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Contents

1

Introduction

This thesis consists of three self-contained essays in microeconomic theory. The common theme of all chapters is the analysis of collective decision problems in which efficient decisions depend on information that is privately held by individuals. Chapter 2 studies optimal provision of public goods. In Chapter 3 we propose a class of mechanisms for coupled binary decision problems. Chapter 4 studies efficient information aggregation in a setting where individuals have both private preferences and private information that is relevant for everyone.

The theory of mechanism design provides a framework to derive optimal decision rules when strategic individuals hold private information. The goal is to aggregate private preferences toward a single joint decision while maximizing an underlying objective such as overall welfare. To prevent that individuals misreport their private preferences the mechanism must provide the right incentives. For the general setting without money, the Gibbard (1973) - Satterthwaite (1975) Theorem shows that all incentive compatible mechanisms are dictatorial if there are more than two alternatives over which all preference orderings are possible.

The introduction of money puts more structure on the theoretical problem and might help to provide incentives. For example, requiring only incentive constraints, the Vickrey (1961) - Clarke (1971) - Groves (1973) mechanism implements efficient decisions. However, the benefit of using money decreases when more constraints are imposed on a mechanism. Participation constraints and the exclusion of external payments can prevent efficient implementation as demonstrated by Myerson and Satterthwaite (1983) for example.

The research questions of Chapter 2 and 3 lie in between settings with and without money. The public good provision setting in Chapter 2 allows for monetary transfers. However, we show that upon imposing incentive, participation and budget balance constraints, money cannot be used to fine-tune incentives anymore. This makes the problem equivalent to one without monetary transfers. In the derivation of the optimal mechanism, we apply the Gibbard (1973) - Satterthwaite (1975) Theorem to characterize all mechanisms that fulfill the constraints.

The imposed constraints in Chapter 2 prevent using money to provide incentives in the public good setting. In other contexts, monetary transfers are ruled out by assumption: for example, there is a consensus that individuals' impact on a voting outcome should not depend on their material wealth. In Chapter 3, we study mechanisms for coupled binary decision problems without allowing for monetary transfers. The Gibbard (1973) - Satterthwaite (1975) Theorem does not apply because the structure of the problem restricts the set of preference orderings over outcomes, and we work with a relaxed equilibrium concept. This allows us to propose a mechanism that improves in terms of welfare on simple mechanisms such as dictatorship or separate majority voting.

Chapter 2 and 3 share a common conceptual idea: coupling several unrelated decision problems can help to increase efficiency. In Chapter 2, which is joint work with Felix Bierbrauer, we show that coupling the decisions on several public projects facilitates public good provision. Different public goods can be bundled together if there is enough capacity, i.e. resources to pay for all the public goods in the bundle. The analysis focuses on the *all-or-nothing* mechanism: expand provision as much as resources allow if no one vetoes - otherwise stick to the status quo. Individuals might prefer the bundle over the status quo even if they dislike particular projects. In fact, we show that the probability of providing the bundle of public projects converges to one as the capacity becomes unbounded. Further, we provide conditions under which the all-or-nothing mechanism is ex-ante welfare-maximizing - even though, ex-post, it involves an overprovision of public goods.

Chapter 3 is joint work with Kilian Russ. We propose the class of *ranking mechanisms* for coupled binary decisions. Conceptually, a ranking serves as "quasi-money" that makes utility to some extent transferable between decision problems. A ranking mechanism truthfully elicits two parts of individuals' private information. Individuals communicate which alternative they prefer in each decision problem. Additionally, they report a priority ranking over decision problems by ranking each problem according to the absolute difference in utilities between the two proposed alternatives. These rankings are then used to assign weights to individuals' votes in a voting mechanism. Any ranking mechanism is thus implementable as a weighted voting procedure. We derive a closed-form solution for the ex-ante efficient weight vector. The optimal ranking mechanism ex-ante Pareto dominates separate majority voting for an arbitrary number of individuals and decision problems. We extend our results to non-identical distributions of preferences between individuals and across problems.

Chapter 2 and 3 emphasize the importance of voting-like mechanisms when aggregating private preferences. Another strand of the literature discusses the use of voting to aggregate dispersed and privately held information on a state of the world that is relevant for everyone. The classical work of Condorcet (1785) shows that many individuals with independent information can jointly make efficient decisions. Chapter 4, which is joint work with Patrick Lahr, builds on this idea and studies efficient information aggregation when individuals have not only private information, but also private preferences. We demonstrate that there are more efficient ways to aggregate information than by majority voting. Under common interests, the most efficient way is a weighted voting procedure similar to that in Chapter 3. Specialization, i.e. more heterogeneously distributed information, helps to infer the state. When allowing for private interests, complete differentiation of information qualities breaks down: individuals who are not interested in learning the state of the world claim to have very precise information to benefit from a misconception of the state. These *fake experts* prevent optimal discrimination of individuals' information and devalue individuals with actually precise information, mitigating the value of specialization. If preferences are sufficiently heterogeneous, any differentiating weighting of information breaks down, and majority voting becomes the best way to aggregate information.

References

Condorcet, Marquis de. 1785. "Essay on the Application of Analysis to the Probability of Majority Decisions." *Paris: Imprimerie Royale*, [3]

Vickrey, William. 1961. "Counterspeculation, auctions, and competitive sealed tenders." *Journal of Finance* 16 (1): 8–37. [1]

Clarke, Edward H. 1971. "Multipart pricing of public goods." *Public Choice* 11 (1): 17–33. [1]

Gibbard, Allen. 1973. "Manipulation of voting schemes: A general result." *Econometrica* 41 (4): 587–601. [1, 2]

Groves, Theodore. 1973. "Incentives in teams." *Econometrica*, 617–31. [1]

Satterthwaite, Mark A. 1975. "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory* 10 (2): 187–217. [1, 2]

Myerson, Roger B, and Mark A Satterthwaite. 1983. "Efficient mechanisms for bilateral trading." *Journal of Economic Theory* 29 (2): 265–81. [1]

2

All or Nothing: State Capacity and Optimal Public Goods Provision

Joint with Felix Bierbrauer

2.1 Introduction

We study the following situation: There is a status quo with a limited provision of public goods. Moving towards more goods being provided requires both sufficient resources and sufficient political support. Our main result shows that an increase in capacity, i.e. in resources available to finance public goods, makes it possible to overcome all obstacles to increased public goods provision. It eliminates resistance by those who dislike certain public goods and it eliminates incentives to free-ride on the contributions of others. Specifically, with sufficient capacity, providing as many public goods as possible is an incentive-feasible mechanism. We also provide conditions under which this mechanism maximizes expected welfare.

The paper contributes to the literature that studies public goods provision from a mechanism design perspective. By and large, the existing literature, reviewed in more detail below, emphasizes the difficulties that are associated with incentive and participation constraints. The second-best mechanisms that respect these constraints typically involve an underprovision of public goods. Our setting, by contrast, gives rise to a second-best mechanism with an overprovision of public goods.

The capacity to pay for public goods is a key variable in our approach. This relates our analysis to the literature on the expansion of state capacity, originating from Besley and Persson (2009), and also to the observation, sometimes referred to as Wagner's law, see Wagner (1883), that public expenditures, as a share of GDP, have been rising in the 19th century. To be clear, an abstract mechanism design approach cannot identify the specific political forces that have led to increased public spending in the course of economy history. Still, the underprovision results in the literature provoke the question whether public goods provision subject to incentive and participation constraints can ever be compatible with a simultaneous increase in state capacity and public spending. Mailath and Postlewaite (1990), for instance, show that the probability of public good provision goes to zero under any such mechanism as the number of individuals gets large. A tempting conclusion therefore is that the imposition of participation constraints, i.e. of a requirement of unanimity in favor of increased public good provision, makes it impossible to have increasing expenditures. Increasing expenditures can then be reconciled only with a violation of participation constraints or, equivalently, a use of the government's coercive power to finance public goods, against the will of at least some of the people. Against this background, our analysis shows a theoretical possibility to have increasing expenditures on public goods in the presence of participation constraints: An increasing capacity allows to bundle public goods in such a way that moving towards increased expenditures is in everyone's interest.

Sketch of the formal analysis. There are n individuals and there is sufficient capacity to finance m additional public goods. Individuals have private information on their valuations of these goods. For any one else, valuations are taken to be *iid* random variables with a mean that exceeds the per capita provision cost and which take values lower than the cost with positive probability. Thus, it is a priori unclear which public goods should be provided.

A mechanism determines which goods are provided and also what individuals have to pay. Admissible mechanisms satisfy participation, incentive and budget constraints. We require that all these constraints hold ex post. Thus, whatever the state of the economy, ex post, no individual prefers the status quo over the outcome of the mechanism, nor does any one individual regret to have revealed her preferences. In addition, the money that is collected from individuals is exactly what is needed to cover the cost of provision. We also impose a condition of anonymity.

Mailath and Postlewaite (1990) have established an impossibility result for the case $m = 1$: With many individuals, the probability of public goods provision is close to zero under any admissible mechanism. Mailath and Postlewaite employ participation, incentive and resource constraints that are more permissive than ours. In their analysis, participation constraints are satisfied if all individuals' expected utility under the mechanism is higher than in the status quo. Incentive compatibility holds if a truthful revelation of preferences is a Bayes-Nash equilibrium, rather than an ex post or dominant strategy equilibrium. Our analysis shows that the impossibility of public goods provision can be overcome if many public goods are provided simultaneously. An impossibility result in mechanism design gets stronger with weaker constraints. A possibility result gets stronger with stronger constraints. Thus, while for the purposes of Mailath and Postlewaite, it was a natural choice to have constraints that need to hold only in expectation, for us, the natural choice is to have separate participation, incentive and budget constraints for each state of the economy.[1]

The all-or-nothing-mechanism plays a decisive role in our analysis. This mechanism has only two outcomes: Either the status quo prevails, or the capacity for increased public goods provision is exhausted. Costs are shared equally among individuals. Exhausting the capacity requires a consensus. As soon as one individual opts for the status quo, the status quo stays in place. This mechanism is obviously admissible: The veto rights ensure that participation constraints are satisfied. If no one makes use of his veto power, then, whatever the preference profile, the mechanism stipulates the same outcome. This limited use of information on preferences ensures incentive compatibility.

Our first set of results shows that, under the all-or-nothing-mechanism, the probability of the "all-outcome" is an increasing function of the capacity m and converges to 1 as m becomes unbounded. This can be understood as a large numbers effect. The larger the bundle, the closer are individual preferences to the mean of the distribution from which preferences are drawn. As the mean exceeds the per capita cost, the larger the bundle the less likely is a veto. To relate our analysis to Mailath and Postlewaite (1990) we also consider the possibility that both the capacity m and the number of individuals n grow. If this process is such that the ratio $\frac{m}{n}$ converges to a positive constant, the limit probability of the all-outcome is bounded away from zero.

A second set of results establishes conditions under which the all-or-nothing-mechanism is a second-best mechanism, i.e. a mechanism that maximizes the expected surplus over the set of admissible mechanisms. The all-or-nothing-mechanism may not appear as a natural candidate for an optimal mechanism: It gives rise to an overprovision of public goods as the capacity exhausting bundle typically includes public goods with negative surplus. Since the alternatives are only "all" and "nothing", there is no possibility to eliminate those goods from the bundle.

Our analysis invokes the famous impossibility result by Gibbard (1973) and Satterthwaite (1975). According to this result, with an unrestricted preference domain, any mechanism that is ex post incentive compatible and allows for more than two outcomes is dictatorial. We show that, under an

[1] Ex post constraints are attractive also for another reason. Mechanisms that satisfy these constraints are robust in the sense that they reach the intended outcome whatever the individuals' probabilistic beliefs about the types of other individuals, see Bergemann and Morris (2005).

ancillary assumption, this theorem applies to our setup. The implication is that the set of admissible mechanisms becomes small: There can be at most two outcomes. One of the two outcomes has to be the status quo. Otherwise, it would impossible to respect participation constraints. Thus, the only degree of freedom is the choice of the second outcome. The assumption that public goods provision is desirable in expectation, implies that it is desirable to exhaust the capacity to provide public goods. Thus, a second best mechanism gives a choice between two outcomes, "all" or "nothing".

Related Literature. The observation that bundling can alleviate inefficiencies due to incentive or participation constraints is due to Jackson and Sonnenschein (2007) and Fang and Norman (2006). Both papers focus on Bayes-Nash equilibria and on participation constraints that need to hold at the interim stage where individuals know their own type but still face uncertainty about the types of others and hence about the outcome of the mechanism. Moreover, both papers show that bundling a large number of decisions allows to approximate first-best outcomes. Our work differs in that we invoke ex post incentive and participation constraints. As a consequence, first-best outcomes cannot be reached. The second-best outcome is the all-or-nothing-mechanism that gives rise to an overprovision of public goods.

If bundling is not an option, second-best mechanisms give rise to an underprovision of public goods.[2] More specifically, Güth and Hellwig (1986) show that the second-best mechanisms involve underprovision. Mailath and Postlewaite (1990) show that, under any admissible mechanism, the probability of public goods provision goes to zero as the number of individuals becomes unbounded. An important assumption is that the per capita cost of provision remains constant as additional individuals are added to the system. Hellwig (2003), by contrast, allows for scale economies. Welfare-maximizing provision levels then increase with the number of individuals. Still, these second-best provision levels may fall short of first-best levels. For excludable public goods, as shown by Norman (2004), second-best mechanisms involve use restrictions to mitigate the distortions from incentive and participation constraints, again with the implication that second-best provision levels are smaller than first-best levels.

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[2] Some qualifications are in order. With correlated, rather than independent types first-best outcomes can typically be reached in the presence of incentive and participation constraints, see Crémer and McLean (1988). With independent types, and without participation constraints, first best outcomes can typically be implemented as a Bayes-Nash equilibrium, see d'Aspremont and Gérard-Varet (1979), but not as a dominant strategy equilibrium, see Green and Laffont (1977).

Outlook. The following section introduces the formal framework. In Section 2.3, we show that, under the all-or-nothing-mechanism, public expenditures increase in the capacity to provide public goods. Section 2.4 shows that the all-or-nothing-mechanism is a second-best mechanism. The last section contains concluding remarks. Formal proofs are relegated to the Appendix and to a Supplement.

2.2 The Model

The set of individuals is denoted by $I = \{1, \ldots, n\}$. A finite set $K = \{1, \ldots, m\}$ of public projects is available. A mechanism determines which elements of K are implemented and how the costs are shared.

The benefit that individual i realizes if project k is undertaken is denoted by θ_{ik}. We write $\theta_i = \{\theta_{ik}\}_{k \in K}$ for the preference profile of i and denote the set of possible profiles by Θ_i. We write $\theta = (\theta_1, \ldots, \theta_n)$, refer to θ as a *state of the economy* and to $\Theta = \Pi_{i=1}^{n} \Theta_i$ as the set of states. For any project k, individual i privately observes θ_{ik}. For any one else, θ_{ik} is a random variable with *cdf* F_{ik} and density f_{ik}. We assume that these are *iid* across projects and individuals, i.e. there exist F and f so that $F_{ik} = F$ and $f_{ik} = f$, for all i and k. We denote the mean of these random variables by μ and the variance by σ^2.

Let κ be the per capita cost of any one public project k. Without loss of generality, we let $\kappa = 1$. We denote by $s_k(\theta) = \frac{1}{n} \sum_{i=1}^{n} \theta_{ik} - 1$ the per capita surplus that would be generated if public good k was implemented in state θ. We assume that $\mu > 1$, with the implication that the expected value of $s_k(\theta)$ is positive. We also assume that realizations of θ_{ik} strictly smaller than 1 occur with positive probability. Hence, negative values of $s_k(\theta)$ have positive probability.

The revelation principle applies so that we can focus on direct mechanisms. A direct mechanism is a collection of functions $q_k : \Theta \to \{0, 1\}$, $k \in K$, that indicate, for each state of the economy, whether public good k is provided or not. In addition, there is a collection of functions $t_i : \Theta \to \mathbb{R}$, $i \in I$, that specify individual payments as a function of the state of the economy. Under such a mechanism, the payoff of individual i in state θ is given by

$$u_i(\theta) \;=\; \sum_{k \in K} \theta_{ik} \;\; q_k(\theta) - t_i(\theta).$$

We say that a direct mechanism is admissible if it satisfies incentive, participation and budget constraints. Participation constraints hold in an ex post sense if, for all i and θ,

$$u_i(\theta) \;\geq\; 0 \;. \tag{2.1}$$

Incentive compatibility holds provided that truth-telling is an ex post or dominant strategy equilibrium, i.e. if for all i, all $\theta = (\theta_i, \theta_{-i})$ and all $\hat{\theta}_i$,[3]

$$u(\theta_i, \theta_{-i}) \;\geq\; u(\hat{\theta}_i, \theta_{-i}) \quad . \tag{2.2}$$

Budget balance requires that, for all θ,

$$\frac{1}{n}\sum_{i=1}^{n} t_i(\theta) \;=\; \sum_{k=1}^{m} q_k(\theta) \quad . \tag{2.3}$$

Finally, we require a mechanism to be anonymous, i.e. a permutation of individual types must not affect the outcome of the mechanism.

Capacity. We think of state capacity m as the part of national income that can be used to finance public expenditures and are interested in the comparative statics of state capacity: What does a change in state capacity imply for the possibility to finance expenditures on public goods in the presence of participation, incentive and budget constraints? To introduce state capacity into the model, we proceed as follows: Let m be the part of any one individual's income that can be devoted to the financing of public goods. Thus, for any i, and any state θ,

$$t_i(\theta) \;\leq\; m \quad . \tag{2.4}$$

Moreover, for notational convenience, assume that possible values of m are multiples of $\kappa = 1$. Thus, $m = 1$ means that there is capacity for one public project, $m = 2$ means that there is capacity for two public projects, and so on.

The all-or-nothing-mechanism. The all-or-nothing-mechanism is an admissible mechanism. Under this mechanism, all public goods are provided and the costs are shared equally unless there is an individual who prefers the status quo. In this case, the status quo prevails. Formally: If $\frac{1}{m}\sum_{k=1}^{m} \theta_{jk} < 1$ for some $j \in I$, then $q_k(\theta) = 0$, for all k, and $t_i(\theta) = 0$, for all i. Otherwise, $q_k(\theta) = 1$, for all k, and $t_i(\theta) = m$, for all i.

[3] In environments with private values, ex post and dominant strategy equilibria coincide, see e.g. Bergemann and Morris (2005).

2.3 Capacity and Expenditures on Public Goods

Let $\mathbb{P}_{all}(m)$ be the probability of the all-outcome under an all-or-noting mechanism with capacity m. We will use a result from statistics to show that, under a monotone hazard rate assumption, $\mathbb{P}_{all}$ is an increasing function. Thus, the probability of provision is an increasing function of the capacity to provide public goods. We also provide limit results for the case that m becomes unbounded. The limit results hold irrespectively of whether or not the monotone hazard rate assumption is satisfied.

These results can be related to the literatures on state capacity and Wagner's law. Under the all-or-nothing-mechanism, expected expenditures on public goods are given by $\mathbb{P}_{all}(m)\ m$. Thus, we can think of the ratio $\frac{\mathbb{P}_{all}(m)\ m}{y}$, where y is national income per capita, as a proxy for public expenditures as a share of GDP. If we express state capacity m as a fraction of *GDP* so that $m = g\,y$, we can write

$$\frac{\mathbb{P}_{all}(m)\ \ m}{y} \ = \ g\ \ \mathbb{P}_{all}(m).$$

If y grows, so does m if g is held constant. With $\mathbb{P}_{all}$ an increasing function, this implies an increasing expenditure share, in line with Wagner's law. If $\mathbb{P}_{all}$ converges to a positive constant as m and, possibly also n grow without bounds, the only way to increase the expenditure share is to increase g, i.e. the fraction of national income that can be used to finance public goods. The literature on the expansion of state capacity focusses on this variable.

Proposition 1. *Suppose that the density f is symmetric and log-concave. Then $\mathbb{P}_{all}(m)$ increases monotonically in m.*

The result of Mailath and Postlewaite (1990) applies to the case $m = 1$: $\mathbb{P}_{all}(1)$ is close to zero if the number of individuals n is sufficiently large.[4] If the density f is both symmetric and log-concave, then the probability of the all-outcome is larger if the capacity suffices to finance two public projects, $\mathbb{P}_{all}(2) > \mathbb{P}_{all}(1)$ and even larger if it suffices to finance three public projects and so on.[5] According to the Proposition 2 this sequence of probabilities converges to 1, i.e. as m grows without bound, the probability that there is an individual who prefers the status quo over the all-outcome vanishes.

Proposition 2. $\lim_{m\to\infty}\ \mathbb{P}_{all}(m) = 1.$

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[4] The result of Mailath and Postlewaite (1990) applies to any admissible mechanism. Therefore it applies, in particular, to the all-or-nothing-mechanism.

[5] The assumption of log-concavity is satisfied by many well-known probability distributions, including the uniform distribution, the normal distribution or the logistic distribution, see Bagnoli and Bergstrom (2005).

The proposition follows from a straightforward application of Chebychef's inequality.[6] Intuitively, as m grows without bound, for any individual i, $\frac{1}{m}\sum_{k=1}^{m}\theta_{ik}$ converges to μ by a large numbers effect. Providing all public goods is therefore in every one's interest.

Suppose that $m = 1$ and that n is large. The per capita valuation of the public good $\frac{1}{n}\sum_{i=1}^{n}\theta_{i1}$ is then close to μ, i.e. the surplus $s_1(\theta)$ from providing the public good is positive with probability close to one. The probability of a veto is also close to one, however: with probability close to one there are individuals with $\theta_{i1} < 1$. This observation raises the question how $\mathbb{P}_{all}$ behaves if both m and n grow at the same time.

Proposition 3. *Suppose that $n = \gamma\, m$ for $\gamma > 0$. Then* $\lim_{m\to\infty}\ \mathbb{P}_{all}(m) > 0$.

The argument in the proof of Proposition 2 is easily adapted to deal with m and n growing at the same rate. The conclusion is weaker in that case, $\mathbb{P}_{all}(m)$ is bounded from below by a positive constant that may be smaller than 1.[7] The fact that it is bounded away from zero implies that the impossibility result that is obtained for $m = 1$ does not extend to this case.

2.4 On the Optimality of the All-or-Nothing Mechanism

We will now show that, under certain conditions, the all-or-noting-mechanism is a second-best mechanism, i.e. a mechanism which maximizes the expected surplus

$$E\left[\frac{1}{n}\sum_{i=1}^{n}u_i(\theta)\right] = E\left[\sum_{k=1}^{m}s_k(\theta)\ \ q_k(\theta)\right]$$

over the set of mechanisms which satisfy the constraints in (2.1)-(2.4).

In doing so, we will treat n as fixed. As a consequence, the all-or-nothing-mechanism is not a first-best mechanism.[8] For any good k, the probability of the event $s_k(\theta) < 0$ is strictly positive. As a consequence, the all-outcome includes projects with a negative surplus with positive probability. Moreover, for large m, this probability is close to one.

The following assumption greatly simplifies the proof that the all-or-nothing mechanism is a second-best mechanism. We further discuss its role below.

Assumption 1. *There is a fixed order for the implementation of projects. Specifically, $q_l(\theta) = 1$ implies $q_k(\theta) = 1$, for all $k \leq l$.*

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[6] Formal proofs of Propositions 2 and 3 can be found in the Supplement.

[7] In the Supplement, we also show that $\mathbb{P}_{all}(m) \to 1$ if m and n do not grow at the same rate, but $\frac{m}{n} \to \infty$.

[8] As $n \to \infty$, for any k, $\frac{1}{n}\sum_{i=1}^{n}\theta_{ik}$ converges in probability to $\mu > 1$. Hence, the all-outcome converges in probability to a first best outcome.

The assumption means that there is a natural order in which public projects can be undertaken. Project 2 can be undertaken only after project 1 has been implemented, project 3 can be implemented only after project 2 has been implemented and so on. The set of possible public good outcomes therefore becomes smaller. Specifically, the possible outcomes can be represented by the set $K' = \{0, 1, \ldots, m\}$ where outcome $k' \in K'$ indicates that all public goods with an index smaller or equal k' are provided. The role that this assumption plays in our proof will become clear. It ensures that all logically conceivable preferences over the set of outcomes can be represented by an additively separable utility function, i.e. we can satisfy a universal domain requirement without having to introduce utility functions that allow for substitutes or complements in public goods preferences.

Theorem 1. *Suppose that the density f is symmetric and log-concave and that Assumption 1 holds. Then, the all-or-nothing-mechanism is a second-best mechanism.*

In the following, we first explain the key steps in the proof of the theorem, with formal details relegated to the Appendix. We then provide a discussion of Assumption 1.

2.4.1 Proof of Theorem 1

The following lemma implies that, in what follows, we can limit attention to mechanisms that involve equal cost sharing.

Lemma 1. *If a direct mechanism is anonymous and satisfies the incentive constraints in (2.2) and the budget constraints in (2.3) then, for all i and for all θ, $t_i(\theta) = \sum_{k=1}^{m} q_k(\theta)$.*

The lemma and its proof in part 2.A.2 of the Appendix are of independent interest. It is useful for the same reason as the characterization of incentive compatibility via the envelope theorem in Bayesian mechanism design. This characterization yields, for instance, the well-known revenue equivalence result in auction theory. Knowing what individual payments have to look like makes it possible to focus on allocation rules, as opposed to allocation and payment rules. This greatly simplifies the analysis. Here, however, the argument involves not only incentive constraints, but the interplay of incentive constraints, budget constraints and the requirement of anonymity. The Lemma generalizes previous results in the literature.[9] Also note that Lemma 1 holds irrespectively of whether or not Assumption 1 is satisfied.

[9] Kuzmics and Steg (2017) treat the case $m = 1$ and focus on non-anonymous mechanisms. Bierbrauer and Hellwig (2016), again for $m = 1$, invoke an additional requirement of coalition-proofness in their proof that every admissible mechanism involves equal cost sharing.

By Lemma 1 and Assumption 1 individual *i*'s preferences over the outcomes $k' \in K'$ of the mechanism can be represented by the utility function

$$\hat{u}_i(\theta) \;=\; \sum_{l=1}^{k'} (\theta_{il} - 1) \;\; . \tag{2.5}$$

According to the impossibility result by Gibbard (1973) and Satterthwaite (1975), with a universal domain of preferences, any incentive compatible mechanism that has more than two outcomes is dictatorial and therefore violates the requirement of anonymity. By the following lemma, under Assumption 1, all logically conceivable rankings over the set of outcomes can be represented by utility functions that take the form in (2.5); i.e. the universal domain property is satisfied.

Lemma 2. *Let $\mathscr{R}$ be the set of preference relations over the set of outcomes K'. To every $\succ_i \in \mathscr{R}$ there exists a type $\theta_i \in \Theta_i$ so that, for any $k, k' \in K'$, $k' \succ_i k$ if and only if*

$$\sum_{l=1}^{k'} (\theta_{il} - 1) \;>\; \sum_{l=1}^{k} (\theta_{il} - 1).$$

Corollary 1. *Under Assumption 1, admissible mechanisms have at most two outcomes.*

The only way in which we can satisfy the individuals' participation constraints is to have the status quo as one of these two outcomes. Thus, the specification of the alternative to the status quo is only one degree of freedom that is left; i.e. the class of admissible mechanisms is of the form *nothing or all public goods with an index below k'*. Let $S(k')$ be the expected surplus that is generated by such a mechanism. By the following Lemma, the surplus is strictly increases in this index, with the implication that the *all-or-nothing-mechanism* is the optimal mechanism.

Lemma 3. *Let f be symmetric and log-concave. Then, for any k', $S(k') < S(k'+1)$.*

2.4.2 On Assumption 1

The universal domain property is needed to justify our use of the Gibbard and Satterthwaite theorem. Assumption 1 ensures that we can satisfy this property by focussing on a simple class of utility functions, $\hat{u}_i(\theta) = \sum_{l=1}^{k'} (\theta_{il} - 1)$. In the Supplement, we present an example that illustrates that, without this Assumption, there are preference profiles that cannot be represented by an additively separable utility function. If we do not impose Assumption 1, we have to consider a richer class of preferences to satisfy the universal domain property. Once such preferences are allowed for, we can again appeal to the Gibbard and Satterthwaite theorem and focus on mechanisms with at most two outcomes. With Assumption 1, the welfare comparison of all these mechanisms becomes more tractable.

