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Contents

Introduction

This thesis consists of three chapters that analyze stability of financial institutions and redistributive effects in health insurance markets based on the example of Germany. The third chapter is based on joint work with my coauthor Benjamin Schickner. My analysis of financial stability in the first two chapters considers financial institutions which borrow short-term, liquid debt such as demand deposits and invest long-term in illiquid and risky assets (maturity and liquidity transformation). These transformations make the financial institution prone to a liquidity squeeze (run) by uninsured short-term debt investors. Runs by debt investors may be driven by fears about low future asset returns but also by fears that a large group of other debt investors might withdraw their deposits. The latter gives rise to self-fulfilling or panic runs caused by miscoordination. The first two chapters are concerned with modeling such self-fulfilling runs using the methodology of global games. Imposition of a specific type of information structure allows the selection of a unique equilibrium. Ex ante identical agents observe correlated, noisy and private signals before choosing one out of two possible actions. As typical in global games, the equilibrium has the nature of a trigger equilibrium. Agents choose action 'withdraw' if they observe signals below the trigger and choose action 'wait' if they observe a signal above the trigger. The size of the equilibrium trigger determines the expected number of agents choosing either action. If the number of agents choosing action 'withdraw' exceeds a critical threshold, an event is triggered, the financial firm defaults. Chapter one and two are concerned with how the equilibrium trigger and thus ex ante probability of runs change in the primitives of the game.

In the first chapter, I analyze how miscoordination on panic runs among debt investors changes under altering capital structure and market liquidity of firm assets. Investors draw on a finite, common pool of liquidity. In case of a run, repayment to debt investors is only partial and endogenous. Taking this endogeneity into account, I show that investors coordinate in a way such that the probability of a run is in general non-monotone in both debt and liquidity ratio. When liquidity dries up, increasing short-term financing may decrease the probability of runs, more short-term debt can discipline debt investors to better coordinate. If the firm is financed through short-term debt and equity only, the result implies that firm stability may decrease in equity. In detail, more short-term debt financing can alter coordination and hence the probability distribution of debt becoming due in the future in a way that runs become less likely. This implies, the probability of runs is non-monotone in liquidity mismatch between assets and liabilities. As a result, capital and liquidity regulation may harm stability. These results hold under partial asset liquidation or collateralized borrowing and therefore apply to classic commercial banks but also to shadow banks such as structured investment vehicles and asset backed commercial paper conduits.

In the second chapter, I ask the question how national bankruptcy codes and interventions of a lender of last resort impact coordination behavior of debt investors. National bankruptcy codes and potential intervention by national central banks (lender of last resort) during runs on financial firms impact recovery values after bankruptcy. But while bankruptcy proceedings impose fixed costs, the intervention by a lender of last resort depends on the scale of the run. As a consequence, recovery values are ex ante random and endogenous to investors. The second chapter studies how recovery values influence coordination behavior of uninsured debt investors and thus stability of financial firms. In particular, the chapter analyzes how the composition of recovery values changes coordination when recovery value consists of a run-size dependent, endogenous part controlled by the lender of last resort and a fixed component to model national differences in bankruptcy code. I find that the composition of recovery value influences how firm stability changes in capital structure and liquidity mismatch. Run probabilities are monotone in debt or liquidity mismatch as long as recovery values are proportional to the size of the run. When recovery values are independent of the magnitude of the run (no lender of last resort) or include a fixed component independent of the size of the run (intercept), run probabilities are non-monotone and have unique maxima. The non-monotonicity changes in composition of recovery value. As a consequence, drops in funding liquidity or capital can have a stabilizing effect in country A but a destabilizing effect on a company with identical capital structure in country B due to variations in national bankruptcy code. If a lender of last resort intervenes more generous in country G compared to country I, liquidity regulation in country G has to be stricter than in country I to guarantee the same minimum stability level. These results have policy implications for capital and liquidity regulation under Basel 3 since member countries agree on regulation but firms underlie different bankruptcy codes. Further, I show that high recovery values achieved by cost efficient bankruptcy proceedings or generous government interventions are never desirable from a stability perspective and only sometimes desirable from a consumer welfare perspective since high recovery values increase the probability of runs.

In the third chapter, Benjamin and I study redistributional effects of competition between private and public insurance on health insurance markets based on the example of Germany. In Germany, health insurance is obligatory and provided by a budget-balancing public insurance and a revenue-maximizing private insurance. Public insurance is regulated, she may charge a fixed contribution rate from customers income up to a cap and she must operate cost-covering. Public contributions do not depend on customers' health risk types. Customers with high income may opt out of public insurance. The regulations and competition with a more flexible private insurance lead to difficulties for public insurance to find a contribution rate which guarantees a balanced budget. We derive a condition on the health income distribution of customers and regulator thresholds such that a unique public contribution rate exists which balances budget. We show that in equilibrium, healthy, high-income customers insure with private insurance. Further, private insurance cream skims customers if possible, that is she selects good risk types. We identify income redistribution streams in the population and argue that an increase in the opt-out threshold decreases the costs of health insurance for all customers. Analyzing changes in the underlying distribution, we show that the equilibrium contribution rate rises as the positive correlation between health and income increases. We demonstrate, even a systematic improvement of the populations health and income may lead to a higher contribution rate. Welfare effects of switching from the contribution-based German system to a premium- based flat payment system with only one type of insurance are discussed.

Chapter 1

Capital Structure, Liquidity and Miscoordination on Runs

1.1 Motivation

This paper is concerned with stability of financial intermediators ('firms') against runs by debt investors. Runs on financial institutions have been a recurrent phenomenon in economic history up to the present. In September 2007, we witnessed a traditional run on UK bank Northern Rock (Shin, 2009). In September 2008, withdrawals by customers forced a shut down of the US savings and loan Washington Mutual. In summer 2015, Greek banks were closed in a bank holiday for several weeks to prevent a run by depositors.
In a debt run, a large number of short-term debt investors rush to withdraw their funds from the firm. Large cash withdrawals, in response, force the firm to liquidate assets on short notice. If assets are illiquid[1], the firm can sell assets quickly only at a large price discount compared to their fair value. If the firm relies heavily on short-term financing, potential overall withdrawals the firm might face in a run exceed total cash the firm can raise through liquidating all assets in short time. Debt investors' awareness of this potential liquidity squeeze and its implications for firm stability and welfare are at the heart of this paper.

In our model, a financial firm finances an investment in a risky, illiquid asset through equity and short-term debt. The firm promises fixed interest payments to debt investors and the residual value of investment to equity investors.[2] At an interim period, debt investors need to decide whether to stay invested in the firm (roll over debt) or to withdraw their investment. They do so after observing imperfect information about the asset's random return. As debt investors make their roll over decisions at the interim period, at the initial period the firm faces a random

[1] An asset's market liquidity depends on several factor such as market size of the asset, potential information asymmetries and current economic market conditions (Foucault et al., 2013).
[2] This is by the ownership structure and seniority of debt.

withdrawal of short-term debt in the following period. To finance withdrawals of funds, the firm liquidates a corresponding fraction of her investment in the illiquid asset and by this diminishes future gross returns.[3] If funds available through selling the asset (market liquidity) undercut the overall amount of potential short-term debt claims the firm might face, the firm is prone to a liquidity squeeze (run) in the interim period:[4] When the number of debt investors claiming their deposit exceeds a critical threshold, the firm cannot serve all its debt investors and goes into default. The potential of a run gives rise to a coordination problem between debt investors. The roll over decision of debt investors is not only based on inferences about the random asset return, a solvency consideration, but also depends on the expected number of other debt investors rolling over, a liquidity consideration. As a result, a debt investor might decide not to roll over, not because the expected asset return is too low but because she expects a too large number of other investors to not roll over. A panic run or self-fulfilling run occurs if too many investors fear other investors will not roll over, withdraw, and cause the run.

In this setting, we analyze the question how coordination and the probability of a run by debt investors depend on firm capital structure and market liquidity of firm assets. As main contribution of the paper, we demonstrate that the probability of a run is in general non-monotone in short-term debt and that non-monotonicity is in large parts affected by asset liquidity. This implies, increases in equity financing may harm coordination and increase the probability of runs. This stands in contrast to the classic literature on bank regulation (Cohen, 1970; Furlong and Keeley, 1989; Kim and Santomero, 1988) which argues that equity always improves firm stability by reducing insolvency risk. Firm insolvency occurs if asset value falls below value of debt. This literature strand however assumes that the firm's debt structure (maturity and amount outstanding) is exogenous. As a consequence of this assumption, capital regulation decreases insolvency risk since it guarantees a minimum equity cushion against shocks in asset's market value when balance sheets are marked to market.[5] In this paper we make a point the other way around. We assume asset liquidity is deterministic[6] but the debt structure is random and in particular endogenous. The probability distribution of short-term debt becoming due tomorrow depends on the capital structure today. Under these changed assumptions we obtain that capital regulation may alter coordination and thus the probability distribution of debt becoming due in the near future in a way that runs become more likely - the illiquidity risk may increase.[7] The general intuition

[3]Our results are robust to allowing the firm to borrow cash by pledging the asset as collateral. By this, partial liquidation is avoided.

[4]This scenario is satisfied for 'sufficiently' illiquid assets but also for liquid assets if promised interest payments to debt investors are large and if the firm is financed through a proportionally large amount of short-term debt.

[5]Insolvency risk is further reduced by capital regulation since banks respond to more 'skin' (equity) in the game by investing in less risky assets.

[6]The assumption that liquidity is deterministic is the major constraint of our model, similar to (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005).It is however as strong an assumption as assuming that short-term withdrawals of deposits or other liquid forms of debt are exogenous in maturity and magnitude.

[7]In this paper, we follow the definition of illiquidity and insolvency risk by Morris and Shin (2009):

that more short-term funding and hence exposure to investors having short-term claims lead to higher short-term withdrawals and liquidity risk in the future is challenged in this article. We show, more short-term debt can discipline debt investors to better coordinate if assets are illiquid.

A further contribution of our paper is of technical nature. The game structure analyzed here exhibits only one-sided strategic complementarity between actions (Karp et al., 2007; Goldstein and Pauzner, 2005) and hence differs from the classic global games structure where actions are global strategic complements (Vives, 2014; Rochet and Vives, 2004; König et al., 2014; Morris and Shin, 2009).

The first game structure evolves naturally in bank run games when closely modeling the real world fact that in the incidence of a run, cash available by asset liquidation is insufficient to satisfy claims by all debt investors. Investors are only partially repaid, and the payment depends on the endogenous number of investors trying to withdraw, see Goldstein and Pauzner (2005): This is because debt investors have a hard claim and draw on a common pool of liquidity. To withdraw, investors queue in front of the firm and are served one after another. To serve an investor the firm liquidates a fraction of the asset. The place in the queue is random. In a run, not the entire queue can be served, service stops when all cash available from liquidating the firm's asset is distributed. The more investors try to withdraw the longer the queue and hence the larger the probability to queue in a position which cannot be served.

As a consequence, the incentive to withdraw versus wait is not largest when all investors withdraw but when only just as many investors withdraw that put the firm on the edge of a run. Then, the entire queue is just served while investors who wait and roll over receive zero independently of the size of the run.[8] Conditional on a run, actions are strategic substitutes, in particular actions are not global strategic complements.[9]

One-sided strategic complementarity is the key driver for the non-monotonicity results obtained in this article. Comparative statics under global strategic complements (Morris and Shin, 2009; Rochet and Vives, 2004; Vives, 2014) are all monotone in this strand of literature (see explicit discussion of literature and technique below). This article thus demonstrates that in global games, monotonicity of bank run probability in debt is not robust to one-sided strategic complementarity between actions.

'Insolvency risk' is the probability of a default due to deterioration of asset quality conditional on the event that no run occurs. Credit risk is the unconditional probability that the firm cannot repay debt at some point in time. Illiquidity risk is the difference between credit and insolvency risk, that is the probability of a default due to a run if the firm had been solvent in absence of the run. In our setting, illiquidity risk is the risk that current liabilities realize such that they undercut asset value. For further discussion of the distinction between insolvency risk and illiquidity risk, see Davydenko (2012)

[8] Conditional on a run, payoffs to investors who want to withdraw strictly decrease in the number of investors trying to withdraw while payoff to investors who roll over is fixed at zero.

[9] Global strategic complementarity between actions implies, that the incentive for an agent to pick a certain action A versus choosing the other action B increases in the number of other agents choosing that same action A.

We now give an intuition for why a departure from the previous literature leads to non-monotone run probabilities in capital. In detail, the main driver of the non-monotonicity results, and the main departure from the previous literature on the impact of capital structure and asset liquidity on runs, is that we impose uncertainty on the action 'withdraw'. Debt investors who simultaneously decide to withdraw from the firm might receive zero. In the incidence of a run, when debt claims by withdrawing investors exceed liquidation value of the asset, the firm may only distribute the liquidation value of the asset, thus not all investors who want to withdraw can be served. Consequently, the payoff to withdrawing becomes risky and sensitive to changes in capital structure:[10] In the incidence of a run, the more the firm is (proportionally) financed with equity, the fewer debt investors have a claim on liquidation value (the shorter the maximum length of the queue) and the higher the expected payoff from withdrawing. The latter is since positions in the queue are random, the maximum queue length has decreases but the number of positions in the queue that can be served at fixed liquidation value of the asset has remained constant. Thus, equity sweetens withdrawal in uncertain times since it serves as cushion in the incidence of a run. Equity also benefits debt investors who roll over. The change of equilibrium due to a marginal change in equity is thus determined by marginal utilities. The following stylized example demonstrates that debt investors who withdraw might benefit stronger from increases in equity than investors who roll over. Thus, a marginal investor who is initially indifferent between rolling over debt or withdrawing might, after an increase in equity, prefer to withdraw which leads to increases in the run probability, explained now.

Example: A firm raises \$5 in equity and short-term debt to finance a long-term investment in a risky asset. At an interim period, after observing information about the random asset return debt investors decide whether to roll over debt or to withdraw. The asset is illiquid, hence premature, fast liquidation of the asset only yields \$1. The firm promises fixed interest payments to debt investors for every period invested. By the nature of equity, debt investors are paid first and all remaining revenues go to equity investors.
Setting A) In order to finance the asset, the firm collects \$1 equity and \$4 short-term debt raised from 4 different debt investors who each invests \$1. At the interim stage markets can either be up or down. Assume markets are up and the asset pays with high probability. Then, all debt investors roll over debt, collect their interest payments with high probability in the following period and all extra returns go to equity investors. If however markets are down and the asset pays with low probability, all debt investors withdraw which forces the firm to liquidate the asset at \$1. Since 4 debt investors have a claim on this dollar, on average every debt investor receives \$1/4.
Setting B) Now assume, the firm increases her equity ratio by financing the same investment with \$2 equity and \$3 debt collected from 3 debt investors. If markets are up, again all debt investors roll over as they receive promised interest payments with very high probability. If markets are

[10] Under the assumption that withdrawing yields a safe payoff, changes in capital structure do not change payoff from withdrawing, hence the payoff to withdrawing is insensitive to changes in capital.

down however, all debt investors withdraw, the firm liquidates the asset at \$1 and on average, every debt investor receives \$1/3 which is larger than \$1/4.

Since debt investors' interest payments are fixed, they do not benefit from the upside potential of the asset, thus the financing structure of the firm has no impact on debt investors' payoffs if asset markets are up. The capital structure however does matter in bad times. If uncertainty about asset returns is high and debt investors refuse to roll over, by illiquidity of assets cash available through liquidation does not cover all withdrawals. Comparing both settings of the example, conditional on a run, the payoff to withdrawing increases in equity from \$1/4 to \$1/3 since realized liquidation value is allocated among less debt investors. The intuition for this example is related to the value of debt taking the form of an inverted hockey stick at expiry (Holmstrom, 2015): conditional on a run, debt is information sensitive with respect to capital structure and its value increases in equity. Conditional on no run (in good times), debt becomes information insensitivity towards capital structure and its value is constant in equity.
The model we analyze in the paper has a unique equilibrium which is characterized by a trigger signal about the asset return. Debt investors will find it optimal to withdraw when observing signals below the trigger since this signals low asset returns and will roll over debt when observing signals above the trigger, see Figure (**??**). Upon observing the trigger signal a debt investor is indifferent between rolling over or withdrawing (marginal investor). Consider the signal of the marginal investor in setting A). As the firm changes her financing structure from setting A) to setting B), the immediate payoff from rolling over stays constant compared to the previous setting since the financing structure does not impact the asset's return probability and promised interest payments to debt investors remain unchanged. But the payoff from withdrawing increases. Hence, the signal that makes an investor indifferent in setting A) cannot make her indifferent in setting B), see Figure (**??**). Instead, at the same signal in setting B) the investor tends towards withdrawing.

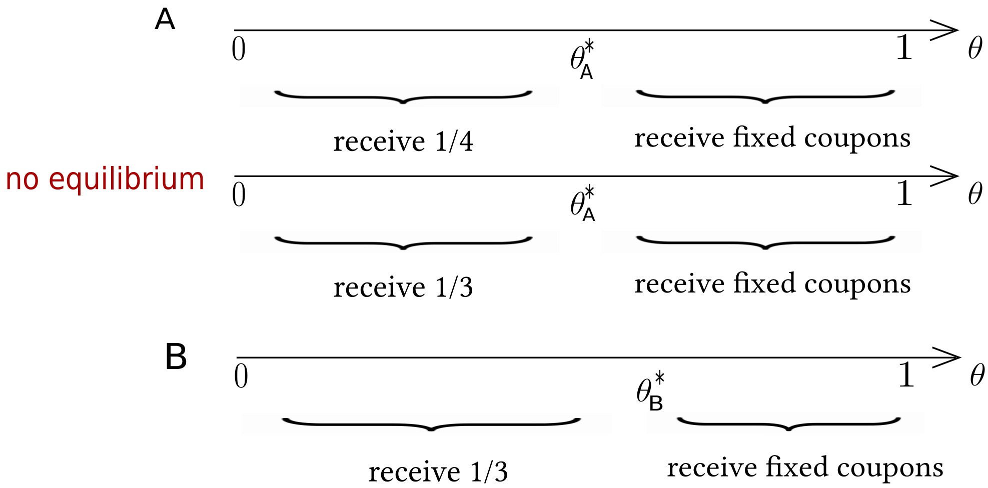


Figure 1.1: Shift of equilibrium trigger signal due to change in equity

The equilibrium trigger in setting B) is thus larger, investors withdraw for a greater range of signals compared to setting A) which increases the ex ante probability of a run although the firm is financed through more equity.[11]

As main result of the article, we obtain that the probability of a run is in general non-monotone in short-term debt and that non-monotonicity is driven by asset illiquidity. When asset liquidity is high, the probability of runs increases monotonically in debt. As liquidity dries up, the probability of runs becomes non-monotone. Probability of runs increases in debt for low debt values but decreases in debt for larger debt values. When the asset becomes perfectly illiquid, monotonicity is restored but tilted: the probability of a run becomes monotone decreasing in debt and increasing in equity.

Second, the non-monotonicity result expands to liquidity mismatch. If we measure liquidity mismatch of assets and liabilities as the ratio of cash the firm can realize by selling the asset over potential short-term withdrawals, we show that the probability of runs is not monotone in liquidity mismatch.[12]

Third, as a consequence of these non-monotonicity results capital and liquidity regulation can have adverse effects on firm stability depending on asset liquidity. We demonstrate, while capital and liquidity regulation of financial institutions can improve stability when market liquidity of assets is high, the identical policy rule can harm stability when liquidity is low or dries up as its implementation deteriorates the coordination problem among short-term debt investors. These results have policy implication with respect to Basel 3. Our results imply that capital regulation should be tailored to particular scenarios for market liquidity or capital regulation should distinguish between firms according to their target asset liquidity. Further we demonstrate, under endogenous panic withdrawals by investors, liquidity mismatch is no reliable measure of liquidity risk since the probability of runs is non-monotone in liquidity mismatch.

Fourth, the non-monotonicity results hold under partial asset liquidation but are robust to collateralized borrowing, where the firm may pledge the asset in the money market and by this prevents partial liquidation. Our results have thus policy implications with respect to regulation of classic commercial banks but also structured investment vehicles (SIVs) and asset-backed commercial paper conduits.

Last, we consider debt investors' welfare from contracts, taking the coordination behavior of investors as given in subgames. We demonstrate, for every contract if asset liquidity is high debt investor suffer from increases in debt ratio. If liquidity is however low, they might benefit from

[11]Note that while in the example game non-symmetric threshold equilibria might exist, the main game introduced later will have a unique equilibrium which is a symmetric threshold equilibrium. Further, debt investors' signals will differ only by a small, random noise term. As the support of the noise becomes small, debt investors observe the same signals and hence choose identical actions. For this example, we have used pro rata shares but the same intuition applies for queuing where conditional on a run agents receive fixed coupons but with varying probability.

[12]A liquidity mismatch exists if overall cash that can be raised through selling all assets on short notice (market liquidity) undercuts the maximum sum of potential short-term cash claims by debt investors. In this case, we define liquidity mismatch as the ratio of asset market liquidity to potential short-term claims. An existing liquidity mismatch gives rise to the possibility of runs on the financial firm.

higher debt ratios since these improve coordination and thus stability.

Related Literature

This paper adds to the growing literature on stability of maturity transforming financial intermediators against runs by short-term debt investors. In a seminal paper, Diamond and Dybvig (1983) analyze coordination behavior of depositors who share consumption risk by entering in demand-deposit contracts with a bank. Due to maturity transformation and risk-sharing two pure equilibria are shown to exist, a bank run and a no run equilibrium. An ex ante probability for the emergence of each equilibrium cannot be calculated within the model. Postlewaite and Vives (1987) analyze demand-deposit contracts using a game structure similar to the Prisoner's dilemma and deduce parameter constellations under which a unique equilibrium evolves with a strictly positive probability of a 'run'. While in Diamond and Dybvig (1983), runs are purely due to panic and always inefficient, Bryant (1980), Chari and Jagannathan (1988) and Jacklin and Bhattacharya (1988) model information-based runs by introducing asset return risk and interim information. Our set-up allows for both, runs caused by panic and self-fulfilling beliefs but also efficient runs driven by bad news about firm solvency. Interim information on the asset return can reveal a low return probability of the asset and running on the firm can be a dominant action. Empirical evidence exists for both types of runs: Evidence for depositors withdrawing when perceived asset risk is too high is provided by Goldberg and Hudgins (1996, 2002). Foley-Fisher et al. (2015) investigate the run on U.S. life insurers during the summer of 2007 and find evidence for self-fulfilling expectations.

To obtain a unique equilibrium, this paper employs technique from global games theory (Carlsson and Van Damme, 1993; Morris and Shin, 1998, 2001). Private, asymmetric but correlated signals serve as coordination device among agents and may lead to equilibrium uniqueness and definite comparative statics.

The model closest to ours is Goldstein and Pauzner (2005) who embed the Diamond and Dybvig (1983) model in a global game and are hence able to show optimality of risk-sharing even though it increases the likelihood of runs. Their paper is the first to show equilibrium existence and uniqueness under only one-sided strategic complementarity with partial, endogenous repayment given default of the bank. We strongly draw on their existence and uniqueness proof in our setting. The question we analyze here however differs. Goldstein and Pauzner (2005) analyze contracts where the bank is fully financed by debt and invests in an asset which is liquid at the interim period. We allow for a general capital structure, general asset market liquidity and focus on the effects changes in capital structure and asset liquidity have on the probability of runs. Further, we concentrate on optimal capital structure from debt investors point of view who take contracts as exogenously given.

The question, how capital structure and asset liquidity impact coordination behavior of debt

investors and probability of runs in a global game has been analyzed before in the context of collateralized funding (Morris and Shin, 2009) and delegated decision making (Rochet and Vives, 2004; Vives, 2014; König et al., 2014). We depart from Rochet and Vives (2004); Vives (2014) and König et al. (2014) in assuming that decisions are made directly by investors not fund managers. We depart from Morris and Shin (2009) by assuming that in the incidence of a run the firm can only partially repay and follows a sequential service constraint as modeled in Goldstein and Pauzner (2005). This modeling feature changes the game structure and implies that in the incidence of a run withdrawing investors do not receive their deposit for sure as they would in Morris and Shin (2009). Hence, the action withdrawing is risky and its payoff becomes sensitive to changes in capital structure. While Morris and Shin (2009), Rochet and Vives (2004); Vives (2014) and König et al. (2014) allow the asset liquidation value to depend on the random state, in our model the liquidation value is exogenous and deterministic.
From a theory perspective, while Morris and Shin (2009); Rochet and Vives (2004); Vives (2014) and König et al. (2014) analyze a classic global game with global strategic complementarity between actions (Bulow et al., 1985), the game analyzed here exhibits only one-sided strategic complements as in Goldstein and Pauzner (2005) and Karp et al. (2007).
Further related papers are Eisenbach (2013) and Szkup (2015) who study roll-over decisions by short-term debt investors in dynamic settings.

1.2 The Model

There are three periods of time 0, 1, 2 and one good (money). We assume no discounting between periods. There is a financial intermediator, called 'the firm', and two types of agents: a continuum of short-term debt investors $[0, \delta]$, of measure $\delta \in (0, 1)$, and a single equity investor. Both types of agents live for two periods.
At period 0, debt investors are symmetric and each endowed with one unit of the good. Debt investors are risk-averse and can consume in either period.[13] Their utility function $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, concave and we normalize $u(0) = 0$.
The equity investor is risk-neutral and can only consume in period 2. At time zero she is endowed with measure $1 - \delta$ units of the good. Hence, at time zero there is an aggregate endowment of measure 1 unit of the good. Debt investors and equity investors finance the firm's investment in a risky asset. Agents are born either as equity investor or debt investor, agents may not split their endowment to finance the firm in both ways.[14]

Investment There exists a storage technology and a risky, illiquid asset in the economy, T. Storage yields the initial investment for sure in every period. For every unit invested in period 0, the asset T yields $0 < l \leq 1$ units if the asset is sold prematurely in period 1. If the investment is

[13]This is in contrast to Diamond and Dybvig (1983) and Goldstein and Pauzner (2005) where a certain proportion of agents has to consume in the first period.
[14]This assumption is for tractability reasons, actions are binary.

continued until period 2, T yields either payoff $H > 1$ with probability p or zero with probability $1 - p$, where $p = p(\theta)$ is random and determined by the random state $\theta \in [0, 1]$ (see information structure below). The asset's probability of return $p(\theta)$ is continuously differentiable in θ, strictly increasing for $\theta \leq \overline{\theta}$ and constant $p(\theta) = 1$ on $[\overline{\theta}, 1]$. $\overline{\theta}$ denotes the boundary to the upper dominance region, introduced below.
We call l the (market) liquidity[15] of the asset. We can also think of l as the fire sales price in the secondary market in times of crises.[16] Liquidity l is exogenously given and deterministic. Debt investors have no access to asset T, only to storage. Debt investors gain indirect access to T through investing in the firm. The expected asset return exceeds the return from storage

$$\mathbb{E}[p(\theta)]H > 1 \tag{1.1}$$

The firm The economy has a representative financial intermediary - the firm. The firm's balance sheet size is normalized to 1. She raises funds of one unit and invests in asset T.[17] The firm finances an endogenous fraction $\delta \in (0, 1)$ of her funds through short-term debt and the remaining fraction $1 - \delta$ with equity. As funds are normalized, we call δ also the firm's capital structure or debt ratio and $1-\delta$ the equity ratio.[18] The firm is in perfect competition for deposits.

Debt contract and firm structure By entering in a debt contract with the firm, debt investors can attain higher returns on their investment than through investing in storage. Every debt contract is characterized by two exogenously given coupons, a period 1 coupon $r > 1$ and period 2 coupon $rk < H$. We henceforth write (r, k) for the contract.

If a debt investor invests in contract (r, k), she hands her endowment to the firm in period 0. The contract is liquid in the sense that a debt investor may decide spontaneously in period 1 whether to claim short-term coupon r in period 1 or whether to wait, roll over debt and claim long-term coupon rk in period 2. For $k < 1$, the game has the dominant action to withdraw early. To keep the analysis interesting, in the remainder of the paper we concentrate on $k > 1$.[19]
In period 1, debt investor i chooses her **action** and decides whether to $\mathcal{A}_i \in$ {withdraw, roll over} her investment. When a debt investor decides to withdraw, we will also say that she 'runs' on the firm. She cannot demand a fraction of her investment. The parameter $k \in (1, H/r)$ can be seen as an implicit forward interest payment which the firm pays to investors for leaving funds

[15] See Brunnermeier and Pedersen (2009)

[16] If the asset as a risky loan, due to information asymmetries a potential buyer is willing to only pay $l < 1$ instead of a price that would reflect the fair value of the loan.

[17] By assumption, the firm commits to investing in the asset no matter how the state realizes. By this, we exclude signaling in a global game and circumvent multiplicity of equilibria.

[18] By this normalization, in the analysis of this paper the firm always holds exactly one asset. But the financing structure of the balance sheet, the composition of equity and short-term debt, changes. By the normalization, an increase in debt (ratio) is always accompanied by a decrease in equity (ratio).

[19] The debt contract can be understood as a one-period zero coupon bond with price 1 and face value r and the option to convert the bond in period 1 to a two period zero coupon bond with face value rk.

invested for another period.[20]

Contract (r, k) and asset return probability function $p(\cdot)$ are such that the expected payoff from rolling over exceeds payoff from withdrawing

$$\mathbb{E}[p(\theta)]u(kr) > u(r) \tag{1.2}$$

Otherwise, running on the firm was a dominant action and the outcome of the game becomes trivial. Note that $r > 1$ implies

$$\mathbb{E}[p(\theta)]u(kr) > u(1) \tag{1.3}$$

that is, the expected period 2 payoff from the contract exceeds utility from storage and participation in the contract is individually rational.

Endogenous Liquidation If the firm has a debt ratio δ and offers contract (r, k) we call (r, k, δ) the *firm structure*. At period 1, a firm with structure (r, k, δ) faces potential withdrawals of short-term debt of value up to δr. As debt is more senior than equity, the firm is committed to make the coupon payments under the premise of solvency. In this paper, we only consider firms which are prone to runs. This is no constraint but keeps the game interesting. If the firm is not prone to runs $\delta r \leq l$, the coordination problem vanishes and the outcome is trivial. For a given contract (r, k), let the proportion of short-term debt funds δ and promised short-term coupon r be high such that ex ante a liquidity squeeze (run) cannot be excluded, i.e. it holds

$$\delta r > l \tag{1.4}$$

Let $n \in [0, 1]$ denote the endogenous, random equilibrium proportion of debt investors who decide to withdraw in period 1 (aggregate action). Given the contract (r, k) and the measure of short-term debt funds $\delta \in (0, 1)$ collected by the firm, in period 1 the firm needs to pay out the ex ante random measure δrn of cash to withdrawing debt investors. The firm finances withdrawals by liquidating the corresponding fraction $n\delta r/l$ of the asset. A **run** occurs in period 1, if the measure of short-term funds claimed back by withdrawing investors exceeds market liquidity of the asset l, that is if n realizes such that

$$n\delta r > l \tag{1.5}$$

Sequential Service Constraint In the incidence of a run, if asset liquidity undercuts debt claims by withdrawing investors, the firm cannot honor her debt and goes into default. In that case, the firm follows a sequential service constraint. Withdrawing investors are served one after another in a queue and paid their promised coupon payments until all cash raised from liquidation

[20]The assumption $k > 1$ is necessary, otherwise we had $r > kr$ and withdrawing early was a dominant action.

is distributed. By definition of a run, there are debt investors in the queue, trying to withdraw, who will not be paid since the firm will run out of cash before it is their turn in the queue.[21] There is mass l in cash available for distribution while there is a claim for cash of mass $\delta rn > l$. **Payoffs** in case of a run to withdrawing debt investors are $u(r)$ with probability $l/(\delta nr)$ (probability of getting served in the queue) or 0 with probability $1 - l/(\delta nr)$. Debt investors who roll over receive zero in case of a run since all debt investors draw on the same pool of liquidity. We assume zero recovery costs.

If the firm stays liquid in period 1, all withdrawing investors receive $u(r)$ and the game proceeds to period 2. In period 2, the return of the asset realizes as either H with probability $p(\theta)$ or zero.[22] In case of zero, remaining debt investors receive zero. Conditional on success, gross return on remaining investment per debt investor equals

$$V(n) = \frac{(1 - \delta nr/l)H}{\delta(1-n)} \tag{1.6}$$

By illiquidity of the asset, liquidation diminishes future gross returns. Thus, gross return per debt investor V may undercut promised long-term coupon kr. Hence, our model allows for the case where the firm is liquid but insolvent at the same time: In period 1, it might be that debt claims and thus liquidation of assets at fire sales prices are so extensive, that the debt service of all claims in the following period becomes a foreseeable impossibility. In period 2, if gross return per debt investor undercuts kr, the firm is insolvent, and again follows a sequential service constraint. Debt investor receive $u(kr)$ only with probability

$$\frac{(1 - \delta nr/l)H}{\delta(1-n)kr} < 1 \tag{1.7}$$

and equity value is zero. If gross return exceeds kr, $kr = \min(V, kr)$, debt investors who roll over receive payoffs $u(kr)$ as promised in the contract, and equity investors obtain the residual value.

Payoffs Debt Investors We assign the following payoffs to agents:

Event/ Action	withdraw	roll-over
no run, $n \in [0, l/(\delta r)]$	$u(r)$	$\begin{cases} u(kr) \cdot q(n), & p(\theta) \\ 0, & 1 - p(\theta) \end{cases}$
run, $n \in [l/(\delta r), 1]$	$\begin{cases} u(r) & \text{, prob. } l/(\delta nr), \\ 0, & \text{, prob. } 1 - l/(\delta nr) \end{cases}$	0

where

$$q(n) = \min\left(1, \frac{(1 - \delta nr/l)H}{\delta(1-n)kr}\right) \tag{1.8}$$

[21] In particular, agents do not receive a pro rata share of their promised coupon for sure but receive their full claims if they are served in the queue (with a probability strictly smaller one). This assumption is for tractability reasons.

[22] For instance, a loan is paid back including interest H or the borrower defaults completely.

is the probability to obtain period 2 coupon when queuing conditional on investment being successful.

Debt investor's random utility difference between withdrawing in period 2 versus withdrawing early in period 1 is given by

$$v(\theta, n) = \begin{cases} p(\theta)u(kr) \cdot q(n) - u(r) & \text{if } n \leq \frac{l}{\delta r} \text{ (no run)} \\ -\frac{l}{\delta nr}\, u(r) & \text{if } n > \frac{l}{\delta r} \text{ (run)} \end{cases} \tag{1.9}$$

Information Structure Here we follow Goldstein and Pauzner (2005). In period zero, the unobservable state $\theta \sim U[0,1]$ realizes and determines the return probability $p(\theta)$ of the asset. Debt investors share a common prior about state θ in period 0. In period 1, debt investors observe private, noisy and asymmetric signals about the state and hence asset return probability

$$\theta_i = \theta + \varepsilon_i,\ i \in [0, \delta]$$

where ε_i are iid random noise terms, independent of θ and distributed according to $U[-\varepsilon, +\varepsilon]$. From the signal structure we see, signals convey information not only about the random asset return probability $p(\theta)$ but also about other investors' signals.

We assume, there exist states which yield dominant actions (dominance regions).[23] There are states $\overline{\theta}$ and $\underline{\theta}$ such that if $\theta < \underline{\theta}$, withdrawing is a dominant action whereas if $\theta > \overline{\theta}$ rolling over is the dominant action to debt investors. We refer to $[0, \underline{\theta}]$ as the lower dominance region and call $[\overline{\theta}, 1]$ the upper dominance region. The bound $\underline{\theta}$ depends on the specific contract (r, k) and is given as the realization of θ such that[24]

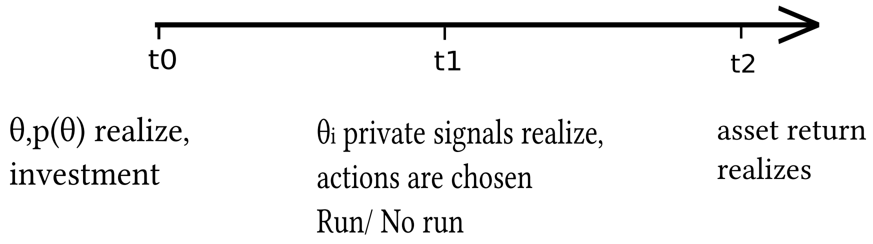
$$u(r) = p(\underline{\theta})\, u(kr)$$

The assumption of existence of the lower dominance region implies that function $p(\cdot)$ takes values below $u(r)/u(kr) > 0$. For high states $\theta \geq \overline{\theta}$, we impose that the asset earns return H already in period 1 with certainty, that is with $p(\theta) = 1$. As a consequence of assumption $H > kr > r$, the coordination problem vanishes for state realizations in the upper dominance region. To ensure that debt investors may receive signals from which they can infer that the state has realized in either of the dominance regions, we assume that noise ε is sufficiently small such that $\underline{\theta}(r, k) > 2\varepsilon$ and $\overline{\theta} < 1 - 2\varepsilon$ hold. In particular, the bounds to the dominance regions are independent of debt ratio and asset liquidity.

[23] Dominance regions are crucial to obtain an equilibrium selection (Morris and Shin, 2001).

[24] Payoff $u(kr)$ is the maximum payoff debt investors who roll over can obtain. By design of the contract, if θ realizes below $\underline{\theta}$, even in the absence of a run the expected payoff to rolling over is smaller than $u(r)$ for every $n \in [0, 1]$, while conditional on a run investors who roll over receive zero.

Timing In period 0, the state θ and payoff probability $p(\theta)$ realize unobservably. Debt and equity investors invest. In period 1, debt investors observe noisy, private signals and subsequently choose actions. The aggregate action n (proportion of withdrawing debt investors) realizes and determines whether the firm defaults due to a run or whether she stays liquid in period 1. In case of a run, all debt investors receive payoffs according to chosen actions and the game ends. If the firm stays liquid, the game proceeds to period 2 after paying debt investors who decide to withdraw. In period 2, the success of the risky investment is determined. Payoffs to equity investors and debt investors who roll over realize. Note that agents choose actions only in period 1.[25]



1.3 Equilibrium

In order to justify the imposition of the information structure described in the outline of the model we briefly discuss equilibrium behavior in the absence of private, noisy and asymmetric signals.

The Common Knowledge Game

Without signals, all debt investors share a common prior about state θ in period zero and receive no further information about the state in period 1. By assumption (2.2), the ex ante expected utility from rolling over exceeds the utility from withdrawing. The presence of the coordination problem in period 1 gives rise to a Diamond and Dybvig (1983) type situation. There are two pure equilibria: In the 'good' equilibrium all debt investors roll over and there is no run. Due to assumption (2.3) welfare in this equilibrium is higher than in the outcome where all debt investors store their endowment. In the bad equilibrium all investors panic and withdraw early which causes a run. In the run equilibrium, due to $u(l) \leq u(1)$, aggregate welfare is lower than in the outcome where all debt investors store their endowment. There is no means to determine the ex ante probability for selection of the Pareto-efficient no-run equilibrium within the model. To achieve an equilibrium selection and definite comparative statics on stability we impose the information structure given in the outline of the model.

[25]Equity investors do not act in this paper since they hold the residual claim in period 2.

1.3.1 The Coordination Game

Let (r, k, δ) the firm's structure and let θ_i an investor's private signal. A mixed strategy for investor i is a measurable function $s_i : [0 - \varepsilon, 1 + \varepsilon] \to [0, 1]$ which assigns a probability that the investor withdraws early (runs) as a function of her signal θ_i. A strategy profile is denoted by $\{s_i\}_{i \in [0,\delta]}$. A fixed strategy profile generates a random variable $\tilde{n}(\theta) \in [0, 1]$ which represents the aggregate action, the proportion of investors who withdraw early, if the unobservable state realizes as θ. The equilibrium concept we use is Bayesian Nash Equilibrium.

Proposition 1.3.1 (Existence and Uniqueness)**.** *The coordination game played by debt investors has a unique equilibrium. The equilibrium is in trigger strategies.*

Denote by $\theta^* = \theta^*(r, k, \delta, l, H, p(\cdot)) \in [\underline{\theta} - \varepsilon, \overline{\theta} + \varepsilon]$ the equilibrium trigger signal. In the trigger equilibrium, if an investor observes a signal $\theta_i < \theta^*$ she withdraws, if she observes a signal $\theta_i > \theta^*$ she rolls over debt. In case $\theta_i = \theta^*$ she is indifferent. For the equilibrium is a symmetric trigger equilibrium played by a continuum of debt investors, the endogenous measure of investors who withdraw is a deterministic function of the random state and the equilibrium trigger signal. Let $n(\theta, \theta^*)$ indicate the endogenous equilibrium proportion of investors demanding early withdrawal in period 1 when the true state is θ and the trigger is θ^*. The function $n(\theta, \theta^*)$ is given by the proportion[26] of investors who observe a signal below the trigger θ^* when the true state is θ. By the uniform distribution of the error term, we have

$$n(\theta, \theta^*) = \begin{cases} \frac{1}{2} + \frac{\theta^* - \theta}{2\varepsilon} & \text{if } \theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon] \\ 1 & \text{if } \theta \leq \theta^* - \varepsilon \\ 0 & \text{if } \theta \geq \theta^* + \varepsilon. \end{cases} \tag{1.10}$$

In Figure (2.2), we have plotted the proportion of investors withdrawing as a function of the state for fixed trigger θ^*. Given state θ, investors observe signals in the range $[\theta - \varepsilon, \theta + \varepsilon]$. For a state below $\theta^* - \varepsilon$, all investors obtain signals smaller than the trigger and hence withdraw, $n = 1$. Vice versa, for a state above $\theta^* + \varepsilon$, all investors observe signals larger than the trigger and hence roll over, $n = 0$.

Having established equilibrium uniqueness, the equilibrium trigger signal is pinned down by the expected payoff difference between actions conditional on having observed the equilibrium trigger $\theta_i = \theta^*$ when all investors use the same trigger θ^*,

$$D(\theta_i, \theta^*) = \frac{1}{2\varepsilon} \int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} v(\theta, n(\theta, \theta^*))\, d\theta \tag{1.11}$$

When observing a signal $\theta_i < \theta^*$, the expected payoff difference $D(\theta_i, \theta^*)$ is negative and the investor withdraws. When instead she observes $\theta_i > \theta^*$, the payoff difference $D(\theta_i, \theta^*)$ is positive and she rolls over. When observing a signal equal to the equilibrium trigger a debt investor's

[26] As the continuum of debt investors has measure δ, the proportion of investors observing signals below the trigger differs from its measure by factor δ.

