Topological and Piecewise Linear Pseudoisotopy Functors

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Abstract

We construct two functors $\mathcal{P}_\partial^{strict} : \mathfrak{Top} \to \mathfrak{Top}$ and $\mathbb{P}_\partial^{strict} : \mathfrak{Top} \to$ Spectra such that for a compact manifold M the space $\mathcal{P}_\partial^{strict}(M)$ has the homotopy type of the stable topological pseudoisotopy space of M and $\mathbb{P}_\partial^{strict}(M)$ has the homotopy type of the topological pseudoisotopy spectrum of M. Both functors also induce homotopy functors that agree with the homotopy functor defined by Hatcher in [Hat78]. The main idea of the construction is to build a homotopy coherent diagram out of induced maps as defined by Hatcher, then strictify the diagram and finally use a left Kan extension to extend the domain of the functor to the whole category $\mathfrak{Top}$ of topological spaces. Our construction generalizes to the piecewise linear category and also yields piecewise linear versions of the two functors.

The functor $\mathcal{P}_\partial^{strict}$ was already used in the work of other authors, although no complete construction of it existed prior to this work. We aim to close this gap in the literature.

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CHAPTER 1

Introduction

Let $P_\partial(M)$ denote the space of topological pseudoisotopies on M, i.e. the space of homeomorphisms $\varphi : M \times [0,1] \to M \times [0,1]$ such that φ is the identity map on the subspace $M \times 0 \cup \partial M \times [0,1]$. The process of assigning to a manifold M the space $P_\partial(M)$ is not functorial for arbitrary continuous maps $f : M \to N$ between manifolds. However, if one assigns to M its stable pseudoisotopy space $\mathcal{P}_\partial(M)$, then the assignment is at least functorial up to homotopy due to induced maps that Hatcher constructed in [Hat78].

After Hatcher's construction was published, people started to ask whether pseudoisotopy spaces can be made into a strict functor. In 1982, Quinn constructed in [Qui82] a functor which he called a pseudoisotopy functor, but he did not check whether his functor evaluated on a manifold is homotopy equivalent to the stable pseudoisotopy space of Hatcher and others. In the same year, Waldhausen claimed in [Wal82] that stable pseudoisotopy spaces can be made functorial. He did not need this result himself and thus ignored the technical difficulties involved in actually constructing such a functor. Nevertheless, people started to cite both Quinn and Waldhausen as sources for the existence of said pseudoisotopy functor (e.g. Farrell and Jones refer in [FJ91] to Quinn and Goodwillie refers in [Goo90] to Waldhausen).

We will close this apparent gap in the literature by constructing a strict pseudoisotopy functor

$$\mathcal{P}_\partial^{strict} : \mathfrak{Top} \to \mathfrak{Top}$$

such that $\mathcal{P}_\partial^{strict}$ induces a homotopy functor which is the same homotopy functor as the one constructed by Hatcher for compact manifolds. We also construct the spectra-valued analogue, i.e. a functor

$$\mathbb{P}_\partial^{strict} : \mathfrak{Top} \to \text{Spectra}$$

such that $\mathbb{P}_\partial^{strict}$ again induces a homotopy functor and for a compact manifold M the spectrum $\mathbb{P}_\partial^{strict}(M)$ has the homotopy type of the pseudoisotopy spectrum of M. The constructions of these two functors are contained in theorems 5.6 and 6.24.

We also construct pseudoisotopy functors for the piecewise linear category. Since the piecewise linear construction is mostly analogous to the topological case, we will focus on the topological case and only remark about the piecewise linear case when there is a difference to the topological construction. The pseudoisotopy functors for both categories will be the same (see remark 5.5), which is a consequence of the fact that the stable pseudoisotopy spaces in both categories are homotopy equivalent. For readers only interested in the topological case remarks that are labeled as PL remarks can be safely skipped.

For the smooth category the construction of the pseudoisotopy functors is quite different from the construction presented here and has been completed by Malte Pieper in [Pie18].

In chapter 2 we will cover basic definitions. Chapter 3 deals with the construction of induced maps between pseudoisotopy spaces over fiber bundle maps $p : E \to M$ between manifolds. In chapter 4 we will define stable pseudoisotopy spaces and use the results of chapter 3 to construct for any continuous map $f : M \to N$ between manifolds an induced maps between the stable pseudoisotopy spaces of M and N. This allows us to give a detailed proof in corollary 4.34 of Hatcher's result in [Hat78] that stable pseudoisotopy spaces are functorial up to homotopy. Chapter 5 then contains the construction of the pseudoisotopy functor. In chapter 6 we will construct pseudoisotopy spectra and the corresponding pseudoisotopy spectrum functor.

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CHAPTER 2

Definitions

In this paper an n-dimensional manifold is considered to be a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$. All maps between manifolds or topological spaces are considered to be continuous.

Definition 2.1. Let X be a finite CW-complex. A *family of manifolds M over X* is a fiber bundle $p : M \to X$ with fiber a fixed manifold F. The boundary of M, denoted by ∂M, is a family of manifolds over X given by the ∂F-subbundle of p. A map between families $p : M \to X$ and $q : N \to X$ of manifolds over X is a continuous map $f : M \to N$ with $p = q \circ f$.

We will often denote a family of manifolds over X only by its total space M and suppress the projection map onto X in the notation. We will also omit to explicitly mention that X is a finite CW-complex and instead use the convention that if we have a family of manifolds over a space X, then X is required to be a finite CW-complex.

The trivial example for a family of manifolds over X is $M \times X \to X$ for a given manifold M. Also, we will regard every manifold as a family of manifolds over a point.

Definition 2.2. A fiber bundle between two families of manifolds E and M over X is a fiber bundle $p : E \to M$ such that p commutes with the projection maps onto X.

Remark 2.3. Let M be a family of manifolds over X and N be a family of manifolds over Y. Then the product $M \times N$ is a family of manifolds over $X \times Y$. If Y is just a point, i.e. N is a manifold, then $M \times N$ is again a family of manifolds over X.

If $p : M \to X$ is a family of manifolds over X and $Y \subseteq X$ is a subcomplex of X, then $p : p^{-1}(Y) \to Y$ is a family of manifolds over Y.

Let $I := [0,1]$ be the unit interval.

Definition 2.4. Let M be a family of manifolds over X. A *pseudoisotopy* over M is a homeomorphism $\varphi : M \times I \to M \times I$ such that φ commutes with the projection onto X (note that $M \times I$ is a family of manifolds over X) and φ restricted to $M \times 0$ is the identity map. We say that a pseudoisotopy φ is a *pseudoisotopy relative boundary* if φ restricted to $M \times 0 \cup \partial M \times I$ is also the identity map.

Denote by $\Delta^n \subseteq \mathbb{R}^n$ the standard n-simplex and by $\underline{\Delta}^n$ the simplicial n-simplex.

Definition 2.5. Let M be a family of manifolds over X. The *pseudoisotopy space* $P(M)$ of M is the singular set of the topological space of of all pseudoisotopies over M equipped with the compact-open topology. That means $P(M)$ is a simplicial set with n-simplices pseudoisotopies of $\Delta^n \times M$ with $\Delta^n \times M$ viewed as a family of manifolds over $\Delta^n \times X$, i.e. the pseudoisotopies commute with the projection onto $\Delta^n \times X$. Denote by $P_\partial(M) \subseteq P(M)$ the subspace of all pseudoisotopies relative boundary over M, i.e. an n-simplex in $P_\partial(M)$ is a self-homeomorphism of $\Delta^n \times M \times I$ commuting with the projection onto $\Delta^n \times X$ such that it is the identity when restricted to $\Delta^n \times M \times 0 \cup \Delta^n \times \partial M \times I$.

Remark 2.6 (Bending around the boundary)**.** The simplicial sets $P(M)$ and $P_\partial(M)$ are isomorphic: Let $h : M \times I \to M \times I$ be a homeomorphism with $h(M \times 0) = M \times 0 \cup \partial M \times I$. Then the map $c_h : P(M) \to P_\partial(M)$, sending an n-simplex φ to $c_h(\varphi) = (\mathrm{id}_{\Delta^n} \times h) \circ \varphi \circ (\mathrm{id}_{\Delta^n} \times h^{-1})$, is an isomorphism. The existence of such a homeomorphism h is an easy consequence of the existence of collars for topological manifolds.

Let $M \subseteq N$ be a codimension-zero-submanifold which is closed as subset of N. The one can extend a pseudoisotopy relative boundary on M to a pseudoisotopy relative boundary on N by taking the identity map on the complement of $M \times I$ inside $N \times I$. Thus we get:

Proposition 2.7. *A codimension-zero-embedding $f : M \to N$ with closed image induces a map $f_* : P_\partial(M) \to P_\partial(N)$.*

If we have an isotopy of codimension-zero-embeddings $h : M \times [0,1] \to N \times [0,1]$ between the embeddings $h_0 = h(-,0)$ and $h_1 = h(-,1) : M \to N$, then we get an isotopy between $(h_0)_*(\varphi)$ and $(h_1)_*(\varphi)$ for $\varphi \in P_\partial(M)$ by extending $\varphi \times \mathrm{id}_{[0,1]}$ with the identity outside the image of the embedding h. This leads to:

Proposition 2.8. *Let $h : M \times [0,1] \to N \times [0,1]$ be an isotopy of codimension-zero-embeddings with closed image. Then h induces a homotopy between the induced maps*

$$(h|_{M\times 0})_*, (h|_{M\times 1})_* : P_\partial(M) \to P_\partial(N).$$

Functoriality for codimension-zero-embeddings is of course not enough for our purposes.

PL Remark 2.9. Denote by $\mathcal{PL}$ the piecewise linear category. We define a family M of manifolds over X to be a $\mathcal{PL}$-family if both M and X have a piecewise-linear structure (e.g. if M and X are polyhedra embedded into some $\mathbb{R}^n$) and the projection $M \to X$ is a piecewise-linear map. For such a $\mathcal{PL}$-family M of manifolds over X we can define the $\mathcal{PL}$-pseudoisotopy space of M as the simplicial subset $P^{\mathcal{PL}}(M) \subseteq P(M)$ of those simplices $\varphi : \Delta^k \times M \times I \to \Delta^k \times M \times I$ that are also $\mathcal{PL}$-isomorphisms. The rest of this chapter carries over to the $\mathcal{PL}$ world if one requires all maps of topological spaces to be piecewise linear maps.

CHAPTER 3

Geometric Transfer

Apart from codimension-zero-embeddings one also gets induced transfer maps $P(M) \to P(E)$ between the pseudoisotopy spaces of a fiber bundle $p : E \to M$. But these induced maps depend on additional choices. In this section we construct these induced maps based on the construction given in [BL82]. We do this in more detail than Burghelea and Lashof, because we need to ensure that the involved choices form a contractible space and that certain constructions with transfer maps work.

Let $p : E \to M$ be a fiber bundle of families of manifolds over X. Denote by $M_X^{[0,1]}$ the topological space of those paths $[0,1] \to M$ that become constant paths after composition with the projection onto X. For $t \in [0,1]$ let

$$\xi_t : M_X^{[0,1]} \to M,\ \omega \mapsto \omega(t)$$

and let $\xi_t^*(p)$ be the pullback bundle of p along ξ_t, i.e. we have a pullback diagram

$$\begin{tikzcd} \xi_t^* E \arrow[r] \arrow[d, "\xi_t^*(p)"] & E \arrow[d, "p"] \\ M_X^{[0,1]} \arrow[r, "\xi_t"] & M \end{tikzcd}$$

with $\xi_t^* E$ given by

$$\xi_t^* E = \{(\omega, e) \in M_X^{[0,1]} \times E \mid \omega(t) = p(e)\}$$

and the bundle map $\xi_t^*(p) : \xi_t^* E \to M_X^{[0,1]}$ is the projection onto $M_X^{[0,1]}$. Because ξ_0 and ξ_1 are homotopic maps and $M_X^{[0,1]}$ is paracompact (because $M_X^{[0,1]}$ is metrizable) the bundles $\xi_0^*(p)$ and $\xi_1^*(p)$ are in fact isomorphic.

There is a canonical section of the projection map $\xi_t^* E \to E$ given by sending $e \in E$ to $(\mathrm{const}_{p(e)}, e) \in \xi_t^* E$, where $\mathrm{const}_{p(e)}$ denotes the constant path at $p(e) \in M$.

Definition 3.1. Let $p : E \to M$ be a fiber bundle of families of manifolds over X. A *parallel transport* over p is a bundle isomorphism $\nu : \xi_0^* E \xrightarrow{\cong} \xi_1^* E$ of bundles over $M_X^{[0,1]}$ which commutes with the canonical sections of $\xi_t^* E \to E$, $t \in \{0,1\}$, i.e. it satisfies $\nu(\mathrm{const}_{p(e)}, e) = (\mathrm{const}_{p(e)}, e)$ for every $e \in E$.

Let $\nu' : \xi_0^* E \to \xi_1^* E$ be an arbitrary bundle isomorphism. Composing it with the canonical section and the projection onto E defines a bundle isomorphism over $E \to M$ given by $e \mapsto \mathrm{pr}_E \circ \nu'(\mathrm{const}_{p(e)}, e)$, which is the identity if ν' is a parallel transport. If not, we can precompose the E-coordinate of ν' with the inverse of this bundle isomorphism to construct a parallel transport $\nu : \xi_0^* E \to \xi_1^* E$ out of ν' given by the formula

$$\nu(\omega, e) \coloneqq \nu'(\omega, \mathrm{pr}_E \circ (\nu')^{-1}(\mathrm{const}_{p(e)}, e)).$$

In particular, this implies that a parallel transport always exists.

PL Remark 3.2. Let $p : E \to M$ be a $\mathcal{PL}$ fiber bundle of $\mathcal{PL}$ families of manifolds over X. A piecewise linear parallel transport over p is a parallel transport ν over p such that for all $k \in \mathbb{N}$ and piecewise linear maps $f : \Delta^k \times [0,1] \to M$ the induced map

$$\Delta^k \times p^{-1}(f(\Delta^k \times 0)) \to p^{-1}(f(\Delta^k \times 1)),\ (x, e) \mapsto \mathrm{pr}_E \circ \nu(f(x, (-)), e)$$

is piecewise linear.

There exists a piecewise linear transport for every $\mathcal{PL}$ fiber bundle of $\mathcal{PL}$ families of manifolds over X, we give a construction in 7.1.

Example/Definition 3.3. Let M be a family of manifolds over X and N be a manifold. Let $p : M \times N \to M$ be the projection map onto M. Then $\xi_t^*(M \times N)$ consists of elements of the form $(\omega, (\omega(t), n))$ with $\omega \in M_X^{[0,1]}$ and $(\omega(t), n) \in M \times N$. Now a parallel transport over p is given by the formula

$$\nu : \xi_0^*(M \times N) \to \xi_1^*(M \times N),\ (\omega, (\omega(0), n)) \mapsto (\omega, (\omega(1), n).$$

We will call this the *trivial parallel transport* over the (trivial) fiber bundle p.

Example/Definition 3.4. Let $p : E \to M$ be a fiber bundle of families of manifolds over X and $p' : E' \to M'$ be a fiber bundle of families of manifolds over Y. Let ν, ν' be parallel transports over p, respectively p'. Then $p \times p' : E \times E' \to M \times M'$ is again a fiber bundle (of families of manifolds over $X \times Y$) and we can define the *product parallel transport*

$$\nu_{p \times p'} : \xi_0^*(E \times E') \to \xi_1^*(E \times E')$$

of the two parallel transports ν and ν' via

$$\nu_{p\times p'}(\omega,(e,e')) := (\omega,(\nu(\mathrm{pr}_M\circ\omega,e),\nu'(\mathrm{pr}_{M'}\circ\omega,e'))).$$

Example/Definition 3.5. Let $p : E \to M$ be a fiber bundle of families of manifolds over X, ν a parallel transport over p and $U \subseteq M$ a family of submanifolds over X. Then the restriction $p' := p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is also a fiber bundle of families of manifolds over X and ν restricts to a bundle isomorphism $\nu' : \xi_0^*(p^{-1}(U)) \xrightarrow{\cong} \xi_1^*(p^{-1}(U))$ since it commutes with p. Thus ν' is a parallel transport for the bundle p' which we will call the *restriction* of ν to the bundle p'.