2.5 Concluding Remarks

We have shown that bundling many public goods facilitates public goods provision in the presence of incentive and participation constraints. Additional public goods come with additional resource requirements. Thus, sufficient state capacity is necessary to reap the benefits from bundling. If bundling is not an option, as Mailath and Postlewaite (1990) have shown, it is impossible to have positive provision levels - unless the government uses its coercive power to collect contributions from individuals who do not value the public good. This also points to a potential drawback of deciding about every public project on a stand-alone-basis. If the benefits from bundling remain unused, there will be an underprovision of public goods if participation constraints are respected, or, if they are not respected, public goods provision will be unnecessarily controversial as it will create winners and losers.

References

Wagner, Adolph. 1883. *Finanzwissenschaft*. Vol. 4. 1. CF Winter. [6]

Proschan, Frank. 1965. "Peakedness of distributions of convex combinations." *Annals of Mathematical Statistics* 36 (6): 1703–6. [17]

Gibbard, Allen. 1973. "Manipulation of voting schemes: A general result." *Econometrica* 41 (4): 587–601. [7, 14]

Satterthwaite, Mark A. 1975. "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory* 10 (2): 187–217. [7, 14]

Green, Jerry, and Jean-Jacques Laffont. 1977. "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods." *Econometrica* 45 (2): 427–38. [8]

d'Aspremont, Claude, and Louis-André Gérard-Varet. 1979. "Incentives and incomplete information." *Journal of Public Economics* 11 (1): 25–45. [8]

Güth, Werner, and Martin Hellwig. 1986. "The Private Supply of a Public Good." *Journal of Economics* Supplement 5: 121–59. [8]

Crémer, Jacques, and Richard McLean. 1988. "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions." *Econometrica* 56: 1247–57. [8]

Mailath, George J., and Andrew Postlewaite. 1990. "Asymmetric information bargaining problems with many agents." *Review of Economic Studies* 57 (3): 351–67. [6–8, 11, 15]

Hellwig, Martin. 2003. "Public-Good Provision with Many Participants." *Review of Economic Studies* 70: 589–614. [8]

Norman, Peter. 2004. "Efficient Mechanisms for Public Goods with Use Exclusion." *Review of Economic Studies* 71: 1163–88. [8]

Bagnoli, Mark, and Ted Bergstrom. 2005. "Log-concave probability and its applications." *Economic theory* 26 (2): 445–69. [11]

Bergemann, Dirk, and Stephen Morris. 2005. "Robust Mechanism Design." *Econometrica* 73: 1771–813. [7, 10]

Fang, Hanming, and Peter Norman. 2006. "Overcoming participation constraints." *Unpublished manuscript*, [8]

Jackson, Matthew O, and Hugo Sonnenschein. 2007. "Overcoming Incentive Constraints by Linking Decisions." *Econometrica* 75 (1): 241–57. [8]

Besley, Timothy, and Torsten Persson. 2009. "The Origins of State Capacity: Property Rights, Taxation, and Politics." *American Economic Review* 99 (4): 1218–44. [5]

Bierbrauer, Felix J., and Martin F. Hellwig. 2016. "Robustly coalition-proof incentive mechanisms for public good provision are voting mechanisms and vice versa." *Review of Economic Studies* 83 (4): 1440–64. [13]

Kuzmics, Christoph, and Jan-Henrik Steg. 2017. "On public good provision mechanisms with dominant strategies and balanced budget." *Journal of Economic Theory* 170: 56–69. [13]

2.A Proofs

2.A.1 Proof of Proposition 1

We seek to show that the probability of the event "$\exists i \in I$: $\frac{1}{m}\sum_{k=1}^{m}\theta_{ik} < 1$" is smaller than the probability of the event " $\exists i \in I$: $\frac{1}{m+1}\sum_{k=1}^{m+1}\theta_{ik} < 1$." Since preferences are *iid*, this holds if and only if, for any given individual i, the probability of $\frac{1}{m}\sum_{k=1}^{m}\theta_{ik} < 1$ is smaller than the probability of $\frac{1}{m+1}\sum_{k=1}^{m+1}\theta_{ik} < 1$. As an implication of Corollary 2.1 in Proschan (1965), the probability of an event $\frac{1}{m}\sum_{k=1}^{m}\theta_{ik} < x$, where $x < \mu$ is strictly decreasing in m.[10] The proposition follows from this fact upon setting $x = 1$.

2.A.2 Proof of Lemma 1

We occasionally use $q(\theta)$ as a shorthand for $\{q_k(\theta)\}_{k\in K}$. Moreover, we will use $v(\theta_i, q(\theta))$ as a shorthand for $\sum_{k\in K}\theta_{ik}\, q_k(\theta)$. For a given state θ, we write $K_0(\theta) = \{k \mid q_k(\theta) = 0\}$ for the set of projects that are not implemented and, analogously, $K_1(\theta) = \{k \mid q_k(\theta) = 1\}$ for the complementary set. Also, for any k, we write $\underline{\theta}_k(\theta) = \min_{i\in I}\ \theta_{ik}$ and $\overline{\theta}_k(\theta) = \max_{i\in I}\ \theta_{ik}$. If this creates no confusion, we will occasionally suppress the dependence of these minima and maxima on the state θ and simply write $\underline{\theta}_k$ and $\overline{\theta}_k$. The following lemma will also prove useful.

Lemma 4. *Consider two states θ and θ' such that the following holds:*

i) $\theta'_{-i} = \theta_{-i}$,

ii) $\theta'_{ik} > \theta_{ik}$ *for all k with* $q_k(\theta) = 1$,

iii) $\theta'_{ik} < \theta_{ik}$ *for all k with* $q_k(\theta) = 0$.

Then, for all k, $q_k(\theta') = q_k(\theta)$ and $t_i(\theta) = t_i(\theta')$.

Proof. The incentive constraints for individual i in state θ' imply

$$t_i(\theta) - t_i(\theta') \ \geq\ v(\theta'_i, q(\theta)) - v(\theta'_i, q(\theta'))\,. \tag{2.6}$$

Note that

[10] Proschan refers to distributions with a log-concave density as *Polya frequency functions of order 2*.

$$\begin{aligned} v(\theta_i', q(\theta)) - v(\theta_i', q(\theta')) &= \sum_{k\in K} \theta_{ik}'(q_k(\theta) - q_k(\theta')) \\ &= \sum_{k\in K} \theta_{ik}(q_k(\theta) - q_k(\theta')) \\ &\quad + \sum_{k\in K_1(\theta)} (\theta_{ik}' - \theta_{ik})(1 - q_k(\theta')) \\ &\quad + \sum_{k\in K_0(\theta)} (\theta_{ik}' - \theta_{ik})(0 - q_k(\theta')) \\ &\geq v(\theta_i, q(\theta)) - v(\theta_i, q(\theta')). \end{aligned}$$

Moreover,

$$v(\theta_i', q(\theta)) - v(\theta_i', q(\theta')) \;>\; v(\theta_i, q(\theta)) - v(\theta_i, q(\theta')) \quad , \tag{2.7}$$

if there is $k \in K_1(\theta)$ with $q_k(\theta') = 0$ or $k \in K_0(\theta)$ with $q_k(\theta') = 1$. Suppose in the following that this is the case. Then, inequalities (2.6) and (2.7) imply that

$$t_i(\theta) - t_i(\theta') \;>\; v(\theta_i, q(\theta)) - v(\theta_i, q(\theta')).$$

Hence, a violation of incentive compatibility for individual i in state θ'. Thus, the assumption that there is $k \in K_1(\theta)$ with $q_k(\theta') = 0$ or $k \in K_0(\theta)$ with $q_k(\theta') = 1$ has led to a contradiction and must be false. Hence, for all k, $q_k(\theta) = q_k(\theta')$. It remains to be shown that $t_i(\theta) = t_i(\theta')$. With $q(\theta) = q(\theta')$, (2.6) becomes

$$t_i(\theta) - t_i(\theta') \;\geq\; 0 \,. \tag{2.8}$$

Analogously, the incentive constraint $t_i(\theta) - t_i(\theta') \leq v(\theta_i, q(\theta)) - v(\theta_i, q(\theta'))$ becomes

$$t_i(\theta) - t_i(\theta') \;\leq\; 0 \,. \tag{2.9}$$

Inequalities (2.8) and (2.9) imply $t_i(\theta) = t_i(\theta')$. □

2.A.2.0.1 Proof of Lemma 1. Consider a state θ and suppose that there exist individuals i and i' with $t_i(\theta) \neq t_{i'}(\theta)$. We show that this leads to a contradiction to the assumption that the given mechanism is anonymous, incentive compatible and satisfies ex post budget balance. Assume without loss of generality that $t_i(\theta) > \sum_{k\in K} q_k(\theta)$. We construct state θ' so that

i) $\theta_{-i}' = \theta_{-i}$,

ii) $\theta_{ik}' = \overline{\theta}_k$ for all $k \in K_1(\theta)$,

iii) $\theta_{ik}' = \underline{\theta}_k$ for all $k \in K_0(\theta)$.

By Lemma 4, $q(\theta) = q(\theta')$ and $t_i(\theta) = t_i(\theta')$. Therefore, $t_i(\theta') > \sum_{k\in K} q_k(\theta')$ and there must exist an individual i' with $t_{i'}(\theta') < \sum_{k\in K} q_k(\theta')$. Otherwise there would be a budget surplus in state θ'. We now construct state θ'' so that

i) $\theta''_{-i} = \theta'_{-i}$,

ii) $\theta''_{ik} = \overline{\theta}_k$ for all $k \in K_1(\theta)$,

iii) $\theta''_{ik} = \underline{\theta}_k$ for all $k \in K_0(\theta)$.

Again, by Lemma 4, $q(\theta') = q(\theta'')$ and $t_{i'}(\theta') = t_{i'}(\theta'')$. Also, by anonymity, $t_i(\theta'') = t_{i'}(\theta'')$. Since $t_i(\theta'') < \sum_{k\in K} q_k(\theta'')$ there must exist $i'' \neq i, i'$ with $t_{i''}(\theta'') > \sum_{k\in K} q_k(\theta'')$. Otherwise there would be budget deficit.

We now repeat this construction until we have a state $\theta^{(n)}$ so that all individuals have the same type, i.e. so that for all $\iota \in I$, $\theta^{(n)}_{\iota k} = \overline{\theta}_k$ for all $k \in K_1(\theta)$ and $\theta^{(n)}_{\iota k} = \underline{\theta}_k$ for all $k \in K_0(\theta)$. By anonymity $t_\iota(\theta^{(n)}) = t_{\iota'}(\theta^{(n)})$, for all ι and ι'. By the arguments above, we either have $t_\iota(\theta^{(n)}) > \sum_{k\in K} q_k(\theta^{(n)})$ or $t_\iota(\theta^{(n)}) < \sum_{k\in K} q_k(\theta^{(n)})$ in this state, a contradiction to budget balance.

2.A.3 Proof of Lemma 2

Given a preference relation $\succ_i$ over K' denote by $r(\succ_i, k)$ the rank of alternative k. Hence, $k' \succ_i k$ if and only if $r(\succ_i, k') < r(\succ_i, k)$. To construct the corresponding type θ_i, we let $\theta_{ik} = d(\succ_i, k) + 1$ where $d(\succ_i, k)$ is the rank difference of two neighbouring alternatives, $d(\succ_i, k) = r(\succ_i, k-1) - r(\succ_i, k)$. We now show that $r(\succ_i, k') < r(\succ_i, k)$ if and only if $\sum_{l=1}^{k'}(\theta_{il} - 1) > \sum_{l=1}^{k}(\theta_{il} - 1)$. To see that this is the case, suppose that $k' > k$ (the case $k' < k$ is analogous) and note that, by construction,

$$
\begin{aligned}
& & \sum_{l=1}^{k'}(\theta_{il} - 1) &> \sum_{l=1}^{k}(\theta_{il} - 1) \\
\Leftrightarrow & & \sum_{l=k+1}^{k'} \theta_{il} &> k' - k \\
\Leftrightarrow & & \sum_{l=k+1}^{k'} (d(\succ_i, l) + 1) &> k' - k \\
\Leftrightarrow & & \sum_{l=k+1}^{k'} d(\succ_i, l) &> 0 \\
\Leftrightarrow & & \sum_{l=k+1}^{k'} r(\succ_i, l-1) - r(\succ_i, l) &> 0 \\
\Leftrightarrow & & r(\succ_i, k) &> r(\succ_i, k').
\end{aligned}
$$

2.A.4 Proof of Lemma 3

Denote by $p_{no}(k')$ the probability that any one individual i opts for the status-quo-outcome – i.e. the probability of the event $\sum_{l=1}^{k'}(\theta_{il}-1)<0$ – under a *nothing or all public goods with an index below k′ mechanism*. From the arguments in the proof of Proposition 1,

$$p_{no}(k') \;<\; p_{no}(k'+1)\ \ . \tag{2.10}$$

Also note that

$$\begin{aligned}
& S(k'+1)\\
&= E\Bigg[\frac{1}{n}\sum_{i=1}^{n}\sum_{l=1}^{k'+1}(\theta_{il}-1)\\
&\quad \cdot \mathbf{1}\left(\sum_{l=1}^{k'+1}(\theta_{il}-1)\;\geq\;0 \quad \text{and} \quad \forall j\neq i, \sum_{l=1}^{k'+1}(\theta_{jl}-1)\;\geq\;0\right)\Bigg]\\
&= p_{no}(k'+1)^{n-1}\frac{1}{n}\sum_{i=1}^{n}E\left[\sum_{l=1}^{k'+1}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'+1}(\theta_{il}-1)\;\geq\;0\right)\right].
\end{aligned} \tag{2.11}$$

where $\mathbf{1}$ is the indicator function. Moreover,

$$\begin{aligned}
& E\left[\sum_{l=1}^{k'+1}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'+1}(\theta_{il}-1)\;\geq\;0\right)\right]\\
&\geq E\left[\sum_{l=1}^{k'+1}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'+1}(\theta_{il}-1)\;\geq\;0 \quad \text{and} \quad \sum_{l=1}^{k'}(\theta_{il}-1)\;\geq\;0\right)\right]\\
&\geq E\left[\sum_{l=1}^{k'+1}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'}(\theta_{il}-1)\;\geq\;0\right)\right].
\end{aligned} \tag{2.12}$$

The first inequality holds because the second expression looks at a smaller set of events among those that satisfy $\sum_{l=1}^{k'+1}(\theta_{il}-1)\geq 0$. The second inequality holds because the sum in the third expression is now both over events with $\sum_{l=1}^{k'+1}(\theta_{il}-1)\geq 0$ and over events with $\sum_{l=1}^{k'+1}(\theta_{il}-1)<0$, among those that satisfy $\sum_{l=1}^{k'}(\theta_{il}-1)\geq 0$. We now rewrite this last expression as

$$
\begin{aligned}
& E\left[\sum_{l=1}^{k'+1}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'}(\theta_{il}-1)\ \geq\ 0\right)\right] \\
&= p_{no}(k')E[\theta_{ik'+1}-1] + E\left[\sum_{l=1}^{k'}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'}(\theta_{il}-1)\ \geq\ 0\right)\right] \\
&= p_{no}(k')(\mu-1) + E\left[\sum_{l=1}^{k'}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'}(\theta_{il}-1)\ \geq\ 0\right)\right] \\
&> E\left[\sum_{l=1}^{k'}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'}(\theta_{il}-1)\ \geq\ 0\right)\right]. \qquad (2.13)
\end{aligned}
$$

Equation (2.11) and the inequalities (2.10), (2.12) and (2.13) imply

$$
\begin{aligned}
S(k'+1) \quad &> \quad p_{no}(k')^{n-1}\frac{1}{n}\sum_{i=1}^{n}E\left[\sum_{l=1}^{k'}(\theta_{il}-1)\mathbf{1}\left(\sum_{l=1}^{k'}(\theta_{il}-1)\ \geq\ 0\right)\right] \\
&= \quad S(k').
\end{aligned}
$$

2.B Supplement

2.B.1 Proof of Proposition 2

We start by bounding the probability that any individual vetoes against the all-outcome with Chebychef's inequality

$$\mathbb{P}\left[\frac{1}{m}\sum_{k=1}^{m}\theta_{ik} \;<\; 1\right] \;\leq\; \mathbb{P}\left[\left|\mu - \frac{1}{m}\sum_{k=1}^{m}\theta_{ik}\right| \;>\; \mu - 1\right] \;\leq\; \frac{\sigma^2}{(\mu-1)^2}\frac{1}{m}. \tag{2.14}$$

Using Inequality (2.14) we bound the probability that no one vetoes from below

$$\begin{aligned}\mathbb{P}_{all}(m) &= \mathbb{P}[\text{no individual vetoes the provision of } m \text{ public projects}]\\ &= \big(1 - \mathbb{P}[\text{i vetoes the provision of } m \text{ public projects}]\big)^n\\ &\geq \left(1 - \frac{\sigma^2}{(\mu-1)^2}\frac{1}{m}\right)^n.\end{aligned}$$

The lower bound goes to 1 as $m \to \infty$. Hence, $\lim_{m\to\infty} \mathbb{P}_{all}(m) = 1$.

2.B.2 Proof of Proposition 3

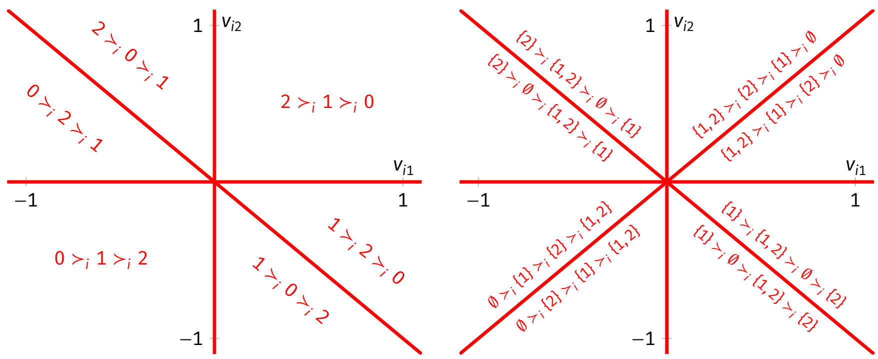
We adapt the argument in the proof of Proposition 2. There it was shown that $\mathbb{P}_{all}(m) \geq \big(1 - \frac{d}{m}\big)^{c\,m}$ for $d = \frac{\sigma^2}{(\mu-1)^2}$. The right hand side of this inequality converges to $e^{-c\,d} > 0$ as $m \to \infty$.

2.B.3 Limit Probability as $\frac{m}{n}$ Becomes Unbounded

Define $h(m) = \frac{n}{m}$. We seek to show that $\mathbb{P}_{all}(m)$ converges to 1 as $h(m)$ converges to 0. We again adapt the argument in the proof of Proposition 2 where it was shown that $\mathbb{P}_{all}(m) \geq \big(1 - \frac{d}{m}\big)^{h(m)\,m}$ for $d = \frac{\sigma^2}{(\mu-1)^2}$. Since $\big(1 - \frac{d}{m}\big)^m$ is an increasing function of m, it is, for all m, (weakly) larger than $1 - d$. Hence,

$$\mathbb{P}_{all}(m) \;\geq\; (1-d)^{h(m)}.$$

The right hand side of this inequality goes to 1 as $h(m) \to 0$.



(a) Assumption 1 **(b)** No restriction

Figure 2.1. Example of preference relations

2.B.4 An Example Illustrating Assumption 1

Consider the case with two public projects that is illustrated in Figure 2.1. For ease of exposition, the figure shows valuations net of the per capita provision costs, $v_{ik} = \theta_{ik} - 1$. With Assumption 1 there are three outcomes: no provision, provision of good 1, and the provision of goods 1 and 2. The number of possible preference orderings over these three outcomes equals $3! = 6$. Figure 2.1(a) illustrates that for every preference ordering $\succ_i$ there is some type θ_i that induces it. For instance, valuations in the upper right quadrant give rise to the following ranking: providing two public goods is preferred over providing one public good. Providing one public good in turn is preferred over no provision at all. As the Figure shows, any one of the 6 possible preference profiles corresponds to some region in Figure 2.1(a). Without Assumption 1, a fourth outcome comes into play, namely to provide the public good with index $k = 2$, but not to provide the public good with index 1. There are now $4! = 24$ preference orderings over these outcomes. As Figure 2.1(b) shows, only eight of these preference relations can be represented in the given type space. For example, a preference relation so that $\{2\} \succ_i \{1\} \succ_i \emptyset \succ_i \{1,2\}$ is incompatible with it.

3

Ranking Mechanisms for Coupled Binary Decisions

Joint with Kilian Russ

3.1 Introduction

Almost all collective decisions in society – be it in committees, parliaments or referenda – are made by means of (simple) Majority Rule. We base decisions on how *many* individuals favor or oppose a reform rather than how *much* everyone cares. This blindness to preference intensities casts doubt on the efficiency of voting as an aggregation mechanism. Consider a scenario in which 49% of individuals oppose a proposed reform with drastic consequences for each individual. A majority of 51% of people marginally benefits and therefore supports the reform. Nevertheless it seems sensible to decide in favor of the minority. This inherent weakness of direct democracy based on Majority Rule has long been recognized as the *Tyranny of the Majority* (De Tocqueville (1835)).

Advocates of Majority Voting point out several desirable properties. First, Majority Voting takes everyone's opinion into account and treats them equally. Second, Majority Voting respects consensus. Third, Majority Voting provides individuals with an incentive to reveal their preferences truthfully. For instance, if one were to naively ask how much everyone cares about a given reform, individuals would certainly like to exaggerate their feelings to sway the decision in their favor regardless of how much they actually cared. So is it at all possible to elicit preference intensities truthfully while maintaining all desirable properties of Majority Voting?

This paper gives an affirmative answer to this question for the practically relevant class of coupled binary decisions. Consider a set of agents who face a fixed agenda of several reforms that have to be approved or rejected. We study a class of Ranking Mechanisms which are sensitive to preference intensities while maintaining all desirable properties mentioned above. Agents communicate which alternative they prefer in each decision problem. Additionally, they report a priority ranking over decision problems by ranking each problem according to the absolute difference in utilities between the reform and the status quo. These rankings are then used to assign weights to agents' votes in a voting mechanism. In each decision problem the reform is implemented if and only if the sum of weighted votes in favor of implementation outweighs the one supporting the status quo. Any Ranking Mechanism is thus implementable as a weighted voting procedure. Rather than deciding upon each reform separately a Ranking Mechanism makes use of the linkage of problems by eliciting cardinal information of preferences. In particular, the approval of any reform depends on *both* the number of agents in favor and their relative preference intensity towards the issue (as reflected by the rank they assigned to that particular problem).

This paper establishes two sets of results. For the case of identical decision problems we prove that the sincere strategy is a Bayes-Nash equilibrium of any Ranking Mechanism. Agents find it optimal to rank problems according to the absolute difference in utilities between the two alternatives as long as all other agents do the same. We then maximize over the class of Ranking Mechanisms and derive a closed form solution for the ex-ante efficient weight vector. The optimal Ranking Mechanism ex-ante Pareto dominates Separate Majority Voting for arbitrary number of agents and decision problems. Further, it achieves full efficiency in the limit as the number of decision problems tends to infinity.

We then extend our idea of ranking to non-identical decision problems. We propose a generalized class of Randomized Ranking Mechanisms all of which induce sincere equilibrium behavior. Intuitively, randomization is such that from the perspective of all other agents each agent reports every priority ranking with equal probability. We derive the ex-ante efficient Randomized Ranking Mechanism and provide a closed form solution for the optimal weight vector. The optimal Randomized Ranking Mechanism ex-ante Pareto dominates Separate Majority Voting for any number of agents and decision problems.

The optimal (Randomized) Ranking Mechanism respects both anonymity and unanimity. Moreover, under mild conditions it allows for strong minorities to overturn weak majorities and therefore mitigates the *Tyranny of the Majority*. Our proposed mechanism is the first to successfully link both identical and non-identical decision problem for any number of agents and problems.

Further we document that the picture changes under a different equilibrium concept. Building on a result by Hortala-Vallve (2010) we show that under mild conditions Separate Majority Voting is ex-ante Pareto efficient in the class of strategy-proof mechanisms. In other words, the requirement that agents find truth-telling a dominant strategy makes it impossible to exploit the coupled structure and precludes any use of cardinal information.

The rest of the paper is organized as follows. Section 3.2 reviews the existing literature. The formal model is presented in Section 3.3. Section 3.4 presents two impossibility results and may be seen as a theoretical justification for our interest in the topic. In Section 3.5 we study the class of Ranking Mechanisms linking identical decision problems. We generalize our results to non-identical decision problems in Section 3.6. Section 3.7 concludes.

3.2 Related Literature

The conceptual idea of evaluating efficiency of voting rules in terms of ex-ante expected welfare goes back to Rae (1969). Our work adds to a series of recent papers studying the ex-ante welfare properties of voting schemes in environments with cardinal preferences (Gershkov, Moldovanu, and Shi (2017), Kim (2017) among others).

The traditional literature on social choice has focused on environments with ordinal preferences over alternatives. Classical impossibility results include the famous *Gibbard-Satterthwaite-Theorem* (Gibbard (1973), Satterthwaite (1975)) and subsequent work demonstrating its robustness on cardinal type spaces with respect to randomization (Hylland (1980)) and Bayesian implementation of ordinal mechanisms (Majumdar and Sen (2004)).

Coupling multiple decisions alone is not sufficient to overcome impossibility results. Barberà, Sonnenschein, and Zhou (1991) study a setting in which agents have separable, ordinal preferences over subsets of objects. In our terminology, an object is a decision problem which is contained in the subset if and only if the reform in that decision problem is implemented. Their main result characterizes the set of strategy-proof mechanisms and implies that only the most preferred subset of each voter can be elicited truthfully. In a cardinal framework with a finite number of binary decisions Hortala-Vallve (2010) shows that any strategy-proof mechanism cannot be both unanimous and sensitive to preference intensities. As shown in Section 3.4 his results imply that (i) Separate Majority Rule is ex-ante efficient in the class of strategy-proof mechanisms and (ii) full efficiency remains unachievable among incentive compatible mechanisms for any finite number of decisions.

We are not the first to show that coupling decision problems may improve efficiency under Bayesian implementation. There exist voting mechanisms that are sensitive to cardinal intensities and Pareto improve upon ordinal mechanisms such as Separate Majority Voting. Most notable examples thereof are a *Rationing Procedure* by Jackson and Sonnenschein (2007), a *Simple Scheme* by Casella and Gelman (2008) and *Qualitative Voting* by Hortala-Vallve (2012).

Jackson and Sonnenschein (2007) demonstrate that as the number of identical decision problems tends to infinity full efficiency is achievable. Their Rationing Procedure works as follows. For any number of decision problems an agent announces his utility type directly, but he has to ration his reported type so as to match the underlying distribution as best as possible. The mechanism then picks the alternative that maximizes reported welfare. However, their Rationing Procedure crucially relies on identical type distributions across problems. It does not readily extend to a finite number of decision problems or continuous type spaces and exact equilibrium strategies are unknown. Our Ranking Mechanism also achieves full efficiency in the limit while simultaneously admitting intuitive equilibrium behavior and welfare improvements for any finite number of decision problems.

In Casella and Gelman (2008), agents are endowed with a single bonus vote, which can be cast in addition to regular votes. The decision is made according to the sum of votes cast for each alternative. Their main result proves that in large populations the Simple Scheme improves upon Separate Majority Voting for small enough bonus votes. Although they restrict attention to large populations our Proposition 6 implies that casting the bonus vote on the decision problem with highest difference in utilities remains an equilibrium for any number of agents. Casella and Gelman (2008) generalize their results to non-identical type distributions across problems but not agents.

Hortala-Vallve (2012) proposes another intuitive voting procedure. Agents are endowed with a fixed number of votes that can be distributed freely among alternatives and problems. A reform is accepted if the total number of votes supporting the reform is larger than the number of votes against it. The main result of the paper shows that in settings with 2 or 3 agents with 4 possible valuations and 2 decision problems Qualitative Voting is ex-ante efficient. Another mechanism motivating much of the recent literature on coupled binary decisions is the Storable Votes procedure by Casella (2005), which applies to a dynamic setup of a committee meeting regularly over time.

