Aspects of Fibers, Fibrations and their Non-Compact Limits in F-theory and Topological String Theory

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Abstract

This thesis presents results about F-theory and topological string theory on genus one fibered Calabi-Yau manifolds and their non-compact limits.

In the first part we work in the fiber based approach to F-theory and study a conjectured relation between the global structure of the non-Abelian gauge group and discrete symmetries that are associated to mirror pairs of fibers. We provide a combinatorial explanation of this phenomenon for the class of hypersurfaces in two-dimensional toric ambient spaces. More generally, we formulate a toric criterion for Mordell-Weil torsion that applies to complete intersection fibers in higher codimension. We then check that the conjecture holds for the much larger class of codimension two complete intersections in three-dimensional toric ambient spaces. Finally we provide a detailed discussion of the geometry of a particular example. As a non-trivial consistency check we show that the non-Abelian anomalies in 6d compactifications are generically cancelled.

We then turn to the question of horizontal fluxes in elliptically fibered Calabi-Yau fourfolds. Determining properly quantized fluxes is an important problem in phenomenological applications of F-theory. We present a technique that uses the relation of topological brane charges under homological mirror symmetry to construct integral elements of the flux lattice. In many cases this leads to a complete basis. For the general class of non-singular elliptic fibrations we construct such a basis explicitly and use it to study the action of certain auto-equivalences of the category of B-branes on the charge lattice. We find that a particular set of auto-equivalences generates a projective action of the modular group and argue that this leads to an expansion of the topological string amplitudes in terms of quasi-modular forms. We then use the integral basis and numerical analytic continuation to perform a preliminary study of the flux vacua for a particular geometry.

In the last part of the thesis we study the non-compact limit of elliptic fibrations from the perspective of topological string theory. To this end we construct a compact embedding of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ and describe the geometry in some detail. In particular we construct an integral basis of periods and use it to obtain a complete set of Picard-Fuchs operators. We can then identify the correct non-compact limit in the moduli space and determine the periods that remain finite. In this way we can relate the genus zero topological string amplitude on the non-compact geometry to the modular parameter of the mirror curve. We then use a theorem by Lockhart to derive a relation between the topological genus one amplitude and a particular product of theta functions. For mirror curves of genus two this leads to the fact that the propagator that appears in the holomorphic anomaly equations is an almost meromorphic Siegel modular form.

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Contents

1 Introduction

This thesis presents work done in the context of string theory and mirror symmetry. In the first chapter we aim to provide brief introductions to the most relevant physical concepts for a reader yet unfamiliar with string theory. It is in no way meant to be comphrehensive. If not cited otherwise, the material in this chapter can be found in [1, 2].

Motivation The standard model of particle physics provides a good approximation to the behaviour of nature at subatomic scales down to at least $l \approx 1\,\mathrm{TeV}^{-1}$. It has been verified in numerous experiments. In fact, with the anomalous magnetic dipole moment of the electron it provides one of the most precise predictions in physics. The theoretical expectation agrees with the measured value to more than 10 significant digits [3].

The standard model describes particles as excitations of quantum fields and forces as exchanges of particles. Matter particles are fermions and have spin-1/2. Force carrying particles are bosons of spin-1 and correspond to excitations of gauge fields. A particle of spin-0 – the Higgs boson – acquires a vacuum expectation value at low energies. This generates a mass term for the matter fields and the carriers of the weak force. Calculations of scattering cross sections are usually carried out in perturbation theory. The order of the expansion is the number of interaction vertices, i.e. points where more than two particles meet, and the expansion parameters are the coupling constants.

The standard model incorporates the electromagnetic, weak and strong interactions but not gravity. However, it is well known how to quantize a field theory of a spin-2 particle [1]. At low energies such a theory is essentially unique and reproduces general relativity in the classical limit. The reason why gravity is usually not considered to be part of the standard model is perhaps twofold. First of all the quantum field theory of gravity is not renormalizable and thus becomes unpredictive at very high energies. The energy at which the theory is expected to break down is called the Planck scale. Second, while at low energies it provides a consistent quantum theory of gravity the predicted deviations from the classical theory are too small to be measurable with current experiments. In any case, interpreting general relativity as the limit of a quantum field theory does not lead to inconsistency. It merely implies that the standard model has to be interpreted as the low-energy effective action of an underlying ultraviolet (UV) theory.

String theory The only known theory that is expected to be valid at all energies and can at low energies be described by an effective quantum field theory that includes spin-2 particles is string theory. Its bosonic version can literally be obtained by quantizing the action of a classical string. Vibrations of the string lead to a discrete spectrum of excitations. In particular the closed string spectrum generically contains a massless spin-2 excitation. When the length of the string is very small, it can be interpreted as a particle and thus yields the graviton.

It is important to note that this procedure leads to quantum field theories *in two ways*. While particles move along a worldline, strings – closed or open – trace out a surface, the so-called *worldsheet* Σ. The path integral in the quantum theory of strings sums over the different "shapes" of the worldsheet and integrates over the possible embeddings into the *target space*. We will elaborate on the meaning of shape when we discuss string perturbation theory. For a fixed shape of the worldsheet the path-integral leads to a two-dimensional quantum field theory with action

$$S_{\mathrm{N.G.}} = -T \int_{\Sigma} d^2\sigma \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X_\mu}\,. \tag{1}$$

Here σ^α are coordinates that parametrize the worldsheet and X^μ are coordinates along the target space. Moreover, T is the tension of the string. We choose the signature of the target space metric to be mostly plus. The action is just the area of the worldsheet with respect to the metric that is induced from the target space. A second quantum field theory arises as the effective description at low energies, where the strings are essentially point-like.

The spectrum of the bosonic string only contains excitations of integral spin. To obtain realistic physics one also needs to describe matter particles and thus fermions. Those can be incorporated by *supersymmetrizing* the bosonic string action. It turns out that superstring theories are tightly constrained. There are only five consistent worldsheet actions, called Type IIA, Type IIB, Heterotic SO(32), Heterotic E8 $\times$ E8 and Type I. All of these theories exhibit some degree of spacetime supersymmetry. Moreover, all of them require the target space to be ten-dimensional. This leads to the concept of *compactification*.

Compactification In the early 20th century, Kaluza and Klein considered general relativity in a five-dimensional spacetime where one dimension is periodic, i.e. a circle. When the circle is small, non-zero momentum modes along the circle become very heavy and can be integrated out. The theory is then effectively described by general relativity in four dimensions that is coupled to electromagnetism and a massless scalar field. Intuitively this can be seen by decomposing the five-dimensional metric g_{IJ} as

$$\begin{gathered} g_{IJ}\, dx^I \otimes dx^J = g_{ij}\, dx^i \otimes dx^j + A_i(dx^i \otimes dx^4 + dx^4 \otimes dx^i) + \phi\, dx^4 \otimes dx^4\,, \\ I, J = 0, ..., 4, \quad i, j = 0, ..., 3, \end{gathered} \tag{2}$$

where g_{ij} is the four-dimensional metric and A_i the electromagnetic gauge field. The massless scalar field ϕ corresponds in five dimensions to the radius of the circle. In this theory the gauge symmetry in four dimensions is a result of invariance under diffeomorphisms of the periodic coordinate. Moreover, if one introduces matter the momentum along the circle translates into electric charge [4].

Analogously, we can assume that six of the ten target space dimensions of a string theory are compact. If the *compact space* is small, the effective low-energy description will be a four-dimensional quantum field theory. In particular, as in Kaluza-Klein theory, the ten dimensional field content will decompose and the effective theory generally exhibits a rich particle spectrum. The effective physics are thus closely related to the choice of compactification.

Another generic phenomenon is that from the decomposition of the metric massless scalar fields arise. As in Kaluza-Klein theory they parametrize the compact space and are called moduli. Since massless scalar particles are not observed in nature, it is necessary to find a mechanism that makes these fields massive. Generating a potential for the moduli and thus stabilizing the compact geometry is in many cases an open problem. We address the issue of moduli stabilization for so-called *complex structure moduli* in the context of F-theory compactifications (see below) in chapter 3 that is based on work published in [5].

A particularly important class of compactification spaces are Calabi-Yau geometries. Calabi-Yau manifolds and hypertori admit Ricci flat metrics that solve the vacuum Einstein equations. Moreover, in general compactification breaks supersymmetry. The only exception being compactifications on hypertori which are essentially products of circles. Calabi-Yau compactifications break only part of the supersymmetry and leave one quarter of the supercharges conserved. Since supersymmetry is not observed in nature, it is then expected to be dynamically broken at some energy below the Kaluza-Klein scale.

While non-supersymmetric compactifications are being studied in the literature, dynamical breaking has the advantage that supersymmetry can be used as a powerful tool for computations. In particular, Calabi-Yau compactifications and supersymmetry are responsible for many fruitful interactions between mathematics and physics.

String perturbation theory To calculate a scattering cross section in string theory we have to integrate over all possible worldsheets that connect the asymptotic states. In particular we have to sum over “holes” in the worldsheet which are the string analogues of loops in Feynman diagrams. This is illustrated in figure 1. The number of holes is

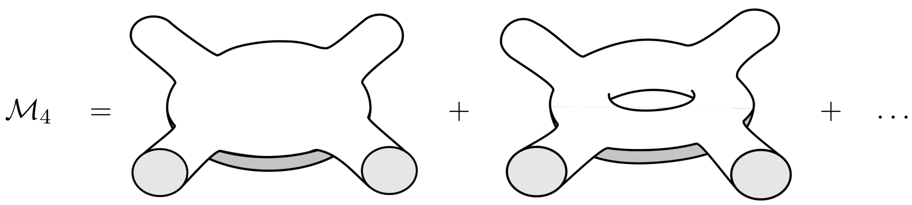


Figure 1: The string path integral involves a sum over the genus of the worldsheet.

called the *genus* of the worldsheet. The fact that interactions of strings are not pointlike but delocalized can be seen as the intuitive reason why string theory does not suffer from ultraviolet divergences.

To make the sum over genera a valid perturbative expansion, we have to introduce an Einstein-Hilbert term on the worldsheet,

$$S = S_{\text{N.G.}} + \frac{\lambda}{4\pi} \int_{\Sigma} d^2\sigma \sqrt{h} R \,. \tag{3}$$

Here λ is a coupling constant, h is the worldsheet metric and R the corresponding Ricci scalar. In two dimensions gravity is non-dynamical and the integral is related to the genus g of the worldsheet via

$$\chi = \frac{1}{4\pi} \int d^2\sigma \sqrt{h} R = 2 - 2g \,. \tag{4}$$

The string path integral can then be written as

$$Z = \sum_{g=0}^{\infty} e^{\lambda(2g-2)} \int \mathcal{D}\Phi \mathcal{D}g \, e^{-S_{N.G.}} \,, \tag{5}$$

where we also integrate over all possible maps

$$\Phi : \Sigma \to M \,, \tag{6}$$

into the target space M and over worldsheet metrics h.

The coupling constant governing string perturbation theory is

$$g_s = e^{\lambda} \,, \tag{7}$$

which is also called the *string coupling*. However, the spectrum of closed strings contains a scalar field ϕ called the dilaton. It turns out that if we allow the string to couple to background dilaton fields, λ can be interpreted as the asymptotic vacuum expectation value of ϕ. The string coupling is therefore dynamically determined by the theory itself. Note that in contrast to quantum field theory the string path integral is intrinsically perturbative. It is an important open problem to find a non-perturbative definition of string theory.

D-branes Until now we have mostly avoided the question whether the strings are closed, that is periodic, or open, i.e. with boundaries. In fact, only closed strings (and thus gravitons) are present in all superstring theories. Open strings can only be found in Type IIA/B and Type I theories.

Open strings can satisfy two different types of boundary conditions along each dimension. One of them forces the end points to be fixed and thus breaks Lorentz invariance. The other type only constrains the momentum flow along the string. It has gradually been realized that a consistent theory of open strings has to include both kinds of boundary condition. The problem of breaking Lorentz invariance is solved by realizing that the spaces to which the end points of the strings are constrained are dynamical objects themselves, called D-branes. Lorentz invariance is then only broken "dynamically" as it is in the presence of ordinary particles.

A D-brane with a $(p+1)$-dimensional world-volume is called Dp-brane. The number thus refers to the spatial dimensions. Type IIA string theory contains Dp-branes with $p = 0, 2, 4, 6, 8$ while in Type IIB there are branes for $p = -1, 1, 3, 5, 7, 9$. The $D(-1)$-brane is pointlike in spacetime and therefore also called the D-instanton.

There are perhaps two most important consequences from the presence of D-branes. First of all, like the excitations of a string, a brane has dynamical fields "living" on its worldvolume. In particular, the spectrum includes a particle of spin one. While a single brane will have a U(1) gauge theory on its worldvolume, a stack of coincident branes will lead to a non-Abelian gauge group. Branes therefore become an important ingredient for phenomenological model building in Type II/I theories.[1]

[1] With the introduction of branes it has been realized that Type I superstrings can be understood as a particular setup in Type IIB string theory. In the following we will only consider the more general Type II theories.

Secondly, D-branes are non-perturbative degrees of freedom. This can be seen from the brane tension which scales as $1/g_s$. When g_s is small string perturbation theory is valid while the branes are strongly coupled and vice versa. Due to the importance of perturbation theory for our understanding of almost any non-trivial dynamical system, this appears to be a serious problem. However, far from rendering perturbation theory useless, it turns out that branes can increase its power via *dualities*.

Dualities We speak of a duality when a hypothetical observer in a physical setup can not perform any experiment to distinguish between two not trivially isomorphic laws of nature. The simplest example of this phenomenon in the context of string theory is *T-duality*. Type IIA string theory compactified on a circle of radius R is physically indistinguishable from Type IIB compactified on a circle of radius α'/R.

A vast generalization of T-duality is *mirror symmetry*. The strong mirror conjecture claims that there are pairs (M, W) of Calabi-Yau manifolds such that Type IIA/B compactified on M is dual to Type IIB/A on W if M and W are odd-dimensional and dual to Type IIA/B on W when the dimensions are even[2]. There is particularly strong evidence for the mirror conjecture in the *topological subsector* of Type II string theory (see below). This duality is at the center of all work that we present in this thesis.

Dualities become powerful when in the context of one theory a calculation can be carried out that maps to a result in the dual theory that is hard or impossible to obtain directly. One example are certain Type IIA correlation functions that encode enumerative invariants associated to Calabi-Yau manifolds. It is in general not known how to calculate these invariants directly. However, mirror symmetry relates these to correlators in the dual theory that can essentially be obtained by solving a system of linear differential equations.

Another example are S-dualities. An S-duality involves a non-trivial identification of coupling constants and can thus map a strongly coupled theory to a weakly coupled one. This makes it possible to apply perturbation theory to strongly coupled problems by mapping them into the weakly coupled dual theory.

F-theory A particularly important example of an S-duality can be found in Type IIB string theory. The effective action contains a scalar field C_0 which combines with the dilaton ϕ into the *axio-dilaton*

$$\tau = C_0 + \frac{i}{e^{\phi}} \,. \tag{8}$$

Type IIB string theory is self-dual under the transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \,, \quad ad - bc = 1 \,, \quad a, b, c, d \in \mathbb{Z} \,. \tag{9}$$

As a combination of ϕ and C_0, τ is a dynamical field. In particular, the expectation value of τ can vary along the compact dimensions. In the presence of a D7-brane, the axio-dilaton

[2]We count the complex dimension, such that e.g. a torus is one-dimensional or in other words a complex curve. In particular Calabi-Yau are complex manifolds and thus of even real dimension.

develops a logarithmic profile along the two transverse dimensions. This implies multi-valuedness of the axion field C_0 but the string coupling $g_S = e^{\phi}$ can still be choosen to be small. Even more general compactifications lead to multi-valued axio-dilaton profiles along the compact dimensions which are inherently non-perturbative.

Due to (9), the axio-dilaton behaves like the *complex structure modulus* of a torus. The latter determines the metric up to an overall factor. We will elaborate on this concept in section 2.1. For now we note that a compact *base* space with a varying axio-dilaton profile can thus be interpreted as a higher-dimensional space with the extra dimensions given by a torus at every point of the base. More generally, a space that is constructed by attaching a particular geometry at (almost) every point of a given base space is called a *fibration*. F-theory can be thought of as an auxilliary twelve-dimensional theory that when compactified on a torus fibered Calabi-Yau manifold leads to Type IIB string theory compactified on the base of the fibration with the axio-dilaton profile given by the complex structure of the fiber tori [6–8].

Given a torus fibered Calabi-Yau fourfold X, F-theory leads to a four-dimensional effective theory with $N = 1$ supergravity. In particular, gauge fields, matter content and Yukawa couplings are determined by the structure of the fibration. It is possible to construct fibers that generically lead to certain properties of the effective theory. A base can then be choosen seperately to further engineer the desired physics. We will study a particular class of fibers in chapter 2. In particular, we explain an interesting relation between the effective physics under mirror symmetry of the fiber.

The C_0 field couples electrically to D-instantons and magnetically to D7-branes. The spectrum of the closed IIB string also contains a 2-form field C_2, that couples electrically to D1-branes and magnetically to D5-branes, and a self-dual 4-form C_4 that couples to D3-branes. The field strengths $H_3 = dB_2$ and $F_3 = dC_2$ can have non-vanishing expectation values along three-cycles in the compact space. This is called *bulk-flux*. In F-theory the flux can be multi-valued and combines with the so called *gauge-flux* localized on the world volume of D7-branes. All three types of fluxes are encoded in a so-called G_4-flux that corresponds to an element $G_4 \in H^4(X)$ in the cohomology of X.

The flux can, among other things, generate a potential for the moduli and is thus of phenomenological importance. However, admissible fluxes are quantized and finding the flux lattice inside the cohomology is in general an unsolved problem [3]. We will provide techniques to determine the lattice for a large class of compactifications in chapter 3 that is based on [5].

M-theory The low energy limits of uncompactified superstring theories are ten-dimensional supergravities. They all arise from compactifications of the *unique* eleven-dimensional supergravity (11d SUGRA). In particular, circle compactification relates the latter to the low energy effective theory of Type IIA string theory. D-branes appear in the 10d supergravity as black branes, i.e. higher-dimensional analogues of the Reissner-Nordström black hole. It

[3]Due to a tadpole cancellation condition, the length of the flux vector is also bounded from above by $\chi/24$ where χ is the Euler characteristic of the given geometry. Estimates that involve the number of fluxes inside this hypersphere lead to the huge numbers (like the famous 10^{500}) that are sometimes found in discussions of the landscape of string vacua.

turns out that 11d SUGRA also contains solitonic objects of dimension three and six. It has therefore been conjectured that it is the low-energy effective description of a quantum theory of three- and six-dimensional $M2-$ and $M5-$branes. M-theory also contains a three form field C_3 that couples electrically to the $M2$ branes and magnetically to the $M5$ branes. Compactification of such a theory along a circle reproduces the non-perturbative spectrum of Type IIA string theory [9].

M-theory lacks a coupling constant and therefore even a perturbative definition although fundamental descriptions of several limits have been found [10, 11]. Its properties can still be inferred using the conjectured dualities to superstring theories. This has led to a plenthora of non-trivial consistency checks that can be seen as evidence for M-theory. On the other hand, assuming the existence of M-theory provides a powerful tool to study string theory.

In particular, a chain of dualities relates M-theory to F-theory. M-theory can be compactified on an elliptic Calabi-Yau n-fold. One of the fiber circles can be interpreted as compactification to Type IIA string theory. Shrinking the other circle leads, via T-duality, to Type IIB string theory that is compactified on the base of the fibration with an axio-dilaton profile given by the complex structure of the fiber. The elliptic fibration that has been an auxilliary object from the Type IIB perspective can therefore be seen as the compact space in a dual M-theory picture. Although M-theory is inherently non-perturbative, this duality has been instrumental in understanding the relation between elliptic fibrations and the F-theory effective physics.

Topological string theory Mirror symmetry is particularly powerful in the topological sub-sector. As was explained above, the string path integral can be written as a sum over genera and an integral over embeddings (eqn. 5). For Calabi-Yau compactifications the string action can be modified such that only embeddings of minimal volume among those that can be continuously deformed into each other contribute. This is called *A-twisting* and the resulting theory is called *A-model*. In other words, the A-twisted string partition function counts certain equivalence classes of curves inside the Calabi-Yau. The counting problem can be made mathematically precise and the corresponding numbers are called enumerative invariants. There are actually several different attempts to formalize the counting which lead to closely related numbers. In our work we are mostly concerned with Gromov-Witten, Gopakumar-Vafa and refined BPS invariants.

There is another way to modify the action that is called *B-twisting*. It makes the path integral localize on constant maps and genus zero amplitudes in the *B-model* are related to classical integrals over the Calabi-Yau. Mirror symmetry identifies A-model and B-model on a mirror pair (M, W) of Calabi-Yau manifolds. The enumerative invariants associated to M that are encoded in the A-model can therefore be related to tractable calculations in the B-model on W.

The twisted theories are also called topological string theories. They are not unrelated to the physical string theories and it can be shown that the twisting procedure amounts to a projection onto a subset of the physical states [12]. In particular the topological string amplitudes calculate terms in the physical string theory effective action [13].

String theory provides several tools to calculate the topological amplitudes at higher genus of the worldsheet. The most general is solving the *holomorphic anomaly equations*. We first express the partition function in terms of generating function

$$F = \log(Z) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(T)\,. \tag{10}$$

In the A-model the generating functions F_g only depend on the *Kähler moduli*. The latter are related to the volume of curves inside the manifold. The B-model does not depend on the Kähler moduli but on the complex structure. In particular, the dependence of F_0 on T is – up to logarithms – holomorphic. It was observed in [14] that the non-holomorphic part of F_g for $g > 1$ can be expressed in terms of the free energies $F_{g'}$ with $g' < g$ and a non-holomorphic object called *propagator*. This recursive relation is called holomorphic anomaly equation. Given $F_{g'}$ for $g' < g$, integrating them determines F_g up to a holomorphic part. The latter is constrained to be a rational function of the moduli that can ideally be determined from contraints on F_g. For non-compact Calabi-Yau manifolds it has been argued [15] that the behaviour of F_g on the boundary of the moduli space indeed fixes the holomorphic ambiguity for all genera. For the quintic Calabi-Yau, the canonical compact example, the authors of [16] integrated the holomorphic anomaly equations up to genus 51.

In chapter 4 we describe how to perform the direct integration for a particular class of non-compact geometries using modular forms or, more generally, products of theta functions. In particular we describe a novel almost meromorphic Siegel modular object that serves as a propagator for so-called mirror curves of genus two and generalize this relation to hypergeometric mirror curves of arbitrary genus. This builds on and extends work that has already been published in [17].

Outline of the thesis

- The second chapter is based on [18],

  Paul-Konstantin Oehlmann, Jonas Reuter, T.S.,
  Mordell-Weil Torsion in the Mirror of Multi-Sections,
  JHEP (2016), `https://arxiv.org/abs/1604.00011`.

  One approach to obtain torus fibered Calabi-Yau manifolds is to construct the fiber as a hypersurface or complete intersection in some ambient space. One can then promote the coefficients of the defining equations to sections of line bundles over the base of the fibration. Certain generic properties of the effective physics in the corresponding F-theory compactification can then be inferred directly from the fiber even without fixing a base.

  The F-theory physics of fibers that are realized as hypersurfaces in two-dimensional toric ambient spaces have been classified in [19]. Unexpectedly the authors observed a relation between the gauge groups of the effective theories associated to mirror pairs of fibers. More concretely, the global structure of the generic, non-Abelian gauge group of a given fiber is related to a discrete gauge symmetry that arises from the mirror dual fiber. It is worthwhile to note that the standard model of particle physics contains both, a non-trivial global structure of the non-Abelian gauge group and discrete symmetries.

In [18] we discussed the mirror phenomenon from the perspective of toric geometry and provided a combinatorial explanation. This led us to a conjecture about the global structure of gauge groups associated to fibers that are more general complete intersections in toric ambient spaces. We therefore studied complete intersection fibers in three-dimensional toric ambient spaces, building on the work of [20]. We showed that the relation between the effective physics associated to mirror pairs as observed by [19] still holds for the much larger class of complete intersections in codimension two. In verifying this relation we found genus one fibers without a section that exhibit Mordell-Weil torsion in the Jacobian. This behaviour has not previously been observed in the F-theory literature and can be used to construct effective theories which have a discrete symmetry and at the same time exhibit a non-simply connected, non-Abelian gauge group.

In the second chapter of this thesis we first review the geometry of elliptic curves and the relation between the structure of torus fibered Calabi-Yau manifolds and effective physics via F-theory. In particular, we discuss the fiber based approach of [19]. This is followed by an introduction to toric geometry that will serve as a basis for the rest of the thesis. We then discuss the work in [18] and then extend it by studying the cancellation of six-dimensional supergravity anomalies in a particular complete intersection fiber.

- As we explained above, for a generic F-theory compactification it is possible to choose non-trivial vacuum expectation values for the higher gauge fields in the effective action. These so-called fluxes are necessary to generate a chiral spectrum, to stabilize the moduli and thus to build phenomenologically viable models. The possible fluxes can however not be inferred from a particular construction of the fiber and depend on the global structure of the geometry. In the third chapter we therefore turn from fibers to fibrations and study the horizontal flux lattice on elliptically fibered Calabi-Yau fourfolds. This is based on work that has been published in [5],

  Cesar Fierro Cota, Albrecht Klemm, T.S.,
  Modular Amplitudes and Flux-Superpotentials on Elliptic Calabi-Yau Fourfolds,
  JHEP (2018), `https://arxiv.org/abs/1709.02820`.

  The lattice of fluxes can be identified with the charge lattice of A-branes on the same Calabi-Yau. This is an instance of open/closed string duality. In particular, the charges of A-branes are given by the period lattice that is associated to an (up to scale) unique holomorphic 4-form [21]. It is well known that the periods can be obtained as solutions to a system of linear differential equations. On the other hand, a set of B-branes that generate the charge lattice on the mirror manifold can often be constructed explicitly and the asymptotic behaviour of the central charge can be calculated. Via mirror symmetry the leading behaviour of the B-brane charges can then be used to fix a set of solutions of the system of differential equations that is a basis for the flux lattice.

This procedure has been used for example in [22] to determine the quantum corrections to the charges of B-branes for certain non-toric constructions of Calabi-Yau manifolds. The authors also reviewed generic formulae for the asymptotic charges of B-branes in codimension zero, two six and eight.

In [5] we utilized the Koszul complex to construct objects in the category of B-branes that have support of codimension four. We argued that for a large class of models this determines a complete basis of fluxes. We then turned towards fibrations with particularly mild, i.e. I_1 singularities in the fiber. These had been studied before by [23, 24] and are somewhat analogous to the threefolds that were studied in [25]. We used our procedure to determine the integral period lattice for general geometries of this type and elaborated on previous results in several ways.

First we clarified the modular properties of the periods that had been studied in [24] and in particular found several recursion relations between them. To this end we also generalized observations that were made by [25]. We also discussed the monodromies of the periods from the perspective of Fourier-Mukai transformations that act on the category of B-branes. Most notably we generalized a calculation by [26, 27] that gives the monodromy associated to a certain involution in the fiber. Finally we discussed the properly quantized fluxes and the associated scalar potentials at various loci in the moduli space of a concrete example.

We start chapter three of this thesis with a review of the moduli spaces and the cohomology of Calabi-Yau three- and fourfolds. In particular we explain how to obtain the system of Picard-Fuchs operators and the mirror maps for Calabi-Yau hypersurfaces in toric ambient spaces. This is followed by an introduction to the admissible fluxes in F-theory and their phenomenological importance. We then discuss the results of [5].

- The moduli space of certain elliptically fibered Calabi-Yau manifolds can contain limits in which the geometry becomes non-compact. Physically this amounts to a decoupling from gravity and the procedure can be used to engineer supersymmetric gauge theories [28, 29]. Non-perturbative corrections to the gauge theory are encoded in the topological sub-sector of physical string theory [30]. In the fourth chapter we study the modular properties of higher genus generating functions for topological string amplitudes on toric, non-compact Calabi-Yau threefolds. This builds on and extends work that has been published in [17],

  Albrecht Klemm, Maximilian Poretschkin, T.S., Martin Westerholt-Raum,
  Direct Integration for Mirror Curves of Genus Two and an
  Almost Meromorphic Siegel Modular Form,
  CNTP, Vol. 10 (2016), `https://arxiv.org/abs/1502.00557`.

  For this class of Calabi-Yau manifolds the mirror is encoded in a curve and a meromorphic differential. Cases where the mirror curve is of genus one have been studied in numerous publications, see e.g. [15, 31–33]. It has been argued that the behaviour of the generating functions on the boundary of the moduli space fixes the holomorphic anomaly at all genera [15]. In particular, it has been shown that the propagator in the holomorphic anomaly equation has modular properties and can be choosen to be the second Eisenstein series E_2 [33].

The analogues of modular forms for curves of genus two are Siegel modular forms. However, an analogue of the quasi-modular Eisenstein series E_2 was missing. In [17] we clarified the modular properties of the topological string amplitudes for mirror curves of genus one in a way that readily generalizes to the case of genus two. This led us to discover a previously unknown almost-meromorphic Siegel modular form that takes the role of the propagator. We also found a generalized Ramanujan identity that is satisfied by this object. We solved the holomorphic anomaly equations for the geometry $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ up to worldsheet genus three and calculated the refined Gopakumar-Vafa invariants. For a second example we checked the corresponding structure up to genus one.

In chapter four we first review topological string theory, local mirror symmetry and the construction of the mirror curves. This is followed by a discussion of refined topological invariants and the holomorphic anomaly equations. We also give an overview of genus two curves and the theory of Siegel modular forms. We then extend the work that has been done in [17] in several ways.

First we construct an elliptically fibered Calabi-Yau with a G_2 singularity in the fiber that turns out to be a compact embedding of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$. We provide a fairly detailed discussion of the geometry and construct the complete system of six Picard-Fuchs operators that depend on four moduli. Along the lines of [31] this enables us to explicitly trace the periods into the non-compact limit. In this way we support and illustrate a general argument made in [17] that relates the genus zero amplitude of topological string theory to the modular parameter of the genus two mirror curve.

We then review a general theorem by Lockhart [34] that relates the polynomial discriminant of a hyperelliptic Riemann surface to a certain product of theta functions. We show that this can be used to provide an alternative derivation of the Siegel modular structure of the propagator that we found in [17]. Moreover, this leads to a generalization of the results to hyperelliptic mirror curves of arbitrary genus.

Mathematical prerequisites Each chapter starts with an introduction of some physical and mathematical preliminaries that are necessary for the work discussed later on. However, many basic mathematical notions will be used without definition. For a primer on complex and Kähler geometry we recommend [35]. There one can also find an introduction to cohomology as well as Calabi-Yau manifolds and the corresponding moduli spaces. For a brief introduction to varieties and divisors we recommend chapter 0 of [36]. Sheafs and line bundles are introduced nicely in [21]. To any reader with a mathematical background we want to appologize in advance for heavy abuse of notation.

2 F-theory from the fiber

In this chapter we explain the fiber based approach to F-theory and shed some light on a conjectured relation between the effective physics of mirror dual fibers.

To start off we review some of the surprisingly rich geometry of two-dimensional tori. The intention is not only to provide the necessary background for F-theory applications of elliptic fibrations. Two-dimensional tori are also the simplest example of Calabi-Yau geometries and provide some intuition for structures that reappear in higher dimensions.

We then continue to introduce F-theory. This is done with an emphasis on the dictionary between the geometry of elliptic fibrations and the effective quantum field theory. For a review see e.g. [37]. However, the current understanding of F-theory is far from complete and mapping out the dictionary is the subject of intensive research activity. For example, the geometric realization of discrete symmetries [38–44] and non-simply connected gauge groups [45–47] has been understood only recently. Since these are the structures that seem to be related under mirror symmetry in the fiber, we try to summerize the most important aspects of the picture that has emerged over the past few years.

To construct explicit examples we use the framework of toric geometry. In fact, toric techniques play such an extensive role in this thesis that we will spend some time to introduce the necessary background. We are then well equipped to tackle the main topic of this chapter.

2.1 Elliptic curves and modular forms

We start with a complex curve $\mathcal{C}$ that is defined by the equation

$$y^2 z = x^3 + f\, xz^2 + g\, z^3\,, \tag{11}$$

where $(x : y : z)$ are homogeneous coordinates in the projective space $\mathbb{P}^2$. The latter consists of the equivalence classes of complex lines in $\mathbb{C}^3$,

$$\mathbb{P}^2 = \frac{\mathbb{C}^3 \backslash (0,0,0)}{\sim}\,, \quad z_1 \sim z_2 \Leftrightarrow z_1 = \lambda z_2 \text{ for some } \lambda \in \mathbb{C}^*\,. \tag{12}$$

From the adjunction formula one can see that this curve has vanishing first Chern class and must therefore be of genus one. Note that $\mathcal{O} = (0 : 1 : 0)$ generically solves the equation (11). A pair that consists of a genus one curve $\mathcal{C}$ and a marked point $\mathcal{O}$ is called an *elliptic curve* and an elliptic curve defined by equation (11) for some f, g is said to be in *Weierstrass form*. Every elliptic curve is isomorphic to a curve in Weierstrass form.

For now we can assume that $f, g \in \mathbb{Q}$ and later replace $\mathbb{Q}$ with the group of sections $\Gamma(B, \mathcal{L})$ of some line bundle $\mathcal{L}$ over a base manifold B or, locally, the field of rational functions on B. A point on the curve for which the coordinates lie in the field of coefficients K is called K-rational. For a general choice of f, g the curve $\mathcal{C}$ will be smooth and the topology is illustrated in figure 2. The homology lattice $H_1(\mathcal{C}, \mathbb{Z})$ is generated by the A- and B-cycle and the oriented intersection numbers between the cycles are given by

$$A \cap B = 1,\; B \cap A = -1,\; A \cap A = B \cap B = 0\,. \tag{13}$$

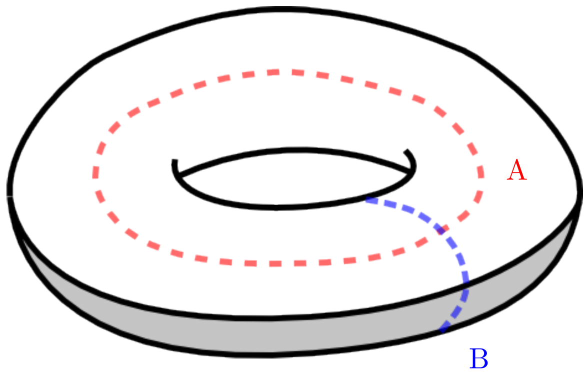


Figure 2: Topology of a smooth torus.

