p-Kazhdan-Lusztig Theory

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Abstract

We describe a positive characteristic analogue of the Kazhdan-Lusztig basis for the Hecke algebra of a crystallographic Coxeter system, called the p-canonical basis. Using Soergel calculus, we present an algorithm to calculate this basis. The p-canonical basis shares strong positivity properties with the Kazhdan-Lusztig basis (similar to the ones described by the Kazhdan-Lusztig positivity conjectures), but it loses many of its combinatorial properties. For this reason, it is much harder to compute the p-canonical basis which is only known in small examples.

Even without explicit knowledge of the p-canonical basis, one may obtain a first approximate description of the multiplicative structure by studying the left, right or two-sided cell preorder with respect to the p-canonical basis. The equivalence classes with respect to these cell preorders lead to the notion of p-cells. Parallel to the very rich theory of Kazhdan-Lusztig cells in characteristic 0, we try to build a similar theory in positive characteristic.

The first properties of p-cells that we prove are the following: Left and right p-cells are related by taking inverses, just like for Kazhdan-Lusztig cells. The set of elements with a fixed left descent set decomposes into right p-cells. The right p-cells satisfy a similar parabolic compatibility as Kazhdan-Lusztig right cells. We show that any right p-cell preorder relation in a finite, standard parabolic subgroup W_I induces right p-cell preorder relations in each right W_I-coset.

In an attempt to explicitly describe p-cells in type A, we study the consequences of the Kazhdan-Lusztig star-operations for the p-canonical basis. We deduce many interesting relations for the structure coefficients of the p-canonical basis and for the base change coefficients between the p-canonical and the Kazhdan-Lusztig basis. These allow us to show that the right star-operations preserve the left cell preorder. Moreover, we explicitly describe the p-cells in type A via the Robinson-Schensted correspondence and show that they coincide with Kazhdan-Lusztig cells for all primes p.

A central question is whether Kazhdan-Lusztig cells decompose into p-cells. Based on the star-operations, we can show that the equivalence classes with respect to Vogan's generalized τ-invariant decompose into left p-cells. Garfinkle showed that Vogan's generalized τ-invariant gives a complete invariant of Kazhdan-Lusztig left cells in type B and C. From this, we deduce that Kazhdan-Lusztig left cells in type B and C decompose into left p-cells for $p > 2$. We show that in type C_3 for $p = 2$, Kazhdan-Lusztig right (resp. two-sided) cells do not decompose into right (resp. two-sided) p-cells. Moreover, we give a criterion that reduces the question about the decomposition of Kazhdan-Lusztig cells to the minimal elements with respect to the weak right Bruhat order.

Recently, Achar, Makisumi, Riche and Williamson proved character formulas for the indecomposable tilting modules of a reductive algebraic group in terms of the p-canonical basis. This further fuels interest in the p-canonical basis because the determination of the tilting characters is a long-standing open problem in modular representation theory. These new character formulas and the geometric Satake equivalence provide two connections between right p-cells in affine Weyl groups and tensor ideals of tilting modules. For this reason, affine Weyl groups of small rank provide intriguing examples of p-cells. We explicitly determine the right p-cell structure in types $\widetilde{A}_1$, $\widetilde{A}_2$ and partly in $\widetilde{C}_2$.

Contents

1 Introduction

Categorification and Canonical Bases

"Canonical" bases often arise in the process of categorification and have many remarkable applications in representation theory and beyond. Among the first examples are the Kazhdan-Lusztig basis of Hecke algebras and the canonical basis of quantum groups. In these cases the canonical bases have geometric origins, encoding the graded dimensions of stalks of intersection cohomology sheaves on a variety. More recently, considerable progress has been made via algebraic categorifications where the canonical basis arises as the character of a simple or indecomposable projective module over an algebra. For example, the canonical basis of the quantum group is realized in simply laced type as the classes in the Grothendieck group of indecomposable projective modules for KLR algebras (see [VV11]). Similarly, the Kazhdan-Lusztig basis arises as the classes of indecomposable Soergel bimodules (see [EW14]). In most cases, the bridge between algebra and geometry is established by realizing the algebra as an extension algebra of geometric origin [Rou12; VV11; Soe01].

From a representation theoretic point of view, the main interest in canonical bases stems from the following two points:

- Multiplicities of interest in representation theory in characteristic 0 are usually encoded by the canonical basis or certain base change coefficients involving the canonical basis. The famous Kazhdan-Lusztig polynomials give the multiplicities of simple modules in Verma modules of a complex semi-simple Lie algebra (Kazhdan-Lusztig conjectures, proved in [BK81; BB81]).

- Based on the hope that the situation in characteristic 0 agrees with the one in characteristic $p \gg 0$, several open conjectures relate multiplicities of interest in modular representation theory to canonical bases. Famous examples include Lusztig's conjecture for simple rational modules for algebraic groups [Lus80b], the LLT conjecture for representations of Hecke algebras at roots of unity [LLT96; Ari96], and the James conjecture on representations of the symmetric group [Jam90].

However, the following way of thinking about problems in modular representation theory has recently emerged: For each prime number p there should be a "p-canonical basis" indexed by the same set as the usual canonical basis. Each p-canonical basis element should coincide with the corresponding canonical basis element for $p \gg 0$, but may differ from it for small p. This p-canonical basis (instead of the canonical basis) should provide the correct answer to questions in representation theory mod p. Examples of this phenomenon include:

(i) The work of Grojnowski, Ariki and others [Gro99; Ari96] identifies the Grothendieck group of the category of representations of all symmetric groups with the basic representation of an affine Lie algebra. In this case, the p-canonical basis is defined via the classes of indecomposable projective modules, and thus contains deep information in representation theory mod p essentially by definition. (We also first learnt the term p-canonical basis from [Gro99].)

(ii) The work of Soergel [Soe00] connects certain multiplicities in the rational representation theory of algebraic groups with indecomposable summands of Bott-Samelson sheaves. Subsequent work ([JMW14b; WB12]) shows that Soergel's results may be restated as giving these multiplicities in terms of coefficients of the p-canonical basis of the Hecke algebra of the finite Weyl group.

(iii) A recent conjecture of Riche and Williamson (see [RW16, Conjecture 5.1]) predicts that the characters of indecomposable tilting modules for reductive algebraic groups should be given in terms of coefficients of the p-canonical basis in the anti-spherical module of the affine Weyl group. The character formulas have recently been proven by Achar, Makisumi, Riche and Williamson in [Ach+17a; Ach+17b]. Explicit knowledge of the tilting characters answers several open questions in modular representation theory. Moreover, this result led to much more efficient algorithms for tilting characters and a new conjecture about tilting characters for SL_3 (see [LW17b]). Current work in progress of Lusztig, Williamson and the author deals with SP_4.

(iv) In recent work by Elias and Losev (see [EL17b]) decomposition numbers in several modular representation categories are expressed in terms of p-Kazhdan-Lusztig polynomials in type A.

This way of thinking suggests the following strategy to tackle a problem in modular representation theory: First one should rephrase it in terms of a p-canonical basis. Then one should calculate the p-canonical basis. Usually the second step is extremely difficult and one should not expect an answer in general (see for example [Wil17c], which shows that questions of arithmetic nature, for example whether a particular Fibonacci number is prime, come up in the calculation in high rank). However, this approach has at least the potential to unify questions in modular representation theory just like many questions have answers given by the same Kazhdan-Lusztig polynomials. In this thesis, we focus on combinatorial constraints on the p-canonical basis which make the situation quite rigid.

The p-Canonical Basis of Hecke algebras

We recall the definition of and study the p-canonical basis for the Hecke algebra of a crystallographic Coxeter system (W, S) (see [JW17]). The definition of the p-canonical basis for the Hecke algebra appears for the first time in [Wil12]. To motivate its definition, we first recall how the Kazhdan-Lusztig basis arises from categorification. Given a based root datum as input, we get a complex reductive algebraic group G together with a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. The crystallographic Coxeter system (W, S) we consider is either the finite Weyl group W_{f} together with the simple reflections S_{f} given by B or the affine Weyl group $W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi^{\vee}$ together with a suitable set of simple reflections (see Section 6.1). Let us focus on the finite setting in this section for the sake of simplicity.

The Hecke algebra $\mathcal{H}$ of $(W_{\mathrm{f}}, S_{\mathrm{f}})$ has two (essentially equivalent) categorifications, often referred to loosely as the Hecke category:

(i) *Geometric:* The additive, monoidal (under convolution) category of B-biequivariant semi-simple complexes on G: the full subcategory of the equivariant derived category $D^b_{B\times B}(G,\mathbb{Q})$ consisting of direct sums of shifts of equivariant intersection cohomology complexes.

(ii) *Algebraic:* The monoidal category of Soergel bimodules: a certain full subcategory of the monoidal category of graded R-bimodules, where R denotes the regular functions on the Lie algebra of the maximal torus T.

In the first setting, the Kazhdan-Lusztig basis arises as the graded dimensions of stalks of the intersection cohomology complexes (see [KL80]). In the second case, the Kazhdan-Lusztig basis is realized as the characters of the indecomposable self-dual Soergel bimodules (see [Soe98; EW14]).

In [EW16] the monoidal category of Soergel bimodules is described by generators and relations, following earlier work by Elias [Eli16], Elias-Khovanov [EK10] and Libedinsky [Lib10] (we recall this description in detail below). The upshot is that there exists a graded monoidal category $\mathbf{H}$, which is defined over the integers, and whose extension of scalars to $\mathbb{Q}$ is equivalent to Soergel bimodules. Hence, one can think of $\mathbf{H}$ as an integral form of the Hecke category. For any field k we can consider the extension of scalars ${}^k\mathbf{H}$ to k and it is proved in [EW16] that one has a canonical "character" isomorphism of $\mathbb{Z}[v,v^{-1}]$-algebras

$$\mathrm{ch} : [{}^k\mathbf{H}] \xrightarrow{\sim} \mathcal{H}$$

between the split Grothendieck group of ${}^k\mathbf{H}$ and the Hecke algebra. Hence, for any field ${}^k\mathbf{H}$ provides a categorification of the Hecke algebra.[1] (Note that while the coefficients of $\mathbf{H}$ change, the Grothendieck group is always the same Hecke algebra over $\mathbb{Z}[v,v^{-1}]$.)

In [EW16] the indecomposable objects of ${}^k\mathbf{H}$ are classified, following Soergel's classification of the indecomposable Soergel bimodules in [Soe07]. It turns out that for all $w \in W_{\mathrm{f}}$ there exists an indecomposable object ${}^kB_w \in {}^k\mathbf{H}$, and that any indecomposable object is isomorphic to a grading shift of ${}^kB_w$ for some $w \in W_{\mathrm{f}}$. The p-canonical basis is defined as the character of this indecomposable object:

$${}^p\underline{H}_w := \mathrm{ch}({}^kB_w)$$

where p denotes the characteristic of k. From basic properties of ${}^k\mathbf{H}$ it is easy to see that $\{{}^p\underline{H}_w \mid w \in W_{\mathrm{f}}\}$ is a basis for the Hecke algebra, which only depends on the characteristic of k, and that its structure constants are positive (see Proposition 3.10). Moreover, $\{{}^0\underline{H}_w \mid w \in W\}$ is the Kazhdan-Lusztig basis, because ${}^{\mathbb{Q}}\mathbf{H}$ is equivalent to Soergel bimodules with $\mathbb{Q}$-coefficients (see [EW14]).

Geometric Origin of the p-Canonical Basis

In this subsection we outline the connection between the p-canonical basis and parity sheaves on (affine) flag varieties. The reader unfamiliar with affine flag varieties may keep the important case of a (finite) flag variety in mind.

To any root datum $(X,\Phi,X^\vee,\Phi^\vee)$ with bases $\Delta \subset \Phi$ and $\Delta^\vee \subset \Phi^\vee$ we can associate a connected, reductive algebraic group scheme G over $\mathbb{Z}$ with Borel

[1] There is a minor additional technical assumption if the characteristic of k is 2 (see Assumption 2.2). We ignore this point in the introduction.

subgroup $B \subseteq G$ and maximal torus $T \subseteq B \subseteq G$. In the finite case, let $\mathcal{X}$ denote the $\mathbb{C}$-points of the flag variety $\mathcal{F}l := G/B$ with its classical metric topology.

In the affine setting, define the loop group LG (resp. positive loop group L^+G) of G as the $\mathbb{Z}$-functor given by $R \mapsto G(R(\!(t)\!))$ (resp. $R \mapsto G(R[\![t]\!])$). Denote by I the Iwahori subgroup, given by the inverse image of B under the morphism $L^+G \to G$ that is induced by $t \mapsto 0$. In this case, we define $\mathcal{X}$ to be the $\mathbb{C}$-points of the affine flag variety $\mathcal{F}l_a := LG/I$, viewed as ind-variety (see [Gör10] for more information).

In both settings, we have an (Iwahori)-Bruhat decomposition expressing the corresponding flag variety as a disjoint union of left $B(\mathbb{C})$ (resp. $I(\mathbb{C})$)-orbits indexed by the (extended affine) Weyl group. The closure relation is given by the Bruhat order. Note that the affine flag variety is isomorphic as an ind-variety to a suitable disjoint union of Kac-Moody flag varieties (see e.g. [Kum02]).

Fix a field k of coefficients. We may consider $D^b_H(\mathcal{X})$ the H-equivariant bounded derived category of k-sheaves where H is either $B(\mathbb{C})$ or $I(\mathbb{C})$ depending on the setting (see [BL94] for more information about equivariant derived categories). In [JMW14b, §4.1] Juteau, Mautner and Williamson introduce and prove the existence of parity sheaves on generalized flag varieties, a class of objects in $D^b_H(\mathcal{X})$ whose stalks satisfy a cohomological parity vanishing condition (for the constant pariversity function). Their work was motivated by Soergel's idea to consider another class of objects as "replacements" for intersection cohomology complexes with positive characteristic coefficients (see [Soe00]). Observe that while $\mathcal{X}$ is still a variety (resp. an ind-variety) over $\mathbb{C}$ equipped with its classical topology, the coefficients of the sheaves we are studying may lie in a field of positive characteristic.

The theory of parity sheaves parallels the theory of perverse sheaves in the following points:

(i) Indecomposable parity sheaves are classified analogously to simple perverse sheaves, being up to isomorphism the unique extension of an irreducible local system on a stratum.

(ii) In our setting, if the coefficients are a field of characteristic zero, the intersection cohomology sheaves are parity sheaves.

But while the decomposition theorem for perverse sheaves fails in positive characteristic, the pushforward along a proper, even, stratified map preserves parity sheaves. Moreover, it is possible to calculate the multiplicities of the occurring indecomposable parity sheaves via intersection forms (see [JMW14b, §3.3.]). Thus, parity sheaves are particularly interesting in the case of positive characteristic coefficients.

Parity sheaves on various varieties have also been used for categorification. In [Mak15], Maksimau realizes Lusztig's integral form of the positive half of the quantum group associated to a Dynkin quiver as a coalgebra geometrically by considering parity sheaves on quiver moduli spaces. In our setting, parity sheaves also give canonical bases of the Hecke algebra. If the coefficients are a field of characteristic zero, then the graded dimensions of stalks of parity sheaves give the Kazhdan-Lusztig basis (as mentioned above, see [KL80; Spr82]). For a field of positive characteristic, the indecomposable parity sheaves realize the p-canonical basis in this way (see [WB12] in the setting of the flag variety and [RW16] for the full generality).

From this perspective, several interesting geometric implications arise when the p-canonical basis differs from the Kazhdan-Lusztig basis. First of all, it allows one to study the failure of the decomposition theorem in the modular setting (see [JMW14b, §3]). Secondly, there are close connections between the decomposition matrix for intersection cohomology complexes and the base change matrix between the Kazhdan-Lusztig and the p-canonical basis. In [AR16, Theorem 2.6] Achar and Riche show that the base change matrix gives a q-refinement of the decomposition matrix on the Langlands dual flag variety. Moreover, Williamson proves in [Wil15] that certain base change coefficients and decomposition numbers coincide. He uses this to give an example of a reducible characteristic variety in type A. Non-trivial decomposition numbers for an intersection cohomology complex can only occur when the characteristic variety of the corresponding regular holonomic $D_{\mathcal{X}}$-module is reducible (see [VW13]).

p-Cells

The original motivation for the Kazhdan-Lusztig basis was to explicitly construct representations of the Hecke algebra (see [Lus17]). This naturally led Kazhdan and Lusztig to study cells in the Hecke algebra with respect to the Kazhdan-Lusztig basis and the corresponding cell modules.

Recall that the Kazhdan-Lusztig left cell preorder $\underset{L}{\leqslant}$ on W can be defined as follows: $x \underset{L}{\leqslant} y$ if and only if there exists an element $h \in \mathcal{H}$ such that ${}^0\underline{H}_x$ occurs with non-trivial coefficient in $h\,{}^0\underline{H}_y$. The right cell preorder $\underset{R}{\leqslant}$ is defined similarly using right multiplication, whereas the two-sided cell preorder $\underset{2}{\leqslant}$ is generated by $\underset{L}{\leqslant}$ and $\underset{R}{\leqslant}$. Kazhdan-Lusztig left (resp. right or two-sided) cells are the equivalence classes with respect to $\underset{L}{\leqslant}$ (resp. $\underset{R}{\leqslant}$ or $\underset{2}{\leqslant}$). Note that this definition immediately generalizes to the p-canonical basis.

The definition shows that cells give a rough approximation of the multiplicative structure. For any Kazhdan-Lusztig left cell $C \subseteq W$, write $w \underset{L}{\leqslant} C$ (resp. $w \underset{L}{<} C$) if there exists an element $y \in C$ such that $w \underset{L}{\leqslant} y$ (and $w \notin C$). Then the definition of the Kazhdan-Lusztig left cell preorder allows us to define left $\mathcal{H}$-modules

$$\mathcal{H}_{\underset{L}{\leqslant} C} \coloneqq \bigoplus_{w \underset{L}{\leqslant} C} \mathbb{Z}[v, v^{-1}]\,{}^0\underline{H}_w \text{ and similarly } \mathcal{H}_{\underset{L}{<} C}.$$

Finally, the left cell module associated to C is defined as the left $\mathcal{H}$-module given by the quotient

$$\mathcal{H}_{\underset{L}{\leqslant} C}/\mathcal{H}_{\underset{L}{<} C}.$$

In [KL79, Theorem 1.4], Kazhdan and Lusztig show that in type A the cell modules give the irreducible modules of the Hecke algebra $\mathcal{H}$ for generic parameter v. In their proof, they introduce the Kazhdan-Lusztig star-operations, generalizing the (dual) Knuth operations for symmetric groups to pairs of simple reflections s and t in a Coxeter group with st of order 3. The study of the consequences of the star-operations for the structure coefficients leads to an explicit description of the Kazhdan-Lusztig cells in symmetric groups (see [KL79, §5]). The Kazhdan-Lusztig cells in type A can be characterized via the Robinson-Schensted correspondence (see [Ari00]), which gives a bijection $w \mapsto (P(w), Q(w))$ between the symmetric group S_n and pairs of standard tableaux of the same shape with n boxes: The Kazhdan-Lusztig right cell of a given element $w \in S_n$ is given by the set of elements in S_n that have the same P-symbol as w under the Robinson-Schensted correspondence.

In an attempt to describe p-cells for symmetric groups explicitly, we also studied the consequences of the Kazhdan-Lusztig star-operations for structure coefficients as well as base change coefficients between the p-canonical and the Kazhdan-Lusztig bases. This led to some remarkable identities generalizing the results in [KL79, §4] and [Lus85, §10.4]. If a p-canonical basis element differs from the corresponding Kazhdan-Lusztig basis element, then these identities often allow to deduce the non-triviality of other p-canonical basis elements (see [LW17b, Remark 5.2. (9)] for an example). Another consequence is that p-cells coincide with Kazhdan-Lusztig cells in symmetric groups for all primes p (see Theorem 5.32) and in particular that they are independent of p. This is particularly interesting since the p-canonical basis does differ from the Kazhdan-Lusztig basis for many primes p (see [Wil17c]).

Thus, in type A the p-canonical basis of each cell module gives after extending scalars to $\mathbb{C}$ and specializing v to 1 a basis of the corresponding complex irreducible representation of the symmetric group. Letting p vary, we obtain a very interesting family of bases that merits further study. The relation between Specht modules, the Kazhdan-Lusztig cell representations, and their corresponding natural bases was further studied for the Hecke algebra as well as for the group algebra of symmetric groups in [Mat94; GM88; MP05; MS08].

Supported by the results in finite type A, one may hope that Kazhdan-Lusztig cells always decompose into p-cells. Unfortunately, this is not the case as we show in Section 4.3.3. However, we still believe that the corresponding statement may still be true for p good for the corresponding algebraic group. In Section 4.2, we develop a simple criterion when Kazhdan-Lusztig right cells decompose into right p-cells, which reduces the question to minimal elements with respect to the weak right Bruhat order in each cell.

In a series of papers [Gar90; Gar92; Gar93; GPM], Garfinkle generalizes the Robinson-Schensted correspondence to types B, C and D. She develops combinatorial algorithms to associate to a Weyl group element w a pair $(T_L(w), T_R(w))$ of standard domino tableaux of the same shape from which w can be reconstructed. (Note that the definition of a standard domino tableau depends on the type.) The main difference to the situation in type A is that the partition of the Weyl group into sets with the same left domino tableau is finer than the partition into Kazhdan-Lusztig left cells (see [McG96, §3]). For this reason, Garfinkle groups the set of dominos in a tableau into *cycles* and classifies them as “open” or “closed”. For each open cycle, she defines an involutive algorithm called “moving a tableau through an open cycle” that changes only the positions of the dominos in the open cycle. Based on this, she defines an equivalence relation on the set of standard domino tableaux by declaring two to be equivalent if one can be obtained from the other by moving through open cycles. One of the main results is that two elements in a Weyl group of type B/C lie in the same Kazhdan-Lusztig left cell if and only if their corresponding left standard domino tableaux are equivalent (see [Gar93, Corollary 3.5.6. and Theorem 3.5.9.]). Garfinkle also announced an extension of this result to type D (see [GPM]) which has not yet appeared in print. Moreover, Garfinkle shows that a generalization of Vogan's τ-invariant gives a complete invariant for Kazhdan-Lusztig left cells. Based on our results on the Kazhdan-Lusztig star operations, we can show that the equivalence classes with respect to this generalized τ-invariant give a refinement of the left p-cells under small assumptions on p. This implies, that Kazhdan-Lusztig left cells decompose into left p-cells in finite types B and C for $p > 2$.

In Section 4, we prove some general results about p-cells most of which are generalizations of well-known results for Kazhdan-Lusztig cells. First, we show that right p-cells give a refinement of the partition of the Weyl group by left descent set (see Lemma 4.4). Then we study which automorphisms of the Hecke algebra are well-behaved with respect to the p-canonical basis (see Proposition 4.7) and show that left and right p-cell preorders are related by taking inverses (see Lemma 4.6). The most important result of this section is a certain compatibility of p-cells with parabolic subgroups. We show that a right p-cell preorder relation in a finite, standard parabolic subgroup W_I induces right p-cell preorder relations in each right W_I-coset (see Theorem 4.9).

Examples of p-Cells

Since p-cell theory is still in its early stages, it is of particular importance to provide interesting examples. For this reason, we developed extensive Magma code that allows us to compute p-cells in small finite type: B_2, B_3, B_4, B_5, C_3, C_4, C_5, D_4, D_5, G_2 and F_4. In this thesis, we only included small examples that are of particular interest in order not to bore the reader with large amounts of data. Based on our extensive calculations, we have developed some interesting conjectures about distinguished involutions in right p-cells which might govern the decomposition behaviour of the Kazhdan-Lusztig right cells.

Extremely interesting examples are provided by affine Weyl groups of small rank. Thus, in the second part of the thesis, we focus on p-cells in affine Weyl groups. Consider $W = W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi^\vee$ and add the affine reflection to S_{f} to obtain a set of simple reflections S for W (as described in Section 6.1). Consider the Hecke algebra $\mathcal{H}$ associated to (W, S).

Fix an algebraically closed field k of characteristic $p > 0$ as coefficients. Let $G^\vee$ denote the split connected reductive group that is defined over k and Langlands dual to G from previous sections. Assume that $G^\vee$ is simply-connected. The category $\mathrm{Rep}(G^\vee)$ of finite dimensional, algebraic $G^\vee$ representations over k forms a highest weight category with Weyl modules as standard and induced modules as costandard objects. In this setting, the notion of *tilting modules* makes sense. The full subcategory $\mathrm{Tilt}(G^\vee)$ of all tilting modules in $\mathrm{Rep}(G^\vee)$ has many favourable properties. First of all, $\mathrm{Tilt}(G^\vee)$ is a Krull-Schmidt category. For every $\lambda \in X_+^\vee$ there exists an indecomposable tilting module $T(\lambda)$ and any indecomposable object in $\mathrm{Tilt}(G^\vee)$ is isomorphic to one of the $T(\lambda)$. Moreover, $\mathrm{Tilt}(G^\vee)$ is closed under the tensor product. This allows us to transfer the definition of cells to this setting, giving *modular weight cells* as equivalence classes in $X_+^\vee$. Then p-cells in W are closely related to modular weight cells for $G^\vee$. Modular weight cells have been studied mainly by Ostrik (see [Ost97]) and Andersen (see [And92; And04]).

The connection between modular weight cells and p-cells is two-fold, providing different pieces of the puzzle. Let us briefly sketch the relationship between modular weight cells and p-cells (see Section 7 for more details).

Denote by $\mathcal{O} := \mathbb{C}[\![t]\!]$ the ring of complex Laurent polynomials. The affine Grassmannian $\mathcal{G}r_a := LG/L^+G$ carries the structure of an ind-projective ind-scheme. Firstly, if the characteristic p of k is good for G, $G(\mathcal{O})$-equivariant parity sheaves on $\mathcal{G}r_a(\mathbb{C})$ with coefficients in k are perverse (see [JMW14a; MR15]). The geometric Satake equivalence (see [MV07]) gives an equivalence of monoidal categories between $G(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r_a(\mathbb{C})$ with coefficients in k (equipped with a convolution product) and $\mathrm{Rep}(G^\vee)$. The $G(\mathcal{O})$-orbits on $\mathcal{G}r_a(\mathbb{C})$ are in bijection with $X_+^\vee$ and for every $\lambda \in X_+^\vee$ there exists up to isomorphism a unique $G(\mathcal{O})$-equivariant parity sheaf $\mathcal{E}(\lambda)$ on $\mathcal{G}r_a(\mathbb{C})$ that extends the constant sheaf $\underline{k}$ on the $G(\mathcal{O})$-orbit corresponding to λ. Moreover, under the geometric Satake equivalence the indecomposable tilting module $T(\lambda)$ corresponds to $\mathcal{E}(\lambda)$. The action of the extended affine Hecke algebra $W^{\mathrm{ext}} = W \rtimes \Omega$ on $X^\vee \otimes \mathbb{R}$ induces a bijection between $X_+^\vee$ and $\bigcup_{\sigma \in \Omega} W_{\mathrm{f}} \backslash W / \sigma(W_{\mathrm{f}})$ where the elements of Ω act as automorphisms of the Coxeter system (W, S). We call an element in W *spherical* if it is maximal in a double coset in $W_{\mathrm{f}} \backslash W / W_{\mathrm{f}}'$ for some standard parabolic subgroup $W_{\mathrm{f}}' \cong W_{\mathrm{f}} \subset W$. It follows that a modular weight cell relation induces a right p-cell relation between the corresponding spherical elements.

Secondly, the recent Riche-Williamson conjecture (see [RW16]) establishes another link. Denote by h the Coxeter number of W. Riche and Williamson conjecture that wall-crossing functors give an action of ${}^k\mathbf{H}$ on the principal block $\mathrm{Rep}_0(G^\vee)$ for $p = \mathrm{char}(k) > h$ and prove their conjecture for GL_n. Let sgn denote the sign module for the finite Weyl group W_{f}. The conjecture implies that two categorifications of the anti-spherical $\mathbb{Z}W$-module $\mathrm{sgn} \otimes_{\mathbb{Z}W_{\mathrm{f}}} \mathbb{Z}W$, namely a diagrammatic one obtained from a quotient of ${}^k\mathbf{H}$ and the category of tilting modules in the principal block $\mathrm{Tilt}_0(G^\vee)$, are equivalent. Thus, the tilting characters for $G^\vee$ can be expressed using affine p-Kazhdan-Lusztig polynomials, just like the tilting characters for a quantum group at a root of unity are linked to affine Kazhdan-Lusztig Polynomials (see [Soe97a, Corollary 7.6 and §8] and [Soe97b, Proposition 3.4 (2)]). Unfortunately, the Riche-Williamson conjecture is still open in other types, but their character formulas have recently been proven by Achar, Makisumi, Riche and Williamson using different techniques (see [Ach+17a; Ach+17b]). Denote by ${}^fW$ the set of representatives of cosets in $W_{\mathrm{f}} \backslash W$ of minimal length. Descent set considerations show that any right p-cell in W that intersects ${}^fW$ non-trivially is fully contained in it (see [AHR17, Lemma 5.6]). These right p-cells are called *anti-spherical* (or *canonical* in the terminology of [LX88]). Following along the lines of [Ost97], Achar, Hardesty and Riche use the new character formulas for tilting modules to show that the modular weight cells completely determine the anti-spherical p-cells and vice versa (see [AHR17, Theorem 7.7]).

The most interesting result on modular weight cells for $G^\vee$ shows that they exhibit some beautiful fractal-like behaviour (which is partly dictated by the tilting tensor product theorem from [Don93]). In [And04, Lemma 13] Andersen shows the following: Define ρ to be the half sum of all positive coroots, set $Y_r := (p^r - 1)\rho + X_+^\vee$ and $X_{p^r}^\vee := \{\lambda \in X_+^\vee \mid \langle \lambda, \alpha \rangle < p^r \text{ for all simple roots } \alpha\}$. Then $\lambda \in Y_r$ can be written uniquely as $\lambda = \lambda_0^r + p^r \lambda_1^r$ with $\lambda_0^r \in (p^r - 1)\rho + X_{p^r}^\vee$ and $\lambda_1^r \in X_+^\vee$.

Proposition 1.1. *Assume $p \geqslant 2h-2$. Let $\lambda, \mu \in Y_r$. Then λ and μ lie in the same modular weight cell if and only if λ_1^r and μ_1^r do.*

In Section 6, we first recall important results about tilting modules, focussing on results that we need for Andersen's proof. We also state some results related to Steinberg modules, which we expect to be important for the further study of p-cells. After reviewing what is known about modular weight cells, we explain Andersen's proof in detail.

Finally, we try to explicitly determine the p-cell structure in affine Weyl groups in small rank in Sections 9 and 10 for $p > 2h-2$. First, we deal with SL_2. This case is special as all tilting characters for SL_2 are known and they also give the full p-canonical basis of the affine Hecke algebra. From this, we can easily deduce the p-cell structure. Moreover, we observe that the two-sided p-cell of an element determines the p-valuation of the dimension of the corresponding indecomposable tilting module.

The situation becomes a lot more complicated in rank 2 where we need to combine all techniques at our disposal. The general scheme is roughly as follows: First, we deduce from our decomposition criterion that all right (resp. two-sided) Kazhdan-Lusztig cells decompose into right (resp. two-sided) p-cells under our assumptions on p. Using the Kazhdan-Lusztig star-operations we reduce the number of Kazhdan-Lusztig right cells to decompose. Then we explicitly determine the decomposition behaviour of the anti-spherical Kazhdan-Lusztig right cells not contained in the lowest Kazhdan-Lusztig two-sided cell. This determines the modular weight cells for $G^\vee$, using the link between anti-spherical right p-cells and modular weight cells and their fractal-like behaviour. Lastly, we try to show that the modular weight cells also govern the decomposition behaviour of the Kazhdan-Lusztig right cells containing a spherical element under the geometric Satake equivalence. We carry out all steps for type $\widetilde{A}_2$, determine the modular weight cells for type $\widetilde{C}_2$ and hope to deal with type $\widetilde{G}_2$ in the future.

1.1 Structure of the Thesis

Section 2 We introduce notation and recall important results about the Hecke algebra and Soergel calculus.

Section 3 After recalling the definition of the p-canonical basis, we explain how to calculate it using intersection forms. In Section 3.3 the elementary properties of the p-canonical basis are stated and proved. The section ends with several new and interesting examples of the p-canonical basis.

Section 4 We define p-cells and prove some of their elementary properties. The most important results are the compatibility of p-cells with parabolic subgroups and a criterion for Kazhdan-Lusztig cells to decompose into p-cells. We also give interesting examples of p-cells and state a conjecture resulting from extensive computer calculations.

Section 5 We introduce the Kazhdan-Lusztig star operations. Then we study in detail consequences for base change and structure coefficients of the p-canonical basis and for p-cells. After introducing Vogan's generalized τ-invariant, we show that left p-cells give a refinement of the τ-equivalence classes under small assumptions on p. In the end, we show that p-cells in finite type A are given by the Robinson-Schensted correspondence.

Section 6 In this section tilting modules for reductive algebraic groups are introduced with special emphasis on the Steinberg modules. We also introduce modular weight cells and explain Andersen's proof showing that they exhibit a beautiful fractal-like structure.

Section 8 This section aims to illustrate why the p-canonical basis is important by sketching its connections to some open problems in modular representation theory.

Section 7 We sketch two existing connections between anti-spherical p-cells and modular weight cells: The first one is based on the new character formulas for tilting modules in terms of the p-canonical basis of the anti-spherical module for the Hecke algebra and was established by Achar, Hardesty and Riche. The second one comes from the geometric Satake equivalence.

Section 9 The p-canonical basis as well as the p-cell structure in type $\widetilde{A}_1$ are explained explicitly.

Section 10 We state a general strategy to determine p-cells in affine rank 2 and describe the p-cells explicitly in type $\widetilde{A}_2$ and $\widetilde{C}_2$.

1.2 Acknowledgements

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2 Background

2.1 Coxeter Systems and Based Root Data

Let S be a finite set and $(m_{s,t})_{s,t\in S}$ be a matrix with entries in $\mathbb{N}\cup\{\infty\}$ such that $m_{s,s}=1$ and $m_{s,t}=m_{t,s}\geqslant 2$ for all $s\neq t\in S$. Denote by W the group generated by S subject to the relations $(st)^{m_{s,t}}=1$ for $s,t\in S$ with $m_{s,t}<\infty$. We say that (W,S) is a *Coxeter system* and W is a *Coxeter group.* The Coxeter group W comes equipped with a length function $l: W\to\mathbb{N}$ and the Bruhat order $\leqslant$ (see [Hum90] for more details). A Coxeter system (W,S) is called *crystallographic* if $m_{s,t}\in\{2,3,4,6,\infty\}$ for all $s\neq t\in S$. We denote the identity of W by e. For $w\in W$ we define its *left descent set* via

$$\mathcal{L}(w) \coloneqq \{s\in S \mid l(sw)<l(w)\}.$$

The *right descent set* of w is given by $\mathcal{R}(w)\coloneqq\mathcal{L}(w^{-1})$.

Define an *expression* to be a finite sequence of elements in S. We denote by

$$\operatorname{Ex}(S)\coloneqq\{\varnothing\}\cup\bigcup_{i\in\mathbb{N}\setminus\{0\}}\underbrace{S\times\cdots\times S}_{i\text{-times}}$$

the set of all expressions in S. For an expression $\underline{w}=(s_1,s_2,\ldots,s_n)$ denote its *length* by $l(\underline{w})=n$. For two expressions $\underline{x}$ and $\underline{y}$ in S, we write $\underline{x}{}^\frown\underline{y}$ for their concatenation. The multiplication gives a canonical map $\operatorname{Ex}(S)\to W$, $\underline{w}\mapsto\underline{w}_\bullet$. An expression $\underline{w}$ in S is called *reduced* if $l(\underline{w})=l(\underline{w}_\bullet)$. For an expression $\underline{w}=(s_1,s_2,\ldots,s_n)$ in S a *subexpression* of $\underline{w}$ is a sequence $\underline{w}^{\underline{e}}=(s_1^{e_1},s_2^{e_2},\ldots s_n^{e_n})$ where $e_i\in\{0,1\}$ for all i. The sequence $\underline{e}=(e_1,e_2,\ldots,e_k)$ is called the *associated* 01*-sequence.* We usually decorate $\underline{e}$ as follows: For $1\leqslant k\leqslant n$ denote by $\underline{w}_{\leqslant k}\coloneqq(s_1,s_2,\ldots,s_k)$ the first k terms and set $w_k\coloneqq(\underline{w}_{\leqslant k}^{\underline{e}_{\leqslant k}})_\bullet$. Assign to e_i a decoration $d_i\in\{U,D\}$ where U stands for *Up* and D for *Down* as follows:

$$d_1\coloneqq U \text{ and } d_i\coloneqq\begin{cases}U & \text{if } w_{i-1}s_i>w_{i-1},\\ D & \text{if } w_{i-1}s_i<w_{i-1}\end{cases}\quad\text{for } 2\leqslant i\leqslant n.$$

We often write the decorated sequence as $(d_1e_1,d_2e_2,\ldots,d_ne_n)$. The sequence of elements $e,w_1,w_2,\ldots,w_n$ is called the *Bruhat stroll* associated to $\underline{w}^{\underline{e}}$. The *defect* of $\underline{e}$ is defined to be

$$\operatorname{df}(e)\coloneqq|\{i\mid d_ie_i=U0\}|-|\{i\mid d_ie_i=D0\}|.$$

The set of subexpressions of $\underline{w}$ comes equipped with a partial order, called the *path dominance order* (see [EW16, §2.4] for the definition).

Example 2.1. To illustrate the definitions, consider for example the case $S=\{s,t\}$ with $m_{s,t}=m_{t,s}=3$ (i.e. type A_2). The reduced expression (s,t,s) admits two decorated 01-sequences expressing s:

$$\begin{array}{ll}(U1,U0,D0) & \text{of defect } 0\\ (U0,U0,U1) & \text{of defect } 2\end{array}$$

Recall that given an abstract root datum $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ and a basis $\Delta \subseteq \Phi$ the quadruple $\Psi_0 = (X, \Delta, X^\vee, \Delta^\vee)$ is called a *based root datum* where $\Delta^\vee$ is the set of simple coroots (see [Spr89, §7.4] for the definition of a root datum). From now on, fix a based root datum Ψ_0. The matrix $(\langle \alpha_s^\vee, \alpha_t \rangle)_{(s,t)\in S\times S}$ is called the *Cartan matrix* associated to this based root datum. By the existence theorem (see [Sga, Exposé XXV, Théorème 1.1]), starting from Ψ_0 we get G, a split connected reductive algebraic group scheme over $\mathbb{Z}$, together with a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B \subseteq G$ such that the root datum determined by (G, T) and the basis given by the simple roots whose root groups are contained in B give the corresponding based root datum Ψ_0. For a coroot $\alpha^\vee \in \Phi^\vee$ we define the corresponding reflection on $X^\vee$ via

$$s_{\alpha^\vee} \lambda := \lambda - \langle \lambda, \alpha \rangle \alpha^\vee \qquad \text{for all } \lambda \in X^\vee.$$

We denote the finite Weyl group of Φ by $W_{\mathrm{f}} := \langle s_{\alpha^\vee} \mid \alpha^\vee \in \Phi^\vee \rangle$ and the affine Weyl group by $W := W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi^\vee$ (see Section 6.1 for more details). Recall that $W = W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi^\vee$ is generated by all the affine reflections $s_{\alpha^\vee,n}$ for $\alpha^\vee \in \Phi_+^\vee$ and $n \in \mathbb{Z}$. The finite as well as the affine Weyl group can be realized as Coxeter groups with the set of simple reflections given by $S_{\mathrm{f}} := \{s_{\alpha^\vee} \mid \alpha^\vee \in \Delta^\vee\}$ and $S = \{s_{\alpha^\vee,0} \mid \alpha^\vee \in \Delta^\vee\} \cup \{s_{\alpha_0^\vee,1}\}$ where $\alpha_0 \in \Phi_+$ is the highest root.

For S_{f} the triple $(X^\vee, \Delta, \Delta^\vee)$ gives a (not necessarily symmetric) faithful realization of the Coxeter system $(W_{\mathrm{f}}, S_{\mathrm{f}})$ over $\mathbb{Z}$ (as defined in the appendix of [Eli16]). Alternatively, we could have used $\mathbb{Z}\Phi^\vee$ instead of $X^\vee$ as realization (as in [RW16, §4.2] in the affine setting).

In the affine setting, we define elements $\widetilde{\alpha}_s \in \mathbb{Z}\Phi$ and $\widetilde{\alpha}_s^\vee \in \mathbb{Z}\Phi^\vee$ as follows:

- for $s \in S_{\mathrm{f}}$, choose $\widetilde{\alpha}_s$ and $\widetilde{\alpha}_s^\vee$ as the simple root and simple coroot associated to s respectively;
- if $s \in S \setminus S_{\mathrm{f}}$, then the image of s under the canonical projection $W \to W_{\mathrm{f}}$ is the reflection $s_{\alpha_0^\vee}$; define $\widetilde{\alpha}_s$ and $\widetilde{\alpha}_s^\vee$ to be $-\alpha_0$ and $-\alpha_0^\vee$ respectively.

Then the triple $(X^\vee, \{\widetilde{\alpha}_s\}_{s\in S}, \{\widetilde{\alpha}_s^\vee\}_{s\in S})$ gives a (not necessarily symmetric) balanced realization of the Coxeter system (W, S) over $\mathbb{Z}$ (see [Eli16, Def. A.3]).

Fix a commutative ring k. In both cases, ${}^kV := X^\vee \otimes_{\mathbb{Z}} k$ yields a (potentially non-faithful) realization of the Coxeter system over k. Set ${}^kV^* := \mathrm{Hom}_k({}^kV, k)$ and note that ${}^kV^*$ is isomorphic to $X \otimes_{\mathbb{Z}} k$. Throughout, we will assume our realization to satisfy:

Assumption 2.2 (Demazure Surjectivity)**.** *The maps $\alpha_s : {}^kV \to k$ and $\alpha_s^\vee : {}^kV^* \to k$ are surjective for all $s \in S$.*

This is automatically satisfied if 2 is invertible in k or if the Coxeter system (W, S) is of simply-laced type and of rank $|S| \geqslant 2$.

We denote by $R = S({}^kV^*)$ the symmetric algebra of ${}^kV^*$ over k and view it as a graded ring with ${}^kV^*$ in degree 2. By choosing $X^\vee$ instead of X in the definition of V, we ensure that R is a polynomial ring in the roots. We may also canonically identify R with $H_T^*(\mathrm{pt}; k)$, the T-equivariant cohomology of a point (see [Bri98, §1 Example 2)]).

The action of W on ${}^kV$ induces an action on R by functoriality. For any $s \in S$ we define the *Demazure operator* $\partial_s : R \to R(-2)$ via

$$\partial_s(f) := \frac{f - s(f)}{\alpha_s}$$

where (1) denotes the grading shift down by one: Given a graded R-bimodule $B = \bigoplus_{i\in\mathbb{Z}} B^i$, we denote by $B(1)$ the shifted bimodule with $B(1)^i = B^{i+1}$. Observe that ∂_s is a well-defined graded R^s-bimodule homomorphism (see [EW16, §3.3] for more details).

2.2 The Hecke Algebra

The Hecke algebra is the free $\mathbb{Z}[v, v^{-1}]$-algebra with $\{H_w \mid w \in W\}$ as basis, called the *standard basis*, and multiplication determined by:

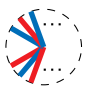
$$\begin{aligned} H_s^2 &= (v^{-1} - v)H_s + 1 && \text{for all } s \in S, \\ H_x H_y &= H_{xy} && \text{if } l(x) + l(y) = l(xy). \end{aligned}$$

There is a unique $\mathbb{Z}$-linear involution $\overline{(-)}$ on $\mathcal{H}$ satisfying $\overline{v} = v^{-1}$ and $\overline{H_x} = H_{x^{-1}}^{-1}$. The Kazhdan-Lusztig basis element $\underline{H}_x$ is the unique element in $H_x + \sum_{y<x} v\mathbb{Z}[v]H_y$ that is invariant under $\overline{(-)}$. This is Soergel's normalization from [Soe97b] of a basis introduced originally in [KL79]. For a sequence $\underline{w} = (s_1, s_2, \ldots, s_n)$ in S, we write $\underline{H}_{\underline{w}}$ for the element $\underline{H}_{s_1}\underline{H}_{s_2}\ldots\underline{H}_{s_n}$. After fixing a reduced expression of every element $w \in W$, the set $\{\underline{H}_{\underline{w}} \mid w \in W\}$ gives a basis of $\mathcal{H}$, called the *Bott-Samelson basis*.

Let ι be the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution on $\mathcal{H}$ satisfying $\iota(H_s) = H_s$ for $s \in S$ and thus $\iota(H_x) = H_{x^{-1}}$.

2.3 Soergel Calculus

We define an *S-graph* to be a finite, decorated, planar graph with boundary properly embedded in the planar strip $\mathbb{R} \times [0, 1]$ whose edges are coloured by S and all of whose vertices are of the following types:

	univalent vertices ("dots")	of degree 1,
	trivalent vertices	of degree -1,
	$2m_{s,t}$-valent vertices	of degree 0,

where we require the $2m_{s,t}$-valent vertex to have exactly $2m_{s,t}$ edges, coloured alternately by s and t around the vertex.

The regions of an S-graph (i.e. the connected components of the complement of the graph in $\mathbb{R} \times [0, 1]$) may be *decorated* by homogeneous elements of R. The degree of a decorated S-graph is defined as the sum of the degrees of its vertices and of the degrees of the polynomials decorating its regions.

Next, we introduce the diagrammatic category of Soergel bimodules. The main reference for this is [EW16] (see also [Eli16] in the dihedral case and [EK10] in type A).

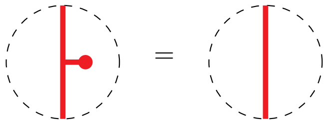
Let BS be the strict monoidal category with $\mathbb{Z}$-graded Hom-spaces which is monoidally generated by the elements in S. Thus the objects of BS are given by $\mathrm{Ex}(S)$ and the monoidal structure on the level of objects is given by concatenation of sequences in S. For any $\underline{x}, \underline{y} \in \mathrm{Ex}(S)$, $\mathrm{Hom}_{\mathrm{BS}}(\underline{x}, \underline{y})$ is defined to be the free R-module generated by isotopy classes of decorated S-graphs with bottom boundary $\underline{x}$ and top boundary $\underline{y}$ modulo the local relations below. The composition (resp. tensor product) of two morphisms is given by vertical (resp. horizontal) concatenation of diagrams.

We now recall the relations defining BS:

2.3.1 One-colour relations

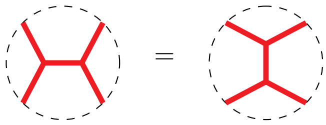
For all $s \in S$ we have:

- Frobenius Unit:



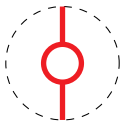
(1)

- Frobenius Associativity:



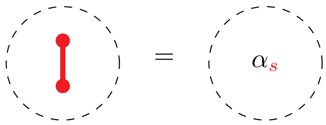
(2)

- Needle Relation:



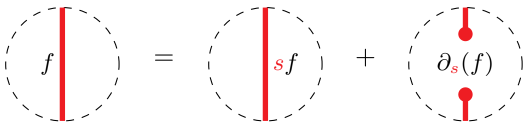
$= \quad 0$ (3)

- Barbell Relation:



(4)

- Nil Hecke Relation:

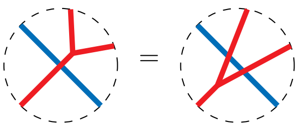


(5)

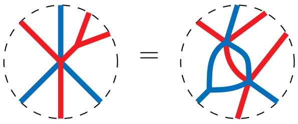
2.3.2 Two-colour relations

There are two colour relations for all pairs $s, t \in S$ such that $m_{s,t} < \infty$ (so that the $2m_{st}$-valent vertex is defined).

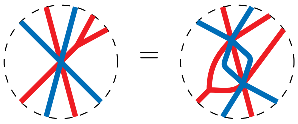
The first two-colour relation is called *Two-colour Associativity* and describes what happens when we pull a trivalent vertex through a $2m_{s,t}$-valent vertex. We give it for $m_{s,t} \in \{2, 3, 4\}$ and let the reader guess the general form (see [Eli16, (6.12)]):



if $m_{s,t} = 2 \quad$ (type $A_1 \times A_1$),

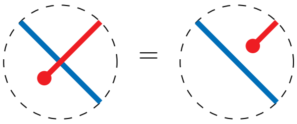


if $m_{s,t} = 3$ (type A_2),

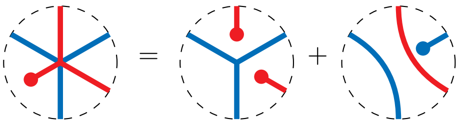


if $m_{s,t} = 4$ (type B_2).

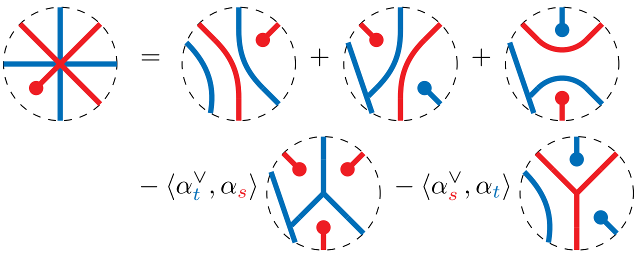
The next two-colour relation is called the *Jones-Wenzl Relation* and expresses a $2m_{s,t}$-valent with a dot on one strand as a linear combination of diagrams in which only dots and trivalent vertices appear. We state it only for $m_{s,t} \in \{2, 3, 4\}$ and refer the reader to [Eli16] for more detail:



if $m_{s,t} = 2$,



if $m_{s,t} = 3$,



if $m_{s,t} = 4$.

2.3.3 Three-colour relations

We do not repeat the definition of the Zamolodchikov relations or "higher braid relations" here. The reader can find them in [EW16, §1.4.3] and is referred to [EW17] for more detail on the topological origins of the Zamolodchikov relations.

2.4 Light Leaves and Double Leaves

In this section we briefly discuss how to describe bases for morphism spaces in BS. Fix an expression $\underline{w}$ and a reduced expression $\underline{x}$. In [EW16, §6.1] it is described how one may associate a "light leaves morphism" $\mathrm{LL}_{\underline{w},\underline{e}} \in \mathrm{Hom}_{\mathrm{BS}}(\underline{w}, \underline{x})$ to each subexpression $\underline{e}$ of $\underline{w}$ such that $(\underline{w}^{\underline{e}})_\bullet = x$. We will not recall the explicit construction here, but the reader is encouraged to consult [EW16, §6.1] to follow our examples. The construction of light leaves follows a construction of Libedinsky for Soergel bimodules [Lib08] and depends on certain additional non-canonical choices.

In the special case of $x = e$, the identity of W, we get (see [EW16, Proposition 6.12]):

Proposition 2.3. *The set of all light leaves indexed by subsequences* $\underline{e}$ *of* $\underline{w}$ *expressing the identity of* W *gives an* R*-basis of* $\mathrm{Hom}_{\mathrm{BS}}(\underline{w}, \varnothing)$*.*

For an S-graph D denote by $\overline{D}$ the S-graph obtained by flipping the diagram upside down. This induces a contravariant equivalence on the monoidal category BS fixing all objects.

Out of light leaves one can construct double leaves as follows. Let $\underline{x}$ and $\underline{y}$ be arbitrary expressions in S. For any subsequences $\underline{e}$ (resp. $\underline{f}$) of $\underline{x}$ (resp. $\underline{y}$) both expressing $w \in W$ define $\mathbb{LL}_{w,\underline{e},\underline{f}} := \overline{\mathrm{LL}_{\underline{y},\underline{f}}} \circ \mathrm{LL}_{\underline{x},\underline{e}}$. The following result can be found in [EW16, Theorem 6.11] (and was proved earlier in the setting of Soergel bimodules by Libedinsky in [Lib15]):

Theorem 2.4. *The set of all double leaves ranging over all* $w \in W$ *and pairs of subsequences* $\underline{e}$ *(resp.* $\underline{f}$*) of* $\underline{x}$ *(resp.* $\underline{y}$*) both expressing* w *gives an* R*-basis of* $\mathrm{Hom}_{\mathrm{BS}}(\underline{x}, \underline{y})$*.*

2.5 The Diagrammatic Category: Properties

Note that all relations in BS are homogeneous for our grading on S-graphs and thus BS is a category enriched in graded left R-modules; multiplying an S-graph D with a homogeneous polynomial $f \in R$ from the left is defined by decorating the leftmost region of D with f.

