On the stability of the blow-up for the wave maps and the cubic wave equation

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To my grandmother Helen,
for her unconditional love no matter what.

“All men’s souls are immortal, but the souls of the righteous are immortal and divine”.

Socrates, Philosopher of the 5th century BC

Contents

Abstract

This thesis is divided into two parts. In the first part, we study co-rotational wave maps from the $(1+d)$-dimensional Minkowski space into the d-sphere for all odd integers $d \geq 3$. This model reads

$$\psi_{tt}(t,r) - \psi_{rr}(t,r) - \frac{d-1}{r}\psi_r(t,r) + \frac{d-1}{2}\frac{\sin(2\psi(t,r))}{r^2} = 0,$$

for $t \in I$, an open interval in $\mathbb{R}$ and $r \geq 0$. Shatah [74], Turok-Spergel [89] (for $d = 3$) and Bizoń-Biernat [7] (for $d \geq 4$), showed that there exist smooth, self-similar, initial data which lead to solutions that blowup in finite time. However, this is an energy-supercritical model meaning that the energy norm is too weak to detect the self-similar break down. Relying on a method developed by Donninger and Schörkhuber [29–32], we prove the asymptotic nonlinear stability of the "ground-state" self-similar solution. Our method is also based on the results of Costin-Donninger-Xia [19] and Costin-Donninger-Glogić [20]. This result constitutes the main result of the first part and is a joint work of the author with Donninger and Glogić [18]. In the second part, we consider the wave equation with a focusing cubic nonlinearity in higher odd space dimensions

$$-u_{tt}(t,x) + \Delta_x u(t,x) + u^3(t,x) = 0,$$

for $t \in I$, an open interval in $\mathbb{R}$ and $x \in \mathbb{R}^d$, without symmetry restrictions on the data. This equation also exhibits finite-time blowup from smooth, compactly supported initial data. Starting from spatially homogeneous solutions which develop a singularity in finite time, we use the symmetries of the equation to construct a larger family of blowup solutions. Donninger and Schörkhuber developed intense research on the stability of the blowup solutions. Their study resulted in a series of papers: in three space dimensions for radial initial data [29, 30], for all space dimensions and radial initial data [33] and in three space dimensions without symmetry restrictions [31]. The latter relies on an integral identity over the 2-sphere that is only valid in three space dimensions. Our main result of the second part completes the picture and concentrates around the stability of the blowup solutions in all odd space dimensions without symmetry restrictions on the data [17]. More precisely, we prove that there exists an open set of initial data for which the solution exists in a backward light-cone and approaches the blowup profile described by the ODE. This is a joint work of the author with Donninger [17].

This thesis is structured into three chapters. In Chapter 1, we introduce the wave maps and the cubic wave equation, the setting in which we work and give a brief overview of some related results that have been obtained. Then, we state Theorem 1.1.1 [18] and Theorem 1.2.1 [17] which are the main results of the present thesis, give an outline of the proofs and discuss the main difficulties encountered. We do not claim originality of any kind concerning the results presented in Chapter 1 besides Theorem 1.1.1 [18] and Theorem 1.2.1 [17].

To the best knowledge of the author, these two results are original. In Chapter 2, we focus on the blowup of co-rotational wave maps from the $(1+d)$-dimensional Minkowski space into the d-sphere in odd space dimensions $d \geq 3$ odd and prove our first result. This chapter contains the result of the paper [18]. In Chapter 3, we turn our attention to the stable blowup for the cubic wave equation in higher dimensions without symmetry restrictions on the data and prove our second result. This chapter contains the result of the paper [17].

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Chapter 1

Introduction and statement of the main results

In this chapter, we begin by introducing the wave maps and the cubic wave equation. Then, we state our main results and discuss the main ideas involved in the proofs.

1.1 Wave Maps

Let (M, g) be a Lorentzian spacetime and (N, h) any curved Riemannian manifold of dimensions $1 + m$ and n respectively. The wave maps equation is a coupled system of non-linear wave equations for a smooth map $u : (M, g) \longrightarrow (N, h)$. This system has attracted a lot of interest due to its pure geometric structure: it involves certain types of non-linearities which contain the Christoffel symbols of the underlying target manifold and consequently information about the curvature. Due to its geometric character, the wave maps equation was proposed as a toy-model for some aspects of the critical behavior in the formation of black holes, see Bizoń-Chmaj-Tabor [9].

1.1.1 Variational formulation

Specifically, wave maps arise naturally as functions for which the action functional

$$S_g[u] := \frac{1}{2} \int_M |d_g u|^2 \, d\mu_g \tag{1.1}$$

is stationary. Here, $d\mu_g$ is the volume form of the domain manifold determined by the metric g. For all $x \in M$, the differential $d_g u(x) : T_x M \longrightarrow T_{u(x)} N$ is a linear map and hence it can be identified with an element of $T_x^\star M \otimes T_{u(x)} N$. Moreover,

$$|d_g u(x)|^2 = |d_g u(x)|^2_{T_x^\star M \otimes T_{u(x)} N} := \operatorname{tr}_g \left(u^\star \left(h \right) \right)$$

that is the trace with respect to g of $u^\star(h)$, the pullback metric on (M,g) via the map u. In local coordinates $\{x_\mu\}_{\mu=0}^m$ on (M,g), this expression reads

$$\mathrm{tr}_g\left(u^\star(h)\right) = g^{\mu\nu}\left(u^\star(h)\right)_{\mu\nu} = g^{\mu\nu}(\partial_\mu u^a)(\partial_\nu u^b)h_{ab}(u).$$

Here and in the following, we adopt the Einstein summation convention. This means that repeated indices are summed over. In particular, we use the Latin alphabet for spatial components only, for example

$$A^j B_{jk} C^k = \sum_{j=1}^n \sum_{k=1}^n A^j B_{jk} C^k, \quad D^{ij}E_{ij} = \sum_{i=1}^n \sum_{j=1}^n D^{ij}E_{ij},$$

whereas the Greek alphabet is used for space and time components, namely

$$a_\mu b^{\mu\nu} c_\nu = \sum_{\mu=0}^m \sum_{\nu=0}^m a_\mu b^{\mu\nu} c_\nu, \quad d^{\mu\nu}e_{\mu\nu} = \sum_{\mu=0}^m \sum_{\nu=0}^m d^{\mu\nu}e_{\mu\nu}.$$

1.1.2 Euler-Lagrangian equations

In this section, we are going to derive the Euler-Lagrangian equations corresponding to the first variation of the action (1.1). To this end, we follow [40,75,81] and reproduce the computation for sake of completeness. We vary a fixed wave map $u : (M,g) \longrightarrow (N,h)$ by allowing it to be a member of an one-parameter family of maps. For any function $\phi \in C_c^\infty(M)$, we define

$$u_\epsilon : (M,g) \longrightarrow (N,h), \quad u_\epsilon := u + \epsilon\phi.$$

Notice that $u_0 = u$ on all of (M,g) and $u_\epsilon = u$ outside a compact set. Let $\Omega \subset M$ be a compact set such that $\mathrm{supp}(\phi) \subseteq \Omega$. Then, u is a wave map if and only if

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_g[u_\epsilon] = 0. \tag{1.2}$$

We compute

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_g[u_\epsilon] &= \frac{1}{2}\int_\Omega \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left(g^{\mu\nu}(\partial_\mu u^a)(\partial_\nu u^b)h_{ab}(u)\right) d\mu_g \\
&= \frac{1}{2}\int_\Omega \Big(g^{\mu\nu}\frac{\partial h_{ab}}{\partial u^c}\phi^c \partial_\mu u^a \partial_\nu u^b + g^{\mu\nu}h_{ab}(u)\partial_\mu\phi^a\partial_\nu u^b \\
&\qquad + g^{\mu\nu}h_{ab}(u)\partial_\mu u^a \partial_\nu \phi^b \Big) d\mu_g \\
&= \frac{1}{2}\int_\Omega g^{\mu\nu}\frac{\partial h_{ab}}{\partial u^c}\phi^c \partial_\mu u^a \partial_\nu u^b d\mu_g + \int_\Omega g^{\mu\nu}h_{ab}(u)\partial_\mu\phi^a\partial_\nu u^b d\mu_g \\
&= \frac{1}{2}\int_\Omega \frac{\partial h_{ab}}{\partial u^c}\phi^c \partial_\mu u^a \partial^\mu u^b d\mu_g + \int_\Omega h_{ab}(u)\partial_\mu\phi^a\partial^\mu u^b d\mu_g,
\end{aligned} \tag{1.3}$$

where the contravariant metric tensor $g^{\mu\nu}$ is used to raise the index, $\partial^\mu := g^{\mu\nu}\partial_\nu$. We focus on the second integral and define the vector field

$$X^\mu := h_{ab}(u)\phi^a\partial^\mu u^b.$$

We compute

$$\nabla_\mu X^\mu = h_{ab}(u)\partial_\mu\phi^a\partial^\mu u^b + h_{ab}(u)\phi^a\Box_g u^b + \partial^\mu u^b\phi^a\frac{\partial h_{ab}}{\partial u^c}\partial_\mu u^c$$

where $\nabla_\mu X^\mu$ stands for the divergence of the vector field X,

$$\nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^\nu_{\mu\nu}X^\mu = \frac{1}{|g|}\partial_\mu\left(|g|X^\mu\right)$$

and $\Box_g$ is the Laplace-Beltrami operator on (M,g),

$$\Box_g := \nabla_\mu\nabla^\mu := \nabla_\mu\partial^\mu := \frac{1}{|g|}\partial_\mu(|g|\partial^\mu), \quad |g| := \sqrt{|\det(g_{\mu\nu})|}.$$

Now, $X=0$ on $\partial\Omega$ since $\phi=0$ on $\partial\Omega$ and Stokes's theorem yields

$$\int_\Omega \nabla_\mu X^\mu d\mu_g = 0.$$

Therefore, we can rewrite the second integral in (1.3) as

$$\int_\Omega h_{ab}(u)\partial_\mu\phi^a\partial^\mu u^b d\mu_g = -\int_\Omega\left(h_{ab}(u)\phi^a\Box_g u^b + \partial^\mu u^b\phi^a\frac{\partial h_{ab}}{\partial u^c}\partial_\mu u^c\right)d\mu_g$$

and using (1.2) we are left with

$$-\int_\Omega\left(-\frac{1}{2}\frac{\partial h_{ab}}{\partial u^c}\phi^c\partial_\mu u^a\partial^\mu u^b + h_{ab}(u)\phi^a\Box_g u^b + \partial^\mu u^b\phi^a\frac{\partial h_{ab}}{\partial u^c}\partial_\mu u^c\right)d\mu_g = 0.$$

Observe that the third term can be written as

$$\partial^\mu u^b\phi^a\frac{\partial h_{ab}}{\partial u^c}\partial_\mu u^c = \partial_\mu u^c\partial^\mu u^b\left(\frac{1}{2}\frac{\partial h_{ab}}{\partial u^c}\phi^a + \frac{1}{2}\frac{\partial h_{ac}}{\partial u^b}\phi^a\right)$$

and therefore

$$\begin{aligned}
0 &= -\int_\Omega\left(h_{ab}(u)\phi^a\Box_g u^b + \frac{1}{2}\partial_\mu u^b\partial^\mu u^c\left(\frac{\partial h_{ab}}{\partial u^c} + \frac{\partial h_{ac}}{\partial u^b} - \frac{\partial h_{bc}}{\partial u^a}\right)\phi^a\right)d\mu_g \\
&= -\int_\Omega\left(h_{af}(u)\phi^a\Box_g u^f + \frac{1}{2}\partial_\mu u^b\partial^\mu u^c\left(\frac{\partial h_{eb}}{\partial u^c} + \frac{\partial h_{ec}}{\partial u^b} - \frac{\partial h_{bc}}{\partial u^e}\right)\phi^a\delta^e_a\right)d\mu_g \\
&= -\int_\Omega\left(h_{af}(u)\phi^a\Box_g u^f + \frac{1}{2}\partial_\mu u^b\partial^\mu u^c\left(\frac{\partial h_{eb}}{\partial u^c} + \frac{\partial h_{ec}}{\partial u^b} - \frac{\partial h_{bc}}{\partial u^e}\right)\phi^a h^{ef}(u)h_{fa}(u)\right)d\mu_g \\
&= -\int_\Omega h_{af}\phi^a\left(\Box_g u^f + \frac{1}{2}\partial_\mu u^b\partial^\mu u^c\left(\frac{\partial h_{eb}}{\partial u^c} + \frac{\partial h_{ec}}{\partial u^b} - \frac{\partial h_{bc}}{\partial u^e}\right)h^{fe}(u)\right)d\mu_g.
\end{aligned}$$

Using the fact that

$$\Gamma^i_{k\ell} := \frac{1}{2} h^{im} (\partial_\ell h_{mk} + \partial_k h_{m\ell} - \partial_m h_{k\ell})$$

are the Christoffel symbols associated to the metric h on the target manifold, we infer

$$-\int_\Omega h_{af} \phi^a \left(\Box_g u^f + \Gamma^f_{bc}(u) \partial_\mu u^b \partial^\mu u^c \right) d\mu_g = 0.$$

We conclude that the Euler-Lagrange equations associated to the functional (1.1) are

$$\Box_g u^a + \Gamma^a_{bc}(u) \partial_\mu u^b \partial^\mu u^c = 0 \tag{1.4}$$

and they constitute a system of semi-linear wave equations

$$\begin{cases} \Box_g u^1 + \Gamma^1_{bc}(u) \partial_\mu u^b \partial^\mu u^c = 0 \\ \Box_g u^2 + \Gamma^2_{bc}(u) \partial_\mu u^b \partial^\mu u^c = 0 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ \Box_g u^n + \Gamma^n_{bc}(u) \partial_\mu u^b \partial^\mu u^c = 0. \end{cases} \tag{1.5}$$

This system of equations is called the wave maps equation.

1.1.3 Equivariant ansatz

The full system of semi-linear wave equations (1.5) can be reduced to a single semi-linear wave equation with a singular non-linear term under the so called equivariant or co-rotational ansatz. To introduce this ansatz, we first need to fix the domain and the target manifolds. Let $(M, g) = (\mathbb{R}^d, g)$ be the Minkowski space and $(N, h) = (\mathbb{S}^d, h)$ the standard round $d-$sphere. In particular, we assume that $d \geq 3$. With respect to spherical coordinates on the Minkowski space

$$(t, r, \omega) \in \mathbb{R} \times [0, \infty) \times [0, \pi)^{d-1} \times [0, 2\pi) \simeq \mathbb{R} \times [0, \infty) \times \mathbb{S}^{d-1} \simeq \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{1+d}$$

the metric on the base manifold is given by

$$g(t, r, \omega) = -dt^2 + dr^2 + r^2 d\omega^2,$$

where $d\omega^2$ stands for the standard round metric on $\mathbb{S}^{d-1}$. In addition, with respect to the hyper-spherical coordinates

$$(\Psi, \Omega) \in [0, \pi) \times [0, \pi)^{d-1} \times [0, 2\pi) \simeq \mathbb{S}^1 \times \mathbb{S}^{d-1} \simeq \mathbb{S}^d$$

the metric on the target is given by

$$h(\Psi, \Omega) = d\Psi^2 + \sin^2(\Psi) d\Omega^2,$$

where $d\Omega^2$ stands for the standard round metric on $\mathbb{S}^{d-1}$. Now, any map from the Minkowski space to the $d-$sphere can be written with respect to these coordinates as

$$u(t,r,\omega) = (\Psi(t,r,\omega), \Omega(t,r,\omega))$$

and the equivariant ansatz suggests

$$\Omega(t,r,\omega) = \omega.$$

Adopting this ansatz, we get

$$\Psi(t,r,\omega) = \psi(t,r)$$

and, most importantly, the wave maps system for functions $u : (\mathbb{R}^{1+d}, g) \longrightarrow (\mathbb{S}^d, h)$ reduces to the single semi-linear wave equation

$$\psi_{tt} - \psi_{rr} - \frac{d-1}{r}\psi_r + \frac{d-1}{2}\frac{\sin(2\psi)}{r^2} = 0. \tag{1.6}$$

Note that the non-linear term involved occurs due to the Christoffel symbols on the $d-$sphere. Due to this singular non-linear term, we ask for the solutions of (1.6) to satisfy the boundary condition $\psi(t,0) = 0$, for all times, so that we can ensure the regularity of the solutions. However, we do not require $\psi_r(t,0) = 0$ due to a special cancellation in the Taylor series expansion.

1.1.4 Blowup solutions

Now, we are interested in the future development of smooth initial data. In other words, we prescribe initial data on the $t = 0$ slice

$$(f,g) = (\psi(0,\cdot), \psi_t(0,\cdot)) \in \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)$$

and consider the Cauchy problem

$$\begin{cases} \psi_{tt}(t,r) - \psi_{rr}(t,r) - \frac{d-1}{r}\psi_r(t,r) = -\frac{d-1}{2}\frac{\sin(2\psi(t,r))}{r^2}, & \text{in } I \times [0,\infty) \\ \psi(0,r) = f(r), \quad \psi_t(0,r) = g(r), & \text{on } \{t=0\} \times [0,+\infty) \\ \psi(t,0) = 0, & \text{on } I \times \{0\}, \end{cases}$$

where $I \subseteq \mathbb{R}$ is an open interval with $0 \in I$. In the study of the Cauchy problem the following questions arise: Does the solution exists? What is the domain in which it is defined? Is the solution unique? Can it become singular in the future? To begin with, we first turn our attention to the symmetries of the equation. Equation (1.6) is invariant under dilations: if $\psi = \psi(t,r)$ is a solution, so is

$$\psi_\lambda(t,r) := \psi\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), \; \lambda > 0. \tag{1.7}$$

Due to this symmetry, it is natural to expect self-similar solutions, that is solutions of the form

$$\psi(t,r) = f\left(\frac{r}{t}\right).$$

Other symmetries of the equation include time translation and reflection symmetry i.e., if $\psi = \psi(t,r)$ is a solution, so is

$$\psi^{\tau}(t,r) := \psi(t+\tau, r), \;\; \tau > 0$$

and

$$\psi^{-}(t,r) := \psi(-t,r),$$

respectively. Taking into account all the symmetries, we see that a generic self-similar solution can be written as

$$\psi(t,r) = f\left(\frac{r}{T-t}\right).$$

Indeed, it is well known that there exist smooth self-similar solutions. To be precise, Shatah [74], Turok-Spergel [89] and Bizoń-Biernat [7] showed that the function

$$\psi^{T}(t,r) := f_0\Big(\frac{r}{T-t}\Big) = 2\arctan\left(\frac{r}{(T-t)\sqrt{d-2}}\right)$$

solves (1.6). However, ψ^T develops a singularity in finite time. Notice that ψ^T is perfectly smooth for all $0 < t < T$ but it breaks down at $t = T$ in the sense that

$$\frac{\partial}{\partial r}\Big|_{r=0}\psi^{T}(t,r) = \frac{2}{\sqrt{d-2}}\frac{1}{T-t} \longrightarrow +\infty, \quad \text{as } t \longrightarrow T^{-}.$$

For small dimensions, $d \in \{3,4,5,6\}$, there exists a sequence of self-similar solutions

$$\left\{ f_n\left(\frac{r}{T-t}\right)\right\}_{n\in\mathbb{N}\cup\{0\}}$$

of (1.6) and ψ^T is the first member of this family, corresponding to $n = 0$, see [3,5]. For this reason, we call ψ^T the "ground-state" solution and it constitutes an one-parameter family of solutions for singularity formation. Furthermore, notice that the break down occurs at a single spacetime event, that is $(T,0)$. By finite speed of propagation and radial symmetry, only information within the backward light-cone

$$C_T := \{(t,r) : 0 < t < T, \;\; 0 \leq r \leq T-t\}$$

can influence this point. Therefore, a natural first step is to restrict our interest to the Cauchy problem

$$\begin{cases} \psi_{tt}(t,r) - \psi_{rr}(t,r) - \frac{d-1}{r}\psi_r(t,r) = -\frac{d-1}{2}\frac{\sin(2\psi(t,r))}{r^2}, & \text{in } C_T \\ \psi(0,r) = f(r), \quad \psi_t(0,r) = g(r), & \text{on } \{t=0\}\times[0,+\infty) \\ \psi(t,0) = 0, & \text{on } (0,T)\times\{0\}. \end{cases} \tag{1.8}$$

1.1.5 Main result

One can use the "ground-state" as initial data to obtain a solution which blows up in finite time as $t \longrightarrow T$. Now, a natural question arises: How generic is this break down? How special are these initial data? Does the singularity occurs only for ψ^T or is it stable with respect to perturbations? The main goal is to establish estimates that prove the latter, namely the existence of an open set of initial data centered at ψ^T which lead to blowup via ψ^T. In other words, we prove the asymptotic non-linear stability of the blowup described by the "ground-state" solution. We formulate our main result in terms of the function ψ as follows.

Theorem 1.1.1 (Chatzikaleas-Donninger-Glogić, [18]) *Fix $T_0 > 0$ and $d \geq 3$ odd. Then there exist constants $M, \delta, \epsilon > 0$ such that for any radial initial data $\psi[0]$ satisfying*

$$\left\| |\cdot|^{-1}\Big(\psi[0] - \psi^{T_0}[0]\Big)\right\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2}_{T_0+\delta})\times H^{\frac{d+1}{2}}(\mathbb{B}^{d+2}_{T_0+\delta})} \leq \frac{\delta}{M}$$

the following statements hold:

1. $T \equiv T_{\psi[0]} \in [T_0 - \delta, T_0 + \delta]$,
2. *the solution* $\psi : C_T \longrightarrow \mathbb{R}$ *satisfies*

$$(T-t)^{k-\frac{d}{2}}\left\| |\cdot|^{-1}\Big(\psi(t,\cdot) - \psi^T(t,\cdot)\Big)\right\|_{\dot{H}^k(\mathbb{B}^{d+2}_{T-t})} \leq \delta(T-t)^{\epsilon}$$

$$(T-t)^{\ell+1-\frac{d}{2}}\left\| |\cdot|^{-1}\Big(\partial_t\psi(t,\cdot) - \partial_t\psi^T(t,\cdot)\Big)\right\|_{\dot{H}^\ell(\mathbb{B}^{d+2}_{T-t})} \leq \delta(T-t)^{\epsilon}$$

   for all $k = 0, 1, 2, \dots, \frac{d+3}{2}$ *and* $\ell = 0, 1, 2 \dots, \frac{d+1}{2}$.

Notice that the Sobolev spaces for the rescaled functions involved in Theorem 1.1.1 are in $d+2$ dimensions. To motivate this, we give an alternative formulation which is more compact and convenient. To do so, we rescale the function ψ and define

$$\chi(t,r) := \frac{1}{r}\psi(t,r).$$

Then, the Cauchy problem transforms into

$$\begin{cases} \chi_{tt}(t,r) - \chi_{rr}(t,r) - \frac{d+1}{r}\chi_r(t,r) = -\frac{d-1}{2}\frac{\sin(2r\chi(t,r)) - 2r\chi(t,r)}{r^3}, & \text{in } C_T \\ \chi(0,r) = \frac{f(r)}{r}, \quad \chi_t(0,r) = \frac{g(r)}{r}, & \text{on } \{t=0\}\times[0,+\infty). \end{cases} \tag{1.9}$$

Note that, since $r\chi(t,r) = \mathcal{O}(r)$ as $r \to 0^+$, the nonlinearity is a smooth function and the radial Laplacian is in $d+2$ dimensions. We also rescale the ground-state solution

$$\chi^T(t,r) := \frac{1}{r}\psi^T(t,r) = \frac{2}{r}\arctan\left(\frac{r}{(T-t)\sqrt{d-2}}\right)$$

and write

$$\chi[t] = (\chi(t,\cdot), \partial_t \chi(t,\cdot))$$

for convenience. Now, Theorem 1.1.1 can be formulated in terms of the variable χ as follows.

Theorem 1.1.2 (Alternative formulation of Theorem 1.1.1, [18]) *Let $d \geq 3$ be an odd integer and fix $T_0 > 0$. There exist constants $M, \delta, \epsilon > 0$ such that for any radial initial data*

$$\chi[0] \in H^{\frac{d+3}{2}}(\mathbb{B}^{d+2}_{T_0+\delta}) \times H^{\frac{d+1}{2}}(\mathbb{B}^{d+2}_{T_0+\delta})$$

satisfying

$$\left\| \chi[0] - \chi^{T_0}[0] \right\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2}_{T_0+\delta}) \times H^{\frac{d+1}{2}}(\mathbb{B}^{d+2}_{T_0+\delta})} \leq \frac{\delta}{M}$$

the following statements hold:

1. $T \equiv T_{\chi[0]} \in [T_0 - \delta, T_0 + \delta]$,
2. *the solution $\chi : C_T \longrightarrow \mathbb{R}$ satisfies the estimates*

$$(T-t)^{k-\frac{d}{2}} \left\| \chi(t,\cdot) - \chi^T(t,\cdot) \right\|_{\dot{H}^k(\mathbb{B}^{d+2}_{T-t})} \leq \delta (T-t)^{\epsilon},$$
$$(T-t)^{\ell+1-\frac{d}{2}} \left\| \partial_t \chi(t,\cdot) - \partial_t \chi^T(t,\cdot) \right\|_{\dot{H}^\ell(\mathbb{B}^{d+2}_{T-t})} \leq \delta (T-t)^{\epsilon},$$

for all $k = 0, 1, 2, \dots, \frac{d+3}{2}$ and $\ell = 0, 1, 2 \dots, \frac{d+1}{2}$.

1.1.6 Outline of the proof

The proof of Theorem 1.1.1 is contained in Chapter 2. Essentially, it is based on suitable perturbation theory around the rescaled ground-state solution χ^T. In other words, we are interested in the evolution of the rescaled perturbation

$$\phi(t,r) := \chi(t,r) - \chi^T(t,r).$$

However, plugging this ansatz into (1.9) yields a second order partial differential equation with respect to the variable ϕ with $T-$dependent coefficients. For this reason, we switch to similarity coordinates. This is a new coordinate system $(t,r) \mapsto \mu(t,r) =: (\tau, \rho)$ which maps the backward light-cone

$$C_T := \{(t,r) : 0 < t < T,\ 0 \leq r \leq T - t\}$$

to the cylinder

$$\mathcal{C} := \{(\tau, \rho) : 0 < \tau < +\infty,\ 0 \leq \rho \leq 1\}.$$

Recall that the break down occurs at $(T,0)$. In particular, this spacetime event is mapped to $\tau=\infty$. Now, we get a second order partial differential equation with respect to the variable $\phi\circ\mu^{-1}$ with $T-$independent coefficients and the blowup time T appears only in the initial data. Then, we transform the second order partial differential equation for the rescaled perturbation $\phi\circ\mu^{-1}$ into a first-order vector-valued evolution equation. We get

$$\begin{cases} \partial_\tau \mathbf{\Phi}(\tau)=\widetilde{\mathbf{L}}\mathbf{\Phi}(\tau)+\mathbf{N}\big(\mathbf{\Phi}(\tau)\big), & \text{for } \tau\in(0,+\infty) \\ \mathbf{\Phi}(0)=\mathbf{U}(\mathbf{v},T), \end{cases}$$

where the linear operator consists of two parts

$$\widetilde{\mathbf{L}}:=\widetilde{\mathbf{L}}_0+\mathbf{L}'.$$

Here, $\widetilde{\mathbf{L}}_0$ stands for the free wave operator, see (2.9), whereas $\mathbf{L}'$ is a compact perturbation containing the linear terms produced from the linearization around the rescaled blowup solution χ^T, see (3.18). The desired estimates in Theorem 1.1.2 follow from a fixed point argument. However, to make the fixed point argument feasible, we must ensure the decay of the solutions.

First, we focus on the free wave operator and study the evolution equation

$$\partial_\tau \mathbf{\Phi}(\tau)=\widetilde{\mathbf{L}}_0\mathbf{\Phi}(\tau).$$

However, the energy norm is not the right candidate to ensure the decay of the solutions for this problem. This fact is a manifestation of the energy-supercritical character of the problem. To explain what this means, we write

$$\psi[t]:=(\psi(t,\cdot),\partial_t\psi(t,\cdot))$$

and recall that equation (1.6) is invariant under dilations: the functions

$$\psi_\lambda(t,r):=\psi\left(\frac{t}{\lambda},\frac{r}{\lambda}\right),\ \lambda>0$$

are solutions provided that $\psi=\psi(t,r)$ is a solution and the scaling property holds

$$\|\psi_\lambda(t,\cdot)\|_{\dot H^s(\mathbb{R}^d)}=\lambda^{\frac{d}{2}-s}\left\|\psi\left(\frac{t}{\lambda},\cdot\right)\right\|_{\dot H^s(\mathbb{R}^d)}.$$

This property defines the space

$$\dot H^{s_c}(\mathbb{R}^d)\times\dot H^{s_c-1}(\mathbb{R}^d),\quad s_c:=\frac{d}{2}$$

as the critical Sobolev space, that is the unique L^2-based homogeneous Sobolev space preserved by this scaling. On the other hand, for any Schwartz function ψ, multiplying (1.6) by $r^{d-1}\psi_t(t,r)$ and integrating by parts yields that the energy

$$E(\psi[t]):=\frac{1}{2}\int_0^\infty\left(\psi_t^2+\psi_r^2+(d-1)\frac{\sin^2(\psi)}{r^2}\right)r^{d-1}dr$$

is conserved in time. Now, the energy defines the energy space

$$\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$$

that is, the space of initial data for which the energy is known to be finite. Indeed, for any $\psi[t]$ in the energy space, the first two terms involved in the energy are obviously bounded. For the third term, we use the condition $\psi(t,0) = 0$ and Hardy's inequality to infer

$$\begin{aligned}\int_0^\infty \frac{\sin^2(\psi(t,r))}{r^2} r^{d-1} dr &\simeq \int_{\mathbb{R}^d} \left(\frac{\sin(\psi(t,|x|))}{|x|} \right)^2 dx \\ &\leq \int_{\mathbb{R}^d} \left(\frac{\psi(t,|x|)}{|x|} \right)^2 dx \\ &\lesssim \int_{\mathbb{R}^d} |\nabla_x \psi(t,|x|)|^2 dx,\end{aligned}$$

which is finite for $\psi(t,\cdot) \in \dot{H}^1(\mathbb{R}^d)$. We call the equation (1.6) energy-supercritical if the critical regularity $s_c = \frac{d}{2}$ is larger than the energy-critical regularity $s_e = 1$. Obviously, our initial restriction on d is equivalent to the the validity of this condition,

$$d \geq 3 \iff s_c > s_e.$$

A consequence of the energy-supercriticality is the fact that the energy norm is too weak to detect the self-similar blowup. To illustrate this phenomenon, we follow [34] and add the term $(d-1)\frac{\psi}{r^2}$ to both sides of (1.6) to smooth out the non-linearity

$$\psi_{tt} - \psi_{rr} - \frac{d-1}{r}\psi_r + (d-1)\frac{\psi}{r^2} = -\frac{d-1}{2}\frac{\sin(2\psi) - 2\psi}{r^2}$$

and we are left with the free wave equation

$$\psi_{tt} - \psi_{rr} - \frac{d-1}{r}\psi_r + (d-1)\frac{\psi}{r^2} = 0. \tag{1.10}$$

Now, for any Schwartz function ψ, multiply (1.10) with $r^{d-1}\psi_t$ and integrate by parts. We infer that the energy

$$E(\psi[t]) := \frac{1}{2}\int_0^\infty \left(\psi_t^2 + \psi_r^2 + (d-1)\frac{\psi^2}{r^2} \right) r^{d-1} dr$$

is conserved in time. As before, the energy space is

$$\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d).$$

We define the local energy

$$E_{\text{loc}}(\psi[t]) := \frac{1}{2}\int_0^{T-t} \left(\psi_t^2 + \psi_r^2 + (d-1)\frac{\psi^2}{r^2} \right) r^{d-1} dr$$

and compute

$$
\begin{aligned}
\left\|\psi_t^T(t,\cdot)\right\|_{L^2(\mathbb{B}^d_{T-t})} &= \left\| f_0'\left(\frac{|\cdot|}{T-t}\right)\frac{|\cdot|}{(T-t)^2}\right\|_{L^2(\mathbb{B}^d_{T-t})} \\
&= (T-t)^{\frac{d}{2}-1}\left\||\cdot| f_0'(|\cdot|)\right\|_{L^2(\mathbb{B}^d_1)} \\
&\simeq (T-t)^{\frac{d}{2}-1}, \\
\left\|\psi^T(t,\cdot)\right\|_{\dot{H}^1(\mathbb{B}^d_{T-t})} &= \left\| f_0\left(\frac{|\cdot|}{T-t}\right)\right\|_{\dot{H}^1(\mathbb{B}^d_{T-t})} \\
&= (T-t)^{\frac{d}{2}-1}\left\| f_0(|\cdot|)\right\|_{\dot{H}^1(\mathbb{B}^d_1)} \\
&\simeq (T-t)^{\frac{d}{2}-1}, \\
\left\|\frac{\psi^T(t,\cdot)}{|\cdot|}\right\|_{L^2(\mathbb{B}^d_{T-t})} &= \left\|\frac{1}{|\cdot|} f_0\left(\frac{|\cdot|}{T-t}\right)\right\|_{L^2(\mathbb{B}^d_{T-t})} \\
&= (T-t)^{\frac{d}{2}-1}\left\|\frac{1}{|\cdot|} f_0(|\cdot|)\right\|_{L^2(\mathbb{B}^d_1)} \\
&\simeq (T-t)^{\frac{d}{2}-1}.
\end{aligned}
$$

Hence,

$$
E_{\text{loc}}(\psi^T[t]) \simeq \left\|\psi_t^T(t,\cdot)\right\|^2_{L^2(\mathbb{B}^d_{T-t})} + \left\|\psi^T(t,\cdot)\right\|^2_{\dot{H}^1(\mathbb{B}^d_{T-t})} + \left\|\frac{\psi^T(t,\cdot)}{|\cdot|}\right\|^2_{L^2(\mathbb{B}^d_{T-t})} \simeq (T-t)^{d-2}.
$$

Since $d \geq 3$, our self-similar blowup solution ψ^T does not blowup in this norm and, on the contrary, it decays, as $t \longrightarrow T^-$. On the other hand, the local energy of any solution to the free wave maps equation (1.10) does not decay. For this fact also, we refer the reader to [34] and we reproduce the result here for sake of completeness. Indeed, we fix a solution ψ to the free wave equation and use the fact that the energy is conserved to obtain

$$
E_{\text{loc}}(\psi[t]) \leq E(\psi[t]) \lesssim 1.
$$

However, this result cannot be strengthened. As pointed out in [34],

$$
\forall \epsilon > 0, \exists \text{ a solution } \psi_\epsilon \text{ to } (1.10) \ : E_{\text{loc}}(\psi_\epsilon[t]) \simeq (T-t)^\epsilon
$$

and consequently

$$
\nexists \gamma > 0 : E_{\text{loc}}(\psi[t]) \lesssim (T-t)^\gamma, \ \forall \text{ solutions } \psi \text{ to } (1.10).
$$

In conclusion, the self-similar blowup solution ψ^T does not blowup in the energy norm, the local energy of ψ^T decays while the same energy of any solution to the free wave equation only stays bounded.

Another way to interpret the energy-supercritical character of the problem is to consider the conserved energy of the original equation (1.6), namely

$$E(\psi[t]) := \int_0^\infty \left(\psi_t^2 + \psi_r^2 + (d-1)\frac{\sin^2(\psi)}{r^2} \right) r^{d-1} dr,$$

for any Schwartz function ψ. As before, the energy space is

$$\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d).$$

However, the minimum regularity required on the initial data

$$(f,g) = (\psi(0,\cdot), \psi_t(0,\cdot)) \in \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)$$

to ensure local well-posedness is

$$s > \frac{d}{2}.$$

For this result, we refer the reader to the works of Klainerman-Machedon [47], Klainerman-Selberg [49] and Keel-Tao [44] for $d \geq 3$, $d = 2$, $d = 1$ respectively. Therefore, for $d \geq 3$, the problem is ill-posed at the energy regularity $s_e = 1$ and consequently the energy cannot be used to control the evolution.

Consequently, we need a stronger topology to detect the blowup. In other words, we need to find a norm such that

$$\|\psi[t]\| \longrightarrow \infty, \quad \text{as } t \longrightarrow T^-.$$

Furthermore, it would be advantageous if this norm follows form a suitable conserved quantity. To motivate the choice of the suitable inner product, we refer the reader to the works of Donninger-Schörkhuber [32] and Donninger-Schörkhuber-Aichelburg [34] and reproduce the result here for sake of completeness. According to [32, 34], the main idea is to map the free part of the equivariant wave maps equation

$$\psi_{tt} - \psi_{rr} - \frac{d-1}{r}\psi_r + (d-1)\frac{\psi}{r^2} = 0$$

to the one-dimensional wave equation

$$\hat{\psi}_{tt} - \hat{\psi}_{rr} = 0. \tag{1.11}$$

This can be done via the transformation

$$\hat{\psi} := D_d \psi := \left(\frac{1}{r}\frac{\partial}{\partial r} \right)^{\frac{d-1}{2}} \left(r^{d-1}\psi \right),$$

see [34] for $d = 3$ and [32] for $d \geq 3$. Now, (1.11) has the conserved energy

$$\hat{E}(\hat{\psi}[t]) := \int_0^\infty \left(\hat{\psi}_t^2 + \hat{\psi}_r^2\right) dr$$

and the local energy

$$\hat{E}_{\text{loc}}(\hat{\psi^T}[t]) := \int_0^{T-t} \left(\hat{\psi^T}_t^2 + \hat{\psi^T}_r^2\right) dr$$

blows up as $t \longrightarrow T$. For more details, see [34] for $d = 3$ and [32] for $d \geq 3$. Now, this "higher energy norm" yields the desired decay for the solutions to the free wave maps equation, see Proposition 1.1.3 below. Then, we turn our attention to the full linear problem

$$\partial_\tau \mathbf{\Phi}(\tau) = \left(\widetilde{\mathbf{L}}_0 + \mathbf{L}'\right) \mathbf{\Phi}(\tau),$$

prove that the operator $\mathbf{L}'$ is in fact a compact perturbation, and that the solution to the full linear evolution exists in the backward light-cone. We summarize these results in the following proposition.

Proposition 1.1.3 (Proposition 2.4.1, [18]) *The operator $\widetilde{\mathbf{L}}_0 : \mathcal{D}(\widetilde{\mathbf{L}}_0) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is closable and its closure $\mathbf{L}_0 : \mathcal{D}(\mathbf{L}_0) \subset \mathcal{H} \longrightarrow \mathcal{H}$ generates a strongly continuous one-parameter semigroup $(\mathbf{S}_0(\tau))_{\tau \geq 0}$ of bounded operators on $\mathcal{H}$ satisfying the growth estimate*

$$\|\mathbf{S}_0(\tau)\| \leq M e^{-\tau} \tag{1.12}$$

for all $\tau \geq 0$ and some constant $M \geq 1$. In addition, the operator $\mathbf{L} := \mathbf{L}_0 + \mathbf{L}' : \mathcal{D}(\mathbf{L}) \subset \mathcal{H} \longrightarrow \mathcal{H}$, $\mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_0)$, is the generator of a strongly continuous semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$ on $\mathcal{H}$ and $\mathbf{L}' : \mathcal{H} \to \mathcal{H}$ is compact.

The proof of this result relies on the Lumer-Phillips theorem (Theorem 3.15, page 83, [36]) and the bounded perturbation theorem (Theorem 1.3, page 158, [36]). Using the result of Proposition 1.1.3, together with Hadamard's equality (Theorem 1.10, p. 55, [36]), we can now locate the spectrum of free linear operator. We obtain

$$\sigma(\mathbf{L}_0) \subseteq \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -1\} \tag{1.13}$$

and see that it consists only of "stable" spectrum points, that is spectrum points with strictly negative real part. On the other hand, the bounded perturbation theorem yields a growth estimate also for the solution operator to the full linear evolution, namely

$$\|\mathbf{S}(\tau)\| \leq M e^{(-1+M\|\mathbf{L}'\|)\tau},$$

for all $\tau \geq 0$ and some constant $M \geq 1$, and Hadamard's equality implies

$$\sigma(\mathbf{L}_0 + \mathbf{L}') \subseteq \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -1 + M\|\mathbf{L}'\|\}.$$

Contrary to (1.13), such a result does not guarantee the decay of the solutions to the full linear problem and hence it is not useful for our purposes.

For this reason, we focus on the spectrum of the generator $\mathbf{L} := \mathbf{L}_0 + \mathbf{L}'$ and note that the full linear operator is highly nonself-adjoint. Consequently, the spectral analysis needed here requires advanced tools from ordinary differential equations as well as asymptotic resolvent estimates. First, we consider the point spectrum and prove the following result.

Proposition 1.1.4 (Proposition 2.4.2, [18]) *We have*

$$\sigma_p(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\} \cup \{1\}.$$

We note that the "unstable" eigenvalue $\lambda = 1$ occurs due the time translation symmetry. Our proof heavily relies on the works of Costin-Donninger-Xia [19] and Costin-Donninger-Glogić [20]. Then, we use the fact that $\mathbf{L}'$ is compact and semigroup theory to pass from the point spectrum to the whole spectrum.

Lemma 1.1.5 (Corollary 2.4.3, [18]) *We have*

$$\sigma(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda < 0\} \cup \{1\}.$$

To establish the desired decay for the solutions to the full linear operator we must ensure that the distance

$$d\left(\sigma(\mathbf{L}), i\mathbb{R}\right) := \inf\left\{|\lambda - \zeta| : \ \lambda \in \sigma(\mathbf{L}), \ \zeta \in i\mathbb{R}\right\},$$

where $i\mathbb{R}$ stands for the imaginary axis, is uniformly bounded from below by a strictly positive number. To this end, we continue our analysis on the spectrum of $\mathbf{L}$ and show absence of spectral points wihtin the region

$$\Omega_{\epsilon,R} := \{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda \geq -1 + \epsilon, \ |\lambda| \geq R\}$$

for $\epsilon > 0$ sufficiently small and $R > 0$ sufficiently large.

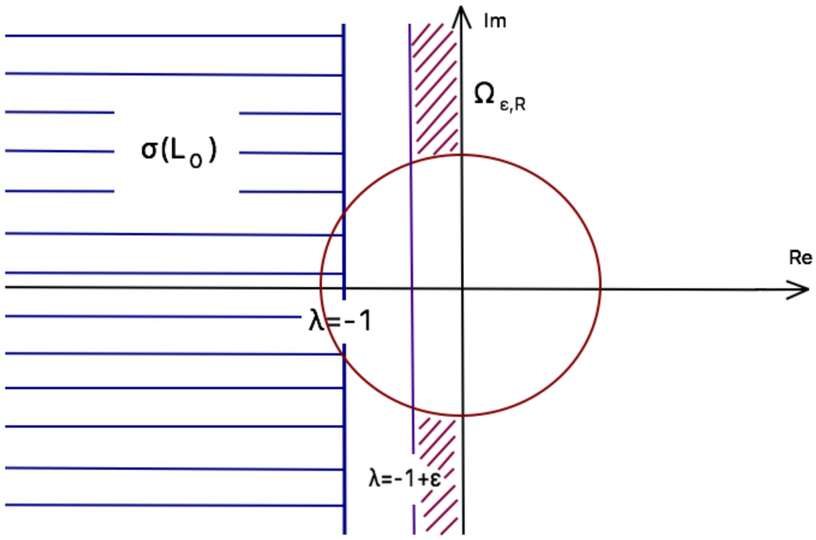

Figure 1.1: The set $\Omega_{\epsilon,R}$.

Such a result follows immediately once we have a uniform bound for the resolvent operator associated to $\mathbf{L}$ on $\Omega_{\epsilon,R}$.

Proposition 1.1.6 (Proposition 2.4.4, [18]) *Let $\epsilon > 0$. Then there exist constants $R_\epsilon, C_\epsilon > 0$ such that the resolvent $\mathbf{R_L}$ exists on $\Omega_{\epsilon,R_\epsilon}$ and satisfies*

$$\|\mathbf{R_L}(\lambda)\| \leq C_\epsilon$$

for all $\lambda \in \Omega_{\epsilon,R_\epsilon}$.