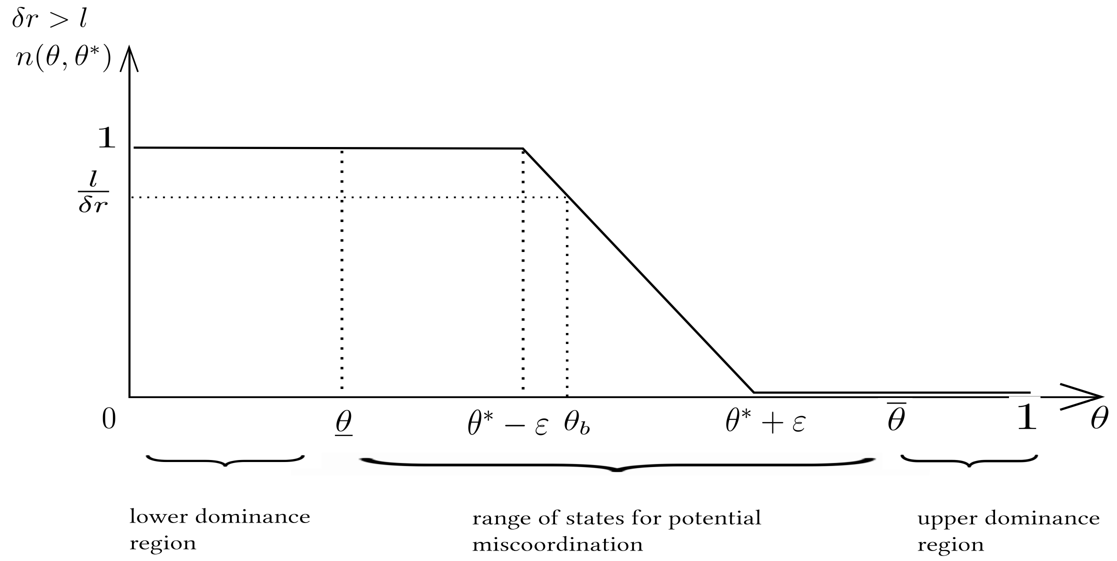


Figure 1.2: Proportion of debt investors who withdraw as a function of the state. Note that while the bounds to the dominance regions, $\overline{\theta}$ and $\underline{\theta}$, and the critical state θ_b are states, the trigger θ^* is a signal. We have included the trigger here, to give some intuition. Further, the trigger θ^* converges to the critical state θ_b as signals become precise, see Lemma 2.3.1.

posterior beliefs on the state and the proportion of withdrawing investors n need to be such that in expectation utility from withdrawing equals utility from rolling over. The trigger is thus implicitly defined by the payoff indifference equality (PIE)

$$D(\theta^*, \theta^*) = 0 \tag{1.12}$$

Graphically, as signals become precise the trigger is located between the dominance regions $[\underline{\theta}, \overline{\theta}]$ in a way such that the area under the curve in Figure (2.1) equals zero in expectation conditional on having observed a signal equal to the trigger. Conditional on observing the trigger signal $\theta_i = \theta^*$, an investor's belief about the proportion of withdrawing agents n is uniform over $[0, 1]$ (Laplacian Belief).[27] Consequently, with slight abuse of notation we can write the PIE using (2.9) and (2.11) as

$$\begin{aligned} 0 = &- \int_{l/(\delta r)}^{1} \frac{l}{\delta r n}\, u(r) dn + \int_{h^*}^{l/(\delta r)} p(\theta(n, \theta^*)) \frac{(1 - nr\delta/l)H}{\delta(1-n)kr}\, u(kr) - u(r)\, dn \\ &+ \int_{0}^{h^*} p(\theta(n, \theta^*))\, u(kr) - u(r)\, dn \end{aligned} \tag{1.13}$$

where $\theta(n, \theta^*) = \theta^* + \varepsilon\,(1 - 2n)$ is the inverse of $n(\theta, \theta^*)$ for $\theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon]$. The parameter h^* given in (1.20) denotes the proportion of withdrawing investors n for which gross return per remaining debt investor $V(n)$ intersects kr. For low withdrawals $n \leq h^*$ the firm is liquid in

[27] We have $\mathbb{P}(n < z | \theta_i = \theta^*) = \mathbb{P}\left(\frac{1}{2} + \frac{\varepsilon_i}{2\varepsilon} < z\right) = z$ for $z \in [0, 1]$

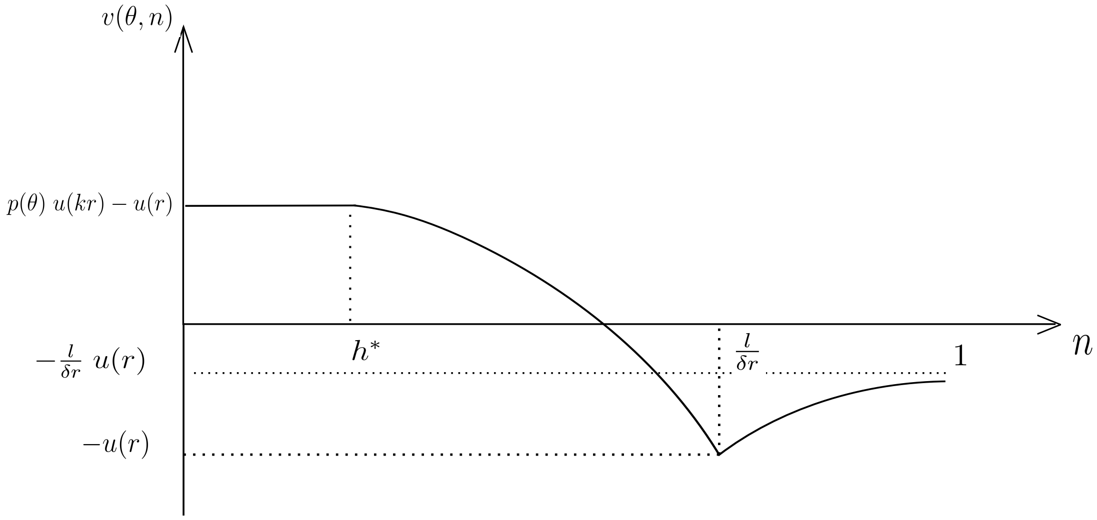


Figure 1.3: Payoff difference function $v(\theta, n)$ from equation (2.9) plotted for fixed θ as function of the endogenous proportion of withdrawing debt investors n. The kink gives rise to non-monotone comparative statics.

period 1 for sure and solvent in period 2 if investment is successful. For all larger proportions of withdrawing investors $n > h^*$, debt investors who roll over cannot be repaid in full and the firm becomes insolvent in period 2. If the proportion of withdrawing investors n is high and lies in interval $[l/(\delta r), 1]$ a run occurs in period 1 and the firm defaults due to illiquidity. For intermediate withdrawals $n \in [h^*, l/(\delta r))$ the firm stays liquid in period 1 but the measure of withdrawn funds is high such that remaining investment cannot earn sufficient interest to pay off all investors in the next period even if the asset pays off. Hence, for withdrawals $n \in [h^*, l/(\delta r))$ the firm is liquid but insolvent in period 1.[28]

Denote by $\tilde{\theta}$ the state at which asset liquidations occur to an extent that puts the firm on the edge of staying solvent in period 2,

$$h^* = n(\tilde{\theta}, \theta^*) \tag{1.14}$$

Then h^* is the critical measure of withdrawing investors at which investors who roll over debt just receive their coupon payment $u(kr)$ for sure conditional on successful investment. If the state realizes such that measure of claimed funds $n\delta r$ just equals available liquidity l, the firm is on the edge of becoming illiquid in period 1. We call this state the *critical state*[29] θ_b,

[28] By assumption, the firm needs to partially sell the asset and has no access to collateralized borrowing. Here, our treatment is different from Morris and Shin (2009) who assume the that the firm may pledge the asset at a hair cut. In a later section, we demonstrate robustness of our results under collateralized borrowing. Also, we do not allow the firm to replace withdrawn deposits with other funds.

[29] Under collateralized borrowing, the critical state θ_b and state $\tilde{\theta}$ would be equal. Hence, if the firm stays liquid she can always repay all investors in period 2 if the asset pays.

$$n(\theta_b, \theta^*) = l/(\delta r) \tag{1.15}$$

As depicted in Figure (2.2), for state realizations smaller than the critical state a run occurs because the value of claimed funds exceeds market liquidity of the asset. In the sequel, we say that *signals become precise* or *noise vanishes*, if the support of the idiosyncratic, random shock collapses to a single point, $\varepsilon \to 0$.
As signals become precise, the critical state converges to the trigger, $\theta_b \to \theta^*$, thus the trigger directly represents the firm's risk to become illiquid or insolvent due to extensive asset liquidations.[30]

Lemma 1.3.1. *As signals become precise, the trigger equals both the probability of a run and the probability of insolvency due to extensive asset liquidations.*

To proof the Lemma - a run occurs if the random state realizes below threshold θ_b. By the uniform distribution of θ and equation (1.15) the probability of a run is hence given as

$$\mathbb{P}\left(\theta < \theta_b\right) = \theta_b = \theta^* + \varepsilon \left(1 - 2\, \frac{l}{\delta r}\right) \tag{1.16}$$

Equivalently, risk of insolvency due to extensive asset liquidations equals

$$\mathbb{P}\left(\theta < \tilde{\theta}\right) = \tilde{\theta} = \theta^* - \varepsilon\left(2h^*(\delta) - 1\right) \tag{1.17}$$

In either case, when signals are precise the trigger converges to both the ex ante probability of a run and to ex ante insolvency risk due to extensive asset liquidations. Further, any partial derivative of the corresponding probability equals the partial derivative of the trigger θ^* plus ε times a constant. As noise ε vanishes, the partial derivative of the probability equals the partial derivative of the trigger. □

As a consequence of Lemma 2.3.1, at the limit state realizations above the trigger lead to successful coordination while realizations below the trigger lead to runs. The size of the equilibrium trigger between the dominance regions determines the quality of coordination in the model. The larger the trigger, the greater the ex ante risk of a run. Runs for signal realizations in the lower dominance region are efficient since they are caused by fears about low asset returns, see Figure (2.3). The range of states between the trigger and the lower dominance region however yields panic or self-fulfilling runs, which cannot be attributed to asset return risk but failure of coordination. We are interested in the behavior of the trigger as capital structure and asset liquidity varies.

We say *stability increases* in debt ratio δ, if the trigger decreases in δ. We say *liquidity risk* increases in δ, if the trigger increases in δ. For given contract (r, k) and fixed liquidity l, a *debt*

[30] Note that for calculating the general insolvency risk we would further need to take into account the probability that the asset does not pay off. Capital structure endogenously affects the risk of insolvency due to extensive asset liquidations but not the payoff probability of the asset.

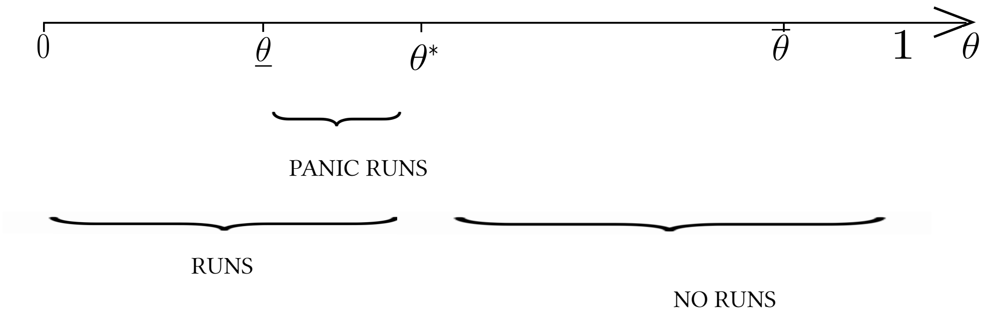


Figure 1.4: The size of the trigger determines the range of states for which panic runs occur

ratio δ yields higher stability than debt ratio $\tilde{\delta}$, if at liquidity l, the ex ante probability of a run is lower under δ than under $\tilde{\delta}$. That is, the trigger under δ is smaller than the trigger under $\tilde{\delta}$.

1.3.2 Stability

We now state our main theorem. By the following result, for every contract (r, k) the change of debt investors' behavior due to a change in debt ratio depends on the according level of asset market liquidity. We only consider debt ratios in the range $(l/r, 1)$ since for ratios below l/r the coordination problem vanishes and the firm is thus not prone to runs.

Theorem 1.3.1 (Stability against Runs). *For given contract (r, k) there exist two contract dependent thresholds $\tilde{l}_B(1)$, $\tilde{l}_A(1) \in [0, 1]$, $\tilde{l}_B(1) \leq \tilde{l}_A(1)$ for liquidity such that*
i) If liquidity is high $l \in (\tilde{l}_A(1), 1]$, firm stability monotonically decreases in short-term debt.
ii) If liquidity is moderate $l \in [\tilde{l}_B(1), \tilde{l}_A(1)]$, there are two disjoint, non-empty intervals for debt ratio such that stability decreases in short-term debt for lower values $\delta \in (l/r, \delta_u)$ and stability is minimized at a higher debt ratio in $[\delta_u, 1)$.
iii) If liquidity is low $l \in [0, \tilde{l}_B(1))$, there exist three non-empty, disjoint intervals for debt ratio such that: stability decreases in short-term debt for small values in $\delta \in (l/r, \delta_u)$ and stability improves in short-term debt for larger values in $\delta \in (\delta_d, 1)$. Stability is smallest at some interior debt ratio in $[\delta_u, \delta_d]$.
(iv) The smaller liquidity, the wider the interval $(\delta_d, 1)$ over which stability improves in short-term debt and the lower the position of the interval $[\delta_u, \delta_d]$ which contains the debt ratio yielding lowest stability.
(v) As liquidity dries up, $l \to 0$, stability monotonically improves in short-term debt and deteriorates in equity.

The proof is conducted using the Implicit Function Theorem on the PIE. Direct comparative statics of the trigger (stability) in debt ratio are non-monotone and depend on the general return probability function $p(\cdot)$ of the asset. To prove Theorem 1.3.1 we proceed by deriving an upper and lower bound for the slope of the trigger to cast off the general function $p(\cdot)$. The bounds for slope have a very similar functional form. By analyzing the cross derivatives of these bounds in liquidity we see that both bounds satisfy single-crossing in debt ratio if liquidity is sufficiently low. [31]

Our main Theorem is in contrast to Morris and Shin (2009), Rochet and Vives (2004), Vives (2014) and König et al. (2014). Rochet and Vives (2004) and Vives (2014) obtain monotone comparative statics in the firm's balance sheet decomposition: The probability of firm failure is strictly decreasing in equity ratio. Similarly, König et al. (2014) obtain a default point that is monotone in debt. Morris and Shin (2009) show that ex ante illiquidity risk decreases in liquidity ratio and thus increases in short-term debt.

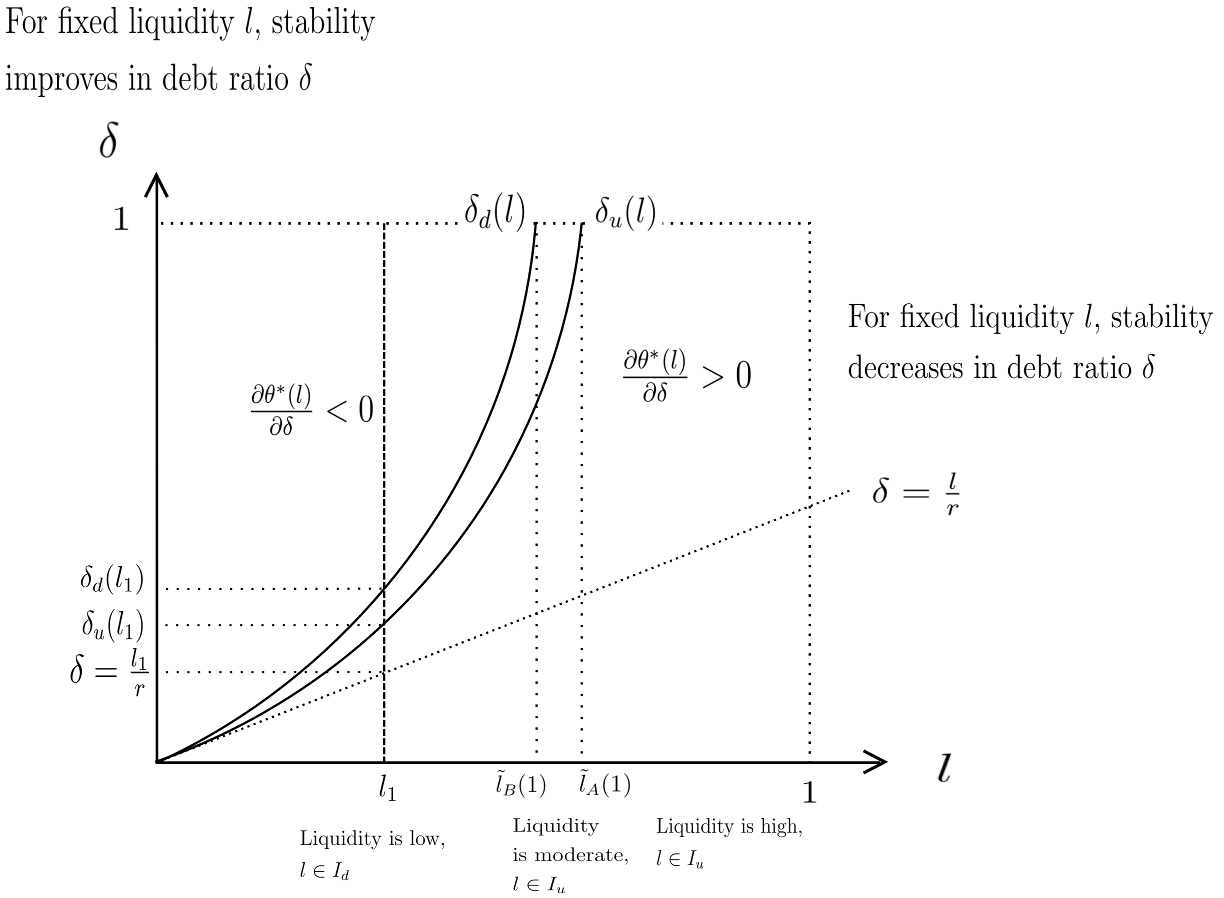


Figure 1.5: Stability as function of liquidity and debt ratio

Figure (1.5) depicts the results of Theorem 1.3.1. We have plotted market liquidity on the horizontal axis and debt ratio on the vertical axis. As we only consider firms which are prone to runs, we are interested in the behavior of the trigger for liquidity-debt combinations (l, δ) which satisfy

[31] As we work with bounds for the slope, a blind spot arises for behavior of the trigger when debt ratio lies in interval $[\delta_u(l), \delta_d(l)]$. In this case, the lower bound for the slope is negative while the upper bound is positive. The blind spot $[\delta_u(l), \delta_d(l)]$ becomes smaller (the range of the interval goes to zero) as k decreases for the bounds converge towards one another. Under collateralized borrowing, the blind spot vanishes and for every asset liquidity we obtain a unique, interior stability minimizing debt ratio as well as two locally stability maximizing debt ratios at the boundaries (see later discussion).

$\delta r > l$. The curves $\delta_d(l)$ and $\delta_u(l)$ play a crucial role. For every liquidity $l \in [0,1]$, the curve $\delta_u(l)$ lies below or at the curve $\delta_d(l)$. When fixing a specific asset liquidity on the horizontal axis, we need to look at the vertical cross-section along the δ-dimension. The curves $\delta_u(l)$ and $\delta_d(l)$ yield the bounds where monotonicity of stability in debt halts ($\delta_u(l)$) or starts ($\delta_d(l)$) at liquidity value l: When at liquidity l the curve $\delta_u(l)$ exceeds value one, then liquidity is high and stability is monotone in debt. If liquidity is sufficiently low $l = l_1$, the line l_1 intersects both curves $\delta_u(l)$ and $\delta_d(l)$. Denote by $\delta_u(l_1)$ and $\delta_d(l_1)$ the corresponding values of debt at the intersection. Then, stability deteriorates in debt for debt values in $(l_1/r, \delta_u(l_1))$ and stability improves in debt for debt values in $(\delta_d(l_1), 1)$. The probability of a run at liquidity l_1 reaches its global maximum (and stability minimum) at some debt ratio in $[\delta_u(l_1), \delta_d(l_1)]$. The functions $\delta_u(l)$ and $\delta_d(l)$ monotonically increase in liquidity[32] and converge to zero as liquidity goes to zero (see Lemma 1.7.3 in Appendix). The lower the asset's market liquidity the greater the range of debt ratios $(\delta_d(l), 1)$ for which stability improves in short-term debt and hence deteriorates in equity. Also, the lower market liquidity, the smaller the debt ratios contained in interval $[\delta_u(l), \delta_d(l)]$. The debt ratio which yields minimum stability is not necessarily monotone in liquidity. But since both interval bounds $\delta_u(l)$, $\delta_d(l)$ increase in liquidity, the minimizing debt ratio lies below a bound which decreases as liquidity dries up. Vice versa, the equity ratio yielding minimum stability lies above a bound which increases as liquidity decreases. Hence, especially for illiquid assets, the intuition that less short-term financing leads to a lower risk of a liquidity squeeze through runs turns out wrong since short-term debt can discipline depositors to coordinate.

Our analysis of a severe decline in liquidity is motivated by empirical evidence; Gorton and Metrick (2009, 2012) document haircuts for structured products used as collateral in repo transactions of 50-100% which corresponds to a sharp plummet in funding and hence market liquidity (Brunnermeier and Pedersen, 2009) in the course of the financial crises 2007-2008.

1.3.3 Policy Implications

To determine the effect of capital regulation on stability,[33] we pick a different, higher liquidity value l_2. To this purpose, assume firm's current debt ratio is δ' and asset risk based capital regulation[34] requires that debt ratio may not exceed $\delta_r < \delta'$. When we decrease short-term debt ratio from δ' to δ_r, the change of stability depends on the level of market liquidity of the asset, see Figure 1.6.

When liquidity l_2 is sufficiently high, the curves $\delta_u(l)$ and $\delta_d(l)$ do not take admissible values

[32] Debt ratios can only take values in $[0,1]$. For larger liquidity values, the functions $\delta_u(l)$ and $\delta_d(l)$ take values above one and are hence not attainable by any debt ratio. For such liquidity values the bounds for monotonicity $\delta_u(l)$ and $\delta_d(l)$ cease to exist and stability is monotone in debt.

[33] Note that once we allow for long-term debt financing, the following discussion may take the form of regulating the amount of short-term debt financing (see later subsection).

[34] Risk-based capital regulation in this paper corresponds to a specific asset return probability function $p(\theta)$ and return H.

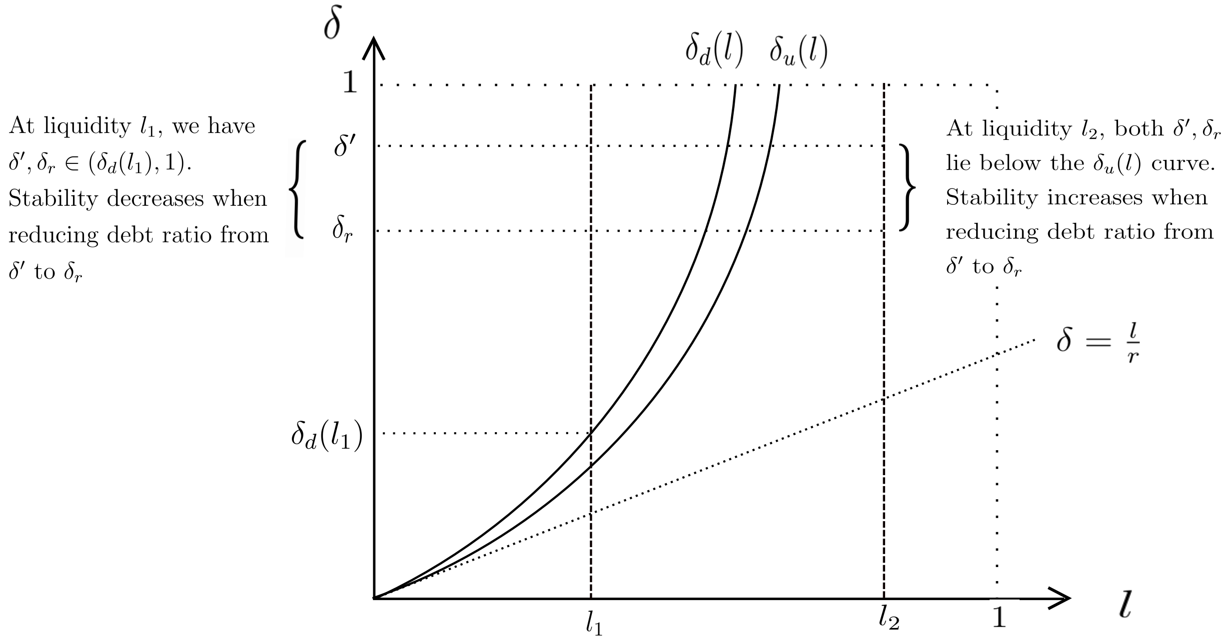


Figure 1.6: Effect of capital regulation on stability depending on asset liquidity

at l_2: Both debt ratios δ' and δ_r lie below the $\delta_u(l)$ curve at l_2, a decrease in debt ratio thus improves stability, $\theta^*(\delta_r) < \theta^*(\delta')$ by Theorem 1.3.1 and Lemma 2.3.1. If instead liquidity is low at l_1 and debt ratios δ' and δ_r lie above value $\delta_d(l_1)$, the decrease in short-term debt will cause a stability loss since coordination would deteriorate, $\theta^*(\delta_r) > \theta^*(\delta')$. By Lemma 2.3.1, the ex ante probability of a run increases when reducing debt ratio from δ' to δ_r.

Lemma 1.3.2. *Assume signals are precise. If liquidity is high, capital regulation can reduce the probability of runs. When liquidity is low, capital regulation can increase the probability of runs.*

Note that Lemma 1.3.2 states, if liquidity is low, capital regulation might change the distribution of debt to be withdrawn on short notice in a way, such that an exceedance of asset's market liquidity becomes more likely although the overall amount of short-term debt financing is decreased: In our example in Figure (1.6), we have $\delta_r < \delta'$. But the probability of a run under debt structure δ_r is higher than under debt structure δ':

$$\mathbb{P}(\theta < \theta_b(\delta')) = \theta^*(\delta') < \theta^*(\delta_r) = \mathbb{P}(\theta < \theta_b(\delta_r)) \tag{1.18}$$

The results demonstrate that capital regulation and regulation of short-term debt financing can have adverse effects on run behavior of agents and hence stability. Capital regulation needs take into account the liquidity profile of firm assets. Further, capital and liquidity regulation should not be separated.

During economic boom times asset liquidity is higher and capital regulation may have a positive effect on coordination behavior and thus stability ex ante. During financial crises however, liquidity dries up. Capital regulation may complicate coordination and thus increase illiquidity risk for ex ante the probability that claimed short-term debt exceeds available liquidity increases although the overall exposure of the firm to short-term debt investors is decreased.

1.3.4 Liquidity Mismatch

Another implication of our result concerns liquidity regulation. To this purpose define the *liquidity ratio* as the ratio of liquidity available through sale of the asset to potential short term debt claims

$$\xi = \frac{l}{\delta r} \in [0, 1] \tag{1.19}$$

Liquidity ratio measures liquidity mismatch between assets and short-term liabilities. The smaller the liquidity ratio, the larger the liquidity mismatch between assets and liabilities. Liquidity mismatch is generally perceived as a source of liquidity risk (Brunnermeier et al., 2014, 2011; Bai et al., 2014; Basel III, 2013). We will next demonstrate that liquidity mismatch does not necessarily aggravate coordination and hence increase probability of runs. When withdrawals are endogenous, liquidity mismatch is no good indicator for liquidity risk driven by miscoordination on runs.

If liquidity mismatch impaired coordination, liquidity ratio would be an indicator for liquidity risk, the smaller liquidity ratio the larger liquidity risk. Theorem (1.3.1) however demonstrate that stability is non-monotone in liquidity ratio: Liquidity ratio as defined in (1.19) decreases in debt ratio δ when keeping liquidity fix. Fix a low liquidity value $l_1 \leq \tilde{l}_B(1)$ and consider the ranges of debt ratios $(l_1/r, \delta_u(l_1))$ and $(\delta_d(l_1), 1)$. For fixed liquidity value l_1, as debt ratio δ increases within the interval $(\delta_d(l_1), 1)$, liquidity ratio decreases by (1.19) while simultaneously stability improves by Theorem (1.3.1). As debt ratio δ increases within the interval $(l_1/r, \delta_u(l_1))$, liquidity ratio still decreases but stability deteriorates by Theorem (1.3.1). A different way of seeing this result is depicted in the Online Appendix.

Lemma 1.3.3. *Stability is non-monotone in liquidity ratio (liquidity mismatch).*

This result is in contrast to Morris and Shin (2009) and Rochet and Vives (2004) where stability is monotone increasing in liquidity ratio. [35]

1.4 Optimal Capital Structure

We next analyze utility debt investors infer from contract (r, k) as a function of firm debt ratio and market liquidity of the asset. As signals become precise, we obtain

[35] Rochet and Vives (2004) have adapted a different notion of liquidity ratio in their model which comes closer to the original definition of Liquidity Coverage Ratio (LCR) as defined in the Basel 3 framework by the Basel Committee on Banking Supervision. The LCR is defined as the ratio of high quality liquid assets (HQLA) over total expected net cash outflow in a stress scenario over 30 days.

Proposition 1.4.1 (Optimality for Debt investors)**.** *Fix contract* (r, k).
When liquidity is high, debt investor's utility from the contract decreases monotone in firm debt ratio. When liquidity is moderate or low, utility decreases in debt ratio for lower values of debt ratio $\delta \in (l/r, \delta_u)$. *If however liquidity is very low, and debt ratio large,* $\delta > \delta_d(l)$, *utility increases in debt for all* $\delta \in (\delta_d(l), 1)$.

Corollary 1.4.1. *If asset liquidity is high, debt investors' utility from the contract reaches the global maximum when the firm is financed through sufficient equity such that panic runs are excluded* $\delta = l/r$. *Utility reaches the global minimum if the firm is financed through debt only.*
If asset liquidity is sufficiently low, debt investors utility from the contract is locally maximized when the firm is financed through debt only.

Intuitively, debt ratio influences debt investors' utility inferred from the contract in two ways. An increase in debt ratio always has a direct, negative impact on immediate utility (assuming the trigger would stay constant) from both withdrawing and rolling over since it goes hand in hand with a decrease in equity ratio and thus a decrease in protective cushion.
A change in debt ratio has a further indirect influence on utility by a manipulation of the trigger and hence a change in stability. Since stability can either improve or deteriorate in debt ratio, the impact of changes in debt ratio on overall utility from the contract is in general not uniquely pinned down.
If stability is impaired by an increase in debt the overall impact on utility is clearly negative. By Theorem (1.3.1), that is the case if liquidity is high or when liquidity is moderate or low and debt ratio sufficiently low. If however stability improves in debt ratio, by Theorem (1.3.1) that is the case when liquidity is low and debt ratio is high, debt investors trade off stability gains against immediate utility losses. When liquidity is sufficiently low and debt is increased, the increase in stability outweighs the decrease in direct utility.[36]

1.5 Extensions

1.5.1 Robustness: Collateralized Borrowing

When allowing the firm to raise cash in the money market by pledging the asset as collateral in a *repurchase agreement* (repo), partial liquidation of assets can be prevented. A repo transaction involves two parties, the firm (the borrower) and a lender. The firm borrows cash from the lender and agrees to repay the amount plus an interest payment (at repo rate) in period 2. In addition, the firm posts a fraction of the asset as collateral which goes into physical possession of the lender but is returned when the amount borrowed is paid back. If the firm cannot repay, she defaults on the repo and the lender in the repurchase agreement may sell the collateral at market price.

[36]By Proposition (1.7.1) the assumption that stability increases in debt ratio for small liquidity is consistent when debt ratio is sufficiently large, i.e. we do not talk about a zero measure set.

The collateral hence reduces the risk of the transaction to the lender. What is the difference to the case where the firm has to sell the asset in the market to raise cash (partial liquidation)? If the firm can repay in period 2 and gains back the fraction of asset posted as collateral, the asset's accrued interest goes to the firm, including the interest which accrued on the collateralized fraction. Under partial liquidation, the interest that accrues on the sold part of the asset goes to the new owner. The payoff structure changes.

The exogenous amount of cash that can be raised when pledging one unit of the asset as collateral is called *funding liquidity* and replaces the notion of market liquidity, the amount of cash that can be raised by selling the asset at market price.[37] Note in particular that funding liquidity is not the market value of the collateral (asset) but the fraction of the 'true' value participants in the money market are willing to pay to accept the asset as collateral (overcollateralization). For the analysis we assume a repo rate of zero but results can be extended to accommodate a general repo rate.

We can show that the non-monotonicity results are robust in this different setting. As a result of the changed payoffs, the blind spot (area between curves $\delta_u(l)$ and $\delta_d(l)$) vanishes and the probability of runs is either monotone for large liquidity or is hump-shaped in debt for lower liquidity values, i.e. is maximized at a unique interior debt ratio. Note that in the following chapter, the setting just described is analyzed as a special case ($a = 0, b = 1$).

1.5.2 Long-term debt

So far the financing structure of the firm is composed of short-term debt and equity. We can extend the structure by adding long-term debt. To keep the balance sheet normalized at 1, let δ again the fraction of short-term debt, let $\tau(1-\delta)$, $\tau \in [0,1]$ the fraction of long-term debt and $(1-\tau)(1-\delta)$ the fraction of equity. Long-term debt investors invest in period 0 and are paid in period 2 prior to equity investors. At the interim period 1, they have no claims.

First, assume that long-term debt investors are less senior than short-term debt investors and are hence paid in period 2 only after short-term debt investors were fully paid. Then long-term debt is like equity to short-term debt investors. Thus, replacing equity with long-term debt has no impact on coordination of short-term debt investors. The non-monotonicity results for short-term debt derived in previous sections hold and have implications for liquidity regulation. The interpretation however changes. In the previous section, when talking about debt we always referred to short-term debt. This was unambiguous since short-term debt was the only form of debt and the remaining financing source was equity. The statements on equity from previous sections now hold for the combined sum of equity and long-term debt $1-\delta$.

If long-term debt is treated equally senior as short-term debt in period 2, long-term debt is no

[37]See Brunnermeier and Pedersen (2009)

longer like equity to short-term investors. Coordination between short-term investors is altered when replacing equity with long-term debt. The immediate payoff (assuming the trigger stayed constant) from withdrawing early stays constant since long-term investors have no claim on payment at the interim period. The payoff from rolling over however decreases since gross return on the asset now has to be sufficiently high to cover remaining short-term investors and more long-term investors. As long-term debt replaces equity, withdrawing becomes more appealing relative to rolling over and the trigger (probability of runs) increases.

1.6 Conclusion

This paper studies stability of financial firms which conduct maturity and liquidity transformation by investing long-term in risky and illiquid assets and financing through equity and liquid short-term debt. We analyze the question, how the probability of runs on such firms depends on capital structure and market liquidity of assets. While this question has been analyzed before (Morris and Shin, 2009; Rochet and Vives, 2004; Vives, 2014)[38] we allow for an incentive structure more generic for settings where uninsured debt investors draw on a common pool of finite liquidity. Since assets are illiquid, cash available through liquidating all assets on short notice is insufficient to cover withdrawals by potentially all short-term debt investors. If the number of withdrawing investors exceeds a critical threshold, the full deposit cannot be paid back. In this case, the firm follows a sequential service constraint. Investors are served one after another until all cash is distributed and some investors who try to withdraw cannot be served. We hence explicitly model partial, endogenous repay in the incidence of a run, see Goldstein and Pauzner (2005). The probability distribution of short-term debt to be withdrawn tomorrow depends on firm capital structure and asset liquidity today. Relative incentives of choosing actions change in capital structure in a way we, to the best of our knowledge, have not observed in the literature before:

As main contribution of the paper, we find that the probability of runs is non-monotone in debt when liquidity dries up. In particular, the run probability is not monotone decreasing in equity but can in fact increase in equity when liquidity is sufficiently low. More debt financing can discipline debt investors to coordinate and may lead to lower probability of runs. Vice versa, decreasing the exposure towards short-term debt investors today may alter the distribution of debt to be withdrawn tomorrow in a way that runs become more likely ex ante. Liquidity risk is increased although liquidity mismatch is lowered. These results stand in contrast to the monotonicity results of previous papers on coordination behavior of debt investors under changes in capital structure and asset liquidity (Rochet and Vives, 2004; Vives, 2014; König et al., 2014; Morris and Shin, 2009)

The non-monotonicity results have consequences for evaluating regulation of capital and liquidity mismatch under Basel 3. By raising quality and quantity of the regulatory capital base, the Basel

[38]Morris and Shin (2009) analyze collateralized borrowing, Rochet and Vives (2004); Vives (2014) model delegated decision making by fund managers. In both cases actions are global strategic complements and comparative statics are monotone.

Committee intends to raise the ”resilience of the banking sector ” and states that ”strong capital requirements are a necessary condition for banking sector stability” (Committee et al., 2010; Basel III, 2013). Our results show, while capital regulation can improve stability when market liquidity is high, the identical policy rule can harm stability when liquidity dries up for the implementation of the rule may deteriorate the coordination problem among short-term debt investors.[39] Our results imply that capital regulation of firms financed by uninsured debt should be tailored to particular scenarios for market liquidity or regulation should distinguish between firms according to their target asset liquidity.

Our results hold under the assumption that the firm needs to partially liquidate assets to raise cash but are robust to assuming that the firm has access to the money market and may borrow by pledging the asset as collateral to avoid partial liquidation. The set-up here thus fits classic commercial banks but also maturity transforming shadow banks such as structured investment vehicles (SIVs) and asset-backed commercial paper conduits (ABCPs), see Gorton et al. (2010); Adrian and Ashcraft (2012).

From a theory perspective, we depart from a model structure exhibiting global strategic complementarity between actions (Rochet and Vives, 2004; Vives, 2014; König et al., 2014; Morris and Shin, 2009) and instead work with one-sided strategic complementarity as modeled in Goldstein and Pauzner (2005) and Karp et al. (2007). One-sided strategic complementarity is the the key driver for non-monotonicity in global games, as we will see in the following chapter.

As for the limitations of the model, we assume market liquidity of the asset is common knowledge among or perfectly anticipated by debt investors prior to making their decisions. We however do not account for correlation between market liquidity and the random state of the economy.

In the model, the firm commits to investing in a particular asset independently of the state realization, which excludes moral hazard by firm managers. We abstract from observable, state dependent investment choices since this would give rise to an endogenous public signal and hence equilibrium multiplicity.

While we discuss optimality of capital structure from a debt investor's point of view when the firm is in perfect competition for deposits, optimal capital structure which maximizes ex ante return on equity remains an interesting question to analyze.

We take contracts as exogenously given since the scope of the paper is on analyzing the impact of capital structure and liquidity on coordination and optimal capital structure from debt investors perspective. Since the non-monotonicity results hold for every contract, they also hold for the optimal contract. Optimal contracts for a fully debt financed firm are analyzed in Goldstein and Pauzner (2005).

[39] Our results are developed in a model lacking deposit insurance. Member countries of the Basel Committee on Banking Supervision have partial deposit insurance in place. Our analysis on capital regulation remains interesting since deposit insurance in the face of a global financial crises might not be perfectly credible. Iyer and Puri (2008) find that deposit insurance is only partially effective in preventing bank runs.

1.7 Appendix

We have implicitly defined h^* such that for $n = h^*$ a debt investor who rolls over receives her full coupon just with probability one given that investment is successful. That is, $\frac{(1-\delta nr/l)H}{\delta(1-n)kr} = 1$, thus

$$h^* = \frac{H - \delta kr}{\delta r(H/l - k)} \tag{1.20}$$

Hence, for all $n \leq h^*$ we have $\min(1, \frac{(1-\delta nr/l)H}{\delta(1-n)kr}) = 1$ and debt investors who roll over receive the full payoff $u(kr)$ for sure if investment is successful. For $n \geq h^*$, $\min(1, \frac{(1-\delta nr/l)H}{\delta(1-n)kr}) = \frac{(1-\delta nr/l)H}{\delta(1-n)kr}$ and debt investors who roll over receive a payoff only with a probability $\frac{(1-\delta nr/l)H}{\delta(1-n)kr} < 1$ while equity holders receive zero.
For $\delta r > l$ the term $\frac{(1-\delta nr/l)H}{\delta(1-n)kr}$ is strictly decreasing in n. Moreover, $h^* < l/(\delta r) < 1$, as $\delta r > l$. We have $h^* > 0$ for by assumption $H > kr > \delta kr$ and $\delta r > l$, thus $H > kl$.

1.7.1 Appendix A: Existence and Uniqueness

Proof. [Theorem 2.3.1]
The existence and uniqueness proof of a trigger equilibrium and the proof that a non-threshold equilibrium cannot exist is as in Goldstein and Pauzner (2005) with $\lambda = 0$. Uniqueness of a threshold equilibrium alternatively holds due to Lemma 2.3 in Morris and Shin (2001) by the single-crossing property of the payoff difference function v from equation (2.9) in the aggregate action n (Figure (2.1) and the monotone likelihood ratio property for the uniform distribution of noise since the function v is strictly decreasing in n whenever v is positive.
We give a short intuition here, why a unique trigger equilibrium exists: Given that all other investors play a trigger strategy around signal θ^*, a trigger equilibrium exists if a single investor also finds it optimal to withdraw for signals $\theta_i < \theta^*$ and to roll over for signals $\theta_i > \theta^*$. That is, we demand (a) $D(\theta_i, n(\cdot, \theta^*)) < 0$ for $\theta_i < \theta^*$ and (b) $D(\theta_i, n(\cdot, \theta^*)) > 0$ for $\theta_i > \theta^*$. Continuity of the integral $D(\theta_i, n(\cdot, \theta^*))$ in signal θ_i holds by Lemma A1 (i) in Goldstein and Pauzner (2005) and ensures indifference in $\theta_i = \theta^*$, $D(\theta_i = \theta^*, n(\cdot, \theta^*)) = 0$ if (a) and (b) hold. Existence of a signal which satisfies $D(\theta_i = \theta^*, n(\cdot, \theta^*)) = 0$ follows by the existence of dominance regions and continuity of $D(\theta_i = \theta^*, n(\cdot, \theta^*))$ in θ^* by Lemma A1 (ii) in Goldstein and Pauzner (2005): If the state realizes high enough in the upper dominance region and ε is small, the investor observes a very high signal such that rolling over is optimal $D(\theta_i, n) > 0$ independently of n, similarly, if the state realizes low enough in the lower dominance region, the investor observes a very low signal such that withdrawing is dominant $D(\theta_i, n) < 0$. Uniqueness of a signal satisfying $D(\theta_i = \theta^*, n(\cdot, \theta^*)) = 0$ holds since by Lemma A1 (iii) in Goldstein and Pauzner (2005), $D(\theta_i = \theta^*, n(\cdot, \theta^*))$ strictly increases in θ^* as long as signal θ^* lies below $\bar{\theta} + \varepsilon$ since the probability function $p(\cdot)$ strictly increases in the state for states below the bound to the upper dominance region. Uniqueness follows since for signals above $\bar{\theta} + \varepsilon$ the definition of the upper

dominance region yields $D(\theta_i, n) > 0$. Therefore, a unique candidate for a threshold equilibrium exists. To show that this candidate also satisfies (a) and (b), Goldstein and Pauzner (2005) decompose the intervals $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and $[\theta^* - \varepsilon, \theta^* + \varepsilon]$, use $D(\theta_i = \theta^*, n(\cdot, \theta^*)) = 0$ and the single crossing property of $v(\theta, n(\theta, \theta^*))$ in θ, see (A8) and (A9) in their proof to Theorem 1 B. The proof why a non-threshold equilibrium cannot exist is less intuitive, and fully given in Goldstein and Pauzner (2005) proof of Theorem 1 C. □

1.7.2 Appendix B: Main Theorem

Proof. [Theorem 1.3.1] Since the unique equilibrium of the game is a trigger equilibrium, the trigger is pinned down by the payoff indifference equation. Upon observing the trigger $\theta_i = \theta^*$ an investors needs to be indifferent between rolling over or withdrawing. The PIE is given by

$$0 = D(\theta_i = \theta^*, n(\cdot, \theta^*)) = \frac{1}{2\varepsilon} \int_{\theta^*-\varepsilon}^{\theta^*+\varepsilon} v(\theta, n(\theta, \theta^*))\, d\theta \tag{1.21}$$

or equivalently by equation (1.13)

$$\begin{aligned} 0 = u\,(kr) \int_0^{h^*} p(\theta(n, \theta^*))\; dn - u(r) \frac{l}{\delta r}\; (1 + \ln(\delta r/l)) \\ + \int_{h^*}^{l/\delta r} p(\theta(n, \theta^*)) \frac{(1 - nr\delta/l)H}{\delta(1-n)kr}\; u(kr)\, dn \end{aligned} \tag{1.22}$$

where

$$\theta(n, \theta^*) = \theta^* + \varepsilon\,(1 - 2n) \tag{1.23}$$

is the inverse of $n(\theta, \theta^*)$ for $\theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon]$. We define the function

$$\begin{aligned} \hat{f}(\theta^*, \delta) \equiv u\,(kr) \int_0^{h^*} p(\theta(n, \theta^*))\; dn - \frac{l}{\delta r}\, u(r)\, (1 + \ln(\delta r/l)) \\ + \int_{h^*}^{l/\delta r} p(\theta(n, \theta^*)) \frac{(1 - nr\delta/l)H}{\delta(1-n)kr}\; u(kr)\, dn \end{aligned} \tag{1.24}$$

The zeros of $\hat{f}$ constitute equilibrium triggers of the game for different parameter constellations. To determine the behavior of the trigger due to parameter changes it is sufficient to look at the set of zeros of $\hat{f}$. We have

$$\begin{aligned} \frac{\partial}{\partial \theta^*} \hat{f}(\theta^*, \delta) &= u\,(kr) \int_0^{h^*(\delta)} p'(\theta(n, \theta^*)) \frac{\partial}{\partial \theta^*} \theta(n, \theta^*)\; dn \\ &+ \int_{h^*(\delta)}^{l/\delta r} p'(\theta(n, \theta^*)) \frac{\partial}{\partial \theta^*} \theta(n, \theta^*)\; \frac{(1 - nr\delta/l)H}{\delta(1-n)kr}\; u(kr)\, dn \\ &= u\,(kr) \int_0^{h^*(\delta)} p'(\theta(n, \theta^*))\; dn + \int_{h^*(\delta)}^{l/\delta r} p'(\theta(n, \theta^*)) \frac{(1 - nr\delta/l)H}{\delta(1-n)kr}\; u(kr)\, dn > 0 \end{aligned} \tag{1.25}$$

since $\frac{\partial}{\partial \theta^*}\theta(n,\theta^*) = 1$ and where h^* is given in (1.20). At the limit $\varepsilon \to 0$, we have $\theta(n,\theta^*) \to \theta^*$. Since $p(\cdot)$ is continuous and defined on a compact interval, $p'(\cdot)$ is bounded. In addition, $n(\theta,\theta^*) \leq 1$, hence with Lebesgue's Dominated Convergence Theorem

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\partial}{\partial \theta^*}\hat{f}(\theta^*,\delta) &\to p'(\theta^*)\,u\,(kr)\,\left(h^* + \frac{H}{\delta kr}\int_{h^*(\delta)}^{l/\delta r}\frac{1-nr\delta/l}{1-n}\,dn\right)\\ &= \frac{p'(\theta^*)}{p(\theta^*)}\,\frac{l}{\delta r}u(r)\left(1+\ln\left(\frac{\delta r}{l}\right)\right)\end{aligned}\tag{1.26}$$

where we have used trigger equation (1.35). Note that $\lim_{\varepsilon\to 0}\frac{\partial}{\partial\theta^*}\hat{f}(\theta^*,\delta) = 0$ if $\theta^* = \overline{\theta}$ as by definition of the function $p(\cdot)$, $p'(\overline{\theta}) = 0$.

