$$\mathcal{A} = \sum_{|\alpha|=k} A_\alpha \partial^\alpha, \quad (4.7)$$

where $A_\alpha \in \text{Lin}(\mathbb{R}^p; \mathbb{R}^q)$, and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ with $|\alpha| := |\alpha_1| + \dots + |\alpha_d|$. We define the \mathcal{A} -Sobolev of Ω as

$$\mathbf{W}^{\mathcal{A}}(\Omega) := \left\{ u \in L^2(\Omega; \mathbb{R}^p) : \mathcal{A}u \in L^2(\Omega; \mathbb{R}^q) \right\}$$

endowed with the norm $\|u\|_{\mathbf{W}^{\mathcal{A}}(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + \|\mathcal{A}u\|_{L^2(\Omega)}^2$. We also define the \mathcal{A} -Sobolev space with zero boundary values in $\partial\Omega$ by letting

$$\mathbf{W}_0^{\mathcal{A}}(\Omega) := \text{cl} \left\{ C_c^\infty(\Omega; \mathbb{R}^p), \|\cdot\|_{\mathbf{W}^{\mathcal{A}}(\Omega)} \right\}.$$

The principal symbol of \mathcal{A} is the positively k -homogeneous map defined as

$$\xi \mapsto \mathbb{A}(\xi) := \sum_{|\alpha|=k} \xi^\alpha A_\alpha \in \text{Lin}(\mathbb{R}^p, \mathbb{R}^q), \quad \xi \in \mathbb{R}^d,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$. One says that \mathcal{A} has the constant rank property if there exists a positive integer r such that

$$\text{rank}(\mathbb{A}(\xi)) = r \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}. \quad (\dagger)$$

Definition 4.6 (Operators of gradient form). *Let \mathcal{A} a homogeneous partial differential operator as in (4.7) with $p = m$ and $q = m \times d^k$. We say that \mathcal{A} is an operator of gradient form if the following properties hold:*

1. Compactness: *There exists a positive constant $C(\Omega)$ for which*

$$\|\varphi\|_{\mathbf{W}^{k,2}(\Omega)}^2 \leq C(\Omega) \left(\|\varphi\|_{L^2(\Omega)}^2 + \|\mathcal{A}\varphi\|_{L^2(\Omega)}^2 \right) \quad (4.8)$$

for all $\varphi \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$. Even more, for every $u \in \mathbf{W}^{\mathcal{A}}(\Omega)$ the following Poincaré inequality holds:

$$\inf \left\{ \|u - v\|_{\mathbf{W}^{k,2}(\Omega)}^2 : v \in \mathbf{W}^{\mathcal{A}}(\Omega), \mathcal{A}v = 0 \right\} \leq C(\Omega) \|\mathcal{A}u\|_{L^2(\Omega)}^2. \quad (4.9)$$

2. Exactness: *There exists an l -th homogeneous partial differential operator*

$$\mathcal{B} := \sum_{|\alpha|=l} B_\alpha \partial^\alpha, \quad (4.10)$$

with coefficients $B_\alpha \in \text{Lin}(Z; \mathbb{R}^n)$ for some positive integer n and a subspace Z of $\mathbb{M}^{m \times d^k}$, such that for every open and simply connected subset $\omega \subset \Omega$ we have the property

$$\left\{ \mathcal{A}u : u \in \mathbf{W}^{\mathcal{A}}(\omega) \right\} = \left\{ v \in L^2(\omega; Z) : \mathcal{B}v = 0 \text{ in } \mathcal{D}'(\omega; \mathbb{R}^n) \right\}.$$

We write \mathcal{A}^* to denote the L^2 -adjoint of \mathcal{A} , which is given by

$$\mathcal{A}^* := (-1)^k \sum_{|\alpha|=k} A_\alpha^T \partial^\alpha.$$

Remark 4.7 (constant rank). Let \mathcal{A} and \mathcal{B} be two linear differential operators satisfying an exactness property as in Definition 4.6. Then both operators \mathcal{A} and \mathcal{B} have the constant rank property (\dagger). This follows from the lower semi-continuity of the rank in any subspace of matrices.

Remark 4.8 (rigidity). The wave cone of an operator \mathcal{A} of the form (4.7) which is defined as

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1} \ker(\mathbb{A}(\xi)) \subset \mathbb{R}^p,$$

contains the admissible amplitudes in Fourier space for which concentration and oscillation behavior is allowed under the constraint $\mathcal{A}u = 0$. As in the case of gradients, it can be seen from the compactness assumption in Definition 4.6 that the wave cone $\Lambda_{\mathcal{A}}$ of a gradient operator \mathcal{A} is the zero space. In particular, there exists a positive constant λ (depending only on the coefficients of \mathcal{A}) such that

$$|\mathbb{A}(\xi)z|^2 \geq \lambda |\xi|^{2k} |z|^2 \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\} \text{ and all } z \in \mathbb{R}^p. \quad (4.11)$$

Remark 4.9 (Poincaré inequality II). It follows from the definition of $W_0^{\mathcal{A}}(\Omega)$ and the compactness assumption of \mathcal{A} that $W_0^{\mathcal{A}}(\Omega) \subset W_0^{k,2}(\Omega; \mathbb{R}^m)$. In particular, $\ker(\mathcal{A}) \cap W_0^{\mathcal{A}}(\Omega) = \{0\} \subset L^2(\Omega; \mathbb{R}^m)$ and $\mathcal{A}[W_0^{\mathcal{A}}(\Omega)]$ is closed in the L^2 norm. Thus, by [10, Theorem 2.21], there exists a constant⁴ $C(\Omega)$

$$\|u\|_{L^2(\Omega)}^2 \leq C(\Omega) \|\mathcal{A}u\|_{L^2(\Omega)}^2 \quad \text{for all } u \in W_0^{\mathcal{A}}(\Omega). \quad (4.12)$$

Elliptic regularity

Let \mathcal{A} be an operator of gradient form as in Definition 4.6 and let $\sigma \in L^\infty\left(\Omega; \mathbb{M}_{\text{sym}}^{(m \times d^k) \times (m \times d^k)}\right)$ be a tensor of variable coefficients satisfying the strong pointwise Gårding inequality (see (4.2))

$$\frac{1}{M} |P|^2 \leq \sigma(x) P \cdot P \leq M |P|^2 \quad \text{for almost every } x \in \Omega \text{ and every } P \in \mathbb{M}^{m \times d^k}. \quad (4.13)$$

If we define

$$\mathbf{A}_{\beta\alpha}^{ij} := (A_\alpha)_{i\beta,j} \quad \text{for } |\alpha| = |\beta| = k, \text{ and } 1 \leq i, j \leq d,$$

then we may write

$$\mathcal{A}\varphi = \mathbf{A} \nabla^k \varphi \quad \text{for every } \varphi \in C^k(\overline{\Omega}; \mathbb{R}^m). \quad (4.14)$$

It is easy to verify, using the compactness assumption of \mathcal{A} , that $\mathbf{C} := (\mathbf{A}^T \sigma \mathbf{A})$ satisfies the weak Gårding inequality

$$\langle \mathbf{C} \nabla^k \varphi, \nabla^k \varphi \rangle \geq \left(\frac{1}{MC} \right) \|\nabla^k \varphi\|_{L^2(\Omega)}^2 - \left(\frac{1}{M} \right) \|\varphi\|_{L^2(\Omega)}^2, \quad (4.15)$$

⁴Possibly abusing the notation, we will denote by $C(\Omega)$ the Poincaré constants from Definition 4.6 and Remark 4.9.

where $C = C(\Omega)$ the constant in the compactness assumption of Definition 4.6; for all smooth, \mathbb{R}^d -valued functions φ in $\overline{\Omega}$.

Lemma 4.10 (Caccioppoli inequality). *Let $\sigma \in L^\infty\left(\Omega; \mathbb{M}_{\text{sym}}^{(m \times d^k) \times (m \times d^k)}\right)$ satisfy the strong point-wise Gårding inequality (4.13) and let $w \in W^{\mathcal{A}}(\Omega)$ be a solution of the state equation*

$$\mathcal{A}^*(\sigma \mathcal{A} u) = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m).$$

Then there exists a positive constant C depending only on M, N, σ and \mathcal{A} such that

$$\int_{B_r(x)} |\nabla^k w|^2 \, dx \leq \frac{C}{(R-r)^{2k}} \int_{B_R(x)} |w|^2 \, dx \quad \text{for every } B_r(x) \subset B_R(x) \subset \Omega.$$

Proof. We may re-write $\mathcal{A}^*(\sigma \mathcal{A} u)$ as the elliptic operator in divergence form

$$(-1)^k \sum \partial^\beta (\mathbf{C}_{\beta\alpha}^{ij} \partial^\alpha u^j),$$

for coefficients $\mathbf{C} = (\mathbf{A}^T \sigma \mathbf{A})$ satisfying a weak Gårding inequality as in (4.15). The assertion then follows from Corollary 22 in [9]. □

Using Lemma 4.10 one can show, by classical methods, the following lemma on the regularizing properties of elliptic operators with constant coefficients:

Lemma 4.11 (constant coefficients). *Let \mathcal{A} be an operator of gradient form and let $\sigma_0 \in \mathbb{M}_{\text{sym}}^{(m \times d^k) \times (m \times d^k)}$ be a tensor satisfying the strong Gårding inequality (4.13). Then the operator*

$$L_{\sigma_0} u := \mathcal{A}^*(\sigma_0 \mathcal{A} u)$$

is hypoelliptic in the sense that if Ω is open and connected, and $w \in L^2(\Omega; \mathbb{R}^m)$, then

$$L_{\sigma_0} w = 0 \quad \Rightarrow \quad w \in C_{\text{loc}}^\infty(\Omega; \mathbb{R}^m).$$

Furthermore, there exists a constant $c = c(M, d) \geq 2^d$ such that

$$\frac{1}{\rho^d} \int_{B_\rho(x)} |\nabla^k u|^2 \, dx \leq \frac{c}{r^d} \int_{B_r(x)} |\nabla^k u|^2 \, dx \quad \text{for all } 0 < \rho \leq \frac{r}{2},$$

$$\frac{1}{\rho^d} \int_{B_\rho(x)} |\mathcal{A} u|^2 \, dx \leq \frac{c}{r^d} \int_{B_r(x)} |\mathcal{A} u|^2 \, dx \quad \text{for all } 0 < \rho \leq \frac{r}{2},$$

for every $B_r(x) \subset \Omega$.

Examples

Next, we gather some well-known differential structures that fit into the definition of operators of gradient form.

- (i) **Gradients.** Let $\mathcal{A} : L^2(\Omega; \mathbb{R}^m) \rightarrow W^{-1,2}(\Omega; \mathbb{M}^{m \times d}) : u \mapsto (\partial_j u^i)$ for $1 \leq i \leq m$ and $1 \leq j \leq d$. In this case

$$A_j z = z \otimes \mathbf{e}_j \quad \text{for every } z \in \mathbb{R}^m.$$

Hence, $W^{\mathcal{A}}(\Omega) = W^{1,2}(\Omega; \mathbb{R}^m)$ and the compactness property is a consequence of the classical Poincaré inequality on Ω .

The exactness assumption is the result of the characterization of gradients via curl-free vector fields.

Let $\mathcal{B} : L^2(\Omega; \mathbb{M}^{m \times d}) \rightarrow W^{-1,2}(\Omega; \mathbb{M}^{m \times (d \times d)})$ be the curl operator

$$\mathcal{B}v = (\text{curl}(v^i))_i := (\partial_l v_{ir} - \partial_r v_{il})_{ilr} \quad 1 \leq i \leq m, \quad 1 \leq l, r \leq d,$$

then condition (4.10) is fulfilled for $\mathcal{B} = \sum_{j=1}^d B_j \partial_j$ with coefficients

$$(B_j)_{ilr,pq} = \delta_{ip}(\delta_{jl}\delta_{rq} - \delta_{jr}\delta_{lq}) \quad 1 \leq l, r, q \leq d, \quad 1 \leq i, p \leq m.$$

Observe that $\mathcal{B}v = 0$ if and only if $\text{curl } v^i = 0$, for every $1 \leq i \leq m$; or equivalently, $v^i = \nabla u^i$ for some function $u^i : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$, for every $1 \leq i \leq m$ (as long as Ω is simply connected). Hence,

$$\{\nabla u : u \in W^{1,2}(\omega; \mathbb{R}^m)\} = \{v \in L^2(\omega; \mathbb{M}^{m \times d}) : \mathcal{B}v = 0\},$$

textfor all Lipschitz, and simply connected $\omega \subset \subset \Omega$.

- (ii) **Higher gradients.** Let $\mathcal{A} : L^2(\Omega) \rightarrow W^{-k,2}(\Omega; \mathbb{R}^{d^k})$ be the linear operator given by

$$u \mapsto \partial^\alpha u, \quad \text{where } |\alpha| = k.$$

Compactness is similar to the case of gradients.

We focus on the exactness condition: Let

$$\mathcal{B}^k : L^2(\Omega; \text{Sym}(\mathbb{R}^{d^k})) \rightarrow W^{-1,2}(\Omega; \mathbb{R}^{d^{k+1}})$$

be the curl operator on symmetric functions defined by the coefficients

$$(B_j^k)_{pq\beta_2 \dots \beta_k, \alpha_1 \dots \alpha_k} := \left(\delta_{jp} \delta_{\alpha_1 q} \prod_{h=2}^k \delta_{\alpha_h \beta_h} - \delta_{jq} \delta_{\alpha_1 p} \prod_{h=2}^k \delta_{\alpha_h \beta_h} \right),$$

where $1 \leq p, q, \beta_h, \alpha_h \leq d$, $h \in \{2, \dots, k\}$.

We write

$$\mathcal{B}^k v := \sum_{i=1}^d B_i^k \partial_i v, \quad v : \Omega \subset \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^{d^k}).$$

It easy to verify that $\mathcal{B}^k v = 0$ if and only if

$$\text{curl}((v_{p\alpha'})_p) = 0 \quad \text{for all } |\alpha'| = k - 1.$$

If Ω is simply connected, then there exists a function $u^{\alpha'} : \Omega \rightarrow \mathbb{R}$ such that $v_{p\alpha'} = \partial_p u^{\alpha'}$ for every $|\alpha'| = k - 1$. Using the symmetry of v under the permutation of its coordinates one can further deduce the existence of a function $u_k : \Omega \rightarrow \text{Sym}(\mathbb{R}^{d^{k-1}})$ with

$$v = \nabla u_k \quad \text{and} \quad (u_k)_{\alpha'} = u^{\alpha'}.$$

Moreover, $\mathcal{B}^{k-1} u_k = 0$. By induction one obtains that

$$v = \nabla^k u_0 \quad \text{for some function } u_0 : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}.$$

(iii) **Symmetrized gradients.** Let $\mathcal{E} : L^2(\Omega; \mathbb{R}^d) \rightarrow W^{-1,2}(\Omega; \text{Sym}(\mathbb{R}^{d^2}))$ be the linear operator given by

$$u \mapsto \mathcal{E}u := \frac{1}{2}(\partial_j u^i + \partial_i u^j)_{ij}, \quad \text{for } 1 \leq i, j \leq d.$$

The compactness property is a direct consequence of Korn's inequality. Consider the second-order homogeneous differential operator $\mathcal{B} : L^2(\Omega; \text{Sym}(\mathbb{R}^{d^2})) \rightarrow W^{-2,2}(\Omega; \mathbb{R}^{d^3})$ defined in the following way

$$\mathcal{B}v = \text{curl}(\text{curl}(v)) = \left(\frac{\partial^2 v_{ij}}{\partial x_i \partial x_l} + \frac{\partial^2 v_{il}}{\partial x_i \partial x_j} - \frac{\partial^2 v_{ii}}{\partial x_j \partial x_l} - \frac{\partial^2 v_{jl}}{\partial x_i \partial x_i} \right)_{1 \leq i, j, l \leq d}. \quad 5$$

Then $\mathcal{B}v = 0$, if and only if $v = \mathcal{E}u$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^d) = \mathbf{W}^{\mathcal{E}}(\Omega)$.

Remark 4.12. In the previous examples, we have omitted the characterization of higher gradients of vector-valued functions; however, the ideas remain the same as in the examples (i) and (ii).

Remark 4.13 (two-dimensional elasticity). In dimension $d = 2$ and provided that Ω is simply connected, the fourth-order equation for pure bending of a thin plate given by

$$\nabla \cdot \nabla \cdot (\mathbf{D}(x) \nabla^2 u(x)) = 0 \quad \text{for } u \in W^{2,2}(\Omega)$$

is equivalent to the in-plane elasticity equation

$$\nabla \cdot (\mathbf{S}(x) \mathcal{E}w(x)) = 0 \quad \text{where } w \in W^{1,2}(\Omega; \mathbb{R}^2),$$

for some tensor \mathbf{S} such that $\mathbf{D} = (\mathbf{R}_{\perp} \mathbf{S}^{-1} \mathbf{R}_{\perp})$, and where \mathbf{R}_{\perp} is the fourth-order tensor whose action is to rotate a second-order tensor by 90° (see, e.g., [28, Chapter 2.3]). Furthermore,

$$\mathbf{S}(x) \mathcal{E}w(x) = \mathbf{R}_{\perp} \nabla^2 u(x) \quad \text{and} \quad \nabla \cdot \nabla \cdot (\mathbf{R}_{\perp} \mathcal{E}w(x)) = 0.$$

For this reason, when working with the linear equations for pure bending of a thin plate we may indistinctly use regularizing properties of any of the equations above in the portions where \mathbf{D} is regular.

⁵Here, \mathcal{B} is a second order operator expressing the Saint-Venant compatibility conditions.

4.2.2 Compensated compactness

The following theorem is a generalized version of the well-known div-curl Lemma.