A Weierstrass curve is equipped with the induced complex structure from the ambient space. Moreover, $\mathcal{C}$ is Kähler and the Hodge numbers are $h^{0,0} = h^{1,0} = h^{0,1} = h^{1,1} = 1$. Note that the up to scale unique, nowhere vanishing holomorphic 1-form Ω varies with the choice of complex structure that is in turn determined by the values of f, g. Another way to parametrize the complex structure of an elliptic curve is by using the quotient of periods

$$\tau = \frac{\int_A \Omega}{\int_B \Omega}\,. \tag{14}$$

It can be shown that for a smooth curve τ will always be an element of the complex upper half-plane $\mathcal{H}_1$, i.e. $\operatorname{Im}\tau > 0$. The curve degenerates when τ is equivalent to $i\infty$, the so-called cusp point. This happens when the *discriminant* $\Delta = 4f^3 + 27g^2$ vanishes. A curve $\mathcal{C}_\tau$ with a given complex structure can be constructed as a quotient of the complex plane $\mathbb{C}$, where two points are identified if the difference lies in the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$,

$$\mathcal{C}_\tau = \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}\,. \tag{15}$$

The complex structure on $\mathcal{C}_\tau$ is then induced from the complex structure on $\mathbb{C}$. The construction of τ as a quotient of periods works for general curves $\mathcal{C}$ of genus one and the resulting elliptic curve is called the Jacobian $J(\mathcal{C})$.

The so-called Abel-Jacobi map provides a morphism $\mathcal{C} \to J(\mathcal{C})$. For elliptic curves this is an isomorphism and the Jacobian construction shows that the rational points form a group. We can choose the Abel-Jacobi map such that $\mathcal{O}$ maps to the equivalence class of the origin in $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Then we can map two rational points into the Jacobian and add them under the usual group law of complex numbers. It can be shown that the pre-image of the result under the Abel-Jacobi map is again rational. The Mordell-Weil theorem states that this group is finitely generated and thus isomorphic to $\mathbb{Z}^r$ times a torsional group. The group of rational points is therefore called the *Mordell-Weil group*.

In the definition of τ we did not provide a canonical choice for the A- and B-cycle. In fact, many values of τ correspond to the same complex structure. The automorphisms of $H_1(\mathcal{C}, \mathbb{Z})$ that are compatible with the intersection matrix act on a basis of cycles (A, B) via

$$\begin{pmatrix} A \\ B \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\,. \tag{16}$$

The corresponding action on τ is

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{17}$$

and it can be shown that τ and τ' describe equivalent complex structures iff they are related via a transformation in $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/(-1)$.

2.2 F-Theory, fibers and physics

As we explained in the introduction, the axio-dilaton $\tau = C_0 + ie^{-\Phi}$ in Type IIB string theory behaves like the complex structure parameter of a torus in that the duality group identifies vacua that are related by $\mathrm{SL}(2,\mathbb{Z})$ transformations. The low energy effective theory corresponding to IIB string theory is Type IIB supergravity. The D-branes appear in the supergravity as black branes, i.e. higher dimensional analogues of the Reissner-Nordström black hole. In the 2-dimensional plane transversal to the 7-brane the equations of motion force the axio dilaton to exhibit a logarithmic profile,

$$\tau \sim \frac{1}{2\pi i}\log(z) + \dots\,, \tag{18}$$

where $z = 0$ corresponds to the location of the brane. Encircling the 7-brane therefore leads to a shift of the axio-dilaton $\tau \to \tau + 1$.

Moreover, the IIB string spectrum contains an anti-symmetric two-form called the Kalb-Ramond field B_2. In the supergravity, B_2 and C_2 transform as a vector under the $\mathrm{SL}(2,\mathbb{Z})$ duality. Since D1-branes couple to the C_2 form and fundamental strings couple to B_2, the corresponding charges have to transform aswell. Consistency requires the presence of $[p,q]$-strings which carry p units of charge under B_2 and q units of charge under C_2. Given that D-branes were introduced as objects on which open (fundamental) strings can end we also expect the presence of $[p,q]$ 7-branes. The latter can be shown to lead to a monodromy that corresponds to the $\mathrm{SL}(2,\mathbb{Z})$ matrix

$$M_{p,q} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}\,. \tag{19}$$

Together this allows for a multi-valued axio-dilaton profile along a compactification space B. Divisors in B that are wrapped by [p,q]-branes induce monodromies acting on τ. We can encode this profile in an elliptic fibration that is obtained by “gluing” to each point of B a *fiber* $\mathcal{C}_\tau$. We will call the resulting space X. Note that it is equipped with a projection map $\pi : X \to B$. Turning the argument around, we can associate to a given elliptically fibered manifold X a Type IIB string theory compactified on the base of the fibration with varying axio-dilaton profile given by the complex structures of the fibers [4]. It can be shown that the effective theory will exhibit $\mathcal{N} = 1$ supergravity iff X is Calabi-Yau [7, 8].

The fibration can sometimes be described by a Weierstrass equation (11), where the coefficients f, g are now sections of line bundles on the base. This is the case if the fibration

[4]Strictly speaking, X will in general be singular and is then an algebraic variety.

admits a rational section, that is a morphism $\hat{s}: B \to X$ such that $\pi \circ s = \text{id}$. Independently of the presence of a section there exists a *discriminant locus*

$$\Delta = 4f + 27g = 0\,, \tag{20}$$

that is a divisor in the base of the fibration. The fiber degenerates over this locus and we expect $[p, q]$ 7-branes that wrap the divisor *and* the non-compact spacetime. On the worldvolume of the branes there will be a gauge theory and it turns out that the gauge group depends on the degeneration of the fiber [48].

By now much is known about the relation between the structure of the fibration and the effective physical theory. We will summarize the entries in this dictionary that are most relevant to our work. This includes degenerations of the fiber in various codimensions, the structure of the Mordell-Weil group of sections and the effect of the absence of sections. For more details see for example [19].

2.2.1 Sections and fiber degenerations in various codimensions

We will first discuss the aspects that are directly related to degenerations of the fiber and non-torsional elements of the Mordell-Weil group. Degenerations in the fibers of elliptic surfaces have been classified by Kodaira [49] and Tate [50] and were shown to follow an ADE classification. Singular fibers can be resolved with a tree of spheres that intersect like the affine Dynkin diagram of an ADE Lie algebra. The F-theory effective action exhibits a corresponding gauge symmetry. For higher dimensional bases more general degenerations of the fiber might occur and the discriminant locus $\Delta = 0$ can itself become singular [48, 51, 52]. This allows for a richer structure of singular fibers in codimension 1 but also enhanced degenerations in higher codimension.

Divisors D^b in the base of the fibration give rise to so-called vertical divisors $D = \pi^*(D^b)$ which are the preimages under the projection map. In general the discriminant locus is the union of irreducible components

$$\mathcal{S}^b_{G_I} = \{\Delta_I = 0\},\ I = 1, ..., N\,, \tag{21}$$

in the base [19]. Over a generic point of $\mathcal{S}^b_{G_I}$ the singularity can be resolved with a tree of $\mathbb{P}^1$'s. The latter intersect like the nodes of a Dynkin diagram that corresponds to a group G_I. [5] We will denote the curve that is related to a simple root α_i by $c^{G_I}_{-\alpha_i}$. The union of curves $c^{G_I}_{-\alpha_i}$ for a given root is fibered over $\mathcal{S}^b_{G_I}$ and we will denote the corresponding *Cartan divisors* by $D^{G_I}_i$. In particular the intersection between divisors and curves is given by the Cartan matrix $C^{G_I}_{ij}$,

$$D^{G_I}_i \cdot c^{G_J}_{-\alpha_j} = -C^{G_I}_{ij}\delta_{IJ}\,. \tag{22}$$

The Cartan divisors are dual to harmonic $(1,1)$-forms $\omega^{G_I}_i$. In the M-theory picture the three form field C_3 can be expaned along the harmonic forms via

$$C_3 = \ldots + \sum_{G_I}\sum_i \omega^{G_I}_i \wedge A_i\,, \tag{23}$$

[5] There are cases where the intersection does not correspond to a Dynkin diagram that is associated to a group [48]. However, we will not encounter those in this thesis.

to give rise to the Z-like vector bosons with gauge fields A_i. On the other hand, $M2$ branes can wrap chains of curves $c^{G_I}_{-\alpha_i}$ and thus lead to the W-like bosons that complete the adjoint representation of G_I. The non-Abelian part of the F-theory effective action is the product

$$G_{\text{n.A.}} = \prod_{i=1}^{N} G_I \,. \tag{24}$$

While codimension one singularities lead to non-Abelian gauge symmetries, the abelian part of the gauge group is related to the free part of the Mordell-Weil group MW of the fibration [8]. An elliptic fibration admits at least one section $\hat{s}_0$ that we can choose to be the identity of the Mordell-Weil group. We will denote the independent generators of the group by $\hat{s}_i$, $i = 1, ..., m$. In general there will also be n-sections $\hat{s}^{(n)}$ and we have associated divisors $S_i, S^{(n)}$ that satisfy

$$S_i \cdot \mathfrak{f} = \frac{1}{n} S^{(n)} = 1 \,, \tag{25}$$

where $\mathfrak{f}$ is the class of the generic fiber.

The $m+1$ divisors of sections correspond to harmonic forms that could be used to reduce the M-theory three form as in (23). However, this is not compatible with the F-theory limit and the divisors have to be orthogonalized. The zero section leads to a Kaluza-Klein mode that is absorbed by the four-dimensional metric. To orthogonalize the other sections one uses the *Shioda map* σ that is a homormorphism

$$\sigma : \, \text{MW} \to H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \,, \tag{26}$$

given by

$$\sigma(\hat{s}_m) = S_m - S_0 + [K_B] - \pi(S_m \cdot S_0) + \sum_{I=1}^{N} ([S_m - S_0] \cdot c^{G_I}_{-\alpha_i})(C^{-1}_{G_I})^{ij} D^{G_I}_j \,. \tag{27}$$

where $[K_B]$ is the canonical class of the base. The harmonic forms that are associated to images of free generators of MW lead to abelian gauge bosons in F-theory [53], i.e.

$$G = U(1)^r \times \prod_{i=1}^{N} G_I \,, \tag{28}$$

where r is the rank of MW. The contribution of Cartan divisors $D^{G_I}_j$ is such that

$$\sigma(\hat{s}_i) \cdot c^{G_I}_{-\alpha_j} = 0 \tag{29}$$

for all i, j, I which means that the non-Abelian gauge bosons are not charged under the $U(1)$'s. In addition to (29) the image of the Shioda map is orthogonal to vertical divisors and to the zero section. This defines it uniquely up to an overall normalization.

In codimension two of the base the singularity in the fiber can enhance. For elliptically fibered fourfolds an irreducible component of this locus is a curve $\mathcal{C}$ and resolving the enhanced singularity introduces $\mathbb{P}^1$'s fibered over $\mathcal{C}$ that do not stem from one of the $c^{G_I}_i$.

Let us denote one such $\mathbb{P}^1$ fibration by c. Then c corresponds to matter in a representation with Dynkin labels

$$\lambda_i^{G_I} = D_i^{G_I} \cdot c\,. \tag{30}$$

The charge under the Abelian factor of the gauge group that corresponds to the section $\hat{s}_i$ is given by

$$q_i^c = c \cdot \sigma(\hat{s}_i)\,. \tag{31}$$

Singularity enhancement in codimension three occurs when three matter curves meet. This leads to Yukawa couplings between the corresponding matter representations in the effective action.

2.2.2 Mordell-Weil torsion and the global structure of the gauge group

A k-torsional section $\hat{s}_m$ satisfies

$$k \cdot \hat{s}_m = 0\,, \tag{32}$$

as an element of the Mordell-Weil group. On the other hand, the group $H^{1,1}(X) \cap H^2(X, \mathbb{Q})$ is torsion free. The image of a torsional section $\hat{s}_m$ under the torsion Shioda map is therefore trivial. This is equivalent to the identity

$$S_m - S_0 + [K_B] - \pi(S_m \cdot S_0) = -\sum_{I=1}^{N} ([S_m - S_0] \cdot c_{-\alpha_i}^{G_I})(C_{G_I}^{-1})^{ij} D_j^{G_I}\,. \tag{33}$$

Since $S_m - S_0 + [K_B] - \pi(S_m \cdot S_0) \in H^2(X, \mathbb{Z})$, it follows that

$$\Xi(\hat{s}_m) = -\sum_{I=1}^{N} ([S_m - S_0] \cdot c_{-\alpha_i}^{G_I})(C_{G_I}^{-1})^{ij} D_j^{G_I} \in H^2(X, \mathbb{Z})\,, \tag{34}$$

has integral intersections with all curves in X. However, the entries of the inverse of the Cartan matrix $C_{G_I}^{-1}$ are not integral. This implies that the condition

$$\Xi(\hat{s}_m) \cdot c \in \mathbb{Z}\,, \tag{35}$$

on matter curves c is not trivially satisfied and restricts the possible representations [46].

In group theoretical terms, the divisor $\Xi(\hat{s}_m)$ corresponds to a generator of the refined coweight lattice $\Lambda^\vee$. The divisors $D_j^{G_I}$ in turn generate the coroot lattice $Q^\vee \subseteq \Lambda^\vee$. The quotient

$$\pi_1(G) = \frac{\Lambda^\vee}{Q^\vee}\,, \tag{36}$$

is the fundamental group of the gauge group G and always a subgroup of the center. Non-torsional sections therefore affect the global structure of the gauge group.

It has subsequently been pointed out that a non-torsional generator $\hat{s}_n$ also leads to a restriction of representations [47]. As we already explained, the charge of a matter curve c under the corresponding $U(1)$ is given by

$$q_n^c = c \cdot \sigma(\hat{s}_n) \,. \tag{37}$$

Note that the charge is quantized but in general not integral. It follows that

$$[q_n^c - \Xi(\hat{s}_n)] \cdot c \in \mathbb{Z} \,, \tag{38}$$

imposes additional non-trivial conditions.

2.2.3 Discrete gauge symmetries and genus one fibrations

Not every torus fibration exhibits a section and in general only a multi-section exists. A k-section $\hat{s}^{(m)}$ is a divisor that intersects the generic fiber k times. The number of intersections is also called the degree of $\hat{s}^{(m)}$. The existence of genus one fibrations without a section has a beautiful physical interpretation via F-theory.

We illustrate the general principle at the example of a 2-section. A 2-section has branch points in the base and moving around these loci permutes the two intersections with the generic fiber. A deformation of the complex structure of X can lead to a collision of branch points such that the monodromy becomes trivial [39]. The result is an elliptic fibration with two sections and the effective theory therefore generates a $U(1)$ gauge symmetry. Physically the deformation of complex structure changes the vacuum expectation value of a charged scalar particle to zero. In other words, it is an unhiggsing process. The charge of the particle is $q = 2$ and turning on the vacuum expectation value leaves a $\mathbb{Z}_2$ subgroup of $U(1)$ unbroken. Via this process multi-sections are associated with discrete symmetries in the effective theory.

There is a dual interpretation involving a non-linear Higgs mechanism with a fluxed circle reduction and also a geometrical interpretation via M-theory [43]. A rich story involves the relation to the Jacobian fibration [39, 43]. Moreover, the genus one fibrations with the same Jacobian fibration form the so-called Weil-Châtelet group [54]. For more information we refer the reader to the original literature [38–44].

2.3 Toric geometry and mirror pairs

Having talked about how properties of compactification geometries relate to effective physics via F-theory we need a way to construct such spaces. Toric geometry provides a combinatorial framework to build ambient spaces from which Calabi-Yau manifolds can then be cut out as the zero locus of a set of equations. In particular the Batyrev-Borisov construction allows to build an abundant number of mirror pairs by relating them to the classification of reflexive lattice polytopes and nef-partitions [55, 56]. We will now give a brief introduction to the Cox construction of toric varieties and the Batyrev-Borisov construction of mirror pairs. Thorough treatments of toric geometry can be found in [36, 57–59]. A detailed discussion of the mirror construction is also given in [59].

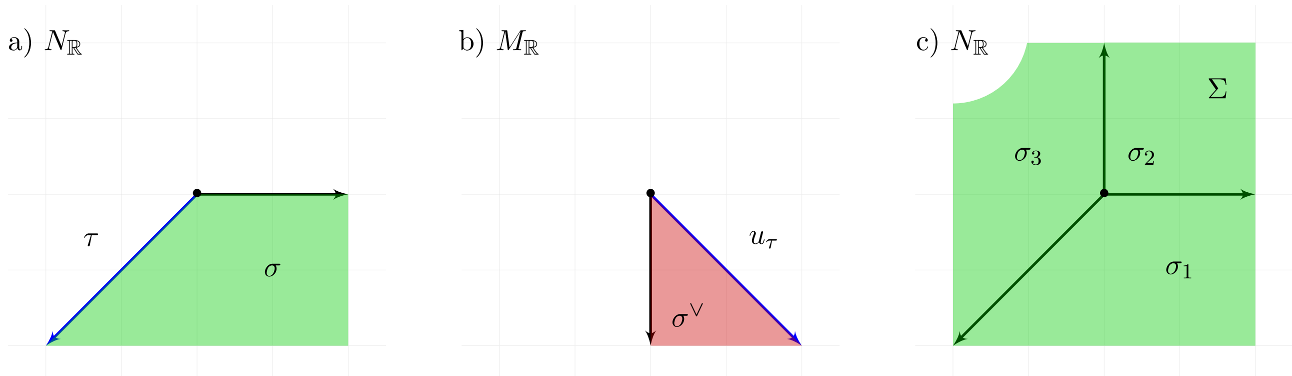


Figure 3: Part a) shows a strongly convex rational polyhedral cone σ and a face τ inside the real extension $N_{\mathbb{R}}$ of a 2-dimensional lattice N. In b) the corresponding dual cone $\sigma^\vee$ and vector u_τ are drawn. A fan Σ consisting of cones $\sigma_1, \sigma_2, \sigma_3$ and the mutual intersections is illustrated in c).

Cox construction of toric varieties We start with a dual pair of d-dimensional lattices N, M and denote the real extensions by $N_{\mathbb{R}}, M_{\mathbb{R}}$ respectively. A *strongly convex rational polyhedral cone* is a subset $\sigma \subset N_{\mathbb{R}}$ such that σ is the positive linear span of a finite set of points $v_i \in N$

$$\sigma = \{\sum_{i=1}^{d} a_i v_i \subset \mathbb{N}_{\mathbb{R}} \,|\, a_1, ..., a_r \geq 0\}\,, \tag{39}$$

and $\sigma \cap (-\sigma) = \{0\}$. In the following we will implicitly assume that cones are strongly convex rational polyhedral. Given a cone σ there is a *dual cone* $\sigma^\vee \subset M_{\mathbb{R}}$,

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \,:\, \langle u, v\rangle = 0 \text{ for all } v \in \sigma\}\,. \tag{40}$$

A *proper face* τ of a cone is a subset $\tau \subset \sigma$ such that

$$\tau = \{v \in \sigma \,:\, \langle u, v\rangle = 0 \text{ for some } u \in \sigma^\vee\}\,. \tag{41}$$

We indicate this by writing $\tau \prec \sigma$. Note that every face is again a strongly convex rational polyhedral cone. A cone is defined to be a (non-proper) face of itself $\sigma \preceq \sigma$. A *fan* Σ is a set of cones such that for $\sigma \in \Sigma$ every face of σ is a cone in Σ and for every pair $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face of each. We say that a cone $\sigma \in \Sigma$ is *maximal* if it is not a proper face of another cone in Σ. Cones, faces and fans are illustrated in figure 3. A fan is completely specified by the set of maximal cones.

For a given fan Σ we denote the set of n-dimensional cones by $\Sigma(n)$. For each 1-dimensional cone $\sigma_i^{(1)} \in \Sigma(1)$ there is a unique $\rho_i \in N$ that generates $\sigma_i^{(i)} \cap N$. The *set of generators* ρ_i, $i = 1, ..., k$ is in general not linearly independent. We can choose a *basis of linear relations* $l^{(j)} \in \mathbb{Z}^k$, $j = 1, ..., m$ such that $\sum_i l_i^{(j)} \rho_i = 0$. In a slight abuse of notation we will sometimes use $\Sigma(1)$ to denote the set of generators instead of the one-dimensional cones. Now there is a *toric variety* $\mathbb{P}_\Sigma$ associated to the fan Σ in the following way:

1. To each generator ρ_i associate a coordinate z_i in a common space $\mathbb{C}^k$.

2. Consider the sets $\{z_{i_1} = \cdots = z_{i_n} = 0\} \in \mathbb{C}^k$ such that $\sigma_{i_1}^{(1)}, \ldots, \sigma_{i_n}^{(1)}$ are not faces of a common cone in Σ. Denote the union of these sets by S. The ideal generated by the corresponding products $z_{i_1} \cdot \cdots \cdot z_{i_n}$ is called the *Stanley-Reisner ideal* $\mathcal{SRI}$.

3. Construct the quotient

$$\mathbb{P}_\Sigma = \frac{\mathbb{C}^k \backslash S}{(\mathbb{C}^*)^m}\,, \tag{42}$$

where $(z_1, \dots, z_k) \sim (\lambda^{l_1^{(j)}} z_1, \dots, \lambda^{l_k^{(j)}} z_k)$ for all $j = 1, ..., m$ and $\lambda \in \mathbb{C}^*$.

Note that the subvariety $V(\mathcal{SRI}) \subset \mathbb{C}^k$ that is associated to the Stanley-Reisner ideal is exactly the set of points that is removed. This is best illustrated by an example:

Example:
Consider the fan Σ in figure 3. The set of generators is

$$\rho_1 = (1, 0)\,, \quad \rho_2 = (0, 1)\,, \quad \rho_3 = (-1, -1)\,, \tag{43}$$

and there is only one independent linear relation $l^{(1)} = (1, 1, 1)$,

$$\sum l_i^{(1)} \rho_i = \rho_1 + \rho_2 + \rho_3 = 0\,. \tag{44}$$

While $\rho_1, \rho_2 \prec \sigma_2$, $\rho_2, \rho_3 \prec \sigma_1$ and $\rho_3, \rho_1 \prec \sigma_3$, ρ_1, ρ_2, ρ_3 do not share a common cone. Therefore $S = \{z_1 = z_2 = z_3 = 0\}$ and

$$\mathbb{P}_\Sigma = \frac{\mathbb{C}^3 \backslash (0, 0, 0)}{\mathbb{C}^*} \quad \text{where} \quad (z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3)\,, \lambda \in \mathbb{C}^*\,. \tag{45}$$

This is the 2-dimensional projective space $\mathbb{P}^2$. We write a point in $\mathbb{P}_\Sigma$ by stating the equivalence class $[z_1 : z_2 : z_3]$. Note that Σ also contains the 0-dimensional cone $\{(0, 0)\} \in N_\mathbb{R}$. Furthermore, the Stanley-Reisner ideal is given by

$$\mathcal{SRI} = \langle z_1 z_2 z_3 \rangle \subset \mathbb{C}[z_1, z_2, z_3]\,. \tag{46}$$

It can be shown that the variety $\mathbb{P}_\Sigma$ is *smooth* if the generators of every maximal cone form a $\mathbb{Z}$ basis of N. More generally, $\mathbb{P}_\Sigma$ is an *orbifold* if the generators of every maximal cone are linearly independent in $N_\mathbb{R}$. We then also call the variety $\mathbb{P}_\Sigma$ and the fan Σ *simplicial*. Moreover, for a simplicial fan with $\Sigma(d) \neq \emptyset$ there are exactly $m = k - d$ linear relations among the generators. The space $\mathbb{P}_\Sigma$ is *compact* with respect to the topology induced from $\mathbb{C}^k$ iff the support $|\Sigma| = \cup_{\sigma \in \Sigma} \sigma$ is equal to $N_\mathbb{R}$.

Divisors and line bundles The generators ρ_i, $i = 1, ..., k$ correspond to Weil divisors $D_i = \{z_i = 0\} \subset \mathbb{P}_\Sigma$. A Weil divisor of the form $D = \sum_i a_i D_i$, $a_i \in \mathbb{Z}$ is called T-Weil divisor. It is T-Cartier iff for every maximal cone $\sigma \in \Sigma$ there is a $u(\sigma) \in M$ such that for all $\rho_i \in \sigma$

$$\langle u(\sigma), \rho_i \rangle = -a_i\,. \tag{47}$$

The line bundle $\mathcal{O}(D)$ is generated by global sections iff the *support function*

$$\Psi_D(v) = \langle u(\sigma), v \rangle\,, \quad \text{for } v \in \sigma\,, \tag{48}$$

is convex, i.e.

$$\Psi_D(\rho_i) \geq -a_i\,, \quad \text{for all } \rho_i \in \Sigma(1)\,. \tag{49}$$

For a smooth toric variety, every Weil divisor is $\mathbb{Q}$-Cartier. Principal Cartier divisors are in one-to-one correspondence with points $m \in M$ via

$$\sum_{i=1}^{k} \langle m, \rho_i \rangle D_i \sim 0 \in \mathrm{Pic}(\mathbb{P}_\Sigma)\,. \tag{50}$$

Now consider the polynomial ring $\mathbb{Z}[z_1, ..., z_k]$. Let $\mathcal{SRI}$ again be the ideal generated by monomials $z_{i_1} \cdot \cdots \cdot z_{i_n}$, where $\rho_1, ..., \rho_n$ are not faces of a common cone in Σ. Furthermore, let $\mathcal{J}$ be the ideal generated by polynomials $\sum_{i=1}^{k} \langle m, \rho_i \rangle z_i$ for all $m \in M$. The Jurkiewicz-Danilov theorem determines the cohomology ring for a smooth complete toric variety [58] as follows:

Theorem:
Given a smooth complete toric variety $\mathbb{P}_\Sigma$, the map $z_i \mapsto [D_i]$ induces a ring isomorphism

$$H^\bullet(\mathbb{P}_\Sigma, \mathbb{Z}) \equiv \frac{\mathbb{Z}[z_1, \ldots, z_k]}{\mathcal{J} + \mathcal{SRI}}\,, \tag{51}$$

that determines the integral cohomology of $\mathbb{P}_\Sigma$.

Every T-Cartier divisor D determines a rational convex polyhedron

$$\Delta_D = \{m \in M_\mathbb{R} \,:\, \langle m, \rho_i \rangle \geq -a_i,\, \forall i = 1, ..., k\}\,. \tag{52}$$

The points $m \in \Delta_D \cap M$ correspond to global sections of $\mathcal{O}(D)$ via

$$\prod_{i=1}^{k} z_i^{\langle m, \rho_i \rangle + a_i} \in \Gamma(\mathbb{P}_\Sigma, \mathcal{O}(D))\,, \tag{53}$$

and a general section $P_D \in \Gamma(\mathbb{P}_\Sigma, \mathcal{O}(D))$ takes the form

$$P_D = \sum_{m \in \Delta_D \cap M} s_m \prod_{i=1}^{k} z_i^{\langle m, \rho_i \rangle + a_i}\,. \tag{54}$$

We will later use the property that $D \mapsto \Delta_D$ is equivariant under scalings, i.e. $\Delta_{nD} = n \cdot \Delta_D$ for $n \in \mathbb{N}$, and a shift of Δ_D by a lattice vector m corresponds to a shift of D by the principal divisor $\sum_i \langle m, \rho_i \rangle D_i$. Moreover, for two T-Cartier divisors D, E the Minkowski sum of the correpsonding polytopes satisfies

$$P_D + P_E \subseteq P_{D+E}\,, \tag{55}$$

with equality if D and E are generated by global sections. If D is generated by global sections and every maximal cone of Σ is d-dimensional, than the divisor is determined entirely in terms of its polytope. Derivations of these statements and more properties of toric divisors can be found in [57], section 3.3.

Curves and nef divisors We say a fan Σ has *convex support of full dimension* if the support $|\Sigma| \subseteq N_\mathbb{R}$ is convex and $\dim |\Sigma| = d$. Let $Z_1(X)$ be the free abelian group generated by irreducible complete curves $C \subseteq X$. Furthermore, let $N_1(X)$ be the quotient $Z_1(X)/\sim$ where $C \sim C'$ iff $(C - C') \cdot D = 0$ for every Cartier divisor D on X. We say that C and C' are *numerically equivalent*. The *toric cone theorem* then provides a description of the cone $\overline{\mathrm{NE}}(\mathbb{P}_\Sigma) \in N_1(X)$ that is generated by irreducible complete curves in $\mathbb{P}_\Sigma$:

Theorem:
A cone $\tau \in \Sigma(d-1)$ is called a *wall* if it is the intersection $\tau = \sigma \cap \sigma'$ of two d-dimensional cones $\sigma, \sigma' \in \Sigma(d)$. Given a fan Σ that has convex support of full dimension, then

$$\overline{NE}(\mathbb{P}_\Sigma) = \sum_{\tau \text{ a wall of } \Sigma} \mathbb{R}_{\geq 0}[V(\tau)]\,, \tag{56}$$

where $V(\tau) = \{z_{i_1} = \cdots = z_{i_n} = 0\}$ such that $\rho_{i_1}, \ldots, \rho_{i_n}$ generate τ and $[V(\tau)]$ is the equivalence class in $N_1(\mathbb{P}_\Sigma)$.

We also call $\overline{NE}(\mathbb{P}_\Sigma)$ the *Mori cone.*

A divisor is trivial iff it has zero intersection with all curves $C \in \overline{NE}(\mathbb{P}_\Sigma)$. In particular, given any $C \in \overline{NE}(\mathbb{P}_\Sigma)$,

$$\sum_{i=1}^{k} \langle m, \rho_i \rangle D_i \cdot C = \langle m, \sum_{i=1}^{k} (D_i \cdot C)\rho_i \rangle = 0\,, \quad \text{for all } m \in M\,. \tag{57}$$

This implies that $l_i^C = D_i \cdot C$ defines a linear relation among the generators ρ_i. In fact, if Σ is simplicial and has convex support of full dimension, then every linear relation corresponds to a curve. Then $N_1(\mathbb{P}_\Sigma)$ can be interpreted as the lattice of linear relations [58]. A Cartier divisor is said to be *numerically effective (nef)* if $D \cdot C \geq 0$ for every irreducible complete[6] curve $C \subseteq X$.

Example:
The fan Σ on the right hand side corresponds to the Hirzebruch surface $\mathbb{F}_1$. It has 4 generators

$$\begin{aligned} \rho_1 &= (-1,-1), \quad \rho_2 = (0,1), \\ \rho_3 &= (1,0), \quad \rho_4 = (0,-1)\,. \end{aligned} \tag{58}$$

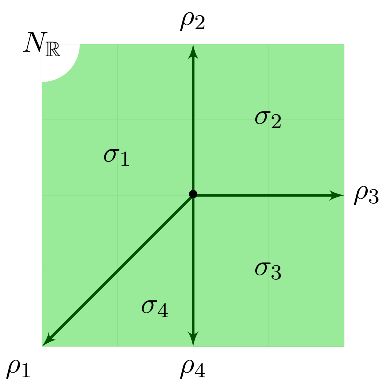


Figure 4: Toric fan corresponding to the Hirzebruch surface $\mathbb{F}_1$

Note that $\mathbb{P}_\Sigma$ is smooth and Σ is in particular simplicial and has convex support of full dimension. Morever, the 1-dimensional cones are simultaneously the walls of Σ. The cone of irreducible compact curves is therefore generated by $[D_i]$, $i = 1, \ldots, 4$.

From the construction of $\mathbb{P}_\Sigma$ in (42), we know that $D_1 \cdot D_2 = D_2 \cdot D_3 = D_3 \cdot D_4 = D_4 \cdot D_1 = 1$ and all other intersections $D_i \cdot D_j$, $i \neq j$ vanish. We can determine self

[6] Complete should be understood as an algebraic equivalent to compact [58].

intersections by using the equivalences among divisors. For example

$$D_4 \cdot D_4 = D_4 \cdot (D_2 - D_1) = -1\,, \tag{59}$$

where we used $\sum_i \langle m, \rho_i \rangle D_i \sim 0$ for $m = (0, -1)$. One finds the relations

$$\begin{aligned} l^{D_1} =& (0,1,0,1)\,, \quad l^{D_2} = (1,1,1,0)\,, \\ l^{D_3} =& (0,1,0,1)\,, \quad l^{D_4} = (1,0,1,-1)\,, \end{aligned} \tag{60}$$

and observes that D_1, D_4 generate the Mori cone. We combine this information into a matrix

$$\left(\begin{array}{rr|rr} -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{array} \right) \tag{61}$$

that we call the toric data associated to $\mathbb{P}_\Sigma$. Note that only for 2-dimensional varieties the divisors are simultaneously curves.

Polytopes and projective toric varieties A *lattice polytope* Δ in $M_\mathbb{R}$ is the convex hull of a finite set of points $p \in M$. We will assume that Δ is full-dimensional in $M_\mathbb{R}$ and that it contains the origin $0 \in \Delta$. The *polar polytope* Δ° is defined to be

$$\Delta^\circ = \{ n \in N_\mathbb{R} \,:\, \langle m, n \rangle \geq -1 \text{ for all } m \in \Delta \}\,. \tag{62}$$

It can be shown that $(\Delta^\circ)^\circ = \Delta$. A *proper face* of a polytope Δ is the intersection with an affine hyperplane that "touches" Δ but does not intersect the interior of Δ. The *normal fan* Σ_Δ associated to Δ is the fan corresponding to the cones over the proper faces of Δ°. A toric variety $\mathbb{P}_\Sigma$ associated to a fan $\Sigma \subset N_\mathbb{R}$ is *projective* iff Σ is the normal fan of a full-dimensional lattice polytope Δ [36]. We then also write $\mathbb{P}_\Delta$. In particular it follows that $\mathbb{P}_\Delta$ is Kähler. Note that by the Jurkiewicz-Danilov theorem the Hodge numbers satisfy $h^{p,q} = 0$ for $p \neq q$.

The Batyrev mirror construction We want to use toric geometry to construct Calabi-Yau varieties. The latter have the defining property that the first Chern class vanishes. The *total Chern class* of a non-singular toric variety is given by

$$c(\mathbb{P}_\Sigma) = \sum_{i=1}^{d} c_i = \prod_{\rho_i \in \Sigma(1)} (1 + D_i)\,. \tag{63}$$

It follows that the first Chern class vanishes iff there is an element $m \in M$ such that

$$\langle m, \rho_i \rangle = 1\,, \quad \text{for all } \rho_i \in \Sigma(1)\,. \tag{64}$$

This implies that all generators lie in a common hyperplane [60]. In particular the toric variety is non-compact. We will deal with topological strings on toric Calabi-Yau varieties in chapter 4. However, toric varieties can also be used to construct compact Calabi-Yau by taking the zero-locus of a section of the anticanonical bundle. It turns out that mirror symmetry then manifests itself on the level of polytopes as the polar involution.