For a given fiber bundle $p : E \to M$ of manifolds one can define a space of parallel transports over p as a topological space by taking the subspace of parallel transports inside $\mathrm{map}(\xi_0^*(E),\xi_1^*(E))$ (with the compact-open topology). Applying the singular set functor to this space one gets a description of a space of parallel transports as a simplicial set with k-simplices given by parallel transports over the bundle $p\times \mathrm{id}_{\Delta^k} : E\times\Delta^k \to M\times\Delta^k$, where $E\times\Delta^k$ and $M\times\Delta^k$ are viewed as families of manifolds over Δ^k via the projection map onto Δ^k. The following proposition uses the description as a topological space:

Proposition 3.6. *The space of parallel transports over $p : E \to M$ is contractible.*

Proof. Fix one parallel transport $\nu_0 : \xi_0^*E \to \xi_1^*E$. For a path $\omega : [0,1] \to M$ denote by $\omega_{[a,b]}$ the path given by $\omega_{[a,b]}(t) = \omega(a + t\cdot(b-a))$. Now define for ν some parallel transport over p and $t\in[0,1]$ the parallel transport $\nu_t : \xi_o^*E \to \xi_1^*E$ by

$$\nu_t(\omega,e) := \big(\omega, \mathrm{pr}_E\circ\nu_0(\omega_{[t,1]}, \mathrm{pr}_E\circ\nu(\omega_{[0,t]},e))\big).$$

Since the assignment $(\nu,t)\mapsto\nu_t$ is continuous in ν and t, it is a contraction of the space of parallel transports to the parallel transport ν_0. □

PL Remark 3.7. Unfortunately, we cannot define a space of piecewise linear parallel transports as a topological subspace of the space of parallel transports, because the resulting topology would be wrong. Instead we define the space of piecewise linear transports as a simplicial subset of the simplicial set of parallel transports consisting of those simplices $\nu : \xi_0^*(E\times\Delta^k) \to \xi_1^*(E\times\Delta^k)$ that are piecewise linear parallel transports over the $\mathcal{PL}$ bundle $p\times\mathrm{id}_{\Delta^k} : E\times\Delta^k \to M\times\Delta^k$ of $\mathcal{PL}$ families of manifolds over X. Fortunately, the proof of 3.6 still induces a contraction (as a simplicial set) of the space of piecewise linear parallel transports.

Definition 3.8. Let $p : E \to M$ be a fiber bundle of families of manifolds over X and let ν' be a parallel transport over p. Denote by

$$\nu := \nu_{p\times \mathrm{id}_I} : \xi_0^*(E\times I) \to \xi_1^*(E\times I)$$

the product parallel transport of ν' with the trivial parallel transport over $\mathrm{id}_I : I \to I$ (with I viewed as a family of manifolds over a point). For $(m,t) \in M \times I$ denote by $\omega_{(m,t)} : [0,1] \to M \times I$ the path given by $\omega_{(m,t)}(s) := (m, t \cdot s)$. Let $\varphi : M \times I \to M \times I$ be a pseudoisotopy. The *geometric transfer* of φ over p with respect to ν' is a pseudoisotopy on E defined by

$$\mathrm{Tr}_{\nu'}(\varphi)(e,t) = \mathrm{pr}_{E \times I} \circ \nu(\varphi \circ \omega_{(p(e),t)}, (e,0))$$

for $(e,t) \in E \times I$.

The above formula certainly defines a continuous map $E \times I \to E \times I$, because it is a composition of continuous maps. Furthermore, $\mathrm{Tr}_{\nu'}(\varphi)$ is a homeomorphism, since an inverse is given by

$$(\mathrm{Tr}_{\nu'}(\varphi))^{-1}(e,t) = (\mathrm{pr}_E \circ \nu^{-1}(\varphi \circ \omega_{\varphi^{-1}(p(e),t)}, (e,t)), \mathrm{pr}_I \circ \varphi^{-1}(p(e),t)).$$

Now for $(e,0) \in E \times 0$ we have $\mathrm{Tr}_{\nu'}(\varphi)(e,0) = (e,0)$, since $\mathrm{pr}_M \circ \varphi \circ \omega_{(p(e),0)}$ is the constant path at $p(e)$ and in the I-coordinate we used the trivial parallel transport. So $\mathrm{Tr}_{\nu'}(\varphi)$ is in fact a pseudoisotopy.

Definition 3.9. Let $p : E \to M$ be a fiber bundle of families of manifolds over X and let ν be a parallel transport over p. The geometric transfer as a map

$$\mathrm{Tr}_\nu : P(M) \to P(E)$$

is defined by sending a k-simplex $\varphi : \Delta^k \times M \times I \to \Delta^k \times M \times I$ to $\mathrm{Tr}_\nu(\varphi) := \mathrm{Tr}_{\nu \times \mathrm{id}_{\Delta^k}}(\varphi)$, where $\nu \times \mathrm{id}_{\Delta^k}$ denotes the product parallel transport of ν with the trivial parallel transport over id_{Δ^k}. Here Δ^k is viewed as a family of manifolds over Δ^k, i.e. $\nu \times \mathrm{id}_{\Delta^k}$ is a parallel transport of families of manifolds over $X \times \Delta^k$.

PL Remark 3.10. If ν is a piecewise linear parallel transport, then Tr_ν restricts to a geometric transfer map

$$\mathrm{Tr}_\nu : P^{\mathcal{PL}}(M) \to P^{\mathcal{PL}}(E)$$

on the piecewise linear pseudoisotopy spaces.

Remark 3.11. If the fiber of the bundle $p : E \to M$ has no boundary and φ is a pseudoisotopy relative boundary, then $\mathrm{Tr}_{\nu'}(\varphi)$ is also a pseudoisotopy relative boundary: In this case we have $\partial E = p^{-1}(\partial M)$ and thus for each point $(e,t) \in \partial E \times I$ the path $\mathrm{pr}_M \circ \varphi \circ \omega_{(p(e),0)}$ is also a constant path. That implies $\mathrm{Tr}_{\nu'}(\varphi)(e,t) = (e,t)$ for each $(e,t) \in \partial E \times I$, so $\mathrm{Tr}_{\nu'}(\varphi)$ is relative boundary.

Definition 3.12. Let $p : E \to M$ and $p' : E' \to E$ be fiber bundles of families of manifolds over X such that $p \circ p' : E' \to M$ is also a fiber bundle of families of manifolds over X. Let ν be a parallel transport for p and ν' a parallel transport for p'. For a path $\omega \in M_X^{[0,1]}$ again denote by $\omega_{[a,b]}$ the path given by $\omega_{[a,b]}(t) = \omega(a + t \cdot (b-a))$. For $\omega \in M_X^{[0,1]}$ and $e' \in (p \circ p')^{-1}(\omega(0))$

define the path $\tilde{\omega}_{e'} \in E_X^{[0,1]}$ by $\tilde{\omega}_{e'}(t) = \mathrm{pr}_E \circ \nu(\omega_{[0,t]}, p'(e'))$. The *composition parallel transport* $\nu' \cdot \nu$ of ν and ν' is a parallel transport over the bundle $p \circ p'$ given by $\nu' \cdot \nu(\omega, e') = \nu'(\tilde{\omega}_{e'}, e')$ for $\omega \in M_X^{[0,1]}$ and $e' \in (p \circ p')^{-1}(\omega(0))$.

The name composition parallel transport comes from the fact that we want to use it to compose geometric transfers defined by these parallel transports:

Proposition 3.13. *Let $p : E \to M$ and $p' : E' \to E$ be fiber bundles of families of manifolds over X such that $p \circ p' : E' \to M$ is also a fiber bundle of families of manifolds over X. Let ν be a parallel transport for p and ν' a parallel transport for p'. Let $\varphi : M \times I \to M \times I$ be a pseudoisotopy on M. Then*

$$\mathrm{Tr}_{\nu' \cdot \nu}(\varphi) = \mathrm{Tr}_{\nu'}(\mathrm{Tr}_{\nu}(\varphi))$$

as pseudoisotopies on E'.

Proof. Denote by $\nu_{p \times \mathrm{id}_I}$ and $\nu'_{p \times \mathrm{id}_I}$ the product parallel transports of ν and ν' with the trivial parallel transport over $\mathrm{id} : I \to I$. Let w denote the path given by $w = \varphi \circ \omega_{(p \circ p'(e'),t)} \in (M \times I)_X^{[0,1]}$ with $\omega_{(p \circ p'(e'),t)}$ as in definition 3.8. Then the path $\tilde{w}_{e'}$ from definition 3.12 with respect to $\nu_{p \times \mathrm{id}_I}$ is given by

$$\begin{aligned}
\tilde{w}_{e'}(s) &= \mathrm{pr}_{E \times I} \circ \nu_{p \times \mathrm{id}_I}(w_{[0,s]}, (p'(e'), 0)) \\
&= \mathrm{pr}_{E \times I} \circ \nu_{p \times \mathrm{id}_I}(\varphi \circ \omega_{(p \circ p'(e'), s \cdot t)}, (p'(e'), 0)) \\
&= \mathrm{Tr}_{\nu}(\varphi)(p'(e'), s \cdot t) \\
&= \mathrm{Tr}_{\nu}(\varphi) \circ \omega_{(p'(e'),t)}(s).
\end{aligned}$$

Thus we can compute

$$\begin{aligned}
\mathrm{Tr}_{\nu' \cdot \nu}(\varphi)(e', t) &= \mathrm{pr}_{E' \times I} \circ (\nu' \cdot \nu)_{p \circ p' \times \mathrm{id}_I}(\varphi \circ \omega_{(p \circ p'(e'),t)}, (e', 0)) \\
&= \mathrm{pr}_{E' \times I} \circ (\nu'_{p' \times \mathrm{id}_I} \cdot \nu_{p \times \mathrm{id}_I})(\varphi \circ \omega_{(p \circ p'(e'),t)}, (e', 0)) \\
&= \mathrm{pr}_{E' \times I} \circ \nu'_{p' \times \mathrm{id}_I}(\mathrm{Tr}_{\nu}(\varphi) \circ \omega_{(p'(e'),t)}, (e', 0)) \\
&= \mathrm{Tr}_{\nu'}(\mathrm{Tr}_{\nu}(\varphi))(e', t).
\end{aligned}$$

Here the proof of the equation $(\nu' \cdot \nu)_{p \circ p' \times \mathrm{id}_I} = \nu'_{p' \times \mathrm{id}_I} \cdot \nu_{p \times \mathrm{id}_I}$ is left as an easy exercise to the reader. □

3.1 Geometric Transfer Relative Boundary

As we have just seen, the geometric transfer does only send pseudoisotopies relative boundary to pseudoisotopies relative boundary if the fiber of the bundle $p : E \to M$ has no boundary. To get a pseudoisotopy relative boundary if the fiber has non-vanishing boundary, one can bend around the boundary as in remark 2.6 after lifting the pseudoisotopy with a geometric transfer. In this section we will develop a fiber-wise version of the bending around the boundary for the special case of D^n-bundles. For technical reasons we also need to choose isotopies of our bending maps to the identity. But that has the upshot, that our choices will again form a contractible space.

Definition 3.14. Let $p : E \to M$ be a fiber bundle of families of manifolds over X and $H : E \times I \times [0,1] \to E \times I \times [0,1]$ be an isotopy of homeomorphisms over X (i.e. H commutes with the projection onto X) such that:

- H starts with the identity homeomorphism on $E \times I$,
- for every time $t \in [0,1]$ the map $H_t = H(-,-,t) : E \times I \to E \times I$ satisfies

$$E \times 0 \subseteq H_t(E \times 0)$$
$$E \times 0 \cup \partial E \times I \subseteq H_t(E \times 0 \cup \partial E \times I).$$

Then we will call H a *bending isotopy* and $h := H_1$ a *partial bending map*. If H also satisfies the condition

- H is fiber preserving in the sense that $H_t(p^{-1}(m) \times I) = p^{-1}(m) \times I$ for all $m \in M$ and $t \in [0,1]$,

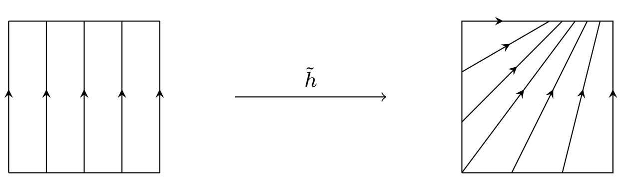
then we will call H a *fiber-wise* bending isotopy and h a *fiber-wise* partial bending map. Furthermore, if h (and thus also H) is fiber-wise and h also satisfies the condition

- for each fiber $F = p^{-1}(m)$ the set $F \times 0 \cup \partial F \times I$ is contained in $h(F \times 0)$,

then h is called a *fiber-wise bending map*.

Next we will construct such a fiber-wise bending map for the special case of D^n-bundles. For that let $\tilde{h} : [0,1] \times I \to [0,1] \times I$ be a homeomorphism with

- $\tilde{h}([0,1] \times 0) = 0 \times I \cup [0,1] \times 0$
- $\tilde{h}|_{1 \times I} = \mathrm{id}_{1 \times I}$
- There exists an isotopy $\tilde{H}$ of homeomorphisms from the identity to $\tilde{h}$ such that the isotopy is relative $1 \times I \subseteq [0,1] \times I$ and for every time t we have $[0,1] \times 0 \subseteq \tilde{H}_t([0,1] \times 0)$ and $0 \times I \cup [0,1] \times 0 \subseteq \tilde{H}_t(0 \times I \cup [0,1] \times 0)$.



As one can easily see in the picture above, such an isotopy $\tilde{H}$ (and thus $\tilde{h}$) exists. Now the homeomorphism $\mathrm{id} \times \tilde{h} : S^{n-1} \times [0,1] \times I \to S^{n-1} \times [0,1] \times I$ induces a homeomorphism $\hat{h} : D^n \times I \to D^n \times I$ via $D^n \cong S^{n-1} \times [0,1]/S^{n-1} \times 1$ which commutes with the obvious $\mathrm{Aut}(S^{n-1})$-action on D^n. Here $\mathrm{Aut}(S^{n-1})$ denotes the group of self-homeomorphisms of S^{n-1}. Analogously, the isotopy $\tilde{H}$ for $\tilde{h}$ induces an isotopy $\hat{H}$ from the identity to $\hat{h}$, which also commutes with the $\mathrm{Aut}(S^{n-1})$-action.

Suppose we have given a D^n-bundle $p : E \to M$. The structure group $\mathrm{Aut}(D^n)$ of this bundle has $\mathrm{Aut}(S^{n-1})$ as deformation retract, so without loss of generality we can assume the structure group to be $\mathrm{Aut}(S^{n-1})$. Let $h : E \times I \to E \times I$ be a fiber-preserving map defined on local charts by $\mathrm{id} \times \hat{h} : U \times D^n \times I \to U \times D^n \times I$ with $U \subseteq M$ a trivializing neighborhood of the bundle. Because $\hat{h}$ and $\hat{H}$ both commute with the $\mathrm{Aut}(S^{n-1})$-action, h is not only well defined but also isotopic to the identity with the isotopy given fiber-wise by $\hat{H}$. Thus h is a fiber-wise bending map for p and we can conclude:

Lemma 3.15. *For every fiber bundle $p : E \to M$ of families of manifolds over X with fiber D^n there exists a fiber-wise bending map.*

Definition 3.16. Let $p : E \to M$ be a fiber bundle of families of manifolds over X, ν a parallel transport over p and H a bending isotopy for p and h the partial bending map defined by H. The geometric transfer with respect to ν and h is a map

$$\mathrm{Tr}_{\nu,h} : P(M) \to P(E)$$

defined by sending a k-simplex $\varphi : \Delta^k \times M \times I \to \Delta^k \times M \times I$ to a pseudoisotopy $\mathrm{Tr}_{\nu,h}(\varphi) := (\mathrm{id}_{\Delta^k} \times h) \circ \mathrm{Tr}_\nu(\varphi) \circ (\mathrm{id}_{\Delta^k} \times h)^{-1} : \Delta^k \times E \times I \to \Delta^k \times E \times I$. If h is a fiber-wise bending map and φ is relative boundary, then $\mathrm{Tr}_{\nu,h}(\varphi)$ is relative boundary and we call the restriction

$$\mathrm{Tr}_{\nu,h} : P_\partial(M) \to P_\partial(E)$$

the *geometric transfer relative boundary.*

Let us check that $\mathrm{Tr}_{\nu,h}(\varphi)$ is in fact relative boundary if h is a fiber-wise bending map and φ is relative boundary. Now the boundary of E consists of $p^{-1}(\partial M)$ and the ∂F-subbundle of p, where F denotes the fiber of the bundle. On $p^{-1}(\partial M) \times I$ the map $\mathrm{Tr}_\nu(\varphi)$ is already the identity, so this also follows for $\mathrm{Tr}_{\nu,h}(\varphi)$, because conjugation with h acts fiber-wise. Let E' denote the total space of the ∂F-subbundle of p. Then by definition of a fiber-wise bending map $h^{-1}(E' \times I \cup E \times 0)$ is contained in $E \times 0$. Since $\mathrm{Tr}_\nu(\varphi)$ is the identity on $E \times 0$, this implies that the map $\mathrm{Tr}_{\nu,h}(\varphi)$ is the identity on $E' \times I \cup E \times 0$. So $\mathrm{Tr}_{\nu,h}(\varphi)$ is a pseudoisotopy relative boundary.