Therefore, for sufficiently small $\epsilon > 0$ and sufficiently large $R > 0$, we have

$$\sigma(\mathbf{L}) \subseteq \mathbb{C} \setminus \Omega_{\epsilon,R_\epsilon}.$$

Now, recall Proposition 1.1.3. The full linear operator $\mathbf{L}$ is closed and consequently $\sigma(\mathbf{L})$ is a closed set. Hence, there exists a sufficiently small $\epsilon > 0$ such that

$$d\left(\sigma(\mathbf{L}), i\mathbb{R}\right) \geq d\left(\sigma(\mathbf{L}), \{it : t \in [-R_\epsilon, R_\epsilon]\}\right) \geq \epsilon > 0.$$

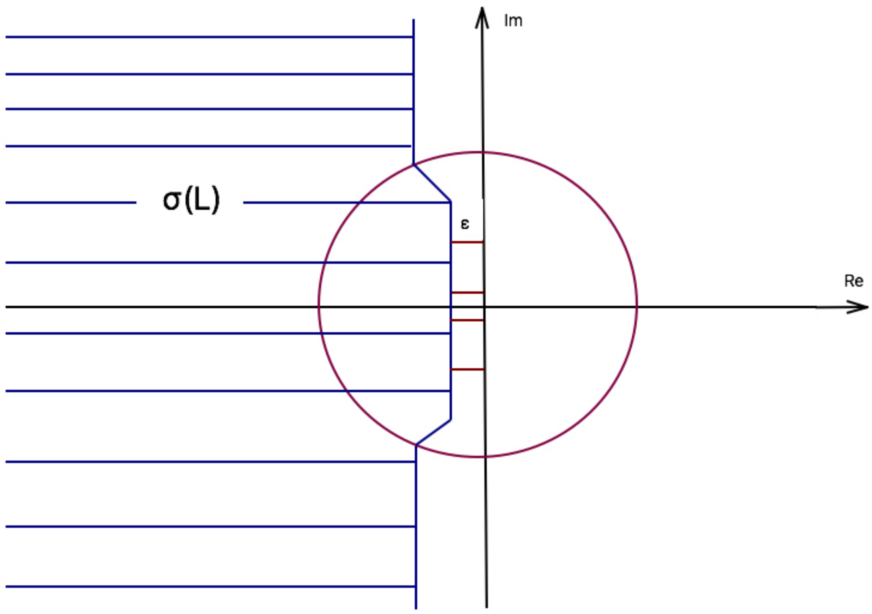

Figure 1.2: Gap between $\sigma(\mathbf{L})$ and the imaginary axis

Finally, we obtain

$$\sigma(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda \leq -\epsilon\} \cup \{1\},$$

for some fixed and sufficiently small $\epsilon > 0$.

Due to the "unstable" eigenvalue $\lambda = 1$, a subspace of the initial data will lead to a solution to the full linear evolution generated by $\mathbf{L}$ which grows exponentially in time whereas all the other initial data will lead to exponential decay. In the following, we study the time evolution of the linearized equation and we prove this result rigorously. First, we introduce a (non-orthogonal) Riesz projection $\mathbf{P}$ on the space of initial data $\mathcal{H}$,

$$\mathbf{P} : \mathcal{H} \longrightarrow \mathcal{H}, \quad \mathbf{P} := \frac{1}{2\pi i} \int_\gamma \mathbf{R}_\mathbf{L}(\mu) d\mu,$$

where $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ is a fixed positively orientated circle around $\lambda = 1$ with sufficiently small radius so that $\gamma\big([0, 2\pi]\big) \subseteq \rho(\mathbf{L})$, see [43]. This projection splits the Hilbert space of initial data into the unstable $\mathrm{rg}\,\mathbf{P}$ and the stable $\mathrm{rg}\,(1 - \mathbf{P})$ space,

$$\mathcal{H} = \mathrm{rg}\,\mathbf{P} \oplus \mathrm{rg}\,(1 - \mathbf{P}).$$

Recall that the full linear operator $\mathbf{L}$ is highly nonself-adjoint. Hence, we do not know appriori that $\mathbf{g}$, the eigenfunction associated to the isolated eigenvalue $\lambda = 1$, is the only unstable direction in $\mathcal{H}$. This fact together with growth estimates on the stable and unstable spaces is the result of the following proposition.

Proposition 1.1.7 (Proposition 2.4.5, [18]) *There exists a projection*

$$\mathbf{P} \in \mathcal{B}(\mathcal{H}), \quad \mathbf{P} : \mathcal{H} \longrightarrow \langle \mathbf{g} \rangle,$$

which commutes with the semigroup $\big(\mathbf{S}(\tau)\big)_{\tau \geq 0}$. *In addition, we have*

$$\mathbf{S}(\tau)\mathbf{P}\mathbf{f} = e^{\tau}\mathbf{P}\mathbf{f}, \tag{1.14}$$

and there exist constants $C, \epsilon > 0$ *such that*

$$\|(1-\mathbf{P})\mathbf{S}(\tau)\mathbf{f}\| \leq Ce^{-\epsilon\tau}\|(1-\mathbf{P})\mathbf{f}\|, \tag{1.15}$$

for all $\mathbf{f} \in \mathcal{H}$ *and* $\tau \geq 0$.

Next, we focus on the non-linear evolution

$$\partial_\tau \mathbf{\Phi}(\tau) = \widetilde{\mathbf{L}}\mathbf{\Phi}(\tau) + \mathbf{N}\big(\mathbf{\Phi}(\tau)\big)$$

and formulate this problem as an anstract integral equation via Duhamel's principle

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}(\tau - s)\mathbf{N}\big(\Phi(s)\big)ds. \tag{1.16}$$

For the purposes of the fixed point argument, we introduce the Banach space

$$\mathcal{X} := \{\Phi \in C([0,\infty);\mathcal{H}) : \quad \|\Phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{\epsilon\tau}\|\Phi(\tau)\| < +\infty\}$$

and the closed ball of radius δ in $\mathcal{X}$,

$$\mathcal{X}_\delta = \left\{\Phi \in C([0,\infty);\mathcal{H}) : \|\Phi(\tau)\| \leq \delta e^{-\epsilon\tau}, \quad \forall \tau > 0\right\}.$$

Notice that ϵ is the decay rate from Proposition 1.1.7. However, due to the one-dimensional subspace $\langle \mathbf{g} \rangle$ from which solutions to the linear problem grow exponentially, a fixed point argument to (1.16) is hopeless. For this reason, we change the initial data

$$\mathbf{u} \longmapsto \mathbf{u}^\star := \mathbf{u} - \mathbf{C}(\Phi, \mathbf{u}).$$

Here, $\mathbf{u}^\star$ is a carefully chosen element defined by subtracting the correction term

$$\mathbf{C}(\Phi,\mathbf{u}) := \mathbf{P}\left(\mathbf{u} + \int_0^\infty e^{-s}\mathbf{N}\big(\Phi(s)\big)ds\right)$$

from the original data. Moreover, notice that this correction term delongs to the unstable space $\operatorname{rg}\mathbf{P}$ and therefore $\mathbf{u}^\star$ is a suitable candidate to stabilize the evolution. Now, we consider the modified integral equation

$$\Phi(\tau) = \mathbf{K}(\Phi,\mathbf{u})(\tau) \tag{1.17}$$

where

$$\mathbf{K}(\Phi,\mathbf{u})(\tau) := \mathbf{S}(\tau)\mathbf{u}^\star + \int_0^\tau \mathbf{S}(\tau - s)\mathbf{N}\big(\Phi(s)\big)ds$$

and together with Lipschitz-type estimates for the non-linear term (which follow from Moser's inequality, see Lemma 2.4.6, [18]) we prove the following result.

Theorem 1.1.8 (Theorem 2.4.7, [18]) *There exist constants $\delta, C > 0$ such that for every $\mathbf{u} \in \mathcal{H}$ with $\|\mathbf{u}\| \leq \frac{\delta}{C}$, there exists a unique $\mathbf{\Phi}(\mathbf{u}) \in \mathcal{X}_\delta$ that satisfies*

$$\mathbf{\Phi}(\mathbf{u}) = \mathbf{K}(\mathbf{\Phi}(\mathbf{u}), \mathbf{u}).$$

Finally, we turn our attention to the initial data we prescribe, $\mathbf{\Phi}(0) = \mathbf{U}(\mathbf{v}, T)$. First, we prove that the original initial data are small provided that the perturbed rescaled initial data

$$|\cdot|^{-1}\mathbf{v} := \frac{1}{|\cdot|}\begin{pmatrix} F \\ G \end{pmatrix} = \frac{1}{|\cdot|}\begin{pmatrix} f - \psi^{T_0}(0,\cdot) \\ g - \partial_0\psi^{T_0}(0,\cdot) \end{pmatrix}$$

are sufficiently small.

Lemma 1.1.9 (Lemma 2.4.8, [18]) *Fix $T_0 > 0$. Let $\delta > 0$ be sufficiently small and $\mathbf{v}$ with $|\cdot|^{-1}\mathbf{v} \in \mathcal{H}^{T_0+\delta}$. Then, the map*

$$\mathbf{U}(\mathbf{v}, \cdot) : [T_0 - \delta, T_0 + \delta] \longrightarrow \mathcal{H}, \quad T \longmapsto \mathbf{U}(\mathbf{v}, T)$$

is continuous. Furthermore, for all $T \in [T_0 - \delta, T_0 + \delta]$,

$$\big\||\cdot|^{-1}\mathbf{v}\big\|_{\mathcal{H}^{T_0+\delta}} \leq \delta \Longrightarrow \big\|\mathbf{U}(\mathbf{v}, T)\big\| \lesssim \delta.$$

Second, given $T_0 > 0$, sufficiently small $\delta > 0$ and any $T \in [T_0 - \delta, T_0 + \delta]$, we can apply Theorem 1.1.8 to $\mathbf{u} = \mathbf{U}(\mathbf{v}, T)$. For all $T \in [T_0 - \delta, T_0 + \delta]$, we get a unique solution to the modified integral equation (1.17). Now, we look at the correction term and use an additional fixed point argument to show that

$$\exists T_{\mathbf{v}} \in [T_0 - \delta, T_0 + \delta] : \quad \mathbf{C}(\Phi_{T_{\mathbf{v}}}, \mathbf{U}(\mathbf{v}, T_{\mathbf{v}})) = 0, \tag{1.18}$$

see (3.61), and the discussion thereafter. In summary, given $T_0 > 0$, sufficiently small $\delta > 0$ and sufficiently large $M > 0$, we have

$$\begin{aligned} \||\cdot|^{-1}\mathbf{v}\| \leq \frac{\delta}{M} &\xRightarrow{\text{Lemma 1.1.9}} \|\mathbf{U}(\mathbf{v}, T)\| \leq \frac{\delta}{C}, \ \forall T \in [T_0 - \delta, T_0 + \delta] \\ &\xRightarrow{\text{Theorem 1.1.8}} \exists! \Phi_T := \Phi(\mathbf{U}(\mathbf{v}, T)) \in \mathcal{X}_\delta \text{ to } (1.17), \ \forall T \in [T_0 - \delta, T_0 + \delta] \\ &\xRightarrow{(1.18)} \exists! \Phi_{T_{\mathbf{v}}} := \Phi(\mathbf{U}(\mathbf{v}, T_{\mathbf{v}})) \in \mathcal{X}_\delta \text{ to } (1.16). \end{aligned}$$

The desired estimates for $\chi(t,\cdot) - \chi^T(t,\cdot)$ and $\partial_t\chi(t,\cdot) - \partial_t\chi^T(t,\cdot)$ follow immediately from the fact that $\Phi(\mathbf{U}(\mathbf{v}, T_{\mathbf{v}})) \in \mathcal{X}_\delta$. This concludes the proof.

1.2 Cubic wave equation

In the second part, we focus on the semi-linear wave equation with a focusing nonlinearity in higher space dimensions without symmetry assumptions on the data. We consider the

Minkowski spacetime $(\mathbb{R}^{1+d}, \eta)$ endowed with the standard Minkowski metric

$$\begin{aligned}\eta(t,x) &:= -dt^2 + |dx|^2 \\ &:= -dt^2 + \sum_{i=1}^{d} (dx^i)^2\end{aligned}$$

with respect to the Cartesian coordinates $(x^0 = t, x^1, \dots, x^d) \in \mathbb{R}^{1+d}$ and study the wave equation

$$\Box_\eta u(t,x) = -\left|u(t,x)\right|^{p-1} u(t,x), \tag{1.19}$$

with $(t,x) \in I \times \mathbb{R}^d$. Here, I is an open interval in $\mathbb{R}$, $p > 1$ and $\Box_\eta$ stands for the Laplace-Beltrami operator with respect to the Minkowski metric η on $\mathbb{R}^{1+d}$,

$$\begin{aligned}\Box_\eta &:= -\partial_t^2 + \Delta_x \\ &:= -\partial_t^2 + \sum_{i=1}^{d} \partial_{x^i}^2.\end{aligned}$$

Equation (1.19) is invariant under conformal transformations [38] if and only if it is of the form

$$\Box_\eta u(t,x) = -\left|u(t,x)\right|^{\frac{4}{d-1}} u(t,x),$$

which defines the conformal exponent

$$p_c := \frac{d+3}{d-1},$$

see [38], and distinguishes the following three cases: we call equation (1.19) subconformal, conformal, or superconformal if $p < p_c$, $p = p_c$, or $p > p_c$, respectively. Furthermore, equation (1.19) is invariant under the following scaling: if $u = u(t,x)$ is a solution, so is

$$u_\lambda(t,x) := \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad \lambda > 0 \tag{1.20}$$

and the scaling property holds

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{\frac{d}{2}-\frac{2}{p-1}-s} \left\| u\left(\frac{t}{\lambda}, \cdot\right)\right\|_{\dot{H}^s(\mathbb{R}^d)}.$$

This property defines the critical Sobolev space

$$\dot{H}^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c-1}(\mathbb{R}^d)$$

where

$$s_c := \frac{d}{2} - \frac{2}{p-1},$$

that is the unique L^2-based homogeneous Sobolev space preserved by this scaling. As before, we write

$$u[t] := (u(t,\cdot), \partial_t u(t,\cdot))$$

for convenience. In addition, for any Schwartz function u, multipling (1.19) by $\partial_t u$ and integrating by parts yields that the energy

$$E(u[t]) := \frac{1}{2}\int_{\mathbb{R}^d} \left(|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2\right) dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1}\, dx. \tag{1.21}$$

is conserved in time. The energy defines the energy space

$$\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \tag{1.22}$$

that is the space of initial data for which the energy is finite. We call equation (1.19) energy-subcritical, critical, or supercritical if the critical regularity is smaller, equal or larger than the energy regularity, namely if $s_c < 1$, $s_c = 1$, or $s_c > 1$, respectively. This is equivalent to $p < p_e$, $p = p_e$, or $p > p_e$, respectively, where

$$p_e := \frac{d+2}{d-2}$$

is the unique exponent for which the energy (1.21) is invariant under the scaling (1.20). In the following, we are interested in the superconformal, energy supercritical wave equation, i.e.,

$$(p,d) \in \mathscr{A}$$

where

$$\mathscr{A} := \left\{(p,d) \in [1,\infty) \times \mathbb{N} : p > p_c := \frac{d+3}{d-1}, \quad p > p_e := \frac{d+2}{d-2}\right\}.$$

For simplicity, we set

$$\boxed{p = 3}$$

and study the equation

$$\Box_\eta u(t,x) = -u^3(t,x), \tag{1.23}$$

for $(t,x) \in I \times \mathbb{R}^d$. Note that $(3,d) \in \mathscr{A}$ if and only if

$$\boxed{d \geq 5}.$$

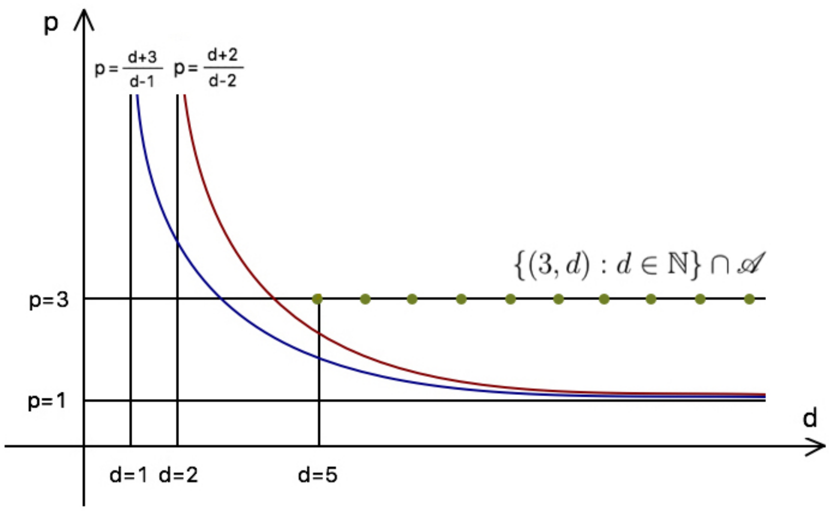

Figure 1.3: The set $\{(3,d) : d \in \mathbb{N}\} \cap \mathscr{A}$.

1.2.1 Blowup solutions

Equation (1.23) admits smooth and compactly supported solutions which blowup in finite time. To construct such blowup solutions, we look for $x-$independent solutions and plug the ansatz

$$u(t,x) = v(t)$$

into (1.23). We obtain the ordinary differential equation

$$\frac{d^2}{dt^2} v(t) = v^3(t)$$

which can be solved explicitly and generates the solution

$$u_1(t,x) = \frac{\sqrt{2}}{1-t}.$$

Obviously, u_1 breaks down at $t = 1$. Now, we use the symmetries of the equation to obtain a much larger family of blowup solutions. Observe that (1.23) enjoys time translation symmetry as well as time reflection symmetry, that is, if $u = u(t,x)$ is a solution, so are

$$u^\tau(t,x) = u(t+\tau, x),$$

for all $\tau > 0$, and

$$u^-(t,x) := u(-t,x),$$

respectively. Hence, for $T \in \mathbb{R}$, the function

$$u_T(t,x) = \frac{\sqrt{2}}{T-t} \tag{1.24}$$

defines a one-parameter family of solutions which blowup at (T, x_0), for all $x_0 \in \mathbb{R}^d$. When studying the evolution in a backward light-cone, we can assume, that the tip of the cone is $(T, 0)$ due to the space translation symmetry i.e., if $u = u(t,x)$ is a solution, so is

$$u^y(t,x) = u(t, x+y),$$

for any fixed $y \in \mathbb{R}^d$. In addition, (1.23) is invariant under the Lorentz transformations

$$\Lambda_T(\alpha) : \mathbb{R}^{1+d} \longrightarrow \mathbb{R}^{1+d},$$

for any $\alpha \in \mathbb{R}^d$. These transformations are similar to rotations in the $d-$dimensional Euclidean space but in the context of a $(1+d)-$dimensional Lorentzian spacetime. In particular, the spacetime event $(T,0)$ is a fixed point and light-cones with vertex $(T,0)$,

$$C_{(T,0)} := \left\{(t,x) \in [0,T) \times \mathbb{R}^d : |x| \leq T - t\right\},$$

remain invariant. For any fixed "angle" of rotation $\alpha = (\alpha^1, \dots, \alpha^d) \in \mathbb{R}^d$ and for all $j = 1, 2, \dots, d$, the Lorentz transformation in the $j-$direction is given by

$$\begin{pmatrix} t \\ x \end{pmatrix} \longmapsto \Lambda_T^j(\alpha^j) \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} s \\ y \end{pmatrix},$$

where

$$y^i := x^i, \quad i = 1, 2, \dots, d, \quad i \neq j$$

and

$$\begin{pmatrix} s - T \\ y^j \end{pmatrix} := \begin{pmatrix} \cosh(\alpha^j) & \sinh(\alpha^j) \\ \sinh(\alpha^j) & \cosh(\alpha^j) \end{pmatrix} \cdot \begin{pmatrix} t - T \\ x^j \end{pmatrix}.$$

Now, the Lorentz transformation is defined by

$$\Lambda_T(\alpha) := \Lambda_T^d(\alpha^d) \circ \Lambda_T^{d-1}(\alpha^{d-1}) \circ \cdots \circ \Lambda_T^1(\alpha^1).$$

Taking into account all the previous symmetries, we infer that, if $u = u(t,x)$ is a solution, so is

$$u_{T,\alpha}(t,x) := u \circ \Lambda_T(\alpha) \begin{pmatrix} t \\ x \end{pmatrix}, \tag{1.25}$$

for $T \in \mathbb{R}$ and $\alpha = (\alpha^1, \cdots, \alpha^d) \in \mathbb{R}^d$. In conclusion, we combine (1.24) with (1.25) and get a $(1+d)-$parameter family of explicit blowup solutions

$$u_{T,\alpha}(t,x) = \frac{\sqrt{2}}{A_0(\alpha)(T-t) - A_j(\alpha)x^j} \tag{1.26}$$

to (1.23), where

$$\begin{cases} A_0(\alpha) := \cosh(\alpha^d) \cdots \cosh(\alpha^3)\cosh(\alpha^2)\cosh(\alpha^1), \\ A_1(\alpha) := \cosh(\alpha^d) \cdots \cosh(\alpha^3)\cosh(\alpha^2)\sinh(\alpha^1), \\ A_2(\alpha) := \cosh(\alpha^d) \cdots \cosh(\alpha^3)\sinh(\alpha^2), \\ \qquad \vdots \\ A_{d-1}(\alpha) := \cosh(\alpha^d)\sinh(\alpha^{d-1}), \\ A_d(\alpha) := \sinh(\alpha^d). \end{cases}$$

Same useful observations are in order. First, it holds $u_{T,0} = u_T$ as well as $A_0(\alpha) = \mathcal{O}(1)$ and $A_j(\alpha) = \mathcal{O}(\alpha)$ for all sufficiently small $\alpha \in \mathbb{R}^d$. Second, notice that

$$u_{T,\alpha}(t,x) = \frac{1}{T-t}\psi_\alpha\left(\frac{x}{T-t}\right)$$

where

$$\psi_\alpha(\xi) := \frac{\sqrt{2}}{A_0(\alpha) - A_j(\alpha)\xi^j}$$

which implies the blowup of the solutions $u_{T,\alpha}$ as $t \longrightarrow T$ in the sense that

$$\begin{aligned}(T-t)^{k-\frac{d}{2}} \|u_{T,\alpha}(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} &= (T-t)^{k-\frac{d}{2}} \left\| \frac{1}{T-t}\psi_\alpha\left(\frac{\cdot}{T-t}\right)\right\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} \\ &\simeq (T-t)^{-1} \|\psi_\alpha\|_{\dot{H}^k(\mathbb{B}^d_1)} \\ &\simeq (T-t)^{-1},\end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$ and $\alpha \neq 0$, see Remark 3.3.4. As before, the energy norm (1.22) is too weak to detect the blowup of $u_{T,\alpha}$. Indeed, the energy norm corresponds to $k=1$ and the energy-super-criticality to $d \geq 5$. We have

$$\|u_{T,\alpha}(t,\cdot)\|^2_{\dot{H}^1(\mathbb{B}^d_{T-t})} \simeq (T-t)^{d-4}.$$

1.2.2 Main result

By finite speed of propagation, one can use the explicit blowup solutions $u_{T,\alpha}$ as initial data to construct a solution to the cubic wave equation which develops a singularity in finite time from smooth and compactly supported initial data. Our main result is concentrated around the stability of the blowup of $u_{T,\alpha}$. We consider the Cauchy problem

$$\begin{cases} \Box u(t,x) + u^3(t,x) = 0, \\ u[0] = (f,g), \end{cases} \tag{1.27}$$

where

$$(f,g) = u_{T_0,\alpha_0}[0] + (\tilde{f},\tilde{g})$$

and

$$T_0 > 0, \quad \alpha_0 \in \mathbb{R}^d, \quad u[t] = (u(t,\cdot), \partial_t u(t,\cdot)).$$

More precisely, we prove the existence of an open, sufficiently small neighbourhood of $u_{T,\alpha}$ from which all initial data lead to the same type of blowup described by the ODE blowup profile.

Theorem 1.2.1 (Chatzikaleas-Donninger, [17]) *Fix $d \in \{5,7,9,11,13\}$, $T_0 > 0$ and $\alpha_0 \in \mathbb{R}^d$. There exist constants $M, \delta > 0$ such that the following holds. Suppose that the initial data*

$$(f,g) \in H^{\frac{d+1}{2}}(\mathbb{B}^d_{T_0+\delta}) \times H^{\frac{d-1}{2}}(\mathbb{B}^d_{T_0+\delta})$$

satisfy

$$\left\| (f,g) - u_{T_0,\alpha_0}[0] \right\|_{H^{\frac{d+1}{2}}(\mathbb{B}^d_{T_0+\delta}) \times H^{\frac{d-1}{2}}(\mathbb{B}^d_{T_0+\delta})} \leq \frac{\delta}{M}.$$

Then, $T = T_{u[0]} \in [T_0 - \delta, T_0 + \delta]$ and there exists an $\alpha \in \mathbb{B}^d_{3M\delta}(\alpha_0)$ such that the solution $u : C_T \longrightarrow \mathbb{R}$ to (1.27) satisfies the estimates

$$\begin{aligned} (T-t)^{k-\frac{d}{2}+1} \left\| u(t,\cdot) - u_{T,\alpha}(t,\cdot) \right\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} &\leq \delta (T-t)^{\frac{1}{2}}, \\ (T-t)^{\ell-\frac{d}{2}+2} \left\| \partial_t u(t,\cdot) - \partial_t u_{T,\alpha}(t,\cdot) \right\|_{\dot{H}^\ell(\mathbb{B}^d_{T-t})} &\leq \delta (T-t)^{\frac{1}{2}}, \end{aligned}$$

for all $k = 0, 1, \cdots, \frac{d+1}{2}$ and $\ell = 0, 1, \cdots, \frac{d-1}{2}$.

1.2.3 Outline of the proof

The proof of Theorem 1.2.1 is contained in Chapter 3. In this section, we will present an outline of the proof and discuss the main ideas involved. Without loss of generality we assume that $T_0 = 1$ and $\alpha_0 = 0$. As before, we are interested in the evolution of the perturbation

$$u(t,\cdot) - u_{T,\alpha}(t,\cdot).$$

First, we switch to similarity coordinates, namely a new coordinate system $(t,x) \mapsto \mu(t,x) =: (\tau,\xi)$ which maps the backward light-cone

$$C_{(T,0)} := \left\{(t,x) \in [0,T) \times \mathbb{R}^d : x \in \mathbb{B}^d_{T-t}\right\}$$

to the cylinder

$$\mathcal{C} := \left\{(\tau,\xi) : 0 \leq \tau < +\infty,\ \xi \in \mathbb{B}^d_1\right\}.$$

Notice in particular that T is mapped to ∞. Second, we obtain a second order partial differential equation for the variable $u \circ \mu^{-1}$ and rescale the function to cancel the $\tau-$dependent factors. Now, the blowup time T appears only in the initial data. Then, we transform the second order partial differential equation for the rescaled variable into a first-order vector-valued evolution equation. We infer

$$\partial_\tau \mathbf{\Phi}(\tau) = \widetilde{\mathbf{L}}\mathbf{\Phi}(\tau) + \mathbf{N}\big(\mathbf{\Phi}(\tau)\big), \quad \tau \in (0,+\infty).$$

Following the same transformations to the blowup solutions (1.26), we obtain the τ-independent blowup solution $\mathbf{\Psi}_\alpha = \mathbf{\Psi}_\alpha(\xi)$, see (3.10). As before, the desired estimates in Theorem 1.2.1 follow from a fixed point argument and hence we must first ensure the decay of the solutions.

First, we start with the linear free evolution generated by $\widetilde{\mathbf{L}}$, namely

$$\partial_\tau \mathbf{\Phi}(\tau) = \widetilde{\mathbf{L}}\mathbf{\Phi}(\tau), \quad \tau \in (0,+\infty)$$

and would like to construct a suitable inner product

$$(\cdot|\cdot) : \left(C^{\frac{d+1}{2}}(\overline{\mathbb{B}^d}) \times C^{\frac{d-1}{2}}(\overline{\mathbb{B}^d})\right)^2 \longrightarrow \mathbb{R}, \quad \big(\mathbf{u}\big|\mathbf{v}\big) := \sum_{i=1}^{d} \big(\mathbf{u}\big|\mathbf{v}\big)_i, \tag{1.28}$$

which yields the desired decay of the solutions to the free linear evolution. However, for a generic $d \geq 5$, the inner products

$$(\cdot|\cdot)_i : \left(C^{\frac{d+1}{2}}(\overline{\mathbb{B}^d}) \times C^{\frac{d-1}{2}}(\overline{\mathbb{B}^d})\right)^2 \longrightarrow \mathbb{R}, \quad i \in \{1,2,\dots,d\}$$

are defined via inconvenient recurrence relations causing technical difficulties and make the proof rather involved. For this reason, we focus on small odd spatial dimensions, $d \in \{5,7,9,11,13\}$, and define the inner products explicitly. For example, for $d=5$, we define

$$(\cdot|\cdot) : \left(C^3(\overline{\mathbb{B}^5}) \times C^2(\overline{\mathbb{B}^5})\right)^2 \longrightarrow \mathbb{R}, \quad \big(\mathbf{u}\big|\mathbf{v}\big) := \sum_{i=1}^{5} \big(\mathbf{u}\big|\mathbf{v}\big)_i$$

where

$$
\begin{aligned}
\left(\mathbf{u}\middle|\mathbf{v}\right)_1 &:= \int_{\mathbb{B}^5} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k v_1(\xi)}d\xi + \int_{\mathbb{B}^5} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j v_2(\xi)}d\xi \\
&\quad + \int_{\mathbb{S}^4} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j v_1(\omega)}d\sigma(\omega), \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_2 &:= \int_{\mathbb{B}^5} \partial_i\partial^k\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial_j v_1(\xi)}d\xi + \int_{\mathbb{B}^5} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j v_2(\xi)}d\xi \\
&\quad + \int_{\mathbb{S}^4} \partial_j u_2(\omega)\overline{\partial^j v_2(\omega)}d\sigma(\omega), \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_3 &:= 5\left(\mathbf{u}\middle|\mathbf{v}\right)_1 + \left(\mathbf{u}\middle|\mathbf{v}\right)_2 + \int_{\mathbb{S}^4} u_2(\omega)\overline{v_2(\omega)}d\sigma(\omega), \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_4 &:= \left(\mathbf{u}\middle|\mathbf{v}\right)_1 + \left(\mathbf{u}\middle|\mathbf{v}\right)_2 + \int_{\mathbb{S}^4} \partial_i u_1(\omega)\overline{\partial^i v_1(\omega)}d\sigma(\omega), \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_5 &:= \left(\int_{\mathbb{S}^4} \zeta\left(\omega, \mathbf{u}(\omega)\right) d\sigma(\omega)\right)\left(\int_{\mathbb{S}^4} \overline{\zeta\left(\omega, \mathbf{v}(\omega)\right)}d\sigma(\omega)\right),
\end{aligned}
$$

for all $\mathbf{u}, \mathbf{v} \in C^3\left(\overline{\mathbb{B}^5}\right) \times C^2\left(\overline{\mathbb{B}^5}\right)$, where

$$
\begin{cases}
\zeta\left(\omega, \mathbf{w}(\omega)\right) : = D_5 w_1(\omega) + \tilde{D}_5 w_2(\omega), \\
\quad D_5 w_1(\omega) : = \omega^i\omega^j\partial_i\partial_j w_1(\omega) + 5\omega^i\partial_i w_1(\omega) + 3w_1(\omega), \\
\quad D_5 w_1(\omega) : = \omega^i\omega^j\partial_i\partial_j w_1(\omega) + 5\omega^i\partial_i w_1(\omega) + 3w_1(\omega).
\end{cases}
$$

For more details, see section 3.5.1 for $d = 5$ and the discussion at the end of section 3.10 for $d \in \{7, 9, 11, 13\}$. To establish the decay of the solutions we use Lumer-Phillips theorem (see Theorem 3.15, page 83, [36]). To proceed, we fix $d = 5$ and prove the following result.

Proposition 1.2.2 (Proposition 3.5.1, [17]) *The free operator $\widetilde{\mathbf{L}} : \mathcal{D}\left(\widetilde{\mathbf{L}}\right) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is densely defined, closable and its closure $\mathbf{L} : \mathcal{D}\left(\mathbf{L}\right) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ generates a strongly continuous one-parameter semigroup of bounded operators $\mathbf{S} : [0, \infty) \longrightarrow \mathcal{B}\left(\mathcal{H}\right)$ which satisfies the decay estimate*

$$
\|\mathbf{S}\left(\tau\right)\| \leq Me^{-\tau}
$$

for all $\tau \geq 0$ and for some constant $M \geq 1$.

The proof of this proposition is divided into three parts. In the first part, we consider the inner products $\left(\cdot\middle|\cdot\right)_i$ for $i \in \{1, 2, 3, 4\}$ and we use an elementary inequality to show the following estimate.

Lemma 1.2.3 (Lemma 3.5.4, [17]) *For all $\mathbf{u} \in \mathcal{D}\left(\widetilde{\mathbf{L}}\right)$ and $i \in \{1, 2, 3, 4\}$, we have*

$$
\operatorname{Re}\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_i \leq -\frac{3}{2}\|\mathbf{u}\|_i^2.
$$

Second, we consider the inner product $\left(\cdot\middle|\cdot\right)_5$ and note that ζ is a carefully chosen function so that the sum (1.28) induces a norm, see Lemma 3.5.3, and the following decay property holds.

Lemma 1.2.4 (Lemma 3.5.5, [17]) *For all* $\mathbf{u} \in \mathcal{D}(\widetilde{\mathbf{L}})$, *we have*

$$\operatorname{Re}\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_5 = -\|\mathbf{u}\|_5^2.$$

The latter is based on the integral identity

$$\int_{\mathbb{S}^4} \zeta\left(\omega, \widetilde{\mathbf{L}}\mathbf{u}(\omega)\right) d\sigma(\omega) = -\int_{\mathbb{S}^4} \zeta\left(\omega, \mathbf{u}(\omega)\right) d\sigma(\omega).$$

The analogous result in $d = 3$ has been proved by Donninger and Schörkhuber [31] but their argument works only in three space dimensions.

Third, we prove the last ingredient of the Lumer-Phillips theorem, namely the density of $\operatorname{rg}(\frac{3}{2} - \widetilde{\mathbf{L}})$ in $\mathcal{H}$, see Lemma 3.5.7 and Lemma 3.5.8. These results constitute the proof of Proposition 1.2.2. As a consequence, we locate the spectrum $\mathbf{L}$, i.e.,

$$\sigma\left(\mathbf{L}\right) \subseteq \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq -1\right\},$$

see (3.34) and Hadamard's equality, [36], p. 55, Theorem 1.10.

Next, we consider the Lorentz symmetry and use a modulation ansatz. We vary the vector $\alpha \in \mathbb{R}^5$ by allowing it to depend on time, set $\alpha(0) = 0$ and assume that the limit $\alpha_\infty := \lim_{\tau\to\infty} \alpha(\tau)$ exists. Later we chose our Banach spaces so that these assumptions are verified, see (3.50) and (3.51). Then, we find an evolution equation for the perturbation

$$\mathbf{\Phi}(\tau) := \mathbf{\Psi}(\tau) - \mathbf{\Psi}_{\alpha(\tau)}.$$

Here, $\mathbf{\Psi}_{\alpha(\tau)}$ are variations in time of $\mathbf{\Psi}_\alpha$, the Lorentz transformations of the static blowup solution solution $\mathbf{\Psi}_0$. We obtain the modulation equation

$$\partial_\tau \mathbf{\Phi}(\tau) - \left(\mathbf{L} + \mathbf{L}'_{\alpha_\infty}\right)\mathbf{\Phi}(\tau) = \hat{\mathbf{L}}_{\alpha(\tau)}\mathbf{\Phi}(\tau) + \mathbf{N}_{\alpha(\tau)}(\mathbf{\Phi}(\tau)) - \partial_\tau \mathbf{\Psi}_{\alpha(\tau)},$$

where $\hat{\mathbf{L}}_{\alpha(\tau)} := \mathbf{L}'_{\alpha(\tau)} - \mathbf{L}'_{\alpha_\infty}$, $\mathbf{L}'_{\alpha(\tau)}$ is the linearized part of the nonlinearity $\mathbf{N}$, see (3.29), and $\mathbf{N}_{\alpha(\tau)}$ stands for the remaining full nonlinearity, see (3.30). Our intention is to formulate the modulation equation as an abstract integral equation via Duhamel's principle and apply a fixed point argument. To do so, we first prove that the linear operator generates a solution for sufficiently small $\alpha \in \mathbb{R}^5$.

Lemma 1.2.5 (Lemma 3.6.1, [17]) *Let* $\alpha \in \mathbb{R}^5$ *be sufficiently small. Then, the operator* $\mathbf{L}'_\alpha$ *defined in* (3.29) *is compact. In particular, the operator*

$$\mathbf{L}_\alpha := \mathbf{L} + \mathbf{L}'_\alpha \tag{1.29}$$

generates a strongly continuous one parameter semigroup of bounded operators $\mathbf{S}_\alpha : [0, \infty) \longrightarrow \mathcal{B}(\mathcal{H})$.

To establish the decay for the solutions to the full linear operator (1.29) we turn our attention to the spectrum of $\mathbf{L}_\alpha$ for sufficiently small $\alpha \in \mathbb{R}^5$.

First, we consider the case where $\alpha = 0$ and prove the following result.

Proposition 1.2.6 (Lemma 3.7.2, [17]) *We have*

$$\sigma\left(\mathbf{L}_0\right) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -1\} \cup \{0, 1\}.$$

To prove this result, it suffices to consider the point spectrum of $\mathbf{L}_0$, see Lemma 3.7.1. We write the spectral equation, switch to spherical coordinates, expand in spherical harmonics and find a decoupled system of ODEs, see (3.35), which has four singular points. First, we reduce the singular points to three, transform the spectral equation into a hypergeometric differential equation and finally rely on the connection formula for the coefficients [68] which is well-known for this class. We note that the eigenvalue $\lambda = 0$ occurs due to Lorentz symmetry whereas the eigenvalue $\lambda = 1$ due to time translation symmetry.

Second, we pass to the spectrum of $\mathbf{L}_\alpha$ for $\alpha \neq 0$ sufficiently small and prove the following result.

Proposition 1.2.7 (Proposition 3.7.5, [17]) *Let $\alpha \in \mathbb{R}^5$ be sufficiently small. Then,*

$$\sigma(\mathbf{L}_\alpha) \subseteq \left\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\frac{3}{4}\right\} \cup \{0, 1\}.$$

For the proof of this result, we define the total projection

$$\mathbf{P}_\alpha^{total} := \frac{1}{2\pi i} \int_{\partial\Omega_{k_0,\omega_0}} \mathbf{R}_{\mathbf{L}_\alpha}(\zeta) d\zeta$$

and rely on Lemma 3.7.4 ($\dim \operatorname{rg} \mathbf{P}_0^{total} = 6$), Lemma 3.7.6 (continuous dependence of $\mathbf{L}'_\alpha$ with respect to α), Lemma 3.7.7 ($\sigma(\mathbf{L}_\alpha) \subseteq \sigma(\mathbf{L}_0)$ for sufficiently small $\alpha \neq 0$), Lemma 3.7.8 (absence of spectrum for $\mathbf{L}_\alpha$ away from the origin) which imply that the ranges of $\mathbf{P}_\alpha^{total}$ are all isomorphic to one another for sufficiently small α and in addition the $\operatorname{rank} \mathbf{P}_\alpha^{total} := \dim \operatorname{rg} \mathbf{P}_\alpha^{total}$ are constant in α, see Lemma 4.10 page 34, [43]. Hence, we infer $\operatorname{rank} \mathbf{P}_\alpha^{total} := \dim \operatorname{rg} \mathbf{P}_\alpha^{total} = \dim \operatorname{rg} \mathbf{P}_0^{total} = 6$.

Third, we study the time evolution for the full linearized problem. As before, we define (non-orthogonal) projections $\mathbf{P}_\alpha$, $\mathbf{Q}_{\alpha,1}$,$\mathbf{Q}_{\alpha,2}$,$\mathbf{Q}_{\alpha,3}$,$\mathbf{Q}_{\alpha,4}$,$\mathbf{Q}_{\alpha,5}$ which split the space of initial data into the stable and unstable spaces and obtain useful growth estimates on the corresponding subspaces.

Proposition 1.2.8 (Proposition 3.7.10, [17]) *Let $\alpha \in \mathbb{R}^5$ be sufficiently small. Then, the projections $\mathbf{P}_\alpha$ and $\mathbf{Q}_{\alpha,j}$ for $j \in \{1, 2, 3, 4, 5\}$ have rank one and commute with the* *semigroup. In addition,*

$$\mathbf{S}_\alpha(\tau)\mathbf{P}_\alpha = e^\tau \mathbf{P}_\alpha,$$
$$\mathbf{S}_\alpha(\tau)\mathbf{Q}_{\alpha,j} = \mathbf{Q}_{\alpha,j},$$
$$\|\mathbf{S}_\alpha(\tau)\widetilde{\mathbf{P}}_\alpha\| \lesssim e^{-\frac{2}{3}\tau}\|\widetilde{\mathbf{P}}_\alpha\|,$$

where $\widetilde{\mathbf{P}}_\alpha := \mathbf{I} - \mathbf{P}_\alpha - \mathbf{Q}_\alpha$*. Furthermore,*

$$\operatorname{rg}(\mathbf{P}_\alpha) = \langle \mathbf{g}_\alpha \rangle,$$
$$\operatorname{rg}(\mathbf{Q}_{\alpha,j}) = \langle \mathbf{h}_{\alpha,j} \rangle, \quad j \in \{1,2,3,4,5\},$$

where $\mathbf{g}_\alpha$ *and* $\mathbf{h}_{\alpha,j}$ *are eigenfunctions of* $\mathbf{L}_\alpha$ *with eigenvalues* 1 *and* 0*, respectively.*

We note that the unstable subspaces occur due to the symmetries of the original equation, i.e. the Lorentz and time-translation symmetry.

Next, we choose our Banach spaces in such a way so that the limit α_∞ exists and in addition

$$|\alpha_\infty| \lesssim \delta,$$

see section 3.9.1 and (3.51). We choose δ sufficiently small and use Lemma 1.2.5 to write the modulation equation as an abstract integral equation

$$\mathbf{\Phi}(\tau) = \mathbf{S}_{\alpha_\infty}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}_{\alpha_\infty}(\tau-\sigma)\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big)d\sigma. \tag{1.30}$$

The desired estimates in Theorem 1.2.1 follow from a fixed point argument to (1.30). However, the solution operator $\mathbf{S}_{\alpha_\infty}$ generated by the linearized operator $\mathbf{L} + \mathbf{L}'_{\alpha_\infty}$ has two unstable subspaces $\operatorname{rg}\mathbf{P}_{\alpha_\infty}$ and $\operatorname{rg}\mathbf{Q}_{\alpha_\infty}$ meaning that the future development of initial data from $\operatorname{rg}\mathbf{P}_{\alpha_\infty}$ and $\operatorname{rg}\mathbf{Q}_{\alpha_\infty}$ do not decay but rather stay constant and grow exponentially in time, respectively, see Proposition 1.2.8. These instabilities render the solvability of (1.30) hopeless from the very beginning. To make the fixed point argument feasible, we proceed as follows. In the case of the Lorentz symmetry, we use a fixed point argument and choose the unknown parameter $\alpha = \alpha(\tau)$ to prevent the development of the instability, see section 3.9.2 and Proposition 3.9.4. In the case of the time translation symmetry, we stabilize the evolution by changing the initial data. This can be done by subtracting the correction term (3.57) from the original data. Then, we obtain a modified integral equation (3.55) and use another fixed point argument to guarantee the existence of the solution, see Proposition 3.9.5. Next, we rely on an additional fixed point argument to ensure that the correction term vanishes, see Lemma 3.9.8 provided that the perturbed initial data are sufficiently small. For more details, see section 3.9.3. All the fixed point arguments are based on a series of Lipschitz-type estimates, Lemma 3.8.1, Lemma 3.8.2, Lemma 3.9.2 and Lemma 3.9.3. In summary, we obtain the following result.

Theorem 1.2.9 (Theorem 3.10.1, [17]) *Let $\delta > 0$ be sufficiently small, c sufficiently large and pick an arbitrary $\mathbf{v} \in H^3(\mathbb{B}^5_{1+\delta/c}) \times H^2(\mathbb{B}^5_{1+\delta/c})$ such that $\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\delta/c}) \times H^2(\mathbb{B}^5_{1+\delta/c})} \leq \frac{\delta}{c^2}$. Then, there exists $T = T_{\mathbf{v}} \in [1 - \frac{\delta}{c}, 1 + \frac{\delta}{c}]$ such that the full, non-corrected equation (3.49) with initial data $\mathbf{u} = \mathbf{U}(T_{\mathbf{v}}, \mathbf{v})$, that is*

$$\mathbf{\Phi}(\tau) = \mathbf{S}_{\alpha_\infty}(\tau)\mathbf{U}(T_{\mathbf{v}}, \mathbf{v}) + \int_0^\tau \mathbf{S}_{\alpha_\infty}(\tau - \sigma)\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big)d\sigma,$$

has a solution $(\Phi, \alpha) = (\Phi_{T_{\mathbf{v}}}, \alpha_{T_{\mathbf{v}}}) \in \mathcal{X}_\delta \times X_\delta$.

Finally, we prove our main result, see section 3.10. The desired estimates follow from the fact that the solution $(\Phi, \alpha) = (\Phi_{T_{\mathbf{v}}}, \alpha_{T_{\mathbf{v}}})$ belongs in $\mathcal{X}_\delta \times X_\delta$.

Chapter 2

On blowup of co-rotational wave maps in odd space dimensions

This chapter contains the result of the paper [18] and is a joint work of the author with Donninger and Glogić.

2.1 Abstract

We consider co-rotational wave maps from the $(1 + d)$-dimensional Minkowski space into the d-sphere for $d \geq 3$ odd. This is an energy-supercritical model which is known to exhibit finite-time blowup via self-similar solutions. Based on a method developed by the second author and Schörkhuber, we prove the asymptotic nonlinear stability of the "ground-state" self-similar solution.