Lemma 4.14. *Let \mathcal{A} be a k -th order operator of gradient form and let $\{\sigma_h\} \subset L^2(\Omega; \mathbb{M}^{m \times d^k} \otimes \mathbb{M}^{m \times d^k})$ be a sequence of symmetric, strongly elliptic tensors as in (4.13). Assume also that $\{u_h\} \subset W^{\mathcal{A}}(\Omega)$ and $\{f_h\} \subset W^{-k,2}(\Omega; \mathbb{R}^m)$ are sequences for which*

$$\mathcal{A}^*(\sigma_h \mathcal{A} u_h) = f_h \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m), \quad \text{for every } h \in \mathbb{N}.$$

Further assume there exist a symmetric tensor $\sigma \in L^2(\Omega; \mathbb{M}^{m \times d^k} \otimes \mathbb{M}^{m \times d^k})$, a function $u \in W^{\mathcal{A}}(\Omega)$, and $f \in W^{-k,2}(\Omega; \mathbb{R}^m)$ for which

$$\begin{aligned} \mathcal{A} u_h &\rightharpoonup \mathcal{A} u \quad \text{in } L^2(\Omega; \mathbb{M}^{m \times d^k}), & f_h &\rightarrow f \quad \text{in } W^{-k,2}(\Omega; \mathbb{R}^m), \\ \text{and } \sigma_h &\rightarrow \sigma \quad \text{in } L^2(\Omega; \mathbb{M}^{m \times d^k} \otimes \mathbb{M}^{m \times d^k}). \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{A}^*(\sigma \mathcal{A} u) &= f \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m), \\ \sigma_h \mathcal{A} u_h \cdot \mathcal{A} u_h &\rightarrow \sigma \mathcal{A} u \cdot \mathcal{A} u \quad \text{in } \mathcal{D}'(\Omega). \end{aligned}$$

In particular,

$$\mathcal{A} u_h \rightarrow \mathcal{A} u \quad \text{in } L^2_{\text{loc}}(\Omega; \mathbb{M}^{m \times d^k}).$$

Proof. For simplicity we denote $\tau_h := \sigma_h \mathcal{A} u_h$, $\tau := \sigma \mathcal{A} u$. It suffices to observe that $\tau_h \rightharpoonup \tau$ in L^2 to prove that

$$\mathcal{A}^* \tau = f \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m).$$

The strong convergence on compact subsets of Ω requires a little bit more effort. Considering that \mathcal{A} is a k -th order linear differential operator, we may find constants $c_{\alpha\beta}$ with $|\alpha| + |\beta| \leq k$, $|\beta| \geq 1$ such that

$$\mathcal{A}(u_h \varphi) = (\mathcal{A} u_h) \varphi + \sum_{\alpha, \beta} c_{\alpha\beta} \partial^\alpha u_h \partial^\beta \varphi \in L^2(\Omega; \mathbb{R}^m) \quad \forall \varphi \in \mathcal{D}(\Omega), \forall h \in \mathbb{N}.$$

Hence,

$$\langle \tau_h \cdot \mathcal{A} u_h, \varphi \rangle = \langle f_h, u_h \varphi \rangle - \langle \tau_h, \sum_{\alpha, \beta} c_{\alpha\beta} \partial^\alpha u_h \partial^\beta \varphi \rangle.$$

By the compactness assumption on \mathcal{A} we may assume without loss of generality that $u_h \rightharpoonup u$ in $W^{k,2}(\Omega; \mathbb{R}^m)$. Thus, passing to the limit we obtain

$$\lim_{h \rightarrow \infty} \langle \tau_h \cdot \mathcal{A} u_h, \varphi \rangle = \langle f, u \varphi \rangle - \langle \tau, \sum_{\alpha, \beta} c_{\alpha\beta} \partial^\alpha u \partial^\beta \varphi \rangle = \langle \tau \cdot \mathcal{A} u, \varphi \rangle,$$

for every $\varphi \in \mathcal{D}(\Omega)$. One concludes that

$$\sigma_h \mathcal{A} u_h \cdot \mathcal{A} u_h \rightarrow \sigma \mathcal{A} u \cdot \mathcal{A} u \quad \text{in } \mathcal{D}'(\Omega). \quad (4.16)$$

Fix $\omega \subset\subset \Omega$ and let $0 \leq \varphi \in \mathcal{D}(\Omega)$ with $\varphi \equiv 1$ on ω . Using the convergence in (4.16) and the uniform ellipticity (4.2) of $\{\sigma_h\}$, one gets

$$\begin{aligned} \lim_{h \rightarrow \infty} \|\mathcal{A} u_h - \mathcal{A} u\|_{L^2(\omega)} &\leq M \cdot \lim_{h \rightarrow \infty} \langle \sigma_h(\mathcal{A}(u_h - u)) \cdot \mathcal{A}(u_h - u), \varphi \rangle \\ &\leq M \cdot \left(\lim_{h \rightarrow \infty} \langle \sigma_h \mathcal{A} u_h \cdot \mathcal{A} u_h, \varphi \rangle \right. \\ &\quad \left. - \lim_{h \rightarrow \infty} 2 \langle \sigma_h \mathcal{A} u_h \cdot \mathcal{A} u, \varphi \rangle + \langle \sigma_h \mathcal{A} u \cdot \mathcal{A} u, \varphi \rangle \right) \\ &= 0. \end{aligned}$$

□

4.2.3 Young measures and lower semi-continuity of integral energies

In this section $\mathcal{B} : L^2(\Omega; Z) \rightarrow W^{-l,2}(\Omega; \mathbb{R}^n)$ is assumed to be a an l -th order homogeneous partial differential operator of the form

$$\sum_{\alpha} B_{\alpha} \partial^{\alpha}, \quad B_{\alpha} \in \text{Lin}(Z; \mathbb{R}^n), \quad \text{with } Z \text{ a linear subspace of } \mathbb{M}^{m \times d^k},$$

satisfying the constant rank condition (†).

Next, we recall some facts about \mathcal{B} -quasiconvexity, lower semi-continuity and Young measures. The results in this section hold for differential operators with coefficients B_{α} in arbitrary spaces $\text{Lin}(\mathbb{R}^p; \mathbb{R}^q)$ for p, q a pair of positive integers; however, we only present versions where the dimensions match our current setting. We start by stating a version of the Fundamental theorem for Young measures due to Ball [8].

Theorem 4.15 (Fundamental theorem for Young measures). *Let $\Omega \subset \mathbb{R}^d$ be a measurable set with finite measure and let $\{v_j\}$ be a sequence of measurable functions $v_j : \Omega \rightarrow Z$. Then there exists a subsequence $\{v_{h(j)}\}$ and a weak* measurable map $\mu : \Omega \rightarrow \mathcal{M}(Z)$ with the following properties:*

1. We denote $\mu_x := \mu(x)$ for simplicity, then $\mu_x \geq 0$ in the sense of measures and $|\mu_x|(Z) \leq 1$ for a.e. $x \in \Omega$.
2. If one additionally assumes that $\{v_{h(j)}\}$ is uniformly bounded in $L^1(\Omega; Z)$, then $|\mu_x|(Z) = 1$ for a.e. $x \in \Omega$.
3. If $F : \mathbb{M}^{m \times d^k} \rightarrow \mathbb{R}$ is a Borel and lower semi-continuous function, and is also bounded from below, then

$$\int_{\Omega} \langle \mu_x, F \rangle \, dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} F(v_{h(j)}) \, dx.$$

4. If $\{v_{h(j)}\}$ is uniformly bounded in $L^1(\Omega; Z)$ and $F : \mathbb{M}^{m \times d^k} \rightarrow \mathbb{R}$ is a continuous function, and bounded from below, then

$$\int_{\Omega} \langle \mu_x, F \rangle \, dx = \liminf_{j \rightarrow \infty} \int_{\Omega} F(v_{h(j)}) \, dx$$

if and only if $\{F \circ v_{h(j)}\}$ is equi-integrable. In this case,

$$F \circ v_{h(j)} \rightharpoonup \langle \mu_x, F \rangle \quad \text{in } L^1(\Omega).$$

In the sense of Theorem 4.15, we say that the sequence $\{v_{h(j)}\}$ generates the Young measure μ .

The following proposition tells us that a uniformly bounded sequence in the L^p norm, which is also sufficiently close to $\ker(\mathcal{B})$, may be approximated by a p -equi-integrable sequence in $\ker(\mathcal{B})$ in a weaker L^q norm. We remark that this rigidity result is the only one where Murat's constant rank condition (\dagger) is used.

Proposition 4.16 ([17, Lemma 2.15]). *Let $1 < p < \infty$. Let $\{v_h\}$ be a bounded sequence in $L^p(\Omega; Z)$ generating a Young measure μ , with $v_h \rightharpoonup v$ in $L^p(\Omega; Z)$ and $\mathcal{B}v_h \rightarrow 0$ in $W^{-1,p}(\Omega; \mathbb{R}^n)$. Then there exists a p -equi-integrable sequence $\{u_h\}$ in $L^p(\Omega; Z) \cap \ker(\mathcal{B})$ that generates the same Young measure μ and is such that*

$$\int_{\Omega} v_h \, dx = \int_{\Omega} u_h \, dx, \quad \|v_h - u_h\|_{L^q(\Omega)} \rightarrow 0, \quad \text{for all } 1 \leq q < p.$$

Let $F : \mathbb{M}^{m \times d^k} \rightarrow \mathbb{R}$ be a lower semi-continuous function with $0 \leq F(P) \leq C(1 + |P|^p)$ for some positive constant C . The \mathcal{B} -quasiconvex envelope of F at $P \in Z \subset \mathbb{M}^{m \times d^k}$ is defined as

$$\begin{aligned} Q_{\mathcal{B}}F(P) := \inf \left\{ \int_{[0,1]^d} F(P + v(y)) \, dy : \right. \\ \left. v \in C_{\text{per}}^{\infty}([0,1]^d; Z), \mathcal{B}v = 0 \text{ and } \int_{[0,1]^d} v \, dy = 0 \right\}. \end{aligned} \quad (4.17)$$

The most relevant feature of $Q_{\mathcal{B}}F$ is that, for $p > 1$, the lower semi-continuous envelope with respect to the weak- L^p topology of the functional

$$v \mapsto \int_{\Omega} F(v) \, dx, \quad \text{where } v \in L^p(\Omega; Z) \text{ and } \mathcal{B}v = 0, \quad (4.18)$$

is given by the functional

$$v \mapsto \int_{\Omega} Q_{\mathcal{B}}F(v) \, dx, \quad \text{where } v \in L^p(\Omega; Z) \text{ and } \mathcal{B}v = 0.$$

If μ is a Young measure generated by a sequence $\{v_h\}$ in $L^p(\Omega; Z)$ such that $\mathcal{B}v_h = 0$ for every $h \in \mathbb{N}$, then we say that μ is a \mathcal{B} -free Young measure.

We recall the following Jensen inequality for \mathcal{B} -free Young measures [17, Theorem 4.1]:

Theorem 4.17. *Let $1 < p < \infty$. Let μ be a \mathcal{B} -free Young measure in Ω . Then for a.e. $x \in \Omega$ and all lower semi-continuous functions that satisfy $|F(P)| \leq C(1 + |P|^p)$ for some positive constant C and all $P \in \mathbb{M}^{m \times d^k}$, one has that*

$$\langle \mu_x, F \rangle \geq Q_{\mathcal{B}} F(\langle \mu_x, \text{id} \rangle).$$

4.2.4 Geometric measure theory and sets of finite perimeter

Most of the facts collected in this section can be found in [27] and [7]; however, some notions as the slicing of sets of finite perimeter are presented there only in a formal way. For a better understanding of such topics we refer the reader to [16].

Let $A \subset \mathbb{R}^d$ be a Borel set. The Gauss-Green measure μ_A of A is the derivative of the characteristic function of A in the sense of distributions, i.e., $\mu_A := \nabla(\mathbb{1}_A)$. We say that A is a set of locally finite perimeter if and only if $|\mu_A|$ is a vector-valued Radon measure in \mathbb{R}^d . We write $A \in \text{BV}_{\text{loc}}(\mathbb{R}^d)$ to express that A is a set of locally finite perimeter in \mathbb{R}^d .

Let $\omega \subset \subset \mathbb{R}^d$ be a Borel set. The perimeter in ω of a set A with locally finite perimeter is defined as

$$\text{Per}(A, \omega) := |\mu_A|(\omega).$$

The Radon-Nikodým differentiation theorem states that the set of points

$$\partial^* A := \left\{ x \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{\text{Per}(A; B_r(x))}{\text{vol}(B'_1) \cdot r^{d-1}} = 1, \right. \\ \left. \text{and } \frac{d\mu_A}{d|\mu_A|}(x) \text{ exists and belongs to } \mathbb{S}^{d-1} \right\}$$

has full $|\mu_A|$ -measure in \mathbb{R}^d ; this set is commonly known as the *reduced boundary* of A . We will also use the notation

$$\nu_A(x) := \frac{d\mu_A}{d|\mu_A|}(x) \quad x \in \partial^* A;$$

the *measure theoretic normal* of A .

In general, for $s \geq 0$, we will denote by \mathcal{H}^s the *s-dimensional Hausdorff measure* in \mathbb{R}^d . The following well-known theorem captures the structure of sets with finite perimeter in terms of the measure \mathcal{H}^{d-1} :

Theorem 4.18 (De Giorgi's Structure Theorem). *Let A be a set of locally finite perimeter. Then*

$$\partial^* A = \bigcup_{j=1}^{\infty} K_j \cup N,$$

where

$$|\mu_A|(N) = 0,$$

and K_j is a compact subset of a C^1 -hypersurface S_j for every $j \in \mathbb{N}$. Furthermore, $\nu_A|_{S_j}$ is normal to S_j and

$$\mu_A = \nu_A \mathcal{H}^{d-1} \llcorner \partial^* A.$$

From De Giorgi's Structure Theorem it is clear that $\text{spt } \mu_A = \overline{\partial^* A}$. Actually, up to modifying A on a set of zero measure, one has that $\partial A = \overline{\partial^* A}$ (see [27, Proposition 12.19]). From this point on, each time we deal with a set A of finite perimeter, we will assume without loss of generality that

$$\partial A = \text{supp } \mu_A = \overline{\partial^* A}. \quad (4.19)$$

For a set of locally finite perimeter A , the deviation from being a *perimeter minimizer* in Ω at a given scale r is quantified by the monotone function

$$\text{Dev}_\Omega(A, r) := \sup \left\{ \text{Per}(A; B_r(x)) - \text{Per}(E; B_r(x)) : E \Delta A \subset\subset B_r(x) \subset \Omega \right\}.$$

The next result, due to Tamanini [33], states that a set of locally finite perimeter with small deviation Dev_Ω at every scale is actually a C^1 -hypersurface up to a lower dimensional set.

Theorem 4.19. *Let $A \subset \mathbb{R}^d$ be a set of locally finite perimeter and let $c(x)$ be a locally bounded function for which*

$$\text{Dev}_\Omega(A, r) \leq c(x)r^{d-1+2\eta} \quad \text{for some } \eta \in (0, 1/2].$$

Then the reduced boundary in Ω , $(\partial^ A \cap \Omega)$, is an open $C^{1,\eta}$ -hypersurface and the singular set $\Omega \cap (\partial A \setminus \partial^* A)$ has at most Hausdorff dimension $(d - 8)$.*

Slicing sets of finite perimeter

Given a Borel set $E \subset \mathbb{R}^d$ and a Lipschitz function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, we shall consider the level set slices

$$E_t := E \cap \{g = t\}, \quad t \in \mathbb{R}.$$

For a set $A \subset \mathbb{R}^d$ of finite perimeter in Ω , the level set slice of the reduced boundary $(\partial^* A)_t$ is \mathcal{H}^{d-2} -rectifiable for almost every $t \in \mathbb{R}$. Furthermore, by the co-area formula, $t \mapsto \mathcal{H}^{d-2}((\partial^* A)_t) \in L^1_{\text{loc}}(\mathbb{R})$.

If the set $\{g = t\}$ is a C^1 -manifold and t is such that $\mathcal{H}^{d-2}((\partial^* A)_t) < \infty$, we shall define the *slice of A in $g^{-1}\{t\}$* as

$$\langle A, g, t \rangle := \mathcal{H}^{d-2} \llcorner (\partial^* A)_t.$$

It turns out that, for $g(x) = |x|$, the level set slice A_t is locally diffeomorphic to a set of finite perimeter in \mathbb{R}^{d-1} . Even more,

$$\mathcal{H}^{d-2} \llcorner \partial^* A_t = \langle A, g, t \rangle \quad \text{for a.e. } t > 0, \text{ and} \quad (4.20)$$

$$\pi_g v_A := (\text{id}_{\mathbb{R}^d} - \nabla g \otimes \nabla g) v_A \neq 0 \quad \text{for } \mathcal{H}^{d-2}\text{-a.e. } x \in (\partial^* A)_t. \quad (4.21)$$

Here, $\partial^* A_t$ is understood as the image, under local diffeomorphisms, of the reduced boundary of a set of finite perimeter. These properties can be inferred from the classical slicing by hyperplanes, see e.g., [27, Chapter 18.3].

We also define the cone extension of a set $E \subset \mathbb{R}$ containing $\{0\}$ by letting

$$D_E := \left\{ \lambda x \in \mathbb{R}^d : \lambda > 0, x \in E \right\}.$$

For a.e. $t > 0$ and $g(x) = |x|$, the cone extension of A_t is a set of locally finite perimeter in \mathbb{R}^d with

$$\partial^* D_{A_t} = D_{(\partial^* A)_t} \quad \text{and} \quad \text{Per}(D_{A_t}; B_\rho) = \left(\frac{1}{d-1} \right) \frac{\rho^{d-1}}{t^{d-2}} \cdot \mathcal{H}^{d-2}((\partial^* A)_t). \quad (4.22)$$

In order to attend different variational problems involving the minimization of perimeter, a well-known technique is to modify a set A within balls B_t without modifying its Gauss-Green measure in $(B_t)^c$.

For almost every $t > 0$, where $\langle A, g, t \rangle$ is well-defined and (4.20)-(4.21) hold, we construct a cone-like comparison set of A by setting

$$\tilde{A} := \mathbf{1}_{B_t} D_{A_t} + \mathbf{1}_{\Omega \setminus B_t} A. \quad (4.23)$$

Exploiting the basic properties of reduced boundaries, it follows by (4.20) that

$$\mu_{\tilde{A}} = \mu_{D_{A_t} \llcorner B_t} + \mu_{A \llcorner (B_t)^c}; \quad (4.24)$$

and, in particular,

$$\text{Per}(\tilde{A}; B_r) = \text{Per}(D_{\partial^* A_t}; B_t) + \text{Per}(A; (B_t)^c \cap B_r) \quad \text{for all } r > t.$$

On the other hand, again by the co-area formula,

$$\mathcal{H}^{d-1}((\partial^* A)_t \cap \{g = t\}) = 0 \quad \text{for almost every } t > 0.$$

Using the monotonicity of $r \mapsto \text{Per}(A; B_r)$ and the general version of the co-area formula (see [16, Theorem 3.2.22]) one can show that the derivative of $r \mapsto \text{Per}(A; B_r)$ exists at almost every $t > 0$; even more, up to a further null set it is given by

$$\frac{d}{dr} \Big|_{r=t} \text{Per}(A; B_r) = |\pi_t \nu_A|^{-1} \mathcal{H}^{d-2}((\partial^* A)_t) \geq \langle A, g, t \rangle(\mathbb{R}^d). \quad (4.25)$$

The previous estimate will play a crucial role in proving the lower bound (LB).