A lattice polytope Δ that contains the origin in its interior is called *reflexive* if the polar polytope Δ° is also a lattice polytope. Then Δ° is also reflexive. If Σ_Δ is smooth, then Δ corresponds to the *anticanonical divisor* $-K$ on $\mathbb{P}_\Sigma$. The zero locus of a generic section $P_\Delta \in \Gamma(\mathbb{P}_\Sigma, -K)$ will be a smooth Calabi-Yau variety. However, in general we have to resolve singularities by refining the fan. Given a reflexive polytope $\Delta \in M_\mathbb{R}$, we call a fan Σ a *projective subdivision* if it refines the normal fan of Δ, $\Sigma(1) \subset \Delta^\circ \cap N - \{0\}$ and $\mathbb{P}_\Sigma$ is projective and simplicial.[7] A projective subdivision is called *maximal* if $\Sigma(1) = \Delta^\circ \cap N - \{0\}$ [59]. For $n \leq 4$ there will always be a projective subdivision such that the zero locus of a generic section $P_\Delta \in \Gamma(\mathbb{P}_\Sigma, -K)$,

$$\sum_{m \in \Delta \cap M} s_m \prod_{i=1}^{k} z_i^{\langle m, \rho_i \rangle + 1} = 0\,, \tag{65}$$

is a smooth Calabi-Yau $(n-1)$-fold (see Appendix A of [59]). We will denote the corresponding fan by $\widehat{\Sigma}$ and the resolved toric variety by $\widehat{\mathbb{P}}_\Delta$.

For $n > 4$ the Calabi-Yau might have singularities that can not be resolved by refining the fan of the ambient space. However, we will not encounter that situation in this thesis. Note that a single polytope admits in general multiple projective subdivisions and the corresponding families of Calabi-Yau varieties are related via flop transitions.

The same construction can be carried out with the role of Δ and Δ° reversed. Let us denote the Calabi-Yau that is obtained from a reflexive lattice polytope Δ by M and the corresponding Calabi-Yau associated to Δ° by W. It has been conjectured that the pair M, W is mirror dual. At the most basic level, mirror symmetry acts by mirroring the Hodge diamond of a variety along the diagonal. Batyrev showed that indeed $h^{n-1,1}(M) = h^{1,1}(W)$ and $h^{n-1,1}(W) = h^{1,1}(M)$ [55]. By now there is plenty of evidence that the varieties are indeed mirror pairs in the sense that there is a corresponding duality of string compactifications. At least for topological string amplitudes of genus zero the duality has been proven in [61–65].

The Batyrev-Borisov construction The above construction has been generalized by Batyrev and Borisov to complete intersections in toric varieties [56]. In addition to the polytope and a projective subdivision that together determine the ambient space, a complete intersection Calabi-Yau is specified by the choice of an *nef-decomposition* of the anticanonical class. The latter is equivalent to a pair of sets of polytopes $\Delta_1, ..., \Delta_n$ and $\nabla_1, ..., \nabla_n$ such that

$$\begin{aligned} \Delta &= \Delta_1 + ... + \Delta_n\,, & \Delta^\circ &= \langle \nabla_1, ..., \nabla_n \rangle\,, \\ \nabla^\circ &= \langle \Delta_1, ..., \Delta_n \rangle\,, & \nabla &= \nabla_1 + ... + \nabla_n\,, \end{aligned} \tag{66}$$

[7] Note that one can check projectivity by finding an ample divisor. Those correspond to strictly convex support functions in the sense of [57].

and $(\Delta_i, \nabla_j) \geq -\delta_{ij}$. Here $\Delta_1 + ... + \Delta_n$ corresponds to the Minkowski sum and $\langle \Delta_1, ..., \Delta_n \rangle$ is the convex hull of the union of polytopes. The inequality has to be satisfied for every pair $v \in \Delta_i$, $v' \in \Delta_j$ with respect to the standard inner product.

The polytopes $\Delta_1, ..., \Delta_n$ define n nef-divisors on the variety associated to $\widehat{\Sigma}_\Delta$. The intersection of zero loci of generic sections of the line bundles associated to $D_{\Delta_1}, ..., D_{\Delta_n}$ is a smooth Calabi-Yau of codimension n in the ambient space. It has been conjectured that the mirror Calabi-Yau is obtained from the analogous construction with Δ replaced by ∇.

Polytopes and nef-partitions can be classified with the PALP software [66] that is also contained in the Sage mathematics system [67]. We will often refer to a particular nef-partition with a pair of numbers that correspond to the polytope id and nef id as defined by PALP.

2.4 Mordell-Weil torsion in the mirror of multi-sections

Genus one curves are Calabi-Yau 1-folds. The Batyrev construction can therefore be used to obtain such curves as hypersurfaces in toric ambient spaces that correspond to reflexive, two-dimensional lattice polytopes. By promoting the coefficients of the hypersurface equation to sections of line bundles over some base variety one obtains a genus one fibration. Under certain conditions on the line bundles this fibration will be Calabi-Yau. In the following we will use *fiber* to refer to a family of curves where the parameters are assumed to be sections of some line bundle over a base.

Even without fixing a base space, certain generic properties of the fibration can be derived from the structure of the fiber. Those were analyzed for genus one hypersurfaces in the 16 two-dimensional reflexive polytopes by [19]. In particular, the authors studied the minimal non-Abelian gauge group, the Mordell-Weil group and multi-sections that arise for a sufficiently generic choice of bundles. Quite unexpectedly they observed that if the curve corresponding to polytope Δ exhibits $\mathbb{Z}_k$ Mordell-Weil torsion the mirror in $\mathbb{P}_{\Delta^\circ}$ will be a genus one curve with k-sections and vice versa.

In this section we study this phenomenon from various perspectives. First we will provide a combinatorial explanation for the observed duality that applies to hypersurfaces in toric ambient spaces. This leads to a more general criterion for toric Mordell-Weil torsion that we will then use to study complete intersections.

2.4.1 Lattice refinement and Shioda maps

A k-torsional section becomes trivial upon multiplication with k and as a group homomorphism the torsion Shioda map[8] must respect this property[46, 68], i.e.

$$k \cdot \sigma_t^{(k)} \sim 0\,. \tag{67}$$

Let us assume that all divisors of the fibration are induced from toric divisors on the ambient space. For hypersurfaces in n-dimensional toric varieties this is guaranteed if $n = 2$ [59].

[8] In a moderate abuse of notation we will in the following use the name Shioda map for the image of a particular section. We will mostly neglect the contribution of vertical divisors that do not correspond to components of the discriminant.

We now consider a fiber $\mathfrak{f}$ corresponding to the two-dimensional reflexive polytope $\Delta \subset N_{\mathbb{R}}$. Using Equation (50) we see that $k \cdot \sigma_t^{(k)}$ corresponds to a dual lattice point $m \in M$ via

$$k \cdot \sigma_t^{(k)} = \sum_{i,\, D_i \cdot \mathfrak{f} \neq 0} \langle m, \rho_i \rangle D_i + \tilde{D}\,, \tag{68}$$

where $\tilde{D}$ is a divisor on the ambient space that is non-trivial but does not intersect the generic fiber. From the discussion in 2.2 of the divisor structure on elliptic fibrations we see that $\tilde{D}$ is a sum of Cartan divisors,

$$\tilde{D} = \sum_{I,i} a_{I,i} D_i^{G_I}\,, \tag{69}$$

for some coefficients $a_{I,i}$. The presence of the torsion Shioda map restricts the admissible matter representations by imposing that intersections of matter curves with $\tilde{D}$ are in $k\mathbb{Z}$, effectively coarse graining the weight lattice.

On the other hand, turning the logic of [46] upside down, a trivial divisor restricts the admissible representations of the gauge group if and only if the coefficients of divisors D_i that intersect the curve have a non-trivial greatest common divisor k that is not a divisor of the coefficients multiplying the resolution divisors.[9] As was already noted in [69] the T-invariant divisors that intersect the curve correspond to the vertices of the polytope Δ°. Moreover, the divisors corresponding to points in the interior of facets resolve singularities of the fiber that generically arise in codimension one of the base and are therefore Cartan divisors. Thus we need an $m \in M$ such that $\langle m,\ \rho_i \rangle \in\ k\mathbb{Z}$ for all vertices ρ_i and $\langle m, \rho_l \rangle \notin k\mathbb{Z}$ for some ρ_l that are not vertices. In other words *the vertices of the polytope span a lattice of index k in N.*

We now want to show that this condition is indeed equivalent to the absence of a section in the dual geometry. First note that if and only if the vertices span a lattice of index k in N, then $\mathbb{P}_\Delta$[10] is covered by $\mathbb{C}^2/\mathbb{Z}_{l \cdot k}$ orbifold patches, i.e. the number of points on any edge of Δ° is $l \cdot k + 1$ for some $l \in \mathbb{N}_{>0}$.

From the perspective of the mirror dual fiber $\hat{\mathfrak{f}}$ the polytope Δ° describes the anticanonical divisor on the ambient space $-K_{\mathbb{P}_{\Delta^\circ}} = D_{\Delta^\circ}$. In two dimensions divisors are curves and by adjunction the curve class of $\hat{\mathfrak{f}}$ is exactly D_{Δ°. Given any vertex $\rho \in \Delta^\circ$ we can shift the polytope $\Delta^\circ - \rho$ so that the origin becomes one of the new vertices. Since every edge of $\Delta^\circ - \rho$ still contains a multiple of $k+1$ points it is proportional to a smaller lattice polytope Δ', i.e. $\Delta^\circ - \rho = k \cdot \Delta'$. In terms of the associated divisors

$$D_{\Delta^\circ} \sim D_{\Delta^\circ - \rho} = k \cdot D_{\Delta'}\,. \tag{70}$$

The intersection of every divisor with $D_{\Delta'}$ is integral and therefore the fiber has no section but only k-sections. Using the toric cone theorem [58] the whole argument can be reversed. This completes the combinatorical explanation why Mordell-Weil torsion is indeed dual to multi-sections for genus one fibers that are constructed from generic hypersurfaces in toric varieties.

[9] This holds of course only up to linear equivalence.

[10] We follow the convention that $\mathbb{P}_\Delta$ is the toric variety associated to the normal fan of Δ. This is standard in the context of mirror symmetry but deviates from [18, 19].

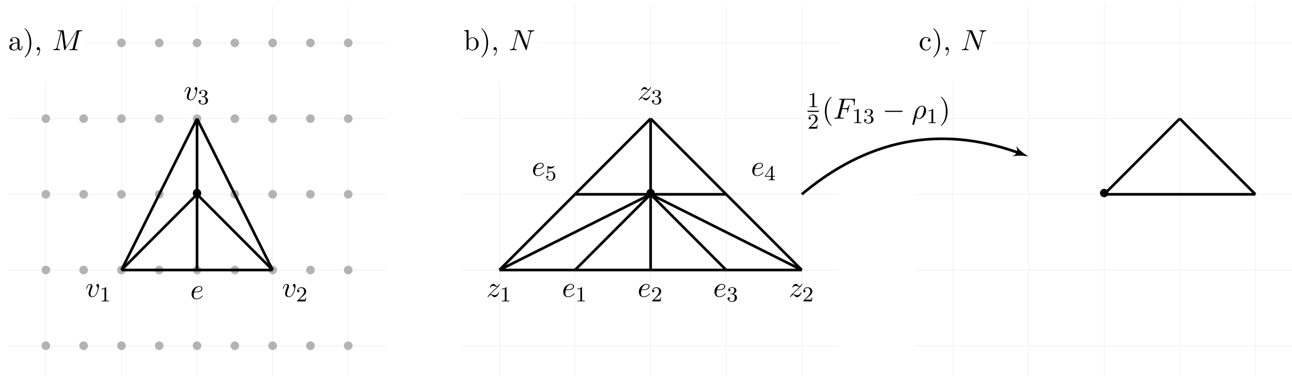


Figure 5: The dual pair of polytopes F_4 and F_{13} is shown in a) and b). We also indicated the toric fan obtained from a complete star triangulation and labelled the homogeneous variables. The gray dots in a) mark the lattice $\tilde{M}$ dual to the sublattice generated by the vertices of F_{13}. A shift followed by a rescaling of F_{13} leads to polytope $\Delta' = \frac{1}{2}(F_{13} - \rho_1)$ shown in c).

Although this finishes the argument, we want to note that the intersections of toric divisors in $\mathbb{P}_\Delta$ with the curve can be directly related to the number of points on the facets of Δ°. Given a toric divisor D_1 that corresponds to a vertex ρ_1 we can choose one of the neighboring generators ρ_2. Then we can calculate the intersection of D_1 with the curve in the patch with variables z_1, z_2 and all other variables set to one. The only monomials that do not vanish for $z_1 = 0$ correspond to points on the facet dual to ρ_1. Moreover, since the facet is orthogonal to ρ_1 it cannot be orthogonal to ρ_2. Therefore imposing $z_1 = 0$ the equation describing the hypersurface in the chosen patch becomes a polynomial in z_2 and every point on the facet dual to ρ_1 corresponds to a different exponent. It is easy to see that it is in fact a polynomial of degree equal to the number of points on the dual facet minus one. The latter is then equal to the number of intersections of D_1 with the curve.

Example: F_4 and F_{13} We illustrate the above with the dual pair of polytopes F_4 and F_{13} shown in Figure 5, a) and b).[11] First look at the toric variety $\mathbb{P}_{F_4}$. The vertices of F_{13} are given by

$$\rho_1 = (-2,\, -1)\,,\; \rho_2 = (2,\, -1)\,,\; \rho_3 = (0,\, 1)\,, \tag{71}$$

and there are five points associated to homogeneous coordinates e_i, $i = 1, ..., 5$ in the relative interior of facets. A generic section of the anti-canonical bundle $\mathcal{O}(D_{F_4})$ is of the form

$$s_1\, z_1^4 e_1^3 e_2^2 e_3 e_5^2 + s_2\, z_1^2 z_2^2 e_1^2 e_2^2 e_3^2 e_4 e_5 + s_3\, z_2^4 e_1 e_2^2 e_3^3 e_4^2 + s_6\, z_1 z_2 z_3 e_1 e_2 e_3 e_4 e_5 + s_9\, z_3^2 e_4 e_4\,. \tag{72}$$

As was explained in [19], given a base variety B we can choose a Cartier divisor $\mathcal{S}_9$ on B and let $s_9 \in \Gamma(B, \mathcal{O}(\mathcal{S}_9))$ be a section of the corresponding line bundle. To obtain an elliptically fibered Calabi-Yau we have to let the coefficients s_1, s_2, s_3, s_6 be sections $s_i \in \Gamma(B, \mathcal{O}(S_i))$, with

$$\mathcal{S}_1 = -3K_B - \mathcal{S}_9\,, \quad \mathcal{S}_2 = -2K_B - \mathcal{S}_9\,, \quad \mathcal{S}_3 = -K_B - \mathcal{S}_9\,, \quad \mathcal{S}_6 = -K_B\,. \tag{73}$$

The discriminant will generically have the components

$$\mathcal{S}^b_{SU(2)_1} = \{s_1 = 0\}\,, \quad \mathcal{S}^b_{SU(2)_2} = \{s_3 = 0\} \quad \text{and} \quad \mathcal{S}^b_{SU(4)} = \{s_9 = 0\}\,. \tag{74}$$

[11] For the numbering of two-dimensional reflexive polytopes we follow the conventions from [19] although this deviates from the ids in the PALP database [66].

Note that the singularity can be enhanced if some of the line bundles have a mutual base point. On the other hand, if any of the $\mathcal{S}_i$, $i = 1, 3, 9$ is principal then not all of these degenerations will occur.

Without fixing a base we can also analyse the toric sections. We denote the divisor associated to a variable z by $[z]$. Then $[z_3]$ leads to a two-section while $[z_1]$ and $[z_2]$ give ordinary sections and the Cartan elements of the $(SU(4) \times SU(2)^2)/\mathbb{Z}_2$ gauge group correspond to $[e_i]$, $i = 1, ..., 5$. The divisors $[z_1]$ and $[z_2]$ are not linearly dependent on the ambient space, so one might expect a $U(1)$ factor in the gauge symmetry. However, one easily sees that the vertices span a sublattice of index 2 in N. The dual lattice $\tilde{M}$ is generated by

$$b_1 = (1/2,\, 0)\,,\, b_2 = (0,\, 1)\,, \tag{75}$$

and we conclude that taking one of the divisors as the zero-section the other becomes a torsional section. In particular we can use the element $b_1 \in \tilde{M}$ to immediately write down the torsion Shioda map

$$\sigma_t^{(2)} = [z_2] - [z_1] + \frac{1}{2}\left(-[e_1] + [e_3] + [e_4] - [e_5]\right)\,, \tag{76}$$

up to base divisors via (68), in agreement with [46, 70].

That the vertices span a lattice of index 2 implies that polytope F_{13} can be shifted and rescaled to the polytope

$$\Delta' = \frac{1}{2}\left(F_{13} - \rho_1\right)\,, \tag{77}$$

shown in Figure 5, c). From the perspective of the ambient space $\mathbb{P}_{F_{13}}$ the polytope F_{13} describes the anticanonical divisor

$$-K = [v_1] + [v_2] + [v_3] + [e]\,. \tag{78}$$

Using the linear equivalences

$$[v_1] \sim [v_2], \quad [v_3] \sim [v_1] + [v_2] + [e]\,, \tag{79}$$

we get

$$-K \sim 4[v_1] + 2[e] = 2 \cdot (2[v_1] + [e])\,, \tag{80}$$

where the divisor $D' = 2[v_1] + [e]$ indeed corresponds to Δ' via (52). This is equivalent to the fact that the fiber in $\mathbb{P}_{F_{13}}$ is a genus-one curve that admits only two-sections.

2.4.2 Complete intersection fibers

Our previous discussion strongly relies on special properties of hypersurfaces in two-dimensional toric ambient spaces. A far larger class of fiber pairs can be obtained using the Batyrev-Borisov construction of complete intersections. This has been applied in [20] to study the fibers that correspond to the 3134 nef-partitions that are admitted by the 4319 three-dimensional reflexive polytopes. The authors studied the generic gauge group and the toric

Mordell-Weil group. Moreover, for every family of curves they calculated the map from the coefficients of the complete intersection equations into the Weierstrass form of the Jacobian.

As a concrete example they studied the complete intersection of two quadrics in $\mathbb{P}^3$. The fiber does not admit a section and they showed that the mirror family again has a torsional Mordell-Weil group. However, this fiber is similar to the hypersurface cases in that it is of Fermat type and the Green-Plesser mirror construction applies. Due to this our combinatorial reasoning can still be used to explain the duality. Nevertheless, we showed in [18] that a family exhibiting Mordell-Weil torsion is dual to a genus one fiber without a section for *every* pair of complete intersections in three dimensional reflexive polytopes. To this end we had to refine the analysis of [20] in several ways.

A naive comparison of the toric Mordell-Weil torsion determined in [20] and the toric intersection numbers led us to 46 counter examples. We reduced this number to 13 mirror pairs by showing that the 3134 families are largely redundant and there are only 1024 inequivalent fibers. Among these we found 12 pairs where both of the fibers did not admit a section. However, for all of these fibers we demonstrated that there is Mordell-Weil torsion in the associated Jacobian. For the single remaining counter example we showed that there is a non-toric section which is consistent with the absence of Mordell-Weil torsion in the dual fiber. We therefore refined the conjecture made by [19] into the following form:

Conjecture 1. *Given a genus-one fiber $\mathfrak{f}$ for which the Mordell-Weil group of the Jacobian contains torsion, the mirror dual is a genus-one fiber $\hat{\mathfrak{f}}$ without a section and vice versa.*

Moreover, while the full derivation given in the previous section does not hold for general complete intersection fibers, we expect the following statement to be true:

Conjecture 2. *A fiber $\mathfrak{f}$, constructed as a complete intersection in a toric ambient space such that all the generic codimension one loci are torically resolved, does exhibit Mordell-Weil torsion of order k in the Jacobian if and only if the one dimensional generators of the fan $\{\rho_i \,:\, D_i \cdot \mathfrak{f} \neq 0\}$ span a sublattice $\tilde{M} \supset M$ of index $k > 1$. Up to base divisors a point $m \in \tilde{M}\backslash M$ corresponds to a torsion Shioda map $\sigma_t^{(k)}$ via*

$$\sigma_t^{(k)} = \sum_{\rho_i \in \Sigma(1)} \langle m,\, \rho_i \rangle \cdot D_i \,. \tag{81}$$

One of the fibers that do not admit a section *and* exhibit Mordell-Weil torsion in the Jacobian corresponds to nef-partition 0 of polytope 122. We studied this example in [18] to show that there is a torsion Shioda map associated to the multi-sections that restricts the matter representations. In particular conjecture 2 applies and the torsion Shioda map can be immediately written down. The discrete Shioda map leads to charge assignments under a discrete symmetry group which is compatible with the observed Yukawa points.

2.5 The fiber (122,0)

We will now extend the analysis of (122,0) in several ways. We embed the fiber into the family of biquadrics in $\mathbb{P}^3$ and parametrize the line bundles of the coefficients in the complete intersection equations by the choice of four line bundles on a base variety. This enables us to show generic cancellation of the six-dimensional non-Abelian gauge anomaly and to derive a generic expression for the Euler characteristic as an integral on the base. The Euler characteristic can in turn be used to calculate the pure gravitational anomaly which does *not* cancel. This indicates that there are additional charged singlets at codimension two loci in the base that do not intersect the gauge divisors.

2.5.1 The biquadric in $\mathbb{P}^3$

The tetrahedron Δ° with vertices

$$\rho_1 = (-1,\,-1,\,-1)\,,\quad \rho_2 = (1,\,0,\,0)\,,\quad \rho_3 = (0,\,1,\,0)\,,\quad \rho_4 = (0,\,0,\,1)\,, \tag{82}$$

admits a unique triangulation that corresponds to the three-dimensional projective space $\mathbb{P}_\Delta = \mathbb{P}^3$. The vertices correspond to the homogeneous coordinates $[w : x : y : z] \in \mathbb{P}^3$. It admits a nef-partition with $\nabla_1 = \langle 0, \rho_1, \rho_2\rangle$, $\nabla_2 = \langle 0, \rho_3, \rho_4\rangle$ that gives the complete intersection $V = \{p_1 = 0\} \cap \{p_2 = 0\} \in \mathbb{P}^3$, where

$$\begin{aligned} p_1 =& s_{1,1}\cdot w^2 + s_{1,2}\cdot wx + s_{1,3}\cdot wy + s_{1,4}\cdot wz + s_{1,5}\cdot x^2 \\ &+ s_{1,6}\cdot xy + s_{1,7}\cdot xz + s_{1,8}\cdot y^2 + s_{1,9}\cdot yz + s_{1,10}\cdot z^2\,, \\ p_2 =& s_{2,1}\cdot w^2 + s_{2,2}\cdot wx + s_{2,3}\cdot wy + s_{2,4}\cdot wz + s_{2,5}\cdot x^2 \\ &+ s_{2,6}\cdot xy + s_{2,7}\cdot xz + s_{2,8}\cdot y^2 + s_{2,9}\cdot yz + s_{2,10}\cdot z^2\,. \end{aligned} \tag{83}$$

The coefficients $s_{i,j}$ parametrize the complex structure (redundantly) and for a generic choice of values V is a smooth curve of genus one. It is easy to see that $[w] = [x] = [y] = [z]$ and $[w]\cdot[p_1]\cdot[p_2] = 4$. Therefore the fiber has no toric sections and one toric four-section.

We now want to consider the coefficients $s_{i,j}$ as sections of line bundles on some base variety B such that the resulting fibration $\pi : X \to B$ is Calabi-Yau. Let us denote the hyperplane class on $\mathbb{P}^3$ by H. Then in general $[w] = H + D_w$ for some divisor D_w on the base and similar relations hold for x, y and z. With the $\mathbb{C}^*$ action on the homogeneous coordinates we can set $D_w = 0$ and the Calabi-Yau condition fixes the class

$$[p_1] + [p_2] = -K_X = [w] + [x] + [y] + [z] + c_1(B)\,. \tag{84}$$

Solving this equation, it turns out that we can parametrize the fibration for a given base B by choosing four divisors $\mathcal{S}_2, \mathcal{S}_6, \mathcal{S}_7, \mathcal{S}_9$ such that

$$\begin{aligned} [w] = H\,,\quad & [x] = H + 2\cdot c_1(B) - (\mathcal{S}_2 + \mathcal{S}_6 + \mathcal{S}_7 + \mathcal{S}_9)\,, \\ [y] = H + c_1(B) - (\mathcal{S}_6 + \mathcal{S}_9)\,,\quad & [z] = H + c_1(B) - (\mathcal{S}_7 + \mathcal{S}_9)\,. \end{aligned} \tag{85}$$

The coefficients are then sections of the line bundles associated to the divisors listed in table 1.

We will now study the non-generic subfamily obtained from imposing $s_{1,5} = s_{1,6} = s_{1,7} = s_{1,8} = s_{1,9} = 0$ and $s_{2,1} = s_{2,3} = s_{2,4} = s_{2,9} = s_{2,10} = 0$. This is torically realized in form of the generic codimension two complete intersections with PALP ids (215,2), (215,5), (122,0), (122,1). While (122,0) and (122,1) as well as (215,2) and (215,5) are identical up to relabelling of the variables, only in (122,0)=(122,1) all the codimension one singularities in the fiber are torically resolved. We will therefore use this realization of the subfamily in the following analysis.

Coefficient	Divisor	Coefficient	Divisor
$s_{1,1}$	$2c_1 - \mathcal{S}_6 - \mathcal{S}_7 - \mathcal{S}_9$	$s_{2,1}$	$3c_1 - \mathcal{S}_2 - \mathcal{S}_6 - \mathcal{S}_7 - 2\mathcal{S}_9$
$s_{1,2}$	$\mathcal{S}_2$	$s_{2,2}$	$c_1 - \mathcal{S}_9$
$s_{1,3}$	$c_1 - \mathcal{S}_7$	$s_{2,3}$	$2c_1 - \mathcal{S}_2 - \mathcal{S}_7 - \mathcal{S}_9$
$s_{1,4}$	$c_1 - \mathcal{S}_6$	$s_{2,4}$	$2c_1 - \mathcal{S}_2 - \mathcal{S}_6 - \mathcal{S}_9$
$s_{1,5}$	$-2c_1 + 2\mathcal{S}_2 + \mathcal{S}_6 + \mathcal{S}_7 + \mathcal{S}_9$	$s_{2,5}$	$-c_1 + \mathcal{S}_2 + \mathcal{S}_6 + \mathcal{S}_7$
$s_{1,6}$	$-c_1 + \mathcal{S}_2 + \mathcal{S}_6 + \mathcal{S}_9$	$s_{2,6}$	$\mathcal{S}_6$
$s_{1,7}$	$-c_1 + \mathcal{S}_2 + \mathcal{S}_7 + \mathcal{S}_9$	$s_{2,7}$	$\mathcal{S}_7$
$s_{1,8}$	$\mathcal{S}_6 - \mathcal{S}_7 + \mathcal{S}_9$	$s_{2,8}$	$c_1 - \mathcal{S}_2 + \mathcal{S}_6 - \mathcal{S}_7$
$s_{1,9}$	$\mathcal{S}_9$	$s_{2,9}$	$c_1 - \mathcal{S}_2$
$s_{1,10}$	$-\mathcal{S}_6 + \mathcal{S}_7 + \mathcal{S}_9$	$s_{2,10}$	$c_1 - \mathcal{S}_2 - \mathcal{S}_6 + \mathcal{S}_7$

Table 1: Divisor classes of the coefficients for the biquadric and nef (122,0).

2.5.2 A torically realized subfamily

The nef-partition (122,0) is determined by

$$\nabla_1 = \langle 0, \rho_0, \rho_3, \rho_4, \rho_6 \rangle\,, \quad \nabla_2 = \langle 0, \rho_1, \rho_2, \rho_5, \rho_8 \rangle\,, \tag{86}$$

with the points ρ_i listed in (89). The corresponding equations are

$$\begin{aligned} p_1 =& s_{1,1} \cdot w^2 e_1^2 e_3 + s_{1,2} \cdot w x e_2^2 e_4 + s_{1,3} \cdot w y e_2 e_4 + s_{1,4} \cdot w z e_1 e_3 + s_{1,10} \cdot z^2 e_3\,, \\ p_2 =& s_{2,2} \cdot w x e_1^2 e_3 + s_{2,5} \cdot x^2 e_2^2 e_4 + s_{2,6} \cdot x y e_2 e_4 + s_{2,7} \cdot x z e_1 e_3 + s_{2,8} \cdot y^2 e_4\,, \end{aligned} \tag{87}$$

and we use the maximal projective subdivision with the maximal cones of the fan $\widehat{\Sigma}_\Delta$ generated by

$$\begin{aligned} &\langle \rho_3, \rho_1, \rho_5 \rangle,\ \langle \rho_3, \rho_2, \rho_7 \rangle,\ \langle \rho_3, \rho_7, \rho_5 \rangle,\ \langle \rho_3, \rho_6, \rho_1 \rangle,\ \langle \rho_6, \rho_1, \rho_4 \rangle,\ \langle \rho_3, \rho_6, \rho_2 \rangle, \\ &\langle \rho_6, \rho_2, \rho_4 \rangle,\ \langle \rho_5, \rho_4, \rho_0 \rangle,\ \langle \rho_1, \rho_5, \rho_4 \rangle,\ \langle \rho_2, \rho_7, \rho_0 \rangle,\ \langle \rho_7, \rho_5, \rho_0 \rangle,\ \langle \rho_2, \rho_4, \rho_0 \rangle\,. \end{aligned} \tag{88}$$

The variables $w, x, y, z, e_1, e_2, e_3, e_4$ are identified under five $\mathbb{C}^*$ actions with the respective charges given by the generators of the Mori cone $l^{(i)}$, $i = 1, ..., 5$:

$$\begin{array}{cc|ccc|ccccc|c} \text{gen.} & \text{coord.} & & \Delta^\circ & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & n \\ \rho_0 & w & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \rho_1 & x & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \rho_2 & y & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ \rho_3 & z & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 2 \\ \rho_4 & e_1 & 1 & -1 & -1 & 0 & -1 & 0 & 1 & 1 & 2 \\ \rho_5 & e_2 & 0 & 2 & 1 & 1 & -1 & 1 & 0 & 0 & 2 \\ \rho_6 & e_3 & 0 & -1 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \\ \rho_7 & e_4 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \tag{89}$$

In the last column of the above data we list the number of intersections of the corresponding toric divisor with a generic fiber. Switching again to the fibration picture, we can use four

$\mathbb{C}^*$ actions to make the divisors $[e_i]$, $i = 1, ..., 4$ sections of the trivial line bundle over the base and obtain

$$\begin{aligned}
[w] =& H\,, \quad [x] = H + 2E_1 - 2E_2 + E_3 - E_4 + 2 \cdot c_1(B) - (\mathcal{S}_2 + \mathcal{S}_6 + \mathcal{S}_7 + \mathcal{S}_9)\,, \\
[y] =& H + 2E_1 - E_2 + E_3 - E_4 + c_1(B) - (\mathcal{S}_6 + \mathcal{S}_9)\,, \\
[z] =& H + E_1 + c_1(B) - (\mathcal{S}_7 + \mathcal{S}_9)\,, \\
[e_i] =& E_i\,, \quad i = 1, ..., 4\,.
\end{aligned} \tag{90}$$

The family can be mapped into Weierstrass form as had been done in [20]. It can be easily checked that the coefficients of the Weierstrass equation

$$y^2 = x^3 + fx + g\,, \tag{91}$$

can be rewritten as

$$f = A_4 - \frac{1}{3}A_2^2\,, \quad g = \frac{1}{27}A_2(2A_2^2 - 9A_4)\,, \quad \Delta = A_4^2(4A_4 - A_2^2)\,, \tag{92}$$

where Δ is the discriminant and

$$\begin{aligned}
A_2 =& 4s_{1,4}s_{1,2}s_{2,8}s_{2,7} + s_{1,3}^2 s_{2,7}^2 - 2s_{1,4}s_{1,3}s_{2,7}s_{2,6} + s_{1,4}^2 s_{2,6}^2 - 4s_{1,10}s_{1,1}s_{2,6}^2 \\
&- 4s_{1,4}^2 s_{2,8}s_{2,5} + 16s_{1,10}s_{1,1}s_{2,8}s_{2,5} - 8s_{1,10}s_{1,2}s_{2,8}s_{2,2} + 4s_{1,10}s_{1,3}s_{2,6}s_{2,2}\,, \\
A_4 =& 16s_{1,10}s_{2,8}(s_{1,2}^2 s_{2,8} - s_{1,3}s_{1,2}s_{2,6} + s_{1,3}^2 s_{2,5})(s_{1,1}s_{2,7}^2 - s_{1,4}s_{2,7}s_{2,2} + s_{1,10}s_{2,2}^2)\,.
\end{aligned} \tag{93}$$

As has been explained for example in [45], this is the generic form of an elliptic curve with a $\mathbb{Z}_2$ torsion element in the Mordell-Weil group.

The Kodaira classification of singular fibers applies at least in codimension one [48] and it is easy to see that the fiber develops I_2 singularities over the divisors

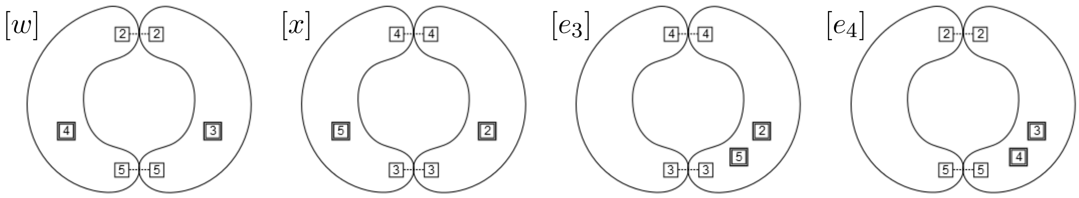
$$\begin{aligned}
L_1 =& \{s_{1,10} = 0\}\,, \quad L_2 = \{s_{2,8} = 0\}\,, \\
L_3 =& \{s_{1,2}^2 s_{2,8} - s_{1,3}s_{1,2}s_{2,6} + s_{1,3}^2 s_{2,5} = 0\}\,, \\
L_4 =& \{s_{1,1}s_{2,7}^2 - s_{1,4}s_{2,7}s_{2,2} + s_{1,10}s_{2,2}^2 = 0\}\,.
\end{aligned} \tag{94}$$

As we explain in [18], the singular fibers over $L_i\,, i = 1, ..., 4$ are torically resolved with the corresponding exceptional divisors $[w], [x], [e_3], [e_4]$. The subvarieties in codimension two of the base where the singularity enhances and matter that transforms in non-trivial representations of the non-Abelian gauge group is located are listed in table 2. The intersection of the corresponding components of the fiber with the 2-sections are shown in figure 7.

2.5.3 Torsional Shioda maps from multi-sections

Since all singular fibers in codimension one of the base are torically resolved, the requirements for conjecture 2 are fulfilled and we can use it to immediately write down "torsion Shioda maps" associated to the bisections. Using the list of non-Abelian matter representations we then argue that it is compatible with the restrictions expected from the non-simply connected global structure of the gauge group. The points

$$\rho_2 = (0,\, 0,\, 1)\,, \quad \rho_3 = (-1,\, -1,\, -1)\,, \quad \rho_4 = (1,\, -1,\, -1)\,, \quad \rho_5 = (0,\, 2,\, 1)\,, \tag{95}$$



Gauge locus L_1 Gauge locus L_2 Gauge locus L_3 Gauge locus L_4

Figure 6: Splitting of the fiber above loci of codimension one. A number n in a box means that the two-section $[\rho_n]$ intersects that component. A double-box is used when the two-section intersects that component twice. Note that $[\rho_2] = [y]$, $[\rho_3] = [z]$, $[\rho_4] = [e_1]$ and $[\rho_5] = [e_2]$. We use this somewhat indirect notation for the labels to be more readable.