Comparative Statics of Trigger in delta

$$\frac{\partial \hat{f}}{\partial \delta} = u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}p(\theta^*)\,\frac{u(kr)}{kr}\,\ln\left(\frac{H-kl}{H}\right)\tag{1.27}$$

since $\int_{h^*}^{l/(\delta r)}\left(-\frac{1}{1-n}\right)\,dn = \ln\left(\frac{1-l/(\delta r)}{1-h^*}\right) = \ln\left(\frac{H-kl}{H}\right) < 0.$

With (2.24) and the Implicit Function Theorem, it follows

$$\frac{\partial\theta^*}{\partial\delta} = -\frac{\frac{\partial\hat{f}}{\partial\delta}}{\frac{\partial\hat{f}}{\partial\theta^*}} \qquad = -\frac{u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}p(\theta^*)\,\frac{u(kr)}{kr}\,\ln\left(\frac{H-kl}{H}\right)}{u\,(kr)\int_0^{h^*(\delta)}p'(\theta)\,dn + \int_{h^*(\delta)}^{l/\delta r}p'(\theta)\frac{(1-nr\delta/l)H}{\delta(1-n)kr}\,u(kr)\,dn}\tag{1.28}$$

The denominator of (1.28) is positive while the numerator can change sign. For the numerator is non-monotone in δ, to analyze the slope $\frac{\partial\theta^*}{\partial\delta}$ we work with boundaries of the numerator instead. If at the limit $\varepsilon \to 0$ $\theta^* \neq \overline{\theta}$, the denominator in (1.28) converges to a constant unequal to zero and using (2.25) we can write

$$\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial\delta} = -\frac{\frac{\partial\hat{f}}{\partial\delta}}{\frac{\partial\hat{f}}{\partial\theta^*}} \qquad = -\frac{u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}p(\theta^*)\,\frac{u(kr)}{kr}\,\ln\left(\frac{H-kl}{H}\right)}{\frac{p'(\theta^*)}{p(\theta^*)}\,\frac{l}{\delta r}u(r)\left(1+\ln\left(\frac{\delta r}{l}\right)\right)}\tag{1.29}$$

Upper boundary: By (1.27) and using $p(\theta)u(kr) > u(r)$ for $\theta > \underline{\theta}$

$$\begin{aligned}\frac{d}{d\delta}\hat{f}(\theta^*,\delta) &= u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}p(\theta^*)\,\frac{u(kr)}{kr}\,\ln\left(\frac{H-kl}{H}\right)\\ &< u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}\frac{u(r)}{kr}\,\ln\left(\frac{H-kl}{H}\right)\\ &= \frac{1}{\delta^2 r}\,u(r)\,\ln\left[\left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{E}\right)^E\right]\end{aligned}\tag{1.30}$$

where $E \equiv H/k > 1$. Thus, $\frac{d}{d\delta}\hat{f}(\theta^*,\delta) < 0$ if $\left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{E}\right)^E < 1.$

Lower boundary:

$$\begin{aligned}
&\frac{d}{d\delta}\hat{f}(\theta^*,\delta) = u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}p(\theta^*)\,\frac{u(kr)}{kr}\,\ln\left(\frac{H-kl}{H}\right) \\
&> u(r)\,\frac{l}{\delta^2 r}\ln(\frac{\delta r}{l}) + \frac{H}{\delta^2}\frac{u(r)}{r}\,\ln\left(\frac{H-kl}{H}\right) \\
&= \frac{u(r)}{r\delta^2}\,\ln\left(\left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{(H/k)}\right)^H\right) \qquad (1.31)
\end{aligned}$$

for $p \le 1$ and $\frac{u(kr)}{kr} < \frac{u(r)}{r}$ by concavity, $k > 1$ and $u(0) = 0$. Thus, $\frac{d}{d\delta}\hat{f}(\theta^*,\delta) > 0$ if $\left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{(H/k)}\right)^H > 1$. We therefore conclude, that

$$\frac{d}{d\delta}\hat{f}(\theta^*,\delta) \in \left[\frac{u(r)}{r\delta^2}\,\ln(B(\delta,l)), \frac{u(r)}{r\delta^2}\,\ln(A(\delta,l))\right] \qquad (1.32)$$

where we define

$$A(\delta,l) = \left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{E}\right)^E \qquad (1.33)$$

$$B(\delta,l) = \left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{(H/k)}\right)^H \qquad (1.34)$$

The functions A and B are both strictly increasing in δ for every fixed liquidity because $1-\frac{l}{E} > 0$ and $1-\frac{l}{(H/k)} > 0$.

The following Lemmata analyze monotonicity behavior of the bounds of $\frac{\partial}{\partial\delta}\hat{f}(\delta,\theta^*)$ in liquidity l for every fixed $\delta \in (0,1]$. In our analysis, we treat δ as the fixed variable and let liquidity vary in the admissible parameter space $(0, \min(1,\delta r))$.

Lemma 1.7.1. *For every $\delta \in (0,1]$ there exists a unique $l_A^*(\delta) \in (0,\min(1,\delta r))$ such that the term $A(\delta,l) := \left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{E}\right)^E$ is strictly increasing for $l \in (0, l_A^*(\delta))$ and strictly decreasing for $l \in (l_A^*(\delta), \min(1,\delta r))$. $l_A^*(\delta)$ strictly increases in δ. For every δ, $A(\delta,l) \to 1$ as $l \to 0$ and hence, there exists a unique $\tilde{l}_A(\delta) \in (l_A^*(\delta),1] \cap (l_A^*(\delta),\delta r)$ such that $A(\delta,l) > 1$ for all $l \in (0,\tilde{l}_A(\delta))$ and $A(\delta,l) < 1$ for all $l \in (\tilde{l}_A(\delta),1] \cap (\tilde{l}_A(\delta),\delta r)$. $\tilde{l}_A(\delta)$ weakly increases in δ.*

Proof. [Lemma 1.7.1] In online Appendix □

Lemma 1.7.2. *For every $\delta \in (0,1]$ there exists a unique $l_B^*(\delta) \in (0,\min(\delta r,1))$ such that the term $B(\delta,l) = \left(\frac{\delta r}{l}\right)^l\left(1-\frac{l}{(H/k)}\right)^H$ is strictly increasing for $l \in (0,l_B^*)$ and strictly decreasing for $l \in (l_B^*, \min(\delta r,1))$. $l_B^*(\delta)$ strictly increases in δ. For every δ we have $B(\delta,l) \to 1$ as $l \to 0$ and there hence exists a unique $\tilde{l}_B(\delta) \in (l_B^*,1] \cap (l_B^*,\delta r)$ such that $B(\delta,l) > 1$ for all $l \in (0,\tilde{l}_B)$ and $B(\delta,l) < 1$ for all $l \in (\tilde{l}_B,1] \cap (\tilde{l}_B,\delta r)$. $\tilde{l}_B(\delta)$ is weakly increasing in δ. It holds $l_B^*(\delta) < l_A^*(\delta)$ and $\tilde{l}_B(\delta) \le \tilde{l}_A(\delta)$.*

Proof. [Lemma 1.7.2] In online Appendix □

Proposition 1.7.1 (Comparative Statics in Equilibrium)**.** *For given contract (r,k) there exist two contract dependent thresholds $\tilde{l}_B(1)$, $\tilde{l}_A(1) \in [0,1]$, $\tilde{l}_B(1) \leq \tilde{l}_A(1)$ for liquidity such that*
i) If $l \in [0, \tilde{l}_B(1))$, there exist two boundaries $\delta_d, \delta_u \in (l/r, 1)$, $\delta_u < \delta_d$ such that the trigger θ^* strictly increases in debt ratio for debt ratio $\delta \in (l/r, \delta_u)$, strictly decreases for debt ratio $\delta \in (\delta_d, 1)$ and takes its global maximum at some $[\delta_u, \delta_d]$.
ii) If $l \in [\tilde{l}_B(1), \tilde{l}_A(1)]$, there exists a boundary $\delta_u \in (l/r, 1)$ such that for $\delta \in (l/r, \delta_u)$ the trigger is strictly increasing in debt ratio and the global maximum of the trigger is reached at some $\delta \in (\delta_u, 1)$.
iii) If $l \in (\tilde{l}_A(1), 1]$, the trigger is strictly increasing in debt ratio δ for all $\delta \in (l/r, 1)$. Hence, the trigger takes its supremum at $\delta = 1$ and its infimum at $\delta = \frac{l}{r}$.

Proof. (Proposition 1.7.1) Fix contract (r,k). By (2.24), $\frac{\partial \hat{f}}{\partial \theta^*} > 0$ for all δ. Hence, by the Implicit Function Theorem the slope of the trigger $\frac{\partial \theta^*}{\partial \delta} = -\frac{\frac{\partial \hat{f}}{\partial \delta}}{\frac{\partial \hat{f}}{\partial \theta^*}}$ equals zero if and only if $\frac{d}{d\delta}\hat{f}(\theta^*, \delta)$ equals zero. As this expression is not easy to handle, we instead work with its boundaries. The upper boundary of $\frac{d}{d\delta}\hat{f}(\theta^*, \delta)$ equals zero if and only if $A(\delta, l) = \left(\frac{\delta r}{l}\right)^l \left(1 - \frac{l}{E}\right)^E = 1$.

Case 1: Let $l \in (\tilde{l}_A(1), 1]$ (assume liquidity is high). Then, by Lemma 1.7.1, $A(\delta = 1, l) = \left(\frac{r}{l}\right)^l \left(1 - \frac{l}{E}\right)^E < 1$. For every fix l the function $A(\delta, l)$ is strictly increasing in δ for admissible values in $(l/r, 1)$. Hence, for all $\delta \in (l/r, 1)$ we have $A(\delta, l) < 1$. As A determines the upper bound of the slope, it follows $\frac{d}{d\delta}\hat{f}(\theta^*, \delta) < 0$ and $\frac{\partial \theta^*}{\partial \delta} > 0$. Thus, the trigger gets minimized and firm stability maximized in $\delta = l/r$. By monotonicity, firm stability deteriorates as δ increases.

Case 2: Assume liquidity is small, that is fix $l \in (0, \tilde{l}_B(1))$. By Lemma 1.7.2, $\tilde{l}_B(\delta) \leq \tilde{l}_A(\delta)$ and thus $B(\delta = 1, l) > 1$ and $A(\delta = 1, l) > 1$.
For any l, the function A is continuous and strictly increasing in δ for all admissible values in $(l/r, 1)$. At $\delta \to l/r$ we obtain $A(\delta = l/r, l) = \left(1 - \frac{l}{E}\right)^E < 1$. Using continuity and strict monotonicity of A in δ, by the Intermediate Value Theorem for fixed l there exists a unique $\delta_u(l) \in (l/r, 1)$ such that in $\delta = \delta_u$ we have $A(\delta_u, l) = \left(\frac{\delta_u r}{l}\right)^l \left(1 - \frac{l}{E}\right)^E = 1$. Moreover, for $\delta < \delta_u(l)$ it holds $A(\delta, l) = \left(\frac{\delta r}{l}\right)^l \left(1 - \frac{l}{E}\right)^E < 1$, $\frac{d}{d\delta}\hat{f}(\theta^*, \delta) < 0$ and $\frac{\partial \theta^*}{\partial \delta} > 0$ as we are considering the upper bound.
Using the same argument on the function B, for l there exists a unique $\delta_d(l) \in (l/r, 1)$ with $B(\delta_d, l) = \left(\frac{\delta_d r}{l}\right)^l \left(1 - \frac{l}{E}\right)^H = 1$. Further, when $\delta > \delta_d(l)$ we have $B(\delta, l) > 1$. As B determines the lower bound of the slope we can follow $\frac{d}{d\delta}\hat{f}(\theta^*, \delta) > 0$ and $\frac{\partial \theta^*}{\partial \delta} < 0$ for $\delta > \delta_d(l)$.
Due to $B(\delta, l) < A(\delta, l)$ for all $\delta \in (0,1)$, when δ_d exists so does δ_u and we have $\delta_d(l) > \delta_u(l)$ for all $l \in I_d$. Therefore, for fixed $l \in I_d$ the global maximum of the trigger in δ and hence the debt ratio minimizing stability lies in interval $[\delta_u(l), \delta_d(l)] \subset (l/r, 1)$.
Case 3: Let liquidity be moderate, fix $l \in (\tilde{l}_B(1), \tilde{l}_A(1))$. Then, by Lemma 1.7.1 and 1.7.2, $\delta_u(l) \in (l/r, 1)$ exists but no δ_d. We can infer $\frac{d}{d\delta}\hat{f}(\theta^*, \delta) < 0$ and $\frac{\partial \theta^*}{\partial \delta} > 0$ for $\delta \in (l/r, \delta_u)$. Thus

the trigger gets maximized at some $\delta \geq \delta_u(l)$.

□

To finish the proof of Theorem 1.3.1, by Lemma 2.3.1, at the limit the trigger equals ex ante risk of runs. By Lemma (1.7.3), δ_u and δ_d are strictly increasing in l if they exist. Apply Proposition 1.7.1. □

Lemma 1.7.3. *Given that $\delta_u(l)$ and $\delta_d(l)$ exist, they are strictly increasing in liquidity. At the limit $l \to 0$, the functions take limits $\delta_u(l) \to 0$, $\delta_d(l) \to 0$.*

Proof. [Lemma 1.7.3] In online Appendix □

Lemma 1.7.4. *The trigger strictly increases in liquidity if and only if*

$$u(r)\,\frac{1}{\delta r}\ln(\frac{\delta r}{l}) - \int_{h^*}^{l/\delta r} p(\theta(n,\theta^*))\,u(kr)\left(\frac{Hn}{l^2k(1-n)}\right) dn > 0$$

.

Proof. (Lemma 1.7.4) Online Appendix □

1.7.3 Appendix C: Triggers explicit

Lemma 1.7.5. *As noise vanishes, the trigger satisfies*

$$\lim_{\varepsilon\to 0} p(\theta^*) = \frac{\frac{l}{\delta r}\,u(r)\,(1+\ln(\delta r/l))}{u(kr)\left(h^* + \int_{h^*}^{l/(\delta r)} \frac{(1-nr\delta/l)H}{\delta(1-n)kr}\,dn\right)} \tag{1.35}$$

$$= \frac{\frac{l}{\delta r}\,u(r)\,(1+\ln(\delta r/l))}{u(kr)\left(1-\frac{H}{\delta rk}(\frac{\delta r}{l}-1)\ln\left(\frac{H}{H-kl}\right)\right)} \tag{1.36}$$

Proof. (Lemma (1.7.5)) Online Appendix □

1.7.4 Appendix D: Optimality - Debt Investors

Lemma 1.7.6 (Optimality for debt investors)**.** *Fix contract (r,k) and liquidity l. As noise vanishes, if the firm's stability decreases in debt ratio then also debt investor's utility from the contract decreases in debt ratio. If however firm stability increases in debt ratio and the asset's liquidity is sufficiently small, $l < \hat{l}$, $\hat{l} \in I_d$, investor's utility from the contract can increase in debt ratio.*

Proof. (Lemma 1.7.6) Online Appendix □

Proof. (Proposition 1.4.1) By Lemma (1.7.6) and Theorem (1.3.1), when liquidity is high, stability decreases in debt ratio for all values of debt ratio in $\delta \in (l/r, 1)$. When liquidity is moderate or low, stability decreases in debt ratio for values of debt ratio in $\delta \in (l/r, \delta_u)$. When liquidity is sufficiently low, i.e. $l \in (0, \hat{l})$, where $\hat{l} \in (0, \tilde{l}_B(1))$ is low, stability improves in debt ratio and investor's utility increases in debt ratio. □

1.7.5 Proofs Lemmata

Proof. (Lemma 1.7.1) Fix contract (r,k) and let $\delta \in (0,1]$. By assumption we only consider firm structures that are prone to runs and hence satisfy $l \in (0, \min(1, \delta r))$.

$$\frac{\partial}{\partial l}A(\delta,l) := \frac{\partial}{\partial l}\left[\left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{E}\right)^E\right]$$
$$= \left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{E}\right)^{E-1} \left[(\ln(\frac{\delta r}{l})-1)(1-\frac{l}{E})-1\right] \tag{1.37}$$

The constant $\left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{E}\right)^{E-1}$ is positive by definition of $E = H/k$, because $H > kr > kl$. Hence, A strictly increases in l if the square bracket in (1.37) is positive. That is the case if and only if

$$\ln\left(\frac{\delta r}{l}\right) > 1 + \frac{E}{E-l} \tag{1.38}$$

The left hand side is positive, continuous and strictly decreasing in l. The right hand side is continuous, bounded and increasing in l. For $l \to 0$ the left hand side tends to infinity while the right hand side approaches value $2 < \infty$. Parameter l is bounded from above by $\min(1, \delta r)$. Let $\min(1, \delta r) = 1$ and $l \to 1$. Then the left hand side undercuts the right hand side, $\ln(\delta r) < \delta r - 1 \leq r - 1 < 1 < 1 + \frac{E}{E-1}$. Let $\min(1, \delta r) = \delta r$ and $l \to \delta r$. Again, the left hand side undercuts the right hand side, $\ln(1) = 0 < 1 + \frac{E}{E-\delta r}$. These inequalities hold for $E - \delta r > 0$ as $H > kr > \delta kr$ and $r \leq 2$ is like the coupon payment of a zero coupon bond (principal + interest) and interest rates are below 100 percent in most economically meaningful situations. Thus, in either case the right hand side exceeds the left hand side at the upper boundary of l. Thus, by strict monotonicity, continuity and the Intermediate Value Theorem for every fixed $\delta \in (0,1]$ there exists a unique $l_A^*(\delta, r, k) \in (0, \min(\delta r, 1))$ for which both sides are equal,

$$\ln\left(\frac{\delta r}{l_A^*(\delta)}\right) - 1 - \frac{E}{E - l_A^*(\delta)} = 0$$

We have $\ln\left(\frac{\delta r}{l}\right) > 1 + \frac{E}{E-l}$ and $\frac{\partial}{\partial l}A > 0$ for $l \in (0, l_A^*(\delta))$ while $\ln\left(\frac{\delta r}{l}\right) < 1 + \frac{E}{E-l}$ and $\frac{\partial}{\partial l}A < 0$ for $l \in (l_A^*(\delta), \min(1, \delta r))$.

$l_A^*(\delta)$ increases in δ for the left hand side in (1.38) decreases in l, increases in δ but the right hand side increases in l and is independent of δ.

Next, observe that for all $\delta \in (0,1]$ the function $A(\delta, l)$ converges to 1 as $l \to 0$: We have $\left(1-\frac{l}{E}\right)^E \to 1$ as $l \to 0$, $l \ln(\delta r/l) \to 0$ and thus by continuity of the exponential function $\left(\frac{\delta r}{l}\right)^l \to 1$. As $A(\delta, l)$ is strictly increasing for $l < l_A^*$ and decreasing for $l \in (l_A^*, 1]$ with $\lim_{l\to 0} A(\delta, l) = 1$, there exists a unique $\tilde{l}_A(\delta) \in (l_A^*, 1]$ such that $A(\delta, l) > 1$ for all $l \in (0, \tilde{l}_A)$, $A(\delta, l) < 1$ for $l \in (\tilde{l}_A, 1]$.

In case $\min(\delta r, 1) = \delta r$, we always have $\tilde{l}_A \in (l_A^*, \min(1, \delta r))$ (interior) by the structure of

$A(\delta,l) := \left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{E}\right)^E$: the factor $\left(1-\frac{l}{E}\right)$ is positive and strictly smaller one for every $l \leq \delta r < 1$. Thus, at $l = \delta r$ we already have $A(\delta, l = \delta r) < 1$. Therefore, $\tilde{l}_A$ has to lie below δr. In case $\min(\delta r, 1) = 1$ we set $\tilde{l}_A = 1$ if $A(\delta,l) > 1$ for all $l \in (0,1]$. In either case, $\tilde{l}_A < \delta r$.
$\tilde{l}_A(\delta)$ is weakly increasing in δ since $A(\delta,l)$ is positive and increasing in δ for every l. Concrete, assume $\tilde{l}_A$ is interior: Then, $A(\delta,\tilde{l}_A(\delta)) = 1$. A is strictly increasing in δ, and $\tilde{l}_A(\delta) > l^*_A$. Hence, A decreases in l at $\tilde{l}_A(\delta)$ for every δ. By the Implicit Function Theorem $A(\delta,\tilde{l}_A(\delta)) = 1$ to keep the function A at value 1, $\tilde{l}_A(\delta)$ increases in δ . If $\tilde{l}_A = 1$, then $A(\delta,l) \geq 1$ for all $l \in (0,1]$. Then $\tilde{l}_A$ is constant (at value one) in δ as A increases in δ because $A(\delta,l) \geq 1$ already for all $l \in (0,1]$ under the smaller δ. □

Proof. (Lemma 1.7.2)

$$\frac{\partial}{\partial l}B(\delta,l) := \frac{\partial}{\partial l}\left[\left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{(H/k)}\right)^H\right]$$
$$= \left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{(H/k)}\right)^{H-1} \left[(\ln(\frac{\delta r}{l})-1)(1-\frac{l}{(H/k)})-k\right]$$

again for $l \in (0,\min(1,\delta r)$ and $H > k$ the constant $\left(\frac{\delta r}{l}\right)^l \left(1-\frac{l}{(H/k)}\right)^{H-1}$ is positive. So B is strictly increasing in l if the bracket is positive. That is the case if and only if

$$\ln\left(\frac{\delta r}{l}\right) > 1 + k\frac{E}{E-l}$$

The remaining proof of the first part of the Lemma is as in Lemma (1.7.1). Next we show, for every δ we have $l^*_B(\delta) < l^*_A(\delta)$: For $k > 1$, $1 + \frac{E}{E-l} < 1 + k\frac{E}{E-l}$ holds for all l and $\ln(\delta r/l)$ is strictly decreasing in l.
For every δ we have $\tilde{l}_B \leq \tilde{l}_A$: For every $\delta \in (0,1)$ and $l \in (0,\min(\delta r,1))$ we have $B(\delta,l) < A(\delta,l)$ as $k > 1$ and $1 - l/E < 1$. Fix δ, and assume $\tilde{l}_B \in (0,\min(\delta r,1))$ (interior). Then $1 = B(\delta,\tilde{l}_B(\delta)) < A(\delta,\tilde{l}_B(\delta))$. By Lemma (1.7.1) above, $A(\delta,l) > 1$ for all $l \in (0,\tilde{l}_A)$. Thus, $\tilde{l}_B \in (0,\tilde{l}_A)$. If $\tilde{l}_B = 1$, then $1 \leq B(\delta,l) < A(\delta,l)$ for all $l \in (0,\min(\delta r,1))$, thus $\tilde{l}_A = 1$. □

Proof. (Lemma 1.7.3) Let $\hat{l} \in (0,\tilde{l}_B(1))$, then by Proposition (1.7.1) $\delta_u(\hat{l})$ and $\delta_d(\hat{l})$ exist and are interior in $(\hat{l}/r,1)$. That is $A(\delta_u(\hat{l}),\hat{l}) = 1$, $B(\delta_d(\hat{l}),\hat{l}) = 1$. The function $A(\delta,l)$ is strictly increasing in δ for any $\delta \in (l/r,1)$ so in particular in $\delta = \delta_u(\hat{l})$. We want to show, that $A(\delta_u(\hat{l}),l)$ strictly decreases in l at $l = \hat{l}$. By Lemma (1.7.1), that is exactly the case if $\hat{l} > l^*_A(\delta_u(\hat{l}))$. By the same Lemma, we know $A(\delta_u(\hat{l}),l) > 1$ for all $l \in (0,l^*_A(\delta_u(\hat{l}))]$. But $A(\delta_d(\hat{l})),\hat{l}) = 1$, hence $\hat{l} > l^*_A(\delta_u(\hat{l}))$. Using the Implicit Function Theorem, for A strictly increases in δ at $\delta = \delta_u(\hat{l})$ and decreases in l at $l = \hat{l}$, $\delta_u(\hat{l})$ strictly increases in $\hat{l}$. By the same argument, $\delta_d(\hat{l})$ strictly increases in $\hat{l}$.
As $\hat{l} \to \tilde{l}_B(1)$ there are two cases: If $\tilde{l}_B(1) < 1$, we know $B(1,\tilde{l}_B(1)) = 1$ by continuity and Lemma (1.7.2). Hence, $\delta_d(\hat{l}) \to 1$ as $\hat{l} \to \tilde{l}_B(1)$. If $\tilde{l}_B(1) = 1$, $B(1,\tilde{l}_B(1)) \geq 1$, therefore $\delta_d(\tilde{l}_B(1)) \leq 1$. By Lemma (1.7.2), $\tilde{l}_B \leq \tilde{l}_A$, therefore $A(1,\tilde{l}_B(1)) > 1$ and $\delta_u(\tilde{l}_B(1)) < 1$.

For $\hat{l} < \tilde{l}_B$ we can explicitly calculate δ_u as a function of $\hat{l}$: $A(\delta_u(\hat{l})), \hat{l}) = \left(\frac{\delta_u r}{l}\right)^l \left(1 - \frac{l}{E}\right)^E = 1$. And hence, $\delta_u(\hat{l}) = \frac{\hat{l}}{r}\left(1 - \frac{\hat{l}}{E}\right)^{-\frac{E}{\hat{l}}}$. By definition of the exponential function we have $\left(1 - \frac{\hat{l}}{E}\right)^{\frac{E}{\hat{l}}} \to \exp(-\frac{1}{E})^E = \frac{1}{e}$ as $\hat{l} \to 0$ and hence $\delta_u \to 0$. Analogously, $\delta_d \to 0$ as $\hat{l} \to 0$.

□

Proof. (Lemma 1.3.3) Another way of seeing the non-monotonicity of stability in liquidity ratio is by looking at the derivative of the implicit function $l_f(\delta)$ which for every debt ratio yields the value of liquidity such that the trigger and hence stability would stay constant. Using the payoff indifference equation (2.23), by the Implicit Function Theorem its derivative is given by

$$\frac{\partial l_f(\delta)}{\partial \delta} = -\frac{\frac{\partial \hat{f}}{\partial \delta}}{\frac{\partial \hat{f}}{\partial l}} = \frac{\frac{1}{\delta r}\ln(\delta r/l)u(r) \cdot \frac{l}{\delta} - \int_{h^*}^{l/(\delta r)} p(\theta)\frac{u(kr)}{kr}\frac{H}{1-n}\left(\frac{1}{\delta^2}\right)\, dn}{\frac{1}{\delta r}\ln(\delta r/l)u(r) - \int_{h^*}^{l/(\delta r)} p(\theta)\frac{u(kr)}{kr}\frac{H}{1-n}\left(\frac{nr}{l^2}\right)\, dn} \tag{1.39}$$

If stability improved monotonically in liquidity ratio, for fixed short-term coupon r the function $l_f(\delta)$ would need to be strictly increasing in debt ratio δ since the liquidity ratio decreases in debt ratio. That is the case if and only if numerator and denominator in (1.39) have the same sign. This is however not necessarily true although numerator and denominator look fairly similar: The integration in (1.39) considers only values $n < l/(\delta r)$ or equivalently $nr/l^2 < 1/(\delta l)$. As a consequence, the following inequality holds

$$\int_{h^*}^{l/(\delta r)} p(\theta)\frac{u(kr)}{kr}\frac{H}{1-n}\frac{1}{\delta l}\, dn > \int_{h^*}^{l/(\delta r)} p(\theta)\frac{u(kr)}{kr}\frac{H}{1-n}\frac{nr}{l^2}\, dn \tag{1.40}$$

which allows an analysis of the numerator and denominator in (1.39). By the proof of Proposition 1.7.1 and the comparative statics result in (1.29), stability strictly decreases in debt if and only if the numerator in (1.39) is negative. Similarly, the denominator is negative if and only if stability is increasing in liquidity, see Lemma (1.7.4). Therefore by (1.39), if stability improves in liquidity then stability also decreases in debt ratio, numerator and denominator in (1.39) are negative, and stability monotonically increases in liquidity ratio for the slope of the function $l_f(\delta)$ is positive. If instead stability improves in debt ratio, stability also decreases in liquidity, both numerator and denominator are positive, and again stability improves in liquidity ratio.

The corresponding reverse directions do not hold. For every contract (r, k) there exist parameters (l, δ) such that stability decreases in debt and liquidity simultaneously. For such parameters the function $l_f(\delta)$ has negative slope and stability decreases in liquidity ratio. By (1.39), this is exactly the case if the numerator in (1.39) is negative but the denominator is positive or equivalently if

$$\frac{u(r)}{\delta r}\ln(\frac{\delta r}{l}) \in \left(\int_{h^*}^{l/(\delta r)} \frac{u(kr)}{kr}\frac{p(\theta)H}{1-n}\frac{nr}{l^2}\,dn, \int_{h^*}^{l/(\delta r)} \frac{u(kr)}{kr}\frac{p(\theta)H}{1-n}\frac{1}{\delta l}\,dn\right) \tag{1.41}$$

By Proposition 1.7.1, for low liquidity $l \in I_d$ all maxima of the trigger in debt ratio are interior in the interval $(\delta_u(l), \delta_d(l))$. By continuity of the derivative $\partial\theta^*/\partial\delta$, the slope equals zero at every maximum point δ^* or equivalently,

$$\int_{h^*}^{l/(\delta^* r)} p(\theta)\frac{u(kr)}{kr}\frac{H}{1-n}\left(\frac{1}{\delta^* l}\right)\,dn = \frac{u(r)}{\delta^* r}\ln(\delta^* r/l) \tag{1.42}$$

For δ^* is a maximum point, the derivative $\partial\theta^*/\partial\delta$ needs to be strictly positive on a small open set (δ_*, δ^*) below the maximum point or equivalently

$$\int_{h^*}^{l/(\delta r)} p(\theta)\frac{u(kr)}{kr}\frac{H}{1-n}\left(\frac{1}{\delta l}\right)\,dn > \frac{u(r)}{\delta r}\ln(\delta r/l) \tag{1.43}$$

By (1.40) and continuity there exists an open subset $(\delta_l, \delta^*) \subset (\delta_*, \delta^*)$ such that (1.41) holds.

□

Proof. (Lemma 1.7.4) By the Implicit Function Theorem $\frac{\partial\theta^*}{\partial l} = -\frac{\frac{\partial\hat{f}}{\partial l}}{\frac{\partial f}{\partial\theta^*}}$ and $\frac{\partial\hat{f}}{\partial\theta^*} > 0$ by equation (2.24) while

$$\frac{d}{dl}\hat{f}(\theta^*, l) = -u(r)\,\frac{1}{\delta r}\ln(\frac{\delta r}{l}) + \int_{h^*}^{l/\delta r} p(\theta(n,\theta^*))\,u(kr)\left(\frac{Hn}{l^2 k(1-n)}\right)\,dn \tag{1.44}$$

□

Proof. (Lemma 1.7.5) The trigger is implicitly defined by PIE (1.22) or equivalently

$$\begin{aligned}\frac{l}{\delta r}u(r)(1-\ln(l/(\delta r))) = &\int_{h^*}^{l/(\delta r)} p(\theta(n,\theta^*))\frac{(1-nr\delta/l)H}{\delta(1-n)kr}\,u(kr)\,dn\\ &+\int_0^{h^*} p(\theta(n,\theta^*))\,u(kr)\,dn\end{aligned} \tag{1.45}$$

Taking limits, for $\varepsilon \to 0$ since $n(\theta) \leq 1$, $p(\theta) \leq 1$ we have by Lebesgue's Dominated Convergence Theorem

$$\int_{h^*}^{l/(\delta r)} p(\theta(n,\theta^*))\frac{(1-nr\delta/l)H}{\delta(1-n)kr}\,dn \to p(\theta^*)\int_{h^*}^{l/(\delta r)}\frac{(1-nr\delta/l)H}{\delta(1-n)kr}\,dn \tag{1.46}$$

and $\int_0^{h^*} p(\theta(n,\theta^*))\,u(kr)\,dn \to h^*\,p(\theta^*)\,u(kr)$. By definition of the dominance regions and the noisy signal, away from the limit the trigger lies in the interval $[\underline{\theta}-\varepsilon, \overline{\theta}+\varepsilon]$. As signals become precise, we have $\theta^* \leq \overline{\theta}$ and

$$\lim_{\varepsilon \to 0} p(\theta^*) = \frac{\frac{l}{\delta r}\, u(r)\,(1 + \ln(\delta r/l))}{u(kr)\left(h^* \,+ \int_{h^*}^{l/(\delta r)} \frac{(1-nr\delta/l)H}{\delta(1-n)kr}\, dn\right)} \tag{1.47}$$

Using the definition of h^* and $\frac{(1-nr\delta/l)}{(1-n)} = 1 + (\delta r/l - 1)(1 - \frac{1}{1-n})$ one may simplify this this expression to the formula given in the Lemma.

□

Proof. (Lemma 1.7.6) Let $\delta r > l$. By Lemma 2.3.1, stability is directly related to the size of the trigger. Let θ_b the state below which the firm defaults in period 1, i.e. $n(\theta_b) = l/\delta r$. Let $\tilde{\theta}$ the state at which debt investors who roll over receive their full payment kr for sure, $n(\tilde{\theta}) = h^*$. We have $0 < \theta^* - \varepsilon \leq \theta_b \leq \tilde{\theta} \leq \theta^* + \varepsilon \leq \overline{\theta} < 1$, therefore ex ante utility away from the limit is given as

$$\begin{aligned} \mathbb{E}[u(DD)] = &\int_0^{\theta_b} u(r)\,\frac{l}{\delta r}\,d\theta \\ &+ \int_{\theta_b}^{\tilde{\theta}} n(\theta,\theta^*)\,u(r) + (1-n(\theta,\theta^*))\,p(\theta)\,\frac{(1-\delta n(\theta,\theta^*)r/l)H}{\delta(1-n(\theta,\theta^*))kr}\,u(kr)d\theta \\ &+ \int_{\tilde{\theta}}^{\theta^*+\varepsilon} n(\theta,\theta^*)\,u(r) + (1-n(\theta,\theta^*))\,p(\theta)\,u(kr)d\theta \\ &+ \int_{\theta^*+\varepsilon}^{\overline{\theta}} p(\theta)\,u(kr)\,d\theta + \int_{\overline{\theta}}^1 u(kr)\,d\theta \end{aligned} \tag{1.48}$$

Trigger θ^* depends on δ, further $\theta_b = \theta^* - 2\varepsilon\left(\frac{l}{\delta r} - \frac{1}{2}\right)$ and $\tilde{\theta} = \theta^* - 2\varepsilon\left(h^*(\delta) - \frac{1}{2}\right)$ where $h^*(\delta) = \frac{H-kr\delta}{\delta r(H/l-k)}$. The bound to the upper dominance region $\overline{\theta}$ is constant in δ. With $n(\tilde{\theta},\theta^*) = h^*$ and Leibniz rule for parameter integrals,

$$\frac{\partial}{\partial\delta}\mathbb{E}[u(DD)]$$

$$= -\int_0^{\theta_b} u(r)\,\frac{l}{\delta^2 r}\,d\theta + \frac{\partial n(\theta,\theta^*)}{\partial\theta^*}\cdot\frac{\partial\,\theta^*}{\partial\delta}\int_{\tilde{\theta}}^{\theta^*+\varepsilon}(u(r) - p(\theta)\,u(kr))d\theta \tag{1.49}$$

$$+ \frac{\partial n(\theta,\theta^*)}{\partial\theta^*}\cdot\frac{\partial\,\theta^*}{\partial\delta}\int_{\theta_b}^{\tilde{\theta}}\left(u(r) - p(\theta)\,\frac{(1-\delta nr/l)H}{\delta(1-n(\theta,\theta^*))kr}\,u(kr)\right)d\theta \tag{1.50}$$

$$+ \frac{H}{\delta kr}\int_{\theta_b}^{\tilde{\theta}} p(\theta)\,u(kr)\cdot\left(\frac{(1-\frac{\delta r}{l})\,\frac{dn(\theta^*)}{d\theta^*}\,\frac{\partial\theta^*}{\partial\delta}}{1-n(\theta,\theta^*)} - \frac{1}{\delta}\right)d\theta \tag{1.51}$$

Here, we could draw the derivative $\frac{\partial\theta^*}{\partial\delta}$ out of the integral since the equilibrium θ^* does not depend on the state realization θ. Also, the derivative $\frac{\partial n(\theta,\theta^*)}{\partial\theta^*}$ is independent of θ. Using trigger condition (1.22), or equivalently

$$0 = \int_{\theta^*-\varepsilon}^{\theta_b} -u(r)\frac{l}{\delta r n(\theta)}\, d\theta + \int_{\theta_b}^{\tilde\theta} p(\theta)\frac{(1-\delta n r/l)H}{\delta(1-n(\theta,\theta^*))kr}\, u(kr) - u(r)\, d\theta + \int_{\tilde\theta}^{\theta^*+\varepsilon} p(\theta)\, u(kr) - u(r)\, d\theta \tag{1.52}$$

equation (1.49) simplifies to

$$\frac{\partial}{\partial\delta}\mathbb{E}[u(DD)] = -\int_0^{\theta_b} u(r)\, \frac{l}{\delta^2 r}\, d\theta - \frac{\partial n(\theta,\theta^*)}{\partial\theta^*}\cdot\frac{\partial\theta^*}{\partial\delta}\, \left(\int_{\theta^*-\varepsilon}^{\theta_b} u(r)\frac{l}{\delta r n(\theta)}\, d\theta\right) + \frac{H}{\delta k r}\int_{\theta_b}^{\tilde\theta} p(\theta)\, u(kr)\cdot\left(\frac{(1-\frac{\delta r}{l})\, \frac{dn(\theta^*)}{d\theta^*}\, \frac{\partial\theta^*}{\partial\delta}}{1-n(\theta,\theta^*)} - \frac{1}{\delta}\right) d\theta \tag{1.53}$$

Plugging in $\frac{\partial}{\partial\theta^*}n(\theta,\theta^*) = \frac{1}{2\varepsilon}$ for $\theta\in[\theta^*-\varepsilon,\theta^*+\varepsilon]$ and changing variables of integration to n, if the limit $\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial\delta}$ exists, that is by equation (1.29) if $\lim_{\varepsilon\to 0}\theta^* \neq \bar\theta$, with $n(\theta)\le 1$, $p(\theta)\le 1$ by Lebesgue's Dominated Convergence Theorem the derivative of expected utility converges to

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\delta}\mathbb{E}[u(DD)] = -\theta^*\cdot u(r)\, \frac{l}{\delta^2 r} - \frac{\partial\theta^*}{\partial\delta}\, \frac{l}{\delta r}\ln(\delta r/l)\, u(r) - \left(\frac{\delta r}{l}-1\right)\, \frac{H}{\delta k r}p(\theta^*)\, u(kr)\, \frac{\partial\theta^*}{\partial\delta}\, \int_{h^*}^{l/(\delta r)}\frac{1}{1-n}\, dn \tag{1.54}$$

Clearly, if $\frac{\partial\theta^*}{\partial\delta} > 0$ then due to $\delta r > l$ the limits of all three terms are negative and $\lim_{\varepsilon\to 0}\frac{\partial}{\partial\delta}\mathbb{E}[u(DD)] < 0$.
The limit $\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial\delta}$ does not exist if the trigger converges to the the bound of the upper dominance region $\lim_{\varepsilon\to 0}\theta^* = \bar\theta$ and thus $p'(\theta^*) = 0$. By continuity of all terms in equation (1.53) in ε for $\varepsilon > 0$, when $\frac{\partial\theta^*(\varepsilon)}{\partial\delta} > 0$ the derivative $\frac{\partial}{\partial\delta}\mathbb{E}[u(DD)]$ however is defined and negative for nonzero but sufficiently small noise ε.

If limit $\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial\delta}$ exists and $\frac{\partial\theta^*}{\partial\delta} < 0$ instead, we rewrite equation (1.54) using

$$-\int_{h^*}^{l/(\delta r)}\frac{1}{1-n}\, dn = \ln\left(\frac{1-l/(\delta r)}{1-h^*}\right) = -\ln\left(\frac{H}{H-kl}\right)$$

and the explicit formula for $p(\theta^*)$ at the limit $\varepsilon\to 0$ given in equation (1.36). Concretely, we replace the term $-\frac{H(\delta r/l-1)}{\delta k r}\, \ln\left(\frac{H}{H-kl}\right)\, p(\theta^*)\, u(kr)$ and obtain

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\delta}\mathbb{E}[u(DD)] = -\theta^*\cdot u(r)\, \frac{l}{\delta^2 r} + \frac{\partial\theta^*}{\partial\delta}\, \left(\frac{l}{\delta r}u(r) - p(\theta^*)u(kr)\right) \tag{1.55}$$

We have $\frac{l}{\delta r}u(r) - p(\theta^*)u(kr) < 0$ for $\delta r > l$ and $u(r) < p(\theta^*)u(kr)$. With $\frac{\partial\theta^*}{\partial\delta} < 0$, the second term in (1.55) is positive and can be estimated from below to obtain a boundary independent of

θ^*. We have

$$\frac{l}{\delta r}u(r) - p(\theta^*)u(kr) < \frac{l}{\delta r}u(r) - u(r) < 0 \tag{1.56}$$

for $l/(\delta r) < 1$. Thus,

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \delta} \mathbb{E}[u(DD)] > -\theta^* \cdot u(r)\, \frac{l}{\delta^2 r} + \frac{\partial \theta^*}{\partial \delta}\, u(r) \left(\frac{l}{\delta r} - 1\right) \tag{1.57}$$

For $l \to 0$ the first term in (1.57) goes to zero since $\theta^* \in [\underline{\theta}, \overline{\theta}] \subset [0,1]$ is uniformly bounded. The second term[40] goes to $-u(r)\left(\lim_{l\to 0} \frac{\partial \theta^*}{\partial \delta}\right)$.
It remains to show, that the limit $\lim_{l\to 0} \frac{\partial \theta^*}{\partial \delta}$ is unequal to zero. This is true, if the limit of the upper bound for $\frac{\partial \theta^*}{\partial \delta}$ derived in the comparative statics part of the Appendix is strictly smaller and bounded away from zero. For fix l and $\frac{\partial \theta^*}{\partial \delta} < 0$, that is $\frac{\partial \hat{f}}{\partial \delta} > 0$, we use the lower bound of $\frac{\partial \hat{f}}{\partial \delta}$ to derive an upper bound for $\frac{\partial \theta^*}{\partial \delta}$. Using (1.32), (2.25) and since by assumption $\lim_{\varepsilon \to 0} \theta^* \neq \bar{\theta}$ for $\varepsilon \to 0$

$$\frac{\partial \theta^*}{\partial \delta} < -\frac{\frac{u(r)}{\delta^2 r}\ \ln\left[\left(\frac{\delta r}{l}\right)^l \left(1 - \frac{l}{H/k}\right)^H\right]}{\frac{p'(\theta^*)}{p(\theta^*)}\,\frac{l}{\delta r} u(r)\left(1 + \ln\left(\frac{\delta r}{l}\right)\right)} = -\frac{\frac{1}{\delta}\ \ln\left[\left(\frac{\delta r}{l}\right)^l \left(1 - \frac{l}{H/k}\right)^H\right]}{\frac{p'(\theta^*)}{p(\theta^*)}\, l\left(1 + \ln\left(\frac{\delta r}{l}\right)\right)} \tag{1.58}$$

$$= -\frac{\frac{l}{\delta}\ \ln\left(\frac{\delta r}{l}\right) + \frac{H}{\delta}\ln\left(1 - \frac{l}{H/k}\right)}{\frac{p'(\theta^*)}{p(\theta^*)}\, l\left(1 + \ln\left(\frac{\delta r}{l}\right)\right)} \tag{1.59}$$

$$= -\frac{1}{\delta}\frac{p(\theta^*)}{p'(\theta^*)}\left[\frac{1}{\left(1 + \frac{1}{\ln\left(\frac{\delta r}{l}\right)}\right)} + \frac{H \ln\left(1 - \frac{l}{H/k}\right)}{l\left(1 + \ln\left(\frac{\delta r}{l}\right)\right)}\right] \tag{1.60}$$

The bracket in (1.60) converges to one as $l \to 0$:

$$\frac{1}{\left(1 + \frac{1}{\ln\left(\frac{\delta r}{l}\right)}\right)} \to 1 \text{ as } l \to 0 \tag{1.61}$$

Further, $\ln\left(1 - \frac{l}{H/k}\right) \to 0$ and $l\left(1 + \ln\left(\frac{\delta r}{l}\right)\right) \to 0$. Therefore, by Hôpital's rule

$$\lim_{l\to 0} \frac{H \ln\left(1 - \frac{l}{H/k}\right)}{l\left(1 + \ln\left(\frac{\delta r}{l}\right)\right)} = \lim_{l \to 0} \frac{\frac{\partial}{\partial l} H \ln\left(1 - \frac{l}{H/k}\right)}{\frac{\partial}{\partial l}\, l\left(1 + \ln\left(\frac{\delta r}{l}\right)\right)} = \lim_{l\to 0} \frac{-\frac{k}{1 - \frac{l}{H/k}}}{\ln(\delta r/l)} = 0 \tag{1.62}$$

By assumptions on the lower dominance region and for $p(\cdot)$ is continuous and strictly increasing with $\theta^* \in [\underline{\theta}, \overline{\theta}]$ for $\varepsilon \to 0$ we have $\lim_{l\to 0} p(\theta^*) \geq p(\underline{\theta}) > 0$ and $\lim_{l \to 0} p(\theta^*) \leq p(\overline{\theta}) = 1$. Therefore, $\lim_{l\to 0} p(\theta^*) = const > 0$.

[40] In particular, by Proposition (1.7.1) the assumption $\frac{\partial \theta^*}{\partial \delta} < 0$ for $l \to 0$ is consistent when debt ratio is sufficiently large, i.e. we do not talk about a zero measure set.

Last, $p'(\cdot)$ is uniformly bounded as $\theta \in [0, 1]$ lies in a compact interval and $p'(\cdot)$ is continuous. Precisely, we have $p'(\theta) \leq c$ for all $\theta \in [0, \overline{\theta})$ and $p'(\theta) = 0$ for $\theta \in [\overline{\theta}, 1]$ by assumption on the upper dominance region. For p' is continuous and positive, $lim_{l \to 0} \, p'(\theta^*) \in (0, c]$ and the fraction $\frac{1}{p'(\theta^*)}$ converges to a constant as by assumption $\lim_{\varepsilon \to 0} \theta^* \neq \bar{\theta}$. Therefore, the upper bound of $\frac{\partial \theta^*}{\partial \delta}$ converges to a negative constant. The limit of $\frac{\partial \theta^*}{\partial \delta}$ is thus bounded away from zero as l approaches zero. We therefore obtain the existence of an $\hat{l}$ such that for all $l < \hat{l}$ we have $\frac{\partial}{\partial \delta} \mathbb{E}[u(DD)] > 0$. For $\hat{l}$ needs to be such that $\frac{\partial \theta^*}{\partial \delta} < 0$, that is stability needs to improve in debt ratio, we can infer $\hat{l} \in (0, \tilde{l}_B(1))$ by Theorem (1.3.1).