2.2 Introduction

Let (M, g) be a Lorentzian spacetime and (N, h) a Riemannian manifold. In this paper, we study wave maps $u : (M, g) \longrightarrow (N, h)$, that is, critical points of the geometric action functional

$$S_g[u] := \frac{1}{2} \int_M |d_g u|^2 \, d\mu_g.$$

Here,

$$|d_g u(x)|^2 \equiv |d_g u(x)|^2_{T^\star_x M \otimes T_{u(x)} N} := \mathrm{tr}_g \left(u^\star \left(h \right) \right)$$

is the trace (with respect to g) of the pullback metric on (M, g) via the map u. The integral is understood with respect to the standard measure $d\mu_g$ on the domain manifold. In local coordinates (x_μ) on (M, g), this expression reads

$$S_g[u] = \int_M g^{\mu\nu} (\partial_\mu u^a)(\partial_\nu u^b) h_{ab} \circ u \, d\mu_g$$

where the Einstein summation convention is used. The Euler-Lagrange equations associated to this functional are

$$\Box_g u^a + g^{\mu\nu}(\Gamma^a_{bc} \circ u)(\partial_\mu u^b)(\partial_\nu u^c) = 0 \tag{2.1}$$

and they constitute a system of semi-linear wave equations. Here, $\Box_g$ is the Laplace-Beltrami operator on (M, g)

$$\Box_g := \frac{1}{|g|}\partial_\mu(g^{\mu\nu}|g|\partial_\nu), \quad |g| := \sqrt{|\det(g_{\mu\nu})|}$$

and Γ^a_{bc} are the Christoffel symbols associated to the metric h on the target manifold. Eq. (2.1) is called the wave maps equation (known in the physics literature as non-linear σ model) and is the analog of harmonic maps between Riemannian manifolds in the case where the domain is a Lorentzian manifold instead. For more details, we refer the reader to [71] and [81].

2.2.1 Intuition

Recently, the wave maps equation has attracted a lot of interest. On the one hand, the wave maps equation is a rich source for understanding nonlinear geometric equations since it is a nonlinear generalization of the standard wave equation on Minkowski space. In addition, the wave maps equation has a pure geometric interpretation: it generalizes the notion of geodesic curves. Notice that, if $M = (\alpha, \beta)$ is an open interval and (N, h) any curved Riemannian manifold, the wave maps equation is the geodesic equation

$$\frac{d^2 u^a}{dt^2}(t) + (\Gamma^a_{bc} \circ u(t))\frac{du^b}{dt}(t)\frac{du^c}{dt}(t) = 0.$$

On the other hand, the Cauchy problem for the wave maps system provides an attractive toy-model for more complicated relativistic field equations. Specifically, wave maps contain many features of the more complex Einstein equations but are simple enough to be accessible for rigorous mathematical analysis. Further details on the correlation between the wave maps system and the Einstein equations can be found in [46, 65, 66, 90].
Being a time evolution equation, the fundamental problem is the Cauchy problem: given specified smooth initial data, does there exist a unique smooth solution to the wave maps equation with this initial data? Furthermore, does the solution exist for all times? On the other hand, if the solution only exists up to some finite time T, how does the solution blow up as t approaches T? The investigation of questions of global existence and formation of singularities for the wave maps equation can give insight into the analogous, but much more difficult, problems in general relativity.

2.2.2 Equivariant wave maps

Now, we turn our attention to the Cauchy problem in the case where the domain is the Minkowski spacetime $(\mathbb{R}^{1+d}, g)$ and the target manifold is the sphere $(\mathbb{S}^d, h)$ for $d \geq 3$.

Hence, we pick g =diag$(-1, 1, \dots, 1)$ and h to be the standard metric on the sphere. Furthermore, we choose standard spherical coordinates on Minkowski space and hyperspherical coordinates on the sphere. The respective metrics are given by

$$g = -dt^2 + dr^2 + r^2 d\omega^2, \quad h = d\Psi^2 + \sin^2(\Psi) d\Omega^2,$$

where $d\omega^2$ and $d\Omega^2$ are the standard metrics on $\mathbb{S}^{d-1}$. Moreover, a map $u : (\mathbb{R}^{1+d}, g) \longrightarrow (\mathbb{S}^d, h)$ can be written as

$$u(t, r, \omega) = \big(\Psi(t, r, \omega), \Omega(t, r, \omega)\big).$$

We restrict our attention to the special subclass known as 1-equivariant or co-rotational, that is

$$\Psi(t, r, \omega) \equiv \psi(t, r), \quad \Omega(t, r, \omega) = \omega.$$

Under this ansatz, the wave maps system for functions $u : (\mathbb{R}^{1+d}, g) \longrightarrow (\mathbb{S}^d, h)$ reduces to the single semi-linear wave equation

$$\psi_{tt} - \psi_{rr} - \frac{d-1}{r}\psi_r + \frac{d-1}{2}\frac{\sin(2\psi)}{r^2} = 0. \tag{2.2}$$

By finite speed of propagation and radial symmetry it is natural to study this equation in backward light-cones with vertex $(T, 0)$, that is

$$C_T := \{(t, r) : 0 < t < T,\ 0 \leq r \leq T - t\}$$

where $T > 0$. Consequently, we consider the Cauchy problem

$$\begin{cases} \psi_{tt}(t, r) - \Delta^{\text{rad}}_{r,d}\psi(t, r) = -\frac{d-1}{2}\frac{\sin(2\psi(t,r))}{r^2}, & \text{in } C_T \\ \psi(0, r) = f(r), \quad \psi_t(0, r) = g(r), & \text{on } \{t = 0\} \times [0, +\infty) \end{cases} \tag{2.3}$$

where $\Delta^{\text{rad}}_{r,d}$ stands for the radial Laplacian

$$\Delta^{\text{rad}}_{r,d}\psi(t, r) := \psi_{rr}(t, r) + \frac{d-1}{r}\psi_r(t, r).$$

To ensure regularity of solutions, equations (2.3) must be supplemented by the boundary condition

$$\psi(t, 0) = 0, \qquad \text{for all } t \in (0, T). \tag{2.4}$$

2.2.3 Self-similar solutions

A basic question for the Cauchy problem (2.3) is whether solutions starting from smooth initial data

$$(f, g) = (\psi(0, \cdot), \partial_t \psi(0, \cdot))$$

can become singular in the future. Note that Eq. (2.2) has the conserved energy

$$E[\psi] := \int_0^\infty \left(\psi_t^2 + \psi_r^2 + (d-1)\frac{\sin^2(\psi)}{r^2} \right) r^2 dr.$$

However, the energy cannot be used to control the evolution since Eq. (2.3) is not well-posed at energy regularity, cf. [77]. Indeed, Eq. (2.2) is invariant under dilations

$$\psi_\lambda(t, r) := \psi\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), \ \lambda > 0 \tag{2.5}$$

and the critical Sobolev space for the pair $(\psi(t, \cdot), \partial_t \psi(t, \cdot))$ is $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$. Consequently, Eq. (2.2) is energy-supercritical for $d \geq 3$.
In fact, due to the scaling (2.5) and the supercritical character it is natural to expect self-similar solutions and indeed, it is well known that there exist smooth initial data which lead to solutions that blowup in finite time in a self-similar fashion. Specifically, Eq. (2.2) admits the self-similar solution

$$\psi^T(t, r) := f_0\Big(\frac{r}{T-t}\Big) = 2 \arctan\left(\frac{r}{\sqrt{d-2}(T-t)}\right), \qquad T > 0.$$

This example is due to Shatah [74], Turok-Spergel [89] for $d = 3$, and Bizoń-Biernat [7] for $d \geq 4$ and provides an explicit example for singularity formation from smooth initial data. Indeed, the self-similar solution ψ^T is perfectly smooth for all $0 < t < T$ but breaks down at $t = T$ in the sense that

$$\partial_r \psi^T(t, r)|_{r=0} \simeq \frac{1}{T-t} \longrightarrow +\infty, \quad \text{as } t \longrightarrow T^-.$$

We note in passing that for $d \in \{3, 4, 5, 6\}$, ψ^T is just one member of a countable family of self-similar solutions, see [3, 5].

2.2.4 The main result

By finite speed of propagation one can use ψ^T to construct smooth, compactly supported initial data which lead to a solution that blows up as $t \longrightarrow T$. Our main theorem is concerned with the asymptotic nonlinear stability of ψ^T. In other words, we prove the existence of an open set of radial data which lead to blowup via ψ^T. In this sense, the blowup described by ψ^T is stable. To state our main result, we will need the notion of the blowup time at the origin. From now on we use the abbreviation $\psi[t] = (\psi(t, \cdot), \partial_t \psi(t, \cdot))$.

Definition 2.2.1 *Given initial data* (ψ_0, ψ_1)*, we define*

$$T_{(\psi_0,\psi_1)} := \sup \left\{ T > 0 \middle| \begin{smallmatrix} \exists \text{ solution } \psi: C_T \longrightarrow \mathbb{R} \text{ to } (2.3) \text{ in the sense of} \\ \text{Definition } 3.9.1 \text{ with initial data } \psi[0] = (\psi_0,\psi_1)|_{\mathbb{B}^d_T} \end{smallmatrix} \right\} \cup \{0\}.$$

In the case where $T_{(\psi_0,\psi_1)} < \infty$*, we call* $T \equiv T_{(\psi_0,\psi_1)}$ *the blowup time at the origin.*

We remark that the effective spatial dimension for the problem (2.3) is $d+2$. To see this, recall that, by regularity, we get the boundary condition (2.4). Therefore, it is natural to switch to the variable $\widehat{\psi}(t,r) := r^{-1}\psi(t,r)$. Then (2.3) transforms into

$$\begin{cases} \widehat{\psi}_{tt}(t,r) - \Delta^{\text{rad}}_{r,d+2}\widehat{\psi}(t,r) = -\frac{d-1}{2}\frac{\sin(2r\widehat{\psi}(t,r)) - 2r\widehat{\psi}(t,r)}{r^3}, & \text{in } C_T \\ \widehat{\psi}(0,r) = \frac{f(r)}{r}, \quad \widehat{\psi}_t(0,r) = \frac{g(r)}{r}, & \text{on } \{t=0\} \times [0,+\infty) \end{cases}$$

Note that the nonlinearity is now generated by a smooth function and the radial Laplacian is in $d+2$ dimensions.

Theorem 2.2.2 *Fix* $T_0 > 0$ *and* $d \geq 3$ *odd. Then there exist constants* $M, \delta, \epsilon > 0$ *such that for any radial initial data* $\psi[0]$ *satisfying*

$$\left\| |\cdot|^{-1} \left(\psi[0] - \psi^{T_0}[0] \right) \right\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2}_{T_0+\delta}) \times H^{\frac{d+1}{2}}(\mathbb{B}^{d+2}_{T_0+\delta})} \leq \frac{\delta}{M}$$

the following statements hold:

1. $T \equiv T_{\psi[0]} \in [T_0 - \delta, T_0 + \delta]$,
2. *the solution* $\psi : C_T \longrightarrow \mathbb{R}$ *satisfies*

$$(T-t)^{k-\frac{d}{2}} \left\| |\cdot|^{-1} \left(\psi(t,\cdot) - \psi^T(t,\cdot) \right) \right\|_{\dot{H}^k(\mathbb{B}^{d+2}_{T-t})} \leq \delta (T-t)^{\epsilon}$$

$$(T-t)^{\ell+1-\frac{d}{2}} \left\| |\cdot|^{-1} \left(\partial_t\psi(t,\cdot) - \partial_t\psi^T(t,\cdot) \right) \right\|_{\dot{H}^\ell(\mathbb{B}^{d+2}_{T-t})} \leq \delta (T-t)^{\epsilon}$$

for all $k = 0, 1, 2, \ldots, \frac{d+3}{2}$ *and* $\ell = 0, 1, 2 \ldots, \frac{d+1}{2}$.

Remark 2.2.3 *Note that the normalizing factors on the left-hand sides appear naturally and reflect the behavior of the self-similar solution* ψ^T *in the respective homogeneous Sobolev norms, i.e.,*

$$\||\cdot|^{-1}\psi^T(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^{d+2}_{T-t})} = \left\| |\cdot|^{-1} f_0\left(\frac{|\cdot|}{T-t} \right) \right\|_{\dot{H}^k(\mathbb{B}^{d+2}_{T-t})} = (T-t)^{\frac{d}{2}-k} \||\cdot|^{-1} f_0\left(|\cdot|\right)\|_{\dot{H}^k(\mathbb{B}^{d+2}_1)}$$

and

$$\||\cdot|^{-1}\partial_t\psi^T(t,\cdot)\|_{\dot{H}^\ell(\mathbb{B}^{d+2}_{T-t})} = (T-t)^{-2} \left\| f_0'\left(\frac{|\cdot|}{T-t} \right) \right\|_{\dot{H}^\ell(\mathbb{B}^{d+2}_{T-t})}$$
$$= (T-t)^{\frac{d}{2}-\ell-1} \|f_0'\left(|\cdot|\right)\|_{\dot{H}^\ell(\mathbb{B}^{d+2}_1)}.$$

2.2.5 Related results

The question of singularity formation for the wave maps equation attracted a lot of interest in the recent past, in particular in the energy-critical case $d = 2$. Bizoń-Chmaj-Tabor [10] were the first to provide numerical evidence for the existence of blowup for critical wave maps with $\mathbb{S}^2$ target. Rigorous constructions of blowup solutions for this model are due to Krieger-Schlag-Tataru [50], Rodnianski-Sterbenz [72], and Raphaël-Rodnianski [69]. Struwe [82] showed that blowup for equivariant critical wave maps takes place via shrinking of a harmonic map. This result was considerably generalized to the nonequivariant setting by Sterbenz-Tataru [79, 80], see also Krieger-Schlag [52] for a different approach to the large-data problem and e.g. [21–23, 39, 53, 73] for more recent results on blowup and large-data global existence.
The energy-supercritical regime $d \geq 3$ is less understood. The small-data theory at minimal regularity is due to Shatah-Tahvildar-Zadeh [77] in the equivariant setting whereas Tataru [86, 87] and Tao [83, 84] treat the general case, see also [48, 51, 67, 76, 88]. Self-similar blowup solutions were found by Shatah [74], Turok-Spergel [89], Cazenave-Shatah-Tahvildar-Zadeh [16], and Bizoń-Biernat [7]. The stability of self-similar blowup was investigated numerically in [3, 7, 9] and proved rigorously in [19, 20, 26, 35] in the case $d = 3$. Furthermore, Dodson-Lawrie [24] proved that solutions with bounded critical norm scatter.
Finally, concerning the method, we remark that our proof relies on the techniques developed in the series of papers [26, 27, 29–32, 35]. However, we would like to emphasize that the present paper is not just a straightforward continuation of these works. In fact, new interesting issues arise, e.g. in the spectral theory part, see Proposition 2.4.5 below.

2.3 Radial wave equation in similarity coordinates

To start our analysis, we rewrite the initial value problem (2.3) as an abstract Cauchy problem in a Hilbert space. First, we rescale the variable $\psi \equiv \psi(t, r)$ and switch to similarity coordinates. Then, we linearize around the rescaled blowup solution and derive the evolution problem satisfied by the perturbation.

2.3.1 Rescaled variables

We define

$$\chi_1(t, r) := \frac{T - t}{r}\psi(t, r), \qquad \chi_2(t, r) := \frac{(T - t)^2}{r}\psi_t(t, r).$$

Using the fact that ψ is a solution to (2.3), we get

$$\begin{aligned}\partial_t\chi_1(t,r) &= -\frac{1}{T-t}\chi_1(t,r) + \frac{1}{T-t}\chi_2(t,r),\\ \partial_t\chi_2(t,r) &= -\frac{2}{T-t}\chi_2(t,r) + (T-t)\Delta^{\text{rad}}_{r,d}\chi_1(t,r) + \frac{2(T-t)}{r}\partial_r\chi_1(t,r)\\ &\quad + (d-1)\frac{T-t}{r^2}\chi_1(t,r) - \frac{d-1}{2}(T-t)^2\frac{\sin\left(\frac{2r}{T-t}\chi_1(t,r)\right)}{r^3}.\end{aligned}$$

We introduce similarity coordinates

$$\mu : C_T \longrightarrow \mathcal{C}, \quad (t,r) \longmapsto \mu(t,r) = (\tau,\rho) := \left(\log\left(\frac{T}{T-t}\right), \frac{r}{T-t}\right),$$

which map the backward light-cone C_T to the cylinder $\mathcal{C} := (0,+\infty)\times[0,1]$. By the chain rule, the derivatives transform according to

$$\partial_t = \frac{e^\tau}{T}(\partial_\tau + \rho\partial_\rho),\ \partial_r = \frac{e^\tau}{T}\partial_\rho,\ \partial_r^2 = \frac{e^{2\tau}}{T^2}\partial_\rho^2,\ \Delta^{\text{rad}}_{r,d} = \frac{e^{2\tau}}{T^2}\Delta^{\text{rad}}_{\rho,d}.$$

Finally, setting

$$\psi_j(\tau,\rho) := \chi_j(t(\tau,\rho), r(\tau,\rho)) = \chi_j(T(1-e^{-\tau}), T\rho e^{-\tau}),$$

for $j = 1, 2$, we obtain the system

$$\begin{aligned}\begin{pmatrix}\partial_\tau\psi_1(\tau,\rho)\\ \partial_\tau\psi_2(\tau,\rho)\end{pmatrix} &= \begin{pmatrix}-\psi_1(\tau,\rho) + \psi_2(\tau,\rho) - \rho\partial_\rho\psi_1(\tau,\rho)\\ \Delta^{\text{rad}}_{\rho,d+2}\psi_1(\tau,\rho) - \rho\partial_\rho\psi_2(\tau,\rho) - 2\psi_2(\tau,\rho)\end{pmatrix}\\ &\quad - \frac{d-1}{2\rho^3}\begin{pmatrix}0\\ \sin(2\rho\psi_1(\tau,\rho)) - 2\rho\psi_1(\tau,\rho)\end{pmatrix},\end{aligned} \tag{2.6}$$

for $(\tau,\rho)\in\mathcal{C}$. Note that the linear part is the free operator of the $(d+2)-$dimensional wave equation in similarity coordinates and the nonlinearity is perfectly smooth. Furthermore, the initial data transform according to

$$\begin{pmatrix}\psi_1(0,\rho)\\ \psi_2(0,\rho)\end{pmatrix} = \frac{1}{\rho}\begin{pmatrix}f(T\rho)\\ Tg(T\rho)\end{pmatrix} = \frac{1}{\rho}\begin{pmatrix}\psi^{T_0}(0,T\rho)\\ T\partial_0\psi^{T_0}(0,T\rho)\end{pmatrix} + \frac{1}{\rho}\begin{pmatrix}F(T\rho)\\ TG(T\rho)\end{pmatrix},$$

for all $\rho\in[0,1]$. Here, $T_0 > 0$ is a fixed parameter and

$$\begin{aligned}&\psi^{T_0}(0,T\rho) = 2\arctan\left(\frac{T}{T_0}\frac{\rho}{\sqrt{d-2}}\right), \quad \rho\equiv\rho(t,r) := \frac{r}{T-t},\\ &F := f - \psi^{T_0}(0,\cdot), \quad G := g - \partial_0\psi^{T_0}(0,\cdot).\end{aligned}$$

We emphasize that the only trace of the parameter T is in the initial data.

2.3.2 Perturbations of the rescaled blowup solution

We linearize around the rescaled blowup solution and use the initial value problem for $(\psi_1, \psi_2)^T$ to obtain an initial value problem for the perturbation as an abstract Cauchy problem in a Hilbert space. For notational convenience we set

$$\mathbf{\Psi}(\tau)(\rho) := \begin{pmatrix} \psi_1(\tau,\rho) \\ \psi_2(\tau,\rho) \end{pmatrix}.$$

The blowup solution is given by

$$\mathbf{\Psi}^{\text{res}}(\tau)(\rho) = \begin{pmatrix} \frac{T-t}{r}\psi^T(t,r) \\ \frac{(T-t)^2}{r}\psi_t^T(t,r) \end{pmatrix}\Bigg|_{(t,r)=\mu^{-1}(\tau,\rho)} = \begin{pmatrix} \frac{1}{\rho}f_0(\rho) \\ f_0'(\rho) \end{pmatrix},$$

i.e., it is static. We linearize around $\mathbf{\Psi}^{\text{res}}$ by inserting the ansatz $\mathbf{\Psi} = \mathbf{\Psi}^{\text{res}} + \mathbf{\Phi}$ into (2.6). For brevity we write

$$\eta(x) := \sin(2x) - 2x, \quad x \in \mathbb{R}$$

and use Taylor's theorem to expand the nonlinearity around $\frac{1}{\rho}f_0(\rho)$. We get

$$\sin(2\rho\psi_1) - 2\rho\psi_1 = \eta(\rho\psi_1) = \eta(f_0 + \rho\phi_1) = \eta(f_0) + \eta'(f_0)\rho\phi_1 + N(\rho\phi_1),$$

where, by definition,

$$N(\rho\phi_1) := \eta(f_0 + \rho\phi_1) - \eta(f_0) - \eta'(f_0)\rho\phi_1.$$

We plug the ansatz and the Taylor expansion into Eq. (2.6) which yields the abstract evolution equation

$$\begin{cases} \partial_\tau \mathbf{\Phi}(\tau) = \widetilde{\mathbf{L}}\big(\mathbf{\Phi}(\tau)\big) + \mathbf{N}\big(\mathbf{\Phi}(\tau)\big), & \text{for } \tau \in (0,+\infty) \\ \mathbf{\Phi}(0) = \mathbf{U}(\mathbf{v},T), \end{cases} \tag{2.7}$$

for the perturbation

$$\mathbf{\Phi}(\tau)(\rho) = \begin{pmatrix} \phi_1(\tau,\rho) \\ \phi_2(\tau,\rho) \end{pmatrix} = \begin{pmatrix} \psi_1(\tau,\rho) - \frac{1}{\rho}f_0(\rho) \\ \psi_2(\tau,\rho) - f_0'(\rho) \end{pmatrix}$$

where

$$\widetilde{\mathbf{L}} := \widetilde{\mathbf{L}}_0 + \mathbf{L}', \tag{2.8}$$

$$\widetilde{\mathbf{L}}_0\mathbf{u}(\rho) := \begin{pmatrix} -\rho u_1'(\rho) - u_1(\rho) + u_2(\rho) \\ \Delta^{\text{rad}}_{\rho,d+2}u_1(\rho) - \rho u_2'(\rho) - 2u_2(\rho) \end{pmatrix}, \tag{2.9}$$

$$\mathbf{L}'\mathbf{u}(\rho) := \begin{pmatrix} 0 \\ -\frac{d-1}{2}\frac{\eta'(f_0(\rho))}{\rho^2}u_1(\rho), \end{pmatrix}, \tag{2.10}$$

$$\mathbf{N}(\mathbf{u})(\rho) := \begin{pmatrix} 0 \\ -\frac{d-1}{2}\frac{N(\rho u_1(\rho))}{\rho^3} \end{pmatrix}, \tag{2.11}$$

for $\mathbf{u} = (u_1, u_2)$ and

$$\eta'(f_0(\rho)) = 2\cos(2f_0(\rho)) - 2 = -16(d-2)\frac{\rho^2}{(\rho^2 + d - 2)^2}.$$

Furthermore, the initial data are given by

$$\mathbf{\Phi}(0)(\rho) = \mathbf{U}(\mathbf{v}, T)(\rho) = \begin{pmatrix} \frac{1}{\rho} f_0(\frac{T}{T_0}\rho) \\ \frac{T^2}{T_0^2} f_0'(\frac{T}{T_0}\rho) \end{pmatrix} - \begin{pmatrix} \frac{1}{\rho} f_0(\rho) \\ f_0'(\rho) \end{pmatrix} + \mathbf{V}(\mathbf{v}, T)(\rho) \tag{2.12}$$

where

$$\mathbf{V}(\mathbf{v}, T)(\rho) := \begin{pmatrix} \frac{1}{\rho} F(T\rho) \\ \frac{T}{\rho} G(T\rho) \end{pmatrix}, \ \mathbf{v} := \begin{pmatrix} F \\ G \end{pmatrix}.$$

2.3.3 Strong light-cone solutions

To proceed, we need to define what it means to be a solution to the evolution problem (2.7). We introduce the Hilbert space

$$\mathcal{H} := H_{\text{rad}}^{\frac{d+3}{2}}(\mathbb{B}^{d+2}) \times H_{\text{rad}}^{\frac{d+1}{2}}(\mathbb{B}^{d+2}).$$

In Section 4.3 we prove that the closure of the operator $\widetilde{\mathbf{L}}$, augmented with a suitable domain, generates a semigroup $\mathbf{S}(\tau)$ on $\mathcal{H}$. This allows us to formulate (2.7) as an abstract integral equation via Duhamel's formula,

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{U}(\mathbf{v}, T) + \int_0^\tau \mathbf{S}(\tau - s)\mathbf{N}\big(\Phi(s)\big)ds. \tag{2.13}$$

Eq. (3.33) yields a natural notion of strong solutions in light-cones.

Definition 2.3.1 *We say that $\psi : C_T \longrightarrow \mathbb{R}$ is a solution to (2.3) if the corresponding $\Phi : [0, \infty) \longrightarrow \mathcal{H}$ belongs to $C\big([0, \infty); \mathcal{H}\big)$ and satisfies (3.33) for all $\tau \geq 0$.*

2.4 Proof of the theorem

2.4.1 Notation

Throughout we denote by $\sigma(\mathbf{L})$, $\sigma_p(\mathbf{L})$ and $\sigma_e(\mathbf{L})$ the spectrum, point spectrum, and essential spectrum, respectively, of a linear operator $\mathbf{L}$. Furthermore, we write $\mathbf{R}_{\mathbf{L}}(\lambda) := (\lambda - \mathbf{L})^{-1}$, $\lambda \in \rho(\mathbf{L})$, for the resolvent operator where $\rho(\mathbf{L}) := \mathbb{C} \setminus \sigma(\mathbf{L})$ stands for the resolvent set. As usual, $a \lesssim b$ means $a \leq cb$ for an absolute, strictly positive constant c which may change from line to line. Similarly, we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

2.4.2 Functional setting

In the following we consider radial Sobolev functions $\hat{u} : \mathbb{B}_R^{d+2} \to \mathbb{C}$, that is, $\hat{u}(\xi) = u(|\xi|)$ for all $\xi \in \mathbb{B}_R^{d+2}$ where $u : (0, R) \to \mathbb{C}$. In particular, we define

$$u \in H^m_{\text{rad}}(\mathbb{B}_R^{d+2}) \iff \hat{u} \in H^m(\mathbb{B}_R^{d+2}) := W^{m,2}(\mathbb{B}_R^{d+2}).$$

The function space $H^m_{\text{rad}}(\mathbb{B}_R^{d+2})$ becomes a Banach space endowed with the norm

$$\|u\|_{H^m_{\text{rad}}(\mathbb{B}_R^{d+2})} = \|\hat{u}\|_{H^m(\mathbb{B}_R^{d+2})}.$$

From now, we shall not distinguish between $u(|\cdot|)$ and $\hat{u}$. In addition, we introduce the Hilbert space

$$\mathcal{H} := H^m_{\text{rad}}(\mathbb{B}^{d+2}) \times H^{m-1}_{\text{rad}}(\mathbb{B}^{d+2}), \quad m \equiv m_d := \frac{d+3}{2} \tag{2.14}$$

associated with the induced norm

$$\|\mathbf{u}\|^2 = \|(u_1, u_2)\|^2 := \|u_1\|^2_{H^m_{\text{rad}}(\mathbb{B}^{d+2})} + \|u_2\|^2_{H^{m-1}_{\text{rad}}(\mathbb{B}^{d+2})}.$$

2.4.3 Well-posedness of the linearized problem

We start with the study of the linearized problem and we convince ourselves that it is well-posed. Recall that the linear operator is given by (3.16). To proceed, we follow [32] and define the domain of the free part by

$$\mathcal{D}(\widetilde{\mathbf{L}}_0) := \Big\{\mathbf{u} \in C^\infty(0,1)^2 \cap \mathcal{H} : w_2 \in C^2([0,1]),\ w_1 \in C^3([0,1]),\ w_1''(0) = 0\Big\},$$

where, for all $\rho \in [0,1]$ and $j = 1, 2$,

$$w_j(\rho) := D_{d+2} u_j(\rho) := \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\frac{d-1}{2}} \left(\rho^d u_j(\rho)\right) = \sum_{n=0}^{\frac{d-1}{2}} c_n \rho^{n+1} u_j^{(n)}(\rho),$$

for some strictly positive constants c_n ($n = 0, 1, \dots, \frac{d-1}{2}$). Note that the density of $C^\infty(\overline{\mathbb{B}^{d+2}})$ in $H^m(\mathbb{B}^{d+2})$ implies the density of

$$\left(C^\infty_{\text{even}}[0,1]\right)^2 := \Big\{\mathbf{u} \in \left(C^\infty[0,1]\right)^2 : \ \mathbf{u}^{(2k+1)}(0) = 0, \quad k = 0, 1, 2, \dots\Big\} \subset \mathcal{D}(\widetilde{\mathbf{L}}_0)$$

in $\mathcal{H}$ which in turn proves the density of $\mathcal{D}(\widetilde{\mathbf{L}}_0)$ in $\mathcal{H}$. In other words, $\overline{\mathcal{D}(\widetilde{\mathbf{L}}_0)} = \mathcal{H}$ and $\widetilde{\mathbf{L}}_0$ is densely defined.

Proposition 2.4.1 *The operator $\widetilde{\mathbf{L}}_0 : \mathcal{D}(\widetilde{\mathbf{L}}_0) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is closable and its closure $\mathbf{L}_0 : \mathcal{D}(\mathbf{L}_0) \subset \mathcal{H} \longrightarrow \mathcal{H}$ generates a strongly continuous one-parameter semigroup $(\mathbf{S}_0(\tau))_{\tau \geq 0}$ of bounded operators on $\mathcal{H}$ satisfying the growth estimate*

$$\|\mathbf{S}_0(\tau)\| \leq M e^{-\tau} \tag{2.15}$$

for all $\tau \geq 0$ and some constant $M \geq 1$. In addition, the operator $\mathbf{L} := \mathbf{L}_0 + \mathbf{L}' : \mathcal{D}(\mathbf{L}) \subset \mathcal{H} \longrightarrow \mathcal{H}$, $\mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_0)$, is the generator of a strongly continuous semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$ on $\mathcal{H}$ and $\mathbf{L}' : \mathcal{H} \to \mathcal{H}$ is compact.

Proof. The fact that $\widetilde{\mathbf{L}}_0$ is closable and its closure generates a semigroup satisfying the growth estimate (2.15) follows from Proposition 4.9 in [32] by replacing d in [32] with $d+2$ and setting $p = 3$. It remains to apply the Bounded Perturbation Theorem to show that $\mathbf{L} := \mathbf{L}_0 + \mathbf{L}'$ is the generator of a strongly continuous semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$. In fact, we prove that $\mathbf{L}' : \mathcal{H} \longrightarrow \mathcal{H}$, defined in (3.18), is compact. We pick an arbitrary sequence $(\mathbf{u}_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}$ that is uniformly bounded. By Lemma 4.2 in [32], $(D_{d+2} u_{1,n})_{n \in \mathbb{N}}$ is uniformly bounded in $H^2(0,1)$ and the compactness of the Sobolev embedding $H^2(0,1) \hookrightarrow H^1(0,1)$ implies the existence of a subsequence, again denoted by $(D_{d+2} u_{1,n})_{n \in \mathbb{N}}$, which is Cauchy in $H^1(0,1)$. Hence, for any $n, m \in \mathbb{N}$ sufficiently large, we get

$$\begin{aligned}
\|\mathbf{L}'\mathbf{u}_n - \mathbf{L}'\mathbf{u}_m\| &\lesssim \sup_{\rho \in (0,1)} \left| \frac{\eta'(f_0(\rho))}{\rho^2} \right| \quad \|D_{d+2} u_{1,n} - D_{d+2} u_{1,m}\|_{H^1(0,1)} \\
&\simeq \sup_{\rho \in (0,1)} \frac{1}{(\rho^2 + d - 2)^2} \|D_{d+2} u_{1,n} - D_{d+2} u_{1,m}\|_{H^1(0,1)} \\
&\simeq \|D_{d+2} u_{1,n} - D_{d+2} u_{1,m}\|_{H^1(0,1)},
\end{aligned}$$

which shows that $(\mathbf{L}'\mathbf{u}_n)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}$. This proves that $\mathbf{L}'$ is compact. □

2.4.4 The spectrum of the free operator

We can use the previous decay estimate for the semigroup $(\mathbf{S}_0(\tau))_{\tau \geq 0}$ to locate the spectrum of the free operator $\mathbf{L}_0$. Indeed, by [36], p. 55, Theorem 1.10, we immediately infer

$$\sigma(\mathbf{L}_0) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -1\}. \tag{2.16}$$

2.4.5 The spectrum of the full linear operator

Next, we need to derive a suitable growth estimate for the semigroup $\mathbf{S}(\tau)$ and therefore turn our attention to the spectrum of the operator $\mathbf{L}$. To begin with, we consider the point spectrum.

Proposition 2.4.2 *We have*

$$\sigma_p(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda < 0\} \cup \{1\}.$$

Proof. We argue by contradiction and assume there exists a $\lambda \in \sigma_p(\mathbf{L}) \setminus \{1\}$ with $\mathrm{Re}\lambda \geq 0$. The latter means that there exists an element $\mathbf{u} = (u_1, u_2) \in \mathcal{D}(\mathbf{L}) \setminus \{0\}$ such that $\mathbf{u} \in \ker(\lambda - \mathbf{L})$. A straightforward calculation shows that the spectral equation $(\lambda - \mathbf{L})\mathbf{u} = 0$ implies

$$(1-\rho^2)u_1''(\rho) + \left(\frac{d+1}{\rho} - 2(\lambda+2)\rho\right)u_1'(\rho) - \left((\lambda+1)(\lambda+2) + \frac{d-1}{2}V(\rho)\right)u_1(\rho) = 0,$$

for $\rho \in (0,1)$, where

$$V(\rho) := \frac{\eta'(f_0(\rho))}{\rho^2} = \frac{-16(d-2)}{(\rho^2+d-2)^2}.$$

Since $\mathbf{u} \in \mathcal{H}$, we see that u_1 must lie in $H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})$. To proceed, we set $v_1(\rho) := \rho u_1(\rho)$. A straightforward computation implies that v_1 solves the second order ordinary differential equation

$$(1-\rho^2)v_1''(\rho) + \left(\frac{d-1}{\rho} - 2(\lambda+1)\rho\right)v_1'(\rho) - \left(\lambda(\lambda+1) + \frac{d-1}{2}\hat{V}(\rho)\right)v_1(\rho) = 0, \quad (2.17)$$

for $\rho \in (0,1)$, where

$$\hat{V}(\rho) := 2\frac{\rho^4 - 6(d-2)\rho^2 + (d-2)^2}{\rho^2(\rho^2+d-2)^2}.$$

We remark that this is the spectral equation studied in [19, 20]. Since all coefficients in (2.17) are smooth functions in $(0,1)$, we immediately get the a priori regularity $v_1 \in C^\infty(0,1)$. We claim that $v_1 \in C^\infty[0,1]$. To prove this, we employ Frobenius' method. The point $\rho = 0$ is a regular singularity with Frobenius indices $s_1 = 1$ and $s_2 = -(d-1)$. Therefore, by Frobenius theory, there exists a solution of the form

$$v_1^1(\rho) = \rho\sum_{i=0}^{\infty} x_i\rho^i = \sum_{i=0}^{\infty} x_i\rho^{i+1},$$

which is analytic locally around $\rho = 0$. Moreover, since $s_1 - s_2 = d \in \mathbb{N}_{\text{odd}}$, there exists a second linearly independent solution of the form

$$v_1^2(\rho) = C\log(\rho)v_1^1(\rho) + \rho^{-(d-1)}\sum_{i=0}^{\infty} y_i\rho^i$$

for some constant $C \in \mathbb{C}$ and $y_0 = 1$. However, $v_1^2(\rho)/\rho$ does not lie in the Sobolev space $H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})$ due to the strong singularity in the second term, no matter the value of the constant C. Consequently, v_1 must be a multiple of v_1^1 and we infer $v_1 \in C^\infty[0,1)$. Similarly, the point $\rho = 1$ is a regular singularity with Frobenius indices $s_1 = 0$ and

$s_2 = \frac{d-1}{2} - \lambda$. Now we need to distinguish different cases. If $\frac{d-1}{2} - \lambda \notin \mathbb{Z}$, we have two linearly independent solutions of the form

$$v_1^1(\rho) = \sum_{i=0}^{\infty} x_i (1-\rho)^i,$$
$$v_1^2(\rho) = (1-\rho)^{\frac{d-1}{2}-\lambda} \sum_{i=0}^{\infty} y_i (1-\rho)^i$$

with $x_0 = y_0 = 1$. The solution $v_1^2(\rho)/\rho$ does not belong to the Sobolev space $H_{\text{rad}}^{\frac{d+3}{2}}(\mathbb{B}^{d+2})$ and thus, $v_1 \in C^\infty[0,1]$. In the case $\frac{d-1}{2} - \lambda := k \in \mathbb{N}_0$, we have two fundamental solutions of the form

$$v_1^1(\rho) = (1-\rho)^k \sum_{i=0}^{\infty} x_i (1-\rho)^i, \qquad x_0 = 1$$
$$v_1^2(\rho) = \sum_{i=0}^{\infty} y_i (1-\rho)^i + C \log(1-\rho) v_1^1(\rho), \qquad y_0 = 1$$

near $\rho = 1$. By assumption, $\text{Re}\lambda \geq 0$ and thus, $k \leq \frac{d-1}{2}$. Hence, $v_1^2(\rho)/\rho$ does not lie in the Sobolev space $H_{\text{rad}}^{\frac{d+3}{2}}(\mathbb{B}^{d+2})$ unless $C = 0$ and we conclude $v_1 \in C^\infty[0,1]$. Finally, if $\frac{d-1}{2} - \lambda =: -k$ is a negative integer, the fundamental system around $\rho = 1$ has the form

$$v_1^1(\rho) = \sum_{i=0}^{\infty} x_i (1-\rho)^i$$
$$v_1^2(\rho) = C \log(1-\rho) v_1^1(\rho) + (1-\rho)^{-k} \sum_{i=0}^{\infty} y_i (1-\rho)^i$$

with $x_0 = y_0 = 1$. Again, $v_1^2(\rho)/\rho$ does not belong to $H_{\text{rad}}^{\frac{d+3}{2}}(\mathbb{B}^{d+2})$ and we infer $v_1 \in C^\infty[0,1]$ also in this case. In summary, we have found a nontrivial solution $v_1 \in C^\infty[0,1]$ to Eq. (2.17) with $\text{Re}\lambda \geq 0$, $\lambda \neq 1$, but this contradicts [19, 20]. □

The fact that $\mathbf{L}'$ is compact implies that the result on the point spectrum from Proposition 2.4.2 is already sufficient to obtain the same information on the full spectrum.

Corollary 2.4.3 *We have*

$$\sigma(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \quad \text{Re}\lambda < 0\} \cup \{1\}.$$

Proof. Suppose there exists a $\lambda \in \sigma(\mathbf{L}) \setminus \{1\}$ with $\text{Re}\lambda \geq 0$. Then $\lambda \notin \sigma(\mathbf{L}_0)$ and thus, $\mathbf{R}_{\mathbf{L}_0}(\lambda)$ exists. From the identity $\lambda - \mathbf{L} = [1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda - \mathbf{L}_0)$ we see that $1 \in \sigma(\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda))$. Since $\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$ is compact, it follows that $1 \in \sigma_p(\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda))$ and thus, there exists a nontrivial $\mathbf{f} \in \mathcal{H}$ such that $[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]\mathbf{f} = 0$. Consequently, $\mathbf{u} := \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f} \neq 0$ satisfies $(\lambda - \mathbf{L})\mathbf{u} = 0$ and thus, $\lambda \in \sigma_p(\mathbf{L})$. This contradicts Proposition 2.4.2. □

Next, we provide a uniform bound on the resolvent. To this end, we define

$$\Omega_{\epsilon,R} := \{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda \geq -1+\epsilon, |\lambda| \geq R\}$$

for $\epsilon, R > 0$.

Proposition 2.4.4 *Let $\epsilon > 0$. Then there exist constants $R_\epsilon, C_\epsilon > 0$ such that the resolvent $\mathbf{R_L}$ exists on $\Omega_{\epsilon,R_\epsilon}$ and satisfies*

$$\|\mathbf{R_L}(\lambda)\| \leq C_\epsilon$$

for all $\lambda \in \Omega_{\epsilon,R_\epsilon}$.

Proof. Fix $\epsilon > 0$ and take $\lambda \in \Omega_{\epsilon,R}$ for an arbitrary $R > 0$. Then $\lambda \in \rho(\mathbf{L}_0)$ and the identity $(\lambda - \mathbf{L}) = [1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda - \mathbf{L}_0)$ shows that $\mathbf{R_L}(\lambda)$ exists if and only if $1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$ is invertible. By a Neumann series argument this is the case if $\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\| < 1$.
To prove smallness of $\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$, we recall the definition of $\mathbf{L}'$, Eq. (3.18),

$$\mathbf{L}'\mathbf{u}(\rho) = \begin{pmatrix} 0 \\ -\frac{d-1}{2}V(\rho)u_1(\rho) \end{pmatrix}, \quad V(\rho) = \frac{\eta'(f_0(\rho))}{\rho^2} = \frac{-16(d-2)}{(\rho^2+d-2)^2}.$$

Let $\mathbf{u} = \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}$ or, equivalently, $(\lambda - \mathbf{L}_0)\mathbf{u} = \mathbf{f}$. The latter equation implies

$$(\lambda+1)u_1(\rho) = u_2(\rho) - \rho u_1'(\rho) + f_1(\rho).$$

Now we use Lemma 4.1 from [32] and $\|V^{(k)}\|_{L^\infty(0,1)} \lesssim 1$ for all $k \in \{0, 1, \ldots, m-1\}$ to obtain

$$\begin{aligned} |\lambda+1|\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}\| = |\lambda+1|\|\mathbf{L}'\mathbf{u}\| &\simeq \left\|V\left(u_2 - (\cdot)u_1' + f_1\right)\right\|_{H^{m-1}_{\mathrm{rad}}(\mathbb{B}^{d+2})} \\ &\lesssim \|u_2\|_{H^{m-1}_{\mathrm{rad}}(\mathbb{B}^{d+2})} + \|(\cdot)u_1'\|_{H^{m-1}_{\mathrm{rad}}(\mathbb{B}^{d+2})} + \|f_1\|_{H^{m-1}_{\mathrm{rad}}(\mathbb{B}^{d+2})} \\ &\lesssim \|u_2\|_{H^{m-1}_{\mathrm{rad}}(\mathbb{B}^{d+2})} + \|u_1\|_{H^{m}_{\mathrm{rad}}(\mathbb{B}^{d+2})} + \|f_1\|_{H^{m-1}_{\mathrm{rad}}(\mathbb{B}^{d+2})} \\ &\simeq \|\mathbf{u}\| + \|\mathbf{f}\| \lesssim \left(\frac{1}{\mathrm{Re}\lambda + 1} + 1\right)\|\mathbf{f}\| \\ &\lesssim \|\mathbf{f}\|, \end{aligned}$$

where we have used the bound

$$\|\mathbf{u}\| = \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}\| \leq \frac{1}{\mathrm{Re}\lambda + 1}\|\mathbf{f}\|$$

which follows from semigroup theory, see [36], p. 55, Theorem 1.10. In other words,

$$\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\| \lesssim \frac{1}{|\lambda+1|} \leq \frac{1}{|\lambda|-1} \leq \frac{1}{R-1}$$

and by choosing R sufficiently large, we can achieve the desired $\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\| < 1$. As a consequence, $[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]^{-1}$ exists and we obtain the bound

$$\begin{aligned}\|\mathbf{R}_{\mathbf{L}}(\lambda)\| &= \|\mathbf{R}_{\mathbf{L}_0}(\lambda)[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]^{-1}\| \\ &\leq \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\|\|[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]^{-1}\| \\ &\leq \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\| \sum_{i=0}^{\infty} \|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\|^i \\ &\leq C_\epsilon.\end{aligned}$$

□

2.4.6 The eigenspace of the isolated eigenvalue

In this section, we convince ourselves that the eigenspace of the isolated eigenvalue $\lambda = 1$ for the full linear operator $\mathbf{L}$ is spanned by

$$\mathbf{g}(\rho) := \begin{pmatrix} g_1(\rho) \\ g_2(\rho) \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho^2+d-2} \\ \frac{2(d-2)}{(\rho^2+d-2)^2} \end{pmatrix}, \quad \rho \in [0,1]. \tag{2.18}$$

Consequently, we are looking for all $\mathbf{u} = (u_1, u_2) \in \mathcal{D}(\mathbf{L}) \setminus \{0\}$ such that $\mathbf{u} \in \ker(1 - \mathbf{L})$. A straightforward calculation shows that the spectral equation $(1 - \mathbf{L})\mathbf{u} = 0$ is equivalent to the following system of ordinary differential equations,

$$\begin{cases} u_2(\rho) = \rho u_1'(\rho) + 2u_1(\rho), \\ \big(1-\rho^2\big)u_1''(\rho) + \Big(\frac{d+1}{\rho} - 6\rho\Big)u_1'(\rho) - \Big(6 + \frac{d-1}{2}\frac{\eta'(f_0(\rho))}{\rho^2}\Big)u_1(\rho) = 0, \end{cases}$$

for $\rho \in (0,1)$. One can verify that a fundamental system of the second equation is given by

$$\left\{\frac{1}{\rho^2+d-2}, \ \frac{Q_{d-1}(\rho)}{\rho^d(\rho^2+d-2)}\right\}$$

where Q_{d-1} is a polynomial of degree $d-1$ with non-vanishing constant term. We can write the general solution for the second equation as

$$u_1(\rho) = C_1 \frac{1}{\rho^2+d-2} + C_2 \frac{Q_{d-1}(\rho)}{\rho^d(\rho^2+d-2)}.$$

We must ensure that $\mathbf{u} \in \mathcal{D}(\mathbf{L})$ which in particular implies that u_1 must lie in the Sobolev space $H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})$. This requirement yields $C_2 = 0$ which in turn gives $\mathbf{u} \in \langle \mathbf{g} \rangle$. In conclusion,

$$\ker(1 - \mathbf{L}) = \langle \mathbf{g} \rangle, \tag{2.19}$$

as initially claimed.