Of the two choices of parallel transport and fiber-wise bending map we need to make to construct a geometric transfer relative boundary, we already know that the choice of parallel transport is contractible. It turns out, that the choice of fiber-wise bending map is also weakly contractible in the following sense:

Proposition 3.17. *Let $p : E \to M$ be a fiber bundle of families of manifolds over X and $h : E \times \partial D^n \times I \to E \times \partial D^n \times I$, $n \geq 1$, a fiber-wise bending map for the bundle $p \times \mathrm{id}_{\partial D^n}$. Then h extends to a fiber-wise bending map $h' : E \times D^n \times I \to E \times D^n \times I$ for the bundle $p \times \mathrm{id}_{D^n}$.*

Proof. Let $g : E \times I \to E \times I$ be a fiber-wise bending map for p and denote by $G : [0,1] \times E \times I \to [0,1] \times E \times I$ the associated bending isotopy from the identity to g. Then

$$(G \times \mathrm{id}_{\partial D^n}) \circ (\mathrm{id}_{[0,1]} \times h) : [0,1] \times E \times \partial D^n \times I \to [0,1] \times E \times \partial D^n \times I$$

is a fiber-wise bending map for the bundle $\mathrm{id}_{[0,1]} \times p \times \mathrm{id}_{\partial D^n}$, because it is fiber-preserving and for every fiber F of the bundle we know

$$F \times 0 \cup \partial F \times I \subseteq \mathrm{id}_{[0,1]} \times h(F \times 0)$$
$$F \times 0 \cup \partial F \times I \subseteq G \times \mathrm{id}_{\partial D^n}(F \times 0 \cup \partial F \times I)$$

by definition, which implies

$$F \times 0 \cup \partial F \times I \subseteq (G \times \mathrm{id}_{\partial D^n}) \circ (\mathrm{id}_{[0,1]} \times h)(F \times 0).$$

If now $H : [1,2] \times E \times \partial D^n \times I \to [1,2] \times E \times \partial D^n \times I$ denotes the bending isotopy associated to h, then

$$(\mathrm{id}_{[1,2]} \times g \times \mathrm{id}_{\partial D^n}) \circ H : [1,2] \times E \times \partial D^n \times I \to [1,2] \times E \times \partial D^n \times I$$

is also a fiber-wise bending map. By combining both bending maps, we get a fiber-wise bending map on $[0,2] \times E \times \partial D^n \times I$ starting with h and ending with $g \times \mathrm{id}_{\partial D^n}$. Thus we get the desired extension to h by collapsing $2 \times \partial D^n$ to a point and using a homeomorphism $[0,2] \times \partial D^n / 2 \times \partial D^n \cong D^n$.

We omitted the constructions of the associated fiber-wise bending isotopies for the fiber-wise bending maps in the proof (which have to exist by the definition of bending maps) and instead leave their construction as an exercise to the reader. □

PL Remark 3.18. This section again carries over to the $\mathcal{PL}$ world if one requires all continuous maps to be piecewise linear.

CHAPTER 4

Stable Pseudoisotopy Space

Let $f : M \to N$ be a map between manifolds. Our next goal is to define an induced map for f between the pseudoisotopy spaces of M and N. The rough idea for that goes as follows: Suppose f is an embedding which admits a disk-bundle neighborhood, i.e. there is a codimension-zero-submanifold $E \subseteq N$ and $p : E \to M$ such that p is a disk-bundle and f is the zero-section of this bundle. Then one could lift a pseudoisotopy on M to a pseudoisotopy on E via a geometric transfer on p and then use the codimension-zero-embedding $E \to N$ to get a pseudoisotopy on N. Of course, in general f may not even be homotopic to an embedding. But the map $f \times 0 : M \to N \times D^k$ is homotopic to such an embedding for high enough k. To utilize this, we will need to replace the pseudoisotopy space by a stabilized version of it.

The resulting induced map for f will depend on many choices (e.g. the choice of an embedding homotopic to $f \times 0$). We will organize these choices in spaces with the goal in mind to show that these so-called choice spaces are contractible.

Let $J := [-1, 1] = D^1$ denote the one-dimensional unit disk. Choose a fiber-wise bending map $h : J \times I \to J \times I$ over the trivial bundle $J \to \{\star\}$ and fix the choice of h together with the corresponding choice of a fiber-wise bending isotopy H once and for all.

PL Remark 4.1. Both h and H should be chosen as piecewise linear maps, so that we can use the same maps in the topological and in the piecewise linear world.

Definition 4.2. Let M be a family of manifolds over X and let ν be the trivial parallel transport over the bundle $\mathrm{pr}_M : M \times J \to M$. The *stabilization map* $s : P_\partial(M) \to P_\partial(M \times J)$ is defined by sending a k-simplex $\varphi \in P_\partial(M)$ to

$$(h \times \mathrm{id}_{\Delta^k \times M}) \circ \mathrm{Tr}_\nu(\varphi) \circ (h \times \mathrm{id}_{\Delta^k \times M})^{-1}.$$

In other words, it is the geometric transfer relative boundary with respect to ν and $h \times \mathrm{id}_{\Delta^k \times M}$.

Furthermore, we define the *stable pseudoisotopy space* of M as

$$\mathcal{P}_\partial(M) := \operatorname*{hocolim}_{k\geq 0} P_\partial(M \times J^k)$$

where the homotopy colimit runs over the stabilization maps with respect to the bundles $M \times J^{k+1} \to M \times J^k$. We will use an explicit model for this homotopy colimit, namely the mapping telescope

$$\mathcal{P}_\partial(M) = \coprod_{k\geq 0} P_\partial(M \times J^k) \times \underline{\Delta}^1 \big/ \sim$$

where the equivalence relation $\sim$ is induced by identifying the simplices of $P_\partial(M \times J^k) \times \partial_1\underline{\Delta}^1$ with simplices in $P_\partial(M \times J^{k+1}) \times \partial_0\underline{\Delta}^1$ via the stabilization map. Here $\underline{\Delta}^1$ denotes the simplicial 1-simplex.

Remark 4.3. If M is a n-dimensional compact smoothable manifold with $n \geq 5$, then the stabilization map $P_\partial(M) \to P_\partial(M \times J)$ is k-connected for $n \geq max\{2k+7, 3k+4\}$. The proof for the piecewise linear case, which uses Igusas stability theorem for smooth pseudoisotopy spaces [Igu88] to derive a $\mathcal{PL}$ stability result if M is smoothable, is due to Burghelea and Goodwillie and can be found in [WJR13, Corollary 1.4.2]. For $n \geq 5$ one then gets the topological case by [BL74, Theorem 6.2].

We need to define choices of induced maps between unstable pseudoisotopy spaces first and then we will later construct choices between the stable pseudoisotopy spaces out of them.

Definition 4.4. Let M, N be families of manifolds over X. A choice for an induced map $P_\partial(M) \to P_\partial(N)$ is a finite tuple

$$(E_1, \dots, E_n, p_1, \dots, p_n, s, \nu_1, H^{(1)}, \dots, \nu_n, H^{(n)})$$

for some $n \in \mathbb{N}$ consisting of:

- a closed subset $E = E_n \subseteq N$ which is a family of codimension-zero-submanifolds over X of N,
- a composition of disk bundles (where we explicitly allow 0-dimensional disks as fibers)

  $$E = E_n \xrightarrow{p_n} E_{n-1} \xrightarrow{p_{n-1}} \dots \xrightarrow{p_2} E_1 \xrightarrow{p_1} E_0 = M$$

  where all E_i are families of manifolds over X (and the bundles are also all over X),
- a zero-section $s : M \to E$ of the composition bundle $p_1 \circ \dots \circ p_n$,
- for each p_i a parallel transport ν_i over p_i and a bending isotopy $H^{(i)}$ for p_i such that $H^{(i)}$ is fiber-wise with respect to the bundle $p_1 \circ \dots \circ p_i$ and the composition

  $$\mathrm{Tr}_{\nu_n, h^{(n)}} \circ \dots \circ \mathrm{Tr}_{\nu_1, h^{(1)}} : P_\partial(M) \to P(E)$$

has image in $P_\partial(E)$, where $h^{(i)}$ denotes the partial bending map defined by $H^{(i)}$. Note that the individual $h^{(i)}$ are not required to be fiber-wise bending maps, but only fiber-wise *partial* bending maps.

The map induced by these choices is the composition

$$P_\partial(M) \xrightarrow{\mathrm{Tr}_{\nu_n,h^{(n)}} \circ \ldots \circ \mathrm{Tr}_{\nu_1,h^{(1)}}} P_\partial(E) \to P_\partial(N)$$

where the last map is induced by the inclusion of the family of codimension-zero-submanifolds E into N, i.e. it is given by extending pseudoisotopies with the identity outside of $E \subseteq N$.

PL Remark 4.5. To define a choice between the piecewise linear pseudoisotopy spaces of $\mathcal{PL}$ families of manifolds we add the condition that everything in the above definition has to be $\mathcal{PL}$, i.e. the E_i are $\mathcal{PL}$ families of manifolds, the p_i, $H^{(i)}$ and s are $\mathcal{PL}$ maps and the ν_i are $\mathcal{PL}$ geometric transfers.

Remark 4.6. Note that a disk bundle $p : E \to M$ with fiber a 0-dimensional disk is just a homeomorphism and that the parallel transport ν for such a bundle is uniquely determined. As a fiber-wise bending isotopy H for such a bundle we can simply choose the constant isotopy with value the identity map on $E \times I$. The transfer map $\mathrm{Tr}_{\nu,h}$ is then just conjugation with the homeomorphism $p \times \mathrm{id}_I : E \times I \to M \times I$. Thus for a family of manifolds M over X we have a canonical identity choice (E, p, s, ν, H) with $E = M$, $p = s = \mathrm{id} : M \to M$ and H the constant isotopy such that the induced map of (E, p, s, ν, H) is the identity on $P_\partial(M)$.

Example 4.7. The stabilization map $s : P_\partial(M) \to P_\partial(M \times J)$ is an induced map for the inclusion $M = M \times \{0\} \subseteq M \times J$ and comes with a preferred choice: The bundle is the trivial bundle $\mathrm{pr}_M : M \times J \to M$ with the inclusion $M = M \times 0 \subseteq M \times J$ as zero-section. The decomposition of pr_M into bundles consist only of pr_M itself and the choices for the geometric transfer and the bending isotopy used for the stabilization map were fixed in definition 4.2.

Remark 4.8. In general, a choice for an induced map $P_\partial(M) \to P_\partial(N)$ does not always exist. For example, if $\dim(M) > \dim(N)$ there cannot exists such a choice. See also proposition 4.20 on the question of existence of choices.

Now that we have defined what a choice for an induced map between pseudoisotopy spaces is, we need to define a *space of choices* for induced maps $P_\partial(M) \to P_\partial(N)$ between the unstable pseudoisotopy spaces. For that we first need to choose a small category of manifolds to work with, because else the choices as defined in 4.4 do not even form a set. Thus from now on we will work with the small category of those families of manifolds $p : M \to X$ over a space X for which M is a subspace of $\mathbb{R}^\infty \times X$. As a convention, when we speak of a family of manifolds over X we always mean an object of this category. Also keep in mind the other convention we use, namely that X is a finite CW-complex.

Definition 4.9. Let M, N be families of manifolds over X. Define $\bar{\mathrm{Ch}}^P(M, N)$ as a simplicial set, where a k-simplex is a choice

$$(E_1, \dots, E_n, p_1, \dots, p_n, s, \nu_1, H^{(1)}, \dots, \nu_n, H^{(n)})$$

of an induced map (as in definition 4.4) from $P_\partial(\Delta^k \times M)$ to $P_\partial(\Delta^k \times N)$ with $\Delta^k \times M$ and $\Delta^k \times N$ considered as families of manifolds over $\Delta^k \times X$ and such that all E_i are families of submanifolds of $\mathbb{R}^\infty \times \Delta^k \times X$ over $\Delta^k \times X$. The structure map of the simplicial set for some are given by pullbacks: Let

$$\sigma = (E_1, \dots, E_n, p_1, \dots, p_n, s, \nu_1, H^{(1)}, \dots, \nu_n, H^{(n)}) \in \bar{\mathrm{Ch}}^P(M, N)$$

be a k-simplex and $\theta : \underline{\Delta}^l \to \underline{\Delta}^k$. Then $\theta^*(\sigma) \in \bar{\mathrm{Ch}}^P(M, N)$ is given by a l-simplex

$$\theta^*(\sigma) = (\theta^*(E_1), \dots, \theta^*(E_n), \theta^*(p_1), \dots, \theta^*(p_n), \theta^*(s), \theta^*(\nu_1), \theta^*(H^{(1)}), \dots, \theta^*(\nu_n), \theta^*(H^{(n)}))$$

where $\theta^*(E_i)$ is the pullback of the diagram

$$E_i \to \Delta^k \times X \xleftarrow{\theta_* \times \mathrm{id}_X} \Delta^l \times X,$$

$\theta^*(p_i)$ is uniquely determined by the commuting diagram

$$\begin{tikzcd}
\theta^*(E_i) \arrow[r, "\theta^*(p_i)"] \arrow[d] & \theta^*(E_{i-1}) \arrow[d] \\
E_i \arrow[r, "p_i"] & E_{i-1}
\end{tikzcd}$$

and analogously for $\theta^*(\nu_i)$, $\theta^*(H^{(i)})$ and $\theta^*(s)$.

Now define the space of choices $\mathrm{Ch}^P(M, N)$ of induced maps from $P_\partial(M)$ to $P_\partial(N)$ as a quotient of $\bar{\mathrm{Ch}}^P(M, N)$ by the equivalence relation generated by the following two relations:

- Let
  $$\sigma = (E_1, \dots, E_n, p_1, \dots, p_n, s, \nu_1, H^{(1)}, \dots, \nu_n, H^{(n)}) \in \bar{\mathrm{Ch}}^P(M, N)$$
  be a choice with $H^{(i)}$ the constant isotopy with value the identity on $E_i \times I$ for some $1 \le i < n$. Then we identify σ with the tuple
  $$\begin{aligned}
  (&E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_n, \\
  &p_1, \dots, p_i \circ p_{i+1}, \dots, p_n, \\
  &s, \\
  &\nu_1, H^{(1)}, \dots, \nu_{i+1} \cdot \nu_i, H^{(i+1)}, \dots, \nu_n, H^{(n)}),
  \end{aligned}$$

that we get by composing p_i with p_{i+1} and ν_{i+1} with ν_i (and removing E_i and $H^{(i)}$ from the tuple). Note that both tuples induce the same map from $P_\partial(\Delta^k \times M)$ to $P_\partial(\Delta^k \times N)$ by proposition 3.13 (and the fact that conjugation with the identity homeomorphism does nothing).

- Let

  $$\sigma = (E_1, \dots, E_n, p_1, \dots, p_n, s, \nu_1, H^{(1)}, \dots, \nu_n, H^{(n)}) \in \bar{\mathrm{Ch}}^P(M, N)$$

  be a choice with $p_i : E_i \to E_{i-1}$ a fiber bundle over a 0-dimensional disk (or in other words a homeomorphism) and $H^{(i)}$ the constant isotopy with value the identity map for some $1 < i \leq n$. Then we identify σ with the tuple

  $$\begin{aligned}&(E_1, \dots, E_{i-2}, E_i, E_{i+1}, \dots, E_n,\\ &p_1, \dots, p_{i-2}, p_{i-1} \circ p_i, p_{i+1} \dots, p_n,\\ &s,\\ &\nu_1, H^{(1)}, \dots, \nu_{i-2}, H^{(i-2)}, \nu_i \cdot \nu_{i-1}, \overline{H^{(i-1)}}, \nu_{i+1}, H^{(i+1)}, \dots, \nu_n, H^{(n)})\end{aligned}$$

  with $\overline{H^{(i-1)}}(-, t) = (p_i \times \mathrm{id}_I)^{-1} \circ H^{(i-1)}(-, t) \circ (p_i \times \mathrm{id}_I)(-)$ for $t \in [0, 1]$. Note that since p_i is a homeomorphism, Tr_{ν_i} acts by conjugation with $p_i \times \mathrm{id}_I$. Thus we have again that both tuples induce the same map from $P_\partial(\Delta^k \times M)$ to $P_\partial(\Delta^k \times N)$.

Remark 4.10. Let

$$(E_1, \dots, E_n, p_1, \dots, p_n, s, \nu_1, H^{(1)}, \dots, \nu_n, H^{(n)})$$

be a representative for a simplex $\sigma \in \mathrm{Ch}^P(M, N)$. Apart from the fact that every representative of σ induces the same map on pseudoisotopy spaces, there are some other invariants that all representatives have in common. For example, all of them have the same composition bundle

$$p_1 \circ \dots \circ p_n : E_n \to M$$

(E_n is by definition a subspace of N) and the composition of all parallel transports yields the same parallel transport $\nu_n \cdot \dots \cdot \nu_1$ over this bundle for every representative of σ. On the other hand, n and $H^{(n)}$ are not invariants, the latter because one can always find a representative with p_n the identity map on E_n and $H^{(n)}$ the constant isotopy with value the identity (see also remark 4.6). But we can always choose a representative of minimal length in the sense that either $n = 1$ or none of the $H^{(i)}$ is the constant isotopy and for representatives of minimal length both n and $H^{(n)}$ are invariants.