If $\lim_{\varepsilon \to 0} \theta^* = \bar{\theta}$, $\frac{1}{p'(\theta^*)}$ diverges to infinity as $\varepsilon \to 0$. The upper bound in (1.60) thus also diverges to minus infinity and $\frac{\partial \theta^*}{\partial \delta}$ cannot converge to zero. For noise sufficiently small but nonzero and l sufficiently small, the derivative of expected utility is thus strictly positive. □

Chapter 2

The Impact of Recovery Value on Coordination in Securitized Banking

2.1 Motivation

When a run on a financial firm takes place, national bankruptcy laws and interventions by central banks (lender of last resort) impact recovery values[1] of debt investments. I analyze how recovery values after bankruptcy influence coordination behavior of uninsured debt investors and stability of financial intermediators (firms) against debt runs. In particular, the paper analyzes how the composition of recovery values changes coordination when recovery value consists of a run-size dependent, endogenous part controlled by the lender of last resort and a fixed component to model national differences in bankruptcy costs.
The set-up discussed is interesting in the light of Basel 3 capital and liquidity regulation since the member countries of the Basel Committee on Banking Supervision have agreed upon following the same regulatory framework on bank capital adequacy and market liquidity risk while corresponding bankruptcy costs differ nationally:[2]

Country specific bankruptcy costs impact debt recovery rates given default of the firm through various channels such as allocating different sets of control rights to creditors, demanding different time periods the firm remains in bankruptcy and varying court-declared expenses (trustees, accountants, attorneys), see Acharya et al. (2003). Chapter 11 of the U.S. bankruptcy code leaves control over firm's assets to some degree with management during debt renegotiations.

[1]Throughout the paper, I use the term 'recovery value' as the average value a debt investor can recover after a run, that is taking into account interventions of a lender of last resort during a run and bankruptcy costs after a successful run.

[2]In 2008, the Basel Committee on Banking Supervision (BCBS) and the International Association of Deposit Insurers (IADI) developed the 'Core Principles for Effective Deposit Insurance Systems' (Basel Committee on Banking Supervision and International Association of Deposit Insurers , 2009) as a voluntary framework. Iyer and Puri (2008) however find that deposit insurance is only partially effective in preventing bank runs.

The Swedish bankruptcy law in contrast foresees a public auction where the firm is liquidated either piecewise or survives as a going concern. Management and shareholders immediately lose their control rights. Thorburn (2000) estimates recovery rates of Swedish firms as proportion of debt's face value[3] at a median of 25% for piecewise liquidation and 38% if the firm is auctioned in bankruptcy as going concerns. For the US, Franks and Torous (1994) report a median recovery rate of 51% for firms reorganizing under Chapter 11, based largely on face values. Analyzing US firms, Bris et al. (2006) show that creditors in Chapter 11 reorganizations fare significantly better than those in Chapter 7 liquidations. They find mean recovery rates[4] of 1% for unsecured creditors of firms under Chapter 7 liquidations and 52% for unsecured creditors of firms under Chapter 11.

Bankruptcy proceedings, the way bankrupt firms are liquidated or restructured, and legal costs are fixed costs that diminish recovery values. Interventions by central banks on the other hand depend on the severity of runs and increase the average value a debt investor may recover. Since in real world the scale of a run is ex ante random and endogenous, in the presence of a lender of last resort recovery values to debt investors are random and endogenous too. Differences in debt recovery rates vice versa lead to an adaption of behavior by creditors ex ante. In an empirical study, Davydenko and Franks (2008) find that differences in creditors' rights across countries cause banks to adapt their lending practices at loan origination to companies in France, Germany and the UK. Still, they find that recovery rates in default remain distinct across countries, due to different levels of creditor protection.

Motivated by the study of Davydenko and Franks (2008), this article aims at answering the question how debt investors (creditors) ex ante adapt their behavior to (not) roll over debt, taking into account endogenous, random recovery values which depend on national differences in bankruptcy fixed costs and generosity of national central banks when intervening as lender of last resort.

In the model, a financial firm[5] finances an investment in a risky, illiquid asset through equity and short-term debt. The firm promises fixed interest payments to debt investors and the residual value of investment to equity investors. At an interim period, debt investors observe noisy, private information about the asset's return and then decide whether to stay invested in the firm (roll over debt) or to withdraw their investment. Since debt investors make their roll over decisions at the interim period, the measure of total short-term withdrawals is random in the initial period. To finance withdrawals, the firm turns to the money market and pledges a proportion of the asset to a third party in form of a repurchase agreement (repo). If funds available through pledging the asset (funding liquidity) undercut the overall amount of potential short-term debt claims the firm might face, the firm is prone to a liquidity squeeze (run): When the number of debt investors demanding their deposit exceeds a critical threshold, the firm cannot serve all debt investors and

[3]Note that face values overstate market values.

[4]Here, recovery rate is measured as fraction of initial claim which is distributed by the court in the case closure.

[5]Examples for such financial firms are asset backed commercial paper conduits, banks or structured investment vehicles.

goes into default. Given a default, not liquidity of the asset is available for distribution among debt investors but a bankruptcy cost applies. After costs are withdrawn, the recovery value of the asset remains for distribution to debt investors. Before choosing actions, debt investors take into account the possibility of a run. The potential of a run gives rise to a coordination problem between debt investors. Debt investors base their roll over decision on inferences about the random asset return (insolvency risk), and also on the expected number of other debt investors rolling over (liquidity risk). The endogenous measure of agents rolling over influences whether a run occurs or not and the size of recovery value if a run occurs. As a result, a debt investor might decide not to roll over, not because the expected asset return is too low but because she expects a too large number of other investors to not roll over. A panic run or self-fulfilling run occurs if too many investors fear other investors will not roll over, withdraw, and cause the run.

A recovery value function determines the payoff of a debt investor given bankruptcy of the firm. I model recovery value as an affine function which linearly depends on the scale of the run plus a constant part (intercept). The intercept symbols a fixed fraction of asset liquidity which is recovered after the firm declares bankruptcy. Acharya et al. (2003) provide empirical evidence that a better liquidity position of industry peers of the defaulted firm implies higher recovery at emergence from bankruptcy. The size dependent part ("slope parameter") takes into account that recovery value might be affected by the scale of the run. A negative slope parameter means that larger runs are more costly and detrimental to recovery value. In this paper, the size of the run directly depends on and is inversely related to the random state of the economy. Acharya et al. (2003) find that recovery in a distressed state of the industry is lower than the recovery in a healthy state of the industry by 10 to 20 cents on a dollar which suggests that scale of run negatively affects recoveries. On the other hand, government interventions (bail-out) and actions taken by the lender of last resort (central bank) such as Emergency Liquidity Assistance (ELA), granted to prevent a financial panic and contagion to other financial firms, increasess debt values during a run and hence average recovery values (pro rata shares) if the run is successful, see (Rochet et al., 2008).

As main contribution of the paper, I find that both composition and size of recovery values after bankruptcy have a large impact on stability of financial firms. Allover, I demonstrate that high recovery values are never desirable from a stability or regulator perspective and only sometimes desirable from a consumer perspective. Increases in recovery value through either increases in slope parameter controlled by a lender of last resort or increases in intercept determined through national bankruptcy proceedings both increase the probability of runs. Generosity of a lender of last resort or more cost efficient bankruptcy proceedings harm financial stability since the anticipation of greater recovery values increases incentives to withdraw.
Concerning the composition of recovery values, the probability of runs and ex ante welfare to debt investors from contracts are monotone in liquidity mismatch if recovery values have no intercept. In this case, I recover the results of Morris and Shin (2009) and Rochet and Vives (2004). With

intercepts, probability of runs become hump-shaped in liquidity ratio. There exists a unique, maximizing liquidity ratio which monotonically decreases in intercept and slope parameter of recovery value.

Exploiting the non-monotonicity results, I demonstrate, that in two countries where intercepts of recovery value differ due to differences in national bankruptcy proceedings, drops in funding liquidity[6] can have ambiguous effects on firm stability. While the drop in funding liquidity may harm firm stability in one country, it may increase stability of a firm with identical capital structure in the other country. Regarding two further countries, where recovery values are purely determined through interventions of a lender of last resort (zero intercept), I show that countries with a more generous lender of last resort need to impose tighter liquidity and capital regulation to guarantee the same level of stability as a country with less generous lender of last resort. These result suggests that capital and liquidity regulation should take into account national differences in bankruptcy costs and potential interventions by a lender of last resort.

Last, I analyze welfare debt investors infer from contracts under different recovery values. Higher recovery values increase both payoffs conditional on a run but also probabilities of a run. Therefore, greater recovery values in general do not lead to higher welfare to debt investors unless liquidity ratio is sufficiently high.

As for the theory contribution, this paper analyzes the cause of the non-monotonicity of probabilities of runs on financial institutions in liquidity coverage as first discovered in Schilling (2015). The intercept of recovery value is responsible for the appearance of the interior maximizer of probability of runs. As long as the intercept of recovery value is positive, conditional on a run the game structure exhibits one-sided strategic substitutability between actions. In addition, the size of the intercept of recovery value controls the extent of one-sided strategic substitutability between actions and hence the size of the maximizer. As main theory contribution of the paper, I show that the extent of strategic substitutability in the model, parametrized by the intercept of recovery value, has an essential impact on the probability of runs and utility debt investors infer from the contract. As the intercept of recovery value goes to zero, the game structure changes, the one-sided strategic substitutability between actions vanishes, actions become global strategic complements and the probability of runs becomes monotone in debt and liquidity mismatch. We hence recover the results from Morris and Shin (2009) and Rochet and Vives (2004). One-sided strategic substitutability between actions drives non- monotonicity.

Related Literature

This paper adds to the literature on stability of maturity transforming financial intermediators against runs by short-term debt investors in the presence of self-fulfilling beliefs. Diamond and

[6]Drops in funding liquidity, the amount of cash that can be borrowed when posting the asset as collateral, were documented in the course of the financial crises, see (Gorton and Metrick, 2012, 2009; Dang et al., 2013).

Dybvig (1983) analyze coordination behavior of depositors who share consumption risk by entering in deposit contracts with a bank. Risk-sharing among depositors yields proneness to panic runs. Postlewaite and Vives (1987) analyze demand-deposit contracts and deduce parameter constellations under which a unique equilibrium evolves with a strictly positive probability of a 'run'. Bryant (1980), Chari and Jagannathan (1988) and Jacklin and Bhattacharya (1988) model information-based runs by introducing risk of asset returns and interim information. Empirical evidence exists for both types of runs: Evidence for depositors withdrawing when perceived asset risk is too high is provided by Goldberg and Hudgins (1996, 2002). Foley-Fisher et al. (2015) investigate the run on U.S. life insurers during the summer of 2007 and find evidence for self-fulfilling expectations.

To obtain a unique equilibrium, this paper employs technique from global games theory (Carlsson and Van Damme, 1993; Morris and Shin, 1998, 2001). The models closest to ours are Goldstein and Pauzner (2005) and Schilling (2015). Goldstein and Pauzner (2005) embed the Diamond and Dybvig (1983) model in a global game. They show that risk-sharing through deposit contracts is ex ante optimal although it increases the probability of runs. In their setting the bank is fully financed by debt and invests in an asset which is liquid at the interim period. Their paper is the first to show equilibrium existence and uniqueness under partial, endogenous repayment given default of the bank. I strongly draw on their proof to show existence and uniqueness in our setting. Schilling (2015) extends Goldstein and Pauzner (2005) to analyze the impact of capital structure and asset liquidity on coordination and financial stability. She finds that under partial, endogenous repayment, the probability of runs is in general non-monotone in short-term debt if asset liquidity is sufficiently small. This paper extends Schilling (2015) by introducing (endogenous) recovery values to discuss the impact of varying national bankruptcy costs and interventions of a lender of last resort on coordination and financial stability. Further this paper looks at financial stability under the interaction between recovery values and liquidity mismatch. As a byproduct, this paper studies emergence and behavior of the non-monotonicity of probability of runs as discovered in Schilling (2015). This paper shows, the non-monotonicity alters in recovery value and may vanish completely if recovery values have no intercept. To the best of my knowledge, this is the first paper that studies coordination behavior of debt investors under varying, random and endogenous recovery values. A further difference to Goldstein and Pauzner (2005) and Schilling (2015) is that here, debt investors are risk-neutral. By this, the interpretation that in case of a run agents queue in front of the financial institution to obtain back their fixed funds with certain probability (sequential service constraint) is equivalent to obtaining a pro rata share for sure which simplifies the analysis.
Morris and Shin (2009); Rochet and Vives (2004); Vives (2014); König et al. (2014) study the impact of capital structure and asset liquidity on coordination behavior of debt investors in a global game in the context of collateralized funding or delegated decision making. While in these papers recovery values after default are fixed to one, I allow for variations and endogenous recovery values and analyze its impact on miscoordination. Rochet and Vives (2004) derive policy recom-

mendations by studying solvency and illiquidity risk of firms. Morris and Shin (2009) partition credit risk in illiquidity and insolvency risk. Vives (2014) relates information structure, balance sheet, and market stress parameters to the degree of strategic complementarity of investors actions and fragility. König et al. (2014) analyze optimal capital structure and portfolio choice. While Morris and Shin (2009), Rochet and Vives (2004); Vives (2014) and König et al. (2014) allow the asset liquidation value to depend on the random state, in our model the liquidation value is exogenous and deterministic. In Rochet and Vives (2004); Vives (2014) and König et al. (2014) debt investors delegate decisions to fund managers while in our model investors decide directly.

From a theory perspective, this paper studies the impact on monotonicity when transitioning from a game with global strategic complementarity between actions (Bulow et al., 1985; Morris and Shin, 2009; Rochet and Vives, 2004; Vives, 2014; König et al., 2014) to a game exhibiting one-sided strategic complementarity between actions (Goldstein and Pauzner, 2005; Karp et al., 2007; Schilling, 2015).
Further related set-ups are Eisenbach (2013) and Szkup (2015) who study roll-over decisions by short-term debt investors in dynamic settings. A different class of dynamic coordination models analyzes strategic uncertainty induced by a time-varying fundamental rather than private noisy signals. He and Xiong (2012) study how asset price volatility, debt maturity and credit lines affect the risk of debt runs in intertemporal coordination problems between creditors of different debt maturities. In a related model, Tourre (2015) studies the impact of portfolio liquidity composition on run behavior of creditors.

2.2 The Model

There are three periods of time 0, 1, 2 an one good (money). There is no discounting between periods. There is a financial intermediary, denoted by 'the firm', and two types of agents: a continuum of short-term debt investors $[0, \delta]$, of measure $\delta \in (0, 1)$, and a single equity investor. Both types of agents live for two periods. In period 0, debt investors are symmetric and born each endowed with one unit of the good. Debt investors are risk-neutral and can consume in either period. The equity investor is risk-neutral and can only consume in period 2. At time zero she is endowed with measure $1 - \delta$ units of the good. Hence, at time zero there is an aggregate endowment of measure 1 units of the good. Debt investors and equity investors finance the firm's investment in a risky asset. Agents are born either as equity or debt investor, agents may not split their endowments to finance the firm in both ways.[7]

Investment and Collateralized Borrowing There exists a storage technology and an illiquid, risky asset in the economy, T. Storage yields the initial investment for sure in every

[7] This assumption is for tractability reasons.

period. The risky asset costs one unit of money at the initial period. For every unit invested, it pays a return H only in period 2 with likelihood $p > 0$ and pays zero with probability $1 - p$.
In period 1, the asset pays no return but can be used to raise cash: The firm can pledge fractions of the asset as collateral to borrow from a third party in the money market. This is done in form of a *repurchase agreement (repo)*:
A repo transaction has two parties, the firm (the borrower) and a lender. The lender lends cash to the borrower, the borrower pays interest (repo rate) on the borrowed amount. To reduce the risk of the transaction to the lender, the borrower posts a collateral which goes into physical possession of the lender. Borrower and lender agree on that the collateral is returned to the borrower at a prespecified date if the borrowed amount and interest are paid back. If the collateral accrues interest during maturity of the repo, and the borrower repays, accrued interest goes to the borrower. If the borrower cannot repay, she defaults on the repo and the lender in the repurchase agreement may sell the collateral at market price.[8]
Let fraction $\psi \in (0, 1]$ the exogenous amount of cash that can be raised (*funding liquidity*) when pledging one unit of the asset as collateral.[9] Set the repo rate to zero.[10] If the firm can repay the counterparty of the repo in period 2, she collects interest on the entire investment including the pledged fraction of the asset. Note that this leads to a major distinction in pay-off structure compared to the case where the firm has no access to the money market and has to sell parts of the asset to raise cash.[11]

The asset's probability of return $p = p(\theta)$ is random and determined by the random state $\theta \in [0, 1]$ (see information structure below). The asset's return function $p(\theta)$ is continuously differentiable in θ, strictly increasing for $\theta \le \overline{\theta}$ and constant $p(\theta) = 1$ on $[\overline{\theta}, 1]$. $\overline{\theta}$ denotes the boundary to the upper dominance region, introduced below.
Debt investors have no access to asset T, only to storage. Debt investors gain indirect access to T through investing in the firm. The expected asset return exceeds the return from storage

$$\mathbb{E}[p(\theta)]H > 1 \tag{2.1}$$

The firm The firm is the representative financial intermediator of the economy. I normalize the firm's balance sheet size to one. Denote by $\delta \in (0, 1)$ the endogenous fraction of firm's funds financed by uninsured short-term debt. The remaining fraction $1 - \delta$ is financed with equity. This simplified capital structure is without loss of generality when allowing for long-term debt investors who invest in period zero, have a claim on payments in period 2 and are less senior

[8] See Brunnermeier and Pedersen (2009)
[9] Note, ψ is not the 'true' asset value of the collateral in period 1 but the fraction of the 'true' value participants in the money market are willing to pay to accept the asset as collateral (overcollateralization). Fraction $1 - \psi$ is called the haircut and corresponds to a safety margin to the lender.
[10] The model can easily be adapted to allow for a strictly positive repo rate, this however out of scope of the paper.
[11] Sold parts of the asset do not accrue interest to the previous owner even if the asset is bought back.

than short-term debt investors.[12] By normalization of funds, call δ the firm's capital structure or *debt ratio*. Collected funds of one unit are invests in the risky asset T.[13] The firm is in perfect competition for short-term debt with other firms and maximizes utility to debt investors.

Debt contract I now describe the contracts between debt investors and the firm. By entering in a debt contract with the firm, debt investors can attain higher returns on their investment than through investing in storage. Every debt contract is characterized by two coupon payments, the period 1 coupon $r > 1$ and period 2 coupon $rk < H$, $k > 1$. Henceforth, write (r, k) for the contract. If a debt investor invests in contract (r, k), she hands her endowment to the firm in period 0. The contract is liquid from the view of debt investors: In period 1, a debt investor chooses her **action** and spontaneously decides whether to pull out ('*withdraw*') her investment and earn coupon r or to roll over (*wait*) and earn coupon rk a period later. As a consequence, in period 0 the number of debt investors who are going to withdraw in the following period is not known to the firm. If a debt investor decides to withdraw, we will also say that she 'runs' on the firm. Debt investors cannot demand a fraction of their investment.[14] The parameter $k \in (1, H/r)$ can be seen as an implicit forward interest payment which the firm pays to investors for leaving funds invested for another period.[15]

The contract (r, k) and asset return probability function $p(\cdot)$ are such that the expected payoff from rolling over exceeds payoff from withdrawing

$$\mathbb{E}[p(\theta)]kr > r \tag{2.2}$$

Otherwise, running on the firm was a dominant action. By $r > 1$ this constraint implies that expected period 2 payoff from the contract exceeds utility from storage,

$$\mathbb{E}[p(\theta)]kr > 1 \tag{2.3}$$

the contract satisfies ex ante individual rationality.

Endogenous Liquidation At period 1, the maximum measure of withdrawals a firm with debt ratio δ faces is δr. By seniority of debt, the firm is committed to make the coupon payments

[12]The capital structure of the firm can be extended to incorporate long-term debt. In this case, the model needs to specify whether long-term debt investors are equally senior or less senior than short-term debt investors in period 2. If they are less senior than short-term investors, that is all short-term investors need to be paid first before long-term investors may be paid, the coordination game remains unchanged since in that case long-term debt is like equity to short-term debt investors. If long-term debt investors are equally senior or even more senior than short-term debt investors in period 2, the coordination game will change compared to the case where long-term funds are financed through equity only since short-term investors compete with long-term investors for repayments.

[13]I assume that the firm commits to investing in the asset no matter how the state realizes. By this assumption, I exclude signaling in a Global Game and circumvent multiplicity of equilibria.

[14]This assumption is for tractability reasons.

[15]The assumption $k > 1$ is necessary, otherwise we had $r > kr$ and withdrawing early was a dominant action.

under the premise of solvency.

Let $n \in [0,1]$ denote the endogenous, ex ante random equilibrium proportion of debt investors who decide to withdraw in period 1 (*aggregate action*). Given the contract (r, k) and the measure of short-term debt funds $\delta \in (0,1)$ collected by the firm, in period 1 the firm needs to pay out measure δrn in cash to withdrawing investors. The firm finances withdrawals by pledging the fraction $n\delta r/\psi$ of the asset in the money market as collateral as part of a repo.

A **run** on the firm occurs, if in period 1 the measure of short-term funds claimed back by withdrawing investors exceeds the amount that can be borrowed using the asset as collateral. That is if $n \in [0,1]$ realizes such that

$$n\delta r > \psi \tag{2.4}$$

If funding liquidity ψ is sufficiently high for a given capital structure δ and contract (r, k), the occurrence of a run can be excluded ex ante. Since the proportion of investors who run on the firm cannot exceed one, runs are excluded if $\delta r \leq \psi$. We call such a firm *run-proof.* If instead a run cannot be excluded ex ante, if $\delta r > \psi$, the firm is *run-prone.*

To shorten notation, define **liquidity ratio** as

$$\xi = \frac{\psi}{\delta r} \tag{2.5}$$

By assumptions, we have $\delta \in (\psi/r, 1]$, $\psi \in (0,1]$ and $\xi \in (0,1)$ for a run-prone firm and a run occurs for $n > \xi$.

Bankruptcy costs and Recovery Value In the incidence of a run, $n \in (\xi, 1]$, the firm cannot borrow enough money to satisfy all debt claims. Thus, she cannot honor her debt, defaults and goes bankrupt. In this case, a bankruptcy cost for unwinding or reorganizing the firm applies. I model bankruptcy cost as a multiplier of funding liquidity ψ.[16] After bankruptcy costs are withdrawn, the remaining value is available for distribution to debt investors. Denote by

$$\gamma_{a,b}(n,\xi) = \frac{a}{\xi}n + b, \quad b \geq 0,\, a \in \mathbb{R}, 0 < a + b \leq \frac{1}{H} < 1 \tag{2.6}$$

the **recovery value function**, where constants a, b and ξ are exogenous and common knowledge to investors but recovery value $\gamma(n,\xi)$ is endogenous and ex ante random since the aggregate action n is random and endogenous. Function $\gamma(n,\xi)$ should be seen as first order Taylor approximation in n of a more complex recovery value function in point zero.

[16] In real world, modeling bankruptcy cost as a multiplier of market liquidity (the amount of cash that can be realized through selling the asset) is more adequate. This can however easily be integrated in the model by assuming that market liquidity is a multiple of funding liquidity, see (Brunnermeier and Pedersen, 2009)

In case of a run, proceeds $\gamma(n,\xi)\psi$ are available for distribution to remaining investors where[17]

$$\gamma(n,\xi)\psi < 1, \qquad n \in [\xi, 1] \tag{2.7}$$

The constant b (intercept) denotes the part of recovery value which can be realized independently of the size of the run n while a/ξ (slope) controls how much the scale of the run affects recovery value. To address a directly, we will call a the *slope parameter* . A negative a indicates that larger runs reduce the value to be recovered after bankruptcy compared to smaller runs. A positive a instead indicates that recovery value increases in size of the run.

Note that recovery value γ might exceed one[18], thus liquidity available given bankruptcy $\gamma\psi$ might exceed funding liquidity of the asset ψ if a is sufficiently large.

For a and b small, recovery value γ undercuts one so that $1-\gamma$ has the interpretation of a *bankruptcy cost* which corresponds to the percentage of funding liquidity that is lost to debt investors due to bankruptcy proceedings and the event of a run. If recovery value exceeds one, bankruptcy cost $1-\gamma$ is negative and has the interpretation of a subsidy to debt investors. The constraint $a+b>0$ guarantees that even for negative a and arbitrary liquidity ratio ξ recovery value is strictly positive $\gamma(n,\xi)>0$ for all values of n which imply the occurrence of a run $n \in (\xi, 1]$.[19]

For the case of a zero intercept $b=0$, the recovery value function $\gamma(n,\xi) = \frac{a}{\xi}n$, $a>0$ is *linear.* In the case with nonzero intercept $b>0$, the function $\gamma(n,\xi) = \frac{a}{\xi}n + b$ is *affine.* The distinction between these two cases will become important later.

Payoffs In the incidence of a run the firm cannot pay the full coupon to withdrawing debt investors but pays *pro rata shares.* Investors have a claim on r, $n\delta$ is the measure of investors who withdraw and $\gamma\psi$ is available for distribution after applying the bankruptcy cost. Thus, in case of a run every withdrawing investor receives the share

$$\frac{\gamma\psi}{\delta n} = \frac{\gamma\xi}{n} r \tag{2.8}$$

[17]In case of a run $1 \geq n \geq \xi = \psi/(\delta r)$, $a \in \mathbb{R}, b \geq 0, a+b>0$ the inequality $\gamma(n,\xi)\psi = a\,n\delta r + b\psi \leq (a+b)\,n\delta r \leq (a+b)r < r/H < 1$ holds since $\delta \in (0,1]$ and $r < rk < H$.

[18]Within the euro area, Emergency Liquidity Assistance (ELA) can be granted to 'solvent financial institutions' which face 'temporary liquidity problems' if refinancing via the interbank market or the facility of the European Central Bank breaks down (European Central Bank). The emergency loan is provided by the according national central bank in exchange for assets as collateral to "prevent or mitigate potential systemic effects as a result of contagion through other financial institutions or market infrastructures". ELA operations can be restricted by the Governing Council of the European Central Bank. Examples for banks which received emergency loans (ELA) are the German bank Hypo Real Estate in 2008/2009, Greek banks in 2015 and Cypriot Banks in 2013. The collateral banks post when using ELA may be of lower average quality than is accepted by the ECB facility. If the liquidity assistance granted for collateral by the national central bank exceeds the factual funding liquidity of the asset (determined by markets) we have $\gamma > 1$.

[19]The constraint $a + b \leq \frac{1}{H} < 1$ guarantees that recovery values in case of runs depend on asset payoffs and cannot become too large, see the paragraph on payoffs below.

by definition of ξ.[20] Note that the pro rata share is independent of short-term coupon r. By $a+b<1$, the pro rata share in case of a run undercuts one, $\gamma(n,\xi)\frac{\xi}{n} = a + b\frac{\xi}{n} < a+b < 1$ since in case of a run $n \in (\xi, 1]$. Hence, withdrawing agents can never recover the full coupon r. Debt investors who roll over receive zero in case of a run.[21]
The firm stays liquid in period 1 if she can borrow a sufficiently large amount in the money market to honour her debt, i.e. if $n \leq \xi$. In that case, all withdrawing investors receive r and the game proceeds to period 2. In period 2, the return of the asset realizes as either H with probability $p(\theta)$ or zero with probability $1-p(\theta)$.[22] In case of zero, remaining debt investors receive zero and the firm defaults on the repo, i.e. the counterparty of the repo is not paid back and may sell the collateral at market price. Conditional on success, the firm earns gross return H and can repay all remaining debt investors and the counterparty of the repo.[23]

Payoffs Debt Investors I assign the following payoffs to agents:

Event/ Action	Withdraw	Wait/roll-over
no run, $n \in [0, \xi]$	r	$\begin{cases} kr & , p(\theta) \\ 0 & , 1-p(\theta) \end{cases}$
run, $n \in (\xi, 1]$	$\gamma_{a,b}(n,\xi)\frac{\xi}{n}r$	0

Note that we require parameters a and b to be such that $\gamma(n,\xi) > 0$ for $n \in (\xi, 1]$, otherwise the game has a dominant strategy to roll over and the coordination game vanishes.[24]

Debt investor's utility difference between withdrawing in period 2 versus withdrawing early in period 1 is given by

$$v(\theta, n) = \begin{cases} p(\theta)\, kr - r & \text{if } n \leq \xi \text{ (no run)} \\ -\frac{\gamma(n,\xi)\xi}{n}\, r & \text{if } n > \xi \text{ (run)} \end{cases} \tag{2.9}$$

Note that for given contract (r, k), payoffs to debt investors are determined by funding liquidity and short-term debt only through ξ, a ratio of funding liquidity and short-term debt.

[20] Compare to Schilling (2015) where agents have to queue but are risk-averse.

[21] Hence, I assume that conditionally on a run bankruptcy law prefers withdrawing investors over those who extend the maturity of their debt by rolling over. Conditionally on a run, if we treated withdrawing investors and investors who roll over equally the coordination problem vanishes. This is the case since conditionally on no run rolling over is always optimal by condition (2.2).

[22] For instance, a loan is paid back including interest H or the borrower defaults completely.

[23] This is, since the firm's net return is $H - \delta\, rn - \delta(1-n)kr > 0$ where she repays δrn to the counterparty of the repo to obtain back possession of the pledged fraction of the asset and repays $\delta(1-n)kr$ to remaining debt investors. We have $H - \delta\, rn - \delta(1-n)kr > 0$ since $H > \delta kr$.

[24] Avoiding dominant strategies in particular means, $a > 0$ if $b = 0$. I assume that the intercept b is greater or equal to zero.

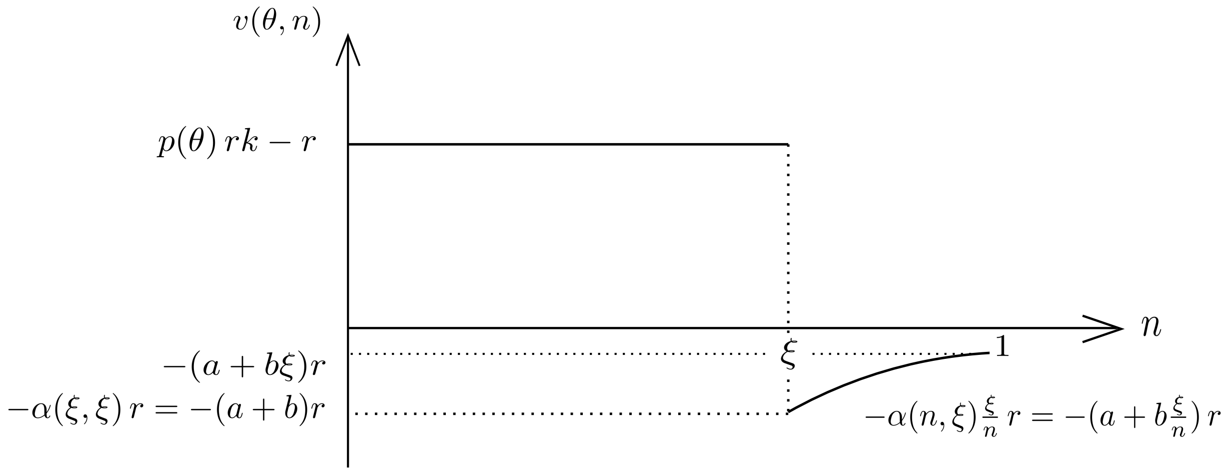


Figure 2.1: Payoff difference function $v(\theta, n)$ from equation (2.9) plotted for fixed θ as function of the endogenous proportion of withdrawing debt investors n.

Information Structure Here, I follow Goldstein and Pauzner (2005). In period zero, the unobservable state $\theta \sim U[0,1]$ realizes and determines the return probability $p(\theta)$ of the asset. Debt investors share a common prior about state θ in period 0. In period 1, debt investors observe private, noisy and asymmetric signals about the state and hence the asset return probability

$$\theta_i = \theta + \varepsilon_i,\ i \in [0, \delta]$$

where ε_i are iid random noise terms, independent of θ and distributed according to $U[-\varepsilon, +\varepsilon]$. From the signal structure we see, signals convey information not only about the random asset return probability $p(\theta)$ but also about other investors' signals.

I assume, there exist states which yield dominant actions (dominance regions).[25] There are states $\overline{\theta}$ and $\underline{\theta}$ such that if $\theta < \underline{\theta}$, withdrawing is a dominant action whereas if $\theta > \overline{\theta}$ rolling over is the dominant action to debt investors. I refer to $[0, \underline{\theta}]$ as the lower dominance region and call $[\overline{\theta}, 1]$ the upper dominance region. The bound $\underline{\theta}$ depends on the specific contract (r, k) and is given as the realization of θ such that[26]

$$r = p(\underline{\theta})\, kr \tag{2.10}$$

The assumption of existence of the lower dominance region implies that function $p(\cdot)$ takes values below $r/kr = 1/k > 0$. For very high states $\theta \geq \overline{\theta}$, I impose that the asset earns return H already in period 1 with certainty[27], that is with $p(\theta) = 1$. By assumption $H > kr > r$, the coordination problem vanishes for state realizations in the upper dominance region. To ensure

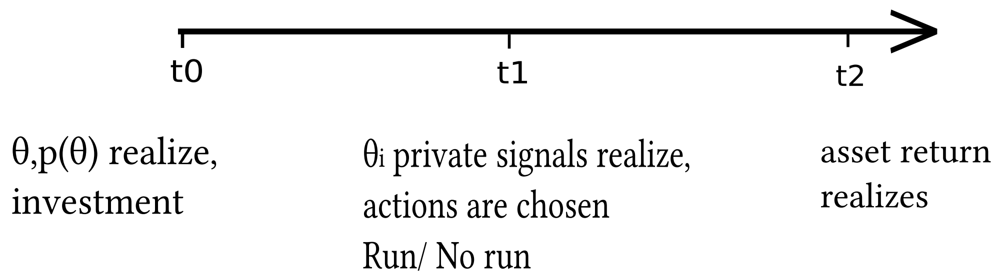
[25] Dominance regions are crucial to obtain an equilibrium selection (Morris and Shin, 2001).

[26] Payoff kr is the maximum payoff debt investors who roll over can obtain. By design of the contract, if θ realizes below $\underline{\theta}$, even in the absence of a run the expected payoff to rolling over is smaller than r for every $n \in [0, 1]$, while conditional on a run investors who roll over receive zero.

[27] This assumption can be justified by assuming that the firm is an investment expert.

that debt investors may receive signals from which they can infer that the state has realized in either of the dominance regions, I assume that noise ε is sufficiently small such that $\underline{\theta}(r, k) > 2\varepsilon$ and $\overline{\theta} < 1 - 2\varepsilon$ hold. In particular, the bounds to the dominance regions are independent of debt ratio and asset liquidity.
Note that the dominance regions are independent of funding liquidity ξ and debt δ and hence independent of liquidity ratio ξ.

Timing In period 0, the state θ and payoff probability $p(\theta)$ realize unobservably. Debt and equity investors invest. In period 1, debt investors observe noisy, private signals and subsequently choose actions. The aggregate action n (proportion of withdrawing debt investors) realizes which determines whether the firm defaults due to a run or whether it stays liquid in period 1. In case of a run, all debt investors receive payoffs according to chosen actions and the game ends. If the firm stays liquid, the game proceeds to period 2 after paying debt investors who decide to withdraw. In period 2, the success of the risky investment is determined. In case of success, the counterparty of the repo and remaining debt investors are repaid, the extra proceeds go to equity investors. Note that agents choose actions only in period 1.



2.3 Equilibrium

In order to justify the imposition of the information structure described in the outline of the model, I briefly discuss equilibrium behavior in the absence of private, noisy and asymmetric signals.

2.3.1 The Common Knowledge Game

Without signals, all debt investors share a common prior about state θ in period zero and receive no further information about the state in period 1. By assumption (2.2), the ex ante expected utility from rolling over exceeds the utility from withdrawing. The presence of the coordination problem in period 1 gives rise to a Diamond and Dybvig (1983) type situation. There are two pure equilibria: In the 'good' equilibrium all debt investors roll over and there is no run. Due to assumption (2.3) welfare in this equilibrium is higher than in the outcome where all debt investors store their endowment. In the bad equilibrium all investors panic and withdraw early which causes a run. In the run equilibrium, due to (2.7) we have $\gamma\psi \leq 1$, that is aggregate welfare is lower than in the outcome where all debt investors store their endowment. There is no means to determine the ex ante probability for selection of the Pareto-efficient no-run equilibrium within the model. To achieve an equilibrium selection and definite comparative statics on stability, I impose the information structure given in the outline of the model.

2.3.2 The Coordination Game

Assume a firm with debt ratio δ offers contract (r, k), faces asset liquidity ψ and recovery value function $\gamma_{a,b}$. Let θ_i an investor's private signal. A mixed strategy for investor i is a measurable function $s_i : [0 - \varepsilon, 1 + \varepsilon] \to [0, 1]$ which assigns a probability that the investor withdraws early (runs) as a function of her signal θ_i. A strategy profile is denoted by $\{s_i\}_{i\in[0,\delta]}$. A fixed strategy profile generates a random variable $\tilde{n}(\theta) \in [0, 1]$ which represents the aggregate action, the proportion of investors who withdraw early, if the unobservable state realizes as θ. The equilibrium concept I use is Bayesian Nash Equilibrium. All proofs can be found in the Appendix.

Proposition 2.3.1 (Existence and Uniqueness)**.** *The coordination game played by debt investors has a unique equilibrium. The equilibrium is in trigger strategies.*

Denote by $\theta^* = \theta^*(r, k, \xi(\delta, \psi), \gamma, H, p(\cdot)) \in [\underline{\theta} - \varepsilon, \overline{\theta} + \varepsilon]$ the equilibrium trigger signal. In the trigger equilibrium, if an investor observes a signal $\theta_i < \theta^*$ she withdraws, if she observes a signal $\theta_i > \theta^*$ she rolls over debt. In case $\theta_i = \theta^*$ she is indifferent. For the equilibrium is a symmetric trigger equilibrium played by a continuum of debt investors, the endogenous measure of investors who withdraw is a deterministic function of the state. The payoff structure to debt investors (2.9) and hence trigger θ^* depend on debt and funding liquidity only through liquidity ratio since also the dominance regions (2.10) are independent of debt and funding liquidity.[28] Let $n(\theta, \theta^*)$ indicate the endogenous equilibrium proportion of investors demanding early withdrawal in period 1 when the true state is θ and the trigger is θ^*. The function $n(\theta, \theta^*)$ is given by the proportion[29] of investors who observe a signal below the trigger θ^* when the true state is θ. By the uniform distribution of the error term, we have

$$n(\theta, \theta^*) = \begin{cases} \frac{1}{2} + \frac{\theta^* - \theta}{2\varepsilon} & \text{if } \theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon] \\ 1 & \text{if } \theta \leq \theta^* - \varepsilon \\ 0 & \text{if } \theta \geq \theta^* + \varepsilon. \end{cases} \tag{2.11}$$

Note that changes in parameters a and b of recovery value change the trigger θ^* and by this function $n(\theta, \theta^*)$. In addition, changes in parameters a and b change recovery value directly but also indirectly through the change in $n(\theta, \theta^*)$.

In figure (2.2), I have plotted the proportion of investors withdrawing as a function of the state

[28] Introducing the liquidity ratio leads to a parameter reduction since it substitutes debt ratio *and* funding liquidity in the model.

[29] As the continuum of debt investors has measure δ, the proportion of investors observing signals below the trigger differs from its measure by factor δ.

for fixed trigger θ^*. Given state θ, investors observe signals in the range $[\theta - \varepsilon, \theta + \varepsilon]$. For a state below $\theta^* - \varepsilon$, all investors obtain signals smaller than the trigger and hence withdraw, $n = 1$. Vice versa, for a state above $\theta^* + \varepsilon$, all investors observe signals larger than the trigger and hence roll over, $n = 0$.

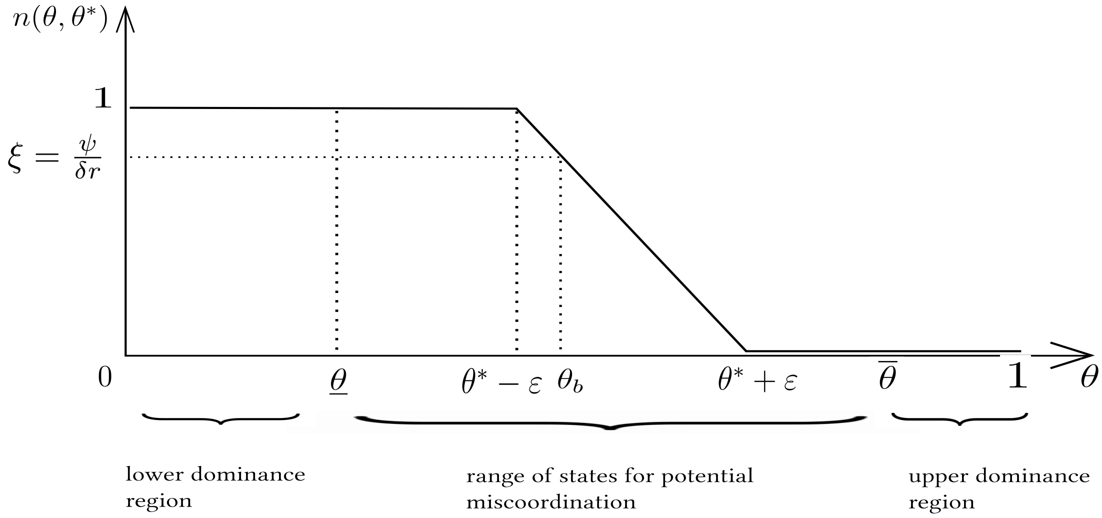


Figure 2.2: Proportion of debt investors who withdraw as a function of the state

Having established equilibrium uniqueness, the equilibrium trigger signal is pinned down using the expected payoff difference between actions when all investors use the same trigger θ^*,

$$D(\theta_i, \theta^*) = \frac{1}{2\varepsilon} \int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} v(\theta, n(\theta, \theta^*))\, d\theta \tag{2.12}$$

When observing a signal $\theta_i < \theta^*$, the expected payoff difference $D(\theta_i, \theta^*)$ is negative and the investor withdraws. When instead she observes $\theta_i > \theta^*$, the payoff difference $D(\theta_i, \theta^*)$ is positive and she rolls over. When observing a signal equal to the equilibrium trigger a debt investor's posterior beliefs on the state and the proportion of withdrawing investors n need to be such that in expectation utility from withdrawing equals utility from rolling over. The trigger is thus implicitly defined by the payoff indifference equality (PIE)

$$D(\theta^*, \theta^*) = 0 \tag{2.13}$$

Graphically, as signals become precise the trigger is located between the dominance regions $[\underline{\theta}, \overline{\theta}]$ in a way such that the area under the curve in figure (2.1) equals zero. Conditional on the observation of the trigger signal $\theta_i = \theta^*$, an investor's belief about the proportion of withdrawing agents n is uniform over $[0, 1]$ (Laplacian Belief).[30] Consequently, with slight abuse of notation I can write the PIE using (2.9) and (2.11) as

[30]We obtain $\mathbb{P}(n < z | \theta_i = \theta^*) = \mathbb{P}\left(\frac{1}{2} + \frac{\varepsilon_i}{2\varepsilon} < z\right) = z$ for $z \in [0, 1]$ by (2.11).

$$0 = -\xi r \int_{\xi}^{1} \frac{\gamma(n,\xi)}{n}\, dn + \int_{0}^{\xi} p(\theta(n,\theta^*))\, kr - r\, dn \tag{2.14}$$

where $\theta(n,\theta^*) = \theta^* + \varepsilon\,(1-2n)$ is the inverse of $n(\theta,\theta^*)$ for $\theta \in [\theta^*-\varepsilon, \theta^*+\varepsilon]$. In period 1, a run takes place if the measure of funds claimed by withdrawing debt investors $n\delta r$ exceeds the measure of funds the firm can raise by pledging the asset as collateral ψ, or equivalently if n realizes such that $n > \xi$. The payoff difference between rolling over and withdrawing conditional on a run is thus negative (integrand of first integral in (2.14)). If instead endogenous withdrawals realize low, $n \leq \xi$, the firm can borrow enough cash in the money market to satisfy all interim debt claims and hence stays liquid. Further, if the asset pays in the second period, all debt claims of investors who rolled over can be satisfied, the counterparty of the repo can be repaid and equity value becomes strictly positive.[31] Thus, conditional on no run, the payoff difference is given by the integrand of the second integral in (2.14).

2.3.3 Probability of Runs

Before stating the main results, I briefly explain why the equilibrium trigger signal θ^* and the ex ante probability of runs coincide when signals become arbitrarily precise.

If the state realizes such that in the interim period claimed withdrawals just equal available liquidity, the firm is on the edge of becoming unable to repay debt investors in period 1. I call this state *the critical state* θ_b implicitly defined by

$$n(\theta_b,\theta^*) = \psi/(\delta r) = \xi \tag{2.15}$$

Since n is a weakly decreasing function of state θ, the larger the critical state θ_b, the smaller the proportion of investors necessary to cause a successful run. Vice versa, if liquidity ratio ξ is large, the firm can bear a larger proportion of investors deciding to withdraw without being subject to a run, hence the critical state must become smaller as ξ increases (see figure 2.2). By equation (2.11) and (2.16), the critical state depends on noise and trigger, and is given as

$$\theta_b = \theta^* + \varepsilon \left(1 - 2\,\frac{\psi}{\delta r}\right) = \theta^* + \varepsilon\,(1 - 2\,\xi) \tag{2.16}$$

As depicted in figure (2.2), for state realizations smaller than the critical state a run occurs because the value of claimed funds exceeds funding liquidity of the asset. By the uniform distribution of states the probability of a run equals

$$\mathbb{P}(\text{run occurs}) = \mathbb{P}(\theta \leq \theta_b) = \theta_b \tag{2.17}$$

[31] I exclude that the firm may replace withdrawn deposits with other funds to simplify the analysis.