2.4.7 Time evolution for the linearized problem

We now focus on the time evolution for the linearized problem (2.7). Due to the presence of the eigenvalue $\lambda = 1$, there exists a one dimensional subspace $\langle \mathbf{g} \rangle$ of initial data for which the solution grows exponentially in time. We call this subspace the unstable space. On the other hand, initial data from the stable subspace lead to solutions that decay exponentially in time. As we will show now, this time evolution estimates can be established using semigroup theory together with the previous results on the spectrum of the linear operators $\mathbf{L}_0$ and $\mathbf{L}$. To make this rigorous, we follow [32] and use the fact that the unstable eigenvalue $\lambda = 1$ is isolated to introduce a (non-orthogonal) projection $\mathbf{P}$. This projection decomposes the Hilbert space of initial data $\mathcal{H}$ into the stable and the unstable space. Most importantly, we must ensure that $\langle \mathbf{g} \rangle$ is the only unstable direction in $\mathcal{H}$. This is the key statement of the following proposition and it is equivalent to the fact that the algebraic multiplicity of the isolated eigenvalue $\lambda = 1$,

$$m_a(\lambda = 1) := \operatorname{rank} \mathbf{P} = \dim \operatorname{rg} \mathbf{P},$$

is equal to one. We denote by $\mathcal{B}(\mathcal{H})$ the set of bounded operators from $\mathcal{H}$ to itself and prove the following result.

Proposition 2.4.5 *There exists a projection*

$$\mathbf{P} \in \mathcal{B}(\mathcal{H}), \quad \mathbf{P} : \mathcal{H} \longrightarrow \langle \mathbf{g} \rangle,$$

which commutes with the semigroup $\big(\mathbf{S}(\tau)\big)_{\tau \geq 0}$. *In addition, we have*

$$\mathbf{S}(\tau)\mathbf{P}\mathbf{f} = e^{\tau}\mathbf{P}\mathbf{f}, \tag{2.20}$$

and there exist constants $C, \epsilon > 0$ *such that*

$$\|(1 - \mathbf{P})\mathbf{S}(\tau)\mathbf{f}\| \leq Ce^{-\epsilon\tau}\|(1 - \mathbf{P})\mathbf{f}\|, \tag{2.21}$$

for all $\mathbf{f} \in \mathcal{H}$ *and* $\tau \geq 0$.

Proof. We argue along the lines of [32]. Since the eigenvalue $\lambda = 1$ is isolated, we can define the spectral projection

$$\mathbf{P} : \mathcal{H} \longrightarrow \mathcal{H}, \quad \mathbf{P} := \frac{1}{2\pi i}\int_{\gamma} \mathbf{R}_{\mathbf{L}}(\mu) d\mu,$$

where $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ is a positively orientated circle around $\lambda = 1$ with radius so small that $\gamma\big([0, 2\pi]\big) \subseteq \rho(\mathbf{L})$, see e.g. [43]. The projection $\mathbf{P}$ commutes with the operator $\mathbf{L}$ and thus with the semigroup $\mathbf{S}(\tau)$. Moreover, $\mathbf{P}$ decomposes the Hilbert space as $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M} := \operatorname{rg} \mathbf{P}$ and $\mathcal{N} := \operatorname{rg}(1 - \mathbf{P}) = \ker \mathbf{P}$. Most importantly, the operator $\mathbf{L}$ is decomposed accordingly into the parts $\mathbf{L}_{\mathcal{M}}$ and $\mathbf{L}_{\mathcal{N}}$ on $\mathcal{M}$ and $\mathcal{N}$, respectively. The spectra of these operators are given by

$$\sigma\left(\mathbf{L}_{\mathcal{N}}\right) = \sigma(\mathbf{L}) \setminus \{1\}, \qquad \sigma\left(\mathbf{L}_{\mathcal{M}}\right) = \{1\}. \tag{2.22}$$

We refer the reader to [43] for these standard results.
To proceed, we break down the proof into the following steps:

Step 1: We prove that $\operatorname{rank}\mathbf{P} := \dim \operatorname{rg}\mathbf{P} < +\infty$. We argue by contradiction and assume that $\operatorname{rank}\mathbf{P} = +\infty$. Using [43], p. 239, Theorem 5.28, the fact that $\mathbf{L}'$ is compact (see Proposition 2.4.1), and the fact that the essential spectrum is stable under compact perturbations ([43], p. 244, Theorem 5.35), we obtain

$$\operatorname{rank}\mathbf{P} = +\infty \Longrightarrow 1 \in \sigma_e(\mathbf{L}) = \sigma_e(\mathbf{L} - \mathbf{L}') = \sigma_e(\mathbf{L}_0) \subseteq \sigma(\mathbf{L}_0).$$

This contradicts (2.16).

Step 2: We prove that $\langle \mathbf{g} \rangle = \operatorname{rg}\mathbf{P}$. It suffices to show $\operatorname{rg}\mathbf{P} \subseteq \langle \mathbf{g} \rangle$ since the reverse inclusion follows from the abstract theory. From Step 1, the operator $1 - \mathbf{L}_{\mathcal{M}}$ acts on the finite-dimensional Hilbert space $\mathcal{M} = \operatorname{rg}\mathbf{P}$ and, from (3.37), $\lambda = 0$ is its only spectral point. Hence, $1 - \mathbf{L}_{\mathcal{M}}$ is nilpotent, i.e., there exists a $k \in \mathbb{N}$ such that

$$\left(1 - \mathbf{L}_{\mathcal{M}}\right)^k \mathbf{u} = 0$$

for all $\mathbf{u} \in \operatorname{rg}\mathbf{P}$ and we assume k to be minimal. Recall (2.19) to see that the claim follows immediately if $k = 1$. We proceed by contradiction and assume that $k \geq 2$. Then, there exists a nontrivial function $\mathbf{u} \in \operatorname{rg}\mathbf{P} \subseteq \mathcal{D}(\mathbf{L})$ such that $(1 - \mathbf{L}_{\mathcal{M}})\mathbf{u}$ is nonzero and belongs to $\ker(1 - \mathbf{L}_{\mathcal{M}}) \subseteq \ker(1 - \mathbf{L}) = \langle \mathbf{g} \rangle$. This means that $\mathbf{u} \in \operatorname{rg}\mathbf{P} \subseteq \mathcal{D}(\mathbf{L})$ satisfies $(1 - \mathbf{L})\mathbf{u} = \alpha \mathbf{g}$, for some $\alpha \in \mathbb{C} \setminus \{0\}$. Without loss of generality we set $\alpha = -1$ and a straightforward computation shows that the first component of $\mathbf{u}$ solves the second order differential equation

$$\left(1 - \rho^2\right) u_1''(\rho) + \left(\frac{d+1}{\rho} - 6\rho\right) u_1'(\rho) - \left(6 + \frac{d-1}{2} \frac{\eta'(f_0(\rho))}{\rho^2}\right) u_1(\rho) = G(\rho),$$

for $\rho \in (0, 1)$, where

$$G(\rho) := \frac{\rho^2 + 5(d-2)}{(\rho^2 + d - 2)^2}, \quad \rho \in [0, 1].$$

In order to find the general solution to this equation, recall (2.18) to see that

$$\hat{u}_1(\rho) := g_1(\rho) = \frac{1}{\rho^2 + d - 2}, \quad \rho \in (0, 1)$$

is a particular solution to the homogeneous equation

$$\left(1 - \rho^2\right) u_1''(\rho) + \left(\frac{d+1}{\rho} - 6\rho\right) u_1'(\rho) - \left(6 + \frac{d-1}{2} \frac{\eta'(f_0(\rho))}{\rho^2}\right) u_1(\rho) = 0.$$

To find another linearly independent solution, we use the Wronskian

$$\mathcal{W}(\rho) := (1-\rho^2)^{\frac{d-5}{2}}\rho^{-d-1}$$

to obtain

$$\hat{u}_2(\rho) := \hat{u}_1(\rho)\int_{\rho_1}^{\rho}(1-x^2)^{\frac{d-5}{2}}x^{-d-1}(x^2+d-2)^2dx,$$

for some constant $\rho_1 \in (0,1)$ and for all $\rho \in (0,1)$. Note that we have the expansion

$$\hat{u}_2(\rho) = \rho^{-d}\sum_{j=0}^{\infty}a_j\rho^j, \quad a_0 \neq 0$$

near $\rho = 0$. Furthermore, if $d \geq 5$, $\hat{u}_2 \in C^\infty(0,1]$ and we choose $\rho_1 = 1$ which yields the expansion

$$\hat{u}_2(\rho) = (1-\rho)^{\frac{d-3}{2}}\sum_{j=0}^{\infty}b_j(1-\rho)^j, \qquad b_0 \neq 0$$

near $\rho = 1$. For $d = 3$, we set $\rho_1 = \frac{1}{2}$ and the expansion of $\hat{u}_2$ near $\rho = 1$ contains a term $\log(1-\rho)$. We invoke the variation of constants formula to see that u_1 can be expressed as

$$\begin{aligned} u_1(\rho) &= c_1\hat{u}_1(\rho) + c_2\hat{u}_2(\rho) \\ &\quad + \hat{u}_2(\rho)\int_0^{\rho}\frac{\hat{u}_1(y)G(y)y^{d+1}}{(1-y^2)^{\frac{d-3}{2}}}dy - \hat{u}_1(\rho)\int_0^{\rho}\frac{\hat{u}_2(y)G(y)y^{d+1}}{(1-y^2)^{\frac{d-3}{2}}}dy, \end{aligned}$$

for some constants $c_1, c_2 \in \mathbb{C}$ and for all $\rho \in (0,1)$. The fact that $u_1 \in H_{\text{rad}}^{\frac{d+3}{2}}(\mathbb{B}^{d+2})$ implies $c_2 = 0$ and we are left with

$$u_1(\rho) = c_1\hat{u}_1(\rho) + \hat{u}_2(\rho)\int_0^{\rho}\frac{\hat{u}_1(y)G(y)y^{d+1}}{(1-y^2)^{\frac{d-3}{2}}}dy - \hat{u}_1(\rho)\int_0^{\rho}\frac{\hat{u}_2(y)G(y)y^{d+1}}{(1-y^2)^{\frac{d-3}{2}}}dy. \tag{2.23}$$

If $d = 3$, $\hat{u}_2(\rho) \simeq \log(1-\rho)$ near $\rho = 1$ and thus, the last term in Eq. (2.23) stays bounded as $\rho \to 1-$ whereas the second term diverges unless

$$\int_0^1\frac{\hat{u}_1(y)G(y)y^{d+1}}{(1-y^2)^{\frac{d-3}{2}}}dy = 0,$$

which, however, is impossible since the integrand is strictly positive on $(0,1)$. This contradicts $u_1 \in H_{\text{rad}}^{\frac{d+3}{2}}(\mathbb{B}^{d+2})$ and we arrive at the desired $k = 1$.
Next, we focus on $d \geq 5$, where the last term in Eq. (2.23) is smooth on $[0,1]$. To analyze the second term, we set

$$\mathcal{I}_d(\rho) := \hat{u}_2(\rho)\int_0^{\rho}\frac{F_d(y)}{(1-y)^{\frac{d-3}{2}}}dy, \;\; F_d(y) := \frac{\hat{u}_1(y)G(y)y^{d+1}}{(1+y)^{\frac{d-3}{2}}} = \frac{y^{d+1}(y^2+5(d-2))}{(1+y)^{\frac{d-3}{2}}(y^2+d-2)^3}. \tag{2.24}$$

Note that $F_5(1) \neq 0$ and thus, the expansion of $\mathcal{I}_5(\rho)$ near $\rho = 1$ contains a term of the form $(1-\rho)\log(1-\rho)$. Consequently, $\mathcal{I}_5'' \notin L^2(\frac{1}{2}, 1)$ and this is a contradiction to $u_1 \in H^4_{\text{rad}}(\mathbb{B}^7)$. The general case is postponed to the appendix (Proposition 2.5.2) where it is shown that the function $\mathcal{I}_d$ is not analytic at $\rho = 1$. This implies that the expansion of $\mathcal{I}_d(\rho)$ near $\rho = 1$ contains a term $(1-\rho)^{\frac{d-3}{2}} \log(1-\rho)$ which again contradicts $u_1 \in H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})$.

Step 3: Finally, we prove the estimates (2.20) and (2.21) for the semigroup. First, note that (2.20) follows immediately from the facts that $\lambda = 1$ is an eigenvalue of $\mathbf{L}$ with eigenfunction $\mathbf{g}$ and $\operatorname{rg} \mathbf{P} = \langle \mathbf{g} \rangle$. Furthermore, from Corollary 2.4.3 and Proposition 2.4.4 we infer the existence of $C, \epsilon > 0$ such that

$$\|\mathbf{R}_{\mathbf{L}}(\lambda)(1 - \mathbf{P})\| \leq C$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -2\epsilon$. Consequently, the Gearhart-Prüss Theorem, see [36], p. 302, Theorem 1.11, yields the bound (2.21). □

2.4.8 Estimates for the nonlinearity

The aim of this section is to establish a Lipschitz-type estimate for the nonlinearity. Recall that the nonlinear term in (2.7) is given by

$$\mathbf{N}(\mathbf{u})(\rho) = \begin{pmatrix} 0 \\ \hat{N}(\rho, u_1(\rho)) \end{pmatrix} := \begin{pmatrix} 0 \\ -\frac{d-1}{2} \frac{N(\rho u_1(\rho))}{\rho^3} \end{pmatrix}.$$

To begin with, we claim that

$$\begin{aligned} &\hat{N}(\rho, u_1(\rho)) \\ &= 4(d-1)u_1^2(\rho) \int_0^1 \int_0^1 \int_0^1 \cos\left(2z\left(f_0(\rho) + xy\rho u_1(\rho)\right)\right) \left(\frac{f_0(\rho)}{\rho} + xyu_1(\rho)\right) xdzdydx. \end{aligned}$$

To see this, we use the fundamental theorem of calculus and the fact that $\eta''(0) = 0$ to write

$$\begin{aligned}
N(\rho u_1(\rho)) &= \eta(f_0(\rho) + \rho u_1(\rho)) - \eta(f_0(\rho)) - \eta'(f_0(\rho))\rho u_1(\rho) \\
&= \int_{f_0(\rho)}^{f_0(\rho)+\rho u_1(\rho)} \eta'(s)ds - \eta'(f_0(\rho))\rho u_1(\rho) \\
&= \rho u_1(\rho) \int_0^1 \eta'(f_0(\rho) + x\rho u_1(\rho))dx - \eta'(f_0(\rho))\rho u_1(\rho) \\
&= \rho u_1(\rho) \int_0^1 \left(\eta'(f_0(\rho) + x\rho u_1(\rho)) - \eta'(f_0(\rho))\right) dx \\
&= \rho u_1(\rho) \int_0^1 \left(\int_{f_0(\rho)}^{f_0(\rho)+x\rho u_1(\rho)} \eta''(s)ds \right) dx \\
&= \rho^2 u_1^2(\rho) \int_0^1 x \int_0^1 \eta''(f_0(\rho) + xy\rho u_1(\rho))dydx \\
&= \rho^2 u_1^2(\rho) \int_0^1 x \int_0^1 \int_0^{f_0(\rho)+xy\rho u_1(\rho)} \eta'''(s)dsdydx \\
&= \rho^2 u_1^2(\rho) \int_0^1 x \int_0^1 \int_0^1 \eta''' \left((f_0(\rho) + xy\rho u_1(\rho))z\right) \left(f_0(\rho) + xy\rho u_1(\rho)\right) dzdydx \\
&= \rho^3 u_1^2(\rho) \int_0^1 x \int_0^1 \int_0^1 \eta''' \left((f_0(\rho) + xy\rho u_1(\rho))z\right) \left(\frac{f_0(\rho)}{\rho} + xyu_1(\rho)\right) dzdydx.
\end{aligned}$$

For later purposes, we note that the function

$$\hat{N}(\rho, \zeta) = 4(d-1)\zeta^2 \int_0^1 \int_0^1 \int_0^1 \cos\left(2z\left(f_0(\rho) + xy\rho\zeta\right)\right) \left(\frac{f_0(\rho)}{\rho} + xy\zeta\right) xdzdydx,$$

defined for all $(\rho, \zeta) \in [0, 1] \times \mathbb{R}$, is perfectly smooth in both variables since

$$\frac{f_0(\rho)}{\rho} = \frac{2}{\rho} \arctan\left(\frac{\rho}{\sqrt{d-2}}\right)$$

is smooth at $\rho = 0$. Moreover, we define

$$M(\rho, \zeta) := \partial_\zeta \hat{N}(\rho, \zeta) = 4(d-1)\left(A(\rho, \zeta) + B(\rho, \zeta) + C(\rho, \zeta) + D(\rho, \zeta)\right), \qquad (2.25)$$

where

$$
\begin{aligned}
A(\rho,\zeta) &:= 2\frac{f_0(\rho)}{\rho}\zeta \int_0^1\int_0^1\int_0^1 \cos\left(2z\left(f_0(\rho)+xy\rho\zeta\right)\right) xdzdydx,\\
B(\rho,\zeta) &:= -2f_0(\rho)\zeta^2 \int_0^1\int_0^1\int_0^1 \sin\left(2z\left(f_0(\rho)+xy\rho\zeta\right)\right) x^2yzdzdydx,\\
C(\rho,\zeta) &:= 3\zeta^2 \int_0^1\int_0^1\int_0^1 \cos\left(2z\left(f_0(\rho)+xy\rho\zeta\right)\right) x^2ydzdydx,\\
D(\rho,\zeta) &:= -2\rho\zeta^3 \int_0^1\int_0^1\int_0^1 \sin\left(2z\left(f_0(\rho)+xy\rho\zeta\right)\right) x^3y^2zdzdydx.
\end{aligned}
$$

We denote by $\mathcal{B}_\delta \subseteq \mathcal{H}$ the ball of radius δ in $\mathcal{H}$ centered at zero, i.e.,

$$
\mathcal{B}_\delta := \left\{ \mathbf{u}\in\mathcal{H}:\ \|\mathbf{u}\| = \|(u_1,u_2)\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})\times H^{\frac{d+1}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq \delta \right\}.
$$

The main result of this section is the following Lipschitz-type estimate.

Lemma 2.4.6 *Let $\delta>0$. Then we have*

$$
\big\|\mathbf{N}(\mathbf{u})-\mathbf{N}(\mathbf{v})\big\| \lesssim (\|\mathbf{u}\|+\|\mathbf{v}\|)\|\mathbf{u}-\mathbf{v}\| \tag{2.26}
$$

for all $\mathbf{u},\mathbf{v}\in\mathcal{B}_\delta$.

Proof. We start by fixing a $\delta>0$, we pick two elements $\mathbf{u},\mathbf{v}\in\mathcal{B}_\delta$ and define the auxiliary function

$$
\zeta(\sigma)(\rho) = \sigma u_1(\rho) + (1-\sigma)v_1(\rho),
$$

for $\rho\in(0,1)$ and $\sigma\in[0,1]$. The triangle inequality implies

$$
\mathbf{u},\mathbf{v}\in\mathcal{B}_\delta \Longrightarrow \|u_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq \delta,\ \|v_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq \delta \Longrightarrow \|\zeta(\sigma)\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq \delta,
$$

for all $\sigma\in[0,1]$. In other words,

$$
\zeta(\sigma)\in\mathscr{B}_\delta := \left\{ f\in H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2}):\ \|f\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq \delta \right\},
$$

for all $\sigma\in[0,1]$. Now, we claim that to show (3.45), it suffices to establish the estimate

$$
\|M(\cdot,f(\cdot))\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \lesssim \|f\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \tag{2.27}
$$

for all $f\in\mathscr{B}_\delta$, where M is given by (2.25). To see this, we use the algebra property

$$
\|fg\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2})} \lesssim \|f\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2})}\|g\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2})},
$$

which holds since $\frac{d+3}{2} > \frac{d+2}{2}$, to estimate

$$
\begin{aligned}
\big\|\mathbf{N}(\mathbf{u})-\mathbf{N}(\mathbf{v})\big\| &= \big\|\hat{N}(\cdot,u_1(\cdot))-\hat{N}(\cdot,v_1(\cdot))\big\|_{H^{\frac{d+1}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \\
&\leq \big\|\hat{N}(\cdot,u_1(\cdot))-\hat{N}(\cdot,v_1(\cdot))\big\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \\
&= \left\|\int_{v_1(\cdot)}^{u_1(\cdot)} \partial_2\hat{N}(\cdot,\zeta)d\zeta\right\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \\
&= \left\|(u_1(\cdot)-v_1(\cdot))\int_0^1 \partial_2\hat{N}(\cdot,\underbrace{\sigma u_1(\cdot)+(1-\sigma)v_1(\cdot)}_{\zeta(\sigma)})d\sigma\right\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \\
&\lesssim \|u_1-v_1\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2})}\left\|\int_0^1 \partial_2\hat{N}(\cdot,\zeta(\sigma))d\sigma\right\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \\
&\lesssim \|u_1-v_1\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2})}\int_0^1 \|M(\cdot,\zeta(\sigma)(\cdot))\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} d\sigma \\
&\lesssim \|u_1-v_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}\int_0^1 \|\zeta(\sigma)\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} d\sigma \\
&\lesssim \|u_1-v_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}\int_0^1 \left(\sigma\|u_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}+(1-\sigma)\|v_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}\right)d\sigma \\
&\lesssim \|u_1-v_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}\left(\|u_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}+\|v_1\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}\right) \\
&\lesssim \|\mathbf{u}-\mathbf{v}\|\left(\|\mathbf{u}\|+\|\mathbf{v}\|\right).
\end{aligned}
$$

It remains to prove (2.27). To this end we use a simple extension argument (see e.g. Lemmas B.1 and B.2 in [32]) and Moser's inequality ([70], p. 224, Theorem 6.4.1) to infer the existence of a smooth function $h : [0,\infty) \to [0,\infty)$ such that

$$\|M(\cdot,f(\cdot))\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq h\left(\|f\|_{L^\infty(\mathbb{B}^{d+2})}\right)\|f\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})}$$

for all $f \in \mathscr{B}_\delta$. By Sobolev embedding we have $\|f\|_{L^\infty(\mathbb{B}^{d+2})} \lesssim \|f\|_{H^{\frac{d+3}{2}}_{\text{rad}}(\mathbb{B}^{d+2})} \leq \delta$ for all $f \in \mathscr{B}_\delta$ and (2.27) follows. This concludes the proof. □

2.4.9 The abstract nonlinear Cauchy problem

In this section, we focus on the existence and uniqueness of solutions to the Cauchy problem (2.7). In fact, by appealing to Definition 3.9.1, we consider the integral equation

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}(\tau-s)\mathbf{N}\big(\Phi(s)\big)ds, \tag{2.28}$$

for all $\tau \geq 0$ and $\mathbf{u} \in \mathcal{H}$. We introduce the Banach space

$$\mathcal{X} := \{\Phi \in C([0,\infty);\mathcal{H}) : \quad \|\Phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{\epsilon\tau}\|\Phi(\tau)\| < +\infty\}$$

with $\epsilon > 0$ from Proposition 2.4.5. Moreover, we denote by $\mathcal{X}_\delta$ the closed ball

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \leq \delta\} = \left\{\Phi \in C([0,\infty);\mathcal{H}) : \|\Phi\| \leq \delta e^{-\epsilon\tau}, \quad \forall\tau > 0\right\}.$$

In the following, we will only sketch the rest of the proof and discuss the main arguments since they are analogous to [26, 27, 29, 30, 32]. To prove the main theorem, we would like to apply a fixed point argument to the integral equation (2.28). However, the exponential growth of the solution operator on the unstable subspace prevents from doing this directly. We overcome this obstruction by subtracting the correction term[1]

$$\mathbf{C}(\Phi,\mathbf{u}) := \mathbf{P}\left(\mathbf{u} + \int_0^\infty e^{-s}\mathbf{N}\big(\Phi(s)\big)ds\right) \tag{2.29}$$

from the initial data. Consequently, we consider the fixed point problem

$$\Phi(\tau) = \mathbf{K}(\Phi,\mathbf{u})(\tau) \tag{2.30}$$

where

$$\mathbf{K}(\Phi,\mathbf{u})(\tau) := \mathbf{S}(\tau)[\mathbf{u} - \mathbf{C}(\Phi,\mathbf{u})] + \int_0^\tau \mathbf{S}(\tau - s)\mathbf{N}\big(\Phi(s)\big)ds. \tag{2.31}$$

This modification stabilizes the evolution as the following result shows.

Theorem 2.4.7 *There exist constants $\delta, C > 0$ such that for every $\mathbf{u} \in \mathcal{H}$ with $\|\mathbf{u}\| \leq \frac{\delta}{C}$, there exists a unique $\mathbf{\Phi}(\mathbf{u}) \in \mathcal{X}_\delta$ that satisfies*

$$\mathbf{\Phi}(\mathbf{u}) = \mathbf{K}(\mathbf{\Phi}(\mathbf{u}),\mathbf{u}).$$

In addition, $\mathbf{\Phi}(\mathbf{u})$ is unique in the whole space $\mathcal{X}$ and the solution map $\mathbf{u} \mapsto \mathbf{\Phi}(\mathbf{u})$ is Lipschitz continuous.

Proof. The proof is based on a fixed point argument and the essential ingredient is the Lipschitz estimate (2.26) for the nonlinearity. Although the proof coincides with the one of Theorem 4.13 in [32], we sketch the main points for the sake of completeness. We pick $\delta > 0$ sufficiently small and fix $\mathbf{u} \in \mathcal{H}$ with $\|\mathbf{u}\| \leq \frac{\delta}{C}$, where $C > 0$ is sufficiently large. First, note that the continuity of the map

$$\mathbf{K}(\Phi,\mathbf{u}) : [0,\infty) \longrightarrow \mathcal{H}, \quad \tau \longmapsto \mathbf{K}(\Phi,\mathbf{u})(\tau)$$

[1] All integrals here exist as Riemann integrals over continuous functions.

follows immediately from the strong continuity of the semigroup $(\mathbf{S}(\tau))_{\tau>0}$. Next, to show that $\mathbf{K}(\cdot, \mathbf{u})$ maps $\mathcal{X}_\delta$ to itself, we pick an arbitrary $\Phi \in \mathcal{X}_\delta$ and decompose the operator according to

$$\mathbf{K}(\Phi, \mathbf{u})(\tau) = \mathbf{P}\mathbf{K}(\Phi, \mathbf{u})(\tau) + (1 - \mathbf{P})\mathbf{K}(\Phi, \mathbf{u})(\tau).$$

The Lipschitz bound (2.26) implies

$$\|\mathbf{N}\left(\Phi(\tau)\right)\| \lesssim \delta^2 e^{-2\epsilon\tau}$$

and together with the time evolution estimates for the semigroup on the unstable and stable subspaces (see Proposition 2.4.5), we get

$$\|\mathbf{P}\mathbf{K}\left(\Phi, \mathbf{u}\right)(\tau)\| \lesssim \delta^2 e^{-2\epsilon\tau}, \quad \|(1 - \mathbf{P})\,\mathbf{K}\left(\Phi, \mathbf{u}\right)(\tau)\| \lesssim (\tfrac{\delta}{C} + \delta^2)e^{-\epsilon\tau}.$$

Clearly, these estimates imply that $\mathbf{K}(\Phi, \mathbf{u}) \in \mathcal{X}_\delta$ for sufficiently small δ and sufficiently large $C > 0$. Finally, we need to show the contraction property. To this end, we pick two elements $\Phi, \widetilde{\Phi} \in \mathcal{X}_\delta$. As before, the Lipschitz estimate (2.26) together with Proposition 2.4.5 imply

$$\left\|\mathbf{P}\left(\mathbf{K}(\Phi, \mathbf{u})(\tau) - \mathbf{K}(\widetilde{\Phi}, \mathbf{u})(\tau)\right)\right\| \lesssim \delta e^{-\epsilon\tau}\left\|\Phi - \widetilde{\Phi}\right\|_{\mathcal{X}},$$
$$\left\|(1 - \mathbf{P})\left(\mathbf{K}(\Phi, \mathbf{u})(\tau) - \mathbf{K}(\widetilde{\Phi}, \mathbf{u})(\tau)\right)\right\| \lesssim \delta e^{-\epsilon\tau}\left\|\Phi - \widetilde{\Phi}\right\|_{\mathcal{X}}$$

and by choosing δ sufficiently small we conclude

$$\left\|\mathbf{K}(\Phi, \mathbf{u}) - \mathbf{K}(\widetilde{\Phi}, \mathbf{u})\right\|_{\mathcal{X}} \leq \frac{1}{2}\left\|\Phi - \widetilde{\Phi}\right\|_{\mathcal{X}}.$$

Consequently, the claim follows by the contraction mapping principle. Uniqueness in the whole space $\mathcal{X}$ and the Lipschitz continuity of the solution map are routine and we omit the details. $\square$

Now we turn to the particular initial data we prescribe. To this end, we define the space

$$\mathcal{H}^R := H^m_{\text{rad}}(\mathbb{B}^{d+2}_R) \times H^{m-1}_{\text{rad}}(\mathbb{B}^{d+2}_R), \quad m \equiv m_d = \frac{d+3}{2}$$

for $R > 0$, endowed with the induced norm

$$\|\mathbf{w}\|^2_{\mathcal{H}^R} = \|(w_1, w_2)\|^2_{\mathcal{H}^R} = \|w_1\|_{H^m_{\text{rad}}(\mathbb{B}^{d+2}_R)} + \|w_2\|_{H^{m-1}_{\text{rad}}(\mathbb{B}^{d+2}_R)}.$$

Recall the definition of the initial data operator $\mathbf{U}(\mathbf{v}, T)$ from Eq. (2.12).

Lemma 2.4.8 *Fix $T_0 > 0$. Let $\delta > 0$ be sufficiently small and $\mathbf{v}$ with $|\cdot|^{-1}\mathbf{v} \in \mathcal{H}^{T_0+\delta}$. Then, the map*

$$\mathbf{U}(\mathbf{v}, \cdot) : [T_0 - \delta, T_0 + \delta] \longrightarrow \mathcal{H}, \quad T \longmapsto \mathbf{U}(\mathbf{v}, T)$$

is continuous. Furthermore, for all $T \in [T_0 - \delta, T_0 + \delta]$,

$$\left\||\cdot|^{-1}\mathbf{v}\right\|_{\mathcal{H}^{T_0+\delta}} \leq \delta \Longrightarrow \left\|\mathbf{U}(\mathbf{v}, T)\right\| \lesssim \delta.$$

Proof. The statements are straightforward consequences of the very definition of $\mathbf{U}(\mathbf{v}, T)$, the smoothness of $\frac{f_0(\rho)}{\rho}$, and the continuity of rescaling in Sobolev spaces. We omit the details. $\square$

Finally, given $T_0 > 0$ and $\mathbf{v} \in \mathcal{H}^{T_0+\delta}$ with $\||\cdot|^{-1}\mathbf{v}\|_{\mathcal{H}^{T_0+\delta}} \leq \frac{\delta}{M}$ for $\delta > 0$ sufficiently small and $M > 0$ sufficiently large, we apply Lemma 2.4.8 to see that $\mathbf{u} := \mathbf{U}(\mathbf{v}, T)$ satisfies the assumptions of Theorem 2.4.7 for all $T \in [T_0 - \delta, T_0 + \delta]$. Hence, for all $T \in [T_0 - \delta, T_0 + \delta]$, the map $\mathbf{K}(\cdot, \mathbf{U}(\mathbf{v}, T))$ has a fixed point $\Phi_T := \mathbf{\Phi}(\mathbf{U}(\mathbf{v}, T)) \in \mathcal{X}_\delta$. In the last step we now argue that for each $\mathbf{v}$, there exists a particular $T_\mathbf{v} \in [T_0 - \delta, T_0 + \delta]$ that makes the correction term vanish, i.e., $\mathbf{C}(\Phi_{T_\mathbf{v}}, \mathbf{U}(\mathbf{v}, T_\mathbf{v})) = 0$. Since $\mathbf{C}$ has values in $\operatorname{rg} \mathbf{P} = \langle \mathbf{g} \rangle$, the latter is equivalent to

$$\exists T_\mathbf{v} \in [T_0 - \delta, T_0 + \delta] : \quad \left\langle \mathbf{C}\left(\Phi_{T_\mathbf{v}}, \mathbf{U}\left(\mathbf{v}, T_\mathbf{v}\right)\right), \mathbf{g} \right\rangle_{\mathcal{H}} = 0. \tag{2.32}$$

The key observation now is that

$$\partial_T \begin{pmatrix} \frac{1}{\rho} f_0(\frac{T}{T_0}\rho) \\ \frac{T^2}{T_0^2} f_0'(\frac{T}{T_0}\rho) \end{pmatrix} \Bigg|_{T=T_0} = \frac{2\sqrt{d-2}}{T_0} \mathbf{g}(\rho)$$

and thus, we have the expansion

$$\left\langle \mathbf{C}\left(\Phi_T, \mathbf{U}(\mathbf{v}, T)\right), \mathbf{g} \right\rangle_{\mathcal{H}} = \frac{2\sqrt{d-2}}{T_0} \|\mathbf{g}\|^2 (T - T_0) + O((T-T_0)^2) + O(\tfrac{\delta}{M} T^0) + O(\delta^2 T^0).$$

Consequently, a simple fixed point argument proves (3.61), see [32], Theorem 4.15 for full details. In summary, we arrive at the following result.

Theorem 2.4.9 *Fix $T_0 > 0$. Then there exist $\delta, M > 0$ such that for any $\mathbf{v}$ with*

$$\||\cdot|^{-1}\mathbf{v}\|_{\mathcal{H}^{T_0+\delta}} \leq \frac{\delta}{M}$$

there exists a $T \in [T_0 - \delta, T_0 + \delta]$ and a function $\Phi \in \mathcal{X}_\delta$ which satisfies

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{U}(\mathbf{v}, T) + \int_0^\tau \mathbf{S}(\tau - s)\mathbf{N}\big(\Phi(s)\big) ds \tag{2.33}$$

for all $\tau \geq 0$. Furthermore, Φ is unique in $C\big([0, \infty); \mathcal{H}\big)$.

2.4.10 Proof of the main theorem

With the results of the previous section at hand, we can now prove the main theorem. Fix $T_0 > 0$ and suppose the radial initial data $\psi[0]$ satisfy

$$\left\| |\cdot|^{-1} \Big(\psi[0] - \psi^{T_0}[0] \Big) \right\|_{H^{\frac{d+3}{2}}(\mathbb{B}^{d+2}_{T_0+\delta}) \times H^{\frac{d+1}{2}}(\mathbb{B}^{d+2}_{T_0+\delta})} \leq \frac{\delta}{M}$$

with $\delta, M > 0$ from Theorem 3.57. We set $\mathbf{v} := \psi[0] - \psi^{T_0}[0]$, cf. Section 2.3. Then we have

$$\||\cdot|^{-1}\mathbf{v}\|_{\mathcal{H}^{T_0+\delta}} = \left\||\cdot|^{-1}\Big(\psi[0] - \psi^{T_0}[0]\Big)\right\|_{\mathcal{H}^{T_0+\delta}} \leq \frac{\delta}{M}$$

and Theorem 3.57 yields the existence of $T \in [T_0 - \delta, T_0 + \delta]$ such that Eq. (2.33) has a unique solution $\Phi \in \mathcal{X}$ that satisfies $\|\Phi(\tau)\| \leq \delta e^{-\epsilon\tau}$ for all $\tau \geq 0$. By construction,

$$\psi(t,r) = \psi^T(t,r) + \frac{r}{T-t}\phi_1\left(\log\frac{T}{T-t}, \frac{r}{T-t}\right)$$

is a solution to the original wave maps problem (2.3). Furthermore,

$$\partial_t\psi(t,r) = \partial_t\psi^T(t,r) + \frac{r}{(T-t)^2}\phi_2\left(\log\frac{T}{T-t}, \frac{r}{T-t}\right).$$

Consequently,

$$\begin{aligned}
(T-t)^{k-\frac{d}{2}} &\left\||\cdot|^{-1}\left(\psi(t,\cdot) - \psi^T(t,\cdot)\right)\right\|_{\dot H^k(\mathbb{B}^{d+2}_{T-t})} \\
&= (T-t)^{k-\frac{d}{2}-1}\left\|\phi_1\left(\log\frac{T}{T-t}, \frac{|\cdot|}{T-t}\right)\right\|_{\dot H^k(\mathbb{B}^{d+2}_{T-t})} \\
&= \left\|\phi_1\left(\log\frac{T}{T-t}, \cdot\right)\right\|_{\dot H^k(\mathbb{B}^{d+2})} \leq \left\|\Phi\left(\log\frac{T}{T-t}\right)\right\| \\
&\leq \delta(T-t)^{\epsilon}
\end{aligned}$$

for all $t \in [0,T)$ and $k = 0, 1, 2, \dots, \frac{d+3}{2}$. Analogously,

$$\begin{aligned}
(T-t)^{\ell-\frac{d}{2}+1} &\left\||\cdot|^{-1}\left(\partial_t\psi(t,\cdot) - \partial_t\psi^T(t,\cdot)\right)\right\|_{\dot H^\ell(\mathbb{B}^{d+2}_{T-t})} \\
&= (T-t)^{\ell-\frac{d}{2}-1}\left\|\phi_2\left(\log\frac{T}{T-t}, \frac{|\cdot|}{T-t}\right)\right\|_{\dot H^\ell(\mathbb{B}^{d+2}_{T-t})} \\
&= \left\|\phi_2\left(\log\frac{T}{T-t}, \cdot\right)\right\|_{\dot H^\ell(\mathbb{B}^{d+2})} \leq \left\|\Phi\left(\log\frac{T}{T-t}\right)\right\| \\
&\leq \delta(T-t)^{\epsilon}
\end{aligned}$$

for all $\ell = 0, 1, 2, \dots, \frac{d-1}{2}$.

2.5 Properties of the function $\mathcal{I}_d$

We first derive a consequence of results from [20] which then leads to the desired statement that $\mathcal{I}_d$ is not analytic at 1. Recall the supersymmetric problem Eq. (4.1) from [20],

$$(1-\rho^2)\tilde u_\lambda'' + \left[\frac{k+1}{\rho} - 2(\lambda+1)\rho\right]\tilde u_\lambda' - \lambda(\lambda+1)\tilde u_\lambda + \frac{2k}{\rho^2}\frac{\rho^2 - k - 2}{\rho^2 + k}\tilde u_\lambda = 0, \qquad (2.34)$$

where $d = k + 2$.

Lemma 2.5.1 *Let $m \in \mathbb{N}$, $m \geq 2$, and $d = 2m + 1$. Then the function*

$$\mathcal{U}_m(\rho) := (1-\rho^2)^{m-1} \int_0^\rho \frac{y^{2m+2}}{(1-y^2)^m} g_1(y)^2 dy, \qquad g_1(y) = \frac{1}{y^2 + d - 2}$$

is not analytic at $\rho = 1$.

Proof. In view of the supersymmetric factorization derived in [20] (or by a direct computation) it follows that $\tilde{u}_1$ satisfies Eq. (2.34) for $\lambda = 1$ if and only if $\tilde{v}_1(\rho) = \rho^m (1-\rho^2)^{-\frac{m}{2}} \tilde{u}_1(\rho)$ satisfies

$$(\partial_\rho - w(\rho))[(1-\rho^2)^2(\partial_\rho + w(\rho))]\tilde{v}_1(\rho) = 0, \tag{2.35}$$

where $w = \frac{v_1'}{v_1}$ and

$$v_1(\rho) = \rho^{m+1}(1-\rho^2)^{1-\frac{m}{2}} g_1(\rho).$$

Observe that the function $1/v_1$ solves Eq. (2.35). Furthermore, the Wronskian of two solutions of Eq. (2.35) is of the form $\frac{c}{(1-\rho^2)^2}$ for some constant c and thus, the reduction formula yields another solution

$$\tilde{v}_1(\rho) = \frac{1}{v_1(\rho)} \int_0^\rho \frac{v_1(y)^2}{(1-y^2)^2} dy = \frac{(1-\rho^2)^{\frac{m}{2}-1}}{g_1(\rho)\rho^{m+1}} \int_0^\rho \frac{y^{2m+2}}{(1-y^2)^m} g_1(y)^2 dy.$$

By construction,

$$\tilde{u}_1(\rho) = \rho^{-m}(1-\rho^2)^{\frac{m}{2}} \tilde{v}_1(\rho) = \frac{(1-\rho^2)^{m-1}}{g_1(\rho)\rho^{2m+1}} \int_0^\rho \frac{y^{2m+2}}{(1-y^2)^m} g_1(y)^2 dy = \frac{\mathcal{U}_m(\rho)}{g_1(\rho)\rho^{2m+1}}$$

is a solution to Eq. (2.34). Clearly, $\tilde{u}_1$ is analytic at $\rho = 0$. Suppose $\tilde{u}_1$ were analytic at $\rho = 1$ also. Then we would have found a nontrivial solution $\tilde{u}_1 \in C^\infty[0,1]$ to Eq. (2.34) with $\lambda = 1$. This, however, contradicts Theorem 4.1 in [20]. We conclude that $\tilde{u}_1$ and hence $\mathcal{U}_m$ must be nonanalytic at $\rho = 1$. $\square$

Proposition 2.5.2 *Let $d \geq 5$ be odd. Then the function $\mathcal{I}_d$ defined in Eq. (2.24) is not analytic at $\rho = 1$.*

Proof. Since $\hat{u}_1 = g_1$ and $G(y) = 2yg_1'(y) + 5g_1(y)$, we have

$$\mathcal{I}_d(\rho) = \hat{u}_2(\rho) \int_0^\rho \frac{y^{d+1}}{(1-y^2)^{\frac{d-3}{2}}} \left[2yg_1(y)g_1'(y) + 5g_1(y)^2\right] dy.$$

To simplify notation, we use the convention from above and write $d = 2m + 1$. Since the order of the zero of $\hat{u}_2(\rho)$ at $\rho = 1$ is $m - 1$, it is enough to prove that

$$\mathcal{J}_m(\rho) := (1-\rho^2)^{m-1} \int_0^\rho \frac{y^{2m+2}}{(1-y^2)^{m-1}} \left[2yg_1(y)g_1'(y) + 5g_1(y)^2\right] dy$$

is nonanalytic at $\rho = 1$. An integration by parts yields

$$\begin{aligned}\mathcal{J}_m(\rho) &= (1-\rho^2)^{m-1} \int_0^\rho \frac{y^{2m-2}}{(1-y^2)^{m-1}} \frac{d}{dy} \left(y^5 g_1(y)^2\right) dy \\ &= \rho^{2m+3} g_1(\rho)^2 - 2(m-1)(1-\rho^2)^{m-1} \int_0^\rho \frac{y^{2m+2}}{(1-y^2)^m} g_1(y)^2 dy\end{aligned}$$

and Lemma 2.5.1 completes the proof. □

Chapter 3

Stable blowup for the cubic wave equation in higher dimensions

This chapter contains the result of the paper [17] and is a joint work of the author with Donninger.

3.1 Abstract

We consider the wave equation with a focusing cubic nonlinearity in higher odd space dimensions without symmetry restrictions on the data. We prove that there exists an open set of initial data such that the corresponding solution exists in a backward light-cone and approaches the ODE blowup profile.