Let $\mathbf{H}$ be the Karoubian envelope of the graded version of the additive closure of BS, in symbols $\mathbf{H} = \mathcal{K}ar(\mathrm{BS})$. We call $\mathbf{H}$ the *diagrammatic category of Soergel bimodules*. In other words, in the passage from BS to $\mathbf{H}$ we first allow direct sums and grading shifts (restricting to degree preserving homomorphisms) and then the taking of direct summands. The following properties can be found in [EW16, Lemma 6.24, Theorem 6.25 and Corollary 6.26]:

Theorem 2.5 (Properties of $\mathbf{H}$)**.**
Let k *be a complete local ring (e.g. a field or the* p*-adic integers* $\mathbb{Z}_p$*).*

(i) $\mathbf{H}$ *is a Krull-Schmidt category.*

(ii) *For all* $w \in W$ *there exists a unique, indecomposable object* $B_w \in \mathbf{H}$ *which is a direct summand of* $\underline{w}$ *for any reduced expression* $\underline{w}$ *of* w *and which is not isomorphic to a grading shift of any direct summand of any expression* $\underline{v}$ *for* $v < w$*. The object* B_w *does not depend up to isomorphism on the reduced expression* $\underline{w}$ *of* w*.*

(iii) *The set* $\{B_w \mid w \in W\}$ *gives a complete set of representatives of the isomorphism classes of indecomposable objects in* $\mathbf{H}$ *up to grading shift.*

(iv) *There exists a unique isomorphism of* $\mathbb{Z}[v, v^{-1}]$*-algebras*

$$\mathrm{ch} : [\mathbf{H}] \longrightarrow \mathcal{H}$$

sending $[B_s]$ *to* $\underline{H}_s$ *for all* $s \in S$*, where* $[\mathbf{H}]$ *denotes the split Grothendieck group of* $\mathbf{H}$*. (We view* $[\mathbf{H}]$ *as a* $\mathbb{Z}[v, v^{-1}]$*-algebra as follows: the monoidal structure on* $\mathbf{H}$ *induces a unital, associative multiplication and* v *acts via* $v[B] := [B(1)]$ *for an object* B *of* $\mathbf{H}$*.)*

It should be noted that we do not have a diagrammatic presentation of $\mathbf{H}$ as determining the idempotents in BS is usually extremely difficult.

Observe that $\overline{(-)}$ extends to a contravariant equivalence of the graded, R-linear, additive, monoidal category $\mathbf{H}$ sending $B_w(n)$ to $B_w(-n)$ for all $n \in \mathbb{Z}$ and $w \in W$.

In order to explicitly give the isomorphism in the last part of Theorem 2.5, we need to introduce some more notation. For $x \in W$, let $\mathbf{H}^{\not< x}$ be the quotient category of $\mathbf{H}$ by the 2-sided ideal of morphisms factoring through any grading shift of a reduced expression $\underline{y}$ for some $y < x$. Write $\mathrm{Hom}_{\not< x}(-,-)$ for homomorphism spaces in $\mathbf{H}^{\not< x}$. In $\mathbf{H}^{\not< x}$ any two reduced expressions for x become canonically isomorphic. We denote the image of any reduced expression for x in $\mathbf{H}^{\not< x}$ by x as well. Under the assumptions of Theorem 2.5 we can define the *diagrammatic character* on an object B of $\mathbf{H}$ as follows:

$$\begin{aligned} \mathrm{ch} : [\mathbf{H}] &\longrightarrow \mathcal{H}, \\ [B] &\longmapsto \sum_{w \in W} \operatorname{grk} \mathrm{Hom}^{\bullet}_{\not< w}(B, w) H_w \end{aligned}$$

and extend $\mathbb{Z}[v, v^{-1}]$-linearly. In the last definition grk denotes the graded rank of the free R-module of homomorphisms of all degrees

$$\mathrm{Hom}^{\bullet}_{\not< w}(B_x, w) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\not< w}(B_x, w(n)).$$

3 The p-Canonical Basis

In this section we recall the definition of the p-canonical basis of the Hecke algebra (see [Wil12]) and explain how to calculate it using intersection forms. Fix a field k of characteristic $p \geqslant 0$ and the realization ${}^kV$ of (W, S). We use this realization to define $\mathbf{H}$.

It is an interesting question what basis of the Hecke algebra the classes of the self-dual indecomposable objects in $\mathbf{H}$ correspond to. The answer is given for $k = \mathbb{R}$ by *Soergel's conjecture* which Elias and Williamson proved in [EW14].

Theorem 3.1 (Elias-Williamson 2013)**.**
$\mathrm{ch}([B_w]) = \underline{H}_w$ *for all* $w \in W$.

This illustrates that the basis of self-dual indecomposable objects in $\mathbf{H}$ gives an extremely interesting basis of $\mathcal{H}$ for $k = \mathbb{R}$ and motivates our definition of the p-canonical basis.

Definition 3.2. Define ${}^p\underline{H}_w = \mathrm{ch}([B_w])$ for all $w \in W$ where $\mathrm{ch} : [\mathbf{H}] \xrightarrow{\cong} \mathcal{H}$ is the isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras introduced earlier, and p denotes the characteristic of k as above.

Theorem 2.5 implies that $\{{}^p\underline{H}_w \mid w \in W\}$ gives a basis of $\mathcal{H}$, called the *p-canonical basis*. As will become clearer later, the p-canonical basis depends only on the type of the root system chosen and on the characteristic p of the field k.

3.1 Calculation using Intersection Forms

Next, we are going to explain how to use intersection forms to explicitly calculate the p-canonical basis. In order to calculate ${}^p\underline{H}_w$ we proceed by induction on $l(w)$. The induction start is given by ${}^p\underline{H}_e = \underline{H}_e = H_e$. Assume that we have already calculated ${}^p\underline{H}_v$ for all $v < w$. Let $\underline{w}$ be an arbitrary reduced expression for w. According to Theorem 2.5, we need to decompose $\underline{w}$ into indecomposable objects $B_x(n)$ for $x \in W$ and $n \in \mathbb{Z}$ in $\mathbf{H}$. For this we need local intersection forms:

Write $\mathrm{Hom}_{\not< x,k}(-,-)$ for the homomorphism spaces in $k \otimes_R \mathbf{H}^{\not< x}$ where we kill the action of the unique maximal ideal of all polynomials of positive degree. Since $\mathrm{Hom}^\bullet_{\not< x}(\underline{w}, x)$ is a graded free R-module with basis (see Section 2.4):

$$\{\mathrm{LL}_{\underline{w},\underline{e}} \mid \underline{e} \text{ is a subexpression of } \underline{w} \text{ expressing } x\},$$

$\mathrm{Hom}^\bullet_{\not< x,k}(\underline{w}, x)$ is a graded k-vector space on the same basis.

For an arbitrary expression $\underline{w}$ in S and $x \in W$, consider the k-bilinear Hom-pairing

$$\begin{aligned} \mathrm{Hom}^\bullet_{\not< x,k}(x, \underline{w}) \times \mathrm{Hom}^\bullet_{\not< x,k}(\underline{w}, x) &\longrightarrow \mathrm{End}^\bullet_{\not< x,k}(x) = k, \\ (f, g) \qquad\qquad &\longmapsto g \circ f. \end{aligned}$$

Observe that $\mathrm{End}^\bullet_{\not< x,k}(x)$ is concentrated in degree 0 and that the duality $\overline{(-)}$ on $\mathbf{H}$ gives an isomorphism between $\mathrm{Hom}^\bullet_{\not< x,k}(x, \underline{w})$ and $\overline{\mathrm{Hom}^\bullet_{\not< x,k}(\underline{w}, x)}$. This allows us to define:

Definition 3.3. The *local intersection form of $\underline{w}$ at x* is the k-bilinear pairing on the graded free k-vector space $\mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x)$ given by

$$\begin{aligned} I_{\underline{w},x} : \mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x) \times \mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x) &\longrightarrow \mathrm{End}_{\not< x,k}(x) = k, \\ (f, g) &\longmapsto g \circ \overline{f}. \end{aligned}$$

The local intersection form of $\underline{w}$ at x can be split up into degree pieces as follows: Since $\mathrm{End}^{\bullet}_{\not< x,k}(x)$ is concentrated in degree 0, a homomorphism in $\mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x(d))$ for some $d \in \mathbb{Z}$ can only pair non-trivially with elements of $\mathrm{Hom}^{\bullet}_{\not< x,k}(\underline{w}, x(-d))$. The *d-th grading piece of the intersection form* can thus be defined as:

$$I^d_{\underline{w},x} : \mathrm{Hom}_{\not< x,k}(\underline{w}, x(-d)) \times \mathrm{Hom}_{\not< x,k}(\underline{w}, x(d)) \longrightarrow \mathrm{End}_{\not< x,k}(x(d)) = k$$

Finally, the *graded rank* of $I_{\underline{w},x}$ is denoted by $n_{x,w} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$ and defined as

$$n_{x,w} := \sum_{d \in \mathbb{Z}} \mathrm{rk}(I^d_{\underline{w},x}) v^d.$$

The following lemma illustrates the importance of intersection forms for the calculation of the p-canonical basis and follows from an argument similar to [JMW14b, Lemma 3.1]:

Lemma 3.4. *The multiplicity of B_x in $\underline{w}$ in $\mathbf{H}$ is given by the graded rank of $I_{\underline{w},x}$.*

After calculating the graded ranks of all $I_{\underline{w},x}$ for $x < w$, we can write for $\underline{w} = s_1 s_2 \dots s_n$:

$$\underline{H}_{s_1} \underline{H}_{s_2} \dots \underline{H}_{s_n} = {}^p\underline{H}_w + \sum_{x < w} n_{x,w} \, {}^p\underline{H}_x.$$

Remark 3.5. By comparing the intersection forms over $\mathbb{Q}$ and k, one may deduce that one only needs to calculate the graded ranks of $I_{\underline{w},x}$ for those x such that $\underline{H}_x$ occurs with a non-trivial coefficient when expressing $\underline{H}_{s_1} \underline{H}_{s_2} \dots \underline{H}_{s_n}$ in terms of the Kazhdan-Lusztig basis.

In order to determine ${}^p\underline{H}_w$ we have to invert the matrix $(n_{x,y})_{x,y \leqslant w}$ which is upper triangular with ones on the diagonal in any total order refining the Bruhat order.

Finally, it should be noted that in practice one calculates the intersection form once over $\mathbb{Z}$ and reduces modulo different primes. Moreover, there is a variant of the algorithm in which one calculates the idempotents of ${}^kB_x$ along the way. For $s \in S$ with $ws < w$ one then decomposes the object ${}^kB_{ws}{}^kB_s$ (instead of $\underline{w}$) into indecomposable objects using intersection forms.

Remark 3.6. There are other ways to calculate the p-canonical basis (which, however, are much more difficult in practice).

(i) If one can describe the geometry of the corresponding Schubert varieties quite explicitly, one can do calculations using parity sheaves (see [WB12, appendix]).

(ii) In [FW14] Fiebig and Williamson show that for a field k of characteristic p (or more generally a complete local PID), the Braden-MacPherson algorithm on the Bruhat graph allows one to compute the p-canonical basis.

3.2 Calculations in the nil Hecke Ring

In [HW15] Xuhua He and Williamson reduce the calculation of certain entries in the intersection form to a simple formula in the nil Hecke ring. Instead of going into too much detail, we will try to give a survey of these results.

First, recall the definition of the nil Hecke ring. Let Q be the field of fractions of R. Denote by $Q * W$ the smash product. In other words, $Q * W$ is the free left Q-module with basis $\{\delta_w \mid w \in W\}$ and multiplication given by

$$(f\delta_x)(g\delta_y) = f(xg)\delta_{xy}$$

for $f, g \in Q$ and $x, y \in W$. Inside $Q * W$, we consider the elements

$$D_s = \frac{1}{\alpha_s}(\delta_e - \delta_s) = (\delta_e + \delta_s)\frac{1}{\alpha_s}$$

which satisfy the following relations:

$$\begin{aligned} D_s^2 &= 0, \\ D_s f &= (sf)D_s + \partial_s(f) \quad \text{for all } f \in Q, \\ \underbrace{D_s D_t D_s \dots}_{m_{s,t} \text{ terms}} &= \underbrace{D_t D_s D_t \dots}_{m_{s,t} \text{ terms}} . \end{aligned}$$

The last relation ensures that for $y \in W$ and any reduced expression $\underline{y} = (s_1, s_2, \dots, s_n)$ of y we get well-defined elements

$$D_y = D_{s_1} D_{s_2} \dots D_{s_n} \in Q * W.$$

The nil Hecke ring $\mathcal{NH}$ is the left R-submodule of $Q*W$ generated by $\{D_y \mid v \in W\}$.

Next, we briefly introduce gobbling morphisms. For any expression $\underline{w} = (s_1, s_2, \dots, s_n)$ in S, consider the following 01-sequence $\underline{e}$ with:

$$e_i = \begin{cases} 1 & \text{if } w_{i-1}s_i > w_{i-1}, \\ 0 & \text{otherwise}, \end{cases}$$

where at each step w_i is defined as in Section 2.1. Note that $(\underline{w}^{\underline{e}})_\bullet$ is the maximal element in W expressible as a subexpression of $\underline{w}$, and that the decoration of $\underline{e}$ consists entirely of $U1$'s and $D0$'s. Therefore any choice of light leaf morphism $\mathrm{LL}_{\underline{w},\underline{e}}$ has degree $l((\underline{w}^{\underline{e}})_\bullet) - l(\underline{w})$ and consists only of $2m_{st}$-valent and trivalent vertices. Denote by $G_{\underline{w}}$ the image of $\mathrm{LL}_{\underline{w},\underline{e}}$ in $\mathbf{H}^{\not< (\underline{w}^{\underline{e}})_\bullet}$. The morphism $G_{\underline{w}}$ is called a *gobbling morphism* and can be characterized as follows (see [HW15, Proposition 3.4]):

Proposition 3.7. *Let $\underline{w}$, $\underline{e}$ be as above. Any morphism $\underline{w} \to (\underline{w}^{\underline{e}})_\bullet$ in $\mathbf{H}^{\not< (\underline{w}^{\underline{e}})_\bullet}$ given by diagrams consisting only of $2m_{st}$-valent vertices and $l(\underline{w}) - l((\underline{w}^{\underline{e}})_\bullet)$ trivalent vertices is equal to $G_{\underline{w}}$.*

From this they deduce the canonicity of any light leaf morphism $\mathrm{LL}_{\underline{w},\underline{f}}$ in $\mathbf{H}^{\not< (\underline{w}^{\underline{f}})_\bullet}$ indexed by a 01-sequence $\underline{f}$ without $D1$'s in its decoration. (This follows because the morphism is given as the composition of a sequence of dots on strands corresponding to $U0$'s followed by a gobbling morphism.)

Finally, we come to their formula in the nil Hecke ring for certain entries of the intersection form. Let $\underline{e^1}$ and $\underline{e^2}$ be two subexpressions of $\underline{w}$. Assume that $\underline{e^1}$ and $\underline{e^2}$ both express the same element $x \in W$ (i.e. $x = (\underline{w}^{\underline{e^1}})_\bullet = (\underline{w}^{\underline{e^2}})_\bullet$) and that their decorations do not contain any $D1$. Define an element of the nil Hecke ring as the product $f(\underline{e^1}, \underline{e^2}) = f_1 f_2 \dots f_m$ where

$$f_i = \begin{cases} \alpha_{s_i} & \text{if } e_i^1 = e_i^2 = U0, \\ 1 & \text{if exactly one of } e_i^1 \text{ and } e_i^2 \text{ is } U0, \\ D_{s_i} & \text{otherwise.} \end{cases}$$

Denote by $d(\underline{e^1}, \underline{e^2}) \in R$ the coefficient of $D_{(\underline{w}^{\underline{e^1}})_\bullet}$ in $f(\underline{e^1}, \underline{e^2})$. The main result is [HW15, Theorem 5.1]:

Theorem 3.8. *For $\underline{e^1}$ and $\underline{e^2}$ as above, we have*

$$I_{\underline{w},x}(\mathrm{LL}_{\underline{w},\underline{e^1}}, \mathrm{LL}_{\underline{w},\underline{e^2}}) = d(\underline{e^1}, \underline{e^2}).$$

This theorem gives a combinatorial formula for some entries in the intersection form. Sometimes one is lucky, and it can be used to calculate the complete intersection form, as we will see in examples below.

3.3 First Properties of the p-Canonical Basis

The goal of this section is to prove elementary properties of the p-canonical basis and to compare it to the Kazhdan-Lusztig basis. For this we need a p-modular system. Let $\mathbb{O}$ be a complete local ring with residue field k of characteristic $p > 0$ and quotient field $\mathbb{K}$ of characteristic 0. Fix the realization ${}^{\mathbb{O}}V$ of (W, S) and use it to define $\mathbf{H}$. For $x \in W$ we will denote by B_x (resp. ${}^kB_x$ or ${}^{\mathbb{K}}B_x$) the indecomposable object in $\mathbf{H}$ (resp. ${}^k\mathbf{H} := \mathbf{H} \otimes_{\mathbb{O}} k$ or ${}^{\mathbb{K}}\mathbf{H} := \mathbf{H} \otimes_{\mathbb{O}} \mathbb{K}$).

The following lemma shows that indecomposable objects in $\mathbf{H}$ remain indecomposable when passing to ${}^k\mathbf{H}$.

Lemma 3.9. *We have for all $w \in W$: $B_w \otimes_{\mathbb{O}} k \cong {}^kB_w$.*

Proof. Assume $B_w \otimes_{\mathbb{O}} k$ is not indecomposable in ${}^k\mathbf{H}$. Then there exists a non-trivial idempotent $e \in \mathrm{End}_{{}^k\mathbf{H}}(B_w \otimes_{\mathbb{O}} k)$. Since $\mathrm{End}_{\mathbf{H}}(B_w)$ is a finitely generated $\mathbb{O}$-module, we can use idempotent lifting techniques for complete local rings (see [Lam01, Theorem 21.31]) and find an idempotent $\tilde{e} \in \mathrm{End}_{\mathbf{H}}(B_w)$ mapping to e in $\mathrm{End}_{{}^k\mathbf{H}}(B_w \otimes_{\mathbb{O}} k) \cong \mathrm{End}_{\mathbf{H}}(B_w) \otimes_{\mathbb{O}} k$. Since B_w is indecomposable, this idempotent has to be trivial, a contradiction. $\square$

Some of the following properties can also be found in [WB12] and [Wil15]:

Proposition 3.10. *For all $x, y \in W$ we have:*

(i) $\overline{{}^p\underline{H}_x} = {}^p\underline{H}_x$, *i.e.* ${}^p\underline{H}_x$ *is self-dual,*

(ii) ${}^p\underline{H}_x = H_x + \sum_{y<x} {}^ph_{y,x} H_y$ *with* ${}^ph_{y,x} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$,

(iii) ${}^p\underline{H}_x = \underline{H}_x + \sum_{y<x} {}^pm_{y,x} \underline{H}_y$ *with self-dual* ${}^pm_{y,x} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$,

(iv) $\iota({}^p\underline{H}_x) = {}^p\underline{H}_{x^{-1}}$ *and thus in particular* ${}^pm_{y,x} = {}^pm_{y^{-1},x^{-1}}$ *as well as* ${}^ph_{y,x} = {}^ph_{y^{-1},x^{-1}}$,

(v) ${}^p m_{y,x} = 0$ *unless* $\mathcal{L}(x) \subseteq \mathcal{L}(y)$ *and* $\mathcal{R}(x) \subseteq \mathcal{R}(y)$,

(vi) ${}^p\underline{H}_x \, {}^p\underline{H}_y = \sum_{z \in W} {}^p\mu^z_{x,y} \, {}^p\underline{H}_z$ *with self-dual* ${}^p\mu^z_{x,y} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$,

(vii) ${}^p\underline{H}_x = \underline{H}_x$ *for* $p \gg 0$ *(i.e. there are only finitely many primes for which* ${}^p\underline{H}_x \neq \underline{H}_x$*).*

Proof. (i) We proceed by induction on $l(x)$. For small x the statement is clear as ${}^p\underline{H}_e = \underline{H}_e$ and ${}^p\underline{H}_s = \underline{H}_s$ for all $s \in S$ and all primes p. Assume that we have shown that ${}^p\underline{H}_y$ is self-dual for $y < x$. Choose $s \in \mathcal{L}(x)$ and set $y = sx$. The characterization of ${}^kB_x$ in Theorem 2.5 implies that ${}^kB_x$ occurs with multiplicity one in ${}^kB_s{}^kB_y$ and that $\overline{{}^kB_x(n)} = {}^kB_x(-n)$. Thus we can write

$$ {}^kB_s{}^kB_y = {}^kB_x \oplus \bigoplus_{\substack{z<x \\ n \in \mathbb{Z}}} ({}^kB_z(n))^{\oplus a_{z,n}} $$

with $a_{y,n} \in \mathbb{Z}_{\geqslant 0}$ for all $y < x$ and $n \in \mathbb{Z}$ and all but finitely many of the $a_{y,n}$ are zero. Applying the duality $\overline{(-)}$ to both sides and using that the left hand side is self-dual yields $a_{z,n} = a_{z,-n}$ for all $z < x$ and $n \in \mathbb{Z}$. This implies

$$ {}^p\underline{H}_s \, {}^p\underline{H}_y = {}^p\underline{H}_x + \sum_{z<x} {}^p\mu_{z,y} \, {}^p\underline{H}_z $$

where ${}^p\mu_{z,y} = \sum_{n \in Z} a_{z,n} v^n \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$ is self-dual for all $z < x$. Therefore the self-duality of the left-hand side and our induction hypothesis imply the self-duality of ${}^p\underline{H}_x$.

(ii) The unicity in the characterization of ${}^kB_x$ in Theorem 2.5 implies that it occurs with multiplicity 1 in any reduced expression $\underline{x}$ of x. In the quotient $\mathbf{H}^{\not< x}$ all other direct summands of x are killed. Thus we get $\operatorname{grk} \operatorname{Hom}_{\not< x}(B_x, x) = 1$. Note that the Laurent polynomials ${}^p h_{y,x}$ have non-negative coefficients as they are given by graded ranks of free R-modules.

(iii) According to (ii), $\underline{H}_x$ occurs precisely with coefficient 1 in ${}^p\underline{H}_x$. The self-duality of the Laurent polynomials ${}^p m_{y,x}$ follows from (i) and the self-duality of the Kazhdan-Lusztig basis. Since $\operatorname{Hom}^\bullet_{\not< w}(B_x, w)$ is a free R-module, we get for $F \in \{k, \mathbb{K}\}$:

$$ \operatorname{Hom}^\bullet_{\not< w}(B_x, w) \otimes_{\mathbb{O}} F \cong \operatorname{Hom}^\bullet_{\mathbf{H}^{\not< w} \otimes_{\mathbb{O}} F}(B_x \otimes_{\mathbb{O}} F, w \otimes_{\mathbb{O}} F). $$

This implies in particular using Lemma 3.9:

$$ \operatorname{ch}({}^kB_x) = \operatorname{ch}(B_x) = \operatorname{ch}(B_x \otimes_{\mathbb{O}} \mathbb{K}). $$

Thus the ${}^p m_{y,x}$ have non-negative coefficients as they come from decomposing $B_x \otimes_{\mathbb{O}} \mathbb{K}$ into indecomposable objects of the form ${}^{\mathbb{K}}B_x$ in ${}^{\mathbb{K}}\mathbf{H}$ whose character is given by $\underline{H}_x$ by Theorem 3.1.

(iv) There is an equivalence on ${}^k\mathbf{H}$ viewed as a k-linear category induced by the horizontal flip of Soergel graphs. It maps $(s_1, s_2, \dots, s_m) \in {}^k\mathbf{H}$ to $(s_m, \dots, s_2, s_1)$ and thus ${}^kB_x$ to ${}^kB_{x^{-1}}$ for all $x \in W$. By slight abuse of notation we will denote this equivalence also by ι. It is easy to see that ι descends to a well-defined $\mathbb{Z}[v, v^{-1}]$-linear anti-involution on $\mathcal{H}$ sending $\underline{H}_x$ to $\underline{H}_{x^{-1}}$ as well as ${}^p\underline{H}_x$ to ${}^p\underline{H}_{x^{-1}}$ for all $x \in W$. From this we deduce that the corresponding anti-involution on $\mathcal{H}$ sends H_s to H_s for $s \in S$ and thus H_x to $H_{x^{-1}}$ showing that it coincides with ι defined in Section 2.2. Therefore we have: $\mathrm{ch} \circ \iota = \iota \circ \mathrm{ch}$. Expressing ${}^p\underline{H}_x$ in the Kazhdan-Lusztig (resp. standard) basis and applying ι proves the last two statements after comparing coefficients.

(v) The statement for left descent sets follows from Lemma 3.11 below and the fact that the Laurent polynomials ${}^p m_{y,x}$ have non-negative coefficients. Using (iv) we can reduce the statement about the right descent sets to the case we have just proven.

(vi) This follows immediately from the analogue of Soergel's categorification theorem (part (iv) of Theorem 2.5). Indeed, in order to express ${}^p\underline{H}_x {}^p\underline{H}_y$ in the p-canonical basis, we need to decompose ${}^kB_x {}^kB_y$ into indecomposable objects in ${}^k\mathbf{H}$ and thus the Laurent polynomial ${}^p\mu_{x,y}^z$ encodes the graded multiplicity of ${}^kB_z$ in this tensor product. Therefore ${}^p\mu_{x,y}^z$ has non-negative coefficients. The self-duality of these Laurent polynomials follows from (i).

(vii) As explained in Section 3.1 we need to calculate the graded rank of finitely many local intersection forms in order to calculate ${}^p\underline{H}_x$. The rank of each of these intersection forms can only decrease for finitely many primes. (vii) now follows.

□

The *multiplication formula* from [KL79, (2.3.a) and (2.3.c)]) reads for $x \in W$ and $s \in S$ as follows:

$$\underline{H}_s \underline{H}_x = \begin{cases} (v + v^{-1})\underline{H}_x & \text{if } sx < x, \\ \underline{H}_{sx} + \sum_{\substack{y<x \text{ s.t.} \\ sy<y}} \mu(y, x)\underline{H}_y & \text{otherwise.} \end{cases} \tag{6}$$

where $\mu(y, x)$ is the coefficient of v in the Kazhdan-Lusztig polynomial ${}^0h_{y,x}$. One remnant of this multiplication formula for the p-canonical basis is the following result:

Lemma 3.11. *For $x \in W$ and $s \in \mathcal{L}(x)$ we have:*

$${}^p\underline{H}_s \, {}^p\underline{H}_x = (v + v^{-1}) \, {}^p\underline{H}_x.$$

Proof. Since $sx < x$, we can write using Proposition 3.10(ii) for $\mathrm{ch}({}^kB_{sx})$:

$$\mathrm{ch}({}^kB_s {}^kB_{sx}) = \underline{H}_s \, \mathrm{ch}({}^kB_{sx}) = H_x + vH_{sx} + \sum_{\substack{y<x \\ y \neq sx}} h_y H_y = \underline{H}_x + \sum_{\substack{y<x \\ y \neq sx}} m_y \underline{H}_y.$$

As ${}^kB_x$ is a summand of ${}^kB_s {}^kB_{sx}$ we can apply Proposition 3.10(iii) to deduce

$$\mathrm{ch}({}^kB_x) = \underline{H}_x + \sum_{\substack{y<x \\ y \neq sx}} {}^p m_{y,x} \underline{H}_y.$$

This implies ${}^p m_{sx,x} = 0$. Next, we calculate as follows:

$$\begin{aligned}
{}^p\underline{H}_s\, {}^p\underline{H}_x &= \underline{H}_s\, {}^p\underline{H}_x \\
&= \underline{H}_s(\underline{H}_x + \sum_{\substack{y<x \\ y\neq sx}} {}^p m_{y,x} \underline{H}_y) \\
&\in (v+v^{-1})\, {}^p\underline{H}_x + \sum_{y<x} \mathbb{Z}_{\geqslant 0}[v,v^{-1}]\, {}^p\underline{H}_y
\end{aligned}$$

where in the last equality we used (6) and the observation ${}^p m_{sx,x} = 0$ to determine the coefficient in front of ${}^p\underline{H}_x$. After evaluating at $v = 1$ and acting on the trivial module we see that no other terms besides $(v+v^{-1})\, {}^p\underline{H}_x$ can occur on the right hand side. □

Next, we will try to understand a little better the base change coefficients between the p-canonical basis and the standard basis and the structure coefficients occurring when multiplying a p-canonical basis element with ${}^p\underline{H}_s$ for $s \in S$. We have the following recursive formula which is not as useful as for the KL basis as it involves a lot of unknown structure coefficients!

Lemma 3.12. *Let $x \in W$ and $s \in S$ such that $sx < x$. Then the following holds:*

$$ {}^p h_{y,x} = {}^p h_{sy,sx} + v^{c_y}\, {}^p h_{y,sx} - \sum_{y\leqslant z<x} {}^p\mu^z_{s,sx}\, {}^p h_{y,z} $$

where $c_y = \begin{cases} 1 & \textit{if } sy > y, \\ -1 & \textit{otherwise.} \end{cases}$

Proof. Simply rewrite the multiplication ${}^p\underline{H}_s\, {}^p\underline{H}_{sx} = \sum_{z\leqslant x} {}^p\mu^z_{s,sx}\, {}^p\underline{H}_z$ in terms of the standard basis and compare coefficients. □

Using the results from [RW16, Section 3] and the fact that the parity sheaves are defined with respect to the constant pariversity, we get the following parity restriction on the degree of the structure coefficients:

Corollary 3.13. *Let x and s be as above. Then we have for all $z \in W$:*

$$\begin{aligned}
\deg({}^p h_{z,x}) &\equiv l(x) - l(z) \quad (\textit{mod } 2) \\
\deg({}^p\mu^z_{s,sx}) &\equiv l(x) - l(z) \quad (\textit{mod } 2)
\end{aligned}$$

Lemma 3.14. *Let $y \leqslant x \in W$ and $s \in \mathcal{L}(x) \cap \mathcal{L}(y)$. Then we have: ${}^p h_{sy,x} = v\, {}^p h_{y,x}$.*

Proof. Simply rewrite ${}^p\underline{H}_s\, {}^p\underline{H}_x = (v+v^{-1})\, {}^p\underline{H}_x$ (see Lemma 3.11) in terms of the standard basis and compare coefficients. □

In the remainder of the section, we will show that the p-canonical basis is compatible with restriction to parabolic subgroups via the induced based subroot datum. For a subset $I \subseteq S$ we have the corresponding parabolic subgroup $W_I \subseteq W$, which may be viewed as a Coxeter system (W_I, I), and its Hecke algebra $\mathcal{H}_{(W_I,I)}$, which is naturally a $\mathbb{Z}[v,v^{-1}]$-subalgebra of $\mathcal{H}$. In order to get elements ${}^p\underline{H}^I_x$, ${}^p h^I_{y,x}$ and ${}^p m^I_{y,x}$ for $x, y \in W_I$, we define the corresponding diagrammatic category using the realization $({}^kV, \{\alpha_s\}_{s\in I}, \{\alpha_s^\vee\}_{s\in I})$ in the finite setting (or $({}^kV, \{\widetilde{\alpha}_s\}_{s\in I}, \{\widetilde{\alpha}_s^\vee\}_{s\in I})$ in the affine setting). In this case Demazure surjectivity is immediate due to our assumption for the original category. This realization is a scalar extension of the realization of the sub-Cartan matrix $(\langle\alpha_s^\vee, \alpha_t\rangle)_{(s,t)\in I\times I}$ over k since we did not change ${}^kV$.

Lemma 3.15. *For $x, y \in W_I$ the following holds:*

(i) ${}^p\underline{H}_x^I = {}^p\underline{H}_x$

(ii) ${}^ph_{y,x}^I = {}^ph_{y,x}$

(iii) ${}^pm_{y,x}^I = {}^pm_{y,x}$

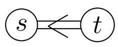
Proof. For the calculation of ${}^p\underline{H}_x$ only the simple reflections occurring in a reduced expression $\underline{x}$ of x and the corresponding induced sub-Cartan matrix matter. This follows from the explicit algorithm introduced in Section 3 and gives (i) which in turn implies (ii) and (iii) using [Lus03, Lemma 9.10 d)]. ☐

3.4 Examples

According to the classification of root systems and connected semi-simple algebraic groups, a Dynkin diagram fixes a semi-simple, adjoint algebraic group G together with a maximal torus $T \subseteq G$ such that the root system determined by (G, T) corresponds to the given Dynkin diagram. In this section we will only give the Dynkin diagram and consider the corresponding root datum of this pair (G, T) together with an arbitrary basis labelled by the nodes of the Dynkin diagram as input.

3.4.1 Type B_2

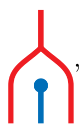
We label the simple reflections as follows:

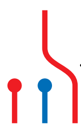


That means that the pairing of simple roots and coroots is given as follows:

$$\langle\alpha_s^\vee, \alpha_t\rangle = -2,$$
$$\langle\alpha_t^\vee, \alpha_s\rangle = -1.$$

Because the Schubert varieties associated to e, s, t, st, ts and $stst$ are smooth, we have ${}^p\underline{H}_x = \underline{H}_x$ for $x \in \{e, s, t, st, ts, stst\}$ and all primes p. (This can also be checked directly.) The remaining two elements are sts and tst. The two subsequences of (s, t, s) expressing s and corresponding light leaves are:

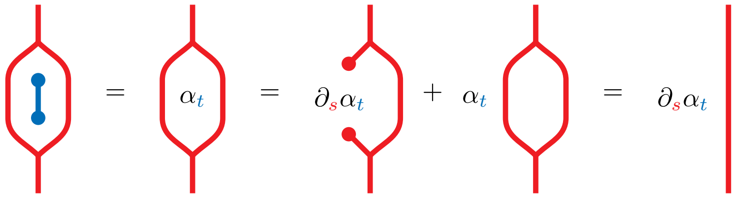
$(U1, U0, D0)$ of defect $0 \rightsquigarrow L_1 =$  ,

$(U0, U0, U1)$ of defect $2 \rightsquigarrow L_2 =$  .

Thus the pairing of the light leaves in $\mathbf{H}^{\not< s}$ is given by

$$(L_i \circ \overline{L_j})_{i,j\in\{1,2\}} = \begin{pmatrix} \langle \alpha_s^\vee, \alpha_t\rangle & \alpha_t \\ \alpha_t & \alpha_s\alpha_t \end{pmatrix}$$

where the top left entry comes from the following calculation:



Therefore the 0-th degree piece of the local intersection form of (s, t, s) at s is $I^0_{sts,s} = (-2)$ (recall that $\langle \alpha_s^\vee, \alpha_t\rangle = \partial_s\alpha_t = -2$). This shows that if $p = 2$, then ${}^kB_s{}^kB_t{}^kB_s$ does not decompose as ${}^kB_{sts} \oplus {}^kB_s$, but remains indecomposable. We get

$${}^p\underline{H}_{sts} = \begin{cases} \underline{H}_{sts} + \underline{H}_s & \text{if } p = 2, \\ \underline{H}_{sts} & \text{otherwise.} \end{cases}$$

Swapping the roles of s and t, the same calculation yields

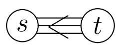
$${}^p\underline{H}_{tst} = \underline{H}_{tst}$$

for all primes p as $\langle \alpha_t^\vee, \alpha_s\rangle = -1$.

Observe that in this case the whole local intersection form of (s, t, s) at s can be calculated using the formula in the nil Hecke ring which we explained in Section 3.2.

3.4.2 Type G_2

We label the simple reflections as follows:



That means that the pairing of simple roots and coroots is given as follows:

$$\langle \alpha_s^\vee, \alpha_t\rangle = -3,$$
$$\langle \alpha_t^\vee, \alpha_s\rangle = -1.$$

For all primes $p > 3$ the Kazhdan-Lusztig basis coincides with the p-canonical basis. Since the Cartan matrix is symmetric modulo 2, the 2-canonical basis is stable under swapping s and t. Here is a summary of the results for $p \in \{2, 3\}$:

$${}^2\underline{H}_{stst} = \underline{H}_{stst} + \underline{H}_{st}$$
$${}^2\underline{H}_{ststs} = \underline{H}_{ststs} + \underline{H}_s$$

$${}^3\underline{H}_{sts} = \underline{H}_{sts} + \underline{H}_s$$
$${}^3\underline{H}_{ststs} = \underline{H}_{ststs} + \underline{H}_{sts}$$

${}^2\underline{H}_x = \underline{H}_x$ for

${}^3\underline{H}_x = \underline{H}_x$ for

$x \notin \{stst, tsts, ststs, tstst\}$ $\qquad\qquad$ $x \notin \{sts, ststs\}$

In this example, all the calculations needed to determine ${}^p\underline{H}_x$ for $x \notin \{ststs, tstst\}$ can be carried out using the formula in the nil Hecke ring from Section 3.2. For (s, t, s, t, s) (resp. (t, s, t, s, t)) there is a subexpression of defect 0 expressing s (resp. t) that contains a $D1$:

$$(U1, U1, U0, D1, D0).$$

We will illustrate how useful the formula in the nil Hecke ring is by calculating the intersection form of (s, t, s, t) at st. For the Kazhdan-Lusztig basis we know:

$$\underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t = \underline{H}_{stst} + 2\underline{H}_{st}.$$

There are two subexpressions of (s, t, s, t) of defect 0 expressing st:

$$\begin{aligned} \underline{e}^1 &:= (U1, U0, D0, U1) \\ \underline{e}^2 &:= (U1, U1, U0, D0) \end{aligned}$$

We need to calculate the coefficient of D_{st} in the following elements of the nil Hecke ring:

$$\begin{aligned} d(\underline{e}^1, \underline{e}^1): &\quad D_s \alpha_t D_s D_t = \partial_s(\alpha_t) D_{st} \\ d(\underline{e}^1, \underline{e}^2): &\quad D_s 1 1 D_t = D_{st} \\ d(\underline{e}^2, \underline{e}^2): &\quad D_s D_t \alpha_s D_t = \partial_t(\alpha_s) D_{st} \end{aligned}$$

Therefore the local intersection form of $stst$ at st is given by

$$\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$$

which implies the result stated above.

3.4.3 Types B_3 and C_3

In the Dynkin diagrams of types B_3 and C_3 we label the simple reflections as follows:

$$\begin{aligned} B_3 : &\quad ① \Leftarrow ② - ③ \\ C_3 : &\quad ① \Rightarrow ② - ③ \end{aligned}$$

The only interesting case is $p = 2$. The following table gives an overview over all the Weyl group elements for which the 2-canonical basis differs from the Kazhdan-Lusztig basis. It illustrates the dependence of the 2-canonical basis on the type of the root system. Even though the combinatorics in types B_3 and C_3 are the same, the corresponding 2-canonical bases are quite different.

	B_3	C_3
${}^2\underline{H}_{212}$	$\underline{H}_{212}$	$\underline{H}_{212} + \underline{H}_{2}$
${}^2\underline{H}_{121}$	$\underline{H}_{121} + \underline{H}_{1}$	$\underline{H}_{121}$
${}^2\underline{H}_{3212}$	$\underline{H}_{3212}$	$\underline{H}_{3212} + \underline{H}_{32}$
${}^2\underline{H}_{2123}$	$\underline{H}_{2123}$	$\underline{H}_{2123} + \underline{H}_{23}$
${}^2\underline{H}_{1321}$	$\underline{H}_{1321} + \underline{H}_{13}$	$\underline{H}_{1321}$
${}^2\underline{H}_{1213}$	$\underline{H}_{1213} + \underline{H}_{13}$	$\underline{H}_{1213}$
${}^2\underline{H}_{32123}$	$\underline{H}_{32123}$	$\underline{H}_{32123} + \underline{H}_{232} + \underline{H}_{3}$
${}^2\underline{H}_{21232}$	$\underline{H}_{21232}$	$\underline{H}_{21232} + \underline{H}_{232}$
${}^2\underline{H}_{23212}$	$\underline{H}_{23212}$	$\underline{H}_{23212} + \underline{H}_{232}$
${}^2\underline{H}_{21321}$	$\underline{H}_{21321} + \underline{H}_{213}$	$\underline{H}_{21321}$
${}^2\underline{H}_{12132}$	$\underline{H}_{12132} + \underline{H}_{132}$	$\underline{H}_{12132}$
${}^2\underline{H}_{232123}$	$\underline{H}_{232123}$	$\underline{H}_{232123} + (v + v^{-1})\underline{H}_{232}$
${}^2\underline{H}_{212321}$	$\underline{H}_{212321}$	$\underline{H}_{212321} + \underline{H}_{2321}$
${}^2\underline{H}_{121321}$	$\underline{H}_{121321} + \underline{H}_{1212} + \underline{H}_{1321} + \underline{H}_{1213} + \underline{H}_{13}$	$\underline{H}_{121321}$
${}^2\underline{H}_{123212}$	$\underline{H}_{123212}$	$\underline{H}_{123212} + \underline{H}_{1232}$
${}^2\underline{H}_{2123212}$	$\underline{H}_{2123212}$	$\underline{H}_{2123212} + \underline{H}_{21232} + \underline{H}_{23212} + \underline{H}_{232}$
${}^2\underline{H}_{1212321}$	$\underline{H}_{1212321} + \underline{H}_{12123}$	$\underline{H}_{1212321}$
${}^2\underline{H}_{1213212}$	$\underline{H}_{1213212} + \underline{H}_{13212}$	$\underline{H}_{1213212}$
${}^2\underline{H}_{21232123}$	$\underline{H}_{21232123}$	$\underline{H}_{21232123} + \underline{H}_{232123}$
${}^2\underline{H}_{12123212}$	$\underline{H}_{12123212} + \underline{H}_{1212}$	$\underline{H}_{12123212}$
${}^2\underline{H}_{12132123}$	$\underline{H}_{12132123} + \underline{H}_{132123}$	$\underline{H}_{12132123}$

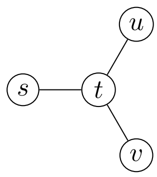
The most interesting entry in the whole table occurs for type C_3 and the element $232123 \in W$ where we have

$$ {}^2\underline{H}_{232123} = \underline{H}_{232123} + (v + v^{-1})\underline{H}_{232} $$

This means that, in the decomposition of $B_{232123} \otimes_{\mathbb{O}} \mathbb{K}$ into indecomposable objects in ${}^{\mathbb{K}}\mathbf{H}$, non-self-dual summands (i.e. with a non-trivial grading shift) occur.

3.4.4 Type D_4

We label the simple reflections as follows:



It turns out that the p-canonical basis and the Kazhdan-Lusztig basis coincide for all primes except for $p = 2$. There are four elements $x \in W$ with ${}^2\underline{H}_x \neq \underline{H}_x$. If $x = suvtsuv$ then we have

$$ {}^2\underline{H}_{t_1 x t_2} = \underline{H}_{t_1 x t_2} + \underline{H}_{t_1 suv t_2} $$

for $t_1, t_2 \in \langle t \rangle$. We will give some more details on how to calculate ${}^2\underline{H}_x$. We start out by decomposing the corresponding Bott-Samelson object into indecomposable objects in ${}^{\mathbb{K}}\mathbf{H}$ to get

$$ H_s H_u H_v H_t H_s H_u H_v = \underline{H}_x + (v^{-2} + 3 + v^2)\underline{H}_{suv}. $$

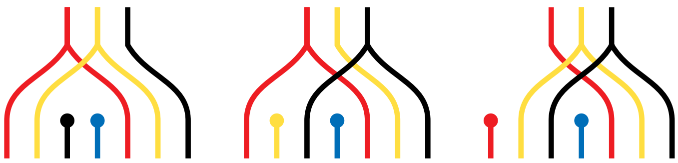
As subexpressions of (s,u,v,t,s,u,v) expressing suv we get

$$\begin{array}{ll}
(U1,U1,U1,U0,D0,D0,D0) & \text{of defect } -2,\\
(U1,U1,U0,U0,D0,D0,U1) & \text{of defect } 0,\\
(U1,U0,U1,U0,D0,U1,D0) & \text{of defect } 0,\\
(U0,U1,U1,U0,U1,D0,D0) & \text{of defect } 0,\\
(U1,U0,U0,U0,D0,U1,U1) & \text{of defect } 2,\\
(U0,U1,U0,U0,U1,D0,U1) & \text{of defect } 2,\\
(U0,U0,U1,U0,U1,U1,D0) & \text{of defect } 2,\\
(U0,U0,U0,U0,U1,U1,U1) & \text{of defect } 4.
\end{array}$$

The light leaf morphism of degree -2 pairs with the three light leaf morphisms of degree 2 to give the matrix

$$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

The light leaf morphisms corresponding to the subexpressions of defect 0 are the following:



Pairing them gives the following degree 0 piece of the intersection form:

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Note that the determinant of this matrix is -2 and its rank in characteristic 2 is 2. Therefore ${}^k B_{suvtsuv} \otimes_{\mathbb{O}} \mathbb{K}$ decomposes as

$${}^{\mathbb{K}}B_{suvtsuv} \oplus {}^{\mathbb{K}}B_{suv}$$

giving the result we stated above. The geometry of this example is discussed in the appendix of [WB12]. Note that all calculations presented in this section can also be carried out using the formula in the nil Hecke ring (see [HW15, §6.2]).

3.4.5 Type A_n

According to [WB12], the p-canonical basis and the Kazhdan-Lusztig basis coincide for all primes p for $n < 7$. Thus, we will describe the case $n = 7$ where the situation is quite remarkable.

For all primes $p \neq 2$ the p-canonical basis and the Kazhdan-Lusztig basis agree. We have ${}^2\underline{H}_x \neq \underline{H}_x$ for exactly 38 out of 40320 elements in S_8 and these examples fall into four classes.

In the following we will denote for a subset $I \subseteq S$ the corresponding parabolic subgroup by $W_I = \langle s \in I \rangle \subseteq W$. If W_I is finite, its longest element will be denoted by w_I. A permutation $\phi \in S_8$ will be displayed as a string $\phi(1)\phi(2)\dots\phi(8)$.

The Kashiwara-Saito singularity ([KS97]): This corresponds to the permutation $w = 62845173$. We have

$$^2\underline{H}_w = \underline{H}_w + \underline{H}_{w_I}$$

with $I = \{1, 3, 4, 5, 7\}$. There is a cluster of 16 elements around the Kashiwara-Saito singularity described as follows. If we let $J = \{2, 6\}$, then we have

$$^2\underline{H}_{uwv} = \underline{H}_{uwv} + \underline{H}_{uw_I v}.$$

for all $u, v \in W_J$ unless $u = v = w_J$ in which case

$$^2\underline{H}_{w_J w w_J} = \underline{H}_{w_J w w_J} + \underline{H}_{w_J w_I w_J} + \underline{H}_{w_K}$$

where $K = \{1, 2, 3, 5, 6, 7\}$.

The Hexagon singularity (Braden's example in the appendix of [WB12]): Consider the permutation $w = 46718235$. We have

$$^2\underline{H}_w = \underline{H}_w + \underline{H}_{w_I}$$

with $I = \{2, 3, 5, 6\}$. In this case we get a cluster of size 4. For any $u, v \in \langle s_4 \rangle$ we have

$$^2\underline{H}_{uwv} = \underline{H}_{uwv} + \underline{H}_{uw_I v}$$

unless $u = v = s_4$ in which case

$$^2\underline{H}_{s_4 w s_4} = \underline{H}_{s_4 w s_4} + \underline{H}_{s_4 w_I s_4} + \underline{H}_{w_K}$$

where $K = \{1, 3, 4, 5, 7\}$.

For the Kashiwara-Saito singularity and the Hexagon singularity the calculation of the the local intersection form of $\underline{w}$ at w_I using the formula in the nil Hecke ring can be found in [HW15, §6.1 and §6.2]. In both cases one can find $\underline{w}$, a reduced expression for w, such that this local intersection form is a 1×1 matrix.

The waterfall: Consider the permutation $w_1 = 67283415$. Then we have

$$^2\underline{H}_{w_1 u} = \underline{H}_{w_1 u} + \underline{H}_{w_I u}.$$

for $I = \{1, 2, 4, 5, 6\}$ and $w_J \neq u \in W_J$ with $J = \{3, 7\}$.

Similarly, if we let $w_2 = 57813462$, then we have

$$^2\underline{H}_{v w_2} = \underline{H}_{v w_2} + \underline{H}_{v w'_I}$$

where $I' = \{2, 3, 4, 6, 7\}$ and $w_{J'} \neq v \in W_J$ with $J = \{1, 5\}$.

The situation becomes more complicated due to the fact that for $u = w_J$ and $v = w_{J'}$ we have $w_1 u = v w_2 =: w$. In this case we get

$$^2\underline{H}_w = \underline{H}_w + \underline{H}_{w_I u} + \underline{H}_{v w'_I}.$$

Note that, unlike the Kashiwara-Saito and hexagon permutations discussed above, the clusters containing w_1 and w_2 are neither swapped nor fixed by the automorphism $s_i \mapsto s_{8-i}$ and thus by applying the graph automorphism one obtains another seven elements for which ${}^2\underline{H}_x \neq \underline{H}_x$. Hence the two "waterfall" clusters contain 14 elements in total.

The basket: Consider the permutation

$$w = 84627351.$$

Then for all $u, v \in \langle s_4 \rangle$ one has

$${}^2\underline{H}_{uwv} = \underline{H}_{uwv} + \underline{H}_{uw_I v}$$

where $I = \{1, 2, 3, 5, 6, 7\}$.

4 General p-Cell Theory

In this section, we want to give the definition of p-cells. This notion is an obvious generalization of a notion introduced by Kazhdan-Lusztig in [KL79].

Definition 4.1. For $h \in \mathcal{H}$ we say that ${}^p\underline{H}_w$ *appears with non-zero coefficient in* h if the coefficient of ${}^p\underline{H}_w$ is non-zero when expressing h in the p-canonical basis.

Define a preorder $\overset{p}{\underset{R}{\leqslant}}$ (resp. $\overset{p}{\underset{L}{\leqslant}}$) on W as follows: $x \overset{p}{\underset{R}{\leqslant}} y$ (resp $x \overset{p}{\underset{L}{\leqslant}} y$) if and only if ${}^p\underline{H}_x$ appears with non-zero coefficient in ${}^p\underline{H}_y h$ (resp. $h\,{}^p\underline{H}_y$) for some $h \in \mathcal{H}$. Define $\overset{p}{\underset{2}{\leqslant}}$ to be the preorder generated by $\overset{p}{\underset{R}{\leqslant}}$ and $\overset{p}{\underset{L}{\leqslant}}$, in other words we have: $x \overset{p}{\underset{2}{\leqslant}} y$ if and only if ${}^p\underline{H}_x$ appears with non-zero coefficient in $h\,{}^p\underline{H}_y h'$ for some $h, h' \in \mathcal{H}$.

For any set of generators of $\mathcal{H}$ as a $\mathbb{Z}[v, v^{-1}]$-algebra, it is easy to see that one gets a set of generating relations for the corresponding p-cell preorders (see [Wil03, Proposition 4.1.1]). The following definition introduces some notation for the relations generating the p-cell preorder obtained from the generating set $\{\underline{H}_s \mid s \in S\}$ which we will use in Section 5.1.

Definition 4.2. Let $x, y \in W$. We write $x \overset{p}{\underset{L}{\leftarrow}} y$ (resp. $x \overset{p}{\underset{R}{\leftarrow}} y$) if ${}^p\mu_{s,y}^x$ (resp. ${}^p\mu_{y,s}^x$) is non-zero for some $s \in S$. In addition, we write $x \overset{p}{\underset{2}{\leftarrow}} y$ if $x \overset{p}{\underset{L}{\leftarrow}} y$ or $x \overset{p}{\underset{R}{\leftarrow}} y$ holds.

For the sake of completeness, we will state explicitly that these elementary relations generate the p-cell-preorder (see [AHR17, Lemma 5.3] for a proof):

Lemma 4.3. *For $x, y \in W$ the following holds: $x \overset{p}{\underset{R}{\leqslant}} y$ if and only if there exists a chain $x = x_0 \overset{p}{\underset{R}{\leftarrow}} x_1 \overset{p}{\underset{R}{\leftarrow}} \dots \overset{p}{\underset{R}{\leftarrow}} x_k = y$. Similarly for the left (resp. two-sided) p-cell preorder.*

In the remainder of the section, we will prove some elementary properties of p-cells. In most cases we will focus on right p-cells and not state the version for left p-cells explicitly.

In [KL79, Proposition 2.4] Kazhdan-Lusztig observed that a Kazhdan-Lusztig right cell preorder relation implies an inclusion of left descent sets. The following result shows that the compatibility between cells and descent sets carries over to the more general setting. More precisely, the set of all elements with a fixed left descent set is a union of right p-cells. The result can also be found in [AHR17, Lemma 5.4]:

Lemma 4.4. *For $x, y \in W$ with $y \overset{p}{\underset{R}{\leqslant}} x$ we have $\mathcal{L}(x) \subseteq \mathcal{L}(y)$. In particular, $x \overset{p}{\underset{R}{\sim}} y$ gives $\mathcal{L}(x) = \mathcal{L}(y)$ and for any $I \subseteq S$ the set $\{w \in W \mid \mathcal{L}(w) = I\}$ is a union of right p-cells.*

Proof. It is enough to consider the case where we multiply ${}^p\underline{H}_x$ with ${}^p\underline{H}_s$ for $s \notin \mathcal{R}(x)$. We have on the one hand:

$$ {}^p\underline{H}_x\, {}^p\underline{H}_s = \sum_y {}^p\mu_{x,s}^y\, {}^p\underline{H}_y $$

On the other hand we can write:

$$ \begin{aligned} {}^p\underline{H}_x\, {}^p\underline{H}_s &= (\sum_{y \leqslant x} {}^p m_{y,x} \underline{H}_y)\underline{H}_s \\ &= \sum_{\substack{y \leqslant x \\ s \in \mathcal{R}(y)}} (v+v^{-1})\, {}^p m_{y,x} \underline{H}_y + \sum_{\substack{y \leqslant x \\ s \notin \mathcal{R}(y)}} {}^p m_{y,x} (\underline{H}_{ys} + \sum_{\substack{z \leqslant y \\ s \in \mathcal{R}(z)}} \mu(z,y) \underline{H}_z) \end{aligned} $$

Proposition 3.10(v) shows that all $y \in W$ occurring with non-zero ${}^p m_{y,x}$ on the right hand side satisfy $\mathcal{L}(x) \subseteq \mathcal{L}(y)$. [KL79, Proposition 2.4] shows that $z \overset{0}{\underset{R}{\leqslant}} y$ implies $\mathcal{L}(y) \subseteq \mathcal{L}(z)$. Observe that the set of $y \in W$ with non-zero structure coefficient ${}^p\mu_{x,s}^y$ is a subset of the set of all $y \in W$ indexing a summand $\underline{H}_y$ with non-zero coefficient on the right hand side (due to Proposition 3.10(iii) and (vi)). Putting all of this together gives the result. ☐

Corollary 4.5. $\{\mathrm{Id}\}$ *is a left, right, and* 2*-sided* p*-cell for all primes* p.