In the sequel, we say that *signals become precise* or *noise vanishes*, if the support of the idiosyncratic, random shock collapses to a single point, $\varepsilon \to 0$. As signals become precise, the critical state converges to the trigger, $\theta_b \to \theta^*$ as $\varepsilon \to 0$, thus as noise vanishes the trigger directly represents the firm's liquidity risk (ex ante probability of runs).

Lemma 2.3.1. *As noise vanishes, the trigger θ^* coincides with the ex ante probability of a run θ_b.*

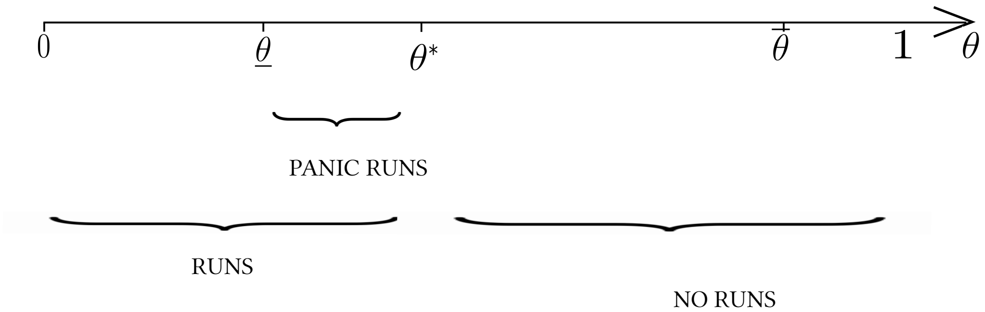


Figure 2.3: The size of the trigger determines the range of states for which panic runs occur

Note that by (2.16) at the limit any partial derivative of the run probability equals the partial derivative of the trigger θ^*.

As a consequence of Lemma 2.3.1, at the limit state realizations above the trigger lead to successful coordination while realizations below the trigger lead to runs. The greater the trigger, the greater the ex ante risk of a run. The bound to the lower dominance region $\underline{\theta}$ is independent of capital structure, liquidity ratio or recovery value. Next, I analyze how the trigger θ^* changes for varying constellations of those three parameters. This is interesting since the range of state realizations between the lower dominance region and the trigger yield panic or self-fulfilling runs, which cannot be attributed to asset return risk.

2.3.4 Recovery Value after Bankruptcy

In Schilling (2015) we have seen that probabilities of runs are non-monotone in liquidity ratio under partial repay if recovery value is fixed at $b = 1$, $a = 0$. In this section I analyze, how probabilities of runs change in recovery value and how probabilities behave in liquidity ratio depending on composition of recovery value as introduced in equation (2.6).
Liquidity ratio ξ, by equation (2.5), measures the liquidity gap (mismatch) between firm assets and short-term liabilities. If liquidity ratio is low, the amount of debt that could be claimed on short notice by withdrawing investors is much higher than the amount of cash the firm can raise through pledging assets. Hence, the smaller the liquidity ratio, the greater the liquidity mismatch, that is the wider the gap between liquidity available by pledging the asset and potential short-term liquidity withdrawals by debt investors. Conversely, if liquidity ratio is one,

the amount of short-term debt that could be claimed equals the asset's funding liquidity and the possibility of a run due to a liquidity squeeze vanishes. In that case the firm is 'run-proof', and the outcome is trivial, investors always roll over unless they observe signals in the lower dominance region. To keep the analysis interesting, in the remaining paper the firm is run-prone, $\xi < 1$.

Monotonicity versus Non-Monotonicity

Proposition 2.3.2 (Probability of runs in recovery value). *Fix contract* (r, k) *and let noise vanish.*
i) If the recovery value function is linear $\gamma(n, \xi) = \frac{n}{\xi}a > 0$, $b = 0$, $a > 0$, *then as noise vanishes the probability of a run is monotone decreasing in liquidity ratio* ξ.
ii) If the recovery value function is affine $\gamma(n, \xi) = \frac{n}{\xi}a + b$ *with* $b > 0$, *the probability of a run is a hump-shaped function of liquidity ratio: the probability of a run takes its unique maximum at interior liquidity ratio* $\xi^*(a, b, r, k, (H, p(\cdot))) \in (0, 1)$, *strictly increases in liquidity ratio on* $(0, \xi^*)$ *and strictly decreases on* $(\xi^*, 1)$.

Proof. Appendix □

Note in particular, that in the affine case the maximizer ξ^* does not depend on debt ratio or funding liquidity. Debt ratio and funding liquidity impact liquidity ratio but not the maximizing liquidity ratio.

Corollary 2.3.1. *Let noise vanish. If the recovery value function is linear, stability improves monotonically in liquidity ratio and deteriorates in liquidity mismatch. The probability of a run is minimized in* $\xi = 1$ *and has its supremum in* $\xi = 0$.

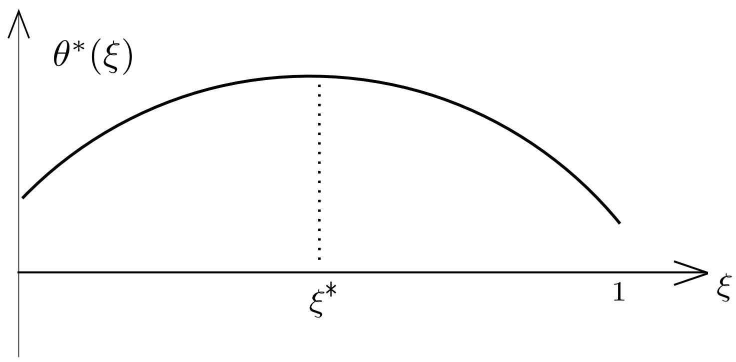


Figure 2.4: For $b > 0$, the trigger and hence probability of runs is a hump-shaped function of liquidity ratio ξ and takes its unique interior maximum in ξ^*

Both liquidity ratio and bankruptcy costs impact coordination. Unless the intercept of recovery value b is zero, by Proposition (2.3.2) there exists a unique, interior, run probability- maximizing liquidity ratio ξ^*. Thus in this case, the intuition that a higher liquidity ratio (lower liquidity mismatch) in general leads to more stability turns out wrong. If liquidity ratio lies below ξ^*, the trigger and hence run probability increase as liquidity mismatch becomes smaller (liquidity ratio increases) since coordination among debt investors is worsened. Only for liquidity ratios above maximizer ξ^* (case $b > 0$) or in the case where the intercept of recovery value is zero $b = 0$, the intuition that smaller liquidity mismatch (larger liquidity ratio) leads to lower run probability holds.

Result ii) in Proposition (2.3.2) was developed in Schilling (2015) for the special case of zero bankruptcy costs $a = 0$, $b = 1$. I briefly give two examples to improve the understanding of the non-monotonicity result and then proceed to explaining why the case of general recovery values is interesting from a theory perspective but also from an applied perspective when thinking about supranational capital and liquidity regulation under varying national bankruptcy law and hence costs.

For a given asset $(H, p(\cdot))$, I call stability θ^* *attainable* at contract (r, k) if there exists a liquidity ratio $\xi \in (0, 1)$ which achieves stability θ^*, $\theta^* = \theta^*(r, k, (H, p(\cdot)), \gamma_{a,b}, \xi)$.

Since ξ is a ratio, every equilibrium θ^* can be attained by infinitely many combinations of debt and funding liquidity. In particular, two assets with same risk profile $(H, p(\cdot))$ but different funding liquidity can achieve the exact same stability level if debt ratios are sufficiently adjusted:

Example 2.3.4.1 (Indeterminacy of stability in funding liquidity)**.** *At contract (r, k) and asset $(H, p(\cdot))$ denote the pairs of debt and funding liquidity by $(\psi_1, \delta_1) = (0.8, 0.72)$ and $(\psi_2, \delta_2) = (0.2, 0.18)$ and assume $\delta_1 r > \psi_1$, $\delta_2 r > \psi_2$ (e.g. $r \geq 1.12$) such that the firm is prone to runs under each pair. Then both pairs yield the same equilibrium and hence stability since*

$$\xi_1 = \frac{0.8}{0.72r} = \frac{0.2}{0.18r} = \xi_2 \tag{2.18}$$

and hence $\theta^*(\xi_1) = \theta^*(\xi_2)$.

By Proposition 2.3.2, for affine recovery value functions $\gamma_{a,b}$ with $b > 0$ the function $\theta^*(\xi)$ is not one-to-one. Therefore, every equilibrium $\theta^*(\xi)$ and its corresponding stability level is not uniquely attainable with respect to liquidity ratio. For given attainable θ^* there can exist liquidity ratios $\xi_1 \neq \xi_2$ with $\theta^*(\xi_2) = \theta^*(\xi_1)$.

Similarly, at contract (r, k) and asset $(H, p(\cdot))$, by Proposition 2.3.2, a decrease in debt ratio alone does not allow a qualified statement about the change of stability if the recovery value function is affine. How the trigger θ^* reacts to changes in debt ratio depends on funding liquidity and whether the change causes liquidity ratio to move towards or away from the trigger maximizing liquidity ratio:

Example 2.3.4.2 (Indeterminacy of stability: Drops in funding liquidity)**.** *Fix contract $(r, k) = (1.03, 1.15)$ and recovery value function $\gamma(n, \xi)$, $b > 0$. Assume, the trigger $\theta^*(\xi)$ is uniquely maximized at $\xi^*(\gamma) = 0.4$ and consider two distinct debt ratios $\delta_1 = 0.6$ and $\delta_2 = 0.3$.*

i) Assume funding liquidity of the asset is $\psi = 0.25$. In this setting the firm is prone to runs

under both debt ratios $\delta_1, \delta_2 > \frac{\psi}{r} = 0.24$. Changing debt ratio from δ_1 to δ_2 causes a change in liquidity ratio from $\frac{0.25}{0.6\times1.03} = 0.41$ to $\frac{0.25}{0.3\times1.03} = 0.81$. Since both values exceed the maximizer ξ^* and the change in debt causes liquidity ratio to increase and move away from the maximizer, by Proposition 2.3.2 the trigger (probability of a run) falls and stability increases.
ii) Now assume instead funding liquidity is $\psi = 0.1$. Maximizer ξ^* is not affected by this change. Again both debt ratios satisfy the new condition $\delta_1, \delta_2 > \frac{\psi}{r} = 0.1$. Changing debt ratios from δ_1 to δ_2 causes a change in liquidity ratio from $\frac{0.1}{0.6\times1.03} = 0.16$ to $\frac{0.1}{0.3\times1.03} = 0.32$. This time both liquidity ratios undercut the maximizer. Hence, the change in debt ratio has led to an increase in the trigger and thus a decrease in stability.[32]

These stylized examples demonstrate that debt ratio or funding liquidity alone are not sufficient to make a statement about firm stability. Only the combination of debt and funding liquidity uniquely pins down equilibrium behavior of debt investors.

The impact of recovery value on stability

I next analyze the interaction of non-monotonicity and recovery values after bankruptcy.

Proposition 2.3.3. *At the limit, for every liquidity ratio $\xi \in (0,1)$ the probability of runs θ^* increases in both parameters of recovery value, slope parameter a and intercept b*

By the Proposition, more cost efficient bankruptcy proceedings lead to ex ante higher run probabilities and lower stability independently of capital structure or asset funding liquidity.
As a consequence of the Proposition, increases in parameters a or b in fact lead to pointwise increases in recovery value function $\gamma(n(\theta,\theta^*),\xi)$ for every state θ and every liquidity ratio ξ if a is nonnegative: An increase in a or b increases recovery value γ directly in a first order effect. In addition, an increase in a or b increases the trigger und thus weakly increases the aggregate action $n(\theta,\theta^*)$ pointwise for every state θ which again increases γ in a second order effect for every θ. Thus, if a and b are positive, increases of these parameters translate to increases in bank run probability and recovery value. In addition, increases in bank run probability translate to increases in recovery value if a is positive.
If a is negative, an increase in b not necessarily leads to an increase in recovery value γ. This is since increasing b on the one hand increases γ directly but also leads to a decrease in γ through an increases of the trigger θ^* and hence function $n(\theta,\theta^*)$ in every point (state) θ.

Corollary 2.3.2. *Let $a \geq 0$ and let noise vanish. An increase in recovery value through either an increase in intercept b or slope parameter a monotonically increases the ex ante probability of runs.*

In particular, the more generous a lender of last resort intervenes, the larger a, the larger the probability of runs since debt investors anticipate larger recovery values which increases incentives

[32]The size of maximizer $\xi^*(\gamma) = 0.4$ is an out of equilibrium assumption.

to run. Similarly, the more efficient national bankruptcy proceedings, i.e. the smaller bankruptcy fixed costs, the larger b and the greater the probability of runs.
Next, we are interested in how maximizer $\xi^*(a, b)$ changes as the recovery value function varies in slope parameter a and intercept b.

Proposition 2.3.4 (Non-monotonicity varies in recovery value)**.** *Fix contract* (r, k)*. Assume the recovery value function is affine. At the limit, the liquidity ratio* $\xi^*(a, b)$ *which maximizes the probability of a run strictly decreases in both recovery value determining parameters* a *and* b*.*

Proof. Appendix □

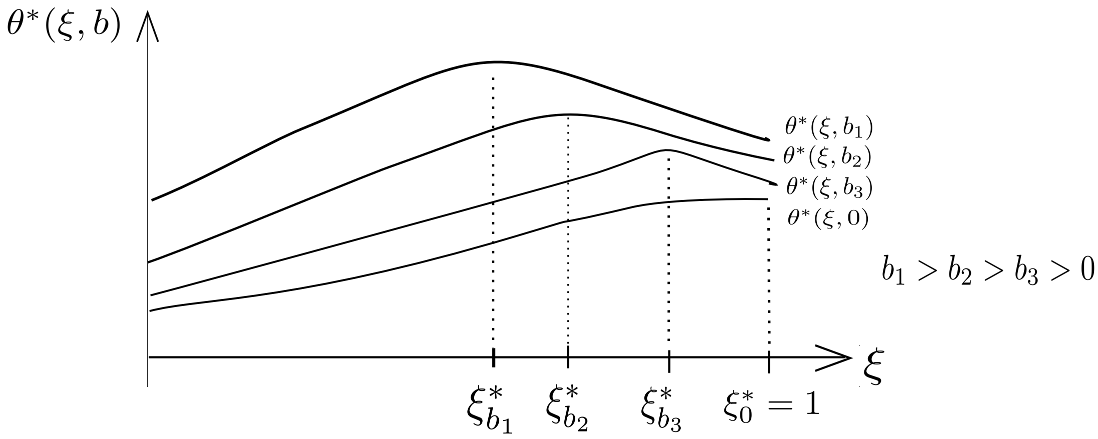


Figure 2.5: For every $b > 0$, the trigger θ^* is maximized at interior liquidity ratio ξ^*_b. As b declines, the maximizer increases. For $b \to 0$, the maximizer converges to the right boundary and the trigger becomes a monotone increasing function in ξ. By Proposition (2.3.3), for every given liquidity ratio ξ the trigger increases pointwise in parameter b of recovery value.

Let us now look at how different bankruptcy laws in countries affect how changes in funding liquidity impact stability:

Example 2.3.4.3 (Stability under distinct bankruptcy laws)**.** *A financial firm offers contract* $(r, k) = (1.05, 1.02)$*, invests in asset* $(H, p(\cdot))$ *and has capital structure* $\delta = 0.6$*. There are two countries, where in country A due to different bankruptcy laws recovery value* γ_A *is smaller than recovery value* γ_B *in country B,* $0 < \gamma_A < \gamma_B$ *with slope parameters* $a_A = a_B$ *but intercepts* $b_B > b_A > 0$*, that is in country B the fixed fraction of recovery value is larger.*
i) Assume the financial firm is based in country A , asset funding liquidity is $\psi_1 = 0.2$ *and the maximizing liquidity ratio is* $\xi^*(\gamma_A) = 0.4$*. The liquidity ratio of the firm is* $\xi_1 = \frac{0.2}{1.05 \times 0.6} = 0.32$ *which is below* $\xi^*(b_A)$*, see Figure 2.6. Now assume funding liquidity drops to* $\psi_2 = 0.1$*. The new liquidity ratio becomes* $\xi_2 = \frac{0.1}{1.05 \times 0.6} = 0.16$ *and has thus moved away from* ξ^**. By Proposition 2.3.2 the firm has become more stable, the bank run probability is decreased.*
ii) Now assume, the firm moves to country B where due to different bankruptcy law recovery value

is increased. By Proposition 2.3.4 the maximizer in country B has to be smaller at $\xi^*(\gamma_B) = 0.1$ than in country A. Before the drop of funding liquidity, the firms' liquidity ratio $\xi_1 = 0.32$ lies above the new maximizer $\xi^*(\gamma_B)$. After the drop of funding liquidity the firm's liquidity ratio $\xi_2 = 0.16$ is still above but has decreases towards maximizer $\xi^*(b_B)$. The bank run probability has increased.[33]

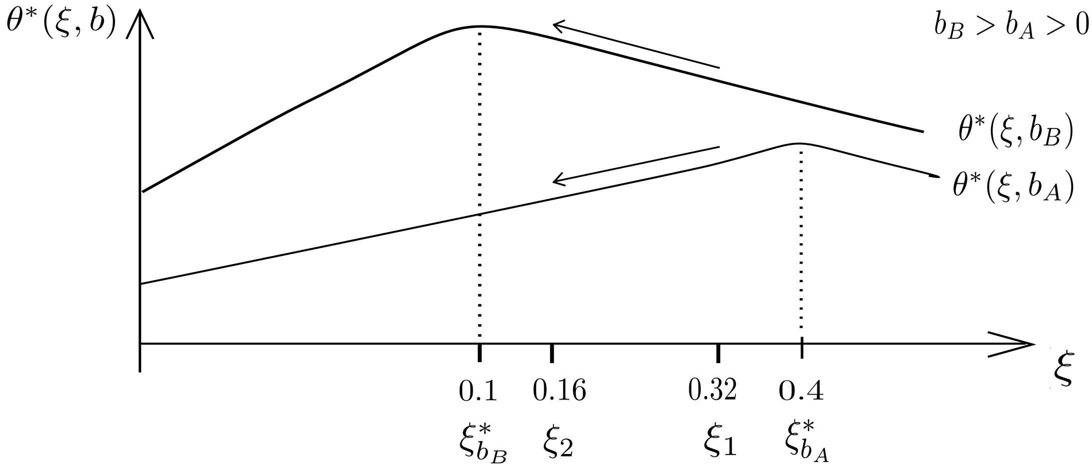


Figure 2.6: The change in stability depends on whether the change in liquidity ratio ξ leads to a move towards or away from the maximizer ξ^*. In country A, the drop in funding liquidity causes liquidity ratio to move away from maximizer ξ^*_A while in country B the same drop in liquidity induces a move towards the maximizer in country B, ξ^*_B.

A similar example can be constructed using changes in debt instead of changes in funding liquidity. Country specific bankruptcy costs affect the size of maximizer ξ^* by Proposition 2.3.4. For fixed funding liquidity, changes in debt influence the size of liquidity ratio. To determine how stability changes in debt the direction of movement and position of liquidity ratio relative to the maximizer are both decisive. Hence, for the same change of debt and hence liquidity ratio it can be that liquidity ratio moves away from the maximizer in country A but towards the maximizer in country B.

The last example in particular demonstrates that capital and liquidity regulation should take into account differences in fixed costs associated with national bankruptcy proceedings. Regulation that is stability enhancing in one country may have a destabilizing effect in another country with different bankruptcy laws. The next example concerns differences in costs that depend on the scale of the run such as interventions by a lender of last resort. The example demonstrates that countries in which a lender of last resort acts more generously should impose tighter liquidity and capital regulation on financial firms.

Example 2.3.4.4. *Imagine two countries G and I where the corresponding national central bank acts as lender of last resort in case of a run. Assume the central bank in country G intervenes*

[33]The size of maximizers $\xi^*(\gamma_A) = 0.4$ and $\xi^*(\gamma_B) = 0.1$ are out of equilibrium assumptions.

more generously during a run than country I, slope parameters satisfy $a_G > a_I > 0$ with intercept $b_G = b_I = 0$. Assume both countries agree on liquidity regulation, that is balance sheets of financial firms in both countries must be composed in a way that liquidity ratio is larger or equal than $\underline{\xi}$. By Propositions 2.3.2 and 2.3.3, at every liquidity ratio ξ the recovery value and probability of runs is higher in country G compared to country I. In particular at the lower bound on liquidity ratio imposed by regulation $\underline{\xi}$, stability level in country G undercuts stability level in country I. Hence, to guarantee the same minimum level of stability in country G as in country I, liquidity regulation in country G needs to be tighter at some liquidity ratio $\underline{\xi}_G$, see Figure 2.7. Assuming that funding liquidity for the specific asset in both countries coincides, the lower bounds $\xi > \underline{\xi}_G > \underline{\xi}_I$ for liquidity ratio by equation (2.5) transfer directly to upper bounds for capital structure $\delta < \overline{\delta}_G < \overline{\delta}_I$. To guarantee the same minimum level of stability in country G as in country I, capital regulation in country G needs to be tighter.

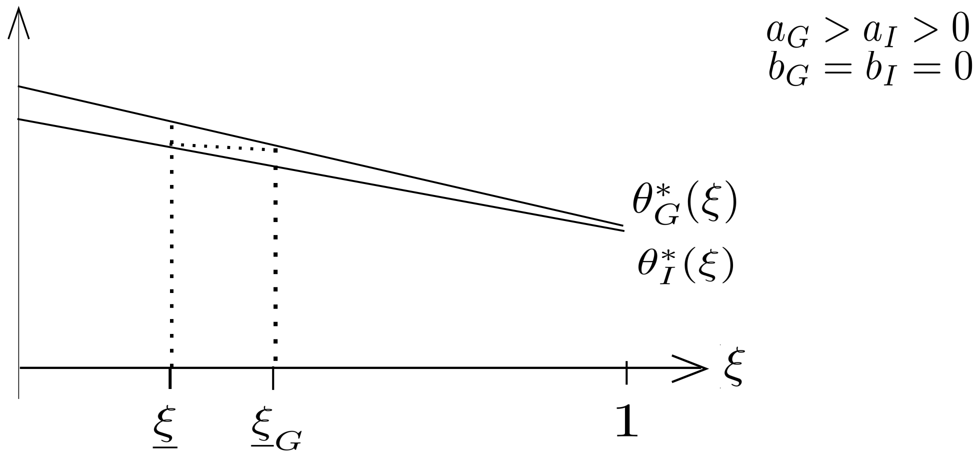


Figure 2.7: Since recovery value after runs is higher in country G compared to country I, country G needs to impose tighter liquidity regulation $\underline{\xi}_G > \underline{\xi}$ to guarantee the same minimum level of stability as in country I, $\theta^* \leq \theta^*_I(\underline{\xi}) = \theta^*_G(\underline{\xi}_G)$.

Capital Structure and Recovery Value

I now connect the results on recovery value with the firm's capital structure. Note that for given contract (r, k) and funding liquidity ψ the value of liquidity ratio $\xi = \frac{\psi}{\delta r}$ is pinned down by capital structure δ. As a corollary of Propositions 2.3.2 and 2.3.4, I obtain a result already seen in Schilling (2015) for the special case of no bankruptcy costs $b = 1, a = 0$.

Corollary 2.3.3 (Probability of runs and debt)**.** *Fix contract (r, k), parameters of recovery value (a, b), funding liquidity ψ and let noise vanish.*
i) If recovery value function is linear $\gamma(n, \xi(\delta)) = \frac{n}{\xi(\delta)}a$ with $b = 0$, $a > 0$, the probability of a run monotonically increases in debt.
ii) If recovery value function is affine $\gamma(n, \xi(\delta)) = \frac{n}{\xi(\delta)}a + b$, $b > 0$, the probability of a run

increases in debt for smaller debt values $\delta \in (\frac{\psi}{r}, \delta_*(a,b))$, decreases in debt for large debt values $\delta \in (\delta_*(a,b), 1)$ and is maximized at interior debt ratio

$$\delta_*(a,b) = \frac{\psi}{\xi^*(a,b)r} \in \left(\frac{\psi}{r}, 1\right)$$

The probability of a run is locally minimized at debt ratios $\delta = 1$ (full debt financing) and $\delta = \frac{\psi}{r}$ (no proneness to runs).

Applying the result on comparative statics of the maximizer $\xi^*(a,b)$ in recovery value, by Proposition 2.3.4, I obtain

Proposition 2.3.5. *Fix (r, k, a, b, ψ) and let noise vanish. If recovery value is affine $\gamma(n, \xi(\delta)) = \frac{n}{\xi(\delta)}a + b$, $b > 0$, the debt ratio which maximizes the probability of a run $\delta_*(a,b)$ increases in both recovery value determining constants a and b.*

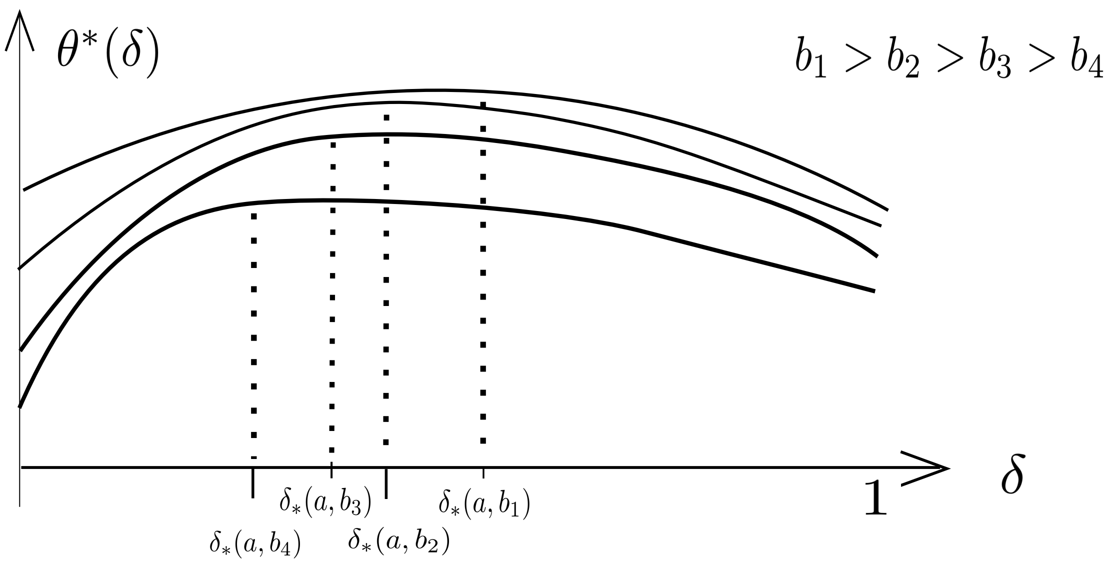


Figure 2.8: For $b > 0$, the trigger and hence probability of runs is a hump-shaped function of debt ratio δ and takes its unique interior maximum in δ_*. Note that by Proposition 2.3.3, for every given debt ratio δ the trigger monotonically increases in parameter b of recovery value. Thus, the curve $\theta^*_{b_1}(\delta)$ for instance lies above curve $\theta^*_{b_2}(\delta)$.

Combining Proposition 2.3.5 with Proposition 2.3.3, we see very nicely that for slope parameter $a \geq 0$ an increase in intercept of recovery value function in a first effect increases the probability of runs at *every* liquidity ratio and hence for given funding liquidity at *every* debt ratio δ. In a second effect, the increase in intercept shifts the maximizing debt ratio δ^* upwards.
If the intercept is zero, the probability of runs is monotone. However, increases in slope parameter also increase the probability of runs at *every* debt ratio as seen in Figure 2.7.

2.3.5 Intuition

By Proposition 2.3.2, a strictly positive intercept of recovery value $b > 0$ is responsible for the occurrence of a non-monotonicity in the probability of a run. This result is due to a change in the game structure when recovery value has zero intercept $b = 0$ compared to a game where recovery value has intercept $b > 0$:
Both games for $b > 0$ and $b = 0$ have in common that conditional on no run $n \leq \xi$ the incentive to withdraw versus roll over is constant in the proportion of other agents who withdraw (aggregate action n), see equation (2.9) and figure (2.1). The game structures differ when conditioning on the occurrence of a run. For $b > 0$, conditional on a run occurring, the payoff difference from withdrawing versus rolling over depends on and strictly decreases in the aggregate action n, $(\gamma\xi/n)\, r = (a + b\frac{\xi}{n})\, r$. This holds since the payoff from rolling over is fixed at zero if a run has happened. For a recovery value function with $b = 0$, the payoff difference from withdrawing from the firm versus rolling over conditional on a run is constant i.e. independent of the aggregate action n: Conditional on a run, a withdrawing agent receives $(\gamma\xi/n)\, r = a\, r$.

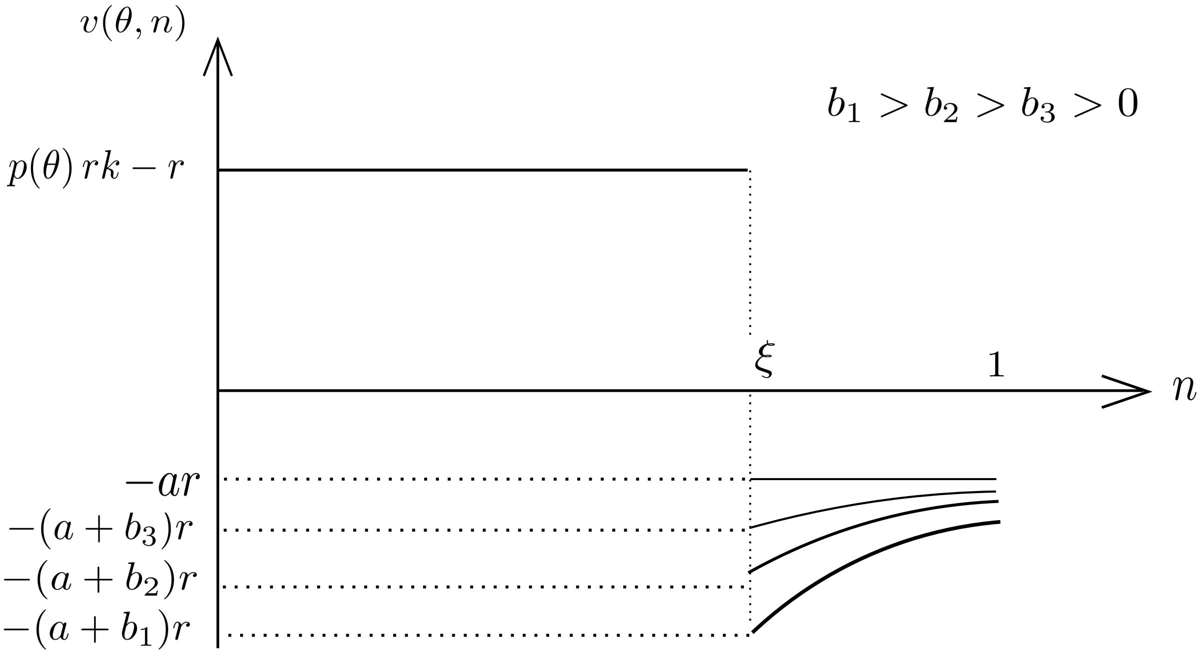


Figure 2.9: Payoff difference function $v(\theta, n)$ from equation (2.9) plotted for varying values of b. For the special case $b = 0$, function v becomes a step function and is particularly constant in n for $n \geq \xi$ which gives rise to a game structure with global strategic complements in actions.

The intercept b of recovery value controls how fast the incentive to withdraw decreases relative to rolling over since it influences the slope of the payoff difference function $v(\theta, n)$ in n as well as the values of v in $n = \xi$ and $n = 1$. The smaller b, the more clinched and flatter the curve v for $n > \xi$ and the slower the change of relative incentives, see Figure 2.9. Further, by Proposition 2.3.4, the smaller b, the larger maximizer ξ^* and hence the larger the range of liquidity values $(0, \xi^*)$

for which the probability of runs increases in liquidity ratio and decreases in liquidity mismatch. As b goes to zero, the curve v for $n > \xi$ approaches the constant $-ar$ and the change of v in n goes to zero.

From a theory perspective, the game with recovery value $b > 0$ is such that there is one-sided strategic substitutability between actions (see Goldstein and Pauzner (2005); Karp et al. (2007); Schilling (2015)) conditional on a run. Further, there is (weak) one-sided strategic complementarity between actions conditional on no run independently of the size of b. In Figure 2.1, we see for $b > 0$, conditional on a run i.e. for $n > \xi$ the payoff difference function $v(n)$ jumps from positive constant $p(\theta)kr - r > 0$ to the negative value $-(a+b)r$ and becomes upward sloping in n.
The intercept of recovery value b controls the extent of strategic substitutability between actions through the slope of v and by this controls the non-monotonicity. As the intercept of recovery value b goes to zero, strategic substitutability between actions vanishes since the payoff difference becomes constant in n while strategic complementarity between actions conditional on no run remains. Actions become global strategic complements. As $b \to 0$ the maximizer ξ^* vanishes and the probability of a run becomes monotone decreasing in liquidity ratio. Thus, for $b \to 0$ I recover the results by Morris and Shin (2009) and Rochet and Vives (2004) which were obtained in games exhibiting global strategic complementarity between actions. [34]

Schilling (2015) suggests that partial repay of deposits to debt investors in case of a run and the corresponding one-sided strategic substitutability in actions is responsible for the arise of the non-monotonicity of the probability of runs in debt. In this paper I show, monotonicity can be achieved under partial repay as long as the recovery value function is such that one-sided strategic substitutability in actions vanishes and actions are global strategic complements. We obtain monotonicity under partial repay as soon as recovery value solely depends (linearly) on size of the run and has no intercept.
In an economic context these results imply, as soon as we believe that recovery values after runs are to some extent determined by a constant independent of the size of the run (intercept $b > 0$), we have to deal with non-monotonic probabilities of runs not only in liquidity ratio but also in debt.

2.3.6 Welfare

In this subsection I analyze, how parameters of recovery value impact debt investors ex ante utility (consumer welfare) inferred from a contract (r, k).

Proposition 2.3.6 (Welfare in Recovery Value)**.** *Let noise vanish. For every contract* (r, k) *and every slope* $a \geq 0$ *there exists a bound for liquidity ratio* $\underline{\xi}_b$ *such that debt investors' utility from*

[34] Note that $b = 0$ requires $a > 0$ since for fixed ξ I demand recovery value γ to be strictly positive for all $n \in [\xi, 1]$.

the contract strictly increases in recovery value intercept b when liquidity ratio exceeds the bound $\xi \geq \underline{\xi}_b$.
For every contract (r, k) and every intercept $b \geq 0$ there exists a bound for liquidity ratio $\underline{\xi}_a$ such that debt investors' utility from the contract strictly increases in recovery value slope parameter a when liquidity ratio exceeds the bound $\xi \geq \underline{\xi}_a$.

In general, greater recovery values do not necessarily improve debt investors' welfare from contracts. Intuitively, this is since recovery values larger in intercept b or slope parameter a make the event of a run more likely, see Proposition 2.3.3. Conditional on a run, however, greater recovery values benefit investors. When determining how recovery values impact welfare, investors trade off higher probability of no run under low recovery values versus higher payoffs conditional on a run under higher recovery values. As the analysis shows, the effect is unambiguous only if liquidity ratios are sufficiently high since in this case the probability of a run becomes close to constant in recovery values.

2.4 Conclusion

This paper is to the best of my knowledge the first to analyze the impact of size and composition of endogenous recovery values after bankruptcy on stability of financial firms against runs by debt investors. When financial firms invest in illiquid, long-term assets and finance by liquid, uninsured debt, the potential of a liquidity squeeze arises: When tomorrow too many investors prematurely demand back their deposits the firm needs to transform illiquid assets to cash quickly. If the number of withdrawing agents is too large, due to asset illiquidity the firm cannot satisfy all claims and goes into default (run). If the firm defaults, agents who did not claim their deposit receive zero. Knowing this in advance, uninsured debt investors face a coordination problem. Invoking the theory of global games, I derive a unique equilibrium of the game which allows us to study ex ante probability of runs as a function of the primitives such as capital structure, liquidity mismatch and composition of recovery value.

After a run, the firm goes into bankruptcy for liquidation or reorganization. This process is in general associated with costs. Therefore, not asset liquidity but the recovered value after bankruptcy is distributed back to debt investors. I model recovery value as an affine function of the endogenous, random size of the run (slope) plus a size independent constant (intercept). While the intercept is associated with fixed costs caused by the bankruptcy proceedings, the slope can be interpreted as an intervention by a lender of last resort in the course of a run on the firm. As main contribution of the paper, I show that high recovery values achieved by cost efficient bankruptcy proceedings or generous government interventions are never desirable from a stability perspective and only sometimes desirable from a consumer perspective. I show, the probability of runs increases in both, slope and intercept of recovery value. Thus, larger recovery values are detrimental to firm stability since it increases incentives to run on the firm. The composition of recovery values effects run probabilities differently. The presence of the intercept makes run probabilities hump-shaped in debt and liquidity ratio, where the non-monotonicity alters in the intercept. With zero intercept, run probabilities are monotone as in Morris and Shin (2009) and Rochet and Vives (2004). Also, greater recovery values in general do not lead to higher welfare to debt investors unless liquidity ratio is sufficiently high, since recovery values increase run probabilities. As a consequence of the non-monotonicity, drops in funding liquidity or changes in capital structure can both decrease or increase the run probability depending on national differences in recovery value after bankruptcy. Countries with a more generously intervening lender of last resort need to impose tighter capital and liquidity regulation to guarantee same stability levels as countries where a lender of last resort intervenes more restrained. These findings are interesting since agreements on supranational capital and liquidity regulation (Basel 3) do not take into account differences in national bankruptcy proceedings and costs.

The most crucial constraint of this paper is, that funding liquidity ψ is assumed to be exogenous and common knowledge. In real world, liquidity varies according to macroeconomic parameters and asymmetric information. Further, slope and intercept of recovery value are assumed to be common knowledge. Knowledge of the intercept can be justified since several empirical studies provide estimates on recovery values such as Thorburn (2000), Franks and Torous (1994), Acharya et al. (2003) and Bris et al. (2006). Information about ELA, in connection to the slope, is more difficult to obtain.[35] I model recovery values as affine function of the size of the run. This choice can be seen as first order Taylor approximation of a more complex recovery value function.

[35]Bank of Cyprus and Laiki Bank obtained a combined volume of around 11 bn euro in 2013, see Attalides et al. (2015). To obtain the slope parameter, knowledge of the volume of withdrawn funds in the course of the runs on both banks and liquidity of assets posted as collateral to obtain ELA is necessary.

2.5 Appendix

Appendix A: Existence and Uniqueness of Equilibrium

Proof. [Proposition 2.3.1] The existence and uniqueness proof of a trigger equilibrium and the proof that a non-threshold equilibrium cannot exist is as in Goldstein and Pauzner (2005) with $\lambda = 0$ and $u(\cdot) = id$. Uniqueness of a threshold equilibrium alternatively holds due to Lemma 2.3 in Morris and Shin (2001) by the following properties: i) The payoff difference function v from equation (2.9) satisfies single-crossing in the aggregate action n (figure (2.1), ii) the monotone likelihood ratio property holds for the uniform distribution of noise, iii) state monotonicity holds, the function $v(\theta, n)$ is monotone in θ, iv) there is limit dominance, either action can be dominant if the state realizes sufficiently high or low, v) the expected payoff difference is continuous in the signal θ_i and vi) it can be shown that there exists a unique signal at which the expected payoff difference is zero.

I give a short intuition here, why a unique trigger equilibrium exists for the general recovery value function $\gamma(n, \xi)$ with $b \geq 0$: Given that all other investors play a trigger strategy around signal θ^*, a trigger equilibrium exists if a single investor also finds it optimal to withdraw for signals $\theta_i < \theta^*$ and to roll over for signals $\theta_i > \theta^*$. That is, we demand (a) $D(\theta_i, \theta^*) < 0$ for $\theta_i < \theta^*$ and (b) $D(\theta_i, \theta^*) > 0$ for $\theta_i > \theta^*$. Continuity of the integral $D(\theta_i, \theta^*)$ in signal θ_i holds by Lemma A1 (i) in Goldstein and Pauzner (2005) and ensures indifference in $\theta_i = \theta^*$, $D(\theta^*, \theta^*) = 0$ if (a) and (b) hold. Existence of a signal which satisfies $D(\theta^*, \theta^*) = 0$ follows by the existence of dominance regions and continuity of $D(\theta^*, \theta^*)$ in θ^* by Lemma A1 (ii) in Goldstein and Pauzner (2005): If the state realizes high enough in the upper dominance region and ε is small, the investor observes a very high signal such that rolling over is optimal $D(\theta_i, n) > 0$ independently of n, similarly, if the state realizes low enough in the lower dominance region, the investor observes a very low signal such that withdrawing is dominant $D(\theta_i, n) < 0$. Uniqueness of a signal satisfying $D(\theta^*, \theta^*) = 0$ holds since by Lemma A1 (iii) in Goldstein and Pauzner (2005), $D(\theta^*, \theta^*)$ strictly increases in θ^* as long as signal θ^* lies below $\bar{\theta} + \varepsilon$ since the probability function $p(\cdot)$ strictly increases in the state for states below the bound to the upper dominance region. Uniqueness follows since for signals above $\bar{\theta} + \varepsilon$ the definition of the upper dominance region yields $D(\theta_i, \theta^*) > 0$. Therefore, a unique candidate for a threshold equilibrium exists. To show that this candidate also satisfies (a) and (b), Goldstein and Pauzner (2005) decompose the intervals $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and $[\theta^* - \varepsilon, \theta^* + \varepsilon]$, use $D(\theta^*, \theta^*) = 0$ and the single crossing property of $v(\theta, n(\theta, \theta^*))$ in θ, see (A8) and (A9) in their proof to Theorem 1 B.
The proof why a non-threshold equilibrium cannot exist is less intuitive, and fully given in Goldstein and Pauzner (2005) proof of Theorem 1 C. We can apply Theorem 1 C since the necessary characteristics of the functions for the proof to hold remain valid: due to $\gamma(n, \xi) > 0$ for all $n \in (\xi, 1]$, the payoff difference function $v(\theta, n)$ satisfies single-crossing and is monotone in n when it is nonnegative for every $b \geq 0$. Further, v remains strictly negative for all $n > \xi$.

A difference to the model in Goldstein and Pauzner (2005) is that here the function v jumps in $n = \xi$ from $p(\theta)rk - r$ to $-(a+b)r$. This however does not impact continuity of the integral over v. □

Appendix B: Comparative Statics

Proof. [Proposition 2.3.2] Let $n(\theta, \theta^*)$ the measure of agents demanding early withdrawal in period 1 when all agents use trigger θ^* and the state of the world is θ. The payoff indifference equality which implicitly determines the trigger $\theta^*(r, \delta)$ as a function of the firm's primitives $(r, k, \delta, \psi, \gamma_{a,b})$ away from the limit is given by $D(\theta^*, \theta^*) = 0$. By the proof of Theorem 1 and changing variables from θ to n using (2.11) I obtain

$$D(\theta^*, \theta^*) = \frac{1}{2\varepsilon} \int_{\theta^*-\varepsilon}^{\theta^*+\varepsilon} v(\theta, n(\theta, \theta^*))\, d\theta = \int_0^1 v(\theta(n, \theta^*), n)\, dn \tag{2.19}$$

where

$$\theta(n, \theta^*) = \theta^* + \varepsilon\,(1 - 2n) \tag{2.20}$$

is the inverse of $n(\theta, \theta^*)$ for $\theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon]$. Hence,

$$0 = D(\theta^*, \theta^*) = \int_0^1 v(\theta(n, \theta^*), n)\, dn \tag{2.21}$$

Plugging in for the function v from (2.9) I obtain the payoff indifference equation

$$0 = -\xi r \int_\xi^1 \frac{\gamma(n, \xi)}{n}\, dn + \int_0^\xi \left(p(\theta(n, \theta^*))\, kr - r\right)\, dn \tag{2.22}$$

Set

$$\hat{f}(\theta^*, \xi) = -\xi r \int_\xi^1 \frac{\gamma(n, \xi)}{n}\, dn + \int_0^\xi p(\theta(n, \theta^*))\, kr - r\, dn \tag{2.23}$$

We have

$$\begin{aligned} \frac{\partial}{\partial \theta^*} \hat{f}(\theta^*, \xi) &= kr \int_0^\xi p'(\theta(n, \theta^*)) \frac{\partial}{\partial \theta^*} \theta(n, \theta^*)\, dn \\ &= kr \int_0^\xi p'(\theta(n, \theta^*))\, dn > 0 \end{aligned} \tag{2.24}$$

since $\frac{\partial}{\partial \theta^*}\theta(n, \theta^*) = 1$. For $\varepsilon \to 0$ we have $\theta(n, \theta^*) \to \theta^*$. Since $p(\cdot)$ is continuous and defined on a compact interval, $p'(\cdot)$ is bounded. Thus, with Lebesgue's Dominated Convergence Theorem,

$$\frac{\partial}{\partial \theta^*} \hat{f}(\theta^*, \xi) \to \xi\, p'(\theta^*)\, kr \tag{2.25}$$

Comparative Statics of Trigger in liquidity ratio

To obtain $\frac{\partial \theta^*}{\partial \xi}$, I use the Implicit Function Theorem and need to calculate $\frac{\partial \hat{f}}{\partial \xi}$. Then,

$$\frac{\partial \theta^*}{\partial \xi} = -\frac{\frac{\partial \hat{f}}{\partial \xi}}{\frac{\partial \hat{f}}{\partial \theta^*}} \tag{2.26}$$

For $\gamma(n,\xi) > 0$ for all $n \in [0,1]$, $\xi \in (0,1)$ using Leibniz' rule

$$\frac{\partial}{\partial \xi}\hat{f}(\theta^*,\xi) = -r\left(1 - \gamma(\xi,\xi) + \int_{\xi}^{1} \frac{\gamma(n,\xi) + \xi\left(\frac{\partial}{\partial \xi}\gamma(n,\xi)\right)}{n} dn\right) \tag{2.27}$$

$$+ p(\theta(\xi,\theta^*))\, kr \tag{2.28}$$

As $\varepsilon \to 0$, we have $\theta(n,\theta^*) \to \theta^*$ and thus $p(\theta(\xi,\theta^*)) \to p(\theta^*)$ independently of ξ. Further, $p(\theta^*)kr \geq r$ by definition of the lower dominance region and since $\theta^* \geq \underline{\theta}$ for $\varepsilon \to 0$. Hence, a sufficient condition for $\lim_{\varepsilon\to 0} \frac{\partial}{\partial \xi}\hat{f}(\theta^*,\xi) > 0$ for all $\xi \in (0,1)$ is

$$\int_{\xi}^{1} \frac{\gamma(n,\xi) + \xi\left(\frac{\partial}{\partial \xi}\gamma(n,\xi)\right)}{n} dn < \gamma(\xi,\xi) \qquad \text{for all } \xi \in (0,1) \tag{2.29}$$

Since $\lim_{\xi\to 1} |\gamma(n,\xi)| \leq |a|n + b \leq |a| + b < \infty$, $a \in \mathbb{R}$, $b > 0$ and $\lim_{\xi\to 1} |\frac{\partial}{\partial \xi}\gamma(n,\xi)| = n|a| \leq |a| < \infty$ and $n \in [\xi,1]$, the integrand on the left hand side of (2.29) is bounded. Hence, using the Intermediate Value Theorem for integrals, we see that for $\xi \to 1$

$$\lim_{\xi\to 1}\int_{\xi}^{1} \frac{\gamma(n,\xi) + \xi\left(\frac{\partial}{\partial \xi}\gamma(n,\xi)\right)}{n} dn = 0 < \lim_{\xi\to 1}\gamma(\xi,\xi) \tag{2.30}$$

since $\gamma(n,\xi)$ is strictly positive for all $n \in [\xi,1]$. Hence, for $\xi \to 1, \varepsilon \to 0$ we have $\frac{\partial \hat{f}}{\partial \xi} > 0$ and therefore $\frac{\partial \theta^*}{\partial \xi} < 0$ for all strictly positive, continuously differentiable recovery value functions $\gamma(n,\xi)$ since $\frac{\partial \hat{f}}{\partial \theta^*} > 0$ for all ξ by (2.24).

