Periodicity in motivic homotopy theory and over BP_*BP

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Böblingen

Bonn, März 2018

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 26.06.2018
Erscheinungsjahr: 2018

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Periodizitätsphänomenen in stabiler motivischer Homotopietheorie. Bei motivischer Homotopietheorie handelt es sich um eine Perspektive auf algebraische Varietäten, die die Anwendung homotopietheoretischer Methoden auf algebro-geometrische Probleme erlaubt, siehe [MV99].

Über dem Grundring $\mathbb{C}$ ist die entsprechende stabile motivische Homotopiekategorie $\mathrm{Sp}_{\mathbb{C}}$ strukturell ähnlich zur klassischen stabilen Homotopiekategorie Sp, und es gibt einen Vergleichsfunktor $\mathrm{Sp}_{\mathbb{C}} \to \mathrm{Sp}$. Diese Beziehung wird sehr erfolgreich eingesetzt um neue Resultate in klassischer stabiler Homotopietheorie zu erhalten, so zum Beispiel von Isaksen in [Isa14].

In klassischer stabiler Homotopietheorie geben die sogenannten Nilpotenz- und Periodizitätstheoreme von Devinatz, Hopkins und Smith [DHS88] umfangreiche Auskunft über periodische Strukturen in Homotopiegruppen. Konkret existiert für jedes kompakte p-lokale Spektrum X eine nicht nilpotente Selbstabbildung $\Sigma^i X \to X$. Diese Selbstabbildung ist dadurch charakterisiert, wie sie auf den komplexen Bordismusgruppen MU_*X von X wirkt. Die auftretenden Arten von Selbstabbildungen gliedern sich nach sogenanntem *Typ*, beschrieben durch eine natürliche Zahl $n \in \mathbb{N}$. Für jedes n existiert eine Homologietheorie $K(n)_*$ (die *n-te Morava K-Theorie*), die genau Selbstabbildungen vom Typ n detektiert. Selbstabbildungen vom Typ n sind eindeutig bis auf Potenzen. Alle Selbstabbildungen von X, die nicht von einem dieser $K(n)_*$ detektiert werden, sind nilpotent.

In motivischer Homotopietheorie existieren zusätzlich zu Analoga dieser sogenannten *v_n-Selbstabbildungen* auch noch andere nicht nilpotente Selbstabbildungen. Ein Beispiel ist die Hopfabbildung $\eta \in \pi_*(S)$, deren Analogon in motivischer Homotopietheorie über $\mathbb{C}$ nicht nilpotent ist, im Gegensatz zur klassischen Situation. Dieses Beispiel wurde von Gheorghe in [Ghe17a] zu einer unendlichen Familie sogenannter *w_i-Selbstabbildungen* erweitert, wobei η zu w_0 korrespondiert. Analog zur klassischen Situation werden diese durch Homologietheorien $K(w_i)_{**}$ detektiert. Andrews konstruierte in [And14] das erste explizite Beispiel einer w_1-Selbstabbildung.

Die vorliegende Arbeit erweitert diese Familie um weitere Typen von Selbstabbildungen β_{ij} für jedes $i > j \geq 0$. Für $j = 0$ stimmt die Familie der $\beta_{i,0}$ mit der unendlichen Familie der w_{i-1} überein, die β_{ij} stellen also höhere Verallgemeinerungen der w_i dar.

Wir definieren motivische Homologietheorien $K(\beta_{ij})_{**}$, die formal ähnlich zu Morava K-Theorien sind, und β_{ij}-Selbstabbildungen detektieren. Das Hauptresultat dieser Arbeit ist Theorem 6.12, aus dem hervorgeht, dass jedes kompakte motivische p-vollständige Spektrum mindestens einen Typ von β_{ij}-Selbstabbildung besitzt. Desweiteren konstruieren wir in Proposition 6.16 nichttriviale Beispiele solcher

Selbstabbildungen für jedes β_{ij}, und diskutieren in Bemerkung 6.17 β_{ij}-periodische Strukturen in motivischen Homotopiegruppen.

Zusätzlich zu den genannten Hauptresultaten in motivischer Homotopietheorie erhalten wir aus der entwickelten Theorie einige interessante Resultate in homologischer Algebra. Zu diesen gehört eine entsprechende Theorie von β_{ij}-Selbstabbildungen in Ext_{BP_*BP}, wobei BP_*BP der zur Brown-Peterson-Homologietheorie assoziierte Hopf-Algebroid ist, siehe Theorem 4.31. Als Anwendung erhalten wir in Kapitel 5 eine Version des klassischen Adams-Periodizitätstheorems für die Kohomologie $\mathrm{Ext}_{\mathcal{A}_*}$ der Steenrod-Algebra bei $p = 2$, allerdings in einer größeren Region als in [Ada66] (siehe Proposition 5.14), sowie ein bisher unbekanntes entsprechendes Periodizitätsresultat für Ext_{BP_*BP} bei $p = 2$, siehe Theorem 5.21. Ein weiterer interessanter, anscheinend bisher nicht in der Literatur vertretener Fakt über Ext_{BP_*BP} ist eine Schranke für den Exponenten der p-Torsion oberhalb einer beliebigen Geraden positiver Steigung, siehe Proposition 5.6.

Contents

1 Introduction

This thesis considers periodicity phenomena in the stable motivic homotopy category over $\mathbb{C}$.

Motivic homotopy theory over the ground ring $\mathbb{C}$ has the particularly nice property that there exists a functor $\mathrm{Sp}_{\mathbb{C}} \to \mathrm{Sp}$, called Betti realization, from the category of motivic spectra to the category of ordinary spectra, roughly given by sending a smooth scheme over $\mathbb{C}$ to its complex points with the smooth topology.

There is a bigraded set of spheres $S^{n,w} \in \mathrm{Sp}_{\mathbb{C}}$, where n is called the *dimension* and w is called the *weight*. This allows one to define bigraded homotopy groups of motivic spectra.

On p-complete objects of $\mathrm{Sp}_{\mathbb{C}}$, the Betti realization functor acts on homotopy groups by inverting a certain element $\tau \in \pi_{0,-1}(S_p^\wedge)$. (See Proposition 6.3 for a precise statement.) Thus, one should think of motivic homotopy groups as a refinement of classical ones: There is an additional grading given by the weight, and there are interesting τ-torsion homotopy classes that are destroyed under Betti realization.

For example, the Hopf map $\eta \in \pi_1(S)$ admits a lift $\eta \in \pi_{1,1}(S^{\mathrm{mot}})$. As it turns out, this class is not nilpotent, meaning that all of the powers $\eta^k \in \pi_{k,k}(S^{\mathrm{mot}})$ are not zero (but beginning with η^4, they are τ-torsion, $\tau\eta^4 = 0$). There are higher-degree examples as well, for example the element $\overline{\kappa}_2 \in \pi_{44,24}(S^{\mathrm{mot}})$ discussed in Example 6.18. Thus, the classical Nishida nilpotence theorem [Nis73], which asserts that positive-degree elements of $\pi_*(S)$ are nilpotent, doesn't admit a direct motivic analogue.

The more general nilpotence theorem due to Devinatz, Hopkins and Smith [DHS88] also fails in the motivic setting: There is a motivic analogue MU^{mot} of MU (more commonly called MGL), but it fails to detect nilpotence. For example, it doesn't see the nonnilpotence of η.

Similarly, the periodicity theorem fails. Classically, the periodicity theorem says that a finite p-local spectrum X admits a non-nilpotent self-map $\Sigma^i X \xrightarrow{f} X$ that acts by some power of v_n on BP_*X. Equivalently, it acts isomorphically on $K(n)_*X$. The n, called the *type* of X (and of f) is characterized as the smallest m such that $K(m)_*X \neq 0$, and any two such self-maps agree after passing to suitable higher powers of them. Motivically, on the other hand, we've already seen that the 2-local sphere has multiple different "types" of non-nilpotent self-maps, e.g. the degree 2-map and the Hopf map η.

It is therefore an interesting question to ask for a classification of types of non-nilpotent self-maps of compact motivic spectra. This thesis gives a partial answer towards that question. We show in Section 6 that for each compact p-complete cellular spectrum, the motivic homotopy groups admit a minimal vanishing line in

(n,w)-grading. Concretely, Theorem 6.11 shows that there are d and c such that

$$\pi_{n,w}(X) = 0$$

for all (n,w) with $w > dn + c$, and that d and c are minimal in the sense that no such statement holds for other d' and c' with $d' < d$, or with $d' = d$ and $c' < c$. We refer to d as the *slope* and to c as the *intercept* of the vanishing line.

The possible slopes of minimal vanishing lines are quite restricted, they all are of the form

$$d_{ij}^{\mathrm{mot}} = \frac{p^{j+1}(p^i - 1)}{2p^{j+1}(p^i - 1) - 2}$$

and correspond to specific May spectral sequence generators through a connection between p-complete motivic homotopy theory over $\mathbb{C}$, the Hopf algebroid BP_*BP, and the Hopf algebra $\mathcal{P}_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \subseteq \mathcal{A}_*$ ($\mathbb{F}_p[\xi_0^2, \xi_1^2, \ldots]$ at $p = 2$) which we call the *even dual Steenrod algebra*.

It turns out that if X has a minimal vanishing line of slope d_{ij}^{mot}, self-maps $\Sigma^{n_0,w_0} X \xrightarrow{f} X$ of slope d_{ij}^{mot}, i.e. $\frac{w_0}{n_0} = d_{ij}^{\mathrm{mot}}$ behave similar to the classical case, in that they always exist and are unique in a suitable sense. This is the statement of Theorem 6.12. We refer to such self-maps as *self-maps parallel to the vanishing line* throughout the document.

Starting with S, we can inductively take cofibres of self-maps parallel to the minimal vanishing line, to obtain a sequence of *generalized Smith-Toda complexes.* The slope of the minimal vanishing line strictly decreases in each step, and one can see that all possible slopes are obtained. That way we obtain nontrivial examples for each d_{ij}^{mot}.

In particular, this shows that if one defines the thick subcategory $\mathcal{C}_{ij}$ of the category of finite p-complete cellular motivic spectra to consist of all those spectra with a vanishing line of slope d_{ij}^{mot}, the $\mathcal{C}_{ij}$ form an infinite, properly nested sequence of thick subcategories whose intersection is trivial. So the slope of the minimal vanishing line behaves a lot like the notion of type in classical p-local finite spectra.

The existence of such generalized Smith-Toda complexes in the motivic setting suggests the possibility of a "β_{ij}-chromatic motivic homotopy theory", where motivic homotopy groups are organized into β_{ij}-periodic information. We discuss this in Remark 6.17.

We also construct motivic spectra $K(\beta_{ij})$, whose homology theories detect precisely self-maps of slope d_{ij}^{mot}. The $K(\beta_{ij})$ also detect the slope of the minimal vanishing line of X, it can be recovered as the maximal d_{ij}^{mot} for which $K(\beta_{ij})_{**}X \neq 0$.

We call these objects *exotic K-theories*, as they share some formal similarities with Morava K-theories. However, their homotopy groups are of the form $\mathbb{F}_p[\alpha, \beta^{\pm 1}]/\alpha^2$ at odd primes, so don't quite look like graded fields. Furthermore, and more strikingly, they are typically not rings. So many of the usual nice properties of Morava K-theories don't carry over.

These $K(\beta_{ij})$ make it possible to discuss β_{ij} self-maps of slope not necessarily equal to the minimal vanishing line. It is there where the main difference to the classical setting appears, as motivic spectra seem to typically have additional self-maps of lower slope. The easiest example of this is on the sphere itself: The minimal vanishing line of the sphere has slope 1, with corresponding $\beta_{1,0}$ self-map η, but the sphere also admits an $\beta_{2,1}$ self-map by explicit computation. This is discussed in Example 6.18.

The existence of self-maps of lower slope shows that the thick subcategories $\mathcal{C}_{ij}$ discussed above do not constitute a full list of thick subcategories. For example, at $p = 2$, the thick subcategory of all X with $K(\beta_{2,1})_{**}X = 0$ does not agree with any of the $\mathcal{C}_{ij}$, see Example 6.18.

It is tempting to conjecture that the thick subcategories characterized by vanishing of a single $K(\beta_{ij})$ or $K(n)$ form a complete list of prime thick subcategories of finite p-complete cellular motivic spectra. The $\mathcal{C}_{ij}$ characterized in terms of slopes are certainly intersections of such vanishing loci, since the slope of the minimal vanishing line is characterized in terms of the $K(\beta_{ij})$.

It is not clear though whether these are actually prime thick subcategories. This is equivalent to the question of whether $K(\beta_{ij})_{**}(X \otimes Y) = 0$ implies $K(\beta_{ij})_{**}X = 0$ or $K(\beta_{ij})_{**}Y = 0$ for compact X and Y.

A corresponding question on comodules over the dual Steenrod algebra is as follows: For M and N finite-dimensional comodules, one can ask whether $H_*(M \otimes N; P_i^j) = 0$ necessarily implies that $H_*(M; P_i^j) = 0$ or $H_*(N; P_i^j) = 0$, where $H_*(-; P_i^j)$ denotes Margolis homology as in Lemma 3.68.

This seems to be closely related to an open question by Margolis, see the conjecture made in Chapter 19 of [Mar83], in the discussion following Proposition 18 there.

In addition, it is an interesting question on how the vanishing loci of the $K(\beta_{ij})$ relate to each other. Classically, $K(n)_*X = 0$ implies that $K(n-1)_*X = 0$ for any compact spectrum, so all the thick subcategories obtained as vanishing loci of the $K(n)$ are linearly contained in each other. In the motivic setting, examples such as $S/\overline{\kappa}_2$ from Example 6.18 show that the situation is more complicated. In the related setting of comodules over the dual Steenrod algebra, Palmieri has some interesting results on the corresponding questions for Margolis homologies, see Theorem A.1 and Proposition 3.10 in Part II of [Pal96].

The results on motivic spectra are obtained through a connection between p-complete motivic cellular spectra and the Hopf algebroid BP_*BP. This was recently proven by Gheorghe, Wang and Xu [GWX], and roughly says that the category of modules over a certain motivic ring spectrum S/τ, obtained as the cofibre of $\tau \in \pi_{**}(S_p^\wedge)$, agrees with a suitable category of derived comodules over BP_*BP, we discuss this relation in Section 6.2 and restate it in Theorem 6.7.

If one works with the actual derived comodule category over BP_*BP here, the correspondence only works between compact S/τ-modules, and derived comodules whose underlying BP_*-module is a perfect complex. This does not extend to the full derived BP_*BP comodule category, as these objects are not actually compact there.

In Section 2, we discuss comodule categories in a very general, ∞-categorical context, namely over arbitrary exact comonads. This allows us to define what we call the *compactly generated comodule category* in Section 2.3, which is built in such a way that it is compactly generated by comodules with compact underlying object. In Theorem 2.44, we give a Barr-Beck like statement for compactly generated comodule categories, suggesting that they naturally appear whenever one deals with adjunctions between compactly generated categories. One noteworthy instance of this phenomenon is Example 2.46, where the category of p-complete spectra is identified with compactly generated comodules over some kind of (nontrivially) coherent version of the dual Steenrod algebra.

In Section 2.4 and the subsequent Section 3, we specialize to comodule categories obtained from coalgebroids, a notion generalizing both Hopf algebroids and coalgebras. Such a coalgebroid Γ over a ring A gives rise, under certain flatness assumptions, to a comonad $\mathcal{D}\Gamma$ on the derived module category $\mathcal{D}\operatorname{Mod}_A$. The corresponding comodule category $\operatorname{Comod}_{\mathcal{D}\Gamma}(\mathcal{D}\operatorname{Mod}_A)$ should be thought of as a derived category of comodules over Γ, but it is built in an automatically derived way. We partially discuss the relation to the classical construction of derived comodule categories in Remark 2.66. In many cases, including over BP_*BP and other connected graded Hopf algebroids, they seem to coincide. Furthermore, the compactly generated $\operatorname{Comod}^{cg}_{\mathcal{D}\Gamma}(\mathcal{D}\operatorname{Mod}_A)$ seem to agree with the *stable comodule categories* Stable(Γ) constructed by Hovey as explicit model categories in [Hov04]. We don't prove these equivalences here, but we strongly suspect that for the interested reader familiar with Hovey's construction, it will be possible to skip ahead and follow the results of sections 4, 5 and 6 by replacing all occurences of $\operatorname{Comod}^{cg}_{\mathcal{D}\Gamma}$ with Stable(Γ), and providing suitable analogues of the tools we use from Section 3. The construction of $\operatorname{Comod}^{cg}_{\mathcal{D}\Gamma}$ was inspired by the alternative construction of Stable(Γ) given in [BH17].

The category $\operatorname{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ is the main category of interest in this thesis, as our version of the GWX-Theorem 6.7 identifies the category of motivic S/τ-modules with a certain full subcategory of $\operatorname{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$. All the motivic results are obtained by establishing them in $\operatorname{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, and then lifting them from S/τ-modules to all motivic spectra via a τ-Bockstein spectral sequence.

The corresponding results are obtained in Section 4 by reducing to smaller Hopf algebroids. Namely, BP_*BP admits a quotient Hopf algebra $\mathcal{P}_*$, which can be identified with a certain subalgebra of the dual Steenrod algebra $\mathcal{A}_*$.

Over $\mathcal{A}_*$, analogues of our results on vanishing lines and self-maps have already been obtained by Palmieri, see [Pal01]. Since we work in a slightly different setting, we provide self-contained proofs of these results over $\mathcal{P}_*$. They are obtained by describing $\mathcal{P}_*$ through a sequence of extensions by particularly small Hopf algebras, each of which leading to an associated Cartan-Eilenberg spectral sequence (which arises as an Adams spectral sequence in our setting). Inductively, one sees that only specific slopes of vanishing lines are possible, and that self-maps parallel to the minimal vanishing line always exist.

Finally, one can lift these results to BP_*BP by a similar process. We do this in two steps, first passing from $\mathcal{P}_*$ to BP_*BP/p through a suitable Adams spectral sequence, and then further from BP_*BP/p to BP_*BP through an Adams spectral sequence that can be identified with the p-Bockstein spectral sequence. For the latter step to work, we require a bound on the p-torsion exponent of Ext_{BP_*BP} along minimal vanishing lines (Lemma 4.30), which is quite interesting in its own right because it leads to a bound on the p-exponent of torsion in Ext_{BP_*BP} above any line of positive slope (Proposition 4.30).

Section 5 contains algebraic applications, namely a version of classical Adams periodicity for $\mathrm{Ext}_{\mathcal{A}_*}$ with strengthened bounds (first established by May in unpublished notes [May]) in Section 5.2, and an analogue for Ext_{BP_*BP} in Section 5.3 that was first conjectured by Isaksen on basis of new computer-assisted computations of Ext_{BP_*BP} by Wang.

We prove these results by carefully analyzing the intercept of vanishing lines for various finite complexes obtained by coning off subsequent self-maps parallel to the vanishing line, in a way that is reminiscent of the classical chromatic filtration on homotopy groups. A discussion of these generalized Smith-Toda complexes and related qualitative generalizations of Adams periodicity results is contained in Section 5.1.

A usual (i.e. based on v_n-periodicity) chromatic homotopy theory of BP_*BP-comodules has been developed in [BH17] already. It is not yet clear how the β_{ij}-based chromatic homotopy theory suggested here relates to this v_n-based theory, but we hope that it will lead to a more complete picture.

Acknowledgements. I would like to thank Peter Teichner and Dan Isaksen for ongoing support and advice on both my research in general, and the creation of this thesis in particular.

Furthermore, I want to thank Bertram Arnold, Tobias Barthel, Lukas Brantner, Jack Davies, Bogdan Gheorghe, Drew Heard, Fabian Henneke, Cary Malkiewich, Lennart Meier, Thomas Nikolaus, Mark Pedron, Nicolas Ricka and Zhouli Xu for many helpful discussions.

Finally, I am very grateful for all the support and encouragement provided by my friends, my family, and my girlfriend.

2 Categorical Preliminaries

The main categories of interest in Section 4 are certain derived categories of comodules, mostly over Hopf algebroids.

The classical approach to constructing these derived comodule categories is through explicit model categories, cf. [Hov04].

We will follow a different, ∞-categorical approach. In sections 2.1 and 2.2, we consider the category of comodules over a comonad, with special attention towards a comonad obtained from an adjunction. This theory is mostly classical, but for completeness, and since most of the literature is phrased in terms of the dual monadic setting (e.g. [Lur16], Section 4.7), we review it here.

This also allows us to exhibit the machinery behind the Barr-Beck Theorem as a very general form of the Adams spectral sequence. Special cases of this "Adams spectral sequence of an adjunction" will play an important role in the computations of Section 4.

In Section 2.3, we introduce the category of compactly generated comodules $\mathrm{Comod}_T^{cg}(\mathcal{D})$. This is a modified version of the comodule category considered in sections 2.1 and 2.2, characterized by the fact that it is compactly generated.

The main motivation for the construction of $\mathrm{Comod}_T^{cg}(\mathcal{D})$ is that a lot of categories one might want to study through the Adams spectral sequence of an adjunction are actually compactly generated. Typically, the Barr-Beck theorem doesn't apply globally in these cases. However, under suitable conditions, there is a variant of the Barr-Beck theorem involving Comod^{cg}, see Theorem 2.44. For example, this applies to p-complete spectra and the adjunction giving rise to the usual $H\mathbb{F}_p$-based Adams spectral sequence, see Example 2.46. Theorem 2.44 will also allow us to give a self-contained proof of a theorem of Gheorghe, Wang and Xu [GWX] on the structure of the stable motivic homotopy category over $\mathbb{C}$, see Section 6.7.

In Section 2.4, we finally specialize to the algebraic setting. We define the notion of a coalgebroid, a common generalization of both Hopf algebroids and coalgebras. A coalgebroid Γ over a ring A gives rise to a comonad on the $(\infty, 1)$-category $\mathcal{D}\,\mathrm{Mod}_A$ of derived A-modules. This allows us to apply the constructions given in sections 2.1 and 2.3 to obtain a derived comodule category and a compactly generated derived comodule category over Γ.

For good enough Hopf algebroids Γ, it seems that the compactly generated derived comodule category we work in agrees with (the $(\infty, 1)$-category associated to the model category) $\mathrm{Stable}(\Gamma)$, defined in [Hov04], see Remark 2.68. The main technical difficulties in Hovey's construction seem to come from the fact that one derives with respect to two things at the same time, the A-module structure and the Γ-comodule structure. For example, it is not clear whether every comodule can be resolved by comodules with underlying projective A-module. Our construction circumvents these difficulties by defining Γ-comodules on top of the already-derived module category $\mathcal{D}\,\mathrm{Mod}_A$.

2.1 Comodule categories

For $\mathcal{D}$ an $(\infty,1)$-category, $\mathrm{Fun}(\mathcal{D},\mathcal{D})$ is monoidal with respect to composition. A comonad on $\mathcal{D}$ is a coherently coassociative comonoid $T \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$. Concretely, let Δ_+ be the category of (possibly empty) finite ordered sets with order-preserving maps. This is monoidal with respect to disjoint union (with order on $S_0 \sqcup S_1$ given by $S_0 < S_1$).

Definition 2.1. *A comonad over $\mathcal{D}$ is a monoidal functor*

$$T \in \mathrm{Fun}^{\otimes}(N(\Delta_+^{\mathrm{op}}), \mathrm{Fun}(\mathcal{D},\mathcal{D}))$$

Informally, such a functor is determined by its value on $\{0\}$, and natural transformations $T(\{0\}) \Rightarrow T(\{0,1\}) \simeq T(\{0\}) \circ T(\{0\})$ and $T(\{0\}) \Rightarrow T(\emptyset) \simeq \mathrm{id}$ together with higher coherences for coassociativity and counitality.

We will typically refer to the value $T(\{0\}) \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$ by T as well, and will use terminology such as "a comonad $T \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$", with the implicit understanding that there is a chosen lift to $\mathrm{Fun}^{\otimes}(N(\Delta_+^{\mathrm{op}}), \mathrm{Fun}(\mathcal{D},\mathcal{D}))$, similar to the classical use of words like "ring spectrum".

For such a comonad, there is an $(\infty,1)$-category of comodules $\mathrm{Comod}_T(\mathcal{D})$, which consists of objects $X \in \mathcal{D}$ together with a coherent coaction $X \to T(X)$. Concretely, let $\Delta_{\max}$ be the category of nonempty finite ordered sets with order-preserving maps which preserve the maximum.

Then $\Delta_{\max}$ is a left Δ_+-module. Since $\mathrm{Fun}(\mathcal{D},\mathcal{D})$ acts on $\mathcal{D}$ from the left (by application), a comonad $T \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$ gives rise to a left action by $N(\Delta_+)$ on $\mathcal{D}$.

Definition 2.2. *For a comonad $T \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$, $\mathrm{Comod}_T(\mathcal{D})$ is the $(\infty,1)$-category of comodules*

$$\mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})}(N(\Delta_{\max}^{\mathrm{op}}), \mathcal{D})$$

of left $N(\Delta_+^{\mathrm{op}})$-module functors, with respect to the $N(\Delta_+^{\mathrm{op}})$-module structure induced on $\mathcal{D}$ by T.

Remark 2.3. This is traditionally called the category of *coalgebras* over T, not comodules. We follow the naming convention used in [Lur16], which we deemed more appropriate for our applications, especially since everything is linear: all the comonads we will consider later are exact, and the comodule categories over them are related to literal categories of comodules over coalgebras and Hopf algebroids.

Since $\Delta_{\max}$ is, on objects, a free Δ_+ module on one generator $\{0\}$, informally such a functor X is determined by its value $X(\{0\}) \in \mathcal{D}$ together with a coaction map $X(\{0\}) \to X(\{0,1\}) \simeq T(X(\{0\}))$, with higher coherences for coassociativity and counitality of this coaction.

The functor $\mathrm{Comod}_T(\mathcal{D}) \to \mathcal{D}$, $X \mapsto X(\{0\})$ will be called the underlying object functor, or forgetful functor. It detects equivalences.

As for comonads, we will notationally identify a comodule X with its underlying object, and typically omit the forgetful functor in formulas.

Remark 2.4. Note that, since $\mathrm{Fun}(\mathcal{D},\mathcal{D})$ acts on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ for any $\mathcal{C}$, we can speak more generally of comodules in those functor categories. However, we have

$$\mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})}(N(\Delta_{\max}^{\mathrm{op}}), \mathrm{Fun}(\mathcal{C},\mathcal{D})) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})}(N(\Delta_{\max}^{\mathrm{op}}), \mathcal{D})),$$

so a comodule in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ is the same as a functor $\mathcal{C} \to \mathrm{Comod}_T(\mathcal{D})$.

Since T is a comonoid in $\mathrm{Fun}(\mathcal{D},\mathcal{D})$, it can in particular be considered a left comodule over itself. Concretely, restricting T along the inclusion $\Delta_{\max} \to \Delta_+$, we obtain a comodule structure on T. By Remark 2.4, this implies that $T : \mathcal{D} \to \mathcal{D}$ factors through a functor $\widetilde{T} : \mathcal{D} \to \mathrm{Comod}_T(\mathcal{D})$. $\widetilde{T}$ is in fact right adjoint to the forgetful functor V: Their composite on $\mathcal{D}$ is just T, with counit $T \Rightarrow \mathrm{id}$, and their composite on $\mathrm{Comod}_T(\mathcal{D})$ sends $X \mapsto \widetilde{T}X$, with unit given by the comodule structure map $X \to TX$.

We can use this adjunction to describe mapping spaces in $\mathrm{Comod}_T(\mathcal{D})$.

Now let $\Delta_{\min}$ denote the category of nonempty finite ordered sets with order-preserving maps which preserve the minimum.

Lemma 2.5. *There is an equivalence $\Delta_{\min}^{\mathrm{op}} \to \Delta_{\max}$, obtained by sending $S \in \Delta_{\min}$ to the set $\mathrm{Map}_{\Delta_{\min}}(S, \{0,1\})$ with the* opposite *pointwise ordering.*

Proof. Since order-reversal yields a covariant equivalence $\Delta_{\min} \to \Delta_{\max}$, it is sufficient to check that the functor

$$\Delta_{\min}^{\mathrm{op}} \to \Delta_{\min}, \quad S \mapsto \mathrm{Map}_{\Delta_{\min}}(S, \{0,1\})$$

with the pointwise ordering (rather than the opposite one), is an equivalence.

First observe that this is well-defined: The minimum in $\mathrm{Map}_{\Delta_{\min}}(S, \{0,1\})$ with respect to the chosen ordering is the constant 0 map, which is obviously preserved by induced maps.

To see that it is an equivalence, we show that it is self-inverse. Namely, there is a natural map

$$S \to \mathrm{Map}_{\Delta_{\min}}(\mathrm{Map}_{\Delta_{\min}}(S, \{0,1\}), \{0,1\}),$$

adjoint to the evaluation map. It is injective, since for any two different $x < y$ in S, there exists an $f : S \to \{0,1\}$ in $\Delta_{\min}$ with $f(x) = 0$, $f(y) = 1$. Finally, note that $\mathrm{Map}_{\Delta_{\min}}(S, \{0,1\})$ has the same cardinality as S. $\square$

Remark 2.6. The reverse pointwise ordering on $\mathrm{Map}_\Delta(S, \{0,1\})$ is more natural when viewed in the perspective of cuts: A cut of S is a decomposition $S = S_0 \sqcup S_1$ with $S_0 < S_1$. The set of all cuts $\mathrm{Cut}(S)$ agrees with $\mathrm{Map}_\Delta(S, \{0,1\})$ by sending S_0 to 0, S_1 to 1, and it is natural to order cuts in such a way that $S \sqcup \emptyset$ is the maximal one, i.e. by inclusion on S_0.

Recall that an augmented cosimplicial object in $\mathcal{C}$ is a functor $N(\Delta_+) \to \mathcal{C}$, and a split cosimplicial object is a functor $N(\Delta_{\min}) \to \mathcal{C}$. Split cosimplicial objects restrict to augmented cosimplicial objects, which restrict further to cosimplicial objects, via the functors $\Delta \to \Delta_+ \to \Delta_{\min}$. Here the first functor is the canonical inclusion, the second is the functor which adds a disjoint minimum to each ordered set. Similarly to how the simplicial index category Δ is generated by boundary maps and degeneracy maps, $\Delta_{\min}$ is generated by the same boundary and degeneracy maps, as well as one “extra degeneracy” s_{-1} on each level.

If $X^\bullet$ is a split cosimplicial object, we refer to the value on the set $\{0, \ldots, n\} \in \Delta_{\max}$ by X^{n-1} (this is compatible with the restrictions to Δ_+ and Δ). The augmentation gives a map

$$X^{-1} \to \lim_{N(\Delta)} X^\bullet,$$

which is an equivalence. This is the dual of Lemma 6.1.3.16 in [Lur09].

Definition 2.7. *For a comodule $Y \in \mathrm{Comod}_T(\mathcal{D})$, the cobar resolution $T^{\bullet+1}Y$ is the split augmented cosimplicial object, natural in Y, obtained from considering the comodule structure diagram in $\mathrm{Fun}_{N(\Delta_+)}(N(\Delta^{\mathrm{op}}_{\max}), \mathcal{D})$ as a covariant functor from $N(\Delta_{\min})$ to $\mathcal{D}$, using Lemma 2.5.*

Informally, the diagram takes the form

$$Y \to \widetilde{T}Y \rightrightarrows \widetilde{T}TY \mathrel{\substack{\to\\[-2pt]\to\\[-2pt]\to}} \cdots,$$

with coboundary maps obtained from the coaction map $T \to TY$ and the comultiplication map $T \to TT$, and codegeneracy maps (and splits, i.e. “extra degeneracies”) obtained from the counit on T.

Since the cobar resolution is split, it exhibits Y as a limit of objects in the essential image of $\widetilde{T}$. We can use this together with the adjunction $\mathrm{Comod}_T(\mathcal{D}) \rightleftarrows \mathcal{D}$ to describe mapping spaces:

Lemma 2.8. *For $X, Y \in \mathrm{Comod}_T(\mathcal{D})$, we have*

$$\begin{aligned} &\mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(X, Y) \\ &\quad = \lim(\mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(X, \widetilde{T}Y) \rightrightarrows \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(X, \widetilde{T}TY) \mathrel{\substack{\to\\[-2pt]\to\\[-2pt]\to}} \cdots) \\ &\quad = \lim(\mathrm{Map}_{\mathcal{D}}(X, Y) \rightrightarrows \mathrm{Map}_{\mathcal{D}}(X, TY) \mathrel{\substack{\to\\[-2pt]\to\\[-2pt]\to}} \cdots) \end{aligned}$$

Given comonads $T \in \mathrm{Fun}(\mathcal{C},\mathcal{C}), T' \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$ and a functor $F : \mathcal{C} \to \mathcal{D}$, we want to understand in what sense F, T and T' need to be compatible to give rise to a functor $\mathrm{Comod}_T(\mathcal{C}) \to \mathrm{Comod}_{T'}(\mathcal{D})$ given by F on underlying objects.

We let Δ_{Cut} be the category whose objects are (possibly empty) finite ordered sets with fixed decomposition $S_0 \sqcup S_1$, $S_0 < S_1$.

Morphisms are order-preserving maps $f : S \to T$ with the requirement that $f(S_1) \subseteq T_1$.

Note that Δ_{Cut} admits a Δ_+-Δ_+-bimodule structure by disjoint union from the left and from the right.

Definition 2.9. *Let $T \in \mathrm{Fun}(\mathcal{C},\mathcal{C})$ and $T' \in \mathrm{Fun}(\mathcal{D},\mathcal{D})$ be comonads. A morphism $F : T \to T'$ is an element of*

$$\mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})\text{-}N(\Delta_+^{\mathrm{op}})\text{-bimod}}(N(\Delta_{\mathrm{Cut}}^{\mathrm{op}}), \mathrm{Fun}(\mathcal{C},\mathcal{D})),$$

where Δ_+^{op} acts on the right through the restriction of the right $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ action on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ along T, and on the left through the restriction of the left $\mathrm{Fun}(\mathcal{D},\mathcal{D})$-action on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ along T'.

Note that Δ_{Cut} is generated as a Δ_+-Δ_+-bimodule by the object $\emptyset \sqcup \emptyset$, and the morphism $\{0\} \sqcup \emptyset \to \emptyset \sqcup \{0\}$. So informally, we can think of a morphism $F : T \to T'$ as consisting of an underlying functor $F(\{0\}) \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$, and a natural transformation $F(\{0\}) \circ T \to T' \circ F(\{0\})$. As for comonads and comodules, we will identify F notationally with its underlying functor.

Now, given a such a morphism $F : T \to T'$ between comonads over $\mathcal{C}$ and $\mathcal{D}$, and a T-comodule X in $\mathcal{C}$, we can tensor the bimodule functor $F : N(\Delta_{\mathrm{Cut}}^{\mathrm{op}}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$ with the module functor $X : N(\Delta_{\max}^{\mathrm{op}}) \to \mathcal{C}$ to obtain a left $N(\Delta_+^{\mathrm{op}})$-module functor

$$N(\Delta_{\mathrm{Cut}}^{\mathrm{op}} \otimes_{\Delta_+^{\mathrm{op}}} \Delta_{\max}^{\mathrm{op}}) \simeq N(\Delta_{\mathrm{Cut}}^{\mathrm{op}}) \otimes_{N(\Delta_+^{\mathrm{op}})} N(\Delta_{\max}^{\mathrm{op}}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D}) \otimes_{\mathrm{Fun}(\mathcal{C},\mathcal{C})} \mathcal{C} \to \mathcal{D}$$

Here $\Delta_{\mathrm{Cut}} \otimes_{\Delta_+} \Delta_{\max}$ can be identified with the category of finite ordered sets with cut $S_0 \sqcup S_1$ with S_1 nonempty and morphisms order-preserving maps $f : S \to T$ with $f(S_1) \subseteq T_1$, and such that f preserves the maximum.

Definition 2.10. *For a morphism of comonads $F : T \to T'$, the corresponding functor $F_* : \mathrm{Comod}_T(\mathcal{C}) \to \mathrm{Comod}_{T'}(\mathcal{D})$ is obtained as the composite*

$$\mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})}(N(\Delta_{\max}^{\mathrm{op}}),\mathcal{C}) \to \mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})}(N(\Delta_{\mathrm{Cut}}^{\mathrm{op}} \otimes_{\Delta_+^{\mathrm{op}}} \Delta_{\max}^{\mathrm{op}}),\mathcal{D}) \to \mathrm{Fun}_{N(\Delta_+^{\mathrm{op}})}(\Delta_{\max}^{\mathrm{op}},\mathcal{D}),$$

where the last map is restriction along the functor $\Delta_{\max} \to \Delta_{\mathrm{Cut}} \otimes_{\Delta_+} \Delta_{\max}$ sending S to $(S \setminus \{\max S\}) \sqcup \{\max S\}$.

Remark 2.11. For a functor $F_* : \mathrm{Comod}_T(\mathcal{C}) \to \mathrm{Comod}_{T'}(\mathcal{D})$ associated to a morphism $F : T \to T'$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Comod}_T(\mathcal{C}) & \xrightarrow{F_*} & \mathrm{Comod}_{T'}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

So F_* acts on underlying objects just as F.

Lemma 2.12. *The natural transformation $FT \Rightarrow T'F$, which is part of the structure of the morphism $F : T \to T'$, refines to a natural transformation*

$$F_*\widetilde{T} \Rightarrow \widetilde{T}'F$$

in $\mathrm{Fun}(\mathcal{C}, \mathrm{Comod}_{T'}(\mathcal{D}))$*, i.e. the natural transformation is compatible with the canonical comodule structures we can put on the individual terms.*

The composite map

$$\begin{aligned} \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{C})}(X, \widetilde{T}Y) &\xrightarrow{F_*} \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(F_*X, F_*\widetilde{T}Y) \\ &\to \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(F_*X, \widetilde{T}'FY) \end{aligned}$$

corresponds under the adjunctions between the forgetful functors and $\widetilde{T}$, $\widetilde{T}'$, to the map

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Map}_{\mathcal{D}}(FX, FY).$$

Proof. The first part follows from the definition of the comodule structure on F_*. For the second, we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{Comod}_T\,\mathcal{C}}(X, \widetilde{T}Y) & \xrightarrow{U} & \mathrm{Map}_{\mathcal{C}}(X, TY) & \xrightarrow{\text{counit}} & \mathrm{Map}_{\mathcal{C}}(X, Y) \\ \downarrow F_* & & \downarrow F & & \downarrow F \\ \mathrm{Map}_{\mathrm{Comod}_{T'}(\mathcal{D})}(F_*X, F_*\widetilde{T}Y) & \xrightarrow{U} & \mathrm{Map}_{\mathcal{D}}(FX, FTY) & \xrightarrow{\text{counit}} & \mathrm{Map}_{\mathcal{D}}(FX, FY) \\ \downarrow & & \downarrow & \simeq & \downarrow \mathrm{id} \\ \mathrm{Map}_{\mathrm{Comod}_{T'}(\mathcal{D})}(F_*X, \widetilde{T}'FY) & \xrightarrow{U} & \mathrm{Map}_{\mathcal{D}}(FX, T'FY) & \xrightarrow{\text{counit}} & \mathrm{Map}_{\mathcal{D}}(FX, FY) \end{array}$$

with upper and lower horizontal composites inducing the adjunction. The commutativity of the lower right square follows from the fact that the natural transformation $FT \Rightarrow T'F$ respects counits by the definition of morphisms $T \to T'$. □

Lemma 2.13. *If $F : T \to T'$ is a morphism of comonads whose underlying functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint G, F_* has a right adjoint $G^* : \mathrm{Comod}_{T'}(\mathcal{D}) \to \mathrm{Comod}_T(\mathcal{C})$. It satisfies $G^*\widetilde{T}'Y = \widetilde{T}GY$ on free comodules, and is computed on arbitrary T'-comodules Y as the totalization of a cosimplicial object*

$$G^*Y = \lim(\widetilde{T}GY \rightrightarrows \widetilde{T}GT'Y \mathrel{\substack{\rightarrow\\[-0.6ex]\rightarrow\\[-0.6ex]\rightarrow}} \cdots).$$

Proof. By Lemma 2.12, the composite map

$$\begin{aligned}\mathrm{Map}_{\mathrm{Comod}_T(\mathcal{C})}(X, \widetilde{T}GY) &\xrightarrow{F_*} \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(F_*X, F_*\widetilde{T}GY)\\ &\to \mathrm{Map}_{\mathrm{Comod}_{T'}(\mathcal{D})}(F_*X, \widetilde{T}'FGY)\\ &\to \mathrm{Map}_{\mathrm{Comod}_{T'}(\mathcal{D})}(F_*X, \widetilde{T}'Y). \end{aligned} \tag{1}$$

corresponds under the adjunction between the forgetful functor and $\widetilde{T}'$ to the equivalence $\mathrm{Map}_{\mathcal{C}}(X, GY) \simeq \mathrm{Map}_{\mathcal{D}}(FX, Y)$. Therefore, it is an equivalence. Since a comodule map $\widetilde{T}'Y_1 \to \widetilde{T}'Y_2$ induces a map, natural in X, between the associated terms on the right side of (1), it does so between the terms on the left. By the Yoneda lemma, this means that it induces a comodule map $\widetilde{T}GY_1 \to \widetilde{T}GY_2$. This shows the existence of a functor G^*, defined on the full subcategory on the essential image of $\widetilde{T}'$, which on objects takes $\widetilde{T}'Y$ to $\widetilde{T}GY$, and is right adjoint to F_* in the sense that there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Comod}_{T'}(\mathcal{D})}(F_*X, Y) \simeq \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{C})}(X, G^*Y)$$

for Y in the full subcategory on the essential image of $\widetilde{T}$.

Using the cobar resolution, we can now define the right adjoint globally. Since the cobar resolution exhibits

$$Y = \lim(\widetilde{T}'Y \rightrightarrows \widetilde{T}'T'Y \mathrel{\substack{\rightarrow\\[-0.6ex]\rightarrow\\[-0.6ex]\rightarrow}} \cdots),$$

we can set

$$G^*Y = \lim(\widetilde{T}GY \rightrightarrows \widetilde{T}GT'Y \mathrel{\substack{\rightarrow\\[-0.6ex]\rightarrow\\[-0.6ex]\rightarrow}} \cdots). \qquad \square$$

Remark 2.14. Note that the d^0 maps in the cosimplial object defining G^*Y are not obvious. They come from natural transformations $\widetilde{T}G \Rightarrow \widetilde{T}GT'$, obtained from functoriality of G^* applied to the transformation $T' \to T'T'$. One can see that this endows $\widetilde{T}G$ with the structure of a *right* T'-comodule, such that the cosimplicial object defining G^*Y can be interpreted as the cobar complex for $(\widetilde{T}G)\square_{T'}Y$.

Roughly speaking, a morphism $H : T \to T'$ intertwines the actions of T on $\mathcal{C}$ and of T' on $\mathcal{D}$ in a lax way, compatible with the respective comonad structures. We will use this notion in section 2.4 to construct adjoint functors between comodule categories.

Moreover, we can also use the construction of Definition 2.10 to lift colimits and monoidal structures from $\mathcal{D}$ to $\mathrm{Comod}_T(\mathcal{D})$.

Lemma 2.15. *If $\mathcal{D}$ has all colimits of shape I,* $\mathrm{Comod}_T(\mathcal{D})$ *has all colimits of shape I too, and the forgetful functor preserves them.*

If $\mathcal{D}$ has all limits of shape I, and T preserves limits of shape I, $\mathrm{Comod}_T(\mathcal{D})$ *has all limits of shape I too, and the forgetful functor preserves them.*

Proof. By assumption, we have a colimit functor $\mathrm{colim} : \mathcal{D}^I \to \mathcal{D}$.

A functor F on $\mathcal{D}$ gives rise to a functor F^I on $\mathcal{D}^I$. Since this construction is natural in F and compatible with composition, T^I is naturally a comonad on $\mathcal{D}^I$.

Now the universal property of colimits shows that there is a transformation $\mathrm{colim} \circ F^I \to F \circ \mathrm{colim}$, natural in F. This endows colim with the structure of a morphism $T^I \to T$.

We obtain a functor $C : \mathrm{Comod}_T(\mathcal{D})^I \simeq \mathrm{Comod}_{T^I}(\mathcal{D}^I) \to \mathrm{Comod}_T(\mathcal{D})$, compatible with colim under the forgetful map. Now it suffices to check that C satisfies the universal property of the colimit.

For a given diagram $(X_i)_{i\in I}$, we thus need to check that the induced map

$$\mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(C((X_i)_{i\in I}), Y) \to \lim \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(X_i, Y)$$

is an equivalence for all Y. Using Lemma 2.8, we can express these mapping spaces as a limit of mapping spaces in $\mathcal{D}$, where it follows from the fact that $C(X_i)$ has as underlying object the colimit $\mathrm{colim}\, X_i$ in $\mathcal{D}$.

The statement for limits is easier, since here, T is assumed to preserve them (whereas for colimits, we only used that any functor preserves colimits in a lax way).

For a diagram $(Y_i)_{i\in I}$ in $\mathrm{Comod}_T(\mathcal{D})$ of shape I, let Y be the limit in $\mathcal{D}$. If T preserves limits of shape I, Y inherits a comodule structure compatible with the maps $Y \to Y_i$.

To see that these exhibit Y as the limit of Y_i in $\mathrm{Comod}_T(\mathcal{D})$, we need to check that

$$\mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(X, Y) \to \lim_{i\in I} \mathrm{Map}_{\mathrm{Comod}_T(\mathcal{D})}(X, Y_i)$$

is an equivalence for any X. This again follows from Lemma 2.8, using that the description given there commutes with limits of shape I in Y (since T commutes with them). □

Corollary 2.16. *It follows in particular that if $\mathcal{D}$ is stable, and T is exact,* $\mathrm{Comod}_T(\mathcal{D})$ *is again a stable $(\infty, 1)$-category.*

We will limit ourselves to the stable case in subsequent sections. Recall that, in a stable $(\infty, 1)$-category $\mathcal{C}$, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ admit canonical deloopings as mapping spectra, which we will denote by $\mathrm{map}_{\mathcal{C}}(X, Y)$.