4.3 Existence of solutions: proof of Theorem 4.1

We show an equivalence between the constrained problem (4.5) and the unconstrained problem (P) for which existence of solutions and regularity properties for minimizers are discussed in the present and subsequent sections. We fix $\mathcal{A} : L^2(\Omega; \mathbb{R}^m) \rightarrow W^{-k,2}(\Omega; \mathbb{M}^{m \times d^k})$ an operator of gradient from as in Definition 4.6. We also fix $A_0 \subset \mathbb{R}^d$, a set of locally finite perimeter.

Recall that, the minimization problem (4.5) under the state constraint (4.4) reads:

$$\text{minimize} \quad \left\{ \int_{\Omega} f w_A + \text{Per}(A; \bar{\Omega}) : A \in \text{BV}_{\text{loc}}(\mathbb{R}^d), A \cap \Omega^c \equiv A_0 \cap \Omega^c \right\},^6$$

where w_A is the unique distributional solution to the state equation

$$\mathcal{A}^*(\sigma_A \mathcal{A} u) = f, \quad u \in \mathbf{W}_0^{\mathcal{A}}(\Omega).$$

On the other hand, the associated saddle point problem (P) reads:

$$\inf \left\{ \sup_{u \in \mathbf{W}_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A) : A \in \text{BV}_{\text{loc}}(\mathbb{R}^d), A \cap \Omega^c \equiv A_0 \cap \Omega^c \right\}, \quad (\text{P})$$

where

$$I_{\Omega}(u, A) := \int_{\Omega} 2f u \, dx - \int_{\Omega} \sigma_A \mathcal{A} u \cdot \mathcal{A} u \, dx + \text{Per}(A; \bar{\Omega}).$$

Theorem 4.1 (existence). *There exists a solution (w, A) of problem (P). Furthermore, there is a one to one correspondence*

$$(w, A) \mapsto (w_A, A)$$

between solutions to problem (P) and the minimization problem (4.5) under the constraint (4.4).

Proof. We employ the direct method. We begin by proving existence of solutions to problem (P). To do so, we will first prove the following:

Claim: 1. For any set $A \subset \mathbb{R}^d$ as in the assumptions, there exists $w_A \in \mathbf{W}_0^{\mathcal{A}}(\Omega)$ such that

$$0 \leq I_{\Omega}(w_A, A) = \sup_{u \in \mathbf{W}_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A) < \infty.$$

The tensor σ_A is a positive definite tensor and therefore the mapping

$$u \mapsto I_{\Omega}(u, A) = \int_{\Omega} 2f u - \sigma_A \mathcal{A} u \cdot \mathcal{A} u \, dx + \text{Per}(A; \bar{\Omega})$$

is strictly concave. Observe that $\sup_{u \in \mathbf{W}_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A) \geq \text{Per}(A; \bar{\Omega})$; indeed, we may take $u \equiv 0 \in \mathbf{W}_0^{\mathcal{A}}(\Omega)$. Hence,

$$\sup_{u \in \mathbf{W}_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A) \geq \text{Per}(A; \bar{\Omega}) \geq 0. \quad (4.26)$$

Because of this, we may find a maximizing sequence $\{w_h\}$ in $\mathbf{W}_0^{\mathcal{A}}(\Omega)$, i.e.,

$$I_{\Omega}(w_h, A) \rightarrow \sup_{u \in \mathbf{W}_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A), \quad \text{as } h \text{ tends to infinity.}$$

⁶As stated in Section 4.2.4, we write $A \in \text{BV}_{\text{loc}}(\mathbb{R}^d)$ to express that A is a Borel set of locally finite perimeter in \mathbb{R}^d .

Even more, one has from (4.2) that

$$-\frac{1}{M} \|\mathcal{A} w_h\|_{L^2(\Omega)}^2 \geq - \int_{\Omega} \sigma_A \mathcal{A} w_h \cdot \mathcal{A} w_h \, dx$$

and consequently from (4.26) and (4.12) one infers that

$$C(\Omega)^{-1} \cdot \limsup_{h \rightarrow \infty} \frac{1}{M} \|w_h\|_{L^2(\Omega)}^2 \leq \limsup_{h \rightarrow \infty} \frac{1}{M} \|\mathcal{A} w_h\|_{L^2(\Omega)}^2 \leq 2 \|f\|_{L^2(\Omega)} \cdot \limsup_{h \rightarrow \infty} \|w_h\|_{L^2(\Omega)}. \quad (4.27)$$

A fast calculation shows that $\|w_h\|_{L^2(\Omega)} \leq 2MC(\Omega)\|f\|_{L^2(\Omega)}$; in return, (4.27) also implies that

$$\limsup_{h \rightarrow \infty} \|\mathcal{A} w_h\|_{L^2(\Omega)}^2 \leq 4C(\Omega)M^2 \|f\|_{L^2(\Omega)}^2.$$

Hence, using again the compactness property of \mathcal{A} , we may pass to a subsequence (which we will not relabel) and find $w_A \in W_0^{\mathcal{A}}(\Omega)$ with

$$w_h \rightarrow w_A \quad \text{in } L^2(\Omega; \mathbb{R}^m), \quad \mathcal{A} w_h \rightharpoonup \mathcal{A} w_A \quad \text{in } L^2(\Omega; \mathbb{M}^{m \times d^k}).$$

The concavity of $-\sigma_A z \cdot z$ is a well-known sufficient condition for the upper semi-continuity of the functional $\mathcal{A} u \mapsto - \int_{\Omega} \sigma_A \mathcal{A} u \cdot \mathcal{A} u$. Therefore,

$$\sup_{u \in W_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A) = \lim_{h \rightarrow \infty} I_{\Omega}(w_h, A) \leq I_{\Omega}(w_A, A).$$

This proves the claim.

Now, we use *Claim 1* to find a minimizing sequence $\{A_h\}$ for $A \mapsto I_{\Omega}(w_A, A)$. Since the uniform bound (4.27) does not depend on A , we may again assume (up to a subsequence) that there exists $\tilde{w} \in W_0^{\mathcal{A}}(\Omega)$ such that

$$w_{A_h} \rightarrow \tilde{w} \quad \text{in } L^2(\Omega; \mathbb{R}^m), \quad \mathcal{A} w_{A_h} \rightharpoonup \mathcal{A} \tilde{w} \quad \text{in } L^2(\Omega; \mathbb{M}^{m \times d^k}), \quad \text{and} \quad \mathcal{A}^*(\sigma_{A_h} \mathcal{A} w_{A_h}) = f.$$

Even more, since $\{A_h\}$ is minimizing, it must be that $\sup_h \{\text{Per}(A_h; B_R)\} < \infty$, for some ball B_R properly containing Ω , and thus (for a further subsequence) there exists a set $\tilde{A} \subset \mathbb{R}^d$ of locally finite perimeter with $\tilde{A} \cap \Omega^c \equiv A_0 \cap \Omega^c$ and such that

$$\mathbb{1}_{A_h} \rightarrow \mathbb{1}_{\tilde{A}} \quad \text{in } L^1(B_R), \quad |\mu_{\tilde{A}}|(B_R) \leq \liminf_{h \rightarrow \infty} |\mu_{A_h}|(B_R).$$

Therefore

$$\begin{aligned} \text{Per}(\tilde{A}; \bar{\Omega}) &= |\mu_{\tilde{A}}|(B_R) - |\mu_{A_0}|(B_R \setminus \bar{\Omega}) \\ &\leq \liminf_{h \rightarrow \infty} |\mu_{A_h}|(B_R) - |\mu_{A_0}|(B_R \setminus \bar{\Omega}) = \liminf_{h \rightarrow \infty} \text{Per}(A_h; \bar{\Omega}) \end{aligned} \quad (4.28)$$

A consequence of Lemma 4.14 is that

$$\mathcal{A}^*(\sigma_{\tilde{A}} \mathcal{A} \tilde{w}) = f \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m), \quad \text{and} \quad \int_{\Omega} \sigma_{A_h} \mathcal{A} w_{A_h} \cdot \mathcal{A} w_{A_h} \rightarrow \int_{\Omega} \sigma_{\tilde{A}} \mathcal{A} \tilde{w} \cdot \mathcal{A} \tilde{w}.^7 \quad (4.29)$$

By taking the limit as h goes to infinity we get from (4.28) and the convergence above that

$$\min_A \sup_{u \in W_0^{\mathcal{A}}(\Omega)} I_{\Omega}(u, A) = \lim_{h \rightarrow \infty} I_{\Omega}(w_{A_h}, A_h) \geq I_{\Omega}(\tilde{w}, \tilde{A}) = I_{\Omega}(w_{\tilde{A}}, \tilde{A}),$$

where the last equality is a consequence of the identity $\tilde{w} = w_{\tilde{A}}$ which can be easily derived by using the equation and the strict concavity of I_{Ω} in the first variable. Thus, the pair $(w_{\tilde{A}}, \tilde{A})$ is a solution to problem (P).

The equivalence of problem (P) and problem (4.5) under the state constraint (4.4) follows easily from (4.29), the strict concavity of $I_{\Omega}(\cdot, A)$, and a simple integration by parts argument. \square

4.4 The energy bound: proof of Theorem 4.2

Throughout this section and for the rest of the manuscript we fix $\mathcal{A} : L^2(\Omega; \mathbb{R}^m) \rightarrow W^{-k,2}(\Omega; \mathbb{M}^{m \times d^k})$ in the class of operators of gradient form. Accordingly, the notations Z and \mathcal{B} shall denote the subspace of $\mathbb{M}^{m \times d^k}$ and the homogeneous operator associated to \mathcal{A} (see Definition 4.6). We will also write (w, A) to denote a particular solution of problem (P).

Consider the energy $J_{\omega} : L^2(\Omega; Z) \times \mathfrak{B}(\Omega) \rightarrow \mathbb{R}$ defined as

$$J_{\omega}(v, E) := \int_{\omega} \sigma_E v \cdot v \, dy + \text{Per}(E; \omega), \quad \text{for } \omega \subset \Omega \text{ an open set.}$$

The goal of this section is to prove a local bound for the map $x \mapsto J_{B_r(x)}(\mathcal{A} w, A)$. More precisely, we aim to prove that for every compactly contained subset K of Ω there exists a positive number Λ_K such that

$$J_{B_r(x)}(\mathcal{A} w, A) \leq \Lambda_K r^{d-1} \quad \text{for all } x \in K \text{ and every } r \in (0, \text{dist}(K, \partial\Omega)). \quad (4.30)$$

Our strategy will be the following. We first define a one-parameter family J^{ε} of perturbations of J_B , in the perimeter term. In Theorem 4.21 we show that, as the perimeter term vanishes, these perturbations Γ -converge (with respect to the L^2 -weak topology) to the relaxation of the energy

$$w \mapsto \int_{\Omega} W(\mathcal{A} w) \, dx,$$

for which we will assume certain regularity properties (cf. property (Reg)). Then, using a compensated compactness argument, we prove Theorem 4.2 (upper bound) by transferring the regularity properties of the relaxed problem to our original problem.

⁷The convergence of the total energy is not covered by Lemma 4.14; however, this can be deduced using integration by parts and the fact that w_h has zero boundary values for every $h \in \mathbb{N}$.

Before moving forward, let us shortly discuss how the higher integrability property (Reg) stands next to the standard assumption that the materials σ_1 and σ_2 are well-ordered.

4.4.1 A digression on the regularization assumption

As commented beforehand in the introduction, a key assumption in the proof of the upper bound (4.30) is that *generalized* local minimizers of the energy

$$u \mapsto \int_{B_1} W(\mathcal{A} u) \, dy, \quad \text{where } u \in W^{\mathcal{A}}(B_1),$$

possess improved decay estimates. More precisely, we require that *local* minimizers \tilde{u} of the functional

$$u \mapsto \int_{B_1} Q_{\mathcal{B}} W(\mathcal{A} u) \, dy, \quad \text{where } u \in W^{\mathcal{A}}(B_1), \quad (4.31)$$

possess a higher integrability estimate of the form

$$[\mathcal{A} \tilde{u}]_{L^{2,d-\delta}(B_{1/2})}^2 \leq c \|\mathcal{A} \tilde{u}\|_{L^2(B_1)}^2 \quad \text{for some } \delta \in [0, 1). \quad (\text{Reg})$$

Only then, we will be able to transfer a decay estimate of order ρ^{d-1} to solutions of our original problem.

Remark 4.20 (the case of gradients). In the case $\mathcal{A} = \nabla$, condition (Reg) boils down to regularity above the critical $C^{0,1/2}$ local regularity. More specifically,

$$\frac{1}{r^{d-\delta+2}} \int_{B_r(x)} |w - (w)_{r,x}|^2 \, dy \leq [\nabla w]_{L^{2,d-\delta}(B_{1/2})}^2 \leq c \|\nabla w\|_{L^2(B_1)}^2 \quad \text{for all } B_r(x) \subset B_{1/2}.$$

By Poincaré's inequality and Campanato's Theorem one can easily deduce (cf. [21]) that

$$w \in C_{\text{loc}}^{0, \frac{1}{2} + \varepsilon}(B_{1/2}).$$

Let us give a short account of some cases where one may find (Reg) to be a *natural assumption*.

The well-ordered case

The notion of well-ordering in Materials Science is not only justified as the comparability of two materials, one being at least *better* than the other. It has also been a consistent assumption when dealing with optimization problems because it allows explicit calculations. See for example [3, 4, 20], where the authors discuss how the well-ordering assumption plays a role in proving the optimal lower bounds of an effective tensor made-up by two materials. If σ_1 and σ_2 are well-ordered, say $\sigma_2 \geq \sigma_1$ as quadratic forms, then $W(P) = \sigma_2 P \cdot P$. Hence, by Lemma 4.11, the desired higher integrability (Reg) holds with $\delta = 0$.

The non-ordered case

Applications for this setting are mostly reserved for the *scalar case* ($m = 1$). In this particular case one can ensure that $Q_{\mathcal{B}}W = W^{**}$, where W^{**} is the convex envelope of W . For example, one may consider an optimal design problem involving the linear conductivity equations for two dielectric materials which happen to be incomparable as quadratic forms. In this setting, it is not hard to see that indeed $QW = W^{**}$ and even that $W^{**} \in C^{1,1}(\mathbb{R}^d, \mathbb{R})$. In dimensions $d = 2, 3$, one can employ a Moser-iteration technique for the dual problem as the one developed in [12] to show better regularity of minimizers of (4.31).

Regarding the case of systems, if no well-ordering of the materials is assumed, it is not clear to us that (Reg) necessary holds (compare to [15, 32]).

4.4.2 Proof of Theorem 4.2

We define an ε -perturbation of $v \mapsto \int_{B_1} \sigma_A v \cdot v$ as follows. Consider the functional

$$(v, A) \mapsto J^\varepsilon(v, A) := \int_{B_1} \sigma_A v \cdot v \, dy + \varepsilon^2 \text{Per}(A; B_1), \quad \text{for } \varepsilon \in [0, 1]; \quad J := J^1. \quad (4.32)$$

By a scaling argument one can easily check that

$$\varepsilon^2 J(v, A) = J^\varepsilon(\varepsilon v, A). \quad (4.33)$$

Furthermore,

$$v \text{ is a local minimizer of } J(\cdot, A) \text{ if and only if } \varepsilon v \text{ is a local minimizer of } J^\varepsilon(\cdot, A). \quad (4.34)$$

We also consider the following one-parameter family of functionals:

$$v \mapsto G^\varepsilon(v) := \begin{cases} \min_{A \in \mathfrak{B}(B_1)} J^\varepsilon(v, A) & \text{if } v \in L^2(\Omega; Z) \text{ and } \mathcal{B}v = 0, \\ \infty & \text{otherwise.} \end{cases} \quad (4.35)$$

The next result characterizes the Γ -limit of these functionals as ε tends to zero.

Theorem 4.21. *The Γ -limit of the functionals G^ε , as ε tends to zero, and with respect to the weak- L^2 topology is given by the functional*

$$G(v) := \begin{cases} \int_{B_1} Q_{\mathcal{B}}W(v) \, dy & \text{if } v \in L^2(\Omega; Z) \text{ and } \mathcal{B}v = 0, \\ \infty & \text{else.} \end{cases} \quad (4.36)$$

Proof. We divide the proof into three steps. First, we will prove the following auxiliary lemma.