Locus	$SU(2)^4 \times \mathbb{Z}_4$	Fiber Components
$L_0 \cap L_1$	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})_{\frac{1}{2}}$	$\mathbb{P}_1^{1,1} = \overline{(\{p_2 = 0\}\backslash\{x = 0\})} \cap \{w = 0\}$ $\mathbb{P}_1^{1,2} = \overline{(\{p_1 = 0\}\backslash\{w = 0\})} \cap \{x = 0\}$ $\mathbb{P}_1^{1,3} = \{w = 0\} \cap \{x = 0\}$ $\mathbb{P}_1^{1,4} = \overline{(\{p_1 = 0\} \cap \{p_2 = 0\})\backslash(\mathbb{P}_1^{1,1} \cup \mathbb{P}_1^{1,2} \cup \mathbb{P}_1^{1,3})}$
$L_0 \cap \{s_{2,7} = 0\}$	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})_1$	$\mathbb{P}_{1,+}^{2,1} \cup \mathbb{P}_{1,-}^{2,1} = \{s_{2,8}y^2 + s_{2,6}xye_2 + s_{2,5}x^2e_2^2\} \cap \{w = 0\}$ $\mathbb{P}_1^{2,2} = \{e_4 = 0\} \cap \{w = 0\}$ $\mathbb{P}_1^{2,3} = \overline{(\{p_1 = 0\} \cap \{p_2 = 0\})\backslash(\mathbb{P}_{1,+}^{2,1} \cup \mathbb{P}_{1,-}^{2,1} \cup \mathbb{P}_1^{2,2})}$
$L_0 \cap \{s_{1,1}s_{2,7} - s_{1,4}s_{2,2} = 0\}$	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})'_0$	$\mathbb{P}_1^{3,1} = \{p_2 = 0\} \cap \{w = 0\}$ $\mathbb{P}_1^{3,2} = \{s_{2,7}z + s_{2,2}we_1 = 0\} \cap \{e_4 = 0\}$ $\mathbb{P}_{1,+}^{3,3} \cup \mathbb{P}_{1,-}^{3,3} = \overline{(\{p_1 = 0\} \cap \{p_2 = 0\})\backslash(\mathbb{P}_1^{3,1} \cup \mathbb{P}_1^{3,2})}$
$L_1 \cap \{s_{1,3} = 0\}$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_0$	$\mathbb{P}_1^{4,1} = \{x = 0\} \cap \{e_3 = 0\}$ $\mathbb{P}_{1,+}^{4,2} \cup \mathbb{P}_{1,-}^{4,2} = \{s_{1,10}z^2 + s_{1,10}wze_1 + s_{1,1}w^2e_1^2\} \cap \{x = 0\}$ $\mathbb{P}_1^{4,3} = \overline{(\{p_1 = 0\} \cap \{p_2 = 0\})\backslash(\mathbb{P}_1^{4,1} \cup \mathbb{P}_{1,+}^{4,2} \cup \mathbb{P}_{1,-}^{4,2})}$
$L_1 \cap \{s_{1,2}s_{2,6} - s_{1,3}s_{2,5} = 0\}$	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})'_1$	$\mathbb{P}_1^{5,1} = \{p_1 = 0\} \cap \{x = 0\}$ $\mathbb{P}_1^{5,2} = \{s_{2,6}y + s_{2,5}xe_2\} \cap \{e_3 = 0\}$ $\mathbb{P}_{1,+}^{5,3} \cup \mathbb{P}_{1,-}^{5,3} = \overline{(\{p_1 = 0\} \cup \{\mathbb{P}_1^{8,1}p_2 = 0\})\backslash(\mathbb{P}_1^{5,1} \cup \mathbb{P}_1^{5,2})}$
$s_{1,10} = 0\,,$ $s_{1,2}^2s_{2,8} - s_{1,3}s_{1,2}s_{2,6} + s_{1,3}^2s_{s,5} = 0$	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{-\frac{1}{2}}$	$\mathbb{P}_1^{6,1} = \{s_{1,3}y + s_{1,2}xe_2 = 0\} \cap \{e_1 = 0\}$ $\mathbb{P}_1^{6,2} = \{s_{1,3}y + s_{1,2}xe_2 = 0\} \cap \{e_3 = 0\}$ $\mathbb{P}_1^{6,3} = \{p_2 = 0\} \cap \{w = 0\}$ $\mathbb{P}_1^{6,4} = \overline{(\{p_1 = 0\} \cup \{p_2 = 0\})\backslash(\mathbb{P}_1^{6,1} \cup \mathbb{P}_1^{6,2} \cup \mathbb{P}_1^{6,3})}$
$s_{2,8} = 0\,,$ $s_{1,1}s_{2,7}^2 - s_{1,4}s_{2,7}s_{2,2} + s_{1,10}s_{2,2}^2 = 0$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\mathbb{P}_1^{7,1} = \{s_{2,7}z + s_{2,2}we_1 = 0\} \cap \{e_2 = 0\}$ $\mathbb{P}_1^{7,2} = \{s_{2,7}z + s_{2,2}we_1 = 0\} \cap \{e_4 = 0\}$ $\mathbb{P}_1^{7,3} = \{p_1 = 0\} \cap \{x = 0\}$ $\mathbb{P}_1^{7,4} = \overline{(\{p_1 = 0\} \cup \{p_2 = 0\})\backslash(\mathbb{P}_1^{7,1} \cup \mathbb{P}_1^{7,2} \cup \mathbb{P}_1^{7,3})}$
$s_{1,2}^2s_{2,8} - s_{1,3}s_{1,2}s_{2,6} + s_{1,3}^2s_{s,5} = 0\,,$ $s_{1,1}s_{2,7}^2 - s_{1,4}s_{2,7}s_{2,2} + s_{1,10}s_{2,2}^2 = 0$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})_{\frac{1}{2}}$	$\mathbb{P}_1^{8,1} = \{s_{1,3}y + s_{1,2}xe_2 = 0\} \cap \{s_{2,7}z + s_{2,2}we_1 = 0\}$ $\mathbb{P}_1^{8,2} = \{s_{1,3}y + s_{1,2}xe_2 = 0\} \cap \{e_3 = 0\}$ $\mathbb{P}_1^{8,3} = \{s_{2,7}z + s_{2,2}we_1 = 0\} \cap \{e_4 = 0\}$ $\mathbb{P}_1^{8,4} = \overline{(\{p_1 = 0\} \cap \{p_2 = 0\})\backslash(\mathbb{P}_1^{8,1} \cup \mathbb{P}_1^{8,2} \cup \mathbb{P}_1^{8,3})}$

Table 2: The vanishing order of the discriminant, components of the fiber and matter representations above codimension two loci with enhanced singularities. A line over a component denotes the closure in the Zariski topology, e.g. $\overline{\{(ay + bxe_2) \cdot z = 0\}\backslash\{z = 0\}} = \{ay + bxe_2 = 0\}$.

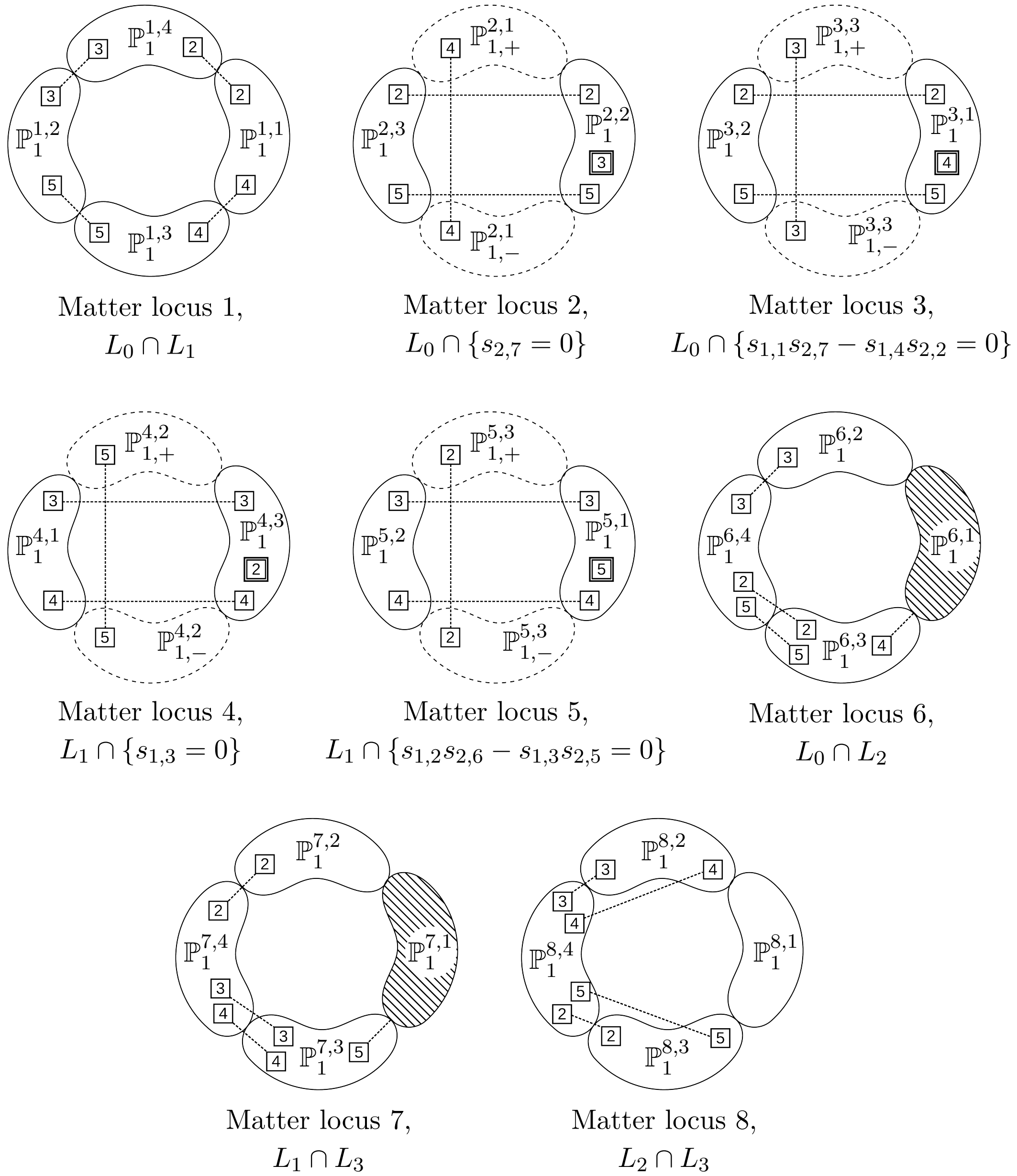


Figure 7: Splitting of the fiber above loci of codimension two. A number i in a box on a component means that the bisection $[\rho_i]$ intersects that component. A double-box is used when the bisection intersects that component twice and the component is shaded when it is wrapped by part of a bisection. Note that $[\rho_2] = [y]$, $[\rho_3] = [z]$, $[\rho_4] = [e_1]$ and $[\rho_5] = [e_2]$. We use this somewhat indirect notation for the labels to be more readable.

are associated to toric divisors that intersect the generic fiber twice. It is easy to see that they generate a lattice of index two in $\mathbb{Z}^3$ and the points

$$b_1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \quad b_2 = (0, 1, 0), \quad b_3 = (0, 0, 1), \tag{96}$$

form a basis of the dual lattice.

Indeed, b_1 corresponds to the $\mathbb{Q}$-Cartier divisor

$$D = \sum_{\rho \in \Sigma(1)} \langle b_1, v_\rho \rangle D_\rho = [e_2] - [z] + \frac{1}{2}\left([w] + [x] - [e_3] + [e_4]\right). \tag{97}$$

Since $2D$ is trivial, we obtain

$$2([e_2] - [z]) \cap C = -\left([w] + [x] - [e_3] + [e_4]\right) \cap C \tag{98}$$

for every curve C. The left hand side is even and therefore

$$\left([w] + [x] - [e_3] + [e_4]\right) \cap C \in 2\mathbb{Z}, \tag{99}$$

where C can be in particular any matter curve that arises in codimension two of the base. Since the intersection of a matter curve with the Cartan divisors corresponds to the Dynkin label of its representation, we see that only representations with label (n_1, n_2, n_3, n_4) such that $n_1 + n_2 - n_3 + n_4 \in 2\mathbb{Z}$ are allowed. This excludes in particular fundamental representations under any single $SU(2)$ factor. A look at table 2 shows that these are indeed absent. We conclude that the fiber generically exhibits Mordell-Weil torsion in the Jacobian and there is an analogue to the torsion Shioda map in terms of bisections that leads to the expected restriction on representations.

2.5.4 6d Anomaly cancellation

At low energies F-theory is expected to be described by an effective supergravity theory. For the latter to be consistent all field theory anomalies have to be cancelled. In general this includes Abelian, non-Abelian, gravitational and mixed anomalies. However, the anomaly constraints from six-dimensional supergravities are particularly strong. These theories arise from compactifications of F-theory on elliptic Calabi-Yau threefolds. It turns out that for a toric family of fibers cancellation of the anomalies can be checked independent of the base [19, 71].

A complete list of 6d anomalies can be found in [19]. Since the Mordell-Weil group of the Jacobian is purely torsional there is no Abelian factor in the gauge group. We therefore only have to check cancellation of the pure non-Abelian, non-Abelian-gravitational and pure gravitational anomalies. The first two read

$$\begin{aligned} \mathrm{tr}\, F_\kappa^2 \mathrm{tr}\, F_\kappa^2 &: \quad \frac{1}{3}\left(\sum_{\mathbf{R}} x_{\mathbf{R}}^\kappa C_{\mathbf{R}} - C_{adj_\kappa}\right) = \left(\frac{b_\kappa}{\lambda_\kappa}\right)^2, \\ \mathrm{tr}\, F_\kappa \mathrm{tr}\, R^2 &: \quad \frac{1}{6}\left(\sum_{\mathbf{R}} x_{\mathbf{R}}^\kappa A_{\mathbf{R}} - A_{adj_\kappa}\right) = a \cdot \left(\frac{b_\kappa}{\lambda_\kappa}\right), \end{aligned} \tag{100}$$

where κ indexes the simple, non-Abelian factors G_κ of the total gauge group and we indicate the corresponding term in the effective action on the left. The sum is over representations of G_κ and the coefficients $x_{\mathbf{R}}$ denote their multiplicity. The relevant group theory factors for $SU(2)$ are $A_{\mathbf{2}} = 1$, $A_{adj} = 4$, $B_{\mathbf{2}} = \frac{1}{2}$, $B_{adj} = 8$ and $\lambda_{\mathbf{2}} = 1$ [19].

To understand how the multiplicities $x_{\mathbf{R}}$ are calculated consider for example the matter curve $L_0 \cap L_1$ in representation $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{\frac{1}{2}}$. The divisor classes of L_0 and L_1 are $-\mathcal{S}_6 + \mathcal{S}_7 + \mathcal{S}_9$ and $c_1 - \mathcal{S}_2 + \mathcal{S}_6 - \mathcal{S}_7$ respectively. If we consider the contribution to the anomaly corresponding to the first $SU(2)$ factor, there is an additional factor of two due to the fact that the matter forms a doublet under the second $SU(2)$. The multiplicity is therefore

$$x^1_{\mathbf{2}} = 2(-\mathcal{S}_6 + \mathcal{S}_7 + \mathcal{S}_9) \cdot (c_1 - \mathcal{S}_2 + \mathcal{S}_6 - \mathcal{S}_7) \,. \tag{101}$$

The multiplicity of adjoint matter from the singularities over codimension in the base is given by the genus of the corresponding curve

$$g_I = 1 + \frac{1}{2} \mathcal{S}^b_{G_I} \cdot (\mathcal{S}^b_{G_I} - c_1) \,. \tag{102}$$

Both expressions can be integrated over a base B given a choice of divisors $\mathcal{S}_i$, $i \in \{2, 6, 7, 9\}$. However, the anomalies cancel already without fixing a base. This is a highly non-trivial check of the matter content listed in 2 *and* the consistency of F-theory.

The pure-gravitational anomalies read

$$\operatorname{tr} R^4 \;:\quad H - V + 29T = 273 \,, \tag{103}$$

$$(\operatorname{tr} R^2)^2 \;:\quad 9 - T = a \cdot a \,, \tag{104}$$

with anomaly coefficient $a = -c_1$ and H, V, T denoting the multiplicity of hyper, vector and tensor multiplets respectively. The second equation is automatically satisfied and relates the number of tensor multiplets to the anti-canonical class on the base. The number of vector multiplets is

$$V = \sum_I \dim(\mathbf{adj}(G_I)) + r \,, \tag{105}$$

where r is the rank of the Mordell-Weil group. The number of hyper multiplets is given by

$$H = H_{\text{codim}=2} + H_{\text{codim}=1} + H_{\text{mod}} \,, \tag{106}$$

where $H_{\text{codim}=1}$ multiplets are in the adjoint representation and H_{mod} counts neutral hyper multiplets that correspond to the complex structure moduli of X [19], i.e.

$$\begin{aligned} H_{\text{codim}=1} =& \sum_{I=1}^{n} g_I (\dim(\mathbf{adj}(G_I)) - \operatorname{rk}(G_I)) \,, \\ H_{\text{mod}} =& h^{(2,1)}(X) + 1 \,. \end{aligned} \tag{107}$$

The latter can be related to the Euler characteristic of X via

$$H_{\text{mod}} = 12 - c_1^2 + \operatorname{rk}(G_X) - \frac{1}{2}\chi(X) \,. \tag{108}$$

Note that $H_{\text{codim}=2}$ also includes the codimension two hyper multiplets that are singlets under the non-Abelian gauge group. This makes the anomaly constraints in six dimensions particularly strong.

To evaluate the anomaly we have to calculate the Euler characteristic $\chi(X)$ in terms of the divisor classes $\mathcal{S}_i$, $i \in \{2, 6, 7, 9\}$ and c_1. This can be done using toric intersection calculus along the lines of [71] and we obtain

$$\begin{aligned}\chi(X) = &-4(3c_1^2 + c_1(-\mathcal{S}_2 - \mathcal{S}_6 - 2\mathcal{S}_7 - 2\mathcal{S}_9) \\ &+ \mathcal{S}_2^2 + 2\mathcal{S}_6^2 + \mathcal{S}_2\mathcal{S}_7 - 2\mathcal{S}_6\mathcal{S}_7 + 2\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)\,.\end{aligned} \tag{109}$$

It turns out that the anomaly does *not* cancel if we only consider the hyper multiplets charged under the non-Abelian gauge group. Since the gauge and gauge-gravity anomalies provide a strong check of the non-Abelian charged matter spectrum, we interpret this as evidence for additional singlets in codimension two of the base. The presence of these singlets is generically expected for geometries with multiple sections and multi-sections [19, 39]. However, there is no general strategy for determining their multiplicity and studying them for the geometry at hand is a subject for future work.

3 Fluxes on elliptically fibered fourfolds

Until now we discussed the geometric realization of gauge symmetries and matter representations in F-theory. Of phenomenological interest is the case of F-theory compactified on elliptic Calabi-Yau fourfolds. In the low energy regime this is expected to be described by a four-dimensional $N = 1$ effective supergravity. The data of such a compactification consists of a geometry X and a choice of flux $G_4 \in H^4(X)$[12] subject to certain constraints. It is in general a difficult problem to determine the set of fluxes that satisfy them. In particular for the set of so-called horizontal fluxes this has been an open problem. In this chapter we will explain how techniques from topological string theory and homological mirror symmetry can be used to find the lattice of horizontal fluxes for a large class of compactifications. This is based on work that has been published in [5].

To set the stage we start with a discussion of the moduli space of Calabi-Yau manifolds, the conditions on G_4 and the phenomenological implications of non-vanishing flux. We also introduce topological branes and their behaviour under mirror symmetry. We will then study the question how for a given fourfold X the set of admissible fluxes can be determined.

3.1 Calabi-Yau manifolds and the mirror map

Calabi-Yau manifolds are solutions to the vacuum Einstein equations that admit exactly one covariantly constant spinor. The latter condition is necessary to preserve a certain amount of supersymmetry. Assuming the Ricci flatness it is for manifolds of complex dimension $d > 1$ equivalent to the requirement that the manifold is Kähler and simply connected. D-branes can be introduced to break half of the remaining supersymmetry as needed for

[12]We assume that $h^3(X) = 0$. Otherwise the additional data is an element in the Deligne cohomology [72].

example to engineer the minimal supersymmetric standard model. In F-theory the Type IIB string theory is compactified on a space that is not Calabi-Yau but can be interpreted as a Calabi-Yau that has been deformed due to backreaction of the D7-branes. However, as we explained above, it is dual to a limit of M-theory compactified on an elliptically fibered Calabi-Yau. From this it can be deduced that the four dimensional effective theory is an $N = 1$ supergravity.

Calabi-Yau manifolds got their name due to a conjecture by Calabi that has subsequently been proven by Yau and allows to study the geometries without actually constructing a metric. A special case of the Calabi-Yau theorem reads as follows. If a compact Kähler manifold M has vanishing first Chern class, then for a given Kähler form ω in the class $[\omega] \in H^{(1,1)}(M, \mathbb{C})$ there is a unique Kähler form $\tilde{\omega}$ with $[\tilde{\omega}] = [\omega]$ such that the corresponding Kähler metric is Ricci flat. Note that in the definition of a Kähler manifold the choice of a complex structure is implicit. The data of a Calabi-Yau manifold therefore consists of a topological type with vanishing first Chern class, a choice of complex structure and a Kähler class $[\omega] \in H^{(1,1)}(M, \mathbb{C})$[13].

3.1.1 Calabi-Yau moduli spaces

For a given topological type the space of complex structures is a Kähler manifold on its own, called the complex structure moduli space. The same holds for the Kähler structure which corresponds to a point in the Kähler moduli space. Both spaces are non-compact with boundary divisors corresponding to various degenerations of the Calabi-Yau that are in general not manifolds but can sometimes be described as algebraic varieties. It is a striking feature of string theory that physics on singular spaces can be non-singular [73].

From the physical perspective we can consider the effective theory expanding around a given complex and Kähler structure. The local coordinates on the moduli spaces appear as scalar fields[14] whose value is dynamically determined. This requires us to study Calabi-Yau manifolds in families. We will now briefly review the structure of the moduli spaces. A more thorough yet accessible introduction can be found for example in [35].

Calabi-Yau 1-folds are just 2-tori that we already discussed above. For $d = 2$ there is only one topological type – the so-called K3 – and the complex structure and Kähler deformation space sit in the same cohomology group. However, we won't need this case in the present work. We therefore restrict ourselves to Calabi-Yau d-folds with $d \geq 3$.

The full moduli space of a Calabi-Yau M is a direct product

$$\mathcal{M}(M) = \mathcal{M}_{\text{c.s}}(M) \times \mathcal{M}_{\text{K.s.}}(M)\,, \tag{110}$$

where $\mathcal{M}_{\text{c.s}}(M)$ is the complex structure and $\mathcal{M}_{\text{K.s.}}(M)$ the Kähler moduli space. Since the effective quantum field theory always corresponds to the perturbative expansion around a given point in $\mathcal{M}(M)$ and the global structure is in general more involved we focus on the local deformations.

[13] We will in general omit the square brackets as the distinction between form and class should be clear from the context.

[14] Since we have target-space supersymmetry by construction each scalar field corresponds to a complete supermultiplet.

The Kähler moduli space consists of the classes of $(1,1)$-forms ω that satisfy the Kähler condition

$$\int_{M_r} \omega^r > 0\,, \tag{111}$$

for every r-dimensional submanifold M_r of M. The classes that satisfy this condition form the so-called Kähler cone. However, physically it makes sense to combine the metric with the Kalb-Ramond field $B \in H^2(M,\mathbb{C})$ into a so-called complexified Kähler form

$$\tilde{\omega} = B + i\omega\,. \tag{112}$$

Due to the shift invariance of the B-field, $\tilde{\omega}$ and $\tilde{\omega}+\alpha$ with $\alpha \in H^2(M,\mathbb{Z})$ are equivalent. The space of sufficiently small deformations to a given complexified Kähler form is therefore an open disc in $H^{(1,1)}(M,\mathbb{C})$.[15]

A vanishing first Chern class (of the tangent bundle) implies that the canonical bundle on M is trivial and there is an up to scale unique $(n,0)$-form Ω that is everywhere non-zero. The spaces $H^{(n,0)}(M,\mathbb{C})$ for different complex structures on M combine into a holomorphic line bundle $\mathcal{L}$ over the complex structure moduli space. This bundle is in general not trivial and given a local trivialization Ω can be treated as a function of the local coordinates.

A perturbation of the complex structure on M turns the same form Ω into an element of $H^{(n,0)}(M,\mathbb{C}) \oplus H^{(n-1,1)}(M,\mathbb{C})$. In local coordinates z_i on $\mathcal{M}_{c.s.}(M)$ this reads

$$\Omega + \delta\Omega = \Omega + z_i \partial_i \Omega + \mathcal{O}(z^2)\,. \tag{113}$$

The Tian-Todorov theorem states that every sufficiently small element of $H^{(n-1,1)}(M,\mathbb{C})$ corresponds in this way to a deformation of complex structure. This implies that Ω, $\partial_i\Omega$ span

$$H^{(n,0)}(M,\mathbb{C}) \oplus H^{(n-1,1)}(M,\mathbb{C})\,. \tag{114}$$

The space of small deformations of complex structure is thus given by an open disc in $H^{(n-1,1)}(M,\mathbb{C})$.

As mentioned above, both $\mathcal{M}_{\text{c.s.}}$ and $\mathcal{M}_{\text{K.s.}}$ are Kähler manifolds. The Kähler metric on $\mathcal{M}_{c.s}$ is called the Weil-Petersson metric. Its Kähler potential has a particularly simple form given by

$$e^{-K} = (-1)^{n(n-1)/2}(2\pi i)^n \int_M \Omega \wedge \bar{\Omega}\,, \tag{115}$$

where n is the complex dimension of the Calabi-Yau M [75]. It can also be shown that $\partial_i + K_i$ with $K_i = \partial_i K$ is a connection on $\mathcal{L}$. More generelly, there are covariant derivatives D_i, $D_{\bar{i}}$ that act on sections $V_{j\bar{j}}$ of $T^*_{1,0}\mathcal{M}_{\text{c.s.}} \otimes T^*_{0,1}\mathcal{M}_{\text{c.s.}} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes m}$ as

$$D_i V_{j\bar{j}} = \partial_i V_{j\bar{j}} - \Gamma^l_{ij} V_{l\bar{j}} + nK_i V_{j\bar{j}}\,, \quad D_{\bar{i}} V_{j\bar{j}} - \Gamma^{\bar{l}}_{\bar{i}\bar{j}} V_{j\bar{l}} + mK_{\bar{i}} V_{j\bar{j}}\,. \tag{116}$$

[15] As we mentioned above, it is physically allowed that the Kähler moduli admit values that are on the boundary of the Kähler cone. This enables smooth transitions between manifolds of different topological type. The idea that such transitions connect all families of Calabi-Yau threefolds is called Raid's fantasy [74]. Wether this fantasy is in fact a reality is still an open question.

3.1.2 Vertical and horizontal cohomology

For a Calabi-Yau threefold the only non-trivial independent hodge numbers are $h^{1,1}$ and $h^{2,1}$. As we explained above, they correspond to the number of Kähler and complex structure moduli respectively. The rest of the Hodge diamond is determined by simply connectedness, implying $h^{0,0} = 1$ and $h^{1,0} = 0$, and a vanishing first Chern that fixes $h^{3,0} = 1$. All other numbers are related to these via complex conjugation and/or Hodge duality.

In the case of Calabi-Yau fourfolds the situation is somewhat more complicated and the independent Hodge numbers are $h^{1,1}, h^{3,1}, h^{2,1}$ and $h^{2,2}$. The first two again correspond to Kähler and complex structure moduli and we will assume that $h^{2,1} = 0$. For information about F-theory on Calabi-Yau fourfolds with $h^{2,1} \neq 0$ see for example [72, 76, 77].

The *horizontal cohomology* of a Calabi-Yau d-fold X is the subset $H^d_H(X, \mathbb{C})$ that is generated by derivatives $\partial_{i_1}...\partial_{i_n}\Omega$ of the holomorphic d-form. For a Calabi-Yau threefold X_3 it is easy to see that $H^3_H(X_3, \mathbb{C}) = H^3(X_3, \mathbb{C})$. However, for a fourfold X_4 the space

$$H^4(X_4, \mathbb{C}) \backslash H^4_H(X_4, \mathbb{C}) \subset H^{2,2}(X_4, \mathbb{C})\,, \tag{117}$$

is not empty. We will in this case denote the dimension of the horizontal part of the cohomology by

$$b^4_H = 2 + 2h^{3,1} + h^{2,2}_H\,, \tag{118}$$

with $h^{2,2}_H = \dim H^4_H \cap H^{2,2}$.

On the other hand, the subspace that is generated by products of classes in $H^{1,1}(X_4, \mathbb{C})$ is called the *vertical cohomology* [16]. Vertical cycles are in particular not horizontal. The complement $H^4(X_4, \mathbb{C}) \backslash H^4_H(X_4, \mathbb{C})$ consists of a vertical part and an additional subspace that is neither vertical nor horizontal [78]. The latter is empty for all cases that we study in this thesis.

3.1.3 The mirror map

Mirror symmetry identifies the physics of Type IIA strings on a Calabi-Yau n-fold M with Type IIA/B string theory on a mirror Calabi-Yau W. It is Type IIA for n even and IIB if n is odd. This in particular implies a map between the effective supergravity degrees of freedom. It turns out that Kähler deformations on M are mapped to complex structure deformations on W and vice versa. Locally the corresponding moduli spaces can therefore be identified. In particular there is an invertible map $t_i(z_i)$ that expresses the Kähler coordinates on M in terms of complex structure coordinates on W. This is called the mirror map.

A detailed review of how the mirror map can be determined for Calabi-Yau threefolds via special properties of the variation of Hodge structure can be found e.g. in [79]. Here we give an explicit description of how to obtain it in the fourfold case at the points of large radius/complex structure for hypersurfaces in toric ambient spaces. For a general discussion see e.g. [80]. The calculation for threefolds is almost identical.

[16] The correct notion of verticality is in general more subtle as was shown in [22]. More concretely, one needs to consider the quantum product in cohomology. For hypersurfaces in toric ambient spaces we can work with the classical notion.

Recall that the data of a Batyev mirror pair is given by a reflexive polytope and a triangulation which is equivalent to the choice of a Mori cone. For concreteness consider the data in table 3 of the elliptically fibered Calabi-Yau fourfold X_{24} with base $\mathbb{P}^3$ and

div.	coord.	$\bar{\nu}_i^\circ \in \Delta^{\circ\prime}$					$l^{(1)}$	$l^{(2)}$
K_M	x_0	0	0	0	0	0	-6	0
D_1	x	-1	0	0	0	0	2	0
D_2	y	0	-1	0	0	0	3	0
E	z	2	3	0	0	0	1	-4
L	u_1	2	3	1	1	1	0	1
L	u_2	2	3	-1	0	0	0	1
L	u_3	2	3	0	-1	0	0	1
L	u_4	2	3	0	0	-1	0	1

Table 3: Toric data of the elliptically fibered Calabi-Yau fourfold hypersurface X_{24}.

mirror $\tilde{X}_{24}$. The rows in $\bar{\nu}_i^\circ$ are the points in $\Delta^{\circ\prime}$ which is in turn the subset of points in Δ° that are not in the interior of facets. The corresponding divisors are listed in the first column of the table and $l^{(1)}$, $l^{(2)}$ are the intersections of those with the two curves generating the Mori cone. We introduce x_0 as an auxilliary coordinate corresponding to the unique interior point of $\Delta^{\circ\prime}$ and the canonical divisor K_M.

Points that are in the interior of facets can be shown to correspond to divisors that generically do not intersect the Calabi-Yau hypersurface. The corresponding monomials in the mirror equation can be removed using toric automorphisms of the ambient space [59]. In the geometry at hand the toric automorphisms have an additional consequence. A discrete remnant of the transformation is dual to the involution that acts on the derived category of X_{24} and together with a large radius monodromy generates a projective representation of the modular group [5, 25, 81].

Let us denote a set of generators for the Kähler cone by

$$J_1 = E + 4L\,, \quad J_2 = L\,, \tag{119}$$

and write a complexified Kähler class J as $J = t_1 J_1 + t_2 J_2$. For the general case let us denote the number of linear relations among the points $\bar{\nu}_i^\circ \in \Delta^{\circ\prime}$ by m. The coefficients t_i, $i = 1, ..., m$ are then coordinates on the complexified Kähler moduli space and the limit point where

$$\mathrm{Im}\, t_i \to \infty \quad \text{for all} \quad i = 1, ..., m\,, \tag{120}$$

is called the *point of large radius.* On the other hand, a redundant set of complex structure coordinates is given by the coefficients a_i of the equation that defines the mirror Calabi-Yau. In particular there is a bijection between the points $\bar{\nu}_i^\circ \in \Delta^\circ$ and the coefficients a_i. It can be shown that the so-called Batyrev variables

$$z_i = (-1)^{l_0^{(i)}} \prod_{\bar{\nu}_j^\circ \in \Delta^{\circ\prime}} a_j^{l_j^{(i)}}\,, \tag{121}$$

provide good complex structure coordinates around the point $z_i = 0$, $i = 1, ..., m$ [59]. This is called the *large complex structure point*.