Definition 2.17. *Suppose $\mathcal{D}$ has an $\mathcal{O}$-monoidal structure for $\mathcal{O}$ a symmetric operad. A lax $\mathcal{O}$-monoidal comonad on $\mathcal{D}$ is a functor*

$$\mathrm{Fun}(N(\Delta_+^{\mathrm{op}}), \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lx}}(\mathcal{D}, \mathcal{D}))$$

Lemma 2.18. *If T is a lax $\mathcal{O}$-monoidal comonad on $\mathcal{D}$, $\mathrm{Comod}_T(\mathcal{D})$ inherits an $\mathcal{O}$-monoidal structure such that the forgetful functor $\mathrm{Comod}_T(\mathcal{D}) \to \mathcal{D}$ is $\mathcal{O}$-monoidal.*

Proof. For such a T, we are given natural transformations

$$\begin{array}{ccc} \mathcal{O}(k) \otimes_{\Sigma_k} \mathcal{D}^k & \longrightarrow & \mathcal{D} \\ {\scriptstyle \mathcal{O}(k)\otimes_{\Sigma_k}(T^n)^{\times k}} \downarrow & \swarrow\!\!\!\swarrow & \downarrow {\scriptstyle T^n} \\ \mathcal{O}(k) \otimes_{\Sigma_k} \mathcal{D}^k & \longrightarrow & \mathcal{D} \end{array}$$

for each n and k, compatible with the comonad structure. This means that we can interpret this data as a morphism from the comonad $\mathcal{O}(k) \otimes_{\Sigma_k} T^{\times k}$ on $\mathcal{O}(k) \otimes_{\Sigma_k} \mathcal{D}^k$, and thus we obtain functors

$$\mathcal{O}(k) \otimes_{\Sigma_k} \mathrm{Comod}_T(\mathcal{D})^k \simeq \mathrm{Comod}_{\mathcal{O}(k)\otimes_{\Sigma_k} T^{\times k}}(\mathcal{O}(k) \otimes_{\Sigma_k} \mathcal{D}^k) \to \mathrm{Comod}_T(\mathcal{D}).$$

Since the morphisms above for different k were compatible with the operadic structure, the resulting structure on $\mathrm{Comod}_T(\mathcal{D})$ is an $\mathcal{O}$-monoidal structure. □

We now want to study the spectral sequence associated to the cobar resolution of Definition 2.8.

Recall that $\Delta_{\leq n}$ is the full subcategory of Δ on ordered sets of cardinality up to n. Similarly, $\Delta_{+,\leq n}$ will denote the full subcategory of Δ_+ on ordered sets of cardinality up to n. For a cosimplicial object X, the totalization $\lim_\Delta X^\bullet$ admits a descending filtration by the $\lim_{\Delta_{\leq n}} X^\bullet$, i.e. a tower

$$\begin{array}{ccc} & & \cdots \\ & & \downarrow \\ F_n(X^\bullet) & \longrightarrow & \lim_{\Delta_{\leq n}} X^\bullet \\ & & \downarrow \\ F_{n-1}(X^\bullet) & \longrightarrow & \lim_{\Delta_{\leq n-1}} X^\bullet \\ & & \downarrow \\ & & \cdots \end{array}$$

with limit $\lim_\Delta X^\bullet$, where we choose to denote the successive fibres by $F_n(X^\bullet)$. Note that $\lim_{\Delta_{\leq -1}} X^\bullet = 0$ since the diagram is empty in this case, and 0 is the terminal object.

The Bousfield-Kan spectral sequence associated to $X^\bullet$ is the homotopy spectral sequence associated to that tower. For an object Y, it has $\mathbb{E}_1$-page given by $\pi_* \operatorname{map}(Y, F_*(X^\bullet))$, and converges conditionally to $\pi_* \operatorname{map}(Y, \lim_\Delta X^\bullet)$

Let $m : \Delta_{\min} \to \Delta_{\min}$ be the functor that sends an ordered set S to $S \sqcup \{\infty\}$, i.e. adjoins a disjoint basepoint. There are analogous functors on Δ_+ and Δ, which we will all denote m.

We can restrict a (split, augmented) cosimplicial space $X^\bullet$ along m to obtain a (split, cosimplicial) space $m^*X^\bullet$. The inclusion map $S \to S \sqcup \{\infty\}$ gives rise to a natural map of (split, augmented) cosimplicial spaces $X^\bullet \to m^*X^\bullet$. Concretely, $m^*X^\bullet$ is in degree n given by X^{n+1}, with structure maps a subset of the structure maps of $X^\bullet$. The remaining coboundary map gives rise to the natural transformation $X^\bullet \to m^*X^\bullet$.

Lemma 2.19. *For the cobar resolution on $Y \in \operatorname{Comod}_T \mathcal{D}$, $m^*(T^{\bullet+1}Y)$ agrees naturally with $T^{\bullet+1}(TY)$, the cobar complex on TY. The natural map*

$$T^{\bullet+1}Y \to m^*(T^{\bullet+1}Y) \simeq T^{\bullet+1}(TY)$$

agrees with the map induced by the comultiplication map $Y \to TY$.

Proof. The comodule structure maps (as in Definition 2.2) for TY can be obtained from the comodule structure maps for Y by restricting along the functor $\Delta_{\max} \to \Delta_{\max}$ that adds a disjoint new maximum. Under the equivalence $\Delta_{\min} \simeq \Delta_{\max}^{\mathrm{op}}$ used in Definition 2.7, this corresponds precisely to the functor m^*.

The natural map $T^{\bullet+1}Y \to m^*T^{\bullet+1}Y$ is induced by the canonical inclusion $S \to S \sqcup \{\infty\}$ in $\Delta_{\min}$. This map is dual to

$$\operatorname{Hom}_{\Delta_{\min}}(S \sqcup \{\infty\}, \{0,1\}) \to \operatorname{Hom}_{\Delta_{\min}}(S, \{0,1\})$$

in $\Delta_{\max}$. This sends both the two largest elements on the left, the constant 0 map and the map that is 1 on ∞ and 0 everywhere else, to the constant 0 map, i.e. the maximum on the right, and is an isomorphism on the remaining elements.

But this latter map corresponds precisely to the map $Y \to TY$ of comodules. Thus, the claim follows. □

The next statement is a standard lemma about cosimplicial objects.

Lemma 2.20. *For $X^\bullet$ an augmented cosimplicial object, we let $\operatorname{fib}_{\leq n} X^\bullet$ denote the fibre of $X^{-1} \to \lim_{\Delta_{\leq n}} X^\bullet$. Then there are natural fibre sequences*

$$\operatorname{fib}_{\leq n} X^\bullet \to \operatorname{fib}_{\leq n-1} X^\bullet \to \operatorname{fib}_{\leq n-1} m^*X^\bullet$$
$$F_nX^\bullet \to F_{n-1}X^\bullet \to F_{n-1}m^*X^\bullet$$

Proof. Let $\mathcal{P}_{\neq\emptyset}(\{0,\ldots,n\})$ denote the partially ordered set of nonempty subsets of $\{0,\ldots,n\}$. There is a functor $\mathcal{P}_{\neq\emptyset}(\{0,\ldots,n\}) \to \Delta_{\leq n}$, which, according to Lemma 1.2.4.17 in [Lur16], is right cofinal.

Since right cofinal functors preserve limits, we can identify $\operatorname{fib}_{\leq n} X^\bullet$ with the fibre of $X^{-1} \to \lim_{\mathcal{P}_{\neq\emptyset}(\{0,\ldots,n\})} X^\bullet$. This is known as the total homotopy fibre of the cubical diagram obtained by restricting $X^\bullet$ to $\mathcal{P}(\{0,\ldots,n\})$ (including the empty set).

We have an analogous initial functor $\mathcal{P}_{\neq\emptyset}(\{0,\ldots,n-1\}) \to \Delta_{\leq n-1}$, and the two functors $\Delta_{\leq n-1} \to \Delta_{\leq n}$ given by the canonical inclusion and m are compatible with the two functors $\mathcal{P}(\{0,\ldots,n-1\}) \to \mathcal{P}(\{0,\ldots,n\})$ given by the canonical inclusion and the functor that adds $\{n\}$ to a subset of $\{0,\ldots,n-1\}$.

Now it is a standard fact about cubical diagrams that the total homotopy fibre of an $n+1$-dimensional cubical diagram indexed over $\mathcal{P}(\{0,\ldots,n\})$ can be computed by first taking the total homotopy fibres of the two n-dimensional cubical subdiagrams indexed over $\mathcal{P}(\{0,\ldots,n-1\})$ and $\{S \subseteq \{0,\ldots,n\} | n \in S\}$, and then taking the homotopy fibre of the map between them, so the first fibre sequence follows.

Taking vertical homotopy fibres in the diagram of fibre sequences

$$\begin{array}{ccccc}
\operatorname{fib}_{\leq n} X^\bullet & \longrightarrow & X^{-1} & \longrightarrow & \lim_{\Delta_{\leq n}} X^\bullet \\
\downarrow & & \downarrow \simeq & & \downarrow \\
\operatorname{fib}_{\leq n-1} X^\bullet & \longrightarrow & X^{-1} & \longrightarrow & \lim_{\Delta_{\leq n-1}} X^\bullet
\end{array}$$

we see that the fibre of $\operatorname{fib}_{\leq n} X^\bullet \to \operatorname{fib}_{\leq n-1} X^\bullet$ coincides with $\Sigma^{-1} T_n X^\bullet$. Now taking vertical homotopy fibres in the diagram of fibre sequences

$$\begin{array}{ccccc}
\operatorname{fib}_{\leq n} X^\bullet & \longrightarrow & \operatorname{fib}_{\leq n-1} X^\bullet & \longrightarrow & \operatorname{fib}_{\leq n-1} m^* X^\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{fib}_{\leq n-1} X^\bullet & \longrightarrow & \operatorname{fib}_{\leq n-2} X^\bullet & \longrightarrow & \operatorname{fib}_{\leq n-2} m^* X^\bullet
\end{array}$$

and suspending once, we get a natural fibre sequence $T_n X^\bullet \to T_{n-1} X^\bullet \to T_{n-1} m^* X^\bullet$. □

Lemma 2.21. *For T exact, let $\overline{T} \in \operatorname{Fun}(\operatorname{Comod}_T(\mathcal{D}), \operatorname{Comod}_T(\mathcal{D}))$ be defined by fibre sequences $\overline{T}Y \to Y \to TY$, where $Y \to TY$ is the coaction map.*

For the cobar resolution of a comodule Y, there are natural equivalences

$$\operatorname{fib}_{\leq n}(T^{\bullet+1}Y) \simeq \overline{T}^{n+1} Y$$
$$F_n(T^{\bullet+1}Y) \simeq \widetilde{T}\overline{T}^n Y$$

Proof. We proceed by induction. For $n = 0$, both claims are clear. Now recall that $m^*T^{\bullet+1}Y$ is naturally equivalent to $T^{\bullet+1}(TY)$, and assume we know the statement of the lemma for $n-1$.

The fibre sequences of 2.20 now read

$$\operatorname{fib}_{\leq n} T^{\bullet+1}Y \to \overline{T}^n Y \to \overline{T}^n TY$$
$$F_n T^{\bullet+1}Y \to \widetilde{T}\overline{T}^n Y \to \widetilde{T}\overline{T}^n TY$$

with maps induced by $Y \to TY$. Thus $\operatorname{fib}_{\leq n} T^{\bullet+1}Y \simeq \overline{T}^{n+1}Y$, and $F_n T^{\bullet+1}Y \simeq T\overline{T}^n Y$. □

Lemma 2.22. *Let T be an exact comonad on $\mathcal{D}$. The Bousfield-Kan spectral sequence associated to the cosimplicial spectrum obtained by applying the functor* $\operatorname{map}_{\operatorname{Comod}_T(\mathcal{D})}(X, -)$ *to the cobar resolution of Y from Definition 2.8 will be called the cobar spectral sequence. It converges conditionally, with E_1-page and abutment of the form*

$$\pi_n(\operatorname{map}_{\mathcal{D}}(X, \overline{T}^k Y)) \Rightarrow \pi_n \operatorname{map}_{\operatorname{Comod}_T(\mathcal{D})}(X, Y)$$

and differential d_r of bidegree $(-1, r)$ in the (n, k)-grading given here.

Remark 2.23. For T not exact, one still obtains a Bousfield-Kan spectral sequence, defined in terms of homotopy groups of a cosimplicial space (instead of a cosimplicial spectrum). However, since homotopy groups of spaces aren't abelian groups in degrees ≤ 1, one would need to take additional care around the edge.

2.2 The Adams spectral sequence of an adjunction

Let Sp denote the classical stable $(\infty, 1)$-category of spectra. For $E \in \mathrm{Sp}$ an $\mathbb{E}_1$-ring, we have a stable $(\infty, 1)$-category Mod_E of E-modules, and a functor $\mathrm{Sp} \to \operatorname{Mod}_E$, $X \mapsto E \otimes X$. This admits a right adjoint, the forgetful functor $\operatorname{Mod}_E \to \mathrm{Sp}$.

The Adams spectral sequence sometimes allows us to compute (homotopy groups of) mapping spectra $\operatorname{map}_{\mathrm{Sp}}(X, Y)$ between spectra X, Y through their E-homologies, i.e. through $E \otimes X$ and $E \otimes Y$. Roughly, there is additional structure on $E \otimes X$ and $E \otimes Y$, namely a coaction of the cooperations of the homology theory E, and we can sometimes recover $\operatorname{map}_{\mathrm{Sp}}(X, Y)$ as maps between the E-modules $E \otimes X$ and $E \otimes Y$ that are compatible with this coaction. On the level of homotopy groups of these mapping spectra, this additional structure leads to Ext-groups if E is sufficiently nice. We will discuss the details in Example 2.36.

In general, let $\mathcal{C}, \mathcal{D}$ be stable $(\infty, 1)$-categories, and let $F \dashv G : \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction of exact functors. For simplicity, assume that $\mathcal{C}$ and $\mathcal{D}$ have all small limits and colimits.

One can ask when this allows us to compute morphism spaces in $\mathcal{C}$ from morphism spaces in $\mathcal{D}$.

The adjunction $F \dashv G$ gives rise to an exact comonad FG on $\mathcal{D}$. Informally, the comultiplication of FG is just given by the unit of the adjunction, $FG \Rightarrow F(GF)G \simeq (FG)(FG)$, and the counit by the counit $FG \Rightarrow \mathrm{id}$ of the adjunction. Moreover, there is a coaction of FG on F, making F into a left comodule. Informally, the coaction map is also given by the unit, $F \Rightarrow F(GF) \simeq (FG)F$. To make these structures fully coherent, it is efficient to describe both the coaction and the comonad structure by a universal property: $FG \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ is universal among all $T \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ together with a natural transformation $F \Rightarrow T \circ F$ of functors in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, i.e. a "co-endomorphism object" of F. One can show that this gives FG the structure of a comonad, and F the structure of a comodule over FG. For details, see [Lur16], Section 4.7.

We can consider the category $\mathrm{Comod}_{FG}(\mathcal{D})$ of FG-comodules in $\mathcal{D}$. Since FG is exact, this is again a stable $(\infty, 1)$-category, by Corollary 2.16.

The functor $F : \mathcal{C} \to \mathcal{D}$ factors through $\mathcal{C} \xrightarrow{\widetilde{F}} \mathrm{Comod}_{FG}(\mathcal{D})$ where $\widetilde{F}X$ is just FX with comodule structure obtained from the left coaction on F. The right adjoint to the forgetful functor $\mathrm{Comod}_{FG}(\mathcal{D}) \to \mathcal{D}$ factors as $\widetilde{F}G$.

Definition 2.24. *We call X complete with respect to the adjunction $F \dashv G$ if the map*

$$\widetilde{F} : \mathrm{map}_{\mathcal{C}}(Z, X) \to \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X)$$

is an equivalence for all Z.

Lemma 2.25. *All objects of the form GY are complete with respect to $F \dashv G$.*

Proof. From the adjunctions between F and G and between the forgetful functor and $\widetilde{F}G$, we get

$$\mathrm{map}_{\mathcal{C}}(Z, GY) \simeq \mathrm{map}_{\mathcal{D}}(FZ, Y) \simeq \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}GY),$$

with composite map induced by $\widetilde{F}$. □

Lemma 2.26. *Given a diagram X_i of shape I in $\mathcal{C}$ with limit $X = \lim_I X_i$, assume that all the X_i are complete with respect to $F \dashv G$.*

If $\widetilde{F}$ preserves the limit, i.e. $\widetilde{F}X \simeq \lim_I \widetilde{F}X_i$, then X is complete with respect to $F \dashv G$.

Proof. We have equivalences

$$\begin{aligned} \mathrm{map}_{\mathcal{C}}(Z, X) &\simeq \lim\nolimits_I \mathrm{map}_{\mathcal{C}}(Z, X_i) \\ &\simeq \lim\nolimits_I \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X_i) \\ &\simeq \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \lim\nolimits_I \widetilde{F}X_i) \end{aligned}$$

and the result follows. □

To analyze whether arbitrary X is complete, we can try to resolve it by objects of the form GY.

Lemma 2.27. *For a comodule $Y \in \mathrm{Comod}_{FG}(\mathcal{D})$, the cobar resolution*

$$FGY \rightrightarrows (FG)^2 Y \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

lifts against $\widetilde{F}$ to a cosimplicial object in $\mathcal{C}$,

$$GY \rightrightarrows G(FG)Y \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots .$$

Let $\widetilde{G}Y$ denote the limit of this cosimplicial diagram. Then $\widetilde{G}$ is right adjoint to $\widetilde{F}$

Proof. The cobar resolution of Y lifts along $\widetilde{F}$ since the objects $GY, GFGY, \ldots$ are all complete with respect to $F \dashv G$, and so $\widetilde{F}$ is fully faithful on the full subcategory on those objects.

We can then directly compute:

$$\begin{aligned}
&\mathrm{map}_{\mathcal{C}}(X, \widetilde{G}Y)\\
&\quad = \lim_{\Delta}(\mathrm{map}_{\mathcal{C}}(X, GY) \rightrightarrows \mathrm{map}_{\mathcal{C}}(X, GFGY) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots)\\
&\quad = \lim_{\Delta}(\mathrm{map}_{\mathrm{Comod}_{FG}(cD)}(\widetilde{F}X, \widetilde{F}GY) \rightrightarrows \mathrm{map}_{\mathrm{Comod}_{FG}(cD)}(\widetilde{F}X, \widetilde{F}GFGY) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots)\\
&\quad = \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}X, Y)
\end{aligned}$$

□

Remark 2.28. Note that we can consider $\mathcal{C}$ as a comodule category $\mathrm{Comod}_{\mathrm{id}}(\mathcal{C})$. The left comodule structure on $\widetilde{F}$ then precisely gives F the structure of a morphism of comonads $\mathrm{id} \to FG$ as in Definition 2.9, with $\widetilde{F} : \mathrm{Comod}_{\mathrm{id}}(\mathcal{C}) \to \mathrm{Comod}_{FG}(\mathcal{D})$ the associated functor. We could then obtain the existence of a right adjoint of $\widetilde{F}$ from Lemma 2.13. In this perspective, however, it is not completely clear how to show that $\widetilde{F}$ turns the defining cosimplicial diagram of $\widetilde{G}Y$ into the cobar resolution of Y.

Definition 2.29. *For $X \in \mathcal{C}$, we will call the cosimplicial object defining $\widetilde{G}(\widetilde{F}X)$ the GF-Adams resolution. In that case, the augmentation lifts against $\widetilde{F}$ as well (but not the splits), so the GF-Adams resolution of X takes the form*

$$X \to GFX \rightrightarrows (GF)^2 X \cdots .$$

We will usually just write it as $(GF)^{\bullet+1}X$.

Remark 2.30. As the terminology suggests, the GF-Adams resolution doesn't depend on the full adjunction $F \dashv G$ anymore, only on the monad GF. Informally, one sees that the coboundary maps come from the unit $\mathrm{id} \Rightarrow GF$, while the codegeneracy maps come from the counit $FG \Rightarrow \mathrm{id}$, which induces the monad structure map

$$(GF)(GF) \simeq G(FG)F \Rightarrow GF.$$

Definition 2.31. *For $X \in \mathcal{C}$, we define $X^\wedge_{GF} \in \mathcal{C}$ to be the limit (i.e. totalization)*

$$X^\wedge_{GF} := \lim_\Delta (GF)^{\bullet+1} X$$

of the GF-Adams resolution of X.

In particular, there is a canonical map $X \to X^\wedge_{GF}$.

Recall that applying $\widetilde{F}$ to the GF-Adams resolution of X yields the cobar resolution of the FG-comodule $\widetilde{F}X$. Since the cobar resolution is split, the augmentation exhibits $\widetilde{F}X$ as its limit. It follows that there is a natural map

$$\widetilde{F}(X^\wedge_{GF}) \to \widetilde{F}X$$

splitting the map $\widetilde{F}X \to \widetilde{F}(X^\wedge_{GF})$.

Lemma 2.32. *The composite*

$$\mathrm{map}_{\mathcal{C}}(Z, X^\wedge_{GF}) \to \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X^\wedge_{GF}) \to \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X)$$

is always an equivalence.

Proof. We have

$$\begin{aligned}\mathrm{map}_{\mathcal{C}}(Z, X^\wedge_{GF}) &\simeq \lim_{\bullet\in\Delta} \mathrm{map}_{\mathcal{C}}(Z, (GF)^\bullet X)\\ &\simeq \lim_{\bullet\in\Delta} \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{C})}(\widetilde{F}Z, \widetilde{F}(GF)^\bullet X)\\ &\simeq \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X),\end{aligned}$$

since $X^\wedge_{GF}$ is the limit of the GF-Adams resolution, the terms of the GF-Adams resolution are in the image of G and thus complete with respect to $F \dashv G$, and $\widetilde{F}X$ is the limit of the cobar resolution. □

Lemma 2.33. *X is complete with respect to the adjunction $F \dashv G$ if and only if the map $X \to X^\wedge_{GF}$ is an equivalence.*

Proof. Consider the diagram

$$\begin{array}{ccc}\mathrm{map}_{\mathcal{C}}(Z, X) & \longrightarrow & \mathrm{map}_{\mathcal{C}}(Z, X^\wedge_{GF})\\ & \searrow & \downarrow \simeq\\ & & \mathrm{map}_{\mathcal{C}}(\widetilde{F}Z, \widetilde{F}X)\end{array}$$

The vertical map is an equivalence by Lemma 2.32. By the Yoneda lemma, the horizontal map is an equivalence if and only if the map $X \to X^\wedge_{GF}$ is an equivalence, and by Definition 2.24, the diagonal map is an equivalence if and only if X is complete with respect to the adjunction. □

Remark 2.34. The comonadic Barr-Beck theorem ([Lur16], Theorem 4.7.3.5) gives conditions under which the functors $\widetilde{F}$ and $\widetilde{G}$ are inverse equivalences. In that case, $X \to X^{\wedge}_{GF}$ is an equivalence for all X.

However, the Barr-Beck conditions are much too restrictive for our applications, and $X \to X^{\wedge}_{GF}$ is never an equivalence for all X in the cases we consider. For example, even for the adjunction $\mathrm{Sp} \to \mathrm{Mod}_{MU}$, whose Adams spectral sequence (cf. Example 2.36) is known to converge for all connective X, there are still nontrivial nonconnective X with $MU \otimes X = 0$. An example is discussed in [Rav92], Section 7.4, and was first constructed in Section 3 of [Rav84].

We will see in Section 2.3 how to embed $\mathcal{C}$ into a certain refinement of $\mathrm{Comod}_{FG}(\mathcal{D})$ under conditions which are more appropriate for our purposes.

Since the GF-Adams resolution of X consists levelwise of $(F \dashv G)$-complete objects, and turns into the FG-cobar resolution of $\widetilde{F}X$ if we apply $\widetilde{F}$ to it, we obtain the following:

Lemma 2.35. *The Bousfield-Kan spectral sequence for the cosimplicial spectrum obtained by applying* $\mathrm{map}_{\mathcal{C}}(Z, -)$ *to the GF-Adams resolution of X agrees with the cobar spectral sequence for the FG-comodule $\widetilde{F}X$. It converges conditionally to*

$$\pi_* \mathrm{map}_{\mathcal{C}}(Z, X^{\wedge}_{GF}) = \pi_* \mathrm{map}_{\mathrm{Comod}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X).$$

We will also refer to this spectral sequence as the Adams spectral sequence of the adjunction $F \dashv G$ for $\mathrm{map}_{\mathcal{C}}(Z, X)$.

In good cases, this E_1-page can further be identified with the cobar complex for comodules over a comonad on some abelian category. In that case, the E_2-page admits a description as Ext of such comodules.

Example 2.36. The Adams spectral sequence for an $\mathbb{E}_1$-ring spectrum E arises by taking $\mathcal{C} = \mathrm{Sp}$, $\mathcal{D} = \mathrm{Mod}_E$, $F(X) = E \otimes X$ and G the forgetful functor from E-modules to spectra.

Then the comonad FG sends a module Y to $(E \otimes E) \otimes_E Y$, and its comultiplication comes from the map

$$E \otimes E \simeq E \otimes S \otimes E \to E \otimes E \otimes E \simeq (E \otimes E) \otimes_E (E \otimes E).$$

For a spectrum X, the canonical GF-resolution takes the form

$$X \to E \otimes X \rightrightarrows E \otimes E \otimes X \cdots ,$$

which is precisely the canonical cosimplicial Adams resolution. The GF-completion $X^{\wedge}_{GF}$ is therefore exactly the E-nilpotent completion $X^{\wedge}_{E}$, as defined in [Bou79].

(The definition of $X_E^\wedge$ in terms of the canonical Adams tower can be related to the cosimplicial one by Lemma 2.38.)

Proposition 2.32 therefore says that

$$\mathrm{map}_{\mathrm{Sp}}(Z, X_E^\wedge) \simeq \mathrm{map}_{\mathrm{Comod}_{E\otimes E}(\mathrm{Mod}_E)}(E \otimes Z, E \otimes X).$$

and the Bousfield-Kan spectral sequence gives a spectral sequence to compute $[Z, X_E^\wedge]$ from the cobar complex in $E \otimes E$-comodules.

Now if $E_*E = \pi_*(E \otimes E)$ is flat over $E_* = \pi_*E$, the comonad $E \otimes E$ on Mod_E induces, on the level of homotopy groups, a comonad $M \mapsto E_*E \otimes_{E_*} M$ on the category of E_*-modules. In that case, the cobar resolution agrees, after passing to homotopy groups, with the algebraic cobar resolution of $E_*X = \pi_*(E\otimes X)$ over E_*E. If additionally E_*Z is projective as E_*-module, the E_1 page of the Bousfield-Kan spectral sequence agrees with the cobar complex for $\mathrm{Ext}_{E_*E}(E_*Z, E_*X)$.

Remark 2.37. In the case of an adjunction $\mathrm{Sp} \rightleftarrows \mathrm{Mod}_E$ for an $\mathbb{E}_1$-ring spectrum E as discussed in Example 2.36, note that the notion of being complete with respect to the adjunction (as in Definition 2.24) is generally not equivalent to the classical notion of being E-local.

Recall that, by Lemma 2.33, X is complete with respect to the adjunction $\mathrm{Sp} \rightleftarrows \mathrm{Mod}_E$ if and only if the map $X \to X_E^\wedge$ is an equivalence. Since $X_E^\wedge$ is a limit of E-local spectra, it is E-local. So if X is complete with respect to the adjunction, it is E-local.

However, the other direction doesn't necessarily hold.

For any X, observe that since $X \to L_E X$ is an E-homology isomorphism, it induces an equivalence $X_E^\wedge \to (L_E X)_E^\wedge$. The natural map $L_E X \to (L_E X)_E^\wedge$ therefore factors through a map $L_E X \to X_E^\wedge$, which we can describe as the unique map obtained from $X \to X_E^\wedge$ and the fact that $X_E^\wedge$ is E-local.

We see that $L_E X$ is complete with respect to the adjunction $Sp \rightleftarrows \mathrm{Mod}_E$ if and only if $L_E X \to X_E^\wedge$ is an equivalence. However, there are examples of X and E such that $L_E X$ and $X_E^\wedge$ are not equivalent, see Theorem 6.7 in [Bou79]. For such an example, $L_E X$ is therefore E-local but not complete with respect to the adjunction $Sp \rightleftarrows \mathrm{Mod}_E$.

The next statement will be used in the graded setting together with connectivity assumptions to derive fairly general convergence statements for Adams spectral sequences of adjunctions.

Lemma 2.38. *Let* $\mathrm{fib}_{\le n}(GF)^{\bullet+1}X$ *denote the fibre of the map*

$$X \to \lim_{\Delta_{\le n}} (GF)^{\bullet+1} X,$$

and let $F_n((GF)^{\bullet+1}X)$ *denote the fibre of*

$$\lim_{\Delta_{\le n}} (GF)^{\bullet+1} X \to \lim_{\Delta_{\le n-1}} (GF)^{\bullet+1} X$$

Denote by $\overline{GF}$ *the fibre of the natural transformation* $\mathrm{id} \to GF$*. Then there are natural equivalences*

$$\mathrm{fib}_{\leq n}(GF)^{\bullet+1}X \simeq \overline{GF}^{n+1}X$$
$$F_n((GF)^{\bullet+1}X) \simeq GF\overline{GF}^{n}X$$

Furthermore, X *is complete with respect to* $F \dashv G$*, i.e.* $X \simeq \lim_{\Delta}((GF)^{\bullet+1}X) = X^{\wedge}_{GF}$*, if and only if*

$$\lim_n \overline{GF}^{n}X = 0.$$

Proof. Since the cosimplicial resolution $(GF)^{\bullet+1}X$ was defined to be the lift of the FG-cobar resolution of $\widetilde{F}X$ along $\widetilde{F}$, Lemma 2.19 implies that $m^*(GF)^{\bullet+1}X$ is the lift of the FG-cobar resolution of $\widetilde{F}GFX$ along $\widetilde{F}$. As the natural map between the two cobar restrictions agrees with the one induced by the coaction map $\widetilde{F}X \to \widetilde{F}GFX$, which lifts to the unit $X \to GFX$, we obtain that the natural map

$$(GF)^{\bullet+1}X \to m^*(GF)^{\bullet+1}X$$

agrees with the map

$$(GF)^{\bullet+1}X \to (GF)^{\bullet+1}(GFX)$$

induced by the unit.

Since $\mathrm{fib}_{\leq 0}(GF)^{\bullet+1}X \simeq \overline{GF}X$ is clear, inductively assume we have natural isomorphisms $\mathrm{fib}_{\leq n-1}(GF)^{\bullet+1}X \simeq \overline{GF}^{n}X$. The first fibre sequence of Lemma 2.20 then reads

$$\mathrm{fib}_{\leq n}(GF)^{\bullet+1}X \to \overline{GF}^{n}X \to \overline{GF}^{n}GFX,$$

so we get a natural equivalence $\mathrm{fib}_{\leq n}(GF)^{\bullet+1}X \simeq \overline{GF}^{n+1}X$.

Similarly, $F_0((GF)^{\bullet+1}X) \simeq GFX$ is clear, and if we inductively have natural equivalences $F_{n-1}((GF)^{\bullet+1}X) \simeq GF\overline{GF}^{n-1}X$, the second fibre sequence of Lemma 2.20 reads

$$F_n((GF)^{\bullet+1}X) \to GF\overline{GF}^{n-1}X \to GF\overline{GF}^{n-1}GFX,$$

from which we get a natural equivalence $F_n((GF)^{\bullet+1}X) \simeq GF\overline{GF}^{n}X$.

The final statement follows from the fact that homotopy fibres commute with limits. So

$$\mathrm{fib}(X \to \lim_{\Delta}(GF)^{\bullet+1}X) \simeq \lim_n \mathrm{fib}_{\leq n}((GF)^{\bullet+1}X) \simeq \lim_n \overline{GF}^{n+1}X. \qquad \square$$

Lemma 2.39. *For an operad* $\mathcal{O}$ *(typically* $\mathbb{E}_1$ *or* $\mathbb{E}_\infty$*), if* $\mathcal{C}$ *and* $\mathcal{D}$ *are* $\mathcal{O}$*-monoidal categories, and* $F : \mathcal{C} \to \mathcal{D}$ *a* $\mathcal{O}$*-monoidal functor with right adjoint* G*,* FG *is a lax* $\mathcal{O}$*-monoidal comonad, so the comodule category* $\mathrm{Comod}_{FG}\,\mathcal{D}$ *admits a* $\mathcal{O}$*-monoidal structure. Furthermore, the functors* $\widetilde{F}$ *and the forgetful functor* $\mathrm{Comod}_{FG}(\mathcal{D}) \to \mathcal{D}$

are $\mathcal{O}$-monoidal, and the cobar resolution of a monoid object Y in $\mathrm{Comod}_{FG}(\mathcal{D})$ is multiplicative, so the associated cobar spectral sequence is a multiplicative spectral sequence.

Proof. If F is monoidal, G is lax monoidal, with structure induced by the adjoint of

$$F(G(X)\otimes G(Y))\simeq F(G(X))\otimes F(G(Y))\to X\otimes Y.$$

It follows that FG is naturally an element of $\mathrm{Fun}^{\otimes,lx}_{\mathcal{O}}(\mathcal{D},\mathcal{D})$, and similarly for GF. Furthermore, both the counit and the unit of the adjunction can be given the structure of natural transformations between lax $\mathcal{O}$-symmetric functors.

It follows that $FG\in\mathrm{Fun}^{\otimes,lx}_{\mathcal{O}}(\mathcal{D},\mathcal{D})$ is a co-endomorphism object of $F\in\mathrm{Fun}^{\otimes,lx}_{\mathcal{O}}(\mathcal{C},\mathcal{D})$ (cf. [Lur16], Section 4.7.1, and Lemma 4.7.3.1). So the comonad structure and the coaction on F are compatible with the structures of lax monoidal functors. □

2.3 Compactly generated comodules

For an exact comonad T on $\mathcal{D}$, we have considered the category $\mathrm{Comod}_T(\mathcal{D})$. This is, for many purposes, not the correct category of comodules to work in. The issue is that $\mathrm{Comod}_T(\mathcal{D})$ is rarely compactly generated.

For the rest of the section, we assume that T preserves filtered colimits. Then its lift $\widetilde{T}:\mathcal{D}\to\mathrm{Comod}_T(\mathcal{D})$ preserves filtered colimits, too, since the forgetful functor $\mathrm{Comod}_T(\mathcal{D})\to\mathcal{D}$ preserves colimits and detects equivalences.

In that case, compact objects $X\in\mathrm{Comod}_T(\mathcal{D})$ have underlying compact objects, as we see from the adjunction

$$\begin{aligned}\mathrm{map}_{\mathcal{D}}(X,\mathrm{colim}_I Y_i)&\simeq \mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X,\mathrm{colim}_I \widetilde{T}Y_i)\\&\simeq \mathrm{colim}_I\,\mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X,\widetilde{T}Y_i)\\&\simeq \mathrm{colim}_I\,\mathrm{map}_{\mathcal{D}}(X,Y_i)\end{aligned}$$

for any filtered diagram $(Y_i)_{i\in I}$ in $\mathcal{D}$.

The converse is almost never true. If we assume $X\in\mathrm{Comod}_T(\mathcal{D})$ any comodule with compact underlying object in $\mathcal{D}$, and $(Y_i)_{i\in I}$ a filtered diagram in $\mathrm{Comod}_T(\mathcal{D})$, we get natural equivalences

$$\begin{aligned}\mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X,\mathrm{colim}_I Y_i)&\simeq \lim_\Delta \mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X,T^{\bullet+1}(\mathrm{colim}_I Y_i))\\&\simeq \lim_\Delta \mathrm{map}_{\mathcal{D}}(X,T^{\bullet}(\mathrm{colim}_I Y_i))\\&\simeq \lim_\Delta \mathrm{colim}_I\,\mathrm{map}_{\mathcal{D}}(X,T^{\bullet}Y_i)\\&\simeq \lim_\Delta \mathrm{colim}_I\,\mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X,T^{\bullet+1}Y_i)\end{aligned}$$

and

$$\mathrm{colim}_I\, \mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X, Y_i) \simeq \mathrm{colim}_I\, \lim_\Delta \mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X, T^{\bullet+1}Y_i).$$

So the cosimplicial cobar resolution for $\mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X, -)$ commutes levelwise with filtered colimits. However, totalization of cosimplicial objects typically never commutes with filtered colimits.

We want to define an alternative category of T-comodules in $\mathcal{D}$ that preserves compactness, in the sense that its compact objects are precisely given by comodules with compact underlying objects in $\mathcal{D}$. Furthermore, it will be generated by its compact objects.

Definition 2.40. *Let* $\mathrm{Comod}_T^c(\mathcal{D})$ *denote the full subcategory of* $\mathrm{Comod}_T(\mathcal{D})$ *on comodules that have compact underlying objects in* $\mathcal{D}$*. We define the* compactly generated comodule category

$$\mathrm{Comod}_T^{cg}(\mathcal{D}) := \mathrm{Ind}(\mathrm{Comod}_T^c(\mathcal{D})).$$

Recall that, for a stable $(\infty,1)$-category $\mathcal{C}$, the stable $(\infty,1)$-category $\mathrm{Ind}(\mathcal{C})$ is defined as the full subcategory of the presheaf category $\mathrm{PSh}(\mathcal{C})$ on filtered colimits of representable presheaves, i.e. presheaves in the image of the Yoneda embedding $X \mapsto \mathrm{map}_{\mathcal{C}}(-, X)$.

In particular, $\mathrm{Comod}_T^{cg}(\mathcal{D})$ is generated by the objects of $\mathrm{Comod}_T^c(\mathcal{D})$ under filtered colimits, and the objects of $\mathrm{Comod}_T^c(\mathcal{D})$ are compact in $\mathrm{Comod}_T^{cg}(\mathcal{D})$.

The image of the Yoneda embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ consists of compact objects, this is [Lur09], Proposition 5.3.5.5.

From the universal property of the Ind-category, if $\mathcal{D}$ has filtered colimits, any functor $\mathcal{C} \to \mathcal{D}$ gives rise to a functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ extending the original functor and commuting with finite colimits. This just sends a filtered colimit of representables in $\mathrm{PSh}(\mathcal{C})$ to the colimit of the associated diagram in $\mathcal{D}$. In other words, it is the left Kan extension of $\mathcal{C} \to \mathcal{D}$ along the embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

By Corollary 5.3.5.4 of [Lur09], a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ lies in $\mathrm{Ind}(\mathcal{C})$ if and only if F preserves finite limits in $\mathcal{C}^{\mathrm{op}}$ (i.e. finite colimits in $\mathcal{C}$).

If $F : \mathcal{C} \to \mathcal{D}$ preserves finite colimits, then the unique filtered colimit-preserving extension $F' : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ has a right adjoint G. G is necessarily defined by sending Y to the presheaf $X \mapsto \mathrm{map}(FX, Y)$, which is in $\mathrm{Ind}(\mathcal{C})$ since F preserves finite colimits (cf. [Lur09], Proposition 5.3.5.13).

In our case, we obtain

Lemma 2.41. *There is an adjunction* $L \dashv I : \mathrm{Comod}_T^{cg}(\mathcal{D}) \to \mathrm{Comod}_T(\mathcal{D})$*, uniquely characterized by requiring that the left adjoint fixes* $\mathrm{Comod}_T^c(\mathcal{D})$*.*

We will denote by U the composite $\mathrm{Comod}_T^{cg}(\mathcal{D}) \to \mathcal{D}$, and still refer to UX it as the "underlying object" of X. Note, however, that U does not detect equivalences on $\mathrm{Comod}_T^{cg}(\mathcal{D})$ anymore, so it shouldn't be considered a forgetful functor.

Proposition 2.42. *If the composite Yoneda map*

$$\mathrm{Comod}_T(\mathcal{D}) \to \mathrm{PSh}(\mathrm{Comod}_T(\mathcal{D})) \to \mathrm{PSh}(\mathrm{Comod}_T^c(\mathcal{D}))$$

is still fully faithful, the adjunction $L \dashv I$ exhibits $\mathrm{Comod}_T(\mathcal{D})$ as the reflexive localization of $\mathrm{Comod}_T^{cg}(\mathcal{D})$ at those morphisms which are mapped to equivalences in $\mathcal{D}$ by the underlying object functor U.

Proof. Since the forgetful functor $\mathrm{Comod}_T(\mathcal{D}) \to \mathcal{D}$ detects equivalences, we equivalently claim that the adjunction exhibits $\mathrm{Comod}_T(\mathcal{D})$ as reflexive localization of $\mathrm{Comod}_T^{cg}(\mathcal{D})$ at those morphisms which are mapped to equivalences in $\mathrm{Comod}_T(\mathcal{D})$ by L. This is equivalent to saying that I is fully faithful with image L-local objects, and the unit $\mathrm{id} \Rightarrow IL$ is the unique L-localization, i.e. has L-acyclic fibre.

By assumption, I is fully faithful. We get natural equivalences

$$\mathrm{map}_{\mathrm{Comod}_T(\mathcal{D})}(X,Y) \simeq \mathrm{map}_{\mathrm{Comod}_T^{cg}(\mathcal{D})}(IX, IY) \simeq \mathrm{map}_{\mathrm{Comod}_T^{cg}(\mathcal{D})}(LIX, Y).$$

Here the composite is induced by the counit $LI \Rightarrow \mathrm{id}$, and by the co-Yoneda lemma, it follows that $LI \Rightarrow \mathrm{id}$ is an equivalence of functors.

Now call $X \in \mathrm{Comod}_T^{cg}(\mathcal{D})$ L-acyclic if $LX \simeq 0$, and $Y \in \mathrm{Comod}_T^{cg}(\mathcal{D})$ L-local if $\mathrm{Map}_{\mathrm{Comod}_T^{cg}(\mathcal{D})}(X,Y) \simeq 0$ whenever $LX \simeq 0$.

Objects in the image of I are clearly L-local because of the adjunction.

Since applying L to the unit $\mathrm{id} \to IL$ yields an equivalence $L \Rightarrow LIL \simeq L$, using that $LI \simeq \mathrm{id}$, we see that $\mathrm{id} \to IL$ has L-acyclic fibre. It follows that $\mathrm{id} \to IL$ is the localization with respect to L. □

Lemma 2.43. *Let $F \dashv G : \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction between stable $(\infty, 1)$-categories. Assume that G preserves filtered colimits, and that $\mathcal{C}$ is compactly generated. Then the functor $\mathcal{C} \to \mathrm{Comod}_{FG}(\mathcal{D})$ factors through a functor $\mathcal{C} \to \mathrm{Comod}_{FG}^{cg}(\mathcal{D})$.*

Proof. Since G preserves filtered colimits, the left adjoint F preserves compact objects. The functor $\mathcal{C} \to \mathrm{Comod}_{FG}(\mathcal{D})$ therefore restricts to a functor

$$\mathcal{C}^c \to \mathrm{Comod}_{FG}^c(\mathcal{D}),$$

where $\mathcal{C}^c$ denotes the compact objects in $\mathcal{C}$. Note again, however, that, $\mathrm{Comod}_{FG}^c(\mathcal{D})$ does not denote compact objects in $\mathrm{Comod}_{FG}(\mathcal{D})$, but comodules with underlying compact objects in $\mathcal{D}$.

Since $\mathcal{C}$ is compactly generated, $\mathrm{Ind}(\mathcal{C}^c) = \mathcal{C}$. So by applying the Ind-construction, we get a functor

$$\mathcal{C} \to \mathrm{Comod}^{cg}_{FG}(\mathcal{D}).$$

(cf. Proposition 5.3.5.11 in [Lur09]) □

We now want to describe conditions under which the functor $\mathcal{C} \to \mathrm{Comod}^{c}_{FG}(\mathcal{D})$ is an equivalence.

Theorem 2.44. *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be an adjunction of exact functors between stable $(\infty, 1)$-categories. Assume that G preserves filtered colimits, and that $\mathcal{C}$ has compact generators K_i that are complete with respect to $F \dashv G$ in the sense of Definition 2.24.*

Then the functor $\mathcal{C} \to \mathrm{Comod}^{cg}_{FG}(\mathcal{D})$ from Lemma 2.43 is fully faithful. Its essential image is precisely generated by the $\widetilde{F}K_i$ under colimits. In particular, it is an equivalence if the $\widetilde{F}K_i$ constitute a set of compact generators for $\mathrm{Comod}^{cg}_{FG}(\mathcal{D})$.

Proof. We first need to show that all compact objects $K \in \mathcal{C}$ are complete with respect to the adjunction $F \dashv G$, i.e. satisfy $K \xrightarrow{\simeq} K^{\wedge}_{GF}$.

Since the K_i are generators, any compact $K \in \mathcal{C}$ can be written as a colimit over copies of the K_i. By filtering the corresponding diagram by its finite subdiagrams, we can write K as a filtered colimit over X_j, where each of the X_j is a finite colimit of copies of the K_i.

As K is compact, the identity $K \to K$ factors through a finite stage of this colimit, so K is a retract of a finite colimit of copies of the K_i.

But the full subcategory on $F \dashv G$-complete objects is closed under finite colimits and retracts. For finite colimits, this is seen from the fact that it is closed under finite limits, by Lemma 2.26.

To see this for retracts, assume that $K \to X \to K$ is homotopic to the identity. We get an induced diagram of retracts

$$\begin{array}{ccc}
\mathrm{map}_{\mathcal{C}}(Z, K) & \longrightarrow & \mathrm{map}_{\mathrm{Comod}^{cg}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}K) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{C}}(Z, X) & \xrightarrow{\simeq} & \mathrm{map}_{\mathrm{Comod}^{cg}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}X) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{C}}(Z, K) & \longrightarrow & \mathrm{map}_{\mathrm{Comod}^{cg}_{FG}(\mathcal{D})}(\widetilde{F}Z, \widetilde{F}K)
\end{array}$$

for any Z, proving that the horizontal maps are all equivalences, since equivalences are closed under retracts in Sp.

So all compacts are $(F \dashv G)$-complete. With $\mathcal{C}^c$ denoting the full subcategory on compact objects in $\mathcal{C}$, the functor

$$\mathcal{C}^c \to \mathrm{Comod}^c_{FG}(\mathcal{D})$$

is therefore fully faithful. It follows that the functor

$$\mathcal{C} \to \mathrm{Comod}^{cg}_{FG}(\mathcal{D})$$

induced on Ind-categories, is fully faithful.

As it preserves colimits, and the K_i generate $\mathcal{C}$, the essential image is generated by the $\widetilde{F}K_i$. □

Example 2.45. In the case of the adjunction $\mathrm{Sp} \rightleftarrows \mathrm{Mod}_E$, we can't typically hope for $X \to X^\wedge_E$ to be an equivalence for general X, without imposing connectiveness conditions on E and on X (together with completeness conditions on X if $\pi_0(E)$ is of positive characteristic).