Definition 4.11. Let M, N be families of manifolds over X. There is a realization map

$$r : \mathrm{Ch}^P(M, N) \to \mathrm{map}(P_\partial(M), P_\partial(N))$$

defined as follows: Let $\sigma \in \mathrm{Ch}^P(M,N)$ be a k-simplex. Then the map induced by σ is a map $\psi : P_\partial(\Delta^k \times M) \to P_\partial(\Delta^k \times N)$. One can assign to ψ a map $\psi' : \underline{\Delta}^k \times P_\partial(M) \to P_\partial(N)$ as follows: Let $(\alpha : [l] \to [k], \varphi : \Delta^l \times M \times I \to \Delta^l \times M \times I)$ be an l-simplex in $\underline{\Delta}^k \times P_\partial(M)$. Then $\psi(\mathrm{id}_{\Delta^k} \times \varphi)$ is an l-simplex in $P_\partial(\Delta^k \times N)$. Take the pullback of $\psi(\mathrm{id}_{\Delta^k} \times \varphi)$ along the map $\alpha_* \times \mathrm{id}_{\Delta^l \times N \times I} : \Delta^l \times N \times I \to \Delta^k \times \Delta^l \times N \times I$ to get an l-simplex $\psi'(\alpha,\varphi) \in P_\partial(N)$. Thus we can assign to the k-simplex $\sigma \in \mathrm{Ch}^P(M,N)$ the k-simplex $\psi' \in \mathrm{map}(P_\partial(M), P_\partial(N))$.

Definition 4.12. Let M, N, K be families of manifolds over X and let $\sigma \in \mathrm{Ch}^P(M,N)$ and $\sigma' \in \mathrm{Ch}^P(N,K)$ be choices given by:

$$\begin{aligned}\sigma =&(E_1,\dots,E_n,p_1,\dots,p_n,s,\nu_1,H^{(1)},\dots,\nu_n,H^{(n)})\\ \sigma' =&(E_1',\dots,E_k',p_1',\dots,p_k',s',\nu_1',H'^{(1)},\dots,\nu_k',H'^{(k)}).\end{aligned}$$

Define $\bar{E}_1 := (p_1')^{-1}(E_n)$, $\bar{p}_1 := p_1'|_{\bar{E}_1}$ and inductively $\bar{E}_i := (p_i')^{-1}(\bar{E}_{i-1})$, $\bar{p}_i := p_i'|_{\bar{E}_i}$. Analogously, we name the restrictions of ν_i' and $H'^{(i)}$ to the bundle $\bar{p}_i$ as $\bar{\nu}_i$ and $\bar{H}^{(i)}$. Note that $\bar{H}^{(i)}$ only exists because $H'^{(i)}$ is fiber-wise with respect to $p_1' \circ \dots \circ p_i'$.

Define the composition $\sigma' \circ \sigma \in \mathrm{Ch}^P(M,K)$ as

$$\begin{aligned}\sigma' \circ \sigma = (&E_1,\dots,E_n,\bar{E}_1,\dots\bar{E}_k,\\ &p_1,\dots p_n,\bar{p}_1,\dots,\bar{p}_k,\\ &s'\circ s,\\ &\nu_1,H^{(1)},\dots,\nu_n,H^{(n)},\bar{\nu}_1,\bar{H}^{(1)},\dots,\bar{\nu}_k,\bar{H}^{(k)}).\end{aligned}$$

The so defined map $- \circ - : \mathrm{Ch}^P(N,K) \times \mathrm{Ch}^P(M,N) \to \mathrm{Ch}^P(M,K)$ is called the *composition map*.

Remark 4.13. This composition of choices is associative, because the process of taking restrictions of bundles and maps is associative.

Remark 4.14. Let M be a family of manifolds over X. The identity choice on M defined in remark 4.6 is a zero-simplex in $\mathrm{Ch}^P(M,M)$, which we will denote by $\sigma_{\mathrm{id}} \in \mathrm{Ch}^P(M,M)$. As the name suggests, σ_{id} is a neutral element with respect to the composition of choices.

Proposition 4.15. *The composition map is compatible with the realization map, i.e. for k-simplices $\sigma \in \mathrm{Ch}^P(M,N)$ and $\sigma' \in \mathrm{Ch}^P(N,K)$ we have*

$$r(\sigma' \circ \sigma) = r(\sigma') \circ r(\sigma) \in \mathrm{map}(P_\partial(M), P_\partial(K)).$$

Proof. Suppose σ' is given by (E',p',s',ν',H') and σ is given by (E,p,s,ν,H). For a k-simplex $\varphi \in P_\partial(M)$, the pseudoisotopy $r(\sigma)(\varphi) = \mathrm{Tr}_{\nu,h}(\varphi)$ is the identity outside of $E \times I \subseteq \Delta^k \times N \times I$. Thus by definition of the geometric transfer, $r(\sigma') \circ r(\sigma)(\varphi) = \mathrm{Tr}_{\nu',h'}(\mathrm{Tr}_{\nu,h}(\varphi))$ is the identity map outside of $(p')^{-1}(E) \times I \subseteq \Delta^k \times K \times I$. On $(p')^{-1}(E)$ the two maps $\mathrm{Tr}_{\nu',h'}(\mathrm{Tr}_{\nu,h}(\varphi))$ and $\mathrm{Tr}_{\bar{\nu},\bar{h}}(\mathrm{Tr}_{\nu,h}(\varphi)) = r(\sigma' \circ \sigma)(\varphi)$ coincide by definition of the restriction parallel transport (see 3.5). Thus both maps coincide on the whole space $\Delta^k \times K \times I$ and the claim follows. For general σ and σ' the proof is analogous. □

The stabilization map comes with a preferred choice as mentioned in example 4.7, which we will denote by $s \in \mathrm{Ch}^P(M, M \times J)$. Composition with s now yields two types of stabilization maps for the space of choices:

$$\begin{aligned}(-) \circ s : \mathrm{Ch}^P(M \times J^{n+1}, N \times J^k) &\to \mathrm{Ch}^P(M \times J^n, N \times J^k)\\ s \circ (-) : \mathrm{Ch}^P(M \times J^n, N \times J^k) &\to \mathrm{Ch}^P(M \times J^n, N \times J^{k+1}).\end{aligned}$$

Definition 4.16. Let M, N be families of manifolds over X. Define the space of choices of induced maps from $P_\partial(M)$ to $\mathcal{P}_\partial(N)$ as

$$\mathrm{Ch}^{P,\mathcal{P}}(M,N) := \coprod_n \mathrm{Ch}^P(M, N \times J^n) \times \underline{\Delta}^1 / \sim$$

where the equivalence relation is induced by identifying $\mathrm{Ch}^P(M, N \times J^n) \times 1$ with its image in $\mathrm{Ch}^P(M, N \times J^{n+1}) \times 0$ under the stabilization map $s \circ (-) : \mathrm{Ch}^P(M, N \times J^n) \to \mathrm{Ch}^P(M, N \times J^{n+1})$.

We will sometimes refer to $\mathrm{Ch}^{P,\mathcal{P}}(M,N)$ as the space of semistable choices from M to N.

Definition 4.17. We have again a realization map

$$r : \mathrm{Ch}^{P,\mathcal{P}}(M,N) \to \mathrm{map}(P_\partial(M), \mathcal{P}_\partial(N)),$$

where the realization of a k-simplex $(\sigma, \alpha) \in \mathrm{Ch}^{P,\mathcal{P}}(M,N)$ with $\sigma \in \mathrm{Ch}^P(M, N \times J^n)$ and $\alpha \in \underline{\Delta}^1$ is given by the composition

$$r(\sigma,\alpha) : P_\partial(M) \times \underline{\Delta}^k \xrightarrow{r(\sigma)\times\alpha} P_\partial(N \times J^n) \times \underline{\Delta}^1 \hookrightarrow \mathcal{P}_\partial(N).$$

We can also use the unstable composition map to precompose these choices with unstable choices and get a composition map

$$\begin{aligned}(-) \circ (-) : \mathrm{Ch}^{P,\mathcal{P}}(N,K) \times \mathrm{Ch}^P(M,N) &\to \mathrm{Ch}^{P,\mathcal{P}}(M,K)\\ ((\sigma',\alpha'),\sigma) &\mapsto (\sigma' \circ \sigma, \alpha')\end{aligned}$$

which is also compatible with the realization maps, i.e. the formula

$$r(\sigma',\alpha') \circ r(\sigma) = r((\sigma',\alpha') \circ \sigma) : P_\partial(M) \to \mathcal{P}_\partial(K)$$

also holds for this composition.

Definition 4.18. Let $f : M \to N$ be a map of families of manifolds over X and let $i, j \in \mathbb{N}$. Define the subspace $\mathrm{Ch}^P_f(M \times J^i, N \times J^j) \subseteq \mathrm{Ch}^P(M \times J^i, N \times J^j)$ of unstable choices with respect to f the following way: Let $\sigma \in \mathrm{Ch}^P(M \times J^i, N \times J^j)$ be a k-simplex and let $s_\sigma : M \times J^i \to N \times J^j$ be the bundle section map of σ. Then $\sigma \in \mathrm{Ch}^P_f(M \times J^i, N \times J^j)$ if and only if the following diagram commutes:

$$\begin{tikzcd}
\Delta^k \times M \times J^i \arrow[r, "s_\sigma"] \arrow[d, "\mathrm{pr}_M"] & \Delta^k \times N \times J^j \arrow[d, "\mathrm{pr}_N"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

We define the semistable space of choices with respect to f, denoted by $\mathrm{Ch}^{P,\mathcal{P}}_f(M \times J^i, N \times J^j)$, as the subspace of those simplices $(\sigma, \alpha) \in \mathrm{Ch}^{P,\mathcal{P}}(M \times J^i, N \times J^j)$ for which $\sigma \in \mathrm{Ch}^P_f(M \times J^i, N \times J^{j+n})$ for some $n \in \mathbb{N}$.

We denote the special cases $i = 0$ or $j = 0$ in the definition simply by $\mathrm{Ch}^{P,\mathcal{P}}_f(M, N \times J^j)$ or $\mathrm{Ch}^{P,\mathcal{P}}_f(M \times J^i, N)$ respectively.

PL Remark 4.19. In the piecewise linear case we define $\mathrm{Ch}^P_f(M, N)$ and $\mathrm{Ch}^{P,\mathcal{P}}_f(M, N)$ only for piecewise linear maps $f : M \to N$. Also in the following definitions and theorems the piecewise linear analogue has always the added condition that all maps between $\mathcal{PL}$ families of manifolds are assumed to be piecewise linear. We stress this fact because in some cases the statements would also make sense if we allowed all continuous maps, but the proofs would not.

Proposition 4.20. *Let $f : M \to N$ be a map between families of compact manifolds over X with X a finite CW-complex and $f(M) \cap \partial N = \varnothing$. Then the space $\mathrm{Ch}^{P,\mathcal{P}}_f(M \times J^i, N \times J^j)$ is non-empty for all $i, j \in \mathbb{N}$.*

Proof. Choose an embedding $e : M \times J^i \to J^k$ for some large $k \in \mathbb{N}$ such that $e(M) \cap \partial(J^k) = \varnothing$. Then the map $(f \times 0, e) : M \times J^i \to N \times J^j \times J^k$ is an embedding. By theorem 7.7 we can assume without loss of generality that we have chosen k large enough such that the embedding $(f \times 0, e)$ also admits a normal disk bundle

$$p : E \to M \times J^i, \quad E \subseteq N \times J^j \times J^k.$$

Now choose an arbitrary parallel transport ν and a fiberwise bending isotopy H for the disk bundle p. So together with the section map $s := (f \times 0, e)$ we get a choice

$$(E, p, s, \nu, H) \in \mathrm{Ch}^P_f(M \times J^i, N \times J^{j+k})$$

and thus the space $\mathrm{Ch}^{P,\mathcal{P}}_f(M \times J^i, N \times J^j)$ is also non-empty. $\square$

Theorem 4.21. *Let $f : M \to N$ be a map between families of compact manifolds over X with X a finite CW-complex and $f(M) \cap \partial N = \varnothing$. Let $j \in \mathbb{N}$. Then $\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)$ is weakly contractible.*

Proof. We will show that for every finite subcomplex $Y \subseteq \mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)$ the embedding $|Y| \hookrightarrow |\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)|$ is contractible in $|\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)|$. We do that in several steps. In each step, we extend the choice of an induced map $P_\partial(M \times |Y|) \to \mathcal{P}_\partial(N \times J^j \times |Y|)$ over $X \times |Y|$ given by the subcomplex Y to a choice of an induced map over $X \times |Y| \times |\underline{\Delta}^1| \supseteq X \times |Y| \times 0$ such that the choices over $X \times |Y| \times 1$ have some special properties. The simplices of $Y \times \underline{\Delta}^1$ are choices of induced maps $P_\partial(M \times \Delta^k) \to \mathcal{P}_\partial(N \times J^j \times \Delta^k)$, so they are also simplices in $\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)$ and we get a homotopy $|Y| \times |\underline{\Delta}^1| \to |\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)|$ starting with the embedding of Y and ending with a map into a subcomplex which has these special properties. Thus we can assume without loss of generality that Y had these properties in the first place and continue.

Step 1. First, $|\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)|$ is a mapping telescope over the spaces $|\mathrm{Ch}_f^{P}(M, N \times J^k)|$, $k \geq j$, so without loss of generality we can assume that Y is a subcomplex of $\mathrm{Ch}_f^{P}(M, N \times J^k)$ for some fixed $k \geq j$.

Step 2. We can assume that each simplex in Y is of the form (E, p, s, ν, H), i.e. each simplex has a representative where the decomposition into disk bundles consists only of one bundle for each simplex. This is achieved as follows:

Choose representatives

$$\sigma = (E_1^\sigma, \ldots, E_{m_\sigma}^\sigma, p_1^\sigma, \ldots, p_{m_\sigma}^\sigma, s^\sigma, \nu_1^\sigma, H_\sigma^{(1)}, \ldots, \nu_{m_\sigma}^\sigma, H_\sigma^{(m_\sigma)})$$

for each simplex in Y, then take the bundles $p_1^\sigma \circ \ldots \circ p_{m_\sigma}^\sigma : E_{m_\sigma}^\sigma \to M \times \Delta^l$ and glue them together to one bundle $p' : E' \to M \times |Y|$ over $X \times |Y|$ with $E' \subseteq N \times J^k \times |Y|$. Without loss of generality we can assume that we have chosen the representatives in a way such that we can also glue the $H_\sigma^{(m_\sigma)}$ together to get a fiber-wise bending isotopy H for the bundle p. For example, one can choose representatives for which each $H_\sigma^{(m_\sigma)}$ is the constant isotopy with value the identity (see remark 4.10).

Choose a fiber-wise bending isotopy $H' : E' \times I \times [0,1] \to E' \times I \times [0,1]$ for the bundle p' such that the corresponding h' is a fiber-wise bending map for p'. By using the isotopy from H to $H' \circ H$ and from $H' \circ H$ to H' as in the proof of proposition 3.17 we can reduce Y to a subcomplex where for each simplex σ the map $h_\sigma^{(m_\sigma)}$ corresponding to $H_\sigma^{(m_\sigma)}$ is a fiber-wise bending map. This means that the composition

$$\mathrm{Tr}_{\nu_{m_\sigma}^\sigma, h_\sigma^{(m_\sigma)}} \circ \ldots \circ \mathrm{Tr}_{\nu_1^\sigma, h_\sigma^{(1)}} : P_\partial(M \times \Delta^l) \to P(E_{m_\sigma}^\sigma)$$

has image in $P_\partial(E_{m_\sigma}^\sigma)$ regardless of the other partial bending maps $h_\sigma^{(i)}$ for $i < m$. So for each

$H_\sigma^{(i)}$, $i < m$ we can use the isotopy from $H_\sigma^{(i)}$ to the constant identity isotopy given by

$$\begin{aligned} E_i^\sigma \times I \times [0,1] \times [0,1] &\to E_i^\sigma \times I \times [0,1] \\ (e,t,r,s) &\mapsto H_\sigma^{(i)}(m,t,r\cdot(1-s)) \end{aligned}$$

Because these isotopies are compatible with the structure maps in the simplicial set $\mathrm{Ch}_f^P(M, N \times J^k)$, this reduces Y to a subcomplex, where each simplex has only one bending isotopy that is not the constant identity isotopy, which in turn implies that each simplex in Y is of the form (E,p,s,ν,H). This concludes step 2.