It is well known for Kazhdan-Lusztig cells that left and right cells are closely related via taking inverses. Using the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution ι on $\mathcal{H}$ together with Proposition 3.10(iv) we obtain the corresponding result for p-cells which will allow us to pass from left to right p-cells:

Lemma 4.6. *For all* $x, y \in W$ *we have:*

$$ x \overset{p}{\underset{L}{\leqslant}} y \iff x^{-1} \overset{p}{\underset{R}{\leqslant}} y^{-1}, $$
$$ x \overset{p}{\underset{LR}{\leqslant}} y \iff x^{-1} \overset{p}{\underset{LR}{\leqslant}} y^{-1}. $$

Next, we want to consider the question which automorphisms of our Coxeter system induce automorphisms on $\mathcal{H}$ that are well-behaved with respect to the p-canonical basis. Choose a total order on S to define the (generalized) Cartan matrix $M = (\langle \alpha_s^\vee, \alpha_t \rangle)_{(s,t) \in S \times S}$. Let $\phi : (W,S) \xrightarrow{\sim} (W,S)$ be an automorphism of Coxeter systems (in particular we have $\phi(S) = S$) which leaves M invariant when permuting simultaneously the corresponding rows and columns (i.e. $\langle \alpha_t^\vee, \alpha_s \rangle = \langle \alpha_{\phi(t)}^\vee, \alpha_{\phi(s)} \rangle$ for all $s, t \in S$). Then ϕ induces a $\mathbb{Z}[v, v^{-1}]$-linear automorphism of $\mathcal{H}$ via $H_x \mapsto H_{\phi(x)}$ for $x \in W$ which we will also denote by ϕ by slight abuse of notation. Therefore, ϕ maps $\underline{H}_x$ to $\underline{H}_{\phi(x)}$ by the defining property of the Kazhdan-Lusztig basis.

Proposition 4.7. *In the setting given above we have for all* $x, y \in W$:

(i) $\phi({}^p\underline{H}_x) = {}^p\underline{H}_{\phi(x)}$,

(ii) ${}^p m_{y,x} = {}^p m_{\phi(y),\phi(x)}$ *and* ${}^p h_{x,y} = {}^p h_{\phi(x),\phi(y)}$,

(iii) ${}^p\mu_{x,y}^z = {}^p\mu_{\phi(x),\phi(y)}^{\phi(z)}$,

(iv) $x \overset{p}{\underset{L}{\leqslant}} y \Leftrightarrow \phi(x) \overset{p}{\underset{L}{\leqslant}} \phi(y)$ *and* $x \overset{p}{\underset{R}{\leqslant}} y \Leftrightarrow \phi(x) \overset{p}{\underset{R}{\leqslant}} \phi(y)$.

Proof. Observe that ϕ induces a monoidal, k-linear equivalence of BS and thus of $\mathbf{H}$ which on the Hom-spaces merely permutes the colours in the diagrams (given by S) and the variables of the polynomials in R decorating the regions according to the action of ϕ. Since the numerical input for the algorithm to calculate the p-canonical basis (as described in Section 3) reduces to M we see immediately that this equivalence sends ${}^kB_x$ to ${}^kB_{\phi(x)}$ and thus on the level of Grothendieck groups ${}^p\underline{H}_x$ to ${}^p\underline{H}_{\phi(x)}$. This proves (i).

Recall that ϕ maps $\underline{H}_x$ to $\underline{H}_{\phi(x)}$ and H_x to $H_{\phi(x)}$ for all $x \in W$. For this reason, (ii) follows from (i) by rewriting ${}^p\underline{H}_x$ in the Kazhdan-Lusztig basis (resp. standard basis), applying ϕ, using (i) and comparing coefficients in the Kazhdan-Lusztig basis (resp. standard basis). (i) implies (iii) in a similar way and (iv) follows from (iii). □

Suppose that our based root datum is irreducible. In this case, the last proposition can be applied to all automorphisms of the (extended) Dynkin diagram of our root system. In finite type conjugation by the longest element in the finite Weyl group is also covered by the last proposition. Indeed, it follows from [Dav08, Remark 13.1.8] that for irreducible finite Coxeter groups the longest element w_0 is central except in types A_n for $n \geqslant 2$, D_n with n odd, E_6, and $I_2(m)$ for m odd where $I_2(m)$ denotes the dihedral group of order $2m$. In all these cases, conjugation by w_0 gives the obvious automorphism of the corresponding Coxeter graph. After restricting to crystallographic Coxeter systems, only simply-laced types remain and so any automorphism of the Coxeter graph gives an automorphism of the Dynkin diagram of the same type in the obvious way (as the graphs are isomorphic).

Definition 4.8. Let $I \subseteq S$ be a subset. Call I *finitary* if the corresponding parabolic subgroup $\langle I \rangle \subseteq W$ is finite. Define W^I to be the set of representatives of minimal length of cosets in W/W_I.

The following result is the main result of this section and generalizes the parabolic compatibility for Kazhdan-Lusztig cells (see [Lus03, Proposition 9.11]) to the setting of p-cells:

Theorem 4.9 (Parabolic compatibility of right p-cells)**.**
Let $I \subseteq S$ be a finitary subset. Then for $y, z \in W_I$ the following holds:

$$z \overset{p}{\underset{R}{\leqslant}} y \text{ in } W_I \Leftrightarrow \forall x \in W^I : xz \overset{p}{\underset{R}{\leqslant}} xy \text{ in } W$$

As a corollary to the proof of Theorem 4.9 we get:

Corollary 4.10. *In the setting of Theorem 4.9 we have:*

$${}^ph_{xy,xz} = {}^ph_{y,z}$$

4.1 Algebraic Proof of Theorem 4.9 and Corollary 4.10

First we will prove Theorem 4.9:

For all elements w in $W_I \cup W^I$ choose a reduced expression $\underline{w}$. We have a bijection

$$\begin{aligned} W^I \times W_I &\longrightarrow W \\ (x, y) &\longmapsto xy \end{aligned}$$

such that $l(xy) = l(x) + l(y)$ (see [BB05, Proposition 2.4.4]). Therefore, for $x \in W^I$ and $y \in W_I$ the concatenation of the corresponding reduced expressions $\underline{x}$ and $\underline{y}$ gives a reduced expression $\underline{x}^\frown \underline{y}$ of xy. Choose $x \in W^I$ and $y \in W_I$ arbitrarily.

Lemma 4.11. *We have a decoration- and defect-preserving bijection:*

$$\begin{aligned} g : \{\textit{subexpr. of } \underline{y} \textit{ expressing } z\} &\xrightarrow{\sim} \{\textit{subexpr. of } \underline{xy} \textit{ expressing } xz\} \\ \underline{e} &\longmapsto \underbrace{(1, \dots, 1)}_{l(x)\ ones}{}^\frown \underline{e} \end{aligned}$$

Proof. It is easy to see that g is well-defined and injective. Leaving out any letter of $\underline{x}$ in a subexpression of $\underline{xy}$ leads to a Bruhat stroll ending in a right coset $\widetilde{x}W_I$ with $\widetilde{x} < x$ as x is of minimal length in xW_I. Thus, g is surjective.

For the definition of the Bruhat graph we refer the reader to [Dye91, Definition 1.1]. The induced Bruhat graph on xW_I (as a subgraph of the Bruhat graph of W) is isomorphic to the one of W_I (see [Dye91, Theorem 1.4]). This follows for example from the subword property (see [BB05, Theorem 2.2.2]). Thus any subexpression $g(\underline{e})$ of $\underline{xy}$ expressing xz will have a decoration starting with $l(x)$ symbols $U1$ (as $\underline{x}$ is a reduced expression) and the remaining expression $\underline{e}$ will be decorated in the same way as $\underline{e}$ would be decorated as a subexpression of $\underline{y}$ expressing z. Since the ones in a subexpression do not contribute to the defect, this immediately implies:

$$\mathrm{df}(g(\underline{e})) = \mathrm{df}(\underline{e})$$

where on the left (resp. right) hand side the defect is calculated as a subexpression of $\underline{xy}$ (resp. $\underline{y}$). ☐

This bijection matches up the combinatorial data used to define the light leaves and thus allows us to compare the corresponding local intersection forms. Consider the local intersection forms $I_{\underline{xy},xz}$ of $\underline{xy}$ at xz (resp. $I_{\underline{y},z}$ of $\underline{y}$ at z) and the matrices representing them with respect to the light leaves bases (see Section 3). For two subexpressions $\underline{e}, \underline{f}$ of $\underline{y}$ expressing z we get in ${}^k\mathbf{H}^{\not< xz} \otimes_R k$:

$$(I_{\underline{xy},xz})_{g(\underline{e}),g(\underline{f})} = \mathrm{Id}_{\mathrm{BS}(\underline{x})}(I_{\underline{y},z})_{\underline{e},\underline{f}}$$

This implies that the multiplicity of ${}^kB_{xz}$ and its grading shifts in $\mathrm{BS}(\underline{xy})$ which is given by $\mathrm{grk}(I_{\underline{xy},xz})$ coincides with the multiplicity of ${}^kB_z$ and its grading shifts in $\mathrm{BS}(\underline{y})$ which is given by $\mathrm{grk}(I_{\underline{y},z})$.

Choose any total order on W refining the Bruhat order and preserving elements in the same coset in W/W_I as blocks of adjacent elements. Note that our choices above have fixed a reduced expression $\underline{w}$ for any element $w \in W$. Denote by A the base change matrix from the Bott-Samelson basis $\{\underline{H}_{\underline{w}} \mid w \in W_I\}$ to the p-canonical basis $\{{}^p\underline{H}_w \mid w \in W_I\}$ of $\mathcal{H}_{(W_I,I)}$. A is an upper-triangular, invertible matrix with entries in $\mathbb{Z}[v, v^{-1}]$ and ones on the diagonal. The above considerations show that the base change matrix from the Bott-Samelson to the p-canonical basis of $\mathcal{H}$ looks as follows:

$$\begin{array}{c|c} & \begin{matrix} W_I & \dots & xW_I & \dots \end{matrix} \\ \begin{matrix} W_I \\ \vdots \\ xW_I \\ \vdots \end{matrix} & \begin{pmatrix} A & & & \\ & \ddots & & * \\ & & A & \\ 0 & & & \ddots \end{pmatrix} \end{array}$$

Note that this form is preserved by taking inverses. This means that when expressing ${}^p\underline{H}_{xy}$ in terms of the Bott-Samelson basis A^{-1} gives all coefficients for terms indexed by $\underline{xz}$ for $z \in W_I$. Using this allows us to partly decouple the terms $\underline{H}_{\underline{x}}$ and $\underline{H}_{\underline{y}}$. When calculating ${}^p\underline{H}_{xy}\,{}^p\underline{H}_w$ for $w \in W_I$ we can simply express both terms in the Bott-Samelson basis, perform the calculation where only the structure coefficients for the Bott-Samelson basis of $\mathcal{H}_{(W_I,I)}$ come into play and rewrite it in terms of the p-canonical basis. This immediately implies the following result as we have full control over the situation in the top coset xW_I:

Corollary 4.12. *For all $y, z, w \in W_I$ and a minimal coset representative x of W/W_I we have:*

$${}^p\mu^{xz}_{xy,w} = {}^p\mu^{z}_{y,w}$$

The last corollary proves Theorem 4.9 since elementary relations obtained from ${}^p\underline{H}_w\,{}^p\underline{H}_s$ for $w \in W_I$ and $s \in I$ generate the right p-cell preorder in W_I. Next, we will prove Corollary 4.10:

Recall the following lemma which in the case of a reduced expression $\underline{w}$ describes how to express the Bott-Samelson basis element $\underline{H}_{\underline{w}}$ in terms of the standard basis (see [EW16, Lemma 2.10]):

Lemma 4.13. *For any expression $\underline{w}$ in S we have:*

$$\underline{H}_{\underline{w}} = \sum_{\substack{\underline{e} \text{ subexpression} \\ \text{of } \underline{w}}} v^{\mathrm{df}(\underline{e})} H_{(\underline{w}^{\underline{e}})}.$$

Recall that we have chosen a total order on W in the proof of Theorem 4.9. Denote by B the base change matrix from the Bott-Samelson basis $\{\underline{H}_{\underline{w}} \mid w \in W_I\}$ to the standard basis $\{H_w \mid w \in W_I\}$ of $\mathcal{H}_{(W_I,I)}$. Then B is an upper-triangular, invertible matrix with entries in $\mathbb{Z}[v, v^{-1}]$ and ones on the diagonal. The defect-preserving bijection from Lemma 4.11 shows that the base change matrix from the Bott-Samelson to the standard basis of $\mathcal{H}$ looks as follows:

$$\begin{array}{c|c} & \begin{matrix} W_I & \dots & xW_I & \dots \end{matrix} \\ \begin{matrix} W_I \\ \vdots \\ xW_I \\ \vdots \end{matrix} & \begin{pmatrix} B & & & \\ & \ddots & & * \\ & & B & \\ 0 & & & \ddots \end{pmatrix} \end{array}$$

Multiplying the base change matrix from the p-canonical to the Bott-Samelson basis with the base change matrix from the Bott-Samelson to the standard basis finishes the proof of Corollary 4.10.

4.2 Decomposition Criterion for Kazhdan-Lusztig Cells

In this section, we want to study the interplay between the weak right Bruhat order (see [BB05, Definition 3.1.1] for the definition) and the right p-cell preorder. This will allow us to formulate a simple criterion as to when p-cells decompose into Kazhdan-Lusztig cells. This criterion will be quite useful in affine rank 2.

In the next few results we will focus on right cells, but a similar version for left cells can easily be formulated. For $x \in W$, let $\underline{x} = s_1 s_2 \dots s_k$ be a reduced expression. Set $x_i := s_1 s_2 \dots s_i$ for all $0 \leqslant i \leqslant k$. Since ${}^p\underline{H}_e = \underline{H}_e$ there exists a maximal $0 \leqslant m \leqslant k$ such that for all $y \leqslant x_m$ with ${}^p m_{y,x_m} \neq 0$ we have $y \underset{R}{\overset{0}{\leqslant}} x_m$. In this setting we have the following result:

Lemma 4.14. *All $y \leqslant x$ with ${}^p m_{y,x}$ non-zero satisfy: $y \underset{R}{\overset{0}{\leqslant}} x_m$.*

Proof. The claim follows as ${}^p\underline{H}_{x_l}$ for $l \geqslant m$ is a linear combination of Kazhdan-Lusztig basis elements indexed by elements in $\{w \in W \mid w \underset{R}{\overset{0}{\leqslant}} x_m\}$. □

Observe that $x_m \underset{R}{\overset{p}{\geqslant}} x$ always holds. So if $x_m \underset{R}{\overset{p}{\leqslant}} x$, then x and x_m lie in the same Kazhdan-Lusztig right cell.

Corollary 4.15. *If x_m and x lie in the same Kazhdan-Lusztig right cell, then for $y \leqslant x$ with ${}^p m_{y,x}$ non-zero we have $y \underset{R}{\overset{0}{\leqslant}} x$.*

Definition 4.16. Let $C \subseteq W$ be an arbitrary subset. C is called *right-connected* if for every two elements $x, y \in C$, there exists a sequence $x = x_0, x_1, \dots, x_k = y$ in C such that $x_i^{-1} x_{i-1} \in S$ (i.e. x_{i-1} and x_i differ by a simple reflection on the right) for all $1 \leqslant i \leqslant k$. It follows that C decomposes as a disjoint union of its right-connected components, i.e. the maximal right-connected subsets.

Call an element $x \in C$ *right-minimal* if x cannot be reached from any other element $y \in C \setminus \{x\}$ via a sequence $y = x_0, x_1, \dots, x_k = x$ in C as above satisfying in addition $y < x_1 < x_2 < \dots < x_k = x$. Observe that an element is right-minimal if and only if it is minimal with respect to the weak right Bruhat order.

Similarly we define *left-connected* and *left-minimal* using multiplication by simple reflections on the left, as well as 2-*connected* and 2-*minimal*.

The following observation follows immediately from the definition of a right-minimal element, but shows their most important property:

Lemma 4.17. *Let $C \subseteq W$ be an arbitrary subset. For all $y \in C$ there exists a right-minimal element $x \in C$ such that $y \underset{R}{\overset{0}{\leqslant}} x$ and $y \underset{R}{\overset{p}{\leqslant}} x$.*

At this point we should mention the following conjecture by Lusztig which he originally formulated for (affine) Weyl groups in [HK83, p. 14]. It still appears to be open in finite type D_n and in general affine type:

Conjecture 4.18.
Every Kazhdan-Lusztig right cell in a Coxeter group is right-connected.

In finite type, the conjecture is known to hold for all dihedral groups, in type A_n (see [KL79, §5]), B_n (see [Gar93, Theorem 3.5.9]) and in all exceptional types H_3, H_4, F_4, E_6, E_7 and E_8 (see [GH15, Example 7.3]). In affine type, it has been verified in affine rank 2 (see [Lus85, Theorem 11.3]), for W of type $\widetilde{A}_n$ with $n \geqslant 1$ (see [Shi86, Theorem 18.2.1]), for Kazhdan-Lusztig right cells contained in the lowest Kazhdan-Lusztig two-sided cell (see [Shi88, Corollary 1.2]), for Kazhdan-Lusztig right cells contained in the Kazhdan-Lusztig two-sided cell of elements with a unique reduced expression (see [Lus83b, Proposition 3.8]) and some other special cases (see for example [Xi89], [Shi02, Theorem 4.8] and [Shi06]).

In the rest of the section we want to apply these notions to compare Kazhdan-Lusztig and p-cells by looking at minimal elements. We will focus on right cells even though there are similar results about left (resp. two-sided) cells. Corollary 4.15 implies the following result:

Corollary 4.19. *Let C be a Kazhdan-Lusztig right cell and C_{min} the set of right-minimal elements in C. Assume for all $x \in C_{min}$ and $y \leqslant x$ the following:*

$$ {}^p m_{y,x} \neq 0 \Rightarrow y \overset{0}{\underset{R}{\leqslant}} x $$

Then for all $x \in C$ and $y \leqslant x$ with ${}^p m_{y,x} \neq 0$ we have $y \overset{0}{\underset{R}{\leqslant}} x$.

Definition 4.20. Let X be a set equipped with a preorder $\leqslant$. A subset $Y \subseteq X$ is called a *lower set* if for $y \in Y$ and any $x \in X$ with $x \leqslant y$ we have $x \in Y$ as well.

Observe that any lower set in the right p-cell preorder can be written as a union of right p-cells. For this reason the following result is the starting point of our criterion:

Lemma 4.21. *Let C be a Kazhdan-Lusztig right cell that satisfies the assumptions of Corollary 4.19 and $x \in C$. Then $y \overset{p}{\underset{R}{\leqslant}} x$ implies $y \overset{0}{\underset{R}{\leqslant}} x$. If C is minimal in the Kazhdan-Lusztig right cell preorder, then C is a lower set in the right p-cell preorder and we have*

$$ \bigoplus_{x \in C} \mathbb{Z}[v, v^{-1}]\, {}^p\underline{H}_x = \bigoplus_{x \in C} \mathbb{Z}[v, v^{-1}] \underline{H}_x $$

as $\mathbb{Z}[v, v^{-1}]$-submodules of $\mathcal{H}$.

Proof. Assume that ${}^p\underline{H}_y$ occurs with non-zero coefficient in ${}^p\underline{H}_x h$ for some $h \in \mathcal{H}$. Write ${}^p\underline{H}_x h = (\underline{H}_x + \sum_{z<x} {}^p m_{z,x} \underline{H}_z)h$. If $\underline{H}_y$ occurs with non-zero coefficient in the product $\underline{H}_x h$ then we have by definition $y \overset{0}{\underset{R}{\leqslant}} x$. If $\underline{H}_y$ occurs with non-zero coefficient in one of the products ${}^p m_{z,x} \underline{H}_z h$, then we have ${}^p m_{z,x} \neq 0$ and by Corollary 4.19 that $y \overset{0}{\underset{R}{\leqslant}} z \overset{0}{\underset{R}{\leqslant}} x$.

Denote by $C_{\min}$ the set of right-minimal elements in C. Our arguments above show that we have the following inclusions:

$$ \{ \overset{p}{\underset{R}{\leqslant}} C\} = \bigcup_{x \in C_{\min}} \{ \overset{p}{\underset{R}{\leqslant}} x\} \subseteq \bigcup_{x \in C_{\min}} \{ \overset{0}{\underset{R}{\leqslant}} x\} = \{ \overset{0}{\underset{R}{\leqslant}} C\} $$

The equalities on the left and right hand side follow from Lemma 4.17. In particular, C is contained in the left hand side. Thus, if C is minimal in the Kazhdan-Lusztig right cell preorder, we actually have equality which implies the claim as the left hand side obviously is a lower set in the p-cell preorder. In this case, Corollary 4.19 shows that for any $x \in C$ the p-canonical basis element ${}^p\underline{H}_x$ can be written in terms of Kazhdan-Lusztig basis elements indexed by elements in C which implies the statement about the span of the p-canonical and the Kazhdan-Lusztig basis elements. □

Lemma 4.22. *Let C be a Kazhdan-Lusztig right cell. Assume that all Kazhdan-Lusztig right cells smaller or equal than C in the Kazhdan-Lusztig right cell preorder satisfy the assumption of Corollary 4.19. Then $\{\overset{0}{\underset{R}{\leqslant}} C\}$ is a lower set in the right p-cell preorder and we have*

$$\bigoplus_{x \in \{\overset{p}{\underset{R}{\leqslant}} C\}} \mathbb{Z}[v, v^{-1}]\, {}^p\underline{H}_x = \bigoplus_{x \in \{\overset{0}{\underset{R}{\leqslant}} C\}} \mathbb{Z}[v, v^{-1}]\underline{H}_x$$

as $\mathbb{Z}[v, v^{-1}]$-submodules of $\mathcal{H}$. Moreover, C decomposes as a union of right p-cells.

Proof. We proceed by induction on the height of C in the Kazhdan-Lusztig right cell preorder. Note that [Lus87, Theorem 2.2 (a)] implies that any affine Weyl group has only finitely many Kazhdan-Lusztig right cells. Thus the height of any cell in the right cell preorder is finite.

Lemma 4.21 gives the induction start. Let C be of height $\geqslant 2$. By induction for all predecessors of C in the Kazhdan-Lusztig right cell preorder we know our claim holds for $\{\overset{0}{\underset{R}{<}} C\}$. Therefore, we may pass to the quotient

$$\mathcal{H} / \bigoplus_{x \in \{\overset{0}{\underset{R}{<}} C\}} \mathbb{Z}[v, v^{-1}]\underline{H}_x$$

where C becomes the smallest cell. Note that this quotient is a right $\mathcal{H}$-module that is free as a $\mathbb{Z}[v, v^{-1}]$-module and admits a p-canonical as well as a Kazhdan-Lusztig basis. Denote by

$$\pi : \mathcal{H} \to \mathcal{H} / \bigoplus_{x \in \{\overset{0}{\underset{R}{<}} C\}} \mathbb{Z}[v, v^{-1}]\underline{H}_x$$

the projection to the quotient. Applying π amounts to forgetting all basis elements indexed by elements in $\{\overset{0}{\underset{R}{<}} C\}$ when expressing an element in the Hecke algebra in the p-canonical or Kazhdan-Lusztig basis. For this reason, the action of $\mathcal{H}$ on this right module captures a lot of information about the p-cell as well as the Kazhdan-Lusztig cell structure in the following sense: For $x, y \in \{\overset{0}{\underset{R}{\not<}} C\}$ we have $y \overset{p}{\underset{R}{\leqslant}} x$ if and only if there exists an element $h \in \mathcal{H}$ such that $\pi({}^p\underline{H}_y)$ occurs with non-trivial coefficient in $\pi({}^p\underline{H}_x)h$. A similar statement holds for the Kazhdan-Lusztig cell structure. This allows us to conclude as in the proof of Lemma 4.21. □

4.3 (Counter-)Examples

In this section, we will give some examples of the p-cell structure of finite Weyl groups. We will follow the conventions and notation of Section 3.4, but restrict to those examples that give counterexamples to obvious generalizations of known results from Kazhdan-Lusztig cell theory. All results in this section were obtained using computer calculations. Denote by w_0 the longest element in the corresponding finite Weyl group.

4.3.1 Type B_2

The following diagrams show the right (resp. two-sided) cells and the corresponding preorders in type B_2:

<table>
<tr><th></th><th>right cells</th><th>two-sided cells</th></tr>
<tr><td>KL-cells:</td><td>$$\begin{tikzcd} & \{\mathrm{Id}\} \arrow[dl, dash] \arrow[dr, dash] & \\ \{s, st, sts\} \arrow[dr, dash] & & \{t, ts, tst\} \arrow[dl, dash] \\ & \{w_0\} & \end{tikzcd}$$</td><td>$$\begin{tikzcd} \{\mathrm{Id}\} \arrow[d, dash] \\ \{s, t, st, ts, sts, tst\} \arrow[d, dash] \\ \{w_0\} \end{tikzcd}$$</td></tr>
<tr><td>2-cells:</td><td>$$\begin{tikzcd} & \{\mathrm{Id}\} \arrow[dl, dash] \arrow[ddr, dash] & \\ \{s\} \arrow[d, dash] & & \\ \{st, sts\} \arrow[dr, dash] & & \{t, ts, tst\} \arrow[dl, dash] \\ & \{w_0\} & \end{tikzcd}$$</td><td>$$\begin{tikzcd} \{\mathrm{Id}\} \arrow[d, dash] \\ \{s\} \arrow[d, dash] \\ \{t, st, ts, sts, tst\} \arrow[d, dash] \\ \{w_0\} \end{tikzcd}$$</td></tr>
</table>

Lusztig showed in [Lus83b, Proposition 3.8] that the set C of non-trivial elements in a Coxeter group that have a unique reduced expression always forms a Kazhdan-Lusztig two-sided cell and for any $s \in S$ the set ${}_sC := \{w \in C \mid \mathcal{L}(w) = \{s\}\}$ gives a Kazhdan-Lusztig right cell. The example above shows that both statements do not hold for p-cells in general.

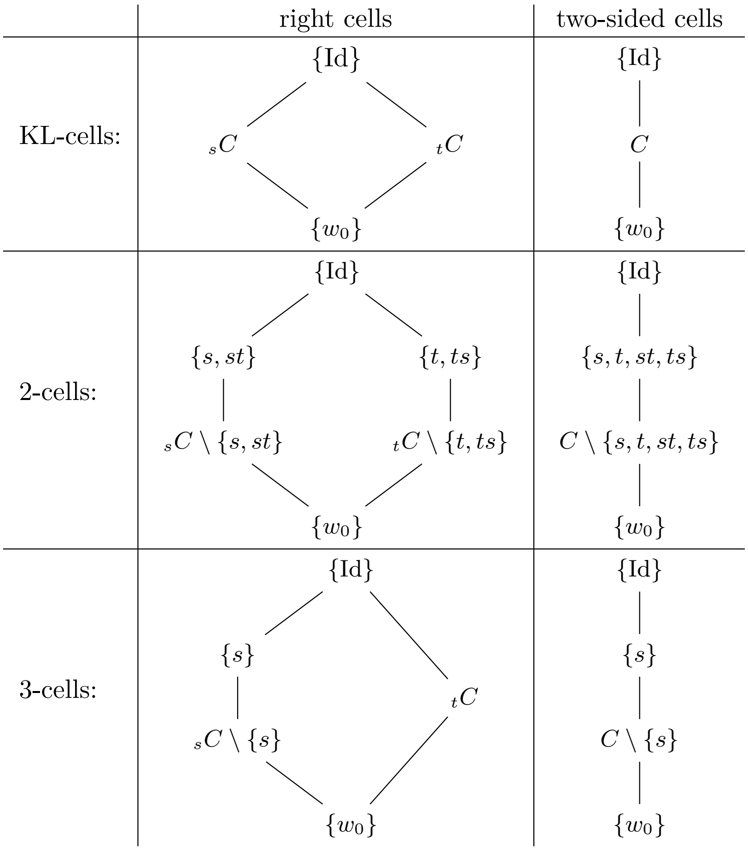
Observe that in characteristic 0 we have for $x, y \in W_{\mathrm{f}}$ (see [Lus03, Corollary 11.7]) the following equivalences

$$x \overset{0}{\underset{R}{\leqslant}} y \Leftrightarrow yw_0 \overset{0}{\underset{R}{\leqslant}} xw_0 \Leftrightarrow w_0y \overset{0}{\underset{R}{\leqslant}} w_0x$$

and the same statement for the left and two-sided cell preorder. The last example also shows that the analogous statement does not hold for p-cells.

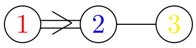
4.3.2 Type G_2

The following diagrams show the right (resp. two-sided) cells and the corresponding preorders in type G_2 using the notation from the last subsection. In particular, C, ${}_sC$ and ${}_tC$ are defined as in the last subsection.



4.3.3 Kazhdan-Lusztig cells do not decompose into p-cells

In this section, we will present the smallest example where Kazhdan-Lusztig cells do not decompose into p-cells. This happens in type C_3 for $p = 2$. We label the simple reflections as follows (also note their colours):



$$\Leftrightarrow \text{Cartan matrix: } \begin{pmatrix} 2, & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Explicit computer calculation gives the following Kazhdan-Lusztig right cells:

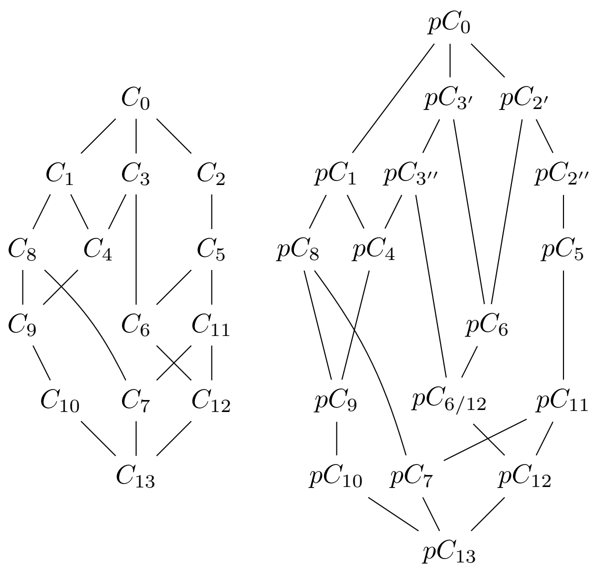
$$\begin{aligned}
C_0 &= \{\mathrm{Id}\} \\
C_1 &= \{1, 12, 121, 123\} \\
C_2 &= \{2, 21, 23, 212, 2123\} \\
C_3 &= \{3, 32, 321, 3212, 32123\} \\
C_4 &= \{13, 132, 1321\} \\
C_5 &= \{213, 2132, 21321\} \\
C_6 &= \{232, 2321, 23212\} \\
C_7 &= \{2121, 21213, 212132, 2121321, 21213213\}
\end{aligned}$$

$$
\begin{aligned}
C_8 &= \{1213, 12132, 121321\}\\
C_9 &= \{1232, 12321, 123212\}\\
C_{10} &= \{13212, 132123, 1213212, 1232123, 12132123\}\\
C_{11} &= \{21232, 212321, 21232121\}\\
C_{12} &= \{232123, 232121, 2321213, 23212132\}\\
C_{13} &= \{w_0\}
\end{aligned}
$$

For $p = 2$ these right Kazhdan-Lusztig cells exhibit the following decomposition behaviour into right p-cells:

$$
\begin{aligned}
C_2 &= \underbrace{\{2, 21\}}_{pC_{2'}} \cup \underbrace{\{23, 212, 2123\}}_{pC_{2''}}\\
C_3 &= \underbrace{\{3, 32\}}_{pC_{3'}} \cup \underbrace{\{321, 3212, 32123\}}_{pC_{3''}}\\
C_6 \cup C_{12} &= \underbrace{\{232\}}_{pC_6} \cup \underbrace{\{2321, 23212, 232123\}}_{pC_{6/12}} \cup \underbrace{\{232121, 2321213, 23212132\}}_{pC_{12}}\\
C_i &= pC_i \text{ for } i \in \{0 \ldots 13\} \setminus \{2, 3, 6, 12\}
\end{aligned}
$$

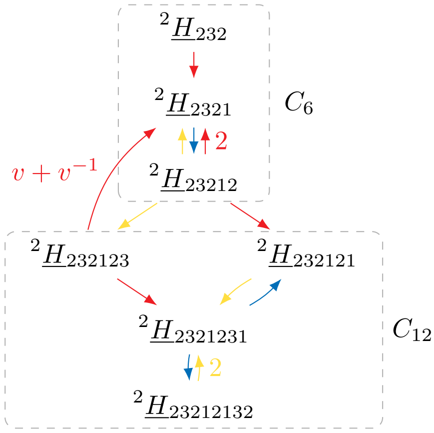
The Hasse-diagrams of the cell preorders look as follows. We display Kazhdan-Lusztig right cells on the left and right p-cells on the right. In these diagrams the cells that are depicted at one height form a two-sided cell.



Finally, let us try to explain the non-trivial decomposition behaviour. For the elements in $C_6 \cup C_{12}$ we have:

$$
\begin{aligned}
{}^2\underline{H}_{23212} &= \underline{H}_{23212} + \underline{H}_{232}\\
{}^2\underline{H}_{232123} &= \underline{H}_{232123} + (v + v^{-1})\underline{H}_{232}\\
{}^2\underline{H}_{23212132} &= \underline{H}_{23212132} + \underline{H}_{232123}\\
{}^2\underline{H}_x &= \underline{H}_x \text{ for } x \in (C_6 \cup C_{12}) \setminus \{23212, 232123, 23212132\}
\end{aligned}
$$

The subquotient $\mathcal{H}_{\overset{p}{\underset{R}{\leqslant}} C_6}/\mathcal{H}_{\overset{p}{\underset{R}{<}} C_{12}}$ is a module for the Hecke algebra and the action on the 2-canonical basis of this module can be encoded in the following graph:



Note that we omitted all edge labels equal to 1 and all loops labelled with $v + v^{-1}$. The strongly connected components of this graph give the right p-cells pC_6, $pC_{6/12}$ and pC_{12}. From this we see that neither two-sided nor right Kazhdan-Lusztig cells decompose into the corresponding p-cells in this example.

In this case we cannot apply the decomposition criterion from Section 4.2 because the Kazhdan-Lusztig right cell C_{12} does not satisfy the assumptions of Corollary 4.19.

In type B_3 the right (and two-sided) Kazhdan-Lusztig do decompose into right (resp. two-sided) p-cells, whereas in type B_4 they do not. This calculation together with Corollary 5.25 completely settles the question of when right Kazhdan-Lusztig cells decompose into right p-cells in types B and C. In summary, right Kazhdan-Lusztig cells in types B_n and C_n decompose into right p-cells for $p > 2$ or $p = 2$ and $n \leqslant 3$ in type B_n.

4.4 A Conjecture

In this section, we present a conjecture that is based on extensive computer calculations (mostly for p-cells in finite type).

Using Lusztig's a-function, one can show that Kazhdan-Lusztig right cells within the same two-sided cell are incomparable. This follows from [Lus03, Conjectures 14.2 P10]. Our calculations support the following generalization:

Conjecture 4.23.
The right p-cells within the same two-sided p-cell are incomparable.

We have verified this in types B_n, C_n for $n \leqslant 5$, D_n for $n \leqslant 5$, F_4 and G_2 for all primes p.

5 Left and Right Star Operations

In the section, we will prove consequences of the Kazhdan-Lusztig star-operations for the p-canonical basis. The star-operations were originally introduced in [KL79, §4], generalizing (dual) Knuth operations from the symmetric group to pairs of simple reflections $r, t \in S$ in general Coxeter groups with $m_{r,t} = 3$. In the literature, there doesn't seem to exist a consensus on how to generalize the star-operations to the case $3 < m_{r,t} < \infty$. We propose the following generalization as in [BG15, Remark 4.3]:

Definition 5.1. Let $r, t \in S$ be two simple reflections. Define:

$$\mathcal{D}_L(r,t) \coloneqq \{w \in W \mid |\mathcal{L}(w) \cap \{r,t\}| = 1\}$$
$$\mathcal{D}_R(r,t) \coloneqq \{w \in W \mid |\mathcal{R}(w) \cap \{r,t\}| = 1\}$$

Set $m \coloneqq m_{r,t}$. For $1 \leqslant k \leqslant m$ denote by ${}_r\hat{k} = rtrt\dots$ the alternating word in r and t starting in r of length k. Recall that $W^{\{r,t\}}$ denotes the set of representatives of minimal length of cosets in $W/\langle r,t\rangle$ (see Definition 4.8). Any coset in $W/\langle r,t\rangle$ contains a unique element $\widetilde{w} \in W^{\{r,t\}}$ and can be partitioned in the following sets:

$$\begin{cases} \{\widetilde{w}\} \cup \{\widetilde{w} \cdot {}_r\hat{k} \mid 1 \leqslant k < m\} \cup \{\widetilde{w} \cdot {}_t\hat{k} \mid 1 \leqslant k < m\} \cup \{\widetilde{w} \cdot {}_t\hat{m}\} & \text{if } m < \infty, \\ \{\widetilde{w}\} \cup \{\widetilde{w} \cdot {}_r\hat{k} \mid 1 \leqslant k\} \cup \{\widetilde{w} \cdot {}_t\hat{k} \mid 1 \leqslant k\} & \text{if } m = \infty. \end{cases}$$

For $m < \infty$ the element $\widetilde{w} \cdot {}_t\hat{m}$ is the unique element of maximal length in the coset. The set $\{\widetilde{w} \cdot {}_x\hat{k} \mid 1 \leqslant k < m\}$ for some $x \in \{r,t\}$ is called a *right* $\langle r,t\rangle$*-string* (also for $m = \infty$) and contained in $\mathcal{D}_R(r,t)$. The element $\widetilde{w} \cdot {}_x\hat{k}$ is the k*-th element* in this string.

It is easy to see that actually any element $w \in \mathcal{D}_R(r,t)$ lies in a right $\langle r,t\rangle$-string and can thus be written as $\widetilde{w} \cdot {}_x\hat{k}$ for some $x \in \{r,t\}$ and $1 \leqslant k < m$ where $\widetilde{w}$ is the element of minimal length in the right coset $w\langle r,t\rangle \in W/\langle r,t\rangle$.

Assume $3 \leqslant m < \infty$. Then the right star operation $(-)^*$ is an involution on $\mathcal{D}_R(r,t)$ sending $w = \widetilde{w} \cdot {}_x\hat{k}$ for x and k as above to $\widetilde{w} \cdot {}_x\widehat{(m-k)}$. The left star operation ${}^*(-)$ is an involution on $\mathcal{D}_L(r,t)$ defined analogously.

It follows immediately that the left and right star operations are related via:

$${}^*w = ((w^{-1})^*)^{-1}.$$

Fix for the rest of the section two simple reflections $r, t \in S$ with $3 \leqslant m \coloneqq m_{r,t} < \infty$. The multiplication formula for the Kazhdan-Lusztig basis (6) does not easily generalize to the p-canonical basis. However it will still be important to understand the structure coefficients of the p-canonical basis a little bit better. The next lemma states a crucial observation that will be used frequently below.

Lemma 5.2. *Let $x, z \in \mathcal{D}_R(r,t)$ with $r \in \mathcal{R}(x)$. The coefficient of $\underline{H}_z$ in ${}^p\underline{H}_x\underline{H}_t$ is given by*

$$\delta_{zr\in\mathcal{D}_R(r,t)}{}^pm_{zr,x} + \delta_{zt\in\mathcal{D}_R(r,t)}{}^pm_{zt,x}$$

where $\delta_{zr\in\mathcal{D}_R(r,t)}$ is the Kronecker delta.

Proof. Rewrite the product ${}^p\underline{H}_x\underline{H}_t$ as follows:

$$
\begin{aligned}
{}^p\underline{H}_x\underline{H}_t &= (\underline{H}_x + \sum_{v<x} {}^p m_{v,x}\underline{H}_v)\underline{H}_t \\
&= \sum_{\substack{z\leqslant xt\\ zt<z}} \mu(z,x)\underline{H}_z + \sum_{\substack{v<x\\ vt>v}} {}^p m_{v,x}\left(\sum_{\substack{z\leqslant vt\\ zt<z}} \mu(z,v)\underline{H}_z\right) \\
&\quad + \sum_{\substack{z<x\\ zt<z}} (v+v^{-1}){}^p m_{z,x}\underline{H}_z
\end{aligned}
$$

If t is not in the right descent set of z, the coefficient in front of $\underline{H}_z$ has to be zero. By Proposition 3.10(v) the formula stated above also gives zero in this case. Thus, we assume $t \in \mathcal{R}(z)$ from now on. Since z lies in $\mathcal{D}_R(r,t)$, this implies that r does not lie in the right descent set of z. Consider an element $v \in W$ with ${}^p m_{v,x} \neq 0$ such that $\underline{H}_z$ occurs with non-zero coefficient in $\underline{H}_v\underline{H}_t$. Observe that v could be x. By Proposition 3.10(v) ${}^p m_{v,x} \neq 0$ implies $r \in \mathcal{R}(x) \subseteq \mathcal{R}(v)$. Since the right descent sets differ, v and z cannot coincide. In particular, v does not have t in its right descent set and thus also lies in $\mathcal{D}_R(r,t)$. (Otherwise $\underline{H}_z$ could not occur with non-zero coefficient in $\underline{H}_v\underline{H}_t$.) Recall the following important fact about the μ-coefficients (from [KL79, (2.3.f)]):

Lemma 5.3. *Let $z < v \in W$ and $r \in \mathcal{R}(v) \setminus \mathcal{R}(z)$. Then we have:*

$$\mu(z,v) \neq 0 \Leftrightarrow v = zr$$

Moreover, $\mu(z,v) = 1$ in this case.

If $z < v$ holds, then we may apply this lemma to $z < v$ and the simple reflection r to get that $v = zr$. Otherwise, we have $z = vt > v$ (due to the multiplication formula). In both cases, we see that z and v lie in the same right $\langle r,t\rangle$-string and the coefficient of $\underline{H}_z$ in ${}^p\underline{H}_x\underline{H}_t$ is

$$\delta_{zr\in\mathcal{D}_R(r,t)}{}^p m_{zr,x} + \delta_{zt\in\mathcal{D}_R(r,t)}{}^p m_{zt,x}. \qquad \square$$

Definition 5.4. The *weak right Bruhat graph* of (W,S) is the labelled directed graph with vertex set W and edge set

$$\{(w,ws) \mid w \in W, s \in S \setminus \mathcal{R}(w)\}.$$

For $w \in W$ and $s \in S \setminus \mathcal{R}(w)$ the edge (w,ws) is labelled by α_s.

The reader may picture this as follows: Consider the subgraph of the weak right Bruhat graph on the vertices $\mathcal{D}_{r,t} \cap (z\langle r,t\rangle)$ and only edges labelled by α_r or α_t. In order to get the coefficient of $\underline{H}_z$ in ${}^p\underline{H}_x\underline{H}_t$ we simply have to slide the coefficients ${}^p m_{?,x}$ up along an edge labelled by α_t and down along an edge labelled by α_r and sum them up if two coefficients collide at a vertex in the process. Here, up and down are meant with respect to the weak right Bruhat order.

For the rest of the section, we will assume:

$$p > \begin{cases} 1 & \text{if } m = 3, \\ 2 & \text{if } m = 4, \\ 3 & \text{if } m = 6. \end{cases}$$

This ensures that for $w \in \langle r,t\rangle$ we have ${}^p\underline{H}_w = \underline{H}_w$.

Proposition 5.5. *Let $\sigma_x = \{x_1 < x_2 < \dots\}$ and $\sigma_z = \{z_1 < z_2 < \dots\}$ be two right $\langle r,t\rangle$-strings consisting of $m-1$ elements. Then we have the following relations between the coefficients ${}^p m_{z_j,x_i}$:*

$$m = 3 \Rightarrow \begin{cases} {}^p m_{z_1,x_1} = {}^p m_{z_2,x_2} \\ {}^p m_{z_2,x_1} = {}^p m_{z_1,x_2} \end{cases} \tag{7}$$

$$m = 4 \Rightarrow \begin{cases} {}^p m_{z_1,x_1} = {}^p m_{z_3,x_3} \\ {}^p m_{z_2,x_1} = {}^p m_{z_1,x_2} = {}^p m_{z_3,x_2} = {}^p m_{z_2,x_3} \\ {}^p m_{z_3,x_1} = {}^p m_{z_1,x_3} \\ {}^p m_{z_2,x_2} = {}^p m_{z_1,x_1} + {}^p m_{z_3,x_1} \end{cases} \tag{8}$$

$$m = 6 \Rightarrow \begin{cases} {}^p m_{z_1,x_1} = {}^p m_{z_5,x_5} \\ {}^p m_{z_2,x_1} = {}^p m_{z_1,x_2} = {}^p m_{z_5,x_4} = {}^p m_{z_4,x_5} \\ {}^p m_{z_3,x_1} = {}^p m_{z_1,x_3} = {}^p m_{z_5,x_3} = {}^p m_{z_3,x_5} \\ {}^p m_{z_4,x_1} = {}^p m_{z_1,x_4} = {}^p m_{z_5,x_2} = {}^p m_{z_2,x_5} \\ {}^p m_{z_5,x_1} = {}^p m_{z_1,x_5} \\ {}^p m_{z_2,x_2} = {}^p m_{z_4,x_4} = {}^p m_{z_1,x_1} + {}^p m_{z_3,x_1} \\ {}^p m_{z_3,x_2} = {}^p m_{z_2,x_3} = {}^p m_{z_4,x_3} = {}^p m_{z_3,x_4} = {}^p m_{z_2,x_1} + {}^p m_{z_4,x_1} \\ {}^p m_{z_4,x_2} = {}^p m_{z_2,x_4} = {}^p m_{z_3,x_1} + {}^p m_{z_5,x_1} \\ {}^p m_{z_3,x_3} = {}^p m_{z_1,x_1} + {}^p m_{z_3,x_1} + {}^p m_{z_5,x_1} \end{cases} \tag{9}$$

Proof. Comparing Laurent polynomials coefficient-wise induces a partial order which we will use in the following. For $x \in \mathcal{D}_R(r,t)$ with $r \in \mathcal{R}(x)$, rewrite ${}^p\underline{H}_x \underline{H}_t$ in terms of the p-canonical basis to get:

$$\sum_{v \leqslant xt} {}^p\mu_{x,t}^v \, {}^p\underline{H}_v$$

Express this in the Kazhdan-Lusztig basis and use Lemma 5.2 to see that we have the following inequality for $z \in \mathcal{D}_R(r,t)$:

$$\sum_{z \leqslant v \leqslant xt} {}^p\mu_{x,t}^v \, {}^p m_{z,v} \leqslant \delta_{zr \in \mathcal{D}_R(r,t)} {}^p m_{zr,x} + \delta_{zt \in \mathcal{D}_R(r,t)} {}^p m_{zt,x} \tag{10}$$

We want to use a weaker form of this inequality together with the fact that we understand the structure coefficients in the right $\langle r,t\rangle$-coset of x (see Corollary 4.12). Let $s_i \in \{r,t\}$ be the simple reflection such that $x_i s_i > x_i$ for $1 \leqslant i \leqslant m-1$. Write x_0 (resp. x_m) for the shortest (resp. longest) element in the right $\langle r,t\rangle$-coset of x. Similarly for z_0 and z_m. We can restrict the sum on the left hand side to $v \in \sigma_x$ and $v \leqslant xt$. From Corollary 4.12 and the explicit knowledge of the structure constants of the Kazhdan-Lusztig basis in the dihedral case we deduce:

$${}^p\underline{H}_{x_i} \, {}^p\underline{H}_{s_i} = \begin{cases} {}^p\underline{H}_{x_2} & \text{if } i = 1, \\ {}^p\underline{H}_{x_{i+1}} + {}^p\underline{H}_{x_{i-1}} & \text{otherwise.} \end{cases} \qquad \left(\text{mod} \sum_{\substack{w < x_{i+1} \\ w \notin \sigma_x}} \mathbb{Z}[v, v^{-1}] \, {}^p\underline{H}_w \right)$$

Using this in inequality (10) and letting x and z in their right $\langle r,t\rangle$-string vary, we obtain for $1 \leqslant i,j \leqslant m-1$:

$$\delta_{x_{i+1}\in\sigma_x}{}^{p}m_{z_j,x_{i+1}} + \delta_{x_{i-1}\in\sigma_x}{}^{p}m_{z_j,x_{i-1}} \leqslant \delta_{z_{j+1}\in\sigma_z}{}^{p}m_{z_{j+1},x_i} + \delta_{z_{j-1}\in\sigma_z}{}^{p}m_{z_{j-1},x_i} \quad (*_{i,j})$$

Surprisingly enough, any solution to this system of inequalities satisfies all inequalities with equality. We will solve this system of linear inequalities for $m = 6$ and leave the cases $m \in \{3,4\}$ to the reader. To simplify notation, write $a_{j,i} = {}^{p}m_{z_j,x_i}$ for all $1 \leqslant i,j \leqslant m-1$ and view them as indeterminates. The set of inequalities can be partitioned into two sets of inequalities which can be solved completely independently:

$$\{(*_{i,j}) \mid i+j \text{ even}\} \cup \{(*_{i,j}) \mid i+j \text{ odd}\}$$

First, let us consider $\{(*_{i,j}) \mid i+j \text{ even}\}$:

$$a_{1,2} \leqslant a_{2,1} \tag{i}$$
$$a_{3,2} \leqslant a_{2,1} + a_{4,1} \tag{ii}$$
$$a_{5,2} \leqslant a_{4,1} \tag{iii}$$
$$a_{2,3} + a_{2,1} \leqslant a_{1,2} + a_{3,2} \tag{iv}$$
$$a_{4,3} + a_{4,1} \leqslant a_{3,2} + a_{5,2} \tag{v}$$
$$a_{1,4} + a_{1,2} \leqslant a_{2,3} \tag{vi}$$
$$a_{3,4} + a_{3,2} \leqslant a_{2,3} + a_{4,3} \tag{vii}$$
$$a_{5,4} + a_{5,2} \leqslant a_{4,3} \tag{viii}$$
$$a_{2,5} + a_{2,3} \leqslant a_{1,4} + a_{3,4} \tag{ix}$$
$$a_{4,5} + a_{4,3} \leqslant a_{3,4} + a_{5,4} \tag{x}$$
$$a_{1,4} \leqslant a_{2,5} \tag{xi}$$
$$a_{3,4} \leqslant a_{2,5} + a_{4,5} \tag{xii}$$
$$a_{5,4} \leqslant a_{4,5} \tag{xiii}$$

Consider the following chain of inequalities:

$$a_{2,1} \overset{\text{(iv)}}{\leqslant} a_{1,2} + a_{3,2} - a_{2,3} \overset{\text{(vii)}}{\leqslant} a_{1,2} + a_{4,3} - a_{3,4} \overset{\text{(x)}}{\leqslant} a_{1,2} + a_{5,4} - a_{4,5} \overset{\text{(xiii)}}{\leqslant} a_{1,2} \overset{\text{(i)}}{\leqslant} a_{2,1}$$

This implies that the Equations (i), (iv), (vii), (x) and (xiii) are all satisfied with equality, which in turn implies:

$$a_{2,3} = a_{3,2} \qquad a_{3,4} = a_{4,3}$$

Next, consider the following chain:

$$a_{4,1} \overset{\text{(v)}}{\leqslant} a_{3,2} + a_{5,2} - a_{4,3} \overset{\text{(vii)}}{=} a_{2,3} - a_{3,4} + a_{5,2} \overset{\text{(ix)}}{\leqslant} a_{1,4} - a_{2,5} + a_{5,2} \overset{\text{(xi)}}{\leqslant} a_{5,2} \overset{\text{(iii)}}{\leqslant} a_{4,1}$$

This shows that the Equations (iii), (v), (ix) and (xi) are also satisfied with equality, from which we deduce:

$$a_{2,3} = a_{3,2} = a_{3,4} = a_{4,3}$$

Finally, we have

$$a_{1,4} + a_{1,2} \overset{\text{(vi)}}{\leqslant} a_{2,3} = a_{3,2} \overset{\text{(ii)}}{\leqslant} a_{2,1} + a_{4,1}$$

and

$$a_{5,4} + a_{5,2} \overset{\text{(viii)}}{\leqslant} a_{4,3} = a_{3,4} \overset{\text{(xii)}}{\leqslant} a_{2,5} + a_{4,5}$$

which imply using $a_{1,2} = a_{2,1}$ and $a_{5,4} = a_{4,5}$ respectively

$$a_{1,4} \leqslant a_{4,1} \qquad a_{5,2} \leqslant a_{2,5}$$

Using $a_{1,4} = a_{2,5}$ and $a_{4,1} = a_{5,2}$ finishes the argument.