In good cases, for example $E = MU$, we do know that $X \to X^\wedge_{MU}$ is an equivalence for all connective X. In particular, all compact X are complete with respect to the adjunction $\mathrm{Sp} \rightleftarrows \mathrm{Mod}_{MU}$. By Lemma 2.44, we get an equivalence

$$\mathrm{Sp} \simeq \mathrm{Comod}^{cg}_{MU\otimes MU}(\mathrm{Mod}_{MU}).$$

Example 2.46. In the case $E = H\mathbb{F}_p$, the Adams spectral sequence doesn't converge for all connective spectra. However, if we work with p-complete spectra $\mathrm{Sp}^\wedge_p$ instead, i.e. consider the adjunction

$$\mathrm{Sp}^\wedge_p \rightleftarrows \mathrm{Mod}_{H\mathbb{F}_p},$$

the associated Adams spectral sequence converges conditionally for all compact objects. Theorem 2.44 gives an equivalence

$$\mathrm{Sp}^\wedge_p \to \mathrm{Comod}^{cg}_{H\mathbb{F}_p\otimes H\mathbb{F}_p}(\mathrm{Mod}_{H\mathbb{F}_p}).$$

The $(\infty,1)$-category $\mathrm{Mod}_{H\mathbb{F}_p}$ admits an algebraic model, namely as the ($(\infty,1)$-categorical refinement of the) derived category $\mathcal{D}\,\mathrm{Mod}_{\mathbb{F}_p}$. This means that we have identified $\mathrm{Sp}^\wedge_p$ with a category of comodules over a comonad on the algebraically defined category of derived $\mathbb{F}_p$-modules!

However, the comonad doesn't admit a purely algebraic description. One can represent its underlying functor $\mathcal{D}\,\mathrm{Mod}_{\mathbb{F}_p} \to \mathcal{D}\,\mathrm{Mod}_{\mathbb{F}_p}$ as tensoring with an explicit $\mathbb{F}_p$-bimodule, namely $\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p) = \mathcal{A}_*$, the dual Steenrod algebra. However, the comonad structure on that functor does not coincide with the comonad structure we can induce from the strictly coassociative coalgebra structure on $\mathcal{A}_*$ (cf. Section 2.4), as that would lead to a degenerating Adams spectral sequence.

Instead, there are coherences of arbitrary height, corresponding precisely to differentials in the Adams spectral sequence, with the strictly coassociative comultiplication on $\mathcal{A}_*$ just the truncation on the level of homotopy categories of the comonad represented by $\Gamma := H\mathbb{F}_p \otimes H\mathbb{F}_p$.

The first interesting coherence structure is the "coassociator"

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma \otimes_{H\mathbb{F}_p} \Gamma \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow \\ \Gamma \otimes_{H\mathbb{F}_p} \Gamma & \longrightarrow & \Gamma \otimes_{H\mathbb{F}_p} \Gamma \otimes_{H\mathbb{F}_p} \Gamma \end{array}$$

which can be seen to be nontrivial: it appears in the dual context in the form of non-trivial composition Toda products in $\mathcal{A} = \pi_* \operatorname{Map}(H\mathbb{F}_p, H\mathbb{F}_p)$, first discovered by Adams in his celebrated solution of the Hopf invariant 1 problem [Ada60]. Work by Baues [Bau06] improves on these results by fully describing the coherences up to this "secondary" level of truncation, and by describing the E_3-page of the Adams spectral sequence in terms of these coherences.

Lemma 2.47. *Let T and T' be exact comonads over stable $(\infty, 1)$-categories $\mathcal{C}$ and $\mathcal{D}$, and let $H : T \to T'$ be a morphism of comonads. Then if the underlying functor $H : \mathcal{C} \to \mathcal{D}$ preserves compact objects, the adjunction*

$$H_* \dashv H^* : \operatorname{Comod}_T(\mathcal{C}) \rightleftarrows \operatorname{Comod}_{T'}(\mathcal{D})$$

induces an adjunction between $\operatorname{Comod}_T^{cg}(\mathcal{C})$ and $\operatorname{Comod}_{T'}^{cg}(\mathcal{D})$, which we will also denote by $H_* \dashv H^*$.

The right adjoint H^* preserves all colimits (and in fact admits a further right adjoint, which we will not discuss here).

Proof. Since H preserves compact objects, H_* induces a functor $\operatorname{Comod}_T^c(\mathcal{C}) \to \operatorname{Comod}_{T'}^c(\mathcal{D})$, and therefore a functor on Ind-categories. The right adjoint arises from the characterization of Ind as finite colimit-preserving presheaves: Since H_* preserves any colimit, precomposition with H_* sends finite colimit-preserving presheaves to finite colimit-preserving presheaves (cf. [Lur09], Proposition 5.3.5.13).

Since colimits are formed levelwise in presheaves (and the colimits taken in the subcategory of finite colimit-preserving presheaves agree with the ones formed in the full category of all presheaves), precomposition with H_* preserves colimits. It follows that H^* preserves colimits. □

Lemma 2.48. *Assume $\mathcal{D}$ is $\mathcal{O}$-monoidal, with compact objects closed under $\otimes$, and $\otimes$ commuting with filtered colimits.*

Then, given a lax $\mathcal{O}$-monoidal comonad T on $\mathcal{D}$, $\operatorname{Comod}_T^{cg}(\mathcal{D})$ inherits a $\mathcal{O}$-monoidal structure such that the functor $L : \operatorname{Comod}_T^{cg}(\mathcal{D}) \to \operatorname{Comod}_T(\mathcal{D})$ is $\mathcal{O}$-monoidal.

Proof. Since T is a lax $\mathcal{O}$-monoidal comonad, we get a $\mathcal{O}$-monoidal structure on $\mathrm{Comod}_T(\mathcal{D})$. By assumption, $\mathrm{Comod}_T^c(\mathcal{D})$ is closed under $\otimes$, so we get an induced monoidal structure on $\mathrm{Comod}_T^{cg}(\mathcal{D})$. Finally, since L preserves colimits, every object in $\mathrm{Comod}_T^{cg}(\mathcal{D})$ is a filtered colimit of objects in Comod_T^c, and L is $\mathcal{O}$-monoidal on the subcategory Comod_T^c, it is $\mathcal{O}$-monoidal. □

2.4 Coalgebroids

In this section, we use the abstract theory of stable $(\infty, 1)$-categories of comodules over an exact comonad to construct derived comodule categories over Hopf algebroids and coalgebras.

To that end, we first define a mutual generalization of Hopf algebroids and coalgebras. Recall that, for a ring A, the category of bimodules Bimod_A has a monoidal structure $M \otimes_A N$, where the tensor product balances the right A-module structure on M with the left A-module structure on N.

Definition 2.49. *For a commutative ring A, a coalgebroid over A is a coassociative and counital comonoid in* Bimod_A*. Concretely, this is given by an A-A-bimodule* Γ*, together with maps*

$$\Gamma \to \Gamma \otimes_A \Gamma$$
$$\Gamma \to A$$

of A-A-bimodules, satisfying the usual coassociativity and counitality conditions.

Note that, if the left and right action of A on Γ agree, this is precisely a coalgebra over A. Also, any Hopf algebroid has an underlying coalgebroid.

Definition 2.50. *A coalgebroid Γ over A is called flat if it is flat as a right A-module.*

Lemma 2.51. *For a coalgebroid Γ, the functor $M \mapsto \Gamma \otimes_A M$ is naturally a comonad on (the 1-category)* Mod_A*. This construction gives a bijection between coalgebroids over A and colimit-preserving comonads on* Mod_A*. Under this bijection, flat coalgebroids correspond to exact comonads.*

Proof. More generally, for any A-A-bimodule N, $N \otimes_A (-)$ defines a colimit-preserving endofunctor of Mod_A. This defines a monoidal functor $\mathrm{Bimod}_A \to \mathrm{Fun}^{\mathrm{cocont}}(\mathrm{Mod}_A, \mathrm{Mod}_A)$. In the other direction, we can send a colimit-preserving endofunctor F to the A-module $F(A)$. Since

$$A \simeq \mathrm{Map}_{\mathrm{Mod}_A}(A, A) \xrightarrow{F} \mathrm{Mod}_A(F(A), F(A))$$

defines a right A-action on $F(A)$, $F(A)$ is naturally an A-A-bimodule, so this construction defines a functor $\mathrm{Fun}^{\mathrm{cocont}}(\mathrm{Mod}_A, \mathrm{Mod}_A) \to \mathrm{Bimod}_A$.

These two constructions are inverse to each other. In one direction, this is trivial. In the other, we have to construct a natural equivalence $F(A) \otimes_A (-) \Rightarrow F(-)$.

On A, we can take this to be the isomorphism

$$F(A) \otimes_A A \to F(A),$$

which is natural in endomorphisms of A (here it is crucial that the tensor product $F(A) \otimes_A (-)$ uses the right module structure on $F(A)$). But since every module M is naturally a colimit over copies of A (e.g. by the canonical Bar resolution associated to the adjunction $\mathrm{Mod}_A \to \mathrm{Set}$), and F as well as the tensor product commute with colimits, the equivalence extends as desired.

Since coalgebroids and comonads are just comonoid objects in the respective categories, we get a bijection between coalgebroids over A and comonads on Mod_A. By definition of flatness, for a bimodule N, the functor $N \otimes_A (-)$ is exact precisely when N is flat as a right A-module.

□

Over a coalgebroid Γ, we can define comodules as A-modules with coaction map $M \to \Gamma \otimes_A M$ satisfying the obvious coassociativity and counitality conditions.

Note that the analogue of Definition 2.2 on a 1-category immediately corresponds to this notion under Lemma 2.51.

For Γ a coalgebroid over A, and Σ a coalgebroid over B, note that $\Gamma \otimes \Sigma$ is naturally a coalgebroid over $A \otimes B$.

Definition 2.52. *A coalgebroid is called multiplicative if there is a commutative ring structure on Γ such that the multiplication map $\Gamma \otimes \Gamma \to \Gamma$ is a map of $(A \otimes A)$-$(A \otimes A)$-bimodules, and compatible with the coalgebroid structure maps on both $\Gamma \otimes \Gamma$ and Γ.*

Note that, when written out explicitly, the structure of a multiplicative coalgebroid is precisely the structure of a Hopf algebroid without antipode.

On the 1-category Mod_A, the notion of a lax monoidal comonad (cf. Definition 2.17) specialises to the following:

Definition 2.53. *A lax symmetric-monoidal comonad T on Mod_A consists of a comonad T together with a natural transformation*

$$T(N) \otimes_A T(M) \to T(N \otimes_A M),$$

compatible with the braiding, such that the diagrams

$$\begin{array}{ccccc} T(N) \otimes_A T(M) & & \longrightarrow & & T(N \otimes_A M) \\ \downarrow & & & & \downarrow \\ TT(N) \otimes_A TT(M) & \longrightarrow & T(T(N) \otimes_A T(M)) & \longrightarrow & TT(N \otimes_A M) \end{array}$$

and

$$\begin{array}{ccc} T(N) \otimes_A T(M) & \longrightarrow & T(N \otimes_A M) \\ \downarrow & & \downarrow \\ N \otimes_A M & \xrightarrow{\text{id}} & N \otimes_A M \end{array}$$

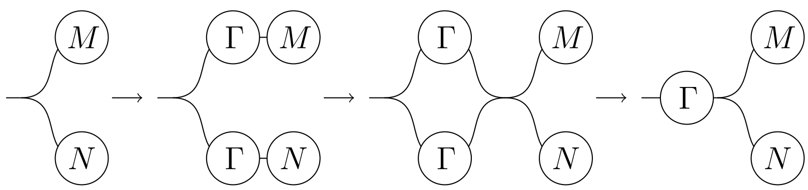
commute.

Lemma 2.54. *Under the correspondence between A-A-bimodules and colimit-preserving endofunctors of* Mod_A *discussed in Lemma* 2.51*, multiplicative coalgebroids correspond precisely to lax symmetric monoidal comonads.*

Proof. For Γ a multiplicative coalgebroid, the lax monoidal structure on $\Gamma \otimes_A (-)$ comes from the map

$$(\Gamma \otimes_A M) \otimes_A (\Gamma \otimes_A N) \to \Gamma \otimes_A (M \otimes_A N)$$

obtained by sending $(x_1 \otimes m) \otimes (x_2 \otimes n) \mapsto (x_1 x_2) \otimes (m \otimes n)$. This is easily checked to be well-defined. Note that one has to carefully keep track of which module structures are balanced by which tensor product in the above formula: some of the tensor products balance a left module structure with a right module structure, i.e. refer to the monoidal structure in A-A-bimodules, while others balance two left module structures, i.e. refer to the monoidal structure in left A-modules.



The composite coaction on $M \otimes_A N$. Lines correspond to A-module structures, which are balanced whenever they connect.

This map is compatible with the braiding and comultiplication, since the multiplication on Γ is commutative and compatible with the coalgebroid structure.

In the other direction, if T is a lax symmetric monoidal comonad, the representing bimodule $T(A)$ inherits a multiplicative structure from

$$T(A) \otimes_A T(A) \to T(A \otimes_A A) \simeq T(A),$$

which is compatible with the coalgebroid structure on $T(A)$ as one sees from the corresponding diagrams for the lax symmetric monoidal comonad T. Since the map is compatible with the braiding, the multiplication on $T(A)$ is commutative.

One easily checks the two constructions relating multiplicative structures on coalgebroids and lax symmetric monoidal structures on comonads are inverse to each other. □

Next, we consider morphisms of coalgebroids.

Definition 2.55. *For Γ a coalgebroid over A, and Σ a coalgebroid over B, a morphism $(A,\Gamma) \to (B,\Sigma)$ consists of a ring morphism $A \to B$, together with a map $\Gamma \to \Sigma$ of A-A-bimodules (where A acts on Σ on either side by restricting), and such that the following diagrams commute*

$$\begin{array}{ccccc}
\Gamma & \longrightarrow & & \longrightarrow & \Sigma \\
\downarrow & & & & \downarrow \\
\Gamma \otimes_A \Gamma & \longrightarrow & \Sigma \otimes_A \Sigma & \longrightarrow & \Sigma \otimes_B \Sigma
\end{array}$$

$$\begin{array}{ccc}
\Gamma & \longrightarrow & \Sigma \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}$$

On 1-categories, the notion of a morphism between comonads T and T' on Mod_A and Mod_B with underlying functor given by the induction functor $\mathrm{Mod}_A \to \mathrm{Mod}_B$, $M \mapsto B \otimes_A M$ (cf. Definition 2.9) specializes to the following:

Definition 2.56. *For a ring map $A \to B$ and comonads T on Mod_A, T' on Mod_B, a morphism $T \to T'$ compatible with the induction functor $\mathrm{Mod}_A \to \mathrm{Mod}_B$ is given by a natural transformation*

$$B \otimes_A T(M) \to T'(B \otimes_A M)$$

such that the following diagrams commute:

$$\begin{array}{ccccc}
B \otimes_A T(M) & \longrightarrow & & \longrightarrow & T'(B \otimes_A M) \\
\downarrow & & & & \downarrow \\
B \otimes_A TT(M) & \longrightarrow & T'(B \otimes_A T(M)) & \longrightarrow & T'T'(B \otimes_A M)
\end{array}$$

$$\begin{array}{ccc}
B \otimes_A T(M) & \longrightarrow & T'(B \otimes_A M) \\
\downarrow & & \downarrow \\
B \otimes_A M & \xrightarrow{\mathrm{id}} & B \otimes_A M
\end{array}$$

Lemma 2.57. *Under the correspondence of Lemma 2.51, morphisms between coalgebroids Γ and Σ over A and B with given underlying ring map $A \to B$ in the sense of Definition 2.55 correspond precisely to morphisms of colimit-preserving comonads in the sense of Definition 2.56.*

Proof. An A-A-bimodule map $\Gamma \to \Sigma$ gives rise to a B-A-bimodule map $B \otimes_A \Gamma \to \Sigma$ by the adjunction between induction and restriction.

This map gives rise to a natural transformation

$$B \otimes_A \Gamma \otimes_A M \to \Sigma \otimes_A M \simeq \Sigma \otimes_B (B \otimes_A M).$$

Vice versa, a natural transformation $B \otimes_A T(M) \to T'(B \otimes_A M)$ can be evaluated at A to give a B-module map $B \otimes_A T(A) \to T'(B \otimes_A A) \simeq T'(B)$, and since the right A-module action on both sides can be described as the action of $\mathrm{Map}_{\mathrm{Mod}_A}(A, A)$ through functoriality, this is a map of B-A-bimodules. The adjunction now gives an A-A-bimodule map from $T(A) \to T'(B)$. The two constructions are inverse to each other.

Furthermore, one can directly check that the diagrams in the respective definitions 2.55 and 2.56 correspond to each other under this construction. □

Recall that any ring A admits a derived stable $(\infty, 1)$-category $\mathcal{D}\,\mathrm{Mod}_A$, which is obtained from the category of chain complexes in Mod_A by localizing with respect to quasi-isomorphisms.

Lemma 2.58. *An exact comonad T on Mod_A induces a comonad $\mathcal{D}T$ on $\mathcal{D}\,\mathrm{Mod}_A$, computed on chain complex representatives by applying T levelwise. If T is a lax symmetric-monoidal comonad in the sense of Definition 2.53, $\mathcal{D}T$ is a lax symmetric-monoidal comonad in the sense of Definition 2.17.*

For a ring homomorphism $A \to B$ and a compatible morphism $T \to T'$ of exact comonads over Mod_A and Mod_B in the sense of Definition 2.56, assume furthermore that T preserves flat modules. Then we get a morphism $\mathcal{D}T \to \mathcal{D}T'$ in the sense of Definition 2.9, with underlying functor the derived induction functor $B \otimes_A^L (-)$.

Proof. Since all the functors $(\mathcal{D}T)^n$ are computed on chain complex representatives by applying T^n levelwise, the strictly coassociative comonad structure on T gives rise to a coherent comonad structure in the sense of Definition 2.1 where the higher coherences are constant. Similarly, the structure of a lax symmetric-monoidal comonad on T gives rise to the structure of a coherently lax-symmetric monoidal comonad on $\mathcal{D}T$.

For the morphisms, observe that $\mathcal{D}\,\mathrm{Mod}_A$ can equivalently be described as obtained from the category of chain complexes of flat A-modules, localized at quasi-isomorphisms. This is because we can compatibly resolve arbitrary chain complexes by levelwise projective ones. But on levelwise flat chain complexes, the functors $(T')^n \circ (B \otimes_A (-)) \circ T^m$ are all exact (using that T preserves flat modules), so on those, $(\mathcal{D}T')^n \circ (B \otimes_A^L (-)) \circ (\mathcal{D}T)^m$ can be computed levelwise. It follows that a morphism $T \to T'$ in the sense of Definition 2.56 gives rise to a morphism $\mathcal{D}T \to \mathcal{D}T'$ with underlying functor $B \otimes_A^L (-)$ and constant higher coherences. □

Definition 2.59. *For a commutative ring A and a flat coalgebroid Γ over A, we define the derived comodule category*

$$\mathrm{Comod}_{\mathcal{D}\Gamma} := \mathrm{Comod}_{\mathcal{D}(\Gamma\otimes_A(-))}(\mathcal{D}\,\mathrm{Mod}_A)$$

Lemma 2.60. *If Γ is a multiplicative flat coalgebroid,* $\mathrm{Comod}_{\mathcal{D}\Gamma}$ *inherits a symmetric-monoidal structure.*

If Γ is a flat coalgebroid over A that's also flat as a left module, and Σ is a flat coalgebroid over B, a morphism $\Gamma \to \Sigma$ gives rise to a pair of adjoint functors

$$\mathrm{Comod}_{\mathcal{D}\Gamma} \to \mathrm{Comod}_{\mathcal{D}\Sigma}$$

where the left adjoint acts on underlying objects by $B \otimes^L_A (-)$.

Proof. All of this follows from Lemma 2.58, we only need to check that $\Gamma \otimes_A (-)$ preserves flat objects if Γ is flat as a left and a right module.

To see this, let N be a right A-module and M a left A-module. Let $C_\bullet$ be a resolution of N, i.e. a complex of nonnegatively graded chain complex of (not necessarily projective) right A-modules with $H_*(C_\bullet) = N$, concentrated in degree 0. Analogously, let $D_\bullet$ be a resolution of M.

Then if either one of $C_\bullet$ or $D_\bullet$ consist levelwise of flat modules,

$$H_*(C_\bullet \otimes_A D_\bullet) \simeq \mathrm{Tor}^A_*(N, M).$$

This is immediately seen, for example, from the bicomplex spectral sequence associated to the bicomplex $C_\bullet \otimes_A D_\bullet$.

Now let N, M be right and left A-modules again, and let both $C_\bullet$ and $D_\bullet$ be levelwise flat resolutions. Since Γ is flat from both sides, tensoring with Γ from either side is an exact functor.

It follows that $C_\bullet \otimes_A \Gamma$ is a resolution of $N \otimes_A \Gamma$ (but a priori not flat anymore). Since $D_\bullet$ is flat, we see that

$$H_*(C_\bullet \otimes_A \Gamma \otimes_A D_\bullet) \simeq \mathrm{Tor}^A_*(N \otimes_A \Gamma, M).$$

Analogously, we see that

$$H_*(C_\bullet \otimes_A \Gamma \otimes_A D_\bullet) \simeq \mathrm{Tor}^A_*(N, \Gamma \otimes_A M).$$

Together, we have obtained that $\mathrm{Tor}^A_*(N \otimes_A \Gamma, M) \simeq \mathrm{Tor}^A_*(N, \Gamma \otimes_A M)$. If M is flat, then the left hand side of this equivalence vanishes in positive degrees, for all N. From the right hand side, we then see that $\Gamma \otimes_A M$ is flat.

It follows that $\Gamma \otimes_A (-)$ preserves flat modules if Γ is flat from both sides. □

Remark 2.61. As an example for the adjoint functors obtained from morphisms of coalgebroids, note that the counit $\varepsilon : \Gamma \to A$ is always a morphism of coalgebroids over A, where A is endowed with the structure of a coalgebroid through the identity map $A \simeq A \otimes_A A$.

The associated comonad of A is the identity comonad. We obtain a pair of adjoint functors

$$\varepsilon_* \dashv \varepsilon^* : \mathrm{Comod}_{\mathcal{D}\Gamma} \rightleftarrows \mathrm{Comod}_{\mathcal{D}A} \simeq \mathcal{D}\,\mathrm{Mod}_A$$

which coincides with the adjunction between the forgetful and the cofree functor.

Analogously, the adjunction between $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ (cf. Definition 2.67) and $\mathcal{D}\,\mathrm{Mod}_A$ can be described as the adjunction between $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ and $\mathrm{Comod}^{cg}_{\mathcal{D}A} \simeq \mathcal{D}\,\mathrm{Mod}_A$, where the latter equivalence comes from the fact that $\mathcal{D}\,\mathrm{Mod}_A$ is already compactly generated.

We start analyzing the category $\mathrm{Comod}_{\mathcal{D}\Gamma}$ by computing its mapping spaces.

Lemma 2.62. *For Γ a flat coalgebroid and $X, Y \in \mathrm{Comod}_{\mathcal{D}\Gamma}$ represented by chain complexes C_*, D_* of Γ-comodules (cf. Remark 2.66). If C_* has levelwise underlying projective A-modules, then $\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(X, Y)$ is represented by the total complex of the cobar resolution*

$$\underline{\mathrm{Hom}}_{\mathrm{Ch}(\mathrm{Mod}_A)}(C_*, D_*) \rightrightarrows \underline{\mathrm{Hom}}_{\mathrm{Ch}(\mathrm{Mod}_A)}(C_*, \Gamma \otimes_A D_*) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

under the Dold-Kan correspondence.

Proof. By Lemma 2.8, we can compute $\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(X, Y)$ as

$$\lim(\mathrm{map}_{\mathcal{D}\,\mathrm{Mod}_A}(X, Y) \rightrightarrows \mathrm{map}_{\mathcal{D}\,\mathrm{Mod}_A}(X, \Gamma \otimes_A Y) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots)$$

Now the $(\infty, 1)$-category structure on $\mathcal{D}\,\mathrm{Mod}_A$ is such that $\mathrm{map}_{\mathcal{D}\,\mathrm{Mod}_A}(C_*, D_*)$, with C_* levelwise projective, is equivalent to the spectrum obtained from the chain complex $\underline{\mathrm{Hom}}_{\mathrm{Ch}(\mathrm{Mod}_A)}(C_*, D_*)$ through the Dold-Kan correspondence. Since the total complex of a cosimplicial chain complex turns into the homotopy limit of the associated spectra under the Dold-Kan correspondence, the statement follows. □

Proposition 2.63. *With notation as in Lemma 2.62, with C_* having levelwise underlying projective A-modules, we have*

$$\pi_s(\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(X, Y)) = \mathrm{Ext}^{-s}_{\Gamma}(C_*, D_*).$$

Proof. For M a Γ-comodule with underlying projective A-module, $\mathrm{Ext}^*_\Gamma(M, \Gamma \otimes_A N)$ vanishes in positive degrees for any A-module N, cf. Lemma A1.2.8 of [Rav86].

So if $C_\Gamma(D_*)$ is the cobar complex of D_*, and I_* any injective resolution of D_* with an injective map $C_\Gamma(D_*) \to I_*$ under D_*, we see that applying $\mathrm{map}(C_*, -)$ gives an equivalence $\underline{\mathrm{Hom}}_\Gamma(C_*, C_\Gamma(D_*)) \simeq \underline{\mathrm{Hom}}_\Gamma(C_*, I_*)$, where the homology groups of the latter compute Ext. □

Assume $f : \Gamma \to \Sigma$ is a morphism of coalgebroids over A and B. One obtains a right Σ-coaction on $\Gamma \otimes_A B$ through

$$\Gamma \otimes_A B \to \Gamma \otimes_A \Gamma \otimes_A B \to \Gamma \otimes_A \Sigma \simeq (\Gamma \otimes_A B) \otimes_B \Sigma.$$

We can use this structure to describe the right adjoint $f^* : \mathrm{Comod}_{\mathcal{D}\Sigma} \to \mathrm{Comod}_{\mathcal{D}\Gamma}$ in a way similar to Lemma 2.62.

Lemma 2.64. *If $f : \Gamma \to \Sigma$ is a morphism of coalgebroids over A and B, and $X \in \mathrm{Comod}_{\mathcal{D}\Sigma}$ is represented by the chain complex D_* of comodules, then $f^*X \in \mathrm{Comod}_{\mathcal{D}\Gamma}$ is represented by the total complex of the cosimplicial chain complex*

$$(\Gamma \otimes_A B) \otimes_B D_* \rightrightarrows (\Gamma \otimes_A B) \otimes_B \Sigma \otimes_B D_* \overset{\textstyle\rightarrow}{\rightrightarrows} \cdots$$

where the coboundary maps are obtained from the coaction maps, and the codegeneracies from the augmentation $\Sigma \to B$.

Proof. From the composite morphism $\Gamma \xrightarrow{\nu} B$ we obtain an adjunction between $\mathrm{Comod}_{\mathcal{D}\Gamma}$ and $\mathcal{D}\,\mathrm{Mod}_B$, where the right adjoint ν^* sends a derived B-module represented by a chain complex C_* to an object represented by $\Gamma \otimes_A C_* \simeq (\Gamma \otimes_A B) \otimes_B C_*$. Since ν factors as $\Gamma \xrightarrow{f} \Sigma \xrightarrow{\varepsilon} B$, the right adjoint f^* sends an object represented by a chain complex of the form $\Sigma \otimes_B C_* = \varepsilon^* C_*$ to $(\Gamma \otimes_A B) \otimes_B C_*$.

For an arbitrary chain complex D_* of Σ-comodules, the result now follows by considering the Σ-cobar resolution of D_* and using that f^* commutes with limits. □

For a coalgebroid Σ over B, a right Σ-comodule N and a left Σ-comodule M, there is the so-called cotensor product defined as the equalizer

$$N \Box_\Sigma M \to N \otimes_B M \rightrightarrows N \otimes_B \Sigma \otimes_B M.$$

The description of the right adjoint in Lemma 2.64 can be interpreted as the derived functor of $(\Gamma \otimes_A B)\Box_\Sigma(-)$. So we obtain:

Lemma 2.65. *Assume that, in the situation of Lemma 2.64, $\Gamma \otimes_A B$ is right Σ-cofree, or, more generally, that $(\Gamma \otimes_A B)\Box_\Sigma(-)$ is exact.*

Then if $X \in \mathrm{Comod}_{\mathcal{D}\Sigma}$ is represented by a chain complex D_* of comodules, f^*X is represented by the chain complex $(\Gamma \otimes_A B)\Box_\Sigma D_*$.

Proof. For N a right comodule, the cobar resolution gives an exact sequence

$$0 \to N \to N \otimes_B \Sigma \to N \otimes_B \Sigma \otimes_B \Sigma \to \cdots$$

In particular, it exhibits N as the equalizer defining $N\square_\Sigma\Sigma$. More generally, if N is flat as a right B-module, all terms in the cobar resolution are flat, and so

$$0 \to N \otimes_B M \to N \otimes_B \Sigma \otimes_B M \to N \otimes_B \Sigma \otimes_B \Sigma \otimes_B M \to \cdots$$

is also exact. This exhibits $N \otimes_B M \simeq N\square_\Sigma(\Sigma \otimes_B M)$.

In our case we get a natural isomorphism $(\Gamma\otimes_A B)\square_\Sigma(\Sigma\otimes_B M) \simeq (\Gamma\otimes_A B)\otimes_B M$. Applying $(\Gamma\otimes_A B)\square_\Sigma(-)$ to the cobar complex of D_*, we thus obtain the augmented cosimplicial chain complex

$$(\Gamma \otimes_A B)\square_\Sigma D_* \to (\Gamma \otimes_A B) \otimes_B D_* \rightrightarrows (\Gamma \otimes_A B) \otimes_B \Sigma \otimes_B D_* \overset{\rightarrow}{\rightrightarrows} \cdots$$

and thus an equivalence between $(\Gamma \otimes_A B)\square_\Sigma D_*$ and f^*X by Lemma 2.64. □

Remark 2.66. For T an exact comonad on Mod_A, the associated 1-category of comodules $\mathrm{Comod}_T(\mathrm{Mod}_A)$ is abelian.

We get a functor

$$\mathrm{Ch}\,\mathrm{Comod}_T(\mathrm{Mod}_A) = \mathrm{Comod}_T(\mathrm{Ch}\,\mathrm{Mod}_A) \to \mathrm{Comod}_{\mathcal{D}T}(\mathcal{D}\,\mathrm{Mod}_A),$$

since T is exact, and thus a chain complex of comodules over T gives rise to a $\mathcal{D}$-comodule structure on the object represented by this chain complex in the derived category.

We can form a derived category $\mathcal{D}\,\mathrm{Comod}_T(\mathrm{Mod}_A)$ by localizing the category $\mathrm{Ch}\,\mathrm{Comod}_T(\mathrm{Mod}_A)$ of chain complexes of strict comodules with respect to quasi-isomorphisms of chain complexes of comodules. Since these are quasi-isomorphisms of the underlying chain complexes, we get an induced functor

$$\mathcal{D}\,\mathrm{Comod}_T(\mathrm{Mod}_A) \to \mathrm{Comod}_{\mathcal{D}T}(\mathcal{D}\,\mathrm{Mod}_A).$$

The left category is what is usually called the derived comodule category in the literature. In general, it doesn't seem clear that the functor is an equivalence: Note that a comodule in $\mathcal{D}\,\mathrm{Mod}_A$ is represented by a chain complex C_* with coherently coassociative comultiplication morphisms to TC_*, T^2C_* and so on. Even if C_* is cofibrant, such that all those maps can be represented by actual chain complex maps, there is still coherence data, and it is not clear that this can be strictified.

In the other direction, note that a priori, we're inverting less morphisms on the left. Concretely, if given two chain complexes C_* and D_* of T-comodules, such that there is a quasi-isomorphism between the underlying objects, it is unclear whether there is necessarily a chain of quasi-isomorphisms of T-comodule maps between the two.

For A a ring, compact objects in $\mathcal{D}\,\mathrm{Mod}_A$ are precisely perfect complexes, i.e. these chain complexes which are quasi-isomorphic to a bounded chain complex of finitely generated projectives.

Definition 2.67. *For Γ an exact coalgebroid over A, we define*

$$\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma} := \mathrm{Comod}^{cg}_{\mathcal{D}(\Gamma\otimes_A(-))}(\mathcal{D}\,\mathrm{Mod}_A)$$

Remark 2.68. In [Hov04], Hovey constructs a model category Stable(Γ) of comodules for nice enough Hopf algebroids Γ ("Adams Hopf algebroids"). Its two main features are that dualizable objects are compact generators, and that homotopy classes of maps between dualizable objects can be described as Ext-groups.

There is a monoidal "underlying A-module" functor $\mathrm{Stable}(\Gamma) \to \mathcal{D}\,\mathrm{Mod}_A$ with right adjoint, and the induced comonad on $\mathcal{D}\,\mathrm{Mod}_A$ is precisely given by $\mathcal{D}\Gamma$. As dualizable objects go to compact objects in $\mathcal{D}\,\mathrm{Mod}_A$, we get a functor of stable $(\infty, 1)$-categories $\mathrm{Stable}(\Gamma) \to \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$.

The fact that Map on both sides can be interpreted in terms of Ext-groups suggests strongly that this functor is an equivalence. Compare [BH17], where a construction is given for $\mathrm{Stable}(BP_*BP)$ that is very similar to our construction of $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$.

Lemma 2.69. *For Γ a multiplicative coalgebroid,* $\mathrm{Comod}^{cg}(\mathcal{D}\Gamma)$ *inherits a symmetric-monoidal structure, such that the functor* $\mathrm{Comod}^{cg}(\mathcal{D}\Gamma) \to \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}(\mathcal{D}\,\mathrm{Mod}_A)$ *is symmetric-monoidal.*

Proof. This follows from the fact that in $\mathcal{D}\,\mathrm{Mod}_A$, tensor products of compacts are compact again. □

Lemma 2.70. *For $f : \Gamma \to \Sigma$ a morphism of coalgebroids, we obtain a pair of adjoint functors*

$$(f_* \dashv f^*) : \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma} \to \mathrm{Comod}^{cg}_{\mathcal{D}\Sigma},$$

with f^* preserving all colimits.

If Γ and Σ are multiplicative coalgebroids and f is a ring homomorphism, f_* is symmetric-monoidal.

Proof. If $A \to B$ is a map of commutative rings, the derived induction functor $B\otimes^L_A(-) : \mathcal{D}\,\mathrm{Mod}_A \to \mathcal{D}\,\mathrm{Mod}_B$ clearly takes perfect complexes to perfect complexes, so it preserves compact objects. □

3 Homotopy theory of derived comodules

In this section, we specialize the setting developed in Section 2.4 further to the situation of graded coalgebroids. This makes homotopy groups in the categories $\mathrm{Comod}_{\mathcal{D}\Gamma}$ and $\mathrm{Comod}_{\mathcal{D}\Gamma}^{cg}$ bigraded. Under suitable connectivity assumptions, there are analogues of cell structures, the Hurewicz theorem and Postnikov decompositions, which we discuss in Section 3.1.

Furthermore, under these connectivity assumptions, it is possible to describe an explicit system of compact generators for $\mathrm{Comod}_{\mathcal{D}\Gamma}^{cg}$ (see Proposition 3.17), and to give a quite satisfying characterization of $\mathrm{Comod}_{\mathcal{D}\Gamma}^{cg}$ as a right t-completion of $\mathrm{Comod}_{\mathcal{D}\Gamma}$ under a suitable t-structure (see Theorem 3.18).

In Section 3.2, we study the Adams spectral sequences associated to adjunctions between various $\mathrm{Comod}_{\mathcal{D}\Gamma}^{cg}$. Two particular special cases are the classical Cartan-Eilenberg spectral sequence and the classical Bockstein spectral sequence, both associated to different kinds of Hopf algebroid extensions.

Section 3.3 then introduces the main comodule category of interest for this work, $\mathrm{Comod}_{\mathcal{D}(BP_*BP)}^{cg}$. We discuss the structure of the Hopf algebroid BP_*BP and of various quotients, including the *even dual Steenrod algebra* $\mathcal{P}_*$.

Finally, we focus on certain minimal quotient coalgebras of BP_*BP in Section 3.4. The associated adjunctions lead to particularly nice (co)homology theories in $\mathrm{Comod}_{\mathcal{D}(BP_*BP)}^{cg}$, which we will call *exotic K-theories* due to their formal similarities to Morava K-theories in classical stable homotopy. Even though these exotic K-theories typically don't have a ring structure, and have no relation to formal group laws as far as we can tell, they will play a prominent role in Section 4, where they detect certain types of periodic self-maps.

3.1 Graded connected coalgebroids

Definition 3.1. *A graded coalgebroid over a graded-commutative ring A is a graded A-A-bimodule Γ with structure maps as in the ungraded case.*

All the constructions described in the ungraded case carry over to the graded case, for example we obtain derived comodule categories $\mathrm{Comod}_{\mathcal{D}\Gamma}$ and $\mathrm{Comod}_{\mathcal{D}\Gamma}^{cg}$ whenever Γ is flat as a graded right A-module. We will only consider flat Γ here.

The additional structure obtained from the grading is a $\mathbb{Z}$-action on $\mathcal{D}\,\mathrm{Mod}_A$, $\mathrm{Comod}_{\mathcal{D}\Gamma}$, and $\mathrm{Comod}_{\mathcal{D}\Gamma}^{cg}$, obtained from the degree shift on graded modules. Together with usual suspension (i.e. homotopy cofibre of the map to the zero object), we obtain a $\mathbb{Z}\times\mathbb{Z}$-action. We choose coordinates for $\mathbb{Z}\times\mathbb{Z}$ such that the degree shift acts through $(1,0)\in\mathbb{Z}\times\mathbb{Z}$, and usual suspension acts through $(1,-1)\in\mathbb{Z}\times\mathbb{Z}$. We write $\Sigma^{n,s}$ for the functor obtained through action with $(n,s)\in\mathbb{Z}\times\mathbb{Z}$.

Throughout this section Γ always refers to a graded coalgebroid over a graded-commutative ring A.

Definition 3.2. *A graded coalgebroid Γ is called connected if A is concentrated in nonnegative degrees, and the augmentation $\Gamma \to A$ is an isomorphism in degrees ≤ 0. More generally, we will call Γ d-connected if the map $\Gamma \to A$ is an isomorphism in degrees $\leq d$.*

For connected Γ, note that there is a comodule structure on A uniquely characterized by the property that the composite $A \to \Gamma \to A$ is 1. If Γ is a multiplicative coalgebroid, this comodule A is the monoidal unit of the corresponding symmetric-monoidal structure on comodules over Γ.

For Γ connected, we refer to the object represented by the comodule A in $\mathrm{Comod}_{\mathcal{D}\Gamma}$ by S, even in the absence of a monoidal structure. We denote by $S^{n,s}$ the shifted object $\Sigma^{n,s}S$. Note that, since the underlying derived A-module of $S^{n,s}$ is compact, $S^{n,s}$ lies in $\mathrm{Comod}^c_{\mathcal{D}\Gamma}$ and thus also represents an object $S^{n,s}$ in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$.

We introduce the corresponding notations $[X,Y]_{n,s} = [\Sigma^{n,s}X, Y]$ (which refers to $\pi_0 \operatorname{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(\Sigma^{n,s}X, Y))$, and $\pi_{n,s}(X) = [S,X]_{n,s} = [S^{n,s}, X]$ for homotopy classes of maps in $\mathrm{Comod}_{\mathcal{D}\Gamma}$, and analogously in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. The category we work in will usually be understood from context.

By Proposition 2.63, $\pi_{n,s}(S) = \operatorname{Ext}^{s,n+s}_\Gamma(A,A)$ in both $\mathrm{Comod}_{\mathcal{D}\Gamma}$ and, since S has compact underlying object, also in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$.

Remark 3.3. We have an equivalence

$$[X,Y]_{n,s} = \pi_0 \operatorname{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(\Sigma^{-s,s}\Sigma^{n+s,0}X, Y) = \pi_{-s} \operatorname{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(\Sigma^{n+s,0}X, Y),$$

which for X and Y represented by chain complexes C_* and D_*, with C_* consisting of projective underlying A-modules, can be described as $\operatorname{Ext}^{s,n+s}(C_*, D_*)$ by Proposition 2.63. If we write $n = t - s$, this becomes

$$[X,Y]_{t-s,s} = \operatorname{Ext}^{s,t}(C_*, D_*),$$

which means our grading convention is compatible with the classical Adams grading convention.

Recall that the adjunction $\mathrm{Comod}_{\mathcal{D}\Gamma} \to \mathcal{D}\operatorname{Mod}_A$ between the forgetful and the cofree functor can be identified with the adjunction $\varepsilon_* \dashv \varepsilon^* : \mathrm{Comod}_{\mathcal{D}\Gamma} \to \mathrm{Comod}_{\mathcal{D}A} \simeq \mathcal{D}\operatorname{Mod}_A$.

Definition 3.4. *For $X \in \mathrm{Comod}_{\mathcal{D}\Gamma}$, call $H_{**}(X;A) := \pi_{**}(\varepsilon_* X)$ its underlying homology. More generally, for $A \to B$ any homomorphism of graded commutative rings, let $H_{**}(X;B)$ denote $\pi_{**}(\varphi_* X)$, where φ is the composite $\Gamma \to A \to B$.*

Since left adjoints commute with all colimits, $H_{**}(X;B)$ preserves filtered colimits, using that the corepresenting objects of π_{**} in $\mathcal{D}\,\mathrm{Mod}_B$ are compact. Note that π_{**} formed in $\mathrm{Comod}_{\mathcal{D}\Gamma}$ does not preserve filtered colimits.

In $\mathrm{Comod}_{\mathcal{D}\Gamma}$ over a graded coalgebroid Γ, the cobar spectral sequence of Lemma 2.22 takes the following form:

Lemma 3.5. *The cobar spectral sequence for $\pi_{**}\,\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(X,Y)$ is trigraded, with additional grading k corresponding to the filtration of the $\lim_{\Delta_{\leq k}}$-tower. It converges conditionally, and its E_1-page and abutment are given by*

$$\pi_{n,s}\,\mathrm{map}_{\mathcal{D}\,\mathrm{Mod}_A}(X,\overline{\Gamma}^{(\otimes_A)k}\otimes_A Y)\Rightarrow\pi_{n,s}\,\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(X,Y)$$

with differentials d_r acting on tridegree (n,s,k) by $(-1,1,r)$. Here $\overline{\Gamma}$ is given by $\Sigma^{-1}\,\mathrm{fib}(\Gamma\to A)$, which inherits a bimodule structure. More precisely, the functor $\overline{\Gamma}\otimes_A(-)$ is given by

$$\Sigma^{-1}\,\mathrm{fib}((\Gamma\otimes_A(-))\Rightarrow\mathrm{id})$$

in $\mathrm{Fun}(\mathcal{D}\,\mathrm{Mod}_A,\mathcal{D}\,\mathrm{Mod}_A)$.

Proof. A priori, Lemma 2.22 describes the E_1-page in terms of the $\overline{T}^kY$, with $\overline{T}X$ the fibre of $X\to\Gamma\otimes_A X$ in the present setting.

However, since the composite $X\to\Gamma\otimes_A X\to X$ is the identity, $\Gamma\otimes_A X\simeq X\oplus\mathrm{fib}(\Gamma\otimes_A X\to X)$, and we can equivalently describe $\overline{T}X$ as $\Sigma^{-1}\,\mathrm{fib}(\Gamma\otimes_A X\to X)$. □

Definition 3.6. *On bidegrees $(n,s)\in\mathbb{Z}\times\mathbb{Z}$, we introduce a partial ordering by saying $(n',s')\leq_d(n,s)$ if and only if $s'\leq s$ and $(n-n')\geq d(s-s')$.*

Note that this is defined in such a way that for Γ d-connected, the tensor powers of $\overline{\Gamma}$ are concentrated in the degrees (n,s) with $(0,0)\leq_d(n,s)$. The positive tensor powers of $\overline{\Gamma}$ are even concentrated in the degrees (n,s) with $(d,1)\leq_d(n,s)$.

This means that only the homology groups $H_{n,s}(X;A)$ for $(n,s)\leq_d(n_0,s_0)$ contribute to the E_1-page of the cobar spectral sequence in tridegrees (n_0,s_0,k) with $k\geq 0$. More generally, for given $k_{\min}\in\mathbb{Z}_{\geq 0}$, only the homology groups $H_{n,s}(X;A)$ with $(n,s)\leq_d(n_0-dk_{\min},s_0-k_{\min})$ contribute to tridegrees (n_0,s_0,k) with $k\geq k_{\min}$.

We can now obtain an interesting bigraded variant of the Hurewicz theorem in $\mathrm{Comod}(\mathcal{D}\Gamma)$:

Lemma 3.7. *Assume Γ is d-connected and $X\in\mathrm{Comod}_{\mathcal{D}\Gamma}$. Fix (n_0,s_0), and consider the Hurewicz homomorphism $\pi_{n_0,s_0}(X)\to H_{n_0,s_0}(X;A)$ induced by ε_*.*

1. *If $H_{n,s}(X;A)=0$ for all $(n,s)\leq_d(n_0-d,s_0-1)$, the Hurewicz homomorphism is injective.*

2. *If $H_{n,s}(X;A) = 0$ for all $(n,s) \leq_d (n_0-1-d, s_0)$, the Hurewicz homomorphism is surjective.*

Proof. We can characterize the Hurewicz homomorphism $\pi_{n_0,s_0}(X) \to H_{n_0,s_0}(X;A)$ as the edge homomorphism of the cobar spectral sequence. In particular, it is injective if there are no elements of degrees (n_0, s_0, k) with $k > 0$ on the E_∞ page, and surjective if none of the elements in degree $(n_0, s_0, 0)$ support nontrivial differentials.

Since tensor powers of $\overline{\Gamma}$ contribute only to the E_1 page in degrees (n_0, s_0, k) with $k > 0$ if there exists $(n,s) <_d (n_0 - d, s_0 - 1)$ with $H_{n,s}(X;A) \neq 0$, under the conditions of (1), the E_1 page vanishes in those degrees and we get that the Hurewicz homomorphism is injective.