While there has been considerable effort the literature has not yet proposed a mechanism which is *both* intuitive and predictable – at least for the practically relevant case of finitely many problems and agents. Simplicity as well as predictability are prerequisites for any real world application. Our paper fills this gap and applies the idea of ranking to a social choice setting without monetary transfers, namely coupled binary decisions.[1]

Our results for the optimal (Randomized) Ranking Mechanism identify expected order statistics as important moments of the underlying type distribution. In this spirit our work is related to a recent paper by Kim (2017), who studies a social choice problem with K alternatives. Kim (2017) proposes a mecha-

[1] The idea of ranking alternatives or objects has also been studied in the multidimensional cheap talk literature, for example in the context of coordination in auctions Campbell (1998) and Pesendorfer (2000) or biased expert advise Chakraborty and Harbaugh (2007) among others.

nism based on expected order statistics which improves upon Majority Voting by partly eliciting cardinal information on preferences. However, Kim's mechanism is not applicable in our setting due to the impossibility result by Majumdar and Sen (2004). Relatedly, Apesteguia, Ballester, and Ferrer (2011) rely on expected order statistics to characterize the ordinal mechanism that maximizes ex-ante expected utility in a social choice problem with cardinal preferences. In contrast to our work, Apesteguia, Ballester, and Ferrer (2011) abstract from incentive considerations.

3.3 The Model

There are $n \in \mathbb{N}$ agents, who have to decide on $d \in \mathbb{N}$ binary decisions. Each decision problem $k \in D = \{1, \ldots, d\}$ consists of two alternatives $\{0, 1\}$. We interpret 0 as maintaining the status quo and 1 as implementing a reform. The overall outcome is a vector $x \in X = [0, 1]^d$ where the kth component x_k represents the probability of implementing the reform in decision $k \in D$.

We normalize the utility of maintaining the status quo to 0 for every agent and every decision problem. Each agent $i \in N = \{1, \ldots, n\}$ draws a private von Neumann-Morgenstern utility vector (or type) $u_i = (u_i^1, \ldots, u_i^d)$ representing his cardinal utility if the reform is implemented in each of the different decision problems. We refer to the sign of u_i^k as agent i's ordinal type and to $|u_i^k|$ as his preference intensity in decision k. Throughout the paper we refer to u_i^k as the random variable and its realization interchangeably. The random variable u_i^k takes on values in $U_i^k \subset \mathbb{R}$ and is independently distributed between agents and across problems. Formally, u_i^k is independent of u_j^l for all $i, j \in N$ with $i \neq j$ and all $k, l \in D$ with $k \neq l$. It has a finite first absolute moment and its continuous pdf ρ_i^k is symmetric around zero. For notational convenience let $\mathscr{U}_i = (U_i^1, \ldots, U_i^d)$, $\mathscr{U} = (\mathscr{U}_i)_{i \in N}$ and $\mathscr{U}_{-i} = (\mathscr{U}_1, \ldots, \mathscr{U}_{i-1}, \mathscr{U}_{i+1}, \ldots, \mathscr{U}_n)$.

The distribution of types is common knowledge among agents. We assume that agents' utility is separable across problems and write the overall utility of agent i with utility type $u_i \in \mathscr{U}_i$ for outcome $x \in X$ as $V_i(x) = \sum_{k=1}^{d} u_i^k \cdot x_k$.[2] The above environment is entirely separable implying that there is no a priori reason to link decision problems at all.

An indirect mechanism $G = (\mathscr{M}, g(\cdot))$ consists of a message space $\mathscr{M} = M^{\otimes n}$, which encompasses a message or action set M for each agent and a decision rule $g : \mathscr{M} \to X$, which maps into the set of possible outcomes. Again, we refer to g_k as the random outcome in decision k as well as to its realization. Unless made

[2] We follow most of the literature by assuming that preferences are additively separable across problems. Ahn and Oliveros (2012) demonstrate the importance of the separability assumption for equilibrium predictions even under Separate Majority Voting.

explicit all expectation operators are meant to include the randomness of the mechanism. We restrict attention to mechanisms that treat all agents equally.

Definition 1. *An indirect mechanism $G = (\mathscr{M}, g)$ is anonymous if for all permutations σ on N and all $m \in \mathscr{M}$ it holds that $g(m_1, ..., m_n) = g(m_{\sigma(1)}, ..., m_{\sigma(n)})$.*

Agent i's strategy $s_i : \mathscr{U}_i \to M$ maps i's utility vector into a message in his action set. A collection of strategies for all agents $s = (s_1, ..., s_n)$ is called a strategy profile and the strategy profile of all but agent i is denoted by s_{-i}. Agent i evaluates a strategy s_i given an indirect mechanism $(\mathscr{M}, g)$ and a strategy profile of the other agents s_{-i} by taking expectations over all other agents' utility types (and the potentially random mechanism), i.e. according to $\mathbb{E}_{-i}[V_i(g(s_i, s_{-i}))]$.[3]

Definition 2. *A strategy profile $\hat{s}$ is a Bayes-Nash equilibrium of $(\mathscr{M}, g)$ if for every agent i the strategy $\hat{s}_i$ is in expectation a best response to the strategy profile $\hat{s}_{-i}$ of the other agents. Formally, $\mathbb{E}_{-i}[V_i(g(\hat{s}_i, \hat{s}_{-i}))] \geq \mathbb{E}_{-i}[V_i(g(s_i, \hat{s}_{-i}))]$ for all i and s_i.*

A mechanism is direct if $\mathscr{M} = \mathscr{U}$, i.e. agents report their utility type directly.

Definition 3. *A direct mechanism $(\mathscr{U}, g)$ is*

1. *strategy-proof, if for every agent i with type u_i the truthful strategy is a best response to any strategy profile of all other agents. Formally, $V_i(g(u_i, u_{-i})) \geq V_i(g(\tilde{u}_i, u_{-i}))$ for all i, u_i, $\tilde{u}_i$ and u_{-i}.*
2. *incentive compatible, if for every agent i with type u_i the truthful strategy is in expectation a best response to the truthful strategy profile of all other agents. Formally, $\mathbb{E}_{-i}[V_i(g(u_i, u_{-i}))] \geq \mathbb{E}_{-i}[V_i(g(\tilde{u}_i, u_{-i}))]$ for all i, u_i and $\tilde{u}_i$.*

While the revelation principle guarantees that there is theoretically no loss in restricting attention to direct mechanisms, it might still be simpler to communicate indirect mechanisms in practice. The Ranking Mechanisms we introduce below are examples for which an indirect representation facilitates understanding and offers an intuitive implementation.

Throughout the paper we measure efficiency at the ex-ante stage. Agent i's ex-ante expected utility under a mechanism $(\mathscr{M}, g)$ and strategy profile s is $\mathbb{E}[V_i(g(s))]$ where the expectation is taken w.r.t. to all random variables.

Definition 4. *A mechanism (ex-ante Pareto) dominates another mechanism if it generates at least as high levels of ex-ante expected utility for all agents. A mechanism is (ex-ante Pareto) efficient if it is not dominated.*

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[3] Throughout $\mathbb{E}_i$ and $\mathbb{E}_{-i}$ denote expectations taken with respect to all random variables with subscript i and subscripts $j \neq i$ (including the randomness of the mechanism), respectively.

Full efficiency refers to the highest ex-ante utility level achievable under any (not necessarily incentive compatible) mechanism. Ex-ante expected welfare is the sum over all agents' ex-ante expected utility levels. A necessary requirement for ex-ante efficiency is unanimity.

Definition 5. *A mechanism is unanimous if in every decision problem it implements the alternative preferred by all agents whenever such alternative exists.*

One unanimous mechanism is Separate Majority Voting, which serves as a benchmark throughout this work. Every agent casts a single vote on every problem and the decision is made by simple Majority Rule or in case of a tie by a fair coin toss separately for each problem. Separate Majority Voting is strategy-proof but makes no use of the fact that there are multiple problems.

3.4 Impossibility Results

The first result of this section shows that if one restricts attention to strategy-proof mechanisms Separate Majority Voting is ex-ante Pareto efficient. This result is a consequence of an impossibility result by Hortala-Vallve (2010). We borrow the following definition.

Definition 6. *(Hortala-Vallve (2010)) The preference domain is unrestricted if there exists* $\epsilon > 0$ *such that* $(-\epsilon, \epsilon) \subseteq U_i^k$ *for all* $i \in N$ *and all* $k \in D$.

With this definition we have the following proposition.

Proposition 4. *Among anonymous, strategy-proof mechanisms Separate Majority Voting is efficient in an unrestricted domain.*

Proposition 4 follows from the impossibility result established in Hortala-Vallve (2010): Among strategy-proof mechanisms unanimity implies non-sensitivity and separability. A mechanism is separable on d coupled decision problems if the outcome implemented in each decision problem only depends on agents' utilities for that problem. A mechanism is sensitive if there exist two utility profiles of the same ordinal type but with different intensities, which result yet in a different outcome for at least one decision problem. In other words, any strategy-proof and unanimous mechanism elicits only *ordinal* types and cannot *link* decision problems. Proposition 4 follows because on a single decision problem (Separate) Majority Voting is ex-ante efficient among all anonymous, strategy-proof mechanisms in symmetric environments.[4]

Proposition 4 implies that from an ex-ante welfare perspective there is no advantage in *linking* decision problems in the class of strategy-proof mechanisms. Moreover the efficient mechanism elicits only ordinal types. Our environment

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[4] For a general proof of the optimality of majority voting see Schmitz and Tröger (2012).

with cardinal utility and randomization allows us to work with the different implementation concept of incentive compatibility. The weaker requirement of incentive compatibility does not make the problem trivial. Incentive constraints still present a non-negligible restriction as full efficiency remains unachievable.

Proposition 5. *Full efficiency is unachievable among incentive compatible mechanisms.*

Proof. See Appendix. □

Incentive compatibility implies that only proportional types can be elicited which prevents full efficiency. Together the above results raise the following question: Is it at all possible to find an incentive compatible mechanism that improves upon Separate Majority Voting? In the remainder of this paper we give an affirmative answer to this question.

3.5 Ranking Identical Decision Problems

Our improvement upon Separate Majority Voting is centered around the intuitive idea of *ranking* decision problems. Concretely, we would like agents to not only communicate which alternative they prefer in each decision problem – as in Separate Majority Voting – but also express which problem they care most about, which second, and so on.

To formalize the idea we define the following message space.

Definition 7. *The (ranking) message space is defined as* $M = \{(a, \pi) \mid a \in \{0,1\}^d, \pi \in \sigma(D)\}$*, where* $\sigma(D)$ *denotes the set of all permutations over* D*. We denote the profile of message spaces for all agents by* $\mathcal{M} = M^{\otimes n}$*.*

Note that any message $m \in M$ can canonically be separated across problems, i.e. $m = (m^k)_{k=1,\dots d}$. For decision problem $k \in D$ we interpret message $m^k = (a^k, \pi^k)$ in two parts. The ordinal part a^k encodes whether an agent is in favor of the reform $a^k = 1$ or prefers the status quo $a^k = 0$. The cardinal part $\pi^k \in D$ corresponds to the rank an agent assigns to problem k. Importantly, an agent can assign every rank exactly once.

We are interested in eliciting *one particular* ranking over decision problems, namely, the one in which an agent ranks each decision problem according to the absolute difference in utilities between the two proposed alternatives. Using Definition 7 of the message space we formulate the following strategy.

Definition 8. *A strategy* $\overset{*}{s}_i : \mathscr{U}_i \rightarrow M$ *is sincere if agent* i *reports his favored alternative for every problem* k *and a ranking* $\overset{*}{\pi}_i$ *which sorts all problems by their preference intensities. Formally,* $\overset{*}{a}{}_i^k = 1$ *if and only if* $u_i^k > 0$ *and* $\overset{*}{\pi}{}_i^k > \overset{*}{\pi}{}_i^l$ *only if* $|u_i^k| \geq |u_i^l|$.[5]

Our goal is to induce sincere equilibrium behavior. The motivation to focus attention on the sincere strategy is twofold. First, it is intuitive and easy to understand from the perspective of an agent thereby making it a likely equilibrium outcome in practice. This is especially important because we work with the weaker Bayes-Nash equilibrium concept. Second, the sincere strategy contains *strictly* more information than what is elicited by Separate Majority Voting. Therefore the sincere strategy is a promising starting point both from a practical as well as theoretical perspective.

However, it is not obvious how to construct (non-trivial) mechanisms for which the sincere strategy profile is a Bayes-Nash equilibrium. We first present our main idea under the following simplifying assumption.

Assumption 2. *The random variable* u_i^k *is identically and independently distributed between agents and across problems.*

Section 3.6 generalizes our results to settings with different type distributions between agents and across problems. The next subsection introduces a class of simple mechanisms all of which induce sincere equilibrium behavior under Assumption 2.

3.5.1 Ranking Mechanisms

In this section we introduce the class of *Ranking Mechanisms*. Ranking Mechanisms correspond to a generalization of standard voting procedures like Separate Majority Voting. For every decision problem an agent communicates whether or not he is in favor of implementing the reform. A Ranking Mechanism then assigns a weight to every vote of an agent. For each decision problem the weight assigned to an agent's vote *solely* depends on the agent's reported rank of that particular problem. Therefore two agents may be assigned different weights in the same decision problem if they rank it differently. However, weights are not agent-specific and since every agent gets to report every rank exactly once a Ranking Mechanism remains anonymous.

Formally, we define a Ranking Mechanism $(\mathscr{M}, g^{RM,w})$ with $\mathscr{M}$ defined in Definition 7 as follows. Every agent reports an ordinal type as well as a priority ranking. The decision rule $g^{RM,w} : \mathscr{M} \rightarrow X$ is parametrized by a weight vector $w = (w^1, ..., w^d) \in W$, where $W \subset \mathbb{R}^d_{++}$ denotes the set of strictly positive weight vectors with d non-decreasing entries. For every decision problem $k \in D$ and

[5] Since types are continuously distributed the sincere strategy is unique with probability one.

every agent $i \in N$ the Ranking Mechanism $(\mathcal{M}, g^{RM,w})$ translates the report $m_i^k = (a_i^k, \pi_i^k)$ into a signed weight $(2 \cdot a_i^k - 1) \cdot w^{\pi_i^k}$. The ordinal part maps into the sign of the weight such that $a_i^k = 0, 1$ corresponds to a negative and a positive sign, respectively. The cardinal part $\pi_i^k \in D$ determines the entry of the weight vector $w \in W$. In particular, higher reported ranks map into (weakly) higher weights. Formally, we define Ranking Mechanisms as the maximization of the resulting sum of signed weights.

Definition 9. *The Ranking Mechanism $(\mathcal{M}, g^{RM,w})$ with weight vector $w \in W$ implements the outcome that maximizes the sum of signed weights. Ties are broken by a fair coin toss. Formally, the decision rule is defined as*

$$g^{RM,w}(m) \in \operatorname*{arg\,max}_{x \in \{0, \frac{1}{2}, 1\}^d} \left\{ \sum_{i=1}^{n} \sum_{k=1}^{d} \left(2 \cdot a_i^k - 1\right) \cdot w^{\pi_i^k} \cdot x_k \right\}.$$

Note that $g^{RM,w}$ is identical for multiples of w, i.e. $g^{RM,w} \equiv g^{RM,\lambda \cdot w}$ for all $\lambda > 0$.

The class of Ranking Mechanisms has several desirable properties. First, Ranking Mechanisms are both anonymous and unanimous. Second, as we show in Section 3.5.3 all Ranking Mechanisms induce sincere equilibrium behavior. Third, any Ranking Mechanism corresponds to a simple voting procedure: Every agent is endowed with d votes of pre-specified weights $w^1, ..., w^d$. Agents are allowed to cast one weighted vote in each decision problem. For every problem the alternative with the higher sum of weighted votes is implemented. Ties are broken by a fair coin toss. This interpretation offers a simple implementation of any Ranking Mechanism in practice. Further it identifies Separate Majority Voting as belonging to the class of Ranking Mechanisms with weights $(1, ..., 1)$.[6] Lastly, from a theoretical perspective the results of Section 3.5.4 imply that on the ranking message space there is no loss in restricting attention to Ranking Mechanisms. If agents report sincerely the ex-ante efficient outcome is implementable by a Ranking Mechanism. The next subsection illustrates the class of Ranking Mechanisms by means of an example.

3.5.2 Example

Distribution of Types. There are three agents, who have to decide on two binary decisions. For every decision problem we normalize every agent's utility to 0 if the status quo is maintained. We denote by u_i^k the utility of agent $i \in \{1, 2, 3\}$ in decision problem $k \in \{I, II\}$ if the corresponding reform is implemented. Utility

[6] Note that the *Simple Scheme* by Casella and Gelman (2008) also belongs to the class of Ranking Mechanisms with weights $(1, ..., 1, 1 + \theta)$.

Table 3.1. Example of a Ranking Mechanism

Decision problem	Agent 1 (10, 21)	Agent 2 (00, 21)	Agent 3 (11, 12)	Sum of weights	Outcome RM	Outcome SMV
I	+3	+3	+1	7	1	1
II	−1	−1	+3	1	1	0

u_i^k is drawn from a standard normal distribution. It may thus be positive or negative implying that an agent is either in favor or against the proposed reform, respectively. Further the absolute value of u_i^k encodes his preference intensity towards the decision.

Sincere Strategies. Suppose agents 1, 2 and 3 draw utility vectors $u_1 = (4, -1)$, $u_2 = (3, -2)$ and $u_3 = (1, 4)$, respectively. Assume further that all agents find it optimal to report sincerely. Agents 1's sincere report $\overset{*}{s}_1$ is given by $(\overset{*}{a}_1, \overset{*}{\pi}_1) = (10, 21)$. Agent 1 is in favor of the reform in the first decision problem and against it in the second $\overset{*}{a}_1 = (10)$. By reporting $\overset{*}{\pi}_1 = (21)$ agent 1 assigns a higher rank to his vote in decision problem I and a lower priority to problem II. The sincere reports of agent 2 and 3 are given by $\overset{*}{s}_2 = (10, 21)$ and $\overset{*}{s}_3 = (11, 12)$, respectively.

Ranking Mechanism. We illustrate the Ranking Mechanism with weight vector $w = (1, 3)$. For decision problem I the Ranking Mechanism translates agent 1's report $m_1^1 = (a_1^1, \pi_1^1) = (1, 2)$ into the signed weight $(2 \cdot a_1^1 - 1) \cdot w^{\pi_1^1} = +3$. Intuitively, agent 1 is in favor of the reform (positive sign) and indicates a high priority (weight 3) in decision problem I. For decision problem II agent 1 reports $(a_1^2, \pi_1^2) = (0, 1)$ and the assigned weight equals $(2 \cdot a_1^2 - 1) \cdot w^{\pi_1^2} = -1$. The assigned weights for decision problem I and II for agent 1 are summarized in column 2 in Table 3.1. Analogously the Ranking Mechanism assigns signed weights to agent 2 and 3 as summarized in column 3 and 4. After translating all agents' reports into signed weights the Ranking Mechanism calculates the problem wise sum of signed weights (column 5) and implements the reform if and only if the sum is positive. The resulting outcome of the Ranking Mechanism is illustrated in column 6. For comparison column 7 contains the outcome under Separate Majority Voting.

A few points are worth noting. First, the Ranking Mechanism respects unanimity in decision problem I. Second, by overturning the majority of agents the Ranking Mechanism deviates from Separate Majority Voting in decision problem II. Agent 3 ranks problem II highest and thereby sways the decision in his favor albeit being a minority. In our case this is indeed a desirable outcome. Utility vectors are drawn such that the sum of utilities increases from 8 under Separate Majority Voting to 9 under the Ranking Mechanism. Third, the sincere strategy is not a dominant strategy for every agent. Agent 1, for example, prefers to deviate to report $(10, 12)$ thereby changing the outcome of problem

II while not affecting that of problem I. Although it is not a dominant strategy, the next section proves that the sincere strategy profile constitutes a Bayes-Nash equilibrium.

3.5.3 Sincere Equilibrium

This section establishes the most important property of the class of Ranking Mechanisms, namely that agents find it optimal to report sincerely. Apart from its theoretical appeal the existence and characterization of an equilibrium is indispensable to any further efficiency analysis.

Proposition 6. *Under Assumption 2, the sincere strategy profile is a Bayes-Nash equilibrium of any Ranking Mechanism.*

Proof. See Appendix. □

The proof consists of two parts essentially separating ordinal and cardinal incentives. For every agent the sincere ordinal report is weakly optimal independently of the reported priority ranking and the message profile of all other agents. Having reported the sincere ordinal type an agent finds it optimal to rank decision problems sincerely. Since all reports by the other agents are equally probable, an agent has no incentive to strategically rank decision problems.

Importantly, Proposition 6 allows us to make precise welfare predictions and to compare the performance of different Ranking Mechanisms. In the remainder of this section we assume that agents report sincerely whenever we evaluate the performance of a Ranking Mechanism.

3.5.4 The Optimal Ranking Mechanism

This section compares the ex-ante welfare of different Ranking Mechanisms. For any fixed number of agents and decision problems we solve for the ex-ante efficient Ranking Mechanism. In particular, we derive a closed form solution for the ex-ante efficient weight vector. A consequence of our derivation is that the optimal Ranking Mechanism ex-ante dominates Separate Majority Voting.

What is the best outcome a mechanism can implement given that agents report sincerely? Intuitively, the efficient mechanism should decide in favor of a reform if the sum of expected utilities from doing so is positive. Therefore it should assign to each agent's vote a weight that corresponds to that agent's expected utility from implementing the reform. More precisely, the signed weight should equal the agent's expected utility conditional on all information contained in his report. By Definition 8 the ordinal part of an agent's message is informative about the sign of his utility and should therefore only determine the sign of the weight. The cardinal part - i.e. the rank assigned to a problem - contains information about an agent's preference intensity. Concretely, an agent ranks a problem at position $l \in D$ if he has his l-th highest preference intensity in that problem. Therefore the l-th weight should correspond to the expected value of an agent's l-th order statistic of his preference intensity. Building on this logic we define the following weight vector.

Definition 10. *The efficient weight vector* $\overset{*}{w} \in W$ *is given by*

$$\overset{*}{w} = \left(\mathbb{E}_i\left[\left|u_i^k\right|_{(1:d)}\right], \ldots, \mathbb{E}_i\left[\left|u_i^k\right|_{(d:d)}\right]\right)$$

for some $i \in N$ *and* $d \in D$, *where* $\left|u_i^k\right|_{(k:d)}$ *denotes the k-th (out of d) order statistic of the preference intensity* $\left|u_i^k\right|$.[7] *Under Assumption 2, the above definition is independent of the choice of* $i \in N$ *and* $k \in D$ *which justifies the notation. We refer to* $(\mathscr{M}, g^{RM,\overset{*}{w}})$ *as the optimal Ranking Mechanism.*

The next proposition justifies Definition 10.

Proposition 7. *Under Assumption 2, the optimal Ranking Mechanism is ex-ante Pareto efficient in the class of Ranking Mechanisms.*

Proof. See Appendix. □

In the Appendix we prove a stronger result. The optimal Ranking Mechanism is ex-ante Pareto efficient among *all* indirect mechanisms that are defined on the ranking message space and induce sincere equilibrium behavior. Put differently, there is no better way to make use of the information elicited through sincere equilibrium behavior than to assign weights to agents' votes. For any sincere message profile the Ranking Mechanism maximizes ex-ante expected welfare conditional on all agents' reports.

Definition 10 characterizes the efficient weight vector in terms of agents' type distributions. For $u_i^k \sim ii\mathcal{N}(0,1)$ as in the example in Section 3.5.2 the efficient weights are approximately $(0.467, 1.128)$.[8] We round all numbers to three digits throughout this paper. A consequence of Proposition 7 is that the optimal Ranking Mechanism ex-ante dominates Separate Majority Voting.

Corollary 2. *Under Assumption 2, the optimal Ranking Mechanism ex-ante dominates Separate Majority Voting.*

The optimal Ranking Mechanism dominates Separate Majority Voting in the weak sense of Definition 4. Inspection of the proof of Proposition 7 shows that the optimal Ranking Mechanism generates at least as high levels of conditional ex-ante expected welfare message profile by message profile. It is thus sufficient

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[7] Since u_i^k is Lebesgue-integrable the weights $\overset{*}{w}$ are well-defined, see Ahsanullah, Nevzorov, and Shakil (2013), page 76.

[8] $\mathbb{E}\left[\left|u_i^k\right|_{(l:d)}\right] = \frac{d!}{(l-1)!\cdot(d-l)!}\int_{-\infty}^{\infty} |x| \cdot (F(x))^{l-1} \cdot (1-F(x))^{d-l} \mathrm{d}F(x)$, see for example chapter 7 in Ahsanullah, Nevzorov, and Shakil (2013).

to ensure the existence of one message profile that results in different outcomes to guarantee a strict improvement. Note that the two mechanisms differ if there exists a message profile such that a strong minority overturns a weak majority (see Section 3.5.2). This occurs if the largest minority of $\lfloor \frac{n}{2} \rfloor$ agents all with the highest assigned weight of $\overset{*}{w}^d = \mathbb{E}_i \left[\left| u_i^k \right|_{(d:d)} \right]$ overturn the smallest majority of $\lceil \frac{n}{2} \rceil$ agents all with the smallest assigned weight $\overset{*}{w}^1 = \mathbb{E}_i \left[\left| u_i^k \right|_{(1:d)} \right]$. The following condition is sufficient for the optimal Ranking Mechanism to strictly increase ex-ante expected utility upon Separate Majority Voting.

Remark 1. *The optimal Ranking Mechanism strictly increases ex-ante expected welfare over Separate Majority Voting if the number of agents $n \in \mathbb{N}$, the number of decision problems $d \in \mathbb{N}$ and the distribution of types is such that*

$$\left\lfloor \frac{n}{2} \right\rfloor \cdot \mathbb{E}_i \left[\left| u_i^k \right|_{(d:d)} \right] > \left\lceil \frac{n}{2} \right\rceil \cdot \mathbb{E}_i \left[\left| u_i^k \right|_{(1:d)} \right]. \tag{3.1}$$

Ceteris paribus Condition (3.1) is more likely to hold the higher the number of agents or problems or the more dispersed the type distribution. If the number of agents is even and there are at least two decision problems $d \geq 2$ the existence of some cardinal information - i.e. $Var\left(\left|u_i^k\right|\right)$ is nonzero - is sufficient for Condition (3.1) to be satisfied. Condition (3.1) in Remark 1 ensures that the optimal Ranking Mechanism dominates Separate Majority Voting not merely by more efficient resolution of ties, but also the more substantive change of allowing strong minorities to overturn weak majority. Put differently, the optimal Ranking Mechanism strictly improves upon Separate Majority Voting whenever it mitigates the *Tyranny of the Majority*. Note that in the example in Section 3.5.2 both the weight vector $(1, 3)$ and the efficient weight vector $(0.467, 1.128)$ satisfy Condition (1) for three agents and two decision problems.

In the remainder of this section we provide a limiting result reminiscent of Jackson and Sonnenschein (2007). As the number of decision problems goes to infinity, the optimal Ranking Mechanism achieves full efficiency.

Proposition 8. *Under Assumption 2, if the support of the type distribution is bounded, the ex-ante utility levels under the optimal Ranking Mechanism converge to full efficiency as the number of decision problems tends to infinity.*

Proof. See Appendix. □

Proposition 8 is driven by the insight that as the number of decision problems becomes arbitrarily large agents are able to perfectly communicate their underlying utility vector. Apart from being theoretically appealing the above result offers a strong rationale for linking decision problems. Note that by symmetry of the environment any Ranking Mechanism trivially converge to full efficiency as the number of agents tends to infinity.

3.6 Ranking Non-Identical Decision Problems

In this section we relax Assumption 2 and allow for different type distributions between agents and across problems. We impose that utility types are independently but not necessarily identically distributed.

We first show that the sincere strategy profile is - in general - no longer a Bayes-Nash equilibrium. However, there exist special cases for which it is. Motivated by this observation we propose a shuffling procedure based on randomization which restores sincere equilibrium behavior of all agents. We then derive the ex-ante efficient Randomized Ranking Mechanism and prove that it ex-ante dominates Separate Majority Voting.