Plugging in for the function $\gamma(n,\xi) = \frac{n}{\xi}a + b$ I obtain

$$\frac{\partial}{\partial \xi}\hat{f}(\theta^*,\xi) = -r\left(1 - (a+b) + b\int_{\xi}^{1}\frac{1}{n}dn\right) + p(\theta(\xi,\theta^*))\, kr$$

$$= -r\,(1 - (a+b) - b\ln(\xi)) + p(\theta(\xi,\theta^*))\, kr \tag{2.31}$$

and taking the limit noise to zero

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi) = -r\ (1 - (a+b) - b\ln(\xi)) + p(\theta^*)\, kr \tag{2.32}$$

The sufficient condition for $\lim_{\varepsilon \to 0} \frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi) > 0$ for all $\xi \in (0,1)$ becomes

$$-b\ln(\xi) < a + b \qquad \text{for all } \xi \in (0,1) \tag{2.33}$$

As the intercept of recovery value b goes to 0, the sufficient condition becomes

$$0 < a \tag{2.34}$$

Hence, for $\gamma(n,\xi) = \frac{n}{\xi}a$, $a > 0$ we have $\lim_{\varepsilon \to 0} \frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi) > 0$ for all $\xi \in (0,1)$ and thus $\frac{\partial \theta^*}{\partial \xi} < 0$.

For any arbitrary small $b > 0$, there exists a $\xi^*(b,a)$ such that in (2.32) $\frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi) < 0$ and $\frac{\partial \theta^*}{\partial \xi} > 0$ for $\xi \in (0, \xi^*(b,a))$ since $\ln(\xi)$ goes to minus infinity as $\xi \to 0$. In addition, for $b > 0$ the function $\frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi)$ is strictly increasing and continuous in ξ. Thus, by (2.29) and the Intermediate Value Theorem $\frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi)$ satisfies single-crossing in ξ which implies that for every $b > 0$, $\xi^*(b,a)$ is the unique maximizer of the function $\theta^*(\xi)$. □

Proof. [Lemma 2.3.3] I show that the probability of runs strictly increases in recovery value parameters a and b: Plugging $\gamma(n,\xi)$ into (2.23), we have

$$\hat{f}(\theta^*, a, b) = -r\ (\xi + (1-\xi)a - \xi b \ln(\xi)) + kr \int_0^{\xi} p(\theta(n, \theta^*))\, dn \tag{2.35}$$

And thus for $\xi \in (0,1)$

$$\frac{\partial \hat{f}(\theta^*, a, b)}{\partial a} = -(1-\xi)\, r < 0 \tag{2.36}$$

$$\frac{\partial \hat{f}(\theta^*, a, b)}{\partial b} = \xi \ln(\xi)\, r < 0 \tag{2.37}$$

Thus, with (2.25) and the Implicit Function Theorem we have for $\xi \in (0,1)$

$$\lim_{\varepsilon \to 0} \frac{\partial \theta^*}{\partial b} = -\frac{r\ln(\xi)}{krp'(\theta^*)} = -\frac{\ln(\xi)}{k\,p'(\theta^*)} > 0 \tag{2.38}$$

$$\lim_{\varepsilon \to 0} \frac{\partial \theta^*}{\partial a} = \frac{r(1-\xi)}{\xi kr\, p'(\theta^*)} = \frac{(1-\xi)}{\xi k\, p'(\theta^*)} > 0 \tag{2.39}$$

□

Proof. [Proposition 2.3.4] Let $\gamma(n,\xi) = \frac{n}{\xi}a + b > 0$ with $b > 0$. We know by the single crossing property of function $\lim_{\varepsilon \to 0} \frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi)$ in ξ for $b > 0$, shown in proof of Proposition 2.3.2, that at the limit $\varepsilon \to 0$ for given $b > 0$, $a \in \mathbb{R}, 0 < a + b < 1/H < 1$ the maximizer $\xi^*(b, a)$ of the trigger θ^* exists, is unique and is implicitly defined as the zero of equation (2.32):

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \xi} \hat{f}(\theta^*, \xi) = -r\ (1 - (a + b) - b \ln(\xi)) + p(\theta^*)\, kr = 0$$

Applying the Implicit Function Theorem to equation (2.31) (away from the limit) and taking the limit $\varepsilon \to 0$ gives us the behavior of the maximizer ξ^* in parameters a, b: We have

$$\frac{\partial \xi^*}{\partial b} = -\frac{\frac{\partial}{\partial b}\frac{\partial \hat{f}}{\partial \xi}}{\frac{\partial}{\partial \xi}\frac{\partial \hat{f}}{\partial \xi}}\Big|_{\xi=\xi^*}, \qquad \frac{\partial \xi^*}{\partial a} = -\frac{\frac{\partial}{\partial a}\frac{\partial \hat{f}}{\partial \xi}}{\frac{\partial}{\partial \xi}\frac{\partial \hat{f}}{\partial \xi}}\Big|_{\xi=\xi^*}$$

Here, away from the limit using (2.31) I obtain with (2.20)

$$\begin{aligned}
\frac{\partial}{\partial \xi}\frac{\partial \hat{f}}{\partial \xi}\Big|_{\xi=\xi^*} &= \frac{rb}{\xi^*} + kr\, p'(\theta(n, \theta^*)) \left(\frac{\partial \theta(n, \theta^*)}{\partial n}\Big|_{n=\xi^*} + \frac{\partial \theta(n, \theta^*)}{\partial \theta^*} \frac{\partial \theta^*}{\partial \xi}\Big|_{\xi=\xi^*} \right) \\
&= \frac{rb}{\xi^*} + kr\, p'(\theta(n, \theta^*)) \left(-2\varepsilon + \frac{\partial \theta^*}{\partial \xi}\Big|_{\xi=\xi^*} \right)
\end{aligned}$$

Since ξ^* exists and maximizes θ^* when $\varepsilon \to 0$, we have $\frac{\partial \theta^*}{\partial \xi^*} = 0$ and thus taking the limit

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \xi}\frac{\partial \hat{f}}{\partial \xi}\Big|_{\xi=\xi^*} = \frac{rb}{\xi^*}$$

Further, with (2.38) and (2.39) at the limit

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{\partial}{\partial b}\frac{\partial \hat{f}}{\partial \xi}\Big|_{\xi=\xi^*} &= r\ (1 + \ln(\xi^*)) + kr\ p'(\theta^*) \lim_{\varepsilon \to 0} \frac{\partial \theta^*}{\partial b} \\
&= r\ (1 + \ln(\xi^*)) + kr\ p'(\theta^*) \left(-\frac{\ln(\xi^*)}{kp'(\theta^*)} \right) \\
&= r
\end{aligned}$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\partial}{\partial a}\frac{\partial \hat{f}}{\partial \xi}\Big|_{\xi=\xi^*} &= r + kr\, p'(\theta^*)\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial a}\\
&= r + kr\, p'(\theta^*)\frac{(1-\xi^*)}{\xi^* k\, p'(\theta^*)}\\
&= r\left(1+\frac{1-\xi^*}{\xi^*}\right)\\
&= \frac{r}{\xi^*}
\end{aligned}
$$

Finally,

$$
\lim_{\varepsilon\to 0}\frac{\partial \xi^*}{\partial b} = -\frac{\lim_{\varepsilon\to 0}\frac{\partial}{\partial b}\frac{\partial \hat{f}}{\partial \xi}}{\lim_{\varepsilon\to 0}\frac{\partial}{\partial \xi^*}\frac{\partial \hat{f}}{\partial \xi}} = -\frac{r}{rb/\xi^*} = -\frac{\xi^*}{b} < 0
$$

$$
\lim_{\varepsilon\to 0}\frac{\partial \xi^*}{\partial a} = -\frac{\lim_{\varepsilon\to 0}\frac{\partial}{\partial a}\frac{\partial \hat{f}}{\partial \xi}}{\lim_{\varepsilon\to 0}\frac{\partial}{\partial \xi^*}\frac{\partial \hat{f}}{\partial \xi}} = -\frac{\frac{1}{\xi^*}r}{rb/\xi^*} = -\frac{1}{b} < 0
$$

□

Proof. [Proposition 2.3.3] By equation (2.22), the trigger depends on debt ratio δ and funding liquidity ψ only through liquidity ratio. By its definition (2.5), for every recovery value function $\gamma(n,\xi) = \frac{n}{\xi}a + b$ with a, b such that $\gamma(n,\xi) > 0$ liquidity ratio $\xi = \frac{\psi}{\delta r}$ strictly increases in funding liquidity ψ and decreases in debt.
Fix a, b and (r, k, ψ). If $b = 0$, by Proposition 2.3.2 the trigger monotonically decreases in liquidity ratio ξ an thus for given ψ monotonically increases in debt δ.

If $b > 0$, the trigger maximizing liquidity ratio $\xi^*(a,b) \in (0,1)$ is uniquely pinned down as a function of a, b, r, k. We have $\xi(\delta) < \xi^*(a,b)$ if and only if

$$
\delta > \frac{\psi}{r\xi^*(a,b)} \tag{2.40}
$$

Thus, as δ increases within $[\frac{\psi}{r\xi^*(a,b)}, 1)$, $\xi(\delta) = \frac{\psi}{\delta r}$ decreases and moves away from $\xi^*(a,b)$. Since $\xi^*(a,b)$ uniquely maximizes the probability of a run, the probability of a run has to decrease in δ. Equivalently, we have $\xi(\delta) > \xi^*(a,b)$ if and only if

$$
\delta < \frac{\psi}{r\xi^*(a,b)} \tag{2.41}
$$

As δ increases within $(\frac{\psi}{l}, \frac{\psi}{r\xi^*(a,b)})$, $\xi(\delta)$ decreases and approaches $\xi^*(a,b)$ from above. Thus, the probability of a run increases in δ for δ in $(\frac{\psi}{l}, \frac{\psi}{r\xi^*(a,b)})$. □

Appendix C: Welfare

Proof. [Proposition 2.3.6] Debt investors' ex ante utility from the contract equals

$$EU(\xi) = \int_0^{\theta_b(\xi)} n(\theta, \theta^*(\xi)) \, \frac{\gamma(n(\theta, \theta^*), \xi)\xi}{n(\theta, \theta^*(\xi))} r + (1 - n(\theta, \theta^*(\xi))) \cdot 0 \, d\theta \tag{2.42}$$

$$+ \int_{\theta_b(\xi)}^{\theta^*(\xi)+\varepsilon} n(\theta, \theta^*(\xi))r + (1 - n(\theta, \theta^*(\xi)))p(\theta) \, kr \, d\theta \tag{2.43}$$

$$+ \int_{\theta^*(\xi)+\varepsilon}^{\overline{\theta}} p(\theta) \, kr \, d\theta + \int_{\overline{\theta}}^{1} kr \, d\theta \tag{2.44}$$

$$= \int_0^{\theta_b(\xi)} \gamma(n(\theta, \theta^*), \xi) \cdot \xi \, r \, d\theta \tag{2.45}$$

$$+ \int_{\theta_b(\xi)}^{\theta^*(\xi)+\varepsilon} n(\theta, \theta^*(\xi))r + (1 - n(\theta, \theta^*(\xi)))p(\theta) \, kr \, d\theta \tag{2.46}$$

$$+ \int_{\theta^*(\xi)+\varepsilon}^{\overline{\theta}} p(\theta)kr \, d\theta + (1 - \overline{\theta})kr \tag{2.47}$$

Using Leibniz rule and $\gamma(n(\theta, \theta^*), \xi) = \frac{a}{\xi} n(\theta, \theta^*) + b$, the change in utility due to a change in slope parameter a is

$$\frac{\partial}{\partial a} EU = \frac{\partial}{\partial a} \int_0^{\theta_b(\xi)} \left(\frac{a}{\xi} n(\theta, \theta^*) + b \right) \cdot \xi \, r \, d\theta \tag{2.48}$$

$$+ \frac{\partial}{\partial a} \int_{\theta_b(\xi)}^{\theta^*(\xi)+\varepsilon} n(\theta, \theta^*(\xi))r + (1 - n(\theta, \theta^*(\xi)))p(\theta) \, kr \, d\theta \tag{2.49}$$

$$= \int_0^{\theta_b(\xi)} \frac{\partial}{\partial a} \left(\frac{a}{\xi} n(\theta, \theta^*) + b \right) \cdot \xi \, r \, d\theta \tag{2.50}$$

$$+ \frac{\partial \theta_b}{\partial a} (a + b) \, \xi r \tag{2.51}$$

$$+ \int_{\theta_b(\xi)}^{\theta^*(\xi)+\varepsilon} \frac{\partial}{\partial a} \left[n(\theta, \theta^*(\xi))r + (1 - n(\theta, \theta^*(\xi)))p(\theta) \, kr \right] \, d\theta \tag{2.52}$$

$$- \frac{\partial \theta_b}{\partial a} \left[\xi r + (1 - n(\theta_b, \theta^*))p(\theta_b)kr \right] \tag{2.53}$$

$$= \int_0^{\theta_b(\xi)} \left(\frac{1}{\xi} n(\theta, \theta^*) + \frac{a}{\xi} \frac{\partial n(\theta, \theta^*)}{\partial \theta^*} \frac{\partial \theta^*}{\partial a} \right) \cdot \xi \, r \, d\theta \tag{2.54}$$

$$+ \frac{\partial \theta_b}{\partial a} (a + b) \, \xi r - \frac{\partial \theta_b}{\partial a} \left[\xi r + (1 - \xi)p(\theta_b)kr \right] \tag{2.55}$$

$$+ \int_{\theta_b(\xi)}^{\theta^*(\xi)+\varepsilon} \frac{\partial n(\theta, \theta^*)}{\partial \theta^*} \frac{\partial \theta^*}{\partial a} \left[r - p(\theta) \, kr \right] \, d\theta \tag{2.56}$$

since $n(\theta_b, \theta^*) = \xi$. Next, due to definition of θ_b, since $n(\theta^* - \varepsilon, \theta^*) = 1$, $n(\theta^* + \varepsilon, \theta^*) = 0$, and $\xi \in (0, 1)$ $\theta_b > \theta^* - \varepsilon$. Also, for all $\theta \leq \theta^* - \varepsilon$ we have $n(\theta, \theta^*) = 1$ and hence $\frac{\partial n(\theta, \theta^*)}{\partial \theta^*} = 0$. Further $\frac{\partial n(\theta, \theta^*)}{\partial \theta^*} = \frac{1}{2\varepsilon}$ for $\theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon]$ by (2.11). The integral (2.54) therefore simplifies to

$$\int_0^{\theta^*-\varepsilon} \left(\frac{1}{\xi}n(\theta,\theta^*) + \frac{a}{\xi}\frac{\partial n(\theta,\theta^*)}{\partial\theta^*}\frac{\partial\theta^*}{\partial a}\right)\cdot \xi\, r\, d\theta \tag{2.57}$$

$$+\int_{\theta^*-\varepsilon}^{\theta_b(\xi)} \left(\frac{1}{\xi}n(\theta,\theta^*) + \frac{a}{\xi}\frac{\partial n(\theta,\theta^*)}{\partial\theta^*}\frac{\partial\theta^*}{\partial a}\right)\cdot \xi\, r\, d\theta \tag{2.58}$$

$$= (\theta^*-\varepsilon)r + \int_{\theta^*-\varepsilon}^{\theta_b(\xi)} \left(\frac{1}{\xi}n(\theta,\theta^*) + \frac{a}{\xi}\frac{1}{2\varepsilon}\frac{\partial\theta^*}{\partial a}\right)\cdot \xi\, r\, d\theta \tag{2.59}$$

Changing variables to n with (2.11), (2.59) becomes

$$(\theta^*-\varepsilon)r + \int_\xi^1 \left(\frac{1}{\xi}n\, 2\varepsilon + \frac{a}{\xi}\frac{\partial\theta^*}{\partial a}\right)\cdot \xi\, r\, dn$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \theta^* r + \int_\xi^1 a\frac{\partial\theta^*}{\partial a}\cdot r\, dn$$

For integral (2.56), changing variables to n and then applying the PIE (2.22) in a second step, the integral becomes

$$\frac{\partial\theta^*}{\partial a}\int_0^\xi [r - p(\theta(n,\theta^*))\, kr]\; dn$$
$$= -\frac{\partial\theta^*}{\partial a}\int_\xi^1 \frac{\gamma(n,\xi)\xi}{n}\, r\; dn = -\frac{\partial\theta^*}{\partial a}\int_\xi^1 (a + \frac{b}{n}\xi)\, r\; dn$$

where I can draw out $\frac{\partial\theta^*}{\partial a}$ since the trigger only depends on the primitives of the game, not the random state and $\theta(n,\theta^*)$ is as in (2.20). Altogether, with $\theta_b \to \theta^*$ for $\varepsilon \to 0$ and (2.16)

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial a}EU = \theta^* r - \lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial a}\int_\xi^1 \frac{b}{n}\xi\, r\; dn \tag{2.60}$$

$$+\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial a}\left[(a+b-1)\,\xi r - (1-\xi)p(\theta^*)kr\right] \tag{2.61}$$

$$= \theta^* r + \lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial a}\left[(a+b-1+b\ln(\xi))\,\xi r - (1-\xi)p(\theta^*)kr\right] \tag{2.62}$$

We have $\theta^* r \geq \underline{\theta} r > 0$. Since $a+b<1$ and $\ln(\xi)<0$ for $\xi\in(0,1)$, the bracket is always strictly negative. By Proposition 2.3.3, $\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial a} > 0$ for $\xi\in(0,1)$. However, by (2.39) as $\xi\to 1$,

$$\lim_{\xi\to 1}\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial a} = \lim_{\xi\to 1}\frac{(1-\xi)}{\xi k\, p'(\theta^*)} = 0 \tag{2.63}$$

where $\lim_{\xi\to 1} p'(\theta^*)$ is bounded from below, in particular $\lim_{\xi\to 1} p'(\theta^*) \geq p'(\underline{\theta}) > 0$ since $p(\cdot)$ is strictly increasing for all $\theta \in [0,1]$, $p'(\cdot)$ is continuous and $\lim_{\xi\to 1}\theta^* > \underline{\theta} > 0$. In addition, the bracket is bounded for $\xi \to 1$ as $p(\cdot)$ is bounded. Therefore,

$$\lim_{\xi\to 1}\lim_{\varepsilon\to 0}\frac{\partial}{\partial a}EU = \lim_{\xi\to 1}\theta^* r > \underline{\theta} r > 0 \tag{2.64}$$

By continuity of $\lim_{\varepsilon\to 0}\frac{\partial}{\partial a}EU$, there exists $\underline{\xi}_a$ such that for all $\xi \geq \underline{\xi}_a$, we have $\lim_{\varepsilon\to 0}\frac{\partial}{\partial a}EU \geq 0$.

By an identical argument,

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial b}EU = \theta^*\xi r - \lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial b}\int_{\xi}^{1}\frac{b}{n}\xi\, r\; dn \tag{2.65}$$

$$+ \lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial b}\left[(a+b-1)\,\xi r - (1-\xi)p(\theta^*)kr\right] \tag{2.66}$$

$$= \theta^*\xi r + \lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial b}\left[(a+b-1+b\ln(\xi))\,\xi r - (1-\xi)p(\theta^*)kr\right] \tag{2.67}$$

Note that the bracket equals the bracket in (2.62) and is hence negative. By Proposition 2.3.3, $\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial b} > 0$ for $\xi \in (0,1)$ but by (2.38) as $\xi \to 1$,

$$\lim_{\xi\to 1}\lim_{\varepsilon\to 0}\frac{\partial\theta^*}{\partial b} = \lim_{\xi\to 1} -\frac{\ln(\xi)}{k\,p'(\theta^*)} = 0 \tag{2.68}$$

where still $\lim_{\xi\to 1}p'(\theta^*)$ is bounded from below, $\lim_{\xi\to 1}p'(\theta^*) \geq p'(\underline{\theta}) > 0$. Therefore,

$$\lim_{\xi\to 1}\lim_{\varepsilon\to 0}\frac{\partial}{\partial b}EU = \lim_{\xi\to 1}\theta^*\xi r = \lim_{\xi\to 1}\theta^* r > \underline{\theta} r > 0 \tag{2.69}$$

By continuity of $\lim_{\varepsilon\to 0}\frac{\partial}{\partial b}EU$, there exists $\underline{\xi}_b$ such that for all $\xi \geq \underline{\xi}_b$, we have $\lim_{\varepsilon\to 0}\frac{\partial}{\partial b}EU \geq 0$. □

Chapter 3

Redistributional Effects of Health Insurance in Germany: Private and Public Insurance, Premia and Contribution Rates

3.1 Motivation

In a model of obligatory health insurance, we study redistributional effects when public and private insurances coexist and compete for profitable customers. In Germany, systemic competition between public and private insurance is regulated by requiring public insurance to operate cost covering[1] and to finance health expenditures via an only income dependent contribution[2] with a price cap. Private insurance maximizes profits and premia may depend on health risk. Public insurance is available to every citizen. Sufficiently wealthy customers may opt out of the public system and insure privately.

The opportunity to opt out of the redistributive, public system was originally granted to enhance consumer choice and stimulate competition between insurers (Wissenschaftlicher Beirat beim BMF, 2004; Jacobs and Schulze, 2004; Fehr et al., 2006). Thomson and Mossialos (2006) however find that choice of public or private health insurance coverage (systemic competition) creates incentives for private insurers to select risks (cream skimming[3]) and leads to risk segmentation, thereby increasing the financial risk borne by public insurers. In Germany, cream skimming by private insurance is a result of regulation. Solidary public insurance may condition the contribution rate not on health risk but on income only (Pauly, 1984). As a result, healthy

[1] That is she balances budget: Public insurance charges a contribution such that health expenses payable to customers on average equal overall collected contributions.

[2] The current contribution ("Beitragssatz") is 14.6% of income in 2015. The contribution is split equally between employer and employee such that 7.3% of income are payable by the employee.

[3] Selection of customers who in expectation cause a profit to the insurer.

and wealthy individuals opt out attracted by low, private health premia. Since private insurance operates profit maximizing, health premia paid by privately insured customers are lost for the redistributive public system which affects the health contribution charged by the budget balancing public insurance. The public contribution rate vice versa impacts the customers' final choice for a contract and thus the selection of risk by private insurance.

In a model where customers are characterized by health and income, we study how the German opt-out option[4] and the price cap[5] on public health premia affect cream skimming of profitable customers by private insurances and thus health premia of all customers, publicly and privately insured.
We further analyze, how improvements in the health income distribution or increases in correlation between health and income affect redistribution streams between customer groups and health premia when opt-out and price caps exist. Our analysis is motivated by a study by Deaton and Paxson (1998). Using US data from the National Health Interview Survey (NHIS) and the Panel Study of Income Dynamics (PSID), they provide empirical evidence that for more recently born cohorts the correlation between income and health is increased. Health shocks may have a larger impact on future income and social security, vice versa income increasingly affects choice of lifestyle and according risk factors such as smoking, drinking and obesity.

The question how to optimally organize the market for health insurance provision remains relevant in many countries - not only in Germany. Political parties in Germany discuss changes of the health insurance system to a system financed by flat premia ('Kopfpauschale'). This system demands that private and public insurances offer equal health benefits at flat, income and health independent premiums to every agent, see Worz and Busse (2005). Insurances would compete in price for customers and customers may choose between insurances unrestrictedly.
When changing the model to a flat premium system, we analyze the customer groups who win and lose compared to the current German system. Last and most interesting, we analyze optimal health and income dependent premia under a budget balancing constraint. Further, we analyze optimal premiums constrained to depend not on health risk but on income only.

To conduct our analysis, we model three types of agents: A population of customers, a public and a private insurance. Customers are characterized by a two-dimensional random but observable type for health h and income e. Health and income are assumed to be positively correlated. Taking the health-income distribution of types as given, the public insurer endogenously sets the percentage of income she charges as the premium. We assume that the public insurer commits to operate cost-covering instead of maximizing profits. In contrast, the private insurer sets a

[4]The income threshold to opt out ("Versicherungspflichtgrenze") is 56,250 euro in yearly income in 2016.

[5]The income threshold to determine the maximum public premium ("Beitragsbemessungsgrenze") is 50,850 euro in yearly income in 2016.

profit-maximizing premium as a function of the customer's health and income type, taking the public premium as given. We assume both insurance contracts offer equivalent maximum health benefits and hence equally high partial coverage.
A customer decides on her insurance contract after observing her type. Customers hence face no uncertainty about future net income. The motive to insure is imposed by regulation of obligatory health insurance. Also, since customer types are observable by insurances, the model features no asymmetric information and hence no adverse selection problem.

As first main result we demonstrate, under voluntary health insurance no insurance company offers a contract to any potential customer and the market collapses. This result is not driven by adverse selection and is hence fundamentally different from the collapse of insurance markets as described in Rothschild and Stiglitz (1976). Instead, regulation of public insurances to condition public insurance premiums on income only and the budget balancing constraint in combination with continuity and multi-dimensionality of types make it impossible for public insurance to finance the system. Public insurance contributions are based on income but actual health costs depend on customers' health types. Only those agents will insure voluntary whose health costs to the insurance company will exceed benefits receivable from her insurance. As a consequence, running a balanced budget becomes impossible to the public insurance.
Under obligatory insurance, we give a constraint under which a unique public insurance contribution exists charged as percentage of income. In addition, we give a closed form solution of the private premium offered to customers. Existence requires that average income of customers who must insure with public insurance exceeds health costs caused by the entire customer population. The constraint thus reflects the difficulty for public insurance to run a balanced budget in the face of the opt-out opportunity for rich customers. Monopolistic private insurance discriminates between profitable and unprofitable customers, i.e. customers for whom the health premium payable exceeds or undercuts expected health costs. She attracts healthy customers by setting a premium slightly below the public insurance premium and tries to chase away unhealthy customers by setting the maximum premium possible (cream skimming). Given this behavior, a sufficient condition for existence of an equilibrium is that the opt-out threshold is high enough such that the public insurance insures sufficiently many healthy customers and thereby can operate cost-covering. In that case, the public insurance contribution is unique and publicly insured customers pool along the health dimension in the sense that customers with equal income type but varying health type pay the same premium. The public insurance system is solidary (Hinrichs, 1995), publicly insured customers with equal health type but varying income pay different premiums.
Looking at variations of opting out, we demonstrate that increasing the opt-out threshold up to the level of the contribution cap decreases health premiums for all customers, public and private. This is since the private insurance can cream skim customers with earnings between these cutoffs. By equating them, cream skimming is prevented.

As next main result, we show that systematic improvements of the population's health and income distribution in the sense of first-order stochastic dominance, not necessarily lead to decreases in public contribution rates. Contribution may increase due to migration from public to private insurance since private insurance pockets the gains instead of redistributing them.
Increasing correlation between health and income may lead to an increase in public contribution rate. This result is again due to the opt-out threshold. Under higher positive correlation, publicly insured, low income customers on average will cause higher health costs while higher income customers have improved health but opt out. To quantify increases in correlation we use the notion of supermodular stochastic order (Shaked and Shanthikumar, 2007).

Last, we apply our model to study changes in welfare[6] when changing the health system to a flat, health and income independent premium system. If the opt-out threshold is sufficiently high, the current German system yields higher welfare than the premium system. This is, since in the German, income tax resembling system customers with high income pay more than customers with low income which accounts for the concave utility function. Introducing a simple, budget-balanced income redistribution scheme (income tax) into the flat premium system however allows to obtain the same level of welfare as the current German system may achieve.[7] This implies that a change of the current system to a premium system should be accompanied by an appropriate change in income taxation.
Going one step further, we derive the welfare-maximizing pricing scheme which may depend on health and income. The optimal pricing scheme requires every customer to pay the health costs she imposes on the system plus the deviation of her net income from average net income of the population if there was no insurance. By this, the optimal pricing scheme entails a redistribution but not in form of an income tax but by deviation from mean.
Analyzing the welfare-maximizing pricing scheme constrained to depend on income only, we show that the current public redistribution system financed via an income tax fails to be optimal since it does not take into account the correlation between health and income.

Literature

The classic paper by Rothschild and Stiglitz (1976) demonstrates that insurance markets can fail as a consequence of asymmetric information about risk types and adverse selection. In the context of adverse selection, Neudeck and Podczeck (1996) analyze regulation of health insurance markets in a Rothschild Stiglitz type model where agents can opt out of social public insurance and insure privately.
As Neudeck and Podczeck (1996), our paper analyzes competition between public and private

[6]We study utilitarian welfare, where health enters a customer's utility function since health costs and insurance affect her comsumption.

[7]Note that factually in Germany, income taxation is in addition to income dependent health premiums while here in our model we do not model income taxation in addition to income dependent premiums. In Germany, a budget balancing income dependent redistribution scheme is thus already in place.

health insurance under regulation and opt out. Our paper differs, since in our model types are revealed before the choice of insurance is made. The motive for agents to insure is thus not by uncertainty and desire for risk-sharing but by regulation. In addition, types are publicly observable, information is symmetric which excludes adverse selection. In particular, our results on market failure under voluntary insurance and cream skimming under obligatory insurance are not driven by adverse selection but by regulation of the health insurance market.

Our analysis focuses on systemic competition between public and private insurance under regulation. As opposed to Neudeck and Podczeck (1996), our model distinguishes agents not only by their health risk type but also by an income type. In Neudeck and Podczeck (1996) social public insurance is financed by a lump-sum tax on endowments, all agents pay the same flat premium. In our model, agents' premium payment for the same public insurance coverage may vary since agents differ in income types and public insurance charges an income dependent contribution. As a consequence here, not only high risk types are subsidized by low risk types but also healthy very low income types are subsidized by healthy higher income types. The latter is, since our model contains the feature that the maximum compensation payment agents may obtain from their insurance cannot be larger than the health costs they impose on the system. In Neudeck and Podczeck (1996) all agents may opt out of public insurance, while in our model only agents with income above the opt-out threshold have this option. Pooling is thus enforced on low income types by regulation and the obligation to insure.

Our model features cream-skimming by private insurance, but not driven by asymmetric information and flat premiums as in Barros (2003) but by regulation as described in Pauly (1984). In our model, public insurance premiums may only depend on income while private insurance may condition the premium on a customer's risk type.

Kemnitz (2013) studies differences in consumer welfare between an income tax financed and a flat premium financed health insurance system under competition and switching costs. As opposed to our setting, the model does not allow for opt-out and regulated, systemic competition between public and private insurance.

Our analysis of optimal social insurance premiums when individuals differ in productivity and health is related to the work by Blomqvist and Horn (1984), Rochet (1991) and Cremer and Pestieau (1996). Rochet (1991) and Cremer and Pestieau (1996) study welfare under social, income tax financed insurance and budget balancing in a model synthesizing Mirrlees (1971) and Rothschild and Stiglitz (1976). As in our paper, agents have two-dimensional types that affect consumption. As opposed to our model, agents face uncertainty about falling ill and in Rochet (1991) private insurance may be chose in addition to public insurance. In our model, all uncertainty is resolved before customers choose contracts, hence when calculating optimal premiums we also condition on health types.

Besley (1989), Blomqvist and Johansson (1997) and Petretto (1999) study efficiency of systems with coexisting public and private insurance under moral hazard when public insurance is compulsory and agents buy additional private insurance to top up services. While in our paper public and private insurance coexist, insurances compete for customers and are mutually exclusive. Further, our model does not feature moral hazard.

The paper is organized as follows. In Section 2 we give a formal description of the organizational structure of the health insurance market. Section 3 starts by describing the general insurance problem and then proceeds with solving the benchmark case with equal benefits. Thereafter, comparative statics in the primitives and distribution are discussed.

3.2 Model

In the health insurance market a population of customers purchases health insurance contracts from either of two insurances: a private health insurance and a public health insurance.

3.2.1 Population

The population consists of a unit mass continuum of customers. Every customer is characterized by her health type h and her income type e; a high value of h corresponds to a good state of health. Types are distributed according to distribution function $F(h, e)$ with compact and connected support $\mathcal{H}_b \times \mathcal{E}_b = [0, \bar{h}] \times [0, \bar{e}]$. The distribution has a strictly positive, twice differentiable density $f(h, e)$. Health and income types are affiliated,[8] that is, for all points (h_1, e_1) and (h_2, e_2), the density f satisfies

$$f(\max(h_1, h_2), \max(e_1, e_2)) \cdot f(\min(h_1, h_2), \min(e_1, e_2)) \geq f(h_1, e_1)f(h_2, e_2) \tag{3.1}$$

The condition means that large values for health make income more likely to be large than small and vice versa. Associated with a customer's health type h are health costs $c(h)$ where $c(\cdot)$ is a continuous, positive, and strictly decreasing function. Customers' utility function is strictly increasing, strictly concave, and twice continuously differentiable in consumption w, where for an uninsured customer consumption is the difference between income and health costs

$$w = e - c(h). \tag{3.2}$$

Customers' health and income types are observable by all agents in the market. In the course of the paper we refer to high income types as *wealthy* and high health types as *healthy.*

[8] Affiliation is a strong form of positive correlation and widely used in Auction Theory, see Milgrom (1982).

Choice of Insurance

Health insurance is compulsory, every customer *must* choose a contract. A customer has to insure with the public insurance if her income is less than the *opt-out threshold* K_1; otherwise, she is eligible to choose between private and public insurance. Neither insurance is allowed to exclude customers from their services, both insurances have to offer a contract to every eligible customer (open enrollment).

Contracts

A health insurance contract is defined by its payment $p(h, e)$ and the maximum monetary reimbursement of health costs, *benefit level* L. As a consequence, health insurance provides only partial coverage. We assume that both insurances offer the same fixed benefit level L.[9] Denote by $\mathcal{C}(h, e)$ the set of contracts available to a type-(h, e) customer. Upon signing a contract, a customer pays $p(h, e)$ and receives a monetary, health type dependent reimbursement of $\min(c(h), L)$, referred to as *health benefit*. Applying the minimum function allows us to model overinsurance, i.e., the case of $L > c(h)$. A customers *net benefit* from contract $\mathcal{C}(h, e)$ is then given as $\min(c(h), L) - p(h, e)$. Formally, type-(h, e)'s decision problem is to choose $(L, p(h, e))$ such that

$$(L, p(h, e)) \in \underset{(L,p)\in\mathcal{C}(h,e)}{\arg\max}\ u(e + \min(L - c(h), 0) - p(h, e)). \tag{3.3}$$

3.2.2 Public Health Insurance

The public health insurance (PU) charges its customers a fixed percentage α, the *contribution rate*, of their income. Above income threshold K_2, however, the payment to PU remains constant; we refer to K_2 as the *contribution cap*. The public payment is therefore

$$p_{\text{PU}}(e) = \alpha \min(K_2, e). \tag{3.4}$$

and does not explicitly depend on a customer's health type.

PU commits to operate with a balanced budget, i.e., she equates revenues and expenditures. Formally, PU's objective is to set $\alpha \in [0, 1]$ such that

$$\alpha\, \mathbb{E}[\min(K_2, e)|(h, e) \in PU(\alpha)] = \mathbb{E}[\min(L, c(h))|(h, e) \in PU(\alpha)] \tag{3.5}$$

where $PU(\alpha)$ denotes the set of PU's customers, and the expectation is taken with respect to $F(\cdot)$.

[9]See a later section for a relaxation of this assumption.

3.2.3 Private Health Insurance

The private health insurance (PR) charges each customer a payment $p_{\text{PR}}(h,e)$ that may depend on the customer's health type and income. We call the function $p_{\text{PR}}(\cdot)$ PR's payment scheme. PR is by assumption obliged to set a payment below αK_2, $p_{\text{PR}}(h,e) \leq \alpha K_2$, for all health and income types. We refer to a payment $p_{\text{PR}}(h,e)$ satisfying this requirement as *feasible payment* and to $p_{\text{PR}}(\cdot)$ as *feasible payment scheme.* PR aims at maximizing profit, i.e., payments collected less health benefits to pay. Thus, PR's objective is to choose $p_{\text{PR}}(\cdot)$ such that

$$p_{\text{PR}}(\cdot) \in \underset{p(\cdot)\text{ feasible}}{\arg\max}\ \mathbb{E}\big[(p(h,e) - \min(L, c(h)))\ \mathbf{1}_{PR(\alpha)}\ \big] \tag{3.6}$$

where $PR(\alpha)$ denotes the set of PR customers.

3.3 Equilibrium

3.3.1 Timing and Equilibrium Concept

We study the health insurance market as a two-period game. In the first period PU and PR simultaneously devise payments for all customer types; in the second period every customer chooses from her set of contracts. We solve for pure-strategy subgame-perfect equilibria:

Definition 3.3.1. *For given distribution $F(h,e)$ and benefit level L, a pure-strategy subgame-perfect equilibrium of the health insurance game is a tuple $(\alpha^*, p^*_{PR}(\cdot), (L, p^*(h,e)))$ such that*

(i) α^* satisfies (3.5) *given $p^*_{PR}(\cdot)$ and $(L, p^*(h,e))$,*

(ii) $p^*_{PR}(\cdot)$ solves (3.6) *given α^* and $(L, p^*(h,e))$,*

(iii) $(L, p^*(h,e))$ solves (3.3)*, for all $\alpha \in [0,1]$, feasible $p_{PR}(\cdot)$, and (h,e).*

In the sequel, we refer to a tripel $(\alpha^*, p^*_{pr}, (L, p^*(h,e)))$ satisfying the conditions of Definition 3.3.1 simply as an *equilibrium* of the health insurance market.

3.3.2 Voluntary Health Insurance

Before establishing equilibrium existence, we study the health insurance market when insurance is voluntary. In this case, instead of purchasing contracts from PU or PR, every customer may choose to be uninsured and bear health costs herself. As is not uncommon in models of insurance, voluntary insurance leads to a complete unraveling of the health insurance market:

Proposition 3.3.1. *Under voluntary insurance, for any positive benefit level there exists no equilibrium in the health insurance market.*

Note that this result is not due to adverse selection since customer types are observable. Instead, the combination of budget balancing, multi-dimensional continuous types and regulation that public insurance must condition her contribution rate on income only (Pauly, 1984) leads to market collapse. We provide intuition for Proposition 3.3.1 here, all proofs can be found in the Appendix.

If health insurance is voluntary, a customer is willing to insure only if a contract offers her a net benefit, that is, if the difference between health benefits and payment is weakly positive. All customers whose contract set contains only contracts with negative net benefit decide to remain uninsured. A net benefit for the customer translates one-to-one in a loss in profits for the insurance. Hence, only customers who inflict a (weak) loss on a health insurance company choose to be insured. PR can avoid the loss on most parts of the population as it can finetune its contract to customer's health and income. PR may only incur a loss if the upper bound on its payment binds. As a consequence, PR might fail to deter unprofitable customers who are rich but unhealthy.

PU is less flexible than PR since it only discriminates along the income dimension. As a consequence, for strictly positive benefit level PU cannot avoid a loss on comparatively poor and unhealthy customers who want to insure. Since such customers' income is sufficiently low, PR can deter these from insuring privately. Thus, they insure with PU. Customers who are indifferent between insuring or not, with zero net benefit, might decide to insure with PU but do not generate profits either. In addition, since types are continuous, for every contribution rate PU might set, customers with negative net benefit who would prefer insuring with PU over remaining uninsured do exist. This causes PU to be unable to run a balanced budget and consequently to a failure of equilibrium existence.

To sum up, if health insurance is voluntary, customers who are attractive from the insurances' perspective remain uninsured, leaving insurances with unprofitable customers. This makes health insurance non-viable. Hence, Proposition 3.3.1 provides a rationale for why health insurance is obligatory in Germany and more generally in many health insurance markets.

3.3.3 Equilibrium Existence

Retaining obligatory health insurance, we prove existence of equilibrium in the health insurance market. We proceed backward from the second period, first studying customers' optimal insurance choice.

Lemma 3.3.1. *Given any contribution rate set by PU and any feasible payment scheme of PR, it is optimal for customers to choose the insurance which offers the contract with the lowest payment.*

This result is immediate since both insurances offer contracts with equivalent benefit level L. Customers whose income is below the opt-out threshold can only choose PU's contract. All other customers have the choice between PU and PR. As the utility function is strictly increasing, every

customer chooses the contract which offers her the largest net benefit. Since the benefit level is fixed and equal for PU and PR, it is the contract's payment that determines the net benefit and, thereby, its attractiveness for customers.
In the following, we assume that customers choose PR when they are indifferent, i.e., if both insurances charge the same payment.[10]

Having determined the population's optimal insurance choice, we analyze PR's optimal payment scheme. We call a customer *profitable* at a given PU contribution rate, if the payment charged by PU exceeds health benefits payable to the customer; otherwise, we call the customer unprofitable.[11]

Lemma 3.3.2. *Given customer's optimal contract choice and an arbitrary contribution rate set by PU, it is optimal for PR to set its payment equal to PU's payment if a customer is profitable and to set the highest possible payment if a customer is unprofitable.*

For a profitable customer, PR faces the trade-off between attracting the customer and charging a high payment. If PR's payment exceeds PU's payment, the customer turns down PR's contract and chooses PU. If PR's payment is strictly less than PU's payment, PR can increase profits by increasing its payment slightly without losing the customer. Hence, it is optimal for PR to set its payment exactly equal to PU's payment for all profitable customers.
If a customer is unprofitable and PR sets a payment below PU's payment, PR incurs a loss since she attracts the customer. Since PR may not reject customers, PR tries to deter unprofitable customers by setting its payment as high as possible. Note that PR may not deter all unprofitable customers because of the upper bound on its payment (feasibility constraint).

In contrast to PU, PR sets a flexible payment and discriminates based on both health and income. The above argument shows that PR exploits its greater flexibility to cream skim all profitable customers with sufficiently high income, i.e., with income exceeding the opt-out threshold. In fact, without an opt-out threshold, PR would cream skim all profitable customers in the population which would make it impossible for PU to run a balanced budget. Hence, the opt-out threshold is essential for the existence of equilibrium in the health insurance market.

This observation motivates the following assumption which we maintain throughout the paper.

Assumption 1. *(Viable health insurance market.) The aggregated income of customers with income below the opt-out threshold and below the contribution cap exceeds the entire population's health benefits:*

$$\mathbb{E}[\min(L, c(h))] < \mathbb{E}[e\mathbf{1}_{\{e<\min(K_1,K_2)\}}].$$

[10] See also the remarks following Theorem 3.3.1.

[11] Note that such a situation may only arise since insurance is obligatory.

Roughly, Assumption 1 says that the total income of all customer who must insure with PU covers the health costs of the whole population. It guarantees that the population structure is such that at least potentially PU can run a balanced budget. There are several reasons why Assumption 1 may be satisfied; some of those may be under direct control of an exogenous regulator (benefit level, opt-out threshold, contribution cap) but some of those may not (health costs). In particular, for a given health income distribution F Assumption 1 holds if the benefit level is sufficiently low or the opt-out threshold and contribution cap are sufficiently high. With this assumption in place, we obtain the following theorem:

Theorem 3.3.1. *Assume that the health insurance market is viable. Then the health insurance market has an equilibrium and the equilibrium contribution rate α^* is unique.*

The proof of Theorem 3.3.1 relies on the intermediate value theorem. The key step is to establish continuity of PU's objective in the contribution rate. See the Appendix for details.
In equilibrium, customers with income below the opt-out threshold choose PU. Above the opt-out threshold customers who are profitable insure with PR; unprofitable customers insure with PU. However, all customers, profitable or unprofitable, with income above the contribution cap and above the opt-out threshold choose PR. See Figure 3.1 for a graphical illustration.