Step 3. We can assume that each simplex in Y is of the form

$$(E_0 \times \Delta^l, p_0 \times \mathrm{id}_{\Delta^l}, s_0 \times \mathrm{id}_{\Delta^l}, \nu, H)$$

for some fixed bundle $p_0 : E_0 \to M$ with $E_0 \subseteq N \times J^k$ and zero-section $s_0 : M \to E_0$.

For this step we need to be able to stabilize the bundles $p : E \to M \times \Delta^l$ of each simplex. We can achieve that by moving from $|\mathrm{Ch}_f^P(M, N \times J^k)|$ to $|\mathrm{Ch}_f^P(M, N \times J^{k+k'})|$ in the telescope direction of $|\mathrm{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)|$ (which induces k' times composition of the choices with the stabilization map) and then repeating step 2 so that each simplex is again of the form (E,p,s,ν,H). This replaces a simplex $\sigma = (E,p,s,\nu,H)$ with a simplex $\sigma' = (E',p',s',\nu',H')$ such that $E' = E \times J^{k'}$ as well as $p' = p \circ \mathrm{pr}_E : E \times J^{k'} \to M \times \Delta^l$ and the zero-section s' is given by s composed with the inclusion $E = E \times 0 \to E \times J^{k'}$.

Choose an embedding $\iota : M \to J^k$ of M into the interior of J^k. If necessary, increase k by stabilizing to find such an embedding. Then the embedding $s_0 := f \times \iota : M \to N \times J^k$ admits a stable normal disk bundle $p_0 : E_0 \to M$, $E_0 \subseteq N \times J^k$. Again, one may have to increase k to find E_0. For the existence of stable normal bundles see theorem 7.7. Denote by $p : E \to M \times |Y|$ again the bundle over $X \times |Y|$ obtained by gluing together the bundles of each simplex in Y. We want to extend p to a bundle $p' : E' \to M \times |Y| \times [0,1]$ over $X \times |Y| \times [0,1]$ such that the bundle restricted to $M \times |Y| \times 1$ is given by $p_0 \times \mathrm{id}_{|Y|} : E_0 \times |Y| \to M \times |Y|$. First, the zero-sections s and s_0 are homotopic as embeddings into $N \times J^k \times |Y|$ (both are homotopic to $f \times 0 \times \mathrm{id}_{|Y|}$) so by increasing k we can find an isotopy over $X \times |Y|$ between the two maps. This isotopy extends the zero-sections to an embedding $s' : M \times |Y| \times [0,1] \to N \times J^k \times |Y| \times [0,1]$ over $X \times |Y| \times [0,1]$. Second, we can extend the normal disk bundles given over the embedding $s'|_{M \times |Y| \times \{0,1\}}$ by $p : E \to M \times |Y|$ and $p_0 \times \mathrm{id}_{|Y|} : E_0 \times |Y| \to M \times |Y|$ to a normal disk bundle $p' : E' \to M \times |Y| \times [0,1]$, $E' \subseteq N \times J^k \times |Y| \times [0,1]$ using theorem 7.7 (and again increasing k by stabilizing if necessary). As always, one may have to increase k for this. Lastly, we have to extend ν and H. Now p' is a concordance between the two bundles p and $p_0 \times \mathrm{id}_{|Y|}$, thus both bundles are isotopic. Use this isotopy to extend ν and H over $M \times [0,1]$.

Step 4. We can assume that each simplex in Y is of the form

$$(E_0 \times \Delta^l, p_0 \times \mathrm{id}_{\Delta^l}, s_0 \times \mathrm{id}_{\Delta^l}, \nu_0 \times \mathrm{id}_{\Delta^l}, H_0 \times \mathrm{id}_{\Delta^l})$$

for some parallel transport ν_0 over p_0 (where $\nu_0 \times \mathrm{id}_{\Delta^l}$ denotes the product parallel transport of ν_0 with the trivial parallel transport over $\mathrm{id} : \Delta^l \to \Delta^l$) and some fiberwise bending isotopy H_0 over p_0. This step follows directly from propositions 3.6 and 3.17.

With step 4 we have reduced Y to a single point in $|\operatorname{Ch}_f^{P,\mathcal{P}}(M, N \times J^j)|$, which finishes the proof. □

Corollary 4.22. *Let $f : M \to N$ be a map of families of compact manifolds over X with X a finite CW-complex and $f(M) \cap \partial N = \varnothing$. Let $i, j \in \mathbb{N}$. Then $\operatorname{Ch}_f^{P,\mathcal{P}}(M \times J^i, N \times J^j)$ is weakly contractible.*

Proof. We will modify the proof of theorem 4.21 for this. We start again with an arbitrary finite subcomplex $Y \subseteq \operatorname{Ch}_f^{P,\mathcal{P}}(M \times J^i, N \times J^j)$ and then use steps 1 and 2 as above to reduce Y to a subcomplex where every choice is of the form (E, p, s, ν, H) and lies in $\operatorname{Ch}^P(M \times J^i, N \times J^k)$ for some $k \geq j$.

Step 3. We can assume that Y is contained in the subcomplex

$$\operatorname{Ch}_{f\circ \mathrm{pr}_M}^{P,\mathcal{P}}(M \times J^i, N \times J^j) \subseteq \operatorname{Ch}_f^{P,\mathcal{P}}(M \times J^i, N \times J^j).$$

Gluing all the bundles of our choices in Y together, we get an element (E, p, s, ν, H) in $\operatorname{Ch}_g^{P,\mathcal{P}}(M \times J^i \times |Y|, N \times J^j \times |Y|)$ for some map $g : M \times J^i \times |Y| \to N \times |Y|$ whose image does not intersect $\partial N \times |Y|$ (and with everything viewed as families of manifolds over $X \times |Y|$). Let h be a homotopy from g to the map $(f \circ \mathrm{pr}_M) \times \mathrm{id}_{|Y|}$ and let

$$c \in \operatorname{Ch}_h^{P,\mathcal{P}}(M \times J^i \times |Y| \times [0,1], N \times J^j \times |Y| \times [0,1])$$

with the spaces viewed as families of manifolds over $X \times |Y| \times [0,1]$. Then c yields a homotopy from a choice in $|\operatorname{Ch}_g^{P,\mathcal{P}}(M \times J^i \times |Y|, N \times J^j \times |Y|)|$ to a choice in

$$\left|\operatorname{Ch}_{(f\circ \mathrm{pr}_M)\times \mathrm{id}_{|Y|}}^{P,\mathcal{P}}(M \times J^i \times |Y|, N \times J^j \times |Y|)\right|$$

Since $\operatorname{Ch}_g^{P,\mathcal{P}}(M \times J^i \times |Y|, N \times J^j \times |Y|)$ is weakly contractible by theorem 4.21, this means that we can deform our choice (E, p, s, ν, H) to a choice living in $\operatorname{Ch}_{f\circ \mathrm{pr}_M}^{P,\mathcal{P}}(M \times J^i \times |Y|, N \times J^j \times |Y|)$. Respectively, this deforms our subcomplex $|Y| \subseteq |\operatorname{Ch}_f^{P,\mathcal{P}}(M \times J^i, N \times J^j)|$ into a subcomplex contained in $|\operatorname{Ch}_{f\circ \mathrm{pr}_M}^{P,\mathcal{P}}(M \times J^i, N \times J^j)|$.

Step 4. $\operatorname{Ch}_{f\circ \mathrm{pr}_M}^{P,\mathcal{P}}(M \times J^i, N \times J^j)$ is weakly contractible by theorem 4.21 and thus the claim follows. □

PL Remark 4.23. In the piecewise linear case the proofs of 4.21 and 4.22 can be simplified a bit, because for the existence and uniqueness of tubular neighborhoods for piecewise linear embeddings one does not need to stabilize the embedding first.

Now we want to define spaces of stable choices. Because our previous choice spaces are not Kan and thus do not behave particularly well with respect to certain constructions like infinite products, we will need to switch from simplicial sets to the category of compactly generated topological spaces using the geometric realization of simplicial sets $|-|: \mathrm{sSet} \to \mathrm{cg}\mathfrak{Top}$. We will implicitly use the fact that geometric realization as a functor to $\mathrm{cg}\mathfrak{Top}$ preserves finite products and is well behaved with respect to mapping spaces in the following sense: For X, Y simplicial sets we have a map

$$|\operatorname{map}(X,Y)| \to \operatorname{map}(|X|,|Y|),$$

natural in X and Y, which is given on a k-cell of $|\operatorname{map}(X,Y)|$ indexed by $\sigma : X \times \underline{\Delta}^k \to Y$ by the map $\Delta^k \to \operatorname{map}(|X|,|Y|)$ adjoint to $|\sigma| : |X| \times \Delta^k \to |Y|$. These maps commute with composition, i.e. the diagram

$$\begin{tikzcd}
|\operatorname{map}(Y,Z)| \times |\operatorname{map}(X,Y)| \arrow[r, "|-\circ-|"] \arrow[d] & |\operatorname{map}(X,Z)| \arrow[d] \\
\operatorname{map}(|Y|,|Z|) \times \operatorname{map}(|X|,|Y|) \arrow[r, "-\circ-"] & \operatorname{map}(|X|,|Z|)
\end{tikzcd}$$

commutes. Using that, the geometric realization of the composition maps of choice spaces again yields composition maps

$$-\circ- : |\operatorname{Ch}^P(N,K)| \times |\operatorname{Ch}^P(M,N)| \to |\operatorname{Ch}^P(M,K)|$$
$$-\circ- : |\operatorname{Ch}^{P,\mathcal{P}}(N,K)| \times |\operatorname{Ch}^P(M,N)| \to |\operatorname{Ch}^{P,\mathcal{P}}(M,K)|$$

which are again compatible with realization as before, where realization is now defined as the compositions

$$r : |\operatorname{Ch}^P(M,N)| \xrightarrow{|r|} |\operatorname{map}(P_\partial(M), P_\partial(N))| \to \operatorname{map}(|P_\partial(M)|, |P_\partial(N)|)$$
$$r : |\operatorname{Ch}^{P,\mathcal{P}}(M,N)| \xrightarrow{|r|} |\operatorname{map}(P_\partial(M), \mathcal{P}_\partial(N))| \to \operatorname{map}(|P_\partial(M)|, |\mathcal{P}_\partial(N)|).$$

Definition 4.24. Let M, N be families of manifolds over X and denote $\Delta^1 := |\underline{\Delta}^1| = [0,1]$. Define the space of choices $\mathbf{Ch}^{\mathcal{P}}(M,N)$ of induced maps from $|\mathcal{P}_\partial(M)|$ to $|\mathcal{P}_\partial(N)|$, also called the space of stable choices, as a subspace

$$\mathbf{Ch}^{\mathcal{P}}(M,N) \subseteq \prod_{n \in \mathbb{N}} \operatorname{map}\left(\Delta^1, |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^n, N)|\right)$$

such that a tuple $(\varphi_0, \varphi_1, \dots)$ with $\varphi_i \in \operatorname{map}(\Delta^1, |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^i, N)|)$ is in $\mathbf{Ch}^{\mathcal{P}}(M, N)$ if and only if for each $i \in \mathbb{N}$ we have

$$\varphi_i(1) = \varphi_{i+1}(0) \circ |s| \in |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^i, N)|$$

where $s \in \operatorname{Ch}^P(M \times J^i, M \times J^{i+1})$ again denotes the preferred choice for the stabilization map.

Definition 4.25. There is again a realization map

$$r : \mathbf{Ch}^{\mathcal{P}}(M, N) \to \operatorname{map}\big(|\mathcal{P}_\partial(M)|, |\mathcal{P}_\partial(N)|\big)$$

defined as follows: Composition with the realization map in each factor gives a map

$$\begin{aligned} \mathbf{Ch}^{\mathcal{P}}(M, N) &\to \prod_{n \in \mathbb{N}} \operatorname{map}(\Delta^1, \operatorname{map}\big(|\mathcal{P}_\partial(M \times J^n)|, |\mathcal{P}_\partial(N))|\big) \\ &\cong \prod_{n \in \mathbb{N}} \operatorname{map}\big(|\mathcal{P}_\partial(M \times J^n)| \times \Delta^1, |\mathcal{P}_\partial(N)|\big). \end{aligned}$$

This map has image in the subset

$$U \subseteq \prod_{n \in \mathbb{N}} \operatorname{map}\big(|P_\partial(M \times J^n)| \times \Delta^1, |\mathcal{P}_\partial(N)|\big)$$

such that $(f_0, f_1, \dots) \in U$ if and only if for all $i \in \mathbb{N}$ the restriction of f_i to $|P_\partial(M \times J^i)| \times 1$ equals the composition of f_{i+1} restricted to $|P_\partial(M \times J^{i+1})| \times 0$ with the stabilization map $|s|$. Since $|\mathcal{P}_\partial(M)| = \bigcup_{k \in \mathbb{N}} |P_\partial(M \times J^k)|$ glued along the stabilization maps $|s|$, we can glue such a tuple $(f_0, f_1, \dots) \in U$ together to a map $|\mathcal{P}_\partial(M)| \to |\mathcal{P}_\partial(N)|$, which defines a map

$$U \to \operatorname{map}\big(|\mathcal{P}_\partial(M)|, |\mathcal{P}_\partial(N)|\big).$$

Now $r : \mathbf{Ch}^{\mathcal{P}}(M, N) \to \operatorname{map}\big(|\mathcal{P}_\partial(M)|, |\mathcal{P}_\partial(N)|\big)$ is given as the composition

$$\mathbf{Ch}^{\mathcal{P}}(M, N) \to U \to \operatorname{map}\big(|\mathcal{P}_\partial(M)|, |\mathcal{P}_\partial(N)|\big).$$

For the composition map we first define another composition map of the form

$$(-) \circ (-) : \mathbf{Ch}^{\mathcal{P}}(N, K) \times |\operatorname{Ch}^{P,\mathcal{P}}(M, N)| \to |\operatorname{Ch}^{P,\mathcal{P}}(M, K)|$$

for families of manifolds M, N and K over X as follows: Using

$$|\operatorname{Ch}^{P,\mathcal{P}}(M, N)| = \bigcup_{k \in \mathbb{N}} |\operatorname{Ch}^P(M, N \times J^k)| \times \Delta^1 \big/ \sim$$

we define for

$$\begin{aligned}(\sigma, t) &\in |\operatorname{Ch}^P(M, N\times J^i)| \times \Delta^1 \subseteq |\operatorname{Ch}^{P,\mathcal{P}}(M,N)|\\ (\varphi_0, \varphi_1, \dots) &\in \mathbf{Ch}^{\mathcal{P}}(N,K)\end{aligned}$$

their composition as

$$(\varphi_0, \varphi_1, \dots) \circ (\sigma, t) := \varphi_i(t) \circ \sigma \in |\operatorname{Ch}^{P,\mathcal{P}}(M,K)|.$$

This is continuous in φ_i, σ and t and well defined for $t \in \{0,1\}$ since $\varphi_i(1)\circ\sigma = \varphi_{i+1}(0)\circ(|s|\circ\sigma)$ by definition of $\mathbf{Ch}^{\mathcal{P}}(N,K)$.

Definition 4.26. Let M, N and K be families of manifolds over X. The composition map for choices of stable maps

$$(-)\circ(-) : \mathbf{Ch}^{\mathcal{P}}(N,K) \times \mathbf{Ch}^{\mathcal{P}}(M,N) \to \mathbf{Ch}^{\mathcal{P}}(M,K)$$

is defined by

$$(c', c) \mapsto (c'\circ\varphi_0(-), c'\circ\varphi_1(-), \dots)$$

with $c' \in \mathbf{Ch}^{\mathcal{P}}(N,K)$ and $c = (\varphi_0, \varphi_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(M,N)$.

Proposition 4.27. *The composition map for choices of stable maps is associative and compatible with the realization map, i.e.* $r(c')\circ r(c) = r(c'\circ c)$ *for* $c \in \mathbf{Ch}^{\mathcal{P}}(M,N)$ *and* $c' \in \mathbf{Ch}^{\mathcal{P}}(N,K)$.