3.6.1 Strategic Ranking

Consider a modified version of our example from Section 3.5.2. Suppose agent 1 draws his utility type in problem II from a uniform distribution with support $[-1, 1]$ instead of from a standard normal distribution. Formally, all $u_i^k \sim ii\mathcal{N}(0, 1)$ with the exception of u_1^2 which is independently drawn from Uniform$[-1, 1]$.

Under these conditions agent 1 no longer reports every priority ranking with the same probability when following the sincere strategy. Agent 1 is more likely to rank problem I as his first ranked problem, i.e. report priority ranking $\pi_1 = (21)$. Concretely, the probability that agent 1 reports priority ranking $\pi_1 = (21)$ under the sincere strategy is $\mathbb{P}_1[\overset{*}{\pi}_1 = (21)] = \mathbb{P}_1\left[\left|u_1^1\right| > \left|u_1^2\right|\right] = 0.631 \neq 0.5$. So agent 1 ranks problem I over problem II with probability 63.1% when reporting sincerely.

So, if agent 1 and 2 were to report sincerely, agent 3 would anticipate that agent 1 is likely to rank problem I highest and thus might have an incentive to strategically misreport his priority ranking. Since agent 1 is more likely to prioritize problem I agent 3 might prefer ranking $\pi_3 = (12)$ in order to influence the decision in problem II with higher probability. Agent 3 will find such deviations desirable if he has similar preference intensities for problem I and II. Straightforward calculations show that this is indeed the case in our example and agent 3 deviates from the sincere strategy.[9] Therefore the sincere strategy profile is no longer a Bayes-Nash equilibrium.

[9] W.l.o.g consider the case of $u_3^k > 0$ for $k = 1, 2$, i.e. agent 3 is in favor of implementing the reform in both decision problems. Let $p := \mathbb{P}_1\left[\left|u_1^1\right| > \left|u_1^2\right|\right]$. It is straightforward to verify that

$$\begin{aligned}\mathbb{E}_{-3}\left[V_3\left(g^{RM,w}\left(\overset{*}{s}_3, \overset{*}{s}_{-3}\right)\right)\right] &= \mathbb{E}_{-3}\left[V_3\left(g^{RM,w}\left((11), (21), \overset{*}{s}_{-3}\right)\right)\right] \\ &< \mathbb{E}_{-3}\left[V_3\left(g^{RM,w}\left((11), (12), \overset{*}{s}_{-3}\right)\right)\right]\end{aligned}$$

for all $u_3 \in \left\{(u_3^1, u_3^2) \in \mathbb{R}^2_{++} \middle| \frac{u_3^1}{u_3^2} < \frac{1+p}{2-p}\right\}$.

The example above demonstrates that Proposition 6 does not hold when we allow for differently distributed types between agents and across problems. Agent 3 deviates from the sincere ranking, because agent 1 is more likely to rank problem I over problem II when following the sincere strategy. Conversely, as long as agent 1 reports every priority ranking with the same probability, agent 3 has no incentive to strategically rank problems. This implies that it is not necessary that agent 1 has the same distribution of types across all problems. It is merely necessary that all agents have type distributions which result in a uniform distribution over all possible priority rankings under the sincere strategy. Formally, we define the following property of an agent's type distribution.

Assumption 3. *For every agent i the type distribution is Ranking Uniform, i.e. $\mathbb{P}_i[\overset{*}{\pi}_i = \pi_i] = \mathbb{P}_i[\overset{*}{\pi}_i = \pi_i']$ for all $\pi_i, \pi_i' \in \sigma(D)$ and all $i \in N$.*

Assumption 3 is violated in the example above. But suppose agent 1 drew his utility in decision problem II from a uniform distribution with support $[-c, c]$ for some $c \in \mathbb{R}_{++}$. Then for $c \approx 1.470$ it holds that $\mathbb{P}_1\left[\left|u_1^1\right| > \left|u_1^2\right|\right] = \frac{1}{2}$. Agent 1 reports each priority ranking with equal probability and type distributions are Ranking Uniform. As this example illustrates there exist Ranking Uniform type distributions that are not identical across problems. For these the following corollary generalizes Proposition 6.

Corollary 3. *Under Assumption 3, the sincere strategy profile is a Bayes-Nash equilibrium of any Ranking Mechanism.*

The corollary follows from inspection of the proof of Proposition 6. The next section builds on the above insight and defines a shuffling procedure based on randomization. Motivated by Corollary 3 the procedure guarantees that from the perspective of every agent all reports of the other agents are equally probable.

3.6.2 Shuffling Rankings

To illustrate the idea of our shuffling procedure consider the example from the previous Section 3.6.1. Recall that all $u_i^k \sim \mathcal{N}(0, 1)$ with the exception of $u_1^2 \sim \text{Uniform}[-1, 1]$. Under the sincere strategy agent 1 is more likely to rank problem I over problem II, i.e. $\overset{*}{\pi}_1 = (21)$ with probability 0.631 and $\overset{*}{\pi}_1 = (12)$ with probability $1 - 0.631 = 0.369$.

Suppose agent 1 reported sincerely and consider the following shuffling procedure that turns every reported ranking of agent 1 into a shuffled ranking as follows. With probability 0.208 the reported ranking is changed to the less probable ranking (12) and with probability $(1 - 0.208)$ the reported ranking remains unchanged. Then, the probability that agent 1's shuffled ranking equals ranking (21) is given by $0.631 \cdot (1 - 0.208) = 0.500$ and the probability for it to be (12) equals $0.369 + 0.631 \cdot 0.208 = 0.500$. If the mechanism were to use the shuffled ranking of agent 1 there would be no incentive for agent 2 and 3 to strategically rank decision problems. From their point of view all shuffled rankings of agent 1 are equally likely. Further, agent 1 has no incentive to strategically rank decision problems since shuffling occurs with equal probability after any report. In the remainder of this section we generalize the above shuffling procedure to an arbitrary number of agents and problems.

A shuffling procedure is a (random) mapping from agents' reported rankings into the set of all possible rankings. It is characterized by two parts. First, for every agent i we define a shuffling probability $\alpha_i \in [0, 1]$ which corresponds to the probability with which every reported ranking of agent i is shuffled. Second, for every agent i we specify the shuffling lottery $\beta_i \in \Delta(\sigma(D))$ where $\beta_i^{\pi_i} \in [0, 1]$ is the probability with which the reported ranking is changed to ranking π_i in case it does get shuffled. Formally, we define a shuffling procedure as follows.

Definition 11. *A shuffling procedure for agent i is a random mapping $\gamma_i : \sigma(D) \to \sigma(D)$ defined as*

$$\gamma_i(\pi_i) \;=\; \begin{cases} \pi_i & \text{with probability } 1 - \alpha_i \\ \pi_i' & \text{with probability } \alpha_i \cdot \beta_i^{\pi_i'} \end{cases}$$

for $\pi_i, \pi_i' \in \sigma(D)$, where $\alpha_i \in [0, 1]$ and $\beta_i \in \Delta(\sigma(D))$ are referred to as agent i's shuffling probability and shuffling lottery, respectively. We refer to the image $\gamma_i(\pi_i)$ as agent i's shuffled ranking.

The goal is to construct a shuffling procedure – that is choose α and β – such that every agent's shuffled ranking is uniformly distributed under the sincere strategy profile. We formalize this point in the following remark.

Remark 2. *For any agent i the choice of α_i and β_i is such that it leads to a uniform distribution of shuffled rankings under the sincere strategy. Formally, we choose $\alpha_i \in [0, 1]$ and $\beta_i \in \Delta(\sigma(D))$ such that*

$$\mathbb{P}_i\big[\gamma_i(\overset{*}{\pi}_i) \;=\; \pi_i\big] \;=\; (1 - \alpha_i) \cdot p_i^{\pi_i} + \alpha_i \cdot \beta_i^{\pi_i} \;=\; \frac{1}{d!} \text{ for all } \pi_i \;\in\; \sigma(D), \tag{3.2}$$

where $p_i^{\pi_i} := \mathbb{P}_i\big[\overset{*}{\pi}_i = \pi_i\big]$ is agent i's probability of ranking π_i under the sincere strategy.

The intuition behind Equation (3.2) is straightforward. There are two ways a shuffled ranking takes on one particular ranking: either the agent sincerely reports that ranking and it does not get shuffled, or the agent's reported ranking does get shuffled in which case the shuffling lottery picks the ranking.

Equation (3.2) immediately places a lower bound on the shuffling probabilities α_i. To see this, consider the ranking that an agent is most likely to report under the sincere strategy and suppose the shuffling lottery β_i places probability zero on this ranking. Plugging this into Equation (3.2) gives the lower bound for the shuffling probability α_i. Intuitively, even if the shuffling lottery places probability zero on the most probable sincerely reported ranking the shuffling procedure still needs to bring down its probability to $\frac{1}{d!}$. Since all other rankings are by definition less likely the minimal level of shuffling is pinned down by the probability of an agent's most probable reported ranking under the sincere strategy profile. Formally, we define the shuffling probabilities for all agents as follows.

Definition 12. *The (minimal) shuffling probability for agent i is given by*

$$\alpha_i \;=\; 1 - \frac{1}{d! \cdot p_i^{\max}},$$

where $p_i^{\max} := \max_{\pi_i \in \sigma(D)} \mathbb{P}_i\left[\overset{*}{\pi}_i = \pi_i\right]$ *is the probability of the ranking which agent i is most likely to reported under the sincere strategy.*[10] *Let* $\alpha = (\alpha_i)_{i \in N}$ *correspond to the collection of shuffling probabilities for all agents.*

Note that if (and only if) an agent's type distribution is Ranking Uniform in the sense of Assumption 3 his shuffling probability is zero. The shuffling procedure does not introduce randomization if an agent already reports all rankings with equal probability. For nonzero shuffling probabilities Equation (3.2) implies the following choice for the shuffling lottery.

Definition 13. *The shuffling lottery of agent i with nonzero shuffling probability* α_i *(defined in Definition 12) is given by*

$$\beta_i^{\pi_i} \;=\; \frac{1}{\alpha_i}\left(\frac{1}{d!} - (1-\alpha_i)\cdot p_i^{\pi_i}\right) \text{ for } \pi_i \;\in\; \sigma(D),$$

where $p_i^{\pi_i} := \mathbb{P}_i\left[\overset{*}{\pi}_i = \pi_i\right]$ *is the probability with which agent i reports ranking* π_i *under the sincere strategy. For consistency we choose* $\beta_i^{\pi_i} = p_i^{\pi_i}$ *for all* $\pi_i \in \sigma(D)$ *if agent i's shuffling probability is zero in Definition 12. Let* $\beta = (\beta_i)_{i \in N}$ *denote the collection of shuffling lotteries for all agents.*

A shuffling procedure with shuffling probabilities and shuffling lotteries as in Definition 12 and Definition 13 – henceforth referred to as *the* shuffling procedure – leads to a uniform distribution of shuffled rankings by all agents. From the perspective of any one agent all other agents' shuffled rankings are equally

[10] While our shuffling procedure also works for larger choices of α_i the next section shows that in the context of our Ranking Mechanisms the minimal choice in Definition 12 is desirable from an ex-ante welfare perspective.

likely and there is no incentive to strategically rank decision problems. In the next section we integrate the shuffling procedure into our Ranking Mechanism.

3.6.3 Randomized Ranking Mechanisms

Equipped with the shuffling procedure from the previous section, we define the new class of Randomized Ranking Mechanisms. A Randomized Ranking Mechanism corresponds to a Ranking Mechanism on the shuffled message profile. It uses the message space from Definition 7 and its decision rule $g^{RRM,w} : \mathscr{M} \to \Delta(X)$ is parametrized by a weight vector $w = (w^1, ..., w^d) \in W$, where $W \subset \mathbb{R}^d_{++}$ denotes the set of strictly positive weight vectors with non-decreasing components. Formally, we define a Randomized Ranking Mechanism as follows.

Definition 14. *The Randomized Ranking Mechanism $(\mathscr{M}, g^{RRM,w})$ with weight vector $w \in W$ implements the same outcome as the Ranking Mechanism $g^{RM,w}$ on the shuffled message profile. Formally,*

$$g^{RRM,w}(m) \;=\; \left(g^{RM,w} \circ \gamma\right)(m)$$

where $\gamma(m) = (a_i, \gamma_i(\pi_i))_{i \in N}$ denotes the profile of shuffled messages of all agents and γ_i is the shuffling procedure defined in the previous section (Definition 11, 12 and 13).

Every Randomized Ranking Mechanism is a composition of the corresponding Ranking Mechanism (with the same weight vector) and the shuffling procedure. Definition 14 immediately implies that for any strategy profile the distribution over outcomes under the Randomized Ranking Mechanism is identical to that of the corresponding Ranking Mechanism under the shuffled strategy profile. Formally, we have the following remark.

Remark 3. *For any strategy profile $\hat{s}$, the corresponding shuffled strategy profile $\gamma(\hat{s}) = \left(\hat{a}, \gamma(\hat{\pi})\right)$ and any $w \in W$ we have*

$$g^{RRM,w}(\hat{s}) \;\sim\; g^{RM,w}(\gamma(\hat{s})).$$

Definition 14 and Remark 3 allow Randomized Ranking Mechanisms to inherit many of the properties of Ranking Mechanisms. In particular, we have the following proposition analogous to Proposition 6.

Proposition 9. *The sincere strategy profile is a Bayes-Nash equilibrium of any Randomized Ranking Mechanism.*

Proof. See Appendix. □

The logic of the proof is as follows. First, every agent finds it optimal to sincerely report his ordinal type because shuffling only affects the reported ranking. Second, neither the shuffling probability nor the shuffling lottery depend on an agent's reported ranking. Therefore every agent only considers the case in which his reported ranking is not shuffled. But in expectation the shuffled report profile of the sincere rankings of all other agents is uniformly distributed and an agent has no incentive to deviate from the sincere strategy by the same logic as in the proof of Proposition 6.

It is straightforward to see that Proposition 9 continues to hold for choices of shuffling probabilities larger than the minimal choice defined in Defintion 12, as long as we also adjust the shuffling lotteries as in Definition 13. The following remark illustrates the optimality of the minimal choice in Definition 12 from an ex-ante welfare perspective.

Remark 4. *To illustrate the optimality of a minimal choice of α (together with corresponding shuffling lottery defined in Definition 13) rewrite ex-ante expected welfare as*

$$\sum_i \mathbb{E}_{u,\gamma}\left[V_i(g^{RRM,w}(\overset{*}{s}))\right] = \sum_i \mathbb{E}_{u,\gamma}\left[V_i(g^{RM,w}(\gamma(\overset{*}{s})))\right]$$
$$= \sum_i \mathbb{E}_{u_i}\Bigg[(1-\alpha_i)\underbrace{\mathbb{E}_{-i}\left[V_i(g^{RM,w}(\overset{*}{s}_i,\tilde{s}_{-i}))\right]}_{(*)} + \alpha_i \sum_{\pi_i'} \beta_i^{\pi_i'} \underbrace{\mathbb{E}_{-i}\left[V_i(g^{RM,w}((\overset{*}{a}_i,\pi_i'),\tilde{s}_{-i}))\right]}_{(**)}\Bigg],$$

where $\tilde{s}_{-i} = (\overset{*}{a}_i, \gamma_i(\overset{*}{\pi}_i))_{-i}$ denotes the shuffled strategy profile of all agents but agent i. For all i and $u_i \in \mathcal{U}_i$ expression $(*)$ is weakly larger than expression $(**)$ by Proposition 6 and both are independent of α_{-i} and β_{-i} as long as $\tilde{s}_{-i} \sim Uniform(\mathcal{M}_{-i})$. Thus α chosen minimally (subject to achieving uniformity) maximizes ex-ante expected welfare for any weight vector.

The shuffling probabilities in Definition 12 are not only desirable in terms of ex-ante expected welfare, but also ensure that if an agent's type distribution is Ranking Uniform no randomization is introduced. In particular, we have the following remark.

Remark 5. *Under Assumption 3, any Randomized Ranking Mechanism collapses to the corresponding Ranking Mechanism.*

In the next section we turn to the optimal choice of the weight vector. We implicitly assume that agents report sincerely whenever we evaluate the performance of any Randomized Ranking Mechanism.

3.6.4 The Optimal Randomized Ranking Mechanism

Having established sincere equilibrium behavior this section compares the ex-ante welfare of different Randomized Ranking Mechanisms. For any fixed number of agents and decision problems we derive the ex-ante efficient Randomized Ranking Mechanism and provide a closed form solution for the associated weight vector. The optimal Randomized Ranking Mechanism ex-ante dominates Separate Majority Voting.

The intuition is analogous to that in Section 3.5.4. What weight should the mechanism assign to an agent's vote based on his shuffled ranking? Intuitively, the weight should correspond to an agent's expected utility from implementing the reform conditional on his report. However, the assigned weights can neither discriminate between agents or problems nor can they depend on the realization of the randomization or the choice of the shuffle lottery. The following definition generalizes Definition 10.

Definition 15. *The efficient weight vector* $\overset{**}{w} \in W$ *is given by*

$$\overset{**}{w}^{l} = \frac{1}{n} \cdot \sum_{i} \Big((1-\alpha_i) \cdot \sum_{k} \mathbb{P}_i \big[\overset{*}{\pi}_i^k = l \big] \cdot \mathbb{E}\Big[\big| u_i^k \big|_{(l:d)} \Big] + \alpha_i \cdot \sum_{k} \mathbb{P}_{\beta_i} \big[\tilde{\pi}_i^k = l \big] \cdot \mathbb{E}\big[\big| u_i^k \big| \big] \Big)$$

for $l = 1, \dots, d$*, where* $\left| u_i^k \right|_{(l:d)}$ *denotes the l-th (out of d) order statistic of the preference intensity* $\left| u_i^k \right|$ *and* $\mathbb{P}_{\beta_i}\left[\tilde{\pi}_i^k = l \right] := \sum_{\pi_i : \pi_i^k = l} \beta_i^{\pi_i}$ *is defined as the probability that problem k is ranked on l-th position through shuffling lottery* β_i*. We refer to* $(\mathcal{M}, g^{RRM, \overset{**}{w}})$ *as the optimal Randomized Ranking Mechanism.*

Before providing intuition we justify Definition 15 by the following proposition.

Proposition 10. *The optimal Randomized Ranking Mechanism is ex-ante Pareto efficient in the class of Randomized Ranking Mechanisms.*

Proof. See Appendix. □

Proposition 10 follows from suitably rewriting ex-ante expected welfare. The intuition behind Proposition 10 and Definition 15 is as follows. For every agent i the efficient l-th weight trades-off two cases. First, with probability $1 - \alpha_i$ agent i's report is sincere and did not get shuffled. In this case it is efficient to set the l-th weight for agent i to the expected value of his l-th highest preference intensity under the sincere strategy. Because we allow for different distributions across problems the expected value of the l-th highest order statistic for a fixed decision problem may vary across problems. The ex-ante expected value of the

l-th highest preference intensity under the sincere strategy thus weights all expected l-th order statistics by their respective sincere probability, that is, by the probability that an agent sincerely ranks that problem at position l. Second, with probability α_i agent i's ranking gets shuffled. In this case – since the shuffle lottery is uninformative about the preference intensity – the efficient l-th weight corresponds to the unconditional expected preference intensity. Again, the expected preference intensity may vary across problems and needs to be weighted by the probability that a problem is ranked at the l-th position by the shuffling lottery β_i of agent i. Lastly, since we restrict attention to anonymous mechanism the efficient weight vector cannot depend on an agent's identity. It is therefore efficient to take the average over all "agent-specific" efficient weights outlined above.[11]

It is further instructive to consider the following special case. Under Assumption 3, the shuffling probabilities are zero and Definition 15 simplifies.

Remark 6. *Under Assumption 3, the optimal Randomized Ranking Mechanism corresponds to the Ranking Mechanism with weight vector $\overset{**}{w} \in W$ given by*

$$\overset{**}{w}^l = \frac{1}{n \cdot d} \cdot \sum_i \sum_k \mathbb{E}\left[|u_i^k|_{(l:d)} \right]$$

for $l = 1, \ldots, d$.

Remark 6 generalizes Definition 10 of the optimal weight vector in the identical case of Section 3.5 by allowing for different type distributions between agents and across problems as long as Assumption 3 is satisfied. In this case the efficient l-th weight corresponds to the expected value of the l-th highest preference intensity averaged across all agents and problems. By Definition 14 of the Randomized Ranking Mechanism the randomization procedure only shuffles the reported ranking. Thus it has no effect for constant weight vectors for which the order is irrelevant. This implies that the Randomized Ranking Mechanism with weight vector $w = (1, \ldots, 1)$ implements the same outcome as Separate Majority Voting. Therefore Proposition 10 implies that the optimal Randomized Ranking Mechanism ex-ante dominates Separate Majority Voting.

Corollary 4. *The optimal Randomized Ranking Mechanism ex-ante dominates Separate Majority Voting.*

Analogously to Section 3.5.4 the optimal Randomized Ranking Mechanism dominates Separate Majority Voting in the weak sense of Definition 4. The following remark follows from the same logic as Remark 1 in Section 3.5.4 and guarantees the welfare improvement to be strict. The Randomized Ranking

[11] All our results readily extend to the case in which we drop the anonymity requirement and allow for agent-specific weights.

Mechanism and Separate Majority Voting differ if there exists a report profile such that a strong minority overturns a weak majority. Formally, we have the following remark.

Remark 7. *The optimal Randomized Ranking Mechanism strictly increases ex-ante expected welfare over Separate Majority Voting if the number of agents $n \in \mathbb{N}$, the number of decision problems $d \in \mathbb{N}$ and the distribution of types is such that*

$$\left\lfloor \frac{n}{2} \right\rfloor \cdot \overset{**}{w}^{d} > \left\lceil \frac{n}{2} \right\rceil \cdot \overset{**}{w}^{1}. \tag{3.3}$$

The intuition is analogous to Section 3.5.4. Condition (3.3) in Remark 7 ensures that the optimal Randomized Ranking Mechanism dominates Separate Majority Voting not merely by more efficient resolution of ties, but also by allowing strong minorities to overturn weak majorities thereby mitigating the *Tyranny of the Majority*.

Note that in our modified example in Section 3.6.1 the efficient weight vector equals $(0.455, 1.041)$. It allows for overturning by satisfying Condition (3.3) in Remark 7 for three agents and two decision problems and therefore strictly improves upon Separate Majority Voting.

3.7 Concluding Remarks

In this paper we show that among strategy-proof mechanisms Separate Majority Voting is ex-ante efficient and there is no benefit in coupling binary decisions. When moving to the class of incentive compatible mechanisms full efficiency remains unachievable for a finite number of decision problems but one can improve upon Separate Majority Voting.

In order to do so, we study a class of Ranking Mechanisms. A Ranking Mechanism corresponds to a simple weighted voting procedure, in which agents are free to distribute weights across problems and alternatives. For the case of identically distributed preferences over problems any Ranking Mechanism admits an intuitive equilibrium strategy. Agents rank problems according to the absolute difference in utilities between alternatives, i.e. by their preference intensities. We solve for the ex-ante efficient Ranking Mechanism and give a close-form solution for the corresponding optimal weight vector. The optimal Ranking Mechanism ex-ante dominates Separate Majority Voting and achieves full efficiency in the limit as the number of decision problems goes to infinity. In the case of non-identically distributed problems we introduce a randomization procedure which sustains sincere equilibrium behavior. Incentives are preserved by ensuring that from the perspective of every agent all priority rankings of all other agents are equally likely. We provide a closed-form solution for the ex-ante efficient weight vector and prove that the optimal Randomized Ranking Mechanism ex-ante dominates Separate Majority Voting.

All our results hold for an arbitrary number of agents and decisions thereby complementing mechanisms in the previous literature, which work well for an infinite number of decisions (Jackson and Sonnenschein (2007)) or a large enough number of agents (Casella and Gelman (2008)). Moreover the optimal (Randomized) Ranking Mechanism represents - to the best of our knowledge - the first mechanism which successfully couples non-identically distributed binary decision problems, induces intuitive equilibrium behavior and dominates Separate Majority Voting for any number of agents and problems.

Throughout this work we restricted attention to anonymous mechanisms. From an ex-ante welfare perspective it might be desirable to discriminate between agents. All of our analysis readily extends to the case of allowing for agent-specific weights. The optimal (Randomized) Ranking Mechanism is not without its weaknesses. In particular, it relies on a strong knowledge assumption regarding the underlying type distributions.

References

De Tocqueville, Alexis. 1835. *Democracy in america.* Saunders, and Otley. [25]

Rae, Douglas W. 1969. "Decision-rules and individual values in constitutional choice." *American Political Science Review* 63 (01): 40–56. [27]

Gibbard, Allen. 1973. "Manipulation of voting schemes: A general result." *Econometrica* 41 (4): 587–601. [27]

Satterthwaite, Mark A. 1975. "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory* 10 (2): 187–217. [27]

Hylland, Aanund. 1980. "Strategy proofness of voting procedures with lotteries as outcomes and infinite sets of strategies." [27]

Barberà, Salvador, Hugo Sonnenschein, and Lin Zhou. 1991. "Voting by committees." *Econometrica* 59 (3): 595–609. [27]

Campbell, Colin M. 1998. "Coordination in auctions with entry." *Journal of Economic Theory* 82 (2): 425–50. [28]

Pesendorfer, Martin. 2000. "A study of collusion in first-price auctions." *Review of Economic Studies* 67 (3): 381–411. [28]

Majumdar, Dipjyoti, and Arunava Sen. 2004. "Ordinally Bayesian incentive compatible voting rules." *Econometrica* 72 (2): 523–40. [27, 29]

Casella, Alessandra. 2005. "Storable votes." *Games and Economic Behavior* 51 (2): 391–419. [28]

Chakraborty, Archishman, and Rick Harbaugh. 2007. "Comparative cheap talk." *Journal of Economic Theory* 132 (1): 70–94. [28]

Jackson, Matthew O, and Hugo Sonnenschein. 2007. "Overcoming Incentive Constraints by Linking Decisions." *Econometrica* 75 (1): 241–57. [27, 38, 48]

Casella, Alessandra, and Andrew Gelman. 2008. "A simple scheme to improve the efficiency of referenda." *Journal of Public Economics* 92 (10): 2240–61. [27, 28, 34, 48]

Hortala-Vallve, Rafael. 2010. "Inefficiencies on linking decisions." *Social Choice and Welfare* 34 (3): 471–86. [26, 27, 31, 50]

Apesteguia, Jose, Miguel A Ballester, and Rosa Ferrer. 2011. "On the justice of decision rules." *Review of Economic Studies* 78 (1): 1–16. [29]

Ahn, David S, and Santiago Oliveros. 2012. "Combinatorial voting." *Econometrica* 80 (1): 89–141. [29]

Hortala-Vallve, Rafael. 2012. "Qualitative voting." *Journal of Theoretical Politics* 24 (4): 526–54. [27, 28]

Schmitz, Patrick W, and Thomas Tröger. 2012. "The (sub-) optimality of the majority rule." *Games and Economic Behavior* 74 (2): 651–65. [31]

Ahsanullah, Mohammad, Valery B Nevzorov, and Mohammad Shakil. 2013. *An introduction to order statistics.* Springer. [37, 56]

Gershkov, Alex, Benny Moldovanu, and Xianwen Shi. 2017. "Optimal voting rules." *Review of Economic Studies* 84 (2): 688–717. [27]

Kim, Semin. 2017. "Ordinal versus cardinal voting rules: A mechanism design approach." *Games and Economic Behavior* 104: 350–71. [27–29]

3.A Appendix: Proofs

Proof of Proposition 5. The following corollary is a straightforward adoption of Proposition 1 in Hortala-Vallve (2010) for the case of incentive compatibility.

Definition 16. *The expected indirect utility function $\mathcal{V}_i : \mathcal{U}_i \to \mathbb{R}$ of agent i under mechanism $(\mathcal{U}, g)$ is defined by $\mathcal{V}_i(u_i) = \sum_{k=1}^{d} u_i^k \cdot \mathbb{E}_{-i}[g_k(u_i, u_{-i})]$.*

Corollary 5 (Hortala-Vallve (2010))**.** *A mechanism $(\mathcal{U}, g)$ is incentive compatible if and only if agents' expected indirect utilities are homogeneous of degree one and convex.*
That is, $\mathcal{V}_i(\lambda \cdot u_i) = \lambda \cdot \mathcal{V}_i(u_i)$ for $\lambda \geq 0$, $u_i \in \mathcal{U}_i$ and is convex in u_i.