Lemma 4.22. *Let $\omega \subset \mathbb{R}^d$ be an open and bounded domain. Let $p > 1$ and let $F : \mathbb{M}^{m \times d^k} \rightarrow [0, \infty)$*

be a continuous integrand with p -growth, i.e.,

$$0 \leq F(P) \leq C(1 + |P|^p), \quad P \in \mathbb{M}^{m \times d^k}.$$

If $v \in L^p(\omega; Z)$ and $\mathcal{B}v = 0$, then there exists a p -equi-integrable recovery sequence $\{v_h\} \subset L^p(\omega; Z)$ for v such that

$$\mathcal{B}v_h = 0 \quad \text{and} \quad F(v_h) \rightharpoonup Q_{\mathcal{B}}F(v) \quad \text{in } L^1(\omega).$$

Proof. Since $v \mapsto \int_{\omega} Q_{\mathcal{B}}F(v)$ is the lower semi-continuous envelope of $v \mapsto \int_{\omega} F(v)$ (see (4.17)-(4.18)) with respect to the weak- L^p topology, we may find a sequence $\{v_h\}$ with the following properties:

$$\mathcal{B}v_h = 0, \quad v_h \xrightarrow{L^p} v,$$

and

$$\int_{\omega} Q_{\mathcal{B}}F(v) \, dx \geq \int_{\omega} F(v_h) \, dx - \frac{1}{h}.$$

Passing to a subsequence if necessary, we may assume that the sequence $\{v_h\}$ generates a \mathcal{B} -free Young measure which we denote by μ . We then apply [17, Lemma 2.15] to find a p -equi-integrable sequence $\{v'_h\}$ (with $\mathcal{B}v'_h = 0$) generating the same Young measure μ . On the one hand, the Fundamental Theorem for Young measures (Theorem 4.15) and the fact that $\{v_h\}$ generates μ yield

$$\liminf_{h \rightarrow \infty} \int_{\omega} F(v_h) \, dx \geq \int_{\omega} \langle \mu_x, F \rangle \, dx.$$

On the other hand, due to the same theorem and the equi-integrability of the sequence $\{|v'_h|^p\}$ one gets the convergence $F(v'_h) \rightharpoonup \langle \mu_x, F \rangle \in L^1$. In other words,

$$\lim_{h \rightarrow \infty} \int_{\omega} F(v'_h) \, dx = \int_{\omega} \langle \mu_x, F \rangle \, dx.$$

The three relations above yield

$$\int_{\omega} Q_{\mathcal{B}}F(v) \, dx \geq \limsup_{h \rightarrow \infty} \int_{\omega} F(v_h) \geq \int_{\omega} \langle \mu_x, F \rangle \, dx = \lim_{h \rightarrow \infty} \int_{\omega} F(v'_h) \, dx \geq \int_{\omega} Q_{\mathcal{B}}F(v) \, dx. \quad (4.37)$$

We summon the characterization for \mathcal{B} -free Young measures from Theorem 4.17 to observe that

$$\langle \mu_x, F \rangle \geq Q_{\mathcal{B}}F(\langle \mu_x, \text{id} \rangle) = Q_{\mathcal{B}}F(v(x)) \quad \text{a.e. } x \in \omega.$$

This inequality and (4.37) imply

$$\langle \mu_x, F \rangle = Q_{\mathcal{B}}F(v(x)) \quad \text{a.e. } x \in \omega.$$

We conclude by recalling that $F(v'_h) \rightharpoonup \langle \mu_x, F \rangle$ in $L^1(\omega)$. □

The lower bound. Let $v \in L^2(B_1; Z)$ and let $\{v_\varepsilon\}$ be a sequence in $L^2(B_1; Z)$ such that $v_\varepsilon \rightharpoonup v$ in

$L^2(B_1; Z)$. We want to prove that

$$\liminf_{\varepsilon \downarrow 0} G^\varepsilon(v_\varepsilon) \geq G(v).$$

Notice that, we may reduce the proof to the case where $\mathcal{B}v_\varepsilon = 0$ for every ε . From the inequality $\sigma_A \geq W \geq Q_{\mathcal{B}}W$ (as quadratic forms), we infer that

$$J^\varepsilon(v_\varepsilon) \geq \int_{B_1} Q_{\mathcal{B}}W(v_\varepsilon) dy.$$

Next, we recall that $v \mapsto \int_{B_1} Q_{\mathcal{B}}W(v)$ is lower semi-continuous in $\{v \in L^2(\Omega; Z) : \mathcal{B}v = 0\}$ with respect to the weak- L^2 topology. Hence,

$$\liminf_{\varepsilon \downarrow 0} G^\varepsilon(v_\varepsilon) \geq \int_{B_1} Q_{\mathcal{B}}W(v) dy.$$

This proves the lower bound inequality.

The upper bound. We fix $v \in L^2(B_1; Z)$, we want to show that there exists a sequence $\{v_\varepsilon\}$ in $L^2(B_1; Z)$ with $v_\varepsilon \rightharpoonup v$ in $L^2(B_1; Z)$ and such that

$$\limsup_{\varepsilon \downarrow 0} G^\varepsilon(v_\varepsilon) \leq G(v).$$

We may assume that $\mathcal{B}v = 0$, for otherwise the inequality occurs trivially. Lemma 4.22 guarantees the existence of a 2-equi-integrable sequence $\{v_h\}_{h=1}^\infty$ for which

$$\mathcal{B}v_h = 0, \quad v_h \rightharpoonup v \quad \text{in } L^2(B_1; Z), \quad \text{and} \quad W(v_h) \rightharpoonup Q_{\mathcal{B}}W(v) \quad \text{in } L^1(B_1). \quad (4.38)$$

Next, we define an h -parametrized sequence of subsets of B_1 in the following way:

$$A_h := \left\{ x \in B_1 : (\sigma_1 - \sigma_2)v_h \cdot v_h \leq 0 \right\}.$$

Using the fact that smooth sets are dense in the broader class of subsets with respect to measure convergence, we may take a smooth set $A'_h \subset B_1$ such that the following estimates hold for some strictly monotone function $L : \mathbb{N} \rightarrow \mathbb{N}$ (with $\lim_{h \rightarrow \infty} L(h) = \infty$):

$$|(A'_h \Delta A_h) \cap B_1| = O(h^{-1}), \quad \text{Per}(A'_h; B_1) \leq L(h). \quad (4.39)$$

Observe that, by the 2-equi-integrability of $\{v_h\}$, one gets that

$$\|(\sigma_{A_h} - \sigma_{A'_h})v_h \cdot v_h\|_{L^2(B_1)} \leq M \|v_h\|_{L^2(S_h)}^2 = O(h^{-1}), \quad \text{where } S_h := A'_h \Delta A_h. \quad (4.40)$$

The next step relies, essentially, on stretching the sequence $\{v_h\}$. Define the ε -sequence

$$\bar{v}_\varepsilon := v_{K(\varepsilon)}, \quad \varepsilon \leq \frac{1}{L(1)},$$

where $K : \mathbb{R}_+ \rightarrow \mathbb{N}$ is the piecewise constant decreasing function defined as

$$K := \sum_{h=1}^{\infty} h \cdot \mathbb{1}_{R_h}, \quad R_h := \left(\frac{1}{L(h+1)}, \frac{1}{L(h)} \right].$$

Claim:

1. $L \circ K(\varepsilon) \leq \varepsilon^{-1}$, if $\varepsilon \in (0, L(1)^{-1}]$.
2. $K(\varepsilon) = h$, where h is such that $\varepsilon \in R_h$.

Proof. To prove (1), observe from the strict monotonicity of L that $\cup_{h=1}^{\infty} R_h = (0, L(1)^{-1}]$. A simple calculation gives

$$L(K(\varepsilon)) = L\left(\sum_{h=1}^{\infty} h \cdot \mathbb{1}_{R_h}(\varepsilon)\right) = \sum_{h=1}^{\infty} L(h) \cdot \mathbb{1}_{R_h}(\varepsilon) = L(h_0) \cdot \mathbb{1}_{R_{h_0}}(\varepsilon) \leq \frac{1}{\varepsilon}, \quad (4.41)$$

where h_0 is such that $\varepsilon \in R_{h_0}$. The proof of (2) is an easy consequence of the definition of K and the fact that $\{R_h\}$ is a disjoint family of sets. Indeed, if $\varepsilon \in R_h$ then $K(\varepsilon) = h \cdot \mathbb{1}_{R_h}(\varepsilon) = h$. \square

Since K is a decreasing function and $K(\mathbb{R}_+) = \mathbb{N} \cup \{0\}$, it remains true that

$$\bar{v}_{K(\varepsilon)} \rightharpoonup v \text{ in } L^2(B_1; \mathbb{M}^{m \times d^k}), \quad \text{as } \varepsilon \rightarrow 0.$$

We are now in position to calculate the lim sup inequality:

$$\begin{aligned} G^\varepsilon(v_{K(\varepsilon)}) &= \min_{A \in \mathfrak{B}(B_1)} \int_{B_1} \sigma_A v_{K(\varepsilon)} \cdot v_{K(\varepsilon)} + \varepsilon^2 \text{Per}(A; B_1) \leq \int_{B_1} \sigma_{A'_{K(\varepsilon)}} v_{K(\varepsilon)} \cdot v_{K(\varepsilon)} + \varepsilon^2 \text{Per}(A'_{K(\varepsilon)}; B_1) \\ &\leq \int_{B_1} \sigma_{A_{K(\varepsilon)}} v_{K(\varepsilon)} \cdot v_{K(\varepsilon)} + O(K(\varepsilon)^{-1}) + \varepsilon^2 L(K(\varepsilon)) \leq \int_{B_1} W(v_{K(\varepsilon)}) + O(\varepsilon) + \varepsilon. \end{aligned}$$

Hence, by (4.38)

$$\limsup_{\varepsilon \downarrow 0} G^\varepsilon(\bar{v}_\varepsilon) \leq \limsup_{\varepsilon \downarrow 0} \int_{B_1} W(v_{K(\varepsilon)}) = \lim_{h \rightarrow \infty} \int_{B_1} W(v_h) = \int_{B_1} Q_{\mathfrak{B}} W(v).$$

This proves the upper bound inequality. \square

Corollary 4.23. *Let $\{w_\varepsilon\} \subset W^{\mathcal{A}}(B_1)$ be a sequence of almost local minimizers of the sequence of functionals*

$$\{u \mapsto G^\varepsilon(\mathcal{A}u)\}.$$

Assume that $\{\mathcal{A}w_\varepsilon\}$ is 2-equi-integrable in B_s for every $s < 1$. Assume also that there exists $w \in W^{\mathcal{A}}(B_1)$ such that

$$\mathcal{A}w_\varepsilon \rightharpoonup \mathcal{A}w \text{ in } L^2(B_1; \mathbb{M}^{m \times d^k}).$$

Then,

$$Q_{\mathcal{B}}W(\mathcal{A} w_{\varepsilon}) \rightharpoonup Q_{\mathcal{B}}(\mathcal{A} w) \quad \text{in } L^1_{\text{loc}}(B_1).$$

Moreover, w is a local minimizer of $u \mapsto G(\mathcal{A} u)$.

Proof. The first step is to check that

$$Q_{\mathcal{B}}W(\mathcal{A} w_{\varepsilon}) \rightharpoonup Q_{\mathcal{B}}(\mathcal{A} w) \quad \text{in } L^1(B_s), \text{ for every } s < 1. \quad (4.42)$$

The sequence $\mathcal{A} w_{\varepsilon}$ generates (up to taking a subsequence) a \mathcal{B} -free Young measure $\mu : B_1 \rightarrow Z$ so that by Theorem 4.15, Theorem 4.17 and the local 2-equi-integrability assumption,

$$W(\mathcal{A} w'_{\varepsilon}) \rightharpoonup \langle \mu_x, W \rangle \geq Q_{\mathcal{B}}W(\mathcal{A} w) \quad \text{in } L^1_{\text{loc}}(B_1). \quad (4.43)$$

Fix $s \in (0, 1)$ and consider the rescaled functions

$$w_{\varepsilon}^s := \frac{w_{\varepsilon}(sy)}{s^{k-\frac{1}{2}}}, \quad w^s := \frac{w(sy)}{s^{k-\frac{1}{2}}}.$$

It is not hard to see that, because of the (almost) minimization properties of $\{w_{\varepsilon}\}$, the rescaled sequence $\{w_{\varepsilon}^s\}$ is also a sequence of almost local minimizers of the sequence of functionals $\{u \mapsto G(\mathcal{A} u)\}$.⁸ Moreover, $\mathcal{A} w_{\varepsilon}^s \rightharpoonup \mathcal{A} w^s$ in $L^2(B_1; Z)$.

From the proof of the lower bound in Theorem 4.21, we may find a 2-equi-integrable recovery sequence $\{v'_{\varepsilon}\}$ for v , i.e., such that $v'_{\varepsilon} \rightharpoonup \mathcal{A} w^s$ and

$$\lim_{\varepsilon \downarrow 0} G^{\varepsilon}(v'_{\varepsilon}) = G(\mathcal{A} w^s).$$

Recall that, by the exactness assumption of \mathcal{A} and \mathcal{B} , there are functions $w'_{\varepsilon} \in W^{\mathcal{A}}(B_1)$ such that

$$v'_{\varepsilon} = \mathcal{A} w'_{\varepsilon} \quad \text{for every } \varepsilon > 0.$$

A recovery sequence with the same boundary values. The next step is to show that one may assume, without loss of generality, that $\text{supp}(w'_{\varepsilon} - w_{\varepsilon}^s) \subset\subset B_1$.

We may further assume (without loss of generality) that $\{w_{\varepsilon}^s\}$ and $\{w'_{\varepsilon}\}$ are $W^{k,2}$ -uniformly bounded, and that $w_{\varepsilon}^s - w'_{\varepsilon} \rightharpoonup 0$ in $W^{k,2}(B_1; \mathbb{R}^m)$.

Define

$$\tilde{v}_{h,\varepsilon} := \mathcal{A}(\varphi_h w'_{\varepsilon} + (1 - \varphi_h) w_{\varepsilon}^s) = \varphi_h \mathcal{A} w'_{\varepsilon} + (1 - \varphi_h) \mathcal{A} w_{\varepsilon}^s + \overbrace{\sum_{\substack{|\beta| \geq 1 \\ |\alpha| + |\beta| = k}} c_{\alpha\beta} \partial^{\alpha} (w'_{\varepsilon} - w_{\varepsilon}^s) \partial^{\beta} \varphi_h}}^{g(h)}$$

where, for every $h \in \mathbb{N}$, $\varphi_h \in C^{\infty}(B_1; [0, 1])$ with $\varphi_h \equiv 1$ in $B_{1-1/h}$. Since $\|g(h)\|_{L^2(B_1)} \rightarrow 0$ as $\varepsilon \rightarrow 0$,

⁸This scaling has the property that $s^{d-1}J(\mathcal{A} w^s, A^s) = J_{B_s(x)}(\mathcal{A} w, A)$.

we infer that

$$\limsup_{\varepsilon \downarrow 0} \|\tilde{v}_{h,\varepsilon} - \mathcal{A} w'_\varepsilon\|_{L^2(B_1)} \leq \limsup_{\varepsilon \downarrow 0} \|\mathcal{A} w'_\varepsilon\|_{L^2(B_1 \setminus B_{1-1/h})} + \limsup_{\varepsilon \downarrow 0} \|\mathcal{A} w_\varepsilon\|_{L^2(B_1 \setminus B_{1-1/h})}.$$

We now let $h \rightarrow \infty$ and use the 2-equi-integrability of $\{\mathcal{A} w_\varepsilon^s\}$ and $\{\mathcal{A} w'_\varepsilon\}$ to get

$$\limsup_{h \rightarrow \infty} \limsup_{\varepsilon \downarrow 0} \|\tilde{v}_{h,\varepsilon} - \mathcal{A} w'_\varepsilon\|_{L^2(B_1)} = 0.$$

Thus, we may find a diagonal sequence $\tilde{v}_\varepsilon = \tilde{v}_{h(\varepsilon),\varepsilon} = \mathcal{A} \tilde{w}_\varepsilon^s$ which is 2-equi-integrable, $\text{supp}(\tilde{w}_\varepsilon^s - \tilde{w}_\varepsilon) \subset\subset B_1$, and such that

$$\lim_{\varepsilon \downarrow 0} \|\mathcal{A} w'_\varepsilon - \mathcal{A} \tilde{w}_\varepsilon\|_{L^2(B_s)} = O(\varepsilon).$$

In particular, the (almost) local minimizing property of $\{\mathcal{A} w_\varepsilon^s\}$ gives

$$\limsup_{\varepsilon \downarrow 0} \int_{B_1} W(\mathcal{A} w_\varepsilon^s) \leq \limsup_{\varepsilon \downarrow 0} G^\varepsilon(\mathcal{A} w_\varepsilon^s) \leq \limsup_{\varepsilon \downarrow 0} G^\varepsilon(\mathcal{A} \tilde{w}_\varepsilon) \leq \lim_{\varepsilon \downarrow 0} G^\varepsilon(\mathcal{A} w'_\varepsilon) = G(\mathcal{A} w^s).$$

Rescaling back, the inequality above yields

$$\limsup_{\varepsilon \downarrow 0} \int_{B_s} W(\mathcal{A} w_\varepsilon) \leq \int_{B_s} Q_{\mathcal{B}} W(\mathcal{A} w),$$

which together with (4.43) proves (4.42).