By the same reasoning, under the conditions in (2), the E_1 page vanishes in degrees $(n_0 - 1, s_0 + 1, k)$ with $k > 0$. It follows that elements in degree (n_0, s_0) can't support differentials, and the Hurewicz homomorphism is surjective. □

Corollary 3.8. *If Γ is d-connected, $\pi_{**}S$ is concentrated in the region defined by $(n,s) \geq_d (0,0)$, i.e. $s \geq 0$, $n \geq ds$. In particular, if Γ is connected, $\pi_{**}(S)$ is concentrated in the first quadrant, and if Γ is d-connected for $d > 0$, $\pi_{**}(S)$ is concentrated below the line $s = \frac{1}{d}n$ and above the line $s = 0$.*

Lemma 3.9. *Assume Γ is d-connected and X_i is a filtered diagram in $\mathrm{Comod}_{\mathcal{D}\Gamma}$. If, for each (n_0, s_0), for all but finitely many (n,s) with $(n,s) \leq_d (n_0, s_0)$ we have $H_{n,s}(X_i; A) = 0$ for all i, then*

$$\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, \mathrm{colim}\, X_i) \simeq \mathrm{colim}\, \mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, X_i)$$

for all $K \in \mathrm{Comod}^c_{\mathcal{D}\Gamma}$, i.e. comodules with underlying compact object in $\mathcal{D}\,\mathrm{Mod}_A$.

Proof. Since the entry at (n_0, s_0, k) in the E_1-page of the cobar spectral sequence for computing $\mathrm{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, X_i)$ is of the form $\mathrm{Map}_{\mathcal{D}\,\mathrm{Mod}_A}(K, \overline{\Gamma}^{(\otimes_A)k} \otimes_A X_i)$, and K is compact in $\mathcal{D}\,\mathrm{Mod}_A$, it commutes with filtered colimits in X_i. So the whole E_1-page of the cobar spectral sequence commutes with filtered colimits. As a result, every finite page commutes with filtered colimits as well. Note that a priori, this does not follow for the E_∞ page.

By assumption, for given K and (n_0, s_0), there exists $k_{\min}$ such that the E_1-pages of the cobar spectral sequences for computing $\mathrm{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, X_i)$ or $\mathrm{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, \mathrm{colim}\, X_i)$ all vanish in tridegrees (n_0, s_0, k) for $k \geq k_{\min}$. So to any bidegree (n_0, s_0), only finitely many terms contribute.

It follows that, around a fixed bidegree (n_0, s_0), the spectral sequences all degenerate on the same finite page. So the E_∞ entries at (n_0, s_0, k) also commute with the colimit of the X_i, and since the $\pi_{n_0,s_0}\, \mathrm{Map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, X_i)$ are built from those through a finite sequence of natural extensions, the claim follows. □

Essentially, this means that even though $K \in \mathrm{Comod}^c_{\mathcal{D}\Gamma}$ is not a compact object in $\mathrm{Comod}_{\mathcal{D}\Gamma}$, it behaves as such with respect to "uniformly bounded-below filtered colimits".

Definition 3.10. *We say X' is obtained from X by attaching an (n,s)-cell along $S^{n-1,s} \to X$ if there is a cofibre sequence*

$$S^{n-1,s} \to X \to X'.$$

More generally, we will say X' is obtained from X by attaching a set I of cells of dimensions (n_i, s_i) along $\bigvee S^{n_i-1,s_i} \to X$ if there is a cofibre sequence

$$\bigvee S^{n_i-1,s_i} \to X \to X'.$$

Definition 3.11. *A CW complex in $\mathrm{Comod}_{\mathcal{D}\Gamma}$ is an object $X \in \mathrm{Comod}_{\mathcal{D}\Gamma}$ together with a $\mathbb{Z}$-indexed filtration*

$$\cdots \to X^n \to X^{n+1} \to \cdots$$

with $X^n = 0$ for $n < N$ for some N, $\operatorname{colim} X^n = X$ and such that X^n is obtained from X^{n-1} by attaching a set of cells of dimensions of the form (n, s_i). The X^n are called the skeleta of X. X will be called a finite-dimensional CW complex if $X = X^n$ for some n, and a finite cell complex if it is built from finitely many cells in total.

Definition 3.12. *A subset $C \in \mathbb{Z} \times \mathbb{Z}$ is called* bounded-below *if there is $n_{\min}$ such that for all $(n,s) \in C$, $n \geq n_{\min}$. We will sometimes call C* bounded-below by $n_{\min}$ *as well. We will call the maximum such $n_{\min}$ the connectivity of C.*

Furthermore, C is called strongly bounded-below *if it is bounded-below, and there is a sequence s_n such that for all $(n,s) \in C$ we have $s \geq s_n$.*

We will say things like "X has strongly bounded-below homology" to mean "the set of all $(n,s) \in \mathbb{Z} \times \mathbb{Z}$ with $\pi_{n,s}(X) \neq 0$ is strongly bounded-below".

Note that, for a CW complex X, the dimensions of cells of X are by definition bounded-below. We will call X a strongly bounded-below CW complex if the set of dimensions of its cells is strongly bounded-below.

Observe that for C bounded-below, the set $\{(n,s) \in C | (n,s) \leq_d (n_0, s_0)\}$ is finite if either C is strongly bounded-below, or $d \geq 1$. In the light of statements such as the Hurewicz theorem (Lemma 3.7) and Lemma 3.9, this means that over 1-connected graded coalgebroids Γ, we will often consider bounded-below objects, whereas for Γ only 0-connected, we will typically be required to restrict attention to strongly bounded-below objects.

Lemma 3.13. *Assume that* Γ *is connected. Let* $X \in \mathrm{Comod}_{\mathcal{D}\Gamma}$*, and assume that the homology of* X *is bounded-below by* $n_{\min}$*. Then* X *has a CW complex structure, with dimensions of cells bounded below by* $n_{\min}$*.*

If $H_{**}(X; A)$ *is strongly bounded-below,* X *can be chosen as a strongly bounded-below CW complex.*

Proof. Let $n_{\min}$ be the connectivity of $H_{**}(X; A)$ and set $X_{n_{\min}} := X$.

We now inductively assume that we have constructed X_k with $H_{n,s}(X_k; A) = 0$ for all (n, s) with $n < k$.

Then we can choose a set of generators for $H_{k,s}(X_k; A)$ for all s. As condition (2) of Lemma 3.7 is satisfied, all of these elements can be represented by maps $S^{k,s} \to X_k$.

We define X_{k+1} as the corresponding cofibre

$$\bigvee S^{k,s_i} \to X_k \to X_{k+1}.$$

From the long exact sequence on $H_{**}(-; A)$, we see that X_{k+1} satisfies all the inductive assumptions. Since homology commutes with filtered colimits, and homology detects equivalences, we have $\operatorname{colim} X_k = 0$.

We now set $X^k = \operatorname{fib}(X \to X_{k+1})$. It follows that X^k is obtained from X^{k-1} by attaching cells of dimension of the form (k, s), that $X^{N-1} = 0$, and that $\operatorname{colim} X^k = X$.

If $H_{**}(X; A)$ is strongly bounded bounded-below, inductively, we can choose the X_k such that $H_{**}(X_k; A)$ is strongly bounded-below, and in each step, we only have to attach cells above some $s_{\min}$. So the resulting cell structure has cells in dimensions strongly bounded-below. □

Lemma 3.14. *If* Γ *is connected and* $H_{**}(X; A)$ *is bounded-below with connectivity* $n_{\min}$*,* X *can be written as a filtered colimit over finite CW complexes* K_i*, each of which has cells only in dimensions of the form* (n, s) *with* $n \geq n_{\min}$*. If* $H_{**}(X; A)$ *is strongly bounded-below, the dimensions of cells appearing in these* K_i *are also strongly bounded-below.*

Proof. A CW structure in particular exhibits X as a colimit of copies of $S^{N,s}$ for all s. Since every colimit is the filtered colimit of the colimits over its finite sub-diagrams, we are reduced to checking that finite colimits of copies of $S^{N,s}$ are finite CW complexes.

It suffices to check that a pushout of finite CW complexes has again a finite CW complex structure. This is established as soon as we check that for $X \to Y$ a map of finite CW complexes, the map $X^n \to Y$ factors (up to homotopy) through Y^n.

Thus, we need to check that $[X^n, Y/Y^n] = 0$. But Y/Y^n is a finite CW complex built from cells of dimensions (k, s) with $k > n$, and X^n a finite CW complex built from cells of dimensions (k, s) with $k \leq n$, so this follows from the fact that $\pi_{k,s}(S) = 0$ if $k < 0$. □

Remark 3.15. For Γ connected, analogous CW approximation results hold in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ as well. Namely, if $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is such that $\pi_{**}(X)$ is (strongly) bounded-below by N, then there is a CW structure with cells of dimensions (strongly) bounded-below by N. Furthermore, any such X is a filtered colimit of finite CW complexes. The proofs are completely analogous, using that π_{**} commutes with filtered colimits and that $\pi_{n,s}(S)$ is concentrated in the region determined by $n \geq 0$.

Lemma 3.16. *Assume Γ is connected.*

Then all $K \in \mathrm{Comod}^{c}_{\mathcal{D}\Gamma}$, i.e. comodules with underlying compact object in $\mathcal{D}\,\mathrm{Mod}_A$, are retracts of finite CW complexes.

Proof. Since the homology $H_{**}(K; A)$ is the homology of a perfect complex over A, it is strongly bounded-below. Lemma 3.14 produces a filtered system of finite cell complexes K_i such that K is their colimit, with the set of all dimensions of cells strongly bounded-below.

So we can apply Lemma 3.9 to obtain that

$$\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, \mathrm{colim}\, K_i) \simeq \mathrm{colim}\, \mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(K, K_i),$$

i.e. the identity $K \to K$ factors through a finite stage $K \to K_i$. Thus, K is a retract of K_i. □

Proposition 3.17. *If Γ is connected, $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is compactly generated by the $S^{0,s}$, with s ranging over all $s \in \mathbb{Z}$.*

Proof. The full subcategory of $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ generated by the $S^{0,s}$ contains all finite CW complexes, and because of Lemma 3.16, it contains all of $\mathrm{Comod}^{c}_{\mathcal{D}\Gamma}$. So the result follows. □

Theorem 3.18. *Assume Γ is 1-connected.*

We define a t-structure on $\mathrm{Comod}_{\mathcal{D}\Gamma}$ by letting $(\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$ be the full subcategory on all X with $H_{**}(X; A)$ bounded-below by 0.

Similarly, we define a t-structure on $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ by letting $(\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$ be the full subcategory on all X with $\pi_{**}(X; A)$ bounded-below by 0.

Then

1. *The functors $L \dashv I : \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma} \rightleftarrows \mathrm{Comod}_{\mathcal{D}\Gamma}$ restrict to equivalences between $(\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$ and $(\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$.*

2. *$\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is exhibited as the right t-completion of $\mathrm{Comod}_{\mathcal{D}\Gamma}$ with respect to this t-structure. (cf. [Lur16], Section 1.2.1)*

Proof. By Remark 3.15, $X \in (\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$ admits a description as filtered colimit of finite CW complexes K_i with cells in degrees of the form (n, s) with $n \geq 0$.

Since L preserves colimits and sends $S^{n,s}$ to $S^{n,s}$, it follows that $LX \in (\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$.

As the right adjoint I preserves homotopy groups, and objects $Y \in (\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$ have their homotopy bounded-below by 0 by the Hurewicz theorem, we also see that $IY \in (\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$. So L and I restrict to adjoint functors on the (≥ 0)-subcategories.

Next, we check that L is fully faithful: As mentioned before, given $X, Y \in (\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$, X can be written as filtered colimit of finite cell complexes $K_i \in (\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$, and similarly we write $Y = \operatorname{colim} L_i$. Now

$$\mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(X, Y) \simeq \lim_i \operatorname{colim}_j \mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(K_i, L_j)$$

and, because of Lemma 3.9 (which applies to uniformly bounded-below colimits since Γ is 1-connected),

$$\mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(LX, LY) \simeq \lim_i \operatorname{colim}_j \mathrm{map}_{\mathrm{Comod}_{\mathcal{D}\Gamma}}(L(K_i), L(L_j)).$$

Since L is fully faithful on the full subcategory of finite cell complexes by definition of $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, L is fully faithful on all of $(\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$.

The essential image of L contains all the $S^{n,s}$ with $n \geq 0$, therefore it contains all finite cell complexes, and by Lemma 3.13, it contains all of $(\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$. So L is an equivalence. By the adjunction, I is the inverse equivalence.

For the second claim, it is sufficient to check that the t-structure on $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is right t-complete. This means that for every sequence

$$\cdots \to X_k \to X_{k-1} \to \cdots \tag{2}$$

indexed over $\mathbb{Z}$, with $X_k \in (\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq k}$ and $X_k \to \tau_{\geq k} X_{k-1}$ an equivalence for all k, we have that $X := \operatorname{colim} X_k$ has $\tau_{\geq k} X \simeq X_k$.

To see this, note that $(\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq k}$ is generated under colimits by the compact $S^{k,s}$, where s ranges over all $s \in \mathbb{Z}$. Since the homotopy groups of the $S^{k,s}$ vanish in degrees (n, s) for $n < k$, this is true for all $\tau_{\geq k} X$.

For any X, $\tau_{\leq k-1} X$ has $\pi_{n,s}(\tau_{\leq k-1} X) = 0$ for all $n \geq k$, because $S^{n,s} \in (\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq k}$.

Together, the fibre sequence

$$\tau_{\geq k} X \to X \to \tau_{\leq k-1} X$$

shows that the homotopy groups of $\tau_{\geq k} X$ vanish in degrees (n, s) with $n < k$, and agree with the ones of X otherwise.

But since homotopy groups commute with filtered colimits in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, we see that

$$X_k \to \tau_{\geq k}(\operatorname{colim} X_k)$$

is an equivalence for X_k as in (2). □

Remark 3.19. We can give an explicit example to illustrate that $\mathrm{Comod}_{\mathcal{D}\Gamma}$ is not right t-complete. Let Γ be an exterior Hopf algebra $\mathbb{F}_2[x]/x^2$ with x in degree 2. Then $\pi_{**}(S) = \mathbb{F}_2[\alpha]$ for some element α in degree $|\alpha| = (|x| - 1, 1) = (1, 1)$. The sequence

$$S \xrightarrow{\alpha} \Sigma^{-|\alpha|}S \xrightarrow{\alpha} \Sigma^{-2|\alpha|}S \xrightarrow{\alpha} \cdots$$

is of the form (2) discussed in the proof of Theorem 3.18, but its colimit is 0, since α acts as 0 on $H_{**}(-;\mathbb{F}_2)$.

In $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, the colimit of the corresponding diagram has homotopy groups $\mathbb{F}_2[\alpha^{\pm 1}]$, and we can recover the diagram by taking k-connected covers as claimed.

Remark 3.20. If Γ is only 0-connected, a statement similar to Theorem 3.18 still holds. However, we have to replace the t-structure by a marginally more subtle construction. We choose $0 < \lambda < 1$, and define $(\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma})_{\geq 0}$ to consist of those X with $\pi_{n,s}(X) = 0$ whenever $s \geq 0$ and $n < 0$, or $s < 0$ and $n < -\lambda s$. Similarly, we define $(\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$ by an analogous condition on homology.

This ensures that, for $X \in (\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$, we have that $H_{n,s}(X; A)$ is strongly bounded-below, so we can still apply Lemma 3.9.

Theorem 3.18 (and Remark 3.20) give a satisfying characterization of $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ for connected Γ, as obtained from the connective derived comodule category $(\mathrm{Comod}_{\mathcal{D}\Gamma})_{\geq 0}$ by passing to "formal Postnikov systems". This is analogous to the relationship between the full category of spectra Sp and the category of connective spectra $\mathrm{Sp}_{\geq 0}$ in classical stable homotopy theory.

As a result, the machinery developed for $\mathrm{Comod}_{\mathcal{D}\Gamma}$, especially the cobar spectral sequence and its various implications (such as the Hurewicz theorem), carry over to bounded-below objects in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. We will therefore exclusively work in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ from now on.

We will close this section with a number of more precise statements about cell structures and homology.

Definition 3.21. *A connected coalgebroid Γ is* of finite type *if Γ and A are finitely generated as an A_0-module in each degree, and A_0 is a coherent ring in the sense that finitely generated A_0-modules are finitely presented.*

For Γ of finite type, $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is of finite type *if $\pi_{**}(X)$ is bounded-below (strongly bounded-below if Γ is only 0-connected), and all $\pi_{n,s}(X)$ are finitely generated A_0-modules.*

Lemma 3.22. *If Γ is of finite type, then $S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is of finite type. Furthermore, if $\pi_{**}(X)$ is of finite type, it admits a cell structure with finitely many cells in each degree.*

Proof. The first statement follows immediately from the cobar spectral sequence. For the second, note that if X in the proof of Lemma 3.13 is assumed to be of finite type, only finitely many cells are required in each step. □

Lemma 3.23. *Assume Γ is connected. Consider a surjective ring map $A \to B$, and assume $\pi_{**}X$ is bounded-below by $n_{\min}$.*

Then the Hurewicz map $\pi_{n_{\min},s}(X) \to H_{n_{\min},s}(X;B)$ is surjective for all $s \in \mathbb{Z}$.

If Γ is 1-connected, it exhibits $H_{n_{\min},s}(X;B)$ as $\pi_{n_{\min},s}(X) \otimes_{A_0} B_0$.

If Γ is only 0-connected, and $s_{\min}$ is such that $\pi_{n_{\min},s}(X) = 0$ for $s < s_{\min}$, we still get an isomorphism

$$H_{n_{\min},s_{\min}}(X;B) \simeq \pi_{n_{\min},s_{\min}}(X) \otimes_{A_0} B_0$$

in that degree.

Proof. Assume $X \neq 0$. Then let $n_{\min}$ be the connectivity of $\pi_{**}(X)$.

After choosing a cell structure on X with dimensions of cells bounded-below by $n_{\min}$, we see that $H_{n_{\min},s}(X;A)$ is the lowest-degree part of the cokernel of a map between free A-modules, and $H_{n_{\min},s}(X;B)$ the cokernel of a map between the associated free B-modules. So we see that

$$H_{n_{\min},s}(X;B) \simeq H_{n_{\min},s}(X;A) \otimes_{A_0} B_0.$$

Now the first statement follows from the fact that the Hurewicz homomorphism $\pi_{n_{\min},s}(X) \to H_{n_{\min},s}(X;A)$ is surjective. The other two follow from the fact that the Hurewicz homomorphism is an isomorphism in these cases. □

This has a number of nice applications:

Proposition 3.24. *If Γ is 1-connected, assume $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ with $\pi_{**}(X)$ bounded-below. If Γ is only 0-connected, assume $\pi_{**}(X)$ strongly bounded-below.*

Then the connectivities of $H_{**}(X;A_0)$ and $\pi_{**}(X)$ agree.

Proof. This follows immediately from Lemma 3.23. □

If A_0 is local, we can further simplify the homology theory involved:

Proposition 3.25. *Assume Γ and X to be of finite type. Furthermore, assume A_0 is local with maximal ideal $\mathfrak{m}$ and residue field $k = A_0/\mathfrak{m}$. Then the connectivities of $H_{**}(X;k)$ and X agree.*

Proof. Since X is of finite type, $\pi_{**}(X)$ is degreewise a finitely generated A_0-module.
Letting $n_{\min}$ be the connectivity of X, assume first that Γ is 1-connected.
Lemma 3.23 implies that the maps

$$\pi_{n_{\min},s}(X) \otimes_{A_0} k \to H_{n_{\min},s}(X;k)$$

are isomorphisms. By the Nakayama Lemma, we get that $\pi_{n_{\min},s}(X) = 0$ if and only if $H_{n_{\min},s}(X;k) = 0$. Thus there has to be s with $H_{n_{\min},s}(X;k) \neq 0$ and the connectivities agree.

If Γ is only 0-connected, we let s be minimal such that $\pi_{n_{\min},s}(X) \neq 0$. By Lemma 3.23, the map

$$\pi_{n_{\min},s}(X) \otimes_{A_0} k \to H_{n_{\min},s}(X;k)$$

is an isomorphism in that degree. This shows that $H_{n_{\min},s}(X;k) \neq 0$, and the connectivities agree. □

So we can detect equivalences between bounded-below objects on $H_{**}(X;A_0)$ $H_{**}(X;k)$. For example, when constructing a cell structure as in Lemma 3.13, we can kill generators in $H_{**}(X;A_0)$. Since A_0 has typically small projective dimension, this makes it possible to give very precise statements about the number and dimension of cells required, similar to the results discussed for CW complexes in classical homotopy theory in Section 4.C of [Hat02].

We will give a nice special case of this technique:

Proposition 3.26. *Let Γ be a connected coalgebroid of finite type, and $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ of finite type.*

Then X has a cell structure with cells of dimension (n,s) in bijection to a basis of $H_{**}(X;k)$.

Proof. We can carry out the construction in the proof of Lemma 3.13, but attach cells to kill basis elements of $H_{**}(X;k)$ instead of generators of $H_{**}(X;A)$. This is possible since the connectivity of the intermediate X_k is detected on $H_{**}(X;k)$ by Proposition 3.25, and the generalized Hurewicz homomorphism from Lemma 3.23 is surjective in the required degrees.

Since $H_{**}(S;k)$ is just a single copy of k, concentrated in degree $(0,0)$, attaching a cell to kill an element in $H_{**}(X;k)$ doesn't introduce any new elements. It follows that the cells arising from this construction are in bijection with a basis for $H_{**}(X;k)$. □

We will now consider some facts specific to multiplicative coalgebroids, over which the comodule categories are symmetric-monoidal.

First recall the following "push-pull" relation:

Lemma 3.27. *For $\Gamma \xrightarrow{f} \Sigma$ a multiplicative morphism of connected multiplicative coalgebroids, we have equivalences*

$$f^*(X) \otimes Y \simeq f^*(X \otimes f_*(Y))$$

for all $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}$, $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$.

Proof. We obtain a natural transformation between the two as the adjoint of

$$f_*(f^*(X) \otimes Y) \simeq f_* f^*(X) \otimes f_*(Y) \to X \otimes f_*(Y).$$

The resulting map is clearly an equivalence for Y the monoidal unit $S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ (represented by the comodule A). Since both sides commute with colimits, and S generates $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, the result follows. □

If $\Gamma \xrightarrow{f} \Sigma$ is a map of multiplicative coalgebroids, the homology theory $\pi^{\Sigma}_{**}(f_*(-))$ is actually represented by an object $H\Sigma \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, which we will describe now.

Definition 3.28. *For $\Gamma \xrightarrow{f} \Sigma$ a morphism of coalgebroids, we define*

$$H\Sigma := f^* S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}\,.$$

From the adjunction between $f_* \dashv f^*$ and the fact that $f_*(S) = S$, we see that $H\Sigma$ has homotopy groups

$$\pi^{\Gamma}_{**}(H\Sigma) \simeq \pi^{\Sigma}_{**}(S).$$

Lemma 3.29. *If $\Gamma \to \Sigma$ is a map between connected coalgebroids over A and B, and $\Gamma \otimes_A B$ is right Σ-cofree (or more generally $(\Gamma \otimes_A B)\Box_{\Sigma}(-)$ is exact), then $H\Sigma \in \mathrm{Comod}_{\mathcal{D}\Gamma}$ is represented by the comodule $(\Gamma \otimes_A B)\Box_{\Sigma} B$. In particular, it has homology $H_{**}(H\Sigma; A)$ concentrated in degrees $(n, 0)$ with $n \geq 0$.*

Proof. Since $S \in \mathrm{Comod}_{\mathcal{D}\Sigma}$ is represented by the chain complex of comodules with B in degree 0, $H\Sigma = f^* S$ is represented by the comodule $(\Gamma \otimes_A B)\Box_{\Sigma} B$ according to Lemma 2.65. □

Lemma 3.30. *If Γ and Σ are multiplicative coalgebroids and $\Gamma \xrightarrow{f} \Sigma$ is a morphism, there is a natural equivalence*

$$f^* f_* X \simeq H\Sigma \otimes X$$

for each $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. In particular,

$$\pi^{\Sigma}_{**}(f_* X) \simeq \pi^{\Gamma}_{**}(H\Sigma \otimes X) = H\Sigma_{**} X.$$

Proof. This follows immediately from the push-pull relation from Lemma 3.27:

$$f^* f_* X \simeq f^*(S \otimes f_*(X)) \simeq f^*(S) \otimes X = H\Sigma \otimes X. \qquad \square$$

This equivalence will be used in Section 3.2 to interpret the E_2-pages of some Adams spectral sequences associated to adjunctions $f_* \dashv f^*$. Concretely, it allows us to identify the Adams resolution with respect to the adjunction $f_* \dashv f^*$ with the Adams resolution with respect to the ring $H\Sigma$.

Lemma 3.31. *Assume we are given a diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{h} & \Gamma' \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & \Sigma \end{array}$$

with Γ and Γ' multiplicative coalgebroids, and h a multiplicative morphism. We obtain objects $H\Sigma = f^*S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ and $H\Sigma = g^*S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma'}$.

Then we have an equivalence

$$f^*S \otimes X \simeq h^*(g^*S \otimes h_*X),$$

so in particular

$$\pi^{\Gamma}_{**}(H\Sigma \otimes X) \simeq \pi^{\Gamma'}_{**}(H\Sigma \otimes h_*X).$$

Proof. After writing $f^*S = h^*g^*S$, the first equivalence follows from the push-pull relation from Lemma 3.27. The second is just the corresponding statement on homotopy, using that

$$\pi^{\Gamma}_{**}(h^*(g^*S \otimes h_*X) \simeq \pi^{\Gamma'}_{**}(g^*S \otimes h_*X).$$

by the adjunction. $\square$

For Γ a graded coalgebroid over a ring A, and $A \to B$ a map of rings, the composite morphism $\Gamma \to B$ to the identity coalgebroid over B gives rise to an object $HB \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. This has homotopy groups given by B in the single degree $(0, 0)$.

There is an analogue of the Postnikov tower for objects of $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. Note that the t-structure defined on $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ previously already leads to something like Postnikov sections: For every X, we have produced a map $X \to \tau_{\leq k}X$, isomorphic on $\pi_{n,s}$ for $n \leq k$, and such that $\pi_{n,s}(\tau_{\leq k}X) = 0$ for $n > k$.

The following lemma yields Postnikov sections in a stronger sense, concentrated in a single degree.

Lemma 3.32. *Assume* Γ *is connected. For* $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ *with* $\pi_{**}(X)$ *weakly bounded below by* $n_{\min}$*, assume furthermore that* $s_{\min} \in \mathbb{Z}$ *is such that* $\pi_{n_{\min},s}(X)$ *vanishes for all* $s < s_{\min}$*.*

Then there exists $H \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ *and a map* $X \to H$ *inducing an isomorphism on* $\pi_{n_{\min},s_{\min}}$*, and furthermore* $\pi_{n,s}(H) = 0$ *for* $(n,s) \neq (n_{\min}, s_{\min})$*.*

Proof. We can kill homotopy groups $\pi_{n_{\min},s}(X)$ for $s > s_{\min}$ by attaching cells. Since $\pi_{**}(S)$ is concentrated in degrees (n,s) with $n \geq 0$, $s \geq 0$, attaching a cell along a map $S^{n_{\min},s_0} \to X$ only introduces new homotopy groups in degrees of the form (n,s) with $n \geq n_{\min}+1$ and $s \geq s_0 - 1$, and only kills elements in degrees of the form (n,s) with $n \geq n_{\min}$ and $s \geq s_0$. In particular, we can kill all homotopy groups $\pi_{n_{\min},s}$ for $s > s_{\min}$ without changing $\pi_{n_{\min},s_{\min}}$. Then, in a further step, we can successively attach cells of dimension of the form (n,s) with $n \geq n_{\min}+2$ to kill all homotopy groups $\pi_{n,s}$ for $n \geq n_{\min}+1$ (this is just the construction $\tau_{\leq n_{\min}}$). □

Similar to classical Eilenberg-MacLane spaces and spectra, the H appearing in Lemma 3.32 are uniquely characterized by their homotopy groups:

Lemma 3.33. *Let* $H_1, H_2 \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ *be two objects with homotopy groups concentrated in a single degree* (n,s)*. Given an isomorphism* $\pi_{n,s}(H_1) \simeq \pi_{n,s}(H_2)$*, there is an equivalence* $H_1 \to H_2$ *inducing that isomorphism, unique up to homotopy.*

Proof. Denote $M = \pi_{n,s}(H_1)$, and identify $M \simeq \pi_{n,s}(H_2)$ along the given isomorphism. Under these identifications we are reduced to show that there is a map $H_1 \to H_2$ inducing the identity on homotopy groups.

Fix a presentation of M as an $A_0 = \pi_{0,0}(S)$-module. This describes a cell complex K, with cells of dimension (n,s) and $(n+1,s-1)$, such that $\pi_{n,s}(K) = M$. Furthermore, the maps $M \to \pi_{n,s}(H_i)$ give rise to maps $K \to H_i$ inducing them: The behaviour on the chosen generators defines a map from the n-skeleton $K^n \to H_i$, and since the defining relations of M hold in $\pi_{**}(H_i)$, the map extends over all of K.

We now apply the construction of Lemma 3.32 to K, obtaining a new complex K' with homotopy M in the single degree (n,s). Since we only attach cells along elements of dimensions (n',s') with $n' > n$ or $n' = n$ and $s' > s$, and the homotopy groups of H_1 and H_2 vanish in these degrees, we obtain corresponding maps $K' \to H_i$.

But now both of these maps are equivalences, so we get a composite equivalence $H_1 \to H_2$, which on homotopy induces the identity. Furthermore, similar obstruction theory shows that the equivalences $K' \to H_i$ are uniquely characterized up to homotopy by their behaviour on $\pi_{n,s}$, from which the uniqueness statement follows. □

For M any A_0-module, we will denote the object with homotopy groups a single M concentrated in degree $(0,0)$ by HM. It is well-defined up to unique homotopy equivalence by Lemma 3.33, and exists since we can construct it as Postnikov section (in the sense of Lemma 3.32 of a finite cell complex obtained from a presentation of M.

Again by Lemma 3.33, for B an A-algebra concentrated in degree 0, the HB described in Definition 3.28 is equivalent to HB in the sense given here.

Remark 3.34. The theory of cell structures, Postnikov sections and a corresponding obstruction theory in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ could be pushed much further. Indeed, due to the bigrading of homotopy groups, there is a lot of flexibility in the choice of order in which to kill homotopy groups, i.e. for every appropriately shaped line, there is a Postnikov section functor killing all homotopy groups beyond the line. The admissible shapes of those lines depend on the region in which $\pi_{**}(S)$ is concentrated, so by Corollary 3.8, on the connectivity of Γ.

The analogue of the Hurewicz theorem given in Lemma 3.7 illustrates these phenomena.

Lemma 3.35. *Let Γ be a coalgebra over $\mathbb{F}_p$. Then there is an equivalence of algebras*

$$\pi_{**} \operatorname{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(H\mathbb{F}_p, H\mathbb{F}_p) \simeq \Gamma^*,$$

where Γ^* is the $\mathbb{F}_p$-linear dual of Γ, with algebra structure dual to the comultiplication on Γ, and the left hand side has the algebra structure obtained from the composition product.

Proof. There is an action of Γ^* on Γ through maps of left Γ-comodules, by sending $\theta \in \Gamma^*$ to the composite map

$$\Gamma \to \Gamma \otimes \Gamma \xrightarrow{\mathrm{id} \otimes \theta} \Gamma \otimes \mathbb{F}_p \simeq \Gamma.$$

Since $H\mathbb{F}_p = \varepsilon^* S$ for the augmentation $\varepsilon : \Gamma \to \mathbb{F}_p$ can be represented by the comodule Γ (e.g. by computing that the homotopy groups of the object represented by Γ are $\mathbb{F}_p$ concentrated in degree $(0,0)$), this defines an algebra map

$$\Gamma^* \to \pi_{**} \operatorname{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(H\mathbb{F}_p, H\mathbb{F}_p).$$

From the adjunction $\varepsilon_* \dashv \varepsilon^*$, we obtain an isomorphism

$$\pi_{**} \operatorname{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(H\mathbb{F}_p, H\mathbb{F}_p) \simeq \pi_{**} \operatorname{map}_{\mathcal{D}\,\mathrm{Mod}_{\mathbb{F}_p}}(\varepsilon_* H\mathbb{F}_p, S) \simeq \operatorname{Hom}_{\mathbb{F}_p}(\Gamma, \mathbb{F}_p) = \Gamma^*.$$

inverse to the action map. □

Lemma 3.36. *For Γ a Hopf algebra over $\mathbb{F}_p$, and X an object of* $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ *represented by a Γ-comodule M, finite-dimensional as an $\mathbb{F}_p$-module, the mod p homology and cohomology can be identified with*

$$(H\mathbb{F}_p)_{**}X = \pi^{\Gamma}_{**}(H\mathbb{F}_p \otimes X) \simeq M$$
$$H\mathbb{F}_p^{**}X = \pi_{**}\,\mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(X, H\mathbb{F}_p) \simeq \mathrm{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p) = M^*.$$

For $\theta \in \Gamma^*$, the induced action on $(H\mathbb{F}_p)_{**}X$ and $H\mathbb{F}_p^{**}X$ (via Lemma 3.35) is as follows:

1. *On cohomology, θ acts through the action map $\Gamma^* \otimes M^* \to M^*$ dual to the coaction map, i.e. the dual Γ^*-action on M^*.*
2. *On homology, θ acts through the map $M \to \Gamma \otimes M \xrightarrow{\chi(\theta)\otimes \mathrm{id}} M$, i.e. the conjugate of the right Γ^*-action on M.*

Proof. The first two statements are just the push-pull formula (Lemma 3.30) together with the adjunction.

For the action, observe that the natural maps

$$\mathrm{Hom}_\Gamma(\mathbb{F}_p, \Gamma \otimes M) \to \pi^{\Gamma}_{**}(H\mathbb{F}_p \otimes X) = (H\mathbb{F}_p)_{**}X$$
$$\mathrm{Hom}_\Gamma(M, \Gamma) \to \pi_{**}\,\mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(X, H\mathbb{F}_p) = H\mathbb{F}_p^{**}X$$

are isomorphisms. Since the action of $\theta \in \Gamma^*$ comes from a map $\Gamma \to \Gamma$ of comodules, we can therefore determine this action completely in the world of ordinary comodules.

For cohomology, note that the isomorphism

$$\mathrm{Hom}_\Gamma(M, \Gamma) \simeq \mathrm{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p)$$

is induced in one direction by postcomposing a map $M \to \Gamma$ with the augmentation $\Gamma \xrightarrow{\varepsilon} \mathbb{F}_p$, and in the other direction by sending a map $\varphi : M \to \mathbb{F}_p$ to the composite

$$M \to \Gamma \otimes M \xrightarrow{\mathrm{id}\otimes\varphi} \Gamma \otimes \mathbb{F}_p \simeq \Gamma.$$

One therefore computes the action of $\theta \in \Gamma^*$ on $\varphi \in M^*$ by forming the composite

$$M \to \Gamma \otimes \Gamma \otimes M \xrightarrow{\varepsilon\otimes\theta\otimes\varphi} \mathbb{F}_p,$$

or equivalently, since ε is the counit for the comultiplication on Γ, the composite

$$M \to \Gamma \otimes M \xrightarrow{\theta\otimes\varphi} \mathbb{F}_p.$$

So the action map $\Gamma^* \otimes M^* \to M^*$ is the dual of the coaction map $M \to \Gamma \otimes M$ as claimed.

On homology, note that the isomorphism

$$\mathrm{Hom}_\Gamma(\mathbb{F}_p, \Gamma \otimes M) \simeq M$$

is given in one direction by postcomposing with $\Gamma \otimes M \xrightarrow{\varepsilon \otimes \mathrm{id}} M$. In the other direction, it works by sending $m \in M$ to the primitive element $\sum_i \chi(x_i) \otimes m_i$, where $\psi(m) = \sum_i x_i \otimes m_i$ for $\psi : M \to \Gamma \otimes M$ the coaction map.

From this one explicitly computes that $\theta \in \Gamma^*$ acts on $M \simeq \mathrm{Hom}_\Gamma(\mathbb{F}_p, \Gamma \otimes M)$ by the map

$$M \to \Gamma \otimes M \xrightarrow{\chi(\theta)} \mathbb{F}_p \otimes M \simeq M.$$

□

3.2 Adams spectral sequences

For $f : \Gamma \to \Sigma$ a morphism of coalgebroids over commutative rings A and B, we obtain a pair of adjoint functors

$$f_* \dashv f^* : \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma} \to \mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}.$$

As discussed for an abstract adjunction $F \dashv G$ in Section 2.2, this gives rise to a cosimplicial $f^* f_*$-Adams resolution for objects in $\mathrm{Comod}_{\mathcal{D}\Gamma}$. It takes the form

$$X \to f^* f_* X \rightrightarrows (f^* f_*)^2 X \Rrightarrow \cdots$$

and can be used to compute mapping spaces in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ from those in $\mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}$ in good cases.

Lemma 3.37. *Suppose $f : \Gamma \to \Sigma$ is a morphism of graded connected coalgebroids. Consider the induced adjunction $f_* \dashv f^*$ on* Comod^{cg}.

1. *If the induced map $\pi^\Gamma_{n,s}(S) \to \pi^\Sigma_{n,s}(S)$ is an isomorphism in degrees $(0, s)$, and surjective in degrees $(1, s)$, all X with $\pi_{**}(X)$ bounded below are complete with respect to the adjunction $f_* \dashv f^*$.*

2. *If the induced map $\pi^\Gamma_{n,s}(S) \to \pi^\Sigma_{n,s}(S)$ is an isomorphism in degree $(0, 0)$ and surjective in degrees $(0, s)$, all X with $\pi_{**}(X)$ strongly bounded below are complete with respect to the adjunction $f_* \dashv f^*$.*

Proof. As $\pi^{\Gamma}_{**}(f^*Y) = \pi^{\Sigma}_{**}(Y)$, the unit $X \Rightarrow f^*f_*X$ induces the map $\pi^{\Gamma}_{**}(X) \to \pi^{\Sigma}_{**}(f_*X)$ on homotopy groups.

Let $\overline{f^*f_*}$ be the fibre of $\mathrm{id} \Rightarrow f^*f_*$.

By Lemma 2.38, X is complete with respect to the adjunction $f_* \dashv f^*$ if and only if $\lim \overline{f^*f_*}^n X = 0$.

We discuss the first statement first. Under the assumptions stated there, the homotopy groups of $\overline{f^*f_*}(S)$ are bounded below by 1.

Both f_* and f^* preserve colimits, and so f^*f_* does, too. Since in the stable setting, homotopy fibres also preserve colimits, $\overline{f^*f_*}$ does, too. Since $\overline{f^*f_*}$ increases the (homotopical) connectivity on S, it therefore increases connectivity on bounded-below X as well, since X admits a CW structure with cells of bounded-below dimensions.

Inductively, we see that for any fixed (n, s), $\pi_{n,s}\overline{f^*f_*}^k X$ vanishes for large k. We thus see $\lim \overline{f^*f_*}^n X = 0$.

Under the assumptions of the second statement, the homotopy groups of $\overline{f^*f_*}(S)$ are concentrated in the first quadrant, with $\pi_{0,0}(\overline{f^*f_*}(S)) = 0$. Since $\pi_{**}(X)$ is strongly bounded-below, there is a CW structure with cells in strongly bounded-below dimensions. Inductively, one sees again that for fixed (n, s), $\pi_{n,s}\overline{f^*f_*}^k X = 0$ for large k. So again, we obtain $\lim \overline{f^*f_*}^n X = 0$. □

Example 3.38. For $A \to B$ a map of rings which is an isomorphism in degree 0, as well as a compatible morphism $\Gamma \xrightarrow{f} \Sigma$ of connected coalgebroids which is surjective in degree 1, Lemma 3.37 shows that the f^*f_*-Adams resolution converges for all X with $\pi_{**}X$ strongly bounded-below.

Proposition 3.39. *Let $\Gamma \xrightarrow{f} \Sigma$ be a morphism of graded connected coalgebroids such that $\pi^{\Gamma}_{**}(S) \to \pi^{\Sigma}_{**}(S)$ is an isomorphism in degree $(0, 0)$, and surjective in degrees $(0, s)$. Then there is an equivalence*

$$\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma} \simeq \mathrm{Comod}^{cg}_{f_*f^*}(\mathrm{Comod}^{cg}_{\mathcal{D}\Sigma})$$

Proof. This follows from Theorem 2.44, using that all compact objects are $f_* \dashv f^*$-complete by Lemma 3.37, and that $\mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}$ is generated by S, which is in the image of the left adjoint f_* □

Remark 3.40. Under the conditions of Proposition 3.39, if f is a morphism of multiplicative coalgebroids, one also gets an equivalence

$$\mathrm{Comod}^{cg}_{\mathcal{D}\Sigma} \simeq \mathrm{Mod}_{H\Sigma}(\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}),$$

by applying the monadic Barr-Beck theorem. The monad f^*f_* can be identified with $H\Sigma \otimes (-)$ through Lemma 3.30.

Remark 3.41. There are interesting cases not covered by Lemma 3.37, for example maps between coalgebroids where the underlying map $A \to B$ is not an isomorphism on $A_0 \to B_0$. If that map is still required to be surjective, $\pi_{**}(\overline{f^* f_*}(S))$ is still concentrated in the first quadrant.

Under suitable additional conditions, one can identify $X^\wedge_{f^* f_*}$ with some completion of X. For instance, if $\pi^\Gamma_{0,0}(S) \to \pi^\Sigma_{0,0}(S)$ is the map $\mathbb{Z} \to \mathbb{F}_p$, $X^\wedge_{f^* f_*}$ agrees with the p-completion $X^\wedge_p = \lim X/p^k$ for connective X. We won't prove this more general statement here, as the only instance of it we need is Lemma 3.43, which we can easily analyze by hand.

We will need to identify the structure of the E_1 or E_2-pages of two special cases of the resulting Adams spectral sequences, for later use in Section 4.

The first example coincides with the classical Cartan-Eilenberg spectral sequence for Hopf algebras:

Lemma 3.42. *Let $\Gamma \xrightarrow{f} \Sigma$ be a normal quotient of connected Hopf algebras over $\mathbb{F}_p$, in the sense that*

$$\Gamma \Box_\Sigma \mathbb{F}_p = \mathbb{F}_p \Box_\Sigma \Gamma =: \Phi$$

as subgroups (and indeed sub Hopf algebras) of Γ. Assume that Φ is 1-connected. Then the Adams spectral sequence associated to the adjunction

$$f_* \dashv f^* : \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma} \leftrightarrows \mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}$$

converges for all X with $\pi_{**}(X)$ bounded-below, and has E_2 page and abutment described by

$$\mathrm{Ext}_\Phi(\mathbb{F}_p, \pi^\Sigma_{**}(f_* X)) \Rightarrow \pi^\Gamma_{**}(X).$$

If X is a ring, the spectral sequence is multiplicative, and if X is a ring and Y a module over X, the spectral sequence for Y is a module over the one for X.

Proof. By Lemma 3.30, we can write $f^* f_*(X) = H\Sigma \otimes X$.

As $S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}$ is just represented by the comodule $\mathbb{F}_p$, and $\Gamma \Box_\Sigma(-)$ is exact since Γ is right Σ-cofree, we can represent $H\Sigma = f^*(S)$ through $\Gamma \Box_\Sigma \mathbb{F}_p = \Phi$, with comodule structure obtained as a left Γ sub-comodule of Γ.

From normality, it follows now that $f_*(\Phi)$, i.e. Φ with left Σ-coaction obtained by corestriction from the left Γ-coaction, has trivial comodule structure. We get natural equivalences

$$f_* f^* f_*(X) \simeq \Phi \otimes f_*(X),$$

where Φ has the trivial comodule structure, due to normality. On homotopy groups, this says that

$$\pi^\Sigma_{**} f_* f^* f_*(X) \simeq \Phi \otimes \pi^\Sigma_{**} f_*(X).$$

It follows that we can identify the cobar complex for the f_*f^*-comodule f_*X on homotopy groups with the cobar complex for the Φ-comodule $\pi^{\Sigma}_{**}f_*(X)$. The result follows.

The statement about multiplicative structures follows from the fact that the Bousfield-Kan spectral sequence of a cosimplicial ring is multiplicative, and similarly for module structures. □

Another important special case is the classical Bockstein spectral sequence:

Lemma 3.43. *Let Γ be a Hopf algebroid over A, $x \in A$ an invariant non-zero divisor, and let $\Sigma := \Gamma/x\Gamma$ be the corresponding quotient Hopf algebroid over A/x. Denote the quotient map $f : \Gamma \to \Sigma$.*

Let X have $\pi^{\Gamma}_{**}(X)$ bounded-below. Then the f^*f_*-Adams spectral sequence has E_1-page of the form

$$\pi^{\Sigma}_{**}(f_*X)[\xi],$$

with ξ detecting x. It converges to $\pi^{\Gamma}_{**}(X)$ if $|x| \geq 2$ and $\pi^{\Gamma}_{**}(X)$ is bounded-below, or $|x| \geq 1$ and $\pi^{\Gamma}_{**}(X)$ strongly bounded-below. Generally it converges to the completion $\pi^{\Gamma}_{**}(X^{\wedge}_x)$.

If X is a ring, the spectral sequence is multiplicative, and if X is a ring and Y is a module over it, the spectral sequence for Y is a module over the one for X.

Proof. Using Lemma 3.30, we get $f^*f_*(X) = H\Sigma \otimes X$. Now since $S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Sigma}$ is represented by A/x, $H\Sigma = f^*(S)$ is represented by A/x as well, but considered a left Γ-comodule (through the map $A/x \to \Gamma/x\Gamma \simeq \Gamma \otimes_A A/x$).

Equivalently, we can represent $f^*(S)$ as the cofibre of

$$S \xrightarrow{x} S$$

in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. So the natural map $X \to f^*f_*X$ is the quotient map $X \to X/x$, and thus the fibre $\overline{f^*f_*}(X)$ agrees with $\Sigma^{|x|,0}X$. The natural transformation $\overline{f^*f_*X} \to X$ agrees with the multiplication by x-map $\Sigma^{|x|,0}X \xrightarrow{x} X$.

From Lemma 2.38, we obtain fibre sequences

$$\Sigma^{(k+1)|x|,0}X \xrightarrow{x^{k+1}} X \to \lim_{\Delta_{\leq k}}(f^*f_*)^{\bullet+1}X.$$

This allows us to identify $X^{\wedge}_{f^*f_*} \simeq \lim_k X/x^k = X^{\wedge}_x$. For $|x| \geq 1$ and X strongly bounded-below, or for $|x| \geq 2$ and X bounded-below, the map $X \to X^{\wedge}_x$ is an equivalence since for every (n,s), there exists $k_{\min}$ such that $\pi_{n,s}(X) \to \pi_{n,s}(X/x^k)$ is an equivalence for all $k \geq k_{\min}$.