For any incentive compatible mechanism (U, g) it follows that

$$\mathbb{E}_{-i}\left[g_k(u_i, u_{-i})\right] \;=\; \mathbb{E}_{-i}\left[g_k(\lambda \cdot u_i, u_{-i})\right] \tag{3.4}$$

for every $i \in \{1, ..., n\}$, every $k \in \{1, ..., d\}$, every $u_i \in \mathcal{U}_i$ and every $\lambda \geq 0$.

Equation (3.4) states that from agent i's perspective the expected outcome of the mechanism is identical on proportional utility types. In other words, proportional types of agent i are bunched in expectation.

To prove Proposition 5 suppose for sake of contradiction that g is incentive compatible and achieves full efficiency. By the above g bunches proportional types in expectation. Consider two cases depending on whether or not Equation (3.4) holds pointwise:

Case 1: Equation (3.4) holds pointwise everywhere, that is, proportional types are bunched type by type. It is enough to consider the case of two agents and one decision problem. For one decision problem all possible types are proportional and hence g must be constant, which is not optimal. The same line of reasoning extends to settings with more agents and more decision problems.

Case 2: Equation (3.4) does not hold pointwise everywhere, implying that there exist $i \in \{1, ..., n\}$, $k \in \{1, ..., d\}$, $u_i \in \mathcal{U}_i$ with $u_i^k \neq 0$, $\lambda \in \mathbb{R}_{++} \setminus \{1\}$ and $u'_{-i} \in \mathcal{U}_{-i}$ such that $g_k(u_i, u'_{-i}) \neq g_k(\lambda \cdot u_i, u'_{-i})$. Consider the case $u_i^k > 0$ and $g_k(u_i, u'_{-i}) > g_k(\lambda \cdot u_i, u'_{-i})$. All other cases follow by an analogous argument. In order for Equation (3.4) to be satisfied, there must exist $u''_{-i} \in \mathcal{U}_{-i}$ such that $g_k(u_i, u''_{-i}) < g_k(\lambda \cdot u_i, u''_{-i})$. The fact that g achieves full efficiency necessitates that for fixed $u_{-i} \in \mathcal{U}_{-i}$ the function $g_k(\cdot, u_{-i})$ depends only on the value of u_i^k and not on the other components of u_i. Further $g_k(\cdot, u_{-i})$ has to be non-decreasing in $u_i^k > 0$. For $\lambda > 1$ this contradicts the first inequality, for $\lambda < 1$ the second. □

Proof of Proposition 6. Fix $w \in W$ and denote $g^{RM,w}$ by g. Formally, we need to show that for all agents $i \in N$, all $u_i \in \mathcal{U}_i$ and $\overset{*}{s}_{-i} = \left(\overset{*}{a}_{-i}, \overset{*}{\pi}_{-i}\right)$ it holds that

$$\overset{*}{s}_i(u_i) \;=\; \left(\overset{*}{a}_i, \overset{*}{\pi}_i\right)(u_i) \;\in\; \underset{(a_i,\pi_i)}{\arg\max}\left\{\mathbb{E}_{-i}\left[V_i(g((a_i,\pi_i),\overset{*}{s}_{-i}))\right]\right\}. \tag{3.5}$$

The proof consists of two parts.

Part 1: For any message profile of the other agents $m_{-i} = (a_{-i}, \pi_{-i})$ and any fixed priority ranking π_i agent i finds the strategy $s_i = (\overset{*}{a}_i, \pi_i)$ weakly optimal. Note that

$$V_i(g((a_i,\pi_i),m_{-i})) \;=\; \sum_k u_i^k \cdot g_k\left(\left(a_i^k, \pi_i^k\right), m_{-i}^k\right) \tag{3.6}$$

for $u_i \in \mathcal{U}_i$ and $(a_i, \pi_i) \in M$. Since for all $k \in D$, all $\pi_i^k \in D$ and all $m_{-i} \in \mathcal{M}_{-i}$

$$g_k\left(\left(1, \pi_i^k\right), m_{-i}^k\right) \;\geq\; g_k\left(\left(0, \pi_i^k\right), m_{-i}^k\right),$$

it follows that Equation (3.6) is maximized for

$$a_i^k \;=\; \overset{*}{a}{}_i^k \;=\; \begin{cases} a_i^k \;=\; 1 & \text{if } u_i^k \;>\; 0 \\ a_i^k \;=\; 0 & \text{if } u_i^k \;\leq\; 0. \end{cases}$$

Part 2: For the sincere strategy profile of the other agents $\overset{*}{s}_{-i} = (\overset{*}{a}_{-i}, \overset{*}{\pi}_{-i})$ and the sincere $\overset{*}{a}_i$ agent i finds the sincere priority ranking $\overset{*}{\pi}_i$ weakly optimal. Exploiting uncorrelated types we have

$$\begin{aligned} &\mathbb{E}_{-i}\left[V_i(g((\overset{*}{a}_i,\pi_i),\overset{*}{s}_{-i}))|u_i\right] \;=\; \mathbb{E}_{-i}\left[V_i(g((\overset{*}{a}_i,\pi_i),\overset{*}{s}_{-i}))\right] \\ &= \sum_{k:u_i^k>0} u_i^k \cdot \mathbb{E}_{-i}\left[g_k\left(\left(1,\pi_i^k\right),\overset{*}{s}{}_{-i}^k\right)\right] + \sum_{k:u_i^k<0} u_i^k \cdot \mathbb{E}_{-i}\left[g_k\left(\left(0,\pi_i^k\right),\overset{*}{s}{}_{-i}^k\right)\right] \end{aligned} \tag{3.7}$$

for $u_i \in \mathcal{U}_i$ and $(\overset{*}{a}_i, \pi_i) \in M$. We decompose the message space $\mathcal{M}_{-i}$ of all other agents according to the outcome that would be implemented in the absence of agent i. Formally, we define

$$\mathcal{M}_{-i}^{k,q} \;:=\; \left\{m_{-i} \;\in\; \mathcal{M}_{-i} \middle| g_k\left(\frac{1}{2}, \pi_i^k, m_{-i}^k\right) \;=\; q\right\} \text{ for } q \;=\; 0, \frac{1}{2}, 1.$$

The set $\mathcal{M}_{-i}^{k,q}$ encompasses all message profiles of the other agents such that without agent i outcome $q \in \left\{0, \frac{1}{2}, 1\right\}$ is implemented in problem k. Note that by definition of g_k the set $\mathcal{M}_{-i}^{k,q}$ is independent of π_i^k which justifies the notation. Using the fact that for all $k \in D$, all $\pi_i^k \in D$ and all $m_{-i} \in \mathcal{M}_{-i}$

$$g_k\left(\left(1, \pi_i^k\right), m_{-i}^k\right) \;\geq\; g_k\left(\left(\frac{1}{2}, \pi_i^k\right), m_{-i}^k\right) \;\geq\; g_k\left(\left(0, \pi_i^k\right), m_{-i}^k\right)$$

with strict inequalities if $g_k\left(\left(\frac{1}{2},\pi_i^k\right),m_{-i}^k\right)=\frac{1}{2}$. We write (3.7) as

$$
\begin{aligned}
&\sum_{k:u_i^k>0} u_i^k\cdot\Big\{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,0}\Big]\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,0}\Big]\\
&\qquad+\overbrace{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,1}\Big]}^{=1}\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,1}\Big]\\
&\qquad+\underbrace{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,\frac{1}{2}}\Big]}_{=1}\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,\frac{1}{2}}\Big]\Big\}\\
&+\sum_{k:u_i^k<0} u_i^k\cdot\Big\{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,1}\Big]\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,1}\Big]\\
&\qquad+\overbrace{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,0}\Big]}^{=0}\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,0}\Big]\\
&\qquad+\underbrace{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,\frac{1}{2}}\Big]}_{=0}\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,\frac{1}{2}}\Big]\Big\}\\
=&\sum_{k:u_i^k>0} u_i^k\cdot\Big\{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,0}\Big]\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,0}\Big]\Big\}\\
&+\sum_{k:u_i^k<0} u_i^k\cdot\Big\{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\Big|\mathscr{M}_{-i}^{k,1}\Big]\cdot\mathbb{P}_{-i}\Big[\mathscr{M}_{-i}^{k,1}\Big]\Big\}+C,
\end{aligned}
\tag{3.8}
$$

with C independent of π_i. We further decompose the type space of all other agents depending on whether or not agent i with priority ranking π_i^k changes the outcome in decision problem k. We split $\mathscr{M}_{-i}^{k,0}$ into three disjoint sets of reports of other agents: the set $\mathscr{M}^{k,0}\left(\pi_i^k\right)$ such that the inclusion of agent i voting in favor of the reform with a priority ranking π_i^k does not change the outcome and 0 is still implemented in decision problem k and the sets $\mathscr{T}^{k,0}\left(\pi_i^k\right)$ and $\mathscr{P}^{k,0}\left(\pi_i^k\right)$ on which the outcome changes to $\frac{1}{2}$ and 1, respectively. Formally,

$$
\begin{aligned}
\mathscr{M}_{-i}^{k,0}=&\underbrace{\Big\{m_{-i}\in\mathscr{M}_{-i}\Big|0=g_k\left(0.5,\pi_i^k,m_{-i}^k\right)\wedge g_k\left(1,\pi_i^k,m_{-i}^k\right)=0\Big\}}_{=:\mathscr{M}^{k,0}(\pi_i^k)}\\
&\biguplus\underbrace{\Big\{m_{-i}\in\mathscr{M}_{-i}\Big|0=g_k\left(0.5,\pi_i^k,m_{-i}^k\right)\wedge g_k\left(1,\pi_i^k,m_{-i}^k\right)=\frac{1}{2}\Big\}}_{=:\mathscr{T}^{k,0}(\pi_i^k)}\\
&\biguplus\underbrace{\Big\{m_{-i}\in\mathscr{M}_{-i}\Big|0=g_k\left(0.5,\pi_i^k,m_{-i}^k\right)\wedge g_k\left(1,\pi_i^k,m_{-i}^k\right)=1\Big\}}_{=:\mathscr{P}^{k,0}(\pi_i^k)}.
\end{aligned}
$$

As made explicit by the notation the decomposition depends on π_i^k. Analogously, we define $\mathscr{M}^{k,1}\left(\pi_i^k\right)$, $T^{k,1}\left(\pi_i^k\right)$ and $P^{k,1}\left(\pi_i^k\right)$ by

$$\begin{aligned}
\mathscr{M}_{-i}^{k,1} = & \underbrace{\left\{m_{-i} \in \mathscr{M}_{-i} \middle| 1 = g_k\left(0.5, \pi_i^k, m_{-i}^k\right) \wedge g_k\left(0, \pi_i^k, m_{-i}^k\right) = 1\right\}}_{=:\mathscr{M}^{k,1}\left(\pi_i^k\right)} \\
& \biguplus \underbrace{\left\{m_{-i} \in \mathscr{M}_{-i} \middle| 1 = g_k\left(0.5, \pi_i^k, m_{-i}^k\right) \wedge g_k\left(0, \pi_i^k, m_{-i}^k\right) = \frac{1}{2}\right\}}_{=:\mathscr{T}^{k,1}\left(\pi_i^k\right)} \\
& \biguplus \underbrace{\left\{m_{-i} \in \mathscr{M}_{-i} \middle| 1 = g_k\left(0.5, \pi_i^k, m_{-i}^k\right) \wedge g_k\left(0, \pi_i^k, m_{-i}^k\right) = 0\right\}}_{=:\mathscr{P}^{k,1}\left(\pi_i^k\right)}.
\end{aligned}$$

Since for all $k \in D$, all $m_{-i} \in \mathscr{M}_{-i}$ and $\tilde{\pi}_i^k < \pi_i^k \in D$

$$g_k\left(1, \tilde{\pi}_i^k, m_{-i}^k\right) \leq g_k\left(1, \pi_i^k, m_{-i}^k\right) \text{ and } g_k\left(0, \tilde{\pi}_i^k, m_{-i}^k\right) \geq g_k\left(0, \pi_i^k, m_{-i}^k\right)$$

with strict inequalities if $g_k\left(0, \tilde{\pi}_i^k, m_{-i}^k\right) = \frac{1}{2}$. It follows that

$$\mathscr{T}^{k,q}\left(\tilde{\pi}_i^k\right) \subseteq \mathscr{P}^{k,q}\left(\pi_i^k\right) \text{ and } \mathscr{P}^{k,q}\left(\tilde{\pi}_i^k\right) \subseteq \mathscr{P}^{k,q}\left(\pi_i^k\right) \tag{3.9}$$

for all $k \in D$, all $\tilde{\pi}_i^k < \pi_i^k \in D$ and $q = 0, 1$. By definition of g we have for all $k, l, \pi_i^k \in D$:

$$\begin{aligned}
& \left|\mathscr{P}^{k,1}\left(\pi_i^k\right)\right| = \left|\mathscr{P}^{l,1}\left(\pi_i^k\right)\right| = \left|\mathscr{P}^{l,0}\left(\pi_i^k\right)\right| \\
\text{and } & \left|\mathscr{T}^{k,1}\left(\pi_i^k\right)\right| = \left|\mathscr{T}^{l,1}\left(\pi_i^k\right)\right| = \left|\mathscr{T}^{l,0}\left(\pi_i^k\right)\right|.
\end{aligned} \tag{3.10}$$

Using the above construction we write (3.8) as

$$
\begin{aligned}
&\sum_{k:u_i^k>0} u_i^k \cdot \Big\{ \underbrace{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\,\Big|\,\mathscr{P}^{k,0}\big(\pi_i^k\big)\Big]}_{=1} \cdot \mathbb{P}_{-i}\big[\mathscr{P}^{k,0}\big(\pi_i^k\big)\big] \\
&\qquad + \underbrace{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\,\Big|\,\mathscr{M}^{k,0}\big(\pi_i^k\big)\Big]}_{=0} \cdot \mathbb{P}_{-i}\big[\mathscr{M}^{k,0}\big(\pi_i^k\big)\big] \\
&\qquad + \underbrace{\mathbb{E}_{-i}\Big[g_k\big((1,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\,\Big|\,\mathscr{T}^{k,0}\big(\pi_i^k\big)\Big]}_{=\frac{1}{2}} \cdot \mathbb{P}_{-i}\big[\mathscr{T}^{k,0}\big(\pi_i^k\big)\big] \Big\} \\
&+ \sum_{k:u_i^k<0} u_i^k \cdot \Big\{ \underbrace{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\,\Big|\,\mathscr{P}^{k,1}\big(\pi_i^k\big)\Big]}_{=0} \cdot \mathbb{P}_{-i}\big[\mathscr{P}^{k,1}\big(\pi_i^k\big)\big] \\
&\qquad + \underbrace{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\,\Big|\,\mathscr{M}^{k,1}\big(\pi_i^k\big)\Big]}_{=1} \\
&\qquad \cdot \Big(1 - \mathbb{P}_{-i}\big[\mathscr{P}^{k,1}\big(\pi_i^k\big)\big] - \mathbb{P}_{-i}\big[\mathscr{T}^{k,1}\big(\pi_i^k\big)\big]\Big) \\
&\qquad + \underbrace{\mathbb{E}_{-i}\Big[g_k\big((0,\pi_i^k),\overset{*}{s}{}_{-i}^k\big)\,\Big|\,\mathscr{T}^{k,1}\big(\pi_i^k\big)\Big]}_{=\frac{1}{2}} \cdot \mathbb{P}_{-i}\big[\mathscr{T}^{k,1}\big(\pi_i^k\big)\big] \Big\} + C \\
= &\sum_{k:u_i^k>0} u_i^k \cdot \Big(\mathbb{P}_{-i}\big[\mathscr{P}^{k,0}\big(\pi_i^k\big)\big] + \frac{1}{2} \cdot \mathbb{P}_{-i}\big[\mathscr{T}^{k,0}\big(\pi_i^k\big)\big]\Big) \\
&+ \sum_{k:u_i^k<0} u_i^k \cdot (-1) \cdot \Big(\mathbb{P}_{-i}\big[\mathscr{P}^{k,1}\big(\pi_i^k\big)\big] + \frac{1}{2} \cdot \mathbb{P}_{-i}\big[\mathscr{T}^{k,1}\big(\pi_i^k\big)\big]\Big) + \tilde{C} \qquad (3.11)
\end{aligned}
$$

with $\tilde{C}$ independent of π_i. Exploiting symmetry and independence assumptions the crucial step in the proof is to realize that every report profile of other agents is equally probable. Formally, the fact that ρ_i^k is centered around zero for all $k \in D$ and independence of $\{u_i^k\}_k$ imply that $\overset{*}{a}_i \sim \text{Uniform}\big(\{0,1\}^d\big)$ for all $i \in N$. From $\big\{u_i^k\big\}_k$ independent and identically distributed it follows that $\overset{*}{\pi}_i \sim \text{Uniform}\,(\sigma(D))$ for all $i \in N$. By Definition 8 we have $\overset{*}{a}_i \perp \overset{*}{\pi}_i$, which implies $\overset{*}{s}_i \sim \text{Uniform}\,(M)$ for all $i \in N$. Independence of the family $\big\{u_i^k\big\}_i$ guarantees $\overset{*}{s}_{-i} \sim \text{Uniform}\big(\mathscr{M}_{-i}\big)$ for all $i \in N$, which together with (3.10) allows us to rewrite (3.11) dropping superscripts

$$
\sum_k \big|u_i^k\big| \cdot \Big(\mathbb{P}_{-i}\big[\mathscr{P}\big(\pi_i^k\big)\big] + \frac{1}{2} \cdot \mathbb{P}_{-i}\big[\mathscr{T}\big(\pi_i^k\big)\big]\Big) + \tilde{C}. \qquad (3.12)
$$

From (3.9) it follows that for all $\tilde{\pi}_i^k < \pi_i^k \in D$

$$
\mathbb{P}_{-i}\big[\mathscr{P}\big(\tilde{\pi}_i^k\big)\big] + \frac{1}{2} \cdot \mathbb{P}_{-i}\big[\mathscr{T}\big(\tilde{\pi}_i^k\big)\big] \;\le\; \mathbb{P}_{-i}\big[\mathscr{P}\big(\pi_i^k\big)\big] + \frac{1}{2} \cdot \mathbb{P}_{-i}\big[\mathscr{T}\big(\pi_i^k\big)\big]
$$

which implies that (3.12) is maximized for

$$\pi_i = \overset{*}{\pi}_i \in \left\{\pi_i \in \sigma(D) : \pi_i^k < \pi_i^l \text{ only if } \left|u_i^k\right| \leq \left|u_i^l\right| \text{ for all } k,l \in D\right\}.$$

□

Proof of Proposition 7. Let $\mathcal{M}$ be defined as in Definition 7. For any mechanism $(\mathcal{M}, g)$, not necessarily a Ranking Mechanism, ex-ante expected welfare is given by

$$\begin{aligned}
&\mathbb{E}\left[\sum_i V_i\left(g(\overset{*}{s})\right)\right] \\
&= \sum_{m \in \mathcal{M}^{RM}} \mathbb{P}[\overset{*}{s} = m] \cdot \mathbb{E}\left[\sum_i \sum_k u_i^k \cdot g_k(m) \middle| \overset{*}{s} = m\right] \\
&= \sum_m \mathbb{P}[\overset{*}{s} = m] \cdot \sum_i \sum_k \mathbb{E}_i\left[u_i^k \middle| \overset{*}{s}_i = m_i\right] \cdot g_k(m) \\
&= \sum_m \mathbb{P}[\overset{*}{s} = m] \cdot \sum_i \sum_k \left(2 \cdot a_i^k - 1\right) \cdot \mathbb{E}_i\left[\left|u_i^k\right| \middle| \overset{*}{\pi}_i = \pi_i\right] \cdot g_k(m) \\
&= \sum_m \mathbb{P}[\overset{*}{s} = m] \cdot \sum_i \sum_k \left(2 \cdot a_i^k - 1\right) \cdot \mathbb{E}_i\left[\left|u_i^k\right|_{(\pi_i^k:d)}\right] \cdot g_k(m),
\end{aligned}$$

which implies that the is no loss in restricting attention to separable mechanisms and $g \equiv g^{RM,\overset{*}{w}}$ from Definition 9 and 10 is efficient. □

Proof of Proposition 8. We rewrite the sum of ex-ante expected utilities in problem k under $g_k^{RM,\overset{*}{w}}$ as

$$\begin{aligned}
&\sum_i \mathbb{E}\left[u_i^k \cdot g_k^{RM,\overset{*}{w}}\left(\overset{*}{s}^k\right)\right] \\
&= \sum_{m^k}\left(\mathbb{P}\left[\overset{*}{s}^k = m^k\right] \cdot g_k^{RM,\overset{*}{w}}\left(m^k\right) \cdot \sum_i \mathbb{E}_i\left[u_i^k \mid \overset{*}{s}_i^k = m_i^k\right]\right) \\
&= \sum_{(a^k,\pi^k)}\left(\mathbb{P}\left[\overset{*}{s}^k = \left(a^k,\pi^k\right)\right] \cdot \max\left\{0, \sum_i\left(2a_i^k - 1\right) \cdot \mathbb{E}_i\left[\left|u_i^k\right|_{(\pi_i^k:d)}\right]\right\}\right) \\
&= \frac{1}{2^n \cdot d^n} \cdot \sum_{a^k \in \{0,1\}^n} \sum_{\pi_1^k=1}^{d} \cdots \sum_{\pi_n^k=1}^{d} \max\left\{0, \sum_i\left(2a_i^k - 1\right) \cdot \mathbb{E}_i\left[\left|u_i^k\right|_{(\pi_i^k:d)}\right]\right\} \\
&= \frac{1}{2^n} \cdot \sum_{a^k \in \{0,1\}^n} \int_0^1 \cdots \int_0^1 \max\left\{0, \sum_i\left(2a_i^k - 1\right) \cdot \mathbb{E}_i\left[\left|u_i^k\right|_{(\lceil d \cdot \pi_i^k \rceil:d)}\right]\right\} \mathrm{d}\pi_1^k \ldots \mathrm{d}\pi_n^k,
\end{aligned}$$

where we used the fact that $\overset{*}{s}^k \sim \text{Uniform}\left(\mathscr{M}^k\right)$. Exploiting boundedness of all integrals and continuity of the maximum-operator we obtain

$$\begin{aligned}
&\lim_{d\to\infty} \frac{1}{2^n} \cdot \sum_{a^k\in\{0,1\}^n} \int_0^1 \cdots \int_0^1 \max\left\{0, \sum_i \left(2a_i^k - 1\right) \cdot \mathbb{E}_i\left[\left|u_i^k\right|_{\left(\lceil d\cdot\pi_i^k\rceil:d\right)}\right]\right\} \mathrm{d}\pi_1^k \ldots \mathrm{d}\pi_n^k \\
=&\frac{1}{2^n} \cdot \sum_{a^k\in\{0,1\}^n} \int_0^1 \cdots \int_0^1 \max\left\{0, \sum_i \left(2a_i^k - 1\right) \cdot \mathbb{E}_i\left[\lim_{d\to\infty} \left|u_i^k\right|_{\left(\lceil d\cdot\pi_i^k\rceil:d\right)}\right]\right\} \mathrm{d}\pi_1^k \ldots \mathrm{d}\pi_n^k \\
=&\frac{1}{2^n} \cdot \sum_{a^k\in\{0,1\}^n} \int_0^1 \cdots \int_0^1 \max\left\{0, \sum_i \left(2a_i^k - 1\right) \cdot \Phi^{-1}\left(\pi_i^k\right)\right\} \mathrm{d}\pi_1^k \ldots \mathrm{d}\pi_n^k \\
=&\mathbb{E}\left[\max\left\{0, \sum_{i=1}^n u_i^k\right\}\right],
\end{aligned} \tag{3.13}$$

where Φ^{-1} denotes the inverse cdf of the absolute value of agents' valuations. The crucial step in the proof makes use of the following result on the asymptotic convergence of order statistics. For any random variable X with cdf F and pdf f and any $p \in [0, 1]$ it holds that $X_{(\lceil d\cdot p\rceil:d)} \sim AN\left(F^{-1}(p), \frac{p\cdot(1-p)}{d\cdot f(F^{-1}(p))^2}\right)$ at all points such that $f\left(F^{-1}(p)\right) \neq 0$, see Ahsanullah, Nevzorov, and Shakil (2013), page 111. Since (3.13) is the sum of ex-ante expected utility levels that correspond to full efficiency for problem k this concludes the proof.

□

Proof of Proposition 9. Fix $w \in W$. We need to show that for all agents $i \in N$, all $u_i \in \mathscr{U}_i$ and $\overset{*}{s}_{-i} = \left(\overset{*}{a}_{-i}, \overset{*}{\pi}_{-i}\right)$ it holds that

$$\overset{*}{s}_i(u_i) \ \in \ \arg\max_{s_i} \left\{\mathbb{E}_\gamma\left[\mathbb{E}_{-i}\left[V_i(g^{RRM}(s_i, \overset{*}{s}_{-i}))\right]\right]\right\}.$$

We define the profile of shuffled sincere strategies of all other agents as $\tilde{s}_{-i} = (\overset{*}{a}_{-i}, \gamma_{-i}(\overset{*}{\pi}_{-i}))$ and rewrite the above expression as

$$\begin{aligned}
&\mathbb{E}_{\gamma_i}\left[\mathbb{E}_{u_{-i},\gamma_{-i}}\left[V_i(g^{RRM}(s_i, \overset{*}{s}_{-i}))\right]\right] \\
&= \mathbb{E}_{\gamma_i}\left[\mathbb{E}_{u_{-i},\gamma_{-i}}\left[V_i(g^{RM}(\gamma_i(s_i), \tilde{s}_{-i}))\right]\right] \\
&= (1 - \alpha_i) \cdot \underbrace{\mathbb{E}_{u_{-i},\gamma_{-i}}\left[V_i(g^{RM}(s_i, \tilde{s}_{-i}))\right]}_{(*)} \\
&\quad + \alpha_i \cdot \sum_{\pi_i'} \beta_i^{\pi_i'} \cdot \underbrace{\mathbb{E}_{u_{-i},\gamma_{-i}}\left[V_i(g^{RM}((a_i, \pi_i'), \tilde{s}_{-i}))\right]}_{(**)}.
\end{aligned}$$

By construction $\tilde{s}_{-i} \sim \text{Uniform}(\mathscr{M}_{-i})$ and therefore expression $(*)$ is maximized by $\overset{*}{s}_i$ by Proposition 6. Further expression $(**)$ is independent of the reported π_i and maximized by $\overset{*}{a}_i$ by the same logic as Part 1 of Proposition 6.