Next, we solve $\{(*_{i,j}) \mid i + j \text{ odd}\}$:

$$a_{2,2} \leqslant a_{1,1} + a_{3,1} \tag{i'}$$
$$a_{4,2} \leqslant a_{3,1} + a_{5,1} \tag{ii'}$$
$$a_{1,3} + a_{1,1} \leqslant a_{2,2} \tag{iii'}$$
$$a_{3,3} + a_{3,1} \leqslant a_{2,2} + a_{4,2} \tag{iv'}$$
$$a_{5,3} + a_{5,1} \leqslant a_{4,2} \tag{v'}$$
$$a_{2,4} + a_{2,2} \leqslant a_{1,3} + a_{3,3} \tag{vi'}$$
$$a_{4,4} + a_{4,2} \leqslant a_{3,3} + a_{5,3} \tag{vii'}$$
$$a_{1,5} + a_{1,3} \leqslant a_{2,4} \tag{viii'}$$
$$a_{3,5} + a_{3,3} \leqslant a_{2,4} + a_{4,4} \tag{ix'}$$
$$a_{5,5} + a_{5,3} \leqslant a_{4,4} \tag{x'}$$
$$a_{2,4} \leqslant a_{1,5} + a_{3,5} \tag{xi'}$$
$$a_{4,4} \leqslant a_{3,5} + a_{5,5} \tag{xii'}$$

In this case we argue as follows:

$$a_{1,1} \overset{\text{(iii')}}{\leqslant} a_{2,2} - a_{1,3} \overset{\text{(vi')}}{\leqslant} a_{3,3} - a_{2,4} \overset{\text{(ix')}}{\leqslant} a_{4,4} - a_{3,5} \overset{\text{(xii')}}{\leqslant} a_{5,5}$$

We use this in the last inequality of the following chain:

$$a_{3,1} \overset{\text{(iv')}}{\leqslant} a_{2,2} + a_{4,2} - a_{3,3} \overset{\text{(vii')}}{\leqslant} a_{2,2} + a_{5,3} - a_{4,4} \overset{\text{(x')}}{\leqslant} a_{2,2} - a_{5,5} \overset{\text{(i')}}{\leqslant} a_{1,1} + a_{3,1} - a_{5,5} \leqslant a_{3,1}$$

This implies that $a_{1,1} = a_{5,5}$ and the Equations (i'), (iii'), (iv'), (vi'), (vii'), (ix'), (x') and (xii') are satisfied with equality. Moreover, we have the following equivalences:

$$\begin{aligned} \text{(i')} = \text{(iii')} &\Leftrightarrow a_{1,3} = a_{3,1} \Leftrightarrow \text{(iv')} = \text{(vi')} \\ &\Leftrightarrow a_{2,4} = a_{4,2} \Leftrightarrow \text{(vii')} = \text{(ix')} \\ &\Leftrightarrow a_{5,3} = a_{3,5} \Leftrightarrow \text{(x')} = \text{(xii')} \end{aligned}$$

The last four inequalities can be used as follows:

$$a_{5,3} + a_{5,1} \overset{\text{(v')}}{\leqslant} a_{4,2} \overset{\text{(ii')}}{\leqslant} a_{3,1} + a_{5,1}$$
$$a_{1,5} + a_{1,3} \overset{\text{(viii')}}{\leqslant} a_{2,4} \overset{\text{(xi')}}{\leqslant} a_{1,5} + a_{3,5}$$

This gives $a_{5,3} \leqslant a_{3,1} = a_{1,3} \leqslant a_{3,5} = a_{5,3}$ and finishes the argument.

Finally, observe that the space of solutions for these inequalities is a free $\mathbb{Z}[v, v^{-1}]$-module of rank $m-1$ is and we can choose $\{a_{1,1}, a_{2,1}, \dots, a_{m-1,1}\}$ as a basis. In other words, the solution is uniquely determined after fixing these Laurent polynomials. Of course, not every solution gives a possible set of base change coefficients $\{{}^p m_{z_j,x_i} \mid 1 \leqslant i, j \leqslant m-1\}$ as these coefficients have to satisfy more constraints: Proposition 3.10(iii) shows that these coefficients are self-dual and have non-negative integers as coefficients. Moreover, due to Proposition 3.10(v) these coefficients satisfy parity vanishing for fixed i and arbitrary $1 \leqslant j \leqslant m-1$. This is also the underlying reason why we could partition the set of inequalities in two sets. In practice the set of variables involved in at most one of these sets is non-zero for fixed strings σ_x and σ_z.

From all these equalities, the reader easily deduces the relations given in the proposition where we expressed each coefficient in terms of our chosen basis of the solution space. □

Corollary 5.6. *For $z \leqslant x \in \mathcal{D}_R(r,t)$ one has:*

$$ {}^p m_{z,x} = {}^p m_{z^*,x^*} $$

Proof. Note that ${}^p m_{z,x} = {}^p m_{z^*,x^*}$ asks only for $a_{i,j} = a_{m-i,m-j}$ for $1 \leqslant i, j \leqslant m-1$ whereas we have shown many more relations among these coefficients in the last proposition. □

In the proof of Proposition 5.5 we have seen that when translating ${}^k B_x$ by ${}^k B_t$ for $x \in \mathcal{D}_R(r,t)$ with $xt > x$ the available coefficient of $\underline{H}_z$ in ${}^p\underline{H}_x\underline{H}_t$ for $z \in \mathcal{D}_R(r,t)$ is completely subsumed by the neighbouring elements of x in its right $\langle r,t\rangle$-string. This implies the following about the structure coefficients:

Corollary 5.7. *Let $x, z \in \mathcal{D}_R(r,t)$ with $xt > x$. Then ${}^p\mu_{x,t}^z$ vanishes unless x and z are neighbouring elements in the same right $\langle r,t\rangle$-string.*

The next result will allow us to apply the star-operations to the study of p-cells. It is a generalization of [Lus85, (10.4.1), (10.4.2) and (10.4.3)]:

Proposition 5.8. *Let $\sigma_x = \{x_1 < x_2 < \dots\}$ (resp. $\sigma_z = \{z_1 < z_2 < \dots\}$) be two right $\langle r,t\rangle$-strings consisting of $m-1$ elements. For any $s \in S \setminus \mathcal{L}(x_1)$ all the relations stated in Proposition 5.5 hold with ${}^p m_{z_j,x_i}$ replaced by ${}^p\mu_{s,x_i}^{z_j}$.*

Proof. Lemma 5.10 shows that all elements in σ_x lie in the same right p-cell. We therefore deduce from Lemma 4.4 that all elements in σ_x have the same left descent set and satisfy $sx_i > x_i$ for $1 \leqslant i \leqslant m-1$.

Write x_0 (resp. x_m) for the shortest (resp. longest) element in the right $\langle r,t\rangle$-coset of x. Similarly for z_0 and z_m. We are interested in the structure coefficients ${}^p\mu_{s,x_i}^{z_j}$ for $1 \leqslant i, j \leqslant m-1$. To simplify notation, write $a_{j,i} = {}^p\mu_{s,x_i}^{z_j}$.

Fix $1 \leqslant i \leqslant m-1$ arbitrary. We may assume without loss of generality $x_i t > x_i$ and thus $r \in \mathcal{R}(x_i)$. The main idea is to express ${}^p\underline{H}_s\,{}^p\underline{H}_{x_i}\,{}^p\underline{H}_t$ in the p-canonical basis in different ways and to analyze the coefficients in front of basis elements indexed by elements in σ_z. Depending on which multiplication in ${}^p\underline{H}_s\,{}^p\underline{H}_{x_i}\,{}^p\underline{H}_t$ we first carry out, we get two ways to express this product in terms of the p-canonical basis. On the one hand we have:

$$ {}^p\underline{H}_s\,{}^p\underline{H}_{x_i}\,{}^p\underline{H}_t = \Big(\sum_{v \leqslant sx_i} {}^p\mu_{s,x_i}^v\,{}^p\underline{H}_v\Big)\,{}^p\underline{H}_t $$

$$= \sum_{\substack{v \leqslant sx_i \\ vt < v}} (v + v^{-1})^p\mu^v_{s,x_i} \, {}^p\underline{H}_v + \sum_{\substack{v \leqslant sx_i \wedge w \leqslant vt \\ vt > v}} {}^p\mu^v_{s,x_i} \, {}^p\mu^w_{v,t} \, {}^p\underline{H}_w$$

Consider an element $v \leqslant sx_i$ with ${}^p\mu^v_{s,x_i} \neq 0$. The version of Lemma 4.4 for left cells implies $r \in \mathcal{R}(v)$ and in particular v is not minimal in its right $\langle r, t\rangle$-coset. This implies:

$$v \in \mathcal{D}_R(r,t) \Leftrightarrow vt > v$$

Assume $v \in \mathcal{D}_R(r,t)$. Combining Corollary 5.7 and Corollary 4.12 we have full control over the p-canonical basis elements indexed by elements in $\mathcal{D}_R(r,t)$ that occur in ${}^p\underline{H}_v \, {}^p\underline{H}_t$. Therefore, the only p-canonical basis elements that are indexed by elements in σ_z and that occur with non-trivial coefficient in this product are the following:

$$\sum_{1 \leqslant j \leqslant m-1} a_{j,i}(\delta_{z_{j+1} \in \sigma_z} \, {}^p\underline{H}_{z_{j+1}} + \delta_{z_{j-1} \in \sigma_z} \, {}^p\underline{H}_{z_{j-1}})$$

On the other hand we can rewrite the product as follows:

$$\begin{aligned} {}^p\underline{H}_s \, {}^p\underline{H}_{x_i} \, {}^p\underline{H}_t &= {}^p\underline{H}_s \Big(\sum_{v \leqslant x_i t} {}^p\mu^v_{x_i,t} \, {}^p\underline{H}_v \Big) \\ &= \sum_{\substack{w \leqslant x_i t \\ sw < w}} (v + v^{-1})^p\mu^w_{x_i,t} \, {}^p\underline{H}_w + \sum_{\substack{v \leqslant x_i t \wedge w \leqslant sv \\ sv > v}} {}^p\mu^v_{x_i,t} \, {}^p\mu^w_{s,v} \, {}^p\underline{H}_w \end{aligned}$$

Consider an element $v \leqslant x_i t$ with ${}^p\mu^v_{x_i,t} \neq 0$. It follows that v has t in its right descent set. The version of Lemma 4.4 for left cells shows that ${}^p\underline{H}_s \, {}^p\underline{H}_v$ can only contribute p-canonical basis elements indexed by elements in $\mathcal{D}_R(r,t)$ if v lies in $\mathcal{D}_R(r,t)$ itself. Using Corollary 5.7 and Corollary 4.12 again, we see that the only p-canonical basis elements that are indexed by elements in σ_z and that occur with non-trivial coefficient in this product are the following:

$$\sum_{1 \leqslant j \leqslant m-1} (\delta_{x_{i+1} \in \sigma_x} a_{j,i+1} + \delta_{x_{i-1} \in \sigma_x} a_{j,i-1}) \, {}^p\underline{H}_{z_j}$$

Comparing coefficients in front of ${}^p\underline{H}_{z_j}$ we get:

$$\delta_{z_{j-1} \in \sigma_z} a_{j-1,i} + \delta_{z_{j+1} \in \sigma_z} a_{j+1,i} = \delta_{x_{i+1} \in \sigma_x} a_{j,i+1} + \delta_{x_{i-1} \in \sigma_x} a_{j,i-1}$$

Letting i and j vary, we see that the $a_{j,i}$'s satisfy precisely the inequalities (i) - (xiii) and (i') - (xii') (with equality). □

Corollary 5.9. *Let $x, z \in \mathcal{D}_R(r,t)$ and $s \in S$ such that $sx > x$. Then we have*

$${}^p\mu^y_{s,x} = {}^p\mu^{y^*}_{s,x^*}$$

where $(-)^*$ is the right star-operation with respect to $\{r,t\}$.

Using the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution ι on $\mathcal{H}$ we can translate all the results in this section about right strings and right star-operations into results about left strings and left star-operations.

5.1 Consequences for p-Cells

As we have seen in the last section one needs some assumptions on p for the left and right star-operations to be well-behaved. Therefore, we keep these assumptions throughout this subsection. Fix for the rest of the section two simple reflections $r, t \in S$ with $3 \leqslant m := m_{r,t} < \infty$. Throughout the section we consider the right star-operation $(-)^*$ with respect to $\{r, t\}$.

Under these assumptions, the p-cells in any finite Weyl group of rank 2 coincide with the Kazhdan-Lusztig cells (see Section 4.3). Therefore, Theorem 4.9 implies for $I = \{r, t\} \subseteq S$:

Lemma 5.10. *Let σ be any right $\langle r, t\rangle$-string. Then all elements of σ lie in the same right p-cell of W.*

In particular, we have the following result:

Corollary 5.11. *For all $x \in \mathcal{D}_R(r,t)$, x and x^* lie in the same right p-cell.*

The following important result follows from Corollary 5.9:

Theorem 5.12. *For $x, y \in \mathcal{D}_R(r,t)$ and $s \in S$ we have:*

$$x \overset{p}{\underset{L}{\leqslant}} y \Leftrightarrow x^* \overset{p}{\underset{L}{\leqslant}} y^*$$

In particular, if x and y lie the same left p-cell, then the same holds for x^* and y^*.

Definition 5.13. For $r, t \in S$ with $rt \neq tr$ and $x \in \mathcal{D}_R(r,t)$ we denote by σ_x the right $\langle r, t\rangle$-string through x. Define $\mathfrak{T}_{r,t}(x) := \{xr, xt\} \cap \mathcal{D}_R(r,t)$ to be the neighbouring elements of x in σ_x.

View $\mathfrak{T}_{r,t}$ as a map $\sigma_x \to \mathcal{P}(\sigma_x)$ where $\mathcal{P}(\sigma_x)$ denotes the power set of σ_x. We define $\mathfrak{T}^2_{r,t} : \sigma_x \to \mathcal{P}(\sigma_x)$ to be the map sending $y \in \sigma_x$ to $\bigcup_{z \in \mathfrak{T}_{r,t}(y)} \mathfrak{T}_{r,t}(z)$. For $l \geqslant 2$, the map $\mathfrak{T}^l_{r,t}$ is defined inductively in a similar fashion.

Actually, one can characterize precisely the possible left p-cell preorder relations between elements in right $\langle r, t\rangle$-strings:

Proposition 5.14. *Let $\sigma = \{x_1 < x_2 < \dots\}$ (resp. $\sigma' = \{y_1 < y_2 < \dots\}$) be two right $\langle r, t\rangle$-strings consisting of $m-1$ elements. Up to possibly interchanging the roles of σ and σ', the set of left p-cell preorder relations between the elements of these two strings is one of the following:*

$$\textit{no relation: } \{\} \tag{$\varnothing$}$$

$$\textit{trivial case: } \{x_i \overset{p}{\underset{L}{\leqslant}} y_i \mid \forall 1 \leqslant i \leqslant m-1\} \tag{T}$$

$$\textit{permuted case: } \{x_i \overset{p}{\underset{L}{\leqslant}} y_{\pi(i)} \mid \forall 1 \leqslant i \leqslant m-1\} \tag{P}$$

$$\textit{neighbour case: } \{x_i \overset{p}{\underset{L}{\leqslant}} y \mid \forall 1 \leqslant i \leqslant m-1, y \in \mathfrak{T}^k_{r,t}(y_i)\} \tag{N_k}$$

$$\textit{permuted neighour case: } \{x_i \overset{p}{\underset{L}{\leqslant}} y_{\pi(j)} \mid \forall 1 \leqslant i \leqslant m-1, y_j \in \mathfrak{T}^l_{r,t}(y_i)\} \tag{PN_l}$$

$$\textit{zig-zag case: } \{x_{i-1} \overset{p}{\underset{L}{\leqslant}} y_{i+1}, x_i \overset{p}{\underset{L}{\leqslant}} y_i, x_{i+1} \overset{p}{\underset{L}{\leqslant}} y_{i-1} \mid \forall 2 \leqslant i \leqslant m-2\} \tag{Z}$$

for $1 \leqslant k \leqslant m-2$, $1 \leqslant l \leqslant m-4$ *where* $\pi = (1, m-1)(2, m-2)$ *is a permutation of* $\{1, 2, \ldots, m-1\}$.

Proof. The proof strategy is as follows: By applying Proposition 5.8 to the relation $y \xrightarrow[L]{p} x$ via $s \in S$, we get other elementary left p-cell relations between the elements in σ_x and σ_y. The idea is to encode them in a function $f : \sigma_y \to \mathcal{P}(\sigma_x)$ such that

$$ {}^p\mu^{x_j}_{s,y_i} \neq 0 \Leftrightarrow x_j \in f(y_i). $$

The claim of the proposition is that there is a normal form of arbitrary finite compositions of such functions. If $g : \sigma_x \to \mathcal{P}(\sigma_z)$ is another such function, their composition $g \circ f : \sigma_y \to \mathcal{P}(\sigma_z)$ sends y_i to $\bigcup_{x_j \in f(y_i)} g(x_j)$. This is simply the composition of multi-valued functions.

In order to simplify notation, we will identify the j-th element in a right $\langle r, t\rangle$-string with its position j. This allows us to view any such f as a map $\{1, 2, \ldots, m-1\} \to \mathcal{P}(\{1, 2, \ldots, m-1\})$. The composition of such functions is to be understood in a similar fashion. One needs to keep track of the start and end string of the whole composition in order to retranslate the function into the set of left p-cell relations. Using Corollary 5.9 we see that such a map is already fully determined by the images of $1, 2, \ldots, \lceil \frac{m-1}{2} \rceil$.

We will prove the statement for $m = 6$ and leave the other cases to the reader. Apart from the identity $\mathrm{Id} : i \mapsto \{i\}$ consider the following maps:

$$ a : \begin{aligned} 1 &\mapsto \{3\} \\ 2 &\mapsto \{2, 4\} \\ 3 &\mapsto \{1, 3, 5\} \end{aligned} \qquad b : \begin{aligned} 1 &\mapsto \{2\} \\ 2 &\mapsto \{1, 3\} \\ 3 &\mapsto \{2, 4\} \end{aligned} $$

$$ c : \begin{aligned} 1 &\mapsto \{4\} \\ 2 &\mapsto \{3, 5\} \\ 3 &\mapsto \{2, 4\} \end{aligned} \qquad p : i \mapsto \{\pi(i)\} $$

Using Proposition 5.8 it is easy to check that any elementary left p-cell relation implies relations encoded by one of the functions above, for example $3 \xrightarrow[L]{p} 3$ implies relations encoded by either Id, a or p. Analyzing the relations among compositions of these functions, one has:

$$ \begin{aligned} a \circ p = p \circ a &= a \\ b \circ p = p \circ b &= c \\ b \circ c &= c \circ b \\ p^2 &= \mathrm{Id} \\ a \circ b = b \circ a &= b^3 \\ a^k = a^2 = b^4 = b^{2k} &= b^{2k} \circ p \text{ for } k \geqslant 2 \\ b^3 = b^{2k+1} &= b^{2k+1} \circ p \text{ for } k \geqslant 1 \end{aligned} $$

Using these relations, we see that any finite composition of these maps can be reduced to one of the following compositions

$$ \mathrm{Id}, p, b^k \text{ for } 1 \leqslant k \leqslant 4, b^l \circ p \text{ for } 1 \leqslant l \leqslant 2, a. $$

These correspond precisely to the cases with at least one relation stated in the proposition. □

Remark 5.15. It should be noted that for $m < 6$ some of these cases coincide. For example for $m = 3$, the permutation π is trivial and there are only three distinct cases: The permuted case coincides with the trivial case. Moreover, the zig-zag case does not contain any relations and the permuted neighbour case does not exist. For $m = 4$, there are four distinct cases: The zig-zag case reduces to the permuted case and the permuted neighbour case does not exist.

The normal forms given in the proof of Proposition 5.14 allow us to deduce the following equivalences, which show how rigid the combinatorics in this situation are. The reader should compare these equivalences with Proposition 5.8 which only deals with the generating relations for the p-cell preorder. It is a generalization of [Shi94, Proposition 4.6 and Remark 4.7]:

Corollary 5.16. *Let $\sigma = \{x_1 < x_2 < \dots\}$ (resp. $\sigma' = \{y_1 < y_2 < \dots\}$) be two right $\langle r,t\rangle$-strings consisting of $m-1$ elements. In addition to $x \overset{p}{\underset{L}{\leqslant}} y \Leftrightarrow x^* \overset{p}{\underset{L}{\leqslant}} y^*$ for $x \in \sigma$ and $y \in \sigma'$ we get the following equivalences:*

$$m = 4 \Rightarrow \begin{cases} x_1 \overset{p}{\underset{L}{\leqslant}} y_2 \Leftrightarrow x_2 \overset{p}{\underset{L}{\leqslant}} y_1 \\ x_2 \overset{p}{\underset{L}{\leqslant}} y_2 \Leftrightarrow (x_1 \overset{p}{\underset{L}{\leqslant}} y_1 \vee x_1 \overset{p}{\underset{L}{\leqslant}} y_3) \end{cases} \tag{11}$$

$$m = 6 \Rightarrow \begin{cases} x_1 \overset{p}{\underset{L}{\leqslant}} y_2 \Leftrightarrow x_2 \overset{p}{\underset{L}{\leqslant}} y_1 \\ x_1 \overset{p}{\underset{L}{\leqslant}} y_3 \Leftrightarrow x_3 \overset{p}{\underset{L}{\leqslant}} y_1 \\ x_1 \overset{p}{\underset{L}{\leqslant}} y_4 \Leftrightarrow x_4 \overset{p}{\underset{L}{\leqslant}} y_1 \\ x_2 \overset{p}{\underset{L}{\leqslant}} y_2 \Leftrightarrow (x_1 \overset{p}{\underset{L}{\leqslant}} y_1 \vee x_1 \overset{p}{\underset{L}{\leqslant}} y_3) \\ x_2 \overset{p}{\underset{L}{\leqslant}} y_3 \Leftrightarrow x_3 \overset{p}{\underset{L}{\leqslant}} y_2 \Leftrightarrow (x_1 \overset{p}{\underset{L}{\leqslant}} y_2 \vee x_1 \overset{p}{\underset{L}{\leqslant}} y_4) \\ x_2 \overset{p}{\underset{L}{\leqslant}} y_4 \Leftrightarrow (x_1 \overset{p}{\underset{L}{\leqslant}} y_3 \vee x_1 \overset{p}{\underset{L}{\leqslant}} y_5) \\ x_3 \overset{p}{\underset{L}{\leqslant}} y_3 \Leftrightarrow (x_1 \overset{p}{\underset{L}{\leqslant}} y_1 \vee x_1 \overset{p}{\underset{L}{\leqslant}} y_3 \vee x_1 \overset{p}{\underset{L}{\leqslant}} y_5) \end{cases} \tag{12}$$

We can now generalize [Lus85, Proposition 10.7] as follows:

Proposition 5.17. *Let $r, t \in S$ and Γ be a union of left p-cells such that $\Gamma \subseteq \mathcal{D}_R(r,t)$. Then the following holds:*

(i) *$\widetilde{\Gamma} := (\bigcup_{w \in \Gamma} \sigma_w) \setminus \Gamma$ is a union of left p-cells, where σ_w denotes the right $\langle r,t\rangle$-string through w.*

(ii) *If Γ is a left p-cell, then $\widetilde{\Gamma}$ is a union of at most $m_{r,t} - 2$ left p-cells.*

(iii) *If Γ is a left p-cell, then $\Gamma^* := \{w^* \mid w \in \Gamma\}$ is a left p-cell as well.*

Proof. For two left p-cells Γ_1 and Γ_2 contained in $\mathcal{D}_R(r,t)$ we have:

$$\begin{aligned}\widetilde{(\Gamma_1 \cup \Gamma_2)} &= \left(\bigcup_{w \in \Gamma_1 \cup \Gamma_2} \sigma_w \right) \setminus (\Gamma_1 \cup \Gamma_2) \\ &= \begin{cases} \widetilde{\Gamma_1} \cap \widetilde{\Gamma_2} & \text{if } \widetilde{\Gamma_1} \cap \widetilde{\Gamma_2} \neq \varnothing, \\ \widetilde{\Gamma_1} \cup \widetilde{\Gamma_2} & \text{otherwise.} \end{cases}\end{aligned}$$

Therefore, we may assume without loss of generality that Γ is a left p-cell. It is enough to prove that $\bigcup_{w \in \Gamma} \sigma_w$ is a union of left p-cells. Let $x \in \bigcup_{w \in \Gamma} \sigma_w$ and $y \in W$ such that $x \overset{p}{\underset{L}{\sim}} y$. Then there exists a sequence

$$P : x = x_0 \overset{p}{\underset{L}{\to}} x_1 \overset{p}{\underset{L}{\to}} \dots \overset{p}{\underset{L}{\to}} x_k = y \overset{p}{\underset{L}{\to}} x_{k+1} \overset{p}{\underset{L}{\to}} \dots \overset{p}{\underset{L}{\to}} x_l = x$$

of elements in $\mathcal{D}_R(r,t)$ that all lie in the same left p-cell as x and y. Consider the right $\langle r,t \rangle$-strings of all the elements in the sequence and note that σ_x contains an element $\bar{x} \in \Gamma$. Since each $\langle r,t \rangle$-string is contained in a right p-cell, the elements have the same left descent set and thus in each step we can apply the simple reflection used to get from x_i to x_{i+1} to all the elements in σ_{x_i} for $1 \leqslant i \leqslant l-1$. The given sequence gives for any $z \in \sigma_x$ a set $f_P(z)$ of elements in σ_x such that $z \overset{p}{\underset{L}{\geqslant}} z'$ if and only if $z' \in f_P(z)$. Using Proposition 5.14 we see that for $n \gg 1$ the image of f_P^{2n} stabilizes in the sense that $z \in f_P^{2n}(z)$. The element $\bar{x}$ shows that there is an element in $\sigma_y \cap \Gamma$ which implies $y \in \bigcup_{w \in \Gamma} \sigma_w$ and finishes the proof of (i).

We claim that any left p-cell Γ' in $\bigcup_{w \in \Gamma} \sigma_w$ intersects σ_x non-trivially. Let $y \in \Gamma'$ and $\bar{y} \in \sigma_y \cap \Gamma$. Use a sequence P as above relating $\bar{x}$ and $\bar{y}$ to construct $f_P : \sigma_x \to \mathcal{P}(\sigma_x)$. Proposition 5.14 shows that there exist $x' \in \sigma_x$ such that $x' \overset{p}{\underset{L}{\geqslant}} y$. Using the stabilization of f_P as above, we get $x' \in f_P^{2n}(x')$ for $n \gg 1$ and in particular x' lies $\Gamma' \cap \sigma_x$. This finishes the proof of the claim and shows that $\widetilde{\Gamma}$ is a union of at most $m_{r,t} - 2$ left p-cells giving (ii).

(iii) is an immediate consequence of Theorem 5.12. ☐

Remark 5.18. Analyzing the situation more carefully allows us to say more about the number of left p-cells in $\widetilde{\Gamma}$. Let $x, y \in \Gamma$. Consider f_P^n constructed as in the last proof for a sequence P relating x and y. The proof of Proposition 5.14 shows that f_P^{2n} for $n \gg 1$ stabilizes to one of the following maps:

$$\begin{array}{lll} f_{\text{triv}} : \sigma_x \longrightarrow \mathcal{P}(\sigma_x) & \text{or} & f_{\text{nontriv}} : \sigma_x \longrightarrow \mathcal{P}(\sigma_x) \\ x_i \longmapsto \{x_i\} & & x_{2l-1} \longmapsto \{x_{2k-1} \mid 1 \leqslant k \leqslant \left\lfloor \frac{m}{2} \right\rfloor\} \\ & & x_{2l} \longmapsto \{x_{2k} \mid 1 \leqslant k \leqslant \left\lfloor \frac{m-2}{2} \right\rfloor\} \end{array}$$

Finally, let P vary over all potential sequences and y vary over the elements in Γ. If there exists an element $y \in \Gamma$ and a sequence P relating x and y such that f_P^n stabilizes to f_{nontriv}, then $\widetilde{\Gamma}$ is a left p-cell (note that $\widetilde{\Gamma}$ is always non-empty as all elements in Γ have the same right descent set and this is not the case for all elements in σ_x).

Another situation, in which we can say more about the number of p-cells in $\widetilde{\Gamma}$ is the following: If $m_{r,t} \geqslant 4$ and there exists a string σ such that $\sigma \cap \Gamma$ contains only one odd-numbered element of σ, then $\widetilde{\Gamma}$ decomposes into at least 2 left p-cells for descent set reasons. Therefore, under these assumptions $\widetilde{\Gamma}$ contains precisely two right p-cells if $m_{r,t} = 4$.

The definition of a W-graph from [KL79, §1] describes a based representation of the Hecke algebra. A typical example is given by a Kazhdan-Lusztig cell module equipped with the Kazhdan-Lusztig basis, for which the W-graph describes the action of the generating set $\{H_s \mid s \in S\}$ of the Hecke algebra. In order to allow W-graphs to also describe p-cell modules, we need to slightly generalize the original definition, in which we use the generating set $\{\underline{H}_s \mid s \in S\}$ of the Hecke algebra instead:

Definition 5.19. A p-W-graph is a directed graph with vertices $\mathcal{V}$ and edges $\mathcal{E}$ together with the following decorations:

- for each vertex $x \in \mathcal{V}$ a subset I_x of S ,
- for each edge $(x, y) \in \mathcal{E}$ a family of Laurent polynomials

$$\{\mu^y_{s,x} \mid s \in I_y \setminus I_x\} \subset \mathbb{Z}[v, v^{-1}]$$

subject to the conditions below. Let V be the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\mathcal{V}$. For $s \in S$ define an $\mathbb{Z}[v, v^{-1}]$-linear endomorphism of V as follows

$$\tau_s(x) = \begin{cases} (v + v^{-1})x & \text{if } s \in I_x, \\ \sum_{\substack{(x,y)\in\mathcal{E} \\ s \in I_y}} \mu^y_{s,x} y & \text{otherwise.} \end{cases}$$

where the sum is finite due to the second condition below. Then a p-W-graph is required to satisfy:

(i) the morphism $\mathcal{H} \to \operatorname{End}(V)$, $\underline{H}_s \mapsto \tau_s$ extends to a morphism of $\mathbb{Z}[v, v^{-1}]$-algebras,

(ii) for each pair $(x, s) \in \mathcal{V} \times S$ there are only finitely many edges $(x, y) \in \mathcal{E}$ with $\mu^y_{s,x} \neq 0$.

It is immediate that we can associate to each left p-cell C a p-W-graph Γ_C as defined above with C as vertex set. For each $x \in C$ we set $I_x \coloneqq \mathcal{L}(x)$ and use edges to encode the structure coefficients for the p-canonical basis ${}^p\mu^y_{s,x}$ for $x, y \in C$. Observe that right star operations do not modify the left descent set. This combined with Proposition 5.17 and Corollary 5.9 implies the following result:

Lemma 5.20. *Let C be a left p-cell contained in $\mathcal{D}_R(r,t)$. Then the left p-cell module associated to C is isomorphic to the left p-cell module associated to C^* where $*$ is the right star-operation associated to r and t. In particular, the p-W-graphs Γ_C and Γ_{C^*} are isomorphic as decorated, directed graphs.*

5.2 Vogan's Generalized τ-Invariant

Vogan defined in [Vog79, Definition 3.10] an invariant of Kazhdan-Lusztig left cells in the setting of primitive ideals for semi-simple Lie algebras. This became known as Vogan's generalized τ-invariant and was generalized in [BG15, Definition 5.1] to arbitrary Coxeter groups. (Observe that only pairs of simple reflections $\{r, t\} \in W$ with $m_{r,t} \in \{3, 4\}$ matter in the following definition.)

Definition 5.21. Observe that $\mathfrak{T}_{r,t}(x)$ for $r, t \in S$ consists of one or two elements. We use the following convention: We consider $\mathfrak{T}_{r,t}(x)$ as a multiset with two identical elements in the following if $\{xr, xt\} \cap \mathcal{D}_R(r, t)$ is of cardinality 1.

We define a sequence of equivalence relations $\approx_n$ for $n \in \mathbb{N}$ as follows. For $x, y \in W$ we write:

$$\begin{aligned} x \approx_0 y &\text{ if } \mathcal{R}(x) = \mathcal{R}(y), \\ x \approx_{n+1} y &\text{ if } x \approx_n y \text{ and for any pair } r, t \in S \text{ such that } m_{r,t} \in \{3, 4\} \text{ and} \\ &\quad x, y \in \mathcal{D}_R(r, t) \text{ with } \mathfrak{T}_{r,t}(x) = \{x_1, x_2\} \text{ and } \mathfrak{T}_{r,t}(y) = \{y_1, y_2\} \\ &\quad \text{we have: } x_1 \approx_n y_1,\, x_2 \approx_n y_2 \text{ or } x_1 \approx_n y_2,\, x_2 \approx_n y_1 \end{aligned}$$

We say that x and y have the same *generalized τ-invariant* if $x \approx_n y$ holds for all $n \geqslant 0$. We call the set

$$\{w \in W \mid x \approx_n w \text{ for all } n \geqslant 0\}$$

the τ-equivalence class of x.

Remark 5.22. Observe that the last definition admits an obvious generalization allowing the case $m_{r,t} = 6$. For our current applications, we do not need this generality. Thus we exclude this case just like Vogan did in the original definition.

The following result shows that left p-cells give a refinement of the τ-equivalence classes. It is a generalization of [BG15, Theorem 5.2]:

Theorem 5.23. *Assume $p > 2$ if G has a simple factor of type B_n or C_n. Let Γ be a left p-cell. Then all elements in Γ have the same generalized τ-invariant. In particular, any τ-equivalence class decomposes into left p-cells.*

Proof. The proof of [BG15, Theorem 5.2] works after replacing all Kazhdan-Lusztig related constructions by their p-canonical analogues with the following modifications: First add a zeroth case in which there exists a right $\langle s, t\rangle$-string σ such that $\Gamma \cap \sigma = \{x's, x'sts\}$ for some minimal element x' in its right $\langle s, t\rangle$-coset. In this case, $\widetilde{\Gamma}$ consists of a single left p-cell, which allows us to conclude. Finally in the second case, use the left p-cell preorder relations for the roles of y and w swapped instead of appealing to [Lus85, Corollary 6.3]. ☐

Corollary 5.24. *Assume $p > 2$ if G has a simple factor of type B_n or C_n. Assume that two elements in W_f have the same generalized τ-invariant if and only if they belong to the same Kazhdan-Lusztig left cell (i.e. the generalized τ-invariant gives a complete invariant of Kazhdan-Lusztig left cells). Then Kazhdan-Lusztig left cells decompose into left p-cells.*

In [Vog79, Theorem 6.5], Vogan shows that the generalized τ-invariant gives a complete invariant in finite type A. The same holds in finite types B/C (see [Gar93, Theorem 3.5.9] based on [Gar92; Gar93, Definitions 2.1.3. - 2.1.7., 3.2.1. and 3.4.1.]). Therefore, we have:

Corollary 5.25. *The Kazhdan-Lusztig left cells in finite type A decompose into left p-cells. The same holds in finite types B and C for $p > 2$.*

Even though we currently can only calculate the full p-canonical basis in types B_n and C_n for $n \leqslant 5$ and in these groups only 2-torsion occurs, the last result is in particular of interest due to [Wil17c; Wil17b]. In these papers Williamson shows that there is torsion in the local integral intersecion cohomology of Schubert varieties in the flag variety of the general linear group that grows exponentially in the rank. This implies that the p-canonical basis does not coincide with the Kazhdan-Lusztig basis for arbitrarily large primes in type A_n (and that the primes for which this occurs grow exponentially in the rank). Since type A_n embeds into type B_{n+1} and C_{n+1} this gives many interesting examples for which the p-canonical basis and the Kazhdan-Lusztig basis differ for large primes p.

In [Vog79, Remark following Proposition 4.4] Vogan mentions that the generalized τ-invariant is not a complete invariant in type F_4. Similarly in type D_n for $n \geqslant 6$ (see the introduction of [Gar90]).

As proposed by [Vog80, §4] and [BG15, Remark 5.3] we could have defined a potentially weaker invariant as follows:

Definition 5.26. As in Definition 5.26 we define a sequence of equivalence relations $\approx'_n$ for $n \in \mathbb{N}$ as follows. For $x, y \in W$ we write:

$$\begin{aligned} x \approx'_0 y &\text{ if } \mathcal{R}(x) = \mathcal{R}(y), \\ x \approx'_{n+1} y &\text{ if } x \approx'_n y \text{ and for any pair } r, t \in S \text{ such that } \infty > m_{r,t} \geqslant 3 \text{ and} \\ &\qquad x, y \in \mathcal{D}_R(r,t) \text{ we have: } x^* \approx'_n y^* \end{aligned}$$

where $(-)^*$ is the right star-operation with respect to $\{r, t\}$. As above we say that x and y have the same *generalized $\widetilde{\tau}$-invariant* if $x \approx'_n y$ holds for all $n \geqslant 0$. We call the set

$$\{w \in W \mid x \approx'_n w \text{ for all } n \geqslant 0\}$$

the $\widetilde{\tau}$-equivalence class of x.

In this case, Theorem 5.12 immediately implies that the partition of W into left p-cells gives a refinement of the $\widetilde{\tau}$-equivalence classes:

Corollary 5.27. *Assume $p > 2$ if G has a simple factor of type B_n or C_n and $p > 3$ if G has a simple factor of type G_2. Let Γ be a left p-cell. Then all elements in Γ have the same generalized $\widetilde{\tau}$-invariant. In particular, any $\widetilde{\tau}$-equivalence class decomposes into left p-cells.*

In [GH15, Conjecture 6.9] Geck and Halls propose a slight variation of Vogan's original conjecture (see [Vog79, Conjecture 3.11]) which goes as follows:

Conjecture 5.28.
For any finite Coxeter group W_f two elements $x, y \in W_f$ belong to the same Kazhdan-Lusztig left cell if and only if x and y lie in the same Kazhdan-Lusztig two-sided cell and in the same $\widetilde{\tau}$-equivalence class.

They verified their conjecture in all types BC_n and D_n for $n \leqslant 9$ and in all exceptional types. Moreover, they mention that in type F_4 the Kazhdan-Lusztig left cells are precisely the $\widetilde{\tau}$-equivalence classes. From this, we deduce that for $p \geqslant 3$ the Kazhdan-Lusztig left cells in type F_4 decompose into left p-cells. Our explicit computer calculations show that the p-canonical basis and the Kazhdan-Lusztig basis in type F_4 only differ for $p \in \{2,3\}$ and for $p = 2$ the Kazhdan-Lusztig left cells do not decompose into left p-cells.

5.3 p-Cells in Type A

Kazhdan-Lusztig cells in type A can be characterized using the Robinson-Schensted correspondence. This result is usually attributed to [KL79, §4] even though the result is not stated in this form and depends on results of Joseph and Vogan in the setting of primitive ideals. The first combinatorial proof is due to [GM88] and [Ari00]. In this section we transfer almost verbatim Ariki's proof to the modular setting. Since we feel that the proof is not as well documented as it should be, we decided to give the proof here.

Throughout this section we assume that we used the root datum of some PGL_n together with a Borel and a split maximal torus as input. In this case (W_f, S_f) can be identified with $(S_n, \{s_1, \dots, s_{n-1}\})$ the symmetric group together with the set of simple transpositions. Letting S_n act on $\{1, 2, \dots, n\}$ on the left, we can write any element $w \in S_n$ uniquely as $w = w(1)w(2)\dots w(n)$ which we call string notation.

Recall the definition of the elementary Knuth transformation K_i for $1 < i < n$: Let $x = x_1x_2\dots x_n, y = y_1y_2\dots y_n \in S_n$ be two elements of the symmetric group in string notation. Write $x \underset{K_i}{\approx} y$ if x and y differ only on the substrings $x_{i-1}x_ix_{i+1}$ and $y_{i-1}y_iy_{i+1}$ and these substrings are related to each other in either of the following two ways

$$bca \leftrightarrow bac \text{ or } cab \leftrightarrow acb$$

where $a < b < c$. We say that x and y are Knuth equivalent if they are related by a sequence of elementary Knuth transformations K_i for various i. The following result follows immediately from the definitions:

Lemma 5.29. *$\mathcal{D}_R(s_{i-1}, s_i)$ is the subset of elements in S_n to which K_i can be applied. Moreover, $K_i(w) = w^*$ for all $w \in \mathcal{D}_R(s_{i-1}, s_i)$ where $(-)^*$ is the right star-operation defined with respect to $\{s_{i-1}, s_i\}$.*

The Robinson-Schensted correspondence (see [BB05, §A.3.3] or [Ful97, §4.1]) gives a bijection between between the symmetric group S_n and pairs of standard tableaux of the same shape with n boxes. The row-bumping algorithm gives a way to explicitly calculate the image $(P(w), Q(w))$ of $w \in S_n$ under the Robinson-Schensted correspondence. We will need the following important classical result about the Robinson-Schensted correspondence (see [Ful97, §4.1, Corollary to Symmetry Theorem]):

Theorem 5.30 (Symmetry Theorem for S_n)**.**
If $w \in S_n$ corresponds to $(P(w), Q(w))$, then w^{-1} corresponds to $(Q(w), P(w))$ under the Robinson-Schensted correspondence.

Moreover, we need the following result by Knuth (see [Knu70, Theorem 6]):

Theorem 5.31. *Let $x, y \in S_n$. Then x and y are Knuth equivalent if and only if $P(x) = P(y)$.*

The goal is to prove the following theorem which is known for Kazhdan-Lusztig cells (see [Wil03, Theorem 5.4.1 and Corollary 5.4.2] or [BB05, Chapter 6, Exercise 11]):

Theorem 5.32. *For $x, y \in S_n$ we have:*

$$\begin{aligned} x \underset{L}{\overset{p}{\sim}} y &\Leftrightarrow Q(x) = Q(y) \\ x \underset{R}{\overset{p}{\sim}} y &\Leftrightarrow P(x) = P(y) \\ x \underset{LR}{\overset{p}{\sim}} y &\Leftrightarrow Q(x) \textit{ and } Q(y) \textit{ have the same shape} \end{aligned}$$

In particular, Kazhdan-Lusztig cells and p-cells of S_n coincide.

Proof. We will first deal with the statement about left p-cells:

$\Leftarrow$ By Theorem 5.30 we have $P(x^{-1}) = Q(x) = Q(y) = P(y^{-1})$ which implies by Theorem 5.31 that x^{-1} and y^{-1} are related by a sequence of elementary Knuth transformations (i.e. right start operations with respect to different subsets of S consisting of two neighbouring simple reflections). Successive applications of Corollary 5.11 show that $x^{-1} \underset{R}{\overset{p}{\sim}} y^{-1}$. This is equivalent to $x \underset{L}{\overset{p}{\sim}} y$ by Lemma 4.6.

$\Rightarrow$ Denote the shape of $Q(x)$ (resp. $Q(y)$) by π_x (resp. π_y) and let P_x (resp. P_y) be the column superstandard tableau (see [BB05, §A3.5] for the definition) of shape π_x (resp. π_y). The Robinson-Schensted correspondence gives elements $\hat{x}, \hat{y} \in S_n$ with P and Q-symbols $(P_x, Q(x))$ and $(P_y, Q(y))$ respectively. The implication we proved above gives $x \underset{L}{\overset{p}{\sim}} \hat{x}$ and $y \underset{L}{\overset{p}{\sim}} \hat{y}$ which implies by our assumption $\hat{x} \underset{L}{\overset{p}{\sim}} \hat{y}$. In order to show $Q(x) = Q(y)$ consider the elements $x', y'' \in S_n$ corresponding to (P_x, P_x) and (P_y, P_y) respectively (under the Robinson-Schensted correspondence). Theorem 5.31 implies that the elements $\hat{x}$ and x' as well as $\hat{y}$ and y'' are related by sequences of Knuth moves:

$$\begin{aligned} x' &= K_{i_r} \circ \cdots \circ K_{i_1}(\hat{x}) \\ y'' &= K_{j_s} \circ \cdots \circ K_{j_1}(\hat{y}) \end{aligned}$$

Lemma 4.4 for the left p-cell preorder shows that the elements $\hat{x}$ and $\hat{y}$ have the same right descent set, so the same right star-operations or Knuth moves (see Lemma 5.29) can be applied to both elements. By Theorem 5.12 we have $K_{i_1}(\hat{x}) \underset{L}{\overset{p}{\sim}} K_{i_1}(\hat{y})$ and $K_{j_1}(\hat{x}) \underset{L}{\overset{p}{\sim}} K_{j_1}(\hat{y})$. Therefore, we can repeat the argument to see that the following elements are well-defined:

$$\begin{aligned} x'' &= K_{j_s} \circ \cdots \circ K_{j_1}(\hat{x}) \\ y' &= K_{i_r} \circ \cdots \circ K_{i_1}(\hat{y}) \end{aligned}$$

Moreover, we have $x' \overset{p}{\underset{L}{\sim}} y'$ as well as $x'' \overset{p}{\underset{L}{\sim}} y''$ and thus $\mathcal{R}(x') = \mathcal{R}(y')$ and $\mathcal{R}(x'') = \mathcal{R}(y'')$. Using Theorem 5.31 we see that $P(x'') = P(\hat{x}) = P_x$ and $P(y') = P(\hat{y}) = P_y$.

Denote the column lengths of π_x (resp. π_y) by $l_1, l_2, \dots$ (resp. $k_1, k_2, \dots$). It follows from the row-bumping algorithm that x' is the longest element in the parabolic subgroup $S_{l_1} \times S_{l_2} \times \dots$ of S_n, i.e. in string notation the element

$$l_1, l_1 - 1, \dots, 1, l_1 + l_2, l_1 + l_2 - 1, \dots, l_1 + 1, \dots.$$

From $\mathcal{R}(x') = \mathcal{R}(y')$ and the characterization of right descent sets for elements in S_n in terms of inversions, we deduce that in the string notation for y' the first l_1 letters are decreasing as well as the next l_2 letters and so on. Similarly, we may use that y'' is the longest element in the parabolic subgroup $S_{k_1} \times S_{k_2} \times \dots$ of S_n and $\mathcal{R}(x'') = \mathcal{R}(y'')$ to deduce that in the string notation of x'' the first k_1 letters are decreasing, the next k_2 letters are in decreasing order, etc.

Applying the row-bumping algorithm to y' to calculate $P(y') = P_y$, we obtain the inequality $k_1 \geqslant l_1$. Using x'' instead, we get the opposite inequality giving $l_1 = k_1$. Moreover, this shows that when inserting the next l_2 letters of y', no row bumping occurs in the first column (otherwise we would have $k_1 > l_1$) and thus we have $k_2 \geqslant l_2$. Again, we may use x'' to get the opposite inequality and to show $k_2 = l_2$. Repeating the argument, we get $\pi_x = \pi_y$ and thus $Q(y') = P_y = P_x = Q(x'')$ (by the definition of the column superstandard tableau and the fact that $Q(-)$ encodes the order in which boxes are added in the course of the row-bumping algorithm). This shows $y' = x' = y'' = x''$ as well as $\hat{x} = \hat{y}$ by unravelling the sequences of Knuth moves. Finally, the Robinson-Schensted correspondence gives $Q(x) = Q(y)$ and finishes the proof of the characterization of left p-cells in terms of Q-symbols.

Using Theorem 5.30 and Lemma 4.6 we obtain the version for right p-cells. Finally, we prove the statement about two-sided p-cells:

$\Leftarrow$ Theorem 5.31 shows that any two elements of S_n with the same P-symbol can be related by a sequence of elementary Knuth transformations. Dually, any two elements with the same Q-symbol are linked by a sequence of elementary dual Knuth transformations, which we did not introduce, but which correspond to left star-operations. Given an element of $x \in S_n$ we can thus transform its P and Q symbols using Knuth transformations and their duals into any pair of given standard tableaux of the same shape. Denote by π the shape of $Q(x)$ and let P_π be the column superstandard tableau of shape π. The statement about left and right p-cells shows that x lies in the same two-sided p-cell as the element w_π corresponding to (P_π, P_π) under the Robinson-Schensted correspondence. From this, the reader easily deduces the direction $\Leftarrow$.

$\Rightarrow$ Given an element $x \in S_n$, denote by π the shape of $Q(x)$. Let w_π be as defined above. We claim that $y \overset{p}{\underset{LR}{\leqslant}} x$ implies $y \overset{0}{\underset{LR}{\leqslant}} x$.

Note that w_π is the longest element in a standard parabolic subgroup of S_n. Thus we have

$$^{p}\underline{H}_{w_\pi} = \underline{H}_{w_\pi}. \tag{13}$$

As $Q(w_\pi)$ and $Q(x)$ have the same shape, the direction $\Leftarrow$ implies $w_\pi \underset{LR}{\overset{p}{\sim}} x$ and $w_\pi \underset{LR}{\overset{0}{\sim}} x$. The relation $x \underset{LR}{\overset{p}{\leqslant}} w_\pi$ together with (13) gives us for all $z \leqslant x$:

$$^p m_{z,x} \neq 0 \Rightarrow z \underset{LR}{\overset{0}{\leqslant}} w_\pi$$

Therefore, any element $y \underset{LR}{\overset{p}{\leqslant}} x$ satisfies $y \underset{LR}{\overset{0}{\leqslant}} x$.
Finally, this finishes the proof of the direction $\Rightarrow$ by using the characterization of the Kazhdan-Lusztig two-sided cells in terms of the shape of the Q-symbols. □

Theorem 5.30 implies that the involutions in S_n are precisely those elements w that satisfy $P(w) = Q(w)$. This is the only missing observation for the next result:

Corollary 5.33. *Each left p-cell contains a unique involution. Each two-sided p-cell contains the longest element in a standard parabolic subgroup. Let C_L (resp. C_R) be a left (resp. right) p-cell. Then we have:*

$$|C_L \cap C_R| = \begin{cases} 1 & \text{if } C_L \text{ and } C_R \text{ lie in the same two-sided p-cell,} \\ 0 & \text{otherwise.} \end{cases}$$

Let π be a permutation of n. Recall Frame, Robinson and Thrall's hook length formula for the number of standard tableaux of shape π ([Ful97, §4.3]):

$$f^\pi = \frac{n!}{\prod_{(i,j)\in\pi} h_\pi(i,j)}$$

where $h_\pi(i,j)$ denotes the number of boxes in the hook of (i,j) in π, i.e. in formulas $h_\pi(i,j) = |\{(a,b) \in \pi \mid (a = i \text{ and } b \geqslant j) \text{ or } (a \geqslant i \text{ and } b = j)\}|$. The following corollary shows that the hook length formula gives the answer to some counting problems related to p-cells:

Corollary 5.34. *Let C be a two-sided p-cell. Denote by π the shape of the P and Q-symbols of the elements in C. Then the following holds:*

(i) *The number of left (or right) p-cells in C is given by f^π.*

(ii) *For any left (or right) p-cell contained in C, the corresponding p-cell module is free of rank f^π over $\mathbb{Z}[v, v^{-1}]$.*

Observe that preorder $\underset{LR}{\overset{p}{\leqslant}}$ is by definition generated by $\underset{L}{\overset{p}{\leqslant}}$ and $\underset{R}{\overset{p}{\leqslant}}$. It is in general not clear though whether $\underset{LR}{\overset{p}{\sim}}$ is also generated by $\underset{L}{\overset{p}{\sim}}$ and $\underset{R}{\overset{p}{\sim}}$. For the Kazhdan-Lusztig cell preorder this follows using certain properties of Lusztig's a-function (see [Lus03, Conjectures 14.2 P9 and P10]).

Corollary 5.35. *The cell preorder $\underset{LR}{\overset{p}{\sim}}$ is generated by $\underset{L}{\overset{p}{\sim}}$ and $\underset{R}{\overset{p}{\sim}}$ in type A.*

As a consequence of Lemma 5.20, we get the following result which is known for Kazhdan-Lusztig left cell modules (see [BB05, Theorem 6.5.2]) and for which the proof works exactly as in characteristic 0:

Corollary 5.36. *Let C_1 and C_2 be left p-cells in the same two-sided p-cell. Then the corresponding left cell modules are isomorphic.*

We want to close the section with some interesting questions that merit further study:

- In type A, the p-canonical basis for various primes p gives a family of interesting bases of each Kazhdan-Lusztig cell module. Which bases of the corresponding irreducible representation of S_n do they specialize to? In [Wil15, §1 and §2.3] Williamson explains that any p-canonical basis element of a right (or two-sided) cell module that differs from the corresponding Kazhdan-Lusztig basis element provides an example of a reducible characteristic variety of a simple highest weight module for $sl_n(\mathbb{C})$.
- In Type D, the generalized τ-invariant can be strengthened to give a complete invariant of Kazhdan-Lusztig cells. This is shown in [GPM] which Professor McGovern kindly provided a preliminary version of. Can one show that all elements of a left p-cell have the same strengthened generalized τ-invariant to get that Kazhdan-Lusztig cells decompose into p-cells in type D?

6 Tilting Modules and Modular Weight Cells

In this section we will recall some results about tensor ideals in the category of tilting modules for a reductive algebraic group scheme. Throughout this section we work in the following setting: Given the based root datum $(X, \Delta, X^\vee, \Delta^\vee)$ we used as input for ${}^k\mathbf{H}$ (see Section 2.1). We consider the (Langlands) dual root datum and associate to it a split connected reductive group scheme $G^\vee_\mathbb{Z}$ together with a maximal torus $T^\vee_\mathbb{Z}$ and a Borel $B^\vee_\mathbb{Z}$ such that $T^\vee_\mathbb{Z} \subseteq B^\vee_\mathbb{Z} \subseteq G^\vee_\mathbb{Z}$. The bases Δ and $\Delta^\vee$ give choices of positive roots Φ_+ and coroots $\Phi^\vee_+$. We assume $B^\vee_\mathbb{Z}$ to be chosen such that the coroots whose root subgroups are contained in $B^\vee_\mathbb{Z}$ correspond to the *negative* coroots $-\Phi^\vee_+$. For a field k of characteristic $p > 0$, we denote by $T^\vee \subseteq B^\vee \subseteq G^\vee$ the algebraic group schemes over k obtained from $T^\vee_\mathbb{Z} \subseteq B^\vee_\mathbb{Z} \subseteq G^\vee_\mathbb{Z}$ via extension of scalars. For simplicity, we will assume $G^\vee$ to be almost simple, semi-simple, simply-connected and k to be algebraically closed. Our main reference for this section is [Jan03]. An excellent survey over the theory with a slightly different focus can be found in [Wil17a, §1].