From Lemma 2.38, we also see that the E_1 page is given by the terms

$$\pi^{\Gamma}_{**}(f^*f_*\overline{f^*f_*}^{k+1}X) \simeq \pi^{\Gamma}_{**}(f^*f_*\Sigma^{(k+1)|x|,0}X) = \pi^{\Sigma}_{**}(\Sigma^{(k+1)|x|,0}f_*X),$$

from which we obtain the claimed description.

If X is a ring, the cosimplicial Adams resolution is a cosimplicial ring, so the Bousfield-Kan spectral sequence is multiplicative. A similar statement holds for modules.

To see the multiplicative structure on the E_1-page, we can instead identify the cobar complex on homotopy. Since $f^* f_* X$ agrees with the cofibre of $\Sigma^{|x|,0} X \xrightarrow{x} X$, but x is trivial after applying f_*, we can identify $f_* f^* f_*(X) \simeq X \oplus \Sigma^{|x|+1,-1} X$. On homotopy groups, we can identify

$$\pi_{**}^{\Sigma}(f_* f^* f_* X) = \Lambda(\sigma) \otimes \pi_{**}^{\Sigma}(f_* X),$$

multiplicatively, where $\Lambda(\sigma)$ is an exterior Hopf algebra on one primitive generator σ in degree $(|x|+1,-1)$. The E_1-page can therefore also be described as the algebraic reduced cobar complex of the $\Lambda(\sigma)$-comodule algebra $\pi_{**}^{\Sigma}(f_* X)$, which is precisely polynomial over $\pi_*(X)$ on one generator.

□

3.3 BP_*BP and the even dual Steenrod algebra

In this section, we review the structure of the Hopf algebroid BP_*BP, and a related Hopf algebra $\mathcal{P}_*$ over $\mathbb{F}_p$. This Hopf algebra $\mathcal{P}_*$ is closely related to the dual Steenrod algebra, and will be referred to as the *even dual Steenrod algebra*.

Lemma 3.44. *Fix a prime p. For BP the p-local Brown-Peterson spectrum (see for example [Rav86], Section 4), the BP-homology Hopf algebroid*

$$(BP_*, BP_*BP)$$

is flat, and has the following properties:

1. $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ *with* $|v_i| = 2(p^i - 1)$.
2. $BP_*BP = BP_*[t_1, t_2, \ldots]$ *as a BP_*-algebra (through the left unit $\eta_L : BP_* \to BP_*BP$), with* $|t_i| = 2(p^i - 1)$.
3. $\eta_R v_n = v_n$ *mod* $I_n BP_*BP$, *where* $I_n := (p, v_1, \ldots, v_{n-1})$.
4. $\Delta t_n = \sum_{i+j=n} t_i^{p^j} \otimes t_j$ *mod* $I_n BP_*BP$.
5. *The natural map $BP_* \otimes BP_* \to BP_*BP$, given by η_L and η_R, is an equivalence after tensoring with $\mathbb{Q}$.*

Proof. See [Rav86], in particular Theorem A2.1.27. □

This partial description of the Hopf algebroid structure will be sufficient for us.

Note that, inductively, (3) implies that I_n is an invariant ideal, since if I_{n-1} is an invariant ideal, $(BP_*/I_n, BP_*BP/I_n)$ is a Hopf algebroid, and (3) shows that v_n is an invariant element there. Since v_n is invariant modulo I_n, $I_{n+1} = I_n + (v_n)$ is invariant.

Definition 3.45. *With $I_\infty := (p, v_1, v_2, \ldots)$ the union of the I_n, the quotient Hopf algebroid $(BP_*/I_\infty, BP_*BP/I_\infty)$ is a commutative Hopf algebra over $\mathbb{F}_p$, which we will denote $\mathcal{P}_*$ and refer to as the* even dual Steenrod algebra.

As follows immediately from part (4) of Lemma 3.44, $\mathcal{P}_*$ has the form $\mathbb{F}_p[t_1, t_2, \ldots]$, with comultiplication given by

$$\Delta t_n = \sum_{i+j=n} t_i^{p^j} \otimes t_j.$$

Letting χ denote the Hopf algebra antipode, we get the corresponding formula

$$\Delta\chi(t_n) = \sum_{i+j=n} \chi(t_j) \otimes \chi(t_i)^{p^j},$$

which exactly agrees with the comultiplication on the ξ_n in the dual Steenrod algebra $\mathcal{A}_*$.

At odd primes p, we can thus identify the Hopf algebra $\mathcal{P}_*$ with the sub Hopf algebra of $\mathcal{A}_*$ generated by the ξ_n, by sending $t_n \mapsto \chi(\xi_n)$. At $p = 2$, we send $t_n \mapsto \chi(\xi_n^2)$, which defines an isomorphism of $\mathcal{P}_*$ onto the subalgebra of the dual Steenrod algebra generated by the ξ_n^2. This is why we call $\mathcal{P}_*$ the even dual Steenrod algebra.

Note that, at $p = 2$, the map $\mathcal{A}_* \to \mathcal{P}_*$ sending $\xi_i \mapsto \xi_i^2$ is an (ungraded) isomorphism of Hopf algebras. Thus, at $p = 2$, $\mathrm{Ext}_{\mathcal{P}_*}$ can be identified with $\mathrm{Ext}_{\mathcal{A}_*}$ after regrading. This is why $\mathcal{P}_*$ is often referred to as *doubled dual Steenrod algebra* at $p = 2$.

Proposition 3.46. *We can identify $\mathcal{P}_*$ with the image of BP_*BP in $\mathcal{A}_*$ under the natural map*

$$BP \otimes BP \to H\mathbb{F}_p \otimes H\mathbb{F}_p$$

Proof. The natural map $BP_*BP \to H\mathbb{F}_{p*}H\mathbb{F}_p = \mathcal{A}_*$ sends $t_i \mapsto \chi(\xi_i)$ for p odd, and $t_i \mapsto \chi(\xi_i^2)$ for $p = 2$. This is established, for example, in the proof of Lemma 3.7 of [Zah72].

Also, the map $BP_* \to H\mathbb{F}_{p*}$ sends $p \mapsto 0$, and $v_i \mapsto 0$ for degree reasons. Thus, the image is precisely the subalgebra of $\mathcal{A}_*$ generated by the $\chi(\xi_i)$ ($\chi(\xi_i^2)$ for $p = 2$). The result follows. $\square$

To treat the cases of $p = 2$ and odd p uniformly, we introduce the notation $\bar{t}_i = \chi(t_i)$. So the $\bar{t}_i$ form a system of generators of $\mathcal{P}_*$ which map to the $\xi_i \in \mathcal{A}_*$ for odd p, and to $\xi_i^2 \in \mathcal{A}_*$ for $p = 2$.

The quotient map $BP_*BP \to \mathcal{P}_*$ has been used extensively to compute Ext_{BP_*BP}. Since it factors through a sequence of quotient maps

$$BP_*BP/I_n \to BP_*BP/I_{n+1},$$

each of which is just a quotient by the invariant non-zero divisor v_n.

Each of these steps has an associated Bockstein spectral sequence. This can be used to inductively compute Ext_{BP_*BP} from $\mathrm{Ext}_{\mathcal{P}_*}$ in a range. In Section 4.4 of [Rav86], this is carried out for various primes.

In Section 4, we will use similar techniques to lift periodicity results from $\mathcal{P}_*$ to BP_*BP.

We now review the classical Milnor basis multiplication formula. This is essentially [Mil58], Theorem 4b.

Definition 3.47. *We denote the $\mathbb{F}_p$-dual of $\mathcal{P}_*$ by $\mathcal{P}^*$. It admits a basis dual to the monomial basis of $\mathcal{P}_*$, where, for a sequence $I = (i_1, i_2, \ldots)$ of nonnegative integers, almost all zero, we let $P^{(I)}$ denote the basis element dual to $\bar{t}^I := \prod \bar{t}_k^{i_k}$.*

Proposition 3.48. *The multiplication in $\mathcal{P}^*$ dual to the comultiplication on $\mathcal{P}_*$ is given by the following formula:*

$$P^{(I)}P^{(J)} = \sum_T c(T) \cdot P^{(K(T))}$$

where the sum ranges over all tables $T = (a_{ij})$ of nonnegative integers a_{ij} indexed over $i \geq 0$, $j \geq 0$, with

$$\begin{aligned} a_{00} &= 0 \\ \sum_{j\geq 0} p^j a_{ij} &= I_i \quad \text{for all} \quad i \geq 1 \\ \sum_{i\geq 0} a_{ij} &= J_j \quad \text{for all} \quad j \geq 1 \end{aligned}$$

For each of these tables, $K(T) = (k_1, k_2, \ldots)$ is given by

$$k_n = \sum_{i+j=n} a_{ij},$$

and the coefficient $c(T)$ is given by a product of multinomial coefficients

$$c(T) = \prod_{n\geq 1} \frac{k_n!}{\prod_{i+j=n} a_{ij}!}$$

Proof. For sequences I, J, K, the coefficient of $P^{(K)}$ in $P^{(I)}P^{(J)}$ agrees with the coefficient of $\bar{t}^I \otimes \bar{t}^J$ in $\Delta(\bar{t}^K)$.

We describe how to expand $\Delta\bar{t}^K$. $\bar{t}^K$ is a product of various $\bar{t}$, namely k_n copies of $\bar{t}_n$. So $\Delta\bar{t}^K$ is a product of k_n copies of $\Delta\bar{t}_n = \sum_{i+j=n} \bar{t}_i^{p^j} \otimes \bar{t}_j$.

Expanding this product leads to a sum, with each summand corresponding to a choice, selecting one summand from each of the individual factors.

To such a choice, we assign a table T, with a_{ij} given by the number of times we picked the summand $\bar{t}_i^{p^j} \otimes \bar{t}_j$ among the k_n factors of the form $\Delta\bar{t}_n$. Then clearly, $\sum_{i+j=n} a_{ij} = k_n$, and the summand in the expanded sum corresponding to this choice is of the form $\bar{t}^{I(T)} \otimes \bar{t}^{J(T)}$, with $I(T)_i = \sum_j p^j a_{ij}$ and $J(T)_j = \sum_i a_{ij}$.

Of course, the table T doesn't uniquely determine the choice. The number of choices leading to the same table T is

$$\prod_{n\geq 1} \frac{k_n!}{\prod_{i+j=n} a_{ij}!} = c(T)$$

as this is the number of ways to partition the k_n identical factors accordingly, for each n. In total, we see that

$$\Delta\bar{t}^K = \sum_{\substack{T \text{ with} \\ K(T)=K}} c(T)\bar{t}^{I(T)} \otimes \bar{t}^{J(T)}$$

The coefficient of an arbitrary $P^{(K)}$ in $P^{(I)}P^{(J)}$ corresponds, by duality, to the coefficient of $\bar{t}^I \otimes \bar{t}^J$ in $\Delta\bar{t}^K$. So, we obtain

$$P^{(I)}P^{(J)} = \sum_{\substack{T \text{ with} \\ I(T)=I \\ J(T)=J}} c(T)\bar{t}^{K(T)}.$$

□

Definition 3.49. *Let $P_k(n) \in \mathcal{P}^*$ denote the Milnor basis element dual to $\bar{t}_k^n$. Furthermore, let $P_k^s := P_k(p^s)$.*

Lemma 3.50. *Fix k. If $n, m < p^k$, then*

$$P_k(n)P_k(m) = \binom{n+m}{n} P_k(n+m).$$

Proof. Under the given conditions, there is only one table $T = (a_{ij})$ which contributes to the formula of Proposition 3.48.

Namely, all a_{ij} with $i \notin \{0, k\}$ or $j \notin \{0, k\}$ are zero. But then, since $a_{k,0} + p^k a_{k,k} = n < p^k$, we get $a_{k,k} = 0$ and $a_{k,0} = n$. From $a_{0,k} + a_{k,k} = m$ we then see $a_{0,k} = m$.

For this table T, $K(T)$ is the sequence with entry $n+m$ at position k, and zeros everywhere else. Thus, $P^{(K(T))} = P_k(n+m)$. The corresponding product of multinomial coefficients takes the form of a single binomial coefficient $\binom{n+m}{n}$, and the result follows. □

The following statement is helpful in computations with binomial coefficients modulo p. It is called Lucas' theorem, but it is a standard fact used a lot in algebraic topology (often unnamed).

Lemma 3.51. *If $n = \sum_{e\geq 0} n_e p^e$ and $k = \sum_{e\geq 0} k_e p^e$ are the base p expansions of nonnegative integers (i.e. $0 \leq n_e < p$ and $0 \leq k_e < p$ for all e), then*

$$\binom{n}{k} = \prod_{e\geq 0} \binom{n_e}{k_e} \quad \textit{mod } p.$$

Proof. See Lemma 3C.6 in [Hat02]. □

There is a straightforward generalization to multinomial coefficients, but we will not need it here.

Lemma 3.52. *The linear span of the $P_i(n)$ with $n < p^i$ forms a commutative subalgebra of $\mathcal{P}^*$, which is of the form*

$$\bigotimes_{0\leq j<i} \mathbb{F}_p[P_i^j]/(P_i^j)^p.$$

In particular, if $j < i$, the subalgebra of $\mathcal{P}^*$ generated by P_i^j is of the form $\mathbb{F}_p[P_i^j]/(P_i^j)^p$.

Proof. By Lemma 3.50, we have

$$P_i(n) \cdot P_i(m) = \binom{n+m}{n} P_i(n+m),$$

for $n, m < p^i$. In particular all these $P_i(n)$ for $n < p^i$ commute with each other. From Lucas' Theorem 3.51 and the fact that for $k, l < p$, $\binom{k+l}{k} = 0$ modulo p if and only if $k + l \geq p$, we see that $\binom{n+m}{n} \neq 0$ modulo p if and only if the sum $n + m$ is formed without carry in base p.

It follows that for $n = \sum_{e\geq 0} n_e p^e$ the base p expansion of a nonnegative integer n with $n < p^i$, the elements $P_i(n)$ and

$$\prod_{j\geq 0} (P_i^j)^{n_j}$$

agree up to a nonzero coefficient. Furthermore, since $\binom{p^{j+1}}{p^j} = 0$ modulo p, we have

$$P_i^j \cdot P_i((p-1)p^j) = 0,$$

and therefore $(P_i^j)^p = 0$ for all $j < i$. So the span of the $P_i(n)$ with $n < p^i$ forms a subalgebra generated by the P_i^j with $j < i$ subject to the condition $(P_i^j)^p = 0$. Since the $\prod_{j\geq 0}(P_i^j)^{n_j}$ are linearly independent, it is of the claimed form. □

Note that this fails if $j \geq i$. For example, using Lemma 3.50, $(P_i^i)^p$ can be seen to agree with $-P^{(I)}$ for I the sequence with $(p^{i+1} - p^i - 1)$ at position i, 1 at position $2i$, and 0 everywhere else. The P_i^j with $j < i$ will play a special role in sections 3.4 and 4.

We note another nice property of the P_i^j with $j < i$:

Lemma 3.53. *On the elements P_i^j with $j < i$, the antipode χ of the Hopf algebra $\mathcal{P}_*$ acts by*

$$\chi(P_i^j) = -P_i^j.$$

More generally, $\chi(P_i(n)) = (-1)^n P_i(n)$ for each $n < p^i$.

Proof. For an element $x \in \mathcal{P}^*$ with comultiplication given by $\Delta x = \sum_k x'_k \otimes x''_k$, one of the axioms of Hopf algebras states that

$$\sum_k \chi(x'_k)x''_k = 0.$$

Isolating the terms corresponding to $x \otimes 1$ and $1 \otimes x$ from that sum, we obtain the formula

$$\chi(x) = -x - \sum{}' \chi(x'_k)x''_k,$$

where $\sum'$ denotes that the sum is now only taken over the remaining summands.

Fix $l < i$, and, inductively, assume that we know $\chi(P_i^j) = -P_i^j$ for all $j < l$. We want to show $\chi(P_i^l) = -P_i^l$.

As in the proof of Lemma 3.52, we note that with $n = \sum_{e\geq 0} n_e p^e$ the base p expansion of a nonnegative integer n with $n < p^l$, the elements $P_i(n)$ and

$$\prod_{j\geq 0}(P_i^j)^{n_j}$$

agree up to a nonzero coefficient.

Since all these terms commute, we see that

$$\chi\left(\prod_{j\geq 0}(P_i^j)^{n_j}\right) = (-1)^{\sum n_j} \prod_{j\geq 0}(P_i^j)^{n_j},$$

and thus $\chi(P_i(n)) = (-1)^{\sum n_j} P_i(n)$. Since $(-1)^{n_j} = (-1)^{n_j p^j}$ modulo p, either because p^j is odd or because we work in characteristic 2, we can write this as $\chi(P_i(n)) = (-1)^n P_i(n)$.

For P_i^l, we now get

$$\begin{aligned}\chi(P_i(p^l)) &= -P_i(p^l) - \sum_{0<k<p^l} \chi(P_i(k)) \otimes P_i(p^l - k) \\ &= -P_i(p^l) - \sum_{0<k<p^l} (-1)^k P_i(k) \otimes P_i(p^l - k),\end{aligned}$$

and since we can see from Lucas' Theorem 3.51 that $\binom{p^l}{k} = 0$ modulo p for all $0 < k < p^l$ (essentially because computing the sum $k + (p^l - k)$ in base p will always lead to a carry), all summands on the right hand side except for $-P_i(p^l)$ vanish. □

It follows that if we consider the dual basis to the t^I instead of the $\bar{t}^I$, the basis element dual to $t_i^{p^j}$ for $j < i$ agrees with $-P_i^{p^j}$.

We end this section with a computation in $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$. It will be used in Section 4 to deduce bounds on the p-torsion exponent in certain regions of $\pi_{**}^{BP_*BP}$.

Proposition 3.54. *Let (A, Γ) be the Hopf algebroid BP_*BP. Then $\pi^\Gamma_{0,0}(S) = \mathbb{Z}_{(p)}$, and $\pi^\Gamma_{**}(S)$ is torsion in all other degrees.*

Proof. $\pi_{0,0}(S) = \mathbb{Z}_{(p)}$ is clear, because this is just the degree 0 part of the ground ring BP_*.

According to Lemma 3.29, $H(H(\mathbb{Z}_{(p)}); BP_*)$ is given by $BP_*BP \otimes_{BP_*} \mathbb{Z}_{(p)}$, i.e. $BP_*BP/(\eta_R(v_1), \eta_R(v_2), \ldots)$. The Postnikov section $S \to H(\mathbb{Z}_{(p)})$ induces the left BP_*-module map

$$BP_* \to BP_*BP \otimes_{BP_*} \mathbb{Z}_{(p)}$$

on homology. By part 5 of Lemma 3.44, this is an equivalence after rationalization, so this is true on homotopy as well. □

Lemma 3.55. *Let F be the fibre of the Postnikov section map $S \to H\mathbb{Z}_{(p)}$. For each l, there exists an $F_l \to F$, such that $F_l \to F$ has l-connected cofibre, F_l is built from finitely many cells of dimension (n, s) with $n \leq l + 1$ and $s \leq 1$, and F_l is rationally trivial.*

Proof. As $H\mathbb{Z}_{(p)}$ has homology $H_{**}(H\mathbb{Z}_{(p)}; BP_*)$ concentrated in degrees $(n, 0)$ with $n \geq 0$, there is a cell structure with cells in dimensions (n, s) with $n \geq 0$ and $s \leq 0$. This is because attaching a cell to kill generators in degree (n_0, s_0) can introduce new homology in degrees of the form $(n, s_0 - 1)$ with $n \geq n_0 + 1$. Furthermore, for each l, this only requires finitely many cells of the form (n, s) with $n \leq l$.

It follows that the fibre F admits a cell structure with cells in degrees (n, s) with $n \geq 0$ and $s \leq 1$.

From Proposition 3.54, we get that F has torsion homotopy. We let F^l be the l-skeleton of F, such that $F^l \to F$ induces an isomorphism on $\pi_{n,s}$ for $n < l$ and all s, and an epimorphism on $\pi_{l,s}$ for all s. Furthermore, again from Proposition 3.54, we see that the homotopy groups of F^l are torsion in all degrees except possibly those of the form $\pi_{l,s}$ with $s \leq 1$. We can attach cells along rational generators in the kernel of the map $\pi_{l,s}(F^l) \to \pi_{l,s}(F)$ for each s to obtain a new object F_l with map to F and the desired properties. □

3.4 Exotic K-theories

In section 4, we want to analyze vanishing regions and non-nilpotent self-maps for compact objects in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, for Γ a connected Hopf algebroid. We will detect these self-maps and vanishing lines using cohomology theories similar to the Morava K-theories in classical homotopy theory.

These will be coinduced from particularly small coalgebroids.

The following terminology follows [MW81]:

Definition 3.56. *A simple Hopf algebra of type D over $\mathbb{F}_p$ is a Hopf algebra of the form*

$$D(x) = \mathbb{F}_p[x]/x^p$$

with $\Delta(x) = x \otimes 1 + 1 \otimes x$ (and necessarily $|x|$ even if p is odd).

A simple Hopf algebra of type E over $\mathbb{F}_p$ is a Hopf algebra of the form

$$E(x) = \mathbb{F}_p[x]/x^2$$

with $\Delta(x) = x \otimes 1 + 1 \otimes x$ (and necessarily $|x|$ odd if p is odd).

Note that over $p = 2$, the types D and E agree.

Lemma 3.57. *For $|x| \neq 0$, any comodule over $D(x)$ is a direct sum of copies of the subcomodules $N_k := \mathbb{F}_p\{1, \dots, x^k\} \subseteq D(x)$, with $0 \leq k \leq p - 1$.*

Similarly, any comodule over $E(x)$ with $|x| \neq 0$ is a direct sum of copies of the trivial comodule $N_0 := \mathbb{F}_p$ and the cofree comodule $N_1 := E(x)$.

Proof. Let N be any $D(x)$-comodule, and let $D(x)^*$ be the $\mathbb{F}_p$-dual of $D(x)$. Then one easily sees that $D(x)^* = \mathbb{F}_p[y]/y^p$ with y the dual basis element to x with respect to the monomial basis, i.e. $D(x)^*$ is again a simple Hopf algebra of type D.

By definition, $\langle y, x \rangle = 1$, and inductively, one can determine the pairing on higher powers:

$$\langle y^k, x^k \rangle = \langle y \otimes y^{k-1}, \Delta x^k \rangle = k \langle y^{k-1}, x^{k-1} \rangle,$$

so

$$\langle y^k, x^k \rangle = k!$$

(This is closely related to the standard fact that the dual of a primitively generated polynomial algebra is a divided power algebra.)

Now the left coaction of $D(x)$ on N induces a right $D(x)^*$-action on N: for $\theta \in D(x)^*$ we define the action of θ on N by the composite

$$N \xrightarrow{\psi} D(x) \otimes N \xrightarrow{\theta^*} \mathbb{F}_p \otimes N \simeq N.$$

Explicitly, y^k sends $n \in N$ to $k!$ times the coefficient of x^k in $\psi(n)$.

Vice-versa, a right $D(x)^*$-module structure determines a left $D(x)$-comodule structure, through the formula

$$\psi(n) = \sum_{k=0}^{p-1} \frac{1}{k!} x^k \otimes (n \cdot y^k).$$

But $\mathbb{F}_p[y]/y^p$ is a graded principal ideal ring, so all modules are direct sums of modules of the form $\mathbb{F}_p[y]/y^k$. The claim follows.

For $E(x)$, the argument is completely analogous. □

Lemma 3.58. *For p odd, we have*

$$\mathrm{Ext}^{**}_{D(x)}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p[\alpha, \beta]/\alpha^2$$

with generators α and β in degrees $|\alpha| = (|x| - 1, 1)$ and $|\beta| = (p|x| - 2, 2)$ in Adams grading.

Furthermore, we have

$$\mathrm{Ext}^{**}_{D(x)}(N_k, \mathbb{F}_p) \simeq \mathbb{F}_p[\beta] \cdot 1 \oplus \mathbb{F}_p[\beta] \cdot \alpha_k$$

as $\mathbb{F}_p[\beta]$-modules, for all $1 \leq k \leq p-2$. Here α_k is an element in degree $|\alpha_k| = ((k+1)|x| - 1, 1)$. For $k < p-2$, α acts trivially, for $k = p-2$, $\alpha\alpha_{p-2} = \beta \cdot 1$.

$\mathrm{Ext}^{**}_{D(x)}(N_{p-1}, \mathbb{F}_p)$ *is $\mathbb{F}_p$ concentrated in degree $(0,0)$.*

Over $E(x)$, we have (for p now any prime):

$$\mathrm{Ext}^{**}_{E(x)}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p[\alpha]$$

with generator α in degree $|\alpha| = (|x| - 1, 1)$.

Proof. This is easily computed explicitly through minimal resolutions. Note that N_{p-1} is already a cofree $D(x)$-comodule, for the others, one obtains periodic resolutions. □

Definition 3.59. *For Λ a simple Hopf algebra of type D, we denote by β_Λ the generator β in*

$$\mathrm{Ext}^{**}_{D(x)}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p[\alpha, \beta]/\alpha^2.$$

For Λ a simple Hopf algebra of type E, we denote by β_Λ the element α^2 in

$$\mathrm{Ext}^{**}_{E(x)}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p[\alpha]$$

This notation is chosen in such a way as to make a more uniform treatment of the cases $\Lambda = D(x)$ and $\Lambda = E(x)$ possible. For example, both $\mathrm{Ext}_{D(x)}(\mathbb{F}_p, \mathbb{F}_p)$ and $\mathrm{Ext}_{E(x)}(\mathbb{F}_p, \mathbb{F}_p)$ are of the form $\mathbb{F}_p[\alpha, \beta]/(\alpha^2 - \varepsilon\beta)$, with $\varepsilon = 0$ or 1.

Also, for both $\Lambda = D(x)$ and $E(x)$, β_Λ is the essentially unique non-nilpotent self-map of the monoidal unit S in $\mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}$, which will play a central role in Section 4.

Remark 3.60. Note that β_Λ has only been characterized up to a unit multiple through Lemma 3.58. It is easy to fix a specific representative (depending only on the choice of generator of $D(x)$), for example by writing down an explicit cobar complex cocycle. However, all our statements remain equally valid if one replaces β_Λ by a unit multiple.

Observe now that the statements made above about the structure of comodules and Ext-groups over $D(x)$ and $E(x)$ all only rely on the coalgebra structure of $D(x)$ and $E(x)$. This motivates the following definition:

Definition 3.61. *Let Γ be a graded coalgebroid over A. A simple coalgebra quotient of Γ of type D or E is given by a surjective map $A \to \mathbb{F}_p$ together with a compatible surjective coalgebroid map $\Gamma \to D(x)$ or $\Gamma \to E(x)$.*

Example 3.62. Consider $\mathcal{P}_* = \mathbb{F}_p[\bar{t}_1, \bar{t}_2, \ldots]$ the even dual Steenrod algebra considered in Section 3.3, with $\bar{t}_i \in \mathcal{A}_*$ the element ξ_i for odd p, and ξ_i^2 at $p = 2$. We saw in Lemma 3.52 that there are subalgebras $\mathbb{F}_p[P_i^j]/(P_i^j)^p$ for $j < i$. Dually, we get that the maps

$$\mathcal{P}_* \to \mathbb{F}_p\{1, \bar{t}_i^{p^j}, \bar{t}_i^{2p^j}, \ldots, \bar{t}_i^{(p-1)p^j}\} =: \Lambda_{ij},$$

given by sending all other monomials to 0, are simple coalgebra quotient maps if $j < i$.

As a result, the composites

$$BP_*BP \to \mathcal{P}_* \to \Lambda_{ij}$$

define simple coalgebra quotient maps if $j < i$.

Given a simple coalgebra quotient $\Gamma \to \Lambda$, we obtain an object $H\Lambda \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ by coinduction. The homotopy groups of $H\Lambda$ agree with $\pi^{\Lambda}_{**}(S) = \mathrm{Ext}_{\Lambda}(\mathbb{F}_p, \mathbb{F}_p)$, which was described in Lemma 3.58.

Lemma 3.63. *For $\Gamma \xrightarrow{f} \Lambda$ a simple coalgebra quotient, $H\Lambda$ has an action by $\mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}}(S,S)$, which exhibits $\pi_{**}(H\Lambda)$ as a free module of rank 1 over $\pi^{\Lambda}_{**}(S) = \mathrm{Ext}_{\Lambda}(\mathbb{F}_p, \mathbb{F}_p)$.*

Proof. Since the coinduction $f^* : \mathrm{Comod}^{cg}_{\mathcal{D}\Lambda} \to \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a functor, it gives rise to a map

$$\mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}}(S,S) \to \mathrm{map}_{\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}}(H\Lambda, H\Lambda),$$

compatible with the ring structure obtained on both spectra through composition.

Given an element $a \in \pi^{\Lambda}_{**}(S)$, it acts on $x \in \pi^{\Gamma}_{**}(H\Lambda)$ through the composition

$$\begin{array}{ccc} S & \xrightarrow{x} & H\Lambda \\ & {\scriptstyle a\cdot x}\searrow & \downarrow{\scriptstyle f^*(a)} \\ & & H\Lambda \end{array}$$

which is adjoint to the composition

$$\begin{array}{ccc} S & \xrightarrow{x_\flat} & S \\ & {\scriptstyle (a\cdot x)_\flat}\searrow & \downarrow{\scriptstyle a} \\ & & S \end{array}$$

in $\mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}$, where $(-)_\flat$ is the adjunction isomorphism

$$(-)_\flat : \mathrm{map}_{\mathcal{D}\Gamma}(S, H\Lambda) \xrightarrow{\simeq} \mathrm{map}_{\mathcal{D}\Lambda}(S,S).$$

Under this isomorphism, the action of $\pi^{\Lambda}_{**}(S)$ on $\pi^{\Gamma}_{**}H\Lambda$ defined above therefore corresponds to the action of $\pi^{\Lambda}_{**}(S)$ on itself through composition. This is obviously free. □

Note that although this exhibits $\pi_{**}(H\Lambda)$ as a free module of rank 1 over the ring $\pi^{\Lambda}_{**}(S)$, there is in general no ring structure on the object $H\Lambda$, cf. Remark 3.69.

We can use the action of $\pi^{\Lambda}_{**}(S)$ on $H\Lambda$ to obtain a map $H\Lambda \to H\Lambda$ through the action of β_Λ.

Definition 3.64. *We set $K\Lambda = \beta_\Lambda^{-1} H\Lambda$, where β_Λ acts as described in Lemma 3.63.*

Lemma 3.65. *If p is odd and Λ of type D, $K\Lambda$ has homotopy groups*

$$\pi_{**}(K\Lambda) = \mathbb{F}_p[\alpha, \beta^{\pm 1}]/\alpha^2$$

as a module over $\pi^{\Lambda}_{**}(S)$.

If Λ is of type E, $K\Lambda$ has homotopy groups

$$\pi_{**}(K\Lambda) = \mathbb{F}_p[\alpha^{\pm 1}].$$

If Λ is of type E, $K\Lambda$ looks similar to the classical Morava K-theories, with homotopy groups of the form $\mathbb{F}_p[v^{\pm 1}]$. If Λ is of type D, we still have the exterior α around, but it is not possible to obtain a comodule over Λ whose homotopy is just $\mathbb{F}_p[\beta^{\pm 1}]$, because β can be expressed as a p-fold Toda bracket of copies of α. So, in a sense, the homotopy groups of $K\Lambda$ are minimal such that they contain β and an inverse β^{-1}.

We will thus call these objects, or the homology theories represented by them, exotic K-theories. "Exotic", because in the category where our main results live, $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, there are also algebraic analogues of the classical Morava K-theories, with homotopy groups $\mathbb{F}_p[v_n^{\pm 1}]$. Contrary to our exotic K-theories, the elements v_n here come out of the underlying ring BP_*, so the construction is quite different.

Furthermore, our exotic K-theories differ from the classical ones by two important factors. In the case of Λ a simple coalgebra of type D over odd p, the homotopy of $K\Lambda$ can be slightly more complicated than for the classical Morava K-theories.

More importantly, the $K\Lambda$ are rarely rings. Typically, they only admit a ring structure if the map $\Gamma \to \Lambda$ is compatible with the given multiplicative structure on Γ, and the multiplicative structure on Λ coming from the description as a Hopf algebra $\mathbb{F}_p[x]/x^p$ or $\mathbb{F}_p[x]/x^2$. We will discuss this in Remark 3.69.

Proposition 3.66. *Let Γ be a coalgebroid over $\mathbb{F}_p$, and let $\Lambda = D(x)$ be a simple coalgebra quotient. The object $H\Lambda \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ admits a periodic Postnikov decomposition, with k-invariants between successive stages given alternatingly by*

$$H\mathbb{F}_p \xrightarrow{y} \Sigma^{(|x|,0)} H\mathbb{F}_p$$
$$H\mathbb{F}_p \xrightarrow{y^{p-1}} \Sigma^{(p-1)|x|,0)} H\mathbb{F}_p$$

where $y \in \Lambda^*$ is the dual basis element to x, and the maps are given through the algebra map $\Lambda^* \to \Gamma^*$ and Lemma 3.35.

Similarly, for $\Lambda = E(x)$ a simple coalgebra quotient, there is a periodic Postnikov decomposition for $H\Lambda$ with all k-invariants between successive stages given by $H\mathbb{F}_p \xrightarrow{y} \Sigma^{(|x|,0)} H\mathbb{F}_p$.

Proof. It is sufficient to see that such a Postnikov decomposition exists for $S \in \mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}$, since we can then coinduce it up to $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$.

This in turn arises explicitly from the minimal cofree resolution

$$\Lambda \xrightarrow{y} \Lambda \xrightarrow{y^{p-1}} \Lambda \xrightarrow{y} \cdots$$

of $\mathbb{F}_p$ as a Λ-comodule. □

Definition 3.67. *For M a module (not comodule) over an algebra of the form $D(y) = \mathbb{F}_p[y]/y^p$, we define the Margolis homology groups*

$$H_*(M; y^i) := \ker y^i / \operatorname{im} y^{p-i}$$

for each $0 < i < p$. Similarly, for M a module over an algebra of the form $E(y)$, we define the Margolis homology groups

$$H_*(M; y) := \ker y / \operatorname{im} y.$$

Lemma 3.68. *Assume Γ is a Hopf algebra over $\mathbb{F}_p$, $\Lambda = D(x)$ is simple coalgebra quotient of type D, and $X \in \mathrm{Comod}_{\mathcal{D}\Gamma}$ an object represented by an explicit comodule M. Denote the image of the dual generator $y \in \Lambda^*$ in Γ^* by θ. Then there are isomorphisms*

1. $$H\Lambda_{n,s}X \simeq \begin{cases} \ker(\chi(\theta))_n \text{ if } s = 0 \\ H_{n-\frac{s}{2}(p|\theta|-2)}(M; \chi(\theta)) \text{ if } s > 0 \text{ even} \\ H_{n-\frac{s-1}{2}(p|\theta|-2)+(|\theta|-1)}(M; \chi(\theta)^{p-1}) \text{ if } s > 0 \text{ odd} \end{cases}$$

2. $$H\Lambda^{n,s}X \simeq \begin{cases} \ker(\theta)_n \text{ if } s = 0 \\ H_{n-\frac{s}{2}(p|\theta|-2)}(M^*; \theta) \text{ if } s > 0 \text{ even} \\ H_{n-\frac{s-1}{2}(p|\theta|-2)+(|\theta|-1)}(M^*; \theta^{p-1}) \text{ if } s > 0 \text{ odd} \end{cases}$$

For $\Lambda = E(x)$ a simple coalgebra quotient of type E, we similarly have

1. $$H\Lambda_{n,s}X \simeq \begin{cases} \ker(\chi(\theta))_n \text{ if } s = 0 \\ H_{n-s(|\theta|-1)}(M; \chi(\theta)) \text{ if } s > 0 \end{cases}$$

2. $$H\Lambda^{n,s}X \simeq \begin{cases} \ker(\theta)_n \text{ if } s = 0 \\ H_{n-s(|\theta|-1)}(M^*; \theta) \text{ if } s > 0 \end{cases}$$

where θ acts on M^* and $\chi(\theta)$ acts on M as in Lemma 3.36.

Proof. We discuss the case $\Lambda = D(x)$, the case of type E follows similarly.

The Postnikov filtration on $H\Lambda$ gives rise to an Atiyah-Hirzebruch spectral sequence converging to $H\Lambda_{**}X$ or $H\Lambda^{**}X$. As $\pi_{n,s}(H\Lambda)$ is concentrated in degrees with $s \geq 0$ and either s even and $n = \frac{s}{2}(p|x|-2)$, or s odd and $n = \frac{s-1}{2}(p|x|-2) + (|x|-1)$, and is given by a single $\mathbb{F}_p$ in each of these degrees, the E_2-page consists of an appropriately shifted copy of $H_{**}(X;\mathbb{F}_p) \simeq M$ or $H^{**}(X;\mathbb{F}_p) \simeq M^*$ for each $s \geq 0$.

For degree reasons, this spectral sequence collapses after the d_2-differentials, and there is no room for nontrivial extensions. (Both of these statements would fail if $H_{**}(X;\mathbb{F}_p)$ wasn't concentrated in degrees of the form $(n,0)$.)

The d_2-differential is induced by the k-invariants of the Postnikov tower. From Proposition 3.66 and Lemma 3.36, these act by θ and θ^{p-1} alternatingly in the cohomology case, and by $\chi(\theta)$ and $\chi(\theta)^{p-1}$ alternatingly in the homology case. The result follows. □

We obtain corresponding descriptions (without the $s \geq 0$ restriction) for $K\Lambda_{**}X$ and $K\Lambda^{**}X$. In particular, for X represented by a comodule M, $K\Lambda^{**}X = 0$ if and only if the corresponding Margolis homologies vanish. Thus, one can see the $K\Lambda$ as a derived variant of Margolis homology groups, and many of the results discussed in Section 4 are derived analogues of classical results involving vanishing of Margolis homology groups, see [AM71], [MP72].

Remark 3.69. Let Γ be a Hopf algebroid and $\Gamma \to \Lambda$ a simple coalgebroid quotient of type D or E.

If $\Gamma \to \Lambda$ is multiplicative (with respect to the product structure discussed on Λ as part of the Hopf algebra structure), $K\Lambda$ and $H\Lambda$ are rings.

Typically, the converse holds, too: Observe that for Γ a Hopf algebra over $\mathbb{F}_p$, $\Gamma \to \Lambda$ is multiplicative if and only if the dual $\Lambda^* \to \Gamma^*$ preserves the coalgebra structure, which is the case if and only if the image $\theta \in \Gamma^*$ of the generator of Λ^* is primitive in Γ^*. If θ is not primitive, it is usually possible to find a compact comodule such that M has a trivial action by θ, but some tensor power $M^{\otimes k}$ has nontrivial action by θ.

Now assume we are given such a comodule. Then $H\Lambda_{**}M$ is a free $H\Lambda_{**}$-module of rank equal to the $\mathbb{F}_p$-rank of M by Lemma 3.68.

If $H\Lambda$ were a ring (note that if $K\Lambda$ is a ring, $H\Lambda$ inherits a ring structure since it is the connective cover $\tau_{\geq 0}K\Lambda$), we would obtain a Künneth theorem

$$H\Lambda_{**}X \otimes_{H\Lambda_{**}} H\Lambda_{**}Y \simeq H\Lambda_{**}(X \otimes Y)$$

whenever $H\Lambda_{**}X$ or $H\Lambda_{**}Y$ are free, as in classical stable homotopy theory. In particular, we get

$$H\Lambda_{**}(M^{\otimes k}) \simeq (H\Lambda_{**}M)^{(\otimes_{H\Lambda_{**}})k}$$

However, since there is some power $M^{\otimes k}$ with nontrivial action by θ, we get a contradiction by looking at the ranks.

The following statement is analogous to the classical observation that any compact spectrum X has $K(n)_*(X) \neq 0$ for large enough n.

Proposition 3.70. *If Γ is a connected coalgebroid over A, and A is local with residue field $\mathbb{F}_p$, assume $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is compact. Then there is N such that the following holds:*

For each simple coalgebra quotient $\Gamma \to \Lambda$ of the form $\Lambda = D(x)$ or $E(x)$ with $|x| > N$, $K\Lambda_{**}Y \neq 0$.

Proof. Since Y is compact, it has finite nonzero $\mathbb{F}_p$-homology, i.e. there exist $n_- < n_+$ such that the $\mathbb{F}_p$-homology of Y vanishes in degrees $(n, *)$ for $n \leq n_-$ and $n \geq n_+$. Now let $N = n_+ - n_-$.

Then, filtering $H\Lambda$ by its Postnikov tower, we get a spectral sequence describing $H\Lambda_{**}Y$ by shifted copies of $(H\mathbb{F}_p)_{**}Y$, and if $|x| > N$, the individual filtrations can't interact for degree reasons. So β acts injectively on $H\Lambda_{**}Y$, and $K\Lambda_{**} \neq 0$. □

We now state a few observations about thick subcategories. These will be required to prove that exotic K-theories detect the slope of minimal vanishing lines (see Proposition 4.22).

A thick subcategory of $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a full subcategory closed under cofibres, retracts and arbitrary shifts $\Sigma^{n,s}$. Note that this means that in addition to the classical definition of thick subcategory in a stable $(\infty, 1)$-category (or triangulated category), we also require to be closed under the $\mathbb{Z}$-action through the $\Sigma^{0,s}$.

Lemma 3.71. *Let Λ be of the form $D(x)$ or $E(x)$ with $|x| \neq 0$. Consider β as a morphism in $\mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}$,*

$$S^{|\beta|} \xrightarrow{\beta} S,$$

and let S/β denote its cofibre.

Then S/β and $H\mathbb{F}_p$ generate the same thick subcategory of $\mathrm{Comod}^{cg}_{\mathcal{D}\Lambda}$.

Proof. The homotopy of S/β is given by $\mathbb{F}_p[\alpha]/\alpha^2$, as seen from the long exact sequence associated to the defining cofibre sequence, and the description of

$$\pi_{**}(S) = \mathrm{Ext}_\Lambda(\mathbb{F}_p, \mathbb{F}_p).$$

in Lemma 3.58.

It follows that the Postnikov section $S/\beta \to H\mathbb{F}_p$ has fibre $\Sigma^{|x|-1,1}H\mathbb{F}_p$. So S/β is contained in the thick subcategory generated by $H\mathbb{F}_p$.

For the other direction, observe that $H\mathbb{F}_p$ admits a finite cell structure, i.e. a filtration by finitely many copies of S. This is obtained explicitly from the degree filtration on the representing comodule Λ, or from Lemma 3.26.

So $H\mathbb{F}_p/\beta$ admits a filtration by finitely many copies of S/β, i.e. is contained in the thick subcategory generated by S/β. Since $H\mathbb{F}_p/\beta = H\mathbb{F}_p \oplus \Sigma^{|\beta|+(1,-1)}H\mathbb{F}_p$, $H\mathbb{F}_p$ is a retract of $H\mathbb{F}_p/\beta$ and the result follows. □

Now consider a Hopf algebra $\Gamma = \mathbb{F}_p[x]/x^{p^k}$ with primitive generator x. Letting $y_i \in \Gamma^*$ denote the dual basis element to x^{p^i}, the underlying algebra structure of Γ^* is easily identified to be $\bigotimes_{0\leq i\leq k-1} \mathbb{F}_p[y_i]/y_i^p$. This is related to the fact that the dual of a primitively generated polynomial algebra is a divided power algebra.

Dually, there is a coalgebra isomorphism (incompatible with the ring structure)

$$\Gamma \simeq \bigotimes_{0\leq i\leq k-1} D(x_i) \tag{3}$$

where x_i corresponds to $x^{p^i} \in \Gamma$.

We obtain an isomorphism

$$\begin{aligned}\mathrm{Ext}_\Gamma(\mathbb{F}_p,\mathbb{F}_p) &\simeq \bigotimes_{0\leq i\leq k-1} \mathrm{Ext}_{D(x_i)}(\mathbb{F}_p,\mathbb{F}_p)\\ &\simeq \mathbb{F}_p[\alpha_0,\ldots,\alpha_{k-1},\beta_0,\ldots,\beta_{k-1}]/(\alpha_1^2,\ldots,\alpha_{k-1}^2),\end{aligned}$$

which, even though the isomorphism (3) is not multiplicative, is still compatible with the ring structure, because the ring structure on both sides can be obtained from the composition product on Ext. Note however that it will not necessarily be compatible with higher product structure, i.e. Steenrod power operations.

Proposition 3.72. *Let Γ be the Hopf algebra $\mathbb{F}_p[x]/x^{p^k}$ with x primitive. It admits an elementary coalgebra quotient of type D, by*

$$\mathbb{F}_p[x]/x^{p^k} \to \mathbb{F}_p\{1, x^{p^{k-1}}, x^{2p^{k-1}}, \ldots, x^{(p-1)p^{k-1}}\} =: \Lambda_{k-1}$$

We have $\mathrm{Ext}_\Gamma(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p[\alpha_0,\ldots,\alpha_{k-1},\beta_0,\ldots,\beta_{k-1}]/(\alpha_1^2,\ldots,\alpha_{k-1}^2)$.

Furthermore, $H\Lambda_{k-1}$ and the iterated cofibre $S/(\beta_1,\ldots,\beta_{k-2})$ generate the same thick subcategory.

Proof. According to Lemma 3.29, $H\Lambda_{k-1}$ is represented by the comodule $\Gamma\square_{\Lambda_{k-1}}\mathbb{F}_p$. Under the equivalence (3), this corresponds to the sub-comodule of the form $\bigotimes_{0\leq i\leq k-2} D(x_i)$.

Generally, given a family of objects $K_i \in \mathrm{Comod}^{cg}_{\mathcal{D}(D(x_i))}$, the equivalence (3) gives us an object in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ as an external tensor product

$$\boxtimes_{0\leq i\leq k-1} K_i.$$

So with respect to this notation, $H\Lambda_{k-1}$ corresponds to the external tensor product of the family with $K_{k-1} = S$ and $K_i = H\mathbb{F}_p$ for all $i < k-1$. Similarly, $S/(\beta_0, \ldots, \beta_{k-2})$ corresponds to the external tensor product of the family with $K_{k-1} = S$ and $K_i = S/\beta_i$ for all $i < k-1$.