□

Proof of Proposition 10. We write ex-ante expected welfare under the Randomized Ranking Mechanism $(\mathscr{M}, g^{RRM,w})$ as

$$
\begin{aligned}
\mathbb{E}\left[\sum_i V_i(g^{RRM,w}(\overset{*}{s}))\right] &= \mathbb{E}\left[\sum_i V_i(g^{RM,w}(\tilde{s}))\right] \\
= \sum_{m\in\mathscr{M}} \mathbb{P}[\tilde{s} = m] \cdot \sum_i \sum_k \mathbb{E}\left[u_i^k | \tilde{s} = m\right] \cdot g_k^{RM,w}(m) \\
= \frac{1}{(2^d \cdot d!)^n} \cdot \sum_m \sum_k \sum_i \mathbb{E}\left[u_i^k | \tilde{s}_i^k = m_i^k\right] \cdot g_k^{RM,w}\left(m^k\right).
\end{aligned} \tag{3.14}
$$

Let $\bar{M}$ denote the set of all possible report profiles of all agents for a single decision problem, i.e. $\bar{M} = \left(\{0,1\}, D\right)^{\otimes n}$. Further, we divide $\bar{M}$ into anonymous equivalence classes. Formally, for $\bar{m} \in \bar{M}$ define $[\bar{m}] = \left\{\tilde{m} \in \bar{M} | \left(\tilde{m}_{\sigma(1)}, ..., \tilde{m}_{\sigma(n)}\right) = \bar{m} \text{ for some } \sigma \in \sigma(N)\right\}$. We refer to $[\bar{m}]$ as the equivalence class and its representative interchangeably and denote by $[\bar{M}]$ the set of all equivalence classes. We write (3.14) as

$$
\begin{aligned}
&\frac{1}{(2^d \cdot d!)^n} \cdot \sum_k \sum_{\bar{m}\in\bar{M}} \sum_i \mathbb{E}\left[u_i^k | \tilde{s}_i^k = \bar{m}_i\right] \cdot g_k^{RM,w}(\bar{m}) \\
&= \frac{1}{(2^d \cdot d!)^n} \cdot \sum_k \sum_{[\bar{m}]\in[\bar{M}]} \sum_{\bar{m}\in[\bar{m}]} \sum_i \mathbb{E}\left[u_i^k | \tilde{s}_i^k = \bar{m}_i\right] \cdot g_k^{RM,w}(\bar{m}) \\
&= \frac{1}{(2^d \cdot d!)^n} \cdot \sum_k \sum_{[\bar{m}]} g_k^{RM,w}\left([\bar{m}]\right) \cdot \left((n-1)! \cdot \sum_j \sum_i \mathbb{E}\left[u_i^k | \tilde{s}_i^k = [\bar{m}]_j\right]\right),
\end{aligned} \tag{3.15}
$$

where we used that $g^{RM,w}$ is anonymous and that for all $i, j \in N$ there exist $(n-1)!$ permutations in $\sigma(N)$ sending i onto j. After suitably rearranging (3.15) Definition 9 gives us

$$
\begin{aligned}
&\frac{1}{(2^d \cdot d!)^n} \cdot \sum_k n! \cdot \sum_{[\bar{m}]} \sum_j \frac{1}{n} \cdot \sum_i \mathbb{E}\left[u_i^k | \tilde{s}_i^k = [\bar{m}]_j\right] \cdot g_k^{RM,w}\left([\bar{m}]\right) \\
=&\frac{1}{(2^d \cdot d!)^n} \cdot \sum_k \sum_{\bar{m}\in\bar{M}} \sum_j \left(\frac{1}{n} \cdot \sum_i \mathbb{E}\left[u_i^k | \tilde{s}_i^k = \bar{m}_j\right]\right) \cdot \left(\frac{1}{d} \cdot \sum_l g_l^{RM,w}(\bar{m})\right) \\
=&\frac{1}{(2^d \cdot d!)^n} \cdot \sum_{\bar{m}\in\bar{M}} \sum_j \sum_l \left(\frac{1}{n} \cdot \sum_i \sum_k \frac{1}{d} \cdot \mathbb{E}\left[u_i^k | \tilde{s}_i^k = \bar{m}_j\right]\right) \cdot g_l^{RM,w}(\bar{m}) \\
=&\sum_{m\in\mathcal{M}} \mathbb{P}[\tilde{s} = m] \cdot \sum_j \sum_l \left(2 \cdot a_j^l - 1\right) \\
&\cdot \left(\frac{1}{n} \cdot \sum_i \sum_k \mathbb{P}\left[\tilde{\pi}_i^k = \pi_j^l\right] \cdot \mathbb{E}\left[\left|u_i^k\right| \middle| \tilde{\pi}_i^k = \pi_j^l\right]\right) \cdot g_l^{RM,w}\left(m^l\right) \\
\leq&\sum_{m\in\mathcal{M}} \mathbb{P}[\tilde{s} = m] \cdot \sum_j \sum_l \left(2 \cdot a_j^l - 1\right) \\
&\cdot \left(\frac{1}{n} \cdot \sum_i \sum_k \mathbb{P}\left[\tilde{\pi}_i^k = \pi_j^l\right] \cdot \mathbb{E}\left[\left|u_i^k\right| \middle| \tilde{\pi}_i^k = \pi_j^l\right]\right) \cdot g_l^{RM,\overset{**}{w}}\left(m^l\right) \\
=&\mathbb{E}\left[\sum_j V_j(g^{RM,\overset{**}{w}}(\tilde{s}))\right] = \mathbb{E}\left[\sum_j V_j(g^{RRM,\overset{**}{w}}(\overset{*}{s}))\right],
\end{aligned}
$$

with weight vector $\overset{**}{w} \in W$ given by

$$
\begin{aligned}
\overset{**}{w}{}^r =&\frac{1}{n} \cdot \sum_i \sum_k \mathbb{P}\left[\tilde{\pi}_i^k = r\right] \cdot \mathbb{E}\left[\left|u_i^k\right| \middle| \tilde{\pi}_i^k = r\right] \\
=&\frac{1}{n} \cdot \sum_i \sum_k \Bigg((1-\alpha_i) \cdot \mathbb{P}_i\left[\overset{*}{\pi}_i^k = r\right] \cdot \mathbb{E}\left[\left|u_i^k\right| | \overset{*}{\pi}_i^k = r\right] \\
&+ \alpha_i \cdot \sum_{\hat{\pi}_i : \hat{\pi}_i^k = r} \beta_i^{\hat{\pi}_i} \cdot \mathbb{E}\left[\left|u_i^k\right|\right] \Bigg) \\
=&\frac{1}{n} \cdot \sum_i \sum_k \Bigg((1-\alpha_i) \cdot \mathbb{P}_i\left[\overset{*}{\pi}_i^k = r\right] \cdot \mathbb{E}\left[\left|u_i^k\right|_{(r:d)}\right] \\
&+ \alpha_i \cdot \mathbb{P}_{\beta_i}\left[\tilde{\pi}_i^k = r\right] \cdot \mathbb{E}\left[\left|u_i^k\right|\right] \Bigg)
\end{aligned}
$$

for $r = 1, \ldots, d$ as in Definition 15.

□

4

Fake Experts

Joint with Patrick Lahr

4.1 Introduction

Consider a set of physicians and researchers who help to formulate a new guideline on basic life support.[1] All members of this scientific advisory board have the same goal, to make recommendations that help to save as many lives as possible. Some physicians on the board might have more relevant information on the best procedure than others due to their research activities or their specialty. To formulate the guideline it is in everyone's interest that all members report their privately known competence and to weight members by their expertise.

As a second example consider a scientific advisory board that recommends to a regulatory body whether or not to approve a drug designed to prevent heart attacks. Besides varying information on the efficacy of the drug board members might have privately known commercial interests due to relations to pharmaceutical companies.[2] These board members have an incentive to present their assessment in a way that sways the decision in their preferred direction. Giving board members more weight based on self-reported expertise can lead to manipulation due to private interests.

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[1] For example, the sequence of steps for cardiopulmonary resuscitation has been changed in the 2010 American Heart Association guidelines (see Field et al. (2010)) from A(irway)-B(reathing)-C(hest compressions) to C-A-B.

[2] Piller (2018) discusses in an article in *Science* the specific case of approval of a drug designed to prevent heart attacks and strokes. He finds that one physician in the admission panel received more than $2 million for various purposes from the drug manufacturing pharmaceutical company. Further, Piller (2018) documents that the majority of 107 physicians who advised the Food and Drug Administration in the United States on the approval of 28 drugs from 2008 to 2014 received payments from makers of the drugs or from competing firms.

We analyze decision problems of this kind with and without private interests in a multi-sender cheap talk game in which senders have private signals of heterogeneous informativeness about a binary state of the world. The senders represent the scientific advisory board. The receiver processes the senders' messages and takes an action that aims to match the state of the world. She represents the regulatory body which is uncertain about private interests of senders. In many fields, such bodies regularly seek scientific advice, for example in economics, medicine, agriculture or ecology.

We show that under common interests the receiver can weight all qualities of signals differently. In the receiver-optimal equilibrium, senders transmit all information and the receiver acts as if senders' information were public. Further, specialization, i.e. more heterogeneous signal quality, improves the receiver's ability to match the state. In other words, it is better for the receiver to face few senders with precise and many with imprecise signals instead of facing senders who all have medium precise information.

We continue to study the effect of private interests, in the sense that senders might prefer a decision irrespective of the state of the world. We assume that conflicts of interests are private information. Other senders or the receiver are only aware that these conflicts occur with a certain probability. We show that under private interests, complete differentiation of signal qualities breaks down. Senders whose preferences are not aligned with the receiver's claim to have highly informative signals about the state of the world. These fake experts prevent optimal signal discrimination and devalue messages sent by senders with the most informative signals. This diminishes the gains from specialization and average individual precision becomes more important.

If privately known preferences are sufficiently heterogeneous, any differentiating weighting of messages breaks down, and the receiver uses only two weights in the optimal equilibrium. Private interests create incentives to exaggerate recommendations in a way that prevents any transmission of information that is finer than the mere direction of the recommended decision. An information aggregation setting with only two messages can be interpreted as voting. In a world with strong private preferences voting constitutes the best mechanism to aggregate information because it is robust to manipulation.

The rest of the paper is organized as follows. Section 4.2 reviews related literature on cheap talk and information aggregation. The formal model is presented in Section 4.3. Section 4.4 studies the common interests case. We solve for the receiver optimal equilibrium and introduce the concept of specialization. We analyze the case with private interests in Section 4.5. We solve for the receiver optimal equilibrium and study the effect of private preferences. We find that the role of specialization decreases and that the role of average individual precision increases when preferences become more heterogeneous. Further, we show that voting becomes the best way to aggregate information if preferences are sufficiently heterogeneous. Section 4.6 concludes and discusses open questions and possible future lines of research building on our work.

4.2 Related Literature

Our work is connected to the literature on cheap talk and to the literature on information aggregation in voting. The former builds on the seminal work of Crawford and Sobel (1982) and analyzes strategic communication between a better-informed sender and a receiver whose action determines the payoff of both. In their original setup, the sender has private and perfect information on a one-dimensional state of the world and a bias known to the receiver.

The approach has been generalized to settings with multiple senders. Gilligan and Krehbiel (1989) study a model in which two privately and perfectly informed senders with publicly known biases communicate with a receiver. The focus of their analysis is the comparison of three communication protocols that comprise different forms of cheap talk. Similarly, Krishna and Morgan (2001) study a setting with two senders that sequentially send public messages to a receiver. The degree of information revelation depends on whether the senders have aligned or opposing biases.

Austen-Smith (1990) is the first who studies a cheap talk problem in which senders are imperfectly informed about a binary state of the world. While he identifies circumstances under which a cheap talk phase alters the decision, Wolinsky (2002) solves for the most efficient communication structure.

We build on these approaches by studying a multi-sender cheap talk game with imperfectly informed senders. However, a key difference between all mentioned papers and our work is that we do not assume that biases are known to the receiver. In contrast, we understand biases as a private characteristic of senders. We are not the first who model bias as private information. However, in other models the bias is often directly payoff relevant for the receiver whereas it only obfuscates useful information in our setting. For example Alonso, Dessein, and Matouschek (2008) study under which circumstances a corporation should delegate certain decisions to its regional subsidiaries in order to make it more responsive to privately held information by the subsidiaries. Similarly, Hummel, Morgan, and Stocken (2013) model a firm that engages in market research. Each individual respondent wants the firm to match his personal type while the firm wants to match the average type of respondents. Misalignment of preferences arises endogenously and is unknown to the receiver.

The literature on information aggregation in voting goes back to Condorcet (1785) and his famous jury theorem, stating that large groups of independently informed agents select the correct alternative with almost certainty. He assumes that agents vote sincerely while Feddersen and Pesendorfer (1997) establish a similar result for strategic agents. They show that privately held information leads to the same decision as under public information.

Some papers allow for a bigger message space to aggregate information. For example, McMurray (2017) studies a common interests election of ex-ante symmetric candidates by a fixed number of heterogeneously informed agents. In equilibrium voters coordinate on specific candidates to transmit information. His model can be interpreted as a cheap talk game with a restricted number of messages. If the number of candidates becomes large the model converges to our common interests setting.

The above papers focus on information aggregation in large electorates. They do not shed light on the (in)efficiency with which information is aggregated, particularly when the electorate is finite and heterogeneous. Azrieli (2018b) studies this question. He analyzes the loss of anonymous voting rules if agents are publicly known to be differently well informed. The common-value analysis is closely related to ours. However, we assume that signals are private information and are interested in the interplay with private interests.

4.3 The Model

There is a set of senders $\{1, ..., n\}$ and a receiver indexed by 0. Each sender i receives an independently and identically distributed signal about an unknown state of the world $\omega = \{0, 1\}$. There is a common prior $p_0 = \mathbb{P}[\omega = 1] \in (0, 1)$ that the state of the world is 1. Each signal induces a posterior $p_i = \mathbb{P}[\omega = 1|\text{signal}]$ distributed according to a probability mass function μ_ω conditional on the state of the world being ω. The ex-ante posterior distribution $\mu = (1 - p_0)\mu_0 + p_0\mu_1$ is consistent with the common prior p_0. We assume that the information structure is such that it leads to a finite number of possible posteriors $\mathscr{P} = \text{supp}\,\mu$. For some results we assume that no signal is not informative at all, i.e. $p_0 \notin \mathscr{P}$. We call a distribution μ that fulfills this assumption *never ignorant*. The receiver gets no signal, but shares the prior.

In addition to different signals players are also heterogeneous with respect to their preferences as described by a preference parameter $\lambda \in \{0, \lambda_0, 1\}$ with $\lambda_0 \in (0, 1)$. Each sender i independently draws a preference parameter λ_i that is also independent of the posteriors and distributed according to probability mass function γ. The decision maker has commonly known preference parameter λ_0. We call the tuple (p_i, λ_i) the type of sender i and $\mu \times \gamma$ the distribution over types.

After observing the signal each sender i simultaneously sends a cheap talk message $t_i \in [0, 1]$ to the receiver. We denote the potentially mixed strategy by $m_i : \mathscr{P} \times \{0, \lambda_0, 1\} \to \Delta[0, 1]$ where $\Delta[0, 1]$ denotes the set of all probability measures over $[0,1]$. We denote the probability that sender i with type (p_i, λ_i) sends message t_i by $m_i(p_i, \lambda_i)(t_i)$. We call a strategy *truthful* if $m_i(p_i, \lambda_i)(p_i) = 1$ for all types (p_i, λ_i). The tuple of messages of all senders is denoted by $t = (t_1, ..., t_n)$.

The receiver processes the messages of all senders according to Bayes' rule. We denote the belief of the receiver accounting only for sender i's message t_i by $q(t_i)$ and call it the virtual posterior of sender i.[3] The virtual posterior of the receiver incorporating the messages t of all senders is denoted by $q(t)$. After getting and processing all messages the receiver takes an action $a \in \{0, 1\}$. Utilities for senders and the receiver are given by

$$u(a, \omega, \lambda_i) = (1 - \lambda_i)1\{a = \omega = 0\} + \lambda_i 1\{a = \omega = 1\},$$

where $1\{A\}$ is the indicator function that is 1 if event A is *true* and 0 otherwise.

A player i prefers action 1 if and only if his belief that the state of the world is one is larger or equal $1 - \lambda_i$. A higher preference parameter λ_i leads to a higher utility of player i given that action and state are equal to one. Senders with preference parameters 0 and 1 weakly prefer the action that matches their preference parameter irrespective of the posterior. We call senders with these preference parameters *partisans*. The remaining senders with $\lambda_i = \lambda_0$ have the same interests as the receiver. We call these senders *advisors*.

Before we proceed, we summarize the timing of the game. First, nature draws a state of the world ω. Second, every sender i randomly draws a type (p_i, λ_i) according to the conditional type distribution $\mu_\omega \times \gamma$. Third, each sender i sends a message t_i to the receiver. Last, the receiver takes an action a and payoffs realize. We assume that the receiver does not have commitment, i.e. she can not credibly commit to a decision rule before getting the messages of the senders.[4] Consequently, we solve for perfect Bayesian equilibria.

The distribution of posteriors μ is a key object in our setting. We are interested to explore the effect of the signal structure on the receiver's ability to match the state of the world. We introduce two concepts that allow us to compare posterior distributions with respect to the utility of the receiver. The first definition describes an incomplete order of distributions that is well known in the literature.

[3] Anticipating that senders play symmetric strategies in the optimal equilibrium, we drop the subscript i of the virtual posterior $q_i(\cdot)$ to simplify notation.

[4] In particular, this excludes equilibria of the kind discussed in Gerardi, McLean, and Postlewaite (2009).

Definition 1. *Let μ and ν two posterior distributions with cdfs F and G, respectively. We say that μ is more informative than ν, denoted by $\mu \succ \nu$, if*

$$\int_0^y F(x)\mathrm{d}x \;\geq\; \int_0^y G(x)\mathrm{d}x \quad \textit{for all } y \;\in\; [0,1]. \tag{4.1}$$

Blackwell (1951) established the concept of informativeness[5] in the context of decision problems. We remind the reader on some of his results in Appendix 4.B. By the common prior assumption both distribution have an expected value of p_0.

A second method to compare posterior distributions is by means of their average individual precision.

Definition 2. *The average individual precision $\pi(\mu)$ of a sender's posterior distribution μ is*

$$\pi(\mu) \;=\; \mathbb{E}\left[|p_i - p_0|\right].$$

The average individual precision of a distribution measures the expected distance of the posterior from the prior p_0. Mathematically, it is the first absolute central moment. The greater the difference between prior and posterior the more precise is the information of a sender. A distribution μ with $\pi(\mu) = 0$ does not contain any information at all whereas the maximal average individual precision is $2p_0(1-p_0)$. Note that a higher average individual precision of μ compared to ν is a necessary but not a sufficient condition for μ being more informative than ν. In fact, in Section 4.4.2 we fix average individual precision and specify distributions that are maximally and minimally informative.

In the following we split the analysis in two parts. We start to study the common interests case in Section 4.4. In the common interests case all players have aligned preferences, i.e. $\gamma(\lambda_0) = 1$ and $\gamma(0) = \gamma(1) = 0$. This special case of our setting serves as a benchmark and allows us to get familiar with the concepts introduced in Definitions 1 and 2. In Section 4.5 we proceed to the general case in which we allow for private interests.

4.4 Common Interests Analysis

In this section we assume that players have common interests. In Subsection 4.4.1 we derive the receiver optimal perfect Bayesian equilibrium. We introduce the concept of specialization in Subsection 4.4.2 and derive maximally and minimally specialized distributions.

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[5] The concept is also know in the literature as second order stochastic domination, i.e. μ is second order stochastic dominated by ν if Condition (4.1) is fulfilled.

4.4.1 Receiver Optimal Equilibrium

There are many perfect Bayesian equilibria in cheap talk games. In this subsection we derive an equilibrium that maximizes the utility of the receiver. In the common interests case such an equilibrium maximizes the utility of the senders too. We define a class of strategies for the receiver that plays a crucial role in equilibrium.

Definition 3. *A receiver follows a weighted majority rule if her strategy* $a : [0,1]^n \to \{0,1\}$ *is of the form*

$$a(t) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} w(t_i) > \tau \\ 0 & \text{else} \end{cases}$$

for messages $t = (t_1, ..., t_n)$ *of senders, a weighting function* $w : [0,1] \to \mathbb{R}$*, and a threshold* τ*.*

In a weighted majority rule the receiver transforms every message t_i into a weight $w(t_i)$ and takes decision 1 if the sum of weighted messages is larger than a threshold τ. The next proposition derives the optimal equilibrium and the corresponding weighting function.

Proposition 1. *The following describes a receiver optimal perfect Bayesian equilibrium:*

- *Senders play the truthful strategy, i.e.* $m_i(p_i, \lambda_0)(p_i) = 1$ *for all types* (p_i, λ_0) *and all* $i \in \{1, ..., n\}$
- *The receiver calculates the virtual posterior* $q(t_i) = t_i$ *for a message* $t_i \in \mathscr{P}$ *by Bayes' rule*
- *The receiver's virtual posterior of off-equilibrium messages* $t_i \notin \mathscr{P}$ *is given by* $q(t_i) = p_0$
- *The receiver follows a weighted majority rule with weighting function*

  $$w(x) = \begin{cases} \ln \frac{x}{1-x} - \ln \frac{p_0}{1-p_0} & \text{if } x \in \mathscr{P} \\ 0 & \text{else} \end{cases}$$

  and threshold $\tau = -\left(\ln \frac{\lambda_0}{1-\lambda_0} + \ln \frac{p_0}{1-p_0}\right)$

Proof. See Appendix 4.A. □

In the optimal equilibrium senders play the truthful strategy to transmit all information to the receiver. The receiver behaves as if all private signals were public and can base the decision on all available information. Hence, there cannot be any equilibrium with higher payoffs for the receiver.[6]

[6] McLennan (1998) studies optimality of equilibria in common interests games more generally.

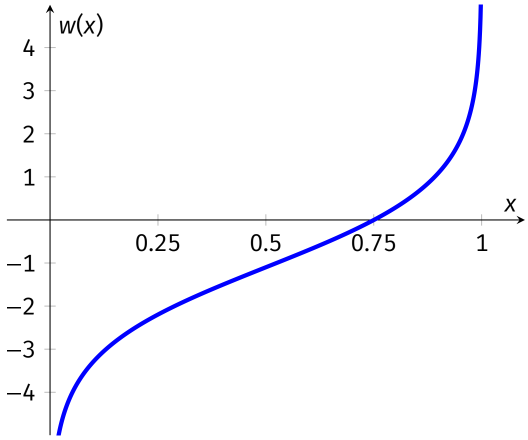


Figure 4.1. Weighting function for the common interests case

The strategy of the receiver is a generalization of the decision rule derived in Theorem 1 in Nitzan and Paroush (1982). They study a non-strategic setting with a symmetric prior $p_0 = \frac{1}{2}$. In the proof of Proposition 1 the common prior and the conditional iid posteriors allow to write the updating process of the receiver as a product formula of posteriors. Applying the logarithm to the equation gives the problem an additive structure. Every posterior can be mapped into a weight that is the log-likelihood ratio of the posterior $\ln \frac{p_i}{1-p_i}$ minus the log-likelihood ratio of the prior $\ln \frac{p_0}{1-p_0}$. The threshold is the log-likelihood ratio of the receiver's preference parameter $\ln \frac{\lambda_0}{1-\lambda_0}$ minus the log-likelihood ratio of the prior $\ln \frac{p_0}{1-p_0}$. In this way the prior p_0 is taken into account only once (in the threshold) and all other weights are taken net of the information from the prior. The log-likelihood ratio of the preference parameter $\ln \frac{\lambda_0}{1-\lambda_0}$ guarantees that action 1 is taken if and only if the final posterior $q(t)$[7] is larger than $(1-\lambda_0)$. Thus, the decision rule in the receiver optimal equilibrium can be interpreted as a weighted majority rule with weighting function $w(x) = \ln \frac{x}{1-x} - \ln \frac{p_0}{1-p_0}$ and threshold $\tau = -\left(\ln \frac{\lambda_0}{1-\lambda_0} + \ln \frac{p_0}{1-p_0}\right)$.

Figure 4.1 illustrates the weighting function with prior $p_0 = \frac{3}{4}$ for the common interests case. A posterior p_i of sender i that equals the prior p_0

[7] By similar steps as in the proof of Proposition 1 we can write the virtual posterior $q(t)$ of the receiver as

$$\begin{aligned} q &= \frac{\exp\left(\sum_{i=1}^{n}\left(\ln \frac{p_i}{1-p_i} - \ln \frac{p_0}{1-p_0}\right) + \ln \frac{p_0}{1-p_0}\right)}{1 + \exp\left(\sum_{i=1}^{n}\left(\ln \frac{p_i}{1-p_i} - \ln \frac{p_0}{1-p_0}\right) + \ln \frac{p_0}{1-p_0}\right)} \\ &= \frac{(1-p_0)^{n-1}\prod_{i=1}^{n} p_i}{(p_0)^{n-1}\prod_{i=1}^{n}(1-p_i) + (1-p_0)^{n-1}\prod_{i=1}^{n} p_i}. \end{aligned}$$

gets weight 0 because it does not transmit any additional information. In contrast, a posterior $p_i \in \{0, 1\}$ means that sender i perfectly knows the state of the world. The information of this sender is sufficient to make an optimal decision and he should outweigh all other senders. Thus, as p_i goes to 1 (0) the corresponding weight tends to ∞ ($-\infty$). The unrestrictedly high weight encodes the extraordinary value of perfect information.

For the rest of this section we refer to the receiver optimal equilibrium when we assess different distributions of sender types. The ex-ante expected utility of the receiver $u^\star(q(t))$ with the virtual posterior $q(t)$ in the receiver optimal equilibrium is given by

$$u^\star(q) \;=\; \begin{cases} q\lambda_0 & \text{if } q \;>\; 1-\lambda_0 \\ (1-q)(1-\lambda_0) & \text{else.} \end{cases}$$

We now turn to comparative statics building on the receiver optimal equilibrium. We employ Definition 1 and 2 to introduce the concept of specialization.

4.4.2 Specialization

In Definition 2 we define average individual precision as the expected distance of the posterior from the prior p_0. In this subsection we study distributions that have the same average individual precision. We show that there are distributions with the same average individual precision but yet result in different expected utilities for the receiver. More concretely, we hold average individual precision fixed and construct distributions of different informativeness according to Definition 1. If two distributions have the same average individual precision, we call a distribution that is more informative than the other one more *specialized*. A specialized distribution has the feature that there are senders who have a very precise signal about the state. On the downside, there are also senders with a very imprecise signal. With other words, there is a small probability of knowing much and a high probability of knowing little. This is in contrast to not specialized distributions where all senders have medium precise information.

We start to derive the most specialized distribution with a given average individual precision. Then we repeat this exercise for the least specialized distribution. Both distributions take simple forms so that we can use them to bound the utility of the receiver.

Consider posterior distribution μ with cdf F. We derive the most specialized distribution $\hat{\mu}$ given average individual precision $\pi(\mu)$. To determine the most specialized distribution we sequentially apply the operation of *mean-preserving spreads*. Performing a mean-preserving spread makes a distribution more informative as proven in Theorem 12.2.2 (5) in Blackwell and Girshick (1979). The average individual precision does not change if a mean preserving spread does not shift mass from the interval $[0, p_0)$ to the interval $(p_0, 1]$ or vice versa. Taking this into account we apply mean-preserving spreads until all mass is distributed on the set $\{0, p_0, 1\}$. If a distribution has mass on the open interval $(0, p_0)$ ($(p_0, 1)$) it is possible to perform a mean-preserving spread that distributes the mass on 0 and p_0 (p_0 and 1). The most specialized distribution $\hat{\mu}$ with average individual precision $\pi(\mu)$ has to fulfill the following three conditions.

1. $\hat{\mu}(0) + \hat{\mu}(p_0) + \hat{\mu}(1) = 1$
2. $0 \cdot \hat{\mu}(0) + p_0 \cdot \hat{\mu}(p_0) + 1 \cdot \hat{\mu}(1) = p_0$
3. $p_0 \cdot \hat{\mu}(0) + (1 - p_0) \cdot \hat{\mu}(1) = \pi(\mu)$

The first condition guarantees that all mass is distributed on $\{0, p_0, 1\}$, the second condition requires that the distribution is consistent with the common prior p_0 and the third condition ensures that the average individual precision is $\pi(\mu)$. The following lemma contains the solution to this system of equations.

Lemma 1. *Let μ be a posterior distribution. The most specialized posterior distribution $\hat{\mu}$ with an average individual precision $\pi(\mu)$ is characterized by*

$$\begin{aligned} \hat{\mu}(0) &= \frac{\pi(\mu)}{2p_0}, \\ \hat{\mu}(p_0) &= 1 - \frac{\pi(\mu)}{2p_0(1-p_0)}, \quad \textit{and} \\ \hat{\mu}(1) &= \frac{\pi(\mu)}{2(1-p_0)}. \end{aligned}$$

In the optimal equilibrium the most specialized distribution $\hat{\mu}$ leads to a specific kind of information aggregation. The message of a sender counts nothing unless he knows the state of the world with certainty. The information of one such sender is sufficient for the receiver to take an action that matches the state.

The derivation of the least specialized posterior distribution $\bar{\mu}$ is similar. A sequential application of *garblings* leads to the least informative distribution by Theorem 12.2.2 in Blackwell and Girshick (1979) (see Proposition A in Appendix 4.B). The process terminates if there are only two mass points p_l and p_r, one point left of the prior and one point right of it. $\bar{\mu}(l)$ and $\bar{\mu}(r)$ denote the probability mass on points p_l and p_r, respectively. To keep average individual precision $\pi(\mu)$ constant no garbling can combine mass from the two intervals $[0, p_0)$ and $(p_0, 1]$. To simplify the analysis, we assume that μ is never ignorant. Without this assumption the least specialized posterior distribution is not unique. In fact, there is a continuum of distributions that are consistent with the common prior p_0, have individual precision $\pi(\mu)$, and have only two mass points.