To identify the mirror map we have to consider integrals over the holomorphic 4-form on $\tilde{X}_{24}$. There are $b^4_H(\tilde{X}_{24})$ linearly independent *periods*

$$\Pi_\delta(z) = \int_\delta \Omega\,, \quad \delta \in H_4(\tilde{X}_{24}, \mathbb{Z})\,. \tag{122}$$

In particular, there is exactly one regular period

$$\Pi^0 = 1 + \mathcal{O}(z)\,, \tag{123}$$

and there are $h^{3,1}(\tilde{X}_{24})$ periods Π^i, $i = 1, ..., h^{3,1}(\tilde{X}_{24})$ of the form

$$\Pi^i = \frac{1}{2\pi i}\Pi^0 \cdot \log(z_i) + \mathcal{O}(z)\,. \tag{124}$$

The quotients Π^i/Π^0 form another complete set of complex structure coordinates. Under mirror symmetry they are identified with the Kähler coordinates t_i introduced above. In other words, the mirror map in the vicinity of the points of large radius/complex structure is given by [80]

$$t_i = \frac{\Pi^i(z)}{\Pi^0(z)}\,, \quad i = 1, ..., h^{3,1}(\tilde{X}_{24}) = h^{1,1}(X_{24})\,. \tag{125}$$

3.1.4 Picard-Fuchs operators

The periods of the holomorphic n-form on a Calabi-Yau n-fold are annihilated by a set of differential operators, the Picard-Fuchs system. For Calabi-Yau varieties constructed as hypersurfaces in a toric ambient space it is easy to write down differential equations for which the solution set is in general larger than that spanned by the periods. However, in many cases the solution sets are equal and it is sufficient to study the so-called GKZ-system. How to derive the GKZ-system from the toric data and the relation to the Picard-Fuchs system is explained e.g. in [59, 82]. Here we summarize the algorithm starting from the Mori cone vectors $l^{(i)}$, $1, ..., m$:

1. Write down the m differential operators

$$\mathcal{D}'_i = \left(\prod_{l_j^{(i)}>0} \partial_{a_j}^{l_j^{(i)}} - \prod_{l_j^{(i)}<0} \partial_{a_j}^{-l_j^{(i)}} \right) a_0^{-1}\,. \tag{126}$$

2. Express the derivatives ∂_{a_j} in terms of logarithmic derivatives $\Theta_{a_j} = a_j\partial_{a_j}$ to obtain

$$\mathcal{D}'_i = a_0^{-1} \prod_{l_j^{(i)}>0} a_j^{-l_j^{(i)}} \prod_{k=0}^{l_j^{(i)}-1} (\Theta_{a_j} - k) - a_0^{-1} \prod_{l_j^{(i)}<0} a_j^{l_j^{(i)}} \prod_{k=0}^{l_j^{(i)}-1} (\Theta_{a_j} - k - \delta_{j,0}) \tag{127}$$

3. Since we only care about the kernel of the differential system we can multiply the operator with a product of a_j and get

$$\begin{aligned}\mathcal{D}_i'' &= \prod_{l_j^{(i)}>0} \prod_{k=0}^{l_j^{(i)}-1} (\Theta_{a_j} - k) - \left(\prod_j a_j^{l_j^{(i)}}\right) \prod_{l_j^{(i)}<0} \prod_{k=0}^{l_j^{(i)}-1} (\Theta_{a_j} - k - \delta_{j,0}) \\ &= \prod_{l_j^{(i)}>0} \prod_{k=0}^{l_j^{(i)}-1} (\Theta_{a_j} - k) - (-1)^{l_0^{(i)}} z_i \prod_{l_j^{(i)}<0} \prod_{k=0}^{l_j^{(i)}-1} (\Theta_{a_j} - k - \delta_{j,0})\,.\end{aligned} \tag{128}$$

4. Now express the logarithmic derivatives Θ_{a_j} in terms of the corresponding derivatives $\Theta_i = z_i \partial_{z_i}$ with respect to the invariant variables z_i via

$$\Theta_{a_j} = \sum_i l_j^{(i)} \Theta_i\,. \tag{129}$$

The resulting operators are

$$\begin{aligned}\mathcal{D}_i = &\prod_{l_j^{(i)}>0} \prod_{k=0}^{l_j^{(i)}-1} \left(\sum_m l_j^{(m)} \Theta_i - k\right) \\ &- (-1)^{l_0^{(i)}} z_i \prod_{l_j^{(i)}<0} \prod_{k=0}^{l_j^{(i)}-1} \left(\sum_m l_j^{(m)} \Theta_i - k - \delta_{j,0}\right)\,.\end{aligned} \tag{130}$$

As we already mentioned above, the Kernel of the GKZ system will in general be larger than the solution space of the Picard-Fuchs equations. It is often possible to obtain the Picard-Fuchs system from the GKZ system by factorizing operators in the latter. More concretely, given GKZ operators $\mathcal{D}_i$, $i = 1, ..., m$ one tries to find operators D_i such that

$$p_i(\Theta) D_i = \sum_{j=1}^m q_j(\Theta) \mathcal{D}_j\,, \tag{131}$$

for some polynomials $p_i(\Theta)$, $q_j(\Theta)$ in the logarithmic derivatives $\Theta_i = z_i \partial_{z_i}$. Another way to enlarge the GKZ system is to construct operators $\mathcal{D}$ associated to linear combinations of the Mori cone vectors. This can be done by carrying out the algorithm above with $l_j^{(i)}$ replaced by the corresponding linear combination $\sum c_i l_j^{(i)}$ and z_i in the second line of (128) is replaced by $\prod (z_i)^{c_i}$.

As an example we will now discuss the Picard-Fuchs system of the mirror of X_{24} where factorizing the GKZ system is necessary. We will denote the mirror by X_{24}^*. Later in section 4 we will show how techniques from homological mirror symmetry can be used to find the correct linear combinations of Mori cone vectors and factorizations thereof to obtain the Picard-Fuchs system associated to a particular four parameter family.

The Mori vectors for X_{24} are

$$l^{(1)} = (-6, 3, 2, 1, 0, 0, 0, 0)\,, \quad l^{(2)} = (0, 0, 0, -4, 1, 1, 1, 1)\,. \tag{132}$$

Using formula (130) we find the GKZ operators

$$\begin{aligned}\mathcal{D}_1 =&(3\Theta_1-2)(3\Theta_1-1)3\Theta_1(2\Theta_1-1)2\Theta_1(\Theta_1-4\Theta_2)\\&-z_1(6\Theta_1+6)(6\Theta_1+5)(6\Theta_1+4)(6\Theta_1+3)(6\Theta_1+2)(6\Theta_1+1)\,,\\\mathcal{D}_2 =&\Theta_2^4-z_2(\Theta_1-4\Theta_2-3)(\Theta_1-4\Theta_2-2)(\Theta_1-4\Theta_2-1)(\Theta_1-4\Theta_2)\,.\end{aligned} \tag{133}$$

The first operator satisfies the relation

$$\mathcal{D}_1 = 6(3\Theta_1-2)(3\Theta_1-1)(2\Theta_1-1)\Theta_1 D_1\,, \tag{134}$$

with

$$D_1 = \Theta_1(\Theta_1-4\Theta_2)-12z_1(6\Theta_1+5)(6\Theta_1+1)\,. \tag{135}$$

One can obtain a basis for the kernel of $D_1, \mathcal{D}_2$ to any given order in z_1, z_2 by making an ansatz and solving the corresponding equations. We find eight independent solutions which agrees with the dimension of the horizontal cohomology on X_{24}^* that via mirror symmetry is equal to the dimension of the vertical cohomology on X_{24}. The latter will be discussed in 3.5. We conclude that $D_1, \mathcal{D}_2$ is the Picard-Fuchs system associated to X_{24}^*.

3.1.5 The discriminant locus

In the previous chapter we introduced the discriminant locus in the base of an elliptic fibration as the codimension one subvariety over which the fiber degenerates. At certain points in the complex structure moduli space the Calabi-Yau variety itself becomes singular. A hypersurface $\{P(x_1,...,x_d)=0\}\subset\mathbb{C}^d$ is singular at points $\vec{x}^s$ where

$$P(x_1^s,...,x_d^s)=\partial_{x_1}P(x_1^s,...,x_d^s)=\cdots=\partial_{x_d}P(x_1^s,...,x_d^s)=0\,. \tag{136}$$

It is clear that a general set of $d+1$ equations in d variables does not admit a simultaneous solution. Only at a locus of codimension one in the moduli space of a general polynomial P does the hypersurface contain singular points. A similar argument holds for more general complete intersections and the corresponding locus in the moduli space is called the discriminant locus. It is defined by $\Delta=0$ where Δ is some reduced polynomial in the complex structure moduli. While possible it is in general impractical to determine Δ directly from this definition. An easier method is to calculate the Yukawa couplings (see e.g. [24]) and take the reduced product of the denominators.

3.2 Fluxes in F-theory

As we explained above, the moduli of a Calabi-Yau manifold are promoted to dynamical scalar fields in the effective low-energy quantum field theory. This also happens to the complex structure moduli of the elliptically fibered Calabi-Yau and the Kähler moduli of the base of the fibration in F-theory. However, until now we have not discussed how to generate a potential for the moduli fields. If the potential and its derivatives vanish, the moduli fields are massless and their expectation values unconstrained.

A potential for the moduli fields corresponding to a four-dimensional F-theory compactification on an elliptically fibered Calabi-Yau M can be generated by choosing a flux $G_4 \in H^4(M,\mathbb{C})$ such that $G_4 + \frac{c_2}{2} \in H^4(M,\mathbb{Z})$.[17] The quantization condition has been derived in [83]. In addition, G_4 has to have exactly one leg along the fiber. For a fibration with a section $[Z]$ this is equivalent to the conditions

$$\int_M G_4 \wedge \pi^* \beta_4 = 0\,, \quad \int_M G_4 \wedge [Z] \wedge \pi^* \beta_2 = 0\,, \tag{137}$$

for all $\beta_4 \in H^4(B)$ and $\beta^2 \in H^2(B)$ [72, 84].

The superpotential generated by a given choice of flux has been derived in [85] and reads

$$W = \int_M \Omega \wedge \frac{G_4}{2\pi} + \int_M \tilde{\omega}^2 \wedge \frac{G_4}{4\pi}\,. \tag{138}$$

As the authors of [85] note, this expression in general receives corrections due to D5-brane and D-string instantons ending on D7-branes. However, these contributions are suppressed by the volume of M. One can therefore choose a complexified Kähler form $\tilde{\omega}$ such that the volume of M is large and G_4 flux that satisfies

$$\tilde{\omega} \wedge G_4 = 0\,, \tag{139}$$

i.e. G_4 is primitive. Then we can assume the contributions to the superpotential that depend on the complex and Kähler structure to decouple and study minima of the scalar potential that is associated to

$$W_{c.s.} = \int_M \Omega \wedge \frac{G_4}{2\pi}\,. \tag{140}$$

A supersymmetric vacuum corresponds to a point in moduli space where

$$W = D_i W = 0\,. \tag{141}$$

In the decoupling limit this is equivalent to the requirement that G_4 is primitive and $G_4 \in H^{(2,2)}(M)$. Moreover, the scalar potential associated to a given superpotential is

$$v = e^K \left[(D_i W)(D_{\bar{j}} \bar{W}) G^{i\bar{j}} - 3W\bar{W} \right]\,, \tag{142}$$

and the complex structure moduli are stabilized at a point where v has a local minimum.

Introducing G_4 flux is also necessary to generate a chiral matter spectrum [86]. A review of this phenomenologically important issue can be found e.g. in [37]. At this point we only note that the chiral index for matter corresponding to a $\mathbb{P}^1$ fibration c over a curve $\mathcal{C}$ in the base is given by

$$\chi = \int_c G_4\,. \tag{143}$$

The expressions for bulk matter that is freely propagating on discriminant components of codimension one in the base are more involved.

[17]For $h^3(M) \neq 0$ there are additional degrees of freedom that can be combined with the choice of G_4 flux into an element of the Deligne-cohomology on M, see e.g. [72]. We will only consider fourfolds with $h^3(M) = 0$.

3.3 D-branes and mirror symmetry

How do we find primitive fluxes that are properly quantized? We will for simplicity assume that c_2 is even. One source of primitive classes is the horizontal cohomology. As we discussed above, this is the subspace $H^4_H(X,\mathbb{C})$ that is generated by derivatives of the holomorphic 4-form Ω with respect to local complex structure coordinates. It can be shown that

$$\int_X \omega \wedge \alpha = 0\,, \tag{144}$$

for $\alpha \in H^4_H$ and ω the Kähler class on X [75]. Note that a choice of horizontal flux $G_4 \in H^4_H$ contributes to the superpotential via

$$W_{\text{c.s.}} = \int_X \Omega \wedge \frac{G_4}{2\pi} = \frac{1}{2\pi} n^i \Pi_i\,, \qquad n^i \in \mathbb{Z}\,, \tag{145}$$

where in the last step we expanded G_4 along an integral basis of H^4_H.

A 4-cycle Σ dual to an element in $H^4_H(W,\mathbb{C}) \cap H^4(W,\mathbb{Z})$ is calibrated symplectically, i.e.

$$\operatorname{Re} e^{i\theta}\Omega\Big|_\Sigma = 0\,, \tag{146}$$

and the Kähler class restricts to zero $\omega|_\Sigma = 0$. In other words, Σ is a *special lagrangian cycle* and those can be wrapped by a topological A-brane L. The central charge of this brane is then given by the period

$$Z_A(L) = \int_\Sigma \Omega\,. \tag{147}$$

Note that this is equal to the superpotential generated by a flux quantum along Σ.

Now assume that W is mirror dual to a Calabi-Yau manifold M. By homological mirror symmetry [87, 88], the topological A-branes on W are related to B-branes on M. The latter correspond to elements in $D^b(M)$, the bounded derived category of coherent sheaves on M. Given a B-brane that corresponds to a complex $\mathcal{E}^\bullet \in D^b(M)$, the asymptotic behaviour of the central charge is

$$Z_B^{\text{asy}}(\mathcal{E}^\bullet) = \int_M e^J \Gamma_{\mathbb{C}}(M)\,(\operatorname{ch}\mathcal{E}^\bullet)^\vee\,, \tag{148}$$

where J is the Kähler class on M. The details of this formula will be discussed in the next section. The crucial fact is that the central charges of A- and B-branes are identified via the mirror map. Moreover, while a construction for all objects in $D^b(M)$ is in general not available, the central charge only depends on the K-theory charge of a complex of sheaves.

Our approach to fix an integral basis for the period lattice will be to construct elements $\mathcal{E}^\bullet$ in $D^b(M)$ that generate the algebraic K-theory group $K^0_{\text{alg}}(M)$ and calculate the asymptotic behaviour of the central charges. Using the mirror map, these can be interpreted as the leading logarithmic terms of generators of the period lattice. The subleading terms are given by the corresponding solutions to the Picard-Fuchs equations.

3.4 Fixing a basis of B-branes

For a Calabi-Yau manifold M, the topological B-branes and the open string states stretched between them are encoded in the bounded derived category of coherent sheaves $D^b(M)$. The objects of this category are equivalence classes of bounded complexes of coherent sheaves

$$\mathcal{E}^\bullet = \ldots \xrightarrow{d^{\mathcal{E}}_{-2}} \mathcal{E}^{-1} \xrightarrow{d^{\mathcal{E}}_{-1}} \mathcal{E}^0 \xrightarrow{d^{\mathcal{E}}_0} \mathcal{E}^1 \xrightarrow{d^{\mathcal{E}}_1} \ldots . \tag{149}$$

A set of maps $f_i : \mathcal{E}^i \to \mathcal{F}^i$, such that the f_i commute with the coboundary maps, corresponds to an element $f \in \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. Objects as well as morphisms are identified under certain equivalence relations but a more detailed discussion of topological branes and $D^b(M)$ is outside the scope of this thesis and can be found e.g. in [21].

However, we note that if there is an exact sequence

$$\ldots \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{F} \longrightarrow 0 \,, \tag{150}$$

where $\mathcal{F}$ is a coherent sheaf and $\mathcal{E}^i$ are locally free sheaves, i.e. equivalent to vector bundles, then the complex

$$\mathcal{E}^\bullet = \ldots \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow 0 \,, \tag{151}$$

is equivalent to $\mathcal{F}$ inside $D^b(M)$.

Now given the Kähler class J, the asymptotic charge of a B-brane that corresponds to the complex $\mathcal{E}^\bullet$ is given by

$$Z^{\mathrm{asy}}(\mathcal{E}^\bullet) = \int_M e^J \Gamma_{\mathbb{C}}(M) \left(\mathrm{ch}\,\mathcal{E}^\bullet\right)^\vee . \tag{152}$$

The characteristic class $\Gamma_{\mathbb{C}}(M)$ can be expressed in terms of the Chern classes of M and for a Calabi-Yau manifold the expansion reads

$$\Gamma_{\mathbb{C}}(M) = 1 + \frac{1}{24} c_2 - \frac{i\zeta(3)}{8\pi^3} c_3 + \frac{1}{5760}(7c_2^2 - 4c_4) + \ldots . \tag{153}$$

The Chern character of the complex is given by

$$\mathrm{ch}(\mathcal{E}^\bullet) = \ldots - \mathrm{ch}(E^{-1}) + \mathrm{ch}(E^0) - \mathrm{ch}(E^1) + \mathrm{ch}(E^2) - \ldots \,, \tag{154}$$

where E^i is the vector bundle corresponding to the locally free sheaf $\mathcal{E}^i$ and the involution $(\ldots)^\vee$ acts on an element $\beta \in H^{2k}(M)$ as $\beta^\vee = (-1)^k \beta$.

A general basis of 0-, 2-, 6- and 8-branes has been constructed in [22]. The 8-brane corresponds to the structure sheaf $\mathcal{O}_M$ and the 6-branes are generated by locally free resolutions of sheaves $\mathcal{O}_{J_i}$, where the divisors J_i generate the Kähler cone. The 0-brane is represented by the skyscraper sheaf $\mathcal{O}_{\mathrm{pt.}}$. A basis of 2-branes was constructed as

$$\mathcal{C}_a^\bullet = \iota_! \mathcal{O}_{\mathcal{C}^a}(K_{\mathcal{C}^a}^{1/2}) \,, \tag{155}$$

where ι is the inclusion of the curve $\mathcal{C}^a$ that is part of a basis for the Mori cone and $K_{\mathcal{C}^a}^{1/2}$ is a spin structure on $\mathcal{C}^a$. The asymptotic charges have been calculated in [22] and for the readers convenience they are reproduced below.

As we described in [5] there is a construction of 4-branes which in many cases leads to an integral basis. Given effective divisors D_i, $i \in I$ that correspond to codimension one subvarieties of M and $S = \bigcap_{i\in I} D_i$, the Koszul sequence

$$
\begin{array}{l}
0 \longrightarrow \mathcal{O}_M\left(-\sum_{i\in I} D_i\right) \longrightarrow \bigoplus_{j\in I} \mathcal{O}_M\left(-\sum_{i\in I\setminus\{j\}} D_i\right) \longrightarrow \dots \\
\longrightarrow \bigoplus_{i\in I} \mathcal{O}_M\left(-D_i\right) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_S \longrightarrow 0
\end{array}
\tag{156}
$$

is exact and provides a locally free resolution of the coherent sheaf $\mathcal{O}_S$. When I contains only one element, this is just the familiar short exact sequence

$$
0 \longrightarrow \mathcal{O}_M(-D) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_D \longrightarrow 0\,. \tag{157}
$$

The latter implies the equivalence

$$
0 \longrightarrow \mathcal{O}_M(-D) \longrightarrow \mathcal{O}_M \longrightarrow 0 \quad \sim \quad 0 \longrightarrow \mathcal{O}_D \longrightarrow 0\,, \tag{158}
$$

of complexes in $D^b(M)$. This is the locally free resolution employed in [22] to calculate the central charges for a basis of 6-branes.

More generally, we can use the Koszul sequence to describe branes wrapped on arbitrary cycles that are intersections of subvarieties of codimension one. If a basis of $H_V^{2,2}(M,\mathbb{C}) \cap H^4(M,\mathbb{Z})$ can be constructed this way, then, as we described above, this leads to an integral basis of the period lattice in the mirror. In particular the asymptotic behaviour then uniquely singles out a solution to the Picard-Fuchs system. For a Calabi-Yau hypersurface M in a toric variety $\mathbb{P}_\Delta$, the cohomology of the ambient spaces is generated by elements in $H^{1,1}(\mathbb{P}_\Delta)$. As was pointed out by the authors of [22], the quantum Lefschetz hyperplane theorem then guarantees that $H_V^{2,2}(M,\mathbb{C})$ is generated by restrictions of elements in $H^{2,2}(\mathbb{P}_\Delta,\mathbb{C})$.

The formula for the asymptotic central charge gives the following results:

- **8-brane:**

$$
\begin{aligned}
Z^{\text{asy}}\left(\mathcal{O}_M\right) &= \int_M e^J \Gamma_{\mathbb{C}}(M) = \frac{1}{4!} C^0_{ijkl} t^i t^j t^k t^l + \frac{1}{2} c_{ij} t^i t^j + c_i t^i + c_0\,, \\
C^0_{ijkl} &= \int_M J_i J_j J_k J_l\,, \quad c_{ij} = \frac{1}{24} \int_M c_2(M) J_i J_j\,, \\
c_i &= -\frac{i\zeta(3)}{8\pi^3} \int_M c_3(M) J_i\,, \quad c_0 = \frac{1}{5760} \int_M \left[7 c_2(M)^2 - 4 c_4(M)\right]
\end{aligned}
\tag{159}
$$

- **6-brane wrapped on J_a:**

$$
\begin{aligned}
Z^{\text{asy}}\left(\mathcal{O}_{J_a}\right) &= \int_M e^J \Gamma_{\mathbb{C}}(M) \left[1 - \operatorname{ch}\left(\mathcal{O}_M(J_a)\right)\right] \\
&= -\frac{1}{3!} C^0_{aijk} t^i t^j t^k - \frac{1}{4} C^0_{aaij} t^i t^j - \left(\frac{1}{6} C^0_{aaai} + \frac{1}{24} c^a_i\right) t^i \\
&\quad - \left(\frac{1}{24} C^0_{aaaa} + c^a_0\right)\,, \\
c^a_i &= \int_M c_2(M) J_a J_i\,, \quad c^a_0 = \frac{1}{48} \int_M c_2(M) J_a^2 - \frac{\zeta(3)}{(2\pi i)^3} \int_M c_3(M) J_a
\end{aligned}
\tag{160}
$$

- **4-brane wrapped on $H = D_a \cap D_b$:**

$$
\begin{aligned}
Z^{\text{asy}}\left(\mathcal{O}_{D_a \cap D_b}\right) =& \frac{1}{2}\int_M h_{ij} t^i t^j + h_i t^i + h\,, \\
h_{ij} =& \int_M D_a D_b J_i J_j\,, \quad h_i = \frac{1}{2}\int_M D_a D_b (D_a + D_b) J_i\,, \\
h =& \frac{1}{12}\int_M D_a D_b (2D_a^2 + 3D_a D_b + 2D_b^2) + \frac{1}{24}\int_M c_2(M) D_a D_b
\end{aligned}
\tag{161}
$$

- **2-brane wrapped on $\mathcal{C}^a$ dual to J_a:**

$$
Z_{\text{asy}}(\mathcal{C}_a^\bullet) = -t_a \tag{162}
$$

The charge of the **0-brane** is universally $Z^{\text{asy}}(\mathcal{O}_{\text{pt.}}) = 1$. We denoted the generators of the Kähler cone by J_i and the Kähler form is given by $J = t^i J_i$.

Finally we need the intersection matrix of the 4-cycles mirror dual to the B-branes. They are not given by the classical intersection numbers in the A-model but rather by the open string index

$$
\chi(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \int_M \text{Td}(M)\,(\text{ch}\,\mathcal{E}^\bullet)^\vee\,\text{ch}\,\mathcal{F}^\bullet\,. \tag{163}
$$

The Todd class $\text{Td}(M)$ is for a Calabi-Yau fourfold given by

$$
\text{Td}(M) = 1 + \frac{c_2(M)}{12} + 2V\,, \tag{164}
$$

where V is the volume form. Note that if we construct a basis of B-branes

$$
\vec{v} = (\mathcal{E}_1^\bullet, \ldots, \mathcal{E}_n^\bullet), \tag{165}
$$

and introduce the intersection matrix $\eta_{ij} = \chi(v_i, v_j)$, the inverse matrix η^{-1} will act on the period vector Π corresponding to the mirror dual cycles. For example

$$
\int_W \Omega \wedge \Omega = 0 \quad \rightarrow \quad \Pi^T \eta^{-1} \Pi = 0\,. \tag{166}
$$

3.5 Geometry of non-singular elliptic Calabi-Yau fourfolds

We will now consider non-singular elliptically fibered Calabi-Yau fourfolds $\pi:\, M \to B$ with one section E. For this class of geometries we showed in [5] that there is a canonical basis of branes.

We denote by D'_k, $k = 1, ..., h^{1,1}(B)$ a basis for the Kähler cone of the base B. The dual basis of the Mori cone will be called $\mathcal{C}'^k$, $k = 1, ..., h^{1,1}(B)$ and satisfies

$$
D'_i \cdot \mathcal{C}'^j = \delta_i^j\,. \tag{167}
$$

A basis of the Mori cone on M is then given by the generic fiber $\tilde{\mathcal{C}}^e$ and pullbacks of curves in the base that are intersected with the section,

$$\tilde{\mathcal{C}}^k = E \cdot \pi^* \mathcal{C}'^k \,. \tag{168}$$

To construct the Kähler cone on M note that for elements $\alpha_1, \alpha_2, \alpha_3 \in H^{*,*}(B)$

$$\int_M E \cdot \pi^* \alpha_1 \cdot \pi^* \alpha_2 \cdot \pi^* \alpha_3 = \int_B \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \,. \tag{169}$$

Furthermore, non-singular elliptic fourfolds are isomorphic to the corresponding Weierstrass model which implies that [89]

$$E \cdot (E + \pi^* c_1(B)) = 0 \,. \tag{170}$$

An orthogonal basis of the Kähler cone on M is therefore given by

$$\tilde{D}_e = E + \pi^* c_1(B) \,, \quad \tilde{D}_k = \pi^* D'_k \,, k = 1, ..., h^{1,1}(B) \,. \tag{171}$$

We also define the topological invariants of the base

$$a = c_1(B)^3 \,, \quad a_i = c_1(B)^2 \cdot D'_i \,, \quad a_{ij} = c_1(B) \cdot D'_i \cdot D'_j \,, \quad c_{ijk} = D'_i \cdot D'_j \cdot D'_k \,, \tag{172}$$

and denote the k-th degree component of $\mathrm{ch}(\mathcal{F}^\bullet)$ by $\mathrm{ch}_k(\mathcal{F}^\bullet)$.

The definitions above are straightforward extensions of the corresponding threefold expressions introduced in [25]. For Calabi-Yau fourfolds a basis of middle-dimensional cycles has to be specified as well. It turns out that for elliptically fibered fourfolds with at most I_1 singularities in the fibers such a basis is given by

$$H_k = E \cdot \pi^{-1} D'_k = E \cdot \tilde{D}_k \,, \quad H^k = \pi^{-1} \tilde{\mathcal{C}}'^k \,, \tag{173}$$

with

$$H_i \cdot H_j = -a_{ij} \,, \quad H_i \cdot H^j = \delta_i^j \,, \quad H^i \cdot H^j = 0 \,. \tag{174}$$

We call the 4-cycles $H^k = \pi^{-1} \tilde{\mathcal{C}}'^k$, $k = 1, \ldots, h_{11}(B)$ that result from lifting a curve in the base to a 4-cycle in M the *π–vertical 4-cycles*. Using the Koszul sequence (156) we calculate

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_{H_i}) =& H_i - \frac{1}{2} \tilde{\mathcal{C}}^k (c_{kii} - a_{ki}) + \frac{1}{12} V (2a_i - 3a_{ii} + 2c_{iii}) \,, \\ \mathrm{ch}(\mathcal{O}_{H^i}) =& H^i - \tilde{\mathcal{C}}^e \cdot h^i \,, \end{aligned} \tag{175}$$

with the volume form V and

$$h^i = \int_M E \,\mathrm{ch}_3(\mathcal{O}_{H^i}) = \sum_{a,b} \frac{1}{2} \lambda_{a,b} E \cdot (D_a \cdot D_b) \cdot (D_a + D_b) \,, \tag{176}$$

where we assume that

$$H^i = \sum_{a,b} \lambda_{a,b} \bar{D}_a \cdot \bar{D}_b \,, \tag{177}$$

for effective divisors $\bar{D}_a$. The Chern characters of the 6-branes are given by

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_{\tilde{D}_i}) =& \tilde{D}_i - \frac{1}{2} H^k c_{kii} + \frac{1}{6} \tilde{\mathcal{C}}^e c_{iii} \,, \\ \mathrm{ch}(\mathcal{O}_E) =& E + \frac{1}{2} H_i \cdot a^i + \frac{1}{6} \tilde{\mathcal{C}}^i a_i + \frac{1}{24} V \cdot a \,. \end{aligned} \tag{178}$$

Moreover, $\mathrm{ch}(\mathcal{O}_M) = 1$, $\mathrm{ch}(\tilde{\mathcal{C}}^{e\bullet}) = \tilde{\mathcal{C}}^e$ and $\mathrm{ch}(\tilde{\mathcal{C}}^{k\bullet}) = \tilde{\mathcal{C}}^k$.

3.6 Fourier-Mukai transformations and the modular group

The B-model periods are multi-valued and experience monodromies along paths encircling special divisors in the complex structure moduli space. Homological mirror symmetry [87] implies that the corresponding monodromies in the A-model lift to auto-equivalences of the derived category [87, 90, 91]. Furthermore, an important theorem by Orlov states that every equivalence of derived categories of coherent sheaves of smooth projective varieties is a Fourier-Mukai transform.

A Fourier-Mukai transform $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ is determined by an object $\mathcal{E} \in D^b(X \times Y)$ and acts as [90, 91][18]

$$\mathcal{F}^\bullet \mapsto R\pi_{1*}(\mathcal{E} \otimes_L L\pi_2^* \mathcal{F}^\bullet)\,, \tag{179}$$

where π_1 and π_2 are the projections from $X \times Y$ to Y and X respectively. The object $\mathcal{E}$ is called the kernel and R and L indicate that one has to take the left- or right derived functor in place of π_*, π^* or $\otimes$.

For our purpose the nice property of this picture is that certain general monodromies correspond to generic Fourier-Mukai kernels. This allows us to write down closed forms not only for the large complex structure monodromies but also for a certain generic conifold monodromy and a third type that is special to elliptically fibered Calabi-Yau.

Let D be one of the generators of the Kähler cone and C the dual curve. The limit in which C becomes large corresponds to a divisor in the Kähler moduli space. It is well known [90] that the Fourier-Mukai transform corresponding to the monodromy around this large radius divisor acts as

$$\mathcal{E}^\bullet \mapsto \mathcal{O}(D) \otimes \mathcal{E}^\bullet\,. \tag{180}$$

We choose a basis of branes

$$\left(\mathcal{O}_M, \mathcal{O}_E, \mathcal{O}_{D_i}, \mathcal{O}_{H_i}, \mathcal{O}_{H^i}, \tilde{\mathcal{C}}^i, \tilde{\mathcal{C}}^e, \mathcal{O}_{\text{pt.}}\right)\,, \tag{181}$$

and calculate the monodromy for the large radius divisor corresponding to D_j,

$$\tilde{T}_j = \begin{pmatrix} 1 & 0 & -\delta_j^k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\delta_j^k & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_i^k & 0 & -c_{jik} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_i^k & 0 & -c_{jik} & 0 & \frac{1}{2}(c_{jii} + c_{jji} - a_{ji}) \\ 0 & 0 & 0 & 0 & \delta_k^i & 0 & -\delta_j^i & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_k^i & 0 & -\delta_j^i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{182}$$

acting on the vector of charges. One can obtain a similar expression for the monodromy $\tilde{T}_e$, corresponding to $\tilde{D}_e$.

[18] An accessible explanation for physicists of how these calculations are performed can be found in [92].

Another auto-equivalence, the Seidel-Thomas twist, corresponds to the locus where, given a suitable loop based on the point of large radius, the D8-brane becomes massless. Its action on the brane charges is given by

$$Z(\mathcal{E}^\bullet) \mapsto Z(\mathcal{E}^\bullet) - \chi(\mathcal{E}^\bullet, \mathcal{O}_M) Z(\mathcal{O}_M)\,. \tag{183}$$

As was explained in [75], for a Calabi-Yau fourfold $\chi(\mathcal{O}_M, \mathcal{O}_M) = 2$. This implies that $Z(\mathcal{O}_M)$ transforms into $-Z(\mathcal{O}_M)$ and this monodromy is of order two.

Elliptically fibered Calabi-Yau manifolds with at most I_1 singularities exhibit yet another type of auto-equivalence. Physically it corresponds to T-duality along both circles of the fiber torus. The corresponding action Φ on the derived category was first studied by Bridgeland [93] in the context of elliptic surfaces. Calculations for Calabi-Yau threefolds can be found in [26] and were elaborated on in the subsequent review [27]. In full generality the auto-equivalences and their implications for the modularity of the amplitudes on elliptic Calabi-Yau threefolds with I_1 singularities [94] have been presented in [95].

We can decompose the Chern character of a general brane $\mathcal{E}^\bullet$ as

$$\begin{aligned}
\mathrm{ch}_0(\mathcal{E}^\bullet) =& n,\\
\mathrm{ch}_1(\mathcal{E}^\bullet) =& n_E\, E + F_1,\\
\mathrm{ch}_2(\mathcal{E}^\bullet) =& E \cdot B_1 + F_2,\\
\mathrm{ch}_3(\mathcal{E}^\bullet) =& E \cdot B_2 + n_e\, \tilde{\mathcal{C}}^e,\\
\mathrm{ch}_4(\mathcal{E}^\bullet) =& s\, V.
\end{aligned} \tag{184}$$

Here we introduced $n, n_E, n_e, s \in \mathbb{Q}$, and F_i, B_i are pullbacks of forms in $H^{i,i}(B, \mathbb{C})$. The volume form on M is denoted by V. Adapting the calculation in [27] to Calabi-Yau fourfolds, we find that the Chern character of the transformed brane is given by

$$\begin{aligned}
\mathrm{ch}_0(\Phi(\mathcal{E}^\bullet)) =& n_E\,,\\
\mathrm{ch}_1(\Phi(\mathcal{E}^\bullet)) =& B_1 - \frac{1}{2} n_E\, c_1 - n \cdot E\,,\\
\mathrm{ch}_2(\Phi(\mathcal{E}^\bullet)) =& B_2 - \frac{1}{2} B_1 \cdot c_1 + \frac{1}{12} n_E\, c_1^2 - F_1 \cdot E + \frac{1}{2} n\, c_1 \cdot E\,,\\
\mathrm{ch}_3(\Phi(\mathcal{E}^\bullet)) =& -\frac{1}{2} B_2 \cdot c_1 + \frac{1}{12} B_1 \cdot c_1^2 + s\, \tilde{\mathcal{C}}^e + \frac{1}{2}\, c_1 \cdot F_1 \cdot E - F_2 \cdot E - \frac{1}{6} n\, c_1^2 \cdot E\,,\\
\mathrm{ch}_4(\Phi(\mathcal{E}^\bullet)) =& -n_e\, V - \frac{1}{6}\, c_1^2 \cdot F_1 \cdot E + \frac{1}{2} c_1 \cdot F_2 \cdot E + \frac{1}{24} n\, c_1^3 \cdot E\,,
\end{aligned} \tag{185}$$

with $c_1 = \pi^* c_1(B)$. Using the formulae for the Chern characters of the basis of branes introduced above, this translates into the matrix

$$\tilde{S} = \begin{pmatrix}
0 & -1 & 0 & a^k & 0 & \frac{1}{2}\left(c_{kii} a^i - a_k\right) & 0 & \frac{1}{12}(3 a_{ii} a^i - 2 c_{iii} a^i - a)\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\delta_i^k & 0 & a_{ki} & 0 & -\frac{1}{2} a_{ii}\\
0 & 0 & \delta_i^k & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\delta_k^i & 0 & h^i + \frac{1}{2} a^i\\
0 & 0 & 0 & 0 & \delta_k^i & 0 & h^i - \frac{1}{2} a^i & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \tag{186}$$

for the corresponding monodromy.