Over the coalgebra $D(x_i)$, Lemma 3.71 implies that there is an object in $\mathrm{Comod}^{cg}_{\mathcal{D}(D(x_i))}$ which is filtered by finitely many shifted copies of S/β_i, and admits $H\mathbb{F}_p \in \mathrm{Comod}^{cg}_{\mathcal{D}(D(x_i))}$ as a retract. Taking the external product of those for all $i < k-1$ (and $S \in \mathrm{Comod}^{cg}_{\mathcal{D}(D(x_{k-1}))}$)), we obtain an object in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ which is filtered by finitely many shifted copies of $S/(\beta_0, \ldots, \beta_{k-2})$ and admits $H\Lambda_{k-1}$ as a retract.

Similarly, there is an object in $\mathrm{Comod}^{cg}_{\mathcal{D}(D(x_i))}$ which is filtered by shifted copies of $H\mathbb{F}_p$ and admits S/β_i as a retract, so by taking the corresponding external tensor product, we obtain an object in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ which is filtered by finitely many shifted copies of $H\Lambda_{k-1}$ and has $S/(\beta_0, \ldots, \beta_{k-2})$ as a retract.

In total, we have that $S/(\beta_0, \ldots, \beta_{k-2})$ and $H\Lambda_{k-1}$ generate the same thick subcategory of $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$. $\square$

4 Vanishing lines and self-maps

A result of Palmieri (see [Pal92], [Pal01]) shows that, for $\mathcal{A}_*$ the dual Steenrod algebra and M a comodule finitely generated as an $\mathbb{F}_2$-vector space, $\mathrm{Ext}_{\mathcal{A}_*}(M,M)$ vanishes above a line of one of a specific set of slopes, and furthermore, there is a non-nilpotent element in $\mathrm{Ext}_{\mathcal{A}_*}(M,M)$ of precisely that slope.

In this section, we want to prove analogues of these statements over the Hopf algebroid BP_*BP.

We start reviewing Palmieri's results. Instead of the dual Steenrod algebra $\mathcal{A}_*$, we will work over the even dual Steenrod algebra $\mathcal{P}_*$ (see Definition 3.45), but the techniques we employ easily apply to both. Since we work in $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, while Palmieri works in Hovey's Stable(Γ), and Remark 2.68 only conjecturally identifies the two, we give a self-contained account of Palmieri's results here. This also makes it easier to follow the ideas that go into lifting the results to BP_*BP.

Definition 4.1. *For $X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$, we will say that X has a* vanishing line of slope d and intercept c *if $\pi_{n,s}(X) = 0$ whenever*

$$s > dn + c$$

We will also sometimes say that X has a vanishing line of slope d through (a, b) *if $\pi_{n,s}(X) = 0$ whenever*

$$(s - b) > d(n - a)$$

i.e. if it has a vanishing line of slope d and intercept $b - da$.

Remark 4.2. The usual convention in the literature seems to be that in this algebraic context, vanishing lines are usually described in the (s,t)-grading, and the notion of slope is with axes reversed. For example, in [MW81] or [Pal01], a vanishing line of slope d refers to vanishing whenever

$$t < ds + c.$$

In our language, this would be a vanishing line of slope $\frac{1}{d-1}$.

This leads to awkward fractions for the slopes and intercepts, but is chosen in such a way as to correspond directly to the usual Adams spectral sequence charts. (For example, most topologists would say the Adams periodicity statement in the classical Adams spectral sequence is about a line of slope $\frac{1}{2}$, not 3.)

In the situation of Definition 4.1, we will also say that $\pi_{**}(X)$ vanishes in the region above the line of slope d and intercept c, or that π_{**} is concentrated in the region below that line.

We will now work over a Hopf algebra Γ over $\mathbb{F}_p$. The strategy to obtain control over vanishing lines and self-maps will be to write Γ as a sequence of normal extensions by simple Hopf algebras of type D or E.

Lemma 4.3. *Let Γ be a connected Hopf algebra over $\mathbb{F}_p$, and let $\Gamma \to \Sigma$ be a normal Hopf algebra quotient map with $\Phi = \Gamma \square_\Sigma \mathbb{F}_p$ finite.*

Then there is a composition series interpolating between the two, i.e. a sequence of normal quotients

$$\Gamma = \Gamma_n \to \Gamma_{n-1} \to \ldots \to \Gamma_0 = \Sigma$$

with each map a normal extension by a simple Hopf algebra Λ of type D or E.

Proof. It is sufficient to prove that, if $\Gamma \neq \Sigma$, $\Gamma \to \Sigma$ factors through a quotient $\Gamma \to \Gamma'$ with $\Gamma \square_{\Gamma'} \mathbb{F}_p$ a simple Hopf algebra of type D or E.

To see this, let $x \in \Phi$ be a nonzero primitive element. These exist due to connectedness, as all elements of minimal positive degree are primitive for degree reasons.

If p is odd and x has odd degree, then it squares to 0. So $\Gamma \to \Gamma/x =: \Gamma'$ is a normal Hopf algebra quotient map, with $\Gamma \square_{\Gamma/x} \mathbb{F}_p = E(x)$. Since $x \in \Phi$, the map $\Gamma \to \Sigma$ factors through Γ/x.

Otherwise, by replacing x with a suitable power x^{p^i} (which is still primitive since we are in characteristic p), we can assume that $x^p = 0$, but x itself is nonzero. Then, since

$$\Delta(x^k) = x^k \otimes 1 + 1 \otimes x^k + \sum_{i=1}^{k-1} \binom{k}{i} x^i \otimes x^{k-i}$$

for all k, all the x^k with $k < p$ are inductively seen to be nonzero. So x generates a sub Hopf algebra of the form $D(x)$ in Γ.

It follows that $\Gamma \to \Gamma/x =: \Gamma'$ is a normal Hopf algebra quotient map, with $\Gamma \square_{\Gamma/x} \mathbb{F}_p = D(x)$. Again, $\Gamma \to \Sigma$ factors through Γ/x.

As the dimension of $\Gamma' \square_\Sigma \mathbb{F}_p$ is smaller than the dimension of $\Gamma \square_\Sigma \mathbb{F}_p$, iterating this process eventually terminates, yielding a finite sequence of normal extensions as claimed. □

Associated to such extensions, we have Adams spectral sequences that behave essentially like "β-Bockstein spectral sequences". By this we mean that all pages roughly look like a collection of (possibly truncated) towers originating in low filtration, and differentials precisely connect non-truncated towers:

Lemma 4.4. *Let $\Gamma \to \Sigma$ be a normal extension of connected Hopf algebras by a simple Hopf algebra Λ of type D or E. We consider the associated Adams spectral sequence (cf. Lemma 3.42), whose E_2 page and abutment take the form*

$$\mathrm{Ext}_\Lambda(\mathbb{F}_p, \pi^\Sigma_{**}(X)) \Rightarrow \pi^\Gamma_{**}(X).$$

It has the following properties:

1. *There is an action by $\mathrm{Ext}_\Lambda(\mathbb{F}_p, \mathbb{F}_p)$, commuting with differentials.*

2. *For all r, the E_r page is generated as an $\mathbb{F}_p[\beta]$-module by elements in filtration 0 and 1, and β acts isomorphically on filtrations $(\geq r-1)$.*

3. *The E_∞-page is generated as an $\mathbb{F}_p[\beta]$-module by elements in filtration 0 and 1. The β^k-torsion part of the E_∞ page agrees with the β^k-torsion part of the E_{2k+2}-page.*

Proof. The first statement follows from the multiplicative structure of the Adams spectral sequence.

We prove the second statement by induction. On the E_2-page, it follows from lemmas 3.57 and 3.58.

Assume the statement is true on the E_r page, then all elements are of the form $\beta^k x$, with x in filtration 0 or 1. As $d_r(\beta^k x) = \beta^k d_r(x)$, and $d_r(x)$ is in filtration $(\geq r)$, where β acts already isomorphically, we see that $\beta^k x$ is in the kernel of d_r if and only if x is. Thus, the kernel of d_r, and therefore the next page E_{r+1}, are generated as $\mathbb{F}_p[\beta]$-module by elements in filtration 0 or 1.

As the kernel has β acting periodically in degrees $(\geq r-1)$, and the image of d_r is freely generated as an $\mathbb{F}_p[\beta]$-module by elements in filtration r and $r+1$, the E_{r+1}-page has β acting periodically in degrees $(\geq r)$. This completes the inductive step.

For the third statement, observe that if a permanent cycle is represented by $\beta^k x$ on some page, x itself is a permanent cycle, because on the E_r-page, d_r takes values in the β-periodic part of E_r for filtration reasons. So the E_∞ page is generated as an $\mathbb{F}_p[\beta]$-module by elements in filtration 0 and 1.

Note that, inductively, the β-torsion on the E_r page is of the form $\mathbb{F}_p[\beta]/\beta^k$ for $k \leq \frac{r}{2}$, on generators in filtration 0 and 1. From the above observations, torsion elements don't support nontrivial differentials, and can't be hit. So the torsion submodule of E_r embeds into E_∞. Furthermore, since all differentials are determined by their values on filtration 0 and 1 elements, and so the image of d_r is generated by filtration r and $r+1$-elements, the d_r differential introduces precisely additional β^k-torsion with $k = \frac{r}{2}$ is even, and $k = \frac{r\pm 1}{2}$-torsion if r is odd. So all the β^k-torsion has to have appeared on the E_{2k+2}-page already. □

For finite Hopf algebras, we can use the existence of composition series as in Lemma 4.3 together with the Adams spectral sequence described in Lemma 4.4 to inductively obtain information about vanishing lines and self-maps. This will be the subject of sections 4.1 and 4.2.

4.1 Minimal vanishing lines

In this section, we will show that, over a finite connected Hopf algebra Γ over $\mathbb{F}_p$, every compact object has a minimal vanishing line, and the possible slopes of minimal vanishing lines are a subset of the slopes of classes β corresponding to the various simple Hopf algebras appearing in a composition series of Γ.

Definition 4.5. *We order vanishing lines lexicographically by slope, then intercept. In other words, the vanishing line of slope d_1 and intercept c_1 is smaller than the one with slope d_2 and intercept c_1 if $d_1 < d_2$, or $d_1 = d_2$ and $c_1 < c_2$.*

We will say Y has a minimal vanishing line of slope d and intercept c *if it has no smaller one.*

Remark 4.6. Not every Y has a minimal vanishing line. For example, if Y has a nonlinear vanishing curve $s(n)$ of slope tending to 0 as n goes to ∞, Y is bounded by vanishing lines of arbitrarily low positive slope. For example, take

$$Y = \bigvee \Sigma^{i^2,i} H\mathbb{F}_p.$$

However, we will see that for compact Y, there are minimal vanishing lines.

We first note that the existence of minimal vanishing lines essentially boils down to the existence of minimal *slopes* of vanishing lines:

Lemma 4.7. *Assume Y has a vanishing line of slope d, and does not admit a vanishing line of slope d' for any $d' < d$. Then it has a minimal vanishing line of slope d.*

Proof. Let

$$c_0 = \inf\{c \mid \pi^{\Gamma}_{**}Y \text{ admits a vanishing line of slope } d \text{ and intercept } c\}.$$

Then the line of slope d and intercept c_0 is a minimal vanishing line for Y. □

Lemma 4.8. *For $\Gamma \xrightarrow{f} \Sigma$ a quotient map of connected Hopf algebras over $\mathbb{F}_p$, with if $\pi^{\Gamma}_{**}(Y)$ has a vanishing line, $\pi^{\Sigma}_{**}(f_*Y)$ has the same vanishing line.*

Proof. For example from the comodule version of Theorem 4.4 in [MM65], one sees that a quotient map of connected Hopf algebras $\Gamma \to \Sigma$ exhibits Γ as Σ-cofree. So we can apply Lemma 3.29 to describe $H_{**}(H\Sigma; \mathbb{F}_p)$ as $\Gamma \square_{\Sigma} \mathbb{F}_p$.

By Lemma 3.26, we see that $H\Sigma$ admits a cell structure with cells in degree $(n, 0)$ and $n \geq 0$.

So $H\Sigma \otimes Y$ is filtered by copies of $\Sigma^{n,0}Y$ for $n \geq 0$. All of these are bounded by the vanishing line of Y.

Since $\pi^{\Sigma}_{**}(Y) \simeq \pi^{\Gamma}_{**}(H\Sigma \otimes Y)$, the result follows. □

Lemma 4.9. *Assume $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a compact object, and $f : \Sigma^{|f|}Y \to Y$ a self-map.*

Then the following two are equivalent:

1. *The map f acts nilpotently on $\pi_{**}(Y)$, meaning that for each $x \in \pi_{**}(Y)$, there exists a k such that $f^k x = 0$.*

2. *The map f acts uniformly nilpotently on $\pi_{**}(Y)$, meaning that there exists k such that $f^k x = 0$ for all $x \in \pi_{**}(Y)$.*

Proof. If f acts nilpotently on all $x \in \pi^{\Gamma}_{**}(Y)$, the colimit

$$f^{-1}Y = \operatorname{colim}\left(Y \xrightarrow{f} \Sigma^{-|f|}Y \xrightarrow{f} \dots\right)$$

has trivial homotopy groups, thus is zero. As Y is compact, a finite composite $Y \xrightarrow{f^k} \Sigma^{-k|f|}Y$ is therefore nullhomotopic. □

Lemma 4.10. *Let $\Gamma \xrightarrow{f} \Sigma$ be a normal extension of Hopf algebras over $\mathbb{F}_p$, by a simple Hopf algebra Λ of type D or E.*

Then $\beta_\Lambda \in \operatorname{Ext}_\Lambda(\mathbb{F}_p, \mathbb{F}_p)$ gives rise to an element of $\operatorname{Ext}_\Gamma(\mathbb{F}_p, \mathbb{F}_p)$, which we will also denote by β_Λ. It acts on $\pi^{\Gamma}_{**}(Y)$ for all Y.

Assume Y is compact, and $\pi^{\Sigma}_{**}(f_*Y)$ has a minimal vanishing line of slope d.

1. *If the slope of β_Λ is not bigger than d, then $\pi^{\Gamma}_{**}(Y)$ has a minimal vanishing line of slope d.*

2. *If the slope of β_Λ is bigger than d, but β_Λ acts nilpotently on $\pi^{\Gamma}_{**}(Y)$, then $\pi^{\Gamma}_{**}(Y)$ has a minimal vanishing line of slope d.*

3. *If the slope of β_Λ is bigger than d, and β_Λ does not act nilpotently on $\pi^{\Gamma}_{**}(Y)$, then $\pi^{\Gamma}_{**}(Y)$ has a minimal vanishing line of slope equal to the slope of β_Λ.*

Proof. If β_Λ has slope not bigger than d, the E_2 page of the Adams spectral sequence from Lemma 4.4 lies entirely beneath a line of slope d, so $\pi^{\Gamma}_{**}(Y)$ admits a vanishing line of slope d. From Lemma 4.8, it follows that $\pi^{\Gamma}_{**}(Y)$ can't have a smaller vanishing line, from which the first claim follows.

For β_Λ of higher slope, the E_2 page is typically not bounded by a slope d vanishing line, but still certainly by one of slope parallel to β_Λ.

If β_Λ acts nilpotently on $\pi^{\Gamma}_{**}(Y)$, by Lemma 4.9, there is n such that β_Λ^n already acts trivially on $\pi^{\Gamma}_{**}(Y)$. From Lemma 4.4, we see that the Adams spectral sequence degenerates on the E_{2n+2}-page, and is generated as an $\mathbb{F}_p[\beta_\Lambda]/\beta_\Lambda^n$-module by elements in filtration 0 and 1 there.

As the nonzero elements of filtration (≤ 1) on the E_2-page all lie below a line of slope d, we thus obtain that all elements on the E_∞ page lie below a translate of that line (determined by the degree of β_Λ^n). The second claim follows.

Now if β_Λ acts non-nilpotently on $\pi^{\Gamma}_{**}(Y)$, there is an element x in $\pi^{\Gamma}_{**}(Y)$ for which all $\beta_\Lambda^k x$ are nonzero on the E_∞ page, so there can't be a vanishing line of slope less than the slope of β_Λ. As there is a vanishing line of the slope of β_Λ, we see that there is a minimal one by Lemma 4.7. □

Remark 4.11. Note that we have some explicit control over the intercept of the resulting vanishing lines: In the case where β_Λ has slope ($\leq d$), the new vanishing line is equal to the old one, or a translate of the old one (by $|\alpha|$) if p is odd, Λ is of type D, and $\alpha \in \mathrm{Ext}_\Lambda(\mathbb{F}_p, \mathbb{F}_p)$ has slope bigger than d.

In the case where the slope of β_Λ is bigger than d, but β_Λ acts nilpotently, the new vanishing line is a translate of the old one, determined by the degree of β_Λ^n, where n is such that β_Λ^n acts trivially on $\pi_{**}^\Gamma(Y)$.

The key observation to make from Lemma 4.10 is that there is a minimal vanishing line, and that its slope is either equal to the slope of β or to the slope of the vanishing line over Σ. The nilpotence criterion to decide between the two cases is not very practical, and we will later give a better criterion in terms of exotic K-theories.

Lemma 4.12. *Let $\Gamma \to \Sigma$ be a normal extension of Hopf algebras over $\mathbb{F}_p$, by a simple Hopf algebra Λ of type D or E. Assume $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a compact object such that Y has a minimal vanishing line of slope d, and assume further that d is different from the slope of β.*

Then for any given line of slope d and c, there is an upper bound on the filtration of nontrivial elements in the region above that line on the E_∞ page of the Adams spectral sequence

$$\mathrm{Ext}_\Lambda(\mathbb{F}_p, \pi_{**}^\Sigma(Y)) \Rightarrow \pi_{**}^\Gamma(Y),$$

and the E_r-page agrees with the E_∞ in that region for some r.

Proof. By the previous result, a minimal vanishing line for $\pi_{**}^\Gamma(Y)$ of slope different from the slope of β can only arise in two cases.

Either some power β^n already acts trivially on $\pi_{**}^\Gamma(Y)$, in which case the Adams spectral sequence degenerates on the E_{2n+2}-page, and admits a global bound on filtration by $2n$.

Otherwise, β has slope stricly lower than d (since we assumed their slopes to be distinct). Then, for a fixed line slope d, there is some k such that the image of the multiplication by β^k map is concentrated below the line, so no nonzero elements of filtration $\geq 2k$ lie above the line. □

From Lemma 4.3, any finite Hopf algebra Γ admits a composition series

$$\Gamma = \Gamma_k \to \Gamma_{k-1} \to \ldots \to \Gamma_1 \to \Gamma_0 = \mathbb{F}_p.$$

By inductive application of Lemma 4.10, we see that

Lemma 4.13. *Assume* Γ *is a finite connected Hopf algebra over* $\mathbb{F}_p$*. Fix a composition series* Γ_i *for* Γ*, with* $\Gamma_i \to \Gamma_{i-1}$ *a normal extension by a simple Hopf algebra* Λ_i *of type* D *or* E*. For each* i*, denote the corresponding element* $\beta_{\Lambda_i} \in \mathrm{Ext}_{\Gamma_i}(\mathbb{F}_p, \mathbb{F}_p)$ *by* β_i*, and let* d_i *be its slope.*

Then $\pi_{**}^{\Gamma}(Y)$ *for compact* Y *has a minimal vanishing line of slope equal to* 0 *or one of the* d_i*.*

For example, the possible slopes of minimal vanishing lines over the quotient Hopf algebra

$$\mathcal{P}_*(n) := \mathbb{F}_p[\bar{t}_1, \ldots, \bar{t}_n]/(\bar{t}_1^{p^n}, \bar{t}_2^{p^{n-1}}, \ldots)$$

of $\mathcal{P}_*$, is a subset of the $d_{ij} = \frac{1}{p^{j+1}(p^i-1)-1}$ for $i \geq 1$, $j \geq 0$, $i + j \leq n$.

We now want to study which of the possible slopes d_i are actually attained.

Definition 4.14. *We let* $\mathrm{Slopes}(\Gamma)$ *be the set of slopes that are attained by minimal vanishing lines of compact objects in* $\mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$*.*

The previous result shows that for Γ a finite connected Hopf algebra, $\mathrm{Slopes}(\Gamma)$ is contained in the set of 0 and the d_i arising as slopes of the elements $\beta_i \in \mathrm{Ext}_{\Lambda_i}(\mathbb{F}_p, \mathbb{F}_p)$ associated to the factors Λ_i in a composition series of Γ. We can characterize this subset more precisely:

Lemma 4.15. *Assume* Γ *is a finite connected Hopf algebra, and fix a composition series* Γ_i*. Assume all the* d_i *are distinct. Then if the element* β_i *is nilpotent in* $\pi_{**}^{\Gamma_i}(S)$*,* $d_i \notin \mathrm{Slopes}(\Gamma)$*.*

Proof. Since the slope of the minimal vanishing line of Y over Γ_{i-1} is one of the d_j with $j < i$, and all the d_j are distinct, the slope of the minimal vanishing line of Y over Γ_{i-1} is not d_i. By Lemma 4.10, the slope of the minimal vanishing line of Y over Γ_i agrees with the one over Γ_{i-1}, since β_i is nilpotent.

Applying Lemma 4.10 to the rest of the composition series $\Gamma \to \cdots \to \Gamma_i$, it follows that the slope of the minimal vanishing line of $\pi_{**}^{\Gamma}(Y)$ is one of the d_j with $j \neq i$, so $d_i \notin \mathrm{Slopes}(\Gamma)$. □

For the dual Steenrod algebra over $\mathbb{F}_p$, this observation appeared first in [MW81].

Proposition 4.16. *For* $\mathcal{P}_*(n) := \mathbb{F}_p[\bar{t}_1, \ldots, \bar{t}_n]/(\bar{t}_1^{p^n}, \bar{t}_2^{p^{n-1}}, \ldots)$*,* $\mathrm{Slopes}(\mathcal{P}_*(n))$ *consists of* 0 *and those*

$$d_{ij} = \frac{1}{p^{j+1}(p^i - 1) - 1}$$

with $i > j \geq 0$*,* $i + j \leq n$*.*

Proof. Fix $i \leq j$ with $i+j \leq n$. In particular, $2i \leq n$. Now

$$\Sigma := \mathbb{F}_p[\bar{t}_{di} | di \leq n]/(\bar{t}_i^{p^{j+1}}, \xi_{2i}^{p^{j-i+1}}, \xi_{3i}^{p^{j-2i+1}}, \ldots)$$

is a normal Hopf algebra quotient of $\mathcal{P}_*(n)$. By Lemma 4.3, we can find a composition series interpolating between $\mathcal{P}_*(n)$ and Σ, and then further between $\Sigma/\bar{t}_i^{p^j}$ and $\mathbb{F}_p$. Thus, we obtain a composition series for Γ with intermediate terms $\Gamma_k = \Sigma$ and $\Gamma_{k-1} = \Sigma/\bar{t}_i^{p^j}$.

In $\mathrm{Ext}_\Sigma(\mathbb{F}_p, \mathbb{F}_p)$, the cobar element $[\bar{t}_i^{p^i}|\bar{t}_i]$ is zero, killed in the cobar complex by $[\bar{t}_{2i}]$. This corresponds to a differential in the Adams spectral sequence for the normal quotient map $\Sigma \to \Sigma/\xi_i$. Power operations can be used to deduce from this a differential killing a power of the element $[\bar{t}_i^{p^j}]$, for details see the proof of Proposition 4.1 in [MW81]. (Recall that $\bar{t}_i$ corresponds to $\xi_i \in \mathcal{A}_*$ at odd p, and to $\xi_i^2 \in \mathcal{A}_*$ at $p=2$.)

It follows that the β_k associated to the normal extension $\Gamma_k \to \Gamma_{k-1}$ is nilpotent in $\mathrm{Ext}_{\Gamma_k}(\mathbb{F}_p, \mathbb{F}_p)$. So by Lemma 4.15, for $i \leq j$ and $i+j \leq n$, we see $d_{ij} \notin \mathrm{Slopes}(\Gamma)$.

To show that the d_{ij} for $j < i$ are contained in $\mathrm{Slopes}(\Gamma)$, note that $H\Lambda_{ij}$, for Λ_{ij} the corresponding coalgebra quotient of $\mathcal{P}_*(n)$, has a minimal vanishing line of slope d_{ij} (and is compact). □

There is, in fact, a stronger way in which $K\Lambda_{ij}$ detects minimal vanishing lines.

Proposition 4.17. *If a compact object $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*(n))}$ has a vanishing line of slope d_{ij}, it has a minimal vanishing line of slope d_{ij} if and only if $(K\Lambda_{ij})_{**}Y \neq 0$.*

Proof. First observe that, if $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*(n))}$ has a minimal vanishing line of slope smaller than d_{ij}, $(K\Lambda_{ij})_{**}Y = 0$.

To see this, observe that, since $\mathcal{P}_*$ is right-cofree over Λ_{ij}, as one can check on duals using Proposition 3.48, Lemma 3.29 implies that $H\Lambda_{ij}$ is represented by the comodule $\Gamma\Box_{\Lambda_{ij}}\mathbb{F}_p$. So by Lemma 3.26, it admits a cell structure with cells in dimensions $(n,0)$ with $n \geq 0$. It follows that the minimal vanishing line of Y also bounds $H\Lambda_{ij} \otimes Y$.

But since $K\Lambda_{ij}$ is obtained from $H\Lambda_{ij}$ by inverting an element of slope d_{ij}, $K\Lambda_{ij} \otimes Y = 0$.

For the other direction, assume $(K\Lambda_{ij})_{**}Y = 0$.

We have a quotient Hopf algebra $\mathbb{F}_p[\bar{t}_i]/\bar{t}_i^{p^{j+1}}$ of $\mathcal{P}_*(n)$. Thus, $\mathcal{P}_*(n)$ admits a decomposition series with $\Gamma_k = \mathbb{F}_p[\bar{t}_i]/\bar{t}_i^{p^{j+1}}$, and $\Gamma_{k-1} = \Gamma_k/\xi_i^{p^j}$. As Λ_{ij} is a coalgebra quotient of Γ_k, there is an analogous exotic K-theory object $K\Lambda_{ij} \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma_k}$, and due to Lemma 3.31, we are reduced to work over Γ_k.

So we know that $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma_k}$ has trivial $K\Lambda_{ij}$-homology, and want to deduce that it has a vanishing line strictly smaller than d_{ij}. Since all the d_{ij} are distinct, the slope of the minimal vanishing line of Y over Γ_{k-1} is not equal d_{ij}, so it has slope strictly smaller than d_{ij}. It is thus sufficient to check that β_{ij} acts nilpotently on $\pi^{\Gamma_k}_{**}(Y)$.

We now apply Proposition 3.72 to $\Gamma_k = \mathbb{F}_p[\bar{t}_i]/\bar{t}_i^{p^{j+1}}$. Since $H\Lambda_{ij}$ and the iterated cofibre $S/(\beta_{i,0}, \ldots, \beta_{i,j-1})$ generate the same thick subcategory, there is an object filtered by finitely many shifted copies of $H\Lambda_{ij}$, admitting $S/(\beta_{i,0}, \ldots, \beta_{i,j-1})$ as a retract. By inverting β_{ij}, we obtain an object filtered by finitely many copies of $K\Lambda_{ij}$, admitting $\beta_{ij}^{-1}S/(\beta_{i,0}, \ldots, \beta_{i,j-1})$ as a retract. This shows that $\beta_{ij}^{-1}S/(\beta_{i,0}, \ldots, \beta_{i,j-1})$ is contained in the thick subcategory generated by $K\Lambda_{ij}$, and similarly for the other direction.

So $K\Lambda_{ij}$ and $\beta_{ij}^{-1}S/(\beta_{i,0}, \ldots, \beta_{i,j-1})$ generate the same thick subcategory. We thus know that

$$\beta_{ij}^{-1}S/(\beta_{i,0}, \ldots, \beta_{i,j-1})_{**}Y = 0.$$

Now denote by K_l for $0 \leq l \leq j-1$ the iterated cofibre

$$S/(\beta_{i,0}, \ldots, \beta_{i,l}),$$

and set $K_{-1} = S$.

Since the K_l are finite cell complexes, $K_l \otimes Y$ admits a vanishing line of slope d_{ij}. So $\beta_{ij}^{-1}K_l \otimes Y$ does, too. Now consider the cofibre sequences

$$\Sigma^{|\beta_{il}|}K_{l-1} \otimes Y \xrightarrow{\beta_{il}} K_{l-1} \otimes Y \to K_l \otimes Y,$$

and, obtained from these by inverting β_{ij}, the cofibre sequences

$$\Sigma^{|\beta_{il}|}\beta_{ij}^{-1}K_{l-1} \otimes Y \xrightarrow{\beta_{il}} \beta_{ij}^{-1}K_{l-1} \otimes Y \to \beta_{ij}^{-1}K_l \otimes Y.$$

If $\beta_{ij}^{-1}K_l \otimes Y = 0$, this implies that β_{il} acts on $\beta_{ij}^{-1}K_{l-1} \otimes Y$ as an isomorphism. But since $\beta_{ij}^{-1}K_{l-1} \otimes Y$ has a vanishing line of slope d_{ij}, and β_{il} has slope different from d_{ij}, this implies $\beta_{ij}^{-1}K_{l-1} \otimes Y = 0$.

By downward induction over l, we see that $\beta_{ij}^{-1}Y = 0$. Thus, β_{ij} acts nilpotently, and the claim follows. □

Proposition 4.18. *The slope of the minimal vanishing line of compact* $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*(n))}$ *is the largest* d_{ij} *with* $i > j$, $i + j \leq n$, *for which* $(K\Lambda_{ij})_{**}Y \neq 0$. *In particular,* Y *has a horizontal minimal vanishing line if and only if* $(K\Lambda_{ij})_{**}Y$ *vanishes for all* $i > j$ *with* $i + j \leq n$.

Remark 4.19. Since compact objects with horizontal vanishing line can be seen to have a finite Postnikov tower, and $H\mathbb{F}_p \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*(n)))}$ is represented by the cofree comodule $\mathcal{P}_*(n)$, the property of having a horizontal vanishing line can be regarded as a derived version of the property of being a cofree comodule.

As Lemma 3.68 essentially exhibits the $(K\Lambda_{ij})_{**}$ as a derived analogue of the classical Margolis homologies, the last sentence of Proposition 4.18 can be considered an analogue of the classical Adams-Margolis theorem [AM71].

We now want to lift these results to the infinite Hopf algebra $\mathcal{P}_*$.

Lemma 4.20. *Let $\Gamma \xrightarrow{f} \Sigma$ be a quotient map of Hopf algebras over $\mathbb{F}_p$ which is an isomorphism in degrees below l, with $l \geq 2$. For $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ bounded below, if $\pi^{\Sigma}_{**}(f_* Y)$ has a vanishing line of slope d, and $d \geq \frac{1}{l-1}$, then $\pi^{\Gamma}_{**}(Y)$ has that same vanishing line.*

Furthermore, if the intercept of the vanishing line for $\pi^{\Sigma}_{**}(f_* Y)$ is c, the map $\pi^{\Gamma}_{**}(Y) \to \pi^{\Sigma}_{**}(f_* Y)$ is an isomorphism above a line of slope d and intercept

$$c + 1 - d(l-1) < c.$$

Proof. Lemma 3.30 tells us that $f^* f_* Y = H\Sigma \otimes Y$. Since a surjective map $\Gamma \to \Sigma$ between connected Hopf algebras always exhibits Γ as cofree over Σ (see e.g. Theorem 4.4 in [MM65] for a slightly stronger statement in the dual case), Lemma 3.29 implies that the homology $H_{**}(H\Sigma; \mathbb{F}_p)$ is $\Gamma \square_{\Sigma} \mathbb{F}_p$.

Thus it is given by $\mathbb{F}_p$ in degree $(0,0)$, and otherwise concentrated in degrees of the form $(n, 0)$ with $n \geq l$.

So $\overline{H\Sigma}$, the fibre of $S \to H\Sigma$, has homology concentrated in degrees of the form $(n, 1)$ with $n \geq l - 1$. By Proposition 3.26, it admits a cell structure with cells in those dimensions.

We now consider the Adams tower of Y with respect to the adjunction $f_* \dashv f^*$. According to Lemma 2.38, it converges if $\lim_n \overline{H\Sigma}^n Y = 0$, and has successive fibres of the form

$$F_k((f^* f_*)^{\bullet+1} Y) = H\Sigma \otimes \overline{H\Sigma}^k \otimes Y$$

Since Y is bounded-below, and $l \geq 2$, $\lim_n \overline{H\Sigma}^n Y = 0$ follows for connectivity reasons.

As $\overline{H\Sigma}^k$ admits a cell structure with cells in dimensions of the form (n, s) with $n \geq k(l-1)$, $s \leq k$, we see that $\overline{H\Sigma}^k \otimes Y$ is filtered by copies of $\Sigma^{n,s} Y$ with $n \geq k(l-1)$, $s \leq k$.

From $\pi^{\Gamma}_{**}(H\Sigma \otimes \overline{H\Sigma}^k \otimes Y) = \pi^{\Sigma}_{**}(f_*(\overline{H\Sigma}^k \otimes Y))$ and the assumption $d \geq \frac{1}{l-1}$, we then see that the whole E_1 page of the $f^* f_*$-Adams spectral sequence for Y is bounded by the vanishing line for $\pi^{\Sigma}_{**}(f_* Y)$.

For the statement about the isomorphism range, consider the fibre sequence

$$\overline{H\Sigma} \otimes Y \to Y \to H\Sigma \otimes Y.$$

As $\overline{H\Sigma}$ admits a cell structure with cells in dimensions of the form (n, s) with $n \geq l - 1$ and $s \leq 1$, and thus is filtered by shifts $\Sigma^{n,s} Y$ by these dimensions, the claim follows. □

Not every commutative Hopf algebra can be approximated by finite quotient Hopf algebras. However, $\mathcal{P}_*$ can, for example by the previously described quotients $\mathcal{P}_*(n)$. The map $\mathcal{P}_* \to \mathcal{P}_*(n)$ induces an isomorphism below degree $2p^{n+1} - 2$. We obtain:

Proposition 4.21. *If $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*)}$ is compact, it admits a minimal vanishing line of slope*

$$d_{ij} = \frac{1}{p^{j+1}(p^i - 1) - 1}$$

for some $i > j \geq 0$ (in particular, it is never zero). This slope is characterized as the largest d_{ij} for which $(K\Lambda_{ij})_{**}Y \neq 0$.

Proof. Since a vanishing line of Y over $\mathcal{P}_*(n)$ also bounds Y over $\mathcal{P}_*(n-1)$ by Lemma 4.8, the sequence of slopes of minimal vanishing lines of Y over $\mathcal{P}_*(n)$ is nondecreasing. Assume first that Y has a minimal vanishing line of positive slope d over some $\mathcal{P}_*(n)$.

Then choosing n' large enough that the minimal vanishing line of Y has slope $(\geq d)$ over $\mathcal{P}_*(n')$ and such that

$$d > \frac{1}{2p^{n'+1} - 3},$$

we see that the minimal vanishing line of Y over $\mathcal{P}_*(n')$ and the quotient map $\mathcal{P}_* \to \mathcal{P}_*(n')$ satisfy the requirements of Lemma 4.20, and we see that Y has the same minimal vanishing line over $\mathcal{P}_*$ as over $\mathcal{P}_*(n')$.

It remains to prove that the minimal vanishing line of Y over $\mathcal{P}_*(n)$ has positive slope for n large enough. But from Proposition 3.70, we see that $(K\Lambda_{ij})_{**}Y$ can't vanish for all $i > j$. By Proposition 4.18, we are done. □

Now, we lift these results up to BP_*BP along the map $BP_*BP \to \mathcal{P}_*$.

Proposition 4.22. *For $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ compact, there is a minimal vanishing line of slope d_{ij} for some $i > j \geq 0$, which coincides with the largest d_{ij} such that $(K\Lambda_{ij})_{**}Y \neq 0$.*

Proof. It suffices to show that the vanishing lines of Y over BP_*BP and $\mathcal{P}_*$ agree. We can obtain the map $BP_*BP \to \mathcal{P}_*$ as the composite

$$BP_*BP \xrightarrow{f} BP_*BP/p \xrightarrow{g} \mathcal{P}_*.$$

To see that the vanishing lines of Y over BP_*BP and BP_*BP/p agree, observe first that, since $f^*S \simeq S/p$, a vanishing line for $\pi^{BP_*BP}_{**}(Y)$ also bounds $\pi^{BP_*BP}_{**}(f^*S \otimes Y) \simeq \pi^{BP_*BP/p}_{**}(Y)$.

For the other direction, we can use the associated Adams spectral sequence from Lemma 3.43. A vanishing line for $\pi_{**}^{BP_*BP/p}(Y)$ bounds the E_1 page of that spectral sequence, and therefore also the homotopy groups $\pi_{**}^{BP_*BP}(Y_p^\wedge)$ of the p-completion.

But since Y is assumed to be compact, it is of finite type. Thus, the map $\pi_{**}(Y) \to \pi_{**}(Y_p^\wedge)$ is injective.

To see that the vanishing lines of Y over BP_*BP/p and $\mathcal{P}_*$ agree, note first that by Lemma 3.29, the homology of $H(\mathcal{P}_*) = g^*S \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP/p)}$ with coefficients in BP_*/p is given by $\mathbb{F}_p$ concentrated in degree $(0,0)$.

We thus find a cell structure for $H(\mathcal{P}_*)$ with cells in dimensions (n,s) with $n \geq 0$ and $s \leq 0$.

So a vanishing line for $\pi_{**}^{BP_*BP/p}(Y)$ also bounds $\pi_{**}^{BP_*BP/p}(H\mathcal{P}_* \otimes Y) = \pi_{**}^{\mathcal{P}_*}(Y)$.

Next, we see that with $\overline{H\mathcal{P}_*}$ the fibre of $S \to H\mathcal{P}_*$, its BP_*/p-homology is concentrated in degrees $(n,0)$ with $n \geq 2$. So for $\overline{H\mathcal{P}_*}$, we find a cell structure with cells of dimensions (n,s) with $s \leq 0$ and $n \geq 2$.

Now consider the g^*g_*-Adams resolution of Y. It converges to $\pi_{**}^{BP_*BP/p}(Y)$ by Lemma 3.37, with fibre $F_k((g^*g_*)^{\bullet+1}Y)$ of the form $H(\mathcal{P}_*) \otimes \overline{H(\mathcal{P}_*)}^{\otimes k} \otimes Y$. Since

$$\pi_{**}^{BP_*BP}(H(\mathcal{P}_*) \otimes \overline{H(\mathcal{P}_*)}^{\otimes k} \otimes Y) = \pi_{**}^{\mathcal{P}_*}(\overline{H(\mathcal{P}_*)}^{\otimes k} \otimes Y),$$

and $\overline{H(\mathcal{P}_*)}^{\otimes k} \otimes Y$ is filtered by copies of $\Sigma^{n,s}Y$ with $n \geq 2k$ and $s \leq 0$ due to the aforementioned cell structure on $\overline{H(\mathcal{P}_*)}$, the whole Adams spectral sequence is bounded by the vanishing line of Y over $\mathcal{P}_*$. □

4.2 Self-maps

We are now ready to discuss the existence of self-maps.

Instead of studying self-maps in the strict sense of maps of suspensions of an object Y to itself, one can more generally consider elements in $\pi_{**}(R)$ for any $\mathbb{E}_1$-ring R. For a compact object Y, self-maps $\Sigma^n Y \to Y$ can be recovered from this perspective by setting $R = \mathit{End}(Y) := Y \otimes DY$, with DY the $\otimes$-dual of Y.

Definition 4.23. *Let $R \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ be a compact ring. A $\theta \in \pi_{n,s}(R)$ of nonzero degree will be called a $K\Lambda_{ij}$ self-map if $(K\Lambda_{ij})_{**}(R)$ is nonzero, and θ induces an isomorphism on it.*

Note that, due to compactness of R, $(K\Lambda_{ij})_{**}R$ is constrained to a finite width strip of slope d_{ij}. So any $K\Lambda_{ij}$ self-map necessarily has slope d_{ij}. Note furthermore that such θ are clearly not nilpotent.

Also, since R is necessarily bounded below, the degree of θ is necessarily positive.

For self-maps parallel to the minimal vanishing line of R, there are other characterisations:

Proposition 4.24. *Given $R \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ a compact ring with minimal vanishing line of slope d_{ij}, and $\theta \in \pi_{**}R$ an element of nonzero degree, the following are equivalent:*

1. *The element θ is a $K\Lambda_{ij}$ self-map.*
2. *The cofibre R/θ has a vanishing line of slope smaller d_{ij}.*
3. *The element θ induces an isomorphism above a line of slope smaller than d_{ij}.*

Proof. Since R has a minimal vanishing line of slope d_{ij}, $(K\Lambda_{ij})_{**}R$ is nonzero. Since R/θ has a smaller-slope vanishing line if and only if it has vanishing $K\Lambda_{ij}$, the equivalence between (1) and (2) follows from the long exact sequence in $K\Lambda_{ij}$-homology associated to the cofibre sequence $\Sigma^{|\theta|}R \xrightarrow{\theta} R \to R/\theta$. The equivalence between (2) and (3) follows from the corresponding long exact sequence in homotopy. □

In this case, we will often just call θ a self-map parallel to the minimal vanishing line. The characterization in terms of vanishing lines makes sense more generally, even if we don't have a corresponding exotic K-theory to detect it.

Definition 4.25. *For Γ a connected graded multiplicative coalgebroid, assume $R \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a compact ring such that $\pi_{**}(R)$ has a minimal vanishing line of slope d.*

An element $\theta \in \pi_{**}(R)$ of nonzero degree is called a self-map parallel to the vanishing line if it has slope d and acts isomorphically on $\pi_{**}(R)$ above some line of slope strictly smaller than d.

We want to prove here that for any compact $R \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, a self-map parallel to the minimal vanishing line exists. As for existence of minimal vanishing lines in Section 4.1, our main tool will be an inductive approach through iterated Adams spectral sequences.

Lemma 4.26. *Say $R \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a ring with minimal vanishing line of slope d. Assume $\Gamma \to \Sigma$ is a normal extension of connected Hopf algebras over $\mathbb{F}_p$ along Λ simple of type D or E, and the associated β_Λ has slope different from d.*

Assume $\theta \in \pi^{\Sigma}_{**}R$ is a self-map parallel to the minimal vanishing line. Then, after replacing θ by some power, it lifts to an element of $\pi^{\Gamma}_{**}R$, and this lift is a self-map parallel to the vanishing line. Furthermore, if θ commutes with all elements of $\pi^{\Sigma}_{**}R$ above a line of slope d and intercept c, some power of it can be lifted to an element of $\pi^{\Gamma}_{**}R$ that commutes with all elements of $\pi^{\Gamma}_{**}R$ above that line. If the intercept of that line can be chosen such that $c < 0$, any two such lifts agree after up to taking powers of them.

Proof. From Lemma 4.12, it follows that in the Adams spectral sequence for $\Gamma \to \Sigma$, the spectral sequence degenerates on some finite page above any line of slope d, either because $\beta_\Lambda \in \mathrm{Ext}_\Lambda(\mathbb{F}_p, \mathbb{F}_p)$ acts nilpotently on $\pi^\Gamma_{**}(R)$ or because it has slope strictly smaller than d.

If θ survives to the E_r-page, then θ^p survives to the E_{r+1}-page, and so on, and since all those powers lie above the same line of slope d, where the spectral sequence degenerates after finitely many pages, some power θ^{p^k} is a permanent cycle.

If θ commutes with all elements of $\pi^\Sigma_{**}R$ above a line of slope d and intercept c, then on the E_2-page of the Adams spectral sequence, and therefore on the E_∞ page, it commutes with everything in that region.

To simplify notation, replace θ by a power which is a permanent cycle, and let $\widetilde{\theta} \in \pi^\Gamma_{**}(R)$ be a lift. Then, since $\widetilde{\theta}$ commutes up to higher filtration with any $x \in \pi^\Gamma_{**}(R)$ above the line of slope d and intercept c, the commutator $[\widetilde{\theta}, x] = \widetilde{\theta}x - x\widetilde{\theta}$ has strictly bigger filtration than x.

Now since, on the E_∞ page, filtration of nontrivial elements is bounded in the region above the line of slope d and intercept c, say by r, this means that $\widetilde{\theta}$ commutes with all elements of filtration r.

But then $\widetilde{\theta}^p$ commutes with all elements of filtration $(\geq r-1)$: Indeed,

$$[\widetilde{\theta}^p, x] = \widetilde{\theta}^{p-1}[\widetilde{\theta}, x] + \widetilde{\theta}^{p-2}[\widetilde{\theta}, x]\widetilde{\theta} + \ldots + [\widetilde{\theta}, x]\widetilde{\theta}^{p-1},$$

and if x has filtration $r-1$, $[\widetilde{\theta}, x]$ has filtration r. But then $\widetilde{\theta}$ commutes with $[\widetilde{\theta}, x]$, and the expression simplifies to

$$[\widetilde{\theta}^p, x] = p\widetilde{\theta}^{p-1}[\widetilde{\theta}, x] = 0$$

since we are in characteristic p. Inductively, $\widetilde{\theta}^{p^k}$ commutes with all elements of Adams filtration $(\geq r-k)$ above the line of slope d and intercept c. So $\widetilde{\theta}^{p^r}$ commutes with all elements in that range.

Finally, a different lift of θ^{p^r} differs by an element u of positive filtration from $\widetilde{\theta}^{p^r}$, which is nilpotent. But if $c < 0$, since $\widetilde{\theta}^{p^r}$ then commutes with u, some p^k-th powers of $\widetilde{\theta}^{p^r} + u$ and $\widetilde{\theta}^{p^r}$ agree.

It remains to establish that the lift of θ to $\pi^\Gamma_{**}(R)$ is a self-map parallel to the vanishing line. For this we have to establish that the cofibre $R/\widetilde{\theta}$ has a vanishing line of slope strictly less than d. By Lemma 4.10, the slope of the minimal vanishing line of $\pi^\Gamma_{**}(R/\widetilde{\theta})$ either agrees with the slope of the minimal vanishing line of $\pi^\Sigma_{**}(R/\theta)$, which is strictly smaller than d, or it agrees with the slope of β. However, the slope of β was assumed to be distinct from d, and since $\pi^\Gamma_{**}(R)$ has a vanishing line of slope d, $\pi^\Gamma_{**}(R/\widetilde{\theta})$ has a vanishing line of slope d. So this case can only occur if the slope of β is actually smaller than d. □

Lemma 4.27. *Let Γ be a finite Hopf algebra over $\mathbb{F}_p$, with composition series*

$$\Gamma = \Gamma_k \to \cdots \to \Gamma_0 = \mathbb{F}_p,$$

I.e. all $\Gamma_j \to \Gamma_{j-1}$ are normal extensions along a simple Hopf algebra Λ_j of type D or E. For all j, let d_j be the slope of the element $\beta_j \in \mathrm{Ext}_{\Lambda_j}(\mathbb{F}_p, \mathbb{F}_p)$. Now assume that $R \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ is a compact ring, with minimal vanishing line of slope d_i.