3.2 Introduction

3.2.1 Cubic wave equation

In this paper we study the wave equation with a focusing cubic nonlinearity

$$\Box u(t,x) + u^3(t,x) = 0, \tag{3.1}$$

with $(t,x) \in \mathbb{R}^{1+d}$. Here, $\Box$ stands for the Laplace-Beltrami operator on Minkowski space with signature $(-+++)$, i.e.,

$$\Box := -\partial_t^2 + \Delta_x.$$

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Equation (3.1) has the conserved energy

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x u(t,x)|^2\,dx - \frac{1}{4}\int_{\mathbb{R}^d} |u(t,x)|^4\,dx.$$

Obviously, equation (3.1) is invariant under time-translations. In addition, other symmetries of the equation that are relevant in our context are Lorentz boosts, namely, if u is a solution to (3.1), so is

$$u_{T,\alpha}(t,x) := u \circ \Lambda_T(\alpha)\begin{pmatrix} t \\ x \end{pmatrix}, \tag{3.2}$$

for $T \in \mathbb{R}$ and $\alpha = (\alpha^1, \cdots, \alpha^d) \in \mathbb{R}^d$. Here, we define the Lorentz transformations in a way that resembles circular rotations in d-dimensional space using hyperbolic functions, that is

$$\Lambda_T(\alpha) := \Lambda_T^d(\alpha^d) \circ \Lambda_T^{d-1}(\alpha^{d-1}) \circ \cdots \circ \Lambda_T^1(\alpha^1)$$

where the boost in the $j-$direction is given by

$$\Lambda_T^j(\alpha^j)\begin{pmatrix} t \\ x^1 \\ \vdots \\ x^j \\ \vdots \\ x^d \end{pmatrix} := \begin{pmatrix} (t-T)\cosh(\alpha^j) + x^j\sinh(\alpha^j) + T \\ x^1 \\ \vdots \\ (t-T)\sinh(\alpha^j) + x^j\cosh(\alpha^j) \\ \vdots \\ x^d \end{pmatrix}.$$

A Lorentz boost can be thought of as a hyperbolic rotation of spacetime coordinates of the $(1+d)-$dimensional Minkowski space. The parameter $\alpha \in \mathbb{R}^d$ (called rapidity) is the hyperbolic angle of rotation, analogous to the ordinary angle for circular rotations. Note in particular that the spacetime event $(T,0)$ is a fixed point of the transformation $\Lambda_T(\alpha)$ and the light-cones emanating from $(T,0)$ are invariant under $\Lambda_T(\alpha)$.

3.2.2 Blowup solutions

Equation (3.1) exhibits finite-time blowup from smooth, compactly supported initial data. This fact is most easily seen by looking at spatially homogeneous blowup solutions. In other words, we ignore the Laplacian in the space variable in the equation and the remaining ordinary differential equation can be solved explicitly. This leads to the solution

$$u_1(t,x) := \frac{\sqrt{2}}{1-t}.$$

Using the symmetries of the equation we get a larger family of blowup solutions. Namely, time translation symmetry yields

$$u_T(t,x) := \frac{\sqrt{2}}{T-t} \tag{3.3}$$

and Lorentz symmetry implies that

$$u_{T,\alpha}(t,x) = \frac{\sqrt{2}}{A_0(\alpha)(T-t) - A_j(\alpha)x^j} \tag{3.4}$$

is also a solution, see (3.2). Here and in the following, we adopt the Einstein summation convention, namely

$$a_j b^j = \sum_{j=1}^{d} a_j b^j$$

and

$$\begin{cases} A_0(\alpha) := \cosh(\alpha^d)\cdots\cosh(\alpha^3)\cosh(\alpha^2)\cosh(\alpha^1), \\ A_1(\alpha) := \cosh(\alpha^d)\cdots\cosh(\alpha^3)\cosh(\alpha^2)\sinh(\alpha^1), \\ A_2(\alpha) := \cosh(\alpha^d)\cdots\cosh(\alpha^3)\sinh(\alpha^2), \\ \qquad\vdots \\ A_{d-1}(\alpha) := \cosh(\alpha^d)\sinh(\alpha^{d-1}), \\ A_d(\alpha) := \sinh(\alpha^d). \end{cases}$$

Observe that $A_0(\alpha) = \mathcal{O}(1)$ whereas $A_j(\alpha) = \mathcal{O}(\alpha)$ for all sufficiently small $\alpha \in \mathbb{R}^d$.

3.2.3 The Cauchy problem

Our intention is to study the future development of small perturbations of u_{T_0,α_0} under (3.1) for fixed $T_0 \in \mathbb{R}$ and $\alpha_0 \in \mathbb{R}^d$. Hence, we consider the Cauchy problem

$$\begin{cases} \Box u(t,x) + u^3(t,x) = 0, \\ u[0] = (f,g), \end{cases} \tag{3.5}$$

where

$$(f,g) = u_{T_0,\alpha_0}[0] + (\tilde{f},\tilde{g}). \tag{3.6}$$

Here, we use the abbreviation $u[t] = (u(t,\cdot), \partial_t u(t,\cdot))$ for convenience, u_{T_0,α_0} is defined in (3.4) and $(\tilde{f},\tilde{g})$ are small in a suitable sense. Furthermore, we restrict the evolution to the backward light-cone

$$C_T := \{(t,x) : 0 \le t < T,\ |x| \le T-t\} = \bigcup_{t\in[0,T)} \{t\} \times \mathbb{B}^d_{T-t}.$$

3.2.4 Related results

There is a lot of activity in the study of blowup for wave equations. The interest in (3.1) stems from the fact that this equation contains many features common to a whole range of blow-up problems arising in mathematical physics, as for example in nonlinear optics [8] and general relativity [28].
By definition, u is a solution to (3.5) if and only if it satisfies the equation in the integral form using Duhamel's principle, namely

$$u(t,\cdot) = \cos(t|\nabla|) f + \frac{\sin(t|\nabla|)}{|\nabla|} g + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} u^3(s,\cdot) ds,$$

for initial data

$$(f,g) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d).$$

Using this formula, one can show that (3.5) is locally well-posed for initial data in $\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)$ for $s > \frac{d}{2}$, see [85]. On the one hand, equation (3.1) is invariant under the scaling transformation

$$u_\lambda(t,x) := \frac{1}{\lambda} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \ \lambda > 0 \tag{3.7}$$

and

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{\frac{d}{2}-1-s} \left\| u\left(\frac{t}{\lambda},\cdot\right)\right\|_{\dot{H}^s(\mathbb{R}^d)}.$$

This scaling property is closely related to the existence of a suitable local theory for the problem and distinguishes the space $\dot{H}^{s_3}(\mathbb{R}^d) \times \dot{H}^{s_3-1}(\mathbb{R}^d)$, $s_3 := \frac{d}{2} - 1$ as the critical Sobolev space, the unique L^2-based homogeneous Sobolev space preserved by the scaling (3.7). Indeed, Strichartz theory shows that (3.5) is locally well-posed for initial data in the critical Sobolev space $\dot{H}^{s_3}(\mathbb{R}^d) \times \dot{H}^{s_3-1}(\mathbb{R}^d)$, [78], [55]. On the other hand, equation (3.1) has the conserved energy

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x u(t,x)|^2 dx - \frac{1}{4}\int_{\mathbb{R}^d} |u(t,x)|^4 dx$$

which distinguishes the space $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ as the energy space, that is, the space of initial data for which the energy is known to be finite. For $d \geq 5$, the critical regularity $s_3 = \frac{d}{2} - 1$ is larger than the energy-critical regularity $s = 1$ and equation (3.1) is energy-supercritical.

The one-dimensional case has been completely understood, see [59], [60], [61], [62] where Merle and Zaag exhibited a universal one-parameter family of functions which yields the blowup profile in self-similar variables for general initial data. In higher dimensions, the situation is less clear. In three space dimensions, Bizoń together with Breitenlohner, Maison and Wasserman in [12], [4] showed that equation (3.1) admits infinitely many radial self-similar blowup solutions of the form

$$\frac{1}{T-t} f_n\left(\frac{|x|}{T-t}\right).$$

Here, the ground-state solution (3.3) corresponds to $f_0 = \sqrt{2}$. Levine [54] used energy methods and a convexity argument to show that initial data with negative energy and finite L^2-norm lead to blowup in finite time, see also [45] for generalizations to the Klein-Gordon equation. We also mention the works of Alinhac [2] and Caffarelli and Friedman [15], [14] where more blowup results can be found. The stability of the ground-state has been studied extensively by Schörkhuber and the second author in three space dimensions (in [29], [30] for radial initial data and in [31] without symmetry restrictions) and later in [33] for all space dimensions and for radial initial data. Some numerical results are available in a series of papers by Bizoń, Chmaj, Tabor and Zenginoğlu, see [6], [11], [13]. Furthermore, in the superconformal and Sobolev subcritical range, an upper bound on the blowup rate was proved by Killip, Stoval and Vişan in [45], then refined by Hamza and Zaag in [41]. In a series of papers [58], [64], [63], [57], [56], Merle and Zaag obtained sharp upper and lower bounds on the blowup rate of the H^1-norm of the solution inside cones that terminate at the singularity, see also the work of Alexakis and Shao [1]. We also mention the recent work by Dodson-Lawrie [25] on large-data scattering for the cubic equation in five dimensions.

3.3 The main result

By finite speed of propagation one can use $u_{T,\alpha}$ to construct smooth, compactly supported initial data which lead to a solution that blows up as $t \longrightarrow T$. In the present work, we study the asymptotic nonlinear stability of $u_{T,\alpha}$. As a matter of fact, we prove that all initial data from an open, sufficiently small region centered at $u_{T,\alpha}$ lead to the same type of blowup described by the ODE blowup profile. First, we need a definition for our notion of the blowup time.

Definition 3.3.1 *Given initial data (f, g), we define*

$$T_{(f,g)} := \sup\left\{T>0 \,\middle|\, \substack{\exists \text{ solution } u:C_T\longrightarrow\mathbb{R} \text{ to } (3.5) \text{ in the sense of} \\ \text{Definition } 3.9.1 \text{ with initial data } u[0]=(f,g)|_{\mathbb{B}^d_T}}\right\} \cup \{0\}.$$

In the case where $T_{(f,g)} < \infty$, we call $T = T_{(f,g)}$ the blowup time at the origin.

The main result of this work is the following.

Theorem 3.3.2 *Fix $d \in \{5, 7, 9, 11, 13\}$, $T_0 > 0$ and $\alpha_0 \in \mathbb{R}^d$. There exist constants $M, \delta > 0$ such that the following holds. Suppose that the initial data*

$$(f, g) \in H^{\frac{d+1}{2}}(\mathbb{B}^d_{T_0+\delta}) \times H^{\frac{d-1}{2}}(\mathbb{B}^d_{T_0+\delta})$$

satisfy

$$\left\|(f,g)-u_{T_0,\alpha_0}[0]\right\|_{H^{\frac{d+1}{2}}(\mathbb{B}^d_{T_0+\delta})\times H^{\frac{d-1}{2}}(\mathbb{B}^d_{T_0+\delta})}\leq\frac{\delta}{M}.$$

Then, $T=T_{u[0]}\in[T_0-\delta,T_0+\delta]$ *and there exists an* $\alpha\in\mathbb{B}^d_{3M\delta}(\alpha_0)$ *such that the solution* $u:C_T\longrightarrow\mathbb{R}$ *to* (3.5) *satisfies the estimates*

$$(T-t)^{k-\frac{d}{2}+1}\left\|u(t,\cdot)-u_{T,\alpha}(t,\cdot)\right\|_{\dot{H}^k(\mathbb{B}^d_{T-t})}\leq\delta(T-t)^{\frac{1}{2}},$$
$$(T-t)^{\ell-\frac{d}{2}+2}\left\|\partial_t u(t,\cdot)-\partial_t u_{T,\alpha}(t,\cdot)\right\|_{\dot{H}^\ell(\mathbb{B}^d_{T-t})}\leq\delta(T-t)^{\frac{1}{2}},$$

for all $k=0,1,\cdots,\frac{d+1}{2}$ *and* $\ell=0,1,\cdots,\frac{d-1}{2}$.

Remark 3.3.3 *Theorem 3.3.2 shows that the future development of small perturbations of the blowup solution* u_{T_0,α_0} *defined in* (3.4) *converge back to* u_{T_0,α_0} *up to symmetries of the equation.*

Remark 3.3.4 *Note that the normalizing factors on the left-hand sides appear naturally and reflect the behavior of the solution* $u_{T,\alpha}$ *in the respective homogeneous Sobolev norms. Namely, for*

$$\psi_\alpha(\xi):=\frac{\sqrt{2}}{A_0(\alpha)-A_j(\alpha)\xi^j}\tag{3.8}$$

we have

$$(T-t)^{k-\frac{d}{2}+1}\left\|u_{T,\alpha}(t,\cdot)\right\|_{\dot{H}^k(\mathbb{B}^d_{T-t})}=(T-t)\left\|u_{T,\alpha}(t,(T-t)\cdot)\right\|_{\dot{H}^k(\mathbb{B}^d_1)}\simeq\left\|\psi_\alpha\right\|_{\dot{H}^k(\mathbb{B}^d_1)},$$
$$(T-t)^{l-\frac{d}{2}+2}\left\|\partial_t u_{T,\alpha}(t,\cdot)\right\|_{\dot{H}^\ell(\mathbb{B}^d_{T-t})}=(T-t)^2\left\|\partial_t u_{T,\alpha}(t,(T-t)\cdot)\right\|_{\dot{H}^\ell(\mathbb{B}^d_1)}\simeq\left\|\nabla\psi_\alpha\right\|_{\dot{H}^\ell(\mathbb{B}^d_1)},$$

for all $k,\ell\in\mathbb{N}_0$ *and* $\alpha\neq0$.

Remark 3.3.5 *We strongly believe that the result holds true in all odd dimensions* d *and the restriction on* d *is not essential and for technical reasons only. Similarly, the restriction to the cubic power is for the sake of simplicity only. Similar results are true for any focusing power and can be proved by straightforward adaptations of our method.*

Remark 3.3.6 *The corresponding result in* $d=3$ *was proved in [31] and relied on a delicate identity that only holds in 3 dimensions. In this paper we show that our method is robust enough to extend to all odd dimensions.*

3.4 Formulation as a first-order system in time

Without loss of generality we assume that $T_0=1$ and $\alpha_0=0$.

3.4.1 First-order system

To start our analysis, we write the Cauchy problem (3.5) as a first-order system in time. First, we change coordinates and map the backward light-cone

$$C_T = \{(t,x) : 0 \le t < T,\ |x| \le T-t\} = \bigcup_{t\in[0,T)} \{t\} \times \mathbb{B}^d_{T-t}$$

diffeomorphically into the cylinder

$$\mathcal{C} := \{(\tau,\xi) : 0 \le \tau < +\infty,\ |\xi| \le 1\} = [0,\infty) \times \mathbb{B}^d.$$

Specifically, we introduce the similarity coordinates

$$(t,x) \longmapsto \mu(t,x) := (\tau(t,x), \xi(t,x)) := \left(\log\left(\frac{T}{T-t}\right), \frac{x}{T-t}\right)$$

and derivatives translate according to

$$\begin{aligned}
\partial_t &= \frac{e^\tau}{T}\left(\partial_\tau + \xi^j\partial_{\xi^j}\right), \\
\partial_t^2 &= \frac{e^{2\tau}}{T^2}\left(\partial_\tau^2 + \partial_\tau + 2\xi^j\partial_{\xi^j}\partial_\tau + \xi^j\xi^k\partial_{\xi^i}\partial_{\xi^k} + 2\xi^j\partial_{\xi^j}\right), \\
\partial_{x^j} &= \frac{e^\tau}{T}\partial_{\xi^j}, \\
\partial_{x^j}\partial_{x_j} &= \frac{e^{2\tau}}{T^2}\partial_{\xi^j}\partial_{\xi_j}.
\end{aligned}$$

Notice in particular that the blowup time T is mapped to ∞. Now, equation (3.1) can be written equivalently as

$$\frac{e^{2T}}{T^2}\Big(- \partial_\tau^2 - \partial_\tau - 2\xi^j\partial_{\xi^j}\partial_\tau + \left(\delta^{jk} - \xi^j\xi^k\right)\partial_{\xi^j}\partial_{\xi^k} - 2\xi^j\partial_{\xi^j}\Big)U(\tau,\xi) = -U^3(\tau,\xi),$$

for $U := u \circ \mu^{-1}$. Next, we remove the $\tau-$dependent weight on the left hand side by rescaling,

$$\psi(\tau,\xi) := Te^{-\tau}U(\tau,\xi),$$

which implies

$$\Big(\partial_\tau^2 + 3\partial_\tau + 2\xi^j\partial_{\xi^j}\partial_\tau - (\delta^{jk} - \xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k} + 4\xi^j\partial_{\xi^j} + 2\Big)\psi(\tau,\xi) = \psi^3(\tau,\xi).$$

Finally, we set

$$\left(\mathbf{\Psi}(\tau)\right)(\xi) := \begin{pmatrix}\psi_1(\tau,\xi)\\ \psi_2(\tau,\xi)\end{pmatrix} := \begin{pmatrix}\psi(\tau,\xi)\\ \partial_\tau\psi(\tau,\xi) + \xi^j\partial_{\xi^j}\psi(\tau,\xi) + \psi(\tau,\xi)\end{pmatrix}$$

which yields

$$\partial_\tau \mathbf{\Psi}(\tau) = \widetilde{\mathbf{L}}\left(\mathbf{\Psi}(\tau)\right) + \mathbf{N}\left(\mathbf{\Psi}(\tau)\right) \tag{3.9}$$

where

$$\widetilde{\mathbf{L}}\left(\mathbf{u}\right)(\xi) := \begin{pmatrix} -\xi \cdot \nabla u_1(\xi) - u_1(\xi) + u_2(\xi) \\ \Delta^{\mathbb{R}^d} u_1(\xi) - \xi \cdot \nabla u_2(\xi) - 2u_2(\xi) \end{pmatrix},$$

$$\mathbf{N}\left(\mathbf{u}\right)(\xi) := \begin{pmatrix} 0 \\ u_1^3(\xi) \end{pmatrix}.$$

3.4.2 Static blowup solution

Now, starting from (3.4), we switch to similarity coordinates and rescale the function appropriately as before to find a $d-$parameter family $\mathbf{\Psi}_\alpha$ of static blowup solutions to (3.9), i.e.,

$$\mathbf{\Psi}_\alpha(\xi) := \begin{pmatrix} \psi_\alpha(\xi) \\ \xi^j \partial_j \psi_\alpha(\xi) + \psi_\alpha(\xi), \end{pmatrix} \tag{3.10}$$

where ψ_α is defined in (3.8). We emphasize that there is no trace of the blowup time T in the definition of ψ_α.

3.5 The linear free evolution in the backward light-cone

In this section, we focus on the evolution of the free linear equation and obtain a useful decay estimate for the solution operator. To this end, we need to find a norm

$$\|\cdot\| : \mathcal{H} \longrightarrow \mathbb{R}$$

on the function space

$$\mathcal{H} := H^{\frac{d+1}{2}}\left(\mathbb{B}^d\right) \times H^{\frac{d-1}{2}}\left(\mathbb{B}^d\right)$$

which yields the sharp decay for the free evolution. Specifically, we define

$$\mathcal{D}\big(\widetilde{\mathbf{L}}\big) := C^{\frac{d+3}{2}}\left(\overline{\mathbb{B}^d}\right) \times C^{\frac{d+1}{2}}\left(\overline{\mathbb{B}^d}\right)$$

and work towards proving the following result.

Proposition 3.5.1 *The free operator $\widetilde{\mathbf{L}} : \mathcal{D}\big(\widetilde{\mathbf{L}}\big) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is densely defined, closable and its closure $\mathbf{L} : \mathcal{D}\big(\mathbf{L}\big) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ generates a strongly continuous one-parameter semigroup of bounded operators $\mathbf{S} : [0,\infty) \longrightarrow \mathcal{B}\left(\mathcal{H}\right)$ which satisfies the decay estimate*

$$\|\mathbf{S}\left(\tau\right)\| \leq M e^{-\tau}$$

for all $\tau \geq 0$ and for some constant $M \geq 1$.

To proceed, we fix $d = 5$ and construct a suitable inner product on $\mathcal{H} = H^3\left(\mathbb{B}^5\right) \times H^2\left(\mathbb{B}^5\right)$.

3.5.1 Inner Product

We define

$$\widetilde{\mathcal{H}} = C^3\left(\overline{\mathbb{B}^5}\right) \times C^2\left(\overline{\mathbb{B}^5}\right)$$

and consider the sesquilinear forms

$$\begin{aligned}
(\mathbf{u}|\mathbf{v})_1 &:= \int_{\mathbb{B}^5} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k v_1(\xi)}d\xi + \int_{\mathbb{B}^5} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j v_2(\xi)}d\xi + \int_{\mathbb{S}^4} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j v_1(\omega)}d\sigma(\omega),\\
(\mathbf{u}|\mathbf{v})_2 &:= \int_{\mathbb{B}^5} \partial_i\partial^k\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial_j v_1(\xi)}d\xi + \int_{\mathbb{B}^5} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j v_2(\xi)}d\xi + \int_{\mathbb{S}^4} \partial_j u_2(\omega)\overline{\partial^j v_2(\omega)}d\sigma(\omega),\\
(\mathbf{u}|\mathbf{v})_3 &:= 5\,(\mathbf{u}|\mathbf{v})_1 + (\mathbf{u}|\mathbf{v})_2 + \int_{\mathbb{S}^4} u_2(\omega)\overline{v_2(\omega)}d\sigma(\omega),\\
(\mathbf{u}|\mathbf{v})_4 &:= (\mathbf{u}|\mathbf{v})_1 + (\mathbf{u}|\mathbf{v})_2 + \int_{\mathbb{S}^4} \partial_i u_1(\omega)\overline{\partial^i v_1(\omega)}d\sigma(\omega),
\end{aligned}$$

for all $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{H}}$. All these sesquilinear forms are derived from a higher energy of the free wave equation but neither of them defines an inner product on $\mathcal{H}$. To fix this, we also define

$$(\mathbf{u}|\mathbf{v})_5 := \left(\int_{\mathbb{S}^4} \zeta\left(\omega, \mathbf{u}(\omega)\right) d\sigma(\omega)\right)\left(\int_{\mathbb{S}^4} \overline{\zeta\left(\omega, \mathbf{v}(\omega)\right)}d\sigma(\omega)\right)$$

where

$$\zeta\left(\omega, \mathbf{w}(\omega)\right) := D_5 w_1(\omega) + \tilde{D}_5 w_2(\omega)$$

and

$$\begin{aligned}
D_5 w_1(\omega) &:= \omega^i\omega^j\partial_i\partial_j w_1(\omega) + 5\omega^i\partial_i w_1(\omega) + 3w_1(\omega),\\
\tilde{D}_5 w_2(\omega) &:= \omega^j\partial_j w_2(\omega) + 3w_2(\omega).
\end{aligned}$$

Finally, let

$$(\cdot|\cdot) : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \longrightarrow \mathbb{R}, \quad (\mathbf{u}|\mathbf{v}) := \sum_{i=1}^{5} (\mathbf{u}|\mathbf{v})_i \tag{3.11}$$

and

$$\|\cdot\| : \widetilde{\mathcal{H}} \longrightarrow \mathbb{R}, \quad \|\cdot\| := \sqrt{(\cdot|\cdot)}. \tag{3.12}$$

Now, we will show that the norm (3.12) induced by the inner product (3.11) defines indeed a norm equivalent to $\|\cdot\|_{H^3(\mathbb{B}^5)\times H^2(\mathbb{B}^5)}$. However, we first need the following technical result.

Lemma 3.5.2 *For all* $(u_1, u_2) \in \widetilde{\mathcal{H}}$, *we have*

$$\|u_1\|_{H^3(\mathbb{B}^5)} \simeq \|\partial^3 u_1\|_{L^2(\mathbb{B}^5)} + \|\partial^2 u_1\|_{L^2(\mathbb{S}^4)} + \|\partial u_1\|_{L^2(\mathbb{S}^4)} + \|u_1\|_{L^2(\mathbb{S}^4)},$$
$$\|u_2\|_{H^2(\mathbb{B}^5)} \simeq \|\partial^2 u_2\|_{L^2(\mathbb{B}^5)} + \|\partial u_2\|_{L^2(\mathbb{S}^4)} + \|u_2\|_{L^2(\mathbb{S}^4)}.$$

Proof. The process is the same for both estimates and so we illustrate it on the second estimate only. Note that, for a generic function $f \in L^2(\mathbb{B}^5)$, we have

$$\|f\|^2_{L^2(\mathbb{B}^5)} = \int_0^1 \int_{\mathbb{S}^4} r^4 |f(r\omega)|^2 d\sigma(\omega) dr.$$

By density, it suffices to consider $u_2 \in C^\infty(\overline{\mathbb{B}^5})$. Now, the fundamental theorem of calculus, Jensen's inequality and integration by parts yield

$$\begin{aligned}
r^4|u_2(r\omega)|^2 &= \left|\int_0^r \partial_s \left(s^2 u_2(s\omega)\right) ds\right|^2 \leq \left(\int_0^r \left|\partial_s \left(s^2 u_2(s\omega)\right)\right| ds\right)^2 \\
&\leq r \int_0^r \left|\partial_s \left(s^2 u_2(s\omega)\right)\right|^2 ds \leq \int_0^1 \left|\partial_s \left(s^2 u_2(s\omega)\right)\right|^2 ds \\
&= \int_0^1 \left|2s u_2(s\omega) + s^2 \partial_s u_2(s\omega)\right|^2 ds \\
&= \int_0^1 \Big(4s^2|u_2(s\omega)|^2 + s^4 \left|\partial_s u_2(s\omega)\right|^2 + 2s^3 \Big(u_2(s\omega)\overline{\partial_s u_2(s\omega)} + \overline{u_2(s\omega)}\partial_s u_2(s\omega)\Big)\Big) ds \\
&= \int_0^1 \left(4s^2|u_2(s\omega)|^2 + s^4|\partial_s u_2(s\omega)|^2 + 2s^3\partial_s|u_2(s\omega)|^2\right) ds \\
&= 2|u_2(\omega)|^2 + \int_0^1 \Big(-2s^2|u_2(s\omega)|^2 + s^4|\partial_s u_2(s\omega)|^2\Big) ds \\
&\leq 2|u_2(\omega)|^2 + \int_0^1 s^4|\partial_s u_2(s\omega)|^2 ds \\
&= 2|u_2(\omega)|^2 + \int_0^1 s^4|\omega^j \partial_j u_2(s\omega)|^2 ds \\
&\leq 2|u_2(\omega)|^2 + \int_0^1 s^4|\partial u_2(s\omega)|^2 ds.
\end{aligned}$$

Integrating this inequality with respect to $r \in [0,1]$ and $\omega \in \mathbb{S}^4$ yields the estimate

$$\|u_2\|_{L^2(\mathbb{B}^5)} \lesssim \|\partial u_2\|_{L^2(\mathbb{B}^5)} + \|u_2\|_{L^2(\mathbb{S}^4)}.$$

Replacing u_2 by $\partial_i u_2$, we find

$$r^4|\partial_i u_2(r\omega)|^2 \leq 2|\partial_i u_2(\omega)|^2 + \int_0^1 s^4|\partial\partial_i u_2(s\omega)|^2 ds,$$

for all $i \in \{1,2,3,4,5\}$, and hence

$$\|\partial u_2\|_{L^2(\mathbb{B}^5)} \lesssim \left\|\partial^2 u_2\right\|_{L^2(\mathbb{B}^5)} + \|\partial u_2\|_{L^2(\mathbb{S}^4)}.$$

In summary, we get

$$\|u_2\|_{H^2(\mathbb{B}^5)} \lesssim \left\|\partial^2 u_2\right\|_{L^2(\mathbb{B}^5)} + \|\partial u_2\|_{L^2(\mathbb{S}^4)} + \|u_2\|_{L^2(\mathbb{S}^4)}.$$

This concludes the proof since the reverse inequality is a direct consequence of the trace inequality (see Theorem 1, page 258, [37]). □

Lemma 3.5.3 *The sesquilinear form $(\cdot|\cdot)$ in (3.11) defines an inner product on $\widetilde{\mathcal{H}}$. Furthermore, the completion of $\widetilde{\mathcal{H}}$ is a Hilbert space which is equivalent to $\mathcal{H}$.*

Proof. From (3.11) and (3.12), we get

$$\begin{aligned}\|\mathbf{u}\|^2 \simeq & \int_{\mathbb{B}^5} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k u_1(\xi)}d\xi + \int_{\mathbb{B}^5} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j u_2(\xi)}d\xi \\ & + \int_{\mathbb{S}^4} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}d\sigma(\omega) + \int_{\mathbb{S}^4} \partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}d\sigma(\omega) \\ & + \int_{\mathbb{B}^5} \partial_i\partial_j\partial^j u_1(\xi)\overline{\partial^i\partial^k\partial_k u_1(\xi)}d\xi + \int_{\mathbb{S}^4} \partial_j u_2(\omega)\overline{\partial^j u_2(\omega)}d\sigma(\omega) \\ & + \int_{\mathbb{S}^4} |u_2(\omega)|^2\, d\sigma(\omega) + \left|\int_{\mathbb{S}^4} \zeta\left(\omega, \mathbf{u}(\omega)\right) d\sigma(\omega)\right|^2,\end{aligned}$$

for all $\mathbf{u} \in \widetilde{\mathcal{H}}$. We need to show that $\|\mathbf{u}\| \simeq \|\mathbf{u}\|_{H^3(\mathbb{B}^5)\times H^2(\mathbb{B}^5)}$, for all $\mathbf{u} \in \widetilde{\mathcal{H}}$. First, note that it suffices to prove $\|\mathbf{u}\|_{H^3(\mathbb{B}^5)\times H^2(\mathbb{B}^5)} \lesssim \|\mathbf{u}\|$ since the reverse inequality is a direct consequence of the trace theorem (see Theorem 1, page 258, [37]) and the embedding $L^2(\mathbb{S}^4) \hookrightarrow L^1(\mathbb{S}^4)$. From Lemma 3.5.2, we get

$$\|\mathbf{u}\|_{H^3(\mathbb{B}^5)\times H^2(\mathbb{B}^5)} \lesssim \|\mathbf{u}\| + \|u_1\|_{L^2(\mathbb{S}^4)}$$

and the Poincare inequality on the 4−sphere (see Theorem 2.10, page 40, [42]),

$$\left\|u_1 - \frac{2}{\pi^2}\int_{\mathbb{S}^4} u_1(\omega)d\sigma(\omega)\right\|_{L^2(\mathbb{S}^4)} \lesssim \|\nabla u_1\|_{L^2(\mathbb{S}^4)},$$

together with the embedding $L^2(\mathbb{S}^4) \hookrightarrow L^1(\mathbb{S}^4)$ yield

$$
\begin{aligned}
\|u_1\|_{L^2(\mathbb{S}^4)} &\lesssim \|\nabla u_1\|_{L^2(\mathbb{S}^4)} + \left| \int_{\mathbb{S}^4} u_1(\omega) d\sigma(\omega) \right| \\
&\leq \|\mathbf{u}\| + \left| \int_{\mathbb{S}^4} u_1(\omega) d\sigma(\omega) \right| \\
&\lesssim \|\mathbf{u}\| + \left| \int_{\mathbb{S}^4} \zeta(\omega, \mathbf{u}(\omega))\, d\sigma(\omega) \right| + \left| \int_{\mathbb{S}^4} \omega^i \omega^j \partial_i \partial_j u_1(\omega) d\sigma(\omega) \right| \\
&+ \left| \int_{\mathbb{S}^4} \omega^j \partial_j u_1(\omega) d\sigma(\omega) \right| + \left| \int_{\mathbb{S}^4} \omega^j \partial_j u_2(\omega) d\sigma(\omega) \right| + \left| \int_{\mathbb{S}^4} u_2(\omega) d\sigma(\omega) \right| \\
&\lesssim \|\mathbf{u}\| + \left(\int_{\mathbb{S}^4} \left| \partial^2 u_1(\omega) \right|^2 d\sigma(\omega) \right)^{\frac{1}{2}} + \left(\int_{\mathbb{S}^4} \left| \partial u_1(\omega) \right|^2 d\sigma(\omega) \right)^{\frac{1}{2}} \\
&+ \left(\int_{\mathbb{S}^4} \left| \partial u_2(\omega) \right|^2 d\sigma(\omega) \right)^{\frac{1}{2}} + \left(\int_{\mathbb{S}^4} \left| u_2(\omega) \right|^2 \, d\sigma(\omega) \right)^{\frac{1}{2}} \\
&\lesssim \|\mathbf{u}\|,
\end{aligned}
$$

which concludes the proof. □

3.5.2 Free evolution and decay in time

Now, we focus on the proof of Proposition 3.5.1 and show that a semigroup (solution operator) is generated and decays in time with a sharp decay estimate. We specify the domain of $\widetilde{\mathbf{L}}$,

$$
\mathcal{D}\big(\widetilde{\mathbf{L}}\big) := C^4\big(\overline{\mathbb{B}^5}\big) \times C^3\big(\overline{\mathbb{B}^5}\big). \tag{3.13}
$$

To prove Proposition 3.5.1, we intend to apply the Lumer-Phillips theorem (see Theorem 3.15, page 83, [36]). The following two Lemmas constitute the key property of the sesquilinear forms defined above and verify the first part of the hypothesis of the Lumer-Phillips theorem. First, we define

$$
\|\cdot\|_j : \widetilde{\mathcal{H}} \longrightarrow \mathbb{R}, \quad \|\cdot\|_j := \sqrt{\big(\cdot\big|\cdot\big)_j},
$$

for all $j \in \{1,2,3,4,5\}$, where the sesquilinear forms $\big(\cdot\big|\cdot\big)_j$ are defined in section 3.5.1.

Lemma 3.5.4 *For all* $\mathbf{u} \in \mathcal{D}\big(\widetilde{\mathbf{L}}\big)$ *and* $i \in \{1,2,3,4\}$*, we have*

$$
\operatorname{Re}\big(\widetilde{\mathbf{L}}\mathbf{u}\big|\mathbf{u}\big)_i \leq -\frac{3}{2}\|\mathbf{u}\|_i^2.
$$

Proof. To begin with, fix an arbitrary $\mathbf{u} \in C^4(\overline{\mathbb{B}^5}) \times C^3(\overline{\mathbb{B}^5})$. On the one hand, the divergence theorem implies

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{B}^5} \partial_i\partial_j\partial_k\big(\widetilde{\mathbf{L}}\mathbf{u}\big)_1(\xi)\overline{\partial^i\partial^j\partial^k u_1(\xi)}d\xi &= -\frac{3}{2}\int_{\mathbb{B}^5} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k u_1(\xi)}d\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{S}^4} \partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial^k u_1(\omega)}d\sigma(\omega) \\
&\quad + \operatorname{Re}\int_{\mathbb{B}^5} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k u_2(\xi)}d\xi, \\
\operatorname{Re}\int_{\mathbb{B}^5} \partial_i\partial_j\big(\widetilde{\mathbf{L}}\mathbf{u}\big)_2(\xi)\overline{\partial^i\partial^j u_2(\xi)}d\xi &= -\frac{3}{2}\int_{\mathbb{B}^5} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j u_2(\xi)}d\xi \\
&\quad - \operatorname{Re}\int_{\mathbb{B}^5} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k u_2(\xi)}d\xi \\
&\quad + \operatorname{Re}\int_{\mathbb{S}^4} \omega^k\partial_k\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_2(\omega)}d\sigma(\omega) \\
&\quad - \frac{1}{2}\int_{\mathbb{S}^4} \partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_2(\omega)}d\sigma(\omega)
\end{aligned}$$

and, on the other hand, we have

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{S}^4} \partial_i\partial_j\big(\widetilde{\mathbf{L}}\mathbf{u}\big)_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}d\sigma(\omega) &= -3\int_{\mathbb{S}^4} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}d\sigma(\omega) \\
&\quad - \operatorname{Re}\int_{\mathbb{S}^4} \omega^k\partial_k\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}d\sigma(\omega) \\
&\quad + \operatorname{Re}\int_{\mathbb{S}^4} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_2(\omega)}d\sigma(\omega).
\end{aligned}$$

Hence, we obtain

$$\operatorname{Re}\,\big(\widetilde{\mathbf{L}}\mathbf{u}\big|\mathbf{u}\big)_1 + \frac{3}{2}\|\mathbf{u}\|_1^2 = -\int_{\mathbb{S}^4} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}d\sigma(\omega) + \int_{\mathbb{S}^4} A(\omega)d\sigma(\omega),$$

where

$$\begin{aligned}
A(\omega) &:= -\frac{1}{2}\partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial^k u_1(\omega)} - \frac{1}{2}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_2(\omega)} - \frac{1}{2}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)} \\
&\quad + \operatorname{Re}\Big(\omega^k\partial_k\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_2(\omega)}\Big) + \operatorname{Re}\Big(\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_2(\omega)}\Big) \\
&\quad - \operatorname{Re}\Big(\omega^k\partial_k\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\Big).
\end{aligned}$$

Now, we use the inequality

$$\operatorname{Re}(a\overline{b}) + \operatorname{Re}(a\overline{c}) - \operatorname{Re}(b\overline{c}) \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 + \frac{1}{2}|c|^2, \tag{3.14}$$

which holds for all $a, b, c \in \mathbb{C}$, together with Cauchy-Schwarz inequality

$$\left|\sum_k \omega^k \partial_k \partial_i \partial_j u_1(\omega)\right|^2 \leq \sum_k \left(\omega^k\right)^2 \sum_k |\partial_k \partial_i \partial_j u_1(\omega)|^2 = \sum_k |\partial_k \partial_i \partial_j u_1(\omega)|^2$$

to obtain $A(\omega) \leq 0$ for all $\omega \in \mathbb{S}^4$ and the desired estimate for $\big(\widetilde{\mathbf{L}}\mathbf{u}\big|\mathbf{u}\big)_1$ follows. For the second estimate, the divergence theorem yields

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{B}^5} \partial_i \partial^k \partial_k \big(\widetilde{\mathbf{L}}\mathbf{u}\big)_1(\xi)\overline{\partial_i \partial^j \partial_j u_1(\xi)}d\xi &= -\frac{3}{2}\int_{\mathbb{B}^5} \partial_i \partial^j \partial_j u_1(\xi)\overline{\partial^i \partial_k \partial^k u_1(\xi)}d\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{S}^4} \partial_i \partial^j \partial_j u_1(\omega)\overline{\partial^i \partial_k \partial^k u_1(\omega)}d\sigma(\omega) \\
&\quad + \operatorname{Re}\int_{\mathbb{B}^5} \partial^i \partial^j \partial_j u_1(\xi)\overline{\partial_i \partial^k \partial_k u_2(\xi)}d\xi, \\
\operatorname{Re}\int_{\mathbb{B}^5} \partial_i \partial_j \big(\widetilde{\mathbf{L}}\mathbf{u}\big)_2(\xi)\overline{\partial^i \partial^j u_2(\xi)}d\xi &= -\frac{3}{2}\int_{\mathbb{B}^5} \partial_i \partial_j u_2(\xi)\overline{\partial^i \partial^j u_2(\xi)}d\xi \\
&\quad - \frac{1}{2}\int_{\mathbb{S}^4} \partial_i \partial_j u_2(\omega)\overline{\partial^i \partial^j u_2(\omega)}d\sigma(\omega) \\
&\quad - \operatorname{Re}\int_{\mathbb{B}^5} \partial^i \partial^k \partial_k u_1(\xi)\overline{\partial_i \partial^j \partial_j u_2(\xi)}d\xi \\
&\quad + \operatorname{Re}\int_{\mathbb{S}^4} \omega^j \partial_i \partial_j u_2(\omega)\overline{\partial^i \partial^k \partial_k u_1(\omega)}d\sigma(\omega),
\end{aligned}$$

and, in addition, we have

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{S}^4} \partial_j \big(\widetilde{\mathbf{L}}\mathbf{u}\big)_2(\omega)\overline{\partial^j u_2(\omega)}d\sigma(\omega) &= -3\int_{\mathbb{S}^4} \partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}d\sigma(\omega) \\
&\quad - \operatorname{Re}\int_{\mathbb{S}^4} \omega^k \partial_k \partial_j u_2(\omega)\overline{\partial^j u_2(\omega)}d\sigma(\omega) \\
&\quad + \operatorname{Re}\int_{\mathbb{S}^4} \partial^i \partial_i \partial_j u_1(\omega)\overline{\partial^j u_2(\omega)}d\sigma(\omega).
\end{aligned}$$

Therefore, we get

$$\operatorname{Re}\,\big(\widetilde{\mathbf{L}}\mathbf{u}\big|\mathbf{u}\big)_2 + \frac{3}{2}\|\mathbf{u}\|_2^2 = -\int_{\mathbb{S}^4} \partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}d\sigma(\omega) + \int_{\mathbb{S}^4} B(\omega)d\sigma(\omega),$$

where

$$\begin{aligned}
B(\omega) := &-\frac{1}{2}\partial_i \partial^j \partial_j u_1(\omega)\overline{\partial^i \partial^k \partial_k u_1(\omega)} - \frac{1}{2}\partial_i \partial_j u_2(\omega)\overline{\partial^i \partial^j u_2(\omega)} - \frac{1}{2}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)} \\
&+ \operatorname{Re}\Big(\omega^j \partial_j \partial_i u_2(\omega)\overline{\partial^i \partial^k \partial_k u_1(\omega)}\Big) \\
&+ \operatorname{Re}\Big(\partial^j u_2(\omega)\overline{\partial^i \partial_i \partial_j u_1(\omega)}\Big) - \operatorname{Re}\Big(\omega^k \partial_k \partial_j u_2(\omega)\overline{\partial^j u_2(\omega)}\Big).
\end{aligned}$$

As before, we use (3.14) together with Cauchy-Schwarz inequality

$$\left|\sum_k \omega^k \partial_k \partial_i u_2(\omega)\right|^2 \leq \sum_k \left(\omega^k\right)^2 \sum_k |\partial_k \partial_i u_2(\omega)|^2 = \sum_k |\partial_k \partial_i u_2(\omega)|^2$$

to get $B(\omega) \leq 0$ for all $\omega \in \mathbb{S}^4$ and the claim for $\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_2$ follows. For the third estimate, we use the previous estimates together with Cauchy-Schwarz inequalities

$$\left|\sum_i \partial_i \partial^i u_1(\omega)\right|^2 \leq \sum_i 1^2 \sum_i \left|\partial_i \partial^i u_1(\omega)\right|^2 \leq 5 \sum_{i,j} |\partial_i \partial_j u_1(\omega)|^2,$$

$$\left|\sum_k \omega^k \partial_k u_2(\omega)\right|^2 \leq \sum_k \left(\omega^k\right)^2 \sum_k |\partial_k u_2(\omega)|^2 = \sum_k |\partial_k u_2(\omega)|^2,$$

and Young's inequality to obtain

$$\begin{aligned}
\text{Re } \left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_3 + \frac{3}{2}\|\mathbf{u}\|_3^2 = 5\left(\text{Re } \left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_1 + \frac{3}{2}\|\mathbf{u}\|_1^2\right) + \text{Re } \left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_2 + \frac{3}{2}\|\mathbf{u}\|_2^2 + \\
+ \text{Re } \int_{\mathbb{S}^4} \left(\left(\widetilde{\mathbf{L}}\mathbf{u}(\omega)\right)_2 \overline{u_2(\omega)} + \frac{3}{2}\left|u_2(\omega)\right|^2\right) d\sigma(\omega) \\
\leq -5 \int_{\mathbb{S}^4} \partial_i \partial_j u_1(\omega) \overline{\partial^i \partial^j u_1(\omega)} d\sigma(\omega) - \int_{\mathbb{S}^4} \partial_i u_2(\omega) \overline{\partial^i u_2(\omega)} d\sigma(\omega) \\
+ \text{Re } \int_{\mathbb{S}^4} \left(\partial^i \partial_i u_1(\omega) \overline{u_2(\omega)} - \omega^k \partial_k u_2(\omega) \overline{u_2(\omega)} - \frac{1}{2}\left|u_2(\omega)\right|^2\right) d\sigma(\omega) \\
\leq - \int_{\mathbb{S}^4} \partial_i \partial^i u_1(\omega) \overline{\partial_j \partial^j u_1(\omega)} d\sigma(\omega) - \int_{\mathbb{S}^4} \partial_i u_2(\omega) \overline{\partial^i u_2(\omega)} d\sigma(\omega) \\
+ \text{Re } \int_{\mathbb{S}^4} \left(\partial^i \partial_i u_1(\omega) \overline{u_2(\omega)} - \omega^k \partial_k u_2(\omega) \overline{u_2(\omega)} - \frac{1}{2}\left|u_2(\omega)\right|^2\right) d\sigma(\omega) \\
= -\frac{1}{2} \int_{\mathbb{S}^4} \partial_i \partial^i u_1(\omega) \overline{\partial_j \partial^j u_1(\omega)} d\sigma(\omega) - \frac{1}{2} \int_{\mathbb{S}^4} \partial_i u_2(\omega) \overline{\partial^i u_2(\omega)} d\sigma(\omega) \\
- \text{Re } \int_{\mathbb{S}^4} \partial^i \partial_i u_1(\omega) \overline{\omega^k \partial_k u_2(\omega)} d\sigma(\omega) + \int_{\mathbb{S}^4} C(\omega) d\sigma(\omega) \\
\leq \int_{\mathbb{S}^4} C(\omega) d\sigma(\omega)
\end{aligned}$$

where

$$\begin{aligned}
C(\omega) := -\frac{1}{2}\partial_i \partial^i u_1(\omega) \overline{\partial_j \partial^j u_1(\omega)} - \frac{1}{2}\partial_i u_2(\omega) \overline{\partial^i u_2(\omega)} - \frac{1}{2}\left|u_2(\omega)\right|^2 \\
+ \text{Re}\left(u_2(\omega) \overline{\partial^i \partial_i u_1(\omega)}\right) + \text{Re}\left(\partial^i \partial_i u_1(\omega) \overline{\omega^k \partial_k u_2(\omega)}\right) - \text{Re}\left(\omega^k \partial_k u_2(\omega) \overline{u_2(\omega)}\right).
\end{aligned}$$

Inequality (3.14) implies $C(\omega) \leq 0$ for all $\omega \in \mathbb{S}^4$ and the claim for $\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_3$ follows. Finally, for the last estimate, we use the previous estimates together Cauchy-Schwarz inequality

$$\left|\sum_k \omega^k \partial_k \partial_i u_1(\omega)\right|^2 \leq \sum_k \left(\omega^k\right)^2 \sum_k |\partial_k \partial_i u_1(\omega)|^2 = \sum_k |\partial_k \partial_i u_1(\omega)|^2$$

and Young's inequality once more to obtain

$$\begin{aligned}
\text{Re } \left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_4 + \frac{3}{2}\|\mathbf{u}\|_4^2 &= \text{Re } \left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_1 + \frac{3}{2}\|\mathbf{u}\|_1^2 + \text{Re } \left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_2 + \frac{3}{2}\|\mathbf{u}\|_2^2 + \\
&+ \text{Re } \int_{\mathbb{S}^4} \left(\partial_i \left(\widetilde{\mathbf{L}}\mathbf{u}\right)_1(\omega)\overline{\partial^i u_1(\omega)} + \frac{3}{2}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\right) d\sigma(\omega) \\
&\leq - \int_{\mathbb{S}^4} \partial_i \partial_j u_1(\omega)\overline{\partial^i \partial^j u_1(\omega)} d\sigma(\omega) - \int_{\mathbb{S}^4} \partial_i u_2(\omega)\overline{\partial^i u_2(\omega)} d\sigma(\omega) \\
&+ \text{Re } \int_{\mathbb{S}^4} \left(\partial_i u_2(\omega)\overline{\partial^i u_1(\omega)} - \omega^k \partial_k \partial_i u_1(\omega)\overline{\partial^i u_1(\omega)} - \frac{1}{2}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\right) d\sigma(\omega) \\
&= -\frac{1}{2}\int_{\mathbb{S}^4} \partial_i \partial_j u_1(\omega)\overline{\partial^i \partial^j u_1(\omega)} d\sigma(\omega) - \frac{1}{2}\int_{\mathbb{S}^4} \partial_i u_2(\omega)\overline{\partial^i u_2(\omega)} d\sigma(\omega) \\
&- \text{Re } \int_{\mathbb{S}^4} \partial_i u_2(\omega)\overline{\omega^k \partial_k \partial^i u_1(\omega)} d\sigma(\omega) + \int_{\mathbb{S}^4} D(\omega) d\sigma(\omega) \\
&\leq \int_{\mathbb{S}^4} D(\omega) d\sigma(\omega),
\end{aligned}$$

where

$$\begin{aligned}
D(\omega) := &-\frac{1}{2}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)} - \frac{1}{2}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)} - \frac{1}{2}\partial_i \partial_j u_1(\omega)\overline{\partial^i \partial^j u_1(\omega)} \\
&+ \text{Re}\left(\partial_i u_2(\omega)\overline{\partial^i u_1(\omega)}\right) + \text{Re}\left(\partial_i u_2(\omega)\overline{\omega^k \partial_k \partial^i u_1(\omega)}\right) - \text{Re}\left(\omega^k \partial_k \partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\right).
\end{aligned}$$

As before, (3.14) implies $D(\omega) \leq 0$ for all $\omega \in \mathbb{S}^4$ and the claim for $\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_4$ follows. ☐

Lemma 3.5.5 *For all* $\mathbf{u} \in \mathcal{D}\left(\widetilde{\mathbf{L}}\right)$, *we have*

$$\text{Re}\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_5 = -\|\mathbf{u}\|_5^2.$$

Proof. Fix $\mathbf{u} \in C^4\left(\overline{\mathbb{B}^5}\right) \times C^3\left(\overline{\mathbb{B}^5}\right)$. A long but straight-forward calculation yields

$$\zeta\left(\omega, \widetilde{\mathbf{L}}\mathbf{u}(\omega)\right) = -\zeta\left(\omega, \mathbf{u}(\omega)\right) + \Delta_\omega^{\mathbb{S}^4}\left(u_1(\omega) + \omega^j \partial_j u_1(\omega)\right), \tag{3.15}$$

where $\Delta_\omega^{\mathbb{S}^4}$ stands for the Laplace-Beltrami operator on the 4−sphere, namely

$$\Delta_\omega^{\mathbb{S}^4} = \left(\delta^{jk} - \omega^j \omega^k\right) \partial_{\omega^j} \partial_{\omega^k} - 4\omega^j \partial_{\omega^j}.$$

Now, Stoke's theorem yields

$$\int_{\mathbb{S}^4} \Delta_\omega^{\mathbb{S}^4} \Big(u_1(\omega) + \omega^j \partial_j u_1(\omega) \Big) = 0$$

which implies the initial claim. □

Summarizing the results of the two previous Lemmas, we get

Corollary 3.5.6 *For all* $\mathbf{u} \in \mathcal{D}\big(\widetilde{\mathbf{L}}\big)$*, we have*

$$\mathrm{Re}\big(\widetilde{\mathbf{L}}\mathbf{u}\big|\mathbf{u}\big) \le -\|\mathbf{u}\|^2.$$

Next, we prove that the range of $\lambda - \widetilde{\mathbf{L}}$ is dense in $\mathcal{H}$ for some $\lambda > -1$ which verifies the second and last hypothesis of the Lumer-Phillips theorem. However, we will first need a technical result.