Proof. We start with the compatibility of composition with the realization map. Let M, N, K be families of manifolds over X, $c = (\varphi_0, \varphi_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(M,N)$, $c' = (\varphi'_0, \varphi'_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(N,K)$ and let

$$(x,t) \in |P_\partial(M\times J^k)| \times \Delta^1 \subseteq |\mathcal{P}_\partial(M)| = \bigcup_{n\in\mathbb{N}} |P_\partial(M\times J^n)| \times \Delta^1 \big/ \sim$$

for some $k \in \mathbb{N}$. Then

$$\begin{aligned} r(c'\circ c)(x,t) &= r(c'\circ\varphi_k(t))(x)\\ &= r(\varphi'_l(t')\circ\psi)(x)\end{aligned}$$

with $(\psi, t') := \varphi_k(t) \in |\operatorname{Ch}^P(M\times J^k, N\times J^l)| \times \Delta^1 \subseteq |\operatorname{Ch}^{P,\mathcal{P}}(M\times J^k, N)|$. Also denote $(\psi', t'') := \varphi_l(t') \in |\operatorname{Ch}^P(N\times J^l, K\times J^m)| \times \Delta^1 \subseteq |\operatorname{Ch}^{P,\mathcal{P}}(N\times J^l, K)|$. Then

$$\begin{aligned} r(\varphi'_l(t')\circ\psi)(x) &= r(\psi'\circ\psi, t'')(x)\\ &= \big(r(\psi')\circ r(\psi)(x), t''\big)\\ &= r\big(\varphi'_l(t')\big)\big(r(\psi)(x)\big)\\ &= r(c')\big(r(\varphi_k(t))(x)\big) = r(c')\circ r(c)(x,t).\end{aligned}$$

Thus realization is compatible with composition. Note that in this computation we used the compatibility of realization with unstable composition proven in proposition 4.15.

For associativity let $c = (\varphi_0, \varphi_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(M,N)$, $c' = (\varphi_0', \varphi_1', \dots) \in \mathbf{Ch}^{\mathcal{P}}(N,K)$ and $c'' = (\varphi_0'', \varphi_1'', \dots) \in \mathbf{Ch}^{\mathcal{P}}(K,L)$. It suffices to show $(c'' \circ c') \circ \varphi_i = c'' \circ (c' \circ \varphi_i)$ for arbitrary $i \in \mathbb{N}$. So let $t \in \Delta^1$ and define

$$\begin{aligned}(\psi', t') &:= \varphi_i(t) \in |\mathrm{Ch}^P(M \times J^i, N \times J^j)| \times \Delta^1 \subseteq |\mathrm{Ch}^{P,\mathcal{P}}(M \times J^i, N)|\\ (\psi'', t'') &:= \varphi_j'(t') \in |\mathrm{Ch}^P(N \times J^j, K \times J^k)| \times \Delta^1 \subseteq |\mathrm{Ch}^{P,\mathcal{P}}(N \times J^j, K)|\\ (\psi''', t''') &:= \varphi_k''(t'') \in |\mathrm{Ch}^P(K \times J^k, L \times J^l)| \times \Delta^1 \subseteq |\mathrm{Ch}^{P,\mathcal{P}}(K \times J^k, L)|.\end{aligned}$$

Then

$$\begin{aligned}(c'' \circ c') \circ \varphi_i(t) &= (c'' \circ \varphi_j'(t')) \circ \psi' = (\varphi_k''(t'') \circ \psi'') \circ \psi'\\ &= (\psi''' \circ \psi'' \circ \psi', t''') = \varphi_k''(t'') \circ (\psi'' \circ \psi')\\ &= c'' \circ (\varphi_j'(t') \circ \psi') = c'' \circ (c' \circ \varphi_i(t)).\end{aligned}$$

Again we needed the unstable analogue of associativity that was already mentioned in remark 4.13. □

Definition 4.28. Let M be a manifold over X. For $k \geq 0$ let

$$\varphi_k^{\mathrm{id}} : \Delta^1 \to |\mathrm{Ch}^P(M \times J^k, M \times J^k)| \times \Delta^1 \subseteq |\mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, M)|$$

be the map given by $\varphi_k^{\mathrm{id}}(t) = (|\sigma_{\mathrm{id}}|, t)$, where σ_{id} is the unstable identity choice defined in remarks 4.6 and 4.14. We can combine the φ_k^{id} for all $k \in \mathbb{N}$ to get a stable choice

$$c_{\mathrm{id}} := (\varphi_0^{\mathrm{id}}, \varphi_1^{\mathrm{id}}, \dots) \in \mathbf{Ch}^{\mathcal{P}}(M,M)$$

which we call the *stable identity choice* over M.

Lemma 4.29. *Let M be a manifold over X. Then the stable identity choice $c_{\mathrm{id}} \in \mathbf{Ch}^{\mathcal{P}}(M,M)$ is a neutral element with respect to the composition of stable choices and the realization $r(c_{\mathrm{id}})$ is the identity map on $|\mathcal{P}_\partial(M)|$.*

Proof. Denote by σ_{id} the unstable identity choice (see 4.6 and 4.14). For $c = (\varphi_0, \varphi_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(M,N)$ we get

$$c \circ \varphi_k^{\mathrm{id}}(t) = c \circ (\sigma_{\mathrm{id}}, t) = \varphi_k(t) \circ \sigma_{\mathrm{id}} = \varphi_k(t)$$

and thus $c \circ c_{\mathrm{id}} = c$. For $c = (\varphi_0, \varphi_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(K,M)$ let

$$\varphi_k(t) =: (\psi, t') \in |\mathrm{Ch}^P(K \times J^k, M \times J^l)| \times \Delta^1 \subseteq |\mathrm{Ch}^{P,\mathcal{P}}(K \times J^k, M)|.$$

Then

$$c_{\mathrm{id}} \circ \varphi_k(t) = \varphi_l^{\mathrm{id}}(t') \circ \psi = (\sigma_{\mathrm{id}} \circ \psi, t') = (\psi, t') = \varphi_k(t)$$

and thus $c_{\mathrm{id}} \circ c = c$. We also get $r(c_{\mathrm{id}}) = \mathrm{id}_{|\mathcal{P}_\partial(M)|}$ as a direct consequence of the definitions and the fact that $r(\sigma_{\mathrm{id}})$ is the identity on the unstable pseudoisotopy space. □

Remark 4.30. Combining proposition 4.27 and lemma 4.29 we can view the stable choice spaces for families of manifolds over X as the morphism spaces of a category with objects families of submanifolds of $X \times \mathbb{R}^\infty$ over X. The realization maps then form a functor from this category to the category of compactly generated topological spaces.

Definition 4.31. Let $f : M \to N$ be a map between families of manifolds over X and let $i, j \in \mathbb{N}$. Define the *space of stable choices over f* as

$$\mathbf{Ch}_f^{\mathcal{P}}(M \times J^i, N \times J^j) := \mathbf{Ch}^{\mathcal{P}}(M \times J^i, N \times J^j) \cap \prod_{n \in \mathbb{N}} \mathrm{map}\left(\Delta^1, |\,\mathrm{Ch}_f^{P,\mathcal{P}}(M \times J^{i+n}, N \times J^j)|\right)$$

Lemma 4.32. *Let $f : M \to N$, $g : N \to K$ maps of families of manifolds over X and let $c \in \mathbf{Ch}_f^{\mathcal{P}}(M, N)$, $c' \in \mathbf{Ch}_g^{\mathcal{P}}(N, K)$. Then $c' \circ c \in \mathbf{Ch}_{g \circ f}^{\mathcal{P}}(M, K)$.*

Proof. For the corresponding unstable composition of choices, which were defined in 4.18, this follows from the commutativity of the diagram

$$\begin{tikzcd}
\Delta^k \times M \times J^i \arrow[r, "s_\sigma"] \arrow[d, "\mathrm{pr}_M"] & \Delta^k \times N \times J^j \arrow[r, "s_{\sigma'}"] \arrow[d, "\mathrm{pr}_N"] & \Delta^k \times K \times J^l \arrow[d, "\mathrm{pr}_K"] \\
M \arrow[r, "f"] & N \arrow[r, "g"] & K
\end{tikzcd}$$

for choices $\sigma \in \mathrm{Ch}_f^P(M \times J^i, N \times J^j)$, $\sigma' \in \mathrm{Ch}_g^P(N \times J^j, K \times J^l)$ and corresponding zero-sections s_σ, $s_{\sigma'}$. For the semistable and stable compositions of choices we have that a choice is with respect to $g \circ f$ if and only if the underlying unstable choices are with respect to $g \circ f$, thus the claim follows from the unstable case. □

Corollary 4.33. *Let $f : M \to N$ be a map between families of compact manifolds over X with X a finite CW-complex and $f(M) \cap \partial N = \varnothing$. Then $\mathbf{Ch}_f^{\mathcal{P}}(M \times J^i, N \times J^j)$ is contractible for all $i, j \in \mathbb{N}$.*

Proof. The spaces $|\,\mathrm{Ch}_f^{P,\mathcal{P}}(M \times J^{i+n}, N \times J^j)|$, $n \in \mathbb{N}$ are contractible by theorem 4.21 and corollary 4.22, thus this is also true for the spaces

$$\mathrm{map}(\Delta^1, |\,\mathrm{Ch}_f^{P,\mathcal{P}}(M \times J^{i+n}, N \times J^j)|).$$

Using this it is now easy to construct a contraction of $\mathbf{Ch}_f^{\mathcal{P}}(M \times J^i, N \times J^j)$. □

The functor mentioned in remark 4.30 is still far away from the pseudoisotopy functor we want to define. But constructing induced maps out of geometric transfers and bending maps is already enough to show that pseudoisotopy spaces are functorial up to homotopy. Hatcher sketched a proof of this fact in [Hat78] and thanks to our choice spaces we can give a precise proof here:

The homotopy functor sends a manifold M again to $|\mathcal{P}_\partial(M)|$, whereas for morphisms we do the following: For $f : M \to N$ a map between manifolds choose a map $g : M \to N$ such that f is homotopic to g and $g(M) \cap \partial N = \varnothing$. Let $x \in \mathbf{Ch}_g^{\mathcal{P}}(M, N)$ be any point. Then assign to the homotopy class of f the homotopy class of $r(x) : |\mathcal{P}_\partial(M)| \to |\mathcal{P}_\partial(N)|$. This is independent of x via corollary 4.33. If $g' : M \to N$ is another map with $f \simeq g'$ and $g'(M) \cap N = \varnothing$, we get a homotopy $h : M \times [0,1] \to N \times [0,1]$ from g to g'. If you view $M \times [0,1]$ and $N \times [0,1]$ as families of manifolds over $[0,1]$, then the elements of $\mathbf{Ch}_h^{\mathcal{P}}(M \times [0,1], N \times [0,1])$ are by definition homotopies from points in $\mathbf{Ch}_g^{\mathcal{P}}(M, N)$ to points in $\mathbf{Ch}_{g'}^{\mathcal{P}}(M, N) \subseteq \mathbf{Ch}^{\mathcal{P}}(M, N)$. Because $\mathbf{Ch}_h^{\mathcal{P}}(M \times [0,1], N \times [0,1])$ is contractible by corollary 4.33, this in particular implies that the homotopy class of $r(x)$ is independent of the choice of g (and does only depend on the homotopy class of f). Compatibility with composition is a consequence of proposition 4.27. To show that the identity is sent to the homotopy class of the identity we take for a manifold M an isotopy $h : M \times [0,1] \to M$ from the identity on M to a codimension-zero-embedding $g : M \to M$ with image in the interior of M. We get a stable choice $c \in \mathbf{Ch}^{\mathcal{P}}(M \times [0,1], M \times [0,1])$ out of h by crossing it with id_{J^k} analogous to the construction of the stable identity choice in 4.6 and 4.28. Now c is a homotopy from the stable identity choice to a choice in $\mathbf{Ch}_g^{\mathcal{P}}(M, M)$ and by lemma 4.29 we conclude that our homotopy functor sends the identity to the homotopy class of the identity. Thus we have proven:

Corollary 4.34 (Hatcher, [Hat78])**.** *The assignment sending a manifold M to $|\mathcal{P}_\partial(M)|$ and the homotopy class of a map $f : M \to N$ to the homotopy class of $r(x)$ with $r(x)$ as above is a functor from the homotopy category of compact manifolds to the homotopy category of compactly generated topological spaces.*

CHAPTER 5

Functoriality

We already know by corollary 4.34 that the pseudoisotopy spaces are functorial up to homotopy. In this chapter we will improve on that by constructing strict pseudoisotopy functors using the strictification process described by Cordier and Porter in [CP86].

We obviously require that a pseudoisotopy functor has to agree, up to homotopy, with the homotopy functor given in corollary 4.34. Furthermore, we want it to be defined on the category of all topological spaces, not just compact manifolds, and it should still induce a homotopy functor. But apart from that there are no other requirements on a pseudoisotopy functor and in fact we will not prove any uniqueness property for the functor that we will construct.

For the convenience of the reader we recall the relevant theorems for the strictification process from [CP86] here.

Definition 5.1. Let $\mathfrak{C}$ be a small category and $\mathfrak{D}$ be a simplicially enriched category. Denote by $\mathrm{Ner}(\mathfrak{C})$ the nerve of the category $\mathfrak{C}$. A *homotopy coherent diagram*

$$F \in \mathrm{Coh}(\mathfrak{C}, \mathfrak{D})$$

consists of the following data:

- For each object $C \in \mathfrak{C}$ an object $F(C) \in \mathfrak{D}$
- For each $(f_n, f_{n-1}, \dots, f_0) \in \mathrm{Ner}(\mathfrak{C})_{n+1}$ with $f_n \circ \dots \circ f_0 \in \mathrm{map}_{\mathfrak{C}}(C, C')$ a map

  $$F(f_n, \dots, f_0) : (\underline{\Delta}^1)^n \to \mathrm{map}_{\mathfrak{D}}(F(C), F(C'))$$

  such that
  - if $n = 0$ and $f_0 = \mathrm{id}_C$ then $F(f_0) = F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$,
  - if $f_0 = \mathrm{id}$ then $F(f_n, \dots, f_0) = F(f_n, \dots, f_1) \circ \mathrm{pr}$, where $\mathrm{pr} : (\underline{\Delta}^1)^n \to (\underline{\Delta}^1)^{n-1}$ is the projection onto the last $n-1$ factors,

– if $f_i = \mathrm{id}$, $0 < i < n$, then

$$F(f_n, \dots, f_0) = F(f_n, \dots f_{i+1}, f_{i-1}, \dots f_0) \circ \mathrm{id}_{(\underline{\Delta}^1)^{i-1}} \times m \times \mathrm{id}_{(\underline{\Delta}^1)^{n-i-1}}$$

with $m : \underline{\Delta}^1 \times \underline{\Delta}^1 \to \underline{\Delta}^1$ the multiplicative structure on $\underline{\Delta}^1$ defined by $m(0,0) = 0$ and $m(0,1) = m(1,0) = m(1,1) = 1$,

– if $f_n = id$ then $F(f_n, \dots, f_0) = F(f_{n-1}, \dots, f_0) \circ \mathrm{pr}$, where $\mathrm{pr} : (\underline{\Delta}^1)^n \to (\underline{\Delta}^1)^{n-1}$ is the projection onto the first $n-1$ factors,

– $F(f_n, \dots, f_0)|_{(\underline{\Delta}^1)^{i-1} \times 0 \times (\underline{\Delta}^1)^{n-i}} = F(f_n, \dots, f_i \circ f_{i-1}, \dots, f_0)$ for $0 < i \leq n$,

– $F(f_n, \dots, f_0)|_{(\underline{\Delta}^1)^{i-1} \times 1 \times (\underline{\Delta}^1)^{n-i}} = F(f_n, \dots, f_i, \mathrm{id}, \dots, \mathrm{id}) \circ F(\mathrm{id}, \dots \mathrm{id}, f_{i-1}, \dots, f_0)$ for $0 < i \leq n$.

Coherent diagrams can also be defined in a different way using the coherent nerve of a category. For that we refer the interested reader to [CP86], which also discusses the underlying ideas of coherent diagrams.

A strict functor $F : \mathfrak{C} \to \mathfrak{D}$ can be seen as a special case of a coherent diagram where the map $(\underline{\Delta}^1)^n \to \mathrm{map}_{\mathfrak{D}}(F(C), F(C'))$ corresponding to a tuple $(f_n, \dots, f_0)$ is a constant map with value $F(f_n \circ \dots \circ f_0)$. Similarly, we also get natural transformations of functors as special cases of natural transformations of coherent diagrams, where the latter are defined as coherent diagrams from the category $\mathfrak{C} \times [1]$ to $\mathfrak{D}$ with $[1]$ the category with two objects $0, 1$ and one morphism $0 \to 1$.

Theorem 5.2 (Cordier, Porter, [CP86, Corollary 4.5, 4.6]). *Let $\mathfrak{C}$ be a small category and $\mathfrak{D}$ be a simplicially enriched category. Assume that $\mathfrak{D}$ is locally Kan, i.e. all homomorphism sets are Kan sets. Let $F \in \mathrm{Coh}(\mathfrak{C}, \mathfrak{D})$ be a coherent diagram. Then there exists a (strict) functor $\hat{F} : \mathfrak{C} \to \mathfrak{D}$ and a natural isomorphism*

$$F \xrightarrow{\cong} \hat{F}$$

of coherent diagrams.