We can now explicitly calculate that

$$\left(\prod_{i=1}^{h_{11}(B)} \tilde{T}_i^{-a^i}\right) \tilde{S} \cdot \tilde{S} = -\mathbb{I}\,, \tag{187}$$

and another careful calculation reveals

$$(\tilde{S} \cdot \tilde{T}_e^{-1})^3 = -\mathbb{I}\,. \tag{188}$$

It follows that

$$S = \left(\prod_{i=1}^{h^{11}(B)} \tilde{T}_i^{-a^i/2}\right) \tilde{S}\,, \quad T = \left(\prod_{i=1}^{h^{11}(B)} \tilde{T}_i^{a^i/2}\right) \tilde{T}_e^{-1}\,, \tag{189}$$

generate a group isomorphic to $PSL(2,\mathbb{Z})$, the modular group. In particular, Q^k, q are invariant under T, while some of the $\tilde{Q}^k$ obtain a sign under T-transformations if the canonical class of the base is not even. As was already noted by [25], this makes Q^k and q the correct expansion parameters for the topological string amplitudes to exhibit modular properties.

Indeed we have shown in [5] that the genus zero and genus one free energies of topological strings on certain non-singular elliptic Calabi-Yau fourfolds admit an expansion in terms of quasi modular forms. We also derived certain modular anomaly equations that relate the various amplitudes. The results support a general conjecture by Georg Oberdieck and Aaron Pixton [96, 97] about the modular structure of Gromov-Witten invariants of non-singular elliptically fibered Calabi-Yau manifolds.

3.7 Horizontal flux vacua for X_{24}^*

We will now use the integral period basis for the mirror X_{24}^* of $X_{24}(1,1,1,1,8,12)$ to study the admissible horizontal fluxes and the corresponding vacua. To this end we analytically continue the basis to various special loci in the complex structure moduli space. Note that the structure of the moduli space is similar to that of the mirror of the threefold $X_{18}(1,1,1,6,9)$ which has been studied in [81].

The toric ambient space of X_{24} corresponds to the data given in Table 3. The intersection ring on X_{24} is generated by the restrictions of the divisors E, L to X_{24}, where L is the pullback $L = \pi^* H$ of the hyperplane class H on the base $\mathbb{P}^3$ and E is the class of the section. They satisfy

$$\int_{X_{24}} E^{k+1} \cdot L^{3-k} = 4^k\,, \quad k = 0, ..., 3\,, \tag{190}$$

and the generators of the Mori cone on X_{24} are given by $\tilde{C}^e = L^3$, $\tilde{C} = E \cdot L^2$. Furthermore we have $\pi^* c_1(B) = 4L$ and the total Chern class of X_{24} can be obtained from the adjuction formula

$$c(X_{24}) = \frac{(1+E)(1+2[E+4L])(1+3[E+4L])(1+L)^4}{1+6E+24L}\,. \tag{191}$$

Using the formulae (159)-(162) we calculate that the asymptotic central charges of the basis of branes outlined in [5] are given by

$$
\begin{aligned}
Z_8^{\mathrm{asy}} =& \frac{1}{6}\cdot t^3\tau + t^2\tau^2 + \frac{8}{3}\cdot t\tau^3 + \frac{8}{3}\cdot\tau^4 + t^2 + \frac{91}{12}\cdot t\tau + \frac{91}{6}\cdot\tau^2 \\
& - i\frac{120\zeta(3)}{\pi^3}\cdot t - i\frac{965\zeta(3)}{2\pi^3}\cdot\tau - \frac{37}{6}\,, \\
Z_{6,1}^{\mathrm{asy}} =& -\frac{1}{6}\cdot t^3 + t^2 - \frac{9}{4}\cdot t + i\frac{5\zeta(3)}{2\pi^3} + \frac{11}{6}\,, \\
Z_{6,2}^{\mathrm{asy}} =& -\frac{1}{2}\cdot t^2\tau - 2\cdot t\tau^2 - \frac{8}{3}\cdot\tau^3 - \frac{1}{2}\cdot t\tau - \tau^2 - 2\cdot t \\
& - \frac{31}{4}\cdot\tau + i\frac{120\zeta(3)}{\pi^3} - 1\,, \\
Z_{4,1}^{\mathrm{asy}} =& \frac{1}{2}\cdot t^2 - \frac{3}{2}\cdot t + \frac{17}{12}\,, \\
Z_{4,2}^{\mathrm{asy}} =& t\tau + 2\cdot\tau^2 + \tau + 2\,, \\
Z_{2,1}^{\mathrm{asy}} =& -t\,, \\
Z_{2,2}^{\mathrm{asy}} =& -\tau\,, \\
Z_0^{\mathrm{asy}} =& 1\,.
\end{aligned}
\tag{192}
$$

Here $Z_d, Z_{d,j}$ denote the charges of d-branes and τ, t are Kähler parameters of the fiber and a curve in the base. Using the mirror map and making an ansatz we can determine the corresponding periods annihilated by (133) to arbitrary order in the complex structure parameters.

Consider now the defining equation of X_{24}^*,

$$
z_b\, u_1^{24} + u_2^{24} + u_3^{24} + u_4^{24} + (u_1u_2u_3u_4)^6 + u_1u_2u_3u_4xy + z_e^{\frac{1}{2}}\, x^3 + y^2 = 0\,. \tag{193}
$$

The two components of the discriminant are given by the vanishing loci of

$$
\Delta_1 = 1 - 2^8\cdot z_b\,,\quad \Delta_2 = 2^{24}3^{12}\cdot z_e^4 z_b - \left(1 - 2^4 3^3\cdot z_e\right)^4\,. \tag{194}
$$

First we introduce a new set of complex structure variables by rescaling the homogeneous coordinates on $\mathbb{P}(1,1,1,1,8,12)$. Equation (193) becomes

$$
u_1^{24} + u_2^{24} + u_3^{24} + u_4^{24} + 4\,\phi\,(u_1u_2u_3u_4)^6 + 2\sqrt{3}\,\psi\, u_1u_2u_3u_4xy + x^3 + y^2 = 0\,, \tag{195}
$$

and the new complex structure variables ϕ, ψ are related to z_e, z_b via

$$
z_b = \frac{1}{256}\frac{1}{\phi^4}\,,\quad z_e = \frac{1}{432}\frac{\phi}{\psi^6}\,. \tag{196}
$$

In these variables the components of the conifold become

$$
\Delta_1' = (\phi-1)(\phi+1)(1+\phi^2)\,,\quad \Delta_2' = (\phi'-1)(\phi'+1)(1+\phi'^2)\,, \tag{197}
$$

where we introduced $\phi' = \phi - \psi^6$.

The general structure of the moduli space is sketched in figure 8. Note that z_e and z_b are the Batyrev variables and the large complex structure divisors LR_1 and LR_2 correspond to

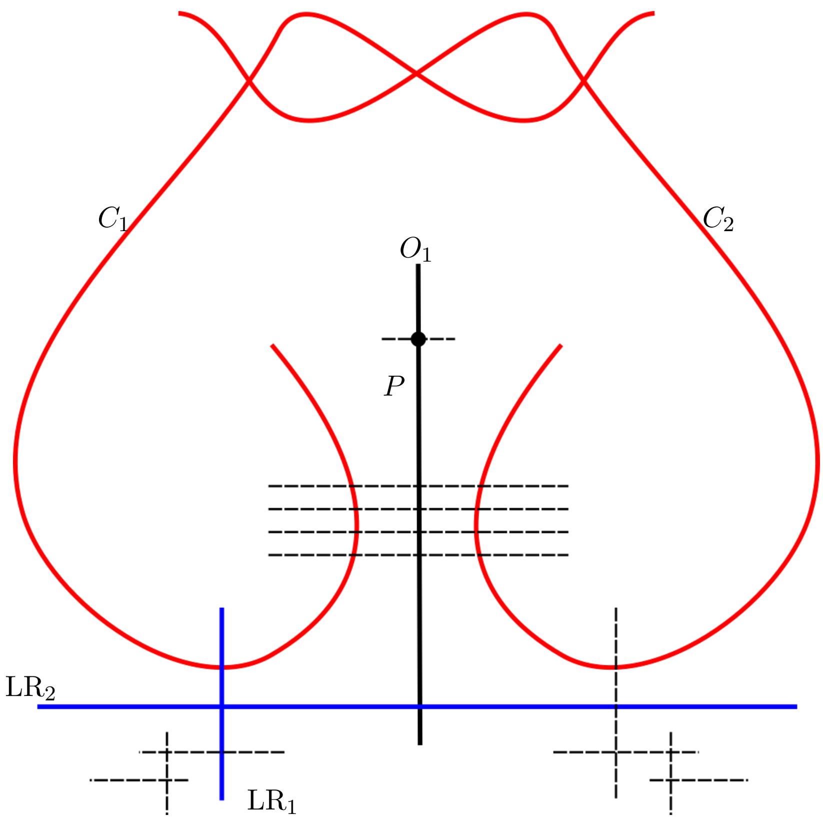


Figure 8: Schematic structure of the resolved complex structure moduli space of X_{24}^*. The large complex structure divisors are shown in blue and the conifold components are red. Exceptional divisors resolving non-normal crossing intersections are indicated with dashed lines.

$z_e = 0$ and $z_b = 0$ respectively. On the other hand, using ϕ and ψ as variables, both $\Delta_1' = 0$ and $\Delta_2' = 0$ have a forth-order tangency with LR_2. Only after resolving $\mathrm{LR}_2 \cap \{\Delta_1' = 0\}$ we get LR_1 as one of the exceptional divisors. This is reflected in the fact that the point $\{z_e = 0\} \cap \{z_b = 0\}$ corresponds to a double-scaling limit in ϕ and ψ. The two divisors that correspond to the components of the conifold are labelled with C_1 and C_2 respectively. Furthermore, we will analyze solutions around the orbifold divisor O_1 that is given by $\psi = 0$.

Finally note that Δ_1 and Δ_2 as well as LR_1 and LR_2 are exchanged under the involution

$$z_e = 2^{-4}3^{-3} - z_e', \quad z_b = \left(\frac{2^4 3^3 z_e'}{1 - 2^4 3^3 z_e'}\right)^4 z_b'. \tag{198}$$

Physically this involution can be seen as the result of T-dualizing along both cycles of the fiber and the corresponding transformation of the A-brane charges is given by $\tilde{S}$, (186).

3.7.1 Conifold C_1

First we study the possible fluxes around $C_1 \cap \mathrm{LR}_1$. To this end we choose local coordinates

$$c_1 = z_b + \frac{1}{256}, \tag{199}$$

and z_e. We transform and solve the Picard-Fuchs equations (133),(135) to obtain a vector of eight solutions with asymptotic behaviour given by

$$\Pi_c = \left(1,\, c_1,\, z_e,\, \log\left(z_e\right),\; \log^2\left(z_e\right),\, \log^3\left(z_e\right),\, \log^4\left(z_e\right),\, c_1^{3/2}\right) + \mathcal{O}(c^2, z^2)\,. \tag{200}$$

We demand that the leading monomial of each period is absent from the other solutions to specify the vector uniquely.

This is related to the integral basis at large complex structure via

$$\Pi_{LR} = T_c \cdot \Pi_c\,. \tag{201}$$

The matrix T_c can be obtained by numerical analytic continuation and is given by

$$\begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \frac{54\pi^6 r_4^2-91}{24\pi^2} & r_4 & \frac{1}{6\pi^4} & 0 \\ f_{2,1} & \left(1+i\sqrt{2}\right) r_3 & f_{2,3} & 0 & 0 & 0 & 0 & \frac{10240 i\sqrt{2}}{3\pi^2} \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \frac{1-6i\pi^3 r_4}{4\pi^2} & -\frac{i}{3\pi^3} & 0 & 0 \\ r_1+ir_2+1 & \frac{1}{4}\left(2+3i\sqrt{2}\right) r_3 & f_{4,3} & 0 & 0 & 0 & 0 & \frac{2048 i\sqrt{2}}{\pi^2} \\ f_{5,1} & f_{5,2} & f_{5,3} & -\frac{3\pi^3 r_4+i}{2\pi} & -\frac{1}{2\pi^2} & 0 & 0 & \frac{1024 i\sqrt{2}}{3\pi^2} \\ \frac{2ir_2}{3} & \frac{ir_3}{\sqrt{2}} & f_{6,3} & 0 & 0 & 0 & 0 & \frac{4096 i\sqrt{2}}{3\pi^2} \\ f_{7,1} & -\frac{i\left(\sqrt{2}\pi r_3-256\right)}{8\pi} & f_{7,3} & \frac{i}{2\pi} & 0 & 0 & 0 & -\frac{1024 i\sqrt{2}}{3\pi^2} \\ 1 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the algebraic constraint

$$\int \Omega\wedge\Omega = 0 \quad\Leftrightarrow\quad \Pi_c^T T_c^T \eta^{-1} T_c \Pi_c = 0\,, \tag{202}$$

and the integral monodromies corresponding to LR_1 and C_1[19] we reduced the numerical uncertainty to five real values r_i, $i = 1, ..., 5$. Due to their size we omitted the explicit expressions for the elements $f_{*,*}$ and refer the reader to the appendix of [5].

To further simplify the analysis we will move away from $z_e = 0$ and introduce

$$c_2 = z_e - \frac{1}{1728}\,. \tag{203}$$

The corresponding vector of solutions is given by

$$\Pi_{c'} = \begin{pmatrix} 1 - 3840 c_1 c_2 + 430080 c_1^2 c_2 \\ c_1 - 1920 c_1^2 c_2 \\ c_1^2 \\ c_1^3 \\ c_2 + 32 c_1 c_2 - 29568 c_2^2 c_1 - \frac{13216}{3} c_1^2 c_2 \\ c_2^2 + \frac{1}{18} c_1 c_2 + 64 c_2^2 c_1 - \frac{40}{9} c_1^2 c_2 \\ c_2^3 + \frac{1}{12} c_2^2 c_1 + \frac{1}{432} c_1^2 c_2 \\ c_1^{3/2} - \frac{2024}{9} c_1^{5/2} \end{pmatrix} + \mathcal{O}(c^4)\,. \tag{204}$$

[19] Since we performed the analytic continuation to very high precision, the integral monodromy matrix corresponding to $c_1 \to e^{2\pi i} c_1$ is essentially determined by the numerical value.

This is related to the integral basis at large complex structure via

$$\Pi_{LR} = T_c \cdot T_{c'} \cdot \Pi_{c'} \,. \tag{205}$$

The numerical value of $T_{c'}$ as well as those of the other continuation matrices in this section are provided in a Mathematica worksheet that can be downloaded from [98].

We now obtain the monodromy action

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & 0 & 20 & 0 & -10 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 13 & 0 & -6 & 0 & -6 \\ 0 & -1 & 0 & 2 & 1 & -1 & 0 & -1 \\ 0 & -4 & 0 & 8 & 0 & -3 & 0 & -4 \\ 0 & 1 & 0 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{206}$$

on Π_{LR} when transported along a lasso wrapping C_1. Using the algebraic constraints

$$\int \Omega \wedge \Omega = 0 \,, \quad \int \Omega \wedge \partial_{c_1} \Omega = 0 \,, \quad \int \Omega \wedge \partial_{c_2} \Omega = 0 \,, \tag{207}$$

we find the analytic expression for $(T_c T_{c'})^T \eta^{-1} T_c T_{c'}$. Unfortunately we are unable to solve the resulting equation for $T_{c'}$.

However, note that

$$M_c = \mathbb{I} - \vec{v} \cdot \vec{v}^T \cdot \eta^{-1} \,, \tag{208}$$

where $\vec{v} = \pm\,(0,\, 10,\, 0,\, 6,\, 1,\, 4,\, -1,\, 0)$. In other words, the monodromy M_c corresponds to a Seidel-Thomas twist, where the charge of the shrinking brane is given by

$$\pi_c = \vec{v} \eta^{-1} T_c T_{c'} \Pi_{c'} = \frac{2048\sqrt{2}}{3\pi^2} \left(c_1^{\frac{3}{2}} - \frac{2024}{9} c_1^{\frac{5}{2}} + \mathcal{O}(c^4) \right) . \tag{209}$$

To obtain a vanishing superpotential at $c_1 = 0$, we can turn on $n \in \mathbb{Z}$ units of flux along the cycle with period π_c. For this to be a supersymmetric minimum we also have to check that $D_i W = 0$. In flat coordinates t_c^i this condition reads

$$(\partial_i + K_i) W = 0 \,, \tag{210}$$

where $K_i = \partial_i K$ and K is the Kähler potential

$$e^{-K} = \int \bar{\Omega} \wedge \Omega = \Pi_{LR}^\dagger \eta^{-1} \Pi_{LR} \,. \tag{211}$$

As flat coordinates we can use the normalized periods

$$\begin{aligned} t_c^1 =& \frac{\Pi_{c',2}}{\Pi_{c',1}} = c_1 + 1920 c_1^2 c_2 + \mathcal{O}(c^4) \,, \\ t_c^2 =& \frac{\Pi_{c',5}}{\Pi_{c',1}} = c_2 + 32 c_1 c_2 - \frac{13216}{3} c_1^2 c_2 - 25728 c_1 c_2^2 + \mathcal{O}(c^4) \,. \end{aligned} \tag{212}$$

In terms of these, the vanishing period reads

$$\pi_c = \frac{2048\sqrt{2}}{3\pi^2}\left[(t_c^1)^{\frac{3}{2}} - \frac{2024}{9}(t_c^1)^{\frac{5}{2}} + \mathcal{O}(t_c^4)\right]. \tag{213}$$

Using the numerical result for $T_c \cdot T_{c'}$ we find that $\partial_i K$ are regular at $c_1 = 0$ and therefore $D_i\pi_c \sim (t_c^1)^{i-1/2}$.

The scalar potential is given by

$$v = e^K \left[(D_i W)(D_{\bar{j}}\bar{W})G^{i\bar{j}} - 3W\bar{W}\right], \tag{214}$$

where $G^{i\bar{j}}$ is the inverse of the metric $G_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$. We restrict to $t_c^2 = 0$ and introduce $\mathrm{Re}(t_c^2) = x$, $\mathrm{Im}(t_c^2) = y$. Then the leading terms of the scalar potential are

$$v = 0.020174\sqrt{x^2+y^2} + 0.31715x^2 + 0.31715y^2 - 2.8019x\sqrt{x^2+y^2} + \mathcal{O}(x^3, y^3). \tag{215}$$

A plot is shown in figure 9. We checked that this is the dominant contribution at least up

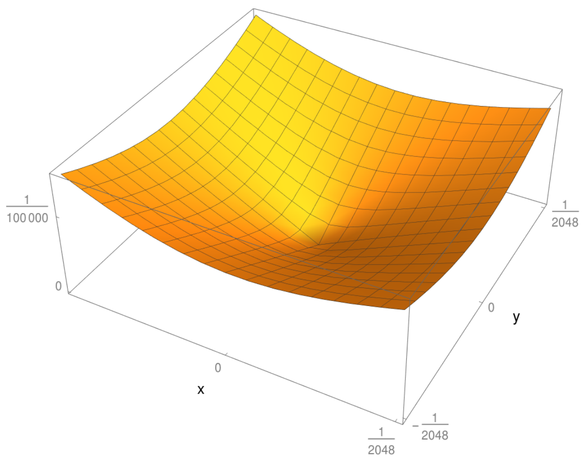


Figure 9: The scalar potential generated by aligned flux, depending on the distance to the conifold C_1 in flat coordinates $t_c^1 = x + Iy$, $t_c^2 = 0$.

to order seven, where we calculated the coefficients to a precision of twenty digits. Deep inside the radius of convergence $|t_c^1| \approx |c_1| < 1/256$ the potential is well approximated by the leading order $v \approx 0.020174 \cdot |c_1|$. Our findings are in agreement with [75] where it was argued that for Calabi-Yau fourfolds the Conifold is generically stabilized by aligned flux.

3.7.2 Orbifold O_1

To expand around $O_1 \cap \mathrm{LR}_2$ we use the variables z_b and

$$o_1 = \frac{1}{z_b^6}. \tag{216}$$

We find a vector of solutions to the transformed Picard-Fuchs system with leading terms

$$\begin{aligned}\Pi_o = \big(&o_1^5,\, o_1^5 \log(z_b),\, o_1^5 \log^2(z_b),\, o_1^5 \log^3(z_b),\\ &o_1,\, o_1 \log(z_b),\, o_1 \log^2(z_b),\, o_1 \log^3(z_b)\big) + \mathcal{O}(o_1^7, z).\end{aligned} \tag{217}$$

It is related to the integral basis at large complex structure via

$$\Pi_{\mathrm{LR}} = T_o \cdot \Pi_o \,. \tag{218}$$

However, in contrast to the analytic continuation matrix to the conifold, T_o can be determined exactly with the help of the Barnes integral method. The latter has been discussed for one-parameter models in [16] and can be adapted to this two-parameter model. We give the analytic expression in the Mathematica worksheet that can be found online [98]. The monodromy acting on Π_{LR} when transported along a lasso wrapping O_1 is of order six and given by

$$M_o = \begin{pmatrix} 1 & 1 & 0 & -4 & 0 & 6 & 0 & 2 \\ -1 & 0 & -4 & 0 & -10 & 0 & -20 & 0 \\ 0 & 0 & 1 & 1 & 0 & -4 & 0 & 2 \\ 0 & 0 & -1 & 0 & -4 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} . \tag{219}$$

To analyze the possible fluxes we will again move away from the large complex structure divisor and introduce the variable

$$o_2 = z_b - \frac{1}{512} \,. \tag{220}$$

Solutions in the new variables are

$$\Pi_{o'} = \left(o_1,\, o_1^7,\, o_1^{12},\, o_1^{19},\, o_1^5,\, o_1^{11},\, o_1^{17},\, o_1^{23}\right) + \mathcal{O}(o_1^2, o_2) \,, \tag{221}$$

We demand that the leading monomial of each period is absent from the other solutions to specify the vector uniquely. It is related to the previous basis via

$$\Pi_o = T_{o'} \cdot \Pi_{o'} \,, \tag{222}$$

where the numerical expression for $T_{o'}$ has been calculated with a precision of around fifty digits.

From the solution vector it follows that every choice of flux leads to a vanishing superpotential at $o_1 = 0$. If one chooses the flux superpotential

$$\begin{aligned} W =& T_{o'}^{-1} T_o^{-1} \Pi_{\mathrm{LR},0} = -(0.237201\ - 0.907908i) o_1 \\ &+ (97.5605\ - 9.49343i) o_1 o_2 - (24181.7 + 1211.32i) o_1 o_2^2 + \mathcal{O}(o^4) \,, \end{aligned} \tag{223}$$

this leads to the scalar potential

$$v = 0.0111391615585497 87439 + \mathcal{O}(x^2, y^2) \,. \tag{224}$$

in terms of $o_1 = x + Iy$ at $o_2 = 0$. A plot of the potential, expanded to order eleven, is shown in figure 10. Note that the radius of convergence is $o_1 < 216 \cdot (2 - 2^{3/4}) \approx 69$.

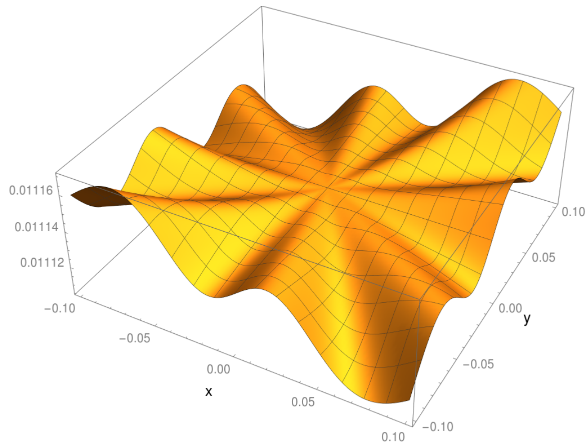


Figure 10: The scalar potential generated by a generic choice of flux, depending on the distance to the orbifold O_1 in coordinates $o_1 = x + Iy$, $o_2 = 0$.

We did a Monte Carlo scan over non-vanishing flux vectors and found that the scalar potential was always positive at $x = y = 0$. Moreover, the behaviour close to the origin was qualitatively the same in that the gradient vanished at $x = y = 0$ but the Hessian was undefined.

We also performed an analytic continuation to the special locus P where the Calabi-Yau becomes a Gepner model. However, the behaviour of the scalar potential was qualitatively the same as for a generic point on $\mathcal{O}_1$.

4 Local mirror symmetry and Siegel modular forms

In the previous chapter we used charges of BPS branes to determine properly quantized horizontal fluxes on Calabi-Yau fourfolds. The charges depend only on the complex structure respectively Kähler structure of the Calabi-Yau and the class of the brane. A- and B-branes are but one example of the topological subsector of physical string theory. Topological strings can be obtained by twisting the worldsheet theory in one of two ways. The corresponding theories are called the topological A- and B-model. They encode different geometric quantities of the target space and are related via mirror symmetry.

Topological string theory has particularly nice properties on non-compact Calabi-Yau varieties where it can be solved exactly. Recently the solution has been related to spectral theory of trace class operators and novel structures in quantum mechanics [99, 100]. The mirror of a non-compact toric Calabi-Yau variety X is essentially encoded in a Riemann surface Σ and a meromorphic differential λ on Σ. This data is called the mirror curve. For Riemann surfaces of genus $g = 1$ it is known that the partition function of the topological A-model can be expressed in terms of modular forms. In [17] we showed that in cases where the mirror curve has genus two the correct building blocks are Siegel modular forms. Moreover, we used the structure of topological string theory to derive the existence of a novel almost meromorphic, vector valued Siegel modular form. To demonstrate this we calculated higher genus topological string amplitudes on the non-compact toric Calabi-Yau

$\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ and $\widehat{\mathbb{C}^3/\mathbb{Z}_6}$.

In this chapter we start with a non-technical introduction to topological string theory. This is followed by a review of the geometrical structure that is encoded in the B-model and the enumerative invariants that are counted by A-model correlation functions. We also provide an introduction to local mirror symmetry and the theory of Siegel modular forms. We then build on the work in [17] and start by constructing a compact embedding of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$. After discussing the geometry of the compact Calabi-Yau we identify the non-compact limit and use this to explain a relation between the generating function of topological genus zero amplitudes and the modular parameter of the curve. We then use this structure and a theorem by Lockhart to relate the propagator that is necessary to integrate the holomorphic anomaly equations to a certain product of theta functions. This provides a generalisation of the results in [17] to hyperelliptic mirror curves of genus $g > 2$.

4.1 Topological string theory

Perturbative topological string theory can be obtained by twisting the physical string worldsheet action such that the path integral localizes. Here we will give a relatively non-technical introduction and refer the reader to [12, 79, 101, 102] for more information.

The worldsheet theory of Type II strings compactified on a d-dimensional space X factorizes. One factor is a superconformal field theory (SCFT) of $10-d$ free bosons and their superpartners. The other is a quantum field theory of maps from the worldsheet into X. If X is a Kähler manifold the latter will be an $\mathcal{N}=(2,2)$ superconformal field theory. One copy of the $\mathcal{N}=2$ algebra is described by the operator product expansions (OPE)

$$\begin{aligned}
G^{\pm}(z)G^{\mp}(w) =& \frac{\frac{2}{3}c}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial_w J(w)}{(z-w)} + \dots\,, \\
J(z)G^{\pm}(w) =& \pm \frac{G^{pm}(w)}{(z-w)} + \dots\,, \\
J(z)J(w) =& \frac{\frac{1}{3}c}{(z-w)^2} + \dots\,, \\
T(z)J(w) =& \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \dots\,, \\
T(z)G^{\pm}(w) =& \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial_w G^{\pm}(w)}{(z-w)} + \dots\,, \\
T(z)T(w) =& \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots\,,
\end{aligned} \tag{225}$$

Here T is the left-moving energy momentum tensor, $G^{\pm}$ are the two superpartners of conformal weight $3/2$, J is a $U(1)$ current and c is the so-called central charge. Coordinates on the worldsheet are denoted by z, w and terms that are regular in the limit $z \to w$ have been omitted. Another copy of this algebra is generated by the right moving fields $\bar{T}, \bar{G}^{\pm}, \bar{J}$.

The left- and right-moving R-symmetries can be combined into a vector or axial symmetry

$$U(1)_V = U(1)_L + U(1)_R\,, \quad U(1)_A = U(1)_L - U(1)_R\,. \tag{226}$$

Note that a worldsheet parity transformation exchanges the left- and right moving sector such that $U(1)_V$ is even and $U(1)_A$ is odd under this operation. While $U(1)_V$ is anomaly free, the axial R-symmetry is generically broken at the quantum level except when $c_1(TX) = 0$. We will therefore assume that X is a Calabi-Yau manifold. Moreover, we will in the following restrict to the case where X is a Calabi-Yau threefold.

Physicists have learned rules to derive physical properties and observable quantities from the infinite dimensional path integral of general quantum field theories but it is not mathematically well defined yet. Supersymmetric theories on the other hand can exhibit a property called localization. Here the supertransformation of bosons can be used to substract fermionic insertions from the action such that the path integral annihilates the integrand. The only non-vanishing contributions are then from field configurations that make the supertransformation of the bosons vanish. In this way the infinite dimensional path integral reduces to a finite dimensional sum over solutions of differential equations.

The two supercharges and correspondingly the parameters of the supertransformations of the $\mathcal{N} = (2,2)$ symmetry are sections of non-trivial bundles on the worldsheet. For the path integral to localize at least one of the bundles has to admit non-zero global sections. This is only the case for worldsheet genus $g = 1$. However, it is possible to gauge an R-symmetry and couple the conserved current to the spin connection. The effect of this so-called twisting is that the spins of the fields are shifted by their R-charge. A linear combination of the supercharges can thus be turned into a scalar and used to localize the theory.

Another effect of twisting is that the ring of observables becomes cohomological with boundary operator given by the now scalar supercharge Q. An operator $\mathcal{O}$ is called Q-closed if $\{Q, \mathcal{O}\} = 0$ and Q-exact if $\mathcal{O} = \{Q, \mathcal{O}'\}$ for some operator $\mathcal{O}'$. Correlation functions involving Q-exact operators vanish and we can restrict to the ring of closed operators modulo exact operators. A finite subring of operators corresponds to a geometrical cohomology theory on the target space X. The ring is graded by the charge $(q, \bar{q})$ under the left- and right moving R-symmetries with currents $J, \bar{J}$.

Twisting by the $U(1)_V$ symmetry is called the A-twist. The path integral localizes on holomorphic maps from the worldsheet into the target space X and the partition function can be written as

$$\log\, Z_A = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^{(g)}\,. \tag{227}$$

The *free energies* $\mathcal{F}^{(g)}$ contain all contributions from a given worldsheet genus g. The expansion parameter $\lambda = g_S F$ is the product of the string coupling g_S and the vacuum expectation value F of the self-dual graviphoton field strength [13, 103]. We denote a basis of the Kähler cone on X by $J_1, ..., J_m$ and expand the Kähler form ω as $\omega = \sum_{j=1}^{m} t^i J_i$ with Kähler parameters t^i, $i = 1, ..., m$. Then anti-holomorphic deformations $\propto \bar{t}$ of the worldsheet theory are Q-exact and should decouple from the theory. It was shown in [14, 104] that worldsheet gravity leads to an anomaly in the decoupling. In the *holomorphic limit* $F^{(g)} = \lim_{\bar{t} \to i\infty} \mathcal{F}^{(g)}$ the free energies encode so called Gromov-Witten invariants $N_{g,\beta}$, $\beta \in$

$H_2(X,\mathbb{Z})$ via

$$\mathcal{F}^{(g)} = \sum_{\beta \in H_2(X,\mathbb{Z})} N_{g,\beta} e^{2\pi i \int_\beta \omega} . \tag{228}$$

Here we denote the Kähler class on X by ω. For $g = 0, 1$ this equation only holds up to a cubic respectively linear polynomial in the flat coordinates. The Gromov-Witten invariants $N_{g,\beta}$ “count” stable holomorphic maps from the genus g worldsheet into X such that the image is in the homology class of β. Due to mathematical subtleties the invariants are in general rational numbers and not integral. However, they can be related to various integral invariants as we will outline below. The cohomology of a finite subring of operators in the A-twisted theory corresponds to the quantum deformed vertical cohomology on X.

Twisting by the $U(1)_A$ symmetry is called the B-twist. The theory localizes on constant maps into the target space and the operator cohomology corresponds to the horizontal cohomology of X. Correlation functions correspond to classical integrals and they only depend on the complex structure of X. The A-twisted theory on a Calabi-Yau M is related to the B-twisted theory on the mirror Calabi-Yau W via the mirror map.

4.1.1 Special geometry in the B-model

The periods on a compact Calabi-Yau threefold satisfy the so-called special geometry relations [79]. Consider a symplectic basis $A^I, B_J \in H_3(W,\mathbb{Z})$ and a dual basis $\alpha^I, \beta_I \in H^3(W,\mathbb{Z})$ such that

$$A^I \cap B_J = \delta^I_J , \quad A^I \cap A^J = B_I \cap B_J = 0 , \tag{229}$$

for $I, J = 0, ..., h^{2,1}(W)$. The holomorphic 3-form Ω can now be expanded as

$$\Omega = X^I \alpha_I + \mathcal{F}_J \beta^J , \tag{230}$$

and $X^I, \mathcal{F}_J$ form a basis of periods. In particular $\mathcal{F}_J$ are derivatives $\mathcal{F}_J = \partial_{X^J}\mathcal{F}$ of a function $\mathcal{F}(X^I)$ that is homogeneous of weight 2. For this reason the normalized function $F^{(0)} = (X^0)^{-2}\mathcal{F}$ is also called the prepotential. Introducing the special coordinates

$$t^a = \frac{X^a}{X^0} , \quad a = 1, ..., h^{2,1}(W) , \tag{231}$$

we obtain the normalized periods

$$\vec{\Pi} = \Big(1,\, t^a,\, \partial_{t^a} F^{(0)},\, (2F^{(0)} - t^b \partial_{t^b} F^{(0)})\Big) . \tag{232}$$

At the point of large complex structure and using Batyrev variables the period vector is filtered by the maximum power of logarithms that appear in a solution. We denote the highest power of a logarithm in a series of the form

$$f(\vec{z}) = \sum_{\vec{i},\vec{j}=\vec{0}}^{\infty} c_{\vec{i},\vec{j}} \prod_{m=1}^{h^{2,1}(W)} (z^m)^{i_m} (\log z^m)^{j_m} , \tag{233}$$

by

$$\mathrm{mlo}(f) = \max\{n \mid \exists \vec{i}, \vec{j} \in \mathbb{N}^{h^{2,1}(W)} \text{ s.t. } c_{\vec{i},\vec{j}} \neq 0 \text{ and } |\vec{j}| = n\}\,. \tag{234}$$

Then for Batyrev variables z^i

$$\mathrm{mlo}(\vec{\Pi}) = (0, 1, 2, 3)\,. \tag{235}$$

Moreover, after inserting the mirror map $F^{(0)}(t)$ is the genus zero free energy of the A-model on the mirror Calabi-Yau M. It encodes the Gromov-Witten invariants of M via the expansion given in (228). On the other hand the so-called Yukawa couplings $C_{abc} = \partial_{t^a}\partial_{t^b}\partial_{t^c}F^{(0)}(t)$ can be transformed back into Batyrev variables via

$$C_{ijk}(z) = \left[(X^0)^2 \frac{\partial t^a}{\partial z^i}\frac{\partial t^b}{\partial z^j}\frac{\partial t^c}{\partial z^k} C_{abc}\right](t(z)) = \int_W \Omega \wedge \partial_i\partial_j\partial_k\Omega\,, \tag{236}$$

and are rational functions in z. They can be singular at the large complex structure divisors and the *discriminant locus* but are regular everywhere else in the moduli space.