This setting will allow us to relate the tensor ideals of tilting modules for $G^\vee$ to p-cells in the affine Weyl group W. Even though our exposition makes the notation more complicated, we hope that it is less confusing to introduce the notions in the setting in which we need them. Let us start by introducing the geometric realization of the affine Weyl group.

We denote by

$$X^\vee_+ := \{\lambda \in X^\vee \mid \langle \lambda, \alpha \rangle \geqslant 0 \text{ for all } \alpha \in \Phi_+\}$$

the set of *dominant coweights*. Similarly X_+ denotes the set of *dominant weights*.

6.1 Geometric Realization of the Affine Weyl Group

We will need the geometric realization of the affine Weyl group. All results in this section are classical and can be found in [Bou68, Chapter V, §3], [Kan01, Chapter 11] and [IM65].

For this define $E := X^\vee \otimes \mathbb{R}$ where $X^\vee$ is the cocharacter lattice we used as input datum. The affine Weyl group $W = W_\mathrm{f} \ltimes \mathbb{Z}\Phi^\vee$ can be realized as affine transformations on E where $\mathbb{Z}\Phi^\vee$ acts by translations and W_f acts linearly (see Section 2.1). Denote by t_λ the translation by $\lambda \in \mathbb{Z}\Phi^\vee$. The lattice $\mathbb{Z}\Phi^\vee$ is invariant under the action of W. It follows that W is also generated by all affine reflections of the form

$$s_{\alpha^\vee,m}(\lambda) := \lambda - (\langle \lambda, \alpha \rangle - m)\alpha^\vee$$

for $\alpha \in \Phi_+$ and $m \in \mathbb{Z}$ as $t_{m\alpha^\vee} = s_{\alpha^\vee,m} \circ s_{\alpha^\vee,0}$. Observe that $s_{\alpha^\vee,m}$ fixes the hyperplane $H_{\alpha^\vee,m} = \{e \in E \mid \langle e, \alpha \rangle = m\}$. The connected components of

$$E \setminus \bigcup_{\alpha \in \Phi_+, n \in \mathbb{Z}} H_{\alpha^\vee,n}$$

are called *alcoves*. An example of an alcove is

$$A_0 = \{\lambda \in E \mid 0 < \langle \lambda, \alpha \rangle < 1 \text{ for all } \alpha \in \Phi_+\} \subseteq E,$$

called the *fundamental alcove*. For an alcove A, we call the subset of all points in $\overline{A}$ with a fixed stabilizer in W a *facet*. It follows that each facet is open in its closure and that $\overline{A}$ is a disjoint union of facets. The facets of codimension 1 of $\overline{A}$ are called *walls*. The affine reflections in those hyperplanes that intersect $\overline{A}_0$ in the walls of A_0 give the set of simple reflections S realizing W as a Coxeter Group. We have

$$S = \{s_{\alpha^\vee,0} \mid \alpha \in \Delta\} \cup \{s_{\alpha_0^\vee,1}\}$$

where $\alpha_0 \in \Phi_+$ is the highest root, so the affine reflection coincides with $t_{\alpha_0^\vee} \circ s_{\alpha_0^\vee,0}$ and fixes the hyperplane $H_{\alpha_0^\vee,1}$ (see [IM65, §1.4]).The group W acts simply transitively on the set of alcoves (see [IM65, Proposition 1.2]). Therefore, we may colour the walls of any alcove by an element in S in such a way that the action of W on E preserves the wall colours.

A point $\mathtt{v} \in E$ is called *special* if there are $\nu = |\Phi_+|$ hyperplanes in $\{H_{\alpha^\vee,m} \mid \alpha \in \Phi_+, m \in \mathbb{Z}\}$ passing through $\mathtt{v}$. Note that the origin 0 is a special point. The following lemma follows immediately from the definitions and shows that the assumption $G^\vee$ simply-connected ensures that every special point lies in $X^\vee$:

Lemma 6.1. *For a point $\mathtt{v} \in E$ the following conditions are equivalent:*

(i) $\mathtt{v}$ *is special.*

(ii) $\langle \mathtt{v}, \alpha\rangle \in \mathbb{Z}$ *for all* $\alpha \in \Delta$.

(iii) $\mathtt{v} \in \bigoplus_{\alpha\in\Delta} \mathbb{Z}\omega_\alpha^\vee$ *where* $\{\omega_\alpha^\vee \mid \alpha \in \Delta\}$ *are the fundamental coweights* (*i.e. the dual basis to* Δ *of* $\mathbb{Z}\Phi$).

Therefore, the action of W^{ext} on E gives the following bijections:

$$\begin{array}{ccccc} W^{\mathrm{ext}}/W_{\mathrm{f}} & \xleftrightarrow{\sim} & X^\vee & \xleftrightarrow{\sim} & \{\text{special vertices in } E\} \\ wW_{\mathrm{f}} & \longleftrightarrow & w(0) & \longleftrightarrow & w(0) \\ W_{\mathrm{f}}\backslash W^{\mathrm{ext}}/W_{\mathrm{f}} & \xleftrightarrow{\sim} & X_+^\vee & \xleftrightarrow{\sim} & \{\text{special vertices in } X_+^\vee \otimes_{\mathbb{Z}} \mathbb{R}_{\geqslant 0}\} \end{array}$$

The definition of a spherical element from Section 7.4 can be reformulated as follows: An element $w \in W$ is called *spherical* if there exist special vertices $\mathtt{v}, \tilde{\mathtt{v}} \in E$ such that $\mathcal{L}(w) = S_{\mathtt{v}}$ and $\mathcal{R}(w) = S_{\tilde{\mathtt{v}}}$.

Let $\mathcal{C}_0 := \{e \in E \mid \langle e, \alpha\rangle > 0 \text{ for all } \alpha \in \Phi_+\}$ be the *dominant cone*. For any special point $\mathtt{v} \in E$ let $\mathcal{C}_{\mathtt{v}}$ be the unique cone such that $\mathcal{C}_{\mathtt{v}}$ is a translate of $\mathcal{C}_0$ with $\mathtt{v}$ as vertex (i.e. extremal point). Let $A_{\mathtt{v}}$ be the unique alcove contained in $\mathcal{C}_{\mathtt{v}}$ with closure containing $\mathtt{v}$. The connected components of

$$E \setminus \bigcup_{\alpha\in\Delta, n\in\mathbb{Z}} H_{\alpha^\vee,n}$$

are called *boxes*. Let $\Pi_{\mathtt{v}}$ be the unique box containing $A_{\mathtt{v}}$.

In addition to the left action of W on the set of alcove induced by the continuous action of W on E, there is another action of W on the set of alcoves: For $s \in W$ and A an arbitrary alcove, denote by As the alcove whose closure shares an s-coloured wall with the closure of A. Denote by $(W_{\mathtt{v}}, S_{\mathtt{v}})$ the standard parabolic subgroup of the affine Weyl group W that leaves the set of alcoves with $\mathtt{v}$ in their closure invariant under the right action. Observe that $W_{\mathtt{v}}$ is isomorphic to the finite Weyl group and let $w_{\mathtt{v}}$ be its longest element. For example w_0 is the longest element in the finite Weyl group $W_0 = W_{\mathrm{f}}$.

The extended affine Weyl group $W^{\mathrm{ext}} = W_{\mathrm{f}} \ltimes X^\vee$ can also be realized as a subgroup of affine transformations on E where again $X^\vee$ acts by translations and W_{f} acts linearly. The lattice $X^\vee$ is invariant under the action of W^{ext}. The group W^{ext} has a length function $l : W^{\mathrm{ext}} \to \mathbb{Z}_{\geqslant 0}$ given by:

$$l(x) = |\{H_{\alpha^\vee,n} \mid H_{\alpha^\vee,n} \text{ separates } A_0 \text{ from } xA_0 \text{ for } \alpha \in \Phi_+, n \in \mathbb{Z}\}|$$

Since W is a subgroup in W^{ext}, this length function can be restricted to W and one can check that it coincides with the length function given by the minimal number of simple reflections needed to express an element of W (see [IM65, §1.5]). Consider the set of *length zero elements*:

$$\Omega := \{w \in W^{\mathrm{ext}} \mid l(w) = 0\} = \{w \in W^{\mathrm{ext}} \mid wA_0 = A_0\}$$

Since W acts simply transitively on the set of alcoves, we see that $W^{\mathrm{ext}} = \Omega \ltimes W$. This implies that Ω is isomorphic to $X^\vee/\mathbb{Z}\Phi^\vee$ and the corresponding diagonalizable group sceme is isomorphic to the center of $G^\vee$ (see [Jan03, Part II, §1.6]). In order to shed some light on the mysterious length zero elements, we want to summarize the most important results about Ω. First, we have a nice parametrization of Ω given in [IM65, Proposition 1.8]:

Proposition 6.2. *The set $\{0\} \cup \{\omega_\alpha^\vee \mid \alpha \in \Delta$ and $\langle \omega_\alpha^\vee, \alpha_0 \rangle = 1\}$ maps bijectively to Ω via $0 \mapsto 1, \omega_\alpha^\vee \mapsto t_{\omega_\alpha^\vee} v_\alpha w_0$ where v_α is the longest element in the parabolic subgroup of (W_f, S_f) generated by $\{s_{\beta^\vee,0} \mid \beta \in \Delta \setminus \{\alpha\}\}$.*

Since the set $\{0\} \cup \{\omega_\alpha^\vee \mid \alpha \in \Delta$ and $\langle \omega_\alpha^\vee, \alpha_0 \rangle = 1\}$ is precisely the set of elements in the intersection of $\overline{A_0}$ with the coweight lattice (see [IM65, Proposition 1.17]), the following result follows from the last proposition and Lemma 6.1:

Corollary 6.3. *The order of Ω is the number of special points in $\overline{A_0}$.*

The following result describes the action of Ω on W by conjugation (see [IM65, §1.8]) and will be quite important for us:

Proposition 6.4. *The action of $\sigma \in \Omega$ on W given by $w \mapsto \sigma w \sigma^{-1}$ preserves the set of simple reflections S and is given by an automorphism of the affine Dynkin diagram which is realized as the Dynkin diagram of $\{-\alpha_0\} \cup \{\alpha \mid \alpha \in \Delta\}$. The element $t_{\omega_\alpha^\vee} v_\alpha w_0 \in \Omega$ maps the affine reflection $s_{\alpha_0^\vee,1}$ to $s_{\alpha^\vee,0}$.*

6.2 More Notation

Set $\rho := \frac{1}{2} \sum_{\alpha^\vee \in \Phi_+^\vee} \alpha^\vee$. In the representation theory of $G^\vee$ in characteristic p it is necessary to dilate the action of the affine Weyl group by p and shift the origin to $-\rho$. For this reason, we consider the *p-dilated dot action* of W on $X^\vee$ defined for $\mu \in X^\vee$ via

$$x \cdot_p \mu := x(\mu + \rho) - \rho \text{ for } x \in W \text{ and} \tag{14}$$
$$t_\lambda \cdot_p \mu := \mu + p\lambda \text{ for } \lambda \in \mathbb{Z}\Phi^\vee. \tag{15}$$

A fundamental domain for the p-dilated dot action of affine Weyl group on $X^\vee$ is given by the closure of:

$$C_p = \{\lambda \in X^\vee \mid 0 < \langle \lambda + \rho, \alpha \rangle < p \text{ for all } \alpha \in \Phi_+\}$$

We define facets of $\overline{C}_p$ as in Section 6.1 using the p-dilated dot action of W (see [Jan03, Part II §6.2] for an equivalent definition). Denote the *p^r-fundamental box* by

$$X^\vee_{p^r} \coloneqq \{\lambda \in X^\vee_+ \mid \langle \lambda, \alpha \rangle < p^r \text{ for all } \alpha \in \Delta\}.$$

The elements of $X^\vee_{p^r}$ are called *p^r-restricted coweights* (resp. p^r-restricted weights for $G^\vee$). When expressing an element of $X^\vee_{p^r}$ in the fundamental coweights, all coefficients are $< p^r$.

Set $h = \max\{\langle \rho, \alpha \rangle \mid \alpha \in \Phi_+\} + 1$. Since our coroot system is assumed to be indecomposable, h coincides with its *Coxeter number*.

The weight $(p^r - 1)\rho$ for $r \in \mathbb{N}$ is called the *r-th Steinberg weight* for $G^\vee$. Since $G^\vee$ is assumed to be simply-connected, we have $(p^r - 1)\rho \in X^\vee$. For $r \in \mathbb{N}$ define $Y_r \coloneqq (p^r - 1)\rho + X^\vee_+$ to be the cone of dominant coweights shifted by the r-th Steinberg weight. Any $\lambda \in Y_r$ can be uniquely written as $\lambda = \lambda_0 + p^r \lambda_1$ with $\lambda_0 \in (p^r - 1)\rho + X^\vee_{p^r}$ and $\lambda_1 \in X^\vee_+$.

Since $G^\vee$ has an $\mathbb{F}_p$-structure (as it even arises from $G^\vee_{\mathbb{Z}}$ through base change), we can view the Frobenius F as an endomorphism of $G^\vee$ (see [Jan03, Part I, §9.2]). Denote by $G^\vee_r$ the kernel of F^r. The normal subgroup functor $G^\vee_r$ is called the *r-th Frobenius kernel*. For a $G^\vee$-module M, denote by $M^{[r]}$ its *r-th Frobenius twist*.

To each dominant coweight $\lambda \in X^\vee_+$ we can associate an induced representation $\nabla(\lambda) \coloneqq \operatorname{ind}_{B^\vee}^{G^\vee} k_\lambda$. Its dual $\Delta(\lambda)$ is called the *Weyl module* with highest weight λ. We also have a simple module $L(\lambda)$ sitting in the sequence $\Delta(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow \nabla(\lambda)$ given by projection to the head and inclusion of the socle. Moreover, the set $\{L(\lambda) \mid \lambda \in X^\vee_+\}$ gives a complete set of representatives for the isomorphism classes of simple, rational $G^\vee$-modules (see [Jan03, Part II, Corollary 2.7]). The category $\operatorname{Rep}(G^\vee)$ of rational representations of $G^\vee$ forms a highest-weight category. Its standard (resp. costandard) modules are the Weyl (resp. induced) modules.

6.3 Tilting Modules

In this section, we will collect the most important results about tilting modules. The main sources for this are [Don93; And98] and [Jan03, Part II, §E].

A rational representation of $G^\vee$ is called *tilting* if it admits two filtrations, one with successive quotients isomorphic to Weyl modules and the other one with successive quotients isomorphic to induced modules. We denote by $\operatorname{Tilt}(G^\vee)$ the full subcategory of tilting modules in $\operatorname{Rep}(G^\vee)$. The following vanishing statement is fundamental to the theory of tilting modules (see [Jan03, Part II, Proposition 4.13]):

Proposition 6.5. *For $\lambda, \mu \in X^\vee_+$ we have:*

$$\operatorname{Ext}^i_{G^\vee}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Without explicit mention we will often use the following result (see [Jan03, Part II, Lemma E.6]):

Lemma 6.6. $\operatorname{Tilt}(G^\vee)$ *is a k-linear Krull-Schmidt category.*

In [Don93, Theorem 1.1] Donkin classifies indecomposable tilting modules for $G^\vee$:

Theorem 6.7. *For each $\lambda \in X_+^\vee$ there exists up to isomorphism a unique indecomposable tilting module $T(\lambda)$ of highest weight λ. Moreover, λ occurs with multiplicity one as a weight of $T(\lambda)$. Every indecomposable tilting module is isomorphic to $T(\lambda)$ for some $\lambda \in X_+^\vee$.*

The following difficult result shows that the category of tilting modules is monoidal (see [Mat90] for a proof using Frobenius splitting techniques):

Theorem 6.8. *If Q_1 and Q_2 are tilting modules, then so is $Q_1 \otimes Q_2$.*

The strong linkage principle implies that we know the characters of the tilting modules with highest weight in $\overline{C}_p$:

Lemma 6.9. *For all $\lambda \in \overline{C_p} \cap X_+^\vee$ we have:*

$$T(\lambda) = L(\lambda) = \Delta(\lambda) = \nabla(\lambda)$$

Donkin's tilting tensor product theorem (see [Don93, Proposition 2.1]) will have important consequences for p-cells:

Proposition 6.10. *Suppose $p \geqslant 2h - 2$ and $n \in \mathbb{N} \setminus \{0\}$. Write $\lambda \in Y_n$ as $\lambda = \lambda_0 + p^n \lambda_1$ with $\lambda_0 \in (p^n - 1)\rho + X_{p^n}^\vee$ and $\lambda_1 \in X_+^\vee$. Then the following holds:*

$$T(\lambda) \cong T(\lambda_0) \otimes T(\lambda_1)^{(n)}$$

For a semi-simple, simply-connected algebraic group, its weight and character lattice coincide. Thus, all the fundamental coweights $\omega_i^\vee$ lie in $X^\vee$. Write a dominant weight $\lambda \in Y_1$ as a sum $\sum_i a_i \omega_i^\vee$ with $a_i \in \mathbb{Z}_{\geqslant 0}$. Then consider for all i the twisted p-adic decompostion $a_i = \sum_{j=0}^{m_i} a_{i,j} p^j$ where $p - 1 \leqslant a_{i,j} \leqslant 2p - 2$ for all $0 \leqslant j < m_i$, $0 \leqslant a_{i,m_i} \leqslant p - 1$ and m_i minimal for all i (see [EH02, Lemma 5]). Let m (resp. M) be the minimum (resp. maximum) of all m_i's. Set $\lambda_l := \sum_i a_{i,l} \omega_i^\vee$ for all $l < m$ and $\lambda_m = \sum_i (\sum_{l=m}^M p^{l-m} a_{i,l}) \omega_i^\vee$ where $a_{i,l} = 0$ for $l > m_i$. Then we have

$$\lambda = \sum_{l=0}^{m} p^l \lambda_l \quad \text{and} \quad \sum_{0 \leqslant l < r} p^l \lambda_l \in (p^r - 1)\rho + X_{p^r}^\vee \text{ for all } r \leqslant m$$

as $p^r - 1 \leqslant \langle \sum_{0 \leqslant l < r} p^l \lambda_l, \alpha \rangle \leqslant 2p^r - 2$ for all simple roots $\alpha \in \Delta$. Observe that λ_m lies in $X_+^\vee \setminus Y_1$ and thus within a p-strip of the walls of the dominant cone. In this situation we may successively apply Proposition 6.10 to get:

Corollary 6.11. *Under the same assumptions as in Proposition 6.10, for a dominant weight $\lambda \in Y_1$ write $\lambda = \sum_{l=0}^m p^l \lambda_l$ as above. Then we have:*

$$T(\lambda) = \bigotimes_{l=0}^{m} T(\lambda_l)^{(l)}$$

Observe also that Donkin's tilting tensor product theorem does not make any statement about the indecomposable tilting modules with highest weight in $X_+^\vee \setminus Y_1$ (i.e. those weights which lie too close to the walls of the dominant cone). It follows that if the characters of the indecomposable tilting modules with highest weight in $X_+^\vee \setminus Y_1 \cup (p-1)\rho + X_p^\vee$ are known, then one can deduce all indecomposable tilting characters.

The following definition is taken from [Jan80, §4.2]:

Definition 6.12. A weight $\mu \in X^\vee$ is called *p^r-bounded* if μ satisfies

$$\langle \mu, \alpha \rangle < 2p^r(h-1)$$

for all $\alpha \in \Phi \cap X_+$. A $G^\vee$-module N is called *p^r-bounded* if each weight μ of N is p^r-bounded.

For Section 8 we will need the following result from [And98, Proposition 2.6]:

Proposition 6.13. *Suppose $p \geqslant 2h-2$. For $\lambda \in X_p^\vee$ and $\mu \in X_+^\vee$ p-bounded we have:*

$$[\Delta(\mu) : L(\lambda)] = [T(\widetilde{\lambda}) : \Delta(\mu)]$$

where $\widetilde{\lambda} = 2(p-1)\rho + w_0\lambda$ and $w_0 \in W_f$ is the longest element.

6.4 Steinberg Modules

For $r \in \mathbb{N}$ the simple, rational $G^\vee$-module with the r-th Steinberg weight as heighest weight is called the *r-th Steinberg module* and denoted by:

$$\mathrm{St}_r := L((p^r-1)\rho)$$

The following result can be found in [Jan03, Part II, Proposition 3.19]:

Proposition 6.14. *For every $B^\vee$-module M and $i \in \mathbb{N}$ we have an isomorphism:*

$$H^i((p^r-1)\rho \otimes M^{[r]}) \cong St_r \otimes H^i(M)^{[r]}$$

By setting $M = k$, $i = 0$ and using $H^0(k) = k$ we get the first equality in the following corollary, which in turn implies the others:

Corollary 6.15. *The following holds:*

$$St_r = \nabla((p^r-1)\rho) = \Delta((p^r-1)\rho) = T((p^r-1)\rho)$$

In particular, St_r is self-dual and lies in $\mathrm{Tilt}(G^\vee)$*.*

Corollary 6.16. *St_r is a direct summand of $St_r \otimes St_r \otimes St_r$.*

Proof. Recall that in a k-linear, rigid, symmetric monoidal category $\mathcal{C}$ for every object $c \in \mathcal{C}$ there exists an object $c^\vee \in \mathcal{C}$, called the dual, together with maps $\mathrm{ev}_c : c^\vee \otimes c \to k$ and $\mathrm{coev}_c : k \to c \otimes c^\vee$ such that the composition

$$c \xrightarrow{\mathrm{coev}\otimes c} c \otimes c^\vee \otimes c \xrightarrow{c \otimes \mathrm{ev}} c$$

gives the identity on c. Since St_r is self-dual, we have an isomorphism $\varphi : \mathrm{St}_r^\vee \to \mathrm{St}_r$. Define:

$$\begin{array}{rcll} \iota = \mathrm{St}_r \otimes \varphi \otimes \mathrm{St}_r \ \circ \ \mathrm{coev} \otimes \mathrm{St}_r : & \mathrm{St}_r & \to & \mathrm{St}_r \otimes \mathrm{St}_r \otimes \mathrm{St}_r \\ \mathrm{pr} = \mathrm{St}_r \otimes \mathrm{ev} \ \circ \ \mathrm{St}_r \otimes \varphi^{-1} \otimes \mathrm{St}_r : & \mathrm{St}_r \otimes \mathrm{St}_r \otimes \mathrm{St}_r & \to & \mathrm{St}_r \end{array}$$

Now ι and pr obviously realize St_r as a direct summand of $\mathrm{St}_r \otimes \mathrm{St}_r \otimes \mathrm{St}_r$. □

By Weyl's dimension formula we get the following result:

Lemma 6.17. *The dimension of St_r is p^{rN} where $N = |\Phi_+^\vee|$ is the number of positive roots.*

The character of the Steinberg modules can also be computed using Weyl's character formula:

Lemma 6.18. *The character of St_r is given by*

$$\operatorname{ch} St_r = e^{(p^r-1)\rho} \prod_{\alpha \in \Phi_+^\vee} \frac{1 - e^{-p^r \alpha}}{1 - e^{-\alpha}}.$$

We are interested in the tensor ideals of tilting modules for $G^\vee$. For this, it will be important to know whether St_r is injective and projective for the l-th Frobenius kernel $G_l^\vee$. Combining [Jan03, Part II, Lemma 9.4 and Lemma E.8] we get an answer to this question:

Lemma 6.19. *Let $\lambda \in X_+^\vee$ and $r \in \mathbb{N} \setminus \{0\}$. Then $T(\lambda)$ is projective and injective for $G_r^\vee$ if and only if $\langle \lambda, \alpha \rangle \geqslant p^r - 1$ for all simple roots $\alpha \in \Delta$.*

Corollary 6.20. *The following holds: St_r is projective and injective for $G_l^\vee$ if and only if $1 \leqslant l \leqslant r$.*

In general, injective $G^\vee$-modules live in a ind-completion of $\mathrm{Rep}(G^\vee)$. In order to bypass this difficulty one usually truncates $\mathrm{Rep}(G^\vee)$ suitably. We want to find a full subcategory of $\mathrm{Rep}(G^\vee)$ in which the r-th Steinberg module St_r is injective. Combining the results in [Jan03, Part II, §11.11 and §E.9] (or in [Jan80, §4.4-4.5] as explained in [Don93, §2 Example 1]) we get:

Lemma 6.21. *Assume $p \geqslant 2h - 2$. Then for all $\lambda \in X_{p^r}^\vee$ the indecomposable tilting module $T(2(p^r - 1)\rho + \omega_0\lambda)$ is injective among the p^r-bounded modules where ω_0 denotes the longest element in the Weyl group of $G^\vee$.*

Corollary 6.22. *St_r is injective among the p^r-bounded modules.*

6.5 Translation Functors

In this section, we briefly introduce translation functors and recall the most important results about them.

For a subset $b \subset X_+$ denote by $\mathrm{Rep}(G^\vee)_b$ the full subcategory of all rational $G^\vee$-modules whose composition factors are indexed by elements in b. The following well-known result shows that $\mathrm{Rep}(G^\vee)_{W \cdot_p \lambda}$ is a direct sum of blocks of $Rep(G^\vee)$ (see [Jan03, Part II, Corollary 6.17]):

Theorem 6.23 (The Linkage Principle)**.**
Let $\lambda, \mu \in X_+$ be two dominant weights. If $Ext^1_{G^\vee}(L(\lambda), L(\mu))$ is non-zero, then $\mu \in W \cdot_p \mu$.

For $\lambda \in \overline{C_p}$ denote by pr_λ (resp. incl_λ) the projection to (resp. inclusion of) the subcategory $\mathrm{Rep}(G^\vee)_{W \cdot_p \lambda} \subset \mathrm{Rep}(G^\vee)$. For $\lambda, \mu \in \overline{C_p}$ there exists a unique dominant weight $\nu \in W(\mu - \lambda) \cap X_+^\vee$. The *translation functor*

$$T_\lambda^\mu : \mathrm{Rep}(G^\vee)_{W \cdot_p \lambda} \to \mathrm{Rep}(G^\vee)_{W \cdot_p \mu}$$

is defined via
$$T_\lambda^\mu := \mathrm{pr}_\mu(T(\nu) \otimes \mathrm{incl}_\lambda(-)).$$
Note that we obtain an isomorphic functor if we replace $T(\nu)$ by any $G^\vee$-module whose extremal weights are $W(\mu - \lambda)$ (see [Jan03, Part II, §7.6, Remark 1)]). One of the first result about translation functors is the following (see [Jan03, Part II, Lemma 7.6]):

Lemma 6.24. *For $\lambda, \mu \in \overline{C}_p$ we have:*

(i) *The functor T_λ^μ is exact.*

(ii) *The functors T_λ^μ and T_μ^λ are adjoint to each other.*

One of the most important results about translation functors is the "translation principle" (see [Jan03, Part II, Proposition 7.9]):

Proposition 6.25. *Suppose that $\lambda, \mu \in \overline{C}_p$ belong to the same facet. Then T_μ^λ induces an equivalence of categories $\mathrm{Rep}(G^\vee)_{W \cdot_p \mu} \to \mathrm{Rep}(G^\vee)_{W \cdot_p \lambda}$. The functor $T_\mu^\lambda \circ T_\lambda^\mu$ is isomorphic to pr_λ.*

The following result from [Jan03, Part II, Proposition E.11] describes how translation functors act on tilting modules:

Proposition 6.26. *Let $\lambda, \mu \in \overline{C}_p$ be such that μ belongs to the closure of the facet of λ. Let $w \in W$ be such that $w \cdot_p \mu \in X_+^\vee$ and such that $w \cdot_p \lambda$ is maximal among all $wx \cdot_p \lambda$ with $x \in \mathrm{Stab}_{W \cdot_p}(\mu)$. Then the following holds:*

(i) $T_\mu^\lambda T(w \cdot_p \mu) \cong T(w \cdot_p \lambda)$

(ii) $T_\lambda^\mu T(w \cdot_p \lambda) \cong \bigoplus_{i=1}^l T(w \cdot_p \lambda)$ *where* $l = (\mathrm{Stab}_{W \cdot_p}(\mu) : \mathrm{Stab}_{W \cdot_p}(\lambda))$

6.6 Modular Weight Cells

We will assume in this section $p \geqslant h$. This ensures that $0 \in C_p$ and thus C_p is non-empty (see [Jan03, Part II, §6.1 (8)]).

Definition 6.27. Let $\lambda, \mu \in X_+^\vee$. We write $\lambda \underset{\mathcal{T}}{\overset{p}{\leqslant}} \mu$ if there exists a tilting module $Q \in \mathrm{Tilt}(G^\vee)$ such that $T(\lambda)$ is a direct summand in $T(\mu) \otimes Q$. If both $\lambda \underset{\mathcal{T}}{\overset{p}{\leqslant}} \mu$ and $\mu \underset{\mathcal{T}}{\overset{p}{\leqslant}} \lambda$ hold, then we write $\lambda \underset{\mathcal{T}}{\overset{p}{\sim}} \mu$. The equivalence classes for $\underset{\mathcal{T}}{\overset{p}{\sim}}$ are called *modular weight cells* (after [And04] and [Ost01]).

Any alcove C with respect to the p-dilated dot action of the affine Weyl group consists of all weights $\lambda \in X^\vee$ satisfying a family of inequalities of the form $n_\alpha p < \langle \lambda + \rho, \alpha \rangle < (n_\alpha + 1)p$ for $\alpha \in \Phi_+$. Denote the set of weights $\lambda \in X^\vee$ such that $n_\alpha p \leqslant \langle \lambda + \rho, \alpha \rangle < (n_\alpha + 1)p$ holds for all $\alpha \in \Phi_+$ by ${}^{\leqslant}\overline{C}$. This set is called the *lower closure of* C. Combining Proposition 6.25 and Proposition 6.26 it is easy to see that the set

$$\{{}^{\leqslant}\overline{C} \mid C \text{ alcove w.r.t. the } p\text{-dilated dot action of } W \text{ with } C \cap X_+ \neq \emptyset\}$$

gives a refinement of modular weight cells (see [AHR17, Lemma 7.6] and [And04, §4]):

Lemma 6.28. *If $\lambda, \mu \in X_+^\vee$ belong to the lower closure of the same alcove, then $\lambda \underset{\mathcal{T}}{\overset{p}{\sim}} \mu$. In particular, each modular weight cell is a union of lower closures of alcoves for the dot-action of the p-dilated affine Weyl group intersected with $X_+^\vee$.*

The definition of translation functors immediately implies the following result:

Lemma 6.29. *For $\lambda, \mu \in \overline{C}_p$ and $w, w' \in W$ we have:*

$$T(w' \cdot_p \mu) \overset{\oplus}{\subseteq} T_\lambda^\mu T(w \cdot_p \lambda) \Rightarrow w' \cdot_p \mu \underset{\mathcal{T}}{\overset{p}{\leqslant}} w \cdot_p \lambda$$

The following result gives the first example of a modular weight cell. It can be deduced from results in [AP95, §2] or [GM94]:

Proposition 6.30. *For $p \geqslant h$ the set C_p is a modular weight cell, more precisely the highest one with respect to the preorder $\underset{\mathcal{T}}{\overset{p}{\leqslant}}$.*

Proof. Recall that the assumption $p \geqslant h$ assures that 0 lies in C_p. We want to show that for any $\lambda \in C_p$ we have $T(0) \overset{\oplus}{\subseteq} T(\lambda)^\vee \otimes T(\lambda)$. Since $\mathrm{Rep}(G^\vee)$ is a rigid, symmetric monoidal category, we have maps $\mathrm{coev}_{T(\lambda)} : k \to T(\lambda) \otimes T(\lambda)^\vee$, $\mathrm{ev}_{T(\lambda)} : T(\lambda)^\vee \otimes T(\lambda) \to k$ and the dimension of $T(\lambda)$ can be calculated as follows (see [Eti+15, Proposition 8.10.14]):

$$k \xrightarrow{\mathrm{coev}_{T(\lambda)}} T(\lambda) \otimes T(\lambda)^\vee \cong T(\lambda)^\vee \otimes T(\lambda) \xrightarrow{\mathrm{ev}_{T(\lambda)}} k$$

[AHR17, Proposition 7.9] shows that for $\lambda \in X_+^\vee$ the dimension of $T(\lambda)$ is not divisible by p if and only if λ lies in C_p. Therefore the composition above is non-zero and realizes k as a direct summand of $T(\lambda)^\vee \otimes T(\lambda)$. It follows that C_p is contained in a modular weight cell. [GM94, Lemma 2.7] reads in our setting as follows:

Lemma 6.31. *Let $M, N \in \mathrm{Rep}(G^\vee)$. Assume that M is indecomposable and of dimension divisible by p. Then the dimension of any direct summand of $M \otimes N$ is divisible by p.*

Combining this result with [AHR17, Proposition 7.9] gives us that C_p is a modular weight cell. Since $0 \in C_p$ corresponds to the trivial module, it is clear that C_p is the highest modular weight cell with respect to $\underset{\mathcal{T}}{\overset{p}{\leqslant}}$. □

Lemma 6.32. *There are infinitely many modular weight cells.*

Proof. From Corollary 6.20 it follows that St_{r+1} is injective for the $(r+1)$-th Frobenius kernel $G_{r+1}^\vee$ of $G^\vee$, but St_r is not. Therefore St_r cannot be a direct summand of $\mathrm{St}_{r+1} \otimes Q$ for some tilting module $Q \in \mathrm{Tilt}(G)$ as all direct summands of $\mathrm{St}_{r+1} \otimes Q$ are injective $G_{r+1}^\vee$-modules. This implies

$$(p^{r+1} - 1)\rho \underset{\mathcal{T}}{\overset{p}{<}} (p^r - 1)\rho.$$

□

Remark 6.33. In [Lus87, Theorem 2.2] Lusztig shows that there are only finitely many Kazhdan-Lusztig right cells in affine Weyl groups. As soon as we have established a connection between modular weight cells and right p-cells, it follows from the last result that affine Weyl groups have infinitely many right p-cells for $p \geqslant h$.

As a corollary to the proof and Lemma 6.19 we get:

Corollary 6.34. *Let $\lambda \in X_+^\vee$ and Q be a tilting module for $G^\vee$. If $T(\lambda)$ is a direct summand in $\mathrm{St}_r \otimes Q$, then λ lies in $(p^r-1)\rho + X_+^\vee$.*

6.7 Fractal-Like Structure of Modular Weight Cells

In this section, we will explain the beautiful fractal-like structure of modular weight cells. The proof is due to Andersen (see [And04, §4]) and based on the tilting tensor product theorem (see Proposition 6.10). As in the tilting tensor product theorem, we write $\lambda \in Y_r$ as $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in (p^r-1)\rho + X_{p^r}^\vee$ and $\lambda_1 \in X_+^\vee$.

Lemma 6.35. *Assume $p \geqslant 2h-2$. For $\lambda, \mu \in Y_r$ we have:*

$$\lambda \underset{\mathcal{T}}{\overset{p}{\leqslant}} \mu \text{ if and only if } \lambda_1 \underset{\mathcal{T}}{\overset{p}{\leqslant}} \mu_1.$$

Proof. First, we show that $\lambda_0 \underset{\mathcal{T}}{\overset{p}{\sim}} (p^r-1)\rho$. For all $\nu_1, \nu_2 \in X_+^\vee$ we know that $T(\nu_1+\nu_2)$ is a direct summand of $T(\nu_1) \otimes T(\nu_2)$ and thus $\nu_1 + \nu_2 \underset{\mathcal{T}}{\overset{p}{\leqslant}} \nu_1$. In particular, we have $\lambda_0 \underset{\mathcal{T}}{\overset{p}{\leqslant}} (p^r-1)\rho$. To see that the other inequality also holds, recall that under the assumption $p \geqslant 2h-2$ the Steinberg module St_r is injective among $G^\vee$-modules with highest weight in $C_{2p^r(h-1)}$ (see Corollary 6.22). Write $\lambda_0 = (p^r-1)\rho + \nu$ for some $\nu \in X_{p^r}^\vee$. Since St_r is simple, it is enough to check that there is a non-zero $G^\vee$-module homomorphism from St_r to $T(\lambda_0) \otimes T(\nu)$. As tilting modules are self-dual, we have by the definition of ν:

$$\mathrm{Hom}_{G^\vee}(\mathrm{St}_r, T(\lambda_0) \otimes T(\nu)) \cong \mathrm{Hom}_{G^\vee}(\mathrm{St}_r \otimes T(\nu), T(\lambda_0)) \neq 0$$

This finishes the proof of $\lambda_0 \underset{\mathcal{T}}{\overset{p}{\sim}} (p^r-1)\rho$.

Next, we claim that we may assume $\lambda_0 = (p^r-1)\rho = \mu_0$. Indeed, recall the tilting tensor product theorem which gives us for $p \geqslant 2h-2$:

$$T(\lambda) \cong T(\lambda_0) \otimes T(\lambda_1)^{(r)}$$

where $(-)^{(r)}$ denotes the r-th Frobenius twist. From this we deduce

$$\lambda \underset{\mathcal{T}}{\overset{p}{\sim}} (p^r-1)\rho + p^r\lambda_1.$$

Finally, we prove the claim of the lemma.

- $\Leftarrow$ Suppose that $T(\lambda_1)$ occurs as a direct summand of $T(\mu_1) \otimes Q$ for some tilting module $Q \in \mathrm{Tilt}(G^\vee)$. Then $T(\lambda) \cong \mathrm{St}_r \otimes T(\lambda_1)^{(r)}$ is a direct summand of $\mathrm{St}_r \otimes T(\mu_1)^{(r)} \otimes Q^{(r)}$. Using the fact that St_r is a summand of $\mathrm{St}_r \otimes \mathrm{St}_r \otimes \mathrm{St}_r$ (see Corollary 6.16) we see that $\mathrm{St}_r \otimes T(\mu_1)^{(r)} \otimes Q^{(r)}$ and thus $T(\lambda)$ is a direct summand of

$$\underbrace{\mathrm{St}_r \otimes T(\mu_1)^{(r)}}_{T(\mu)} \otimes \underbrace{\mathrm{St}_r \otimes \mathrm{St}_r \otimes Q^{(r)}}_{\in \mathrm{Tilt}(G^\vee)}.$$

$\Rightarrow$ Suppose that $T(\lambda)$ is a direct summand of $T(\mu) \otimes Q$ for some tilting module $Q \in \mathrm{Tilt}(G^\vee)$. First consider $\mathrm{St}_r \otimes Q$. By Corollary 6.34 all indecomposable direct summands of this tilting module are indexed by elements in Y_r. By the tilting tensor product theorem we have:

$$T(\mu) \otimes Q \cong \bigoplus_{\nu \in Y_r} (T(\nu_0) \otimes (T(\mu_1) \otimes T(\nu_1))^{(r)})^{\oplus c_\nu}$$

for suitable $c_\nu \in \mathbb{N}$ almost all of which are 0. Since $T(\lambda) \cong \mathrm{St}_r \otimes T(\lambda_1)^{(r)}$ occurs as a direct summand of a term on the right hand side, there has to be an element $\nu \in Y_r$ such that $c_\nu > 0$, $\nu_0 = (p^r - 1)\rho$ and $T(\lambda_1)$ is a direct summand of $T(\mu_1) \otimes T(\nu_1)$. This implies $\lambda_1 \overset{p}{\underset{\mathcal{T}}{\leqslant}} \mu_1$. $\square$

The last result together with Proposition 6.30 gives us:

Proposition 6.36. *For each $r \in \mathbb{N}$ the finite set $\underline{c}_1^r := (p^r - 1)\rho + X_{p^r}^\vee + p^r C_p$ is a modular weight cell.*

Moreover, Lemma 6.35 reduces the description of all modular weight cells in X_+ to the problem of decomposing $Y_1 \setminus C_p$ into modular weight cells.

The modular weight cell $\underline{c}_1^0 = C_p$ gives rise to a rigid, symmetric monoidal category with fusion rule described in [AP95, Proposition 2.10] and [GM94, Theorem 4.10].

7 Modular Weight Cells and p-Cells

In this section we will recall some results about the (anti-)spherical module for the affine Hecke algebra. References for this section are [Soe97b] and [Deo87]. We continue to work in the setting of Section 6 and preserve the additional assumptions made there as well. In particular, $\mathcal{H}$ denotes the Hecke algebra associated to the affine Weyl group (W, S) (as introduced in Section 6.1).

Denote by $\mathcal{H}_\mathrm{f} = \mathcal{H}_{(W_\mathrm{f}, S_\mathrm{f})} \subset \mathcal{H}$ the Hecke algebra of the finite Weyl group $(W_\mathrm{f}, S_\mathrm{f})$. Set $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$. Let ${}^fW$ be the set of representatives of minimal lengths of the cosets $W_\mathrm{f}\backslash W$. Denote by w_0 the longest element in W_f.

Recall the quadratic relation $(H_s + v)(H_s - v^{-1}) = 0$ in the Hecke algebra. Thus for $u \in \{-v, v^{-1}\}$ we may define a surjection of $\mathcal{L}$-algebras $\phi_u : \mathcal{H}_\mathrm{f} \twoheadrightarrow \mathcal{L}$ via $H_s \mapsto u$ for all $s \in S_\mathrm{f}$. The morphism ϕ_u turns $\mathcal{L}$ into a $\mathcal{H}_\mathrm{f}$-module which will be denoted by $\mathcal{L}(u)$. Observe that $\mathcal{L}(v^{-1})$ (resp. $\mathcal{L}(-v)$) is a deformation of the *trivial* (resp. *sign*) representation of W_f.

Define the *spherical (right) module* $\mathcal{M}$ and the *anti-spherical (right) module* $\mathcal{N}$ for the Hecke algebra $\mathcal{H}$ as follows:

$$\mathcal{M} = \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}$$
$$\mathcal{N} = \mathcal{L}(-v) \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}$$

One may define a bar involution on $\mathcal{M}$ and $\mathcal{N}$ via $f \otimes H \mapsto \overline{f} \otimes \overline{H}$. $\mathcal{M}$ comes equipped with *a standard basis* $\{M_x := 1 \otimes H_x \mid x \in {}^fW\}$, and *a Kazhdan-Lusztig basis* $\{\underline{M}_x \mid x \in {}^fW\}$ (see [Soe97b, Theorem 3.1]). Similarly for $\mathcal{N}$ using the notation N_x, $\underline{N}_x$ respectively.

7.1 The p-Canonical Basis of $\mathcal{M}$ and Its Categorification

There exists an embedding of $\mathcal{H}_\mathrm{f}$-modules $\mathcal{L}(v^{-1}) \hookrightarrow \mathcal{H}_\mathrm{f}$ defined by $1 \mapsto \underline{H}_{w_0}$ (which is well-defined due to the multiplication formula for the Kazhdan-Lusztig basis). This induces an embedding of right $\mathcal{H}$-modules $\eta : \mathcal{M} \hookrightarrow \mathcal{H}$ which is compatible with the $\overline{(-)}$ operator. Due to $\underline{H}_{w_0} = \sum_{y \leqslant w_0} v^{l(w_0)-l(y)} H_y$ we get (see [Soe97b, Proposition 3.4]):

$$\eta(M_x) = \sum_{y \in W_\mathrm{f}} v^{l(w_0)-l(y)} H_{yx}$$
$$\eta(\underline{M}_x) = \underline{H}_{w_0 x}$$

where, for the second equality, one uses the characterization of $\underline{M}_x$ as the unique self-dual element in $M_x + \sum_{y \leqslant x} v\mathbb{Z}[v] M_y$ and similarly in $\mathcal{H}$.

Recall that any element $w \in W$ with $S_\mathrm{f} \subseteq \mathcal{L}(w)$ can be written as $w = w_0 \widetilde{w}$ with $\widetilde{w} \in {}^fW$. It follows from Proposition 3.10(v) that

$${}^p\underline{H}_{w_0 x} = \sum_{y \leqslant x} {}^p m_{w_0 y, w_0 x} \underline{H}_{w_0 y}$$

lies completely in the image of η. Therefore, we define the p-canonical basis of $\mathcal{M}$ via

$${}^p\underline{M}_x := \eta^{-1}({}^p\underline{H}_{w_0 x})$$

for all $x \in {}^fW$. If follows from $\eta(\underline{M}_x) = \underline{H}_{w_0x}$ that we have for all $x, y \in {}^fW$:

$$ {}^p\underline{M}_x = \sum_{y \leqslant x} {}^p m_{w_0 y, w_0 x} \underline{M}_y. $$

The image of η in $\mathcal{H}$ suggests the following categorification of $\mathcal{M}$. Denote by ${}^k\mathbf{M}$ the full additive subcategory of ${}^k\mathbf{H}$ generated by all indecomposable objects ${}^kB_{w_0x}(n)$ for $x \in W$ such that $l(w_0x) = l(w_0) + l(x)$ and $n \in \mathbb{Z}$. Then ${}^k\mathbf{M}$ is naturally a right ${}^k\mathbf{H}$-module category where any indecomposable object ${}^kB_x$ simply acts via $(-) \otimes {}^kB_x$. Since we will not use the following result, we omit its proof.

Lemma 7.1. *${}^k\mathbf{M} \xrightarrow{\sim} \mathcal{M}$ via $[{}^kB_{w_0x}] \mapsto \underline{M}_x$ for $x \in {}^fW$ as right $\mathcal{H}$-modules.*

7.2 The p-Canonical Basis of $\mathcal{N}$ and Its Categorification

Consider the canonical quotient map of right $\mathcal{H}$-modules $\zeta : \mathcal{H} \twoheadrightarrow \mathcal{N}$ given by $h \mapsto 1 \otimes h$. The map ζ is compatible with the $\overline{(-)}$ operator. One can checks that the following holds (see [Soe97b, Proposition 3.4]):

$$ \zeta(H_{yx}) = (-v)^{l(y)} N_x \text{ for } y \in W_{\mathrm{f}}, x \in {}^fW $$
$$ \zeta(\underline{H}_x) = \begin{cases} \underline{N}_x & \text{if } x \in {}^fW \\ 0 & \text{otherwise.} \end{cases} $$

For the last equality one uses:

(i) the characterization of the Kazhdan-Lusztig basis elements $\underline{N}_x$ as the unique self-dual element in $N_x + \sum_{y \leqslant x} v\mathbb{Z}[v] N_y$ and similarly for $\underline{H}_x$,

(ii) for $y \leqslant x$ and $s \in \mathcal{L}(x) \cap \mathcal{L}(y)$, the relation $h_{sy,x} = v h_{y,x}$ as well as the equivalence $sy \leqslant x \Leftrightarrow y \leqslant x$.

For $s \in S$ and $w \in {}^fW$ we have (see [Soe97b, §3]):

$$ N_w \cdot \underline{H}_s = \begin{cases} N_{ws} + vN_w & \text{if } ws \in {}^fW \text{ and } ws > w, \\ N_{ws} + v^{-1}N_w & \text{if } ws \in {}^fW \text{ and } ws < w, \\ 0 & \text{otherwise.} \end{cases} $$

We define the p-canonical basis for $\mathcal{N}$ as follows:

$$ {}^p\underline{N}_x \coloneqq \zeta({}^p\underline{H}_x) = 1 \otimes {}^p\underline{H}_x \text{ for all } x \in {}^fW. $$

Therefore we get:

$$ {}^p\underline{N}_x = \sum_{\substack{y \leqslant x \\ y \in {}^fW}} {}^p m_{y,x} \underline{N}_y. $$

Write ${}^pn_{y,x}$ for the coefficient of N_y when expressing ${}^p\underline{N}_x$ in terms of the standard basis. As for the Kazhdan-Lusztig basis we obtain using Proposition 3.10(v) (see [AHR17, Lemma 5.8]):

Lemma 7.2. *For $x \notin {}^fW$ we have $\zeta({}^p\underline{H}_x) = 0$.*

The surjection ζ suggests the following categorification of $\mathcal{N}$. Denote by ${}^k\mathbf{N}$ the quotient of ${}^k\mathbf{H}$ by the two-sided ideal of all morphisms factoring through a finite direct sum of indecomposable objects of the form ${}^kB_x(n)$ for $x \notin {}^fW$ and $n \in \mathbb{Z}$. Then ${}^k\mathbf{N}$ is naturally a right ${}^k\mathbf{H}$-module category where any indecomposable object ${}^kB_x$ simply acts via $(-) \otimes {}^kB_x$. By definition the diagrammatic character factors as follows:

$$\begin{tikzcd} {[{}^k\mathbf{H}]} \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, two heads] & \mathcal{H} \arrow[d, two heads] \\ {[{}^k\mathbf{N}]} \arrow[r, dotted, "\mathrm{ch}"] & \mathcal{N} \end{tikzcd}$$

Then the following lemma is easy to verify:

Lemma 7.3. ch : ${}^k\mathbf{N} \xrightarrow{\sim} \mathcal{N}$ *via* $[{}^kB_x] \mapsto \underline{N}_x$ *for* $x \in {}^fW$ *as right* $\mathcal{H}$*-modules.*

Denote by ${}^k\mathbf{N}_{\mathrm{deg}}$ the degrading of ${}^k\mathbf{N}$, i.e. the category on the same objects as ${}^k\mathbf{N}$ but with morphisms for $X, Y \in {}^k\mathbf{N}$ given by:

$$\mathrm{Hom}_{{}^k\mathbf{N}_{\mathrm{deg}}}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{{}^k\mathbf{N}}(X, Y(n)).$$

Let $\lambda_0 \in C_p$ and consider $\mathrm{Rep}_0(G^\vee)$, the regular block of the category of rational $G^\vee$-modules corresponding to λ_0. Recall that h is the Coxeter number of $G^\vee$. We refer the reader to [RW16, §3.2 and §3.5] for the definition of wall-crossing functors:

Conjecture 7.4 (Riche-Williamson Conjecture)**.**
The assignment of ${}^kB_s$ *to a wall crossing functor* Ξ_s *for* $s \in S$ *induces a categorical action of* ${}^k\mathbf{H}$ *on* $\mathrm{Rep}_0(G^\vee)$*.*

Actually, this is not the precise form of the conjecture, but it nonetheless captures its spirit. For a precise formulation the reader is referred to [RW16, Conjecture 5.1]. The Riche-Williamson conjecture has the following important consequences for us (see [RW16, Theorem 1.2]):

Theorem 7.5. *Suppose that the Riche-Williamson conjecture holds. Then there is an equivalence of* k*-linear right* ${}^k\mathbf{H}$*-module categories:*

$${}^k\mathbf{N}_{\mathrm{deg}} \xrightarrow{\sim} \mathrm{Tilt}_0(G^\vee)$$

which sends ${}^kB_w$ *to* $T(w \cdot_p \lambda_0)$ *for any* $w \in {}^fW$ *where* $\cdot_p$ *denotes the* p*-dilated dot action of* W *on* $X^\vee$*.*

Theorem 7.5 has the following important corollary, a character formula for indecomposable tilting modules:

Theorem 7.6. *For* $p > h$ *and* $x, y \in {}^fW$ *we have:*

$$[\mathcal{T}(x \cdot_p \lambda_0) : \nabla(y \cdot_p \lambda_0)] = {}^pn_{y,x}(1)$$

Recently, Achar, Makisumi, Riche and Williamson proved Theorem 7.6 in [Ach+17a] and [Ach+17b]. It follows that the p-canonical basis of $\mathcal{N}$ gives all tilting characters of $G^\vee$. The Riche-Williamson conjecture remains open in types other than A.

Remark 7.7. In the papers [RW16] and [Ach+17b] a more general setting is used. $G^\vee$ is allowed to be a connected reductive group over k with simply-connected derived subgroup (for example GL_n).

Lemma 4.4 shows that ${}^fW$ decomposes into right p-cells (see also [AHR17, Lemma 5.6]). The right p-cells contained in ${}^fW$ are called *anti-spherical* (or *canonical* in the terminology of [LX88]). Following along the lines of [Ost97], Achar, Hardesty and Riche use the new character formulas for tilting modules to establish a link between anti-spherical p-cells and modular weight cells (see [AHR17, Theorem 7.7]) which we will use frequently:

Theorem 7.8. *For $p > h$ and $x, y \in {}^fW$ we have:*

$$x \cdot_p 0 \overset{p}{\underset{\mathcal{T}}{\leqslant}} y \cdot_p 0 \Leftrightarrow x \overset{p}{\underset{R}{\leqslant}} y$$

7.3 Anti-spherical Light Leaves and Soergel's Hom-Formula

In this section, we want to introduce anti-spherical light leaves and state Soergel's Hom-formula for the anti-spherical module.

Definition 7.9. Given an expression $\underline{w} = (s_1, s_2, \dots, s_n)$ together with a subexpression $\underline{e}$, set $w_0 := e$ and $w_k := (\underline{w}_{\leqslant k}^{\underline{e}_{\leqslant k}})_\bullet$ for $1 \leqslant k \leqslant n$. For a subset $K \subseteq W$ we say that $\underline{e}$ *avoids* K if $w_{k-1}s_k \notin K$ for $1 \leqslant k < n$.