We are interested in the distribution that can constructed from μ. We call this distribution attainable from μ. The following four conditions characterize the least specialized distribution that is attainable from μ and has average individual precision $\pi(\mu)$.

1. $\bar{\mu}(l) + \bar{\mu}(r) = 1$
2. $p_l \cdot \bar{\mu}(l) + p_r \cdot \bar{\mu}(r) = p_0$
3. $(p_0 - p_l) \cdot \bar{\mu}(l) + (p_r - p_0) \cdot \bar{\mu}(r) = \pi(\mu)$
4. $\bar{\mu}(l) = F(p_0)$

The first two conditions are analogous to the first two conditions of the most specialized distribution. The third and fourth condition ensure that average individual precision is $\pi(\mu)$ and that no mass is shifted between the intervals $[0, p_0)$ and $(p_0, 1]$. A violation of these constraints would lead to a different average individual precision or to a distribution that is not attainable from μ. The next lemma solves the above system of equations.

Lemma 2. *Let μ be a posterior distribution with cdf F. The least specialized posterior distribution $\bar{\mu}$ that is attainable from μ with average individual precision $\pi(\mu)$ is characterized by*

$$\begin{aligned} \bar{\mu}(l) &= F(p_0), \\ p_l &= \frac{2F(p_0)p_0 - \pi(\mu)}{F(p_0)}, \\ \bar{\mu}(r) &= 1 - F(p_0), \quad \textit{and} \\ p_r &= \frac{2p_0 - 2F(p_0)p_0 + \pi(\mu)}{2 - 2F(p_0)}. \end{aligned}$$

A posterior distribution with only two mass points implies that senders use only two messages in the optimal equilibrium. An information aggregation setting with just two messages can be interpreted as voting for one or the other alternative. Thus, in the optimal equilibrium, a posterior distribution with two mass points corresponds to a qualified majority rule. Specifically, in the optimal equilibrium the receiver takes action 1 if $\sum_{i=1}^{n} w(t_i) > \tau$ with $w(t_i)$ and τ as in Proposition 1. If the distribution has only two mass points p_l and p_r there are only two weights $\alpha_l = w(p_l)$ and $\alpha_r = w(p_r)$ in equilibrium. We denote the number of senders whose message is left (right) of the prior by n_l (n_r). The receiver takes action 1 if

$$n_l \alpha_l + n_r \alpha_r \; > \; \tau.$$

This corresponds to a qualified majority rule with threshold $n_1(n)$ where n is the total number of senders. The receiver takes action 1 if and only if

$$n_1(n) > \frac{\tau - n\alpha_l}{\alpha_r - \alpha_l}.$$

In remainder of this subsection we apply the above insights to obtain bounds of the receiver's utility. As discussed, the least and most specialized distributions $\hat{\mu}$ and $\bar{\mu}$ have a particularly simple form. By Theorem 12.2.2 (4) in Blackwell and Girshick (1979) (see Proposition C in Appendix 4.B) the utility of the receiver is increasing in the specialization of posteriors. Thus, the next proposition summarizes Lemma 1 and 2 with respect to the utility of the receiver.

Proposition 2. *Let μ be a posterior distribution. The utility of the receiver facing senders with posterior distribution μ is bounded above by the utility of a receiver facing senders with $\hat{\mu}$ and below by the utility of a receiver facing senders with $\bar{\mu}$.*

We illustrate Proposition 2 idea with help of Figure 4.2. The y-axis represents the expected utility of the receiver and the x-axis the number of senders. The receiver matches the state of the world with probability close to 1 as the number of senders tends to infinity. This holds for any distribution μ with positive average individual precision $\pi(\mu)$. Thus, the blue line (representing the most specialized distribution) and the red line (representing the least specialized distribution) converge to the ex-ante expected utility in a setting where the state of the world is known by the receiver. If the number of senders is finite, the value of specialization is visible. The benefit of a sender that knows the state of the world perfectly is higher than the loss of the reduced probability with which such sender exists. Technically, this is a result of the concavity (convexity) of the weighting function $w(x) = \ln \frac{x}{1-x} - \ln \frac{p_0}{1-p_0}$ for posteriors $p_i < p_0$ $(p_i > p_0)$. On the other extreme, the step-wise form of least specialized distribution (red line) can be explained by the voting analogy. Ignoring a randomly drawn sender if the number of senders is even is the same as fair tie-breaking. Thus, a qualified majority rule with fair tie-breaking and an even number of senders n contains the same information as a qualified majority rule with $n-1$ senders.

We now turn to the general setting that allows for private interests. We return to a discussion of specialization in the general setting in Subsection 4.5.2.

4.5 Fake Experts - Private Interests Analysis

In this section we turn to the case with private interests. In Subsection 4.5.1 we solve for the receiver optimal equilibrium. In Subsection 4.5.2 we proceed to study effects of private interests in the receiver optimal equilibrium. We find

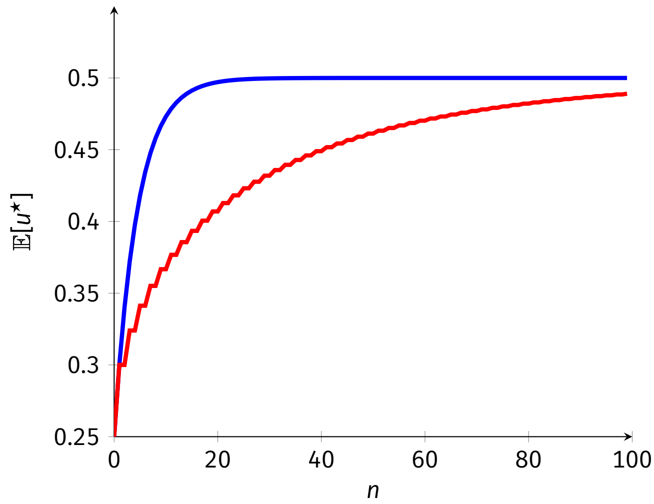


Figure 4.2. Upper and lower bounds for the receiver's utility for n senders with posterior distribution μ that is symmetric around the prior $p_0 = \frac{1}{2}$, average individual precision $\pi(\mu) = 0.1$ and $\lambda_0 = \frac{1}{2}$. The blue line is represents the utility of the most specialized distribution $\hat{\mu}$ and the red line the least specialized distribution $\bar{\mu}$.

that voting is optimal if preferences are sufficiently heterogeneous. Further, average individual precision becomes more and specialization less important as the number of partisans increases.

4.5.1 Receiver Optimal Equilibrium

We start by sketching that the strategies and beliefs in Proposition 1 do no longer form an equilibrium in the presence of private interests. Partisans can only get positive utility if the action of the receiver matches their preference type. Fixing the strategy of the receiver, partisans maximize the probability that their preferred action is taken by sending the message with the highest possible weight in the respective direction. But then the receiver cannot rationally believe that these messages only come from senders who indeed received the according signals. Consequently, players adapt their strategies. The next proposition summarizes the strategies in the receiver optimal equilibrium.

Proposition 3. *The following describes a receiver optimal perfect Bayesian equilibrium. There exists unique expertise bounds $\underline{b}, \overline{b} \in [0,1]$ that describe the lowest and highest possible virtual posterior, respectively.*

- *Advisors message truthfully:*

$$m_i(p_i, \lambda_0)(t_i) = \begin{cases} 1 \text{ if } t_i = p_i \\ 0 \text{ else} \end{cases}$$

- *0-partisans imitate the lowest posteriors $t_i \leq \underline{b}$:*

$$m_i(p_i, 0)(t_i) = \begin{cases} \frac{\gamma(\lambda_0)\mu(t_i)(\underline{b}-t_i)}{\gamma(0)(p_0-\underline{b})} & \text{if } t_i \in \mathcal{P} \wedge t_i \leq \underline{b} \\ 0 & \text{else} \end{cases}$$

- *1-partisans imitate the highest posteriors $t_i \geq \overline{b}$:*

$$m_i(p_i, 1)(t_i) = \begin{cases} \frac{\gamma(\lambda_0)\mu(t_i)(t_i-\overline{b})}{\gamma(1)(\overline{b}-p_0)} & \text{if } t_i \in \mathcal{P} \wedge t_i \geq \overline{b} \\ 0 & \text{else} \end{cases}$$

- *Given message t_i the receiver's virtual posterior $q(t_i)$ is given by*

$$q(t_i) = \begin{cases} \underline{b} & \text{if } t_i \in \mathcal{P} \wedge t_i < \underline{b} \\ t_i & \text{if } t_i \in \mathcal{P} \wedge t_i \in [\underline{b}, \overline{b}] \\ \overline{b} & \text{if } t_i \in \mathcal{P} \wedge t_i > \overline{b} \\ p_0 & \text{else} \end{cases}$$

- *The receiver uses weighted majority rule with weight function*

$$w(x) = \begin{cases} \ln \frac{\underline{b}}{1-\underline{b}} - \ln \frac{p_0}{1-p_0} & \text{if } x \in \mathcal{P} \wedge x < \underline{b} \\ \ln \frac{x}{1-x} - \ln \frac{p_0}{1-p_0} & \text{if } x \in \mathcal{P} \wedge x \in [\underline{b}, \overline{b}] \\ \ln \frac{\overline{b}}{1-\overline{b}} - \ln \frac{p_0}{1-p_0} & \text{if } x \in \mathcal{P} \wedge x > \overline{b} \\ 0 & \text{else} \end{cases}$$

and threshold $\tau = -\left(\ln \frac{\lambda_0}{1-\lambda_0} + \ln \frac{p_0}{1-p_0}\right)$

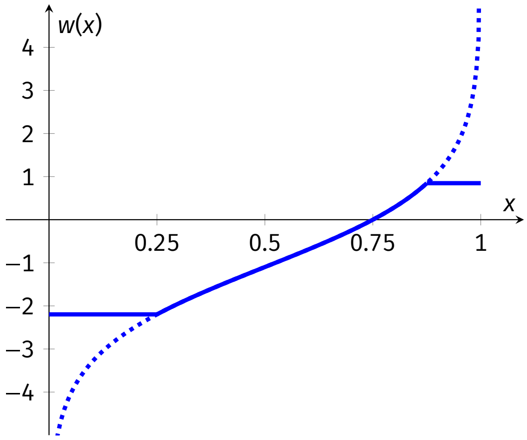


Figure 4.3. Weighting function for the private interests case

In the equilibrium advisors play the truthful strategy as in Proposition 1. It is in their best interest to transmit as much information as possible. Partisans interfere in this communication. Their strategy is independent of their signals. They do not transmit any information to the receiver but maximize their influence by imitating advisors with the most informative signals. Therefore, the receiver needs to discount these messages. In this way expertise bounds $\underline{b}$ and $\overline{b}$ arise. They constitute bounds on the highest possible precision of messages. The weight $w(\underline{b})$ ($w(\overline{b})$) is the lowest (highest) weight used in the weighted majority rule of the receiver. Between the expertise bounds communication between advisors and the receiver is noise-free because partisans do not imitate advisors with imprecise signals. Thus, communication is perfect within these bounds as in the equilibrium from Proposition 1. All off-equilibrium messages are not weighted at all. Figure 4.3 depicts an example of a weighting function of virtual posteriors with lower expertise bound $\underline{b} = \frac{1}{4}$, upper expertise bound $\overline{b} = \frac{7}{8}$, and prior $p_0 = \frac{3}{4}$. The dashed line is the weighting function of the receiver in the absence of partisans.

Some parts of the proof of Proposition 3 provide insights to the general structure of the problem. Therefore, we present the main ideas of the proof in the main text. Technical details and less interesting steps are relegated to Appendix 4.A. Specifically, we split the proof of Proposition 3 into three parts. First, we show that the strategies are indeed a perfect Bayesian equilibrium. This part can be found in Appendix 4.A. Then, we argue in two steps that no other equilibrium is better for the receiver. The first step introduces a technique that allows to compare distributions of virtual posteriors in different equilibria. The second step shows that the introduction of private interests does not change the order of any comparison.

Before we start with the proof of the optimality of the equilibrium we present a lemma that establishes the existence and uniqueness of expertise bounds.

Lemma 3. *For distributions of preference parameters such that there are both types of partisans with positive probability, i.e.* $\gamma(0), \gamma(1) > 0$*, it holds that there is exactly one tuple* $(\underline{b}, \overline{b}) \in [0,1]^2$ *of expertise bounds that fulfills the conditions in Proposition 3.*

Proof. See Appendix 4.A. □

Proof of the optimality of the Equilibrium in Proposition 3. We will show that the equilibrium in Proposition 3 is optimal for the receiver. We proceed in two steps. First, we introduce a technique that allows us to compare equilibria in the common value case. We could have used this technique to prove Proposition 1 but find the proof above more straight forward. Second, we show that the comparison carries over to the case with partisans. More concretely, we show that if an equilibrium in which advisors play the truthful strategy is more informative than another one in the case without partisans, it continues to be more informative than the other one in the presence of partisans.

The receiver bases her decision on the virtual posteriors $q(t_i)$ that she infers from messages t_i of senders $i = \{1, ..., n\}$. The same set of virtual posteriors leads to the same decision. The distribution of virtual posteriors $q(t_i)$ for sender i is determined by the distribution of posteriors μ and the sender i's strategy m_i. In what follows, we focus on the distributions of virtual posteriors.

Definition 4. *Let* μ *be a distribution of posteriors and* m_i *the strategy of sender i. We denote the distribution of virtual posteriors of sender i by* $\mu^{\gamma}_{m_i}$ *and define it by its cdf*

$$F^{\gamma}_{m_i}(x) \;=\; \mathbb{P}[q(t_i) \;\leq\; x],$$

where t_i *is sender i's message. We suppress superscript* γ *in the common interests case, i.e. we write* μ_{m_i} *and* F_{m_i} *if* $\gamma(\lambda_0) = 1$.

In the following we compare the virtual posterior distribution of the equilibrium in which advisors play the truthful strategy with virtual posterior distributions of other equilibria. We know from Proposition 1 that playing the truthful strategy is part of the receiver optimal equilibrium for the common interests case. Using the concept of virtual posterior distributions helps us generalize this observation to the case with partisans.

We start with the analysis of the common interests case. Note that there are other equilibria that lead to the same distribution of virtual posteriors as in Proposition 1. Consequently, these equilibria induce the same ex-ante expected utility of the receiver. For example, consider strategies $\tilde{m}_i$ characterized by $\tilde{m}_i(p_i, \lambda_i) = 1 - m_i(p_i, \lambda_i)$ for senders $i = \{1, ..., n\}$ and a strategy of the receiver in which she uses a weighting function $\tilde{w}$ characterized by $\tilde{w}(x) = w(-x)$. We select the equilibrium in which advisors play the truthful strategy as a representative for all equilibria that induce this distribution of virtual posteriors.

Lemma 4. *Let μ be a posterior distribution, $m_i^\star$ the truthful strategy, and m_i' any other strategy. Then it holds that $\mu_{m_i^\star}$ is more informative than $\mu_{m_i'}$, i.e. $\mu_{m_i^\star} \succ \mu_{m_i'}$.*

Proof. See Appendix 4.A. □

The proof of Lemma 4 builds on the machinery of Blackwell (1953). If $m_i^\star$ is the truthful strategy, any distribution $\mu_{m_i'}$ that is induced by another strategy m_i' can be constructed from the distribution $\mu_{m_i^\star}$ by an application of garblings. The only way in which the virtual posterior distribution $\mu_{m_i'}$ can differ from $\mu_{m_i^\star}$ is that a sender i might send message t_i for two different posteriors p_i and p_i'. The virtual posterior $p(t_i)$ is a weighted average of posteriors that induce sending t_i. Hence, strategy m_i' is a garbling of $m_i^\star$ which implies that $\mu_{m_i'}$ is a garbling of $\mu_{m_i^\star}$. By Theorem 12.3.2 in Blackwell and Girshick (1979) (see Proposition B in Appendix 4.B), the sender-wise comparison suffices to conclude that the virtual posterior distribution $q(t)$ in the equilibrium in which senders play the truthful strategy is more informative than that in any other equilibrium. Proposition 1 follows by Theorem 12.2.2 (4) in Blackwell and Girshick (1979) (see Proposition C in Appendix 4.B), more informative distributions imply higher ex-ante expected utility for the receiver.

In the last step we show that the argument generalizes to the case with partisans. In particular this implies that the virtual posterior distribution $\mu^\gamma_{m_i^\star}$ is more informative than any other distribution $\mu^\gamma_{m_i'}$.

Lemma 5. *Let μ be a posterior distribution, $m_i^\star$ a strategy in which advisors play truthful, and m_i' any other strategy. Then, if $\mu_{m_i^\star}$ is more informative than $\mu_{m_i'}$ it follows that $\mu^\gamma_{m_i^\star}$ is more informative than $\mu^\gamma_{m_i'}$, i.e.*

$$\mu_{m_i^\star} \succ \mu_{m_i'} \Rightarrow \mu^\gamma_{m_i^\star} \succ \mu^\gamma_{m_i'}.$$

Proof. See Appendix 4.A. □

We have shown that the virtual posterior of sender i is most informative if types with $\lambda_i = \lambda_0$ play the truthful strategy. Again, by Theorem 12.3.2 in Blackwell and Girshick (1979) (see Proposition B in Appendix 4.B) the sender-wise comparison carries over to the overall information structure. By Theorem 12.2.2 (4) in Blackwell and Girshick (1979) (see Proposition C in Appendix 4.B) we conclude that there cannot be any better equilibrium for the receiver than the equilibrium described in Proposition 3. This concludes the proof.

□

In the remainder of this article we use the equilibrium in Proposition 3 to compare different distributions of types of senders with respect to the utility of the receiver. Before we proceed we discuss two effects of partisans. The first effect describes the cost of partisans with respect to the average individual precision. A partisan sends a message irrespective of his posterior. Thus, the information held by partisans does not find its way to the receiver. The average individual precision decreases proportionally as the share of partisans increases, i.e. $\pi(\mu^{\gamma}) = \gamma(\lambda_0)\pi(\mu)$. This effect would remain even if preference parameters were publicly known. The second effect emerges because preferences are private information. Partisans imitate well informed senders. This results in a garbling of the most informative advisors with the noise of the partisans. The virtual posterior distribution becomes less informative and average individual precision remains constant.

Having solved for the optimal equilibrium the next subsection studies consequences of the presence of private interests. We demonstrate that sufficiently heterogeneous preferences can prevent any differentiating weighting of messages. Further, we show that the value of specialization vanishes and average individual precision becomes more important as the share of partisans rises.

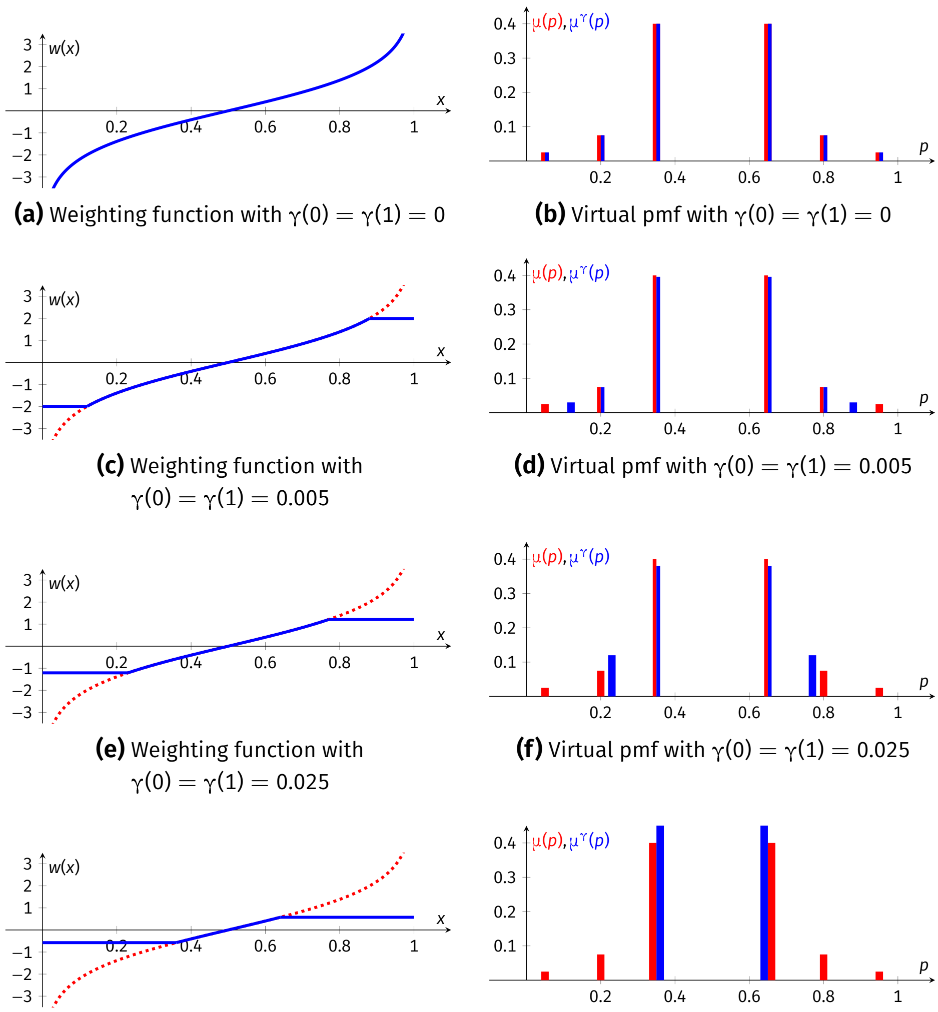
4.5.2 The Effect of Private Interests

In this subsection we illustrate the effect of private interests on information aggregation and the utility of the receiver. We begin with a result that states that sufficiently many partisans can prevent transmission of any information that is finer than the mere direction of the preferred alternative. With other words all senders send one of only two messages in the optimal equilibrium. This small message space can be interpreted as voting.

Proposition 4. *Let μ be never ignorant. Then there exists $c_0, c_1 \in (0,1)$ with $c_0 + c_1 < 1$ s.t. for all γ with $\gamma(0) \geq c_0$, $\gamma(1) \geq c_1$, and $\gamma(\lambda_0) > 0$ the receiver forms only two expected posteriors, i.e. voting is the most informative equilibrium.*

Proof. See Appendix 4.A. □

The proof of Proposition 4 exploits properties of the expertise bounds. We find c_0 (c_1) such that the lower (upper) expertise bound is weakly greater (smaller) than the highest (lowest) possible posterior that is smaller (greater) than the prior. This means that all posteriors on one side of the prior get the same weight. To guarantee that $c_0 + c_1 < 1$ we need to assume that μ is never



(a) Weighting function with $\gamma(0) = \gamma(1) = 0$

(b) Virtual pmf with $\gamma(0) = \gamma(1) = 0$

(c) Weighting function with $\gamma(0) = \gamma(1) = 0.005$

(d) Virtual pmf with $\gamma(0) = \gamma(1) = 0.005$

(e) Weighting function with $\gamma(0) = \gamma(1) = 0.025$

(f) Virtual pmf with $\gamma(0) = \gamma(1) = 0.025$

(g) Weighting function with $\gamma(0) = \gamma(1) = 0.125$

(h) Virtual pmf with $\gamma(0) = \gamma(1) = 0.125$

Figure 4.4. Equilibrium weighting function and virtual probability mass functions for probability mass function

ignorant. Note that if $c_0 + c_1 = 1$ all posteriors get the same weight $w(p_0)$ because all senders are partisans who do not send informative messages.

Figure 4.4 illustrates the effect of different levels of partisans. For all panels the posterior distribution is μ with $\mathbb{P}\left[p = \frac{1}{20}\right] = \mathbb{P}\left[p = \frac{19}{20}\right] = \frac{1}{40}$, $\mathbb{P}\left[p = \frac{1}{5}\right] = \mathbb{P}\left[p = \frac{4}{5}\right] = \frac{3}{40}$, and $\mathbb{P}\left[p = \frac{7}{20}\right] = \mathbb{P}\left[p = \frac{13}{20}\right] = \frac{4}{10}$. This pmf is symmetric around prior $p_0 = \frac{1}{2}$. Panel (a) and (b) depict the case without partisans. The presence of 1% partisans (Panel (c) and (d)) devalues the weight of advisors with the most precise posteriors. 5% partisans (Panel (e) and (f)) prevent the differentiation of advisors with the two most precise posteriors on both sides of the prior. Any differentiation between advisors of one side of the prior breaks down if the share of partisans is 25% (Panel (g) and (h)) or more. Then, only the direction of the posterior can be transmitted. This case corresponds to the situation in Proposition 4.

The next result points out that average individual precision becomes more important as the share of partisans increases. Concretely, a posterior distribution with higher average individual precision is more informative and therefore leads to a higher utility of the receiver if the share of partisans is sufficiently high.

Proposition 5. *Let μ and ν with $\pi(\mu) > \pi(\nu)$ be never ignorant posterior distributions with cdfs F and G, respectively. Then there exist $c_0, c_1 \in (0,1)$ with $c_0 + c_1 < 1$ s.t. for all γ with $\gamma(0) \geq c_0$ and $\gamma(1) \geq c_1$ and any number of senders n the ex-ante expected utility of the receiver is greater under posterior distribution μ than under ν.*

Proof. See Appendix 4.A. □

The proof of Proposition 5 builds on Proposition 4. Suppose there are sufficiently many partisans so that voting is the optimal equilibrium for both distributions. The virtual posterior of any message of senders with the higher average individual precision is more precise, i.e. further away from the prior p_0. This implies that the ex-ante utility of the sender is higher under the posterior distribution with higher average individual precision.

Proposition 5 contrasts the observation on the value of specialization in the common interests case in Section 4.4.2. Without partisans, a more specialized distribution with lower average individual precision can be better for the receiver. This statement is not true if the share of partisans sufficiently increases. Then, a higher individual precision is all that matters and all value from specialization is lost.

4.6 Summary and Discussion

We have studied information aggregation in a cheap talk game with multiple senders who have differently precise information and heterogeneous preferences. We have three main findings. First, under common interests the receiver can utilize all information of senders by discriminating messages based on their informational content.

Second, private interests lead to an information loss that comes from two effects. Senders with private interests do not communicate their information on the state of the world to the receiver. Additionally, these senders imitate advisors with precise information and thereby add noise to the communication with the receiver.

Third, if preferences are sufficiently heterogeneous voting becomes the optimal equilibrium. Senders with private interests prevent transmission of information that is finer than the mere direction of the recommended alternative. Further, average individual precision becomes more important as the share of partisan increases.

In our analysis we assume that all senders are ex-ante symmetric. Posterior and preference distributions do not differ across senders. However, in many important applications there is public information on the efficacy or preferences of senders. In that case the receiver optimally takes that heterogeneity into account and processes messages of heterogeneous senders accordingly. An interesting question building on our analysis is how much efficiency the receiver loses if he is restricted to anonymous procedures. Azrieli (2018a) studies this question in a setting without partisans and gives comparative statics on the size of the inefficiency.

The information loss of private interests is not only caused by wasting the information of partisans. Another significant loss is due to the indistinguishability of partisans and advisors. This deteriorates trust in the most informed advisors. Our analysis might provide a rationale for professional ethics and codes of conduct. These require advisors to inform their clients of all possible conflicts of interest. This is necessary to retain trust in professions in general.

The attempt of third parties to influence political decision has been studied in the literature. Buchanan, Tollison, and Tullock (1980) and Baye, Kovenock, and De Vries (1993) model lobbying as a tournament in which parties pay money to politicians that proportionally increases the probability of influencing the decision. The literature on cheap talk describes an influence over selective information provision as discussed in Section 4.2. In our model there is uncertainty on the preferences of senders. In future research our model can be extended to two lobby groups who try to optimally manipulate the set of senders. Then, lobbyists do not send information by themselves but manipulate experts which creates an uncertainty on the motives of senders that affects the decision of the receiver. The endogeneity of the preference distribution offers a third interpretation of a more indirect influence of lobbyists and opens a set of interesting questions.