Lemma 3.5.7 *For any* $F \in H^2(\mathbb{B}^5)$ *and* $\epsilon > 0$*, there exists* $v \in C^4(\overline{\mathbb{B}^5})$ *such that the function defined by*

$$h(\xi) := -\left(\delta^{ij} - \xi^i \xi^j\right) \partial_i \partial_j v(\xi) + 7\xi^j \partial_j v(\xi) + \frac{35}{4} v(\xi)$$

satisfies $h \in C^2(\overline{\mathbb{B}^5})$ *and* $\|h - F\|_{H^2(\mathbb{B}^5)} < \epsilon$*.*

Proof. To begin with, we pick an arbitrary $F \in H^2(\mathbb{B}^5)$ and $\epsilon > 0$. Since $C^\infty(\overline{\mathbb{B}^5})$ is dense in $H^2(\mathbb{B}^5)$, we pick a function $\tilde{h} \in C^\infty(\overline{\mathbb{B}^5})$ such that $\|F - \tilde{h}\|_{H^2(\mathbb{B}^5)} < \frac{\epsilon}{2}$. We consider the equation

$$-(\delta^{ij} - \xi^i \xi^j)\partial_i \partial_j v(\xi) + 7\xi^j \partial_j v(\xi) + \frac{35}{4} v(\xi) = \tilde{h}(\xi). \tag{3.16}$$

To solve (3.16), we switch to spherical coordinates $\xi = \rho\omega$, where $\rho = |\xi|$ and $\omega = \frac{\xi}{|\xi|}$. Then,

$$\partial_j \rho(\xi) = \omega_j(\xi), \quad \partial_j \omega^k(\xi) = \frac{\delta_j^k - \omega_j(\xi)\omega^k(\xi)}{\rho(\xi)}$$

and derivatives transform according to

$$\begin{aligned}
&\xi^j \partial_j u(\xi) = \rho \partial_\rho u(\rho\omega), \\
&\xi^i \xi^j \partial_i \partial_j u(\xi) = \rho^2 \partial_\rho^2 u(\rho\omega), \\
&\partial^j \partial_j u(\xi) = \left(\partial_\rho^2 + \frac{4}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^4}\right) u(\rho\omega).
\end{aligned}$$

Hence, (3.16) can be written equivalently as

$$\left(-\left(1-\rho^2\right)\partial_\rho^2 + \left(-\frac{4}{\rho}+7\rho\right)\partial_\rho + \frac{35}{4} - \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^4}\right) v(\rho\omega) = \tilde{h}(\rho\omega). \tag{3.17}$$

The Laplace-Beltrami operator $\Delta^{\mathbb{S}^4}$ is self-adjoint on $L^2\left(\mathbb{S}^4\right)$ and its spectrum coincides with the point spectrum

$$\sigma\big(-\Delta^{\mathbb{S}^4}\big) = \sigma_p\big(-\Delta^{\mathbb{S}^4}\big) = \big\{l(l+3) : l \in \mathbb{N}_0\big\}.$$

For each $l \in \mathbb{N}_0$, the eigenspace to the eigenvalue $l(l+3)$ is finite dimensional and spanned by the spherical harmonics $\{Y_{l,m} : m \in \Omega_l\}$ which are obtained by restricting harmonic homogeneous polynomials in $\mathbb{R}^5$ to $\mathbb{S}^4$. Here, $\Omega_l \subseteq \mathbb{Z}$ stands for the set of admissible indices m. Since $\tilde{h} \in C^\infty(\overline{\mathbb{B}^5})$, we can expand

$$\tilde{h}\left(\rho\omega\right) = \sum_{l=0}^{\infty}\sum_{m\in\Omega_l} \tilde{h}_{l,m}(\rho) Y_{l,m}(\omega)$$

and we define $\tilde{h}_N \in C^\infty(\overline{\mathbb{B}^5})$ by

$$\tilde{h}_N\left(\rho\omega\right) = \sum_{l=0}^{N}\sum_{m\in\Omega_l} \tilde{h}_{l,m}(\rho) Y_{l,m}(\omega),$$

for all $N \in \mathbb{N}$. It is well known that

$$\big\|\tilde{h} - \tilde{h}_N\big\|_{H^2(\mathbb{B}^5)} \longrightarrow 0, \quad \text{as } N \longrightarrow \infty$$

and therefore we can pick $N \in \mathbb{N}$ large enough so that $\big\|\tilde{h}-\tilde{h}_N\big\|_{H^2(\mathbb{B}^5)} < \frac{\epsilon}{2}$. Then, (3.17) and the linear independence of $Y_{l,m}$ yield the decoupled system of elliptic ordinary differential equations

$$\left(-\left(1-\rho^2\right)\frac{d^2}{d\rho^2} + \left(-\frac{4}{\rho}+7\rho\right)\frac{d}{d\rho} + \frac{35}{4} + \frac{l(l+3)}{\rho^2}\right) v_{l,m}(\rho) = \tilde{h}_{l,m}(\rho). \tag{3.18}$$

Setting $u_{l,m}(\rho) = \rho v_{l,m}(\rho)$, (3.18) yields an equation for $u_{l,m}$, that is

$$\left(-\left(1-\rho^2\right)\frac{d^2}{d\rho^2} + \left(-\frac{2}{\rho}+5\rho\right)\frac{d}{d\rho} + \frac{15}{4} + \frac{(l+1)(l+2)}{\rho^2}\right) u_{l,m}(\rho) = \rho\tilde{h}_{l,m}(\rho). \tag{3.19}$$

Note that this is a second-order linear ordinary differential equation with four regular singular points, $\rho = -1, 0, 1$ and ∞. By the reflection symmetry, these four singular points can be reduced to three and therefore, (3.19) can be transformed into a hypergeometric differential equation. First, consider the homogeneous version of this equation, namely we set the right hand side equal to zero. Now, we introduce a new dependent variable. The transformation $u_{l,m}(\rho) = \rho^{l+1} w_{l,m}(z)$, $z = \rho^2$ brings (3.19) to a hypergeometric differential equation in its canonical form

$$z(1-z)w''_{l,m}(z) + \Big(c - (a+b+1)z\Big) w'_{l,m}(z) - ab w_{l,m}(z) = 0, \tag{3.20}$$

where

$$a = \frac{5+2l}{4}, \; b = a + \frac{1}{2} = \frac{7+2l}{4}, \; c = 2a = \frac{5+2l}{2}.$$

Then, (3.20) admits two solutions

$$\phi_{0,l}(z) = {}_2F_1\left(a, a+\frac{1}{2}, 2a; z\right), \quad \phi_{1,l}(z) = {}_2F_1\left(a, a+\frac{1}{2}, \frac{3}{2}; 1-z\right),$$

which are analytic around $z = 0$ and $z = 1$ respectively, see [68], page 395, 15.10.2 and 15.10.4. First, notice that both $\phi_{0,l}$ and $\phi_{1,l}$ can be expressed in closed forms as

$$\phi_{0,l}(z) = \frac{1}{\sqrt{1-z}} \left(\frac{2}{1+\sqrt{1-z}}\right)^{\frac{3}{2}+l}, \tag{3.21}$$

$$\phi_{1,l}(z) = \frac{1}{(3+2l)\sqrt{1-z}} \left(\left(\frac{1}{1-\sqrt{1-z}}\right)^{\frac{3}{2}+l} - \left(\frac{1}{1+\sqrt{1-z}}\right)^{\frac{3}{2}+l}\right), \tag{3.22}$$

see [68], page 387, 15.4.18 and [68], page 386, 15.4.9. Second, we argue that $\phi_{0,l}$ and $\phi_{1,l}$ are linearly independent. Indeed, we assume that there exist constants $c_{0,l}, c_{1,l} \in \mathbb{C}$ such that

$$c_{0,l}\phi_{0,l}(z) + c_{1,l}\phi_{1,l}(z) = 0.$$

Now, $c_{1,l} = 0$ since $\lim_{z\to 0^+} \phi_{1,l}(z) = \infty$ whereas $\lim_{z\to 0^+} \phi_{0,l}(z) < \infty$. Furthermore, $c_{0,l} = 0$ since $\lim_{z\to 1^-} \sqrt{1-z}\phi_{1,l}(z) < \infty$. For later reference, we note that the function

$$\tilde{\phi}_{1,l}(z) = (1-z)^{-\frac{1}{2}} \hat{\phi}_1(z)$$

is also a solution to (3.20), see [68], page 395, 15.10.4, where

$$\hat{\phi}_1(z) := {}_2F_1\left(a, a - \frac{1}{2}, \frac{1}{2}; 1-z\right)$$

is analytic around $z = 1$, see [68], page 384, 15.2.1. Since $\{\phi_{0,l}, \phi_{1,l}\}$ is a fundamental system for (3.20), we get that there exist constants $\alpha_l, \beta_l \in \mathbb{C}$ such that

$$\phi_{0,l}(z) = \alpha_l \phi_{1,l}(z) + \beta_l \tilde{\phi}_{1,l}(z).$$

Transforming back, we obtain two linearly independent solutions $\psi_{j,l}(\rho) = \rho^{l+1}\phi_{j,l}(\rho^2)$, $j \in \{0,1\}$ to the homogeneous version of equation (3.19) as well as $\hat{\psi}_{1,l}(\rho) = \rho^{l+1}\hat{\phi}_{1,l}(\rho^2)$. In particular, we get that there exist constants $\alpha_l, \beta_l \in \mathbb{C}$ such that

$$\psi_{0,l}(\rho) = \alpha_l \psi_{1,l}(\rho) + \beta_l (1-\rho)^{-\frac{1}{2}} \hat{\psi}_{1,l}(\rho),$$

where $\hat{\psi}_{1,l}$ is analytic around $\rho = 1$. Moreover, $\psi_{1,l}$ is analytic around $p = 1$ since $\phi_{1,l}$ is analytic around $z = 1$, see [68], page 384, 15.2.1. Next, we find the Wronskian. A straightforward calculation yields

$$W(\psi_{0,l}, \psi_{1,l})(\rho) = 2\rho^{3l+2} W(\phi_{0,l}, \phi_{1,l})(\rho^2) = \frac{-2^{l+\frac{3}{2}}}{\rho^2 (1-\rho^2)^{\frac{3}{2}}}. \tag{3.23}$$

By the variation of constants formula, a particular solution to equation (3.19) is given by

$$u_{l,m}(\rho) = -\psi_{0,l}(\rho) I_{1,l}(\rho) - \psi_{1,l}(\rho) I_{0,l}(\rho), \tag{3.24}$$

where

$$I_{0,l}(\rho) := \int_0^\rho \psi_{0,l}(s) Z_{l,m}(s) ds, \quad I_{1,l}(\rho) := \int_\rho^1 \psi_{1,l}(s) Z_{l,m}(s) ds,$$

and

$$Z_{l,m}(s) := \frac{s \tilde{h}_{l,m}(s)}{(1-s^2) W(\psi_{0,l}, \psi_{1,l})(s)} = (1-s)^{\frac{1}{2}} \xi_{l,m}(s), \quad \xi_{l,m}(s) := -\frac{1}{2^{l+\frac{3}{2}}} (1+s)^{\frac{1}{2}} s^3 \tilde{h}_{l,m}(s).$$

Notice that $\xi_{l,m} \in C^\infty([0,1])$ since $\tilde{h}_{l,m} \in C^\infty([0,1])$. We claim that $u_{l,m} \in C^\infty(0,1]$. To prove this, we first observe that the quantity

$$c_{l,m} := \int_0^1 (1-s)^{\frac{1}{2}} \psi_{0,l}(s) \xi_{l,m}(s) = \alpha_l \int_0^1 (1-\rho)^{\frac{1}{2}} \psi_{1,l}(\rho) \xi_{l,m}(s) ds + \beta_l \int_0^1 \hat{\psi}_{1,l}(s) \xi_{l,m}(s) ds$$

is a real number since both integrands are continuous functions on the closed interval $[0,1]$. Hence, we can write

$$I_{0,l}(\rho) = c_{l,m} - \alpha_l \int_\rho^1 (1-s)^{\frac{1}{2}} \psi_{1,l}(s) \xi_{l,m}(s) ds - \beta_l \int_\rho^1 \hat{\psi}_{1,l}(s) \xi_{l,m}(s) ds.$$

Moreover,

$$\begin{aligned}
\psi_{1,l}(\rho) I_{0,l}(\rho) &= c_{l,m} \psi_{1,l}(\rho) \\
&\quad - \alpha_l \psi_{1,l}(\rho) \int_\rho^1 (1-s)^{\frac{1}{2}} \psi_{1,l}(s) \xi_{l,m}(s) ds - \beta_l \psi_{1,l}(\rho) \int_\rho^1 \hat{\psi}_{1,l}(s) \xi_{l,m}(s) ds, \\
\psi_{0,l}(\rho) I_{1,l}(\rho) &= \alpha_l \psi_{1,l}(\rho) \int_\rho^1 (1-s)^{\frac{1}{2}} \psi_{1,l}(s) \xi_{l,m}(s) ds \\
&\quad + \beta_l (1-\rho)^{-\frac{1}{2}} \hat{\psi}_{1,l}(\rho) \int_\rho^1 (1-s)^{\frac{1}{2}} \psi_{1,l}(s) \xi_{l,m}(s) ds,
\end{aligned}$$

and hence

$$u_{l,m}(\rho) = -c_{l,m}\psi_{1,l}(\rho) \\ + \beta_l\psi_{1,l}(\rho)\int_\rho^1 \hat{\psi}_{1,l}(s)\xi_{l,m}(s)ds - \beta_l(1-\rho)^{-\frac{1}{2}}\hat{\psi}_{1,l}(\rho)\int_\rho^1 (1-s)^{\frac{1}{2}}\psi_{1,l}(s)\xi_{l,m}(s)ds.$$

Obviously, the first and the second terms belong to $C^\infty(0,1]$. Therefore, we focus on the third term and define

$$U_{l,m}(\rho) := (1-\rho)^{-\frac{1}{2}}\hat{\psi}_{1,l}(\rho)\int_\rho^1 (1-s)^{\frac{1}{2}}\psi_{1,l}(s)\xi_{l,m}(s)ds.$$

Now, we choose an arbitrary $N \in \mathbb{N}$ and show that the limit

$$\lim_{\rho\to 1^-}\frac{d^N}{d\rho^N}U_{l,m}(\rho) \tag{3.25}$$

exists. Fix sufficiently small $\delta > 0$, $\rho \in (1-\delta, 1)$. Then, the Taylor series expansion yields

$$\xi_{l,m}(\rho) = \sum_{i=0}^N a_{i,l,m}(1-\rho)^i + R_{N+1}(1-\rho),$$

for some coefficients $a_{i,l,m}$. Here, $R_M(1-\rho)$ stands for a remainder term which may change from line to line and satisfies the estimates

$$|R_M(1-\rho)| \le K(1-\rho)^M, \quad \left|\partial_\rho^k R_M(1-\rho)\right| \le \Lambda(1-\rho)^{M-k},$$

for all $k = 0, 1, \dots, M$ and $\rho \in (1-\delta, 1)$ and for some constants $M \in \mathbb{R}$, $K, \Lambda \ge 0$. Recall that $\psi_{1,l}$ and $\hat{\psi}_{1,l}$ are analytic functions around $\rho = 1$ and hence we can write

$$\psi_{1,l}(\rho) = \sum_{i=0}^\infty b_{i,l}(1-\rho)^i, \quad \hat{\psi}_{1,l}(\rho) = \sum_{i=0}^\infty \epsilon_{i,l}(1-\rho)^i$$

for some coefficients $b_{l,l}$ and $\epsilon_{i,l}$. Then, we have

$$(1-\rho)^{\frac{1}{2}}\psi_{1,l}(\rho)\xi_{l,m}(\rho) = \sum_{k=0}^\infty \gamma_{i,l,m}(1-\rho)^{k+\frac{1}{2}} + R_{N+1+\frac{1}{2}}(1-\rho),$$

$$\int_\rho^1 (1-s)^{\frac{1}{2}}\psi_{1,l}(s)\xi_{l,m}(s) = \sum_{k=0}^\infty \frac{2\gamma_{i,l,m}}{2k+3}(1-\rho)^{k+\frac{3}{2}} + R_{N+2+\frac{1}{2}}(1-\rho),$$

$$(1-\rho)^{-\frac{1}{2}}\int_\rho^1 (1-s)^{\frac{1}{2}}\psi_{1,l}(s)\xi_{l,m}(s) = \sum_{k=0}^\infty \frac{2\gamma_{i,l,m}}{2k+3}(1-\rho)^{k+1} + R_{N+2}(1-\rho),$$

$$\hat{\psi}_{1,l}(\rho)(1-\rho)^{-\frac{1}{2}}\int_\rho^1 (1-s)^{\frac{1}{2}}\psi_{1,l}(s)\xi_{l,m}(s) = \sum_{k=0}^\infty \zeta_{k,l,m}(1-\rho)^{k+1} + R_{N+2}(1-\rho),$$

for some coefficients $\gamma_{i,l,m}$ and $\zeta_{k,l,m}$. Therefore,

$$\frac{d^N}{d\rho^N}U_{l,m}(\rho) = \frac{d^N}{d\rho^N}\left(\sum_{k=0}^{\infty}\zeta_{k,l,m}(1-\rho)^{k+1} + R_{N+2}(1-\rho)\right) = \sum_{i=0}^{\infty}\eta_{i,l,m}(1-\rho)^i + R_2(1-\rho),$$

for some coefficients $\eta_{i,l,m}$. Consequently, the limit (3.25) exists and we get that $u_{l,m} \in C^\infty(0,1]$. Finally, $u \in H^2(\mathbb{B}^5) \cap C^\infty(\overline{\mathbb{B}^5} \setminus \{0\})$ and translating back we get $v \in H^2(\mathbb{B}^5) \cap C^\infty(\overline{\mathbb{B}^5} \setminus \{0\})$. By elliptic regularity, we infer $v \in C^\infty(\mathbb{B}^5) \cap C^\infty(\overline{\mathbb{B}^5} \setminus \{0\})$ which implies $v \in C^\infty(\overline{\mathbb{B}^5})$ as desired. □

Lemma 3.5.8 *The range of $\frac{3}{2} - \widetilde{\mathbf{L}}$ is dense in $\mathcal{H}$.*

Proof. Since $\left(C^\infty(\overline{\mathbb{B}^5})\right)^2$ is dense in $\mathcal{H}$, it suffices to show that

$$\forall\, \mathbf{f} \in \left(C^\infty(\overline{\mathbb{B}^5})\right)^2 \text{ and } \forall \epsilon > 0,\ \exists \mathbf{g} \in \operatorname{rg}\left(\frac{3}{2} - \widetilde{\mathbf{L}}\right) : \|\mathbf{f} - \mathbf{g}\| < \epsilon.$$

First note that, for any $\mathbf{u} \in \mathcal{D}(\widetilde{\mathbf{L}})$, the equation $\left(\frac{3}{2} - \widetilde{\mathbf{L}}\right)\mathbf{u} = \mathbf{g}$ reads

$$\begin{cases} u_2(\xi) = \frac{5}{2}u_1(\xi) + \xi^j\partial_j u_1(\xi) - g_1(\xi), \\ -\partial^j\partial_j u_1(\xi) + \xi^i\partial_i u_2(\xi) + \frac{7}{2}u_2(\xi) = g_2(\xi) \end{cases}$$

Inserting u_2 into the second equation, we obtain

$$-(\delta^{ij} - \xi^i\xi^j)\partial_i\partial_j u_1(\xi) + 7\xi^i\partial_i u_1(\xi) + \frac{35}{4}u_1(\xi) = G(\xi),$$

where

$$G(\xi) = g_2(\xi) + \frac{7}{2}g_1(\xi) + \xi^j\partial_j g_1(\xi).$$

Now, pick an arbitrary $\mathbf{f} \in \left(C^\infty(\overline{\mathbb{B}^5})\right)^2$, $\epsilon > 0$ and apply Lemma 3.5.7 to the function

$$F(\xi) = f_2(\xi) + \frac{7}{2}f_1(\xi) + \xi^j\partial_j f_1(\xi).$$

We infer the existence of a function $v \in C^4(\overline{\mathbb{B}^5})$ such that

$$h(\xi) := -\left(\delta^{ij} - \xi^i\xi^j\right)\partial_i\partial_j v(\xi) + 7\xi^j\partial_j v(\xi) + \frac{35}{4}v(\xi)$$

satisfies $h \in C^2(\overline{\mathbb{B}^5})$ and $\|h - F\|_{H^2(\mathbb{B}^5)} < \epsilon$. Now, define

$$\begin{cases} u_1(\xi) := v(\xi), \\ u_2(\xi) := \frac{5}{2}u_1(\xi) + \xi^j\partial_j u_1(\xi) - f_1(\xi), \end{cases}$$

$$\begin{cases} g_1(\xi) := f_1(\xi), \\ g_2(\xi) := h(\xi) - F(\xi) + f_2(\xi). \end{cases}$$

Then, by construction, we have $\mathbf{u} \in \mathcal{D}(\widetilde{\mathbf{L}})$, $\left(\frac{3}{2} - \widetilde{\mathbf{L}}\right)\mathbf{u} = \mathbf{g}$ and $\|\mathbf{f} - \mathbf{g}\| < \epsilon$. □

Proof of Proposition 3.5.1. It follows immediately from Corollary 3.5.6 and Lemma 3.5.8. □

3.6 Modulation ansatz

To account for the Lorentz symmetry we use a modulation ansatz. To be precise, we allow for the unknown parameter α to depend on τ, set $\alpha(0) = 0$ initially and assume (and later verify) that $\alpha_\infty := \lim_{\tau\to\infty} \alpha(\tau)$ exists. Then, we define

$$\mathbf{\Phi}(\tau) := \mathbf{\Psi}(\tau) - \mathbf{\Psi}_{\alpha(\tau)} \tag{3.26}$$

where $\mathbf{\Psi}_\alpha$ are the Lorentz transformations defined in (3.10) of the static blowup solution solution $\mathbf{\Psi}_0$. This ansatz leads to an equivalent description as an evolution equation for the perturbation term $\mathbf{\Phi}$, that is

$$\partial_\tau \mathbf{\Phi}(\tau) - \big(\mathbf{L} + \mathbf{L}'_{\alpha_\infty}\big)\mathbf{\Phi}(\tau) = \hat{\mathbf{L}}_{\alpha(\tau)}\mathbf{\Phi}(\tau) + \mathbf{N}_{\alpha(\tau)}(\mathbf{\Phi}(\tau)) - \partial_\tau \mathbf{\Psi}_{\alpha(\tau)}, \tag{3.27}$$

where

$$\hat{\mathbf{L}}_{\alpha(\tau)} := \mathbf{L}'_{\alpha(\tau)} - \mathbf{L}'_{\alpha_\infty} \tag{3.28}$$

and $\mathbf{L}'_{\alpha(\tau)}$ denotes the linearized part of the nonlinearity $\mathbf{N}$, i.e

$$\mathbf{L}'_{\alpha(\tau)}(\mathbf{u}(\xi)) := \begin{pmatrix} 0 \\ V_{\alpha(\tau)}(\xi)u_1(\xi)\end{pmatrix}, \quad V_{\alpha(\tau)}(\xi) := \frac{6}{(A_0(\alpha(\tau)) - A_j(\alpha(\tau))\xi^j)^2} \tag{3.29}$$

and $\mathbf{N}_{\alpha(\tau)}$ stands for the remaining full nonlinearity

$$\mathbf{N}_{\alpha(\tau)}(\mathbf{u}) := \mathbf{N}(\mathbf{u} + \mathbf{\Psi}_{\alpha(\tau)}) - \mathbf{N}(\mathbf{\Psi}_{\alpha(\tau)}) - \mathbf{L}'_{\alpha(\tau)}\mathbf{u}. \tag{3.30}$$

The advantage of this formulation is that the left hand side of (3.27) consists (besides $\partial_\tau \mathbf{\Phi}$) only of linear and $\tau-$independent operations on $\mathbf{\Phi}$, whereas the right hand side is expected to be small for large τ. Therefore, the right hand side of the equation (3.27) will be treated perturbatively. Note that, for sufficiently small α, we have $A_0(\alpha) = \mathcal{O}(1)$ whereas $A_j(\alpha) = \mathcal{O}(\alpha)$ which shows that

$$\sup_{j\in\{0,1,2\}} \left\|\partial^j V_\alpha\right\|_{L^\infty(\mathbb{B}^5)} \lesssim 1 \tag{3.31}$$

provided that α is sufficiently small. As we will now prove, this fact, together with the compactness of the Sobolev embedding yields the compactness of the operator $\mathbf{L}'_\alpha$ for sufficiently small α.

Lemma 3.6.1 *Let $\alpha \in \mathbb{R}^5$ be sufficiently small. Then, the operator $\mathbf{L}'_\alpha$ defined in (3.29) is compact. In particular, the operator*

$$\mathbf{L}_\alpha := \mathbf{L} + \mathbf{L}'_\alpha \tag{3.32}$$

generates a strongly continuous one parameter semigroup of bounded operators $\mathbf{S}_\alpha : [0,\infty) \longrightarrow \mathcal{B}(\mathcal{H})$.

Proof. To begin with, we fix α sufficiently small. First, we prove that $\mathbf{L}'_\alpha$ is compact. We pick a bounded sequence $\{\mathbf{u}_n\}_{n=1}^\infty \subseteq \mathcal{H}$. The compactness of the Sobolev embedding $H^3(\mathbb{B}^5) \hookrightarrow H^2(\mathbb{B}^5)$ yields the existence of a subsequence $\{\mathbf{u}_{k_n}\}_{n=1}^\infty$ in $H^3(\mathbb{B}^5)$ which is Cauchy in $H^2(\mathbb{B}^5)$. Now, (3.31) together with Hölder's inequality imply

$$\|\mathbf{L}'_\alpha \mathbf{u}_{k_n} - \mathbf{L}'_\alpha \mathbf{u}_{k_m}\| = \|V_\alpha (u_{1,k_n} - u_{1,k_m})\|_{H^2(\mathbb{B}^5)} \lesssim \|u_{1,k_n} - u_{1,k_m}\|_{H^2(\mathbb{B}^5)}$$

for sufficiently large $n, m \in \mathbb{N}$. This proves that $\{\mathbf{L}'\mathbf{u}_{k_n}\}_{n=1}^\infty$ is Cauchy in $\mathcal{H}$ and the claim follows. It remains to apply the Bounded Perturbation Theorem (see Theorem 1.3, page 158, [36]) to show that $\mathbf{L}_\alpha := \mathbf{L} + \mathbf{L}'_\alpha$ is the generator of a strongly continuous semigroup $(\mathbf{S}_\alpha(\tau))_{\tau>0}$. ☐

3.6.1 Solution to the full linear problem

Due to Lemma 3.6.1, we can write the solution to the linear part of (3.27),

$$\begin{cases} \partial_\tau \mathbf{\Phi}(\tau) = \left(\mathbf{L} + \mathbf{L}'_{\alpha_\infty}\right) \mathbf{\Phi}(\tau), \\ \mathbf{\Phi}(0) = \mathbf{u} \in \mathcal{H}, \end{cases}$$

as

$$\mathbf{\Phi}(\tau) = \mathbf{S}_{\alpha_\infty}(\tau)\mathbf{u}, \tag{3.33}$$

provided that α_∞ is sufficiently small which is verified later, see (3.51). In addition to the existence of the semigroup $\mathbf{S}_{\alpha_\infty}$, we need growth estimates in time. By Proposition 3.5.1 and Lemma 3.6.1, the Bounded Perturbation Theorem (see Theorem 1.3, page 158, [36]) yields

$$\|\mathbf{S}_{\alpha_\infty}(\tau)\| \le M e^{\left(-1+M\|\mathbf{L}'_{\alpha_\infty}\|\right)\tau},$$

as long as α_∞ is sufficiently small. However, such a growth estimate would not suffice and hence we turn our attention to the spectrum of the generator $\mathbf{L}_\alpha$.

3.7 Spectral Analysis

In this section, we intend to establish a useful growth estimate for $\mathbf{S}_\alpha$ for sufficiently small α and therefore we turn our attention to the spectrum of the generator $\mathbf{L}_\alpha$. We start our analysis with the case $\alpha = 0$ where the Lorentz boost $\Lambda(0)$ is the identity. Therefore, the potential V_0 in the definition of $\mathbf{L}'_0$, see (3.29), is constant in ξ. Consequently, the spectral equation can be solved explicitly and solutions belong to the hypergeometric class, as it turns out. The advantage here is that we can use the connection formula which is well known for this class. Then, we proceed to the case where $\alpha \neq 0$ but we are only interested in small α which allows for a perturbative approach, as already explained above.

3.7.1 The spectrum of the free operator.

We can use the decay estimate for the free semigroup $(\mathbf{S}(\tau))_{\tau>0}$ from Proposition 3.5.1 to locate the spectrum of the closure $\mathbf{L}$ of the free operator $\widetilde{\mathbf{L}}$. As a matter of fact, by [36], p. 55, Theorem 1.10, we immediately infer

$$\sigma(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -1\}. \tag{3.34}$$

3.7.2 The spectrum of the full linear operator for $\alpha = 0$.

To begin with, we use the fact that $\mathbf{L}'_\alpha$ is compact for sufficiently small α to see that it suffices to consider the point spectrum of $\mathbf{L}_\alpha$.

Lemma 3.7.1 *Let $\alpha \in \mathbb{R}^5$ be sufficiently small. We have*

$$\sigma(\mathbf{L}_\alpha) \setminus \sigma(\mathbf{L}) \subseteq \sigma_p(\mathbf{L}_\alpha).$$

Proof. Fix $\alpha \in \mathbb{R}^5$ sufficiently small and pick $\lambda \in \sigma(\mathbf{L}_\alpha) \setminus \sigma(\mathbf{L})$. From the identity $\lambda - \mathbf{L}_\alpha = [1 - \mathbf{L}'_\alpha \mathbf{R}_\mathbf{L}(\lambda)](\lambda - \mathbf{L})$ we see that $1 \in \sigma(\mathbf{L}'_\alpha \mathbf{R}_\mathbf{L}(\lambda))$. Since $\mathbf{L}'_\alpha \mathbf{R}_\mathbf{L}(\lambda)$ is compact, it follows that $1 \in \sigma_p(\mathbf{L}'_\alpha \mathbf{R}_\mathbf{L}(\lambda))$ and thus, there exists a nontrivial $\mathbf{f} \in \mathcal{H}$ such that $[1 - \mathbf{L}'_\alpha \mathbf{R}_\mathbf{L}(\lambda)]\mathbf{f} = 0$. Consequently, $\mathbf{u} := \mathbf{R}_\mathbf{L}(\lambda)\mathbf{f} \neq 0$ satisfies $(\lambda - \mathbf{L}_\alpha)\mathbf{u} = 0$ and thus, $\lambda \in \sigma_p(\mathbf{L}_\alpha)$. □

Now, we prove the following result.

Proposition 3.7.2 *We have*

$$\sigma(\mathbf{L}_0) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -1\} \cup \{0, 1\}.$$

Proof. To prove this result, we argue by contradiction. To begin with, fix a spectral point $\lambda \in \sigma(\mathbf{L}_0)$ with $\mathrm{Re}\lambda > -1$ and $\lambda \neq 0, 1$. Then, (3.34) implies that $\lambda \notin \sigma(\mathbf{L})$ and Lemma 3.7.1 yields $\lambda \in \sigma_p(\mathbf{L}_0)$. Consequently, there exists a non-trivial element $\mathbf{v} \in \mathcal{D}\big(\mathbf{L}_0\big) \subseteq \mathcal{H}$ such that $\big(\lambda - \mathbf{L}_0\big)\mathbf{v} = 0$. Then, for $\mathbf{v} = (v_1, v_2)$, we get

$$\begin{cases} v_2(\xi) = (\lambda+1)v_1(\xi) + \xi^j \partial_j v_1(\xi), \\ -\partial^j \partial_j v_1(\xi) + \xi^i \partial_i v_2(\xi) + (\lambda+2)v_2(\xi) - 6v_1(\xi) = 0. \end{cases}$$

Inserting v_2 into the second equation, we obtain

$$-\left(\delta^{ij} - \xi^i \xi^j\right) \partial_i \partial_j v_1(\xi) + 2(\lambda+2)\xi^i \partial_i v_1(\xi) + \Big((\lambda+1)(\lambda+2) - 6\Big) v_1(\xi) = 0.$$

To solve this equation, we switch to spherical coordinates $\xi = \rho\omega$, where $\rho = |\xi|$ and $\omega = \frac{\xi}{|\xi|}$. Then,

$$\partial_j \rho(\xi) = \omega_j(\xi), \quad \partial_j \omega^k(\xi) = \frac{\delta_j^k - \omega_j(\xi)\omega^k(\xi)}{\rho(\xi)}$$

and derivatives transform according to

$$\begin{aligned}
\xi^j\partial_j v_1(\xi) &= \rho\partial_\rho v_1(\rho\omega),\\
\xi^i\xi^j\partial_i\partial_j v_1(\xi) &= \rho^2\partial_\rho^2 v_1(\rho\omega),\\
\partial^j\partial_j v_1(\xi) &= \left(\partial_\rho^2 + \frac{4}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^4}\right) v_1(\rho\omega).
\end{aligned}$$

Hence, the spectral equation above can be written equivalently as

$$\left[-\left(1-\rho^2\right)\partial_\rho^2 - \left(\frac{4}{\rho} - 2(\lambda+2)\rho\right)\partial_\rho + \Big((\lambda+1)(\lambda+2)-6\Big) - \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^4}\right] v_1(\rho\omega) = 0.$$

By elliptic regularity, we infer $v_1 \in C^\infty(\mathbb{B}^5) \cap H^3(\mathbb{B}^5)$. Therefore, we may expand

$$v_1(\rho\omega) = \sum_{l=0}^{\infty}\sum_{m\in\Omega_l} v_{1,l,m}(\rho)Y_{l,m}(\omega).$$

Inserting this ansatz into the spectral equation above, we obtain the decoupled system of ordinary differential equations

$$\left[-\left(1-\rho^2\right)\frac{d^2}{d\rho^2} - \left(\frac{4}{\rho} - 2(\lambda+2)\rho\right)\frac{d}{d\rho} + \left((\lambda+1)(\lambda+2)-6+\frac{l(l+3)}{\rho^2}\right)\right] v_{1,l,m}(\rho) = 0. \tag{3.35}$$

From now on we suppress the subscripts. First note that this is a second order ordinary differential equation with four regular singular points: $-1, 0, 1$ and ∞. Again, by the reflection symmetry, these four singular points can be reduced to three and therefore, (3.35) can be transformed into a hypergeometric differential equation. To do so, we introduce the change of variables $v(\rho) = \rho^l w(z)$ with $z = \rho^2$ and we get

$$z(1-z)w''(z) + \Big(c-(a+b+1)z\Big)w'(z) - abw(z) = 0 \tag{3.36}$$

where

$$a := \frac{1}{2}(\lambda+l-1), \quad b := \frac{1}{2}(\lambda+l+4), \quad c := \frac{5}{2}+l.$$

The functions

$$\begin{aligned}
w_0(z) &:= {}_2F_1\left(a,b;c;z\right),\\
\widetilde{w}_0(z) &:= z^{1-c}{}_2F_1\left(a-c+1,b-c+1;2-c;z\right),\\
w_1(z) &:= {}_2F_1\left(a,b;a+b+1-c;1-z\right),\\
\widetilde{w}_1(z) &:= (1-z)^{c-a-b}{}_2F_1\left(c-a,c-b;c-a-b+1;1-z\right),
\end{aligned}$$

are all solutions to (3.36), see [68]. First, note that $\widetilde{w}_1$ is not admissible since the initial condition $\mathrm{Re}\lambda > -1$ yields

$$\mathrm{Re}(c-a-b) = 1 - \mathrm{Re}\lambda < 2$$

and thus $\widetilde{w}_1 \notin H^3(\frac{1}{2}, 1)$ whereas $\mathcal{D}(\mathbf{L}_0) \subseteq \mathcal{H}$. Similarly, $\widetilde{w}_0$ is not admissible either since it would lead to a solution $v_{l,m}$ that behaves like $\rho^{-\frac{3}{2}}$ as $\rho \to 0+$ which contradicts $v_{l,m} \in C^\infty[0,1)$. Hence, we are left with w_0 and w_1 and since both $\{w_0, \widetilde{w}_0\}$ and $\{w_1, \widetilde{w}_1\}$ are fundamental systems for the hypergeometric equation (3.36) we infer that w_0 and w_1 must be linearly dependent. In view of the connection formula [68],

$$w_0(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} w_1(z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \widetilde{w}_1(z),$$

the linear dependence of w_0 and w_1 implies that

$$\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} = 0.$$

However, the gamma function has no zeros and therefore we see that a or b must be a pole of Γ. The latter means $-a \in \mathbb{N}_0$ or $-b \in \mathbb{N}_0$. The first condition, $-a = n$ for some $n \in \mathbb{N}_0$, yields $2n < 2 - l$ which is possible only if $n = 0$ and $l \in \{0, 1\}$ which in turn implies $\lambda \in \{0, 1\}$ and refutes the initial assumption. The second condition, $-b = m$ for some $m \in \mathbb{N}_0$, yields $\lambda = -2m-4-l$ and the initial hypothesis on λ yields $-1 <\mathrm{Re}\lambda = -2m-4-l$ which is a contradiction, namely $3 < -(2m+l)$. □

Remark 3.7.3 *The spectral equations for $\lambda = 0$ and $\lambda = 1$ respectively read*

$$\begin{aligned} &-\left(\delta^{ij} - \xi^i\xi^j\right)\partial_i\partial_j v_1(\xi) + 4\xi^i\partial_i v_1(\xi) - 4v_1(\xi) = 0, \\ &-\left(\delta^{ij} - \xi^i\xi^j\right)\partial_i\partial_j v_1(\xi) + 6\xi^i\partial_i v_1(\xi) = 0. \end{aligned}$$

It is straightforward to check that, for all fixed $j \in \{1, 2, 3, 4, 5\}$, $v_1(\xi) = \xi^j$ solves the first equation whereas the constant function $v_1(\xi) = 1$ solves the second equation. Consequently, the eigenspaces for the isolated eigenvalues $\lambda = 0$ and $\lambda = 1$ of the operator $\mathbf{L}_0$ are spanned respectively by

$$\begin{aligned} &\mathbf{h}_{0,j}(\xi) = \partial_{\alpha^j}\mathbf{\Psi}_\alpha(\xi)|_{\alpha=0} = \sqrt{2}\begin{pmatrix} \xi^j \\ 2\xi^j \end{pmatrix}, \quad j \in \{1, 2, 3, 4, 5\} \\ &\mathbf{g}_0(\xi) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \end{aligned}$$

and hence $\{0, 1\} \subseteq \sigma_p(\mathbf{L}_0)$. Finally, notice that the above derivation shows that the geometric eigenspaces of 0 and 1 are $5-$dimensional and $1-$dimensional, respectively.

Note that since the operator $\mathbf{L}_0$ is highly non self-adjoint, it is not straightforward to see that the algebraic multiplicity of the isolated eigenvalues $\lambda = 0$ and $\lambda = 1$ are equal to 5 and 1, respectively. Now, we focus on proving this result rigorously. To be precise, we follow [31] and use the fact that the eigenvalues $\lambda = 0$ and $\lambda = 1$ are isolated to introduce two (non-orthogonal) Riesz projections $\mathbf{Q}_0$ and $\mathbf{P}_0$, namely

$$\mathbf{Q}_0 := \frac{1}{2\pi i}\int_{\gamma_0} \mathbf{R}_{\mathbf{L}_0}(\zeta)d\zeta,$$
$$\mathbf{P}_0 := \frac{1}{2\pi i}\int_{\gamma_1} \mathbf{R}_{\mathbf{L}_0}(\zeta)d\zeta,$$

where $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ stand for the circles centered at $\lambda = 0$ and $\lambda = 1$,

$$\gamma_0(s) := \frac{1}{2}e^{2\pi i s}, \quad \gamma_1(s) := 1 + \frac{1}{2}e^{2\pi i s},$$

respectively. These projections decompose the Hilbert space of initial data $\mathcal{H}$ into $\mathrm{rg}(1-\mathbf{Q}_0)$ (stable space for $\lambda = 0$) and $\mathrm{rg}\mathbf{Q}_0$ (unstable space for $\lambda = 0$),

$$\mathcal{H} = \mathrm{rg}(1 - \mathbf{Q}_0) \oplus \mathrm{rg}(\mathbf{Q}_0).$$

Similarly, for $\mathbf{P}_0$. We show that

$$m_a(\lambda = 0) := \operatorname{rank} \mathbf{Q}_0 = \dim \mathrm{rg}\, \mathbf{Q}_0,$$
$$m_a(\lambda = 1) := \operatorname{rank} \mathbf{P}_0 = \dim \mathrm{rg}\, \mathbf{P}_0,$$

are equal to 5 and 1 respectively.