4.2 Local mirror symmetry

A remarkably rich source of physical mathematics are topological strings on non-compact Calabi-Yau varieties. In some cases these theories arise as a limit of topological string theory on compact Calabi-Yau. Physically the decompactification then corresponds to a decoupling of gravity. In this way one can engineer for example the famous 4d $N = 2$ gauge theory that was studied by Seiberg and Witten [28, 105]. In particular, the prepotential of the Seiberg-Witten theory corresponds to the prepotential of the topological string theory on a non-compact Calabi-Yau.

Without refering to a particular embedding the general setup for threefolds can be described as follows [31, 106]. Consider a two-dimensional lattice polytope Δ with g inner points and denote the points by $\vec{p}_i$, $i = 1, ..., n$. Then every triangulation of Δ can be lifted to a three-dimensional fan Σ_Δ with generators $(\vec{p}, 1)$, $i = 1, ..., n$. The corresponding toric variety is non-compact and satisfies the Calabi-Yau condition. Note that the generators satisfy $m = n-3$ independent linear relations $l^{(i)}$, $i = 1, ..., m$ that we choose to correspond to a basis of the Mori cone on $\mathbb{P}_\Sigma$. The local mirror is given by

$$uv = \sum_{i=1}^{m+3} x_i\,, \tag{237}$$

where $u, v \in \mathbb{C}$ and $x_i \in \mathbb{C}^*$, $i = 1, ..., m+3$ are homogeneous coordinates with respect to the action

$$x_i \mapsto \lambda x_i\,, \quad i = 1, ..., m+3\,, \quad \lambda \in \mathbb{C}^*\,. \tag{238}$$

Furthermore, they are related to Batyrev type complex structure coordinates z^i, $i = 1, ..., m$ via

$$z_i = (-1)^{l_0^{(i)}} \prod_{j=1}^{m+3} x_j^{l_j^{(i)}}\,. \tag{239}$$

This form of the mirror can be understood from the Hori-Vafa proof of mirror symmetry as a duality of two-dimensional sigma-models [106].

One can use (238) to set one of the x_i in (237) to one and then replace m of the remaining coordinates using the Batyrev variables (239). This leads to an equation of the form

$$uv = H(x, y, \vec{z}) \,, \tag{240}$$

with $u, v \in \mathbb{C}$, $x, y \in \mathbb{C}^*$ where we denote the remaining two x_i by x, y. The equation $H(x, y, \vec{z}) = 0$ defines a non-compact Riemann surface of genus g. Moreover, it is equipped with a meromorphic differential [28]

$$\lambda = \log(y) \frac{dx}{x} \,, \tag{241}$$

and the periods over λ are annihilated by a set of Picard-Fuchs operators. A GKZ system can be obtained from the relations $l^{(i)}$, $i = 1, ..., m$ in the same way as for compact mirror pairs and the Picard-Fuchs system can often be constructed by factoring the operators as described in section 3.1. The curve $H = 0$ together with a meromorphic differential is called the mirror curve. We illustrate this with the example $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$.

Example:

The polytope Δ corresponding to $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ is shown in figure 11. It is easy to see that the triangulation is unique and there are $g = 2$ inner points. The mirror curve is therefore a Riemann surface of genus two.

The toric data of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ is as follows:

$$\left(\begin{array}{rrr|rr} 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ -1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{array}\right) \tag{242}$$

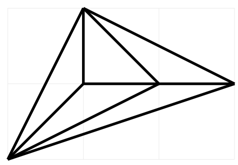


Figure 11: Triangulated polytope Δ that corresponds to $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$.

From this we obtain the local mirror geometry

$$uv = x_1 + x_2 + x_3 + x_4 + x_5 \,, \tag{243}$$

with

$$z_1 = \frac{x_2 x_5}{x_1^2} \,, \quad z_2 = \frac{x_1 x_3 x_4}{x_2^3} \,. \tag{244}$$

We set $x_1 \mapsto 1$ and replace x_5, x_3 to obtain

$$uv = 1 + x + z_2 \frac{x^3}{y} + y + z_1 \frac{1}{x} \,, \tag{245}$$

where we introduce $x = x_2$, $y = x_4$. The mirror curve is therefore given by

$$xy + x^2 y + z_2 x^4 + xy^2 + z_1 y = 0 \,. \tag{246}$$

Via $y \mapsto \frac{y - (x + x^2 + z_1)/2}{x}$ this is birationally equivalent to

$$y^2 = -z_2 x^5 + \frac{1}{4} \left(x - \frac{1}{2}(-1 - \sqrt{1 - 4z_1}) \right)^2 \left(x - \frac{1}{2}(-1 + \sqrt{1 - 4z_1}) \right)^2 \,. \tag{247}$$

The solutions of the Picard-Fuchs system for the mirror curve of a toric Calabi-Yau are of the form

$$\Pi = \left(\begin{array}{c} 1 \\ t_i \\ t_j^m \\ \partial_{t_i} \mathcal{F}^{(0,0)} \end{array} \right) , \quad i = 1, ..., g \,, \quad j = 0, ..., n - g \,, \tag{248}$$

with leading log powers (234)

$$\text{mlo}(\vec{\Pi}) = (0, 1, 1, 2) \,. \tag{249}$$

Here n denotes the total number of points in the affine diagram of the toric Calabi-Yau and g is the number of points that are in the relative interior. In particular g is the genus of the mirror curve. The trivial period and the $n - g$ solutions t_j^m correspond to residues of the meromorphic differential λ. The relation to the corresponding Batyrev variables z_j^m can be given in a closed form.

4.3 Refined invariants

The partition function of the topological A-model on a Calabi-Yau manifold X has been related to a BPS partition function that arises from M-theory on a spacetime [103, 107]

$$X \times S^1 \times \mathbb{R}^4 \,. \tag{250}$$

Assuming that the circle is much larger than the Calabi-Yau this leads to a five dimensional $N = 2$ effective supergravity. Moreover, the metric on $S^1 \times \mathbb{R}^4$ is choosen to be Euclidean and the time dimension is aligned with the circle. This effectively amounts to a calculation at finite temperature [20].

The supercharges of the five dimensional theory decompose under the rotation group $SO(4) = SU(2)_L \times SU(2)_R$ as

$$2\left(\frac{1}{2}, 0\right) + 2\left(0, \frac{1}{2}\right) \,. \tag{251}$$

Due to the extended supersymmetry the theory contains short multiplets that are annihilated either by the $2(1/2, 0)$ or the $2(0, 1/2)$ supercharges. These BPS multiplets arise from $M2$-branes that wrap cycles in X. In particular the mass of the states corresponds to the size of the cycles. One can now define the partition function

$$Z_{GV}(\lambda, \vec{t}) = \text{Tr}(-1)^{2j_L + 2j_R} \exp\left(-\lambda j_L + H(\vec{t})\right) \,, \tag{252}$$

where $\vec{t}$ are the Kähler parameters on X and $H(\vec{t})$ is the Hamiltonian. Due to the factor $(-1)^{2j_R}$ the trace projects out states that are not annihilated by the $2(0, 1/2)$ supercharge.

[20]The technique of evaluating quantum field theory amplitudes at finite temperature by Wick rotating to Euclidean signature and then considering a periodic time variable is called the Matsubara formalism [108].

This includes the long and half of the short multiplets. It was shown in [103] that this partition function is given by

$$\log\, Z_{GV}(\lambda,\vec{t}) = \sum_{g=0}^{\infty}\sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{m=1}^{\infty} n_g^\beta \frac{1}{m}\left(2\sin\frac{m\lambda}{2}\right)^{2g-2} e^{m\int_\beta \omega(\vec{t})}\,, \tag{253}$$

where $\omega(\vec{t})$ is the Kähler class on X. The integral coefficients n_g^β are called Gopakumar-Vafa invariants. They are related to the number of BPS states from $M2$ branes wrapping cycles in the class β. It has been argued in [103] that Z_{GV} can be identified with the topological A-model partition function

$$\log\, Z_A(\lambda,\vec{t}) = \sum_{g=0}^{\infty}\lambda^{2g-2}\mathcal{F}^{(g)}(\vec{t})\,, \tag{254}$$

on X. For $\mathcal{F}^{(0)},\mathcal{F}^{(1)}$ this identification only holds up to a cubic respectively linear polynomial in the Kähler parameters.

The BPS states from $M2$ branes in the class β can be arranged as

$$\left[\left(\frac{1}{2},0\right)\oplus 2\,(0,0)\right]\otimes\sum_{j_L,j_R} N^\beta_{j_L,j_R}(j_L,j_R)\,, \tag{255}$$

where $N^\beta_{j_L,j_R}$ are non-negative integral invariants associated to X. The two sets of invariants are related via

$$\sum_{j_L,j_R} N^\beta_{j_L,j_R}(-1)^{2j_R}(2j_R+1)[j_L] = \sum_g n_g^\beta I_L^g\,, \tag{256}$$

where $I_L^g = \left[\left(\frac{1}{2}\right)+2(0)\right]^{\otimes g}$. It is easy to see that the so-called refined BPS invariants $n^\beta_{j_L,j_R}$ contain more information than the Gopakumar-Vafa invariants. They seem to correspond to a refined partition function

$$\tilde{Z}_{BPS}(\epsilon_L,\epsilon_R,\vec{t}) = \mathrm{Tr}(-1)^{2j_L+2j_R}\exp\left(-\epsilon_L j_L - \epsilon_R j_R + H(\vec{t})\right)\,, \tag{257}$$

which is, however, not supersymmetric in the sense that it does not project out the long multiplets. In the limit where gravity decouples the rigid $N=2$ supersymmetry exhibits an additional R-symmetry $SU(2)_{\mathcal{R}}$. One can then define the supersymmetric partition function [109]

$$Z_{BPS}(\epsilon_L,\epsilon_R,\vec{t}) = \mathrm{Tr}(-1)^{2j_L+2j_R}\exp\left(-\epsilon_L j_L - \epsilon_R j_R - \epsilon_R j_{\mathcal{R}} + H(\vec{t})\right)\,. \tag{258}$$

The authors of [109] generalized the calculation in [103] to Z_{BPS} and found the expansion

$$\log\, Z_{BPS} = \sum_{\substack{j_L,j_R=0\\ k=1}}^{\infty}\sum_{\beta\in H_2(X,\mathbb{Z})}(-1)^{2(j_L+j_R)}\frac{n^\beta_{j_L,j_R}}{k}\frac{\sum\limits_{m_L=-j_L}^{j_L} q_L^{km_L}}{2\sinh\left(\frac{k\epsilon_1}{2}\right)}\frac{\sum\limits_{m_R=-j_R}^{j_R} q_R^{km_R}}{2\sinh\left(\frac{k\epsilon_2}{2}\right)}e^{k\int_\beta\omega}\,, \tag{259}$$

with $\epsilon_1 = \epsilon_+ + \epsilon_-$, $\epsilon_2 = \epsilon_+ - \epsilon_-$ and $q_{L/R} = e^{\epsilon_{L/R}}$.

The refined BPS partition function is conjecturally related to a refinement of the topological A-model partition function and the corresponding free energies

$$\log Z_{BPS}(\epsilon_1, \epsilon_2, \vec{t}) = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} \mathcal{F}^{(n,g)}(\vec{t}) \,. \tag{260}$$

The unrefined partition function Z_A is recovered in the limit $\epsilon_1 = -\epsilon_2$. One can introduce a third set of invariants $n^{\beta}_{g_L, g_R} \in \mathbb{Z}$ that is defined via

$$\sum_{j_L, j_R = 0}^{\infty} N^{\beta}_{j_L, j_R}(j_L, j_R) = \sum_{g_L, g_R}^{\infty} n^{\beta}_{g_L, g_R} I_L^{g_L} \otimes I_R^{g_R} \,. \tag{261}$$

The corresponding expansion reads [110]

$$\log Z_{BPS} = \sum_{\substack{g_L, g_R = 0 \\ k = 1}}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} \frac{n^{\beta}_{g_L, g_R}}{k} \frac{\sin\left(\frac{k(\epsilon_1 - \epsilon_2)}{4}\right)^{2g_L} \sin\left(\frac{k(\epsilon_1 + \epsilon_2)}{4}\right)^{2g_R}}{4 \sin\left(\frac{k\epsilon_1}{2}\right) \sin\left(\frac{k\epsilon_2}{2}\right)} e^{k \int_\beta \omega} \,. \tag{262}$$

It is clear that invariants $n^{\beta}_{g_L, g_R}$ only multiply terms of order greater than $2(g_L + g_R) - 2$ in ϵ_1, ϵ_2. The invariants $n^{\beta}_{g_L, g_R}$ are therefore completely determined by knowledge of the refined free energies $\mathcal{F}^{(n,g)}$ with $n + g \leq g_L + g_R$.

4.4 The holomorphic anomaly equations

The invariants introduced in the previous section have been related to refined Pandharipande-Thomas invariants [109]. The latter are mathematically well defined but, like most enumerative invariants, they are notoriously hard to calculate. Insights from topological string theory have led to several methods that can be used to determine the higher genus free energies and their refinements. The direct integration technique uses the holomorphic anomaly equations that were derived in [14, 104]. It has been generalized to the refined case in [110].

We will summarize the anomaly equations for refined free energies on non-compact Calabi-Yau. For simplicity we will work directly in the holomorphic limit. The genus one free energy $\mathcal{F}^{(1)} = \mathcal{F}^{(0,1)}$ satisfies the ansatz [15]

$$\mathcal{F}^{(0,1)} = \frac{1}{2} \log \det \left(\frac{\partial t_i}{\partial z_j} \right) - \frac{1}{12} \log \Delta + \log \prod_{i=1}^{h^{2,1}} z_i^{b_i} \,, \tag{263}$$

while

$$\mathcal{F}^{(1,0)} = \frac{1}{24} \log \Delta + \log \prod_{i=1}^{h^{2,1}} z_i^{\tilde{b}_i} \,. \tag{264}$$

The exponents $b_i, \tilde{b}_i$ have to be determined from the behaviour of the free energies at the boundary of the moduli space. The anti-holomorphic dependence of the higher genus free energies (before going to the holomorphic limit) is contained in a propagator S^{ij} that satisfies the equations

$$\begin{aligned} D_i S^{kl} =& - C_{inm} S^{km} S^{ln} + f_i^{kl} \,, \\ \Gamma^k_{ij} =& - C_{ijl} S^{kl} + \tilde{f}^k_{ij} \,, \\ \partial_i F^{(0,1)} =& \frac{1}{2} C_{ijk} S^{jk} + A_i \,. \end{aligned} \tag{265}$$

The second equation relates the propagator to the Christoffel symbols Γ^k_{ij} of the Weil-Petersson metric in the holomorphic limit

$$G_i^j = \frac{\partial t^i}{\partial z_j} \,, \tag{266}$$

and the functions $f_i^{kl}, \tilde{f}^k_{ij}, A_i$ are of the form

$$f_i^{kl} = \frac{h(z)}{\Delta^{p_{ikl}} \prod_{j=1}^{h^{2,1}} z_j^{m_j^{ikl}}} \,, \quad \tilde{f}^i_{kl} = \frac{\tilde{h}(z)}{\Delta^{\tilde{p}_{ikl}} \prod_{j=1}^{h^{2,1}} z_j^{\tilde{m}_j^{ikl}}} \,, \tag{267}$$

and

$$A_i = \partial_i \left(\tilde{a} \log \Delta + \sum_{j=1}^{h^{2,1}} \tilde{b}_j \log z_j \right) \,. \tag{268}$$

The numerators $h(z), \tilde{h}(z)$ are polynomials with degree bounded by the degree of the denominator and $p_{ikl}, \tilde{p}_{ikl}, m_j^{ikl}, \tilde{m}_j^{ikl}, b_j, \tilde{b}_j, \tilde{a}$ are some constants.

To solve this system for multi-moduli families of non-compact Calabi-Yau is a daunting task. In particular because the solution is not unique [111]. It is known that for mirror curves of genus one there is a canonical choice for the propagator given by the second Eisenstein series [112]. The ambiguities in equation (265) are then fixed and it can be easily checked that they are of the form (267),(268). We will argue in the next section that for mirror curves of genus two there is also a canonical choice for the propagator [17]. It is given by a novel almost-meromorphic Siegel modular object.

Given a choice for the propagator and the corresponding ambiguities the free energies for $n + g > 1$ satisfy the recursion relation

$$\frac{\partial \mathcal{F}^{(n,g)}}{\partial S^{ij}} = \frac{1}{2} \left(D_i \partial_j \mathcal{F}^{(n,g-1)} + {\sum_{m,h}}' \partial_i \mathcal{F}^{(m,h)} \partial_j \mathcal{F}^{(n-m,g-h)} \right) \,, \tag{269}$$

where the prime indicates that $(m, h) = (0, 0)$ and $(m, h) = (n, g)$ are omitted in the sum. This determines $\mathcal{F}^{(n,g)}$ up to a holomorphic ambiguity

$$\mathcal{A}^{(n,g)} = \frac{a_{n,g}(z)}{\Delta^{2(n+g)-2}} \,, \tag{270}$$

where $a_{n,g}(z)$ is a polynomial with degree bounded by the degree of the denominator. The holomorphic ambiguity can be fixed from the so-called gap condition

$$\log Z_{BPS} = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} \mathcal{F}^{(n,g)} \tag{271}$$

$$= \left[-\frac{1}{12} + \frac{1}{24} (\epsilon_1 + \epsilon_2)^2 (\epsilon_1 \epsilon_2)^{-1} \right] \log(t) \tag{272}$$

$$+ \frac{1}{\epsilon_1 \epsilon_2} \sum_{g=0}^{\infty} \frac{(2g-3)!}{t^{2g-2}} \sum_{m=0}^{g} \hat{B}_{2g} \hat{B}_{2g-2m} \epsilon_1^{2g-2m} \epsilon_2^{2m} + \mathcal{O}(t^0) , \tag{273}$$

where t is a flat coordinate that vanishes at a component of the discriminant. The coefficients $\hat{B}_n$ are related to the Bernoulli numbers B_n via

$$\hat{B}_n = \left(\frac{1}{2^m - 1} - 1 \right) \frac{B_n}{n!} . \tag{274}$$

The latter are defined via

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} B_m = \frac{t}{e^t - 1} . \tag{275}$$

Another boundary condition that has to be satisfied by the free energies is given by the constant map contributions [113]. First we introduce another expansion

$$\sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} \mathcal{F}^{(n,g)} = \sum_{g,m=0}^{\infty} \epsilon_1^{2g-m-1} \epsilon_2^{m} F_{m,g} . \tag{276}$$

Then the constant part of $F_{m,g}$ at the large radius locus is given by

$$F_{m,g} = \frac{\chi}{2} \frac{\hat{B}_{2g-m} \hat{B}_m B_{2g-2}}{2g-2} + \mathcal{O}(t) , \tag{277}$$

where t are flat coordinates around the large radius point.

4.5 Genus two curves and Siegel modular forms

The topology of a genus two Riemann surface C is indicated in figure 12. One can choose a canonical homology basis $\alpha_i, \beta_j \in H_1(C, \mathbb{Z})$, $i, j = 1, 2$ in the sense that

$$\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0 , \quad \alpha_i \cap \beta_j = \delta_{ij} . \tag{278}$$

Two canonical bases are related via an $Sp(4, \mathbb{Z})$ transformation [114]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{279}$$

where the matrices $A, B, C, D \in \mathrm{Mat}(2 \times 2, \mathbb{Z})$ satisfy

$$AB^t = BA^t , \quad CD^t = DC^t , \quad AD^t - BC^t = \mathbb{I} . \tag{280}$$

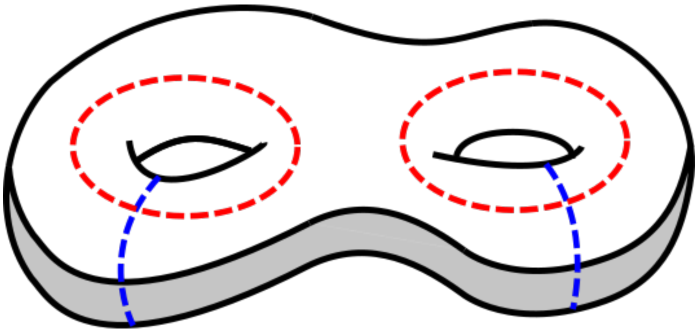


Figure 12: Genus two curve with a choice of A-cycles indicated in red and the corresponding B-cycles indicated in blue.

The hodge numbers are $h^{0,0} = h^{1,1} = 1$, $h^{1,0} = h^{0,1} = 2$ and one can find a *canonical basis* of holomorphic forms $\omega_i \in H^{1,0}(C, \mathbb{C})$ such that

$$\int_{\alpha_j} \omega_i = \delta_{ij} \,. \tag{281}$$

Then the period matrix

$$\tau_{ij} = \int_{\beta_j} \omega_i \,, \tag{282}$$

is symmetric and the imaginary part $\operatorname{Im} \tau$ is positive definite. For later use note that given a general basis $\omega_i' \in H^{1,0}(C, \mathbb{C})$ we obtain

$$\mu_{ij} = \int_{\alpha_j} \omega_i' \,, \quad \mu_{ij}' = \int_{\beta_j} \omega_i' \,, \quad \tau = \mu^{-1}\mu' \,. \tag{283}$$

The space of such matrices is called the Siegel upper half plane

$$\mathcal{H}_2 = \{\tau \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \tau^t = \tau, \, \mathrm{Im}(\tau) > 0\} \,. \tag{284}$$

Under a change of the canonical basis of one-cycles by an element $\gamma \in Sp(4, \mathbb{Z})$ of the form (279) the period matrix transforms as

$$\tau \mapsto \gamma\tau = \frac{A\tau + B}{C\tau + D} \,. \tag{285}$$

Two genus two Riemann surfaces are conformally equivalent iff their period matrix is related by an $Sp(4, \mathbb{Z})$ transformation. Conformal equivalence classes of metrics on a Riemann surface are in one to one correspondence with complex structures.

Every genus two Riemann surface can be described as a double cover $C \to \mathbb{P}^1$

$$y^2 = f(x) = a_6 \prod_{i=1}^{d} (x - r_i) = \sum_{i=0}^{d} a_i x^i \,, \tag{286}$$

with ramification points r_i and $d = 6$ when all r_i are at finite distance or $d = 5$ when one of the roots is at infinity. More generally, a genus g Riemann surface that can be written as a double cover over $\mathbb{P}^1$

$$y^2 = f(x)\,, \tag{287}$$

with $f(x)$ a polynomial of degree $2g+2$ or $2g+1$ is called *hyperelliptic*. A compactification of the moduli space of sextic polynomials $f(x)$ up to automorphisms of $\mathbb{P}^1$ is given by $\bar{\mathcal{U}}_6 = \mathbb{P}^3(2,4,6,10)$. As is explained in [115] the moduli space of genus two Riemann surfaces is a double cover of $\bar{\mathcal{U}}_6$. A map from the space of coefficients a_i, $i = 0, ..., 6$ in (286) into $\bar{\mathcal{U}}_6$ is provided by the relative Igusa invariants $[I_2(a) : I_4(a) : I_6(a) : I_{10}(a)]$. They are defined as

$$\begin{aligned}
I_2 =& a_6^2 \sum_{15} (12)^2(34)^2(56)^2\,,\\
I_4 =& a_6^4 \sum_{10} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2\,,\\
I_6 =& a_6^6 \sum_{60} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2\,,\\
I_{10} =& a_6^{10} \prod_{i<j} (r_i - r_j)\,,
\end{aligned} \tag{288}$$

where the sum is over all permutations of roots $\sigma \in \mathcal{P}_6$ that do not leave the summand invariant and we use $(ij) = (r_{\sigma(i)} - r_{\sigma(j)})$. One can expand the expressions in a basis of symmetric polynomials to obtain the relative Igusa invariants as a function of the coefficients a_i, $i = 0, ..., 6$. The weight 10 Igusa invariant I_{10} is proportional to the discriminant

$$\Delta_2 = \prod_{i<j} (r_i - r_j)\,, \tag{289}$$

which vanishes exactly if to branch points coalesce. The generalization Δ_g of the discriminant for hyperelliptic curves of arbitrary genus is straightforward. It is convenient to define an alternative invariant of weight 6

$$I_6' = \frac{1}{2}(I_2 I_4 - 3I_6)\,. \tag{290}$$

In analogy to the classical theory of modular forms one can define Siegel modular forms that are automorphic with respect to $Sp(4,\mathbb{Z})$ [114]. Consider a finite-dimensional $\mathbb{C}$-vector space V and a representation

$$\rho:\ \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(V)\,. \tag{291}$$

A Siegel modular form of genus two and weight ρ is a holmorphic function $f : \mathcal{H}_2 \to V$ such that

$$f(\gamma\tau) = \rho(C\tau + D) f(\tau)\,, \quad \text{for all} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4,\mathbb{Z})\,. \tag{292}$$

A special case are even powers of the representation

$$\det : GL(2, \mathbb{C}) \to \mathbb{C}^* \,. \tag{293}$$

The corresponding forms are called classical Siegel modular forms and we will denote their weight by the exponent $2k$ of the representation.

The ring of classical Siegel modular forms is given by $M = \mathbb{C}[E_4^{(2)}, E_6^{(2)}, \chi_{10}, \chi_{12}, \chi_{35}]/\chi_{35}^2$. Eisenstein series $E_w^{(2)}$ of weight w are defined via

$$E_w^{(2)} = \sum_{(c,d)} \det(c\tau + d)^{-w} \,, \tag{294}$$

where the sum is over non-associated paris of co-prime symmetric integral matrices [114]. A detailed explanation of how to calculate their coefficients is provided in [17]. The Igusa cusp forms χ_{10}, χ_{12} are of weight $10, 12$ respectively and defined in terms of Eisenstein series

$$\begin{aligned} \chi_{10} =& -\frac{43867}{2^{12}3^5 5^2 7 \cdot 53}\left(E_4^{(2)}E_6^{(2)} - \frac{1}{43867}E_{10}^{(2)}\right) \,, \\ \chi_{12} =& \frac{132 \cdot 593}{2^{13}3^7 5^3 7^2 337}\left[3^2 7^2 \left(E_4^{(2)}\right)^3 + 2 \cdot 5^3 \left(E_6^{(2)}\right)^2 - \frac{1}{132 \cdot 593}E_{12}^{(2)}\right] \,. \end{aligned} \tag{295}$$

An explicit expression for χ_{35} can be found in [114].

4.6 A compact embedding of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$

We will now describe a compact embedding of the toric Calabi-Yau $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ in an elliptically fibered compact Calabi-Yau M. The latter is constructed as a hypersurface in a partial resolution $\mathbb{P}_{\Delta^{\circ\prime}}$ of the weighted projective space $\mathbb{P}(1, 2, 2, 10, 15)$. We provide the toric data in table 4. The points in $\Delta^{\circ\prime}$ admit three different regular star triangulations. Our

div.	coord.	$\bar{\nu}_i^\circ \in \Delta^{\circ\prime}$				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$
$D_1 - 3D_2$	e_3	0	0	−2	−3	1	−3	0	0
	b_1	1	0	−2	−3	−2	1	0	1
D_2	b_2	−1	−1	−2	−3	0	1	0	0
	b_3	0	1	−2	−3	0	1	0	0
	b_4	2	0	−2	−3	0	0	1	−2
	u	1	0	−1	−1	2	0	−2	3
D_4	v	0	0	1	0	0	0	0	1
$D_3 - D_1$	w	0	0	0	1	−1	0	1	0
K_M		0	0	0	0	0	0	0	−3

Table 4: Toric data of the elliptically fibered Calabi-Yau hypersurface that can be decompactified to obtain $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$.

choice is uniquely defined by the Mori cone basis that is encoded in the intersection vectors $l^{(i)}$, $i = 1, ..., 4$. The fibration structure of the toric ambient space can be studied with the techniques that have been developed in [116]. A computer algebra implementation can be found in the Sage mathematics system [67]. Using the latter we find that the total space is a fibration over $\mathbb{P}^2$ with generic fiber $\mathbb{P}(1,2,3)$. The hypersurface equation

$$P = s_1 e_3^6 + s_2 e_3^4 v + s_3 e_3^2 v^2 + s_4 v^3 + s_5 e_3^3 w + s_6 e_3 v w + s_8 w^2 = 0\,, \tag{296}$$

is a specialization of the generalized Tate form that was introduced in [19]. In our case the coefficients $s_1, s_2, s_3, s_4, s_6, s_8$ are functions of b_1, b_2, b_3, b_4 and 208 complex structure moduli.

Around a generic point of the divisor $b_1 = 0$ we can use the torus actions to set $b_2 = b_3 = b_4 = 1$ and obtain

$$s_6 \sim s_3 \sim b_1 + \mathcal{O}(b_1^2)\,, \quad s_5 \sim s_2 \sim b_1^2 + \mathcal{O}(b_1^3)\,, \quad s_1 \sim b_1^3 + \mathcal{O}(b_1^4)\,. \tag{297}$$

The vanishing orders can be compared with the Tate classification in [37] and indicate a G_2 singularity in the generic fiber of the Calabi-Yau over $b_1 = 0$. More precisely, the resolved fiber is composed of spheres that intersect like the affine Dynkin diagram of $SO(8)$. However, the components of the fiber are permuted under monodromies around a codimension two locus in the base. The Dynkin diagram is thus folded into that of the non-simply-laced Lie algebra G_2.

We denote the homogeneous coordinates on $\mathbb{P}_{\Delta^{\circ\prime}}$ by

$$[e_3 : u : b_3 : b_4 : w : b_2 : v : b_1]\,, \tag{298}$$

and the map to the base $\mathbb{P}^2$ is given by

$$[e_3 : u : b_3 : b_4 : w : b_2 : v : b_1] \mapsto [b_2 : b_3 : u b_1 b_4^2]\,. \tag{299}$$

To better understand the geometry we will take a closer look at the singular fiber and how it embeds into the ambient space. Over $u b_1 b_4^2 = 0$ the fiber of the toric ambient space splits into three components. Then homogeneous coordinates on the fiber components including the corresponding $\mathbb{C}^*$ actions, the Stanley-Reisner ideals and maps into $\mathbb{P}_{\Delta^{\circ\prime}}$ are as follows:

$$\begin{array}{c|l}
\mathbb{P}_{F_1} & \begin{array}{l} [z_0 : z_1 : z_2 : z_3] \mapsto [1 : 0 : b_3 : z_2 : z_1 : b_2 : z_0 : z_3] \\ l_{F_1}^{(1)} = (0,1,1,0)\,, \quad l_{F_1}^{(2)} = (1,0,-2,1) \\ \mathcal{SRI} = \langle z_0 \cdot z_3,\; z_1 \cdot z_2 \rangle \end{array} \\
\mathbb{P}_{F_2} & \begin{array}{l} [z_0 : z_1 : z_2] \mapsto [1 : z_2 : b_3 : 0 : 1 : b_2 : z_0 : z_1] \\ l_{F_2}^{(1)} = (1,1,3) \\ \mathcal{SRI} = \langle z_0 \cdot z_1 \cdot z_2 \rangle \end{array} \\
\mathbb{P}_{F_3} & \begin{array}{l} [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : z_4 : b_3 : z_3 : z_2 : b_2 : z_1 : 0] \\ l_{F_3}^{(1)} = (0,0,1,1,-2)\,, \quad l_{F_3}^{(2)} = (0,1,0,-2,3)\,, \quad l_{F_3}^{(3)} = (1,0,-1,0,2) \\ \mathcal{SRI} = \langle z_1 \cdot z_2,\; z_2 \cdot z_3,\; z_0 \cdot z_4,\; z_1 \cdot z_4,\; z_0 \cdot z_3 \rangle \end{array}
\end{array} \tag{300}$$

The homogeneous coordinates b_2, b_3 parametrize the space $b_1 = 0$ in the base of the toric fibration. Note that the $\mathbb{C}^*$ action on b_2, b_3 also acts on the fiber parameters. Each pair of the three components intersects in a $\mathbb{P}^1$,

$$\mathbb{P}_{F_1} \cap \mathbb{P}_{F_2} = [1 : 0 : b_3 : 0 : 1 : b_2 : x : y]\,, \tag{301}$$

$$\mathbb{P}_{F_2} \cap \mathbb{P}_{F_3} = [1 : x : b_3 : 0 : 1 : b_2 : y : 0]\,, \tag{302}$$

$$\mathbb{P}_{F_3} \cap \mathbb{P}_{F_1} = [1 : 0 : b_3 : x : y : b_2 : 1 : 0]\,. \tag{303}$$

We can now restrict the hypersurface equation to the various components and study the solution.