The following lemma (see [RW16, Lemma 4.1]) gives the characters of the anti-spherical Bott-Samelson modules and is the counterpart of [EW16, Lemma 2.10] in the anti-spherical setting:

Lemma 7.10. *For any expression $\underline{w} = (s_1, s_2, \dots, s_n)$, we have in $\mathcal{N}$:*

$$\underline{N}_e \cdot \underline{H}_{s_1} \underline{H}_{s_2} \dots \underline{H}_{s_n} = \sum_{\substack{\underline{e} \text{ subexpression of } \underline{w} \\ \underline{e} \text{ avoids } W \setminus {}^fW}} v^{\mathrm{df}(\underline{e})} N_{\underline{w}^{\underline{e}}}$$

Define $R_{S_\mathrm{f}} := R / \langle \alpha_s \mid s \in S_\mathrm{f} \rangle$. We claim that the morphism spaces in ${}^k\mathbf{N}$ are left R_{S_f}-modules. First, consider any expression $\underline{w}$ starting in a simple reflection $s \in S_\mathrm{f}$. Lemma 4.4 implies that the corresponding Bott-Samelson object in ${}^k\mathbf{H}$ decomposes into a direct sum of grading shifts of indecomposable objects B_x with $s \in \mathcal{L}(x)$. Thus, the image of $\underline{w}$ is zero in ${}^k\mathbf{N}$. Then recall that multiplying an S-graph D with a homogeneous polynomial $f \in R$ from the left is defined by decorating the leftmost region of D with f. Using the Barbell relation (4) it follows that multiplying for $s \in S_\mathrm{f}$ an S-graph with α_s from the left results in a morphism that factors through an expression starting in s. Therefore, the resulting morphism vanishes in ${}^k\mathbf{N}$. This finishes the proof of the claim.

In the following, we will also call a subexpression $\underline{e}$ of $\underline{w}$ that avoids $W \setminus {}^fW$ as well as the corresponding light leaf $\mathrm{LL}_{\underline{e},\underline{w}}$ *anti-spherical.*

Proposition 7.11. *For any expression $\underline{w}$, the graded R_{S_f}-module $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}}(\underline{w}, \varnothing)$ is free. Moreover, we can choose the light leaves morphisms in such a way that the morphisms*

$$\{\mathrm{LL}_{\underline{w},\underline{e}} \mid \underline{e} \text{ anti-spherical subexpression of } \underline{w} \text{ expressing } e\}$$

give a basis of $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}}(\underline{w}, \varnothing)$.

Proof. In [RW16, Proposition 4.7] Riche and Williamson show that the anti-spherical light leaves give a spanning set and the proof of [RW16, Theorem 11.13] implies that they are linear independent. ☐

It should be noted that the necessary choices in Proposition 7.11 are limited to certain braid moves. Since R_{S_f} is a principal ideal domain (as a polynomial ring in one variable over a field), direct summands of free modules are free. Thus Proposition 7.11 implies via adjunction techniques that all Hom-spaces in ${}^k\mathbf{N}$ are free R_{S_f}-modules. Therefore, a $\mathbb{Z}[v,v^{-1}]$-semilinear (in the sense of [EW16, §2.4]) pairing on $\mathcal{N}$ is induced by

$$(\mathrm{ch}(B), \mathrm{ch}(B'))_{\mathbf{N}} = \mathrm{grk}\,\mathrm{Hom}^\bullet_{{}^k\mathbf{N}}(B, B')$$

and it satisfies by adjunction (see [RW16, §5.5]) for $n, m \in \mathcal{N}$:

$$(n \cdot \underline{H}_s, m)_{\mathbf{N}} = (n, m \cdot \underline{H}_s)_{\mathbf{N}} \text{ for all } s \in S. \tag{16}$$

Alternatively, we can define a $\mathbb{Z}[v,v^{-1}]$-bilinear pairing on $\mathcal{N}$ via $(N_x, N_y) = \delta_{x,y}$ and bilinear extension. From the action described above, it follows that this pairing also satisfies the identity in (16). Therefore, both pairings are uniquely determined by the values obtained from pairing $\mathrm{ch}(\underline{w})$ with N_e for arbitrary expressions $\underline{w}$ of elements $w \in {}^fW$. Proposition 7.11 and Lemma 7.10 show that both pairings agree on these values as for a subsequence $\underline{e}$ of $\underline{w}$ we have $\deg(\mathrm{LL}_{\underline{w},\underline{e}}) = \mathrm{df}(\underline{e})$ and therefore we get for $n, m \in \mathcal{N}$:

$$(\overline{n}, m) = (n, m)_{\mathbf{N}}$$

Corollary 7.12 (Soergel's Hom-formula for ${}^k\mathbf{N}$)**.** *For $B, B' \in {}^k\mathbf{N}$ we have:*

$$(\overline{\mathrm{ch}(B)}, \mathrm{ch}\, B') = \mathrm{grk}\,\mathrm{Hom}^\bullet_{{}^k\mathbf{N}}(B, B')$$

Theorem 7.13. *The set of all double leaves ranging over all $w \in W$ and pairs of anti-spherical subsequences $\underline{e}$ (resp. $\underline{f}$) of $\underline{x}$ (resp. $\underline{y}$) both expressing w gives a basis of the free R_{S_f}-module $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}}(\underline{x}, \underline{y})$.*

Sketch of proof. Let $w \leqslant x \in {}^fW$. Choose an arbitary expression $\underline{x}$ for x and a reduced expression $\underline{w}$ for w. We can extend the proof of [RW16, Proposition 4.7] to see that the anti-spherical light leaves indexed by subsequences $\underline{e}$ of $\underline{x}$ expressing w give a spanning set of $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}^{\not< w}}(\underline{x}, \underline{w})$. Following their proof, we deduce the analogous statement of [RW16, (4.8)] for morphisms ending in $\underline{w}$. To prove the claim using induction on the path dominance order, we need the following observation: There is a unique subsequence $\underline{e}$ of $\underline{x}$ expressing w whose decoration consists only of $U0$'s and $U1$'s (see [Pat17, Lemma 1.1.3]). It follows that $\underline{e}$ is minimal with respect to the path dominance order. If $\underline{e}$ does not avoid $W \setminus {}^fW$, then we can choose one of the braid moves in the construction of $\mathrm{LL}_{\underline{x},\underline{e}}$ in such a way that it factors through a reduced expression starting in a simple reflection in S_f and thus $\mathrm{LL}_{\underline{x},\underline{e}}$ vanishes. Note that any light leaf $\mathrm{LL}_{\underline{x},\underline{f}}$ for a subsequence $\underline{f}$ of $\underline{x}$ that is strictly smaller than $\underline{e}$ in the path dominance order or that expresses an element $< w$ in the Bruhat order vanishes in $\mathbf{N}^{\not< w}$.

By definition of the diagrammatic character the graded rank of $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}^{\not< w}}(\underline{x}, \underline{w})$ is the coefficient in front N_w in $\mathrm{ch}(\underline{x})$ and matches the degrees with multiplicities of the anti-spherical light leaves $\underline{x} \to \underline{w}$ (see Lemma 7.10). Therefore, the anti-spherical light leaves give a basis of the free R_{S_f}-module $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}^{\not< w}}(\underline{x}, \underline{w})$. By descending induction on the element w through which the anti-spherical double leaves factor, we see that the set given in the theorem gives a basis of $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}}(\underline{x}, \underline{y})$. □

7.4 The Geometric Satake and the p-Canonical Basis

In this section, we will explain the consequences of results about parity sheaves on the affine Grassmannian for the p-canonical basis.

A very interesting question is under which hypothesis the indecomposable parity sheaves on the affine Grassmannian are perverse. This is equivalent to the p-canonical basis being a $\mathbb{Z}$-linear combination of Kazhdan-Lusztig basis elements. We call an element of $\mathcal{H}$ *perverse* if it is a $\mathbb{Z}$-linear combination of Kazhdan-Lusztig basis elements.

To shorten our notation, write $\mathcal{K} := \mathbb{C}(\!(t)\!)$ and $\mathcal{O} := \mathbb{C}[\![t]\!]$. Define the affine Grassmannian $\mathcal{G}r_a$ to be the $\mathbb{Z}$-functor given by $R \mapsto LG(R)/L^+G(R)$. Its complex points coincide with $G(\mathcal{K})/G(\mathcal{O})$. For $\lambda \in X^\vee$ denote by $t^\lambda \in G(\mathcal{K})$ the image of $t \in \mathcal{K}$ under the following composition

$$\mathcal{K}^* = \mathbb{G}_m(\mathcal{K}) \xrightarrow{\lambda} T(\mathcal{K}) \hookrightarrow G(\mathcal{K}).$$

The Cartan decomposition (see [Zhu16, (2.1.2)])

$$\mathcal{G}r_a(\mathbb{C}) = \bigcup_{\lambda \in X^\vee_+} \underbrace{G(\mathcal{O})t^\lambda G(\mathcal{O})/G(\mathcal{O})}_{\mathcal{G}r_a^\lambda :=}$$

gives a stratification of $\mathcal{G}r_a(\mathbb{C})$ where each stratum $\mathcal{G}r_a^\lambda$ is a vector bundle over a partial flag variety $G(\mathbb{C})t^\lambda G(\mathcal{O})/G(\mathcal{O})$. Since the $G(\mathcal{O})$-orbits are all simply-connected, the indecomposable parity sheaves are all parametrized by $\lambda \in X^\vee_+$ (see [JMW14b, Theorem 4.6]):

Theorem 7.14. *Assume that the characteristic of k is not a torsion prime*[2] *for G. For each $\lambda \in X^\vee_+$ there exists up to isomorphism a unique indecomposable parity complex $\mathcal{E}(\lambda)$ such that* $\mathrm{supp}(\mathcal{E}(\lambda)) = \overline{\mathcal{G}r_a^\lambda}$ *and* $\mathcal{E}(\lambda)|_{\mathcal{G}r_a^\lambda} = \underline{k}_{\mathcal{G}r_a^\lambda}[\dim \mathcal{G}r_a^\lambda]$. *Every indecomposable parity complex is isomorphic to $\mathcal{E}(\lambda)$ for some $\lambda \in X^\vee_+$.*

Denote by $P_{G(\mathcal{O})}(\mathcal{G}r_a(\mathbb{C}), k)$ the $G(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r_a(\mathbb{C})$ with coefficients in k. It comes equipped with a monoidal structure induced by the convolution product $*$.

The Geometric Satake equivalence (see [MV07]) gives a monoidal equivalence

$$(P_{G(\mathcal{O})}(\mathcal{G}r_a(\mathbb{C}), k), *) \xrightarrow{\cong} (\mathrm{Rep}(G^\vee), \otimes).$$

In [JMW14a] Juteau, Mautner and Williamson show that if the characteristic p of k is larger than an explicit bound depending on the root system Φ of G, then the indecomposable k-parity sheaves on the affine Grassmannian are perverse. More precisely, they show that for $\lambda \in X^\vee_+$ the indecomposable tilting module $T(\lambda)$ is mapped to $\mathcal{E}(\lambda)$ under the geometric Satake equivalence for $p > b(\Phi)$ (see below for the definition of $b(\Phi)$).

[2] See [JMW14b, §2.6] for the definition of torsion primes. This restriction can be removed by working in the non-equivariant setting.

Recall that $G^\vee$ is assumed to be simply-connected. The extended affine Weyl group can be written as

$$W^{\mathrm{ext}} = W \rtimes \Omega$$

where Ω is a finite subgroup of "length zero elements" which acts by automorphisms of the Coxeter system (W, S) (see Section 6.1 for more details). The action of W^{ext} on $X^\vee$ gives bijections:

$$\begin{array}{rcl} X^\vee & \overset{\sim}{\longleftrightarrow} & W^{\mathrm{ext}}/W_{\mathrm{f}}, \\ w(0) & \longleftrightarrow & w, \\ X^\vee_+ & \overset{\sim}{\longleftrightarrow} & W_{\mathrm{f}}\backslash W^{\mathrm{ext}}/W_{\mathrm{f}} = \bigcup_{\sigma\in\Omega} W_{\mathrm{f}}\backslash W/\sigma(W_{\mathrm{f}}). \end{array}$$

Given $\lambda \in X^\vee_+$ we denote by $w_\lambda \in W$ the maximal element in the double coset in $\bigcup_{\sigma\in\Omega} W_{\mathrm{f}}\backslash W/\sigma(W_{\mathrm{f}})$ that corresponds to λ under the bijection above.

We call an element in W *spherical* if it is maximal in a double coset in $\sigma_1(W_{\mathrm{f}})\backslash W/\sigma_2(W_{\mathrm{f}})$ for some $\sigma_1, \sigma_2 \in \Omega$. Note that the spherical elements are precisely the elements in W of the form $\sigma(w_\lambda)$ for some $\lambda \in X^\vee_+$ and $\sigma \in \Omega$. We call a p-cell in W spherical if it contains a spherical element.

The Iwahori-Bruhat decomposition (see [Gör10, Theorem 2.18] or [Zhu16, Remark 2.1.22]) of the complex points of the affine flag variety is of the form

$$\mathcal{F}l_a(\mathbb{C}) = \bigcup_{x\in W^{\mathrm{ext}}} I(\mathbb{C})xI(\mathbb{C})/I(\mathbb{C}).$$

In this case, each stratum $I(\mathbb{C})xI(\mathbb{C})/I(\mathbb{C})$ for $x \in W^{\mathrm{ext}}$ is isomorphic to an affine space of dimension $l(x)$. Note that $I(\mathbb{C})$-orbits give a refinement of the $G(\mathcal{O})$-orbits on $\mathcal{F}l_a(\mathbb{C})$. It follows that the projection $\mathcal{F}l_a(\mathbb{C}) \to \mathcal{G}r_a(\mathbb{C})$ is a stratified fibre bundle (in the sense of [JMW14b, Definition 2.32]) with fibres isomorphic to the finite flag variety $\mathcal{F}l(\mathbb{C})$. This implies the following result (see also the next section for some more details):

Theorem 7.15. *Suppose that all parity sheaves on the affine Grassmannian are perverse. Then* ${}^p m_{w_\lambda, w_\nu} \in \mathbb{Z}$ *for all* $\lambda, \nu \in X^\vee_+$.

The explicit bound has later been improved by Mautner and Riche in [MR15]. They show that the parity sheaves on the affine Grassmannian are perverse whenever the characteristic p of k is good for G, thus giving the following bounds $p > b(\Phi)$ if the root system Φ of G is irreducible:

Type of Φ	A_n	B_n, C_n, D_n	E_6, E_7, F_4, G_2	E_8
$b(\Phi)$	1	2	3	5

The following result gives an interpretation of the multiplicities in the p-canonical basis in terms of $\mathrm{Rep}(G^\vee)$ (see [JMW14a, Corollary 4.1] for the first part):

Lemma 7.16. *Suppose that the characteristic* p *of* k *satisfies* $p > b(\Phi)$*. Then we have for* $\lambda, \mu \in X^\vee_+$*:*

(i) ${}^p h_{w_\mu, w_\lambda}(1) = \dim T(\lambda)_\mu$,

(ii) ${}^p m_{w_\mu, w_\lambda} = [T(\lambda) : \Delta(\mu)] = [T(\lambda) : \nabla(\mu)]$ *where* $[T(\lambda) : \Delta(\mu)]$ *denotes the multiplicity of* $\Delta(\mu)$ *in a* Δ*-flag on* $T(\lambda)$*.*

The first part of the last result shows that ${}^p h_{w_\mu, w_\lambda}$ gives a q-analogue of the weight multiplicity of μ in $T(\lambda)$. In characteristic 0 (where $T(\lambda) = L(\lambda)$) this result can be found in [Lus83a, Theorem 6.1]. In particular, knowledge of the characters of indecomposable tilting modules for $G^\vee$ is equivalent to the knowledge of the p-canonical basis elements $\{{}^p \underline{H}_{w_\lambda} \mid \lambda \in X_+^\vee\}$.

7.5 The Geometric Satake and p-Cells

Now, in order to use the geometric Satake equivalence for the study of p-cells we shift our perspective slightly. The first step is to upgrade our setting to a 2-categorical level.

On the geometric side, we want to consider the 2-category $\mathfrak{Parity}(G(\mathcal{K}), k)$ whose objects are subsets J of S_{f}. For each finitary subset $J \subset S_{\mathrm{f}}$ we have a parabolic subgroup $B \subset P_J \subseteq G$ and a corresponding parahoric subgroup $\widetilde{P}_J$ which is the inverse image of P_J under the morphism $L^+G \to G$ induced by $t \mapsto 0$. We define homomorphisms from $J \to K$ in $\mathfrak{Parity}(G(\mathcal{K}), k)$ to be the full subcategory $\mathbf{Parity}_{\widetilde{P}_J(\mathbb{C}) \times \widetilde{P}_K(\mathbb{C})}(G(\mathcal{K}), k)$ in $D^b_{\widetilde{P}_J(\mathbb{C}) \times \widetilde{P}_K(\mathbb{C})}(G(\mathcal{K}), k)$ of equivariant parity sheaves on $G(\mathcal{K})$ with coefficients in k. For three finitary subsets $J, K, L \subseteq S_{\mathrm{f}}$, one can define an associative convolution product

$$D^b_{\widetilde{P}_J(\mathbb{C}) \times \widetilde{P}_K(\mathbb{C})}(G(\mathcal{K}), k) \times D^b_{\widetilde{P}_K(\mathbb{C}) \times \widetilde{P}_L(\mathbb{C})}(G(\mathcal{K}), k) \longrightarrow D^b_{\widetilde{P}_J(\mathbb{C}) \times \widetilde{P}_L(\mathbb{C})}(G(\mathcal{K}), k)$$

denoted by $*_J$ as in [Lus97, §1.1] or [Nad05, §2.2 and §3.3]. Some care needs to be taken as in the process sheaves with infinite dimensional support may occur. The convolution of two parity sheaves will again be a parity sheaf by [JMW14b, Theorem 4.8].

Alternatively, one could use the category of diagrammatic singular Soergel bimodules ${}^k\mathbf{SH}$ as we will do in Section 10, but unfortunately a presentation of this category by generators and relations is only available in rank 2 (see [Eli16; Wil11]).

The category $\mathbf{Parity}_{I(\mathbb{C}) \times I(\mathbb{C})}(G(\mathcal{K}), k)$ gives by [RW16, Part 3] a categorification of the extended affine Hecke algebra $\mathcal{H}_{\mathrm{ext}}$. Therefore, replacing $\mathbf{Parity}_{I(\mathbb{C}) \times I(\mathbb{C})}(G(\mathcal{K}), k)$ by $\mathfrak{Parity}(G(\mathcal{K}), k)$ amounts on the level of Grothendieck groups to replacing $\mathcal{H}_{\mathrm{ext}}$ by the extended affine Hecke algebroid $\mathfrak{H}_{\mathrm{ext}}$. We want to relate it to the affine Hecke algebroid. Let us recall its definition from [Wil11, §2.2] (where it is also called Hecke category; for the extended affine Hecke algebroid replace S by S_{f} and $\mathcal{H}$ by $\mathcal{H}_{\mathrm{ext}}$ in what follows). Denote the longest element in $\langle J \rangle$ by w_J. The affine *Hecke algebroid* $\mathfrak{H} = \mathfrak{H}_{(W,S)}$ is a $\mathbb{Z}[v, v^{-1}]$-linear category with finitary subsets $J \subseteq S$ as objects (as defined in Definition 4.8). For finitary subsets $J, K \subseteq S$ the morphism space $\mathrm{Hom}_{\mathfrak{H}}(J, K)$ is the $\mathbb{Z}[v, v^{-1}]$-module given by the intersection of the left ideal $\underline{H}_{w_J}\mathcal{H}$ with the right ideal $\mathcal{H}\underline{H}_{w_K}$ in $\mathcal{H}$. The composition of morphisms $*_K : \mathrm{Hom}_{\mathfrak{H}}(K, L) \times \mathrm{Hom}_{\mathfrak{H}}(J, K) \to \mathrm{Hom}_{\mathfrak{H}}(J, L)$ for finitary subsets $J, K, L \subseteq S$ is defined via renormalized multiplication:

$$y *_K x = \frac{xy}{\pi_K}$$

where $\pi_K = \sum_{w \in W_K} v^{l(w_K)-2l(w)}$ is the *Poincaré polynomial* of K. This is well-defined due to $\underline{H}_{w_K}^2 = \pi_K \underline{H}_{w_K}$ (see [Wil11, (2.1.6)]). Observe that the affine Hecke algebra $\mathcal{H}$ occurs as the endomorphism space of $\varnothing \subset S$ in $\mathfrak{H}$. Moreover, note that the definition of the Hecke algebroid is independent of the characteristic p of k, as the longest element of any finite parabolic is smooth and thus the Kazhdan-Lusztig and p-canonical basis element indexed by it coincide.

Example 7.17. The affine Hecke algebroid in affine type $\widetilde{A_2}$ can be described as follows (see [Eli16, Proposition 2.20]):

$$\begin{tikzcd}
& \emptyset \arrow[dl, shift right] \arrow[d, shift right] \arrow[dr, shift left] & \\
\{s\} \arrow[ur, shift right] \arrow[d, shift right] \arrow[dr, shift left] & \{t\} \arrow[u, shift right] \arrow[dl, shift left] \arrow[dr, shift right] & \{u\} \arrow[ul, shift left] \arrow[dl, shift right] \arrow[d, shift right] \\
\{s,t\} \arrow[u, shift right] \arrow[ur, shift left] & \{s,u\} \arrow[ul, shift left] \arrow[ur, shift right] & \{t,u\} \arrow[ul, shift right] \arrow[u, shift right]
\end{tikzcd}$$

modulo the following relations for any pair $\{q,r\} \in \{\{s,t\},\{s,u\},\{t,u\}\}$:

- $$\begin{tikzcd} & \emptyset \arrow[dl] \\ \{q\} \arrow[dr] & \\ & \{q,r\} \end{tikzcd} = \begin{tikzcd} \emptyset \arrow[dr] & \\ & \{r\} \arrow[dl] \\ \{q,r\} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \emptyset \\ \{q\} \arrow[ur] & \\ & \{q,r\} \arrow[ul] \end{tikzcd} = \begin{tikzcd} \emptyset & \\ & \{r\} \arrow[ul] \\ \{q,r\} \arrow[ur] & \end{tikzcd}$$

- $$\begin{tikzcd} \emptyset \arrow[d, bend right] \\ \{r\} \arrow[u, bend right, dash] \end{tikzcd} = (v+v^{-1})\,\mathrm{Id}_{\{r\}} \quad \text{and} \quad \begin{tikzcd} \{r\} \arrow[d, bend right] \\ \{q,r\} \arrow[u, bend right, dash] \end{tikzcd} = (v^2+1+v^{-2})\,\mathrm{Id}_{\{q,t\}}$$

- $$\begin{tikzcd} \{r\} \arrow[d, bend right, dash] \\ \{q,r\} \arrow[u, bend right] \end{tikzcd} = \begin{tikzcd} & \emptyset \arrow[dl, bend right] \arrow[dr, bend left, dash] & \\ \{r\} \arrow[ur, bend right, dash] & & \{q\} \arrow[ul, bend left, dash] \end{tikzcd} - \mathrm{Id}_{\{r\}}$$

One also has to add the relations obtained by swapping the roles of q and r.

In order to relate the geometric Satake equivalence to the affine Hecke algebroid, we follow the philosophy given in [Eli17, §6] and reformulate it using the passage from Ω-graded-monoidal categories to Ω-2-categories (see [Eli17, §4.2] for the definitions).

Recall that we have $\Omega \cong X^\vee/\mathbb{Z}\Phi^\vee$ as $G^\vee$ is assumed to be simply-connected. On the representation theoretic side, denote by $\mathrm{Tilt}(G^\vee)^\sigma$ for $\sigma \in \Omega$ the category of all direct sums of indecomposable tilting modules $T(\lambda)$ with $\lambda \in \sigma$. The tensor product then obeys multiplication in Ω as for $\sigma_1, \sigma_2 \in \Omega$ and $\lambda_i \in \sigma_i$ we have that $T(\lambda_1) \otimes T(\lambda_2)$ lies in $\mathrm{Tilt}(G^\vee)^{\sigma_1\sigma_2}$. We view this Ω-graded-monoidal as a k-linear Ω-2-category as follows: The objects are given by Ω and the morphism category from σ to $\sigma\varsigma$ coincides with $\mathrm{Tilt}(G^\vee)^\varsigma$ for $\sigma, \varsigma \in \Omega$. Denote this 2-category by $\mathrm{Tilt}(G^\vee)^\Omega$.

On the geometric side, consider the full 2-category $\mathfrak{Parity}_{\mathrm{msph}}(G(\mathcal{K}), k)$ of $\mathfrak{Parity}(G(\mathcal{K}), k)$ on subsets $\sigma(S_\mathrm{f})$ for $\sigma \in \Omega$ where msph stands for *maximally spherical.* Recall that $\pi_0(LG(\mathbb{C})) = \pi_1(G(\mathbb{C}))$ coincides with Ω as G is assumed to be adjoint. Any indecomposable parity sheaf is supported on a single component. Moreover, indecomposable parity sheaves on different components admit neither morphisms nor extensions. Convolution of parity sheaves corresponds on the level of component support to multiplication in Ω. For this reason,

$$\mathbf{Parity}_{G(\mathcal{O})\times G(\mathcal{O})}(G(\mathcal{K}), k) \cong \mathbf{Parity}_{G(\mathcal{O})}(\mathcal{G}r_a(\mathbb{C}), k)$$

is a Ω-graded-monoidal category. When viewed as a Ω-2-category it gives $\mathfrak{Parity}_{\mathrm{msph}}(G(\mathcal{K}), k)$. (The last equivalence is a special case of the quotient equivalence for equivariant derived categories from [BL94, §2.6.2]).

Now the following reformulation of the Geometric Satake equivalence shows that it governs in the affine Hecke algebroid the morphisms $I \to J$ for finitary subsets $I, J \subseteq S$ such that $\langle I \rangle \cong W_\mathrm{f} \cong \langle J \rangle$:

Corollary 7.18. *There is an equivalence of k-linear Ω-2-categories*

$$\mathfrak{Parity}_{msph}(G(\mathcal{K}), k) \longrightarrow \mathrm{Tilt}(G^\vee)^\Omega$$

mapping $\sigma(S_f)$ to $\sigma \in \Omega$ and for $\lambda \in \sigma$ the indecomposable parity sheaf $\mathcal{E}(\lambda)$ to the indecomposable tilting module $T(\lambda)$ on the level of homomorphisms.

Finally, we want to relate the convolution of $G(\mathcal{O})$-equivariant parity sheaves on $\mathcal{G}r_a(\mathbb{C})$ to the convolution of $I(\mathbb{C})$-equivariant parity sheaves on $\mathcal{F}l_a(\mathbb{C})$. The projection $\pi : \mathcal{F}l_a(\mathbb{C}) \longrightarrow \mathcal{G}r_a(\mathbb{C})$ is a smooth fibration with fibre $\mathcal{F}l(\mathbb{C})$. Consider the following functor:

$$\mathbf{Parity}_{G(\mathcal{O})}(\mathcal{G}r_a(\mathbb{C}), k) \xrightarrow{\pi^*} \mathbf{Parity}_{G(\mathcal{O})}(\mathcal{F}l_a(\mathbb{C}), k) \xrightarrow{\mathrm{Res}} \mathbf{Parity}_{I(\mathbb{C})}(\mathcal{F}l_a(\mathbb{C}), k)$$

where Res is the functor that restricts equivariance from $G(\mathcal{O})$ to the closed subgroup $I(\mathbb{C})$ (see [BL94, §2.6.1]). The whole composition preserves indecomposable parity sheaves (up to shift). This is the reason for Theorem 7.15. For π^* one argues as in [WB12, Proposition 3.5]. Using equivariant formality (see [MR15, Lemma 2.2 (2)]) we see that an equivariant parity sheaf is indecomposable if and only if it is indecomposable after forgetting equivariance (see [MR15, Lemma 2.4] for the non-trivial direction). This implies that Res also preserves indecomposable parity sheaves.

On the decategorified level, this corresponds to:

$$\begin{tikzcd}
\mathrm{End}_{\mathfrak{H}_\mathrm{ext}}(S_\mathrm{f}, S_\mathrm{f}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathfrak{H}_\mathrm{ext}}(S_\mathrm{f}, \varnothing) \arrow[r] \arrow[d, equal] & \mathrm{End}_{\mathfrak{H}_\mathrm{ext}}(\varnothing) \arrow[d, equal] \\
\mathcal{H}_\mathrm{ext}\underline{H}_{w_0} \cap \underline{H}_{w_0}\mathcal{H}_\mathrm{ext} \arrow[r, hook] & \underline{H}_{w_0}\mathcal{H}_\mathrm{ext} \arrow[r, hook] & \mathcal{H}_\mathrm{ext}
\end{tikzcd}$$

Lemma 7.19. *For $\lambda, \mu \in X_+^\vee$ we have:*

$$\lambda \overset{p}{\underset{\mathcal{T}}{\leqslant}} \mu \Rightarrow w_\lambda \overset{p}{\underset{R}{\leqslant}} w_\mu$$

Proof. For $\lambda \overset{p}{\underset{\mathcal{T}}{\leqslant}} \mu$ we have a tilting module T for $G^\vee$ such that $T(\lambda)$ is a direct summand in $T(\mu) \otimes T$. Under the geometric Satake equivalence, this translates into the existence of a $G(\mathcal{O})$-equivariant parity sheaf $\mathcal{E}$ such that $\mathcal{E}(\lambda)$ is a direct summand in $\mathcal{E}(\mu) * \mathcal{E}$. Now, the statement follows as the convolution of $G(\mathcal{O})$-equivariant parity sheaves on $\mathcal{G}r_a(\mathbb{C})$ determines the convolution of their pullbacks under π on $\mathcal{F}l_a(\mathbb{C})$. On the decategorified level, this reduces to the following calculation for $h\underline{H}_{w_0}, \underline{H}_{w_0}\widetilde{h} \in \mathcal{H}_{\text{ext}}\underline{H}_{w_0} \cap \underline{H}_{w_0}\mathcal{H}_{\text{ext}}$:

$$\operatorname{incl}(h\underline{H}_{w_0}) *_{\varnothing} \operatorname{incl}(\underline{H}_{w_0}\widetilde{h}) = \pi_{S_{\mathrm{f}}} \operatorname{incl}(h\underline{H}_{w_0}\widetilde{h}) = \pi_{S_{\mathrm{f}}} \operatorname{incl}(h\underline{H}_{w_0} *_{S_{\mathrm{f}}} \underline{H}_{w_0}\widetilde{h})$$

In particular, for every summand $\mathcal{E}(\lambda)$ of the convolution $\mathcal{E}(\mu) * \mathcal{E}$ on $\mathcal{G}r_a$, its pullback $\pi^*\mathcal{E}(\lambda)$ occurs as well as summand in the convolution $\pi^*\mathcal{E}(\mu) * \pi^*\mathcal{E}$ on $\mathcal{F}l_a(\mathbb{C})$ (even with bigger multiplicity!). □

7.6 Perversity of the p-Canonical Basis in Spherical Cells

As was shown in [JMW14b] and [MR15], the parity sheaves on the affine Grassmannian $\mathcal{G}r_a$ are perverse for p good for G. Thus the p-canonical basis for spherical elements is perverse in these cases.

Lemma 7.20. *Let $x \in W$ be spherical. Choose $s, t \in S$ such that $xs > x$ and $tx > x$. Then for $y \in \{xs, tx, txs\}$ the p-canonical basis element ${}^p\underline{H}_y$ is perverse as well.*

Proof. This follows immediately from Proposition 3.10(iii) and (v) and the multiplication formula in the Kazhdan-Lusztig basis using the fact that there is no element having all elements of S in its left or right descent set. □

In type $\widetilde{A}_2$ the last lemma gives us:

Corollary 7.21. *In type $\widetilde{A}_2$, the p-canonical basis elements indexed by an element in the lowest Kazhdan-Lusztig two-sided cell are perverse.*

8 Open Problems in Modular Representation Theory

The following open questions in modular representation theory have spawned many interesting theories in mathematics. The five questions we will be concerned with are the following: Determine for $n \geqslant m \in \mathbb{N}$ fixed ...

(i) ... the characters of the irreducible modules for the symmetric group S_m in non-semi-simple characteristic $p \leqslant m$.

(ii) ... the dimensions of irreducible S_m-modules for the symmetric group S_m in non-semi-simple characteristic $p \leqslant m$.

(iii) ... the characters of irreducible modules for the finite group of Lie type $SL_n(F_{p^r})$ for all $r, n \in \mathbb{N}$ in defining characteristic p.

(iv) ... the characters of irreducible modules for the algebraic group GL_n or SL_n for all $n \in \mathbb{N}$ in characteristic p.

(v) ... the characters of tilting modules for the algebraic group GL_n or SL_n for all $n \in \mathbb{N}$ in characteristic p.

Most representation theorists have already thought about one of these questions or their characteristic 0-analogues. In this section, we want to explain how the p-canonical basis of the affine Hecke algebra $\mathcal{H}$ in type $\widetilde{A}_{n+1}$ gives an answer to all of these questions via the connections explained in Section 7. Before we mention some implications among these problems, let us assume $G^\vee \in \{GL_n, SL_n\}$ with the standard choice of Borel subgroup and maximal torus to fix our input:

$$G^\vee \supseteq B^\vee := \begin{pmatrix} * & \dots & \dots & * \\ 0 & \ddots & * & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} \supseteq T^\vee := \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & * \end{pmatrix}$$

It should be noted that we only restrict to type $\widetilde{A}_{n+1}$ because this is the best documented case. All the arguments given below should carry over to other types as well.

(v) $\Rightarrow$ (iv) The Steinberg tensor product theorem reduces the task of determining the characters of all irreducible modules for SL_n to those with p-restricted highest weight. Thus by Proposition 6.13, the characters of the indecomposable tilting modules with highest weight in $2(p-1)\rho + w_0 X_p^\vee$ give for $p > 2h - 2$ all irreducible characters.

(iv) $\Rightarrow$ (iii) The Steinberg restriction theorem (see [Ste63]) shows that when restricting the irreducible SL_n-modules with p^r-restricted highest weights to $SL_n(\mathbb{F}_{p^r})$ one obtains a complete set of representatives of the isomorphism classes of simple $SL_n(\mathbb{F}_{p^r})$-modules in defining characteristic (i.e. in characteristic $p = \operatorname{char} k$).

(v) $\Rightarrow$ (ii) Let V be the natural representation for GL_n. Observe that V is for $p \geqslant h$ the indecomposable tilting module with highest weight given by a fundamental coweight. Therefore for all $m \in \mathbb{N}$ the tensor power $V^{\otimes m}$ can be decomposed into indecomposable tilting modules. By modular Schur-Weyl duality (see [KSX01]) we have for $n = \dim V \geqslant m$:

$$kS_m \cong \operatorname{End}_{GL_n}(V^{\otimes m})$$

Write $V^{\otimes m} = \bigoplus_{\lambda \in X_+} T(\lambda)^{\oplus d_\lambda^m}$. From the isomorphism above, it follows that $kS_m / \operatorname{rad}(kS_m)$ is a product of matrix rings

$$\prod_{\substack{\lambda \in X_+ \\ d_\lambda^m \neq 0}} M_{d_\lambda^m}(k).$$

Therefore $T(\lambda)$ corresponds to an irreducible S_m-module of dimension d_λ^m. A more detailed analysis shows that d_λ^m gives the dimension of the irreducible S_m-module indexed by λ for λ p-regular and vanishes otherwise (see [Erd94, Proposition 4.2]).

(v) $\Rightarrow$ (i) The multiplicities of Weyl modules in tilting modules for GL_n give the decomposition numbers of the symmetric group S_m for $n \geqslant m$ (see [Erd94, Lemma 4.5]).

In order to give some more details, we need to recall some facts about the Schur algebra $S_k(n, m)$ and quasi-hereditary algebras in general. The Schur algebra $S_k(n, m)$ is isomorphic to $\operatorname{End}_{kS_m}(V^{\otimes m})$ and the category of $S_k(n, m)$-modules is equivalent to the category of polynomial representations of GL_n that are homogeneous of degree m (see [Gre07, §2.4]). Denote by $\Lambda^+(n, m)$ the set of all partitions of m with at most n parts (which is the set of all partitions of m as soon as $n \geqslant m$). Parshall shows in [Par89, Theorem 4.1] that $S_k(n, m)$ is a quasi-hereditary algebra with simple, standard, costandard modules denoted by $L_S(\lambda)$, $\Delta_S(\lambda)$ and $\nabla_S(\lambda)$ respectively for $\lambda \in \Lambda^+(n, m)$ where $\Lambda^+(n, m)$ is equipped with the usual dominance order.

We set

$$T := V^{\otimes m} \oplus \bigoplus_{\substack{\lambda \in \Lambda^+(n,m) \\ \text{not } p\text{-regular}}} T(\lambda).$$

From [Erd94, Proposition 4.2] it follows that every indecomposable tilting module occurs at least once in T. Thus, $S' := \operatorname{End}_S(T)$ is Morita-equivalent to the Ringel dual of S (as introduced in [Rin92, §6]) due to the strong Ext-vanishing that tilting modules satisfy. (This Ext-vanishing is a consequence of Proposition 6.5.) Denote by $I_S(\lambda)$ the injective hull of $L_S(\lambda)$ for $\lambda \in \Lambda^+(n, m)$ and by $\mathcal{F}_S(\Delta)$ (resp. $\mathcal{F}_S(\nabla)$) the full subcategory of S-modules with a Δ- (resp. ∇-)filtration. Ringel shows in [Rin92, Theorem 6] that the Ringel dual of S and thus S' is again quasi-hereditary with standard modules $\Delta_{S'}(\lambda) = \operatorname{Hom}_S(T, \nabla_S(\lambda))$ for $\lambda \in \Lambda^+(n, m)$ where the partial order on $\Lambda^+(n, m)$ is reversed. Moreover, the functor $\operatorname{Hom}_S(T, -)$ from S-modules to S'-modules induces an equivalence $\mathcal{F}_S(\nabla) \xrightarrow{\sim} \mathcal{F}_{S'}(\Delta)$ mapping $I_S(\lambda)$ to $T_{S'}(\lambda)$.

Therefore, we get the following equalities for $\lambda, \mu \in \Lambda^+(n,m)$ with λ p-regular:

$$\begin{aligned}[T(\lambda) : \Delta(\mu)] &= [T_S(\lambda) : \Delta_S(\mu)] \\ &= [I_{S'}(\lambda) : \nabla_{S'}(\mu)] \\ &= [\Delta_{S'}(\mu) : L_{S'}(\lambda)]\end{aligned}$$

where the second equality comes from the equivalence $\mathcal{F}_{S'}(\nabla) \xrightarrow{\sim} \mathcal{F}_S(\Delta)$ after identifying S with its double Ringel dual S'' and the last one from "BGG-reciprocity" (see [CPS88, Theorem 3.11]).

Finally, Erdmann considers the idempotent $e \in S$ corresponding to the direct summand $V^{\otimes m} \overset{\oplus}{\subseteq} T$ and observes that the canonical morphism $\rho : kS_m \to \mathrm{End}_S(V^{\otimes m})$ induces an isomorphism $kS_m/\ker(\rho) \xrightarrow{\sim} eS'e = \mathrm{End}_S(V^{\otimes m})$ which identifies for $\gamma \in \Lambda^+(n,m)$ the Specht module S^γ with $e\Delta_{S'}(\gamma)$ and for γ p-regular the simple module D^γ with $eL_{S'}(\gamma)$ (see [Erd94, Proposition 4.3]). As λ is assumed to be p-regular, we have that $eL_{S'}(\lambda)$ is non-zero. In particular, $[- : L_{S'}(\lambda)]$ equals $[e(-) : eL_{S'}(\lambda)]$ because the functor $\mathrm{Hom}_{S'}(Ae, -)$ is exact and kills precisely those simple S'-modules $L_{S'}(\gamma)$ whose projective covers do not occur as a direct summand of Ae. Thus, we can conclude:

$$\begin{aligned}[\Delta_{S'}(\mu) : L_{S'}(\lambda)] &= [e\Delta_{S'}(\mu) : eL_{S'}(\lambda)] \\ &= [S^\mu : D^\lambda] = d_{\mu,\lambda}\end{aligned}$$

9 Type $\widetilde{A}_1$

To fix the based root datum as input, we follow the conventions of Section 3.4 and work in the affine setting. We label the simple reflections in S as follows:

$$\textcircled{s} \overset{\infty}{\text{———}} \textcircled{t}$$

That means that the pairing of simple roots and coroots is given as follows:

$$\begin{aligned}\langle \alpha_s^\vee, \alpha_t \rangle &= -2,\\ \langle \alpha_t^\vee, \alpha_s \rangle &= -2.\end{aligned}$$

9.1 The p-Canonical Basis in $\widetilde{A}_1$

As we discussed in Section 7.4, knowledge of the characters of the tilting modules for SL_2 in characteristic p gives us part of the p-canonical basis for the Weyl group of type $\widetilde{A}_1$. Actually in this case two miracles occur:

(i) Donkin's tilting tensor product theorem ([Don93, Proposition 2.1]) together with the knowledge of the characters of the fundamental tilting modules (e.g. see [DH05, Lemma 1.1]) allow us to determine the characters of all indecomposable tilting modules. (This is the only semi-simple group for which all tilting characters are known).

(ii) The set $\{w_\lambda \mid \lambda \in X_+^\vee\}$ together with its image under the automorphism $s \leftrightarrow t$ yields the set $W \setminus \{\mathrm{Id}\}$. In particular, knowledge of the tilting characters gives the whole p-canonical basis for SL_2.

To bypass the calculations, the following result (see [EH02, Lemma 6]) is useful because it gives a combinatorial description of the ∇-multiplicities of an indecomposable tilting module.

Lemma 9.1. *Let $n \in \mathbb{Z}_{\geqslant 0}$. Write n uniquely as $\sum_{i=0}^{l} n_i p^i$ with $p-1 \leqslant n_i \leqslant 2p-2$ for $i < l$ and $0 \leqslant n_l \leqslant p-1$.*

Then $\nabla(m)$ occurs in a ∇-flag of $T(n)$ if and only if $m = \sum_{i=0}^{l} m_i p^i$ where $m_j = n_j$ or $m_j = 2p-2-n_j$ for $j < l$ and $m_l = n_l$. Moreover, the multiplicity $[T(n) : \nabla(m)]$ is at most one.

Observe that in the last lemma for $m = \sum_{i=0}^{l} m_i p^i$ the coefficients are not required to satisfy $p-1 \leqslant m_i \leqslant 2p-2$ for $i < m$. For $n = \sum_{i=0}^{l} n_i p^i$ as in the lemma there are precisely 2^r natural numbers m such that $\nabla(m)$ occurs in a ∇-flag of $T(n)$ where $r = |\{0 \leqslant i < l \mid n_i \neq p-1\}|$.

To give an example how this works, consider $n = 15$ and $p = 3$. This is uniquely written as $15 = 3 \cdot 3^0 + 4 \cdot 3 + 0 \cdot 3^2$ (as described in the last lemma; the last 0 matters!) and thus 4 standard modules occur (with multiplicity one) in a ∇-flag of $T(15)$, namely $\nabla(m)$ for m among the following:

$$\begin{aligned}1 &= 1 \cdot 3^0 + 0 \cdot 3^1 + 0 \cdot 3^2,\\ 3 &= 3 \cdot 3^0 + 0 \cdot 3^1 + 0 \cdot 3^2,\\ 13 &= 1 \cdot 3^0 + 4 \cdot 3^1 + 0 \cdot 3^2,\\ 15 &= 3 \cdot 3^0 + 4 \cdot 3^1 + 0 \cdot 3^2.\end{aligned}$$

See Figure 1 for the multiplicities of $\nabla(m)$ in $T(n)$ for $p = 3$ where each black box represents a one. Using Lemma 7.16 we get for example:

$$\begin{aligned}
{}^3\underline{H}_s &= \underline{H}_s \\
{}^3\underline{H}_{st} &= \underline{H}_{st} \\
{}^3\underline{H}_{sts} &= \underline{H}_{sts} \\
{}^3\underline{H}_{stst} &= \underline{H}_{st} + \underline{H}_{stst} \\
{}^3\underline{H}_{ststs} &= \underline{H}_s + \underline{H}_{ststs} \\
{}^3\underline{H}_{ststst} &= \underline{H}_{ststst} \\
{}^3\underline{H}_{stststs} &= \underline{H}_{ststs} + \underline{H}_{stststs} \\
{}^3\underline{H}_{stststst} &= \underline{H}_{stst} + \underline{H}_{stststst}
\end{aligned}$$

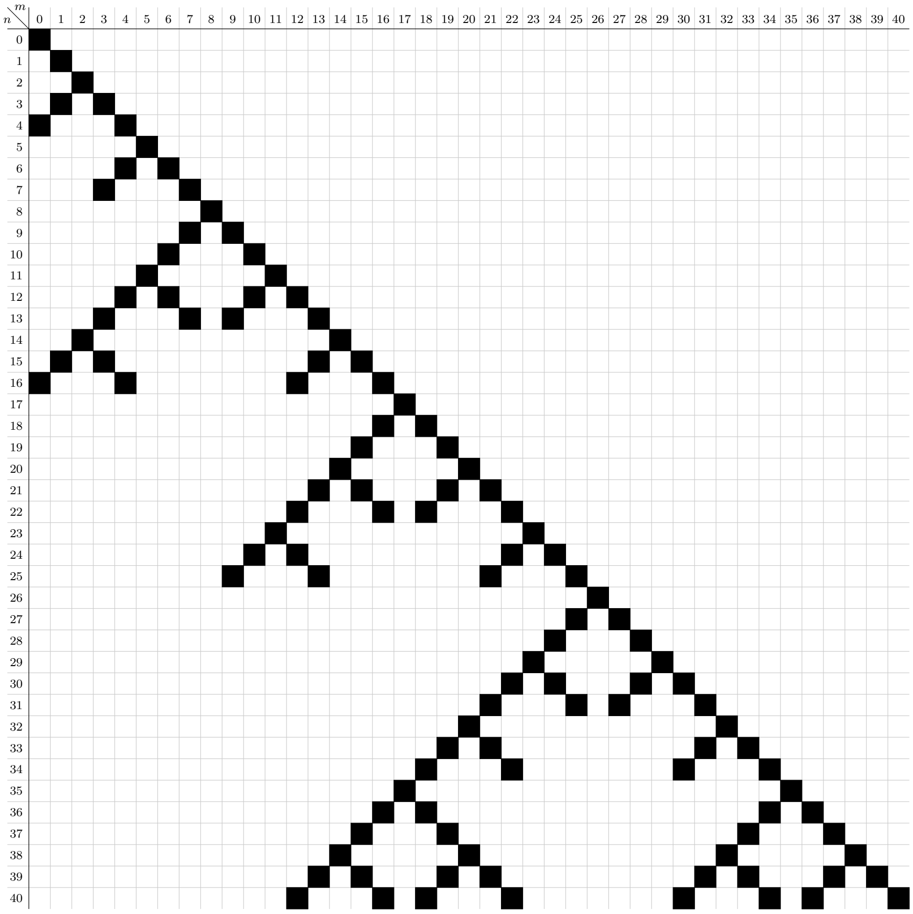


Figure 1: The multiplicities of $\nabla(m)$ in $T(n)$ for $p = 3$.

Note that even in this relatively simple example one sees a beautiful fractal-like structure emerging!

9.2 Cell Structure in $\widetilde{A}_1$

For the infinite dihedral group $(W, \{s,t\})$ the Kazhdan-Lusztig cells look as follows:

Lemma 9.2. (i) *The decomposition of W into Kazhdan-Lusztig right cells is given by $W = \{e\} \cup {}_sC \cup {}_tC$ where ${}_rC := \{w \in W \mid rw < w\}$ for $r \in \{s,t\}$.*

(ii) *There are precisely two Kazhdan-Lusztig two-sided cells:*

$$W = \{e\} \cup (W \setminus \{e\})$$

Proof. Observe that any element of W admits a unique reduced expression. Therefore the statements follow from [Lus83b, Proposition 3.8 (b) and (c)]. □

Recall the definition of strings in Definition 5.1 and observe that ${}_rC$ for $r \in \{s,t\}$ is a right string with respect to $\{s,t\}$. Let ${}_s\widehat{l}$ denote the alternating word in s and t starting in s of length l. The following result describes explicitly the p-cells in this case:

Proposition 9.3. *For $r \in \{s,t\}$ the decomposition of ${}_rC$ into right p-cells is given by*

$$ {}_rC = \bigcup_{l \in \mathbb{Z}_{\geqslant 0}} {}_rC_l$$

where ${}_rC_l = \{{}_r\widehat{i} \mid p^l \leqslant i < p^{l+1}\}$ contains $(p-1)p^l$ elements. The lowest Kazhdan-Lusztig two-sided cell decomposes into two-sided p-cells as follows:

$$W \setminus \{e\} = \bigcup_{l \in \mathbb{N}} C_l$$

where $C_l = {}_sC_l \cup {}_tC_l$ contains $2(p-1)p^l$ elements.

Proof. Recall that the set $\{w_\lambda \mid \lambda \in X_+^\vee\}$ together with its image under the automorphism $s \leftrightarrow t$ yields the set $W \setminus \{\mathrm{Id}\}$. Therefore, the right p-cells are completely determined by Corollary 4.5 and the modular weight cells via the geometric Satake equivalence. Moreover, Proposition 6.36 gives all modular weight cells for SL_2. Finally, observe that each piece ${}_rC_l$ corresponds under Geometric Satake to the modular weight cell $\underline{c}_1^l$. The modular weight cell $\underline{c}_1^l$ contains precisely $(p-1)p^l$ elements and can be described as

$$\{n \in \mathbb{N} \mid p^l - 1 \leqslant n < p^{l+1} - 1\} \subset X_+$$

under the usual identifications $X = \mathbb{Z} = X^\vee$, $\Phi = \Delta = \{2\}$ and $\Phi^\vee = \Delta^\vee = \{1\}$ for SL_2. The two-sided p-cells can be easily deduced from this using Lemma 4.6. □

Remark 9.4. Note that one can also work out the modular weight cells by hand using the explicit knowledge of the characters of tilting modules for SL_2 (see Lemma 9.1) without appealing to Proposition 6.36. For this, one should observe that when tensoring $T(p^{r+1}-2)$ for $r \geqslant 1$ with the standard representation $V =$

$T(1)$ gives a direct sum of indecomposable tilting modules in which $T(p^r-1)$ occurs and all other summands have highest weight $> p^r-1$. One can check:

$$T(l)\otimes V \cong \begin{cases} T(l+1) & \text{if } l=0 \text{ or } l=rp-1 \text{ for some } r\geqslant 1,\\ T(l+1)\oplus T(l-1)^{\oplus m} & \text{if } l\notin\{0, rp-1, (r+2)p-2 \mid r\geqslant 1\}.\end{cases}$$

In the second case the multiplicity m is either 1 or 2. Lastly, fix r and consider $p^r-1\leqslant lp-2 < p^{r+1}-1$ for some $l\geqslant 1$. If $T(m)$ is a summand of $T(lp-2)\otimes V$, then we have $m\geqslant p^r-1$.

Remark 9.5. The last proposition gives the first example of an affine Weyl group with infinitely many right p-cells. Recall that in general affine Weyl groups have finitely many Kazhdan-Lusztig right cells (see [Lus87, Theorem 2.2]).

Lemma 9.6. *Assume $p>2$. For $\lambda\in \underline{c}_1^r$ we have*

$$p^r \mid \dim T(\lambda) \quad \textit{and} \quad p^{r+1} \nmid \dim T(\lambda).$$

Proof. First observe that for $i\in C_p$ the prime p does not divide $\dim T(i) = \dim\nabla(i) = i+1 < p$. By Lemma 6.17 we have $\dim T(p-1)=p$. Combining the knowledge of the irreducible characters (see for example [Bon11, Theorem 10.1.8]) together with Proposition 6.13, we obtain

$$\operatorname{ch} T(p-1+i) = \operatorname{ch}\nabla(p-1+i) + \operatorname{ch}\nabla(p-1-i)$$

for all $1\leqslant i\leqslant p-1$. Thus we have $\dim T(p-1+i) = 2p$ for i in the same range.

For $\lambda\in\mathbb{Z}_{\geqslant 0}$ write $\lambda=\sum_{i=0}^m \lambda_i p^i$ where $p-1\leqslant\lambda_i\leqslant 2p-2$ for $0\leqslant i\leqslant m$ and $0\leqslant\lambda_m\leqslant p-1$. Observe that in our expression for λ the upper bound m of the summation may not be uniquely defined because the sequence of coefficients $(*,\dots,*,p-1)$ of length l may also be viewed as $(*,\dots,*,p-1,0)$ of length $l+1$. The ambiguity in our expression for λ may be eliminated by assuming m to be minimal. Then the above considerations show that the highest power of p dividing $\dim T(\lambda)$ is given by $m+\delta_{\lambda_m,p-1}$ where $\delta_{\lambda_m,p-1}$ is the Kronecker delta.

Finally, check that the natural numbers λ for which the highest power of p dividing $\dim T(\lambda)$ is r are given by

$$\{n\in\mathbb{N} \mid p^r-1\leqslant n\leqslant p^{r+1}-2\}.$$

As this coincides with our description of $\underline{c}_1^r$ given above, the claim is proved. $\square$

Remark 9.7. We assume $p>2$ in the last lemma for the following reason: In the proof we have seen that $\dim T(p-1+i)=2p$ for $1\leqslant i\leqslant p-1$. Thus for $p=2$ the p-valuation of the dimension of the indecomposable tilting modules in a modular weight cell is not constant.

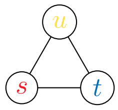
From this the following questions arise:

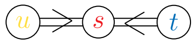
- Can we establish similar connections between modular weight cells and dimensions of indecomposable tilting modules in positive characteristic in other cases?

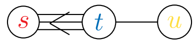
- Can the J-ring be defined in the same way as in the classical case? How do the structure coefficients differ? Recently, Lusztig and Williamson proposed a conjecture about certain tilting characters in SL_3. One of its consequences is that the standard definition of Lusztig's a-function for the p-canonical basis is unbounded (see [LW17b, Remark 5.2(8)]).