Let $G \in \mathrm{Coh}(\mathfrak{C}, \mathfrak{D})$ be another coherent diagram and let $f : F \to G$ be a natural transformation from F to G. Then there exists a natural transformation $\hat{f} : \hat{F} \to \hat{G}$ of strict functors such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\cong"] & G \arrow[d, "\cong"] \\ \hat{F} \arrow[r, "\hat{f}"] & \hat{G} \end{tikzcd}$$

For the construction of a homotopy coherent diagram of pseudoisotopy spaces we need to fix a small source category. This will be the category of compact submanifolds (with or without boundary) of $\mathbb{R}^\infty$, which we will denote by Mfd_c.

Theorem 5.3. *There exists a coherent diagram*

$$\mathcal{P}_\partial^{coh} \in \mathrm{Coh}(\mathrm{Mfd}_c, \mathrm{cg}\mathfrak{Top})$$

such that $\mathcal{P}_\partial^{coh}(M) = |\mathcal{P}_\partial(M)|$ *for every manifold* M *and for every map* $f \in \mathrm{map}_{\mathrm{Mfd}_c}(M,N)$ *there exists a map* $g : M \to N$ *homotopic to* f *and a choice* $c \in \mathbf{Ch}_g^{\mathcal{P}}(M,N)$ *with* $\mathcal{P}_\partial^{coh}(f) = r(c)$.

Proof. Denote by $\mathbf{Ch}^{\mathcal{P}}$ the category with objects the same as Mfd_c and with morphism spaces $\mathbf{Ch}^{\mathcal{P}}(M,N)$ for $M, N \in \mathrm{Mfd}_c$. Denote by $\mathfrak{C}$ the category with objects the same as Mfd_c and morphism spaces

$$\mathrm{map}_{\mathfrak{C}}(M,N) = \mathbf{Ch}^{\mathcal{P}}(M,N) \times \mathrm{map}([0,1], \mathrm{map}(M,N)).$$

Our strategy is to construct a homotopy coherent diagram $F \in \mathrm{Coh}(\mathrm{Mfd}_c, \mathfrak{C})$, then compose it with the forgetful functor to get a diagram $F_{ch} \in \mathrm{Coh}(\mathrm{Mfd}_c, \mathbf{Ch}^{\mathcal{P}})$ and finally compose it with the realization functor $r : \mathbf{Ch}^{\mathcal{P}} \to \mathrm{cg}\mathfrak{Top}$ to get a homotopy coherent diagram in $\mathrm{Coh}(\mathrm{Mfd}_c, \mathrm{cg}\mathfrak{Top})$ which will have the desired properties. Since $\mathfrak{C}$, $\mathbf{Ch}^{\mathcal{P}}$ and $\mathrm{cg}\mathfrak{Top}$ are viewed as simplicially enriched categories by taking the singular sets of their topological mapping spaces, we will construct the maps $(\Delta^1)^n \to \mathrm{map}_{\mathfrak{C}}(M,N)$ (with the corresponding boundary conditions) adjoint to the maps from $(\underline{\Delta}^1)^n$ of the homotopy coherent diagram F.

As a preparation choose for each $M \in \mathrm{Mfd}_c$ an isotopy of codimension-zero-embeddings $\alpha_M : [0,1] \to \mathrm{map}_{\mathrm{Mfd}_c}(M,M)$ starting at the identity to an embedding $\alpha(1) : M \to M$ with $\alpha(1)(M) \cap \partial M = \varnothing$. One can use a collar of M and then push slightly inwards to construct α_M. Remember that codimension-zero-embeddings define unstable choices which are unique if we require the corresponding bending isotopy to be constant (see 4.6). We can glue the choices corresponding to the codimension-zero-embeddings $\alpha_M(t) \times \mathrm{id}_{J^k}$, $k \in \mathbb{N}$, together to a stable choice analogous to the identity choice in 4.28. Thus the isotopy α_M determines a homotopy $c_M : [0,1] \to \mathbf{Ch}^{\mathcal{P}}(M,M)$ which starts at the identity choice in $\mathbf{Ch}^{\mathcal{P}}(M,M)$ and such that for all $t \in [0,1]$ we have $c_M(t) \in \mathbf{Ch}^{\mathcal{P}}_{\alpha_M(t)}(M,M)$.

We use induction over the length of composable tuples to construct F such that for each tuple $(f_n, \ldots, f_0)$ of composable maps in Mfd_c and $t \in (\Delta^1)^n$ the tuple $F(f_n, \ldots, f_0)(t) = (c, \varphi)$ satisfies $\varphi(0) = f_n \circ \ldots \circ f_0$ and $c \in \mathbf{Ch}^{\mathcal{P}}_{\varphi(1)}(M,N)$. Note that these properties already imply that the coherent diagram $\mathcal{P}_\partial^{coh}$ constructed out of F will satisfy $\mathcal{P}_\partial^{coh}(f) = r(c)$ for some $c \in \mathbf{Ch}_g^{\mathcal{P}}(M,N)$ with $g \simeq f$ for all f.

For the induction start we set $F(M) = M$ for each manifold $M \in \mathrm{Mfd}_c$ and $F(\mathrm{id}_M) = (c, \varphi)$ with $c \in \mathbf{Ch}^{\mathcal{P}}(M,M)$ the identity choice and φ the constant homotopy with value id_M. For non-identity maps $f : M \to N$ let $\varphi(-) = \alpha_N(-) \circ f : [0,1] \to \mathrm{map}(M,N)$, choose some $c \in \mathbf{Ch}^{\mathcal{P}}_{\varphi(1)}(M,N)$ and set $F(f) \coloneqq (c, \varphi)$.

For the induction step we assume that we have constructed $F(g_{n-1}, \ldots, g_0)$ for each tuple $(g_{n-1}, \ldots, g_0)$ of composable maps of length n and we want to construct $F(f_n, \ldots, f_0)$ for

$(f_n, \dots, f_0)$ a tuple of composable maps of length $n+1$. If one of the f_i is an identity map, then the boundary conditions in 5.1 already uniquely determine $F(f_n, \dots, f_0)$ from one of the already defined $F(g_{n-1}, \dots, g_0)$. If none of the f_i is an identity map, then the already defined $F(g_{n-1}, \dots, g_0)$ define $F(f_n, \dots, f_0)$ only on the boundary $\partial((\Delta^1)^n)$.

For $t \in \partial((\Delta^1)^n)$ denote

$$F(f_n, \dots, f_0)(t) = (c_t, \varphi_t)$$

with $c_t \in \mathbf{Ch}^{\mathcal{P}}(M, N)$ and $\varphi_t \in \mathrm{map}([0,1], \mathrm{map}(M, N))$. Let

$$\begin{aligned} h_N : [0,1] &\to \mathrm{map}([0,1], \mathrm{map}(N,N))) \\ s &\mapsto r \mapsto \begin{cases} \mathrm{id}_N & \text{for } r \le 1 - 3s \\ \alpha_N(r - (1-3s)) & \text{for } r \ge 1 - 3s \text{ and } s \le \frac{1}{3} \\ \alpha_N(r) & \text{for } s \ge \frac{1}{3} \end{cases} \end{aligned}$$

and use it to extend φ_t to a map

$$\varphi : (\Delta^1)^n = \partial((\Delta^1)^n) \times [0,1] / (\partial((\Delta^1)^n) \times 1) \to \mathrm{map}([0,1], \mathrm{map}(M, N))$$

given by

$$\varphi(t,s)(r) = \begin{cases} h_N(s)(r) \circ \varphi_t(r) & \text{for } s \le \frac{2}{3} \\ h_N(s)(r) \circ \varphi_t(r - 3s + 2) & \text{for } \frac{2}{3} \le s \text{ and } r - 3s + 2 \ge 0 \\ h_N(s)(r) \circ \varphi_t(0) & \text{for } \frac{2}{3} \le s \text{ and } r - 3s + 2 \le 0 \end{cases}$$

which is well-defined, because $\varphi_t(0) = f_n \circ \dots \circ f_0$ and thus it does not depend on t. Note that $\varphi(t,s)$ also satisfies $\varphi(t,s)(0) = f_n \circ \dots \circ f_0$ for all t and s. Now we want to extend c_t to a map

$$c : (\Delta^1)^n = \partial((\Delta^1)^n) \times [0,1] / (\partial((\Delta^1)^n) \times 1) \to \mathbf{Ch}^{\mathcal{P}}(M, N)$$

compatible to φ. For $0 \le s \le \frac{1}{3}$ we extended $\varphi(t,s)(1)$ via postcomposition with $\alpha_N(3s)$, thus we can use postcomposition with $c_N(3s)$ to extend $c(t,s)$ for $s \le \frac{1}{3}$ as

$$c(t,s) = c_N(3s) \circ c_t \in \mathbf{Ch}^{\mathcal{P}}(M, N)$$

using the fact that $c_N(0)$ is the identity choice.

For $\frac{2}{3} \le s \le 1$ we have $\varphi(t,s)(1)(M) \cap \partial(N) = \varnothing$ (since we postcompose with $\alpha_N(1)$). Now for

$$U := \partial((\Delta^1)^n) \times [\frac{2}{3}, 1] / (\partial((\Delta^1)^n) \times 1 \subseteq (\Delta^1)^n$$

we can define

$$\bar{\varphi} : M \times U \to N \times U, \qquad \bar{\varphi}(m,t,s) = \varphi(t,s)(1)(m)$$

and choose an arbitrary element

$$\bar{c} \in \mathbf{Ch}^{\mathcal{P}}_{\bar{\varphi}}(M \times U, N \times U)$$

with $M \times U$ and $N \times U$ viewed as families of manifolds over U. The existence of $\bar{c}$ follows from corollary 4.33. Because we have viewed $M \times U$ and $N \times U$ as families of manifolds over U, $\bar{c}$ defines a corresponding map

$$c : U \to \mathbf{Ch}^{\mathcal{P}}(M, N)$$

which satisfies $c(t,s) \in \mathbf{Ch}^{\mathcal{P}}_{\varphi(t,s)(1)}(M, N)$ by construction. What remains is to extend $c(t,s)$ for $\frac{1}{3} \leq s \leq \frac{2}{3}$. But since $\varphi(t,s) = \varphi(t,s')$ for all $s, s' \in [\frac{1}{3}, \frac{2}{3}]$, this is the same as choosing a homotopy between the corresponding elements in

$$c(-,\frac{1}{3}),\ c(-,\frac{2}{3}) \in \mathbf{Ch}^{\mathcal{P}}_{\bar{\varphi}(-,-,\frac{2}{3})}(M \times \partial(\Delta^1)^n, N \times \partial(\Delta^1)^n)$$

with everything viewed as families of manifolds over $\partial(\Delta^1)^n$, which we get from the already defined $c(t,s)$ for $s \leq \frac{1}{3}$ and $s \geq \frac{2}{3}$. By corollary 4.33, $\mathbf{Ch}^{\mathcal{P}}_{\bar{\varphi}(-,-,\frac{2}{3})}(M \times \partial(\Delta^1)^n, N \times \partial(\Delta^1)^n)$ is contractible and thus we can extend c, which finishes the induction step. □

Remark 5.4. The coherent diagram constructed in theorem 5.3 depends on a lot of choices. One can now extend the proof to also show that for different choices during the construction the coherent diagram only changes up to natural transformation of coherent diagrams. Together with theorem 5.2 this implies some kind of uniqueness property for the strict functor that we will construct. But this uniqueness property only holds as long as one uses our choice spaces for the construction of the coherent diagram. There may be other non-equivalent ways to construct a strict pseudoisotopy functor.

PL Remark 5.5. Let $f : M \to N$ be a piecewise linear map of compact $\mathcal{PL}$ manifolds and let $c \in \mathbf{Ch}^{\mathcal{P}}_{f}(M, N)$ be a stable choice that lies in the subspace of piecewise linear stable choices. Then we get a commutative diagram

$$\begin{tikzcd}
|\mathcal{P}^{\mathcal{PL}}_{\partial}(M)| \arrow[r, "r(c)"] \arrow[d] & |\mathcal{P}^{\mathcal{PL}}_{\partial}(N)| \arrow[d] \\
|\mathcal{P}_{\partial}(M)| \arrow[r, "r(c)"] & |\mathcal{P}_{\partial}(N)|
\end{tikzcd}$$

where $\mathcal{P}^{\mathcal{PL}}_{\partial}(M)$ denotes the stable piecewise linear pseudoisotopy space of M and the columns are the obvious inclusion maps.

Now the inclusion maps $|\mathcal{P}^{\mathcal{PL}}_{\partial}(M)| \to |\mathcal{P}_{\partial}(M)|$ are homotopy equivalences, since starting at dimension 5 they are already homotopy equivalences on unstable pseudoisotopy spaces, see [BL74, Theorem 6.2]. Thus any strict topological pseudoisotopy functor automatically satisfies all properties required for a strict piecewise linear pseudoisotopy functor. For this reason we will only construct the topological version of a strict pseudoisotopy functor and define the strict $\mathcal{PL}$ pseudoisotopy functor to be the same as the topological one.

Using the results of [CP86], note that one can refine the proof of 5.3 to also construct a coherent diagram $\mathcal{P}_\partial^{\mathcal{PL},coh} \in \mathrm{Coh}(\mathrm{Mfd}_c^{\mathcal{PL}}, \mathrm{cg}\mathfrak{Top})$ on the subcategory $\mathrm{Mfd}_c^{\mathcal{PL}} \subseteq \mathrm{Mfd}_c$ of compact $\mathcal{PL}$ submanifolds of $\mathbb{R}^\infty$ with piecewise linear maps as morphisms together with a natural transformation of coherent diagrams to the restriction of $\mathcal{P}_\partial^{coh}$ to the subcategory $\mathrm{Mfd}_c^{\mathcal{PL}}$. But since we do not need to do this, we will leave the details to the interested reader.

Theorem 5.6. *Let $\mathfrak{Top}$ denote the category of topological spaces. There exists a functor*

$$\mathcal{P}_\partial^{strict} : \mathfrak{Top} \to \mathfrak{Top}$$

together with maps $\tau_M : |\mathcal{P}\partial(M)| \to \mathcal{P}_\partial^{strict}(M)$ for each $M \in \mathrm{Mfd}_c$ such that all τ_M are homotopy equivalences and for each map $f \in \mathrm{map}_{\mathrm{Mfd}_c}(M, N)$ with corresponding choice $c \in \mathbf{Ch}_f^{\mathcal{P}}(M, N)$ the diagram

$$\begin{tikzcd}
|\mathcal{P}\partial(M)| \arrow[r, "r(c)"] \arrow[d, "\tau_M"] & |\mathcal{P}\partial(N)| \arrow[d, "\tau_N"] \\
\mathcal{P}_\partial^{strict}(M) \arrow[r, "\mathcal{P}_\partial^{strict}(f)"] & \mathcal{P}_\partial^{strict}(N)
\end{tikzcd}$$

commutes up to homotopy. Furthermore, $\mathcal{P}_\partial^{strict}$ is again functorial up to homotopy in the sense that for $f \simeq g : X \to Y$ the maps $\mathcal{P}_\partial^{strict}(f)$ and $\mathcal{P}_\partial^{strict}(g)$ are also homotopic.

Proof. By strictifying (i.e. applying theorem 5.2) the coherent diagram constructed in theorem 5.3 and postcomposing the resulting functor with a cofibrant replacement functor, we get a functor $F : \mathrm{Mfd}_c \to \mathfrak{Top}$ that satisfies the required properties on objects and morphisms by theorem 5.2 and is functorial up to homotopy by corollary 4.34. To extend it to a functor on the category $\mathfrak{Top}$, we use a homotopy left Kan extension of F along the inclusion functor $\mathrm{Mfd}_c \to \mathfrak{Top}$.

$$\begin{tikzcd}
\mathrm{Mfd}_c \arrow[r, "F"] \arrow[d] & \mathfrak{Top} \\
\mathfrak{Top} \arrow[ur, dotted, "\mathcal{P}_\partial^{strict}"'] &
\end{tikzcd}$$

So on an object $X \in \mathfrak{Top}$ the functor $\mathcal{P}_\partial^{strict}$ is given by

$$\mathcal{P}_\partial^{strict}(X) = \operatorname*{hocolim}_{(M,f)\in \mathrm{Mfd}_c\downarrow X} F(M)$$

where the objects of the comma category $\mathrm{Mfd}_c \downarrow X$ are pairs (M, f) with $M \in \mathrm{Mfd}_c$ and $f \in \mathrm{map}_{\mathfrak{Top}}(M, X)$ and a morphism from (M_1, f_1) to (M_2, f_2) in $\mathrm{Mfd}_c \downarrow X$ is a map $\varphi : M_1 \to M_2$ such that $f_1 = f_2 \circ \varphi : M_1 \to X$. Here we require our homotopy colimit functor to take values in cofibrant spaces to ensure that all weak equivalences are homotopy equivalences.