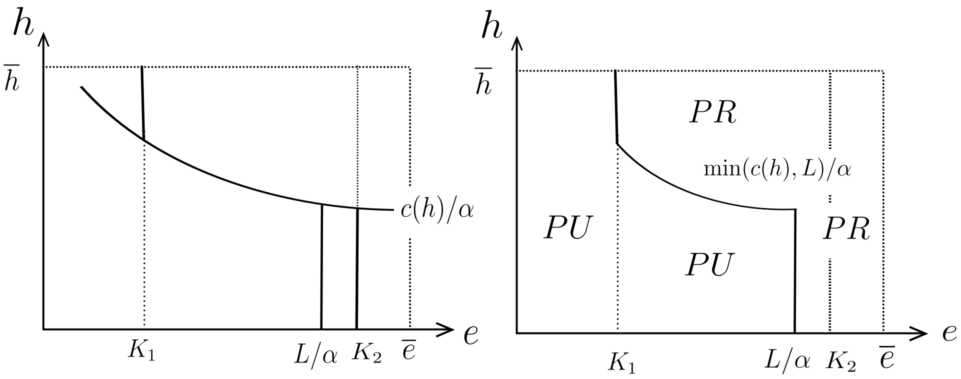


Figure 3.1: Customers' insurance choice by customer type. The function of renormalized health benefits, $\frac{\min(c(h),L)}{\alpha}$, takes values on the e-axis.

Interestingly, independent of their choice of insurance, all PR customers pay the same amount they would pay if they insured with PU. That is, PR's payment is coupled to PU's payment in equilibrium. Intuitively, its monopolistic power allows PR to charge customers a payment that makes them indifferent to choosing their outside option which is insuring with PU.[12] Attracting profitable customers entails two positive effects for PR: Firstly, there is an immediate gain in profits. Secondly, if PU loses profitable customers to PR, PU has to increase the contribution rate leading to a higher payment for all customers. This in turn allows PR to increase payments for all its customers as their outside option has become less attractive. In fact, note that if profitable customers with income above the opt-out threshold would collectively choose to insure with PU

[12]We study the case of competing private insurances in subection 5.3.

instead of PR, PU could adjust the contribution rate downward leading to a lower payment for the entire population. Intuitively, PR prevents this by slightly undercutting PU's payment.

What are the redistributional effects of the health insurance market? Profitable customers with income below the opt-out threshold subsidize all unprofitable customers with income below the contribution cap. Furthermore, the relative profitability of these two customer groups determines the payment for the entire population through their effect on the contribution rate. The surplus of profitable customers with high income above the opt-out threshold is transformed one-to-one into a profit for PR and is lost for the population. PR may incur a loss on unprofitable customers with income above the contribution cap and above the opt-out threshold. However, as a consequence of the organizational structure of the health insurance market, PR obtains an overall profit: PU runs a balanced budget; relative to PU, PR attracts customers with higher income. As health and income are positively correlated, a higher income entails also a better health type. Thus, PR draws upon a more lucrative part of the population and earns positive profits. See the proof of Theorem 3.3.1 for details.

A couple of technical remarks are in order: Firstly, as can be seen from the proof of Theorem 3.3.1, the assumption that health and income are affiliated is not required for the existence of equilibrium.
Secondly, note that PU's contribution rate is only unique given the behavior of PR and customers. However, customers indifference behavior is not unique. Our specification that customers choose PR if they are indifferent resolves existence issues for profitable customers with income exceeding the opt-out threshold: If these customers would choose PU when they are indifferent, PR would like to set a payment arbitrarily close but not equal to PU's payment. However, one could imagine different specifications for unprofitable customers with income above the contribution cap and above the opt-out threshold. For these specifications an analogous analysis applies.
Relatedly, PR's optimal payment scheme may not be unique (even on a set with positive measure): In order to deter unprofitable customers with income between the opt-out threshold and the contribution cap, PR can set any payment that exceeds PU's payment. Note, however, that this does not change customers decisions and thus the equilibrium contribution rate is the same as under Lemma 3.3.2. Furthermore, our specification is particularly robust against tremble-like errors in the behavior of customers.

3.3.4 Comparative Statics in Policy Parameters

Having established existence of equilibrium, we analyze how changes in the opt-out threshold, the contribution cap, and the benefit level affect the equilibrium in the health insurance market. As these three parameters might be controlled by an exogenous regulator, we refer to them as "policy" parameters.

Proposition 3.3.2. *An increase of the opt-out threshold decreases the contribution rate.*

First, consider the case where both the former and the new opt-out threshold are below the contribution cap. Recall that PR cream skims the part of the population with income above the opt-out threshold but below the contribution cap, i.e., profitable customers with income in this range insure with PR whereas unprofitable customers in this range insure with PU. An increase in the opt-out threshold limits PR's possibility to cream skim since some profitable customers are consequently forced to insure with PU under the new opt-out threshold. No additional unprofitable customers join PU because they insured with PU already under the former opt-out threshold. Thus, all new PU customers are profitable, allowing PU to adjust the contribution rate downward.

Next consider the case where the former and the new opt-out thresholds lie above the contribution cap. In this case the cream skimming area for PR has vanished. All customers with income below the the opt-out threshold insure with PU; customers with income above the opt-out threshold insure with PR. An increase in the opt-out threshold forces additional customers to insure with PU. In contrast to the first case, some of these customers might be unprofitable. However, compared to existing PU customers, the new customers have a higher income. As income and health are correlated, a higher income entails (on average) also a better health type. These two factors allow PU to decrease the contribution rate.
Surprisingly here, limiting choices of customers benefits them in that it decreases the contribution rate and thus their payments. As PR's payment is coupled to PU's payment, in equilibrium not only PU customers benefit from the lower contribution rate but all customers do. Intuitively, with a higher opt-out threshold more profitable customers are forced to insure with PU rather than being cream skimmed by PR. These profitable customers' surplus is redistributed to all other customers in the population (including themselves) rather than translated into a profit for PR. PR's profits decrease because PR loses profitable parts of the population to PU and has to charge a lower payment to attract customers. Observe that from the customers' perspective it would be desirable to set the opt-out threshold as high as possible, essentially eliminating PR from the market.

Proposition 3.3.3.

(i) If the contribution cap is above the opt-out threshold, a decrease of the contribution cap to a level that is still above the opt-out threshold decreases the contribution rate.

(ii) If the contribution cap is below the opt-out threshold, a decrease of the contribution cap increases the contribution rate.

Consider first the case where the former and the new contribution cap lie above the opt-out threshold. In this case, lowering the contribution cap decreases the range of income in which PR can cream skim customers because PR faces a lower upper bound on its payment (feasibility constraint). As a result, PR can deter fewer unprofitable customers from its service. Furthermore, PR does not attract any new profitable customers. Regarding it from PU's perspective, PU loses unprofitable customers while retaining all profitable customers. Also, note that the payments all remaining PU customers make are not reduced by the change since these customers have income below the contribution cap. Therefore, PU can adjust the contribution rate downward. Intuitively, after the decrease in the cap profitable customers with income below the opt-out threshold subsidize a smaller number of unprofitable customers which allows for a decrease in the contribution rate. Consequently, all customers pay less, and PR's profits decrease.

Now consider the case where both the former and the new contribution cap lie below the opt-out threshold. In this case, customer sets do not change through the decrease of the contribution cap. PU is however forced to reduce demanded payments for those customers for whom the former contribution cap was binding. To compensate this loss, PU has to adjust the contribution rate upward. Thus, customers' payments increase, and PR's profits increase.

Proposition 3.3.4. *An increase of the benefit level increases the contribution rate.*

An increase in the benefit level L affects PU in two ways. Firstly, existing PU customers for whom the former benefit level was binding[13] become less profitable since the income dependent contribution remains the same. Additionally, if the opt-out threshold is below the contribution cap, there is an income range where PR cream skims. Some of the customers with income in this range are profitable under the former benefit level but become unprofitable under the new benefit level. Under the new benefit level, PR deters these customers who thus join PU. As a result of these two effects, PU has to adjust the contribution rate upward to cover the increased health benefits of its customers. The effect on customers' utility is twofold. On the one hand, all customers face a higher payment; on the other hand, some customers enjoy more health benefits. Accordingly, PR can charge a higher payment but also needs to cover higher health benefits.

3.3.5 Structural Population Changes

Changes in a populations health-income distribution may occur over time due to immigration, advances in technology, better education or rise and fall of national economies. In this section we study how structural changes in the population's health and income affect the health insurance market. To this end, we analyze two different changes in the underlying distribution of health and income: a systematic improvement of health and income and an increased correlation between health and income.

Systematic Improvement of Health and Income

First, we investigate the effect of a systematic improvement of the population's income and health on the contribution rate and PR's profit. Technically, we consider a stochastic improvement of $f(h, e)$ to a distribution with density function $\tilde{f}(h, e)$ in the sense of (multivariate) first-order

[13]These are customers who are underinsured, whose health type causes health costs in equal or larger extent to the maximum benefit level written down in the contract $c(h) \geq L$.

stochastic dominance.[14] Intuitively, as the population's income and health improve, the population should spend a lower percentage of its income on health insurance given that the benefit level stays constant. Indeed, if the entire population was insured with the budget-balancing PU, the contribution rate could be adjusted downward. To account for the precise organizational structure of the insurance market a more thorough analysis is needed.

We start by analyzing how customer sets change as the distribution changes. It is instructive to divide the population into four subgroups and study the effect of a systematic improvement on each of these subgroups separately.
Profitability and unprofitability are defined relative to the original distribution and the corresponding contribution rate α^*. Let $PU^+(\alpha^*)$ be the set of profitable PU customers and $PU^-(\alpha^*)$ the set of unprofitable PU customers. Analogously, let $PR^+(\alpha^*)$ be the set of profitable PR customers and $PR^-(\alpha^*)$ the set of unprofitable PR customers.

Firstly, consider the effect on the subgroup of unprofitable PU customers, $PU^-(\alpha^*)$. As health and income improve, PU's profits on this subgroup unambiguously increase: customers who remain in the group even after the improvement are less unprofitable than before; additionally, some unprofitable customers leave $PU^-(\alpha^*)$ to join one of the other subgroups.
Second, consider how PR's profit is affected on the set of its unprofitable customers, $PR^-(\alpha^*)$. Customers remaining in the group even after the improvement are less unprofitable than before, and some unprofitable customers join $PR^+(\alpha^*)$. This effect increases PR's profit. On the other hand, there is an inflow of new unprofitable customers from $PU^-(\alpha^*)$. These customers are unprofitable before and after the change of distribution but had income lower than the contribution cap before the change and income exceeding the contribution cap after the change. This effect decreases PR's profit. Which of the two effects dominates depends on the precise change in health and income.
Third, we analyze the effect on PR's profit generated from $PR^+(\alpha^*)$: Customers remaining in the group are more profitable than before. Additionally, there is an inflow of new profitable customers from all other subgroups. Thus, PR's profit from this subgroup increases.
Finally, consider $PU^+(\alpha^*)$. Again, customers remaining in this group are more profitable than before. Also, there is an inflow of new profitable customers from $PU^-(\alpha^*)$. These two effects suggest that PU's profit should increase. There is a countervailing effect though. Profitable PU customers whose income exceeds the opt-out threshold after the change are attracted by PR, decreasing PU's profit on $PU^+(\alpha^*)$. Therefore, the overall change in profit depends on the exact changes in health and income.

In general, we cannot determine the sign of the change in profits on $PR^-(\alpha^*)$ and $PU^+(\alpha^*)$. We can however derive an upper bound on a potential loss. In fact, a negative change in profits on

[14] See the Appendix for a definition and technical details.

$PR^{-}(\alpha^{*})$ never outweighs the increase in profits on $PU^{-}(\alpha^{*})$. To see this, observe that the loss in profit on $PR^{-}(\alpha^{*})$ is caused by unprofitable customers switching from $PU^{-}(\alpha^{*})$ to $PR^{-}(\alpha^{*})$. Thus, the loss on $PR^{-}(\alpha^{*})$ corresponds to a one-to-one gain in profit on $PU^{-}(\alpha^{*})$. All other effects increase profit. Put differently, profit on $PR^{-}(\alpha^{*}) \cup PU^{-}(\alpha^{*})$ increases. An analogous argument applies to the change in profit on $PU^{+}(\alpha^{*})$.

Our analysis reveals that the systematic improvement of health and income may affect the overall profit of PU and PR positively or negatively, depending on the precise inflow and outflow in $PU^{+}(\alpha^{*})$ and $PR^{-}(\alpha^{*})$. As noted above, the overall effect is however positive. Thus, it cannot be that both PU's *and* PR's profits decrease. The change in PU's profit determines whether PU adjusts the contribution rate upward or downward in response to the systematic improvement. As PR's profit is increasing in the contribution rate, this effect may reinforce or counteract the initial change in PR's profit. The following proposition summarizes our findings.

Proposition 3.3.5. *Consider a systematic improvement of the population's health and income. Then exactly one of the following three scenarios arises:*

(i) If the loss in PU's profit on $PU^{+}(\alpha^{*})$ outweighs the gain in profit on $PU^{-}(\alpha^{*})$, the contribution rate increases and PR's profit increases.

(ii) If the loss in PR's profit on $PR^{-}(\alpha^{*})$ outweighs the gain in profit on $PR^{+}(\alpha^{*})$, the contribution rate decreases and PR's profit decreases.

(iii) Else the contribution rate decreases and the private insurance may profit or lose.

Proposition 3.3.5 sorts the wide range of possible systematic improvements of health and income into three categories according to their effect on the contribution rate and PR's profit. Given that the class of improvements we consider unambiguously increase health and income for the entire population, these categories are surprisingly manifold. In particular, there exist cases in which customers have to pay a higher percentage of their income for health insurance. Intuitively, this is because an improvement might allow PR to absorb profitable PU customers, urging PU to increase the contribution rate in order to run a balanced budget.
This observation has important implications. The current organization of the health insurance market might mitigate policy programs and campaigns targeted to improve the population's health in order to decrease the contribution rate.

Increase in Correlation Between Health and Income

Motivated by empirical studies (Deaton and Paxson, 1998) which document an increase in correlation between health and income, we investigate how changes in correlation affect the health insurance market. For a meaningful comparison of correlations, we consider distributions ranked by correlation according to the supermodular order which have identical marginal distributions of health and income.[15]

Start with a distribution f and consider a distribution g that is larger than f in the supermodular order. For the case when the opt-out threshold exceeds the contribution cap, we obtain the following clear-cut result.

Proposition 3.3.6. *If the opt-out threshold exceeds the contribution cap, an increase in correlation between health and income increases the contribution rate.*

Note that in Germany, in fact since 2003 the opt-out threshold exceeds the contribution cap and there is hence 'no cream skimming'. To gain intuition for our result, it is instructive to decompose the transition from f to g into several sub steps. Consider the two-dimensional space of health and income types. Start with the health income distribution f. Now fit a rectangle into the health income space and consider a transformation that shifts probability mass from the bottom right corner of the rectangle to the bottom left corner and the same probability mass from the upper left corner to the upper right corner.[16] This transformation increases correlation between health and income while keeping the marginal distributions of health and income fixed. Intuitively, we can construct g from f by applying several of these transformations to f.

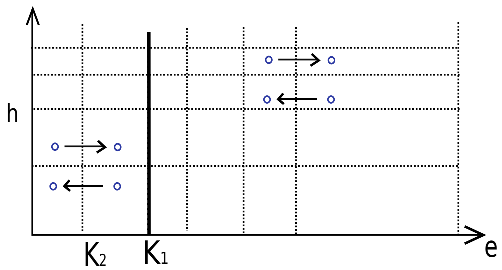


Figure 3.2: Mass shift not affecting the contribution rate

If the correlation-increasing mass transformation is such that all four corners of the rectangle lie within the set of PR customers (see figure **??**), PU is unaffected and the contribution rate remains the same. Similarly, in case the four corners of the rectangle lie within the set of PU customers, PU does not need to adjust the contribution rate because the marginal distributions of health and income conditional on being a PU customer are unchanged.

Lastly, consider the case when the left corners of the rectangle are in the set of PU customers whereas the right corners of the rectangle lie within the set of PR customers (see figure **??**). As a consequence of this transformation, the income distribution of PU customers is not altered since income is on the x-axis and marginals are held constant by the transformation. But the health distribution of PU customers worsens. Therefore, PU has to increase the contribution rate to run a balanced budget. Taking all three cases together, we see that PU increases the contribution rate

[15]See the Appendix for a formal definition.

[16]This probability mass shift corresponds to an 'elementary transformation' as described and analyzed in Meyer and Strulovici (2015) to characterize the supermodular stochastic order.

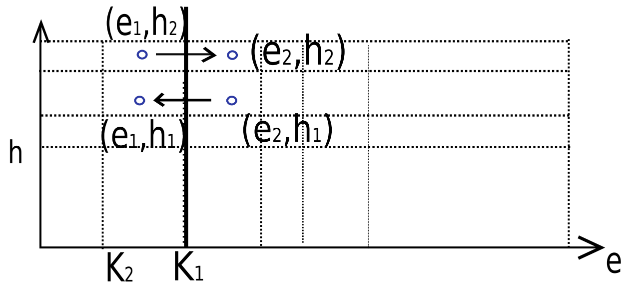


Figure 3.3: Mass shift affecting the contribution rate

if correlation between health and income increases. From a broader perspective, PU customers have comparatively low income whereas PR customer have comparatively high income. If the correlation between health and income increases, PU's low-income customers have also a worse health type, forcing PU to increase the contribution rate.

If the opt-out threshold lies below the contribution cap, there exists an income range where PR cream skims. Graphically, PU and PR customers are not separated any longer by a single cut-off in the income dimension. Thus, we have to consider additional correlation-increasing transformations. Consider the transformation where only the upper right corner of the rectangle is in the set of PR customers; all other corners lie in the set of PU customers. For this transformation there are two conflicting effects. On the one hand, the income distribution of PU customers worsens. On the other hand, unprofitable customers leave PU. It depends on the distribution f which of the two effect dominates, and, consequently, whether PU adjusts the contribution rate upward or downward. All other transformations entail a decrease in the contribution rate. Overall, in this case it depends on the specific increase in correlation and on how much weight is put on which transformation whether PU adjusts the contribution rate upward or downward.

3.4 Applications

We apply our model to address two policy questions. First, we study how customers' welfare changes if the health insurance market changes from the current contribution-based system to a premium-based system. We identify the population subgroups that benefit from such a change and the population subgroups that suffer. Second, we characterize the theoretically welfare-optimal payments. Understanding the properties of welfare-optimal payments yields additional insights into how to adjust the organization of health insurance markets to improve customer welfare.

3.4.1 Health Premia

In recent years, discussions to change the health insurance market in Germany have centered around two ideas. First, an abolishment of the difference between private and public insurances.

Second, a change from an income-dependent contribution-based payment scheme to a premium-based payment scheme, i.e., a scheme in which payments are flat and do not depend on the customer's income or health.

We adjust the model outlined in Section 3.2 to accommodate these two features of a premium-based health insurance market. Subsequently, we apply our two models to compare the premium-based to the contribution-based health insurance market. To make the two models comparable, we alter only the insurance provision sector and leave all other characteristics unchanged such as the population's health and income distribution or the customers' objective.

In the premium-based health insurance market any customer must insure with either of two identical premium insurances, henceforth PM_i, $i \in \{1, 2\}$.[17] Customers can choose freely between PM_1 and PM_2, independently of their income and health. PM_1 and PM_2 offer the same benefit level and face the same health costs. Each PM_i aims at balancing its budget by charging all its customers premium A_i, i.e.,

$$\mathbb{E}[\min(c(h), L)\mathbf{1}_{\{\mathrm{PM}_i(A_i)\}}] = \mathbb{E}[A_i\mathbf{1}_{\{\mathrm{PM}_i(A_i)\}}], \tag{3.7}$$

where $\{\mathrm{PM}_i(A_i)\}$ denotes the set of PM_i's customers given premium A_i.[18] The timing of the game is unchanged. First, PM_1 and PM_2 simultaneously set their premium, then customers choose their preferred insurance. Again, we are interested in subgame-perfect equilibria.

Proposition 3.4.1. *There exists an equilibrium in the premium-based health insurance market. In every equilibrium, all customers pay the premium*

$$A^* = \mathbb{E}[\min(c(h), L)]. \tag{3.8}$$

As before, customers choose the insurance that gives them a higher net benefit. Because the benefit level written down in the contracts is equal, customers choose the insurance with lower payment, i.e., the insurance with lower premium. Thus, if PM_i's premium is strictly lower than PM_{-i}'s premium, all customers choose PM_i. As a result of budget balancing and competition for customers, all PM_i demand the same premium and insure on average identical pools of health risks (identical PM_i) or there is only one PM. In either case, the equilibrium premium is equal to the average health benefit of the population, i.e., (3.8).

How does the change to a premium-based system affect redistribution in the population? Proposition 3.4.1 reveals that in a premium-based system every customer pays the average health benefit of the population, independently of her income. This implies that redistribution occurs only

[17] Analogous results hold if there are more than two premium insurances.

[18] We model premium insurances in the spirit of the public insurance in the contribution-based system. Results are virtually unchanged if we assume that premium insurances maximize profits.

along the health dimension, i.e., customers with a good health type subsidize customers with a bad health type. In contrast to the contribution-based system, there is no redistribution along the income dimension. Thus, the premium-based system disentangles the mixture of redistribution across health and redistribution across income which is inherent to the contribution-based system. As a consequence, we obtain the following corollary

Corollary 3.4.1. *There exists an income threshold such that all customers with income below the threshold have higher utility in the contribution-based system, and all customers with income above the threshold have higher utility in the premium-based system.*

Ceteris paribus, customers with higher income pay more in the contribution-based system and customers with lower income pay less in the contribution-based system. As health benefits remain equal, in the contribution-based system customers with higher income have a lower utility and customers with lower income have a higher utility.
To maintain the current level of income redistribution the introduction of a premium-based system would thus have to be accompanied by an adjustment of income taxation.

We next assess the impact of a change in the insurance system on the population's welfare. We adopt the utilitarian welfare criterion, i.e., our welfare function is the sum (integral) of utilities of all customers in the population. Recall from Proposition 3.3.2 that an increase of the opt-out threshold decreases the contribution rate and consequently the payment of all customers in the population. Thus, as the opt-out threshold increases, all customers enjoy a higher utility, i.e., welfare increases. Welfare in the contribution-based system reaches its maximum once the opt-out threshold is so high that the entire population must insure with PU. We refer to this specific contribution-based system as "contribution-based system without PR".

Proposition 3.4.2.

(i) For high levels of the opt-out threshold, the contribution-based system has higher welfare than the premium-based system.

(ii) In the premium-based system, there exists a budget-balanced income redistribution scheme (income tax) such that welfare is the same as in the contribution-based system without PR.

To understand the result, observe that there are two opposing effects. First, in the contribution-based system PR makes profits and thereby extracts surplus that is not used to cover the population's health benefits. In the premium-based system neither insurance makes profits.[19] This effect reduces welfare in the contribution-based system compared to the premium-based system. Second, concavity of the population's utility function favors the income-dependent payment of the contribution-based system compared to the flat payment of the premium-based system because low-income customers pay relatively less and high-income customers pay relatively more.

[19]Recall, that this is true even if we assume that both premium insurances are profit-maximizing.

With a higher opt-out threshold, PR extracts less surplus which attenuates the first effect. Thus, for a sufficiently high opt-out threshold, the second effect dominates, and the contribution-based system yields higher welfare. Conversely, combining the introduction of a premium-based system with a redistribution of income from high incomes to low incomes compensates for the second effect. Consequently, the premium-based system accompanied by an appropriate income redistribution scheme yields higher welfare than the contribution-based system.

We conclude that an easy-to-implement policy recommendation to improve welfare is to increase the opt-out threshold. If the health insurance market is changed more fundamentally to a premium-based system, an accompanying explicit redistribution of income via an adjustment of income taxation favoring low incomes would increase welfare and make up for the lack of implicit redistribution inherent in the contribution-based system. [20]

3.4.2 Welfare - Optimal Payments

In view of Proposition 3.4.2, we are now interested in welfare-optimal payment schemes to finance a given level of health benefits. Understanding why these payment schemes maximize welfare, gives us further insights into how to adjust the health insurance market in order to increase welfare. Specifically, we consider the following problem: Health insurance is exclusively provided by a benevolent authority that chooses a payment scheme to maximize welfare subject to the constraint of providing a given benefit level. First, we explicitly derive the welfare-optimal payment scheme that may condition on both health and income.

Proposition 3.4.3. *For given health benefits the welfare-maximizing payment scheme that may condition on health and income,* $p_{opt}(h, e)$, *is given by*

$$p_{opt}(h, e) = \min(c(h), L) + e - c(h) - \mathbb{E}\left[e - c(h)\right]. \tag{3.9}$$

Observe that $p_{opt}(h, e)$ conditions on customer's health. Intuitively, $p_{opt}(h, e)$ consists of two components. The first component charges each customer the health benefits she consumes. The second component associates to every customer the difference between her factual and her expected net income, i.e., her income less health costs. If a customer's net income is high relative to the average net income of the population, her payment is augmented by the difference. Otherwise, her payment is reduced by the difference. The first component guarantees that the population's health benefits are covered; the second component is a budget-balanced redistribution scheme from customers with high net income to customers with low net income. The redistribution scheme accounts for the positive effect of equating utility across customers on welfare, which stems from the concavity of customers' utility function.

[20] For arguments in favor of this clear separation of income redistribution and distribution across health types see Wissenschaftlicher Beirat ??.

In the contribution-based system PU's payments only depend on income. Therefore, we are now interested in the properties of welfare-optimal payment schemes which are restricted to only depend on income. Further, we investigate whether PU's payment satisfies these properties. Also, recall that welfare in the premium-based system can be increased, if the introduction of a premium-based system is combined with an appropriate redistribution in the income dimension. Studying characteristics of welfare-optimal payments which depend on income only translates one-to-one into studying the characteristics of welfare-optimal income redistribution schemes in the premium-based system. Technically, we assume for the next result that the density of the distribution of health conditional on income is continuously differentiable.

Proposition 3.4.4. *The welfare-maximizing payment scheme $\hat{p}_{opt}(e)$ restricted to depend on income only satisfies*

$$\frac{\mathrm{d}\hat{p}_{opt}(e)}{\mathrm{d}e} \geq 1.$$

Clearly we have that welfare under $p_{opt}(h, e)$ is higher than welfare under $\hat{p}_{opt}(e)$. Nevertheless, Proposition 3.4.4 shows that $\hat{p}_{opt}(e)$ takes into account the correlation between higher income and better health. An increase in a customer's income by one unit increases her net income by more than a unit since by correlation higher income is associated with better health and thus lower health costs. Payment $\hat{p}_{opt}(e)$ tries to balance net incomes across customers. Thus, it not only neutralizes the increase of income but also balances out the positive effect of an income increase on health. Hence, $\hat{p}_{opt}(e)$ increases faster in income than income itself.

Observe that the last result stands in marked contrast to PU's factual payment: PU's payment increases at a rate equal to the contribution rate, which is less than one, and remains constant above the contribution cap. This indicates that a reform to adjust PU's payment scheme to take the positive correlation of health and income into account has the potential of increasing welfare. On a similar note, if the introduction of a premium-based system is combined with an adjustment of income taxation to compensate for the redistribution that is lost through the abolishment of the contribution-based system, an adjustment of income taxation to account for correlation between health and income would be welfare enhancing.

3.5 Extensions

3.5.1 Health Signals

To single out the effect of the organizational structure of the health insurance market, we assume that insurances perfectly observe customers' characteristics such as their health types. By this we shut down confounding channels like adverse selection as a result of asymmetric information. Nevertheless, to demonstrate robustness of our findings to private information of customers, we consider the following variation of our model. In addition to her income and health type, each customer is characterized by a health signal. We can interpret the signal, as the customer's

answer to a health questionnaire. The customer's health signal is positively correlated with her health type.[21] Insurances observe customers' health signal but not their health type. As PU discriminates only based on income, PU is not directly affected by the change in modeling. PR's ability to discriminate across health types is however hampered; PR has to devise a payment scheme that only depends on income and health signal to maximize profits.
We can reproduce our findings from Section 3.3 following the same steps: As before, customers choose the insurance which offers the lower payment. For insurances, observe that we can replicate our analysis by replacing the health benefit by the expected health benefit conditional on income and health signal. Intuitively, as PR cannot observe customers' health, it estimates health using income and health signal. Due to positive correlation, high income and a favorable health signal are indicative of a good health type. Consequently, PR partially retains its ability to distinguish profitable and unprofitable customers.

3.5.2 Endogenous Health Benefits

So far we have assumed that insurances provide the same maximum benefit level L in their contracts. A careful inspection of the arguments in the proof of Lemma 3.3.1 reveals that customers choose the insurance which offers the higher net benefit, i.e., health benefit minus payment. We had conveniently set equal benefit levels of public and private insurance contracts to focus on payments. The analysis is however unchanged if, PR provides an exogenously higher benefit level. In equilibrium PR will charge higher payments such that the net benefit is unchanged. Does this conclusion remain true if PR chooses the benefit level endogenously?

To answer this question, consider the following variant of our model. PR offers customers two contracts: a simple contract reminiscent of PU's contract and a more elaborate contract tailored to its customers. Specifically, the first contract provides the same benefit level as PU, and the contract's payment corresponds to the highest payment which PU charges, i.e., contribution rate times contribution cap.[22] For the second contract, PR chooses a benefit level and devises an income- and health-dependent payment scheme.

The equilibrium in this health insurance market parallels the equilibrium derived in Section 3.3. The sets of PU and PR customers are unchanged. PR finetunes the elaborate contract to cream skim profitable customers with income above the opt-out threshold. Unprofitable customers with income above the opt-out threshold and the contribution cap choose PR's simple contract. In equilibrium only the net benefit of the elaborate contract is uniquely determined echoing the remarks made at the beginning of this section. Thus, without additional assumptions no prediction about the relative benefit level of PU and PR can be made.

[21] Specifically, we assume that health signal and health type are affiliated.
[22] In the German health insurance system, private insurances have to offer this baseline contract to every customer.

3.5.3 Private Competition

We are now interested in how redistribution streams change if we introduce competition among private insurances. Assume that in addition to PU and the population there are two private insurances, PR_1 and PR_2.[23] All insurances offer contracts with equal maximum benefit level L. PR_1 and PR_2 maximize profits by devising a payment scheme that conditions on customers' health and income. In the first period insurances simultaneously design their type dependent contracts. In the second period customers choose the contract that maximizes their utility. For simplicity assume that customers randomize with equal probability if they are indifferent between PR_1 and PR_2.

In equilibrium, the set of PU customers is unchanged. By budget balancing thus, public health contribution remains the same. Former PR customers split equally between PR_1 and PR_2 and private health premia change. The payments of PR_1 customers and PR_2 customers are equal to the minimum of their health benefit from the contract and the upper bound on PR's payment. In particular, they pay less than before. Intuitively, competition pushes the premiums payable by PR_1 and PR_2 customers down to the cost they impose on the insurance, i.e., their health benefit. Redistribution streams are as follows. Profitable customers with income below the opt-out threshold have to insure with PU. and pay more than their health benefits. They subsidize unprofitable parts of the population insured with PU. Profitable customers with income above the opt-out threshold opt out of the redistribution scheme by insuring with one of the PRs and pay an amount equal to or less than their health benefit. As a consequence, they do not generate a gain to the PRs. Unprofitable customers with income above the opt-out threshold but below the maximum price the PR may set are deterred from entering either of the PRs and insure with PU. Unprofitable customers with income above the maximum price for insurance enter either of the PRs.[24]

3.6 Conclusion

This paper studies redistributional effects of competition between private and public insurance on health insurance markets based on the example of Germany. Public and private insurance co-exist and are mutually exclusive. Private insurance maximizes profits. Public insurance balances budget and is financed by an income tax with a cap. In addition, customers of public insurance have the option to opt out once income is sufficiently large.

On a more abstract level, we study a two-dimensional linear taxation problem with price cap,

[23] Analogous results hold if there are more than two private insurances.

[24] Note that as a consequence, PRs do not survive competition with other PRs since very wealthy but unprofitable customers cannot be deterred from entering and cause a loss to the insurance.

opt-out for high types under a budget balancing constraint and regulation. Public health premia may only depend on income types but health costs depend on health types of customers. This regulation in combination with the potential of opt-out gives rise to cream skimming (risk selection) by a competing private insurance.
Private insurance discriminates between healthy and unhealthy customers. If possible, she deters unprofitable customers while attracting customers who will generate a gain by varying the premium. In the face of cream skimming, opt-out by rich customers and budget balancing, the public insurance sets the public contribution rate.

As first result, we derive a condition under which a unique, redistributive, budget balancing public contribution rate exists. We show, increasing the opt-out threshold up to the level of the public insurance's price cap decreases the premium for all customers, public and private, since the type area where the private insurance may cream skim vanishes. Increasing the opt-out threshold further, leads to even lower public and private premia since health and income types are positively correlated.
Considering a systematic improvement of the population's health and income[25], we show that even though the change clearly improves the population's characteristics, the public contribution rate might increase. Healthy and wealthy customers may opt out and insure privately so that an improvement does not benefit all customers via redistribution in the public insurance but instead is pocketed by private insurance.
Increases in correlation between income and health may increase public health prices to keep a balanced budget: On the one hand, less wealthy types insure publicly and become on average less healthy which causes additional costs to public insurance. On the other hand, higher earning types become more healthy after the increase in positive correlation but may opt out so that the gain in health and decrease in costs is lost to private insurance.

While some characteristics of our model are Germany specific (opt-out and price cap), simpler versions still constitute a contribution to the literature: Health and income types are continuous which in combination with regulation of public insurance and budget balancing leads insurance markets to collapse under voluntary insurance. In particular, this result is not due to adverse selection and may deliver a rationale for why health insurance is compulsory in Germany, France and Switzerland.
In addition, continuous types allow for modeling of maximum and factual health benefit levels and thus over insurance which drives customers' contract choice.
In Germany as in Italy, customers have the choice between public and private insurance. When setting the opt-out threshold at infinity our model corresponds to a completely public, non-profit, earnings-based redistributive health insurance system as in France.

[25] In the sense of first-order stochastic dominance

We formulate the model under the assumption that private insurance perfectly observes customers' health types. We believe this is plausible since private insurances in Germany often require potential customers to fill out binding questionnaires about their medical history. Moreover, insurances can draw on internal statistics to precisely estimate the likelihood that a customer falls sick with a certain disease. We think of an agents' health type as average health over her life time rather than a reflection of a particular moment. By modeling health types as observable and fix over lifetime, we circumvent the moral hazard problem in health insurance: Insured agents do not overuse their insurances. The motive to insure is hence not by risk-sharing but imposed by regulation to redistribute along the income and health dimension. Public health premium as percentage of income is set at an ex ante stage by the public insurance for we implicitly assume that the general income-health risk distribution of the population is known to both public and private insurance.

In our model, the public insurance commits to running a balanced budget rather than maximizing profits. We offer two possible justifications for this behavior. First, we may assume that a benevolent government sets up a health insurance fund to provide large parts of the population with health insurance at lowest possible costs.[26] Second, we can regard public insurance as a representative for an entire competitive public insurance market in which every public health insurance operates at her (identical) costs.[27] Either explanation motivates the objective to balance budget.

[26] In fact, in years in which German public health insurances make significant profits, customers obtain a refund in form of a price deduction.

[27] In Germany, for example, customers can choose from several similar public health insurances, and switching insurances within the public sector is simple.

3.7 Appendix: Proofs

3.7.1 Proofs for Voluntary Health Insurance

Proof of Proposition 3.3.1. If health insurance is voluntary, every customer type's contract set contains, in addition to PU's and PR's contract, contract $(0,0)$, i.e., $(0,0) \in \mathcal{C}(h,e),\ \forall (h,e)$. Assume there exists an equilibrium $(\alpha^*, p^*_{pr}, (L^*, p^*(h,e)))$. Optimal choice of customers requires:

$$u(e + \min(L^* - c(h), 0) - p^*(h,e)) \geq u(e + \min(L' - c(h), 0) - p(h,e)),$$

for all contracts $(L', p(h,e)) \in \mathcal{C}(h,e)$. As the utility function is strictly increasing this is equivalent to

$$\min(L^*, c(h)) - p^*(h,e) \geq \min(L', c(h)) - p(h,e),$$

for all contracts $(L', p(h,e)) \in \mathcal{C}(h,e)$. In particular, with voluntary health insurance we have that

$$\min(L^*, c(h)) - p^*(h,e) \geq 0. \tag{3.10}$$

(3.10) implies that PU and PR incur a weak loss for every insured customer. We will now argue that (3.10) holds with strict inequality on a set of PU customers with positive measure. Thus, PU's equilibrium condition

$$\alpha^*\, \mathbb{E}[\min(K_2, e)\mathbf{1}_{PU(\alpha^*)}] = \mathbb{E}[\min(L, c(h))\mathbf{1}_{PU(\alpha^*)}]$$

does not hold, a contradiction. Let α^* PU's equilibrium contribution rate under voluntary insurance. Consider the part of the population with $e < K_2$ and

$$\alpha^* \min(K_2, e) - \min(L, c(h)) < 0.$$

These are (strictly) unprofitable customers. Because $K_2, L, c(h) > 0$ and the support of income is continuous $[0, \bar{e}]$ this part of the population has positive measure for every α^*. These customers prefer being insured with PU over remaining uninsured. Also, PR does not want to attract this part of the population because PR would need to set $p_{PR}(h,e) \leq \alpha^* \min(K_2, e)$, incurring a loss on these customers. As $e < K_2$, PR can and will set $p^*_{PR}(h,e) \geq \alpha^* \min(K_2, e)$ to deter these unprofitable customers. We have argued that this strictly unprofitable part of the population will insure with PU. The strictly profitable part of the population will decide to remain uninsured. How the part of the poplulation that is indifferent between insuring or not, i.e. for which $\alpha^* \min(K_2, e) - \min(L, c(h)) = 0$ (zero profit zero loss), decides is irrelevant for PU, PR and the outcome since they neither bring a loss or a profit. To sum up, we have shown that for any contribution rate PU might set strictly unprofitable customers exist, insure with PU and cause a loss, while zero profit zero loss customers may insure with PU but do not generate a profit either. Hence, PU cannot run a balanced budget for any contribution rate α it might set.

□

3.7.2 Proofs for Equilibrium Existence

Proof of Lemma 3.3.1. Given any contribution rate $\alpha \in [0,1]$ and any feasible choice of $p_{PR}(\cdot)$, the contract set of a customer with type (h,e) is

$$\mathcal{C}(h,e) = \begin{cases} \{(L, \alpha \min(K_2, e))\} & \text{if } e < K_1, \\ \{(L, \alpha \min(K_2, e)),\ (L, p_{PR}(h,e))\} & \text{else.} \end{cases}$$

It is optimal for a type-(h,e) customer to choose $(L, p^*(h,e)) \in \mathcal{C}(h,e)$ if and only if

$$u(e + \min(L - c(h), 0) - p^*(h,e)) \geq u(e + \min(L - c(h), 0) - p(h,e)),$$

for all $(L, p(h,e)) \in \mathcal{C}(h,e)$. As $u(\cdot)$ is strictly increasing, this is equivalent to

$$\min(L, c(h)) - p^*(h,e) \geq \min(L, c(h)) - p(h,e),$$

for all $(L, p(h,e)) \in \mathcal{C}(h,e)$. Because health benefits L are equal, the latter expression reduces to

$$p^*(h,e) \leq p(h,e),$$

which concludes the proof. □

Proof of Lemma 3.3.2. Fix any contribution rate α. Consider a feasible payment scheme $p(\cdot)$. Optimal customer choice, Lemma 3.3.1, implies that the set of PR customers is given by

$$PR(\alpha) = \{(h,e) : e \geq K_1,\ p(h,e) \leq \alpha \min(K_2, e)\}.$$

Thus, spelling out the expectation, we can rewrite PR's objective as

$$p_{\mathrm{PR}}(\cdot) \in \underset{p(\cdot)\text{ feasible}}{\arg\max} \int_{\mathcal{E}} \int_{\mathcal{H}} \big(p(h,e) - \min(L, c(h)\big) \mathbf{1}_{\{e \geq K_1,\, p(h,e) \leq \alpha \min(K_2,e)\}}(h,e)\, f(h,e)\, dh\, de.$$

Because PR's objective involves no derivatives of $p(h,e)$, we can solve it pointwise. Carefully inspecting

$$\big(p(h,e) - \min(L, c(h)\big) \mathbf{1}_{\{e \geq K_1,\, p(h,e) \leq \alpha \min(K_2,e)\}}(h,e)\, f(h,e)$$

reveals that

$$p_{\mathrm{PR}}(h,e) = \begin{cases} \alpha \min(K_2, e) & \text{if } \alpha \min(K_2, e) \geq \min(L, c(h)), \\ \alpha K_2 & \text{else,} \end{cases}$$

is an optimal policy. □

Proof of Theorem 3.3.1. Let customers' and PR's behavior be as described in Lemma 3.3.1 and Lemma 3.3.2, respectively. Fix a contribution rate α. Formally, the set of PU customers is

$$PU(\alpha) = \{(h,e) : e < K_1\} \dot\cup \{(h,e) : K_1 \le e < K_2,\ \alpha e < \min(L, c(h))\},$$

and the set of PR customers is

$$\begin{aligned} PR(\alpha) =&\{(h,e) : e \ge K_1,\ \alpha \min(K_2, e) \ge \min(L, c(h))\} \\ &\dot\cup \{(h,e) : e \ge \max(K_1, K_2),\ \alpha K_2 < \min(L, c(h))\}. \end{aligned}$$

PU seeks a contribution rate α^* such that

$$\alpha^*\ \mathbb{E}[\min(K_2, e)\mathbf{1}_{PU(\alpha^*)}] = \mathbb{E}[\min(L, c(h))\mathbf{1}_{PU(\alpha^*)}].$$

Reformulating gives

$$\alpha^* = \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU(\alpha^*)}]}{\mathbb{E}[\min(K_2, e)\mathbf{1}_{PU(\alpha^*)}]}.$$

Define the function $T(\alpha)$:

$$T(\alpha) := \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU(\alpha)}]}{\mathbb{E}[\min(K_2, e)\mathbf{1}_{PU(\alpha)}]}.$$

An equilibrium contribution rate α^* corresponds to a fixed point of $T(\cdot)$. First, we show that $T(\cdot)$ is well-defined, i.e., that the denominator cannot become zero:

$$\mathbb{E}\left[\min(K_2, e)\big(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \le e < K_2,\ \alpha e < \min(L, c(h))\}}\big)\right] \ge \mathbb{E}\left[e\mathbf{1}_{\{e<\min(K_1, K_2)\}}\right] > 0,$$

where the last inequality follows from the assumption that $f(h,e)$ has full support and the fact that K_1 and K_2 are strictly positive.

Existence. We prove existence of a fixed point using the intermediate value theorem. Firstly, note

$$T(0) = \frac{\mathbb{E}[\min(L, c(h))\big(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \le e < K_2\}}\big)]}{\mathbb{E}[\min(K_2, e)\big(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \le e < K_2\}}\big)]} > 0.$$

The inequality follows from both numerator and denominator being strictly positive because of full support of $f(h,e)$ and $K_1, K_2, L, c(\cdot) > 0$. Secondly, we have

$$T(1) = \frac{\mathbb{E}[\min(L, c(h))\big(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \le e < K_2,\ e < \min(L, c(h))\}}\big)]}{\mathbb{E}[\min(K_2, e)\big(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \le e < K_2,\ e < \min(L, c(h))\}}\big)]} \le \frac{\mathbb{E}[\min(L, c(h))]}{\mathbb{E}[e\mathbf{1}_{\{e<\min(K_1, K_2)\}}]} < 1,$$

where the last inequality follows from Assumption 1. It remains to be shown that $T(\cdot)$ is continuous.[28] First, consider the numerator of $T(\cdot)$. The first addend does not depend on α, thus, we

[28]Continuity does not follow from standard results for parameter integrals because these require that

only need to check continuity of

$$g(\alpha) := \int_{\mathcal{H}} \int_{\mathcal{E}} \min(L, c(h)) \big(\mathbf{1}_{\{K_2 > e \geq K_1,\ \alpha e < \min(L, c(h))\}} \big) f(h, e)\, de\, dh\,.$$

Fix α and $\tilde{\alpha}$, and assume without loss of generality that $\alpha > \tilde{\alpha}$.

$$\begin{aligned}
&|g(\alpha) - g(\tilde{\alpha})| \\
&\leq \int_{\mathcal{H}} \int_{\mathcal{E}} \min(L, c(h))\, \mathbf{1}_{\{K_2 > e \geq K_1\}} \big| \mathbf{1}_{\{\alpha e < \min(L, c(h))\}} - \mathbf{1}_{\{\tilde{\alpha} e < \min(L, c(h))\}} \big| f(h, e)\, de\, dh \\
&= \int_{\mathcal{H}} \int_{\mathcal{E}} \min(L, c(h))\, \mathbf{1}_{\{K_2 > e \geq K_1\}}\, \mathbf{1}_{\{\tilde{\alpha} < \frac{\min(L, c(h))}{e} \leq \alpha\}}\, f(h, e)\, de\, dh. \qquad (3.11)
\end{aligned}$$

Because $f(h, e)$ has no atoms, the integrand converges pointwise to zero as $\tilde{\alpha} \to \alpha$. Thus, by the dominated convergence theorem, (3.11) converges to zero as $\tilde{\alpha} \to \alpha$. Continuity of the denominator follows from an analogous argument. Hence, $T(\cdot)$ is continuous, and the existence of a fixed point follows from the intermediate value theorem.