Local minimizer of G . The second step is to show that w is a local minimizer of $u \mapsto G(\mathcal{A} u)$. We argue by contradiction: assume that w is not a local minimizer of $u \mapsto G(\mathcal{A} u)$, then we would find $s \in (0, 1)$ and $\eta \in C_c^\infty(B_s; \mathbb{R}^m)$ for which

$$G(\mathcal{A} w) > G(\mathcal{A} w + \mathcal{A} \eta).$$

Again, using a re-scaling argument, this would imply that

$$G(\mathcal{A} w^s) > G(\mathcal{A} w^s + \mathcal{A} \eta^s).$$

Similarly to the previous step, we can find a 2-equi-integrable recovery sequence $\{\mathcal{A}(\varphi_\varepsilon^s + \eta^s)\}$ of $(\mathcal{A} w^s + \mathcal{A} \eta^s)$ with the property that $\text{supp}(\varphi_\varepsilon^s - w_\varepsilon^s) \subset\subset B_1$, for every $\varepsilon > 0$. On the other hand, the (almost) minimizing property of $\mathcal{A} w_\varepsilon^s$ and (4.42) yield

$$G(\mathcal{A} w^s + \mathcal{A} \eta^s) < G(\mathcal{A} w^s) = \lim_{\varepsilon \downarrow 0} G^\varepsilon(\mathcal{A} w_\varepsilon^s) \leq \lim_{\varepsilon \downarrow 0} G^\varepsilon(\mathcal{A} \varphi_\varepsilon^s + \mathcal{A} \eta^s) = G(\mathcal{A} w^s + \mathcal{A} \eta^s),$$

which is a contradiction. This shows that w is a local minimizer of $u \mapsto G(\mathcal{A} u)$. □

Let us recall, for the proof of the next proposition, that the higher integrability assumption (Reg)

on local minimizers \tilde{u} of $u \mapsto G(\mathcal{A} u)$ reads:

$$[\mathcal{A} \tilde{u}]_{L^{2,d-\delta}(B_{1/2})}^2 \leq c \|\mathcal{A} \tilde{u}\|_{L^2(B_1)}^2, \quad \text{for some } \delta \in [0, 1). \quad (\text{Reg})$$

Proposition 4.24. *Let (w, A) be a saddle-point of problem (P). Assume that the higher integrability condition (Reg) holds for local minimizers of $u \mapsto G(\mathcal{A} u)$. Then, for every $K \subset\subset \Omega$ there exists a positive constant $C(K) > 1$ and a smallness constant $\rho \in (0, 1/2)$ such that at least one of the following properties*

1. $J_{B_r(x)}(\mathcal{A} w, A) \leq C(K)r^{d-1}$,
2. $J_{B_{\rho r}(x)}(\mathcal{A} w, A) \leq \rho^{d-(1+\delta)/2} J_{B_r(x)}(\mathcal{A} w, A)$,

holds for all $x \in K$ and every $r \in (0, \text{dist}(K, \partial\Omega))$. Here,

$$J_{B_r(x)}(\mathcal{A} u, A) = \int_{B_r(x)} \sigma_A \mathcal{A} u \cdot \mathcal{A} u \, dy + \text{Per}(A; B_r(x)),$$

Proof. Let (w, A) be a saddle-point of (P) and fix $\rho \in (0, 1)$ (to be specified later in the proof). We argue by contradiction through a blow-up technique: Negation of the statement would allow us to find a sequence $\{(x_h, r_h)\}$ of points $x_h \in K$ and positive radii $r_h \downarrow 0$ for which

$$J_{B_{r_h}(x_h)}(\mathcal{A} w, A) > h r_h^{d-1}, \quad \text{and} \quad (4.44)$$

$$J_{B_{\rho r_h}(x_h)}(\mathcal{A} w, A) > \rho^{d-(1+\delta)/2} J_{B_{r_h}(x_h)}(\mathcal{A} w, A). \quad (4.45)$$

An equivalent variational problem. It will be convenient to work with a similar variational problem: Consider the saddle-point problem

$$\inf \left\{ \sup_{u \in W_0^{\mathcal{A}}(\Omega)} \tilde{I}_{\Omega}(\mathcal{A} u, A) : A \subset \mathbb{R}^d \text{ Borel set, } A \cap \Omega^c \equiv A_0 \cap \Omega^c \right\}, \quad (\tilde{\text{P}})$$

where

$$\tilde{I}_{\Omega}(\mathcal{A} u, A) := \int_{\Omega} 2\tau_A \cdot \mathcal{A} u \, dx - \int_{\Omega} \sigma_A \mathcal{A} u \cdot \mathcal{A} u \, dx + \text{Per}(A; \bar{\Omega}).$$

Here we recall the notation $\tau_A := \sigma_A \mathcal{A} w_A$, where $w_A \in W_0^{\mathcal{A}}(\Omega)$ is the unique maximizer of $u \mapsto I_{\Omega}(u, A)$. It follows immediately from the identity

$$\int_{\Omega} \tau_A \cdot \mathcal{A} u \, dx = \int_{\Omega} f u \, dx \quad u \in W_0^{\mathcal{A}}(\Omega),$$

that saddle-points (w, A) of problem (P) are also saddle-points of $(\tilde{\text{P}})$ and vice versa; hence, in the following we will make no distinction between saddle-points of (P) and $(\tilde{\text{P}})$. A special property of \tilde{I} is that, locally, it is always positive on saddle-points (w, A) of (P). Indeed, in this case $w = w_A$ and therefore

$$\tilde{I}_{B_r(x)}(\mathcal{A} w, A) = \int_{B_r(x)} \sigma_A \mathcal{A} w_A \cdot \mathcal{A} w_A + \text{Per}(A; B_r(x)) = J_{B_r(x)}(\mathcal{A} w, A), \quad B_r(x) \subset \Omega. \quad (4.46)$$

A re-scaling argument. We re-scale and translate $B_r(x)$ into B_1 by letting

$$A^{r,x} := \frac{A}{r} - x, \quad f^{r,x}(y) := r^{k+\frac{1}{2}} f(ry - x) \rightarrow 0 \text{ in } L^\infty(B_1), \quad \text{and} \quad w^{r,x}(y) := \frac{w(ry - x)}{r^{k-\frac{1}{2}}}. \quad (4.47)$$

A further normalization on the sequence takes place by setting

$$\varepsilon(h)^2 := r^{d-1} \cdot J_{B_{r_h}(x_h)}(\mathcal{A} w, A)^{-1} = O(h^{-1}),$$

and defining

$$A_{\varepsilon(h)} := A^{r_h, x_h}, \quad f_{\varepsilon(h)} := \varepsilon(h) \cdot f^{r_h, x_h}, \quad w_{\varepsilon(h)} := \varepsilon(h) \cdot w^{r_h, x_h}, \quad \text{and} \quad \tau_{\varepsilon(h)} := \sigma_{A_{\varepsilon(h)}} \mathcal{A} w_{\varepsilon(h)}.$$

It is easy to check that the scaling rule (4.33), and the relations (4.45) and (4.46) imply

$$J^{\varepsilon(h)}(\mathcal{A} w_{\varepsilon(h)}, A_{\varepsilon(h)}) = 1, \quad \text{and} \quad (4.48)$$

$$\int_{B_\rho} \sigma_{A_{\varepsilon(h)}} \mathcal{A} w_{\varepsilon(h)} \cdot \mathcal{A} w_{\varepsilon(h)} + \varepsilon(h)^2 \text{Per}(A_{\varepsilon(h)}; B_\rho) > \rho^{d-(1+\delta)/2}. \quad (4.49)$$

In particular, due to the coercivity of σ_1 and σ_2 , the norms $\|\mathcal{A} w_{\varepsilon(h)}\|_{L^2(B_1)}^2$ are h -uniformly bounded by M .

Local almost-minimizers of $G^{\varepsilon(h)}$. The next step is to show that $\{w_{\varepsilon(h)}\}$ is $O(\varepsilon)$ -close in L^2 to a sequence $\{\tilde{w}_\varepsilon\}$ of almost minimizers of $\{u \mapsto G^{\varepsilon(h)}(\mathcal{A} u)\}$. Observe that $w_{\varepsilon(h)}$ is the unique solution to the equation

$$\mathcal{A}^*(\sigma_{A_\varepsilon} \mathcal{A} u) = f_{\varepsilon(h)}, \quad u \in W_{w_{\varepsilon(h)}}^{\mathcal{A}}(B_1).$$

Let $\tilde{w}_{\varepsilon(h)}$ be the unique minimizer of $u \mapsto J^{\varepsilon(h)}(\mathcal{A} u, A_{\varepsilon(h)})$ – see (4.32) – in the affine space $W_{w_{\varepsilon(h)}}^{\mathcal{A}}(B_1)$. Thus, in particular, $\tilde{w}_{\varepsilon(h)}$ is the unique solution of the equation

$$\mathcal{A}^*(\sigma_{A_{\varepsilon(h)}} \mathcal{A} u) = 0, \quad u \in W_{w_{\varepsilon(h)}}^{\mathcal{A}}(B_1).$$

A simple integration by parts, considering that $\tilde{w}_\varepsilon(h) - w_\varepsilon(h) \in W_0^{\mathcal{A}}(B_1)$, gives the estimate

$$\|\mathcal{A} w_{\varepsilon(h)} - \mathcal{A} \tilde{w}_{\varepsilon(h)}\|_{L^2(B_1)}^2 \leq C(B_1) \cdot M^2 \|f_{\varepsilon(h)}\|_{L^2(B_1)}^2 = O(h^{-1}), \quad (4.50)$$

where $C(B_1)$ is the Poincaré constant from (4.12); and therefore $\|w_{\varepsilon(h)} - \tilde{w}_{\varepsilon(h)}\|_{W_0^{k,2}(B_1)} = O(h^{-1})$.

Lastly, we use strongly the fact that (w, A) is a saddle-point of (P) to see that $\{(w_{\varepsilon(h)}, A_{\varepsilon(h)})\}$ is also a *local* saddle-point of the energy

$$(u, E) \mapsto \tilde{I}^{\varepsilon(h)}(\mathcal{A} u, E) := \int_{B_1} 2\tau_E \cdot \mathcal{A} u \, dy - \int_{B_1} \sigma_E \mathcal{A} u \cdot \mathcal{A} u \, dy + \varepsilon(h)^2 \text{Per}(E; B_1).$$

Moreover, by (4.33), (4.46) and (4.50) one has that

$$\tilde{I}^{\varepsilon(h)}(\mathcal{A} w_{\varepsilon(h)}, A_{\varepsilon(h)}) = J^{\varepsilon(h)}(\mathcal{A} w_{\varepsilon(h)}, A_{\varepsilon(h)}) = J^{\varepsilon(h)}(\mathcal{A} \tilde{w}_{\varepsilon(h)}, A_{\varepsilon(h)}) + O(h^{-1}). \quad (4.51)$$

An immediate consequence of the two facts above is that $\{\tilde{w}_{\varepsilon(h)}\}$ is a sequence of *local* almost minimizers of the sequence of functionals $\{u \mapsto G^{\varepsilon(h)}(\mathcal{A}u)\}$. The local (almost) minimizing properties of the sequence $\{\tilde{w}_{\varepsilon(h)}\}$ – with respect to the functionals $\{u \mapsto G^{\varepsilon(h)}(\mathcal{A}u)\}$ – are not affected by subtracting \mathcal{A} -free fields; hence, using the compactness assumption of \mathcal{A} once more, we may assume without loss of generality that $\sup_h \|\tilde{w}_{\varepsilon(h)}\|_{W^{k,2}(B_1)} < \infty$. Upon passing to a further subsequence, we may also assume that there exists $\tilde{w} \in W^{k,2}(B_1; \mathbb{R}^m)$ such that

$$\tilde{w}_{\varepsilon(h)} \rightharpoonup \tilde{w} \quad \text{in } W^{k,2}(B_1; \mathbb{R}^m).$$

Equi-integrability of $\{\mathcal{A}\tilde{w}_{\varepsilon(h)}\}$. The last but one step is to show that $\{\mathcal{A}\tilde{w}_{\varepsilon}\}$ is a 2-equi-integrable sequence in B_s , for every $s < 1$.

Since σ_{A_ε} is uniformly bounded, there exists $\tilde{\tau} \in L^2(B_1; \mathbb{M}^{m \times d^k})$ such that (upon passing to a further subsequence)

$$\sigma_{A_{\varepsilon(h)}} \mathcal{A}\tilde{w}_{\varepsilon(h)} =: \tilde{\tau}_{\varepsilon(h)} \rightharpoonup \tilde{\tau} \quad \text{in } L^2(B_1; \mathbb{M}^{m \times d^k}), \quad \mathcal{A}^* \tilde{\tau}_{\varepsilon(h)} = \mathcal{A}^* \tilde{\tau} = 0. \quad (4.52)$$

Let $\varphi \in \mathcal{D}(B_1)$ and fix $\varepsilon > 0$, integration by parts yields

$$\langle \tilde{\tau}_{\varepsilon(h)} \cdot \mathcal{A}\tilde{w}_{\varepsilon(h)}, \varphi \rangle = - \sum_{\substack{|\beta| \geq 1 \\ |\alpha| + |\beta| = k}} c_{\alpha\beta} \langle \tilde{\tau}_{\varepsilon(h)}, \partial^\alpha \tilde{w}_{\varepsilon(h)} \partial^\beta \varphi \rangle \quad c_{\alpha\beta} \in \mathbb{R}.$$

Since the term in the right hand side of the equality depends only on $\nabla^{k-1} \tilde{w}_{\varepsilon(h)}$, the strong convergence $\tilde{w}_{\varepsilon} \rightarrow \tilde{w}$ in $W^{k-1,2}(B_1; \mathbb{R}^m)$ gives

$$\lim_{\varepsilon \rightarrow 0} \langle \tilde{\tau}_{\varepsilon(h)} \cdot \mathcal{A}\tilde{w}_{\varepsilon(h)}, \varphi \rangle = - \sum_{\substack{|\beta| \geq 1 \\ |\alpha| + |\beta| = k}} c_{\alpha\beta} \langle \tilde{\tau}, \partial^\alpha \tilde{w} \partial^\beta \varphi \rangle = \langle \tilde{\tau} \cdot \mathcal{A}\tilde{w}, \varphi \rangle.$$

Therefore,

$$\sigma_{A_{\varepsilon(h)}} \mathcal{A}\tilde{w}_{\varepsilon(h)} \cdot \mathcal{A}\tilde{w}_{\varepsilon(h)} = \tilde{\tau}_{\varepsilon(h)} \cdot \mathcal{A}\tilde{w}_{\varepsilon(h)} \xrightarrow{*} \tilde{\tau} \cdot \mathcal{A}\tilde{w} \in L^1(B_1) \quad \text{weakly* in } \mathcal{M}^+(B_1).$$

The positivity of $\sigma_{A_\varepsilon} \mathcal{A}\tilde{w}_\varepsilon \cdot \mathcal{A}\tilde{w}_\varepsilon$, the Dunford-Pettis Theorem and the convergence above imply that the sequence

$$\{\sigma_{A_\varepsilon} \mathcal{A}\tilde{w}_\varepsilon \cdot \mathcal{A}\tilde{w}_\varepsilon\} \quad \text{is equi-integrable in } B_s; \text{ for every } s < 1.$$

In turn, due to the uniform coerciveness and boundedness of $\{\sigma_{A_\varepsilon}\}$, both sequences $\{\mathcal{A}\tilde{w}_\varepsilon\}$ and $\{\tilde{\tau}_\varepsilon\}$ are 2-equi-integrable in B_s ; for every $s < 1$.

The contradiction. We are in position to apply Proposition 4.23 to the sequence $\{\tilde{w}_\varepsilon\}$, which in particular implies

$$\begin{aligned} \varepsilon(h)^2 \text{Per}(A_{\varepsilon(h)}; B_\rho) &\rightarrow 0, \\ \sigma_{A_{\varepsilon(h)}} \mathcal{A}\tilde{w}_{\varepsilon(h)} \cdot \mathcal{A}\tilde{w}_{\varepsilon(h)} &\rightharpoonup Q_{\mathcal{B}} W(\mathcal{A}\tilde{w}) \leq M |\mathcal{A}\tilde{w}|^2, \quad \text{in } L^1_{\text{loc}}(B_1), \end{aligned} \quad (4.53)$$

and that w is a local minimizer of $u \mapsto G(\mathcal{A} u)$. On the other hand, the higher integrability assumption (Reg) tells us that

$$[\mathcal{A} \tilde{w}]_{L^{2, N-\delta}(B_{1/2})}^2 \leq c \|\mathcal{A} \tilde{w}\|_{L^2(B_1)}^2. \quad (4.54)$$

We set the value of $\rho \in (0, 1/2)$ to be such that $2cM^2\rho^{(1-\delta)/2} \leq 1$. Taking the limit in (4.48) and (4.49), using Fatou's Lemma, (4.50), (4.51), (4.53) and (4.54), we get

$$\begin{aligned} \frac{1}{M} \|\mathcal{A} \tilde{w}\|_{L^2(B_1)}^2 &\leq \liminf_{h \rightarrow \infty} J^{\varepsilon(h)}(\mathcal{A} \tilde{w}_{\varepsilon(h)}, A_{\varepsilon(h)}) = 1 \\ &\leq \left(\frac{1}{\rho^{d-(1+\delta)/2}} \right) \|\mathcal{Q}_{\mathcal{B}} W(\mathcal{A} \tilde{w})\|_{L^1(B_\rho)} \leq \left(\frac{M\rho^{(1-\delta)/2}}{\rho^{d-\delta}} \right) \|\mathcal{A} \tilde{w}\|_{L^2(B_\rho)}^2 \\ &\leq M\rho^{(1-\delta)/2} [\mathcal{A} \tilde{w}]_{L^{2, d-\delta}(B_{1/2})}^2 \leq cM\rho^{(1-\delta)/2} \|\mathcal{A} \tilde{w}\|_{L^2(B_1)}^2 \\ &\leq \frac{1}{2M} \|\mathcal{A} \tilde{w}\|_{L^2(B_1)}^2; \end{aligned}$$

a contradiction. □

Theorem 4.2 (upper bound). *Let (w, A) be a variational solution of problem (P). Assume that the higher integrability condition*

$$[\mathcal{A} \tilde{u}]_{L^{2, d-\delta}(B_{1/2})}^2 \leq c \|\mathcal{A} \tilde{u}\|_{L^2(B_1)}^2, \quad \text{for some } \delta \in [0, 1) \text{ and some positive constant } c,$$

holds for local minimizers of the energy $u \mapsto \int_{B_1} \mathcal{Q}_{\mathcal{B}} W(\mathcal{A} u)$, where $u \in \mathbf{W}^{\mathcal{A}}(B_1)$. Then, for every compactly contained set $K \subset \subset \Omega$, there exists a positive constant Λ_K such that

$$\int_{B_r(x)} \sigma_A \mathcal{A} w \cdot \mathcal{A} w \, dy + \text{Per}(A; B_r(x)) \leq \Lambda_K r^{d-1} \quad \forall x \in K, \forall r \in (0, \text{dist}(K, \partial\Omega)). \quad (4.55)$$

Proof. Let $x \in K$, and set

$$\varphi(r, x) := J_{B_r(x)}(\mathcal{A} w, A),$$

where we recall that

$$J_{B_r(x)}(\mathcal{A} w, A) = \int_{B_r(x)} \sigma_A \mathcal{A} w \cdot \mathcal{A} w \, dy + \text{Per}(A; B_r(x))$$

Proposition 4.24 tells us that there exists a positive constant $\rho \in (0, 1/2)$ such that if $B_r(x) \subset \Omega$, then

$$\varphi(\rho r, x) \leq \rho^{d-(1+\delta)/2} \varphi(r, x) + C(K)r^{d-1}.$$

An application of the Iteration Lemma [19, Lem. 2.1, Ch. III] (stated below) to $r \in (0, \min\{1, \text{dist}(K, \partial\Omega)\})$, and $\alpha_1 := d - (1 + \delta)/2 > \alpha_2 := d - 1$ yields the existence of positive constants $c = c(x)$, and $r = r(K)$ such that

$$\varphi(s, x) \leq cs^{d-1} \quad \forall s \in (0, R(K)).$$

Notice that the constants c and r depend continuously on $x \in \Omega$. Hence, for any $K \subset \subset \Omega$ we may find

$\Lambda_K > 0$ for which

$$J_{B_r(x)}(\mathcal{A} w, A) \leq \Lambda_K r^{d-1} \quad \forall x \in K, \forall r \in (0, \text{dist}(K, \partial\Omega)).$$

□

Lemma 4.25 (Iteration Lemma). *Assume that $\varphi(\rho)$ is a non-negative, real-valued, non-decreasing function defined on the $(0, 1)$ interval. Assume further that there exists a number $\tau \in (0, 1)$ such that for all $r < 1$ we have*

$$\varphi(\tau r) \leq \tau^{\alpha_1} \varphi(r) + C r^{\alpha_2}$$

for some non-negative constant C , and positive exponents $\alpha_1 > \alpha_2$. Then there exists a positive constant $c = c(\tau, \alpha_1, \alpha_2)$ such that for all $0 \leq \rho \leq r \leq R$ we have

$$\varphi(\rho) \leq c \left(\frac{\rho}{r}\right)^{\alpha_2} \varphi(r) + C \rho^{\alpha_2}.$$

Corollary 4.26 (compactness of blow-up sequences). *Let (w, A) be a variational solution of problem (P). Under the assumptions of the upper bound Theorem 4.2, there exists a positive constant C_K such that*

$$[\mathcal{A} w]_{L^{2,d-1}(K)}^2 \leq C_K. \quad (4.56)$$

Proof. The assertion follows directly from the Upper Bound Theorem and the coercivity of σ_1 and σ_2 . □

4.5 The Lower Bound: proof of estimate (LB)

During this section we will write (w, A) to denote a solution of problem (P) under the assumptions of Theorem 4.2. In light of the results obtained in the previous section we will assume, throughout the rest of the paper, that for every compact set $K \subset\subset \Omega$ there exist positive constants C_K , and Λ_K such that

$$\begin{aligned} \text{Per}(A; B_r(x)) &\leq \Lambda_K r^{d-1}, \\ \|\mathcal{A} w^{x,r}\|_{L^2(B_1)}^2 &\leq [\mathcal{A} w]_{L^{2,d-1}(K)}^2 \leq C_K, \end{aligned}$$

for all $x \in K$ and every $r \in (0, \text{dist}(K, \partial\Omega))$.