Then there is a unique self-map parallel to the vanishing line $\theta \in \pi^{\Gamma}_{**}(R)$. It is unique up to taking powers, and for any given $c \in \mathbb{Z}$, can be taken to commute with all elements of $\pi^{\Gamma}_{**}(R)$ above the line of slope d and intercept c.

Proof. In $\pi^{\Gamma_i}_{**}(R)$, the element β_i is a self-map parallel to the vanishing line, and commutes with all elements. It remains to check that any other self-map agrees with β_i up to taking powers.

So let $\theta \in \pi^{\Gamma_i}_{**}(R)$ be a self-map. Since θ acts isomorphically on $\pi^{\Gamma_i}_{**}(R)$ above a line of slope strictly smaller than d_i by Proposition 4.24, it acts isomorphically on all of $\pi^{\Gamma_i}_{**}(\beta_i^{-1}R)$.

Now since S is an $\mathbb{E}_\infty$ ring, $\beta_i^{-1}S$ is an $\mathbb{E}_\infty$-ring (see for example [BNT15], Appendix C, especially Proposition C.5). So $\beta_i^{-1}R \simeq \beta_i S \otimes R$ is an R-algebra.

By taking a power of θ and multiplying with some β_i^{-n}, we obtain a unit θ' in $\pi_{0,0}(\beta_i^{-1}R)$. Since β_i acts isomorphically on $\pi^{\Gamma_i}_{**}(R)$ above a line of slope less than d_i, $\pi_{0,0}(\beta_i^{-1}R)$ agrees with $\pi_{n|\beta_i|}(R)$ for large n. As R is compact and Γ_i a finite Hopf algebra, this group is a finitely generated $\mathbb{F}_p$-vector space.

So θ' is a unit in a finite ring, and therefore some power of it is 1. As β_i commutes with θ, it follows that some power of θ agrees with some power of β_i. Thus we see that self-maps parallel to the vanishing line in $\pi^{\Gamma_i}_{**}(R)$ are unique up to taking powers.

By inductively applying Lemma 4.26, the claim now follows for Γ as well. □

Proposition 4.28. *For $R \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*)}$ a compact ring with minimal vanishing line of slope d_{ij}, there exists a self-map $\theta \in \pi^{\mathcal{P}_*}_{**}(R)$ parallel to the vanishing line, which is unique up to taking powers. For arbitrary c, θ can be chosen such that it commutes with anything above the line of slope d_{ij} and intercept c.*

Proof. From Lemma 4.27, we see that such a self-map exists over any of the $\mathcal{P}_*(n)$. For given c, say with $c < 0$, choose n large enough that the map $\pi^{\mathcal{P}_*}_{**}(R) \to \pi^{\mathcal{P}_*(n)}_{**}(R)$ is an isomorphism above the line of slope d_{ij} and intercept c. In particular, a self-map $\theta \in \pi^{\mathcal{P}_*(n)}_{**}(R)$ lifts to an element $\widetilde{\theta} \in \mathcal{P}_*$ which commutes with elements above the line of slope d_{ij} and intercept c.

Since $\widetilde{\theta}$ is a lift of θ and thus a self-map parallel to the vanishing line over each finite Hopf algebra quotient $\mathcal{P}_*(m)$ with $m > n$, the minimal vanishing line of $R/\widetilde{\theta}$ over each $\mathcal{P}_*(m)$ is strictly smaller than d_{ij}. By choosing m large enough, we see that the minimal vanishing line of $R/\widetilde{\theta}$ over $\mathcal{P}_*$ has slope strictly smaller than d_{ij}. Thus, $\widetilde{\theta}$ is a self-map parallel to the vanishing line. □

Finally, we want to lift self-maps to BP_*BP. This will be done in two steps. First we lift to BP_*BP/p by a connectivity argument.

Proposition 4.29. *For $R \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP/p)}$ a compact ring with minimal vanishing line of slope d_{ij}, there exists a self-map $\theta \in \pi^{BP_*BP/p}_{**}(R)$ parallel to the vanishing line, which is unique up to taking powers. For arbitrary c, can be chosen such that it commutes with anything above a line of slope d_{ij} and intercept c.*

Proof. For $f : BP_*BP/p \to \mathcal{P}_*$, we consider the f^*f_*-Adams resolution of R, which is a cosimplicial ring since f_* is monoidal and f^* is lax monoidal.

As in the proof of Proposition 4.22, we see that $\overline{H(\mathcal{P}_*)}$ admits a cell structure with cells of dimensions (n, s) with $n \geq 2$ and $s \leq 0$.

So $\overline{H(\mathcal{P}_*)}^{\otimes k} \otimes R$ is filtered by copies of $\Sigma^{n,s}R$ with $n \geq 2k$, $s \leq 0$.

By Lemma 2.38, the successive fibres $F_k((f^*f_*)^{\bullet+1}R)$ of the Adams tower are of the form $H(\mathcal{P}_*) \otimes \overline{H(\mathcal{P}_*)}^{\otimes k} \otimes R$. So if $\pi^{\mathcal{P}_*}_{**}(R)$ has a vanishing line of slope d_{ij} and intercept c, the filtration k part of the E_1 page

$$\pi^{BP_*BP/p}_{**}(H(\mathcal{P}_*) \otimes \overline{H(\mathcal{P}_*)}^{\otimes k} \otimes R) \simeq \pi^{\mathcal{P}_*}_{**}(\overline{H(\mathcal{P}_*)}^{\otimes k} \otimes R)$$

has a vanishing line of slope d_{ij} through $(2k, c_0)$, i.e. of intercept $c_0 - 2kd_{ij}$.

We thus obtain that, in the region above any fixed line of slope d_{ij}, the filtration of nonzero elements on the E_1 page is bounded. Thus, in the region above such a line, the spectral sequence degenerates after finitely many pages.

Exactly like in the proof of Proposition 4.26, the result now follows. Note that a lift of a self-map over $\mathcal{P}_*$ is a self-map over BP_*BP/p, because the minimal vanishing line of $R/\widetilde{\theta}$ over BP_*BP/p agrees with the minimal vanishing line of R/θ over $\mathcal{P}_*$ as noted in the proof of Proposition 4.22. □

Finally, we want to lift to BP_*BP. For that, we need a lemma on p-Bockstein filtration:

Lemma 4.30. *Let $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ be compact with minimal vanishing line of slope d_{ij}. There exists an s_0, such that, for any c, all $\pi_{n,s}(Y)$ with $s > s_0$ and (n, s) above the line of slope d_{ij} and intercept c are torsion groups with uniformly bounded exponent, meaning that there is e such that p^e annihilates everything in that range.*

Proof. Since Y is compact, $(H\mathbb{Z}_{(p)})_{**}Y$ is concentrated in finitely many bidegrees. With F the fibre of $S \to H\mathbb{Z}_{(p)}$, we can thus fix s_0 such that $F \otimes Y \to Y$ is an isomorphism in the range of degrees (n, s) with $s > s_0$.

It is thus sufficient to show that the homotopy groups of $F \otimes Y$ above a line of slope d_{ij} and intercept c are p-torsion of uniform exponent, for any c. To do that, observe that Lemma 3.55 guarantees the existence of $F_k \to F$, such that the F_k are compact, that $F_k \to F$ has cofibre with homology in degrees (n, s) with $n \geq k$ and $s \leq 1$, and that all the homotopy groups of F_k are torsion.

The statement about homology of the cofibre of $F_k \to F$ implies that F/F_k has a cell structure with cells in dimensions (n, s) with $n \geq k$ and $s \leq 1$. So it follows that if we choose k large enough, $F_k \otimes Y \to F \otimes Y$ is an isomorphism above the line of slope d_{ij} and intercept c.

But the homotopy groups of $F_k \otimes Y$ are also p-torsion with uniform exponent: Since F_k is compact and its homotopy groups are p-torsion, some power p^i must act trivially on F_k. So it also acts trivially on $F_k \otimes Y$. □

Theorem 4.31. *For $R \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ a compact ring with minimal vanishing line of slope d_{ij}, there exists a self-map θ parallel to the vanishing line, which is unique up to taking powers. For arbitrary c, it can be chosen such that it commutes with anything above a line of slope d_{ij} and intercept c.*

Proof. We use the Adams spectral sequence for R associated to the adjunction $f_* \dashv f^*$, where $f : BP_*BP \to BP_*BP/p$ is the quotient map.

According to Lemma 3.43, it converges to $\pi_{**}(R^\wedge_p)$.

The associated Adams tower was identified with the tower of R/p^k in the proof of that Lemma. One sees that the filtration of an element of $\pi_{**}(R^\wedge_p)$ in the E_∞ page of that spectral sequence corresponds to how far back it pulls in the sequence

$$\cdots \xrightarrow{p} R^\wedge_p \xrightarrow{p} R^\wedge_p \xrightarrow{p} R^\wedge_p.$$

By Lemma 4.30, there is s_0, such that in the region above the horizontal line of height s_0 and the line of slope d_{ij} and intercept c (assume $c < 0$), all of $\pi_{**}(R)$ is p-torsion of uniformly bounded exponent. So in that region, $\pi_{**}(R) \to \pi_{**}(R^\wedge_p)$ is an isomorphism, and the exponent of $\pi_{**}(R^\wedge_p)$ is also bounded in that region.

This shows that the filtration of the spectral sequence is bounded in that region. This is enough to ensure that some power of a self-map $x \in \pi^{BP_*BP/p}_{**}(R)$ of slope d_{ij} is a permanent cycle, as in the proof of Lemma 4.10.

So now assume $\widetilde{\theta}$ is a lift of a self-map which commutes with elements of $\pi^{BP_*BP/p}_{**}(R)$ above a line of slope d_{ij} and intercept c. This still means that $[\widetilde{\theta}, x]$ has strictly larger filtration than x. Now assume inductively that $\widetilde{\theta}$ commutes with all elements of filtration $\geq r$ in the relevant range, and consider x of filtration $r - 1$. Then $\widetilde{\theta}$ commutes with $[\widetilde{\theta}, x]$, and

$$[\widetilde{\theta}^{p^k}, x] = \sum_{i+j=p^k-1} \widetilde{\theta}^i [\widetilde{\theta}, x] \widetilde{\theta}^j = p^k \widetilde{\theta}^{p^k-1} [\widetilde{\theta}, x],$$

which is 0 for k larger than the bound on p-exponent. So the argument given in the proof of Lemma 4.26 for the fact that some power of $\widetilde{\theta}$ commutes with all elements above a line of slope d_{ij} and intercept c still goes through, by replacing p with p^k.

Finally, assume $\widetilde{\theta}$ has already replaced by a suitable power such that it commutes with all elements above a line of slope d and intercept c with $c < 0$. Then a different lift will be of the form $\widetilde{\theta} + pu$. By passing to the p^k-th power, we obtain

$$\widetilde{\theta}^{p^k} + p^{k+1}u'$$

with some u'. This will agree with $\widetilde{\theta}^{p^k}$ for k larger than the bound on p-exponent.

As the minimal vanishing line of $R/\widetilde{\theta}$ over BP_*BP agrees with the minimal vanishing line of R/θ over BP_*BP/p, $\widetilde{\theta}$ is a self-map parallel to the minimal vanishing line. □

We now want to apply Theorem 4.31 to construct self-maps in the strict sense. Given compact $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ with minimal vanishing line of slope d_{ij}, we ask for a map $\theta : \Sigma^{|\theta|}Y \to Y$ which induces an isomorphism on $K(\Lambda_{ij})_{**}Y$, or equivalently, on $\pi_{**}Y$ above a line of slope strictly smaller than d_{ij}.

We want to obtain θ as an element of $\pi_{**}(\mathrm{End}(Y))$, where $\mathrm{End}(Y) = Y \otimes DY$. It is not immediately clear that this works, since we don't know that the minimal vanishing line of $\mathrm{End}(Y)$ is also of slope d_{ij}. Also, it is not obvious that a self-map along the vanishing line in $\mathrm{End}(Y)$ also acts on Y through isomorphisms above a line of slope smaller than d_{ij}.

Proposition 4.32. *Let $Y \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ be compact with minimal vanishing line of slope d_{ij}. Then:*

1. *We also have that $\mathrm{End}(Y)$ has a minimal vanishing line of slope d_{ij}.*
2. *Any self-map $\theta \in \pi_{**}(\mathrm{End}(Y))$ parallel to the vanishing line acts through isomorphisms on $\pi_{**}(Y)$ above a line of slope strictly smaller than Y.*

Proof. Fix a composition series of $\mathcal{P}_*(n)$ for large n, with intermediate terms $\Gamma_k \to \Gamma_{k-1}$ a normal extension by Λ_{ij}.

Since Y has a vanishing line of slope d_{ij}, and the dual DY is compact, i.e. a finite cell complex, we see that $\mathrm{End}(Y) = Y \otimes DY$ has a vanishing line of slope d_{ij}.

As in the proof of Theorem 4.31, we then get that some power of $\beta_{ij} \in \pi^{\Gamma_k}_{**}(\mathrm{End}(Y))$ lifts to an element $\theta \in \pi^{BP_*BP}_{**}(\mathrm{End}(Y))$. (Note that minimality of the vanishing line was only used there for nontriviality of the self-map.)

But as β_{ij} acts isomorphically on $\pi^{\Gamma_k}_{**}(Y)$, we see that θ does so, too. In particular, as the vanishing line of Y was minimal, θ can't be nilpotent. It follows that the vanishing line of slope d_{ij} for $\pi^{BP_*BP}_{**}(\mathrm{End}(Y))$ is minimal, and that θ is a self-map parallel to the vanishing line. As any two self-maps parallel to the vanishing line agree up to taking powers, claim (2) follows. □

Observe that in Section 4.1, we only obtained full descriptions of Slopes(Γ) for $\Gamma = \mathcal{P}_*(n)$. The examples of compact objects with minimal vanishing line of slope d_{ij} with $i > j$ were obtained as $H\Lambda_{ij}$, which is only compact since $\mathcal{P}_*(n)$ is finite. In particular, we haven't yet seen that

$$\mathrm{Slopes}(\mathcal{P}_*) = \{d_{ij} | i > j \geq 0\},$$

only that the left side is contained in the right one.

Lemma 4.33. *We have*

$$\mathrm{Slopes}(BP_*BP) = \mathrm{Slopes}(\mathcal{P}_*) = \{d_{ij} | i > j \geq 0\}$$

Proof. To prove this, we have to provide examples of compact objects in both $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ and $\mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{P}_*)}$ with minimal vanishing line of slope d_{ij} for each $i > j \geq 0$.

The idea is simple: Starting with $K_0 = S$, we inductively define compact objects K_l. According to Theorem 4.31 in the BP_*BP-case, and Proposition 4.28 in the $\mathcal{P}_*$-case, K_l admits a self-map θ_l parallel to the vanishing line (i.e. as an element in the ring $\pi_{**}(DK_l \otimes K_l)$).

Setting K_{l+1} to be the cofibre of $\Sigma^{|\theta_l|} K_l \to K_l$, K_{l+1} has a minimal vanishing line of strictly lower slope.

Now fix a slope d_{ij} with $i > j \geq 0$. We have that $(K\Lambda_{ij})_{**} K_0 \neq 0$, and from the long exact sequences

$$\cdots \to (K\Lambda_{ij})_{**} \Sigma^{|\theta_l|} K_l \xrightarrow{\theta_l} (K\Lambda_{ij})_{**} K_l \to (K\Lambda_{ij})_{**} K_{l+1} \to \cdots,$$

we see that if $(K\Lambda_{ij})_{**} K_l \neq 0$ either θ_l is a self-map of slope d_{ij} or $(K\Lambda_{ij})_{**} K_{l+1} \neq 0$.

Since there are only finitely many possible slopes greater than d_{ij}, we eventually encounter K_l with minimal vanishing line of smaller slope. So we eventually hit some K_l with $(K\Lambda_{ij})_{**} K_l = 0$, and for the minimal such l, θ_{l-1} on K_{l-1} is a self-map of slope d_{ij}, and K_{l-1} will have a minimal vanishing line of that slope. □

The methods used in this section can be applied to a number of other Hopf algebroids. For reference, we state the corresponding results.

Proposition 4.34. *For $p = 2$, the dual Steenrod algebra takes the form $\mathcal{A}_* = \mathbb{F}_2[\xi_i]$, and admits a degree-doubling isomorphism to $\mathcal{P}_*$. We have that every compact $X \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{A}_*)}$ admits a minimal vanishing line, and the set* Slopes($\mathcal{A}_*$) *of slopes of these vanishing lines consists precisely of all*

$$\frac{1}{2^j(2^i - 1) - 1}$$

for $i > j \geq 0$. Note that the exponent is off by 1 *compared to the formulas for $\mathcal{P}_*$.*

For any compact ring R, there is a self-map parallel to the vanishing line in $\pi_{**}^{\mathcal{A}_*}(R)$, which is unique up to taking powers, and can be taken to commute with elements of $\pi_{**}^{\mathcal{A}_*}(R)$ above a line parallel to the minimal vanishing line with arbitrarily small intercept.

Furthermore, there are exotic K-theories $K\Lambda_d$ for each of the $d \in \mathrm{Slopes}(\mathcal{A}_*)$, and they detect minimal vanishing lines and self-maps parallel to the vanishing line.

Proposition 4.35. *For p an odd prime, the dual Steenrod algebra $\mathcal{A}_*$ takes the form*

$$\mathcal{A}_* = \mathbb{F}_p[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_0^2, \tau_1^2, \ldots).$$

Every compact $X \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{A}_*)}$ admits a minimal vanishing line, and the set $\mathrm{Slopes}(\mathcal{A}_*)$ of slopes of these vanishing lines consists precisely of all

$$d_{ij} = \frac{1}{p^{j+1}(p^i - 1) - 1}$$

for $i > j \geq 0$, and all

$$e_i = \frac{1}{2(p^i - 1)}.$$

For any compact ring R, there is a self-map parallel to the vanishing line in $\pi_{**}^{\mathcal{A}_*}(R)$, which is unique up to taking powers, and can be taken to commute with elements of $\pi_{**}^{\mathcal{A}_*}(R)$ above a line parallel to the minimal vanishing line with arbitrarily small intercept.

Furthermore, there are exotic K-theories $K\Lambda_d$ for each of the $d \in \mathrm{Slopes}(\mathcal{A}_*)$, and they detect minimal vanishing lines and self-maps parallel to the vanishing line.

Remark 4.36. For $p = 2$, the Hopf algebra defined as in Proposition 4.35 still makes sense, and the analogous results still hold true. It does not agree with the $p = 2$ Steenrod algebra, but is still important as it appears when quotienting the $p = 2$ *motivic* Steenrod algebra $\mathcal{A}_*^{\mathrm{mot}}$ over $\mathbb{F}_2[\tau]$ by τ.

For example, the E_2 page of the motivic $p = 2$ Adams spectral sequence for $S/2$ admits a minimal vanishing line of slope 1 (in Adams grading, forgetting the weight), as opposed to a slope $\frac{1}{2}$ minimal vanishing line as in the classical $p = 2$ Adams spectral sequence for $S/2$.

Proposition 4.37. *For $B_n = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ and $W_n = B_n[t_1, \ldots, t_n]$ the so-called* n-bud Hopf algebroid *(with structure maps inherited as a sub Hopf-algebroid of BP_*BP), we have*

Every compact $X \in \mathrm{Comod}^{cg}_{\mathcal{D}(W_n)}$ admits a minimal vanishing line, and the set $\mathrm{Slopes}(W_n)$ of slopes of these vanishing lines consists precisely of all

$$d_{ij} = \frac{1}{p^{j+1}(p^i - 1) - 1}$$

with either $2i > n$*, or* $i > j$*.*

For any compact ring R*, there is a self-map parallel to the vanishing line in* $\pi_{**}^{W_n}(R)$*, which is unique up to taking powers, and can be taken to commute with elements of* $\pi_{**}^{W_n}(R)$ *above a line parallel to the minimal vanishing line with arbitrarily small intercept.*

Furthermore, there are exotic K*-theories* $K\Lambda_d$ *for each of the* $d \in \mathrm{Slopes}(\mathcal{A}_*)$*, and they detect minimal vanishing lines and self-maps parallel to the vanishing line.*

Proof. The main additional observation to make is that for $2i \leq n$, the Hopf algebra $W_n/(p, v_1, \ldots, v_n)$ still admits quotients of the form used in the proof of Lemma 4.15, showing that for $2i \leq n$ and $i \leq j$, we get $d_{ij} \notin \mathrm{Slopes}(W_n)$.

If $2i > n$, one can show that there are simple coalgebra quotients Λ_{ij} of type D for each j, not just $j < i$. One way to see this is to characterize the multiplication in the Hopf algebra dual to $W_n/(p, v_1, \ldots, v_n)$ in a similar way to Lemma 3.48. One obtains essentially the same formulas, but quotiented by all $P^{(I)}$ where I has support not contained in $\{1, \ldots, n\}$. From this one sees that $(P_i^j)^p = 0$ for all j if $2i > n$. □

5 Adams periodicity

The classical Adams periodicity statement [Ada66] regards the existence of a vanishing line of slope $\frac{1}{2}$ for the h_0-torsion part of $\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$, and periodic behaviour along that vanishing line.

In this section, we want to explain how the theory of vanishing lines and self-maps developed in section 4 leads to generalizations of that result. In sections 5.2 and 5.3, we will give explicit examples. Section 5.2 improves on the classical Adams periodicity on $\mathrm{Ext}_{\mathcal{A}_*}$, by constructing periodicity operators above a line of slope $\frac{1}{5}$ instead of the $\frac{1}{3}$ provided in [Ada66]. This slope $\frac{1}{5}$ line was first found by May [May].

In Section 5.3, we prove a similar periodicity statement for $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. As far as we can tell, this "BP_*BP-Adams periodicity" has not appeared in the literature so far, and was recently conjectured by Dan Isaksen based on new computer-assisted computations of $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ by Guozhen Wang.

5.1 Qualitative Adams periodicity results

Let Γ be one of the Hopf algebroids BP_*BP, $\mathcal{P}_*$ or $\mathcal{A}_*$. In Section 4, we established that any compact object $K \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ admits a minimal vanishing line, and a self-map $\theta : K \to K$ parallel to that vanishing line, i.e. such that the cofibre K/θ has a minimal vanishing line of strictly smaller slope.

The possible slopes of such vanishing lines are quite restricted, we described the set of all occuring slopes $\mathrm{Slopes}(\Gamma)$ explicitly.

In the examples we discussed, for each slope $d \in \mathrm{Slopes}(\Gamma)$, there was a corresponding simple coalgebra quotient $\Gamma \to \Lambda_d$, such that $K\Lambda_d$ detects slopes and self-maps. Concretely, the slope of the minimal vanishing line of compact K was characterized as the maximal d such that $(K\Lambda_d)_{**}K \neq 0$, and for K with minimal vanishing line of slope d, $\theta : K \to K$ is a self-map parallel to the vanishing line if and only if it acts isomorphically on $(K\Lambda_d)_{**}K$.

Definition 5.1. *For $K \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ compact, a generalized Smith-Toda complex on K is an iterated cofibre of the form*

$$K/(\theta_0, \theta_1, \ldots, \theta_k),$$

where θ_i is a self-map of $K_i := K/(\theta_0, \theta_1, \ldots, \theta_{i-1})$ parallel to the minimal vanishing line.

For $d \in \mathrm{Slopes}(\Gamma)$, if $(K\Lambda_d)_{**}K_i \neq 0$, then $(K\Lambda_d)_{**}K_{i+1} \neq 0$ precisely if the slope of θ_i is different from d.

So the slopes of the θ_i are necessarily the $k+1$ largest elements d_i of $\{d \in \mathrm{Slopes}(\Gamma) | (K\Lambda_d)_{**}K \neq 0\}$.

Lemma 5.2. *Let* $X, K \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$.

Assume that $[K, X]_{**}$ *has a vanishing line of slope* d *and intercept* c. *Let* $\theta : \Sigma^{n_0,s_0} K \to K$ *be a map with slope* $\frac{s_0}{n_0} > d$.

Then

$$[K/\theta, X]_{**} \to [K, X]_{**}$$

is an isomorphism above the line of slope d *and intercept* $c-(s_0-dn_0)$. *In particular,* $[K/\theta, X]_{**}$ *also has a vanishing line of slope* d *and intercept* c.

Proof. The result follows immediate from the long exact sequence associated to the cofibre sequence $\Sigma^{n_0,s_0} K \to K \to K/\theta$:

$$\cdots \to [K, X]_{n+n_0+1,s+s_0-1} \to [K/\theta, X]_{n,s} \to [K, X]_{n,s} \to [K, X]_{n_0+n,s_0+n} \to \cdots \quad \square$$

Lemma 5.3. *Let* $K, X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ *with* K *compact.*

Assume that $[K, X]_{**}$ *has a vanishing line of slope* d *and intercept* c.

Fix a generalized Smith-Toda complex $K/(\theta_0, \theta_1, \ldots, \theta_k)$ *on* K. *Let* $(n_i, s_i) = |\theta_i|$, *and assume that the slopes* $\frac{s_i}{n_i}$ *of the* θ_i *are all bigger than* d.

Then the restriction map

$$[K/(\theta_0, \theta_1, \ldots, \theta_k), X]_{**} \to [K, X]_{**}$$

is an equivalence above a line of slope d *and intercept*

$$c - \min_{0 \le i \le k} (s_i - dn_i)$$

Proof. This follows by inductively applying Lemma 5.2. $\square$

Recall that self-maps parallel to the vanishing line were seen to be unique up to passing to higher powers, and could be taken to commute with elements above any given line parallel to the vanishing line.

Given compact objects $X, Y \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ with minimal vanishing lines of the same slope, and self-maps θ_X and θ_Y on them, both θ_X and θ_Y define a self-map of $DX \otimes Y$. So some power of them agrees, and commutes with given $f \in \pi_{**}(DX \otimes Y)$. This implies that there are exponents a and b making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \theta_X^a}\downarrow & & \downarrow{\scriptstyle \theta_Y^b} \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

We can inductively apply this to show that if X is an object filtered by finitely many compact X_i, and all of them have a minimal vanishing line of the same slope, there are self-maps compatible with the filtration.

Lemma 5.4. *Let $K, X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ with K compact, and let*

$$K/(\theta_0, \ldots, \theta_k)$$

and

$$K/(\theta'_0, \ldots, \theta'_k)$$

be two generalized Smith-Toda complexes. Then for a d smaller than the slope of θ_k, the minimal intercepts for vanishing lines of slope d on $[K/(\theta_0, \ldots, \theta_k), X]_{**}$ and $[K/(\theta'_0, \ldots, \theta'_k), X]_{**}$ agree.

Proof. Let $K_i = K/(\theta_0, \ldots, \theta_{i-1})$, and $K'_i = K/(\theta'_0, \ldots, \theta'_{i-1})$.

We construct a sequence of L_i, such that L_i is filtered by finitely many copies of $\Sigma^{n,s} K_i$ with $\frac{s}{n} > d$, and also filtered by finitely many copies of $\Sigma^{n,s} K'_i$ with $\frac{s}{n} > d$.

It follows that the minimal intercepts for vanishing lines of slope d agree on $[K_i, X]_{**}$ and $[L_i, X]_{**}$. Analogously for $[K'_i, X]_{**}$ and $[L_i, X]$, from which the claim follows.

To define the L_i, we start by setting $L_0 = K$. Now assume that we are given L_i. Due to uniqueness of self-maps, there is a self-map compatible with both filtrations on L_i, which acts on the $\Sigma^{n,s} K_i$ in the first filtration by a power of θ_i, and on the $\Sigma^{n,s} K'_i$ by a power of θ'_i. So we can set L_{i+1} as the cofibre of that self-map. □

Proposition 5.5. *Let $K, X \in \mathrm{Comod}^{cg}_{\mathcal{D}\Gamma}$ with K compact. Given a generalized Smith-Toda complex on K,*

$$K/(\theta_0, \ldots, \theta_k),$$

assume that $[K, X]$ admits a vanishing line of slope d smaller than the slope of θ_{k-1} and intercept c, and that $[K/(\theta_0, \ldots, \theta_{k-1}, \theta_k), X]$ admits a vanishing line of slope $\widetilde{d}$ smaller than the slope of θ_k and intercept $\widetilde{c}$.

Now fix a line of slope d and intercept c. Then there is a map

$$[K, X]_{n,s} \to [K, X]_{n+n_0, s+s_0}$$

for all (n, s) above the line of slope d and intercept c, and it is an isomorphism for (n, s) above the line of slope $\widetilde{d}$ and intercept $\widetilde{c}$.

Proof. Fix a line of slope d and intercept c'. By Lemma 5.3, if we choose a different Smith-Toda complex $K/(\theta'_0, \ldots, \theta'_{k-1})$ with degrees $|\theta_i|$ large enough, the restriction map

$$[K/(\theta'_0, \ldots, \theta'_{k-1}), X]_{**} \to [K, X]_{**}$$

is an isomorphism above the line of slope d and intercept c'.

By Lemma 5.4, if we choose a self-map θ'_k of $K/(\theta'_0, \ldots, \theta'_{k-1})$ parallel to its vanishing line,

$$[K/(\theta'_0, \ldots, \theta'_{k-1}, \theta'_k), X]_{**}$$

will also be bounded by the line of slope $\widetilde{d}$ and intercept $\widetilde{c}$. So θ'_k acts isomorphically on $[K/(\theta'_0, \dots, \theta'_{k-1}), X]_{**}$ above that line. The claim follows. □

The slightly technical condition of Proposition 5.5 on existence of a generalized Smith-Toda complex such that we have a certain vanishing line on $[K/(\theta_0, \dots, \theta_{k-1}, \theta_k), X]_{**}$ is usually satisfied. For example, it is automatic for compact X, as the θ_i automatically form a sequence of self-maps of $DK_i \otimes X$ parallel to the vanishing line.

More generally, it is satisfied if X admits a cell structure with cells concentrated below a line of slope $\widetilde{d}$. In particular, this is automatic if X is represented by an actual comodule.

We close this section with another interesting application of the generalized Smith-Toda complex construction. Namely, we can generalize the bounds on p-torsion in $\pi_{**}^{BP_*BP}$ obtained along the vanishing line in Lemma 4.30 to much more general regions.

Proposition 5.6. *Let $K \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ be compact, and let $d > 0$ and c be arbitrary.*

Then the exponent of p-torsion in $\pi_{**}^{BP_*BP}(K)$ in the region above the line of slope d and intercept c is bounded, i.e. there is e such that every class in that region either has infinite order or is annihilated by p^e.

Proof. Fix K, $d > 0$ and c. We consider a generalized Smith-Toda complex $K/(\theta_0, \dots, \theta_k)$ and denote $K_i := K/(\theta_0, \dots, \theta_{i-1})$. By choosing the degree of the θ_i large enough, we can assume that they are all p-torsion classes, since $\mathrm{End}(K_i) = K_i \otimes DK_i$ is compact and thus the homotopy groups $\pi_{**}(\mathrm{End}(K_i))$ are torsion in all but finitely many degrees. Furthermore, since the slope of the minimal vanishing line of K_i decreases with i, if we choose k large enough, K_{k+1} will have a minimal vanishing line of slope smaller than d.

Thus, in $\pi_{**}(K_{k+1})$, there are only finitely many nontrivial homotopy groups in the region above the line of slope d and intercept c. Thus, $\pi_{**}(K_{k+1})$ trivially admits a bound on the exponent of p-torsion in that region.

We now proceed by downwards induction. Assume that $\pi_{**}K_i$ admits a bound e_i on the exponent of p-torsion in the region above the line of slope d and intercept c.

We then consider the long exact sequence

$$\cdots \to \pi_{(n,s)-|\theta_i|}(K_{i-1}) \xrightarrow{\theta_i} \pi_{n,s}(K_{i-1}) \to \pi_{n,s}(K_i) \to \cdots$$

It follows that $\pi_{n,s}(K_{i-1})$ is an extension of a subgroup of $\pi_{n,s}(K_i)$, and the image of θ_i. For (n, s) in the region above the line of slope d and intercept c, both of these have bounded exponent on their p-torsion subgroups: $\pi_{n,s}(K_i)$ has exponent bounded by e_i due to the inductive assumption, and the image of θ_i is p-torsion with exponent bounded by the order of θ_i. Thus, if $\mathrm{ord}(\theta_i) = p^l$, we can take $e_{i-1} = e_i + l$.

Inductively, the claim follows for $K_0 = K$. □

5.2 Classical Adams periodicity

In this section, we obtain classical Adams periodicity as a special case of Proposition 5.5, and work out the explicit bounds.

Corresponding to the May spectral sequence names for elements in $\mathrm{Ext}_{\mathcal{A}_*}$, we will call a self-map parallel to a minimal vanishing line of slope $1/(2^j(2^i-1)-1)$ an h_{ij} self-map. For example, the usual degree $(8,4)$ self-map of S/h_0 will be referred to as an $h_{2,0}$ self-map.

Lemma 5.7. *let F be the fibre of $S \to h_0^{-1}S$ in $\mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{A}_*)}$. Then the map $F \to S$ induces an isomorphism on $\pi^{\mathcal{A}_*}_{**}$ in degrees (n,s) with $n > 0$, and $\pi_{**}(F)$ has a vanishing line of slope $\frac{1}{2}$ and intercept $\frac{3}{2}$.*

Proof. Observe that the map $S \to H(\mathcal{A}_*(0))$ is represented on comodules by the map $\mathbb{F}_2 \to \mathcal{A}_* \square_{\mathcal{A}_*(0)} \mathbb{F}_2$ of comodules. The cokernel is easily seen to be $\mathcal{A}_*(0)$-cofree. Thus, the map $h_0^{-1}S \to h_0^{-1}H(\mathcal{A}_*(0))$ is an equivalence over $\mathcal{A}_*(0)$. By repeated application of the spectral sequence of Lemma 3.42, it is then an equivalence over $\mathcal{A}_*(n)$ for any n, and thus over $\mathcal{A}_*$ by a connectivity argument.

So $h_0^{-1}S$ is concentrated in degrees $(0,s)$.

The vanishing line for F can be computed over $\mathcal{A}_*(1)$ explicitly. By inductive application of the spectral sequence of Lemma 3.42, it then follows over $\mathcal{A}_*$, using that the slopes of all subsequent β_Λ are smaller than $\frac{1}{2}$. □

Remark 5.8. This is a minimal vanishing line for F, as there is a periodic family of nontrivial classes in degrees $(3+8k, 3+4k)$.

So the F from Lemma 5.7 has a minimal vanishing line. However, it is not compact, and does not admit a self-map of slope $\frac{1}{2}$.

We will show that the homotopy groups of F above any line of slope $\frac{1}{2}$ do still admit periodicity above a line of lower slope, but the period increased as we decrease the intercept of the line of slope $\frac{1}{2}$.

Lemma 5.9. *For*

$$s > \frac{1}{2}n + 3 - k$$

the natural map

$$[S/h_0^k, F]_{n,s} \to [S,F]_{n,s}$$

is an isomorphism.

Proof. This is a special case of Lemma 5.2. □

Since S/h_0^k is a compact object with minimal vanishing line of slope $\frac{1}{2}$, it does admit a self-map of slope $\frac{1}{2}$, i.e. an $h_{2,0}$ self-map $\theta : \Sigma^{|\theta|} S/h_0^k \to S/h_0^k$.

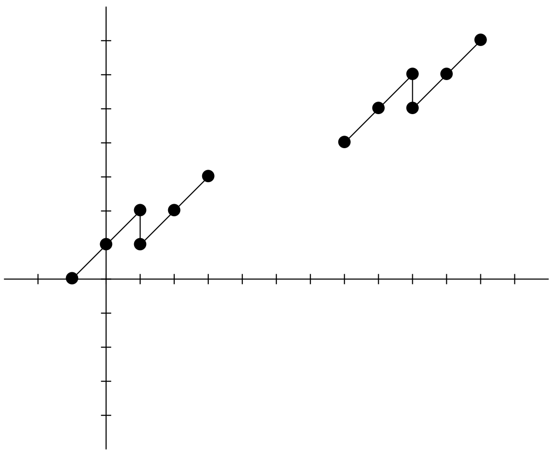
To apply Proposition 5.5 with explicit bounds, we need to compute a vanishing line of $[S/(h_0^k, \theta), K]_{**}$ for one generalized Smith-Toda complex.

Proposition 5.10. *Let θ be an $h_{2,0}^4$-selfmap of S/h_0. Then*

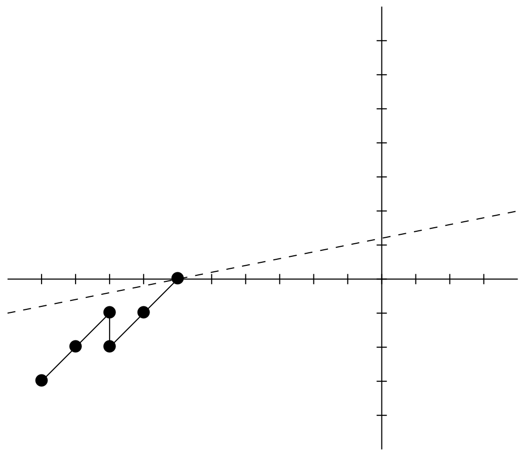
$$[S/(h_0, \theta), S]_{**}$$

has a vanishing line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$.

Proof. It is enough to compute that vanishing line over $\mathcal{A}_*(2)$. We can then successively apply the spectral sequence from Lemma 3.42, and for degree reasons the vanishing line stays the same over arbitrary $\mathcal{A}_*(n)$.



$[S/h_0, S]_{**}^{\mathcal{A}_*(1)}$



$[S/(h_0, \theta), S]_{**}^{\mathcal{A}_*(1)}$. The dashed line indicates the minimal vanishing line of slope $\frac{1}{5}$ for $[S/(h_0, \theta), S]_{**}^{\mathcal{A}_*(2)/\xi_1^4}$.

First, since the $\otimes$-dual of the cofibre sequence

$$S^{0,1} \xrightarrow{h_0} S \to S/h_0$$

is the fibre sequence

$$D(S/h_0) \to S \xrightarrow{h_0} S^{0,-1},$$

we can identify $D(S/h_0)$ with $\Sigma^{-1,0} S/h_0$. So $[S/h_0, S]_{**}^{\mathcal{A}_*(1)} \simeq [S, \Sigma^{-1,0} S/h_0]_{**}$. It follows that $[S/h_0, S]_{**}^{\mathcal{A}_*(1)}$ has a vanishing line of slope $\frac{1}{2}$ and intercept $\frac{3}{2}$, and consists of an $(8,4)$-periodic sequence of "lightning flashes", the first one of which starts in degree $(-1,0)$.

An $h_{2,0}^4$-selfmap necessarily acts by this degree $(8,4)$-periodicity map. The associated long exact sequence reads

$$\cdots \to [S/(h_0,\theta),S]^{\mathcal{A}_*(1)}_{n,s} \to [S/h_0,S]^{\mathcal{A}_*(1)}_{n,s} \xrightarrow{\theta} [S/h_0,S]^{\mathcal{A}_*(1)}_{n+8,s+4} \\ \to [S/(h_0,\theta),S]^{\mathcal{A}_*(1)}_{n-1,s+1} \to \cdots$$

and θ acts injectively. The sequence therefore splits up into short exact sequences, and $[S/(h_0,\theta),S]^{\mathcal{A}_*(1)}_{**}$ consists of the single lightning flash pattern starting in degree $(-10,-3)$. (This is the image of the lightning flash starting in bidegree $(-1,0)$ under the connecting homomorphism from above, which has degree $(-9,-3)$.)

We have a sequence of normal extensions by simple Hopf algebras of type E,

$$\mathcal{A}_*(2) \to \mathcal{A}_*(2)/\xi_1^4 \to \mathcal{A}_*(2)/(\xi_1^4,\xi_2^2) \to \mathcal{A}_*(2)/(\xi_1^4,\xi_2^2,\xi_3) = \mathcal{A}_*(1).$$

The first two of the associated Adams spectral sequences (cf. Lemma 3.42) lead to a vanishing line of slope $\frac{1}{5}$ through $(-6,0)$, since $[\xi_3]$ and $[\xi_2^2]$ have slopes $\frac{1}{6}$ and $\frac{1}{5}$, and the lightning flash pattern starting in degree $(-10,-3)$ lies below the line of slope $\frac{1}{5}$ through $(-6,-0)$.

The final Adams spectral sequence, where we pass to $\mathrm{Ext}_{\mathcal{A}_*(2)}$, adjoins the element $h_2=[\xi_1^4]$. This has slope $\frac{1}{3} > \frac{1}{5}$, but it is nilpotent. Namely, h_2^4 is already zero in $\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2,\mathbb{F}_2)$, so in the E_∞ page in the corresponding Adams spectral sequence every nonzero element is in filtration ≤ 3. It follows that the E_∞ page is bounded by a line of slope $\frac{1}{5}$ through $(-6+3\cdot 3, 3\cdot 1)=(3,3)$, cf. Remark 4.11. This line has intercept $\frac{12}{5}$. □

Remark 5.11. The intercept of the vanishing line established for $[S/(h_0,\theta),S]^{\mathcal{A}_*}_{**}$ in Lemma 5.10 is not minimal. However, obtaining more precise bounds requires the actual computation of $[S/(h_0,\theta),S]^{\mathcal{A}_*(2)}_{**}$.

It follows that for any k and any $h_{2,0}$ self-map θ of S/h_0^k, the corresponding $[S/(h_0^k,\theta),K]_{**}$ has a vanishing line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$, using Lemma 5.4.

Proposition 5.12. *For $k \leq 2^l$, $l \geq 2$, S/h_0^k admits an $h_{2,0}$ self-map of degree $(2\cdot 2^l, 2^l)$, i.e. an $h_{2,0}^{2^l}$ self-map.*

Proof. One can check, using power operations, that

1. For $l \geq 2$, the relations $h_{l+1}h_0^{2^l}=0$ hold in $\mathrm{Ext}_{\mathcal{A}_*}$, where $h_l=[\xi_1^{2^l}]$.
2. Any element u of the cobar complex with $du = h_{l+1}h_0^{2^l}$ maps to $[\xi_2]^{2^l}$ in the cohomology of the $\mathcal{A}_*$-quotient Hopf algebra $\mathbb{F}_2[\xi_2]/\xi_2^2$.

This means that the Massey product $\langle h_{l+1}, h_0^{2^l}, x\rangle$, whenever defined, restricts to $[\xi_2]^{2^l}x$ in the cohomology over $\mathbb{F}_2[\xi_2]/\xi_2^2$. In fact, with $du = h_{l+1}h_0^{2^l}$, $dv = h_0^{2^l}x$, $ux + h_{l+1}v \in \langle h_{l+1}, h_0^{2^l}, x\rangle$ reduces to ux.

In $\pi_{**}(S/h_0^{2^l})$, this Massey product has a universal example $\langle h_{l+1}, h_0^{2^l}, 1\rangle$. This defines an element in $\pi_{2^{l+1},2^l}(S/h_0^{2^l})$ (once we fix a representative), and we ask whether it extends to a map $\Sigma^{2^{l+1},2^l}S/h_0^{2^l} \to S/h_0^{2^l}$. This is equivalent to asking whether $h_0^{2^l}\langle h_{l+1}, h_0^{2^l}, 1\rangle = 0$, or, by juggling, whether $\langle h_0^{2^l}, h_{l+1}, h_0^{2^l}\rangle 1$ is 0.

This Massey product vanishes in $\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ already, for example simply by degree reasons: It is way above the slope $\frac{1}{2}$ vanishing line. Alternatively, there's a general formula for such symmetric Massey products in terms of squaring operations, the so-called *Hirsch formula*.

In total, this shows that there is such a self-map on $S/h_0^{2^l}$ or any S/h_0^k with $k \leq 2^l$, as the required relation $h_0^{2^l} \cdot 1 = 0$ holds there. □

Remark 5.13. This also shows that the partial operation on $\pi_{**}(X)$ given by extending an element over S/h_0^k, precomposing with the $h_{2,0}^{2^n}$ self-map given by Proposition 5.12, and restricting again to the bottom cell precisely coincides with the Massey product operation $\langle h_{n+1}, h_0^{2^n}, -\rangle$ (but has smaller indeterminacy since fixing one self-map corresponds to fixing a choice of u with $du = h_{n+1}h_0^{2^n}$).

This is Adams' original description of the periodicity operator.

Altogether, we have shown the following:

Theorem 5.14. *Fix $l \geq 2$ and u with $du = h_{l+1} \cdot h_0^{2^l}$. The Massey product operation $\langle h_{l+1}, h_0^{2^l}, -\rangle$ (using u) is uniquely defined on $\pi_{n,s}^{\mathcal{A}_*}(S)$ for $n > 0$ and (n, s) above the line of slope $\frac{1}{2}$ and intercept $3 - 2^l$.*

For (n, s) above the line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$, it is an isomorphism

$$\pi_{n,s}(S) \to \pi_{2^{l+1}+n,2^l+s}(S).$$

Remark 5.15. The region in which the Massey product is uniquely defined can be refined a little by analyzing the precise values of $\pi_{**}(F)$ in a region below and close to the vanishing line. This is the original result in [Ada66], which establishes a slightly larger range in which this periodicity holds.

On the other hand, the line established there above which the periodicity operator is periodic only has slope $\frac{1}{3}$, which is what we would have obtained in the proof of proposition 5.10 if we hadn't used the nilpotency of h_2.

This easier bound can be obtained through elementary methods from the fact that $\mathcal{A}_* \to \mathcal{A}_*(1)$ is an isomorphism in degrees below 4.

5.3 BP_*BP Adams periodicity

We now want to obtain a similar statement for $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. This time, the highest slope periodicity of S is with respect to $\alpha_1 \in \pi_{1,1}^{BP_*BP}(S)$, and the

α_1-torsion part is bounded by a vanishing line of slope $\frac{1}{5}$. Parallel to that line, we will establish periodicity operators above a line of slope $\frac{1}{11}$.

Unlike the case over $\mathcal{A}_*$, where the h_0-periodic part of $\mathrm{Ext}_{\mathcal{A}_*}$ was confined to degrees $(0, s)$, the α_1-periodic part is already quite complicated.

In [AM17], $\alpha_1^{-1}\,\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ is computed to be the free $\mathbb{F}_2[\alpha_1^{\pm 1}]$-module on the elements $\alpha_n \in \mathrm{Ext}^{1,2n}_{BP_*BP}(BP_*, BP_*)$.