The restriction of P to $\mathbb{P}_{F_1}$ is a cubic polynomial $P_3(z_0, z_3; b_2, b_3)$ that is parametrized by the base coordinates b_2, b_3 and independent of z_1, z_2. The solution therefore consists of three non-intersecting spheres with coordinates

$$\mathbb{P}^1_{1,i} = \{P = 0\} \cap \mathbb{P}_{F_1} = [1 : 0 : b_3 : x : y : b_2 : 1 : \rho_i(b_2, b_3)] \,, \quad i = 1, ..., 3 \,, \tag{304}$$

where $\rho_i(b_2, b_3)$ are the three roots of $P_3(1, z; b_2, b_3)$. Without loss of generality we assume that $P_3(0, 1; b_2, b_3) \neq 0$. It can be immediately seen that the three spheres intersect $\mathbb{P}_{F_1} \cap \mathbb{P}_{F_2}$ in the three points

$$\{P = 0\} \cap \mathbb{P}_{F_1} \cap \mathbb{P}_{F_2} = [1 : 0 : b_3 : 0 : 1 : b_2 : 1 : \rho_i(b_2, b_3)] \,, \quad i = 1, ..., 3 \,. \tag{305}$$

The intersection $\{P = 0\} \cap \mathbb{P}_{F_1} \cap \mathbb{P}_{F_3}$ is empty as long as the polynomial $P_3(z_0, z_3; b_2, b_3)$ is sufficiently generic. Restricting P to $\mathbb{P}_{F_2}$ we find the $\mathbb{P}^1$

$$\mathbb{P}^1_2 = \{P = 0\} \cap \mathbb{P}_{F_2} = [1 : -P_3(x, y; b_2, b_3)/c_1 : b_3 : 0 : 1 : b_2 : x : y] \,, \tag{306}$$

where c_1 is the coefficient of $w^2 u$ in P. Finally, the restriction of P to $\mathbb{P}_{F_3}$ gives another $\mathbb{P}^1$

$$\mathbb{P}^1_3 = \{P = 0\} \cap \mathbb{P}_{F_3} = [x : -c_2/c_1 : b_3 : y : 1 : b_2 : 1 : 0] \,, \tag{307}$$

where $c_2 = P_3(1, 0)$ is the coefficient of v^3 in P. The five spheres intersect in a tetrahedral form

$$\mathbb{P}^1_{1,i} \cap \mathbb{P}^1_2 = 1 \,, \quad \mathbb{P}^1_{1,i} \cap \mathbb{P}^1_3 = 0 \,, \quad i = 1, ..., 3 \,, \qquad \mathbb{P}^1_2 \cap \mathbb{P}^1_3 = 1 \,. \tag{308}$$

Moreover, the roots $\rho_i(b_2, b_3)$ and therefore the spheres $\mathbb{P}^1_{1,i}$ are permuted under monodromies in the base. The six-dimensional effective theory for F-theory on the generic Calabi-Yau hypersurface in $\mathbb{P}_{\Delta^{\circ\prime}}$ therefore indeed has a non-simply laced gauge group G_2.

We are now in a position to interpret the divisors and the curves that are induced from the toric ambient space. A basis of the Kähler cone is given by

$$D_1 = [e_3] + 3[b_2] \,, \quad D_2 = [b_2] \,, \quad D_3 = [w] + D_1 \,, \quad D_4 = [v] \,. \tag{309}$$

On the other hand $[u]$, $[b_4]$ and $[b_1]$ are in the class of $\cup_{i=1}^3 \mathbb{P}^1_{1,i}$, $\mathbb{P}^1_2$ and $\mathbb{P}^1_3$ respectively fibered over the image of $b_1 = 0$ in the base. We use $\mathcal{C}_i$ to denote the induced curve that corresponds to the intersection vector $l^{(i)}$. Then $\mathcal{C}_1 + \mathcal{C}_3 = \mathbb{P}^1_3$, $\mathcal{C}_3 = \mathbb{P}^1_{1,i}$ and $\mathcal{C}_4 = \mathbb{P}^1_2$. The class of a section is given by $[e_3]$ and $\mathcal{C}_2 = [e_3] \cdot [b_2] = [e_3] \cdot [b_3]$ is a curve in the base. It can be easily checked that $\mathcal{C}_i$, $i = 1, ..., 4$ form a basis of curves dual to D_i, $i = 1, .., 4$.

4.6.1 Picard-Fuchs system and periods

From the intersection vectors $l^{(i)}$, $i = 1, .., 4$ we obtain the Picard-Fuchs operators

$$\begin{aligned}
\mathcal{D}_1 =& (\Theta_1 - 3\Theta_2)(2\Theta_1 - 2\Theta_3 + 3\Theta_4 - 1)(2\Theta_1 - 2\Theta_3 + 3\Theta_4) \\
&+ z_1(\Theta_1 - \Theta_3)(2\Theta_1 - \Theta_2 - \Theta_4)(2\Theta_1 - \Theta_2 - \Theta_4 + 1) \,, \\
\mathcal{D}_2 =& \Theta_2^2(\Theta_2 - 2\Theta_1 + \Theta_4) - z_2(\Theta_1 - 3\Theta_2)(\Theta_1 - 3\Theta_2 - 1)(\Theta_1 - 3\Theta_2 - 2) \,, \\
\mathcal{D}_3 =& (\Theta_3 - \Theta_1)(\Theta_3 - 2\Theta_4) - z_3(2\Theta_1 - 2\Theta_3 + 3\Theta_4)(2\Theta_1 - 2\Theta_3 + 3\Theta_4 - 1) \,, \\
D_4 =& (2\Theta_1 - 2\Theta_3 + 3\Theta_4)(2\Theta_1 - 2\Theta_3 + 3\Theta_4 - 1)(2\Theta_1 - 2\Theta_3 + 3\Theta_4 - 2)(\Theta_2 + \Theta_4 - 2\Theta_1) \\
&- 3z_4(3\Theta_4 + 1)(3\Theta_4 + 2)(2\Theta_4 - \Theta_3)(2\Theta_4 - \Theta_3 + 1) \,,
\end{aligned} \tag{310}$$

where D_4 has already been reduced. However, the Picard-Fuchs system contains additional operators. This can be seen by making a general ansatz and finding the solutions that are annihilated by the operators above to a given order. A basis of branes is given by

$$\vec{B} = (\mathcal{O}_M,\, \mathcal{O}_{D_1},\, \mathcal{O}_{D_2},\, \mathcal{O}_{D_3},\, \mathcal{O}_{D_4},\, \mathcal{C}_1^\bullet,\, \mathcal{C}_2^\bullet,\, \mathcal{C}_3^\bullet,\, \mathcal{C}_4^\bullet,\, \mathcal{O}_{\mathrm{pt.}})\ . \tag{311}$$

Using the formulae for the asymptotic behaviour of the brane charges we know that the leading terms of an integral basis of periods are given by

$$\begin{aligned}
Z_6^{\mathrm{asy}} =& \frac{3}{2}\cdot t_1^3 + \frac{3}{2}\cdot t_1^2 t_2 + \frac{1}{2}\cdot t_1 t_2^2 + 18\cdot t_1^2 t_3 + 12\cdot t_1 t_2 t_3 + 2\cdot t_2^2 t_3 + 63\cdot t_1 t_3^2 \\
&+ 21\cdot t_2 t_3^2 + 71\cdot t_3^3 + 9\cdot t_1^2 t_4 + 6\cdot t_1 t_2 t_4 + t_2^2 t_4 + 63\cdot t_1 t_3 t_4 + 21\cdot t_2 t_3 t_4 \\
&+ \frac{213}{2}\cdot t_3^2 t_4 + 15\cdot t_1 t_4^2 + 5\cdot t_2 t_4^2 + \frac{105}{2}\cdot t_3 t_4^2 + \frac{25}{3}\cdot t_4^3 + \frac{17}{4}\cdot t_1 + \frac{3}{2}\cdot t_2 \\
&+ \frac{27}{2}\cdot t_3 + \frac{41}{6}\cdot t_4 - i\frac{51\zeta(3)}{\pi^3}\,, \\
Z_{4,1}^{\mathrm{asy}} =& -\frac{9}{2}\cdot t_1^2 - 3\cdot t_1 t_2 - \frac{1}{2} t_2^2 - 36\cdot t_1 t_3 - 12\cdot t_2 t_3 - 63\cdot t_3^2 - 18\cdot t_1 t_4 - 6\cdot t_2 t_4 \\
&- 63\cdot t_3 t_4 - 15\cdot t_4^2 - \frac{9}{2}\cdot t_1 - \frac{3}{2}\cdot t_2 - 18\cdot t_3 - 9\cdot t_4 - \frac{23}{4}\,, \\
Z_{4,2}^{\mathrm{asy}} =& -\frac{3}{2} t_1^2 - t_1 t_2 - 12\cdot t_1 t_3 - 4\cdot t_2 t_3 - 21\cdot t_3^2 - 6\cdot t_1 t_4 - 2\cdot t_2 t_4 - 21\cdot t_3 t_4 \\
&- 5\cdot t_4^2 - \frac{1}{2}\cdot t_1 - 2\cdot t_3 - t_4 - \frac{3}{2}\,, \\
Z_{4,3}^{\mathrm{asy}} =& -18\cdot t_1^2 - 12\cdot t_1 t_2 - 2\cdot t_2^2 - 126\cdot t_1 t_3 - 42\cdot t_2 t_3 - 213\cdot t_3^2 \\
&- 63\cdot t_1 t_4 - 21\cdot t_2 t_4 - 213\cdot t_3 t_4 - \frac{105}{2}\cdot t_4^2 - 63\cdot t_1 - 21\cdot t_2 \\
&- 213\cdot t_3 - \frac{213}{2}\cdot t_4 - \frac{169}{2}\,, \\
Z_{4,4}^{\mathrm{asy}} =& -9\cdot t_1^2 - 6\cdot t_1 t_2 - t_2^2 - 63\cdot t_1 t_3 - 21\cdot t_2 t_3 - \frac{213}{2}\cdot t_3^2 - 30\cdot t_1 t_4 - 10\cdot t_2 t_4 \\
&- 105\cdot t_3 t_4 - 25\cdot t_4^2 - 15\cdot t_1 - 5\cdot t_2 - \frac{105}{2}\cdot t_3 - 25\cdot t_4 - \frac{91}{6}\,, \\
Z_{2,1}^{\mathrm{asy}} =& t_1\,, \quad Z_{2,2}^{\mathrm{asy}} = t_2\,, \quad Z_{2,3}^{\mathrm{asy}} = t_3\,, \quad Z_{2,4}^{\mathrm{asy}} = t_4\,, \quad Z_0^{\mathrm{asy}} = -1\,.
\end{aligned} \tag{312}$$

Again, t_i, $i = 1, ..., 4$ are the single logarithmic solutions with leading term

$$t_i = \frac{\log\, z_i}{2\pi i} + \mathcal{O}(z)\,, \tag{313}$$

and z_i are the Batyrev variables. With this knowledge we can choose linear combinations of intersection vectors such that the corresponding Picard-Fuchs operators exclude the additional solutions. It turns out that the operators

$$\begin{aligned}
\mathcal{D}_5 =& (\Theta_1 - 3\Theta_2)(\Theta_3 - 2\Theta_4) - z_1 z_3 (2\Theta_1 - \Theta_2 - \Theta_4)(2\Theta_1 - \Theta_2 - \Theta_4 + 1)\,, \\
D_6 =& (\Theta_3 - \Theta_1)(\Theta_3 - \Theta_1 - 1)(\Theta_3 - \Theta_1 - 2)(\Theta_1 - 3\Theta_2) \\
&- 9 z_1 z_3^4 z_4^2 (3\Theta_4 + 1)(3\Theta_4 + 2)(3\Theta_4 + 4)(3\Theta_4 + 5)\,,
\end{aligned} \tag{314}$$

complete the Picard-Fuchs system $\{\mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_3,\, D_4,\, \mathcal{D}_5,\, D_6\}$. It is noteworthy that D_6 is the reduced operator obtained from $(l^{(1)} + l^{(3)}) + 3\cdot l^{(3)} + 2\cdot l^{(4)}$ which is the intersection vector that correspond to the generic fiber. On the other hand, $\mathcal{D}_5$ is related to the curve $\mathcal{C}_1 + \mathcal{C}_3$ which together with $\mathcal{C}_2$ remains compact in a certain limit that we will study now.

The charges (312) are not yet of the form

$$\vec{\Pi} = (1,\, t_i,\, \partial_{t_i}\mathcal{F},\, 2\mathcal{F} - \sum_i t_i \partial_{t_i}\mathcal{F})\,. \tag{315}$$

To this end we apply the transformation

$$T_{sg} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -45 & -6 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -19 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 54 & -98 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -22 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{316}$$

such that

$$\vec{\Pi}^{\text{asy}} = T_{sg} \cdot \left(Z_6^{\text{asy}},\, Z_{4,1}^{\text{asy}},\, Z_{4,2}^{\text{asy}},\, Z_{4,3}^{\text{asy}},\, Z_{4,4}^{\text{asy}},\, Z_{2,1}^{\text{asy}},\, Z_{2,2}^{\text{asy}},\, Z_{2,3}^{\text{asy}},\, Z_{2,4}^{\text{asy}},\, Z_0^{\text{asy}} \right), \tag{317}$$

has the desired structure (315). The open string index $\eta_{ij} = \chi(T_{sg}^{-1}B_i, T_{sg}^{-1}B_j)$ between the branes in the new basis (317) is given by

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{318}$$

As a non-trivial consistency check we find $(\vec{\Pi}^{\text{asy}})^T \eta^{-1} \vec{\Pi}^{\text{asy}} = 0$ which reflects

$$\int_W \Omega \wedge \Omega = 0\,. \tag{319}$$

4.6.2 The non-compact limit

We will now introduce the new variables

$$z_1' = z_1 z_3\,, \quad t_1' = t_1 + t_3\,, \tag{320}$$

and consider the limit $\mathrm{Im}\, t_3 \to \infty$, $\mathrm{Im}\, t_4 \to \infty$, i.e. $z_3 \to 0$, $z_4 \to 0$ while z_1' stays finite. In this limit the volume of $\mathcal{C}_1 + \mathcal{C}_3 = \mathbb{P}^1_3$ and $\mathcal{C}_2 = [e_3] \cdot [b_2]$ remains finite while the generic fiber decompactifies. The resulting space is a blowup of the bundle $\mathcal{O}(-3) \to \mathbb{P}^2$, the resolution $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$. Since the base remains of finite volume the central charge of the brane wrapping the section $[e_3]$,

$$Z_{4,1}^{\mathrm{asy}'} = \Pi_2^{\mathrm{asy}} - 3 \cdot \Pi_3^{\mathrm{asy}} - 3 \cdot t_3 \,, \tag{321}$$

is finite in the limit. On the other hand, the divisor $[b_1]$ corresponds to $\mathbb{P}^1_3$ fibered over a divisor in the base and the brane with charge

$$Z_{4,2}^{\mathrm{asy}'} = -2 \cdot \Pi_2^{\mathrm{asy}} + \Pi_3^{\mathrm{asy}} + \Pi_4^{\mathrm{asy}} - 59 \cdot t_3 \tag{322}$$

also survives. These are the branes that wrap the divisors of points in the interior of the toric diagram. The open string index between the branes with charges

$$\vec{\Pi}^{\mathrm{asy}'} = (Z_{4,1}^{\mathrm{asy}'}, Z_{4,2}^{\mathrm{asy}'}, Z_{2,1}^{\mathrm{asy}'}, Z_{2,2}^{\mathrm{asy}'}, Z_0^{\mathrm{asy}}) \,, \tag{323}$$

where $Z_{2,1}^{\mathrm{asy}'} = Z_1^{\mathrm{asy}} + Z_3^{\mathrm{asy}}$ and $Z_{2,2}^{\mathrm{asy}'} = Z_2^{\mathrm{asy}}$, is

$$\eta' = \begin{pmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{324}$$

Again we have to ask the question how the double-logarithmic periods are related to a prepotential. From the expressions (322),(321) it follows that

$$-\frac{1}{5}\left(Z_{4,1}^{\mathrm{asy}'} + 3 \cdot Z_{4,2}^{\mathrm{asy}'}\right) = \partial_{t_1'} \mathcal{F}' \,, \tag{325}$$

$$-\frac{1}{5}\left(2 \cdot Z_{4,1}^{\mathrm{asy}'} + Z_{4,2}^{\mathrm{asy}'}\right) = \partial_{t_2} \mathcal{F}' \,, \tag{326}$$

and vice versa

$$Z_{4,i}^{\mathrm{asy}'} = C_i{}^j \partial_j \mathcal{F}' \,, \quad i = 1, 2 \,. \tag{327}$$

where C is the intersection matrix

$$C = \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} , \tag{328}$$

between the curves and divisors that survive the non-compact limit. On the other hand, the charges of branes that together with $\tilde{t}_D^i = Z_{4,i}^{\mathrm{asy}'}$ form a symplectic basis are not $Z_{2,1}^{\mathrm{asy}'}$, $Z_{2,2}^{\mathrm{asy}'}$, but [17]

$$\tilde{t}_i = \left(\mathcal{C}^{-1}\right)_i{}^j Z_{2,j}^{\mathrm{asy}'} \,, \quad i = 1, 2 \,. \tag{329}$$

4.7 The Siegel modular structure

Based on the insights from the compact embedding of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ we are now ready to explain the Siegel modular structure of the topological string amplitudes. We will first discuss the general properties that we assume to hold for arbitrary genus g of the mirror curve C.

4.7.1 Periods on the mirror curve

A key observation is that the derivatives of the meromorphic differential λ with respect to the Batyrev variables z_i, $i = 1, 2$ form a basis

$$\omega_i = \partial_{z_i}\lambda + \text{closed} \in H^{1,0}(C, \mathbb{C}), \quad i = 1, ..., 2, \tag{330}$$

of holomorphic 1-forms on C [30]. Consider a symplectic basis of cycles $A_i, B^i \in H_1(C, \mathbb{Z})$, $i = 1, 2$, such that

$$A_i \cap B^j = \delta_i^j, \quad A_i \cap A_j = 0, \quad B^i \cap B^j = 0. \tag{331}$$

We denote the periods of the meromorphic 1-form λ with respect to this basis by

$$\tilde{t}_j = \int_{A_j} \lambda, \quad \tilde{t}^j_D = \int_{B^j} \lambda, \quad j = 1, 2. \tag{332}$$

It immediately follows that

$$\mu_{ij} = \frac{\partial \tilde{t}_j}{\partial z_i} = \int_{A_j} \omega_i, \quad \mu'_{ij} = \frac{\partial \tilde{t}^j_D}{\partial z_i} = \int_{B^j} \omega_i, \quad i, j = 1, 2, \tag{333}$$

are A- and B-period matrices with respect to the basis of holomorphic 1-forms ω_i, $i = 1, ..., g$. Moreover, the map from the complex structure moduli space to the Siegel upper half plane $\mathcal{H}_g$ is given by $\tau = \mu^{-1}\mu'$,

$$\tau_{ij} = \frac{\partial z_k}{\partial \tilde{t}_i}\frac{\partial \tilde{t}^j_D}{\partial z_k} = \frac{\partial \tilde{t}^j_D}{\partial \tilde{t}_i}. \tag{334}$$

The intersections of 1-cycles on C are identified via homological mirror symmetry with the quantum intersections of vertical cycles in the A-model geometry. If the latter is a toric Calabi-Yau the intersections can be read off from the toric data. In this case there are g toric divisors D_i that correspond to inner points of the toric diagram. We denote a basis of the Mori cone by C^i, $i = 1, ..., g$. Then we can introduce the intersection matrix

$$C_i^{\ j} = D_i \cdot C^j, \tag{335}$$

and, generalizing (327) and (329), it follows that the period matrix $\tau = \mu^{-1}\mu'$ is given by

$$\tau_{ij} = \frac{\partial \tilde{t}^j_D}{\partial \tilde{t}_i} = C_i^{\ m} C_j^{\ n} \partial_m \partial_n \mathcal{F}'. \tag{336}$$

Taking the derivative of this relation with respect to t_k we obtain [17]

$$\frac{\partial \tau_{ij}}{\partial t_k} = C_i^{\ m} C_j^{\ n} C_{kmn}. \tag{337}$$

4.7.2 The propagator for hyperelliptic mirror curves

For hyperelliptic curves there is an interesting relation between the discriminant (289) and a certain product of theta functions [34]. This enables us to derive a general expression for the propagator (265).

For any pair of vectors $\eta, \eta' \in \frac{1}{2}\mathbb{Z}^g$ there is a theta function

$$\vartheta_\eta : \mathbb{C}^g \times \mathcal{H}_g \to \mathbb{C}\,, \tag{338}$$

with characteristic $\eta = \begin{bmatrix} \eta' \\ \eta'' \end{bmatrix}$ defined by

$$\vartheta[\eta](z;\tau) = \sum_{n\in\mathbb{Z}^g} \exp\left(\pi i(\eta'+n)^T\tau(\eta'+n) + 2\pi i(\eta'+n)^T(\eta''+z)\right)\,. \tag{339}$$

For any subset S of $1, 2, ..., 2g+1$ there is a characteristic $\eta_S = \sum_{k\in S}\eta_k \bmod 1$, where

$$\begin{aligned} \eta_{2k-1} &= \begin{bmatrix} (0, & \ldots, & 0, & \frac{1}{2}, & 0, & \ldots, & 0)^T \\ (\frac{1}{2}, & \ldots, & \frac{1}{2}, & 0, & 0, & \ldots, & 0)^T \end{bmatrix}\,, \quad 1 \le k \le g+1\,, \\ \eta_{2k} &= \begin{bmatrix} (0, & \ldots, & 0, & \frac{1}{2}, & 0, & \ldots, & 0)^T \\ (\frac{1}{2}, & \ldots, & \frac{1}{2}, & \frac{1}{2}, & 0, & \ldots, & 0)^T \end{bmatrix}\,, \quad 1 \le k \le g\,, \end{aligned} \tag{340}$$

such that the non-zero entry in the top row occurs in the k-th position.

Now let $\mathcal{T}$ be the set of subsets $T \subset \{1, 2, ..., 2g+1\}$ of cardinality $g+1$, $U = \{1, 3, ..., 2g+1\}$ and denote by $A \ominus B$ the symmetric difference

$$A \ominus B = (A \cup B) \backslash (A \cap B)\,. \tag{341}$$

Then the *modular discriminant* $\phi_g : \mathcal{H}_g \to \mathbb{C}$ is defined by

$$\varphi_g(\tau) = \prod_{T\in\mathcal{T}} \vartheta\left[\eta_{T\ominus U}\right](0;\tau)^8\,. \tag{342}$$

It is a modular form on $\Gamma_g(2) = \{\gamma \in \mathrm{Sp}(2g,\mathbb{Z}) \,|\, \gamma \equiv I_{2g} \bmod 2\}$ of weight $4r$, where

$$r = \begin{pmatrix} 2g+1 \\ g+1 \end{pmatrix}\,. \tag{343}$$

The modular discriminant φ_g is related to the polynomial discriminant Δ (289) via [34, 117]

$$(\Delta)^l = c \cdot (\det\tilde{\mu})^{-4r}\varphi_g(\tau)\,, \quad l = \begin{pmatrix} 2g \\ g+1 \end{pmatrix}\,. \tag{344}$$

Here $\tilde{\mu}$ are the A-periods with respect to a *canonical* basis of holomorphic forms (281). It can be obtained from μ (333) by cancelling the poles at the large complex structure divisors diagonalizing the constant part. Therefore it is related to the topological metric (266) via

$$\frac{1}{z_1 \cdot \ldots \cdot z_g} \det\,\tilde{\mu} = \det\left(C_j{}^k \frac{\partial\tilde{t}_k}{\partial z_i}\right) = \det\left(\frac{\partial t_j}{\partial z_i}\right)\,. \tag{345}$$

There is a constant factor c due to the ambiguity in normalizing the discriminant. In any case we find the relation

$$\frac{1}{2}\partial_k \log \det \left(\frac{\partial t_j}{\partial z_i}\right) = \frac{1}{8r}\partial_k \left(\log \phi_g - l \log \Delta + \sum_{i=1}^{g} \log(z_i)\right) . \tag{346}$$

Now recall from section 4.4 that the general Ansatz for the unrefined genus one free energy $\mathcal{F}^{(0,1)}$ is given by

$$\mathcal{F}^{(0,1)} = \frac{1}{2}\log \det \left(\frac{\partial t_j}{\partial z_i}\right) - \frac{1}{12}\log \Delta + \log \prod_{i=1}^{h^{2,1}} z_i^{b_i} . \tag{347}$$

Furthermore, it satisfies the relation

$$\frac{1}{2\pi i}\partial_i \mathcal{F}^{(0,1)} = \frac{1}{2} C_{ijk} S^{jk} + A_i , \tag{348}$$

for a choice of propagator S^{jk} and the corresponding ambiguity

$$A_i = \partial_i \left(\tilde{a} \log \Delta + \sum_{j=1}^{h^{2,1}} \tilde{b}_j \log z_j\right) . \tag{349}$$

One can choose the ambiguous coefficients $\tilde{a}, \tilde{b}_j$ such that

$$\frac{1}{2\pi i}\frac{1}{4r}\partial_i \log \phi_g = C_{ijk} S^{jk} , \tag{350}$$

or, using (337) and assuming invertibility,

$$S^{ij} = \frac{1}{2\pi i}\frac{1}{4r} C_m{}^i C_n{}^j \frac{\partial}{\partial \tau_{mn}} \log \phi_g . \tag{351}$$

We will now discuss the special cases $g = 1, 2$.

Modular propagator for $g = 1$ The result for mirror curves of genus $g = 1$ has first been observed in [112]. In this case the elements of $\mathcal{T}$ are the sets $T_i = \{1,2,3\}\backslash\{i\}$, $i = 1, ..., 3$. The corresponding theta functions are

$$\begin{aligned} \vartheta[\eta_{T_1}](0;\tau) &= \theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} , \\ \vartheta[\eta_{T_2}](0;\tau) &= \theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} , \\ \vartheta[\eta_{T_3}](0;\tau) &= \theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{\left(n+\frac{1}{2}\right)^2}{2}} , \end{aligned} \tag{352}$$

with $q = e^{2\pi i \tau}$. Note that the theta characteristics $\eta_{T_1}, \eta_{T_2}, \eta_{T_3}$ are the even characteristics for $g = 1$, i.e. $4\eta'_{T_i} \cdot \eta''_{T_i}$ is even. The modular discriminant is related to the Dedekind η-function via [118]

$$\phi_1(\tau) = \theta_2(\tau)^8 \theta_3(\tau)^8 \theta_4(\tau)^8 = 2^8 \eta(\tau)^{24} = 2^8 q \prod_{n=1}^{\infty} (1-q^n)^{24} . \tag{353}$$

. A valid choice for the propagator is therefore

$$S^{tt} = \frac{1}{2\pi i}\frac{c^2}{12}\partial_\tau \log\left(\eta^{24}\right) = \frac{c^2}{12}E_2^{(1)}(\tau)\,, \tag{354}$$

where $E_2^{(1)}(\tau)$ is the quasi modular form

$$E_2^{(1)}(\tau) = 1 - 24q - 72q^2 - \dots\,. \tag{355}$$

More generally, the classical Eisenstein series $E_k^{(1)}(\tau)$ can be defined as

$$E_k^{(1)} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\,, \tag{356}$$

in terms of the Bernoulli numbers B_k (275) and the divisor function

$$\sigma_k(n) = \sum_{m\in\mathbb{N},\, m|n} m^k\,. \tag{357}$$

There is also a concise expression for the topological metric. As we discussed in 2.2, a genus one curve can be brought into Weierstrass form

$$y^2 = x^3 + fx + g\,. \tag{358}$$

The coefficients f, g are projective invariants of weight 4 and 6 respectively. The j-invariant provides an isomorphism of the $\mathrm{SL}(2,\mathbb{Z})$ orbits in the upper half plane $\mathcal{H}_1$ with the 3-punctured sphere

$$j:\ \frac{\mathcal{H}_1}{\mathrm{SL}(2,\mathbb{Z})} \to S^2\backslash\{0,1,\infty\}\,, \tag{359}$$

and is given by

$$j(\tau) = 1728\frac{4f^3}{4f^3+27g^2} = 1728\frac{E_4^3(\tau)}{E_4^3(\tau)-E_6^2(\tau)} = \frac{1}{q} + 744 + \dots\,. \tag{360}$$

Using

$$\eta(\tau)^{24} = \frac{1}{1728}\left(E_4(\tau)^3 - E_6(\tau)^2\right)\,,\quad \Delta = 4f^3 + 27g^2\,, \tag{361}$$

one finds

$$\det\left(\frac{\partial t}{\partial z}\right) \sim \frac{1}{z}\left(\frac{E_4}{f}\right)^{\frac{1}{4}}\,. \tag{362}$$

Modular propagator for $g = 2$ For mirror curves of genus $g = 2$ the ten theta characteristics $\eta_{T\ominus U}$, $T \in \mathcal{T}$ are again the complete set of even characteristics

$$\eta = \left[\begin{array}{c}\eta' \\ \eta''\end{array}\right]\,,\quad \eta',\eta'' \in \frac{1}{2}\mathbb{Z}^2 \quad \text{such that} \quad 4\eta'\cdot\eta'' = 0 \bmod 2\,, \tag{363}$$

for the given genus. It is well known [119] that the product

$$\prod_{T\in\mathcal{T}} \vartheta[\eta_{T\ominus U}](0;\tau)^2 = 2^{14}\chi_{10}(\tau)\,, \tag{364}$$

is proportional to the weight ten cusp form $\chi_{10}(\tau)$. A canonical choice for the propagator is therefore given by

$$S^{ij} = \frac{1}{2\pi i}\frac{1}{10}C_m{}^i C_n{}^j \frac{\partial}{\partial\tau_{mn}} \log\,\chi_{10}(\tau)\,. \tag{365}$$

It can be shown [17] that

$$\tilde{S}^{mn} = \frac{1}{2\pi i}\frac{1}{10}\frac{\partial}{\partial\tau_{mn}} \log\,\chi_{10}(\tau)\,, \tag{366}$$

is a meromorphic quasi Siegel modular form and provides an analogue of the classical Eisenstein series $E_2^{(1)}$. Finding this object and understanding its modular properties has been one of the major results of [17].

The relative Igusa invariants and the Eisenstein series satisfy

$$I_4 = 4\kappa^2 E_4^{(4)} \quad I_6' = 4\kappa^3 E_6^{(2)} \quad I_{10} = -2^{14}\kappa^5\chi_{10}\,, \tag{367}$$

with

$$\kappa = -\frac{1}{24}I_2\frac{\chi_{10}}{\chi_{12}}\,. \tag{368}$$

This leads to a modular expression for the determinant of the topological metric

$$\det\left(\frac{\partial t_i}{\partial z_j}\right) \sim \frac{1}{z_1 z_2}\sqrt{\frac{I_4 E_6^{(2)}}{I_6' E_4^{(2)}}}\,, \tag{369}$$

and other equivalent relations.

5 Conclusion and Outlook

In this thesis we have presented various results about F-theory and topological string theory on elliptically fibered Calabi-Yau manifolds and their non-compact limits.

In the first part we presented evidence for a certain conjecture about F-theory on particular elliptically fibered Calabi-Yau manifolds. After briefly explaining the structure of genus one curves we outlined the dictionary between the effective physics of an F-theory compactification and the geometry of the elliptic fibration. We then spent some time to introduce the machinery of toric geometry and constructions of Calabi-Yau mirror pairs in toric ambient spaces.

The tools were then put to use in the fiber based approach to F-theory compactifications. It had been observed [19, 20] that mirror duality on the fiber relates discrete symmetries and non-simply connected non-Abelian gauge groups in the effective action. Based on [18] we gave a toric explanation of this phenomenon for the fibers that are hypersurfaces in two-dimensional toric ambient spaces. We then discussed the evidence from the much larger class of complete intersection fibers in three-dimensional toric ambient spaces that has also been worked out in [18].

One of the major technical contributions in [18] was the first in depth analysis of a complete intersection fiber including the generic gauge and matter loci and the corresponding discrete charges. In chapter 2.5.4 we went beyond this analysis and studied the anomaly cancellation conditions from generic six-dimensional F-theory compactifications with the given fiber. We showed that the pure non-Abelian and non-Abelian-gravitational anomalies are generically cancelled while the pure gravitational anomaly requires additional contributions from matter in the singlet representation. Determining the loci of singlets requires ideal decompositions that are in general computationally prohibitively expensive. It would be very interesting to try to use the anomaly constraints to guide the decomposition. This however is a question to be addressed in future work.

In the second part we extended the discussion of F-theory to include non-trivial fluxes. To this end we had to study the global properties of elliptically fibered Calabi-Yau manifolds.

We started with a discussion of the moduli space and the cohomology of Calabi-Yau three- and fourfolds. In particular we introduced the periods over the holomorphic d-form and their relation to the mirror map. We also reviewed how the periods can be obtained from the so-called Picard-Fuchs system of differential operators and how the latter can be determined for Calabi-Yau hypersurfaces in toric ambient spaces. We then explained the role of fluxes for phenomenologically viable F-theory compactifications. This led to the problem of determining the lattice of properly quantized horizontal fluxes.

Based on [5] we described a technique that uses homological mirror symmetry and the Gamma class formula to find integral elements of the lattice. We also commented on the question under which circumstances this leads to a basis of fluxes. Again following [5] we then restricted our attention to non-singular elliptically fibered Calabi-Yau fourfolds and describe a general basis for the horizontal flux lattice of these geometries. We used this basis to calculate the action of certain Fourier-Mukai transformations on the dual basis of B-branes. The actions of large radius monodromies and a Bridgeland type involution generate a projective action of the modular group on the brane charges. As we showed in [5] this leads to a modular structure of the topological string amplitudes at genus zero and genus one. The corresponding amplitudes are related via a modular anomaly equation.

For a particular example we then reviewed a preliminary study of the stable vacua that has also been published in [5]. This included numerical analytic continuations of the flux basis and calculations of the scalar potentials.

While the technique that we presented to determine integral horizontal fluxes is fairly general, we performed explicit calculations of the auto-equivalences, topological string amplitudes and concrete vacua only for non-singular models. For some calculations of topological string amplitudes and their modular structure on singular models see for example [120]. The results hint at the existence of a Bridgeland type involution also in these cases. However, the corresponding Fourier-Mukai transformation on elliptically fibered Calabi-Yau varieties with singularities in the fiber has not yet been determined. We hope to come back to this question in future work.

For certain Calabi-Yau varieties it is possible to take a non-compact limit that physically corresponds to the decoupling of gravity. If the the non-compact space is a toric Calabi-Yau threefold the mirror is encoded in the data of a Riemann surface and a meromorphic differential. In the case that the Riemann surface is a torus it is well known [33, 112] that the topological string amplitudes admit an expansion in terms of quasi- or almost holomorphic modular forms. We have argued in [17] that for mirror curves of genus two the corresponding expansion is in terms of a novel quasi- or almost meromorphic Siegel modular form. We performed explicit calculations for the two non-compact models $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ and $\widehat{\mathbb{C}^3/\mathbb{Z}_6}$ to show that this is indeed correct and can be used to facilitate the solution of topological string theory on this class of models.

In the third part of this thesis we built on the work that has been done in [17] and extend the results in several ways. First we gave a brief introduction to topological string theory in the compact and non-compact setting. We then reviewed the holomorphic anomaly equations and the (refined) topological invariants that are encoded in the topological string amplitudes. After this we presented a compact embedding of $\widehat{\mathbb{C}^3/\mathbb{Z}_5}$ and gave a detailed discussion of the geometry and the non-compact limit. To this end we also applied the Gamma class formula for the asymptotic B-brane charges that has been discussed in the second part of this thesis to determine an integral period basis and find a complete set of Picard-Fuchs operators. In this setup we could explicitly trace out the origin of the relation between the modular parameter of the mirror curve and the prepotential of the non-compact geometry that has been described in [17]. The argument for the particular Siegel modular structure of the topological string amplitudes that has been made in [17] relied on a relation between the determinant of the topological metric and the square root of a certain quotient of Igusa invariants and Siegel modular forms. In this thesis we use a theorem by Lockhart [34] to generalize this relation and thus initiate the study of topological strings on geometries with mirror curves of genus $g > 2$.

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