The main result of this section is a lower bound on the density of the perimeter in $\partial^* A$. In other words, there exists a positive constant $\lambda_K = \lambda_K(d, M)$ such that

$$\text{Per}(A; B_r(x)) \geq \lambda_K r^{d-1} \quad \text{for every } 0 < r < \text{dist}(x, \partial\Omega). \quad (\text{LB})$$

There are two major consequences from estimate (LB). The first one (cf. Corollary 4.34) is that the difference between the topological boundary of A and the reduced boundary of A is at most a set of zero \mathcal{H}^{d-1} -measure. In other words, $(\partial A \setminus \partial^* A) = \Sigma$ where $\mathcal{H}^{d-1}(\Sigma) = 0$ (cf. [6, Theorem 2.2]).

The second implication is that (LB) is a necessary assumption for the Height bound Lemma and the Lipschitz approximation Lemma, which are essential tools to prove the flatness excess improvement in the next section.

Throughout this section and the rest of the manuscript we will constantly use the following notations:

The scaled Dirichlet energy

$$D(w; x, r) := \frac{1}{r^{d-1}} \int_{B_r(x)} |\mathcal{A}w|^2 dy,$$

and the excess for γ -weighted energy

$$E_\gamma(w, A; x, r) := D(w; x, r) + \frac{\gamma}{r^{d-1}} \text{Per}(A, B_r(x)).$$

Granted that the spatial-, radius-, or (w, A) - dependence is clear, we will shorten the notations to the only relevant variables, e.g., $D(r)$ and $E_\gamma(r)$. Recall that, up to translation and re-scaling, we may assume

$$0 \in \partial^* A \cap K, \quad \text{and} \quad B_1 \subset K + B_0 \subset \Omega.$$

Bear also in mind that all the constants in this section are universal up to their dependence on Λ_K and C_K .

We will proceed as follows. First we prove in Lemma 4.27 that if the density of the perimeter is sufficiently small, one may regard the regularity properties of solutions as those ones for an elliptic equation with constant coefficients. Then, in Lemma 4.28, we prove a lower bound on the decay of the density of the perimeter in terms of D . Combining these results, we are able to show a discrete monotonicity formula on the decay of E_γ .

The proof of the lower density bound (LB) follows easily from this discrete monotonicity formula, De Giorgi's Structure Theorem, and the upper bound Theorem of the previous section. Finally, we prove that the difference between ∂A and $\partial^* A$ is \mathcal{H}^{d-1} -negligible (Theorem 4.34) as a corollary of the estimate (LB).

Lemma 4.27 (approximative solutions of the constant coefficient problem). *For every $\theta_1 \in (0, 1/2)$, there exist positive constants⁹ $c_1(\theta_1, d, M)$ and $\varepsilon_1(\theta_1, d, M)$ such that either*

$$\int_{B_\rho} |\mathcal{A}w|^2 dy \leq c_1 \rho^d \|f\|_{L^\infty(B_1)}^2,$$

or

$$\int_{B_\rho} |\mathcal{A}w|^2 dy \leq 2c\rho^N \int_{B_1} |\mathcal{A}w|^2 dy \quad \text{for every } \rho \in [\theta_1, 1),$$

where $c = c(d, M)$ is the constant from Lemma 4.11; whenever

$$\text{Per}(A; B_1) \leq \varepsilon_1.$$

⁹As it can be seen from the proof of Lemma 4.27, the constant c_1 does not depend on K .

Proof. Since $c \geq 2^d$, the result holds if we assume $\rho \geq 1/2$, therefore we focus only on the case where $\rho \in (\theta_1, 1/2)$. Fix $\theta_1 \in (0, 1/2]$. We argue by contradiction: We would find a sequence of pairs (w_h, A_h) (locally solving (P) in B_1 for a source function f_h) and constants $\rho_h \in [1/2, \theta_1]$, such that

$$\delta_h^2 := \int_{B_{\rho_h}} |\mathcal{A} w_h|^2 dy > 2c\rho_h^d \int_{B_1} |\mathcal{A} w_h|^2 dy, \quad (4.57)$$

and simultaneously

$$\rho_h^d \cdot \frac{\|f_h\|_{L^\infty(B_1)}^2}{\delta_h^2} \leq \frac{1}{h}, \quad \text{and} \quad \text{Per}(A_h; B_1) \leq \frac{1}{h}.$$

The estimate above yields $\delta_h^{-1} f_h \rightarrow 0$ in $L^2(B_1; \mathbb{R}^m)$. Also, since $\text{Per}(A_h; B_1) \rightarrow 0$, the isoperimetric inequality yields that either $\sigma_{A_h} \rightarrow \sigma_1$ or $\sigma_{A_h} \rightarrow \sigma_2$ in L^2 as h tends to infinity. Let us assume that the former convergence $\sigma_{A_h} \rightarrow \sigma_1$ holds.

Let $u_h := \delta_h^{-1} w_h$, for which

$$\sup_h \|\mathcal{A} u_h\|_{L^2(B_1)} < \infty.$$

We use that w_h is a (local) solution to (P) for A_h as indicator set and f_h as source term, to see that

$$\mathcal{A}^*(\sigma_{A_h} \mathcal{A} u_h) = \delta_h^{-1} f_h \quad \text{in } B_1.$$

Up to passing to a further subsequence, we may assume that $u_h \rightharpoonup u$ in $W^{k,2}(B_1; \mathbb{R}^m)$. We may then apply the compensated compactness result from Lemma 4.14 to obtain that

$$\mathcal{A}^*(\sigma_1 \mathcal{A} u) = 0 \quad \text{in } B_1,$$

and

$$D(u_h; s) \rightarrow D(u; s) \quad \text{where } \rho_h \rightarrow s \in [\theta_1, 1/2].$$

Hence, by (4.57) and Fatou's Lemma one gets

$$2cs^d D(u; 1) \leq \liminf_{h \rightarrow \infty} c\rho_h^d D(u_h; 1) \leq 1 = \lim_{h \rightarrow \infty} D(u_h; \rho_h) = \lim_{h \rightarrow \infty} D(u_h; s) = D(u; s).$$

This is a contradiction to Lemma 4.11 because u is a solution for the problem with constant coefficients σ_1 . The case when $\sigma_{A_h} \rightarrow \sigma_2$ can be solved by similar arguments. \square

The next lemma is the principal ingredient in proving the (LB) estimate. It relies on a cone-like comparison to show that the decay of the perimeter density is controlled by $D(r)/r$: The perimeter density cannot blow-up at smaller scales, while for a fixed scale, the perimeter density is small.

Lemma 4.28 (universal comparison decay). *There exists a positive constant¹⁰ $c_2 = c_2(d, M)$ such that*

$$\frac{d}{dr} \Big|_{\rho=r} \left(\frac{\text{Per}(A; B_\rho)}{\rho^{d-1}} \right) \geq -c_2 \frac{D(r)}{r} \quad \text{for a.e. } r \in (0, 1].$$

¹⁰The constant c_2 is independent of the compact set K ; indeed, this is the result of universal comparison estimates in Ω .

Proof. For a.e. $r \in (0, 1)$ the slice $\langle A, g, r \rangle$, where $g(x) = |x|$, is well defined (see Section 4.2.4). Fix one such r and let \tilde{A} be the cone-like comparison set to A as in (4.23). By minimality of (w, A) and a duality argument, we get

$$\int_{B_r} \sigma_A^{-1} \tau_A \cdot \tau_A \, dy + \text{Per}(A; B_r) \leq \int_{B_r} \sigma_{\tilde{A}}^{-1} \tau_A \cdot \tau_A \, dy + \text{Per}(\tilde{A}; B_r)$$

for $\tau_A = \sigma_A \mathcal{A} w$. Hence,

$$\begin{aligned} \text{Per}(A; B_r) &\leq \text{Per}(\tilde{A}; B_r) + M \int_{B_r} |\mathcal{A} w_A|^2 \, dy \\ &\leq \frac{r}{d-1} \langle A, g, r \rangle(\mathbb{R}^d) + M^3 r^{d-1} D(r). \end{aligned} \tag{4.58}$$

To reach the inequality in the last row we have used that the cone extension \tilde{A} is precisely built (cf. (4.24)) so that the Green-Gauss measures $\mu_{\tilde{A}}$ and μ_A agree in $(B_r)^c$; where, by (4.22),

$$\text{Per}(\tilde{A}; B_\rho) = \frac{1}{(d-1)} \left(\frac{\rho^{d-1}}{r^{d-2}} \right) \mathcal{H}^{d-2}(\partial^* A \cap \{g = r\}) \leq \frac{1}{(d-1)} \left(\frac{\rho^{d-1}}{r^{d-2}} \right) \langle A, g, r \rangle(\mathbb{R}^d)$$

for all $0 < \rho \leq r$. We know from (4.25) that $\frac{d}{d\rho} \Big|_r \text{Per}(A; B_\rho) \geq \langle A, g, r \rangle(\mathbb{R}^d)$ for a.e. $r > 0$. Since (4.58) and the previous inequality are valid almost everywhere in $(0, 1)$, a combination of these arguments yields

$$\frac{d}{dr} \Big|_{\rho=r} \left(\frac{\text{Per}(A; B_\rho)}{\rho^{d-1}} \right) \geq -M^3 (d-1) \frac{D(r)}{r} \quad \text{for a.e. } r \in (0, 1).$$

The result follows for $c_2 := M^3 (d-1)$. □

The following result is a discrete monotonicity for the weighted excess energy E_γ . We remark that, in general, a monotonicity formula may not be expected in the case of systems.

Theorem 4.29 (Discrete monotonicity). *There exist positive constants $\gamma = \gamma(d, M)$, $\varepsilon_2 = \varepsilon_2(\gamma, d) \leq \text{vol}(B'_1) \cdot \gamma/2$, and $\theta_2 = \theta_2(d, M) \in (0, 1/2)$ such that*

$$E_\gamma(\theta_2) \leq E_\gamma(1) + c_1(\theta_2) \|f\|_{L^\infty(B_1)}^2, \quad \text{whenever } E_\gamma(1) \leq \varepsilon_2. \tag{4.59}$$

Proof. We fix γ and θ_1 such that

$$\gamma c_2 \max\{c, c_1(\theta_1)\} \leq \frac{1}{4}, \quad \text{where } 2\theta_1 c \leq \frac{1}{2}.$$

Set $\theta_2 := \theta_1$. Recall that c_2 is the constant from Lemma 4.28, and c is the constant of Lemma 4.11.

Let also $\varepsilon_2 = \varepsilon_2(\gamma, \varepsilon_1)$ be a positive constant with $\varepsilon_2 \leq \min\{\gamma \varepsilon_1(\theta_2), \gamma \cdot \text{vol}(B'_1)/2\}$. This implies

$$\text{Per}(A; B_1) \leq \varepsilon_1(\theta_2),$$

which in turn gives for $c_1 = c_1(\theta_2)$ (see Lemma 4.27)

$$E_\gamma(\theta_2) \leq \frac{\gamma}{\theta_2^{d-1}} \text{Per}(A; B_{\theta_2}) + 2c\theta_2 D(1) + c_1\theta_2 \|f\|_{L^\infty(B_1)}^2.$$

Now, we apply Lemma 4.27 and Lemma 4.28 to $s \in (\theta_2, 1)$ to get

$$\begin{aligned} E_\gamma(\theta_2) &\leq \frac{\gamma}{\theta_2^{d-1}} \text{Per}(A; B_{\theta_2}) + 2c\theta_2 D(1) + c_1\theta_2 \|f\|_{L^\infty(B_1)}^2 \\ &\leq \gamma \text{Per}(A; B_1) + \gamma \int_{\theta_2}^1 -\frac{d}{dr} \Big|_{r=s} \left(\frac{\text{Per}(A, B_r)}{r^{d-1}} \right) ds + \frac{1}{2} D(1) + c_1\theta_2 \|f\|_{L^\infty(B_1)}^2 \\ &\leq \gamma \text{Per}(A; B_1) + \gamma c_2 \int_{\theta_1}^1 \frac{D(s)}{s} ds + \frac{1}{2} D(1) + c_1\theta_2 \|f\|_{L^\infty(B_1)}^2 \\ &\leq \gamma \text{Per}(A; B_1) + 2\gamma c c_2 D(1) + \gamma c_2 c_1 \|f\|_{L^\infty(B_1)}^2 + \frac{1}{2} D(1) + c_1\theta_2 \|f\|_{L^\infty(B_1)}^2 \\ &\leq \gamma \text{Per}(A; B_1) + D(1) + c_1 \|f\|_{L^\infty(B_1)}^2 \\ &= E_\gamma(1) + c_1 \|f\|_{L^\infty(B_1)}^2. \end{aligned}$$

This proves the desired result. \square

Lemma 4.30. *For every $\varepsilon > 0$, there exist positive constants $\theta_0(d, M, K, \varepsilon) \in (0, 1/2)$ and $\kappa(d, M, K, \varepsilon) > 0$ such that*

$$E_\gamma(\theta_0) \leq \varepsilon + c_1 \|f\|_{L^\infty(B_1)}^2;$$

whenever

$$\text{Per}(A; B_1) \leq \kappa.$$

Proof. The result follows by taking θ_0 such that $2c\theta_0 C_K \leq \varepsilon/2$ (recall that, $D(s) \leq C_K$ for every $s \in (0, 1)$) and $\kappa \leq \min \left\{ \frac{\varepsilon \theta_0^{d-1}}{2\gamma}, \varepsilon_1(\theta_0) \right\}$ and then simply applying Lemma 4.27. \square

Lemma 4.31. *Let (w, A) be a saddle-point of (P) and let $x \in K \subset \subset \Omega$. Then, for every $\varepsilon > 0$ there exists a positive radius $r_0 = r_0(d, M, K, \|f\|_{L^\infty(B_1)}, \varepsilon)$ for which*

$$E_\gamma(w, A; x, r) \leq 2\varepsilon;$$

whenever $r \leq r_0$ and $\text{Per}(A; B_{\theta_0^{-1}r}) \leq \kappa(\varepsilon) \cdot \left(\frac{r}{\theta_0}\right)^{d-1}$.

Proof. Let r_0 be a positive constant such that $c_1 r_0^{2k+1} \|f\|_{L^\infty(B_1)}^2 \leq \theta_0^{2k+1} \varepsilon$ and let us set $s := \theta_0^{-1}r$. Since

$$\text{Per}(A^{x,s}; B_1) = s^{-(d-1)} \text{Per}(A; B_s) \leq \kappa(\varepsilon),$$

it follows from the previous lemma and a rescaling argument that

$$E_\gamma(w, A; r) = E_\gamma(w, A; \theta_0 s) \leq \varepsilon + c_1 \|f^s\|_{L^\infty(B_1)}^2 = \varepsilon + c_1 s^{2k+1} \|f\|_{L^\infty(B_1)}^2 \leq 2\varepsilon.$$

\square

Theorem 4.32 (lower bound). *Let (w, A) be a solution of problem (P) in Ω . Let $K \subset\subset \Omega$ be a compact subset. Then, there exist positive constants λ_K and r_K depending only on K , the dimension N , the constant M in the assumption (4.2), and f such that*

$$\text{Per}(A; B_r(x)) \geq \lambda_K r^{d-1}, \quad (\text{LB})$$

for every $r \in (0, r_K)$ and every $x \in \partial^* A \cap K$.