Lemma 3.7.4 *We have* $\dim \mathrm{rg}\, \mathbf{Q}_0 = 5$ *and* $\dim \mathrm{rg}\, \mathbf{P}_0 = 1$.

Proof. Since the process is the same for both quantities, we illustrate it on $\mathbf{Q}_0$ only. We refer the reader to [43] for the following standard results. The projection $\mathbf{Q}_0$ commutes with the operator $\mathbf{L}_0$ and thus with the semigroup $\mathbf{S}_0(\tau)$. Moreover, $\mathbf{Q}_0$ decomposes the Hilbert space as $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M} := \mathrm{rg}\, \mathbf{Q}_0$ and $\mathcal{N} := \ker \mathbf{Q}_0 = \mathrm{rg}(1 - \mathbf{Q}_0)$. Most importantly, the operator $\mathbf{L}_0$ is decomposed accordingly into the parts $\mathbf{L}_{0,\mathcal{M}}$ and $\mathbf{L}_{0,\mathcal{N}}$ on $\mathcal{M}$ and $\mathcal{N}$, respectively. The spectra of these operators are given by

$$\sigma\left(\mathbf{L}_{0,\mathcal{N}}\right) = \sigma(\mathbf{L}_0) \setminus \{0\}, \qquad \sigma\left(\mathbf{L}_{0,\mathcal{M}}\right) = \{0\}. \tag{3.37}$$

Finally, $\mathrm{rg}\, \mathbf{Q}_0 \subseteq \mathcal{D}(\mathbf{L})$. To proceed, we break down the proof into the following steps:

Step 1: We prove that $\operatorname{rank} \mathbf{Q}_0 := \dim \mathrm{rg}\, \mathbf{Q}_0 < +\infty$. We argue by contradiction and assume that $\operatorname{rank} \mathbf{Q}_0 = +\infty$. Using [43], p. 239, Theorem 5.28, the fact that $\mathbf{L}_0'$ is compact (see Lemma 3.6.1), and the fact that the essential spectrum is stable under compact perturbations ([43], p. 244, Theorem 5.35), we obtain

$$\operatorname{rank} \mathbf{Q}_0 = +\infty \Longrightarrow 1 \in \sigma_e(\mathbf{L}_0) = \sigma_e(\mathbf{L}_0 - \mathbf{L}_0') = \sigma_e(\mathbf{L}) \subseteq \sigma(\mathbf{L}),$$

which clearly contradicts (3.34).

Step 2: We prove that $\langle \mathbf{h}_{0,1}, \mathbf{h}_{0,2}, \mathbf{h}_{0,3}, \mathbf{h}_{0,4}, \mathbf{h}_{0,5} \rangle = \operatorname{rg} \mathbf{Q}_0$. It suffices to show $\operatorname{rg} \mathbf{Q}_0 \subseteq \langle \mathbf{h}_{0,1}, \mathbf{h}_{0,2}, \mathbf{h}_{0,3}, \mathbf{h}_{0,4}, \mathbf{h}_{0,5} \rangle$ since the reverse inclusion follows from the abstract theory. From Step 1, the operator $\mathbf{L}_{0,\mathcal{M}}$ acts on the finite-dimensional Hilbert space $\mathcal{M} = \operatorname{rg} \mathbf{Q}_0$ and, from (3.37), $\lambda = 0$ is its only spectral point. Hence, $\mathbf{L}_{0,\mathcal{M}}$ is nilpotent, i.e., there exists a minimal $k \in \mathbb{N}$ such that

$$\left(\mathbf{L}_{0,\mathcal{M}}\right)^k \mathbf{u} = 0$$

for all $\mathbf{u} \in \operatorname{rg} \mathbf{Q}_0$. Now, the claim follows immediately if $k = 1$. Indeed, if $k = 1$, then $\operatorname{rg} \mathbf{Q}_0 = \ker \mathbf{L}_0 = \langle \mathbf{h}_{0,1}, \mathbf{h}_{0,2}, \mathbf{h}_{0,3}, \mathbf{h}_{0,4}, \mathbf{h}_{0,5} \rangle$ which shows that $\dim \operatorname{rg} \mathbf{Q}_0 = 5$. We proceed by contradiction and assume that $k \geq 2$. Then, there exists a nontrivial function $\mathbf{u} \in \operatorname{rg} \mathbf{Q}_0 \subseteq \mathcal{D}(\mathbf{L})$ such that $(\mathbf{L}_{0,\mathcal{M}})\mathbf{u}$ is nonzero and belongs to $\ker(\mathbf{L}_{0,\mathcal{M}}) \subseteq \ker(\mathbf{L}_0)$. This means that $\mathbf{u} \in \operatorname{rg} \mathbf{Q}_0 \subseteq \mathcal{D}(\mathbf{L})$ satisfies $\mathbf{L}_0 \mathbf{u} = \mathbf{f}$, for some $\mathbf{f} \in \ker \mathbf{L}_0$. A straightforward computation shows that the first component of $\mathbf{u}$ solves the second order differential equation

$$-\left(\delta^{ij} - \xi^i \xi^j\right) \partial_i \partial_j u_1(\xi) + 4\xi^i \partial_i u_1(\xi) - 4u_1(\xi) = -f(\xi),$$

where

$$f(\xi) := \xi^j \partial_j f_1(\xi) + 2f_1(\xi) + f_2(\xi)$$

and $\mathbf{f} = (f_1, f_2)$. We switch to hyper-spherical coordinates $\xi = \rho\omega$ where $\rho = |\xi|$ and $\omega = \frac{\xi}{|\xi|}$. Then,

$$\left[-(1-\rho^2)\frac{d^2}{d\rho^2} - \left(\frac{4}{\rho} - 4\rho\right)\frac{d}{d\rho} - 4 - \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^4}\right] u_1(\rho\omega) = f(\rho\omega).$$

Since

$$\mathbf{f} \in \ker(\mathbf{L}_0) = \langle \mathbf{h}_{0,1}, \mathbf{h}_{0,2}, \mathbf{h}_{0,3}, \mathbf{h}_{0,4}, \mathbf{h}_{0,5} \rangle = \langle \sqrt{2}\begin{pmatrix} \xi^1 \\ 2\xi^1 \end{pmatrix}, \cdots, \sqrt{2}\begin{pmatrix} \xi^5 \\ 2\xi^5 \end{pmatrix} \rangle,$$

we infer that

$$f(\xi) = \widetilde{a}_j \xi^j = |\xi| \widetilde{a}_j \omega^j = |\xi| \sum_{m=-2}^{2} a_m Y_{1,m}(\omega).$$

Here, $a_m \neq 0$ for at least one $m \in \{-2, -1, 0, 1, 2\}$. Without loss of generality we assume that $a_0 = 1$. An angular momentum decomposition as in the proof of Proposition 3.7.2 leads to the inhomogeneous ordinary differential equation

$$\left[-(1-\rho^2)\frac{d^2}{d\rho^2} - \left(\frac{4}{\rho} - 4\rho\right)\frac{d}{d\rho} - 4 + \frac{4}{\rho^2}\right] u_{1,1,0}(\rho) = \rho, \tag{3.38}$$

which can be simplified to

$$u_{1,1,0}''(\rho) + \frac{4}{\rho}u_{1,1,0}'(\rho) - \frac{4}{\rho^2}u_{1,1,0}(\rho) = -\frac{\rho}{1-\rho^2}. \tag{3.39}$$

This is a second order ordinary differential equation and one can readily verify that $\{\phi(\rho) = \rho,\ \psi(\rho) = \rho^{-4}\}$ is a fundamental system for the homogeneous version of (3.39). We calculate the Wronskian $W(\phi, \psi)(\rho) = -5\rho^{-4}$ and the variation of constants formula yields

$$u_{1,1,0}(\rho) = \frac{c_1}{\rho^4} + c_0\rho + \frac{\rho}{10}\log\left(1-\rho^2\right) + \frac{1}{10\rho^4}\log\left(\frac{1+\rho}{1-\rho}\right) - \frac{1}{5\rho^4}\left(\rho + \frac{1}{3}\rho^3 + \frac{1}{5}\rho^5\right)$$

for some constants $c_0, c_1 \in \mathbb{C}$. Now, $(\cdot)^{-4} \notin L^2(0,1)$ whereas $u_{1,1,0} \in L^2(0,1)$ and therefore we must have $c_1 = 0$. This fact leaves us with

$$u_{1,1,0}(\rho) = c_0\rho + \frac{\rho}{10}\log\left(1-\rho^2\right) + \frac{1}{10\rho^4}\log\left(\frac{1+\rho}{1-\rho}\right) - \frac{1}{5\rho^4}\left(\rho + \frac{1}{3}\rho^3 + \frac{1}{5}\rho^5\right)$$

which behaves like $(1-\rho)\log(1-\rho)$ near $\rho = 1$ and thus, does not belong to H^3. This contradiction shows that we must have $k = 1$ and thus $\mathbf{Q}_0$ has rank equal to 5. Similarly, one can show that $\mathbf{P}_0$ has rank equal to 1. □

3.7.3 The spectrum of the full linear operator for $\alpha \neq 0$.

Now, we assume that $\alpha \neq 0$ is sufficiently small and we will show that the spectrum $\sigma(\mathbf{L}_\alpha)$ is close to $\sigma(\mathbf{L}_0)$. More precisely, we work towards proving the following result.

Proposition 3.7.5 *Let $\alpha \in \mathbb{R}^5$ be sufficiently small. Then,*

$$\sigma(\mathbf{L}_\alpha) \subseteq \left\{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -\frac{3}{4}\right\} \cup \{0, 1\}.$$

However, we start with some useful properties of $\mathbf{L}_\alpha$. The first crucial observation is that $\mathbf{L}_\alpha'$ depends continuously on α.

Lemma 3.7.6 *There exists $\delta > 0$ sufficiently small such that*

$$\|\mathbf{L}_\alpha' - \mathbf{L}_\beta'\| \lesssim |\alpha - \beta|,$$

for all $\alpha, \beta \in \overline{\mathbb{B}^5_\delta}$.

Proof. It follows from the fundamental theorem of calculus, see Lemma 4.4 in [31]. □

The second observation is that spectrum of $\mathbf{L}_\alpha$ does not differ too much from the spectrum of $\mathbf{L}_0$ when α varies in sufficiently small and compact domains of $\mathbb{R}^5$.

Lemma 3.7.7 *There exists $\delta > 0$ sufficiently small such that*

$$\lambda \in \varrho(\mathbf{L}_0) \Longrightarrow \lambda \in \varrho(\mathbf{L}_\alpha)$$

provided $|\alpha| \leq \delta \min\{1, \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\|^{-1}\}$.

Proof. It follows from Lemma 3.7.6 and the identity

$$\lambda - \mathbf{L}_\alpha = (1 + (\mathbf{L}_0' - \mathbf{L}_\alpha')\, \mathbf{R}_{\mathbf{L}_0}(\lambda))\,(\lambda - \mathbf{L}_0),$$

see Corollary 4.5 in [31]. □

The next result shows absence of spectrum points outside a sufficiently large neighbourhood of the origin. To be precise, we provide a uniform bound on the resolvent operator of $\mathbf{L}_\alpha$ on the set

$$\Omega'_{k_0,\omega_0} := \left\{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda \geq -\frac{3}{4}\right\} \setminus \Omega_{k_0,\omega_0},$$

where

$$\Omega_{k_0,\omega_0} := \left\{\lambda \in \mathbb{C} : \quad \mathrm{Re}\lambda \in \left[-\frac{3}{4}, k_0\right], \quad \mathrm{Im}\lambda \in [-\omega_0, \omega_0]\right\},$$

Lemma 3.7.8 *Let $k_0, \omega_0 > 0$ be sufficiently large and $\delta > 0$ sufficiently small. Then there exists a positive constant C such that the resolvent $\mathbf{R}_{\mathbf{L}_\alpha}$ exists on Ω'_{k_0,ω_0} and satisfies the uniform bound*

$$\|\mathbf{R}_{\mathbf{L}_\alpha}(\lambda)\| \leq C,$$

for all $\lambda \in \Omega'_{k_0,\omega_0}$ and $\alpha \in \overline{\mathbb{B}^5_\delta}$.

Proof. Let $\lambda \in \Omega'_{k_0,\omega_0}$. The identity

$$(\lambda - \mathbf{L}_\alpha) = [1 - \mathbf{L}_\alpha' \mathbf{R}_{\mathbf{L}}(\lambda)](\lambda - \mathbf{L})$$

implies that it suffices to show smallness of $\mathbf{L}_\alpha' \mathbf{R}_{\mathbf{L}}(\lambda)$ which in turn follows from choosing $k_0, \omega_0 > 0$ sufficiently large and the bound

$$\|\mathbf{R}_{\mathbf{L}}(\lambda)\mathbf{f}\| \leq \frac{1}{\mathrm{Re}\lambda + 1}\|\mathbf{f}\|$$

which follows from semigroup theory, see [36], page 55, Theorem 1.10. For more details see Lemma 4.6 in [31]. □

Remark 3.7.9 *A straightforward calculation shows that the eigenspaces for the isolated eigenvalues $\lambda = 0$ and $\lambda = 1$ of the operator $\mathbf{L}_\alpha$ are spanned respectively by*

$$\mathbf{g}_\alpha(\xi) = \begin{pmatrix} A_0(\alpha)\left(A_0(\alpha) - A_j(\alpha)\xi^j\right)^{-2} \\ 2A_0^2(\alpha)\left(A_0(\alpha) - A_j(\alpha)\xi^j\right)^{-3} \end{pmatrix},$$
$$\mathbf{h}_{\alpha,j}(\xi) = \partial_{\alpha^j}\mathbf{\Psi}_\alpha(\xi), \quad j \in \{1,2,3,4,5\}.$$

and hence $\{0,1\} \subseteq \sigma_p(\mathbf{L}_\alpha)$. Finally, the above derivation shows that the algebraic multiplicities of the eigenvalues 0 *and* 1 *are equal to* $5-$*dimensional and* $1-$*dimensional, respectively.*

With these results at hand we can now prove Proposition 3.7.5.

Proof of Proposition 3.7.5. To start with, we choose k_0, ω_0 sufficiently large so that $\overline{\Omega'_{k_0,\omega_0}} \subseteq \rho(\mathbf{L}_\alpha)$ (Lemma 3.7.8) and δ sufficiently small so that $\partial\Omega_{k_0,\omega_0} \subseteq \rho(\mathbf{L}_\alpha)$ for all $|\alpha| \leq \frac{\delta}{M}$ where $M := \max\{1, \sup_{\zeta\in\partial\Omega_{k_0,\omega_0}} \|\mathbf{R}_{\mathbf{L}_0}(\zeta)\|\}$ (Lemma 3.7.7). Now, we define the projection

$$\mathbf{P}_\alpha^{total} := \frac{1}{2\pi i}\int_{\partial\Omega_{k_0,\omega_0}} \mathbf{R}_{\mathbf{L}_\alpha}(\zeta)d\zeta.$$

Lemma 3.7.6 shows that $\mathbf{P}_\alpha^{total}$ depends continuously on α and therefore, from Lemma 4.10 page 34 in [43], it follows that $\operatorname{rg}(\mathbf{P}_\alpha^{total})$ are all isomorphic to one another and the rank $\mathbf{P}_\alpha^{total} = \dim\operatorname{rg}\mathbf{P}_\alpha^{total}$ is constant for all α and Lemma 3.7.4 shows that $\dim\operatorname{rg}\mathbf{P}_0^{total} = 6$. In addition, the total geometric multiplicity of the eigenvalues $\lambda = 0$ and $\lambda = 1$ equals 6 and since $\mathbf{P}_\alpha^{total}$ has rank 6, there can be no other eigenvalues besides $\lambda = 0$ and $\lambda = 1$ in Ω_{k_0,ω_0}. In addition, the algebraic multiplicity of the eigenvalues 0 and 1 must be 5 and 1 respectively. □

3.7.4 Growth estimates for the full linearized problem

The above spectral analysis leads to a description of the full linearised evolution. In particular, we start by partitioning the space of initial data $\mathcal{H}$ into disjoint parts and we establish growth estimates for the semigroup $\mathbf{S}_\alpha$ in each of these parts. Namely, we define the projections

$$\mathbf{Q}_\alpha := \frac{1}{2\pi i}\int_{\gamma_0} \mathbf{R}_{\mathbf{L}_\alpha}(\zeta)d\zeta,$$
$$\mathbf{P}_\alpha := \frac{1}{2\pi i}\int_{\gamma_1} \mathbf{R}_{\mathbf{L}_\alpha}(\zeta)d\zeta,$$

where $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ stand for the circles centered at $\lambda = 0$ and $\lambda = 1$,

$$\gamma_0(s) := \frac{1}{2}e^{2\pi i s}, \quad \gamma_1(s) := 1 + \frac{1}{2}e^{2\pi i s},$$

respectively. By remark 3.7.9, we have

$$\operatorname{rg}\mathbf{Q}_\alpha = \langle \mathbf{h}_{\alpha,1}, \mathbf{h}_{\alpha,2}, \mathbf{h}_{\alpha,3}, \mathbf{h}_{\alpha,4}, \mathbf{h}_{\alpha,5} \rangle$$

and hence we may write

$$\mathbf{Q}_\alpha \mathbf{f} = \sum_{j=1}^{5} a_j \mathbf{h}_{\alpha,j}$$

for coefficients $a_j \in \mathbb{C}$ and for all $\mathbf{f} \in \mathcal{H}$. We define the projection onto the subspace generated by $\mathbf{h}_{\alpha,j}$, that is

$$\mathbf{Q}_{\alpha,j}\mathbf{f} := a_j \mathbf{h}_{\alpha,j},$$

for all $\mathbf{f} \in \mathcal{H}$. We show that the solution operator grows exponentially on $\operatorname{rg}(\mathbf{P}_\alpha)$, is constant in time on $\operatorname{rg}(\mathbf{Q}_{\alpha,j})$ and decays exponentially on the remaining infinite-dimensional subspace.

Lemma 3.7.10 *Let $\alpha \in \mathbb{R}^5$ be sufficiently small. Then, the projections $\mathbf{P}_\alpha$ and $\mathbf{Q}_{\alpha,j}$ for $j \in \{1,2,3,4,5\}$ have rank one and commute with the semigroup. In addition,*

$$\begin{aligned}
&\mathbf{S}_\alpha(\tau)\mathbf{P}_\alpha = e^{\tau}\mathbf{P}_\alpha,\\
&\mathbf{S}_\alpha(\tau)\mathbf{Q}_{\alpha,j} = \mathbf{Q}_{\alpha,j},\\
&\|\mathbf{S}_\alpha(\tau)\widetilde{\mathbf{P}}_\alpha\| \lesssim e^{-\frac{2}{3}\tau}\|\widetilde{\mathbf{P}}_\alpha\|,
\end{aligned}$$

where $\widetilde{\mathbf{P}}_\alpha := \mathbf{I} - \mathbf{P}_\alpha - \mathbf{Q}_\alpha$. Furthermore,

$$\begin{aligned}
&\operatorname{rg}(\mathbf{P}_\alpha) = \langle \mathbf{g}_\alpha \rangle,\\
&\operatorname{rg}(\mathbf{Q}_{\alpha,j}) = \langle \mathbf{h}_{\alpha,j} \rangle, \quad j \in \{1,2,3,4,5\},
\end{aligned}$$

where $\mathbf{g}_\alpha$ and $\mathbf{h}_{\alpha,j}$ are eigenfunctions of $\mathbf{L}_\alpha$ with eigenvalues 1 and 0, respectively.

Proof. The growth estimates follow from the Gearhart-Prüß Theorem ([36], page 302, Theorem 1.11) since Lemma 3.7.5 and Lemma 3.7.8 yield $\sup_{\operatorname{Re}\zeta \geq -\frac{3}{4}} \|\mathbf{R}_{\mathbf{L}_\alpha}(\zeta)\widetilde{\mathbf{P}}_\alpha\| < \infty$. The remaining statements are consequences of Lemma 3.7.5. For more details see Proposition 4.8, page 30, [31]. □

Remark 3.7.11 *It follows that $\mathbf{Q}_{\alpha,j}\mathbf{Q}_{\alpha,k} = \delta_{jk}\mathbf{Q}_{\alpha,j}$ and $\mathbf{Q}_{\alpha,j}\mathbf{P}_\alpha = \mathbf{P}_\alpha\mathbf{Q}_{\alpha,j} = 0$.*

3.8 Non-Linear Estimates

In this section, we establish Lipschitz-type estimates for the eigenfunctions $\mathbf{g}_\alpha$, $\mathbf{h}_{\alpha,j}$, the projections $\mathbf{P}_\alpha$, $\mathbf{Q}_\alpha$, the semigroup $\mathbf{S}_\alpha$ as well as for the nonlinearity $\mathbf{N}_\alpha$. These estimates will be used later for the main fixed point theorem. To begin with, we prove the following result.

Lemma 3.8.1 *For all $\alpha, \beta \in \mathbb{R}^5$ and for all $j \in \{1, 2, 3, 4, 5\}$, we have*

$$\|\mathbf{g}_\alpha - \mathbf{g}_\beta\| + \|\mathbf{h}_{\alpha,j} - \mathbf{h}_{\beta,j}\| \lesssim |\alpha - \beta|, \tag{3.40}$$
$$\|\mathbf{P}_\alpha - \mathbf{P}_\beta\| + \|\mathbf{Q}_\alpha - \mathbf{Q}_\beta\| \lesssim |\alpha - \beta|, \tag{3.41}$$
$$\|\mathbf{S}_\alpha(\tau)\widetilde{\mathbf{P}}_\alpha - \mathbf{S}_\beta(\tau)\widetilde{\mathbf{P}}_\beta\| \lesssim |\alpha - \beta| e^{-\frac{1}{2}\tau}, \tag{3.42}$$

for all $\tau > 0$.

Proof. The estimate (3.40) follows immediately from the fundamental theorem of calculus. Furthermore, the estimate (3.41) follows from a Lipschitz-type estimate for the resolvent operator, namely

$$\|\mathbf{R}_{\mathbf{L}_\alpha}(\lambda) - \mathbf{R}_{\mathbf{L}_\beta}(\lambda)\|\| \lesssim \|\mathbf{R}_{\mathbf{L}_\alpha}(\lambda)\|\|\mathbf{R}_{\mathbf{L}_\beta}(\lambda)\||\alpha - \beta|,$$

which in turn follows from the identity

$$\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{A}^{-1},$$

valid for all invertible operators $\mathbf{A}$ and $\mathbf{B}$. Finally, we establish the estimate (3.42) for the semigroup. To do so, we first observe that the function

$$\Phi_{\alpha,\beta}(\tau) := \frac{\mathbf{S}_\alpha(\tau)\widetilde{\mathbf{P}}_\alpha \mathbf{u} - \mathbf{S}_\beta(\tau)\widetilde{\mathbf{P}}_\beta \mathbf{u}}{|\alpha - \beta|}$$

for $\mathbf{u} \in \mathcal{D}(\mathbf{L}) \subseteq \mathcal{H}$, solves the initial value problem

$$\begin{cases} \partial_\tau \Phi_{\alpha,\beta}(\tau) = \mathbf{L}_\alpha \widetilde{\mathbf{P}}_\alpha \Phi_{\alpha,\beta}(\tau) + \dfrac{\mathbf{L}_\alpha \widetilde{\mathbf{P}}_\alpha - \mathbf{L}_\beta \widetilde{\mathbf{P}}_\beta}{|\alpha - \beta|} \mathbf{S}_\beta(\tau)\widetilde{\mathbf{P}}_\beta \mathbf{u}, \\ \Phi_{\alpha,\beta}(0) = \dfrac{\widetilde{\mathbf{P}}_\alpha - \widetilde{\mathbf{P}}_\beta}{|\alpha - \beta|} \mathbf{u}. \end{cases}$$

The key observation here is that

$$\mathbf{L}_\alpha \widetilde{\mathbf{P}}_\alpha - \mathbf{L}_\beta \widetilde{\mathbf{P}}_\beta = \mathbf{L}'_\alpha - \mathbf{L}'_\beta + \mathbf{P}_\beta - \mathbf{P}_\alpha$$

and therefore the apparently unbounded operator $\mathbf{L}_\alpha \widetilde{\mathbf{P}}_\alpha - \mathbf{L}_\beta \widetilde{\mathbf{P}}_\beta$ is in fact bounded, that is

$$\|\mathbf{L}_\alpha \widetilde{\mathbf{P}}_\alpha - \mathbf{L}_\beta \widetilde{\mathbf{P}}_\beta\| \lesssim |\alpha - \beta|.$$

Now, it remains to apply Duhamel's principle, write down the general solution formula for $\Phi_{\alpha,\beta}(\tau)$ and use the previous estimates. For more details see Lemma 4.9 in [31]. □

Next, we establish a Lipschitz-type estimate for the nonlinearity $\mathbf{N}_\alpha$. To begin with, recall (3.9), (3.29) and (3.8), i.e.,

$$\mathbf{N}\left(\mathbf{u}\right)(\xi) := \begin{pmatrix} 0 \\ u_1^3(\xi) \end{pmatrix}$$

and

$$\mathbf{L}'_\alpha(\mathbf{u}(\xi)) := \begin{pmatrix} 0 \\ V_\alpha(\xi)u_1(\xi) \end{pmatrix}, \quad V_\alpha(\xi) := 3\psi_\alpha^2(\xi), \quad \psi_\alpha(\xi) := \frac{\sqrt{2}}{A_0(\alpha) - A_j(\alpha)\xi^j}.$$

Furthermore, recall that $A_0(\alpha) = \mathcal{O}(1)$ whereas $A_j(\alpha) = \mathcal{O}(\alpha)$ for all sufficiently small $\alpha \in \mathbb{R}^d$. Hence, we find $\epsilon > 0$ small enough so that

$$\sup_{|\alpha|<\epsilon} \sup_{j\in\{0,1,2,3\}} \|\partial^j \psi_\alpha\|_{L^\infty(\mathbb{B}^5)} \lesssim 1. \tag{3.43}$$

A direct calculation shows that the full non-linearity defined in (3.30) can be written as follows

$$\mathbf{N}_\alpha(\mathbf{u}) := \mathbf{N}(\mathbf{u} + \mathbf{\Psi}_\alpha) - \mathbf{N}(\mathbf{\Psi}_\alpha) - \mathbf{L}'_\alpha\mathbf{u} = \begin{pmatrix} 0 \\ \hat{N}(\psi_\alpha, u_1) \end{pmatrix}, \tag{3.44}$$

where

$$\hat{N}(\psi_\alpha(\xi), u_1(\xi)) := 3\psi_\alpha(\xi)u_1^2(\xi) + u_1^3(\xi).$$

Also, we define

$$\hat{M}(\psi_\alpha(\xi), u_1(\xi)) := \partial_2\hat{N}(\psi_\alpha(\xi), u_1(\xi)) := 6\psi_\alpha(\xi)u_1(\xi) + 3u_1^2(\xi).$$

Finally, we write $\|\mathbf{f}\| := \|\mathbf{f}\|_{\mathcal{H}}$ where $\mathcal{H} := H^3\big(\mathbb{B}^5\big) \times H^2\big(\mathbb{B}^5\big)$. We prove the following result.

Lemma 3.8.2 *Fix sufficiently small $\alpha \in \mathbb{R}^5$ and sufficiently small $\delta > 0$. Then, we have*

$$\|\mathbf{N}_\alpha(\mathbf{u}) - \mathbf{N}_\beta(\mathbf{v})\| \lesssim \big(\|\mathbf{u}\| + \|\mathbf{v}\|\big)\|\mathbf{u} - \mathbf{v}\| + (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)|\alpha - \beta|, \tag{3.45}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ with $\|\mathbf{u}\| \leq \delta$ and $\|\mathbf{v}\| \leq \delta$ and for all $\alpha, \beta \in \overline{\mathbb{B}^5_\delta}$.

Proof. To begin with, we fix sufficiently small $\delta > 0$, sufficiently small $\alpha \in \overline{\mathbb{B}^5_\delta}$ and pick any $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ with $\|\mathbf{u}\| \leq \delta$ and $\|\mathbf{v}\| \leq \delta$. First, we show that

$$\big\|\mathbf{N}_\alpha(\mathbf{u}) - \mathbf{N}_\alpha(\mathbf{v})\big\| \lesssim \|\mathbf{u} - \mathbf{v}\| \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right). \tag{3.46}$$

Notice that the function $G(\xi, \zeta) := \hat{M}(\psi_\alpha(\xi), \zeta) = 6\psi_\alpha(\xi)\zeta + 3\zeta^2$, $(\xi, \zeta) \in \mathbb{R}^5 \times \mathbb{R}$ belongs to $C^\infty(\mathbb{R}^5 \times \mathbb{R}; \mathbb{R})$ and $G(\xi, 0) = 0$. Furthermore, for any compact set $K \subseteq \mathbb{R}$, we have

$\partial^{\alpha}_{\xi,\zeta} G \in L^\infty(\mathbb{R}^5 \times K)$, for all multi-indices α with $|\alpha| \leq 4$, due to (3.43). Consequently, Moser's inequality (see [70], p. 224, Theorem 6.4.1) and Sobolev extension imply

$$\|\hat{M}(\psi_\alpha, w)\|_{H^3(\mathbb{B}^5)} \lesssim \|w\|_{H^3(\mathbb{B}^5)}, \tag{3.47}$$

for all $w \in H^3(\mathbb{B}^5)$. For any fixed $\sigma \in [0,1]$, we define $\zeta(\sigma) := \sigma u_1 + (1-\sigma)v_1$. Now, since $3 > \frac{5}{2}$, the algebra property

$$\|fg\|_{H^3(\mathbb{B}^5)} \lesssim \|f\|_{H^3(\mathbb{B}^5)} \|g\|_{H^3(\mathbb{B}^5)} \tag{3.48}$$

holds and we can use this together with (3.47) to estimate

$$\begin{aligned}
\|\mathbf{N}_\alpha(\mathbf{u}) - \mathbf{N}_\alpha(\mathbf{v})\| &= \left\|\hat{N}(\psi_\alpha, u_1) - \hat{N}(\psi_\alpha, v_1)\right\|_{H^2(\mathbb{B}^5)} \\
&\leq \left\|\hat{N}(\psi_\alpha, u_1) - \hat{N}(\psi_\alpha, v_1)\right\|_{H^3(\mathbb{B}^5)} \\
&= \left\|\int_{v_1}^{u_1} \partial_2 \hat{N}(\psi_\alpha, \zeta) d\zeta\right\|_{H^3(\mathbb{B}^5)} \\
&= \left\|(u_1 - v_1)\int_0^1 \partial_2 \hat{N}(\psi_\alpha, \zeta(\sigma)) d\sigma\right\|_{H^3(\mathbb{B}^5)} \\
&\lesssim \|u_1 - v_1\|_{H^3(\mathbb{B}^5)} \left\|\int_0^1 \partial_2 \hat{N}(\psi_\alpha, \zeta(\sigma)) d\sigma\right\|_{H^3(\mathbb{B}^5)} \\
&\lesssim \|u_1 - v_1\|_{H^3(\mathbb{B}^5)} \int_0^1 \left\|\hat{M}(\psi_\alpha, \zeta(\sigma))\right\|_{H^3(\mathbb{B}^5)} d\sigma \\
&\lesssim \|u_1 - v_1\|_{H^3(\mathbb{B}^5)} \int_0^1 \|\zeta(\sigma)\|_{H^3(\mathbb{B}^5)} d\sigma \\
&\lesssim \|u_1 - v_1\|_{H^3(\mathbb{B}^5)} \int_0^1 \left(\sigma \|u_1\|_{H^3(\mathbb{B}^5)} + (1-\sigma)\|v_1\|_{H^3(\mathbb{B}^5)}\right) d\sigma \\
&\lesssim \|u_1 - v_1\|_{H^3(\mathbb{B}^5)} \left(\|u_1\|_{H^3(\mathbb{B}^5)} + \|v_1\|_{H^3(\mathbb{B}^5)}\right) \\
&\lesssim \|\mathbf{u} - \mathbf{v}\| \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right).
\end{aligned}$$

To complete the proof, it suffices show that

$$\|\mathbf{N}_\alpha(\mathbf{u}) - \mathbf{N}_\beta(\mathbf{u})\| \lesssim \|u_1\|^2_{H^3(\mathbb{B}^5)} |\alpha - \beta|,$$

which is a consequence of the fundamental theorem of calculus. Indeed, we fix $\alpha, \beta \in \mathbb{R}^5$ sufficiently small and let $\gamma(t) := t\beta + (1-t)\alpha, \ t \in [0,1]$ be a parametrisation of the line segment $E[\alpha, \beta]$ joining α and β. Then,

$$\psi_\alpha - \psi_\beta = \psi_{\gamma(0)} - \psi_{\gamma(1)} = \int_{E[\alpha,\beta]} \partial\psi_\gamma \cdot d\ell = \sum_{j=1}^{5} (\beta^j - \alpha^j) \int_0^1 \partial_{\gamma^j} \psi_{\gamma(t)} dt,$$

and the triangle inequality implies the bound

$$\|\partial^m(\psi_\alpha - \psi_\beta)\|_{L^2(\mathbb{B}^5)} \lesssim \sum_{j=1}^{5} |\beta^j - \alpha^j| \sup_{s\in E[\alpha,\beta]} \|\partial^m \partial_{\gamma^j}\psi_s\|_{L^2(\mathbb{B}^5)} \lesssim |\beta - \alpha|,$$

for all $m \in \{0,1,2,3\}$, due to (3.43). Therefore, (3.48) yields

$$\begin{aligned}\|\mathbf{N}_\alpha(\mathbf{u}) - \mathbf{N}_\beta(\mathbf{u})\| &= \|3u_1^2(\psi_\alpha - \psi_\beta)\|_{H^2(\mathbb{B}^5)} \le \|3u_1^2(\psi_\alpha - \psi_\beta)\|_{H^3(\mathbb{B}^5)} \\ &\lesssim \|u_1\|^2_{H^3(\mathbb{B}^5)}\|\psi_\alpha - \psi_\beta\|_{H^3(\mathbb{B}^5)} \lesssim \|u_1\|^2_{H^3(\mathbb{B}^5)}|\alpha - \beta|,\end{aligned}$$

which concludes the proof. □

3.9 The modulation equation

To begin with, we apply Duhamel's principle to rewrite the modulation equation (3.27) coupled with initial data in a weak formulation. Due to (3.33), we may write the Cauchy problem

$$\begin{cases} \partial_\tau \mathbf{\Phi}(\tau) - \big(\mathbf{L} + \mathbf{L}'_{\alpha_\infty}\big)\mathbf{\Phi}(\tau) = \hat{\mathbf{L}}_{\alpha(\tau)}\mathbf{\Phi}(\tau) + \mathbf{N}_{\alpha(\tau)}(\mathbf{\Phi}(\tau)) - \partial_\tau \mathbf{\Psi}_{\alpha(\tau)}, \\ \Phi(0) = \mathbf{u} \in \mathcal{H}, \end{cases}$$

as an integral equation, that is

$$\mathbf{\Phi}(\tau) = \mathbf{S}_{\alpha_\infty}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}_{\alpha_\infty}(\tau - \sigma)\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big)d\sigma, \tag{3.49}$$

provided that α_∞ is sufficiently small which we later verify, see (3.51). We use this formulation to define the notion of light-cone solutions.

Definition 3.9.1 *Fix $\alpha \in \mathbb{R}^5$ sufficiently small. We say that $u : C_T \longrightarrow \mathbb{R}$ is a solution to (3.5) if the corresponding $\mathbf{\Phi} : [0,\infty) \longrightarrow \mathcal{H}$ belongs to $C\big([0,\infty);\mathcal{H}\big)$ and satisfies (3.49) for all $\tau \ge 0$.*

Consequently, in order to establish a solution $u = u(t,x)$ to the initial Cauchy problem (3.5) we need to construct a global in τ solution $\mathbf{\Phi}(\tau)$ to (3.49). To prove the existence of a global solution, we would like to apply a fixed point argument to the integral equation (3.49). However, the solution operator $\mathbf{S}_{\alpha_\infty}$ for the linearized equation has two unstable subspaces $\operatorname{rg}\mathbf{Q}_{\alpha_\infty}$, $\operatorname{rg}\mathbf{P}_{\alpha_\infty}$ which appear due to the symmetries of the original equation, namely the Lorentz and time-translation symmetry, respectively (Lemma 3.7.10). Specifically, initial data from $\operatorname{rg}\mathbf{Q}_{\alpha_\infty}$ and $\operatorname{rg}\mathbf{P}_{\alpha_\infty}$ lead to solutions which stay constant or grow exponentially in time, respectively. These growths prevent us from applying a fixed point argument directly. We overcome this obstruction as follows. In the first case, we choose the rapidity parameter $\alpha = \alpha(\tau)$ in such a way that this instability is suppressed. In the second case, we proceed differently and add a correction term to the initial data which stabilizes the evolution. In both cases, we use fixed point arguments to establish existence and uniqueness of the respective modified equations and hence we first introduce the Banach spaces.

3.9.1 Banach spaces

We define the following sets.

$$\mathcal{X} := \{\Phi \in C([0,\infty);\mathcal{H}) : \|\Phi\|_{\mathcal{X}} < \infty\},$$
$$X := \left\{\alpha \in C^1([0,\infty);\mathbb{R}^5) : \alpha(0) = 0 \text{ and } \|\alpha\|_X < \infty\right\},$$

endowed with the norms

$$\|\Phi\|_{\mathcal{X}} := \sup_{\tau>0}\left(e^{\frac{1}{2}\tau}\|\Phi(\tau)\|\right),$$
$$\|\alpha\|_X := \sup_{\tau>0}\left(e^{\frac{1}{2}\tau}|\dot{\alpha}(\tau)| + |\alpha(\tau)|\right),$$

on $\mathcal{X}$ and X respectively. Furthermore, we denote by

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \leq \delta\},$$
$$X_\delta := \left\{\alpha \in X : |\dot{\alpha}(\tau)| \leq \delta e^{-\frac{1}{2}\tau}\right\},$$

the closed subsets of $\mathcal{X}$ and X respectively. Recall that $\mathcal{H} := H^3(\mathbb{B}^5) \times H^2(\mathbb{B}^5)$ and $\|\cdot\| := \|\cdot\|_{H^3(\mathbb{B}^5)\times H^2(\mathbb{B}^5)}$. First, notice that for an element $\alpha \in X_\delta$, the limit $\alpha_\infty := \lim_{\tau\to\infty}\alpha(\tau)$ exists. Indeed, for all $0 < \tau_1 \leq \tau_2$ with $\tau_1, \tau_2 \to \infty$,

$$|\alpha(\tau_2) - \alpha(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |\dot{\alpha}(\tau)|\, d\tau \lesssim \delta\left(e^{-\frac{1}{2}\tau_1} - e^{-\frac{1}{2}\tau_2}\right) \longrightarrow 0.$$

Fixing τ_1 and letting τ_2 go to infinity, we obtain

$$\forall \alpha \in X_\delta : \quad |\alpha_\infty - \alpha(\tau)| \lesssim \delta e^{-\frac{1}{2}\tau}, \quad \forall \tau > 0. \tag{3.50}$$

In particular for $\tau = 0$ we get the smallness condition

$$|\alpha_\infty| \lesssim \delta. \tag{3.51}$$

Furthermore, by Lemma 3.8.2, Lemma 3.7.6, Proposition 3.7.10 and the fact that $\partial_\tau \Psi_{\alpha(\tau)} = \dot{\alpha}^k(\tau)\mathbf{h}_{\alpha(\tau),k}$ we get the following result.

Lemma 3.9.2 *Let $\delta > 0$ be sufficiently small. Then, for all $\Phi \in \mathcal{X}_\delta$ and $\alpha \in X_\delta$,*

$$\left\|\hat{\mathbf{L}}_{\alpha(\tau)}\Phi(\tau)\right\| + \left\|\mathbf{N}_{\alpha(\tau)}(\Phi(\tau))\right\| \lesssim \delta^2 e^{-\tau},$$
$$\left\|\mathbf{P}_{\alpha_\infty}\partial_\tau\Psi_{\alpha(\tau)}\right\| + \left\|(\mathbf{I} - \mathbf{Q}_{\alpha_\infty})\partial_\tau\Psi_{\alpha(\tau)}\right\| \lesssim \delta^2 e^{-\tau}.$$

for all $\tau > 0$.

Proof. The proof coincides with the proof of Lemma 5.4 in [31]. □

We also prove the corresponding Lipschitz bounds.

Lemma 3.9.3 *Let $\delta > 0$ be sufficiently small. Then, for all $\Phi, \Psi \in \mathcal{X}_\delta$ and $\alpha, \beta \in X_\delta$,*

$$\begin{aligned}
&\left\|\hat{\mathbf{L}}_{\alpha(\tau)}\Phi(\tau) - \hat{\mathbf{L}}_{\beta(\tau)}\Psi(\tau)\right\| \lesssim \delta^2 e^{-\tau}\left(\|\Phi - \Psi\|_{\mathcal{X}} + \|\alpha - \beta\|_X\right),\\
&\left\|\mathbf{N}_{\alpha(\tau)}(\Phi(\tau)) - \mathbf{N}_{\beta(\tau)}(\Psi(\tau))\right\| \lesssim \delta^2 e^{-\tau}\left(\|\Phi - \Psi\|_{\mathcal{X}} + \|\alpha - \beta\|_X\right),\\
&\left\|\mathbf{P}_{\alpha_\infty}\partial_\tau \Psi_{\alpha(\tau)} - \mathbf{P}_{\beta_\infty}\partial_\tau \Psi_{\beta(\tau)}\right\| \lesssim \delta^2 e^{-\tau}\|\alpha - \beta\|_X,\\
&\left\|(\mathbf{I} - \mathbf{Q}_{\alpha_\infty})\partial_\tau \Psi_{\alpha(\tau)} - (\mathbf{I} - \mathbf{Q}_{\beta_\infty})\partial_\tau \Psi_{\beta(\tau)}\right\| \lesssim \delta^2 e^{-\tau}\left(\|\Phi - \Psi\|_{\mathcal{X}} + \|\alpha - \beta\|_X\right),
\end{aligned}$$

for all $\tau > 0$.