References

Condorcet, Marquis de. 1785. "Essay on the Application of Analysis to the Probability of Majority Decisions." *Paris: Imprimerie Royale*, [61]

Blackwell, David. 1951. "Comparison of Experiments." In *Proc. Second Berkeley Symp. on Math. Stat. and Probability*. University of California Press, 93–102. [64]

Blackwell, David. 1953. "Equivalent comparisons of experiments." *Annals of Mathematical Statistics*, 265–72. [75, 85]

Blackwell, David A, and Meyer A Girshick. 1979. *Theory of games and statistical decisions*. Courier Corporation. [67, 68, 70, 75, 85, 89, 90]

Buchanan, James M, Robert D Tollison, and Gordon Tullock. 1980. *Toward a theory of the rent-seeking society*. 4. Texas A & M Univ Pr. [79]

Crawford, Vincent P, and Joel Sobel. 1982. "Strategic information transmission." *Econometrica*, 1431–51. [61]

Nitzan, Shmuel, and Jacob Paroush. 1982. "Optimal decision rules in uncertain dichotomous choice situations." *International Economic Review*, 289–97. [66, 82, 84]

Gilligan, Thomas W, and Keith Krehbiel. 1989. "Asymmetric information and legislative rules with a heterogeneous committee." *American Journal of Political Science*, 459–90. [61]

Austen-Smith, David. 1990. "Information transmission in debate." *American Journal of Political Science*, 124–52. [61]

Baye, Michael R, Dan Kovenock, and Casper G De Vries. 1993. "Rigging the lobbying process: an application of the all-pay auction." *American Economic Review* 83 (1): 289–94. [79]

Feddersen, Timothy, and Wolfgang Pesendorfer. 1997. "Voting behavior and information aggregation in elections with private information." *Econometrica*, 1029–58. [62]

McLennan, Andrew. 1998. "Consequences of the Condorcet jury theorem for beneficial information aggregation by rational agents." *American Political Science Review* 92 (2): 413–18. [65]

Krishna, Vijay, and John Morgan. 2001. "A model of expertise." *Quarterly Journal of Economics* 116 (2): 747–75. [61]

Wolinsky, Asher. 2002. "Eliciting information from multiple experts." *Games and Economic Behavior* 41 (1): 141–60. [61]

Alonso, Ricardo, Wouter Dessein, and Niko Matouschek. 2008. "When does coordination require centralization?" *American Economic Review* 98 (1): 145–79. [61]

Borgers, Tilman. 2009. "Notes on Blackwell's Comparison of Experiments." [89]

Gerardi, Dino, Richard McLean, and Andrew Postlewaite. 2009. "Aggregation of expert opinions." *Games and Economic Behavior* 65 (2): 339–71. [63]

Field, John M, Mary Fran Hazinski, Michael R Sayre, Leon Chameides, Stephen M Schexnayder, Robin Hemphill, Ricardo A Samson, John Kattwinkel, Robert A Berg, Farhan Bhanji, et al. 2010. "Part 1: executive summary: 2010 American Heart Association guidelines for cardiopulmonary resuscitation and emergency cardiovascular care." *circulation* 122 (18_suppl_3): S640–S656. [59]

Hummel, Patrick, John Morgan, and Phillip C Stocken. 2013. "A model of flops." *RAND Journal of Economics* 44 (4): 585–609. [61]

McMurray, Joseph. 2017. “Ideology as opinion: a spatial model of common-value elections.” *American Economic Journal: Microeconomics* 9 (4): 108–40. [62]

Azrieli, Yaron. 2018a. “Monitoring Experts.” *Unpublished manuscript*, [79]

Azrieli, Yaron. 2018b. “The price of ‘one person, one vote’.” *Social Choice and Welfare* 50 (2): 353–85. [62]

Piller, Charles. 2018. “Hidden conflicts?” *Science* 361 (6397): 16–20. DOI: 10.1126/science.361.6397.16. eprint: http://science.sciencemag.org/content/361/6397/16.full.pdf. [59]

4.A Appendix: Proofs

Proof of Proposition 1. As noted in the text, in the following we use the proof technique of Theorem 1 in Nitzan and Paroush (1982) who derive the optimal non-strategic processing of signals with a symmetric prior $p_0 = \frac{1}{2}$.

In the main text we use for random variables and their realizations the same notation. For this proof it is useful to introduce a separate notation. We use upper case characters for random variables and lower case characters for their realizations.

The receiver processes messages t to update her posterior. She prefers the action that yields the higher expected utility given her virtual posterior $q(t)$. More precisely, an optimal decision rule selects action 1 if

$$\begin{aligned}
& \lambda_0 \mathbb{P}[\omega = 1|T = t] > (1-\lambda_0)\mathbb{P}[\omega = 0|T = t] \\
\Leftrightarrow \quad & \lambda_0 \frac{\mathbb{P}[P = p|\omega = 1] \cdot \mathbb{P}[\omega = 1]}{\mathbb{P}[P = p]} > (1-\lambda_0)\frac{\mathbb{P}[P = p|\omega = 0] \cdot \mathbb{P}[\omega = 0]}{\mathbb{P}[P = p]} \\
\Leftrightarrow \quad & \lambda_0 p_0 \prod_i \mathbb{P}[P_i = p_i|\omega = 1] > (1-\lambda_0)(1-p_0)\prod_i \mathbb{P}[P_i = p_i|\omega = 0] \\
\Leftrightarrow \quad & \lambda_0 p_0 \prod_i \frac{p_i}{p_0} > (1-\lambda_0)(1-p_0)\prod_i \frac{1-p_i}{1-p_0} \\
\Leftrightarrow \quad & \prod_i \left(\frac{p_i}{1-p_i}\frac{1-p_0}{p_0}\right) > \frac{1-\lambda_0}{\lambda_0}\frac{1-p_0}{p_0} \\
\Leftrightarrow \quad & \sum_i \left(\ln\frac{p_i}{1-p_i} - \ln\frac{p_0}{1-p_0}\right) > -\left(\ln\frac{\lambda_0}{1-\lambda_0} + \ln\frac{p_0}{1-p_0}\right).
\end{aligned}$$

For the first equivalence we apply Bayes' rule and exploit that senders play the truthful strategy. In the second step we use the conditional independence of signals. We arrive at the fourth equation by applying Bayes' rule once again. The fifth equation is a simple reformulation of the fourth. Finally, we obtain the last equation by taking the logarithm on both sides. The resulting decision rule can be interpreted as a weighted majority rule with weighting function

$$w(t_i) = \begin{cases} \ln\frac{t_i}{1-t_i} - \ln\frac{p_0}{1-p_0} & \text{if } t_i \in \mathscr{P} \\ 0 & \text{else,} \end{cases}$$

and threshold $\tau = -\left(\ln\frac{\lambda_0}{1-\lambda_0} + \ln\frac{p_0}{1-p_0}\right)$.

It is optimal for senders to play the truthful strategy since senders and the receiver have the same utility function. With the truthful strategy senders can transmit all available information. Any beneficial transformation of messages can be done by the receiver. □

Lemma 6. *For posterior distribution μ with cdf F and preference distribution γ, the lower expertise bound $\underline{b}$ in the receiver optimal equilibrium is determined by*

$$\gamma(0)\left(p_0 - \underline{b}\right) \;=\; \gamma(\lambda_0) \cdot \int_0^{\underline{b}} F(x)\,\mathrm{d}x,$$

and the upper expertise bound $\overline{b}$ is determined by

$$\gamma(1)\left(\overline{b} - p_0\right) \;=\; \gamma(\lambda_0) \cdot \int_{\overline{b}}^{1} 1 - F(x)\,\mathrm{d}x.$$

Proof. To guarantee that the behavior of 0-partisans in Proposition 3 is actually a strategy, the probabilities of sending any message must be weakly greater than 0 and add up to 1, i.e.

$$\frac{\gamma(\lambda_0)\mu(t_i)(\underline{b} - t_i)}{\gamma(0)(p_0 - \underline{b})} \;\geq\; 0 \quad \text{for all } t_i \;\leq\; \underline{b} \wedge t_i \;\in\; \mathscr{P}, \text{ and}$$

$$\sum_{t_i \leq \underline{b} \wedge t_i \in \mathscr{P}} \frac{\gamma(\lambda_0)\mu(t_i)(\underline{b} - t_i)}{\gamma(0)(p_0 - \underline{b})} \;=\; 1.$$

The first condition is fulfilled for all $t_i \leq \underline{b}$. The second condition implies that

$$\begin{aligned}
\gamma(0)(p_0 - \underline{b}) \;&= \gamma(\lambda_0)\int_0^{\underline{b}} (\underline{b} - x)\mu(x)\mathrm{d}x \\
&= \gamma(\lambda_0) \cdot F(\underline{b}) \cdot \underline{b} - \gamma(\lambda_0) \cdot \left(F(\underline{b}) \cdot \underline{b} - \int_0^{\underline{b}} F(x)\mathrm{d}x\right) \\
&= \gamma(\lambda_0) \cdot \int_0^{\underline{b}} F(x)\mathrm{d}x,
\end{aligned}$$

where we obtain the second equation by integration by parts.

The calculation for the upper expertise bound is analogous. □

Proof of Lemma 3. By Lemma 6 the characterization of the lower expertise bound $\underline{b}$ is given by

$$\gamma(0)\left(p_0 - \underline{b}\right) \;=\; \gamma(\lambda_0) \cdot \int_0^{\underline{b}} F(x)\,\mathrm{d}x.$$

Note that the left side of the equation is strictly decreasing in $\underline{b} \in [0, p_0]$ and is 0 only if $\underline{b} = p_0$. The right side is weakly increasing in $\underline{b}$ and is 0 for $\underline{b} = 0$. Further, both sides are continuous in $\underline{b}$. Thus, there is a unique $\underline{b}$ that fulfills the equation.

The proof for the upper expertise bound is analogous. □

***Proof of the equilibrium in Proposition 3*.** As in the proof of Proposition 1, we use upper case characters for random variables and lower case characters for their realizations.

We start to calculate the virtual posterior $q(t_i)$ of the receiver after receiving message t_i. The only senders that send messages within the expertise bounds are advisors. Thus $q(t_i) = t_i$ for $t_i \in (\underline{b}, \overline{b}) \cap \mathscr{P}$. For messages $t_i \leq \underline{b}$ with $t_i \in \mathscr{P}$ the virtual posterior of the receiver is

$$\begin{aligned} q(t_i) &= \mathbb{P}[\omega = 1 | T_i = t_i] \\ &= \frac{\sum_{\lambda \in \{0, \lambda_0\}} \mathbb{P}[T_i = t_i | \omega = 1 \wedge \Lambda = \lambda] \mathbb{P}[\omega = 1] \mathbb{P}[\Lambda = \lambda]}{\mathbb{P}[T_i = t_i]} \\ &= \frac{\gamma(\lambda_0)\mu(t_i)t_i + \gamma(0)\frac{\gamma(\lambda_0)\mu(t_i)(\underline{b}-t_i)}{\gamma(0)(p_0-\underline{b})}p_0}{\gamma(\lambda_0)\mu(t_i) + \gamma(0)\frac{\gamma(\lambda_0)\mu(t_i)(\underline{b}-t_i)}{\gamma(0)(p_0-\underline{b})}} \\ &= \underline{b}. \end{aligned}$$

The calculation for the virtual posterior of messages $t_i \leq \underline{b}$ with $t_i \in \mathscr{P}$ is $q(t_i) = \overline{b}$ by an analogous calculation. Thus, the receiver's on equilibrium beliefs are consistent with Bayes updating.

The technique of Nitzan and Paroush (1982) and the proof of Proposition 1 teach us how to process a set of (virtual) posteriors optimally. Again, the best response of the receiver can be interpreted as a weighted majority rule with weighting function

$$\begin{aligned} w(x) &= \ln \frac{q(x)}{1-q(x)} - \ln \frac{p_0}{1-p_0} \\ &= \begin{cases} \ln \frac{\underline{b}}{1-\underline{b}} - \ln \frac{p_0}{1-p_0} & x \in \mathscr{P} \wedge x \leq \underline{b} \\ \ln \frac{x}{1-x} - \ln \frac{p_0}{1-p_0} & x \in \mathscr{P} \wedge \underline{b} \leq x \leq \overline{b} \\ \ln \frac{\overline{b}}{1-\overline{b}} - \ln \frac{p_0}{1-p_0} & x \in \mathscr{P} \wedge \overline{b} \leq x \\ 0 & \text{else} \end{cases} \end{aligned}$$

and threshold $\tau = -\left(\ln \frac{\lambda_0}{1-\lambda_0} + \ln \frac{p_0}{1-p_0}\right)$.

We proceed by proving that senders play best responses. Partisans maximize the probability that the receiver takes the action that matches their preference parameter. Given the strategy of advisors and the receiver they send a message with maximal weight in the preferred direction. In the equilibrium strategy 0-(1-)partisans mix over messages with weight $\ln \frac{\overline{b}}{1-\overline{b}} - \ln \frac{p_0}{1-p_0}$ $(\ln \frac{\underline{b}}{1-\underline{b}} - \ln \frac{p_0}{1-p_0})$ which is the highest (lowest) weight assigned by the receiver. Hence, these partisans play best responses.

We proceed to analyze best responses of advisors. Suppose an advisor is pivotal, i.e. two different messages of his induce different actions of the receiver. The advisor and the receiver have the same utility function and prefer the same action given they have the same information. Thus, the best the advisor can do is to reveal all his information to the receiver who processes it optimally. If the advisor is not pivotal any message is a best response.

Taken together, the strategies and updating in Proposition 3 are a perfect Bayesian equilibrium.

□

Proof of Lemma 4. Let $\mu_{m_i^\star}$ be the virtual posterior distribution under the truthful strategy $m_i^\star$. Any distribution $\mu_{m_i'}$ that is induced by another strategy m_i' can be constructed from $\mu_{m_i^\star}$ by an application of garblings. We do not restrict strategies to use only a finite set of messages. Therefore, we apply a result from Blackwell (1953) that generalizes Theorem 12.2.2 in Blackwell and Girshick (1979) (see Proposition A in Appendix 4.B) to the case with continuous signals. Thereby, we conclude that $\mu_{m_i^\star}$ is more informative than $\mu_{m_i'}$. □

Lemma 7. *Let μ and ν with $\mu \succ \nu$ be posterior distributions with cdfs F and G, respectively. Let γ be the distribution of preference parameters. Then, the lower (upper) expertise bound $\underline{b}_\mu$ of μ is weakly smaller (greater) or equal than the lower (upper) expertise bound $\underline{b}_\nu$ of ν in the optimal equilibria with partisans.*

Proof of Lemma 7. Suppose that $\underline{b}_\nu < \underline{b}_\mu$ and use Lemma 6 to see that

$$\begin{aligned}\gamma(0)\left(p_0 - \underline{b}_\nu\right) &= \gamma(\lambda_0)\cdot\int_0^{\underline{b}_\nu} G(x)\mathrm{d}x \\ &\leq \gamma(\lambda_0)\cdot\int_0^{\underline{b}_\nu} F(x)\mathrm{d}x \\ &\leq \gamma(\lambda_0)\cdot\int_0^{\underline{b}_\mu} F(x)\mathrm{d}x = \gamma(0)\left(p_0 - \underline{b}_\mu\right).\end{aligned}$$

Hence $\underline{b}_\mu \leq \underline{b}_\nu$, which is a contradiction. The proof for the upper expertise bound is analogous. □

Proof of Lemma 5. To simplify notation we denote $\mu_{m_i^\star}$ by μ, $\mu_{m_i'}$ by ν, $\mu^\gamma_{m_i^\star}$ by μ^γ, and $\mu^\gamma_{m_i'}$ by ν^γ. Further, we denote $F_{m_i^\star}$ by F, $F_{m_i'}$ by G, $F^\gamma_{m_i^\star}$ by F^γ, $F^\gamma_{m_i'}$ by G^γ.

To show that μ^γ is more informative than ν^γ we show that

$$\int_0^y G^\gamma(x)\mathrm{d}x \leq \int_0^y F^\gamma(x)\mathrm{d}x \quad \text{for all } y \in [0,1].$$

By Lemma 7 it holds that $\underline{b}_\mu \leq \underline{b}_\nu$ and $\overline{b}_\mu \geq \overline{b}_\nu$. This allows us to check the inequality separately on the three intervals $\left[0, \underline{b}_\nu\right]$, $\left[\underline{b}_\nu, \overline{b}_\nu\right]$ and $\left[\overline{b}_\nu, 1\right]$.

For all $y \in \left[0, \underline{b}_\nu\right]$ it holds that

$$\int_0^y G^\gamma(x)\mathrm{d}x \;=\; 0 \;\leq\; \int_0^y F^\gamma(x)\mathrm{d}x.$$

For all $y \in \left[\underline{b}_\nu, \overline{b}_\nu\right]$ it holds that

$$\begin{aligned}
\int_0^y G^\gamma(x)\mathrm{d}x \;&=\; \int_{\underline{b}_\nu}^y \gamma(0) + \gamma(\lambda_0)G(x)\mathrm{d}x \\
&=\; \gamma(0)(y - \underline{b}_\nu) + \gamma(\lambda_0)\int_0^y G(x)\mathrm{d}x - \gamma(\lambda_0)\int_0^{\underline{b}_\nu} G(x)\mathrm{d}x \\
&=\; \gamma(0)(y - p_0) + \gamma(\lambda_0)\int_0^y G(x)\mathrm{d}x \\
&\leq\; \gamma(0)(y - p_0) + \gamma(\lambda_0)\int_0^y F(x)\mathrm{d}x \\
&=\; \int_0^y F^\gamma(x)\mathrm{d}x.
\end{aligned}$$

The first equality follows by the definition of virtual posteriors and the equilibrium strategies. For the third equality we apply Lemma 6. The inequality follows by the assumption that $\mu \succ \nu$.

Since $G^\gamma(x) = 1$ for $x \geq \overline{b}_\nu$ it follows that for all $y \in \left[\overline{b}_\nu, 1\right]$ it holds that

$$\int_y^1 G^\gamma(x)\mathrm{d}x \;\geq\; \int_y^1 F^\gamma(x)\mathrm{d}x.$$

The expected value of both distributions is consistent with the common prior, i.e. $\int_0^1 F^\gamma(x)\mathrm{d}x = \int_0^1 G^\gamma(x)\mathrm{d}x = 1 - p_0$. Thus, we conclude that

$$\int_0^y G^\gamma(x)\mathrm{d}x \;\leq\; \int_0^y F^\gamma(x)\mathrm{d}x,$$

for all $y \in [\overline{b}_\nu, 1]$, which concludes the proof. □

Proof of Proposition 4. We prove the proposition in two steps. We start to show that by monotonicity and continuity of $\underline{b}$ and $\overline{b}$ there exists $c_0, c_1 \in (0, 1)$ such that the receiver can only form two expected posteriors in the optimal equilibrium. Then, we prove that c_0 and c_1 are such that $c_0 + c_1 < 1$. For both parts we use Lemma 6 that characterizes the lower expertise bound by the equation

$$\gamma(0)\left(p_0 - \underline{b}\right) \;=\; \gamma(\lambda_0) \cdot \int_0^{\underline{b}} F(x)\mathrm{d}x.$$

The lower expertise bound can take any value in $\underline{b} \in [0, \max\{x : F(x) = 0\}]$ if $\gamma(0) = 0$. Further, it is p_0 if $\gamma(0) = 1$. Rewriting the above equation yields

$$\frac{\gamma(0)}{\gamma(\lambda_0)} \;=\; \frac{\int_0^{\underline{b}} F(x)\mathrm{d}x}{p_0 - \underline{b}} \tag{4.2}$$

which exhibits that $\underline{b}$ is monotonically increasing in $\gamma(0)$, monotonically decreasing in $\gamma(\lambda_0)$ and continuous in $\gamma(0), \gamma(\lambda_0) \in (0, 1)$.

Since μ is never ignorant there exists a highest type strictly smaller than the prior, $p_L := \max\{x | x < p_0 \wedge x \in \mathscr{P}\}$. The proposition is fulfilled if the lower expertise bound equals this type $\underline{b} = p_L$. Continuity and monotonicity of $\underline{b}$ imply that the right hand side of Equation (4.2) is positive and finite and hence $\gamma(0) < 1$ if $\underline{b} = p_L$. The proof for the upper part with type $p_H := \min\{x | x > p_0 \wedge x \in \mathscr{P}\}$ is analogous so that constants $c_0, c_1 \in (0, 1)$ are implicitly given by

$$\begin{aligned} c_0\left(p_0 - p_L\right) \;&=\; \gamma(\lambda_0) \cdot \int_0^{p_L} F(x)\mathrm{d}x \\ \text{and} \quad c_1\left(p_H - p_0\right) \;&=\; \gamma(\lambda_0) \cdot \int_{p_H}^{1} 1 - F(x)\mathrm{d}x. \end{aligned} \tag{4.3}$$

To see that $c_0 + c_1 < 1$ divide Equations (4.3) by $\left(p_0 - p_L\right)$ and $\left(p_H - p_0\right)$, respectively. Adding both equations yields

$$c_0 + c_1 \;=\; \gamma(\lambda_0) \cdot \frac{\int_0^{p_L} F(x)\mathrm{d}x}{p_0 - p_L} + \gamma(\lambda_0) \cdot \frac{\int_{p_H}^{1} 1 - F(x)\mathrm{d}x}{p_H - p_0}.$$

Since $\frac{\int_0^{p_L} F(x)\mathrm{d}x}{p_0 - p_L}, \frac{\int_{p_H}^{1} 1 - F(x)\mathrm{d}x}{p_H - p_0} > 0$, it follows that $\gamma(\lambda_0) > 0$. This implies that $c_0 + c_1 = \gamma(0) + \gamma(1) = 1 - \gamma(\lambda_0) < 1$ which completes the proof. □

Proof of Proposition 5. Let $c_0^{\mu}, c_0^{\nu} \in (0, 1)$ constants from Proposition 4 for distributions μ and ν with cdfs F and G, respectively. Define $c_0 = \max\left\{c_0^{\mu}, c_0^{\nu}\right\}$ as the smallest constant such that both virtual posteriors μ^{γ} and ν^{γ} have only two mass points. Now we compare the resulting lower expertise bounds $\underline{b}^{\mu^{\gamma}}$ and $\underline{b}^{\nu^{\gamma}}$. The smaller the lower expertise bound the better the signal. Thus, it is sufficient to show that $\underline{b}^{\mu^{\gamma}} < \underline{b}^{\nu^{\gamma}}$.

By Proposition 4 and Lemma 6 it follows that

$$\underline{b}^{\mu^\gamma} = p_0 - \frac{\gamma(\lambda_0)}{\gamma(0)} \int_0^{p_0} F(x)\mathrm{d}x,$$

Lemma 8 implies that $\underline{b}^{\mu^\gamma} < \underline{b}^{\nu^\gamma}$ which completes the proof for the lower expertise bound.

The proof for the upper expertise bounds is analogous. Further, $c_0 + c_1 < 1$ remains to be true despite taking the maximum and minimum, respectively, by the same reason as in the proof of Proposition 4. □

Lemma 8. *Let μ and ν be posterior distributions with cdfs F and G, respectively. Then, $\pi(\mu) > \pi(\nu)$ if and only if*

$$\int_0^{p_0} F(x)\mathrm{d}x > \int_0^{p_0} G(x)\mathrm{d}x \quad \textit{and} \quad \int_{p_0}^1 1 - F(x)\mathrm{d}x > \int_{p_0}^1 1 - G(x)\mathrm{d}x.$$

Proof. We rewrite $\pi(\mu)$ until we arrive at an expression from which the result is immediate:

$$\begin{aligned}
&\pi(\mu) \\
&= \mathbb{E}[|p_i - p_0|] \\
&= \int_0^1 |x - p_0|\ \mu(x)\mathrm{d}x \\
&= \int_0^{p_0} (p_0 - x)\ \mu(x)\mathrm{d}x + \int_{p_0}^1 (x - p_0)\ \mu(x)\mathrm{d}x \\
&= p_0 \cdot F(p_0) - \int_0^{p_0} x\ \mu(x)\mathrm{d}x + \left(p_0 - \int_0^{p_0} x\ \mu(x)\mathrm{d}x\right) - p_0 \cdot \big(1 - F(p_0)\big) \\
&= 2\left(p_0 \cdot F(p_0) - \int_0^{p_0} x\ \mu(x)\mathrm{d}x\right) \\
&= 2\int_0^{p_0} F(x)\mathrm{d}x.
\end{aligned}$$

The fourth equation follows from the common prior $p_0 = \int_0^1 x\ \mu(x)\mathrm{d}x$ and the last equality from integration by parts.

The derivation of the second inequality is analogous. □

4.B Appendix: Blackwell

In this part of the Appendix we introduce tools that allow us to make use of Definition 1. This allows us to compare the receiver's utility with different posterior distributions of senders. All methods and results in this subsection are borrowed from Chapter 12 in Blackwell and Girshick (1979). In order to apply their machinery to our problem we slightly adjust our setting and translate our notation into theirs.

The following results rely on the assumption that the action space of the receiver is a closed bounded convex subset of $\mathbb{R}$. To fulfill this assumption we extend the action space of the receiver from $\{0, 1\}$ to $\Delta\{0, 1\}$ so that her action space is the interval $[0, 1]$. An action $a \in \Delta\{0, 1\}$ corresponds to the probability that the receiver takes action 1. Note that we can use this extended action space throughout the whole paper without changing any result. In all statements on best responses of the receiver one of the two extreme actions $\{0, 1\} \subset \Delta\{0, 1\}$ is optimal. We use the action space $\{0, 1\}$ in the main text of the paper to simplify the exposition.

To present the next results it is also helpful to introduce some of the notation of Blackwell and Girshick (1979)[8] For a posterior distribution μ we define a $2 \times N$ matrix P, where $N = |\mathscr{P}|$ is the number of possible posteriors. The rows represent the two states of the world 0 and 1. Each column represents one possible posterior. The value P_{ij} is the probability of observing the posterior represented by column j in state i. Note that matrix P is Markov which means that $P_{ij} > 0$ for all i and j and that $\sum_{j=1}^{N} P_{ij} = 1$ for all i. With the notation we are equipped to remind the reader of Theorem 12.2.2 in Blackwell and Girshick (1979).

Proposition A (Blackwell and Girshick (1979)). *Let P and Q be two $2 \times N_1$ and $2 \times N_2$ Markov matrices of posterior distributions μ and ν. μ is more informative than ν if and only if there is an $N_1 \times N_2$ Markov matrix M with $PM = Q$.*

Matrix M is said to *garble* information by transforming matrix P to Q. This means that distribution ν can be constructed from distribution μ. This interpretation justifies the statement that μ is more informative than ν.

The next result generalizes the previous proposition by allowing to compare sets of distributions. Each sender sends a conditionally independent posterior. Consider two sets of senders with different posterior distributions. Then, Theorem 12.3.2 in Blackwell and Girshick (1979) allows us to compare the information of both groups in the following sense.

[8] We also enjoyed reading the notes of Borgers (2009) on Chapter 12 of Blackwell and Girshick (1979) and borrow some of his notation.

Proposition B (Blackwell and Girshick (1979)). *Let* $(\mu_i)_{i=1}^n$ *and* $(\nu_i)_{i=1}^n$ *be two sets of posterior distributions. Suppose that* μ_i *is more informative than* ν_i *for every* i*. Then, the combination of posterior distributions* $(\mu_i)_{i=1}^n$ *is more informative than* $(\nu_i)_{i=1}^n$.

The proposition allows to compare the information that is transmitted to the receiver from different distributions. Theorem 12.2.2 (4) in Blackwell and Girshick (1979) allows us to use this result for a statement on the utility of the receiver.

Proposition C (Blackwell and Girshick (1979)). *Let* μ *and* ν *be two posterior distributions such that* μ *is more informative than* ν*. Then, for every continuous convex function* $\phi : [0,1] \to \mathbb{R}$ *we have*

$$\mathbb{E}_{\mu}[\phi(x)] \geq \mathbb{E}_{\mu'}[\phi(x)].$$

Note that the utility function $u^{\star}(q)$ is convex in q. Thus, if there are two posterior distributions with $\mu \succ \nu$ the proposition implies that the expected utility for the receiver with distribution μ is at least as high as with distribution ν.

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