Proof. Let $p(\theta_2) := \sum_{h=0}^{\infty} \theta_2^{(2k+1)h} \in \mathbb{R}$ and define $r_1 \in (0, 1)$ to be a positive constant for which

$$r_1^{2k+1} c_1(\theta_2) p(\theta) \|f\|_{L^\infty(B_1)}^2 \leq \frac{\varepsilon_2}{4}.$$

We argue by contradiction. If the assertion does not hold, we would be able to find a point $x \in \partial^* A$ and a radius $r \leq \min\{r_0, r_1\}$ for which

$$\text{Per}(A; B_{\frac{r}{\theta_0}}(x)) \leq \left(\frac{r}{\theta_0}\right)^{d-1} \kappa(\varepsilon), \quad \varepsilon := \frac{\varepsilon_2}{4}.$$

After translation, we may assume that $x = 0$. The fact that $r \leq r_0$ and Lemma 4.31 yield the estimate

$$E_\gamma(w, A; r) \leq 2\varepsilon \leq \frac{\varepsilon_2}{2};$$

in return, Lemma 4.29 and a rescaling argument give (recall that $f^r(y) = r^{k+\frac{1}{2}} f(ry)$)

$$E_\gamma(w, A; \theta_2 r) \leq E_\gamma(w^r, A^r; 1) + c_1 \|f^r\|_{L^\infty(B_1)}^2 \leq \frac{\varepsilon_2}{2} + c_1 r^{2k+1} \|f\|_{L^\infty(B_1)}^2 \leq \varepsilon_2.$$

A recursion of the same argument gives the estimate

$$E_\gamma(w, A; \theta_2^j r) \leq E_\gamma(w, A; r) + c_1 r^{2k+1} \|f\|_{L^\infty(B_1)}^2 \left(\sum_{h=0}^j \theta_2^{(2k+1)h} \right) \leq \varepsilon_2.$$

Taking the limit as $j \rightarrow \infty$ we get

$$\limsup_{j \rightarrow \infty} \frac{\text{Per}(A; B_{\theta_2^j r})}{\text{vol}(B'_1) \cdot (\theta_2^j r)^{d-1}} \leq \limsup_{j \rightarrow \infty} \frac{E_\gamma(w, A; \theta_2^j r)}{\text{vol}(B'_1) \cdot \gamma} \leq \frac{\varepsilon_2}{\text{vol}(B'_1) \cdot \gamma} \leq \frac{1}{2}.$$

This a contradiction to the fact that $x = 0 \in \partial^* A$ (cf. Section 4.2.4)

□

Corollary 4.33. *Let (w, A) be a solution for problem (P) in Ω . Let $K \subset\subset \Omega$ be a compact subset. Then, there exist positive constants λ_K and r_K depending only on K , the dimension d , and f such that*

$$\text{Per}(A; B_r(x)) \geq \lambda_K r^{d-1},$$

for every $r \in (0, r_K)$ and for every $x \in \partial A \cap K$.

Proof. The property (LB) from the Lower Bound Theorem is a topologically closed property, i.e., it extends to $\overline{\partial^* A} = \text{supp } \mu_A = \partial A$ (cf. (4.19)). \square

Corollary 4.34. *Under the same assumptions of Theorem 4.32, the following characterization for the topological boundary of A holds:*

$$\partial A = \partial^* A \cup \Sigma, \quad \text{where } \mathcal{H}^{d-1}(\Sigma) = 0.$$

Proof. An immediate consequence of the previous corollary is that $\mathcal{H}^{d-1} \llcorner \partial A \ll |\mu_A|$ as measures in Ω . The assertion follows by De Giorgi's Structure Theorem. \square

4.6 Proof of Theorem 4.5

As we have established in the past section, we will assume that for every $K \subset\subset \Omega$ there exist positive constants λ_K, C_K such that $D(w; x, r) \leq C_K$ and

$$\text{Per}(A, B_r(x)) \geq \lambda_K r^{d-1}, \quad \forall x \in (\partial A \cap K), \forall r \in (0, \text{dist}(K, \partial\Omega)). \quad (\text{LB})$$

Half-space regularity. Throughout this section we shall work with the additional assumption for solutions of the half-space problem: let $H := \{x \in \mathbb{R}^d : x_d > 0\}$ and let σ_H be the two-point valued tensor defined in (4.3) for $\Omega = B_1$ (so that $\sigma_H = \sigma_1$ in $H \cap B_1$), then the operator

$$P_H u := \mathcal{A}^*(\sigma_H \mathcal{A} u)$$

is hypoelliptic in $B_1 \setminus \partial H$ in the sense that, if $w \in L^2(B_1; \mathbb{R}^m)$, then

$$P_H w = 0 \quad \Rightarrow \quad w \in C^\infty(\overline{B_r^+}; \mathbb{R}^m) \cup C^\infty(\overline{B_r^-}; \mathbb{R}^m) \quad \text{for every } 0 < r < 1. \quad (4.60)$$

Furthermore, there exists a positive constant $c^* = c^*(d, M, \mathcal{A})$ such that

$$\begin{aligned} \frac{1}{\rho^d} \int_{B_\rho} |\nabla^k w|^2 \, dx &\leq c^* \int_{B_1} |\nabla^k w|^2 \, dx && \text{for all } 0 < \rho \leq \frac{1}{2}, \\ \frac{1}{\rho^d} \int_{B_\rho} |\mathcal{A} w|^2 \, dx &\leq c^* \int_{B_1} |\mathcal{A} w|^2 \, dx && \text{for all } 0 < \rho \leq \frac{1}{2}, \\ \sup_{B_\rho^+ \cup B_\rho^-} |\nabla^{k+1} w|^2 &\leq c^* \int_{B_1} |w|^2 \, dx && \text{for all } 0 < \rho \leq \frac{1}{2}. \end{aligned} \quad (4.61)$$

Remark 4.35 (half-space regularity in applications). For 1-st order operators of gradient form it is relatively simple to show that such estimates as in (4.61) hold. This case includes gradients and symmetrized gradients; while the linear plate equations may be also reduced to this case (cf. Remark 4.13).

A sketch of the proof is as follows: The first step is to observe that the tangential derivatives ($i \neq d$)

¹¹The notation B_r^\pm stands for the upper and lower half ball of radius r : $B_r \cap H$ and $B_r \cap -H$ respectively.

$\partial_i w$ of a solution w of $P_H u = 0$ are also solutions of $P_H u = 0$. The second step is to repeat recursively the previous step and use the Caccioppoli inequality from Lemma 4.10 to estimate

$$\int_{B_{1/2}} |\partial^{\alpha} w|^2 dx \leq C(|\alpha|) \int_{B_1} |w|^2 dx \quad \text{for arbitrary } \alpha \text{ with } \alpha_d \leq 1. \quad (4.62)$$

The third step consists in using the ellipticity of $A_d = \mathbb{A}(\mathbf{e}_d)^{12}$ (cf. Remark 4.8) and the equation to express $\partial_{dd} w$ in terms of the rest of derivatives: The tensor $(A_d^T \sigma A_d)$ is invertible, this can be seen from the inequality $|\mathbb{A}(\mathbf{e}_d)z|^2 \geq \lambda(\mathcal{A})|z|^2$ for every $z \in \mathbb{R}^d$ (cf. 4.8) and the fact that σ_H satisfies Gårding's strong inequality (4.2) with M^{-1} . Hence, using that $P_H w = 0$, we may write

$$\partial_{dd} w = -(A_d^T \sigma_H A_d)^{-1} \sum_{ij \neq dd} (A_i^T \sigma_1 A_j) \partial_{ij} w \quad \text{in } B_1^+, \quad (4.63)$$

from which estimates for $\partial_{dd} w$ of the form (4.62) in the upper half ball easily follow (similarly for the lower half ball). Further ∂_d differentiation of the equation in B_1^\pm and iteration of this procedure together with the Sobolev embedding yield bounds as in (4.61).

For arbitrary higher-order gradients and other general elliptic systems one cannot rely on the same method. However, the Schauder and L^p boundary regularity of such systems has been systematically developed in [1, 2] through the so called *complementing condition*. In the case of strongly elliptic systems (cf. (4.2) and (4.11)) this complementing condition is fulfilled, see [2, pp 43-44]; see also [31] where a closely related *natural notion* of hypoellipticity of the half-space problem is assumed.

Flatness excess. Given a set $A \subset \mathbb{R}^d$ of locally finite perimeter, the *flatness excess* of A at x for scale r and with respect to the direction $\mathbf{v} \in \mathbb{S}^{d-1}$, is defined as

$$e(A; x, r, \mathbf{v}) := \frac{1}{r^{d-1}} \int_{C(x, r, \mathbf{v}) \cap \partial^* A} \frac{|\mathbf{v}_E(y) - \mathbf{v}|^2}{2} d\mathcal{H}^{d-1}(y).$$

Here, $C(x, r, \mathbf{v})$ denotes for the cylinder centered at x with height 2, that is parallel to \mathbf{v} , of radius r .

Intuitively, the flatness excess expresses for a set A , the deviation from being a hyperplane H at a given scale r . Again, up to re-scaling, translating and rotating, it will be enough to work the case $x = 0, \mathbf{v} = \mathbf{e}_d$, and $r = 1$. In this case, we will simply write $e(A)$. The hyper-plane energy excess is defined as

$$H_{\text{ex}}(w, A; x, r, \mathbf{v}) := e(A; x, r, \mathbf{v}) + D(w, A; x, r),$$

and as long as its dependencies are understood we will simply write $H_{\text{ex}}(r) = e(r) + D(r)$.

The following result relies on the (LB) property, a proof can be found in [16, §5.3] or [27, Theorem 22.8].

Lemma 4.36 (Height bound). *There exist positive constants $c_1^* = c_1^*(d)$ and $\epsilon_1^* = \epsilon_1^*(d)$ with the following property. If $A \subset \mathbb{R}^d$ is a set of locally finite perimeter with the (LB) property,*

$$0 \in \partial A \quad \text{and} \quad e(9) \leq \epsilon_1^*,$$

¹²Recall that, for a 1-st order operator as in (4.7), the coefficients A_α can be simply denoted by A_i with $i = 1, \dots, d$.

then

$$\sup \left\{ |y_d| : y \in B'_1 \times [-1, 1] \cap \partial A \right\} \leq c_1^* \cdot e(4)^{\frac{1}{2d-2}}. \quad (\text{HB})$$

The next decay lemma is the half-space problem analog of Lemma 4.27. The proof is similar except that it relies on the half-space regularity assumptions (4.60)-(4.61) (instead of the ones given by Lemma 4.11), and the Height bound Lemma stated above.

Lemma 4.37 (approximative solutions of the half-space problem). *Let (w, A) be a solution of problem (P) in B_1 . Then, for every $\theta_1^* \in (0, 1/2)$ there exist positive constants $c_2^*(\theta_1^*, d, M)$ and $\varepsilon_2^*(\theta_1^*, d, M)$ such that either*

$$\int_{B_\rho} |\mathcal{A}w|^2 \, dx \leq c_2^* \rho^d \|f\|_{L^\infty(B_1)}^2,$$

or

$$\int_{B_\rho} |\mathcal{A}w|^2 \, dx \leq 2c^* \rho^N \int_{B_1} |\mathcal{A}w|^2 \, dx \quad \text{for every } \rho \in [\theta_1, 1),$$

where $c^* = c^*(d, M)$ is the constant from the regularity condition (4.61); whenever

$$\text{Per}(A; B_1) \leq \varepsilon_2^*.$$

Remark 4.38. Let $\delta \in (0, 1)$. Then there exists $\kappa^* = \kappa^*(d, M, \delta)$ such that if $e(1) \leq \kappa^*$, and if one further assumes that the excess function $r \mapsto e(r)$ is monotone increasing, then the scaling $w(ry)/r^{(k-\frac{1}{2})}$ and the Iteration Lemma 4.25 imply that

$$\frac{1}{r^{d-\delta}} \int_{B_r} |\mathcal{A}w|^2 \leq C_\delta (\|\mathcal{A}w\|_{L^2(B_1)}^2 + c_2^* \|f\|_{L^\infty(B_1)}^2 \cdot r^{2k+\delta}) \quad \text{for every } r \in (0, 1/2),$$

for some positive constant $C_\delta = C_\delta(d, M)$.

The next crucial result can be found in [26, Section 5]. We have decided not to include a proof because because the ideas remain the same: the estimate (LB), the Height bound Lemma, the Lipschitz approximation Theorem, the estimates from Lemma 4.37 and the higher integrability for solutions to elliptic equations.¹³

Lemma 4.39 (flatness excess improvement). *Let (w, A) be a saddle point of problem (P) in Ω . There exist positive constants $\eta \in (0, 1]$, c_3^* , and ε_3 depending only on K , the dimension d , the constant M in (4.2), and $\|f\|_{L^\infty}$ with the following properties: If (w, A) is a saddle point of problem (P) in B_9 , and*

$$\text{H}_{\text{ex}}(9) \leq \varepsilon_3^*,$$

then, for every $r \in (0, 9)$, there exists a direction $\mathbf{v}(r) \in \mathbb{S}^{d-1}$ for which

$$|\mathbf{v}(r) - \mathbf{e}_d| \leq c_3^* \text{H}_{\text{ex}}(9) \quad \text{and} \quad \text{H}_{\text{ex}}(r, \mathbf{v}(r)) \leq c_3^* r^\eta \text{H}_{\text{ex}}(9).$$

¹³ $L^{2^*}(\Omega)$ -integrability of $\mathcal{A}w$, for some exponent $2^* > 2$, can be established by standard methods through the use of the Caccioppoli inequality in Lemma 4.10.

Theorem 4.5 (partial regularity). *Let (w, A) be a saddle point of problem (P) in Ω . Assume that the operator $P_H u = \mathcal{A}^*(\sigma \mathcal{A} u)$ is hypoelliptic and regularizing as in (4.60)-(4.61), and that the higher integrability condition*

$$[\mathcal{A} \tilde{u}]_{L^{2,d-\delta}(B_{1/2})}^2 \leq c \|\mathcal{A} \tilde{u}\|_{L^2(B_1)}^2, \quad \text{for some } \delta \in [0, 1),$$

holds for every local minimizer \tilde{u} of the energy $u \mapsto \int_{B_1} Q_{\mathcal{A}} W(\mathcal{A} u)$, where $u \in W^{\mathcal{A}}(B_1)$. Then there exists a positive constant $\eta \in (0, 1]$ depending only on d such that

$$\mathcal{H}^{d-1}((\partial A \setminus \partial^* A) \cap \Omega) = 0, \quad \text{and} \quad \partial^* A \quad \text{is an open } C^{1,\eta/2}\text{-hypersurface in } \Omega.$$

Moreover if \mathcal{A} is a first-order differential operator, then $\mathcal{A} w \in C_{\text{loc}}^{0,\eta/8}(\Omega \setminus (\partial A \setminus \partial^ A))$; and hence, the trace of $\mathcal{A} w$ exists on either side of $\partial^* A$.*

Proof. The reduced boundary is an open hypersurface. The first assertion $\mathcal{H}^{d-1}((\partial A \setminus \partial^* A) \cap \Omega) = 0$ is a direct consequence of Corollary 4.34.

To see that $\partial^* A$ is relatively open in ∂A we argue as follows: De Giorgi's Structure Theorem guarantees that for every $x \in \partial^* A$ there exist $r > 0$ (sufficiently small) and $\nu \in \mathbb{S}^{d-1}$ such that

$$H_{\text{ex}}(w, A; r, x, \nu) \leq \frac{1}{2} \varepsilon_3^*, \quad \text{and} \quad \mu_A(\partial B_r(x)) = 0.$$

The map $y \mapsto \mu_A(B_r(y)) = 0$ is continuous at x , therefore we may find $\delta(x) \in (0, 1)$ such that

$$H_{\text{ex}}(w, A; r, y, \nu) \leq \varepsilon_3^* \quad \text{for every } y \in B_{\delta(x)} \cap \partial A.$$

We may then apply Lemma 4.39 to get an estimate of the form

$$\inf_{\xi \in \mathbb{S}^{d-1}} H_{\text{ex}}(w, A; y, \rho, \xi) \leq c_3^* \rho^\eta H_{\text{ex}}(w, A; y, r, \nu) \quad \text{for all } y \in B_{\delta(x)}, \text{ and all } \rho \in (0, r).$$

This and the first assertion of Lemma 4.39 imply that $y \in \partial^* A$ for every $y \in B_{\delta(x)} \cap \partial A$. Therefore, the reduced boundary $\partial^* A$ is a relatively open subset of the topological boundary ∂A .

We proceed to prove the regularity for $\partial^* A$. It follows from the last equation that

$$D(w; y, \rho) \leq \inf_{\xi \in \mathbb{S}^{d-1}} H_{\text{ex}}(w, A; y, \rho, \xi) \leq c_3^* \varepsilon_3^* \rho^\eta \leq C \rho^\eta \tag{4.64}$$

for every $y \in B_{\delta(x)}$, and every $\rho \in (0, r)$, for some constant $C = C(C_{B_{\delta(x)}}, \Lambda_{B_{\delta(x)}}, d, M)$.

Through a simple comparison, we observe from (4.64) and the property that (w, A) is a local saddle point of problem (P) in $B_{\delta}(x)$, that

$$\text{Dev}_{B_{\delta}(x)}(A, \rho) \leq 2M \rho^{d-1} D(w; y, \rho) \leq 2MC \rho^{d-1+\eta}, \quad \text{for all } \rho \in (0, r) \text{ and every } y \in B_{\delta}(x).$$

We conclude with an application of Tamanini's Theorem 4.19:

$$\partial A = \partial^* A \text{ is a } C^{1,\eta/2}\text{-hypersurface in } B_\delta(x).$$

The assertion follows by observing that the regularity of $\partial^* A$ is a local property.

Jump conditions for the hyper-space problem. Let $\tau \in L^2_{\text{loc}}(B_1; Z) \cap (C^\infty(\overline{B_\rho^+}; Z) \cup C^\infty(\overline{B_\rho^-}; Z))$ for every $\rho \in (0, 1)$, assume furthermore that τ is a solution of the equation

$$\mathcal{A}^* \tau = 0 \quad \text{in } B_1.$$

Let $\eta \in C_c^\infty(B'_1; \mathbb{R}^m)$ be an arbitrary test function and choose a function $\varphi \in C_c^\infty(B_1; \mathbb{R}^m)$ with the following property:

$$\varphi(y', y_d) = \frac{y_d^{k-1}}{(k-1)!} \eta(y') \quad \text{in a neighborhood of } B'_1.$$

Then, integration by parts and Green's Theorem yield that

$$0 = \int_{B_1} \tau \cdot \mathcal{A} \varphi \, dy = \int_{\partial H \cap B_1} [\mathbb{A}(\mathbf{e}_d)^T \cdot \tau] \cdot \eta \, dy',$$

where $[\mathbb{A}(\mathbf{e}_d)^T \cdot \tau] = \mathbb{A}(\mathbf{e}_d)^T \cdot (\tau^+ - \tau^-)$. Here, τ^+ and τ^- are the traces of τ in ∂H from B_1^+ and B_1^- respectively. Since η is arbitrary, a density argument shows that

$$[\mathbb{A}(\mathbf{e}_d)^T \cdot \tau] = 0 \quad \text{in } \partial H \cap B_1, \quad \text{and hence} \quad \mathbb{A}(\mathbf{e}_d)^T \cdot \tau \in W_{\text{loc}}^{1,2}(B_1; \mathbb{R}^m). \quad (4.65)$$

Regularity of $\mathcal{A} w$. From this point and until the end of the proof we further assume that \mathcal{A} is a first-order differential operator of gradient form; we may as well assume that $\partial^* A$ is locally parametrized by $C^{1,\eta/2}$ functions.