10 Cell Structure in Affine Rank 2

Throughout the section, we use the conventions of Section 3.4 to fix the based root datum as input and work in the affine setting. The affine Dynkin diagrams of rank 2 look as follows:

Type $\widetilde{A}_2$: 

Type $\widetilde{C}_2$: 

Type $\widetilde{G}_2$: 

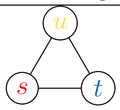
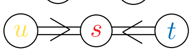
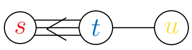
We assume that u is the affine reflection added, thus $S_0 = \{s, t\}$ generates the Weyl group of G. In other words, in the based root datum we used as input the basis Δ equals $\{\alpha_s, \alpha_t\}$.

Next, we want to describe the Coxeter automorphisms of (W, S) given by the elements in Ω more explicitly in affine rank 2. For a special point $\mathtt{v} \in \overline{A_0}$ denote the corresponding automorphism of (W, S) by $\varphi_{\mathtt{v}}$.

- In type $\widetilde{A}_2$ the automorphism $\varphi_{\omega_t^\vee}$ (resp. $\varphi_{\omega_s^\vee}$) acts by a clockwise (resp. counterclockwise) rotation of the Dynkin diagram.
- In type $\widetilde{C}_2$ the automorphism $\varphi_{\omega_t^\vee}$ acts by swapping t and u.
- In type $\widetilde{G}_2$ the group Ω is trivial.

10.1 Kazhdan-Lusztig Cells Decompose into p-Cells

In this section, we will apply the results of Section 4.2 to the affine Weyl groups of rank 2. The necessary assumptions are satisfied for any right cell in the following cases where we have already used the action of Ω to reduce the number of right-minimal elements to check:

Type	Dynkin diagram	Bound on p	Right-minimal elements to check
$\widetilde{A}_2$		none	Id, u, sts, $usts$
$\widetilde{C}_2$		$p \geqslant 3$	Id, s, t, tu, $tstu$, stu, $tsusu$, $tstusts$, $stst$, $stsusu$
$\widetilde{G}_2$		$p \geqslant 5$	Id, s, t, u, su, tsu, $tstsu$, $stsu$, $ststsu$, $utstsu$, tut, $stut$, $ststut$, $tstut$, $tststut$, $utststut$, $ststst$, $ustststst$, $tuststst$, $stuststst$, $tstuststst$, $utstuststst$, $ststuststst$, $uststuststst$, $tuststuststst$, $stuststuststst$, $tstuststuststst$, $utstuststuststst$

In fact, in types $\widetilde{A}_2$ and $\widetilde{C}_2$ the necessary intersection forms can be calculated by hand. For $\widetilde{G}_2$ the result is based on computer calculation. From this, we immediately deduce:

Corollary 10.1. *Kazhdan-Lusztig right cells decompose into right p-cells in affine rank 2 for p larger than the bounds given above.*

In type $\widetilde{C}_2$ the 2-canonical basis differs from the Kazhdan-Lusztig basis only for the following right-minimal elements:

$$\begin{aligned}
{}^2\underline{H}_{tstu} &= \underline{H}_{tstu} + \underline{H}_{tu} \\
{}^2\underline{H}_{tstusts} &= \underline{H}_{tstusts} + \underline{H}_{tusts}
\end{aligned}$$

In type $\widetilde{G}_2$ the p-canonical basis for $p \in \{2, 3\}$ differs from the Kazhdan-Lusztig basis only for the following right-minimal elements:

$$\begin{aligned}
{}^2\underline{H}_{tstsu} &= \underline{H}_{tstsu} + \underline{H}_{tsu} \\
{}^2\underline{H}_{ststsu} &= \underline{H}_{ststsu} + \underline{H}_{su} \\
{}^2\underline{H}_{utstsu} &= \underline{H}_{utstsu} + \underline{H}_{tuts} + \underline{H}_{su} \\
{}^2\underline{H}_{ststut} &= \underline{H}_{ststut} + \underline{H}_{stut} \\
{}^2\underline{H}_{tststut} &= \underline{H}_{tststut} + \underline{H}_{tut} \\
{}^2\underline{H}_{utststut} &= \underline{H}_{utststut} + (v + v^{-1})\underline{H}_{tut} \\
{}^2\underline{H}_{tstuststst} &= \underline{H}_{tstuststst} + \underline{H}_{tuststst} \\
{}^2\underline{H}_{utstuststst} &= \underline{H}_{utstuststst} + (v + v^{-1})\underline{H}_{tuststst} \\
{}^2\underline{H}_{ststuststst} &= \underline{H}_{ststuststst} + \underline{H}_{uststst} \\
{}^2\underline{H}_{uststuststst} &= \underline{H}_{uststuststst} + (v + v^{-1})\underline{H}_{uststst} \\
{}^2\underline{H}_{tuststuststst} &= \underline{H}_{tuststuststst} + (v + v^{-1})\underline{H}_{ststst} \\
{}^2\underline{H}_{stuststuststst} &= \underline{H}_{stuststuststst} + \underline{H}_{uststuststst} + (v^2 + 1 + v^{-1})\underline{H}_{ststst} \\
{}^2\underline{H}_{tstuststuststst} &= \underline{H}_{tstuststuststst} + \underline{H}_{utstuststst} + (v^3 + v^{-3})\underline{H}_{ststst} \\
{}^2\underline{H}_{utstuststuststst} &= \underline{H}_{utstuststuststst} + (v + v^{-1})\underline{H}_{utstuststst} + \\
&\quad \underline{H}_{tuststst} + (v^3 + v^{-3})\underline{H}_{uststst}
\end{aligned}$$

$$\begin{aligned}
{}^3\underline{H}_{tstut} &= \underline{H}_{tstut} + \underline{H}_{tut} \\
{}^3\underline{H}_{tststut} &= \underline{H}_{tststut} + \underline{H}_{tstut} \\
{}^3\underline{H}_{utststut} &= \underline{H}_{utststut} + \underline{H}_{tutstu} + \underline{H}_{stut} \\
{}^3\underline{H}_{tuststuststst} &= \underline{H}_{tuststuststst} + \underline{H}_{utstuststst} \\
{}^3\underline{H}_{tstuststuststst} &= \underline{H}_{tstuststuststst} + \underline{H}_{tuststuststst} \\
{}^3\underline{H}_{utstuststuststst} &= \underline{H}_{utstuststuststst} + \underline{H}_{tutststuststst} + \underline{H}_{uststuststst}
\end{aligned}$$

For any right-minimal element x, consider the difference ${}^p\underline{H}_x - \underline{H}_x$. Observe that it is always a sum of Kazhdan-Lusztig basis elements indexed by elements in a different Kazhdan-Lusztig right cell, but the same or a lower Kazhdan-Lusztig two-sided cell. Therefore, we may apply our criterion to Kazhdan-Lusztig two-sided cells to see that Kazhdan-Lusztig two-sided cells in affine rank 2 decompose into two-sided p-cells for all primes p.

Corollary 10.2. *Kazhdan-Lusztig two-sided cells decompose into two-sided p-cells in affine rank 2 for all primes p.*

Since all elements in one right cell have the same left descent set, we know that

$$\{w \in W \mid \mathcal{L}(w) = S \setminus S_{\mathrm{f}}\} = {}^{f}W \setminus \{\mathrm{Id}\}$$

decomposes into right p-cells. Combining this with the last result we get:

Corollary 10.3. *Anti-spherical Kazhdan-Lusztig right cells decompose into right p-cells in affine rank 2 for all primes p.*

10.2 General Scheme

In this section, we will explain the general idea how to determine the p-cells in $\widetilde{A}_2$ and $\widetilde{C}_2$ for $p > 2h - 2$. We will therefore assume $p > 2h - 2$ throughout the whole section without further mention. In Section 10.1 we showed that the Kazhdan-Lusztig right cells decompose in right p-cells under our assumptions on p.

The first step is to use Proposition 5.17 and Proposition 4.7 to reduce the question of the decomposition behaviour to spherical, anti-spherical and unique reduced expression right Kazhdan-Lusztig cells.

Next, observe that the connection between anti-spherical right p-cells and modular weight cells for $p > h$ together with the fractal-like behaviour of the modular weight cells (see Lemma 6.35) shows that the decomposition behaviour of the anti-spherical Kazhdan-Lusztig right cell contained in the lowest Kazhdan-Lusztig two-sided cell is completely determined by the decomposition behaviour of the the higher anti-spherical Kazhdan-Lusztig right cells.

Then we will determine very explicitly the decomposition behaviour of the remaining Kazhdan-Lusztig right cells. From this, we deduce the modular weight cells using Theorem 7.8.

The last step is to show that via Geometric Satake the spherical right p-cells are determined by modular weight cells. For this, it will suffice to show the following:

- All elements in a box that contains a spherical element belong to the same right p-cell. (Observe that any spherical Kazhdan-Lusztig cell in affine rank 2 decomposes into boxes and each box contains a spherical element).

- There are no more right p-cell relations between the spherical elements than the ones coming from geometric Satake.

10.3 $\widetilde{A}_2$

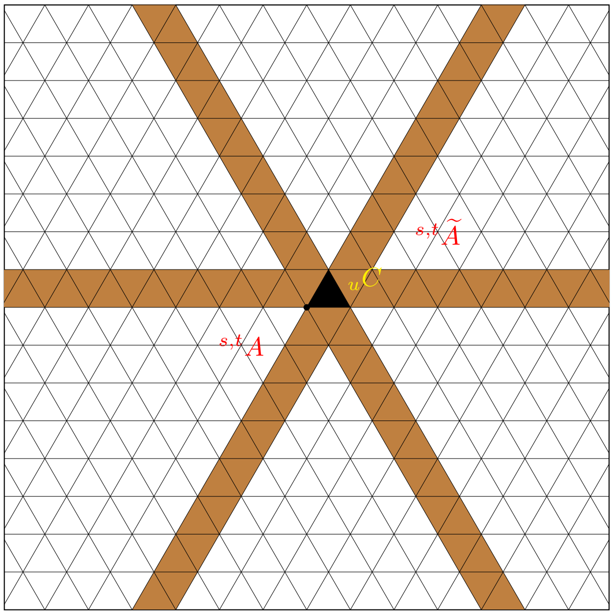
According to the extended Dynkin diagram of type $\widetilde{A}_2$, the pairing of simple roots and coroots is given as follows:

$$\langle \alpha_r^\vee, \alpha_{r'} \rangle = -1 \text{ for } r \neq r' \in S.$$

10.3.1 Reduction step

Figure 2 shows the Kazhdan-Lusztig cells in type $\widetilde{A}_2$ (see [Lus85, Theorem 11.3 and Proposition 11.6]). All alcoves of the same colour belong to one Kazhdan-Lusztig two-sided cell. Each right-connected component among the alcoves of the same colour gives a Kazhdan-Lusztig right cell. The dot marks the origin as chosen special point.

Figure 2: Kazhdan-Lusztig cells in type $\widetilde{A}_2$



Using the action of the elements in $\Omega \cong \mathbb{Z}/3\mathbb{Z}$ and Proposition 4.7 we can reduce the question of the decomposition behaviour to the following Kazhdan-Lusztig right cells:

$$\begin{aligned} C_{\mathrm{triv}} &= \{e\} \\ {}_uC &= \{w \in W \mid w \text{ has a unique rex with } u \in \mathcal{L}(w)\} \\ {}^{s,t}A &= \{w \in W \mid \mathcal{L}(w) = \{s,t\}\} \\ {}^{s,t}\widetilde{A} &= u({}^{s,t}A) \end{aligned}$$

Among these, C_{triv}, ${}_uC$ and ${}^{s,t}\widetilde{A}$ are anti-spherical and ${}^{s,t}A$ is a spherical Kazhdan-Lusztig right cell. In this case, we can even reduce one step further by observing that ${}^{s,t}A$ is obtained from ${}^{s,t}\widetilde{A}$ by applying the left star-operation with respect to $\{s,u\}$ (or $\{t,u\}$) (see Proposition 5.17). Note that ${}^{s,t}A$ belongs to the lowest Kazhdan-Lusztig two-sided cell and thus we only need to determine how ${}_uC$ decomposes into right p-cells.

10.3.2 Unique reduced expression cells

In this section we determine how ${}_uC$ decomposes into p-cells. The following result combined with Corollary 10.1 shows that ${}_uC$ itself is a p-cell.

Lemma 10.4. *Let $w \in {}_uC$. Suppose $\mathcal{R}(w) = \{t\}$ and $ws > w$ for $s \in S$ such that $ws \notin {}_uC$. Consider $(-)^*$ the star-operation defined with respect to $\{s,t\}$. Then ${}^kB_{w^*}$ is a summand in ${}^kB_w {}^kB_s$.*

Proof. Because w lies in ${}_uC$, the element w has a unique reduced expression $\underline{w}$.

First check that ${}^kB_{w^*}$ is a summand in the Bott-Samelson corresponding to $\underline{ws}$. Write $w = vst$ with $l(w) = l(v) + 2$. Then ${}^kB_{w^*} = {}^kB_{vs}$ is a summand in ${}^kB_v {}^kB_s$ which in turn is a summand in the Bott-Samelson corresponding to $\underline{ws}$ (because ${}^kB_v \overset{\oplus}{\subseteq} BS(\underline{v})$ and ${}^kB_s \overset{\oplus}{\subseteq} BS(\underline{sts})$).

In order to see that ${}^kB_{w^*}$ is a summand of ${}^kB_w {}^kB_s$ we claim that it is enough to check that ${}^kB_v$ does not occur in the Bott-Samelson for $\underline{w}$. Indeed, the Bott-Samelson corresponding to $\underline{w}$ contains indecomposable summands for which the length of the indexing element has the same parity as $l(w)$ and among these the only one that may contribute ${}^kB_{vs}$ when tensoring with ${}^kB_s$ are ${}^kB_v$ and ${}^kB_w$.

Our claim holds because there is a unique subexpression of $\underline{w}$ expressing v (just leave out the last two letters) and it is of defect 2 as u also lies on the wall. $\square$

Alternatively, we could have used a similar argument as for type $\widetilde{C}_2$.

10.3.3 Modular weight cells for SL_3

In this section we will use the connection between p-cells in $W = W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi^\vee$ and modular weight cells and their fractal-like behaviour to deduce the modular weight cells for SL_3. The following result follows immediately from Section 10.3.2 and Theorem 7.8:

Proposition 10.5. *The set $\underline{c}_2^0 := X_+^\vee \setminus (Y_1 \cup C_p)$ is a modular weight cell.*

The fractal-like behaviour of modular weight cells (see Lemma 6.35) implies for $p \geqslant 2h-2$ the following result, which is mentioned in [And04, Example 15] without proof:

Corollary 10.6. *For all $r \in N$ the set $\underline{c}_2^r := Y_r \setminus (Y_{r+1} \cup \underline{c}_1^r)$ is a modular weight cell.*

Therefore the modular weight cells in $X_+^\vee$ are given by

$$\{\underline{c}_i^r \mid r \in \mathbb{N}, 1 \leqslant i \leqslant 2\}$$

and they form a descending chain with respect to the preorder $\overset{p}{\underset{\mathcal{T}}{\leqslant}}$ as follows:

$$\underline{c}_1^1 \overset{p}{\underset{\mathcal{T}}{>}} \underline{c}_2^1 \overset{p}{\underset{\mathcal{T}}{>}} \underline{c}_1^2 \overset{p}{\underset{\mathcal{T}}{>}} \underline{c}_2^2 \overset{p}{\underset{\mathcal{T}}{>}} \cdots \overset{p}{\underset{\mathcal{T}}{>}} \underline{c}_1^i \overset{p}{\underset{\mathcal{T}}{>}} \underline{c}_2^i \overset{p}{\underset{\mathcal{T}}{>}} \cdots$$

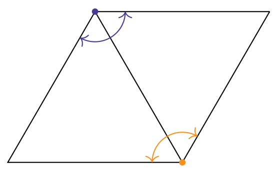
Thus for every modular tensor ideal in $\mathrm{Tilt}(SL_3)$ there exist $i \in \{1,2\}$ and $r \in \mathbb{N}$ such that the modular tensor ideal is the full subcategory of $\mathrm{Tilt}(SL_3)$ consisting of direct sums of indecomposable tilting modules $T(\lambda)$ for $\lambda \overset{p}{\underset{\mathcal{T}}{\leqslant}} \underline{c}_i^r$.

10.3.4 Spherical cells

Figure 3 shows a box containing elements in ${}^{s,t}A$. Observe that any marked point is special. For any marked point $\mathtt{v}$, its stabilizer $W_{\mathtt{v}}$ is a finite parabolic subgroup generated by a pair of simple reflections $S_{\mathtt{v}}$. All alcoves having $\mathtt{v}$ in its closure lie in a right $W_{\mathtt{v}}$-coset. In this coset, the alcoves that intersect non-trivially the arrow of the same colour as the marked point lie in a right $W_{\mathtt{v}}$-string. Due to the position of such a box relative to the fundamental alcove, the corresponding right strings will always look as indicated by the little arrows no matter which box we consider. Therefore, Lemma 5.10 applied to any one of the two marked right strings implies in this setting:

Lemma 10.7. *If a box contains a spherical element, then all elements in the box lie in the same right p-cell of W.*

Figure 3: Box in ${}^{s,t}A$ in type affine $\widetilde{A}_2$



Remark 10.8. Actually, one can understand the situation in this case quite explicitly: For any one of the two marked special points, consider the translation along the string and then to the longest element in the right coset. This can be related under the geometric Satake equivalence to the functor given by tensoring with $T(\omega^\vee) \oplus \mathrm{triv}$ for some fundamental coweight $\omega^\vee$ and the trivial representation triv.

In this section, we will often identify an object $c \in \mathcal{C}$ in a monoidal category $\mathcal{C}$ with the endofunctor $(-) \otimes c : \mathcal{C} \longrightarrow \mathcal{C}$. Singular diagrammatics will only be necessary in the following proof (and in one more proof in type $\widetilde{C}_2$). We denote by ${}^k\mathbf{SH}$ the 2-category of diagrammatic singular Soergel bimodules. Its objects are finitary subsets of S. There is a generating 1-morphism between two finitary subsets if an only if one is contained in the other and their sizes differ by one. The 2-morphisms are given by singular Soergel diagrams. The precise definitions and more details can be found in [Eli17, §3.3] for type $\widetilde{A}_2$ and [Eli16, §6.1.1] for rank 2. For finitary subsetes $J, I_2, I_1 \subset S$ with $I_2 \subset I_1$ write $\mathrm{ind}_{I_1}^{I_2} : \mathrm{Hom}_{{}^k\mathbf{SH}}(J, I_1) \longrightarrow \mathrm{Hom}_{{}^k\mathbf{SH}}(J, I_2)$ for the induction functor on the right hand side. Similarly for finitary subsets $J_2 \subset J_1$ write ${}^{J_2}_{J_1}\mathrm{ind}$ for the induction functor on the left hand side. Set $\mathrm{ind}_I := \mathrm{ind}_I^{\varnothing}$, ${}_J\mathrm{ind} := {}^{\varnothing}_J\mathrm{ind}$ and ${}^{J_2}_{J_1}\mathrm{ind}^{I_2}_{I_1} := {}^{J_2}_{J_1}\mathrm{ind} \circ \mathrm{ind}^{I_2}_{I_1} = \mathrm{ind}^{I_2}_{I_1} \circ {}^{J_2}_{J_1}\mathrm{ind}$.

For $\lambda \in X_+^\vee$ consider $w_\lambda \in W$. For $r \in S_\mathrm{f}$ denote by $P_{\omega_r^\vee}(\lambda)$ the unique sequence of two simple reflections needed to extend any reduced expression of w_λ to a reduced expression for $w_{\lambda+\omega_r^\vee}$. Note that $P_{\omega_r^\vee}(\lambda)$ only depends on the class of λ in $\Omega = X^\vee/\mathbb{Z}\Phi^\vee$. Fix $r \in S_\mathrm{f}$ and let $P_{\omega_r^\vee}(\lambda) = (s_1, s_2)$.

The next result decomposes the identity on the functor

$$B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} : \mathrm{Hom}_{^k\mathbf{SH}}(S_\mathrm{f}, S_\lambda) \longrightarrow \mathrm{Hom}_{^k\mathbf{SH}}(S_\mathrm{f}, \varnothing)$$

as a sum of two idempotents one of which corresponds under geometric Satake to the endofunctor $T(\omega_r^\vee)$ on $\mathrm{Tilt}(G^\vee)$.

Proposition 10.9. *The functor*

$$B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} : \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_\lambda) \longrightarrow \mathrm{Hom}_{^k\mathbf{SH}}(S_f, \varnothing)$$

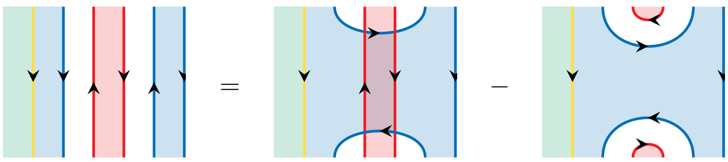
can be decomposed as a direct sum

$$B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} = \mathrm{ind}_{S_\lambda} \oplus \mathrm{ind}_{S_{\lambda+\omega_r^\vee}} \circ T_{\omega_r^\vee}$$

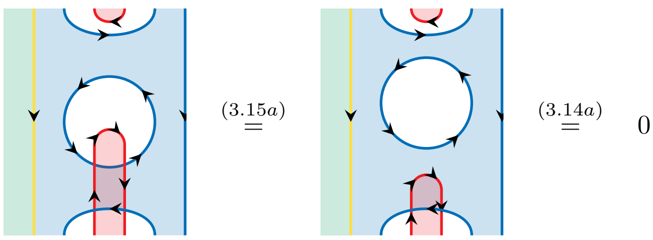
where the functor $T_{\omega_r^\vee} : \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_\lambda) \longrightarrow \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_{\lambda+\omega_r})$ *corresponds under the geometric Satake equivalence to the* 1*-morphism* $T(\omega_r^\vee) \in \mathrm{Tilt}(G^\vee)^{\overline{\omega_r^\vee}}$.

Proof. In the diagrams we will restrict to the case $S_\lambda = \{t, u\}$, $r = t$ and thus $P_{\omega_t^\vee}(\lambda) = (s, t)$. All other cases for $r = t$ can be obtained by applying an arbitrary element of Ω to all colours involved.

The idea is to first induce from S_λ to $S_\lambda \cap S_{\lambda+\omega_r^\vee}$ and then to $\varnothing$. This allows us to use relation [Eli17, (3.16)] to rewrite the identity on $B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda}$ as follows:



Observe that we need the non-quantum case of [Eli17] throughout and thus in all equations q should be specialized to 1. [Eli17, Claim 3.14] shows that the first term on the right hand side is an idempotent. Similary, [Eli17, (3.13) and (3.14a)] give us that the second term on the right hand side (together with the minus sign) is an idempotent as $\partial_x(\alpha_y) = -1$ for all pairs of simple reflections $x \neq y \in S$. Next, we will show that these two idempotents are orthogonal to each other. All equation numbers in the following calculation refer to [Eli17]:



where the last equality holds as $\partial_t(1) = 0$. Thus, we have decomposed the functor $B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda}$ as a direct sum of two functors.

The first idempotent on the right hand side factors through the identity on $\mathrm{ind}_{S_{\lambda+\omega_r^\vee}} \circ T_{\omega_r^\vee}$ (see [Eli17, Theorem 3.21, Claim 3.7 and the paragraph preceding it]), while the second idempotent factors through the identity on ind_{S_λ}. (In order to verify this, recall that Elias in [Eli17] reads 1-morphisms from right to left.) ☐

Remark 10.10. One should think of the decomposition given in Proposition 10.9 as a relation between different paths in the affine Hecke algebroid:

$$\begin{tikzcd}
 & \mathrm{Hom}_{{}^k\mathbf{SH}}(S_{\mathrm{f}}, \varnothing) \arrow[dl, shift left] \arrow[dr, shift left] & & \\
\mathrm{Hom}_{{}^k\mathbf{SH}}(S_{\mathrm{f}}, S_{\lambda+\omega_r^\vee} \setminus S_\lambda) \arrow[ur, shift left] & & \mathrm{Hom}_{{}^k\mathbf{SH}}(S_{\mathrm{f}}, S_\lambda \cap S_{\lambda+\omega_r^\vee}) \arrow[ul, shift left] \arrow[dl, shift left] \arrow[dr, shift left] & \\
 & \mathrm{Hom}_{{}^k\mathbf{SH}}(S_{\mathrm{f}}, S_{\lambda+\omega_r^\vee}) \arrow[ur, shift left] \arrow[uu, dashed, "\mathrm{ind}_{S_{\lambda+\omega_r^\vee}}"'] & & \mathrm{Hom}_{{}^k\mathbf{SH}}(S_{\mathrm{f}}, S_\lambda) \arrow[ul, shift left] \arrow[ll, dashed, "T_{\omega_r^\vee}"'] \arrow[uull, dashed, bend right, "\mathrm{ind}_{S_\lambda}"']
\end{tikzcd}$$

Since in type $\widetilde{A}_2$ both fundamental weights act non-trivially on S we have the following result:

Corollary 10.11. *The only indecomposable summand in ${}^kB_{w_\lambda}{}^kB_{s_1}{}^kB_{s_2}$ indexed by an element $y \in W$ with $\mathcal{R}(y) = S_\lambda$ is w_λ, all other elements y indexing indecomposable summands satisfy $\mathcal{R}(y) = S_{\lambda+\omega_r^\vee}$.*

For any special point $\mathtt{v} \in E$ and $y \in W$ with $A_{\mathtt{v}}y \subset \Pi_{\mathtt{v}}$ we define

$$E_{\mathtt{v},y} = \sum_{\substack{x \leqslant y \\ l(w_{\mathtt{v}}x) = l(w_{\mathtt{v}}) + l(x)}} {}^0h_{w_{\mathtt{v}}x, w_{\mathtt{v}}y} H_x$$

Observe that for $s \in S \setminus S_{\mathtt{v}}$ we have $E_{\mathtt{v},s} = \underline{H}_s$. The following lemma can be found in [Xi90, Lemma 2.7] and is proved using the periodic Hecke module (as introduced in [Lus80a], see also [Soe97b, §4]). For us it is the main reason of interest for the elements $E_{\mathtt{v},y}$:

Lemma 10.12. *Let $x, y \in W$ such that $A_0 x = A_{\mathtt{v}} \subseteq \mathcal{C}_0$ and $A_{\mathtt{v}}y \subset \Pi_{\mathtt{v}}$. Then we have:*

$$\underline{H}_{w_0 x} E_{\mathtt{v},y} = \underline{H}_{w_0 x y}.$$

For us the most interesting case will be the situation when y is the unique simple reflection in $S \setminus S_\lambda = \{s_1\}$.

Corollary 10.13. *The following holds: $\underline{H}_{w_\lambda} \underline{H}_{s_1} = \underline{H}_{w_\lambda s_1}$.*

Lemma 10.14. *We have: ${}^p\underline{H}_{w_\lambda} \underline{H}_{s_1} = {}^p\underline{H}_{w_\lambda s_1}$.*

Proof. When expressing the p-canonical basis element ${}^p\underline{H}_{w_\lambda}$ in terms of the Kazhdan-Lusztig basis, all Kazhdan-Lusztig basis elements with a non-zero coefficient are indexed by spherical elements $y \in W$ with $\mathcal{L}(y) = S_{\mathrm{f}} = \mathcal{L}(w_\lambda)$ and $\mathcal{R}(y) = S_{\mathtt{v}} = \mathcal{R}(w_\lambda)$. Thus Corollary 10.13 is applicable to all these terms and we get:

$${}^p\underline{H}_{w_\lambda} \underline{H}_{s_1} = \sum_{y \leqslant w_\lambda} {}^p m_{y, w_\lambda} \underline{H}_{y s_1}$$

We claim that this element is precisely ${}^p\underline{H}_{w_\lambda s_1}$. Indeed, the claim follows using Corollary 5.7 as all occuring Kazhdan-Lusztig basis elements in ${}^p\underline{H}_{w_\lambda}\underline{H}_{s_1}$ are indexed by an element in a right $\langle S_{\lambda+\omega_r^\vee}\rangle$-string. ☐

Finally, we can show that we do not have any more right p-cell preorder relations between spherical elements than the ones coming from Geometric Satake. In other words, the modular weight cells determine the p-cell decomposition of the Kazhdan-Lusztig right cell ${}^{s,t}A = \{w \in W \mid \mathcal{L}(w) = S_\mathrm{f}\}$.

Corollary 10.15. *For $\lambda, \mu \in X_+^\vee$ we have:*

$$\mu \underset{\mathcal{T}}{\overset{p}{\leqslant}} \lambda \Leftrightarrow w_\mu \underset{R}{\overset{p}{\leqslant}} w_\lambda$$

Proof. The direction $\Rightarrow$ was shown in Lemma 7.19. It remains to prove the direction $\Leftarrow$. Consider a sequence of elementary right p-cell relations:

$$w_\lambda = x_1 \underset{R}{\overset{p}{\to}} x_2 \underset{R}{\overset{p}{\to}} x_3 \underset{R}{\overset{p}{\to}} \dots \underset{R}{\overset{p}{\to}} x_m = w_\mu$$

where for $1 \leqslant i < m$ the structure coefficient ${}^p\mu_{x_i,s_i}^{x_{i+1}}$ is non-zero. Without loss of generality we may assume $x_i s_i > x_i$ for all $1 \leqslant i < m$.

We claim that this right p-cell relation implies a modular weight cell relation $\mu \underset{\mathcal{T}}{\overset{p}{\leqslant}} \lambda$. For this we proceed by induction on m. The induction start and the induction step follow from the same arguments: Lemma 10.14 imples that s_1 is the unique simple reflection in $S \setminus S_\lambda$ and thus $x_2 = x_1 s_1$. It follows that the pair (s_1, s_2) coincides with $P_{\omega_r^\vee}(\lambda)$ for some $r \in S_\mathrm{f}$. From Proposition 10.9 we deduce that $x_3 = w_\zeta$ for some $\zeta \in X_+^\vee$ and that the sequence $w_\lambda = x_1 \underset{R}{\overset{p}{\to}} x_2 \underset{R}{\overset{p}{\to}} x_3 = w_\zeta$ implies $\zeta \underset{\mathcal{T}}{\overset{p}{\leqslant}} \lambda$. More precisely, $T(\zeta)$ is a direct summand in $T(\lambda) \otimes (T(\omega_r^\vee) \oplus \mathrm{triv})$ where triv denotes the trivial $G^\vee$-module. Finally, we may apply induction to the remaining sequence $w_\zeta = x_3 \underset{R}{\overset{p}{\to}} \dots \underset{R}{\overset{p}{\to}} x_m = w_\mu$ to finish the proof. ☐

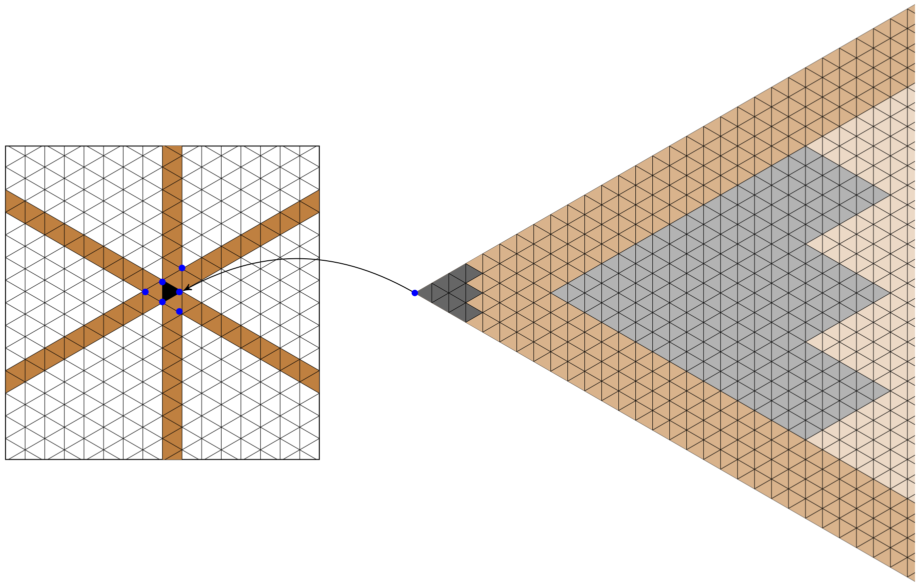
10.3.5 Right p-cell structure

In Figure 4 rotate the cone on the right and place it with its tip at each blue point in such a way that each time it covers precisely one Kazhdan-Lusztig right cell in the lowest Kazhdan-Lusztig two-sided cell. The resulting picture shows the right p-cells in type $\widetilde{A}_2$ for $p = 5$. Note that we could of course only depict a finite region in the cone which shows signs of the fractal-like behaviour of the p-cells.

If follows from Lemma 4.6 that all the alcoves of the same colour in the resulting picture are contained in a two-sided p-cell. Moreover, all Kazhdan-Lusztig two-sided cells apart from the lowest one are two-sided p-cells.

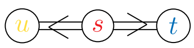
Conjecture 10.16.
All alcoves of the same colour form a two-sided p-cell.

Figure 4: Right p-cells in type $\widetilde{A}_2$



10.4 $\widetilde{C}_2$

As mentioned in Proposition 6.4, the affine Dynkin diagram is realized as the Dynkin diagram of $\{-\alpha_0\} \cup \{\alpha \mid \alpha \in \Delta\}$ where α_0 is the highest root. Taking the Langlands dual amounts to swapping roots and coroots, giving the Dynkin diagram of $\{-\alpha_0^\vee\} \cup \{\alpha^\vee \mid \alpha \in \Delta\}$. In particular, the Langlands dual of the affine Dynkin diagram of type $\widetilde{C}_2$ looks as follows:



That means that the pairing of simple roots and coroots is given as follows for all $r \in \{t, u\}$:

$$\begin{aligned}
\langle \alpha_r^\vee, \alpha_s \rangle &= -2, \\
\langle \alpha_s^\vee, \alpha_r \rangle &= -1, \\
\langle \alpha_t^\vee, \alpha_u \rangle &= \langle \alpha_u^\vee, \alpha_t \rangle = 0.
\end{aligned}$$

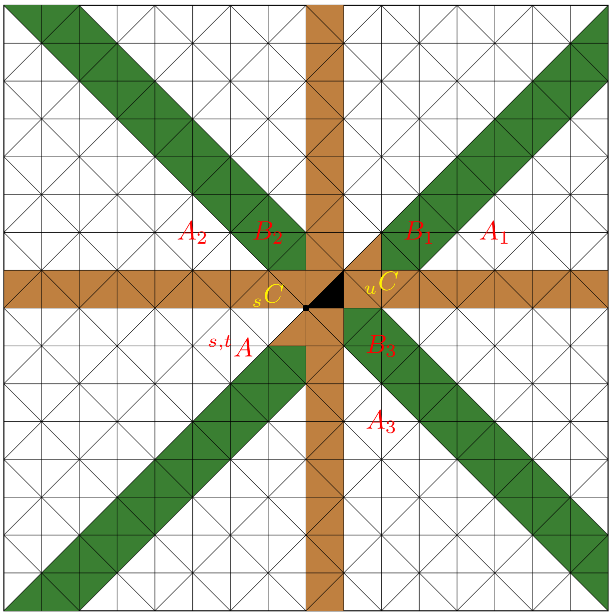
We will use this as input for ${}^k\mathbf{H}$ in this section.

Remark 10.17. We could have also used the pairing of the simple roots and coroots prescribed by the affine Dynkin diagram of type $\widetilde{C}_2$ as input. In the proof of [Ach+17b, Theorem 7.2] an explicit equivalence between the two corresponding diagrammatic categories is constructed for $p > 2$.

10.4.1 Reduction step

Figure 5 shows the Kazhdan-Lusztig cells in type $\widetilde{C}_2$ with the same conventions as for type $\widetilde{A}_2$ (see Section 10.3.1).

Figure 5: Kazhdan-Lusztig cells in type $\widetilde{C}_2$



Using the action of the elements in $\Omega \cong \mathbb{Z}/2\mathbb{Z}$ and Proposition 4.7 we can reduce the question of the decomposition behaviour to the labelled Kazhdan-Lusztig right cells in Figure 5. Next, observe that B_1 and A_1 are contained in $\mathcal{D}_R(s,u)$. We have $\widetilde{B}_1 = B_2 \cup B_3$ and $\widetilde{A}_1 = A_2 \cup A_3$ in the notation of Proposition 5.17. Using descent set considerations as explained in Remark 5.18, this allows us to reduce the question of the decomposition behaviour of these six right Kazhdan-Lusztig cells to B_1 and A_1. Therefore, it is enough to determine how the following Kazhdan-Lusztig right cells decompose into right p-cells:

$$\begin{aligned}
C_{\text{triv}} &= \{e\} \\
{}_uC &= \{w \in W \mid w \text{ has a unique rex with } u \in \mathcal{L}(w)\} \\
{}_sC &= \{w \in W \mid w \text{ has a unique rex with } s \in \mathcal{L}(w)\} \\
{}^{s,t}A &= \{w \in W \mid \mathcal{L}(w) = \{s,t\}\} \\
B_1 &= us(A^{t,u} \setminus (u({}^{s,t}A) \cup t(A^{s,u}))) \\
A_1 &= A^u \setminus ({}_uC \cup B_1)
\end{aligned}$$

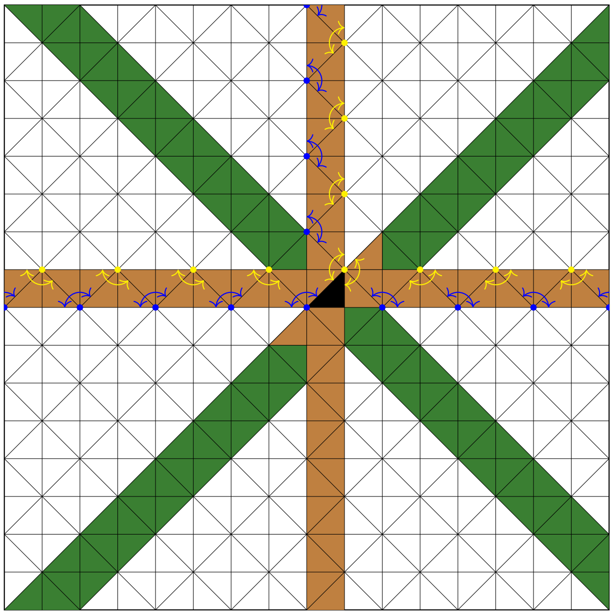
Among these, C_{triv}, ${}_uC$, B_1 and A_1 are anti-spherical, ${}^{s,t}A$ is a spherical Kazhdan-Lusztig right cell and ${}_uC$ as well as ${}_sC$ contain only elements with a unique reduced expression.

10.4.2 Unique reduced expression cells

As in previous pictures, in Figure 6 each point $\mathtt{v}$ marks a right $W_{\mathtt{v}}$-coset in which we are interested in the right $W_{\mathtt{v}}$-string marked by an arrow. Applying Lemma 5.10 to these strings shows the following:

Proposition 10.18. *Each Kazhdan-Lusztig right cell containing elements with a unique reduced expression is a right p-cell for $p > 2$.*

Figure 6: Right strings in the unique reduced expression cells in type $\widetilde{C}_2$



10.4.3 Strategy for the remaining cell on the wall

In order to motivate our strategy to determine the decomposition behaviour of the remaining Kazhdan-Lusztig cell, we need to explain some deep results from Kazhdan-Lusztig cell theory:

We denote by $G^\vee_{\mathbb{C}}$ the simply-connected, semi-simple, connected algebraic group obtained from scalar extension of $G^\vee_{\mathbb{Z}}$ to $\mathbb{C}$. Let g be the Lie algebra of $G^\vee_{\mathbb{C}}$. Denote by $\mathcal{N} \subseteq g$ the nilpotent cone. The group $G^\vee_{\mathbb{C}}$ acts on $\mathcal{N}$ via the adjoint action. The nilpotent cone consists of finitely many $G^\vee_{\mathbb{C}}$-orbits, called *nilpotent orbits* (see [CM93, Theorem 3.5.4]). We define a partial order on the set of nilpotent orbits via $\mathcal{O}' \leqslant \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$.

One of the deepest results of Kazhdan-Lusztig cell theory is the following: In [Lus89, Theorem 4.8] Lusztig constructs a bijection $C \mapsto \mathcal{O}_C$ between the two-sided Kazhdan-Lusztig cells in W and the the set of nilpotent orbits in $\mathcal{N}$ (or equivalently conjugacy classes of unipotent elements in $G^\vee_{\mathbb{C}}$). Moreover, this bijection respects the natural partial orderings, with the trivial two-sided cell

$\{e\}$ corresponding to the regular nilpotent orbit and the lowest two-sided cell corresponding to the zero orbit (see [Bez09, Theorem 4 b)]).

In [LX88, Theorem 1.2], Lusztig and Xi show that any Kazhdan-Lusztig two-sided cell intersects ${}^fW$ in a unique right cell. Combining both results, we obtain a bijection between the anti-spherical Kazhdan-Lusztig right cells and nilpotent orbits in $\mathcal{N}$.

For an anti-spherical Kazhdan-Lusztig right cell C, denote by n_C a representative of the nilpotent orbit $\mathcal{O}_C$. Moreover, denote by F_C the maximal reductive subgroup in the centralizer $C_G(n_C)$. Riche conjectured that anti-spherical Kazhdan-Lusztig cells decompose into right p-cells for $p > h$. Moreover, he believes that the group F_C should govern the decomposition behaviour of the Kazhdan-Lusztig right cell C.

Now, we specialize to $G^\vee_{\mathbb{C}} = SP_4(\mathbb{C})$. Riche's conjecture was the main motivation for our strategy to determine the decomposition behaviour of the remaining anti-spherical Kazhdan-Lusztig right cell B_1. Since Lusztig's bijection respects the natural partial orderings, the Kazhdan-Lusztig right cell B_1 corresponds to the minimal nilpotent orbit $\mathcal{O}_{\min}$ of $sp_4(\mathbb{C})$. For an element $n \in \mathcal{O}_{\min}$, one can check that the maximal reductive subgroup in $C_G(n)$ is isomorphic to $SL_2(\mathbb{C}) \times \{\pm \mathrm{Id}\}$.

The representation theory of SL_2 is closely linked to the Temperley-Lieb category, as we will explain in several remarks in the following section. For this reason, our strategy is to construct a functor from the Temperley-Lieb category to the anti-spherical category. After suitable extension of scalars, the image of this functor will provide approximations of the idempotents of certain indecomposable objects in B_1. This will then allow us to determine the decomposition behaviour of B_1.

10.4.4 Decomposing the remaining cell on the wall

The goal of this section is to show that the decomposition behaviour of the Kazhdan-Lusztig right cell B_1 is governed by SL_2. For this, we will need to introduce the Temperley-Lieb category.

A good reference for the Temperley-Lieb algebra and its graphical calculus is [KT08, §5.7]. For $n, m \geqslant 1$ of the same parity, a *simple* (n,m)*-diagram* D is a disjoint union of $\frac{n+m}{2}$ smoothly embedded, non-intersecting arcs in $\mathbb{R} \times [0,1]$ such that:

- the boundary ∂D of D consists of the points $(1,0)$, $(2,0)$, …, $(n,0)$ and $(1,1)$, $(2,1)$, …, $(m,1)$,
- $D \setminus \partial D \subset \mathbb{R} \times (0,1)$,
- each arc intersects the boundary $\mathbb{R} \times \{0,1\}$ transversally.

We call an isotopy class of a simple (n,m)-diagram relative to the boundary $\mathbb{R} \times \{0,1\}$ an (n,m) crossingless matching.

The uncoloured Temperley-Lieb category **TL** is a monoidal category enriched over $\mathbb{Z}[\delta]$-modules whose objects are natural numbers $n \in \mathbb{N}$, pictured as n distinct points on a horizontal line. The space of morphisms from n to m is a free $\mathbb{Z}[\delta]$-module with the set of all (n,m)-crossingless matchings as basis. The monoidal structure (resp. the composition) is induced by horizontal (resp.

vertical) concatenation of the corresponding diagrams where in the composition of two morphisms each closed loop is resolved to $-\delta$. We will be interested in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL}$, the scalar extension of $\mathbf{TL}$ to $\mathbb{Z}[\frac{1}{2}]$ in which we specialize δ to 2.

For any commutative ring k with a homomorphism $\mathbb{Z}[\frac{1}{2}] \to k$, denote by ${}^{k}\mathbf{N}_{\mathrm{BS}}$ the quotient of ${}^{k}\mathbf{BS}$ by the ideal of all objects indexed by S_{f}-sequences where an S_{f}-sequence is any sequence starting in a simple reflection in S_{f}. Define ${}^{k}\mathbf{N}$ to be the graded, additive Karoubian completion of ${}^{k}\mathbf{N}_{\mathrm{BS}}$.

Note that, in principle, we only have some control over the indecomposable objects in ${}^{k}\mathbf{N}$ if k is a complete local ring. Arguing as in [LW17a, Proposition 3.2], one obtains that for a complete local ring k this category is equivalent to the categorification of the anti-spherical category proposed in Section 7.2.

In both categories ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL}$ and ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$, there exists a duality induced by flipping diagrams upside down. Denote this duality by slight abuse of notation in both cases by $\mathbb{D}$.

If k is a field not of characteristic 2, denote by J_{sph} the two-sided ideal of morphisms in ${}^{k}\mathbf{N}$ factoring through a direct sum of shifts of indecomposable objects indexed by elements in the Kazhdan-Lusztig right cell A_1. The goal of this section is to define a $\mathbb{Z}[\frac{1}{2}]$-linear functor $\mathcal{F} : {}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL} \longrightarrow {}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$ that commutes with $\mathbb{D}$ and whose image consists of degree 0 morphisms. Moreover, we will see that for any field k not of characteristic 2 the composition

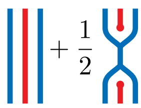
$$ {}^{k}\mathbf{TL} \xrightarrow{\mathcal{F}_k} {}^{k}\mathbf{N} \longrightarrow {}^{k}\mathbf{N}/J_{\mathrm{sph}} $$

is full in degree zero. In this composition $\mathcal{F}_k$ denotes the functor obtained from $\mathcal{F}$ by extension of scalars and ${}^{k}\mathbf{N} \longrightarrow {}^{k}\mathbf{N}/J_{\mathrm{sph}}$ is the quotient functor.

Denote by $\overline{B}_e$ the image in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$ of the Bott-Samelson object corresponding to the empty sequence (i.e. the monoidal unit in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{BS}$). For $n \geqslant 0$ consider the object

$$ B_n := \overline{B}_e B_{usu} B_{tst} B_{usu} \dots \begin{cases} B_{tst} B_u & \text{if } n \text{ is odd,} \\ B_{usu} B_t & \text{otherwise.} \end{cases} \tag{17} $$

in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$ where we act with $n+1$ indecomposable objects of the form B_{usu} or B_{tst} in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{H}$ on $\overline{B}_e$. One easily checks that

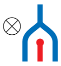
 (18)

is an idempotent in the endomorphism ring of the Bott-Samelson corresponding to the sequence (t, s, t) which is defined in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{H}$. Note that we do not have a classification of the indecomposable objects in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{H}$ and thus B_{tst} above should be understood as the object given by this idempotent (and similarly for B_{usu} after swapping t and u). Denote by e_n the idempotent corresponding to B_n that is obtained by tensoring the idempotents according to (17). Note that e_n lives in the degree 0 piece of the endomorphism ring of the Bott-Samelson corresponding to the sequence

$$ \underline{s}_n := (u, s, u, t, s, t, u, s, u, \dots, \begin{cases} t, s, t, u) & \text{if } n \text{ is odd,} \\ u, s, u, t) & \text{otherwise.} \end{cases} \tag{19} $$

of length $3n+4$. Observe that $\underline{s}_n$ is a reduced expression and denote the corresponding element in W by w_n. A simple calculation shows that e_n postcomposed with any light leaf corresponding to one of the subexpressions

$$(U1,\dots,U1,\underbrace{U1,U0,D0}_{\substack{\text{positions}\\ 3i-2,\,3i-1,\,3i}},U1,\dots,U1) \quad \rightsquigarrow$$

Id $\otimes$  $\otimes$ Id if i is even,

Id $\otimes$  $\otimes$ Id otherwise.

for $1 \leqslant i \leqslant n+1$ is zero (where the diagram on the right hand side corresponds to the three marked positions in the subexpression). The result is the same when precomposing e_n with the flip of one of these light leaves. When referring to this, we say that e_n is killed by a pitchfork.

First, we want to partially calculate the character of B_n. The reader should think of B_n as the analogue of the n-th tensor power of the natural representation for SL_2.

Definition 10.19. For $n, k \geqslant 1$ define the integers c_n^k be the entry in the n-th row and k-th column of the following table which is obtained by setting each entry to be the sum of the one or two entries diagonally below:

$\vdots$							$\iddots$
5		9		5		1	
	5		4		1		
2		3		1			
	2		1				
1		1					
	1						
1							

Remark 10.20. For $SL_2(k)$ we have the following usual identifications $X = \mathbb{Z} = X^\vee$, $\Phi = \Delta = \{2\}$ and $\Phi^\vee = \Delta^\vee = \{1\}$.

Let $V = \nabla(1)$ denote the two-dimensional, standard representation and denote by $L(n)$ the irreducible representation of $SL_2(k)$ with highest weight n. If k is of characteristic 0, then we have the following well-known relations:

$$\begin{aligned} L(n) &\cong \nabla(n) \text{ for all } n \geqslant 0,\\ L(0)\otimes V &\cong L(1) = V,\\ L(n)\otimes V &\cong L(n+1)\oplus L(n-1) \text{ for } n \geqslant 1. \end{aligned}$$

Therefore, the decomposition of $V^{\otimes(n-1)}$ into irreducible representations in characteristic 0 is given by

$$V^{\otimes(n-1)} \cong \bigoplus_{k\leqslant n} L(k-1)^{\oplus c_n^k}.$$

The reader should compare this with the lemma below.

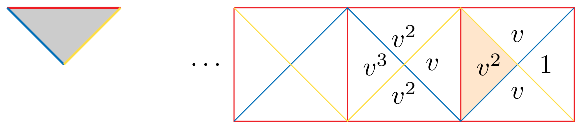
Lemma 10.21. *We have* $\operatorname{ch}(B_n) = \sum_{k\leqslant n} c_{n+1}^{k+1} \underline{N}_{w_k}$ *modulo* $\bigoplus_{x\in A_1} \mathbb{Z}[v,v^{-1}]\underline{N}_x$.

Proof. Using $\operatorname{ch}(B_{usu}) = \underline{H}_u\underline{H}_s\underline{H}_u - \underline{H}_u$ and similarly for $\operatorname{ch}(B_{tst})$, we first verify:

$$\operatorname{ch}(B_0) = \underline{N}_{w_0}$$

$$\mathrm{ch}(B_1) = \underline{N}_{w_1}$$

Calculating the Kazhdan-Lusztig basis in the anti-spherical module, one quickly notices that $\underline{N}_{w_k}$ are given by the same local pattern for $k \geqslant 2$ which we call the *generic pattern*. The following picture illustrates this pattern for k even which we will assume without loss of generality for the rest of this proof. For k odd one simply has to swap the roles of u and t:



(20)

Then we calculate modulo $\bigoplus_{x \in A_1} \mathbb{Z}[v, v^{-1}]\underline{N}_x$:

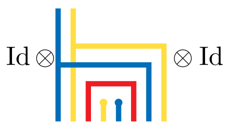
$$\begin{aligned} \underline{N}_{w_k}\underline{H}_s\underline{H}_t &= \underline{N}_{w_k st} + \underline{N}_{w_k} \\ \Rightarrow \underline{N}_{w_k}(\underline{H}_s\underline{H}_t - \underline{H}_e) &= \underline{N}_{w_k st} \\ \underline{N}_{w_k st}\underline{H}_u &= \underline{N}_{w_{k+1}} + \underline{N}_{w_{k-1}} \end{aligned}$$

This shows that the elements $\{\underline{N}_{w_k} \mid k \geqslant 1\}$ exhibit precisely the combinatorics of the Kazhdan-Lusztig basis of the infinite dihedral group under the translation $w_k \rightsquigarrow w_{k+1}$ which is used to populate the defining table for the coefficients c_n^l given above. Therefore, the formula for the characters of e_n follows. $\square$

Next, we want to define certain anti-spherical light leaves. For $k \leqslant n-1$ define LL_n^k to be the light leaf given by the following subexpression of $\underline{s}_n$:

$$(U1, \dots, U1, \underbrace{U1, U1, U1, U0, U0, D1, D0, D0}_{\substack{\text{positions} \\ 3k,\, 3k+1,\, \dots,\, 3k+7}}, U1, \dots, U1)$$

where the marked positions correspond to simple reflections t, u, s, u, t, s, t, u (or u, t, s, t, u, s, u, t) if k is even (resp. odd). Therefore, LL_n^k looks for k even as follows:



where the diagram again corresponds to the marked positions. Observe that LL_n^k ends in s_{n-2} and is of degree 0.

The following proposition summarizes several properties of these light leaves. The reader should think of LL_n^k as a cap connecting the strands k and $k+1$ as a morphism from $n \to n-2$ in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL}$.