Uniqueness. If $K_1 \geq K_2$, $T(\cdot)$ is constant in α and thus the fixed point is unique. For $K_2 > K_1$ we argue that

(i) $T(\cdot)$ is increasing left of the first fixed-point,

(ii) $T(\cdot)$ is decreasing right of the first fixed-point,

together with the existence result above, this yields uniqueness of α^*. Note the following elementary equivalence for $a, b, c, d > 0$

$$\frac{a + c}{b + d} < \frac{a}{b} \Leftrightarrow \frac{a}{b} > \frac{c}{d}. \qquad (3.12)$$

For (i) recall that $T(0) > 0$. Let $\tilde{\alpha}, \alpha$ be left of the first fixed-point and $\tilde{\alpha} > \alpha$, then $T(\tilde{\alpha}) > \tilde{\alpha} > \alpha$. We argue that $T(\tilde{\alpha}) > T(\alpha)$:

$$\begin{aligned}
&T(\tilde{\alpha}) - T(\alpha) \\
&= \frac{\int_{\mathcal{H}} \int_{\mathcal{E}} \min(L, c(h)) \big(\mathbf{1}_{\{e < K_1\}} + \mathbf{1}_{\{K_2 > e \geq K_1,\ \tilde{\alpha} < \frac{\min(L, c(h))}{e}\}} \big) f(h, e)\, de\, dh}{\int_{\mathcal{H}} \int_{\mathcal{E}} e \big(\mathbf{1}_{\{e < K_1\}} + \mathbf{1}_{\{K_2 > e \geq K_1,\ \tilde{\alpha} < \frac{\min(L, c(h))}{e}\}} \big) f(h, e)\, de\, de} \\
&- \frac{\int_{\mathcal{H}} \int_{\mathcal{E}} \min(L, c(h)) \big(\mathbf{1}_{\{e < K_1\}} + \mathbf{1}_{\{K_2 > e \geq K_1\}} \big(\mathbf{1}_{\{\alpha < \frac{\min(L, c(h))}{e} \leq \tilde{\alpha}\}} + \mathbf{1}_{\{\tilde{\alpha} < \frac{\min(L, c(h))}{e}\}} \big) \big) f(h, e)\, de\, dh}{\int_{\mathcal{H}} \int_{\mathcal{E}} e \big(\mathbf{1}_{\{e < K_1\}} + \mathbf{1}_{\{K_2 > e \geq K_1\}} \big(\mathbf{1}_{\{\alpha < \frac{\min(L, c(h))}{e} \leq \tilde{\alpha}\}} + \mathbf{1}_{\{\tilde{\alpha} < \frac{\min(L, c(h))}{e}\}} \big) \big) f(h, e)\, de\, dh}.
\end{aligned} \qquad (3.13)$$

Analyzing the indicator functions, we see that

$$\alpha \leq \frac{\int_{\mathcal{H}} \int_{\mathcal{E}} \min(L, c(h))\, \mathbf{1}_{\{K_2 > e \geq K_1,\ \alpha < \frac{\min(L, c(h))}{e} \leq \tilde{\alpha}\}} f(h, e)\, de\, dh}{\int_{\mathcal{H}} \int_{\mathcal{E}} e \mathbf{1}_{\{K_2 > e \geq K_1,\ \alpha < \frac{\min(L, c(h))}{e} \leq \tilde{\alpha}\}} f(h, e)\, de\, dh} \leq \tilde{\alpha}. \qquad (3.14)$$

---

the integrand is a continuous function of α for almost all h, e.

Using $T(\tilde{\alpha}) > \tilde{\alpha}$ and (3.14), we apply (3.12) to obtain $T(\tilde{\alpha}) > T(\alpha)$.
For (ii) assume that $T(\alpha)$ is not decreasing right of the first fixed-point. Because $T(1) < 1$ and $T(\cdot)$ is continuous, there exist $\tilde{\alpha}, \alpha$, $\tilde{\alpha} > \alpha$, such that $\alpha > \max(\tilde{T}(\tilde{\alpha}), \tilde{T}(\alpha))$ and $\tilde{T}(\tilde{\alpha}) > \tilde{T}(\alpha)$. However, replicating the computations in (3.13) and (3.14), we observe

$$\tilde{T}(\tilde{\alpha}) - \tilde{T}(\alpha) \leq 0,$$

a contradiction. We conclude that there exists a unique contribution rate α^* that balances PU's budget.

Profits of PR. Start by observing that

$$\mathbf{1}_{PR(\alpha)} = \mathbf{1}_{\{e \geq \max\left(K_1, \min\left(\frac{\min(L, c(h))}{\alpha}, K_2\right)\right)\}}$$

is increasing in h and e. Analogously,

$$\mathbf{1}_{PU(\alpha)} = \mathbf{1}_{\{e < \max\left(K_1, \min\left(\frac{\min(L, c(h))}{\alpha}, K_2\right)\right)\}}$$

is decreasing in h and e. Furthermore, $\min(K_2, e)$ is increasing in e, and $\min(L, c(h))$ is decreasing in h. These observations together with the fact that $f(h, e)$ is affiliated, i.e., log-supermodular, allow us to apply the Fortuin-Kasteleyn-Ginibre (FKG) inequality to obtain:

$$\begin{aligned}
\mathbb{E}\left[\min(L, c(h))\mathbf{1}_{PR(\alpha)}\right] &\leq \mathbb{E}\left[\min(L, c(h))\right] \; \mathbb{E}\left[\mathbf{1}_{PR(\alpha)}\right], \\
\mathbb{E}\left[\min(K_2, e)\mathbf{1}_{PR(\alpha)}\right] &\geq \mathbb{E}\left[\min(K_2, e)\right] \; \mathbb{E}\left[\mathbf{1}_{PR(\alpha)}\right], \\
\mathbb{E}\left[\min(L, c(h))\mathbf{1}_{PU(\alpha)}\right] &\geq \mathbb{E}\left[\min(L, c(h))\right] \; \mathbb{E}\left[\mathbf{1}_{PU(\alpha)}\right], \\
\mathbb{E}\left[\min(K_2, e))\mathbf{1}_{PU(\alpha)}\right] &\leq \mathbb{E}\left[\min(K_2, e)\right] \; \mathbb{E}\left[\mathbf{1}_{PU(\alpha)}\right].
\end{aligned}$$

The four inequalities above yield

$$\frac{\mathbb{E}\left[\min(L, c(h))\mathbf{1}_{PR(\alpha)}\right]}{\mathbb{E}\left[\min(K_2, e)\mathbf{1}_{PR(\alpha)}\right]} \leq \frac{\mathbb{E}\left[\min(L, c(h))\right]}{\mathbb{E}\left[\min(K_2, e)\right]} \leq \frac{\mathbb{E}\left[\min(L, c(h))\mathbf{1}_{PU(\alpha)}\right]}{\mathbb{E}\left[\min(K_2, e)\mathbf{1}_{PU(\alpha)}\right]}. \tag{3.15}$$

In equilibrium we have

$$\frac{\mathbb{E}\left[\min(L, c(h))\mathbf{1}_{PR(\alpha^*)}\right]}{\mathbb{E}\left[\min(K_2, e)\mathbf{1}_{PR(\alpha^*)}\right]} \leq \frac{\mathbb{E}\left[\min(L, c(h))\mathbf{1}_{PU(\alpha^*)}\right]}{\mathbb{E}\left[\min(K_2, e)\mathbf{1}_{PU(\alpha^*)}\right]} = \alpha^*.$$

Rearranging terms gives

$$\mathbb{E}\left[\left(\alpha^* \min(K_2, e) - \min(L, c(h))\right) \mathbf{1}_{PR(\alpha^*)}\right] \geq 0,$$

which concludes the proof. □

3.7.3 Proofs for Comparative Statics in Policy Parameters

Proof of Proposition 3.3.2. Consider an increase of K_1 to $\tilde{K}_1$, $K_1 \leq \tilde{K}_1$. For this proof, we make the dependence of $T(\cdot)$ on K_1 explicit and write $T_{K_1}(\cdot)$. Similarly, we denote the set of PU customers by $PU_{K_1}(\alpha)$. Let the contribution rates α^* and $\tilde{\alpha}^*$ be the unique fixed points of $T_{K_1}(\cdot)$ and $T_{\tilde{K}_1}(\cdot)$ respectively. We argue that $T_{K_1}(\tilde{\alpha}^*) \geq T_{\tilde{K}_1}(\tilde{\alpha}^*)$ which implies $\alpha^* \geq \tilde{\alpha}^*$ because the fixed point is unique. Spelling out $T_{K_1}(\tilde{\alpha}^*) \geq T_{\tilde{K}_1}(\tilde{\alpha}^*)$, we obtain

$$\frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{K_1}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(K_2, e))\mathbf{1}_{PU_{K_1}(\tilde{\alpha}^*)}]} \geq \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(K_2, e))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}. \tag{3.16}$$

We distinguish two cases.

Case 1. First, let $K_1 \leq \tilde{K}_1 \leq K_2$. Observe that

$$\begin{aligned}
\mathbf{1}_{PU_{K_1}(\tilde{\alpha}^*)} &= \mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\ \tilde{\alpha}^* e < \min(L, c(h))\}} \\
&= \mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < \tilde{K}_1\}} - \mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}} + \mathbf{1}_{\{\tilde{K}_1 \leq e < K_2,\ \tilde{\alpha}^* e < \min(L, c(h))\}} \\
&= \mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)} - \mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}}.
\end{aligned}$$

Hence, we can rewrite (3.16) as

$$\frac{\mathbb{E}[\min(L, c(h))(\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)} - \mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}})]}{\mathbb{E}[\min(K_2, e)(\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)} - \mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}})]} \geq \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(K_2, e))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}. \tag{3.17}$$

Similarly as in (3.12), we have

$$\frac{a-b}{c-d} \geq \frac{a}{c} \Leftrightarrow \frac{b}{d} \leq \frac{a}{c}$$

for $c - d > 0$, $a, b, c, d \geq 0$. Therefore, (3.17) is equivalent to

$$\frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}}]}{\mathbb{E}[\min(K_2, e)\mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}}]} \leq \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(K_2, e))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}.$$

Exploiting the indicator function of term on the left side of the above inequality and the fact that $\tilde{\alpha}^*$ is a fixed point of $T_{\tilde{K}_1}(\cdot)$, we obtain

$$\frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}}]}{\mathbb{E}[\min(K_2, e)\mathbf{1}_{\{K_1 \leq e < \tilde{K}_1,\ \tilde{\alpha}^* e \geq \min(L, c(h))\}}]} \leq \tilde{\alpha}^* = \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(K_2, e))\mathbf{1}_{PU_{\tilde{K}_1}(\tilde{\alpha}^*)}]}.$$

Thus, (3.16) holds in this case.

Case 2. Second, consider the case $K_2 \leq K_1 \leq \tilde{K}_1$. (3.16) becomes

$$\frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{\{e<K_1\}}]}{\mathbb{E}[\min(K_2, e)\mathbf{1}_{\{e<K_1\}}]} \geq \frac{\mathbb{E}[\min(L, c(h))(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e<\tilde{K}_1\}})]}{\mathbb{E}[\min(K_2, e)(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e<\tilde{K}_1\}})]}.$$

Using (3.12), the latter inequality is equivalent to

$$\frac{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(L, c(h))\mathbf{1}_{\{e<K_1\}} f(h,e)\, dh\, de}{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(K_2, e)\mathbf{1}_{\{e<K_1\}} f(h,e)\, dh\, de} \geq \frac{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(L, c(h))\mathbf{1}_{\{K_1 \leq e\}} f(h,e)\, dh\, de}{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(K_2, e)\mathbf{1}_{\{K_1 \leq e\}} f(h,e)\, dh\, de}. \tag{3.18}$$

Now, we proceed as in the proof Theorem 3.3.1 where we showed that PR's profit is positive. Note that $\mathbf{1}_{\{e<K_1\}}$ is a decreasing function of h, e, and that $\mathbf{1}_{\{K_1 \leq e\}}$ is an increasing function of h, e. Together with the affiliation of $f(h, e)$ and the monotonicity of $\min(L, c(h))$ and $\min(K_2, e)$, these observations imply, using the FKG inequality,

$$\frac{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(L, c(h))\mathbf{1}_{\{K_1 \leq e\}} f(h,e)\, dh\, de}{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(K_2, e)\mathbf{1}_{\{K_1 \leq e\}} f(h,e)\, dh\, de} \leq \frac{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(L, c(h)) f(h,e)\, dh\, de}{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(K_2, e) f(h,e)\, dh\, de}$$

and

$$\frac{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(L, c(h)) f(h,e)\, dh\, de}{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(K_2, e) f(h,e)\, dh\, de} \leq \frac{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(L, c(h))\mathbf{1}_{\{e<K_1\}} f(h,e)\, dh\, de}{\int_0^{\tilde{K}_1}\int_{\underline{h}}^{\overline{h}} \min(K_2, e)\mathbf{1}_{\{e<K_1\}} f(h,e)\, dh\, de}.$$

Thus, (3.18) holds, implying that (3.16) holds also in this case which concludes the proof. □

Proof of Proposition 3.3.3. We proceed similar as in the proof of Proposition 3.3.2. Consider a decrease of K_2 to $\tilde{K}_2$, $\tilde{K}_2 \leq K_2$. For this proof, we make the dependence of $T(\cdot)$ on K_2 explicit and write $T_{K_2}(\cdot)$. Similarly, we denote the set of PU customers by $PU_{K_2}(\alpha)$. Let the contribution rates α^* and $\tilde{\alpha}^*$ be the unique fixed points of $T_{K_2}(\cdot)$ and $T_{\tilde{K}_2}(\cdot)$ respectively.

Proof of (i). First, consider the case $K_1 \leq \tilde{K}_2 \leq K_2$. We argue that $T_{K_2}(\tilde{\alpha}^*) \geq T_{\tilde{K}_2}(\tilde{\alpha}^*)$ which implies $\alpha^* \geq \tilde{\alpha}^*$ because the fixed point is unique. Spelling out $T_{K_2}(\tilde{\alpha}^*) \geq T_{\tilde{K}_2}(\tilde{\alpha}^*)$, we obtain

$$\frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{K_2}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(K_2, e))\mathbf{1}_{PU_{K_2}(\tilde{\alpha}^*)}]} \geq \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}{\mathbb{E}[\min(\tilde{K}_2, e))\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}. \tag{3.19}$$

Observe that

$$\begin{aligned}
\mathbf{1}_{PU_{K_2}(\tilde{\alpha}^*)} &= \mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}} \\
&= \mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e<\tilde{K}_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}} - \mathbf{1}_{\{\tilde{K}_2 \leq e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}} \\
&= \mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)} - \mathbf{1}_{\{\tilde{K}_2 \leq e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}}.
\end{aligned}$$

Hence, we can rewrite (3.19) as

$$\frac{\mathbb{E}[\min(L,c(h))(\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}+\mathbf{1}_{\{\tilde{K}_2\le e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}})]}{\mathbb{E}[e(\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}+\mathbf{1}_{\{\tilde{K}_2\le e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}})]}\ge\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}{\mathbb{E}[e\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]},\quad(3.20)$$

where we used the indicator functions to simplify the denominators. The latter inequality is equivalent to

$$\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{\{\tilde{K}_2\le e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}})]}{\mathbb{E}[e\mathbf{1}_{\{\tilde{K}_2\le e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}}]}\ge\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}{\mathbb{E}[e\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}$$

by (3.12). Exploiting the indicator function of term on the left side of the above inequality and the fact that $\tilde{\alpha}^*$ is a fixed point of $T_{\tilde{K}_2}(\cdot)$, we obtain

$$\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{\{\tilde{K}_2\le e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}})]}{\mathbb{E}[e\mathbf{1}_{\{\tilde{K}_2\le e<K_2,\ \tilde{\alpha}^* e<\min(L,c(h))\}}]}\ge\tilde{\alpha}^*=\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}{\mathbb{E}[e\mathbf{1}_{PU_{\tilde{K}_2}(\tilde{\alpha}^*)}]}.$$

Thus, (3.19) holds.

Proof of (ii). Second, consider the case $\tilde{K}_2\le K_2\le K_1$. Observe that $T(\cdot)$ is constant in α in this case. We argue that $T_{\tilde{K}_2}(\cdot)\ge T_{K_2}(\cdot)$ which implies $\tilde{\alpha}^*\ge\alpha^*$. Spelling out $T_{\tilde{K}_2}(\cdot)\ge T_{K_2}(\cdot)$, we obtain

$$\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{\{e<K_1\}}]}{\mathbb{E}[\min(\tilde{K}_2,e))\mathbf{1}_{\{e<K_1\}}]}\ge\frac{\mathbb{E}[\min(L,c(h))\mathbf{1}_{\{e<K_1\}}]}{\mathbb{E}[\min(K_2,e))\mathbf{1}_{\{e<K_1\}}]},$$

which holds as $\tilde{K}_2<K_2$. □

Proof of Proposition 3.3.4. Consider an increase of L to $\tilde{L}$, $L\le\tilde{L}$. For this proof, we make the dependence of $T(\cdot)$ on L explicit and write $T_L(\cdot)$. Similarly, we denote the set of PU customers by $PU_L(\alpha)$. Let the contribution rates α^* and $\tilde{\alpha}^*$ be the unique fixed points of $T_L(\cdot)$ and $T_{\tilde{L}}(\cdot)$ respectively. We argue that $T_{\tilde{L}}(\alpha^*)\ge T_L(\alpha^*)=\alpha^*$ which implies $\tilde{\alpha}^*\ge\alpha^*$ because the fixed point is unique. $T_{\tilde{L}}(\alpha^*)$ is given by

$$\frac{\mathbb{E}[\min(\tilde{L},c(h))\mathbf{1}_{PU_{\tilde{L}}(\alpha^*)}]}{\mathbb{E}[\min(K_2,e))\mathbf{1}_{PU_{\tilde{L}}(\alpha^*)}]}.$$

Consider the numerator of this fraction

$$\begin{aligned}
&\mathbb{E}[\min(\tilde{L},c(h))(\mathbf{1}_{\{e<K_1\}}+\mathbf{1}_{\{K_1\le e<K_2,\ \alpha^* e<\min(\tilde{L},c(h))\}})]\\
&=\mathbb{E}[\min(\tilde{L},c(h))(\mathbf{1}_{\{e<K_1\}}+\mathbf{1}_{\{K_1\le e<K_2,\ \alpha^* e<\min(L,c(h))\}}+\mathbf{1}_{\{K_1\le e<K_2,\ \min(L,c(h))\le\alpha^* e<\min(\tilde{L},c(h))\}})]\\
&\ge\mathbb{E}[\min(\tilde{L},c(h))(\mathbf{1}_{\{e<K_1\}}+\mathbf{1}_{\{K_1\le e<K_2,\ \alpha^* e<\min(L,c(h))\}})]+\mathbb{E}[\alpha^* e\,\mathbf{1}_{\{K_1\le e<K_2,\ \min(L,c(h))\le\alpha^* e<\min(\tilde{L},c(h))\}})]\\
&\ge\mathbb{E}[\min(L,c(h))(\mathbf{1}_{\{e<K_1\}}+\mathbf{1}_{\{K_1\le e<K_2,\ \alpha^* e<\min(L,c(h))\}})]+\mathbb{E}[\alpha^* e\,\mathbf{1}_{\{K_1\le e<K_2,\ \min(L,c(h))\le\alpha^* e<\min(\tilde{L},c(h))\}})].
\end{aligned}$$

To get from the second to the third line, we exploit the third indicator function. From the third to the fourth line we use $\min(L, c(h)) \leq \min(\tilde{L}, c(h))$. Thus, we obtain

$$
\begin{aligned}
&T_{\tilde{L}}(\alpha^*) \\
&\geq \frac{\mathbb{E}[\min(L, c(h))(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\, \alpha^* e < \min(L, c(h))\}})] + \mathbb{E}[\alpha^* e\, \mathbf{1}_{\{K_1 \leq e < K_2,\, \min(L,c(h)) \leq \alpha^* e < \min(\tilde{L}, c(h))\}})]}{\mathbb{E}[e\,(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\, \alpha^* e < \min(\tilde{L}, c(h)\}})]} \\
&= \frac{\mathbb{E}[\min(L, c(h))(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\, \alpha^* e < \min(L, c(h))\}})] + \mathbb{E}[\alpha^* e\, \mathbf{1}_{\{K_1 \leq e < K_2,\, \min(L,c(h)) \leq \alpha^* e < \min(\tilde{L}, c(h))\}})]}{\mathbb{E}[e\,(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\, \alpha^* e < \min(L, c(h)\}})] + \mathbb{E}[e \mathbf{1}_{\{K_1 \leq e < K_2,\, \min(L,c(h)) \leq \alpha^* e < \min(\tilde{L}, c(h))\}}]}.
\end{aligned}
$$

Observe that

$$
\frac{\mathbb{E}[\min(L, c(h))(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\, \alpha^* e < \min(L, c(h))\}})]}{\mathbb{E}[e\,(\mathbf{1}_{\{e<K_1\}} + \mathbf{1}_{\{K_1 \leq e < K_2,\, \alpha^* e < \min(L, c(h)\}})]} = T_L(\alpha^*) = \alpha^*
$$

and

$$
\frac{\mathbb{E}[\alpha^* e\, \mathbf{1}_{\{K_1 \leq e < K_2,\, \min(L,c(h)) \leq \alpha^* e < \min(\tilde{L}, c(h))\}})]}{\mathbb{E}[e \mathbf{1}_{\{K_1 \leq e < K_2,\, \min(L,c(h)) \leq \alpha^* e < \min(\tilde{L}, c(h))\}}]} = \alpha^*.
$$

Because

$$
\frac{a+b}{c+d} = \frac{a}{c} \Leftrightarrow \frac{a}{c} = \frac{b}{d}.
$$

for $a, b, c, d > 0$, we conclude that

$$
T_{\tilde{L}}(\alpha^*) \geq \alpha^*.
$$

□

3.7.4 Proofs for Structural Population Changes

Proofs for Systematic Improvement of Health and Income

Preliminaries. We make the dependence of the expectation operator on the distribution f explicit and write $\mathbb{E}_f[\cdot]$. Throughout the proof we use the following characterization of (multivariate) first-order stochastic dominance, cf. Shaked and Shanthikumar (2007),

Theorem 3.7.1. *Consider two probability distributions over $\mathbb{R}^n$ with densities $\tilde{f}$ and f respectively. $\tilde{f}$ first-order stochastically dominates f if and only if $\mathbb{E}_{\tilde{f}}[\phi] \geq \mathbb{E}_f[\phi]$ for all increasing functions $\phi : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist.*

Let $\tilde{f}(h, e)$ first-order stochastically dominate $f(h, e)$. Other than that, we assume that $\tilde{f}(h, e)$ satisfies the same assumptions as $f(h, e)$. As before, we are interested in fixed-points of the function

$$
T_f(\alpha) = \frac{\mathbb{E}_f[\min(c(h), L)\mathbf{1}_{PU(\alpha)}]}{\mathbb{E}_f[\min(K_2, e)\mathbf{1}_{PU(\alpha)}]}, \tag{3.21}
$$

where we made the dependence of $T(\cdot)$ on the distribution explicit. By the proof of Theorem 3.3.1, $T_f(\alpha)$ has a unique fixed-point α^* and is increasing for $\alpha \le \alpha^*$ and decreasing for $\alpha \ge \alpha^*$. Denote by α^* the equilibrium contribution rate associated with $f(h,e)$ and by $\tilde{\alpha}^*$ the equilibrium contribution rate associated with $\tilde{f}(h,e)$. If we argue that

$$T_{\tilde{f}}(\alpha^*) \ge (\le) T_f(\alpha^*) = \alpha^*,$$

then we know that $\tilde{\alpha}^* \ge (\le)\alpha^*$.

Start by observing that

$$\mathbb{E}_{\tilde{f}}[\alpha^* \min(K_2, e) - \min(c(h), L)] \ge \mathbb{E}_f[\alpha^* \min(K_2, e) - \min(c(h), L)] = 0. \tag{3.22}$$

because $\alpha^* \min(K_2, e) - \min(c(h), L)$ is an increasing function of (h, e). Hence, if the entire population would insure with PU, the contribution rate could be adjusted downward. Also, note that PR's profit from an (h,e)-type customer, $(\alpha \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha)}$, is an increasing function of α, for all h, e.

Formally, the decomposition outlined in the text is given by

$$PU^+(\alpha^*) = \{(h,e)|\ \alpha^* \min(K_2, e) - \min(c(h), L) \ge 0,\ e < K_1\}, \tag{3.23}$$
$$PU^-(\alpha^*) = \{(h,e)|\ \alpha^* \min(K_2, e) - \min(c(h), L) < 0,\ e < \max(K_1, K_2)\}, \tag{3.24}$$
$$PR^+(\alpha^*) = \{(h,e)|\ \alpha^* \min(K_2, e) - \min(c(h), L) \ge 0,\ e \ge K_1\}, \tag{3.25}$$
$$PR^-(\alpha^*) = \{(h,e)|\ \alpha^* \min(K_2, e) - \min(c(h), L) < 0,\ e \ge \max(K_1, K_2)\}. \tag{3.26}$$

See also Figure ??. Define $\mathbb{E}_{\tilde{f}-f}[\cdot] := \mathbb{E}_{\tilde{f}}[\cdot] - \mathbb{E}_f[\cdot]$. Consider the difference in insurances' profit under $\tilde{f}(h,e)$ and $f(h,e)$ on each set of customers (3.23)-(3.26),

$$\mathbb{E}_{\tilde{f}-f}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PU^+(\alpha^*)}], \tag{3.27}$$
$$\mathbb{E}_{\tilde{f}-f}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PU^-(\alpha^*)}], \tag{3.28}$$
$$\mathbb{E}_{\tilde{f}-f}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR^+(\alpha^*)}], \tag{3.29}$$
$$\mathbb{E}_{\tilde{f}-f}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR^-(\alpha^*)}]. \tag{3.30}$$

By (3.22) we have

$$(3.27) + (3.28) + (3.29) + (3.30) \ge 0. \tag{3.31}$$

We verify the statements about the impact of customers' movements on insurances' profit from each subgroup made in the main body of the text. Checking monotonicity of the appropriate functions and applying Theorem 3.7.1 yields (3.28)$\ge$ 0, (3.28)+(3.30)$\ge$ 0, (3.29)$\ge$ 0, and (3.29)+(3.27)$\ge$ 0.

Proof of Proposition 3.3.5.

Proof of (i). By assumption (3.27)+(3.28)≤0, which is equivalent to

$$\alpha^* = T_f(\alpha^*) \le T_{\tilde{f}}(\alpha^*),$$

hence, $\tilde{\alpha}^* \ge \alpha^*$. Furthermore, by (3.27)+(3.28)≤0 and (3.31), we have 0≤(3.29)+(3.30), i.e.,

$$\mathbb{E}_{\tilde{f}}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha^*)}] - \mathbb{E}_f[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha^*)}] \ge 0. \tag{3.32}$$

(3.32) and $\tilde{\alpha}^* \ge \alpha^*$, together with monotonicity of PR's profit in α show that PR's profit increase under $\tilde{f}$.

Proof of (ii). By assumption (3.29)+(3.30)≤ 0. This implies (3.27)+(3.28)≥ 0, i.e.,

$$\alpha^* = T_f(\alpha^*) \ge T_{\tilde{f}}(\alpha^*),$$

therefore, $\tilde{\alpha}^* \le \alpha^*$. Also, by (3.29)+(3.30)≤ 0,

$$\mathbb{E}_{\tilde{f}}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha^*)}] - \mathbb{E}_f[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha^*)}] \le 0. \tag{3.33}$$

(3.33) and $\tilde{\alpha}^* \le \alpha^*$ show that PR's profit decreases.

Proof of (iii). By assumption (3.27)+(3.28)≥ 0, which is equivalent to

$$\alpha^* = T_f(\alpha^*) \ge T_{\tilde{f}}(\alpha^*),$$

and hence, $\tilde{\alpha}^* \le \alpha^*$. Furthermore, (3.29)+(3.30)≥ 0, i.e.,

$$\mathbb{E}_{\tilde{f}}[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha^*)}] - \mathbb{E}_f[(\alpha^* \min(K_2, e) - \min(c(h), L))\mathbf{1}_{PR(\alpha^*)}] \ge 0. \tag{3.34}$$

The positive effect of the shift from f to $\tilde{f}$ on PR's profit, (3.34), may be mitigated by the decrease of the equilibrium contribution rate. The exact effect on PR's profit depends on the specific shift $\tilde{f}$. □

Proofs for Increase in Correlation Between Health and Income

Preliminaries. We make the dependence of the expectation operator on the distribution f explicit and write $\mathbb{E}_f[\cdot]$. Throughout the proof we use the following characterization of the supermodular order, see Shaked and Shanthikumar (2007).

Definition 3.7.1. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called supermodular if for every two points*

$(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ *it holds*

$$f(\max(x_1, x_2), \max(y_1, y_2)) + f(\min(x_1, x_2), \min(y_1, y_2)) \geq f(x_1, y_1) + f(x_2, y_2) \tag{3.35}$$

Theorem 3.7.2. *Consider two probability distributions over* $\mathbb{R}^n$ *with respective densities* g *and* f *which coincide on their marginal distributions.* g *is larger than* f *in the supermodular order if and only if* $\mathbb{E}_g[\phi] \geq \mathbb{E}_f[\phi]$ *for all supermodular functions* $\phi : \mathbb{R}^n \to \mathbb{R}$ *for which the expectations exist.*

Proof of Proposition 3.3.6. Assume that $g(h, e)$ is larger than $f(h, e)$ in the supermodular order. As before, we are interested in fixed-points of the function

$$T_f(\alpha) = \frac{\mathbb{E}_f[\min(c(h), L)\mathbf{1}_{PU(\alpha)}]}{\mathbb{E}_f[\min(K_2, e)\mathbf{1}_{PU(\alpha)}]}, \tag{3.36}$$

where we made the dependence of $T(\cdot)$ on the distribution explicit. By the proof of Theorem 3.3.1, $T_f(\alpha)$ has a unique fixed-point α^* and is increasing for $\alpha \leq \alpha^*$ and decreasing for $\alpha \geq \alpha^*$, i.e. for every density f the according operator T_f is maximized in the fixed point. Denote by α^* the equilibrium contribution rate associated with $f(h, e)$ and by $\tilde{\alpha}^*$ the equilibrium contribution rate associated with $g(h, e)$. If we argue that

$$T_g(\alpha^*) \geq T_f(\alpha^*) = \alpha^*,$$

then we know that $\tilde{\alpha}^* \geq \alpha^*$. Observe that if $K_1 \geq K_2$, the patient set of PU reduces to the set $\{e < K_1\}$ and we have

$$T_g(\alpha) = \frac{\mathbb{E}_g[\min(c(h), L)\mathbf{1}_{\{e<K_1\}}]}{\mathbb{E}_g[\min(K_2, e)\mathbf{1}_{\{e<K_1\}}]}. \tag{3.37}$$

First, consider the denominator of the latter expression

$$\mathbb{E}_g[\min(K_2, e)\mathbf{1}_{\{e<K_1\}}] = \int_0^{\bar{e}} \min(K_2, e)\mathbf{1}_{\{e<K_1\}} g(e)\, de = \mathbb{E}_f[\min(K_2, e)\mathbf{1}_{\{e<K_1\}}], \tag{3.38}$$

where the last equality follows from the fact that g and f have the same marginals. Second, we analyze the numerator of (3.37). Let $e' \geq e$ and $h' \geq h$, then

$$\min(c(h'), L)\mathbf{1}_{\{e'<K_1\}} + \min(c(h), L)\mathbf{1}_{\{e<K_1\}} \geq \min(c(h'), L)\mathbf{1}_{\{e<K_1\}} + \min(c(h), L)\mathbf{1}_{\{e'<K_1\}}$$

holds since in case $\mathbf{1}_{\{e'<K_1\}} = 0$ we have $\min(c(h), L)\mathbf{1}_{\{e<K_1\}} \geq \min(c(h'), L)\mathbf{1}_{\{e<K_1\}}$ because $\min(c(h), L)$ is decreasing in h and in case $\mathbf{1}_{\{e'<K_1\}} = 1$ both sides are equal as $\mathbf{1}_{\{e'<K_1\}}$ decreases in e. Consequently, $\min(c(h), L)\mathbf{1}_{\{e<K_1\}}$ is a supermodular function, and by definition of the supermodular order we obtain

$$\mathbb{E}_g[\min(c(h), L)\mathbf{1}_{\{e<K_1\}}] \geq \mathbb{E}_f[\min(c(h), L)\mathbf{1}_{\{e<K_1\}}]. \tag{3.39}$$

Putting (3.38) and (3.39) together, we get

$$T_g(\alpha^*) = \frac{\mathbb{E}_g[\min(c(h), L)\mathbf{1}_{\{e<K_1\}}]}{\mathbb{E}_g[\min(K_2, e)\mathbf{1}_{\{e<K_1\}}]} \geq \frac{\mathbb{E}_f[\min(c(h), L)\mathbf{1}_{\{e<K_1\}}]}{\mathbb{E}_f[\min(K_2, e)\mathbf{1}_{\{e<K_1\}}]} = T_f(\alpha^*) = \alpha^*$$

which concludes the proof. □

3.7.5 Proofs for Applications

Proofs for Health Premia

Proof of Proposition 3.4.1. Fix any two premia A_1 and A_2 set by PM_1 and PM_2 respectively. The contract set $\mathcal{C}(h, e)$ of a customer with type (h, e) is

$$\mathcal{C}(h, e) = \{(L, A_1), (L, A_2)\}.$$

Because health benefits are equal, it is optimal for every customer to choose contract (L, A_i) with $A_i = \min(A_1, A_2)$.

Start by observing that the following is an equilibrium: $A_1 \geq A_2$, $A_2 = \mathbb{E}[\min(c(h), L)]$ and all customers choose PM_2. We now deduce more generally that the premium paid by all customers is $\mathbb{E}[\min(c(h), L)]$ in any equilibrium of the premium-based health insurance market. It is convenient to denote by $\beta(h, e) \in \{0, 1\}$ customer-(h, e)'s choice of insurance, where $\beta(h, e) = 1$ means that the customer chooses PM_1, and $\beta(h, e) = 0$ means that the customer chooses PM_2. Let $(A_1^*, A_2^*, \beta^*(h, e))$ be an equilibrium of the premium-based health insurance market.

Case 1. If $A_1^* > A_2^*$, then $\beta^*(h, e) = 0$, for all (h, e). PM_2's equilibrium condition requires $A_2^* = \mathbb{E}[\min(c(h), L)]$. The case $A_1^* < A_2^*$ is symmetric.

Case 2. If $A_1^* = A_2^*$, PM_1's and PM_2's equilibrium condition requires

$$\mathbb{E}\left[\beta^*(h, e) \min(c(h), L)\right] = \mathbb{E}\left[A_1^* \beta^*(h, e)\right] \tag{3.40}$$

and

$$\mathbb{E}\left[(1 - \beta^*(h, e)) \min(c(h), L)\right] = \mathbb{E}\left[A_2^*(1 - \beta^*(h, e))\right]. \tag{3.41}$$

Adding up (3.40) and (3.41) and using $A_1^* = A_2^*$ yields

$$\mathbb{E}\left[\min(c(h), L)\right] = A_1^* = A_2^*,$$

which concludes the proof.[29] □

[29]Strictly speaking, we restrict attention to equilibria where $\beta(\cdot, \cdot)$ is measurable with respect to (h, e).

Proof of Corollary 3.4.1. Comparing the income-increasing payment of the contribution-based system

$$\alpha^* \min(K_2, e)$$

to the income-constant payment of the premium-based system

$$\mathbb{E}[\min(L, c(h))]$$

yields the existence of a threshold $e^* \in [0, \bar{e}]$ such that for all $e < e^*$ we have $\alpha^* \min(K_2, e) < \mathbb{E}[\min(L, c(h))]$, and for all $e > e^*$ we have $\alpha^* \min(K_2, e) > \mathbb{E}[\min(L, c(h))]$. As health benefits are equal in both system customers with income $e > e^*$ enjoy a higher utility and customers with income $e < e^*$ enjoy a lower utility in the premium-based system.

We now argue that $e^* \in (0, \bar{e})$. Firstly, observe that

$$\alpha^* = \frac{\mathbb{E}[\min(L, c(h))\mathbf{1}_{PU(\alpha^*)}]}{\mathbb{E}[\min(K_2, e)\mathbf{1}_{PU(\alpha^*)}]} \geq \frac{\mathbb{E}[\min(L, c(h))]}{\mathbb{E}[\min(K_2, e)]},$$

where the equality follows from α^* being a fixed point of $T(\cdot)$, and the inequality follows from (3.15). Therefore, we can conclude that

$$\alpha^* \min(K_2, \bar{e}) \geq \frac{\mathbb{E}[\min(L, c(h))]}{\mathbb{E}[\min(K_2, e)]} \min(K_2, \bar{e}) > \mathbb{E}[\min(L, c(h))].$$

Secondly, note that

$$\alpha^* \min(K_2, 0) = 0 < \mathbb{E}[\min(L, c(h)],$$

which concludes the proof. □

Proof of Proposition 3.4.2. Fix a payment $p(h, e)$ for each customer type. Given this set of payments, welfare is

$$\mathcal{W}(p(h, e)) = \mathbb{E}\left[u(\min(c(h), L) - c(h) + e - p(h, e))\right]. \tag{3.42}$$

Proof of (i). Set $K_1 = \bar{e}$. Recall that the payment in the contribution-based system is $\alpha^* \min(K_2, e)$, whereas it is $A^* = \mathbb{E}[\min(L, c(h))]$ in the premium-based system. As $K_1 = \bar{e}$, PU insures all customers, $\mathbf{1}_{PU} = 1$ and budget-balancing of PU implies

$$\alpha^* \, \mathbb{E}[\min(K_2, e)] = \mathbb{E}[\min(L, c(h))] = A^*. \tag{3.43}$$

Note that the result still holds if customers are allowed to randomize, i.e., if $\beta(h, e) \in [0, 1]$ denotes the probability that customer-(h, e) chooses PM_1.

To save on notation define $\psi(h,e) = \min(c(h), L) - c(h) + e$ and note that $\psi(\cdot,\cdot)$ is increasing in both arguments. Consider the welfare difference between the premium-based system and the contribution-based system, with (3.43)

$$\begin{aligned} &\mathbb{E}\left[u\left(\psi(h,e) - \alpha^* \, \mathbb{E}[\min(K_2,e)]\right)\right] - \mathbb{E}\left[u\left(\psi(h,e) - \alpha^* \min(K_2,e)\right)\right] \\ &\quad < \mathbb{E}\left[u'\left(\psi(h,e) - \alpha^* \min(K_2,e)\right)\left(\alpha^* \min(K_2,e) - \alpha^* \, \mathbb{E}[\min(K_2,e)]\right)\right], \end{aligned} \tag{3.44}$$

where the inequality follows from strict concavity of $u(\cdot)$. Observe that

1. $u'\left(\psi(h,e) - \alpha^* \min(K_2,e)\right)$ is decreasing in (h,e) because $u'(\cdot)$ is decreasing and $\psi(h,e) - \alpha^* \min(K_2,e)$ is increasing in (h,e) as $\alpha^* \leq 1$.

2. $\alpha^* \min(K_2,e) - \alpha^* \, \mathbb{E}[\min(K_2,e)]$ is weakly increasing in (h,e).

Hence, the FKG inequality implies that (3.44) is bounded from above by the constant

$$\mathbb{E}\left[u'\left(\psi(h,e) - \alpha^* \min(K_2,e)\right)\right] \mathbb{E}\left[\alpha^* \left(\min(K_2,e) - \mathbb{E}[\min(K_2,e)]\right)\right] = 0,$$

where the last equality follows from

$$\mathbb{E}\left[\alpha^* \left(\min(K_2,e) - \mathbb{E}[\min(K_2,e)]\right)\right] = 0. \tag{3.45}$$

Therefore, the contribution-based system with $K_1 = \bar{e}$ gives the population a strictly higher welfare than the premium-based system. Recall that welfare is increasing in K_1. Thus, for sufficiently high K_1 the contribution-based system is welfare-dominant.

Proof of (ii). Consider the income redistribution scheme that is defined by the transfer $\tau(e)$ to agent with income e, where

$$\tau(e) = \alpha^* \, \mathbb{E}[\min(K_2,e)] - \alpha^* \min(K_2,e).$$

By definition the premium-based system together with this income redistribution scheme gives the population the same welfare as the welfare-optimal contribution-based system, i.e., the system with $K_1 = \bar{e}$. Furthermore, (3.45) implies that the income redistribution scheme is budget-balanced. □

Proofs for Welfare-Optimal Payments

Proof of Proposition 3.4.3. Let $A := \mathbb{E}[\min(c(h), L)]$ be the aggregate health benefits of the population. Formally, we consider the problem

$$\max_{p(h,e)} \mathbb{E}\left[u(\min(c(h), L) - c(h) + e - p(h,e))\right], \tag{3.46}$$

$$s.t. \quad A \leq \mathbb{E}\left[p(h,e)\right]. \tag{3.47}$$

The Lagrangian

$$\mathbb{E}\left[u(\min(c(h), L) - c(h) + e - p(h,e)) + \lambda(p(h,e) - A)\right]$$

yields the first-order condition

$$u'(\min(c(h), L) - c(h) + e - p(h,e)) = \lambda. \tag{3.48}$$

Note that $u'(\cdot)$ is strictly decreasing. Solving for $p(h,e)$ and inserting into the constraint, (3.47), gives

$$A = \mathbb{E}\left[-u'^{-1}(\lambda) + \min(c(h), L) - c(h) + e\right].$$

Using the definition of A, we obtain

$$\lambda = u'\left(\mathbb{E}\left[e - c(h)\right]\right). \tag{3.49}$$

Equating (3.48) and (3.49) yields

$$u'(\min(c(h), L) - c(h) + e - p(h,e)) = u'\left(\mathbb{E}\left[e - c(h)\right]\right). \tag{3.50}$$

Again exploiting that $u'(\cdot)$ is strictly decreasing and after rearranging terms we obtain

$$p_{opt}(h,e) = \min(c(h), L) + e - c(h) - \mathbb{E}\left[e - c(h)\right].$$

□

Proof of Proposition 3.4.4. We start by rewriting (3.46) to account for the fact that the payment may not depend on h. For clarity we spell out all expectations explicitly.

$$\max_{p(e)} \int_{\mathcal{E}} \int_{\mathcal{H}} u(\min(c(h), L) - c(h) + e - p(e))\, f(h|e)\, \mathrm{d}h\, f(e)\, \mathrm{d}e, \tag{3.51}$$

$$s.t. \quad A \leq \int_{\mathcal{E}} p(e) f(e)\, \mathrm{d}e. \tag{3.52}$$

The Lagrangian for the problem is

$$\int_{\mathcal{E}} \int_{\mathcal{H}} u(\min(c(h), L) - c(h) + e - p(e)) f(h|e)\, \mathrm{d}h + \lambda(p(e) - A) f(e)\, \mathrm{d}e.$$

Using Leibniz's integral rule we obtain the first-order condition

$$\int_{\mathcal{H}} u'(\min(c(h), L) - c(h) + e - p(e))f(h|e) \,\mathrm{d}h - \lambda = 0. \tag{3.53}$$

(3.53) defines p as an implicit function of e. Denote the left side of (3.53) by $G(e, p)$. Then

$$\frac{\partial G(e,p)}{\partial p} = \int_{\mathcal{H}} -u''(\min(c(h), L) - c(h) + e - p)f(h|e) \,\mathrm{d}h > 0, \tag{3.54}$$

where the last inequality follows from strict concavity of $u(\cdot)$. Furthermore

$$-\frac{\partial G(e,p)}{\partial e} = \frac{\partial G(e,p)}{\partial p} + \int_{\mathcal{H}} -u'(\min(c(h), L) - c(h) + e - p) \,\frac{\partial f(h|e)}{\partial e} \,\mathrm{d}h. \tag{3.55}$$

Rewrite the second term on the right side of inequality (3.55) as

$$\int_{\mathcal{H}} -u'(\min(c(h), L) - c(h) + e - p) \,\frac{\partial \log f(h|e)}{\partial e} \,f(h|e) \,\mathrm{d}h.$$

Observe that:

1. By affiliation $\frac{\partial \log f(h|e)}{\partial e}$ is increasing in h. Indeed, we have

$$0 \leq \frac{\partial^2 \log f(h,e)}{\partial e \,\partial h} = \frac{\partial^2 \log(f(h|e)f(e))}{\partial e \,\partial h} = \frac{\partial}{\partial h}\left(\frac{\partial \log f(h|e)}{\partial e}\right).$$

2. $-u'(\min(c(h), L) - c(h) + e - p)$ is increasing in h because $\min(c(h), L) - c(h)$ is increasing and $-u'(\cdot)$ is increasing by concavity.

Neglecting the argument of $-u'(\cdot)$ for convenience and applying the FKG inequality we get

$$\int_{\mathcal{H}} -u'(\cdot) \,\frac{\partial \log f(h|e)}{\partial e} \,f(h|e) \,\mathrm{d}h \geq \int_{\mathcal{H}} -u'(\cdot) \,f(h|e) \,\mathrm{d}h \,\int_{\mathcal{H}} \frac{\partial \log f(h|e)}{\partial e} \,f(h|e) \,\mathrm{d}h. \tag{3.56}$$

Rewriting the second term on the right hand side of inequality (3.56) and using Lebesgue's dominated convergence theorem we see that

$$\int_{\mathcal{H}} \frac{\partial \log f(h|e)}{\partial e} \,f(h|e) \,\mathrm{d}h = \int_{\mathcal{H}} \frac{\partial f(h|e)}{\partial e} \,\mathrm{d}h = \frac{\partial}{\partial e}\left(\int_{\mathcal{H}} f(h|e) \,\mathrm{d}h\right) = 0.$$

since $\int_{\mathcal{H}} f(h|e) \,\mathrm{d}h = 1$. Consequently, note that

$$\int_{\mathcal{H}} -u'(\cdot) \,\frac{\partial \log f(h|e)}{\partial e} \,f(h|e) \,\mathrm{d}h \geq 0. \tag{3.57}$$

Applying the implicit function theorem, we conclude that

$$\frac{\mathrm{d}\hat{p}_{opt}}{\mathrm{d}e} = \frac{-\frac{\partial G(e,p)}{\partial e}}{\frac{\partial G(e,p)}{\partial p}} = \frac{\frac{\partial G(e,p)}{\partial p}}{\frac{\partial G(e,p)}{\partial p}} + \frac{\int_{\mathcal{H}} -u'(\cdot) \,\frac{\partial \log f(h|e)}{\partial e} \,f(h|e) \,\mathrm{d}h}{\frac{\partial G(e,p)}{\partial p}} \geq 1,$$

where the inequality follows from (3.54) and (3.57). □

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