Due to Campanato's Theorem ($C^{0,\eta/8} \simeq L^{2,d+(\eta/4)}$ on Lipschitz domains), our goal is to show local boundedness of the map

$$x \mapsto \sup_{r \leq 1} \left\{ \frac{1}{r^{d+(\eta/4)}} \int_{B_r(x) \cap A} |\mathcal{A} w - (\mathcal{A} w)_{B_r(x) \cap A}|^2 \, dy \right\} \quad x \in (\Omega \setminus (\partial A \setminus \partial^* A)); \quad (4.66)$$

and a similar result for A^c instead of A .

Also, since Campanato estimates in the interior are a simple consequence of Lemma 4.11, we may restrict our analysis to show only local boundedness at points $x \in \partial^* A$. We first prove the following decay for solutions of the half-space:

Lemma 4.40. *Let $\tilde{w} \in W^{\mathcal{A}}(B_1)$ be such that*

$$\mathcal{A}^*(\sigma_H \mathcal{A} \tilde{w}) = 0 \quad \text{in } B_1. \quad (4.67)$$

Then \tilde{w} satisfies an estimate of the form

$$\frac{1}{\rho^{d+2}} \int_{B_\rho} |R_H \tilde{w} - (R_H \tilde{w})_\rho|^2 dy \leq c(N, \sigma_1, \sigma_2) \int_{B_1} |R_H \tilde{w} - (R_H \tilde{w})_1|^2 dy \quad \text{for all } 0 < \rho \leq 1, \quad (4.68)$$

where we have defined

$$R_A u := (\nabla' u, A_d^T(\sigma_A \mathcal{A} u)), \quad A \subset B_1 \text{ Borel.}$$

Proof. Since for $\rho \geq 1/2$ one can use $c := 2^{-(d+2)}$, we only focus on proving the estimate for $\rho \in (0, 1/2)$. It is easy to verify that $\mathcal{A}^*(\sigma_H \mathcal{A}(\partial_i \tilde{w} - \lambda)) = 0$ in $\mathcal{D}'(B_1; \mathbb{R}^m)$ for all $\lambda \in \mathbb{R}^m$, and every $i = 1, \dots, d-1$. In particular, by (4.61) we know that

$$\frac{1}{\rho^{d+2}} \int_{B_\rho} |\partial_i \tilde{w} - (\partial_i \tilde{w})_\rho|^2 dy \leq \frac{C}{\rho^d} \int_{B_\rho} |\nabla \partial_i \tilde{w}|^2 dy \leq c^* C \int_{B_1} |\partial_i \tilde{w} - (\partial_i \tilde{w})_1|^2 dy, \quad (4.69)$$

for every $\rho \in (0, 1/2)$, and every $i = 1, \dots, d-1$. Here, $C = C(d)$ is the standard scaled Poincaré constant for balls. Summation over $i \in \{1, \dots, d-1\}$ yields an estimate of the form (4.68) for $\nabla' \tilde{w}$.

We are left to calculate the decay estimate for $g_H(\tilde{w}) := A_d^T(\sigma_H \mathcal{A} \tilde{w}) = \mathbb{A}(\mathbf{e}_d) \cdot (\sigma_H \mathcal{A} \tilde{w})$. By the hypoellipticity assumption (4.60) and the jump condition (4.65), we infer that $g_H(\tilde{w}) \in W_{\text{loc}}^{1,2}(B_1; \mathbb{R}^m)$.

Even more, by the same Poincaré's inequality

$$\frac{1}{\rho^{d+2}} \int_{B_\rho} |g(\tilde{w}) - (g(\tilde{w}))_\rho|^2 dy \leq \frac{C}{\rho^N} \int_{B_\rho \setminus \partial H} |\nabla(g(\tilde{w}))|^2 dy \quad (4.70)$$

for every $\rho \in (0, 1/2)$. On the other hand, it follows from the equation in $(B_1 \setminus \partial H)$ and (4.63) that one may write $\nabla g(\tilde{w})$ in terms of $\nabla(\nabla' \tilde{w})$ for almost every $x \in (B_r \setminus \partial H)$. We may then find a constant $C' = C'(\sigma_1, \sigma_2, \mathcal{A})$ such that

$$|\nabla g(\tilde{w}(x))|^2 \leq C' |\nabla(\nabla' \tilde{w})(x)|^2 \quad \text{for every } x \in (B_\rho \setminus \partial H).$$

Using the same calculation as in the derivation of (4.69), it follows from (4.70) that

$$\begin{aligned} \frac{1}{\rho^{d+2}} \int_{B_\rho} |g(\tilde{w}) - (g(\tilde{w}))_\rho|^2 dy &\leq c^* C C' \int_{B_1} |\nabla' \tilde{w} - (\nabla' \tilde{w})_1|^2 dy \\ &\leq c^* C C' \int_{B_1} |R_H \tilde{w} - (R_H \tilde{w})_1|^2 dy, \end{aligned}$$

for every $\rho \in (0, 1/2)$. The assertion follows by letting $c(N, \sigma_1, \sigma_2) := c^* C \max\{1, C'\}$. \square

The next corollary can be inferred from (4.68) by following the strategy of Lin in [26, pp 166-167]:

Corollary 4.41. *Let $\tilde{w} \in W^{\mathcal{A}}(B_2)$ solve the equation*

$$\mathcal{A}^*(\sigma_A \mathcal{A} u) = f \quad \text{in } B_2, \quad \text{with} \quad \|\tilde{w}\|_{L^2(B_2)} \leq 1 \text{ and } \|f\|_{L^\infty(B_2)} \leq 1, \quad (4.71)$$

where $A := \{x \in B_2' \times \mathbb{R} : x_d > \varphi(x')\}$ for some function $\varphi \in C^{1,\eta/2}(B_2')$ with $\varphi(0) = |\nabla \varphi|(0) = 0$,

and $\|\varphi\|_{C^{1,\eta/2}(B'_2)} \leq 1$. Then there exist positive constants $\theta(d, \sigma_1, \sigma_2) \in (0, 1/2)$, and $C(d, \sigma_1, \sigma_2)$ such that either

$$\frac{1}{\theta^{d+1}} \int_{B_\theta} |R_A \tilde{w} - (R_A \tilde{w})_\theta|^2 dy \leq \int_{B_1} |R_A \tilde{w} - (R_A \tilde{w})_1|^2 dy, \quad (4.72)$$

or

$$\int_{B_\theta} |R_A \tilde{w} - (R_A \tilde{w})_\theta|^2 dy \leq C \left(\|\varphi\|_{C^{1,\eta/2}(B'_1)} + \|f\|_{L^\infty(B_1)}^2 \right). \quad (4.73)$$

We are now in the position to prove (4.66). Let $\delta \in (0, \eta/2)$ and let (w, A) be solution of problem (P). Since local regularity properties of the pair (w, A) are inherited to any (possibly rotated and translated) re-scaled pair $(w^{x,r}, A^{x,r})$ – as defined in (4.47), where in particular the source $f^{x,r}$ tends to zero – with $r \leq \text{dist}(x, \partial\Omega)$, we may do the following assumptions without any loss of generality: $B_4 \subset \Omega$ and $x = 0 \in \partial^* A$, ∂A^* is parametrized in B_2 by a function $\varphi \in C^{1,\eta/2}(B'_2)$ such that $\varphi(0) = |\nabla\varphi(0)| = 0$, and $\|\varphi\|_{C^{1,\eta/2}(B'_2)}, \|f\|_{L^\infty(B_2)} \leq \min\{1, \kappa^*\}$ where $\kappa^* = \kappa^*(\delta, d, M)$ is the constant of Remark 4.38. Additionally, since (w, A) is a solution of problem (P), we know that

$$\mathcal{A}^*(\sigma_A \mathcal{A} w) = f \quad \text{in } B_2, \quad (4.74)$$

and

$$\frac{1}{r^{d-\delta}} \int_{B_r} |\mathcal{A} w|^2 dy \leq C_\delta (\|\mathcal{A} w\|_{L^2(B_2)}^2 + \|f\|_{L^\infty(B_1)}^2) \quad \text{for every } r \in (0, 1), \quad (4.75)$$

where $C_\delta(d, M)$ is the constant from Remark 4.38.

Notice that the rescaled functions¹⁴ $w^r(y) := (w(ry) - v_r(ry))/r^{1-(\delta/2)}$ and $\varphi^r(y) := \varphi(ry)/r$ still solve (4.74) for $f^r(y) := r^{1+(\delta/2)} f(ry)$ and $A^r := A/r$ with $\|\varphi^r\|_{C^{1,\eta/2}(B'_2)}, \|f^r\|_{L^\infty(B_2)} \leq \min\{1, \kappa^*\}$. In particular, by (4.75) and Poincaré's inequality

$$\|w^r\|_{L^2(B_1)}^2 \leq C(B_1) \|\mathcal{A} w^r\|_{L^2(B_1)}^2 < \bar{C} := C(B_1) C_\delta (\|\mathcal{A} w\|_{L^2(B_2)}^2 + 1).$$

Thus Recall also that $\|\varphi^r\|_{C^{1,\eta/2}(B'_1)}$ scales as $r^{\eta/2} \|\varphi\|_{C^{1,\eta/2}(B'_2)}$ and, in view of its definition, $\|f^r\|_{L^\infty(B_1)}^2$ scales as $r^{2+\delta}$. In view of these properties, we are in position to apply Corollary 4.41 to $w^r / \max\{1, \bar{C}^{1/2}\}$: We infer that either

$$\frac{1}{\theta^{d+1}} \int_{B_\theta} |R_{A^r} w^r - (R_{A^r} w^r)_\theta|^2 dy \leq \int_{B_1} |R_{A^r} w^r - (R_{A^r} w^r)_1|^2 dy, \quad (4.76)$$

or

$$\int_{B_\theta} |R_{A^r} w^r - (R_{A^r} w^r)_\theta|^2 dy \leq \max\{1, \bar{C}\} \cdot C(d, \sigma_1, \sigma_2) \left(\|\varphi^r\|_{C^{1,\eta/2}(B'_1)} + r^{2+\delta} \right), \quad (4.77)$$

where $\theta = \theta(d, \sigma_1, \sigma_2) \in (0, 1/2)$ is the constant from Corollary 4.41.

It is not difficult to verify, with the aid of the Iteration Lemma 4.25, that re-scaling in (4.76) and

¹⁴Here, v_r is the \mathcal{A} -free corrector function for w in B_r , see Definition 4.6.

(4.77) conveys a decay of the form

$$\frac{1}{r^{d+\eta/2-\delta}} \int_{B_r} |R_A(w - v_r) - (R_A(w - v_r))_r|^2 dy \leq c' \quad \text{for all } r \in (0, 1), \quad (4.78)$$

and some constant $c' = c'(\delta, d, \sigma_1, \sigma_2, \|\mathcal{A}w\|_{L^2(B_2)})$.

The last step of the proof consists in showing that $R_A(w - v_r)$ dominates $\nabla(w - v_r)$. By the definition of R_A , it is clear that $|\nabla'(w - v_r)(x) - (\nabla'(w - v_r))_{B_r \cap A}|^2 \leq |R_A(w - v_r)(x) - (R_A(w - v_r))_{B_r \cap A}|^2$ for all $x \in B_1$ and every $r \in (0, 1)$. We show a similar estimate for $\partial_d(w - v_r)$:

The pointwise Gårding inequality (4.2) and (4.11) imply, in particular, that the tensor $(\mathbb{A}(\mathbf{e}_d)^T \sigma_1 \mathbb{A}(\mathbf{e}_d)) = (A_d^T \sigma_1 A_d) \in \text{Lin}(\mathbb{R}^m; \mathbb{R}^m)$ is invertible (use, e.g., Lax-Milgram in \mathbb{R}^m). Hence,

$$\partial_d(w - v_r) = (A_d^T \sigma_1 A_d)^{-1} \left(g(w - v_r) - \sum_{j \neq d} (A_d^T \sigma_1 A_j) \partial_j(w - v_r) \right) \quad \text{in } B_1 \cap A, \quad (4.79)$$

from where we deduce that

$$\begin{aligned} \frac{1}{r^{d+(\eta/2)-\delta}} \int_{B_r \cap A} |\partial_d(w - v_r) - (\partial_d(w - v_r))_{B_r \cap A}|^2 dy &\leq \\ &\frac{c''}{r^{d+(\eta/2)-\delta}} \int_{B_r \cap A} |R_A(w - v_r) - (R_A(w - v_r))_{B_r \cap A}|^2 dy \end{aligned}$$

for some constant $c'' = c''(\sigma_1) \geq 1$ bounding the right hand side of (4.79) in terms of $\nabla'w$ and $g(w)$.

By (4.45) and the estimate above we obtain

$$\begin{aligned} \frac{1}{r^{d+(\eta/2)-\delta}} \int_{B_r \cap A} |\mathcal{A}w - (\mathcal{A}(w))_{B_r \cap A}|^2 dy &= \\ &\frac{1}{r^{d+(\eta/2)-\delta}} \int_{B_r \cap A} |\mathcal{A}(w - v_r) - (\mathcal{A}(w - v_r))_{B_r \cap A}|^2 dy \\ &\leq \frac{C(\mathcal{A})}{r^{d+(\eta/2)-\delta}} \int_{B_r \cap A} |\nabla(w - v_r) - (\nabla(w - v_r))_{B_r \cap A}|^2 dy \\ &\leq \bar{c}(d, \sigma_1, \sigma_2, \|\mathcal{A}w\|_{L^2(B_2)}) := C(\mathcal{A}) \cdot c' \cdot c'', \end{aligned}$$

for every $r \in (0, 1)$. The assertion follows by taking $\delta = \eta/4$.

Notice that the dependence on $\|\mathcal{A}w\|_{L^2(B_2)}$ is local since we assumed $B_4 \subset \Omega$; this means that in general we may not expect a uniform boundedness of the decay. Similar bounds for A replaced by A^c can be derived by the same method. □

Remark 4.42 (regularity I). In general, for a k -th order operator \mathcal{A} of gradient form, the only feature required to prove the regularity of $\nabla^k w$ up to the boundary ∂^*A by the same methods as for first-order operators of gradient form is to obtain an analog of Lemma 4.40 (and its Corollary 4.41) for higher-order operators.

More specifically, if $\tilde{w} \in \mathbf{W}^{\mathcal{A}}(B_1)$ is a solution of the equation

$$\mathcal{A}^*(\sigma_H \mathcal{A} u) = 0 \quad \text{in } B_1,$$

then \tilde{w} satisfies an estimate of the form

$$\frac{1}{\rho^{d+2}} \int_{B_\rho} |R_H \tilde{w} - (R_H \tilde{w})_\rho|^2 \, dy \leq c(d, \sigma_1, \sigma_2) \int_{B_1} |R_H \tilde{w} - (R_H \tilde{w})_1|^2 \, dy \quad \text{for all } 0 < \rho \leq 1, \quad (4.80)$$

where

$$R_A u := (\nabla' u, \mathbb{A}(\mathbf{e}_d)^T (\sigma_A \mathcal{A} u)), \quad A \subset B_1.$$

Unfortunately, for $2k$ -order systems of elliptic equations (with $k > 1$) it is not clear to us whether one can prove such decay estimates by standard methods. While a decay estimate for $\nabla^{k-1}(\nabla' u)$ can be shown by the very same method as the one in the proof of Theorem 4.5, the main problem centers in proving a decay estimate for the term $\mathbb{A}(\mathbf{e}_d)^T (\sigma \mathcal{A} u) \in \mathbf{W}^{1,2}(B_1)$ – cf. (4.65). Technically, the issue is that one cannot use the equation on half-balls to describe $\partial^{(0,\dots,0,k)} u$ in terms of $\nabla^{k-1}(\nabla' u)$.

Remark 4.43 (regularity II - linear plate theory). In the particular case of models in linear plate theory ($\mathcal{A} = \nabla^2$, $d = 2$, and $m = 1$) it is possible to show a decay estimate as in (4.80) for solutions $w \in \mathbf{W}_0^{2,2}(B_2)$ of the equation

$$\nabla \cdot \nabla (\sigma_H \nabla^2 u) = 0.$$

By Remark 4.13, there exists a field $w \in W^{1,2}(B_2; \mathbb{R}^2)$ which turns out to be a solution of the equation

$$\nabla \cdot (\mathbf{S}_H \mathcal{E} w) = 0,$$

where \mathbf{S} is a positive fourth-order symmetric tensor such that $\sigma_H(x) = \mathbf{R}_\perp \mathbf{S}_H^{-1}(x) \mathbf{R}_\perp$; furthermore, $\mathbf{R}_\perp \mathcal{E} w = \sigma_H \nabla^2 u$. Since $\mathcal{A} = \nabla^2$, it is easy to verify that $A_\alpha = A_{(i,j)} = \mathbf{e}_i \otimes \mathbf{e}_j$ for $i, j \in \{1, 2\}$, a simple calculation shows that

$$g_H(u) := \mathbb{A}(\mathbf{e}_d)^T (\sigma_H \mathcal{A} u) = (\sigma_H \nabla^2 u)_{22} = (\mathbf{R}_\perp \mathcal{E} w)_{22} = \partial_1 w^1;$$

and thus, since \mathcal{E} is an operator of gradient form of order one, it follows from the proof of Theorem 4.5 that an estimate of the form (4.80) indeed holds for $g_H(u)$.

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4.7 Glossary of constants

- d spatial dimension
 M coercivity and bounding constant for the tensors σ_1 and σ_2 (as quadratic forms)
 K an arbitrary compact set in Ω
 λ_K local upper bound constant

Other constants: groups of constants are numbered in non-increasing order, e.g., $c_1^* \geq c_2^* \geq c_3^*$. The following constants play an important role in our calculations:

Constant	Dependence	Description
θ_1	arbitrary in $(0, 1/2)$	ratio constant
c_1	θ_1, d, M	universal constant
ε_1	θ_1, d, M, θ_1	smallness of perimeter density
c_2	M	universal constant
γ	d, M	universal constant
θ_2	d, M	universal constant
ε_2	d, M	smallness of excess energy
c_1^*	λ_K, d	constant in the Height bound Lemma
θ_1^*	arbitrary in $(0, 1/2)$	ratio constant
c_2^*	θ_1^*, d, M	universal constant
ε_2^*	θ_1^*, d, M	smallness of flatness excess
c_3^*	K, d, M, f	flatness excess improvement scaling constant
ε_3^*	K, d, M, f	smallness of flatness excess

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