In particular, all α_1-periodic classes originate on the $s = 1$-line. We obtain:

Lemma 5.16. *For F the fibre of $S \to \alpha_1^{-1}S$, the map $F \to S$ induces an isomorphism on the α_1-torsion part of π_{**} in degrees (n, s) with $s \geq 2$.*

Lemma 5.17. *$\pi_{**}(F)$ has a vanishing line of slope $\frac{1}{5}$ and intercept $\frac{6}{5}$.*

Proof. Over $\mathcal{P}_*$, F corresponds under the degree-halving isomorphism $\mathcal{P}_* \simeq \mathcal{A}_*$ to $\mathrm{fib}(S \to h_0^{-1}S) \in \mathrm{Comod}^{cg}_{\mathcal{D}(\mathcal{A}_*)}$, since the degree-halving isomorphism identifies $h_1 \in \mathrm{Ext}_{\mathcal{P}_*}$ with $h_0 \in \mathrm{Ext}_{\mathcal{A}_*}$. From Lemma 5.7, we obtain a vanishing line for $\pi^{\mathcal{P}_*}_{**}(F)$ of slope $\frac{1}{5}$ and intercept $\frac{6}{5}$ through regrading.

By applying the Adams spectral sequence for $BP_*BP \to \mathcal{P}_*$ as in the proof of Proposition 4.22, we obtain the same vanishing line over BP_*BP. □

Through the degree-halving isomorphism between $\mathcal{P}_*$ and $\mathcal{A}_*$, the classical Adams periodicity results imply that the operator $\langle h_{n+2}, h_1^{2^n}, - \rangle$ is an isomorphism on $\pi^{\mathcal{P}_*}_{**}(F)$ above a line of slope $\frac{1}{11}$ and intercept $\frac{24}{11}$, in the range where it is defined uniquely.

If we had a lift of that operator to BP_*BP, it would also act isomorphically above that same line, again since $BP_*BP \to \mathcal{P}_*$ detects vanishing lines as in Proposition 4.22.

However, the Massey product operator does not immediately lift up to BP_*BP, as the elements $h_n \in \mathrm{Ext}_{\mathcal{P}_*}$ don't lift to Ext_{BP_*BP} for $n \geq 3$.

Lemma 5.18. *There is an element $e_i \in \pi^{BP_*BP}_{**}(S)$ which maps to $h_1 h_i \in \pi^{\mathcal{P}_*}_{**}$.*

Proof. The idea is to write this as a composite of maps in $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$. First of all, notice that in $\pi^{BP_*BP}_{**}(S/2) \simeq \pi^{BP_*BP/2}_{**}(S)$, there actually is a lift of h_i, represented by $[t_1^{2^i}]$ in the cobar complex.

Now observe that the element $\alpha_1 \in \pi^{BP_*BP}_{**}(S)$ is 2-torsion, so it extends to a map $\Sigma^{1,1}S/2 \to S$.

Their composite map $\Sigma^{2^i,2}S \to S$ is the desired class. □

Remark 5.19. These e_i are actually permanent cycles in the Adams-Novikov spectral sequence, and correspond to the Mahowald η_i (cf. [Mah77]) in homotopy.

Lemma 5.20. *e_{n+2} satisfies the relation $e_{n+2}\alpha_1^{2^n-1} = 0$.*

Proof. In the corresponding degree, $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ actually vanishes. This is not quite above the slope $\frac{1}{5}$ vanishing line, but very close - using a refinement of the vanishing region over $\mathcal{A}_*$, we can see that $\mathrm{Ext}_{\mathcal{P}_*}(\mathbb{F}_2, \mathbb{F}_2)$ is zero in this bidegree and to the left of it. The Adams spectral sequence for $BP_*BP \to \mathcal{P}_*$ then shows the same for $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, using arguments similar to the proof of Proposition 4.22. □

This allows us to define the operator $\langle e_{n+2}, \alpha_1^{2^n-1}, -\rangle$ on Ext_{BP_*BP}. Putting together the results of this and the previous section, we obtain

Theorem 5.21. *On the h_1-torsion classes in $\pi_{**}^{BP_*BP}(S)$, the Massey product operator $\langle e_{k+2}, \alpha_1^{2^k-1}, -\rangle$ is uniquely defined (with a fixed choice of nullhomotopy of $e_{k+2}\alpha_1^{2^k-1}$) on $\pi_{n,s}(S)$ for (n,s) above the line of slope $\frac{1}{5}$ and intercept $\frac{12}{5} - \frac{4}{5}(2^k-1)$.*

For (n,s) above the line of slope $\frac{1}{11}$ and intercept $\frac{24}{11}$, it is an isomorphism

$$\pi_{n,s}(S) \to \pi_{n+5\cdot 2^k, s+2^k}(S).$$

Proof. The map $F \to BP_*$ induces an isomorphism on h_1-torsion above the 1-line. So it is sufficient to work in $\pi_{**}^{BP_*BP}(F)$, where we have a vanishing line of slope $\frac{1}{5}$ and intercept $\frac{6}{5}$.

It follows that the Massey product $\langle e_{k+2}, \alpha_1^{2^k}, -\rangle$ is uniquely defined whenever

$$s > \frac{1}{5}n + \frac{12}{5} - \frac{4}{5}(2^k - 1)$$

since in that range,

$$[S/\alpha_1^{2^k-1}, F]_{**} \to [S, F]_{**}$$

is an isomorphism. On $S/\alpha_1^{2^k-1}$, the Massey product $\langle e_{k+2}, \alpha_1^{2^k}, -\rangle$ has a universal example. Since the Massey product $\langle \alpha_1^{2^k-1}, e_{k+2}, \alpha_1^{2^k-1}\rangle \in \mathrm{Ext}_{BP_*BP}$ is α_1-torsion, but well above the $\frac{1}{5}$ line, it vanishes. The universal example therefore extends to a self-map θ of $S/\alpha_1^{2^k-1}$.

It reduces to an $h_{2,1}$ self-map of $S/h_1^{2^k-1}$ over $\mathcal{P}_*$. So $[S/(\alpha_1^{2^k-1}, \theta), F]$ has a vanishing line of slope $\frac{1}{11}$ and intercept $\frac{24}{11}$, as obtained from Proposition 5.10, the degree-doubling isomorphism, and the fact that $BP_*BP \to \mathcal{P}_*$ detects vanishing lines as in Proposition 4.22. □

6 Motivic homotopy theory

In this section, we apply our results on vanishing lines and periodicity in $\mathrm{Comod}_{\mathcal{D}BP_*BP}$ to obtain corresponding results in motivic homotopy theory.

The relevant connection between the two is a theorem of Gheorghe, Wang and Xu [GWX]. This connection was first suggested by Isaksen based on observations made in [Isa14].

It allows us to identify a category of modules over a certain spectrum S/τ in the p-complete cellular stable motivic homotopy category over $\mathbb{C}$ with a full subcategory of $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, as considered in Section 2.4.

We will give a self-contained account of the GWX-Theorem in Section 6.2. The required preliminaries about motivic homotopy theory will be given in Section 6.1.

Finally, in Section 6.3 we will lift the statements obtained in Section 6.2 for S/τ-modules to the whole motivic homotopy category, by considering an associated Bockstein spectral sequence. This leads to new results on vanishing lines and self-maps for motivic spectra.

6.1 A short introduction to motivic homotopy theory

The stable motivic homotopy category over a ring R, constructed by Morel and Voevodsky ([Mor99], [MV99]), is a stable symmetric-monoidal $(\infty, 1)$-category with very interesting connections to both algebraic geometry and algebraic topology.

For a ring R, the category of motivic spaces over R is constructed by first considering the intermediate $(\infty, 1)$-category $\mathrm{Sh}(\mathrm{Sm}_R)$ of simplicial sheaves on the Nisnevich site of smooth schemes over R, with an appropriately homotopy-coherent notion of descent.

This category thus receives a functor from the category $\mathcal{S}$ of spaces, through "constant sheaves", and a functor from the site of smooth schemes Sm_R, through the Yoneda embedding. We can then further localize $\mathrm{Sh}(\mathrm{Sm}_R)$ at all projection maps

$$X \times \mathbb{A}^1 \to X,$$

where $X \in \mathrm{Sh}(\mathrm{Sm}_R)$ is any object, and $\mathbb{A}^1$ is the image of the scheme $\mathbb{A}^1 = \mathrm{Spec}\, R[x]$ under the Yoneda embedding, to obtain the $\mathbb{A}^1$-homotopy category, or $(\infty, 1)$-category of motivic spaces $\mathcal{S}_R$ over R.

We will focus on the case $R = \mathbb{C}$. In this case, there is a functor $\mathrm{Sm}_{\mathbb{C}} \to \mathcal{S}$ that sends a smooth scheme over $\mathbb{C}$ to the set of its complex points with the usual (smooth) topology, locally defined as the subspace topology of the usual topology on $\mathbb{C}^n$.

Through left Kan extension, this induces a functor $\mathrm{Sh}(\mathrm{Sm}_{\mathbb{C}}) \to \mathcal{S}$, which, since it sends $\mathbb{A}_1$ to the contractible homotopy type, factors through a functor $B : \mathcal{S}_{\mathcal{C}} \to \mathcal{S}$.

This is called the Betti realization functor, and it is what makes motivic homotopy theory over $\mathbb{C}$ so useful for studying classical homotopy theory.

For example, there are two very canonical lifts of the homotopy type S^1 along B. One is given by the constant sheaf with value S^1, which we will call $S^{1,0}$. The other one is given by (the image under the Yoneda embedding of) $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, which we will call $S^{1,1}$. Note that $S^{1,1}$ indeed Betti-realizes to $\mathbb{C} \setminus \{0\} \simeq S^1$.

Taking smash powers of $S^{1,0}$ and $S^{1,1}$, we can define spheres $S^{n+k,k}$ for all $n, k \geq 0$. Here $n+k$ will be referred to as the *total dimension* or just dimension, and k will be referred to as the *weight*. We can define homotopy groups $\pi_{n+k,k}(X) = \pi_0 \operatorname{Map}_{\mathcal{S}_\mathbb{C}}(S^{n+k,k}, X)$, these are indeed groups for $n \geq 1$ and abelian for $n \geq 2$, since we can identify S^1 with the suspension of the constant sheaf with value S^0.

Betti realization gives rise to maps $\pi_{n+k,k}(X) \to \pi_{n+k}(BX)$. One can therefore consider the collection of groups $\pi_{n+k,k}(X)$ for fixed $n+k$ as a more refined version of the classical homotopy groups $\pi_{n+k}(BX)$, in the p-complete case see Proposition 6.3 for a precise statement of that form.

From $\mathcal{S}_\mathbb{C}$, we can obtain a closed symmetric-monoidal stable $(\infty,1)$-category of motivic spectra $\mathrm{Sp}_\mathbb{C}$ by stabilizing with respect to $S^{1,0}$ and $S^{1,1}$. We refer the reader to [Hov01], [Jar00] and [Hu03] for different approaches to constructing a structured category of motivic spectra.

Since $S^{1,0}$ and $S^{1,1}$ are invertible in $\mathrm{Sp}_\mathbb{C}$, we can construct spheres of any bidegree $S^{n,w}$. So homotopy groups, as well as homology and cohomology groups represented by a spectrum $E \in \mathrm{Sp}_\mathbb{C}$, are all bigraded.

The homotopy groups of the motivic sphere spectrum $S \in \mathrm{Sp}_\mathbb{C}$ are subtle and contain big uncountable rational summands, for example the so called Milnor-Witt K-theory of $\mathbb{C}$, as discussed in [Mor04] or [Mor12]. Globally, the rational motivic homotopy groups are still not fully understood.

However, the homotopy groups of the p-completed sphere $S_p^\wedge$ can be approached through an Adams spectral sequence.

Proposition 6.1 ([Voe10], [Voe03b], [Voe03a], [Voe11])**.** *There is a spectrum $H\mathbb{F}_p^{\mathrm{mot}} \in \mathrm{Sp}_\mathbb{C}$, with the following properties:*

1. *$H\mathbb{F}_p^{\mathrm{mot}}$ Betti-realizes to the classical $H\mathbb{F}_p$.*

2. *The homotopy groups of $H\mathbb{F}_p^{\mathrm{mot}}$ are given by $\pi_{**}(H\mathbb{F}_p^{\mathrm{mot}}) = \mathbb{F}_p[\tau]$. Here $|\tau| = (0,-1)$.*

3. *The spectrum $H\mathbb{F}_p^{\mathrm{mot}}$ is flat, and the Hopf algebroid of cooperations of $H\mathbb{F}_p^{\mathrm{mot}}$ is a Hopf algebra over $\mathbb{F}_p[\tau]$ of the form*

$$(H\mathbb{F}_p^{\mathrm{mot}})_{**} H\mathbb{F}_p^{\mathrm{mot}} = \mathbb{F}_p[\tau, \tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/I$$

with relations $I = (\tau_0^2, \tau_1^2, \ldots)$ *for odd* p*, and* $I = (\tau_0^2 - \tau\xi_1, \tau_1^2 - \tau\xi_2, \ldots)$ *for* $p = 2$*. Here* $|\tau_i| = (2p^i - 1, p^i - 1)$ *and* $|\xi_i| = (2p^i - 2, p^i - 1)$*.*

The comultiplication is given by

$$\Delta\xi_n = \xi_n \otimes 1 + 1 \otimes \xi_n + \sum_{0<i<n} \xi_i^{p^{n-i}} \otimes \xi_{n-i}$$

and $\Delta\tau_n = \tau_n \otimes 1 + 1 \otimes \tau_n$ *for odd* p*, whereas*

$$\Delta\tau_n = \tau_n \otimes 1 + 1 \otimes \tau_n + \sum_{0<i<n} \xi_{i+1}^{2^{n-i-1}} \otimes \xi_{n-i}$$

for $p = 2$*.*

Proposition 6.2 ([HKO11a], [HKO11b])**.** *Under suitable connectivity assumptions on* X*, the (trigraded)* $H\mathbb{F}_p^{\mathrm{mot}}$*-based Adams spectral sequence for* X *converges to the homotopy groups of the* p*-completion*

$$X_p^\wedge = \lim X/p^k.$$

This makes it quite possible to compute the homotopy groups of the p-completed motivic sphere spectrum through a range, i.e. we have similar control over $\pi_{**}(S_p^\wedge)$ as in the classical case.

In contrast to the classical case, the rational homotopy groups of the motivic sphere spectrum are complicated, and not fully known. Thus, we will work in the category of p-complete motivic spectra. At this point, we will also remark that, contrary to the situation in the classical category of spectra, the motivic spheres $S^{n,w}$ don't generate the stable motivic category, as the equivalences are defined in terms of actual $\mathbb{A}_1$-homotopy equivalences. One can either localize with respect to all weak equivalences, i.e. isomorphisms on homotopy groups, or pass to the full subcategory on so-called *cellular* spectra, which is the subcategory generated by the $S^{n,w}$ under arbitrary colimits.

Given a bigraded $\mathbb{F}_p[\tau]$-module M, we can base-change it along the map $\mathbb{F}_p[\tau] \to \mathbb{F}_p$, $\tau \mapsto 1$, to obtain an $\mathbb{F}_p$-module $\mathbb{F}_p \otimes_{\mathbb{F}_p[\tau]} M$. This has one less grading, as $\mathbb{F}_p[\tau] \to \mathbb{F}_p$ is not homogeneous with respect to weight.

The base-change functor to $\mathbb{F}_p$ is exact. This can be seen most easily by writing it as the composite of two functors. The base-change along $\mathbb{F}_p[\tau] \to \mathbb{F}_p[\tau^{\pm 1}]$, $M \mapsto \tau^{-1}M$, is exact because it can be written in terms of a filtered colimit. The base-change along $\mathbb{F}_p[\tau^{\pm 1}] \to \mathbb{F}_p$ gives rise to an equivalence between bigraded $\mathbb{F}_p[\tau^{\pm 1}]$-modules and graded $\mathbb{F}_p$-modules, so it is also exact.

Since there is a natural transformation

$$\mathbb{F}_p \otimes_{\mathbb{F}_p[\tau]} \pi_{**}(H\mathbb{F}_p^{\mathrm{mot}} \otimes X) \to H_*(BX; \mathbb{F}_p)$$

from the base-changed motivic $\mathbb{F}_p$-homology of X to the classical $\mathbb{F}_p$-homology of the Betti realization BX, both sides are homology theories, and they agree on $S^{n,w}$. It follows that the natural transformation between them is an equivalence for all cellular spectra X.

In particular, we see that the classical $H\mathbb{F}_p$-based Adams spectral sequence for BX can be obtained from the motivic one for X by applying $\mathbb{F}_p \otimes_{\mathbb{F}_p[\tau]} (-)$, i.e. "inverting τ and setting it equal to 1".

Through the Adams spectral sequence, this fact carries over to p-complete homotopy groups. The element $\tau \in (H\mathbb{F}_p)_{**}$ is a permanent Adams spectral sequence cycle, and gives rise to an element $\tau \in \pi_{**}(S_p^\wedge)$. Thus, there is an action of τ on $\pi_{**}(X)$ for each p-complete motivic spectrum X.

Proposition 6.3. *For X a p-completed cellular motivic spectrum, the map*

$$\pi_{**}(X) \to \pi_*(BX),$$

induced by Betti realization, factors through an isomorphism

$$\mathbb{Z}_p^\wedge \otimes_{\mathbb{Z}_p^\wedge[\tau]} \pi_{**}(X) \xrightarrow{\simeq} \pi_*(BX),$$

of graded rings, where τ acts on $\mathbb{Z}_p^\wedge$ by 1.

Proposition 6.4 ([HKO11b]). *There is a p-complete motivic spectrum BP^{mot}, Betti-realizing to classical p-completed BP, with homotopy groups $\mathbb{Z}_p^\wedge[\tau, v_1, v_2, \ldots]$, where $|v_i| = (2p^i - 2, p^i - 1)$.*

The cooperations $BP^{\mathrm{mot}}_{**}BP^{\mathrm{mot}}$ are of the form $BP^{\mathrm{mot}}_{**}[t_1, t_2, \ldots]$, with $|t_i| = (2p^i - 2, p^i - 1)$. The structure maps of the corresponding Hopf algebroid are described by the same formulas as for classical BP, in particular, they don't involve τ.

This spectrum is constructed in terms of a motivic analogue of the Thom spectrum MU. The homotopy groups are only known after p-completion, and are obtained through the $H\mathbb{F}_p$-based Adams spectral sequence.

The statements in Proposition 6.4 about the structure of the $BP^{\mathrm{mot}}_{**}BP^{\mathrm{mot}}$ Hopf algebroid follow easily from Proposition 6.3, using that the elements v_i and t_i Betti-realize to their classical counterparts. One sees that the parts of BP^{mot}_{**}, $BP^{\mathrm{mot}}_{**}BP^{\mathrm{mot}}$ in degrees (n, w) with $n - 2w = 0$ are the subrings $\mathbb{Z}_p^\wedge[v_i]$ and $\mathbb{Z}_p^\wedge[v_i, t_i]$, on which Betti realization is therefore an isomorphism onto the p-completed classical BP_* and BP_*BP. Finally, since the structure maps are homogeneous with respect to bidegree, this determines them completely on the v_i, t_i. The element τ is necessarily invariant, since it comes from the sphere.

Proposition 6.5. *The (trigraded) motivic Adams spectral sequence based on BP^{mot} converges for suitably connective p-complete motivic spectra X. Its E_2 page agrees with*

$$\mathrm{Ext}_{BP^{\mathrm{mot}}_{**}BP^{\mathrm{mot}}}(BP^{\mathrm{mot}}_{**}, BP^{\mathrm{mot}}_{**}X).$$

It follows that $\mathrm{Ext}_{BP^{\mathrm{mot}}_{**}BP^{\mathrm{mot}}}(BP^{\mathrm{mot}}_{**}, BP^{\mathrm{mot}}_{**})$ can be identified with the classical $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)[\tau]$, where $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ is made trigraded such that it is concentrated in degrees (s, t, w) with $t - 2w = 0$.

Since a differential d_r in the motivic Adams-Novikov spectral sequence preserves w, lowers $t - s$ by 1, and increases s by r, we see that d_r increases $t - 2w$ by $r - 1$. Note that, since all terms are concentrated in degrees t even, the only nontrivial differentials are on odd-indexed pages.

As in the proof of Lemma 4.4, one then sees inductively that the E_{2r+1}-page splits as a $\mathbb{Z}_p^\wedge[\tau]$-module into summands of the form $A[\tau]/\tau^i$, with $i < r$, and A in degrees with $t - 2w = 0$. The d_{2r+1}-differential is zero on all τ-torsion classes, and determined on its values on the degree $t - 2w = 0$ part in general. For degree reasons, $d_{2r+1}x$ for x in degree $t - 2w = 0$ is precisely divisible by τ^r, from which it follows that the d_{2r+1} differentials introduce precisely τ^r torsion on generators in degree $t - 2w$.

Since inverting τ (more precisely, base-changing along $\mathbb{Z}_p^\wedge[\tau] \to \mathbb{Z}_p^\wedge$) allows us to compare the motivic Adams-Novikov spectral sequence of the sphere with the classical one, all these differentials are then determined by the classical differentials.

Proposition 6.6. *Every permanent cycle in the classical Adams-Novikov spectral sequence in degree n and filtration s gives rise to an element in motivic homotopy $\pi_{n,w}(S_p^\wedge)$, with weight $w = \frac{n+s}{2}$. The element is τ^r-torsion precisely if it dies on the E_{2r+1}-page.*

In that sense, we can view p-complete motivic homotopy groups as encoding a complete history of classical Adams-Novikov cycles. Since we can also compute motivic homotopy groups through the $H\mathbb{F}_p$-based Adams spectral sequence, this gives a very satisfying mechanism to compare the classical Adams- and Adams-Novikov spectral sequences in a rigorous way.

For an extended discussion of these results, see [Isa14], Chapter 6.

6.2 The GWX-Theorem

From now on, we will work in the category of p-completed cellular motivic spectra over C. We will denote the p-complete motivic sphere by S as well, to simplify notation.

Similar to the situation for the motivic Adams-Novikov spectral sequence for the sphere discussed in propositions 6.5 and 6.6, the E_2-page of the motivic Adams-Novikov spectral sequence for the cofibre S/τ of $\tau : S^{0,-1} \to S$ can be identified with $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, with weight such that it is concentrated in degrees $t-2w=0$. For degree reasons, all differentials here are now zero, and since for fixed $t-s$ and w, there is precisely one s such that the entry at $(t-s,s,w)$ can be nonzero, all extensions are trivial. It follows that there is an isomorphism

$$\pi_{**}(S/\tau) \simeq \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \tag{4}$$

It turns out that S/τ has an $\mathbb{E}_\infty$ ring structure, for degree reasons, see [Ghe17b], so there is a symmetric-monoidal category $\mathrm{Mod}_{S/\tau}$ of S/τ-modules in cellular p-complete motivic spectra.

Since the left-hand side of the isomorphism 4 can be identified with the endomorphisms of the monoidal unit S/τ in the module category $\mathrm{Mod}_{S/\tau}$, and the right hand side can be identified with the endomorphisms of the monoidal unit BP_* in some suitably derived category of BP_*BP-comodules, it is natural to ask whether this isomorphism extends to an equivalence of those categories.

As the category $\mathrm{Mod}_{S/\tau}$ is compactly generated by S/τ and its weight-shifted copies $\Sigma^{0,w}S/\tau$, the correct candidate for the derived comodule category is given by the compactly generated $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$.

Theorem 6.7 (The GWX-Theorem, [GWX])**.** *There is a symmetric-monoidal fully faithful functor of stable* $(\infty,1)$*-categories*

$$\mathrm{Mod}_{S/\tau} \to \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)},$$

where S refers to the p-completed motivic sphere over $\mathbb{C}$, the module category is formed in cellular p-complete motivic spectra, and BP_*BP is considered p-completed. The essential image of this functor is the full subcategory of $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ *generated by the $\Sigma^{n,s}S$ with $n+s$ even.*

Proof. There is an adjunction $\mathrm{Mod}_{S/\tau} \rightleftarrows \mathrm{Mod}_{BP^{\mathrm{mot}}/\tau}$, with symmetric-monoidal left adjoint, whose comonad is given by the BP^{mot}/τ-bimodule

$$(BP^{\mathrm{mot}}/\tau) \otimes_{S/\tau} (BP^{\mathrm{mot}}/\tau) = (BP^{\mathrm{mot}} \otimes BP^{\mathrm{mot}})/\tau.$$

The associated Adams resolution is the BP^{mot}/τ-based Adams-Novikov resolution. Similar to the BP^{mot}-based Adams-Novikov spectra sequence on p-complete motivic spectra, it converges under suitable connectivity assumptions, and in particular, on compact objects in $\mathrm{Mod}_{S/\tau}$.

Thus, Theorem 2.44 applies to give a symmetric-monoidal equivalence

$$\mathrm{Mod}_{S/\tau} \simeq \mathrm{Comod}^{cg}_{(BP^{\mathrm{mot}}\otimes BP^{\mathrm{mot}})/\tau}(\mathrm{Mod}_{BP^{\mathrm{mot}}/\tau}).$$

Since the $\Sigma^{0,w}BP^{\mathrm{mot}}/\tau$ form a subgroup $\mathbb{Z}$ of the Picard group of $\mathrm{Mod}_{BP^{\mathrm{mot}}/\tau}$, the functor

$$\Gamma_* : \mathrm{Mod}_{BP^{\mathrm{mot}}/\tau} \to \mathrm{Sp}^{\mathbb{Z}}, \quad X \mapsto (\mathrm{map}_{\mathrm{Mod}_{BP^{\mathrm{mot}}/\tau}}(\Sigma^{0,w}BP^{\mathrm{mot}}/\tau, X))_{w\in\mathbb{Z}}$$

is lax symmetric-monoidal, where $\mathrm{Sp}^{\mathbb{Z}}$ is endowed with the convolution tensor product.

This gives the image of the monoidal unit, $\Gamma_*(BP^{\mathrm{mot}}/\tau)$, the structure of an $\mathbb{E}_\infty$ ring in $\mathrm{Sp}^{\mathbb{Z}}$.

The functor Γ_* is a right adjoint to the functor that sends $\Sigma^{0,w}S \in \mathrm{Sp}^{\mathbb{Z}}$ to $\Sigma^{0,w}BP^{\mathrm{mot}}/\tau$, where $\Sigma^{0,w}$ on $\mathrm{Sp}^{\mathbb{Z}}$ denotes the degree-shift by w.

Furthermore, Γ_* detects equivalences and preserves geometric realization, since the $\Sigma^{0,w}BP^{\mathrm{mot}}/\tau$ are compact generators. It follows that the conditions of the monadic Barr-Beck theorem are satisfied, and we have a symmetric-monoidal equivalence

$$\mathrm{Mod}_{BP^{\mathrm{mot}}/\tau} \simeq \mathrm{Mod}_{\Gamma_*(BP^{\mathrm{mot}}/\tau)}$$

where the latter is now a module category in $\mathbb{Z}$-graded spectra $\mathrm{Sp}^{\mathbb{Z}}$. But

$$\begin{aligned}\Gamma_0(BP^{\mathrm{mot}}/\tau) &\simeq \mathrm{map}_{BP^{\mathrm{mot}}/\tau}(BP^{\mathrm{mot}}/\tau, BP^{\mathrm{mot}}/\tau)\\ &\simeq \mathrm{map}_{\mathcal{S}_{\mathbb{C}}}(S^{0,0}, BP^{\mathrm{mot}}/\tau) \simeq H\mathbb{Z}\end{aligned}$$

where the last equivalence follows by looking at homotopy groups: As the homotopy groups of BP^{mot}/τ are concentrated in degrees $n-2w=0$, their weight 0 part is concentrated in the single degree $n=0$.

It follows that $\Gamma_*(BP^{\mathrm{mot}}/\tau)$ is actually an $H\mathbb{Z}$-algebra, and that we can form the module category over $\Gamma_*(BP^{\mathrm{mot}}/\tau)$ in $\mathrm{Mod}_{H\mathbb{Z}}^{\mathbb{Z}}$ instead of $\mathrm{Sp}^{\mathbb{Z}}$.

There is a symmetric-monoidal equivalence $\mathrm{Sp}^{\mathbb{Z}} \to \mathrm{Sp}^{\mathbb{Z}}$ which acts by Σ^{-2w} on the degree w-part. It sends $\Gamma_*(BP^{\mathrm{mot}}/\tau)$ to an object with homotopy groups concentrated in degrees $(0,w)$, and $\pi_{0,w}$ equal to $\pi_{2w}BP$, where BP is p-completed classical BP.

So we see that $\Gamma_*(BP^{\mathrm{mot}}/\tau)$-modules correspond to a full subcategory of the usual graded derived category $\mathcal{D}\,\mathrm{Mod}_{BP_*}$ on evenly generated modules. (Alternatively, we could identify it as the graded derived category over a version of BP_* in halved degrees.)

So we have a fully-faithful functor

$$\mathrm{Mod}_{BP^{\mathrm{mot}}/\tau} \to \mathcal{D}\,\mathrm{Mod}_{BP_*} \tag{5}$$

with essential image generated by even shifts of the free module BP_*.

The comonad represented by the bimodule $\Gamma := (BP^{\mathrm{mot}} \otimes BP^{\mathrm{mot}})/\tau$ can be described in terms of maps and coherences in $\mathrm{Map}(\Gamma^k, \Gamma^n)$. Since Γ is a free BP^{mot}/τ-module on generators in degrees $t-2w=0$, there are no higher coherences and the structure of the comonad is detected on homotopy groups (cf. Example 2.46, where the underlying module category was algebraic, but the comonad had interesting coherences). This is done carefully in [GWX], by exhibiting BP^{mot}/τ

and $(BP^{\mathrm{mot}} \otimes BP^{\mathrm{mot}})/\tau$ as contained in the heart of a t-structure on $\mathrm{Mod}_{S/\tau}$, where objects are filtered by the degree $t - 2w$.

From the isomorphism

$$\pi_{**}((BP^{\mathrm{mot}} \otimes BP^{\mathrm{mot}})/\tau) \simeq \pi_*(BP_*BP),$$

we see thus that the functor (5) intertwines the comonads given by the bimodule $(BP^{\mathrm{mot}} \otimes BP^{\mathrm{mot}})/\tau$ on the left, and BP_*BP on the right.

We thus get a fully-faithful functor

$$\mathrm{Comod}^{cg}_{(BP^{\mathrm{mot}} \otimes BP^{\mathrm{mot}})/\tau}(\mathrm{Mod}_{BP^{\mathrm{mot}}/\tau}) \to \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}(\mathcal{D}\,\mathrm{Mod}_{BP_*})$$

as claimed. □

6.3 Vanishing lines and self-maps in motivic homotopy theory

Through the GWX-Theorem 6.7, the statements from section 4 about vanishing lines and self-maps in the category $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ immediately carry over to the category $\mathrm{Mod}_{S/\tau}$.

Note that on $\mathrm{Mod}_{S/\tau}$, $\Sigma^{1,0}$ refers to usual suspension, and $\Sigma^{0,1}$ to weight-shift. This differs slightly from the conventions established for $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, where usual suspension corresponded to $\Sigma^{1,-1}$.

One can check that the weight 1-sphere $S^{0,1}/\tau$ is mapped to the sphere $S^{0,2} \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ under the functor from Theorem 6.7. So generally, $S^{n,w}/\tau$ corresponds to $S^{n,2w-n} \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$. Note that this is compatible with the correspondence explained in Proposition 6.6.

We will continue to write degrees in $\mathrm{Mod}_{S/\tau}$ (and the p-complete cellular motivic category in general) by (n, w), and in $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$. These relate through $s = 2w - n$.

Lemma 6.8. *The category* $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ *splits as a product of categories*

$$\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)} \simeq \mathrm{Comod}^{even}_{\mathcal{D}(BP_*BP)} \times \mathrm{Comod}^{odd}_{\mathcal{D}(BP_*BP)}$$

generated by the $S^{n,s}$ *with* $n + s$ *even, and the* $S^{n,s}$ *with* $n + s$ *odd, respectively.*

Proof. This follows from the fact that $\pi_{**}S = \mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ is concentrated in degrees (n, s) with $n + s$ even.

So for S^{n_1,s_1} and S^{n_2,s_2} such that $n_1 + s_1$ and $n_2 + s_2$ have different parities, $[S^{n_1,s_1}, S^{n_2,s_2}] \simeq 0$. More generally, since the corresponding parity doesn't change under suspension $\Sigma^{1,-1}$, if X^{even} and X^{odd} are colimits of spheres of even and odd parity respectively, $\mathrm{map}(X^{\mathrm{even}}, X^{\mathrm{odd}}) = 0$ and $\mathrm{map}(X^{\mathrm{odd}}, X^{\mathrm{even}}) = 0$. The result follows. □

Theorem 6.7 thus precisely identifies $\mathrm{Mod}_{S/\tau}$ with $\mathrm{Comod}^{\mathrm{even}}_{\mathcal{D}(BP_*BP)}$.

Proposition 6.9. *For each $i > j \geq 0$, there is a $\mathbb{C}$-motivic p-complete cellular S/τ-module $K(\beta_{ij})$, with homotopy groups isomorphic to*

$$\pi_{**}(K(\beta_{ij})) \simeq \mathbb{F}_p[\alpha_{ij}, \beta_{ij}^{\pm 1}],$$

with $|\alpha_{ij}| = (2p^j(p^i-1)-1, p^j(p^i-1))$, and $|\beta_{ij}| = (2p^{j+1}(p^i-1)-2, p^{j+1}(p^i-1))$. For $j = 0$, they admit an E_∞-ring structure.

Under the GWX equivalence, it corresponds to the $K\Lambda_{ij} \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ obtained from the simple coalgebra quotient $BP_*BP \to \Lambda_{ij}$ from Example 3.62.

Proof. We have to prove that the object $K\Lambda_{ij} \in \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$ is contained in $\mathrm{Comod}^{\mathrm{even}}_{\mathcal{D}(BP_*BP)}$. But this is clear, as the homotopy groups are concentrated in even degrees, and so the odd part vanishes.

The degrees of the α_{ij} and β_{ij} are computed using the relation $w = \frac{n+s}{2}$.

As the quotient maps $BP_*BP \to \Lambda_{ij}$ are Hopf algebroid maps for all $j = 0$, the $K\Lambda_{i,0}$ and thus the $K(\beta_{i,0})$ have a commutative ring structure by. □

We let d^{mot}_{ij} denote the slope of β_{ij} in the motivic (n, w)-grading, i.e.

$$d^{\mathrm{mot}}_{ij} = \frac{p^{j+1}(p^i-1)}{2p^{j+1}(p^i-1)-2}.$$

Lemma 6.10. *Let $\mathrm{Sp}_{\mathbb{C}}$ denote p-complete cellular spectra over $\mathbb{C}$. The Adams spectral sequence for X associated to the adjunction*

$$\mathrm{Sp}_{\mathbb{C}} \to \mathrm{Mod}_{S/\tau}$$

is multiplicative, converges conditionally and has E_1-page and abutment of the form

$$\pi_{**}(X/\tau)[t] \Rightarrow \pi_{**}(X^\wedge_\tau),$$

with t detecting τ, and $X^\wedge_\tau$ is the τ-completion $\lim X/\tau^k$.

Proof. As in the proof of Lemma 3.43, one can explicitly identify the associated Adams tower with the tower of X/τ^k. The claim follows. □

We will refer to this spectral sequence as the τ-Bockstein spectral sequence. Note that for compact X we have that for each n, the set of all $w \in \mathbb{Z}$ with $\pi_{n,w}(X) \neq 0$ can be seen to be bounded above, as this is true for spheres. From this one sees that for fixed (n, w), the map $X \to X/\tau^k$ is an isomorphism on $\pi_{n,w}$ for k large enough. Thus, $X \simeq X^\wedge_\tau$.

Theorem 6.11. *For X a compact, p-complete cellular motivic spectrum over $\mathbb{C}$, the homotopy groups $\pi_{n,w}(X)$ admit a minimal vanishing line in (n,w)-grading, i.e. d and c such that*

$$\pi_{n,w}(X) = 0 \quad \text{for} \quad w > dn + w,$$

and such that the same doesn't hold for other d' and c' with $d' < d$, or with $d' = d$ and $c' < c$. The slope of such a minimal vanishing line coincides with one of the d_{ij}^{mot} for some $i > j \geq 0$, characterized as the largest d_{ij}^{mot} for which $K(\beta_{ij})_{**}X \neq 0$.

Proof. First consider the S/τ-module X/τ. Since it corresponds to an object of $\mathrm{Comod}^{\mathrm{even}}_{\mathcal{D}(BP_*BP)} \subseteq \mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, which has a minimal vanishing line by Proposition 4.22, $\pi_{**}(X/\tau)$ has a minimal vanishing line. Since $\pi_{**}(K(\beta_{ij}) \otimes_{S/\tau} X/\tau))$ agrees with $(K\Lambda_{ij})_{**}$ of the corresponding object in $\mathrm{Comod}^{cg}_{\mathcal{D}(BP_*BP)}$, the minimal vanishing line of $\pi_{**}(X/\tau)$ has slope given by the maximal d_{ij}^{mot} with $\pi_{**}(K(\beta_{ij}) \otimes_{S/\tau} X/\tau)) \neq 0$.

Let c be the intercept of the minimal vanishing line for $\pi_{**}(X/\tau)$ in (n,w)-grading. As $|\tau| = (0,-1)$, the filtration k part of the τ-Bockstein spectral sequence of X admits a vanishing line of slope d_{ij}^{mot} and intercept $c - k$. So the E_∞ page, and thus also $\pi_{**}(X^\wedge_\tau) = \pi_{**}(X)$, admits the same minimal vanishing line.

Finally, observe that

$$\pi_{**}(K(\beta_{ij}) \otimes X) \simeq \pi_{**}(K(\beta_{ij}) \otimes_{S/\tau} X/\tau)),$$

so the $K(\beta_{ij})$ detect minimal vanishing lines. □

Theorem 6.12. *For R a compact, p-complete cellular motivic ring spectrum over $\mathbb{C}$, let d_{ij}^{mot} be the slope of the minimal vanishing line for $\pi_{**}(R)$ in (n,w)-grading. Then there is a non-nilpotent element $x \in \pi_{**}(R)$ of slope d_{ij}^{mot}, that acts isomorphically on $K(\beta_{ij})_{**}R$, and acts isomorphically on $\pi_{**}(R)$ above some line of slope stricly smaller than d_{ij}^{mot}.*

Furthermore, any two such θ coincide after raising them to suitable powers, and for any c, some power of θ commutes with all elements of $\pi_{**}(R)$ above the line of slope d_{ij}^{mot} and intercept c.

Proof. As observed in the proof of Theorem 6.11, for any line of slope d_{ij}, the filtration in the τ-Bockstein spectral sequence for R is bounded in the region above that line.

As in the proof of Lemma 4.26, it follows that self-maps in $\pi_{**}(R/\tau)$ parallel to the vanishing line lift to $\pi_{**}(R)$, and have the claimed properties. □

Corollary 6.13. *For X a compact, p-complete cellular motivic spectrum over $\mathbb{C}$, let d_{ij}^{mot} be the slope of the minimal vanishing line of $\pi_{**}(X)$ in (n,w)-grading. Then there is a non-nilpotent self-map $\Sigma^{|\theta|}X \xrightarrow{\theta} X$ of slope d_{ij}^{mot}, which induces an* *isomorphism on $K(\beta_{ij})_{**}X$, and induces an isomorphism on $\pi_{**}(X)$ above a line of slope strictly smaller than d_{ij}^{mot}.*

Proof. This follows as in Proposition 4.32, by applying Theorem 6.12 to the ring $\mathrm{End}(X) = X \otimes DX$. □

Remark 6.14. The $K(\beta_{i,0})$ were first constructed by Gheorghe in [Ghe17a] by obstruction theory, under the name $K(w_{i-1})$. The first example of a $\beta_{2,0}$ self-map, or w_1 self-map, was explicitly constructed by Andrews in [And14], as an explicit w_1^4 self-map of S/η.

We now construct examples of compact motivic spectra with minimal vanishing line of slope d_{ij}^{mot} for each $i > j \geq 0$. Analogously to Definition 5.1, we obtain generalized Smith-Toda complexes in motivic spectra.

Definition 6.15. *For K a compact p-complete cellular motivic spectrum, a generalized Smith-Toda complex on K is an iterated cofibre of the form*

$$K/(\theta_0, \theta_1, \ldots, \theta_k),$$

where θ_i is a self-map of $K_i := K/(\theta_0, \theta_1, \ldots, \theta_{i-1})$ parallel to the minimal vanishing line.

If $K(\beta_{ij})_{**}K_l \neq 0$, then $K(\beta_{ij})_{**}K_{l+1} \neq 0$ precisely if the slope of the self-map θ_l (and the minimal vanishing line of K_l) is distinct from d_{ij}^{mot}.

It follows that the slopes of the θ_l run precisely through the $k+1$ largest elements of the d_{ij}^{mot} for which $K(\beta_{ij})_{**}K \neq 0$.

In particular, if we start with $K = S$, and form a sequence of generalized Smith-Toda complexes on S, every slope d_{ij}^{mot} for $i > j \geq 0$ occurs as the slope of a minimal vanishing line.

Proposition 6.16. *For $i > j \geq 0$, let $\mathcal{C}_{ij}$ be the thick subcategory of finite p-complete cellular motivic spectra consisting of all objects with a vanishing line of slope d_{ij}^{mot}. Then the sequence of $\mathcal{C}_{ij}$ (ordered decreasingly by d_{ij}^{mot}) forms a descending sequence of proper inclusions between thick subcategories, with trivial intersection.*

Proof. This is just the observation that there are examples of compact objects with minimal vanishing lines for each slope d_{ij}^{mot}, obtained as generalized Smith-Toda complexes on S. □

Remark 6.17. For X be a motivic p-complete cellular spectrum, we can organize its homotopy groups $\pi_{**}(X)$ by an inductive process. Namely, for θ_0 a $(\beta_{1,0})$ self-map of S parallel to the vanishing line, an element of $[S, X]_{**}$ is either θ_0-periodic, or extends to a map $S/\theta_0' \to X$ for some power θ_0' of θ_0.

Next, an element of $[S/\theta'_0, X]$ is then either periodic with respect to a $\beta_{2,0}$ self-map θ_1 of S/θ'_0, or extends to a map $S/(\theta'_0, \theta'_1)$ for θ'_1 a power of θ_1. We can continue in this manner, passing through all the β_{ij} in decreasing order of their slopes, and in each step either obtain a β_{ij}-periodic homotopy class or obtain an extension over a further generalized Smith-Toda complex.

This construction is analogous to the chromatic filtration on classical $\pi_*(S)$, which organizes homotopy groups in v_n-periodic families in the same manner. It therefore suggests a "β_{ij}-chromatic motivic homotopy theory". The computation of the η-inverted sphere at $p = 2$ by Andrews and Miller in [AM17] can be regarded as a first step in this direction. At odd primes, partial results on $\beta_{1,0}$-inverted homotopy groups can be found in the forthcoming [Bel].

In contrast to the notion of type in classical stable homotopy theory that the organization of spectra by minimal vanishing lines employed here does not give a full characterization of thick subcategories or self-maps. We want to illustrate this with one example of an additional non-nilpotent element in $\pi_{**}(S)$ discovered in [Isa14].

Example 6.18. At $p = 2$, there is an element $\overline{\kappa}_2 \in \pi_{44,24}(S)$, detected in the motivic Adams spectral sequence by an element of the name g_2 in Adams filtration 4. It is not nilpotent, and one can check that it induces an isomorphism on $K(\beta_{2,1})_{**}$, thus is in fact a $\beta_{2,1}$ self-map of S.

This shows that a compact object may have multiple types of β_{ij} self-maps at the same type.

Also note that it shows that the list of thick subcategories given by the $\mathcal{C}_{ij}$ in Proposition 3.71 is not complete. Namely, the thick subcategory $\mathcal{D}$ consisting of all X with $K(\beta_{2,1})_{**}X = 0$ cannot agree with $\mathcal{C}_{1,0}$, since $\mathcal{C}_{1,0}$ contains S, which has $K(\beta_{2,1})_{**}S \neq 0$. However, $\mathcal{D}$ contains $S/\overline{\kappa}_2$, an object with minimal vanishing line of slope $d_{1,0}^{\mathrm{mot}}$, so can't be contained in any other $\mathcal{C}_{i,j}$.

In addition, not all non-nilpotent self-maps are among the β_{ij} self-maps or v_n self-maps. This again follows from the fact that a compact object may have different types of self-maps at the same time: If two of these self-maps have non-nilpotent product, one obtains "mixed-type" self-maps.

Example 6.19. Consider the motivic $S/2$. By a similar computation as in the classical case, it admits a v_1^4 self-map $\theta : \Sigma^{8,4}S/2 \to S/2$. In addition, η is still not nilpotent on $S/2$.

From the computation of $\eta^{-1}S$ in [AM17], one can extract that the v_1^4 self-map θ still acts non-nilpotently on $\eta^{-1}S/2$. Thus, on $S/2$, any product $\eta^a\theta^b$ is not nilpotent. For $a > 0$ and $b > 0$, this acts trivially on all the K-theories $K(\beta_{ij})_{**}(S/2)$ and $K(n)_{**}(S/2)$, as η only acts nontrivially on $K(\beta_{1,0})_{**}(S/2)$, and θ acts nontrivially only on $K(1)_{**}(S/2)$.

Furthermore, notice that the slope of $\eta^a\theta^b$ is given by $\frac{a+4b}{a+8b}$, which for $a > 0$ and $b > 0$ takes all rational values strictly between $\frac{1}{2}$ and 1. In particular, they provide examples of non-nilpotent self-maps of slope d_{ij}^{mot} for any $i > j \geq 0$ that are no actual β_{ij} self-maps.

The "mixing" of different self-map types can happen in less direct ways, as well:

Example 6.20. At $p = 2$, there is an element $\kappa_1 \in \pi_{32,18}(S)$, detected in the motivic Adams spectral sequence by an element of the name d_1 in Adams filtration. It has slope $\frac{9}{16}$, which is not among the d_{ij}^{mot}. So it is not a β_{ij} self-map.

On S/η, κ_1^2 decomposes as a product of the $\beta_{2,1}$ self-map $\overline{\kappa}_2$ from Example 6.18, and the $\beta_{2,0}$ (or w_1) self-map of S/η from [And14], so up to powers, κ_1 decomposes into β_{ij} self-maps. This decomposition does not lift to S.

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