Proof. The proof coincides with the proof of Lemma 5.5 in [31]. □

3.9.2 The Lorentz symmetry instability

Now, we focus on the instability induced by the Lorentz symmetry and in particular we will choose $\alpha = \alpha(\tau)$ in such a way that this instability is suppressed. To do so, we need an equation for $\alpha = \alpha(\tau)$. By Proposition 3.7.10, we have $\mathbf{Q}_{\alpha_\infty,j}\mathbf{S}_{\alpha_\infty} = \mathbf{Q}_{\alpha_\infty,j}$ and therefore applying $\mathbf{Q}_{\alpha_\infty,j}$ to the weak formulation of the modulation equation, that is (3.49), we infer

$$\mathbf{Q}_{\alpha_\infty,j}\mathbf{\Phi}(\tau) = \mathbf{Q}_{\alpha_\infty,j}\mathbf{u} + \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \mathbf{S}_{\alpha_\infty}(\tau - \sigma)\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big)d\sigma,$$

for all $j \in \{1, 2, 3, 4, 5\}$. To suppress the instability we would like to trivialize the range and set the right-hand side equal to zero. However, this is not possible since for $\tau = 0$ the condition $\mathbf{Q}_{\alpha_\infty,j}\mathbf{u} = 0$ on the initial data would be required which is not true in general. Since we are only interested in the long-term evolution it however suffices to assume that $\mathbf{Q}_{\alpha_\infty,j}\mathbf{\Phi}(\tau)$ vanishes for large τ. Hence, we set

$$\mathbf{Q}_{\alpha_\infty,j}\mathbf{\Phi}(\tau) = \chi(\tau)\mathbf{h}, \quad \mathbf{h} := \mathbf{Q}_{\alpha_\infty,j}\mathbf{u} \in \operatorname{rg}\mathbf{Q}_{\alpha_\infty}$$

where χ is a smooth cut-off function, which equals to 1 on $[0, 1]$, 0 for $\tau \geq 4$ and satisfies $|\dot{\chi}| \leq 1$ everywhere. Now, evaluation at $\tau = 0$ yields $\mathbf{h} = \mathbf{Q}_{\alpha_\infty,j}\mathbf{u}$ which now holds true in general. This ansatz yields an equation for α, namely

$$(1 - \chi(\tau))\mathbf{h} + \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big)d\sigma = \mathbf{0}. \tag{3.52}$$

In particular, we define the auxiliary function

$$\hat{\mathbf{h}}_{\alpha(\tau),k} := \mathbf{h}_{\alpha(\tau),k} - \mathbf{h}_{\alpha_\infty,k},$$

assume that $\alpha(0)=0$ and use the properties of $\mathbf{Q}_{\alpha_\infty,j}$ from remark 3.7.11 to write

$$\begin{aligned}\mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}d\sigma &= \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \dot{\alpha}^k(\sigma)\mathbf{h}_{\alpha(\sigma),k}d\sigma \\ &= \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \dot{\alpha}^k(\sigma)\left(\hat{\mathbf{h}}_{\alpha(\tau),k}+\mathbf{h}_{\alpha_\infty,k}\right)d\sigma \\ &= \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \dot{\alpha}^k(\sigma)\hat{\mathbf{h}}_{\alpha(\tau),k}d\sigma + \alpha^j(\tau)\mathbf{h}_{\alpha_\infty,j}.\end{aligned}$$

Therefore, we can write equation (3.52) as

$$\begin{aligned}\alpha^j(\tau)\mathbf{h}_{\alpha_\infty,j} &= (1-\chi(\tau))\mathbf{Q}_{\alpha_\infty,j}\mathbf{u} \\ &+ \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \left(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma)+\mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma))\right)d\sigma \\ &- \mathbf{Q}_{\alpha_\infty,j}\int_0^\tau \dot{\alpha}^k(\sigma)\hat{\mathbf{h}}_{\alpha(\tau),k}d\sigma \\ &:= \int_0^\tau \mathbf{G}_j(\alpha,\Phi,\mathbf{u})(\sigma)d\sigma. \end{aligned} \tag{3.53}$$

for the functions $\alpha^j=\alpha^j(\tau)\in\mathbb{R}^5$, $j\in\{1,2,3,4,5\}$. Then, we have a fixed point formulation for α,

$$\alpha(\tau)=\int_0^\tau G(\alpha,\Phi,\mathbf{u}) := \widetilde{G}(\alpha,\Phi,\mathbf{u}), \tag{3.54}$$

where $G=(G_1,G_2,G_3,G_4,G_5)$ and

$$G_j(\alpha,\Phi,\mathbf{u})(\sigma) := \frac{1}{\|\mathbf{h}_{\alpha_\infty,j}\|^2}\left(\mathbf{G}_j(\alpha,\Phi,\mathbf{u})(\sigma)|\mathbf{h}_{\alpha_\infty,j}\right).$$

Finally, we use a fixed point argument to show that the function $\alpha:[0,\infty)\to\mathbb{R}^5$ can be chosen in such a way that (3.54) (equivalently (3.53)) holds provided that Φ satisfies a smallness condition. Consequently, the instability induced by the Lorentz symmetry is suppressed.

Proposition 3.9.4 *Let $\delta>0$ be sufficiently small, $c>0$ sufficiently large and suppose that $\Phi\in\mathcal{X}_\delta$. Then, there exists a unique function $\alpha\in X_\delta$ such that equation (3.54) holds for each $j\in\{1,2,3,4,5\}$ provided $\|\mathbf{u}\|\leq\frac{\delta}{c}$. Furthermore, the map $\Phi\longmapsto\alpha$ is Lipschitz continuous.*

Proof. The proof relies on a fixed point argument. The fact that $\widetilde{G}(\cdot,\Phi,\mathbf{u})$ maps X_δ to itself follows from Lemma 3.8.1 and Lemma 3.9.3. Furthermore, the contraction property is a direct consequence of Lemma 3.9.3 and finally the Lipschitz continuity follows from Lemma 3.9.3. For more details see Lemma 5.6 in [31]. □

3.9.3 The time translation instability

Next, we turn our attention to the instability induced by the time translation symmetry. However, this time we proceed differently and we add a correction term to the initial data $\mathbf{\Phi}(0) = \mathbf{u}$ in the equation (3.49) which stabilizes the evolution. In other words, we consider the modified equation

$$\mathbf{\Phi}(\tau) = \mathbf{K}(\Phi, \alpha, \mathbf{u}), \tag{3.55}$$

where

$$\begin{aligned}\mathbf{K}(\Phi, \alpha, \mathbf{u}) &:= \mathbf{S}_{\alpha_\infty}(\tau)\left(\mathbf{u} - \mathbf{C}(\Phi, \alpha, \mathbf{u})\right) \\ &\quad + \int_0^\tau \mathbf{S}_{\alpha_\infty}(\tau - \sigma)\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big) d\sigma,\end{aligned} \tag{3.56}$$

and

$$\mathbf{C}(\Phi, \alpha, \mathbf{u}) := \mathbf{P}_{\alpha_\infty}\mathbf{u} + \mathbf{P}_{\alpha_\infty}\int_0^\infty e^{-\sigma}\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big) d\sigma. \tag{3.57}$$

Here, all integrals exist as Riemann integrals over continuous functions. Now, we can expect that the evolution (3.55) will have a solution provided that the initial data are sufficiently small. This is precisely our next result.

Proposition 3.9.5 *Let $\delta > 0$ be sufficiently small and $c > 0$ sufficiently large. If $\|\mathbf{u}\| \leq \frac{\delta}{c}$, then there exists a unique functions $\alpha \in X_\delta$ and $\Phi \in \mathcal{X}_\delta$ such that equation (3.55) holds for all $\tau > 0$.*

Proof. Here, $\alpha \in X_\delta$ is associated to Φ via Lemma 3.9.4. The proof relies on a fixed point argument. The fact that $\mathbf{K}(\cdot, \alpha, \mathbf{u})$ maps $\mathcal{X}_\delta$ to itself follows from Lemma 3.9.2 and Proposition 3.7.10. Furthermore, the contraction property is a direct consequence of Lemma 3.9.3 and Lemma 3.9.4 and finally the Lipschitz continuity follows from Lemma 3.9.3, Lemma 3.8.1 and Lemma 3.9.4. For more details see Proposition 5.7 in [31]. □

Recall that our initial goal is to solve the modulation equation (3.49) so that we can establish a solution to the initial Cauchy problem (3.5). So far, we can do this only for the modified equation (3.55) where the correction term is included. However, the correction term $\mathbf{C}(\Phi, \alpha, \mathbf{u})$ is closely related to the time translation symmetry and therefore we can choose T in such a way that the correction term vanishes. On the other hand, the blowup time T appears explicitly only in the initial data and not in the equation itself. To be precise, we have that

$$\Phi(0)(\xi) = \Psi(0)(\xi) - \Psi_{\alpha(0)}(\xi) = T\begin{pmatrix}\psi_{0,1}(T\xi) + \tilde{f}(T\xi) \\ \psi_{0,2}(T\xi) + \tilde{g}(T\xi)\end{pmatrix} - \Psi_0(\xi),$$

for some fixed and given functions $(\widetilde{f}, \widetilde{g})$ which stand for a perturbation of the initial data, see (3.6). Note, that we may write the initial data as

$$\Phi(0)(\xi) = \mathbf{U}(T, \mathbf{v}), \tag{3.58}$$

to distinguish between the blowup time T and the perturbation

$$\mathbf{v} := \begin{pmatrix} \widetilde{f} \\ \widetilde{g} \end{pmatrix}, \tag{3.59}$$

where

$$\mathbf{U}(T, \mathbf{v}) := \mathbf{v}^T + \Psi_0^T - \Psi_0. \tag{3.60}$$

Here, we also write

$$\mathbf{w}^T := \begin{pmatrix} Tw_1(T\xi) \\ Tw_2(T\xi) \end{pmatrix},$$

for a generic function $\mathbf{w} = (w_1, w_2) \in \mathcal{H}$. Before describing how one can choose T in such a way that the correction term vanishes, we must ensure that, for all $T \in [1 - \frac{\delta}{c}, 1 + \frac{\delta}{c}]$, the modified equation (3.55) has a solution with initial data $\mathbf{u} = \mathbf{U}(T, \mathbf{v})$ provided that the perturbation $\mathbf{v}$ is sufficiently small. This fact is a direct consequence of Proposition 3.9.5 and the following lemma.

Lemma 3.9.6 *Let $\delta > 0$ be sufficiently small. If $\mathbf{v} \in H^3(\mathbb{B}^5_{1+\delta}) \times H^2(\mathbb{B}^5_{1+\delta})$ such that $\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\delta}) \times H^2(\mathbb{B}^5_{1+\delta})} \leq \delta$ then*

$$\|\mathbf{U}(T, \mathbf{v})\|_{H^3(\mathbb{B}^5_{1+\delta}) \times H^2(\mathbb{B}^5_{1+\delta})} \lesssim \delta,$$

for all $T \in [1 - \delta, 1 + \delta]$. Furthermore, the map $\mathbf{U}(\cdot, \mathbf{v}) \to \mathcal{H}$ is continuous.

Proof. The smallness condition on $\mathbf{U}(T, \mathbf{v})$ follows immediately from the fundamental theorem of calculus since $\psi_{0,1}, \psi_{0,2} \in C^\infty(\mathbb{R}^5)$. Furthermore, the continuity of the map follows from the triangle inequality and an approximation argument using the density of $C^\infty(\overline{\mathbb{B}^5_{1+\delta}})$ in $H^k(\mathbb{B}^5_{1+\delta})$. For a detailed proof see Lemma 5.8 in [31]. ☐

Now, one can apply Proposition 3.9.5 to get the following result.

Corollary 3.9.7 *Let $\delta > 0$ be sufficiently small and c sufficiently large. Furthermore, fix $\mathbf{v} \in H^3(\mathbb{B}^5_{1+\delta/c}) \times H^2(\mathbb{B}^5_{1+\delta/c})$ such that $\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\delta/c}) \times H^2(\mathbb{B}^5_{1+\delta/c})} \leq \frac{\delta}{c}$ and $T \in [1 - \frac{\delta}{c}, 1 + \frac{\delta}{c}]$. Then, the modified equation (3.55) with $\mathbf{u} = \mathbf{U}(T, \mathbf{v})$ has a solution $(\Phi, \alpha) \in \mathcal{X}_\delta \times X_\delta$. Furthermore, the map $T \longmapsto (\Phi, \alpha)$ is continuous.*

Now, we focus on the correction term. To begin with we fix $\delta > 0$ sufficiently small, c sufficiently large and let $\mathbf{v} \in H^3(\mathbb{B}^5_{1+\delta/c}) \times H^2(\mathbb{B}^5_{1+\delta/c})$ such that $\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\delta/c})\times H^2(\mathbb{B}^5_{1+\delta/c})} \leq \frac{\delta}{c^2}$. Furthermore, pick an arbitrary $T = T_{\mathbf{v}} \in [1-\frac{\delta}{c}, 1+\frac{\delta}{c}]$ and let $(\Phi, \alpha) = (\Phi_T, \alpha_T) \in \mathcal{X}_\delta \times X_\delta$ be a solution to the modified equation (3.55) with $\mathbf{u} = \mathbf{U}(T, \mathbf{v})$ by corollary 3.9.7.

Lemma 3.9.8 *There exists $T_{\mathbf{v}} \in [1-\frac{\delta}{c}, 1+\frac{\delta}{c}]$ such that $\mathbf{C}\left(\Phi_{T_{\mathbf{v}}}, \alpha_{T_{\mathbf{v}}}, \mathbf{U}\left(T_{\mathbf{v}}, \mathbf{v}\right)\right) = 0$.*

Proof. Since $\mathbf{C}$ has values in $\operatorname{rg}\mathbf{P}_{\alpha_\infty} = \langle \mathbf{g}_{\alpha_\infty}\rangle$ (see Lemma 3.7.10), the vanishing of the correction term is equivalent to

$$\exists T_{\mathbf{v}} \in [1-\frac{\delta}{c}, 1+\frac{\delta}{c}]: \quad \left\langle \mathbf{C}\left(\Phi_{T_{\mathbf{v}}}, \alpha_{T_{\mathbf{v}}}, \mathbf{U}\left(T_{\mathbf{v}}, \mathbf{v}\right)\right), \mathbf{g}_{\alpha_\infty}\right\rangle_{\mathcal{H}} = 0. \tag{3.61}$$

The key observation here is that

$$\partial_T \Psi_0^T\big|_{T=1} = 2\mathbf{g}_0$$

and thus expanding Ψ_0^T in Taylor with respect to T around $T=1$ we get

$$\mathbf{U}(T, \mathbf{v}) = \mathbf{v}^T + 2\mathbf{g}_0(T-1) + \mathbf{R}_T(T-1)^2,$$

for some remainder term $\mathbf{R}_T$, which we rewrite as

$$\mathbf{U}(T, \mathbf{v}) = \mathbf{v}^T + 2\mathbf{g}_{\alpha_\infty}(T-1) + 2\left(\mathbf{g}_0 - \mathbf{g}_{\alpha_\infty}\right)(T-1) + \mathbf{R}_T(T-1)^2.$$

Now, the fact fact $\alpha(0) = 0$ and (3.50) yield $|\alpha_\infty - \alpha(0)| \lesssim \delta$ and from Lemma 3.8.1 (in particular (3.40)) we get $\|\mathbf{g}_0 - \mathbf{g}_{\alpha_\infty}\| \lesssim \delta$. In addition, $\|\mathbf{R}_T\| \lesssim 1$ for all $T \in [1-\frac{\delta}{c}, 1+\frac{\delta}{c}]$. Hence, using $\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\delta/c})\times H^2(\mathbb{B}^5_{1+\delta/c})} \leq \frac{\delta}{c^2}$ and $\operatorname{rg}\mathbf{P}_{\alpha_\infty} = \langle \mathbf{g}_{\alpha_\infty}\rangle$ from Lemma 3.7.10, we infer

$$\left\langle \mathbf{P}_{\alpha_\infty}\mathbf{U}(T, \mathbf{v}), \mathbf{g}_{\alpha_\infty}\right\rangle_{\mathcal{H}} = O\left(\frac{\delta}{c^2}\right) + 2\left\|\mathbf{g}_{\alpha_\infty}\right\|^2(T-1) + O\left(\frac{\delta^2}{c}\right) + O\left(\frac{\delta^2}{c^2}\right).$$

Moreover, the bounds of Lemma 3.9.2 imply

$$\left\langle \mathbf{P}_{\alpha_\infty}\int_0^\infty e^{-\sigma}\left(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\right)d\sigma, \mathbf{g}_{\alpha_\infty}\right\rangle_{\mathcal{H}} = O\left(\delta^2\right)\left\|\mathbf{g}_{\alpha_\infty}\right\|^2.$$

Finally, summing up we get

$$\left\langle \mathbf{C}\left(\Phi_{T_{\mathbf{v}}}, \alpha_{T_{\mathbf{v}}}, \mathbf{U}\left(T_{\mathbf{v}}, \mathbf{v}\right)\right), \mathbf{g}_{\alpha_\infty}\right\rangle_{\mathcal{H}} = 2\left\|\mathbf{g}_{\alpha_\infty}\right\|^2(T-1) + O\left(\frac{\delta}{c^2}\right).$$

Setting the left hand side equal to zero we obtain the equation

$$T = 1 + F(T)$$

where F is a continuous function in T such that $F(T) = O\left(\frac{\delta}{c}\right)$. We choose c sufficiently large and $\delta = \delta(c)$ sufficiently small so that $|F(T)| \leq \frac{\delta}{c}$. Now, the continuous function $T \longmapsto 1 + F(T)$ maps the closed interval $[1-\frac{\delta}{c}, 1+\frac{\delta}{c}]$ to itself and from Brouwer's fixed point theorem we get a fixed point $T = T_{\mathbf{v}}$. This proves (3.61) and concludes the proof. □

3.10 Proof of the main theorem

To begin with, we summarise the results of the previous section.

Theorem 3.10.1 *Let $\delta > 0$ be sufficiently small, c sufficiently large and pick an arbitrary $\mathbf{v} \in H^3(\mathbb{B}^5_{1+\delta/c}) \times H^2(\mathbb{B}^5_{1+\delta/c})$ such that $\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\delta/c})\times H^2(\mathbb{B}^5_{1+\delta/c})} \leq \frac{\delta}{c^2}$. Then, there exists $T = T_{\mathbf{v}} \in [1 - \frac{\delta}{c}, 1 + \frac{\delta}{c}]$ such that the full, non-corrected equation (3.49) with initial data $\mathbf{u} = \mathbf{U}(T_{\mathbf{v}}, \mathbf{v})$, that is*

$$\mathbf{\Phi}(\tau) = \mathbf{S}_{\alpha_\infty}(\tau)\mathbf{U}(T_{\mathbf{v}}, \mathbf{v}) + \int_0^\tau \mathbf{S}_{\alpha_\infty}(\tau - \sigma)\Big(\hat{\mathbf{L}}_{\alpha(\sigma)}\mathbf{\Phi}(\sigma) + \mathbf{N}_{\alpha(\sigma)}(\mathbf{\Phi}(\sigma)) - \partial_\sigma \mathbf{\Psi}_{\alpha(\sigma)}\Big)d\sigma,$$

has a solution $(\Phi, \alpha) = (\Phi_{T_{\mathbf{v}}}, \alpha_{T_{\mathbf{v}}}) \in \mathcal{X}_\delta \times X_\delta$.

Now, we are in position to prove our main result.

Proof of Theorem 3.3.2 for $d = 5$. Fix $\delta > 0$ sufficiently small and $c > 0$ sufficiently large according to Theorem 3.10.1. Set $\delta' := \frac{\delta}{c}$ and $M := c$. Furthermore, pick any initial data

$$(f, g) \in H^3(\mathbb{B}^5_{1+\delta'}) \times H^2(\mathbb{B}^5_{1+\delta'})$$

satisfying

$$\Big\|(f, g) - u_{1,0}[0]\Big\|_{H^3(\mathbb{B}^5_{1+\delta'})\times H^2(\mathbb{B}^5_{1+\delta'})} \leq \frac{\delta'}{M}.$$

Then, the perturbed initial data $\mathbf{v} := (\tilde{f}, \tilde{g})$ (see (3.6)) satisfy

$$\|\mathbf{v}\|_{H^3(\mathbb{B}^5_{1+\frac{\delta}{c}})\times H^2(\mathbb{B}^5_{1+\frac{\delta}{c}})} = \Big\|(f, g) - u_{1,0}[0]\Big\|_{H^3(\mathbb{B}^5_{1+\delta'})\times H^2(\mathbb{B}^5_{1+\delta'})} \leq \frac{\delta'}{M} = \frac{\delta}{c^2}$$

and Theorem 3.10.1 yields the existence of $T = T_{\mathbf{v}} \in [1-\delta', 1+\delta']$ such that equation (3.49) has a unique solution $(\Phi, \alpha) \in \mathcal{X}_\delta \times X_\delta$ with initial data $\Phi(0) = \mathbf{U}(T_{\mathbf{v}}, \mathbf{v})$. Translating back this statement to the origin setting we obtain a weak solution $\Psi(\tau) = \Psi_{\alpha(\tau)} + \Phi(\tau)$ to the initial system (3.9) with initial data $\Psi(0) = \Psi_0 + \mathbf{U}(T_{\mathbf{v}}, \mathbf{v})$. This means that

$$u(t, x) = \frac{1}{T - t}\psi_1\left(\log\left(\frac{T}{T - t}\right), \frac{x}{T - t}\right)$$

solves the cubic wave equation (3.1) with initial data

$$u(0, x) = \frac{1}{T}\psi_1(0, \frac{x}{T}) = \psi_{1,0}(x) + \tilde{f}(x) = u_{1,0}(x) + \tilde{f}(x)$$
$$\partial_t u(0, x) = \frac{1}{T^2}\psi_2(0, \frac{x}{T}) = \psi_{2,0}(x) + \tilde{g}(x) = \partial_t u_{1,0}(x) + \tilde{g}(x)$$

for all $x \in \mathbb{B}^5_{1+\delta'}$ and therefore is a solution to the Cauchy problem (3.5). Finally, the fact that $\Phi \in \mathcal{X}_\delta$ implies

$$\|\Phi(\tau)\| \leq \delta e^{-\frac{1}{2}\tau}, \quad \forall \tau > 0$$

and hence, for all $t \in [0,T)$ and $k = 0,1,2,3$ we can estimate

$$\begin{aligned}
&(T-t)^{k-\frac{5}{2}+1} \|u(t,\cdot) - u_{T,\alpha_\infty}(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^5_{T-t})} = \\
&(T-t)^{k-\frac{5}{2}+1} \left\| \frac{1}{T-t}\psi_1\left(\log\left(\frac{T}{T-t}\right), \frac{\cdot}{T-t}\right) - \frac{1}{T-t}\psi_{\alpha_\infty,1}\left(\frac{\cdot}{T-t}\right)\right\|_{\dot{H}^k(\mathbb{B}^5_{T-t})} = \\
&(T-t)^{k-\frac{5}{2}} \left\| \psi_1\left(\log\left(\frac{T}{T-t}\right), \frac{\cdot}{T-t}\right) - \psi_{\alpha_\infty,1}\left(\frac{\cdot}{T-t}\right)\right\|_{\dot{H}^k(\mathbb{B}^5_{T-t})} = \\
&\left\| \psi_1\left(\log\left(\frac{T}{T-t}\right), \cdot\right) - \psi_{\alpha_\infty,1}\right\|_{\dot{H}^k(\mathbb{B}^5_1)} \leq \\
&\left\| \psi_1\left(\log\left(\frac{T}{T-t}\right), \cdot\right) - \psi_{\alpha(\log(\frac{T}{T-t})),1}\right\|_{\dot{H}^k(\mathbb{B}^5_1)} + \left\|\psi_{\alpha(\log(\frac{T}{T-t})),1} - \psi_{\alpha_\infty,1}\right\|_{\dot{H}^k(\mathbb{B}^5_1)}.
\end{aligned}$$

For the first term, we get

$$\begin{aligned}
\left\| \psi_1\left(\log\left(\frac{T}{T-t}\right), \cdot\right) - \psi_{\alpha(\log(\frac{T}{T-t})),1}\right\|_{\dot{H}^k(\mathbb{B}^5_1)} &\leq \left\| \psi_1\left(\log\left(\frac{T}{T-t}\right), \cdot\right) - \psi_{\alpha(\log(\frac{T}{T-t})),1}\right\|_{H^3(\mathbb{B}^5_1)} \\
&\leq \left\| \Psi\left(\log\left(\frac{T}{T-t}\right)\right) - \Psi_{\alpha(\log(\frac{T}{T-t}))}\right\| \\
&= \left\| \Phi\left(\log\left(\frac{T}{T-t}\right)\right)\right\| \\
&\lesssim (T-t)^{\frac{1}{2}}.
\end{aligned}$$

For the second term, fix $t \in [0,T)$ and let $\gamma(s) := s\alpha_\infty + (1-s)\alpha\left(\log\left(\frac{T}{T-t}\right)\right),\ s \in [0,1]$ be a parametrisation of the line segment $E[\alpha\left(\log\left(\frac{T}{T-t}\right)\right), \alpha_\infty]$ joining $\alpha\left(\log\left(\frac{T}{T-t}\right)\right)$ and α_∞. Then, the fundamental theorem of calculus yields

$$\begin{aligned}
\psi_{\alpha\left(\log\left(\frac{T}{T-t}\right)\right),1} - \psi_{\alpha_\infty,1} &= \psi_{\alpha\left(\log\left(\frac{T}{T-t}\right)\right)} - \psi_{\alpha_\infty} \\
&= \psi_{\gamma(0)} - \psi_{\gamma(1)} \\
&= \int_{E[\alpha\left(\log\left(\frac{T}{T-t}\right)\right),\alpha_\infty]} \partial\psi_\gamma \cdot d\ell \\
&= \sum_{j=1}^{5} \left(\alpha^j\left(\log\left(\frac{T}{T-t}\right)\right) - \alpha^j_\infty\right) \int_0^1 \partial_{\gamma^j}\psi_{\gamma(t)} dt,
\end{aligned}$$

which implies the bound

$$
\begin{aligned}
\left\|\psi_{\alpha(\log(\frac{T}{T-t})),1}-\psi_{\alpha_\infty,1}\right\|_{\dot{H}^k(\mathbb{B}^5_1)} &= \left\|\partial^k\left(\psi_{\alpha(\log(\frac{T}{T-t}))}-\psi_{\alpha_\infty}\right)\right\|_{L^2(\mathbb{B}^5)} \\
&\lesssim \sum_{j=1}^{5}\left|\alpha^j\left(\log\left(\frac{T}{T-t}\right)\right)-\alpha^j_\infty\right| \sup_{s\in E[\alpha(\log(\frac{T}{T-t})),\alpha_\infty]}\|\partial^k\partial_{\gamma^j}\psi_{\gamma(s)}\|_{L^2(\mathbb{B}^5)} \\
&\lesssim \left|\alpha\left(\log\left(\frac{T}{T-t}\right)\right)-\alpha_\infty\right| \\
&\lesssim (T-t)^{\frac{1}{2}}
\end{aligned}
$$

due to (3.43) and (3.50) since $\alpha\in X_\delta$. The second estimate for $\partial_t\left(u(t,\cdot)-u_{T,\alpha_\infty}(t,\cdot)\right)$ follows similarly. These estimates conclude the proof. □

Proof of Theorem 3.3.2 for $d\in\{7,9,11,13\}$. All the results of the previous sections can be carried on for any $d\in\{7,9,11,13\}$ with slight modifications. The important parts are the function spaces which lead to a sharp decay for the free evolution and the spectral equation for $\alpha=0$.

Referring to the spectral equation for $\alpha=0$ in higher space dimensions, one can readily verify that the potential V_0 in the definition of $\mathbf{L}_0'$, see (3.29), will still turn out to be a constant function. Consequently, the spectral equation will be solved explicitly, solutions will belong to the hypergeometric class as well and we can still use the connection formula which is well known for this class. Then, one can proceed to the case where $\alpha\neq 0$ and since we are only interested in small α we can still apply a perturbative approach. To be precise, all estimates, Lipschitz bounds and decay rates will stay the same in all higher space dimensions since our results are formulated and proved using elements of abstract semigroup theory.

On the other hand, regarding the function spaces in higher space dimensions, one can still define a suitable inner product on

$$\mathcal{H}=H^{\frac{d+1}{2}}\left(\mathbb{B}^d\right)\times H^{\frac{d-1}{2}}\left(\mathbb{B}^d\right)$$

which yields a sharp decay for the "free" evolution operator. To be precise, we let

$$\widetilde{\mathcal{H}}=C^{\frac{d+1}{2}}(\overline{\mathbb{B}^d})\times C^{\frac{d-1}{2}}(\overline{\mathbb{B}^d}),$$

and define

$$(\cdot|\cdot):\widetilde{\mathcal{H}}\times\widetilde{\mathcal{H}}\longrightarrow\mathbb{R},\quad (\mathbf{u}|\mathbf{v}):=\sum_{i=1}^{d}(\mathbf{u}|\mathbf{v})_i\,,$$

where, for $d = 2k+1$, the sesquilinear forms are

$$\begin{aligned}
\left(\mathbf{u}\middle|\mathbf{v}\right)_1 &:= \int_{\mathbb{B}^{2k+1}} \partial_m \partial_{i_1} \cdots \partial_{i_k} u_1(\xi) \overline{\partial^m \partial^{i_1} \cdots \partial^{i_k} v_1(\xi)} d\xi \\
&\quad + \int_{\mathbb{B}^{2k+1}} \partial_{i_1} \cdots \partial_{i_k} u_2(\xi) \overline{\partial^{i_1} \cdots \partial^{i_k} v_2(\xi)} d\xi \\
&\quad + \int_{\mathbb{S}^{2k}} \partial_{i_1} \cdots \partial_{i_k} u_1(\omega) \overline{\partial^{i_1} \cdots \partial^{i_k} v_1(\omega)} d\sigma(\omega), \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_2 &:= \int_{\mathbb{B}^{2k+1}} \partial_m \partial^m \partial_{i_1} \cdots \partial_{i_{k-1}} u_1(\xi) \overline{\partial_n \partial^n \partial^{i_1} \cdots \partial^{i_{k-1}} v_1(\xi)} d\xi \\
&\quad + \int_{\mathbb{B}^{2k+1}} \partial_{i_1} \ldots \partial_{i_k} u_2(\xi) \overline{\partial^{i_1} \cdots \partial^{i_k} v_2(\xi)} d\xi \\
&\quad + \int_{\mathbb{S}^{2k}} \partial_{i_1} \cdots \partial_{i_{k-1}} u_2(\omega) \overline{\partial^{i_1} \cdots \partial^{i_{k-1}} v_2(\omega)} d\sigma(\omega), \\
&\ \ \vdots \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_{3+2q} &:= \sum_{p=1}^{2q+2} A_q^p(d) \left(\mathbf{u}\middle|\mathbf{v}\right)_p + \int_{\mathbb{S}^{2k}} \partial_{i_1} \cdots \partial_{i_{k-2-q}} u_2(\omega) \overline{\partial^{i_1} \cdots \partial^{i_{k-2-q}} v_2(\omega)} d\sigma(\omega), \\
\left(\mathbf{u}\middle|\mathbf{v}\right)_{4+2q} &:= \sum_{r=1}^{2q+3} B_q^r(d) \left(\mathbf{u}\middle|\mathbf{v}\right)_r + \int_{\mathbb{S}^{2k}} \partial_{i_1} \cdots \partial_{i_{k-1-q}} u_1(\omega) \overline{\partial^{i_1} \cdots \partial^{i_{k-1-q}} v_1(\omega)} d\sigma(\omega),
\end{aligned}$$

for some constants $A_q^p(d)$ and $B_q^r(d)$ and for all $q = 0, 1, \ldots, k-2$ and all $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{H}}$. In addition, the missing piece for it to define a norm is given by

$$\left(\mathbf{u}\middle|\mathbf{v}\right)_{2k+1} := \left(\int_{\mathbb{S}^{2k}} \zeta\left(\omega, \mathbf{u}(\omega)\right) d\sigma(\omega) \right) \left(\int_{\mathbb{S}^{2k}} \overline{\zeta\left(\omega, \mathbf{v}(\omega)\right)} d\sigma(\omega) \right)$$

where

$$\zeta\left(\omega, \mathbf{w}(\omega)\right) := D_{2k+1} w_1(\omega) + \tilde{D}_{2k+1} w_2(\omega)$$

and

$$\begin{aligned}
D_{2k+1} w_1(\omega) &:= \sum_{j=1}^{k} a_j \omega^{i_1} \cdots \omega^{i_j} \partial_{i_1} \cdots \partial_{i_j} w_1(\omega) + a_0 w_1(\omega), \\
\tilde{D}_{2k+1} w_2(\omega) &:= \sum_{j=1}^{k-1} b_j \omega^{i_1} \cdots \omega^{i_j} \partial_{i_1} \cdots \partial_{i_j} w_2(\omega) + b_0 w_2(\omega),
\end{aligned}$$

for appropriate constants a_j, b_j, a_0 and b_0. Recall that in all these definitions the Einstein summation convention is assumed. Now, the constants a_j, b_j, a_0 and b_0 are chosen in such a way that the identity

$$\zeta\left(\omega, \widetilde{\mathbf{L}}\mathbf{u}(\omega)\right) = -\zeta\left(\omega, \mathbf{u}(\omega)\right) + \Delta^{\mathbb{S}^{2k}} \left(\tilde{D}_{2k-1}\Big(u_1(\omega) + \omega^j \partial_j u_1(\omega) \Big) \right) \tag{3.62}$$

holds which is the key identity to obtain the decay

$$\text{Re}\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_{2k+1} = -\|\mathbf{u}\|_{2k+1}^2, \tag{3.63}$$

see (3.15). In higher space dimensions, although it is easy to prove that the inner product $(\cdot|\cdot)$ defines indeed a norm equivalent to $\mathcal{H}$, there are two main difficulties. On the one hand, we can find a defining recurrence relation for the coefficients a_j, b_j, a_0 and b_0 which unfortunately is not convenient to write it down nor easy to use and therefore proving (3.62) turns out to be too difficult for us. On the other hand, we can use induction to prove that

$$\text{Re}\left(\widetilde{\mathbf{L}}\mathbf{u}\middle|\mathbf{u}\right)_i \leq -\frac{3}{2}\|\mathbf{u}\|_i^2, \tag{3.64}$$

for all $i \in \{1, 2, \ldots, 2k\}$, but the proof is rather involved.

However, for small d, say $d \in \{7, 9, 11, 13\}$, we can find the coefficients a_j, b_j, a_0 and b_0 explicitly, define D_{2k+1} and $\tilde{D}_{2k+1}$ without recurrence relations and successfully verify (3.64), (3.62) and therefore (3.63). Furthermore, in this case, the proof of (3.64) rely on similar estimates to the ones in Lemma 3.5.4 without any additional tools. Specifically, for $d = 7$, we define

$$\mathcal{D}\big(\widetilde{\mathbf{L}}\big) := C^5\big(\overline{\mathbb{B}^7}\big) \times C^4\big(\overline{\mathbb{B}^7}\big)$$

and

$$(\cdot|\cdot) : \left(C^4(\overline{\mathbb{B}^7}) \times C^3(\overline{\mathbb{B}^7})\right)^2 \longrightarrow \mathbb{R}, \quad \left(\mathbf{u}\middle|\mathbf{v}\right) := \sum_{i=1}^{7} \left(\mathbf{u}\middle|\mathbf{v}\right)_i,$$

where the sesquilinear forms are

$$
\begin{aligned}
(\mathbf{u}|\mathbf{v})_1 &:= \int_{\mathbb{B}^7} \partial_i\partial_j\partial_k\partial_\ell u_1(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell v_1(\xi)}d\xi + \int_{\mathbb{B}^7} \partial_i\partial_j\partial_k u_2(\xi)\overline{\partial^i\partial^j\partial^k v_2(\xi)}d\xi \\
&\quad + \int_{\mathbb{S}^6} \partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial^k v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_2 &:= \int_{\mathbb{B}^7} \partial_i\partial_j\partial^k\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^l\partial_l v_1(\xi)}d\xi + \int_{\mathbb{B}^7} \partial_i\partial_j\partial_k u_2(\xi)\overline{\partial^i\partial^j\partial^k v_2(\xi)}d\xi \\
&\quad + \int_{\mathbb{S}^6} \partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_3 &:= \sum_{j=1}^{2} A_3^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^6} \partial_i u_2(\omega)\overline{\partial^i v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_4 &:= \sum_{j=1}^{3} A_4^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^6} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_5 &:= \sum_{j=1}^{4} A_5^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^6} u_2(\omega)\overline{v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_6 &:= \sum_{j=1}^{5} A_6^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^6} \partial_i u_1(\omega)\overline{\partial^i v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_7 &:= \left(\int_{\mathbb{S}^6} \zeta\left(\omega,\mathbf{u}(\omega)\right)d\sigma(\omega)\right)\left(\int_{\mathbb{S}^6} \overline{\zeta\left(\omega,\mathbf{v}(\omega)\right)}d\sigma(\omega)\right),
\end{aligned}
$$

for some constants A_i^j and for all $\mathbf{u},\mathbf{v} \in C^4(\overline{\mathbb{B}^7}) \times C^3(\overline{\mathbb{B}^7})$. Here,

$$
\begin{aligned}
&\zeta\left(\omega,\mathbf{w}(\omega)\right) := D_7 w_1(\omega) + \tilde{D}_7 w_2(\omega), \\
&D_7 w_1(\omega) := \omega^i\omega^j\omega^k\partial_i\partial_j\partial_k w_1(\omega) + 12\omega^i\omega^j\partial_i\partial_j w_1(\omega) + 33\omega^i\partial_i w_1(\omega) + 15 w_1(\omega), \\
&\tilde{D}_7 w_2(\omega) := \omega^i\omega^j\partial_i\partial_j w_2(\omega) + 9\omega^j\partial_j w_2(\omega) + 15 w_2(\omega).
\end{aligned}
$$

One can prove that this inner product defines indeed a norm equivalent to $H^4\left(\mathbb{B}^7\right)\times H^3\left(\mathbb{B}^7\right)$ and the decay estimates (3.63) and (3.64) hold. Furthermore, for $d = 9$, we define

$$
\mathcal{D}\big(\widetilde{\mathbf{L}}\big) := C^6\big(\overline{\mathbb{B}^9}\big) \times C^5\big(\overline{\mathbb{B}^9}\big).
$$

and

$$
(\cdot|\cdot) : \left(C^5(\overline{\mathbb{B}^9}) \times C^4(\overline{\mathbb{B}^9})\right)^2 \longrightarrow \mathbb{R}, \quad (\mathbf{u}|\mathbf{v}) := \sum_{i=1}^{9}\,(\mathbf{u}|\mathbf{v})_i\,,
$$

where the sesquilinear forms are

$$
\begin{aligned}
(\mathbf{u}|\mathbf{v})_1 &:= \int_{\mathbb{B}^9} \partial_i\partial_j\partial_k\partial_\ell\partial_m u_1(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell\partial^m v_1(\xi)}d\xi + \int_{\mathbb{B}^9} \partial_i\partial_j\partial_k\partial_\ell u_2(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell v_2(\xi)}d\xi \\
&\quad + \int_{\mathbb{S}^8} \partial_i\partial_j\partial_k\partial_\ell u_1(\omega)\overline{\partial^i\partial^j\partial^k\partial^\ell v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_2 &:= \int_{\mathbb{B}^9} \partial_i\partial_j\partial^k\partial_\ell\partial^\ell u_1(\xi)\overline{\partial^i\partial^j\partial_k\partial^m\partial_m v_1(\xi)}d\xi + \int_{\mathbb{B}^9} \partial_i\partial_j\partial_k\partial_\ell u_2(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell v_2(\xi)}d\xi \\
&\quad + \int_{\mathbb{S}^8} \partial_i\partial_j\partial_k u_2(\omega)\overline{\partial^i\partial^j\partial^k v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_3 &:= \sum_{j=1}^{2} B_3^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^8} \partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_4 &:= \sum_{j=1}^{3} B_4^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^8} \partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial_k v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_5 &:= \sum_{j=1}^{4} B_5^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^8} \partial_i u_2(\omega)\overline{\partial^i v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_6 &:= \sum_{j=1}^{5} B_6^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^8} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_7 &:= \sum_{j=1}^{6} B_7^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^8} u_2(\omega)\overline{v_2(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_8 &:= \sum_{j=1}^{7} B_8^j\,(\mathbf{u}|\mathbf{v})_j + \int_{\mathbb{S}^8} \partial_i u_1(\omega)\overline{\partial^i v_1(\omega)}d\sigma(\omega), \\
(\mathbf{u}|\mathbf{v})_9 &:= \left(\int_{\mathbb{S}^6} \zeta\left(\omega,\mathbf{u}(\omega)\right)d\sigma(\omega)\right)\left(\int_{\mathbb{S}^6} \overline{\zeta\left(\omega,\mathbf{v}(\omega)\right)}d\sigma(\omega)\right),
\end{aligned}
$$

for some constants B_i^j and for all $\mathbf{u},\mathbf{v} \in C^5(\overline{\mathbb{B}^9}) \times C^4(\overline{\mathbb{B}^9})$. Here,

$$
\begin{aligned}
&\zeta\left(\omega,\mathbf{w}(\omega)\right) := D_9 w_1(\omega) + \tilde{D}_9 w_2(\omega), \\
&D_9 w_1(\omega) := \omega^i\omega^j\omega^k\omega^\ell\partial_i\partial_j\partial_k\partial_\ell w_1(\omega) + 22\omega^i\omega^j\omega^k\partial_i\partial_j\partial_k w_1(\omega) + 141\omega^i\omega^j\partial_i\partial_j w_1(\omega) \\
&\qquad\qquad + 279\omega^i\partial_i w_1(\omega) + 105 w_1(\omega), \\
&\tilde{D}_9 w_2(\omega) := \omega^i\omega^j\omega^k\partial_i\partial_j\partial_k w_2(\omega) + 18\omega^i\omega^j\partial_i\partial_j w_2(\omega) + 87\omega^j\partial_j w_2(\omega) + 105 w_2(\omega).
\end{aligned}
$$

We can verify that this inner product defines indeed a norm equivalent to $H^5(\mathbb{B}^9) \times H^4(\mathbb{B}^9)$ and the decay estimates (3.63) and (3.64) hold. In addition, for $d = 11$, we define

$$
\mathcal{D}(\widetilde{\mathbf{L}}) := C^7(\overline{\mathbb{B}^{11}}) \times C^6(\overline{\mathbb{B}^{11}}).
$$

and

$$\left(\cdot\middle|\cdot\right):\left(C^6(\overline{\mathbb{B}^{11}})\times C^5(\overline{\mathbb{B}^{11}})\right)^2\longrightarrow\mathbb{R},\quad \left(\mathbf{u}\middle|\mathbf{v}\right):=\sum_{i=1}^{11}\left(\mathbf{u}\middle|\mathbf{v}\right)_i,$$

where the sesquilinear forms are

$$\begin{aligned}
\left(\mathbf{u}\middle|\mathbf{v}\right)_1 &:= \int_{\mathbb{B}^{11}}\partial_i\partial_j\partial_k\partial_\ell\partial_m\partial_n u_1(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell\partial^m\partial^n v_1(\xi)}d\xi+\int_{\mathbb{B}^{11}}\partial_i\partial_j\partial_k\partial_\ell\partial_n u_2(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell\partial^n v_2(\xi)}d\xi\\
&\quad+\int_{\mathbb{S}^{10}}\partial_i\partial_j\partial_k\partial_\ell\partial_n u_1(\omega)\overline{\partial^i\partial^j\partial^k\partial^\ell\partial^n v_1(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_2 &:= \int_{\mathbb{B}^{11}}\partial_i\partial_j\partial_k\partial_\ell\partial_n\partial^n u_1(\xi)\overline{\partial^i\partial^j\partial_k\partial_\ell\partial^m\partial_m v_1(\xi)}d\xi+\int_{\mathbb{B}^{11}}\partial_i\partial_j\partial_k\partial_\ell\partial_n u_2(\xi)\overline{\partial^i\partial^j\partial^k\partial^\ell\partial^n v_2(\xi)}d\xi\\
&\quad+\int_{\mathbb{S}^{10}}\partial_i\partial_j\partial_k\partial_\ell u_2(\omega)\overline{\partial^i\partial^j\partial^k\partial^\ell v_2(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_3 &:= \sum_{j=1}^{2}C_3^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^1}\partial_i\partial_j\partial_k u_2(\omega)\overline{\partial^i\partial^j\partial^k v_2(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_4 &:= \sum_{j=1}^{3}C_4^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}\partial_i\partial_j\partial_k\partial_\ell u_1(\omega)\overline{\partial^i\partial^j\partial_k\partial^\ell v_1(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_5 &:= \sum_{j=1}^{4}C_5^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial_j v_2(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_6 &:= \sum_{j=1}^{5}C_6^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}\partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial_k v_1(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_7 &:= \sum_{j=1}^{6}C_7^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}\partial_i u_2(\omega)\overline{\partial^i v_2(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_8 &:= \sum_{j=1}^{7}C_8^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j v_1(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_9 &:= \sum_{j=1}^{8}C_9^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}u_2(\omega)\overline{v_2(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_{10} &:= \sum_{j=1}^{7}C_{10}^j\left(\mathbf{u}\middle|\mathbf{v}\right)_j+\int_{\mathbb{S}^{10}}\partial_i u_1(\omega)\overline{\partial^i v_1(\omega)}d\sigma(\omega),\\
\left(\mathbf{u}\middle|\mathbf{v}\right)_{11} &:= \left(\int_{\mathbb{S}^{10}}\zeta\left(\omega,\mathbf{u}(\omega)\right)d\sigma(\omega)\right)\left(\int_{\mathbb{S}^{10}}\overline{\zeta\left(\omega,\mathbf{v}(\omega)\right)}d\sigma(\omega)\right),
\end{aligned}$$

for some constants C_i^j and for all $\mathbf{u}, \mathbf{v} \in C^6(\overline{\mathbb{B}^{11}}) \times C^5(\overline{\mathbb{B}^{11}})$. Here,

$$
\begin{aligned}
&\zeta\left(\omega, \mathbf{w}(\omega)\right) := D_{11}w_1(\omega) + \tilde{D}_{11}w_2(\omega),\\
&D_{11}w_1(\omega) := \omega^i\omega^j\omega^k\omega^\ell\omega^m\partial_i\partial_j\partial_k\partial_\ell\partial_m w_1(\omega) + 35\omega^i\omega^j\omega^k\omega^\ell\partial_i\partial_j\partial_k\partial_\ell w_1(\omega) + 405\omega^i\omega^j\omega^k\partial_i\partial_j\partial_k w_1(\omega)\\
&\qquad\qquad + 1830\omega^i\omega^j\partial_i\partial_j w_1(\omega) + 2895\omega^i\partial_i w_1(\omega) + 945 w_1(\omega),\\
&\tilde{D}_{11}w_2(\omega) := \omega^i\omega^j\omega^k\omega^\ell\partial_i\partial_j\partial_k\partial_\ell w_2(\omega) + 30\omega^i\omega^j\omega^k\partial_i\partial_j\partial_k w_2(\omega) + 285\omega^i\omega^j\partial_i\partial_j w_2(\omega)\\
&\qquad\qquad + 975\omega^j\partial_j w_2(\omega) + 945 w_2(\omega).
\end{aligned}
$$

We can verify that this inner product defines indeed a norm equivalent to $H^6\left(\mathbb{B}^{11}\right) \times H^5\left(\mathbb{B}^{11}\right)$ and the decay estimates (3.63) and (3.64) hold. Similarly, we get analogous formulas for the case $d = 13$ and verify (3.63) and (3.64). □

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