Proposition 10.22. *The light leaves LL_n^k for $1 \leqslant k \leqslant n-1$ satisfy the following properties:*

(i) $\mathrm{LL}_n^k \circ e_n \circ (\mathbb{D}\mathrm{LL}_n^k) = -2e_{n-2}$,

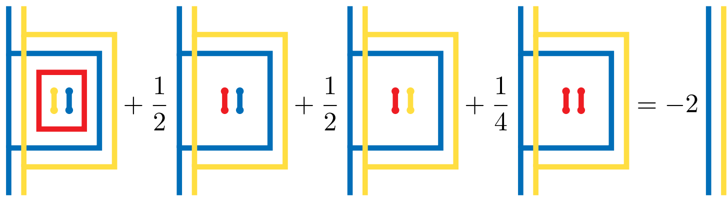
(ii) $\mathrm{LL}_n^k \circ e_n \circ (\mathbb{D}\mathrm{LL}_n^{k-1}) = e_{n-2} = \mathrm{LL}_n^{k-1} \circ e_n \circ (\mathbb{D}\mathrm{LL}_n^k)$ *if* $k \geqslant 2$,

(iii) $\mathrm{LL}_n^k \circ (\mathbb{D}\mathrm{LL}_n^l) = (\mathbb{D}\mathrm{LL}_{n-2}^{l-2}) \circ \mathrm{LL}_{n-2}^k$ *and* $\mathrm{LL}_n^l \circ (\mathbb{D}\mathrm{LL}_n^k) = (\mathbb{D}\mathrm{LL}_{n-2}^k) \circ \mathrm{LL}_{n-2}^{l-2}$ *for* $l \geqslant k+2$,

(iv) $\mathrm{LL}^l_{n-2} \circ \mathrm{LL}^k_n = \mathrm{LL}^k_{n-2} \circ \mathrm{LL}^{l+2}_n$ *for* $l \geqslant k$,

(v) *Any well-defined composition of* l*-terms of the form* LL^k_m *starting in* $\underline{s}_n$ *can be brought in the form* $\mathrm{LL}^{k_l}_{n-2l} \circ \cdots \circ \mathrm{LL}^{k_1}_{n-2} \circ \mathrm{LL}^{k_0}_n$ *with* $k_0 > k_1 > \cdots > k_l$ *and is still an anti-spherical light leaf.*

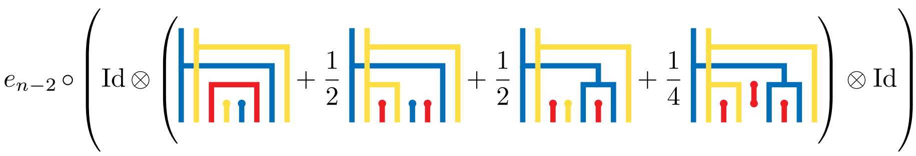
Proof. LL^k_n commutes with any idempotent that one obtains from (18) (or the corresponding version for usu) by tensoring with identities (on the left and on the right) if the idempotent does not occupy the strands $3k+1, 3k+2, \ldots, 3k+6$. For this reason (i) reduces for k even to the following calculation:



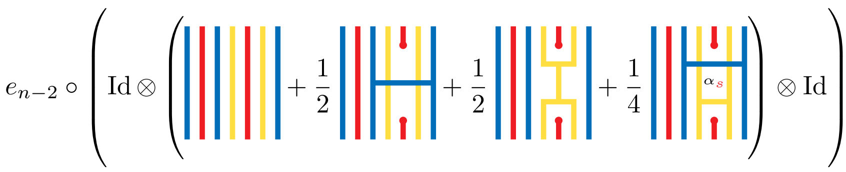
where the end result is based on the following calculation using Demazure operators:

$$\alpha_s\alpha_t \xmapsto{\partial u} -2\alpha_t \xmapsto{\partial t} -4$$
$$\alpha_s^2 \xmapsto{\partial u} -4\alpha_s - 4\alpha_u \xmapsto{\partial t} 8$$

(ii) is a slightly more involved calculation. We will only prove the first equality in (ii), the second one follows from the first one by applying the duality $\mathbb{D}$ and using that all e_n's are self-dual. First, observe that $\mathrm{LL}^k_n \circ e_n$ for k even reduces to:



where the diagrams occupy the strands $3k, 3k+1, \ldots, 3k+7$ of $\underline{s}_n$. Precomposition with $\mathbb{D}\mathrm{LL}^{k-1}_n$ yields:



where the diagrams occupy the strands $3k-2, 3k-1, \ldots, 3k+4$ of s_{n-2}. The third term dies as e_{n-2} is killed by the pitchfork. Sliding α_s through the horizontal u-coloured line in the fourth term, gives another term that dies as e_{n-2} is killed by the resulting pitchfork. The other summand resulting from the application of the nil Hecke relation (see (5)) in the fourth term cancels with the second term as $\partial_u(\alpha_s) = -2$. This finishes the proof of (ii). (iii) and (iv) are immediate from the definitions.

The given normal form in (v) follows from (iv). We prove the last part by induction on the length of this normal form. We may assume that any composition of l-terms of the form LL_m^k are anti-spherical light leaves. Given such a composition of length $l+1$, we bring it into the normal form $\mathrm{LL}_{n-2l}^{k_l} \circ \cdots \circ \mathrm{LL}_{n-2}^{k_1} \circ \mathrm{LL}_n^{k_0}$ with $k_0 > k_1 > \cdots > k_l$.

Note that this composition does not contain any $2m_{s,u}$- or $2m_{s,t}$-valent vertices, but potentially $2m_{t,u}$-valent vertices. Thus, we may speak of the component of a strand, meaning all strands *of the same colour* that are connected to it in a topological sense. It is easy to see that the component of any t or u-coloured strand is a binary tree that, when read from bottom to top, either ends in a dot or intersects non-trivially the top boundary of the diagram.

By the induction hypothesis, the composition of the first l terms gives an anti-spherical light leaf g corresponding to the subexpression $\underline{f}$ of $\underline{s}_n$ ending in $\underline{s}_{n-2l}$ such that

- the first $3k_{l-1}+2$ entries of the decoration of $\underline{f}$ are all $U1$'s,
- for any entry in $\underline{s}_{n-2l}$ following the left- (resp. right-) most branch of the corresponding component in g from the top of the diagram to the bottom, one ends up in a position of $\underline{f}$ decorated with $U1$ (resp. $D0$).

We will explain how to modify $\underline{f}$ to obtain a subexpression of $\underline{s}_n$ that shows that the composition of all $l+1$-terms is still an anti-spherical light leaf. In the case $k_{l-1}-k_l \geqslant 2$, we modify $\underline{f}$ by overwriting the positions $3k_l, 3k_l+1, \ldots 3k_l+7$ (which covers at most the positions $3k_{l-1}$ and $3k_{l-1}+1$) with the following:

$$\ldots, U1, U1, U1, U0, U0, D1, D0, D0, \ldots$$

Note that the remaining decorations do not change as the elements w_{k_l-1} and w_{k_l+1} behave combinatorially in the same way. In other words, for the remaining decorations it does not matter whether we find ourselves after a certain number of steps of the Bruhat stroll in w_{k_l-1} or w_{k_l+1}.

For the case $k_{l-1} = k_l + 1$, the reader should remember that, when adding a cap to a cap-diagram in the Temperley category in such a way that the cap connects two strands that are not neighbouring in the domain, several other caps are "buried" under this cap. In this case, consider the entry at position $3k_l+3 = 3k_{l-1}$ in $\underline{s}_{n-2l}$ and trace the right-most strand of the corresponding component from the top of the diagram representing g to the bottom and denote the resulting position by m. We will change $\underline{f}$ in the positions $m, m+1, \ldots, m+4$ from

$$\ldots, D0, D0, U1, U1, U1, \ldots \quad \text{to} \quad \ldots, D1, D1, D1, D0, D0, \ldots$$

Note that these positions correspond to the simple reflections

$$\begin{cases} \ldots, t, u, s, u, t, \ldots & \text{if } k_l \text{ is odd,} \\ \ldots, u, t, s, t, u, \ldots & \text{otherwise.} \end{cases}$$

Again, the remaining decorations do not change and the result is an anti-spherical light leaf! The reader easily verifies that the three properties stated above are preserved under our modifications in both cases. □

If the isotopy class $[D]$ of a simple (n,k)-diagram contains a representative with only "caps" (local maxima) and no "cups" (local minima), we say that

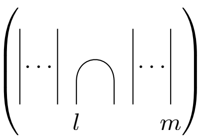
$[D]$ is a caps-diagram. The notion of a cups-diagram is defined similary. The endomorphism ring of any object in the Temperley Lieb category is a cellular algebra as introduced by [GL96]. (Note that there are other cellular structures on the Temperley-Lieb algebra as explained in [AST15, §5.2].) This motivates the following observation: Any crossingless matching $f : n \to m$ can be factored as $f = f_+ \circ f_-$ where $f_- : n \to k$ is a caps-diagram, $f_+ : k \to m$ is a cups-diagram, and $k \leqslant \min\{m, n\}$ is of the same parity as n and m. In this case, f does not factor through any object l with $l < k$ and we say that f has k through-strands. We will need the following easy lemma from [Che14, Corollary 2.2.6]:

Lemma 10.23. *The class $\mathcal{I}_k$ of all morphisms with at most k through-strands forms a two-sided ideal in* $\mathbf{TL}$.

In the quotient $\mathbf{TL}/\mathcal{I}_{l-1}$ the Hom-space $\mathrm{Hom}(n, l)$ is a free $\mathbb{Z}[\delta]$-module with basis given by the set $\mathrm{Caps}(n, l)$ of all caps-diagrams $n \to l$. Similarly, the set $\mathrm{Cups}(l, n)$ of all cups-diagrams is a basis of the free $\mathbb{Z}[\delta]$-module $\mathrm{Hom}(l, n)$ in the quotient $\mathbf{TL}/\mathcal{I}_{l-1}$. The following lemma is immediate:

Lemma 10.24. $\mathbb{D}$ *gives a bijection* $\mathrm{Caps}(n, l) \longrightarrow \mathrm{Cups}(l, n)$.

Finally, we can define the functor $\mathcal{F} : {}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL} \longrightarrow {}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$. $\mathcal{F}$ sends $n \in \mathbb{N}$ to the object B_n in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$. Note that B_n is self-dual. We will define $\mathcal{F}$ on any crossingless matching $f : n \to m$ and extend the functor $\mathbb{Z}[\frac{1}{2}]$-linearly. We write $f = f_+ \circ f_-$ and assume without loss of generality that f has l through-strands. Since we want $\mathcal{F}$ to satisfy $\mathcal{F} \circ \mathbb{D} = \mathbb{D} \circ \mathcal{F}$, it is enough to define $\mathcal{F}$ on $f_- \in \mathrm{Caps}(n, l)$ by Lemma 10.24. Since f_- has an isotopy representative with only caps and no cups, $\mathcal{F}(f_-)$ is uniquely determined by setting

$\mathcal{F}$  $= e_{m-2} \circ \mathrm{LL}^l_m \circ e_m$

for all $l \leqslant m - 1$ and $m \in N$.

Corollary 10.25. *$\mathcal{F}$ is a well-defined, $\mathbb{Z}[\frac{1}{2}]$-linear functor that commutes with $\mathbb{D}$ and whose image consists of degree* 0 *morphisms.*

Proof. All stated properties of $\mathcal{F}$ are clear from the definition. It remains to check that $\mathcal{F}$ is well-defined. Observe that we used a specific representative in the isotopy class of a crossingless matching to define $\mathcal{F}$ on morphisms. For this reason, we need to show that the image of the functor does not depend on this choice and that $\mathcal{F}$ is well-defined under composition. Proposition 10.22(ii), (iii) and (iv) show that all isotopy relations also hold in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$. Composing two crossingless matchings in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL}$ may result in closed components. Proposition 10.22 (i) shows that the removal of closed components in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{TL}$ and in ${}^{\mathbb{Z}[\frac{1}{2}]}\mathbf{N}$ yields the same result. □

Next, we are going to show that, after extending scalars to k, the functor $\mathcal{F}_k$ maps $\mathrm{Caps}(n, l)$ for all $l \leqslant n$ to a linear independent set of morphisms which projects to a generating set of morphisms $B_n \to B_l$ in degree 0 modulo lower terms in the quotient ${}^{k}\mathbf{N}/J_{\mathrm{sph}}$.

Lemma 10.26. *There is a bijection*

$$\mathrm{Caps}(n, l-1) \cup \mathrm{Caps}(n, l+1) \longrightarrow \mathrm{Caps}(n+1, l)$$

given by adding a strand on the right to an element in $\mathrm{Caps}(n, l-1)$ *or adding a strand on the right to an element in* $\mathrm{Caps}(n, l+1)$ *and post-composing with a cap on the strands* $l+1$ *and* $l+2$.

Proof. Given an element in $\mathrm{Caps}(n+1, l)$, consider the isotopy representative D with only caps and no cups. The right-most strand in D is either a through-strand or contains a cap. In the first case, it can be removed to obtain a representative of an isotopy class in $\mathrm{Caps}(n, l-1)$. In the second case, add a strand on the right to D and pre-compose with a cup on the strands $n+1$ and $n+2$. By straightening the corresponding strand, we find in the isotopy class of the resulting simple $(n, l+1)$-diagram a representative in $\mathrm{Caps}(n, l+1)$. This procedure gives an inverse to the map described above and finishes the proof. □

Recall that $\mathrm{Caps}(n, l)$ is empty if n and l are not of the same parity. In addition, $\mathrm{Caps}(0, 0)$ and $\mathrm{Caps}(1, 1)$ both contain a single element - the empty diagram is contained in $\mathrm{Caps}(0, 0)$. These observations together with Lemma 10.26 imply the following result, as the set $\mathrm{Caps}(n, l)$ is governed by the same combinatorics as the coefficients c_{n+1}^{l+1}:

Corollary 10.27. *We have for* $n, l \geqslant 0$*:* $|\mathrm{Caps}(n, l)| = c_{n+1}^{l+1} = |\mathrm{Cups}(l, n)|$

Remark 10.28. Back to the setting of $SL_2(k)$ representations for an algebraically closed field k of characteristic 0 or $p > 2$: Arguing as in [EL17a, Lemma A.7] (or [DPS98, Theorem 6.3] and [Här99]), we get an isomorphism ${}^k\mathrm{TL}_n \cong \mathrm{End}_{SL_2(k)}(V^{\otimes n})$. Therefore, the Temperley Lieb algebra ${}^k\mathrm{TL}_n = \mathrm{End}_{{}^k\mathbf{TL}}(n)$ contains the idempotents corresponding to the indecomposable summands in $V^{\otimes n}$.

In characteristic 0, the canonical idempotent corresponding to $L(n) \overset{\oplus}{\subseteq} V^{\otimes n}$ is known as the Jones-Wenzl projector as it was studied independently by Jones in [Jon86] and Wenzl in [Wen87].

In characteristic $p > 2$, V is a tilting module and thus $V^{\otimes n}$ can be decomposed into indecomposable tilting modules. The idempotents giving the indecomposable summands isomorphic to $T(l)$ in $V^{\otimes n}$ can, for example, be obtained in the following way: Consider the following intersection pairing

$$\mathrm{Hom}_{\mathbf{TL}/\mathcal{I}_{l-1}}(n, l) \times \mathrm{Hom}_{\mathbf{TL}/\mathcal{I}_{l-1}}(n, l) \longrightarrow \mathrm{End}_{\mathbf{TL}/\mathcal{I}_{l-1}}(l) = k$$

that maps (f, g) to $g \circ (\mathbb{D}f)$. Recall that $\mathrm{Caps}(n, l)$ gives a basis of $\mathrm{Hom}_{\mathbf{TL}/\mathcal{I}_{k-1}}(n, k)$. First, we bring the matrix representing this pairing in this basis into the Smith normal form. By keeping track of the bases in the process, we obtain the projections and inclusions and thus the primitive orthogonal idempotents for the indecomposable summands isomorphic to $T(k)$. Finally, the idempotent for $T(n)$ is the complementary idempotent to the sum of all idempotents obtained in this way for $k < n$.

For $x \in {}^fW$ denote by ${}^k\mathbf{N}^{\not< x}$ the quotient of ${}^k\mathbf{N}$ by the two-sided ideal of morphisms factoring through a direct sum of grading shifts of reduced expressions $\underline{y}$ for $y < x$. By slight abuse of notation, we will write $\mathrm{Hom}_{\not< x}(-, -)$ for the homomorphism spaces in ${}^k\mathbf{N}^{\not< x}$ (as we only work with homomorphisms in the anti-spherical module and not the Hecke category in this section).

For two elements $x < y$ in ${}^fW$, let $\underline{y}$ be an arbitrary expression for y and $\underline{x}$ be a reduced expression for x. The definition of light leaves is tailor-made to give bases of the Hom-spaces $\mathrm{Hom}_{\not< x}(\underline{y}, \underline{x})$ that may be translated along as one calculates the character of $\underline{y}$. By definition of the diagrammatic character, the coefficient in front of N_x in $\mathrm{ch}(B)$ gives the graded rank of the free R_{S_f}-module $\mathrm{Hom}_{\not< x}(B, x)$ for an arbitrary object $B \in {}^k\mathbf{N}$.

For $n \geqslant 1$ denote by

$$\underline{\widetilde{s}}_n := (u, s, u, t, s, t, u, s, u, \dots, \begin{cases} u, s, u, t, s) & \text{if } n \text{ is odd,} \\ t, s, t, u, s) & \text{otherwise.} \end{cases}$$

of length $3n+2$. $\underline{\widetilde{s}}_n$ is a reduced expression for the element $\widetilde{w}_n := w_n tu$ of length $l(\widetilde{w}_n) = l(w_n) - 2$. Observe that $\underline{s}_k$ and $\underline{\widetilde{s}}_k$ are starting pieces of $\underline{s}_n$ for $k \leqslant n$.

For any element $x \in {}^fW$ fix a reduced expression $\underline{x}$ such that there is no other reduced expression of x for which a longer starting piece coincides with a starting piece of $\underline{s}_n$ for $n \gg 0$. In order to show that the functor $\mathcal{F}_k$ post-composed with the quotient functor ${}^k\mathbf{N} \to {}^k\mathbf{N}/J_{\mathrm{sph}}$ is full in degree 0, we will construct for $n \geqslant 0$ a set $\mathcal{B}_n$ of morphisms

$$\bigcup_{v \in {}^fW} \{ f_{\underline{e}} : \underline{s}_n \to \underline{v} \mid \underline{e} \text{ subexpression of } \underline{x} \text{ expressing } v \}$$

using the light leaves algorithm. Moreover, we will define for $l \leqslant n$ with $l \equiv n \pmod 2$ a subset of anti-spherical light leaves $\mathbf{LLs}(n,l) \subset \mathcal{B}_n$ from $\underline{s}_n \to \underline{s}_l$ of degree 0. Our construction will ensure that

(i) for any element $v \leqslant w_n$, the set

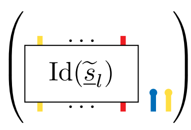
$$\{ f_{\underline{e}} \in \mathcal{B}_n \mid \underline{e} \text{ subexpression of } \underline{s}_n \text{ expressing } v \}$$

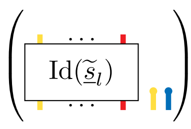
gives a basis of the free R_{S_f}-module $\mathrm{Hom}_{\not< v}(\underline{s}_n, v)$,

(ii) pre-composing $\mathbf{LLs}(n,l)$ with e_n yields $\mathcal{F}_k(\mathrm{Caps}(n,l))$,

(iii) any element $f_{\underline{e}} \in \mathcal{B}_n \setminus \mathbf{LLs}(n,l)$ indexed by a subexpression $\underline{e}$ of $\underline{s}_n$ expressing w_l lies in $\mathcal{J}^\bullet_{\mathrm{sph}}(\underline{s}_n, \underline{s}_l)$ or is zero when pre-composed with e_n.

(iv) Denote by $\widetilde{\mathbf{LLs}}(n,l)$ the following set:


$\circ\, ll \;\Big|\; ll \in \mathbf{LLs}(n,l) \Big\}$ if l is even,


$\circ\, ll \;\Big|\; ll \in \mathbf{LLs}(n,l) \Big\}$ otherwise.

Then any element $f_{\underline{e}} \in \mathcal{B}_n \setminus \widetilde{\mathbf{LLs}}(n,l)$ indexed by a subexpression $\underline{e}$ of $\underline{s}_n$ expressing $\widetilde{w}_l$ lies in $\mathcal{J}^\bullet_{\mathrm{sph}}(\underline{s}_n, \underline{\widetilde{s}}_l)$ or is zero when pre-composed with e_n.

Therefore, properties (ii) and (iii) will immediately imply the following result:

Proposition 10.29. ${}^k\mathbf{TL} \xrightarrow{\mathcal{F}_k} {}^k\mathbf{N} \longrightarrow {}^k\mathbf{N}/J_{sph}$ *is full in degree* 0.

Let us first prove some auxiliary results:

Lemma 10.30. $\mathcal{J}_{sph}$ *is a tensor ideal, i.e. is stable under tensoring with* $B_s \in {}^k\mathbf{H}$ *for* $s \in S$.

Proof. Corollary 10.1 implies that the full subcategory $\langle B_x \mid x \in A_1 \rangle_{(-),\oplus}$ in ${}^k\mathbf{N}$ is stable under the action of the generators $B_s \in {}^k\mathbf{H}$ for $s \in S$. Therefore, the claim follows. □

Recall that, for two expressions $\underline{x}$ and $\underline{y}$ in S, we denote their concatenation by $\underline{x}{}^\frown\underline{y}$. In the process of the light leaves algorithm, we may interrupt the algorithm after completing all the steps for a suitable starting piece of our expression, modify the basis modulo lower terms that we have and continue the algorithm. For any morphism produced by the algorithm we allow at most one of the following modifications in the course of the algorithm:

(i) Let $\mathrm{LL}_{\underline{x},\underline{e}}$ be an anti-spherical light leaf corresponding to a subexpression $\underline{e}$ of $\underline{x}$ of length n such that the subexpression $\underline{e}_{\leqslant n-1}$ of $\underline{x}_{\leqslant n-1}$ avoids A_1, but $\underline{e}$ does not. Let $r \in S$ be the last reflection in $\underline{x}$. This implies that

$$(\underline{x}_{\leqslant n-1}^{\underline{e}_{\leqslant n-1}})_{\bullet} r \in A_1$$

and denote the corresponding element by y. Observe that the last bit of $\underline{e}$ is decorated with a U. Consider the reduced expression $\underline{y}$ of y obtained from $\underline{x}_{\leqslant n-1}^{\underline{e}_{\leqslant n-1}}{}^\frown(r)$ and the idempotent $e_{\underline{y}}$ for ${}^kB_y$ in the corresponding anti-spherical Bott-Samelson. In this case, we will replace $\mathrm{LL}_{\underline{x},\underline{e}}$ by the morphism obtained by completing the last step of the light leaves algorithm using

$$e_{\underline{y}} \circ (\mathrm{LL}_{\underline{x}_{\leqslant n-1},\underline{e}_{\leqslant n-1}} \otimes \mathrm{Id}_{{}^kB_r})$$

instead of $\mathrm{LL}_{\underline{x}_{\leqslant n-1},\underline{e}_{\leqslant n-1}} \otimes \mathrm{Id}_{{}^kB_r}$ (see [EW16, Construction 6.1]).

(ii) Given two elements $x, y \in {}^fW \setminus A_1$ and reduced expressions $\underline{x}$ and $\underline{y}$ for them, let $f : \underline{x} \to \underline{y}$ be a morphism factoring through an indecomposable object indexed by an element in A_1. Assume that we can explicitly express f in terms of the unmodified light leaves basis of $\mathrm{Hom}^{\bullet}_{\not< y}(\underline{x}, \underline{y})$. Assume further that there exists a subexpression $\underline{e}$ of $\underline{x}$ that is the maximum with respect to the path dominance order among all the subexpressions of maximal defect indexing light leaves that occur with non-trivial coefficient in the expression for f. If the coefficients in front of $\mathrm{LL}_{\underline{x},\underline{e}}$ is invertible in $R_{S_{\mathrm{f}}}$ and $\mathrm{LL}_{\underline{x},\underline{e}}$ is still part of the (modified) basis at that stage of the algorithm, then we can replace $\mathrm{LL}_{\underline{x},\underline{e}}$ by f.

The first modification allows us to replace every anti-spherical light leaves $\mathrm{LL}_{\underline{x},\underline{e}} : \underline{x} \to \underline{w}$ corresponding to a subexpression $\underline{e}$ of $\underline{x}$ that does not avoid A_1 by a map in $\mathcal{J}_{\mathrm{sph}}$. Similarly, the second modification allows to replace certain anti-spherical light leaves by maps in $\mathcal{J}_{\mathrm{sph}}$. Using Lemma 10.30 and the definition of an ideal in a category, we see that the corresponding maps remain in $\mathcal{J}_{\mathrm{sph}}$ when continuing the light leaves algorithm.

In the following lemma, we show that both of these modifications preserve the property of being a basis modulo lower terms after continuing the algorithm:

Lemma 10.31. *Let $x, y \in {}^f W \setminus A_1$ and $\underline{x}$ (resp. $\underline{y}$) be a reduced expression for x (resp. y). Performing the anti-spherical light leaves algorithm and allowing modifications as described above still produces a basis of the free R_{S_f}-module $\mathrm{Hom}^{\bullet}_{\not\leq y}(\underline{x}, \underline{y})$.*

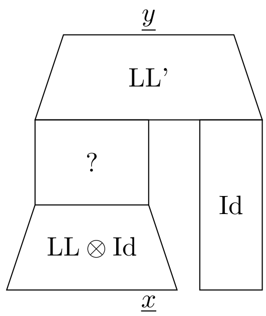
Proof. Let us express the set of morphisms obtained from the algorithm while allowing the modifications in the set of light leaves produced by the light leaves algorithm using the same choices, but not allowing the modifications in between. We want to argue that the corresponding matrix is upper triangular with invertible elements on the diagonal. For this we want to order the basis using a total order refining the lexicographic order on

$$\mathbb{Z} \times \{\underline{e} \mid \underline{e} \text{ subexpression of } \underline{x} \text{ expressing } y\}$$

where the second factor is equipped with the path dominance order and where we associate to each basis element $ll_{\underline{e}}$ the pair $(\deg(ll_{\underline{e}}), \underline{e})$.

For the second type of modification it is clear that after replacing a suitably chosen basis element by a non-trivial linear combination and continuing the light leaves algorithm, the coefficients in the linear combination do not change and only the corresponding light leaves are modified by the algorithm. Thus, the claim follows for this morphism from our assumptions.

For a morphism to which we applied a modification of the first type, we get after continuing the algorithm

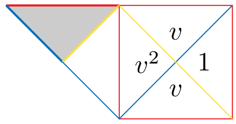


(21)

where the question mark represents the idempotent we introduced. We know that the idempotent is an element of $\mathrm{End}_{{}^{\Bbbk}\mathbf{N}}(\exp z)$ for some reduced expression $\underline{z}$. Observe that the idempotent we introduced corresponds to the "big" indecomposable object in $\underline{z}$, i.e. when expressing it in terms of the anti-spherical double leaves basis the coefficient in front of the identity is 1. We want to argue that, when expressing this morphism in terms of the anti-spherical light leaves, the light leaf corresponding to the same subsequence occurs with coefficient 1 and all other terms are indexed by elements strictly smaller in the path dominance order. Observe that, when expressing the idempotent in the anti-spherical double leaves basis, all occuring terms apart from the identity factor through a sequence of length two smaller than $\underline{z}$. Using "path dominance upper-triangularity" of light leaves (see [EW16, Proposition 6.6]) and localization techniques in ${}^k\mathbf{H}$, we can deduce the claim in ${}^k\mathbf{N}$ following closely the proof of [RW16, Lemma 4.9]. $\square$

In our case, the idempotents e_n are so close to the Bott-Samelson objects $\underline{s}_n \in {}^k\mathbf{N}$ that we can show that the set $\mathcal{F}_k(\mathrm{Caps}(n,l))$ is linear independent in $\mathrm{Hom}_{\not< w_l}(B_n, w_l)$. It is still not clear to us, though, whether the image of the set $\mathcal{F}_k(\mathrm{Caps}(n,l))$ is linear independent in $\mathrm{Hom}(B_n, w_l)$ in the quotient $({}^k\mathbf{N}/\mathcal{J}_{\mathrm{sph}})^{\not< w_l}$.

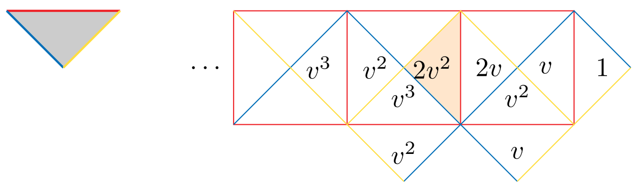
We will construct the sets $\mathcal{B}_n$ and $\mathbf{LLs}(n,l)$ by induction on n. The main idea of the proof is to translate a basis modulo lower terms along as we calculate the character of B_n. For the induction start, the reader easily verifies that the following picture gives $\mathrm{ch}(B_0)$:



(22)

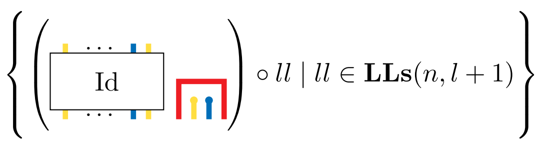
We set $\mathbf{LLs}(0,0) = \{\mathrm{Id}(\underline{s}_0)\}$ (corresponding to the coefficient 1 in (22)) and choose arbitrary anti-spherical light leaves to extend $\mathbf{LLs}(0,0)$ to $\mathcal{B}_0$. It is easy to see that the claims (i) - (iv) hold. For the induction step from n to $n+1$, let us assume that the statement of the claims (i) to (iv) hold for $l \leqslant n$ with $l \equiv n \pmod 2$. Observe that all the claims about $\mathbf{LLs}(n+1,l)$ are clear for $l = n+1$. Thus, fix $l < n+1$ with $l \equiv n+1 \pmod 2$.

First, we deal with the more difficult case of explaining the contribution of $\mathbf{LLs}(n,l+1)$ to $\mathbf{LLs}(n+1,l)$ as in Lemma 10.26. We will illustrate the translation process by translating $\underline{N}_{w_{l+1}}$ along. The reader should imagine that the alcove with the coefficient 1 in (20) corresponds to w_{l+1}. Apart from the alcove corresponding to the identity in W, we will mark another alcove for which we are most interested in the corresponding light leaves. Without loss of generality n and $l+1$ are even (to match the pictures). Applying $\underline{H}_s$ to $\underline{N}_{w_{l+1}}$ yields:



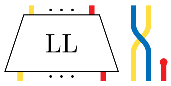
(23)

The set $\widetilde{\mathbf{LLs}}(n,l+1)$ (given by property (iv)) contains anti-spherical light leaves ending in $\widetilde{s}_{l+1}$, which is a reduced expression for the element corresponding to the orange-shaded alcove in (20). We want to translate these along. In the last step, they moved down and contributed to the coefficient in the orange-shaded alcove in (23) in the following form:



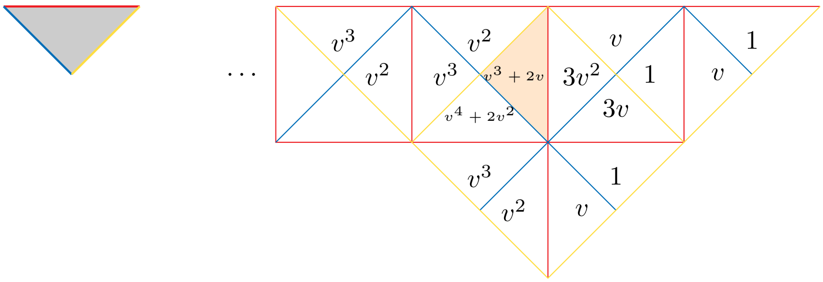
They mingle with (at least) $|\mathbf{LLs}(n,l+1)|$ anti-spherical light leaves of degree

1 in that alcove that are of the form:



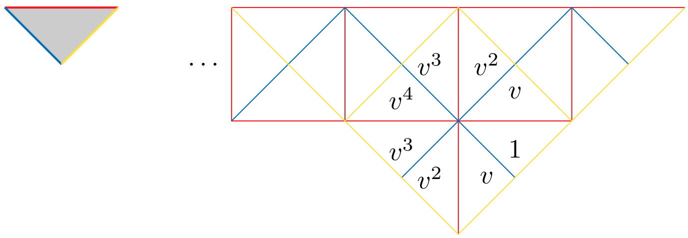
(24)

Multiplying with $\underline{H}_t$ gives:



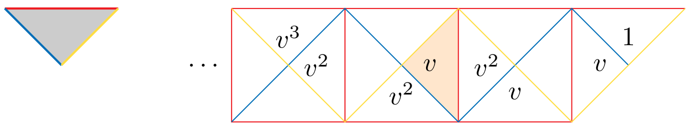
(25)

A calculation shows that $\underline{N}_{w_{l+1}us}$ is given by the following picture:



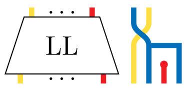
(26)

We will justify the removal of this piece from the picture later. Moreover, since we want to act with B_{tst} and not with the Bott-Samelson corresponding to (t, s, t), we need to remove $\underline{N}_{w_{l+1}}$ on the level of the characters. The result is:



(27)

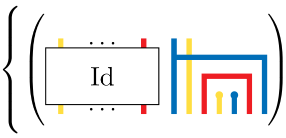
On the level of light leaves, this corresponds to pre-composing with an idempotent that is a horizontal concatenation of idempotents of the form given in (18). As mentioned for e_n, such an idempotent is killed by certain pitchforks. For this reason, among the light leaves that give 0 when pre-composed with such an idempotent are those to which we applied first a $U0$ when translating along s and then a $D0$ for t. Examples of those light leaves are the ones given in (24) which contribute to the orange-shaded alcove in (25) as follows:



(28)

Thus, all light leaves of this form give 0 when pre-composed with the idempotent. And this remains true for any anti-spherical extension of these morphisms using the light leaves algorithm. This argument or degree reasons justify the removal of $\mathrm{ch}(B_{w_{l+1}})$ from the picture. Note that the corresponding morphisms will still be further translated along and added to $\mathcal{B}_{n+1}$ in the end. Since we can identify enough anti-spherical light leaves from $\underline{s}_n{}^\frown(s,t)$ to $\underline{s}_l$ to remove, it will follow that the set $\mathcal{F}_k(\mathrm{Caps}(n+1,l))$ is linear independent in $\mathrm{Hom}_{\not< w_l}(B_{n+1}, \underline{s}_l)$ using the anti-spherical Hom-formula from Section 7.3.

Applying a $D0$ to the set of light leaves we are interested in gives:



$$\circ\, ll \mid ll \in \mathbf{LLs}(n, l+1) \Big\} \tag{29}$$

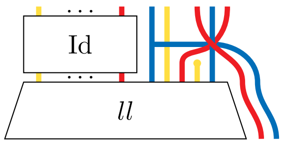
Let us come back to the removal of several copies of $\underline{N}_{w_{l+1}us}$: For every coefficient in (26) there exists a subexpression $\underline{e}$ of $\underline{s}_n{}^\frown(s,t)$ such that the subexpression itself does not avoid A_1, but removing the last two entries results in a subexpression of $\underline{s}_n$ that avoids A_1. According to Lemma 10.31, we may replace the corresponding anti-spherical light leaves by elements in $\mathcal{J}_{\mathrm{sph}}$, preserving the basis property modulo lower terms. As explained in Lemma 10.31, any anti-spherical extension of these morphisms will be in $\mathcal{J}_{\mathrm{sph}}$. It should also be noted that, even though we remove the corresponding coefficients here, we still have to translate the corresponding basis elements along to add to $\mathcal{B}_{n+1}$ in the end. The attentive reader will most likely have noticed that there are the following two minor gaps in our argument:

- For the alcoves corresponding to the elements $w_l s$ and $w_l st$, the subexpressions of $\underline{s}_n{}^\frown(s,t)$ mentioned above that do not avoid A_1 correspond to anti-spherical light leaves that give 0 when pre-composed with an idempotent that is a tensor product of idempotents of the form given in (18).

  We will explain how to proceed for $w_l s$, the situation for $w_l st$ is similar and easier. In this case, we want to calculate the composition of an anti-spherical light leaf LL which we have translated along starting in $\underline{s}_n{}^\frown(s,t)$ and ending in $\underline{s}_l{}^\frown(s,t,s)$, followed by the idempotent for ${}^kB_{w_l sts}$ (in the correct Bott-Samelson) and the unique anti-spherical light leaf from $\underline{s}_l{}^\frown(s,t,s)$ to $\underline{s}_l{}^\frown(s)$. Even though we do not know the idempotent for ${}^kB_{w_l sts}$, one can still compute this composition in ${}^k\mathbf{N}^{\not< w_l s}$ for the following reasons: We want to argue that only the identity term in the idempotent for ${}^kB_{w_l sts}$ can contribute to this composition. Indeed, observe that the element $w_l s$ is of length $l(w_{l+1}us) - 2 = l(w_l + 1)$. Moreover, one can check that the only subexpressions of $\underline{s}_l{}^\frown(s,t,s)$ expressing $w_l s$ of defect zero are of the form

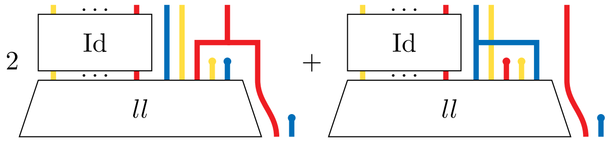
  $$(U1, \ldots, U1, \begin{cases} U1, U0, D0, U1) & \text{or} \\ U1, U1, U0, D0). \end{cases}$$

  Thus, both contain a pitchfork on the $\{s,t\}$-coloured stands among the last five letters. Moreover, the projections (and inclusions) potentially realizing ${}^kB_{w_l s}$ as a summand of the Bott-Samelson corresponding to $\underline{s}_l{}^\frown(s,t,s)$ are linear combinations of the corresponding anti-spherical light leaves (or their flips). By retracing the translation process of LL, we see that it is of the following form



for an element $ll \in \mathbf{LLs}(n, l+1)$ (using a consequence of property (iv)). Thus LL is killed by the pitchforks mentioned above and the claim follows for codimension reasons.

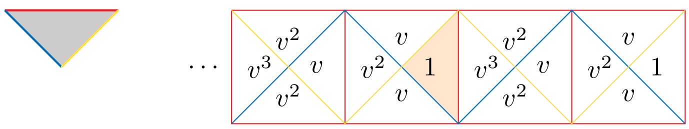
The resulting composition can be explicitly calculated to determine a non-zero morphism factoring through ${}^kB_{w_l sts}$ that we can use to perform a modification of the second type. For this, add dots on the last two strands and calculate modulo lower terms and terms that die when pre-composed with a tensor product of idempotents of the form given in (18). The result is:



The first term is one of anti-spherical light leaves contributing to the coefficient in the alcove of $w_l s$. The second term can be written as a sum of anti-spherical light leaves that are indexed by a subsequence strictly smaller in the path dominance order. To see this, note that we can rewrite the second term without the last red and blue dotted strand in terms of anti-spherical light leaves $\underline{s}_n \longrightarrow \underline{s}_l$ modulo lower terms and use "path dominance upper triangularity" of light leaves and localization in ${}^k\mathbf{H}$.

- Our argument justifying the removal of the coefficients corresponding to $\underline{N}_{w_{l+1}us}$ does not cover the case of translating $\underline{N}_{w_0}$ to $\underline{N}_{w_1}$. In this case, one can explicitly calculate the idempotent for the modular indecomposable object corresponding to $w_1 ts$. Similar arguments as above allow us to perform enough modifications of the second type to justify the removal of the coefficients.

Finally, applying $\underline{H}_u$ to the character shown in (27) yields:



(30)

This last step amounts to applying another $D0$ to the anti-spherical light leaves given in (29), so that we get

$$\{\mathrm{LL}_{l+2}^{l+1} \circ ll \mid ll \in \mathbf{LLs}(n, l+1)\}.$$

This will be the contribution of $\mathbf{LLs}(n, l+1)$ to $\mathbf{LLs}(n+1, l)$. What we have done corresponds on the side of the Temperley-Lieb category precisely to adding another strand on the right and post-composing with a cap on the strands $l+1$ and $l+2$.

Next, we deal with the contribution of $\mathbf{LLs}(n, l-1)$ to $\mathbf{LLs}(n+1, l)$ as in Lemma 10.26. Since n and $l-1$ are of the same parity, the sequence of simple reflections needed to extend $\underline{s}_n$ to $\underline{s}_{n+1}$ is the same sequence needed to extend $\underline{s}_{l-1}$ to $\underline{s}_l$. In the case illustrated above, this sequence was (s, t, u). For this reason, when translating the light leaves in $\mathbf{LLs}(n, l-1)$ along, we have to apply a $U1$ in each step, simply adding a new strand on the right. The resulting anti-spherical light leaves will be the contribution of $\mathbf{LLs}(n, l-1)$ to $\mathbf{LLs}(n+1, l)$ and finish the construction of $\mathbf{LLs}(n+1, l)$. Observe that the whole translation process again corresponds on the side of the Temperley-Lieb category to adding another strand on the right. Therefore, comparing the construction of $\mathbf{LLs}(n+1, l)$ with the proof of Lemma 10.26, we see that property (ii) is preserved. Property (iv) is easily seen to hold, when following the path of the corresponding light leaves in the translation process.

The set $\mathcal{B}_{n+1}$ is obtained by translating the morphisms in $\mathcal{B}_n$ and modifying them as described above for all $l < n+1$ of the same parity. Property (i) follows from the correctness of the light leaves algorithm and property (iii) is clear from the construction.

The reader should compare the following result with Proposition 9.3:

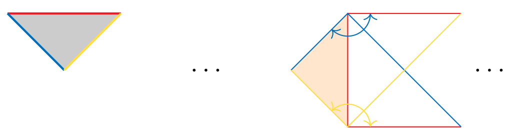
Corollary 10.32. *The Kazhdan-Lusztig right cell B_1 decomposes into right p-cells as follows:*

$$B_1 = \bigcup_{r \in \mathbb{N}} \{w_l, w_l s, w_l st, w_l su \mid p^r - 1 \leqslant l < p^{r+1} - 1\}.$$

Proof. First, observe that the Kazhdan-Lusztig right cell B_1 is partitioned into clusters of four elements as follows:

$$B_1 = \bigcup_{l \geqslant 0} \{w_l, w_l s, w_l st, w_l su\}$$

The alcoves of a cluster are depicted in the following picture where w_l is the orange-shaded alcove:



(31)

Applying Lemma 5.10 to the strings shown in the last picture gives us that the four elements of each cluster lie in the same right p-cell.

Note that the elements $\{w_l \mid l \geqslant 0\}$ are the only elements in B_1 with $\{t, u\}$ as right descent set. Combining this with Proposition 3.10(v) and the results from Section 10.1, we get

$${}^p\underline{H}_{w_n} = \sum_{l \leqslant n} {}^p m_{w_l, w_n} \underline{H}_{w_l} \qquad (\mathrm{mod} \bigoplus_{x \in A_1} \mathbb{Z}[v, v^{-1}] \underline{H}_x).$$

In the quotient ${}^k\mathbf{N}/\mathcal{J}_{\mathrm{sph}}$ an indecomposable object B_x for $x \in {}^fW$ dies if and only if $x \in A_1$. Thus, when decomposing B_n into indecomposable summands to determine the idempotent for B_{w_n} in ${}^k\mathbf{N}$, the quotient ${}^k\mathbf{N}/\mathcal{J}_{\mathrm{sph}}$ gives an approximation of this idempotent up to terms corresponding to indecomposable summands indexed by elements in A_1.

Lemma 10.21 shows that in order to calculate the idempotent for B_{w_n} we need to calculate only the degree 0 local intersection forms of B_n at B_{w_l} for $l < n$. Fix $l < n$. Recall that $\mathcal{F}_k(\mathrm{Caps}(n,l)) \overset{\text{(ii)}}{=} \mathbf{LLs}(n,l) \circ e_n$ is a linear independent subset of $\mathrm{Hom}_{{}^k\mathbf{N}^{\not<w_l}}(B_n, w_l)$. By property (iii) we can extend $\mathbf{LLs}(n,l)$ with elements from $\mathcal{B}_n$ that lie in $\mathcal{J}^\bullet_{\mathrm{sph}}(B_n, B_{w_l})$ to a basis of $\mathrm{Hom}^\bullet_{{}^k\mathbf{N}^{\not<w_l}}(B_n, w_l)$. Denote by $I^0_{B_n,w_l}$ the matrix representing the degree 0 local intersection form of B_n at B_{w_l} with respect to this basis. Denote by A the matrix representing the intersection form

$$\begin{aligned}\mathrm{Hom}_{{}^k\mathbf{TL}/\mathcal{I}_{l-1}}(n,l) \times \mathrm{Hom}_{{}^k\mathbf{TL}/\mathcal{I}_{l-1}}(n,l) &\longrightarrow \mathrm{End}_{{}^k\mathbf{TL}/\mathcal{I}_{l-1}}(l) = k \\ (g,f) &\longmapsto g \circ \mathbb{D}f\end{aligned}$$

with respect to the basis $\mathrm{Caps}(n,l)$. It follows that A gives a $c^{l+1}_{n+1} \times c^{l+1}_{n+1}$-submatrix of $I^0_{B_n,w_l}$. Therefore, we have $[B_n : B_{w_l}] \geqslant [V^{\otimes n} : T(l)]$ where $[V^{\otimes n} : T(l)]$ is the multiplicity of $T(l)$ in $V^{\otimes n}$ for $SL_2(k)$.

We claim that we actually have equality. In other words, the rank of $I^0_{B_n,w_l}$ coincides with the rank of A. Pairing any basis element in $\mathcal{J}^\bullet_{\mathrm{sph}}(B_n, B_{w_l})$ with any other basis element has to give 0. Otherwise, B_{w_l} would be a summand of an object X whose identity lies in $\mathcal{J}_{\mathrm{sph}}$. In particular, B_{w_l} would vanish in ${}^k\mathbf{N}/\mathcal{J}_{\mathrm{sph}}$ contradicting the fact that $w_l \notin A_1$.

We claim:

$$ {}^p m_{w_l,w_n} = [T(n) : \nabla(l)] \text{ for all } 0 \leqslant l \leqslant n \qquad (*_n)$$

where $[T(n) : \nabla(l)]$ is the multiplicity of $\nabla(l)$ in $T(n)$ for $SL_2(k)$ as given in Lemma 9.1. For the proof, we proceed by induction on n. The equality ${}^p\underline{N}_{w_0} = \underline{N}_{w_0}$ gives the induction start. For the induction step, we assume $(*_l)$ to hold for $l < n$. First, express ${}^p\underline{N}_{w_n} = \mathrm{ch}(B_n) - \sum_{l<n}[V^{\otimes n} : T(l)]\, {}^p\underline{H}_{w_l}$ in the Kazhdan-Lusztig basis using Lemma 10.21. Then apply the induction hypothesis and the formula

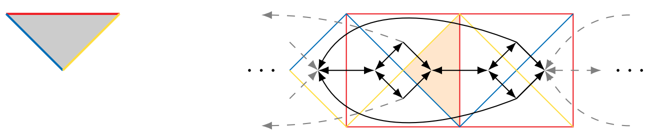
$$[T(n) : \nabla(m)] = c^{m+1}_{n+1} - \sum_{m \leqslant l < n} [V^{\otimes n} : T(l)][T(l) : \nabla(m)]$$

to deduce $(*_n)$.

Since we can determine the modular weight cells for $SL_2(k)$ by hand as mentioned in Remark 9.4, we know at this point that the following partition gives a refinement of the right p-cells into which B_1 decomposes:

$$B_1 = \bigcup_{r \in \mathbb{N}} \{w_l, w_l s, w_l st, w_l su \mid p^r - 1 \leqslant l < p^{r+1} - 1\}$$

We need to check that no sequence of simple reflections taking an element B_1 up introduces additional right p-cell relations between these clusters. Similar calculations as above show that the action of $\underline{H}_s$ for $s \in S$ on the Kazhdan-Lusztig basis of the cell module associated to B_1 can be encoded in the following graph (as in a W-graph) with all μ-coefficients equal to 1:


(32)

It should be noted that this graph does not capture all non-zero μ-coefficients between the Weyl group elements corresponding to its vertices, as non-zero μ-coefficients between elements with the same right descent set do not matter for the action. Moreover, the graph needs to be suitably truncated as there are no long edges from the smallest cluster in B_1 to elements in ${}_uC$.

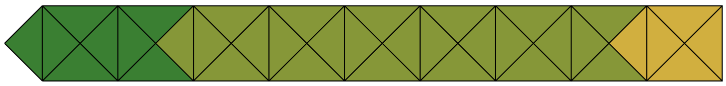
From this graph, we deduce the following identities for $l \geqslant 0$ using Corollary 5.7

$$\begin{aligned} {}^p\underline{H}_{w_l}\underline{H}_s &= {}^p\underline{H}_{w_l s}, \\ {}^p\underline{H}_{w_l s}\underline{H}_r &= {}^p\underline{H}_{w_l sr} + {}^p\underline{H}_{w_l} \text{ for } r \in \{t, u\} \text{ and} \\ {}^p\underline{H}_{w_l sr}\underline{H}_s &= {}^p\underline{H}_{w_l s} \end{aligned}$$

modulo $\bigoplus_{x\in A_1} \mathbb{Z}[v, v^{-1}]\, {}^p\underline{H}_x$. Since the occuring p-canonical basis elements in ${}^p\underline{H}_{w_l st}\underline{H}_u$ and ${}^p\underline{H}_{w_l su}\underline{H}_t$ are completely determined by the SL_2 situation, the claim follows. □

It is quite remarkable that ${}^p\underline{H}_{w_n}$ is perverse modulo $\bigoplus_{x\in A_1} \mathbb{Z}[v, v^{-1}]\underline{H}_x$, but computer calculations show that, in general, the contribution from $\bigoplus_{x\in A_1} \mathbb{Z}[v, v^{-1}]\underline{H}_x$ is not perverse.

The results in this section determine the decomposition behaviour of B_1 for $p > 2$, so we will illustrate the right p-cells in B_1 for $p = 3$ (even though $p = 3 < 2h - 2 = 6$):



. . . (33)

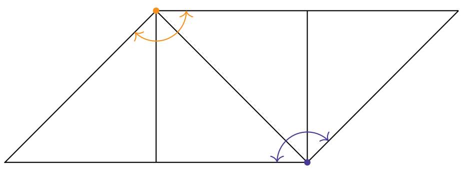
It should be noted that the results of this section together with Section 10.4.2 give modular weight cells for SP_4 using Theorem 7.8 and Lemma 6.35.

10.4.5 Spherical cells

Figure 7 shows a box containing elements in ${}^{s,t}A$. The same comments as in Section 10.3.4 apply in this case. Therefore, Lemma 5.10 applied to the two marked strings in Figure 7 implies in this setting:

Lemma 10.33. *If a box contains a spherical element, then all elements in the box lie in the same right p-cell of W.*

Figure 7: Box in ${}^{s,t}A$ in type affine $\widetilde{C}_2$



For $\lambda \in X^\vee_+$, consider $w_\lambda \in W$. For $r \in S_{\mathrm{f}}$, denote by $P_{\omega_r^\vee}(\lambda)$ the unique sequence of simple reflections needed to extend any reduced expression of w_λ to a reduced expression for $w_{\lambda+\omega_r^\vee}$. Note that $P_{\omega_r^\vee}(\lambda)$ only depends on the class of λ in $\Omega = X^\vee/\mathbb{Z}\Phi^\vee$. For example if $\sigma(S_{\mathrm{f}}) = S_{\mathrm{f}}$, then we have:

$$P_{\omega_r^\vee}(\lambda) = \begin{cases} (u, s, t, s) & \text{if } r = s, \\ (u, s, u) & \text{if } r = t. \end{cases}$$

Fix $r \in S_{\mathrm{f}}$ for the rest of the section.

Proposition 10.34. (i) *For $r = s$ and $P_{\omega_s^\vee}(\lambda) = (s_1, s_2, s_3, s_4)$, the functor*

$$B_{s_4} \circ B_{s_3} \circ B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} : \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_\lambda) \longrightarrow \mathrm{Hom}_{^k\mathbf{SH}}(S_f, \varnothing)$$

can be decomposed as a direct sum

$$B_{s_4} \circ B_{s_3} \circ B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} = (B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda})^{\oplus 2} \oplus \mathrm{ind}_{S_{\lambda+\omega_s}} \circ T_{\omega_s^\vee}.$$

(ii) *For $r = t$ and $P_{\omega_s^\vee}(\lambda) = (s_1, s_2, s_3)$, the functor*

$$B_{s_3} \circ B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} : \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_\lambda) \longrightarrow \mathrm{Hom}_{^k\mathbf{SH}}(S_f, \varnothing)$$

can be decomposed as a direct sum

$$B_{s_3} \circ B_{s_2} \circ B_{s_1} \circ \mathrm{ind}_{S_\lambda} = (B_{s_1} \circ \mathrm{ind}_{S_\lambda})^{\oplus 2} \oplus \mathrm{ind}_{S_{\lambda+\omega_t}} \circ T_{\omega_t^\vee}.$$

In the formulas above, the functor $T_{\omega_r^\vee} : \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_\lambda) \longrightarrow \mathrm{Hom}_{^k\mathbf{SH}}(S_f, S_{\lambda+\omega_r^\vee})$ corresponds under the geometric Satake equivalence to the 1-morphism $T(\omega_r^\vee) \in \mathrm{Tilt}(G^\vee)^{\overline{\omega_r^\vee}}$.

Proof. The proof is completely analogous to the one of Proposition 10.9, we only need to use the non-symmetric version of [Eli16, (6.8) for $m = 4$] which comes from [Eli16, (A.3)]. Note that the complementary idempotent to the singular $2m_{t,s}$- (or $2m_{s,u}$-)valent vertex for $r = s$ (resp. $r = t$) can be further decomposed into two orthogonal idempotents. $\square$

In order to completely determine the right p-cell structure in type $\widetilde{C}_2$, we need to show that there are no more right p-cell preorder relations between the spherical elements than the ones coming from geometric Satake.

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