If X is already an object in Mfd_c, then $\mathrm{Mfd}_c \downarrow X$ has (X, id_X) as a terminal object since the functor $\mathrm{Mfd}_c \to \mathfrak{Top}$ is fully faithful. Thus the the map

$$F(X) \to \operatorname*{hocolim}_{\mathrm{Mfd}_c \downarrow X} F(M)$$

induced by the inclusion of the object (X, id_X) into the category $\mathrm{Mfd}_c \downarrow X$ is a homotopy equivalence. Thus $\mathcal{P}_\partial^{strict}$ satisfies the desired properties on objects and morphisms coming from Mfd_c, because F satisfies them.

To show that $\mathcal{P}_\partial^{strict}$ is functorial up to homotopy, we first show that

$$\mathcal{P}_\partial^{strict}(\mathrm{pr}_X) : \mathcal{P}_\partial^{strict}(X \times [0,1]) \to \mathcal{P}_\partial^{strict}(X)$$

is a homotopy equivalence for all $X \in \mathfrak{Top}$. This is an application of [Dug08, Theorem 6.9] for (using the notation of Dugger)

$$\begin{aligned} I &= \mathrm{Mfd}_c \downarrow X \times [0,1] \\ J &= \mathrm{Mfd}_c \downarrow X \\ \alpha &= (\mathrm{pr}_X)_* : I \to J \end{aligned}$$

with $pr_X : X \times [0,1] \to X$ the standard projection. The functor $J \to \mathfrak{Top}$ that Dugger denotes by X is given by $F \circ \psi$ with $\psi : J \to \mathrm{Mfd}_c$ the forgetful functor sending (M, f) to M. Now for each $(N, g) = j \in J$ the comma category $(\alpha \downarrow j)$ has

$$\big((N \times [0,1], g \times \mathrm{id}_{[0,1]}), \mathrm{pr}_N : N \times [0,1] \to N\big)$$

as a terminal object. Thus the composition

$$\operatorname*{hocolim}_{((M,f),\varphi) \in (\alpha \downarrow j)} F(M) \to \operatorname*{colim}_{((M,f),\varphi) \in (\alpha \downarrow j)} F(M) \to F(N)$$

is a weak equivalence if $F(\mathrm{pr}_N) : F(N \times [0,1]) \to F(N)$ is, which is true due to corollary 4.34. So by 2-out-of-3 for homotopy equivalences the two maps

$$incl_i : X = X \times i \hookrightarrow X \times [0,1]$$

for $i \in \{0,1\}$ induce homotopy equivalences $\mathcal{P}_\partial^{strict}(incl_i)$ that are homotopic to each other since composing them with $\mathcal{P}_\partial^{strict}(\mathrm{pr}_X)$ yields $\mathcal{P}_\partial^{strict}(\mathrm{id}_X)$. Because homotopic maps $f \simeq g : X \to Y$

factorize over $X \times [0,1]$, this implies that $\mathcal{P}_{\partial}^{strict}$ is functorial up to homotopy. $\square$

CHAPTER 6

Pseudoisotopy Spectra

There exists also a spectrum version for pseudoisotopy spaces, whose negative homotopy groups are connected to algebraic K-theory (see remark 6.12). Our results so far carry over: In the following we will construct the pseudoisotopy spectrum for a manifold M, choice spaces for induced maps between pseudoisotopy spectra and finally a pseudoisotopy spectrum functor.

Definition 6.1. Let M be a family of manifolds over X and $n \in \mathbb{N}$. Define the *space of bounded pseudoisotopies* over $M \times \mathbb{R}^n$ (considered as a family of manifolds over X) as the simplicial subset $P^b(M \times \mathbb{R}^n) \subseteq P(M \times \mathbb{R}^n)$ of simplices

$$\varphi : \Delta^k \times M \times \mathbb{R}^n \times I \to \Delta^k \times M \times \mathbb{R}^n \times I$$

that are bounded in the $\mathbb{R}^n$-direction, i.e. the map

$$\|\mathrm{pr}_{\mathbb{R}^n} \circ \varphi - \mathrm{pr}_{\mathbb{R}^n}\| : \Delta^k \times M \times \mathbb{R}^n \times I \to \mathbb{R}$$

is bounded. The space of bounded pseudoisotopies relative boundary is then defined as $P^b_\partial(M \times \mathbb{R}^n) := P^b(M \times \mathbb{R}^n) \cap P_\partial(M \times \mathbb{R}^n)$.

Analogously, define the space of stable, bounded pseudoisotopies relative boundary as

$$\mathcal{P}^b_\partial(M \times \mathbb{R}^n) = \operatorname*{hocolim}_{k \geq 0} P^b_\partial(M \times \mathbb{R}^n \times J^k) \subseteq \mathcal{P}_\partial(M \times \mathbb{R}^n)$$

using the same explicit mapping telescope model for the homotopy colimit as in definition 4.2.

PL Remark 6.2. There is, of course, also a piecewise linear version of the stable, bounded pseudoisotopy space $\mathcal{P}^{b,\mathcal{PL}}_\partial(M \times \mathbb{R}^n) \subseteq \mathcal{P}^b_\partial(M \times \mathbb{R}^n)$ defined as the subspace of piecewise linear pseudoisotopies. In the following definitions and constructions it will again suffice to require everything to be piecewise linear to get the correct piecewise linear version.

Remark 6.3. If the fibers of a family of manifolds $M \to X$ are noncompact manifolds, then bounded pseudoisotopies over $M \times \mathbb{R}^n$ are not necessarily bounded in the M-direction.

Before we can construct the levels of a pseudoisotopy spectrum, we need to choose base points and we need induced maps between the bounded pseudoisotopy spaces. Let us start with base points. The unstable simplicial sets $P^b_\partial(M \times \mathbb{R}^n)$ have a canonical base point given by the 0-simplex representing the identity pseudoisotopy on $M \times \mathbb{R}^n \times I$. In the stable pseudoisotopy space, this base point becomes a 1-dimensional subcomplex, as each $P^b_\partial(M \times \mathbb{R}^n \times J^k) \times \underline{\Delta}^1$ contains a 1-simplex representing the identity on $M \times \mathbb{R}^n \times J^k \times I$. To make the stable pseudoisotopy space well-pointed, we will collapse the subcomplex spanned by these simplices.

Definition 6.4. Let M be a family of manifolds over X and $n \in \mathbb{N}$. Denote by $A \subseteq \mathcal{P}^b_\partial(M \times \mathbb{R}^n)$ the simplicial subcomplex generated by simplices of the form

$$(\mathrm{id}_{\Delta^i \times M \times \mathbb{R}^n \times J^k \times I}, \alpha) \in P^b_\partial(M \times \mathbb{R}^n \times J^k) \times \underline{\Delta}^1$$

for arbitrary $\alpha \in \underline{\Delta}^1$ and $i, k \in \mathbb{N}$. Define the *reduced*, bounded, stable pseudoisotopy space of $M \times \mathbb{R}^n$ relative boundary as a based simplicial set given by the quotient

$$\mathcal{P}^b_\partial(M \times \mathbb{R}^n)_{red} \coloneqq \mathcal{P}^b_\partial(M \times \mathbb{R}^n) \big/ A$$

with base point A. The geometric realization of this base point is the base point of the geometric realization $|\mathcal{P}^b_\partial(M \times \mathbb{R}^n)_{red}|$.

Remark 6.5. Note that all induced maps between pseudoisotopy spaces that we constructed up to now always send the identity pseudoisotopy to the identity pseudoisotopy, because bending maps are conjugation with a homeomorphism, i.e. they send the identity to the identity, and geometric transfer maps as in definition 3.8 were constructed in a way such that they send the identity to the identity. Thus the subcomplex A in definition 6.4 is a contractible subcomplex of $\mathcal{P}_\partial(M)$ and all realization maps in chapter 4 still commute with the composition maps when we replace the targets of the realization maps with the reduced versions of the pseudoisotopy spaces. By abuse of notation, we still use r to denote the various realization maps for the reduced pseudoisotopy spaces.

Also note that realization is not always well defined for *bounded* pseudoisotopy spaces, because the realization of a choice c could send a bounded pseudoisotopy to an unbounded one.

For well-defined realization maps from our choice spaces to the mapping spaces of the bounded pseudoisotopy spaces we need the following definition:

Definition 6.6. Let M, N be families of manifolds over X and let K be a manifold. Consider K as a family of manifolds over a point, i.e. $M \times K$ and $N \times K$ are families of manifolds over X. Define a map

$$- \times \mathrm{id}_K : \mathrm{Ch}^P(M, N) \to \mathrm{Ch}^P(M \times K, N \times K)$$

by taking the cross product component-wise, i.e. for $c \in \mathrm{Ch}^P(M,N)$ we cross each disk bundle, each bending map and the zero-section with id_K and we cross each parallel transport in c with the trivial parallel transport on K as in definitions 3.3 and 3.4.

Furthermore, this unstable cross product with id_K induces maps

$$- \times \mathrm{id}_K : \mathrm{Ch}^{P,\mathcal{P}}(M,N) \to \mathrm{Ch}^{P,\mathcal{P}}(M \times K, N \times K)$$

by taking the union over the unstable products with id_K and

$$- \times \mathrm{id}_K : \mathbf{Ch}^{\mathcal{P}}(M,N) \to \mathbf{Ch}^{\mathcal{P}}(M \times K, N \times K)$$

by composition with $|- \times \mathrm{id}_K|$ in each factor.

Remark 6.7. For the special case $K = \mathbb{R}^n$ we now get induced maps in bounded pseudoisotopy spaces: Let $c \in \mathbf{Ch}^{\mathcal{P}}(M,N)$. Then all bundles and all bending maps in $c \times \mathrm{id}_{\mathbb{R}^n} \in \mathbf{Ch}^{\mathcal{P}}(M \times \mathbb{R}^n, N \times \mathbb{R}^n)$ are obviously bounded with respect to the $\mathbb{R}^n$-direction. Thus $r(c \times \mathrm{id}_{\mathbb{R}^n})$ sends bounded pseudoisotopies to bounded pseudoisotopies and we get an induced map

$$r(c \times \mathrm{id}_{\mathbb{R}^n}) : |\mathcal{P}^b_\partial(M \times \mathbb{R}^n)_{red}| \to |\mathcal{P}^b_\partial(N \times \mathbb{R}^n)_{red}|.$$

Thus we get well-defined realization maps

$$r((-) \times \mathrm{id}_{\mathbb{R}^n}) : \mathbf{Ch}^{\mathcal{P}}(M,N) \to \mathrm{map}(|\mathcal{P}^b_\partial(M \times \mathbb{R}^n)_{red}|, |\mathcal{P}^b_\partial(N \times \mathbb{R}^n)_{red}|)$$

for all $n \in \mathbb{N}$.

For each $t \in \mathbb{R}$ we get a map $\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red} \to \mathcal{P}^b_\partial(M \times \mathbb{R}^n \times \mathbb{R})_{red}$ induced by the codimension-zero-embeddings

$$\begin{aligned} M \times \mathbb{R}^n \times J \times J^k &\to M \times \mathbb{R}^n \times \mathbb{R} \times J^k \\ (m,p,q,r) &\mapsto (m,p,q+t,r). \end{aligned}$$

Since this is continuous in $t \in \mathbb{R}$, it yields a map

$$\mathbb{R} \times |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \to |\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1})_{red}|$$

If we now denote by $S^1 = \mathbb{R} \cup \{\infty\}$ the one point compactification of the real line (with $\{\infty\}$ as base point), we can extend this map to a map

$$\theta : S^1 \wedge |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \to |\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1})_{red}|$$

of pointed topological spaces by sending $\{\infty\} \times |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \cup S^1 \times \{\mathrm{id}\}$ to the base point.

Remark 6.8. The map $|\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \to \Omega|\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1})_{red}|$ adjoint to θ is a weak homotopy equivalence. That the corresponding maps between the unstable pseudoisotopy spaces are weak homotopy equivalences is proven in [WW88, Proposition 1.10], the stable map is then a mapping telescope over weak homotopy equivalences and thus also a weak homotopy equivalence.

The idea is now to compose θ with a map from $|\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1})_{red}|$ to $|\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1} \times J)_{red}|$ and use this composition as structure maps of a spectrum. We will construct this map out of the stabilization maps:

Definition 6.9. Let M be a family of manifolds over X. For $k \geq 0$ let $s \in \mathrm{Ch}^P(M \times J^k, M \times J^{k+1})$ be the choice for the stabilization map, see example 4.7. Then $\{s\} \times \underline{\Delta}^1$ is a one-dimensional simplicial subcomplex of $\mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, M \times J)$ with only one non-degenerate 1-simplex. The geometric realization of the embedding

$$\underline{\Delta}^1 = \{s\} \times \underline{\Delta}^1 \hookrightarrow \mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, M \times J)$$

is a path $\varphi_k : \Delta^1 \to |\,\mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, M \times J)|$ starting at $\varphi_k(0) = (|s|, 0)$ and ending at

$$\varphi_k(1) = (|s| \circ |s|, 0) \in |\,\mathrm{Ch}^P(M \times J^k, M \times J^{k+2})| \times \Delta^1 \subseteq |\,\mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, M \times J)|.$$

So $(\varphi_0, \varphi_1, \dots) \in \mathbf{Ch}^{\mathcal{P}}(M, M \times J)$ is a stable choice, which we will call the *stable stabilization choice* and again denote by $s := (\varphi_0, \varphi_1, \dots)$.

Remark 6.10. Since the stable stabilization choice s is contained in $\mathbf{Ch}^{\mathcal{P}}_{\mathrm{id}_M}(M, M \times J)$, any choice in $\mathbf{Ch}^{\mathcal{P}}_{\mathrm{id}_M}(M \times J, M)$ yields a homotopy inverse to $r(s)$ after realization, see corollary 4.34.

So we want to have the compositions $r(s) \circ \theta$ as the structure maps of our spectrum. Unfortunately, they commute only up to homotopy with the induced maps between the pseudoisotopy spaces of different manifolds. To remedy this, we simply add enough space for these homotopies in the construction of the pseudoisotopy spectrum.

Definition 6.11. Let M be a family of manifolds over X. Define the *pseudoisotopy spectrum* $\mathbb{P}(M)$ of M as follows: The zeroth space of $\mathbb{P}(M)$ is given by $\mathbb{P}(M)_0 := |\mathcal{P}_\partial(M \times J)_{red}|$. Inductively, the $(n+1)$-th space of $\mathbb{P}(M)$ is the homotopy pushout of the diagram of pointed spaces

$$\begin{tikzcd}
S^1 \wedge |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \arrow[r] \arrow[d, "r(s \times \mathrm{id}_{\mathbb{R}^{n+1}}) \circ \theta"] & S^1 \wedge \mathbb{P}(M)_n \\
|\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1} \times J)_{red}| &
\end{tikzcd}$$

where the upper row is the (inductively defined) inclusion map. We again use an explicit model for the homotopy pushout given by

$$\mathbb{P}(M)_{n+1} := (S^1 \wedge \mathbb{P}(M)_n \wedge \{0\}_+) \cup (S^1 \wedge |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \wedge [0,1]_+) \cup |\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1} \times J)_{red}| \wedge \{1\}_+$$

with the $(-)_+$-notation indicating the addition of a disjoint base point.

The structure maps of $\mathbb{P}(M)$ are given by the inclusions $S^1 \wedge \mathbb{P}(M)_n \to \mathbb{P}(M)_{n+1}$ for $n \in \mathbb{N}$.

Remark 6.12. By induction, $\mathbb{P}(M)_n$ is homotopy equivalent to $|\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}|$ for all $n \in \mathbb{N}$. Using remark 6.8 we thus get that the maps $\mathbb{P}(M)_n \to \Omega\mathbb{P}(M)_{n+1}$ adjoint to the structure maps are again homotopy equivalences, so $\mathbb{P}(M)$ is an Ω-spectrum. The negative homotopy groups of $\mathbb{P}(M)$ for M a compact, connected manifold have been computed by Anderson and Hsiang in [AH77, Theorem 3] as

$$\pi_i(\mathbb{P}(M)) = \begin{cases} \mathrm{Wh}(\pi_1(M)) & \text{if } i = -1 \\ \tilde{K}_0(\mathbb{Z}\pi_1(M)) & \text{if } i = -2 \\ K_{i+2}(\mathbb{Z}\pi_1(M)) & \text{if } i \leq -3 \end{cases}$$

where $\mathrm{Wh}(\pi_1(M))$ denotes the Whitehead group, $\tilde{K}_0(\mathbb{Z}\pi_1(M))$ denotes the reduced algebraic K-theory group and $K_{i+2}(\mathbb{Z}\pi_1(M))$ denote the unreduced negative algebraic K-theory groups of $\mathbb{Z}\pi_1(M)$. Here we use that the proof in [AH77] also works for manifolds with boundary, although it is only stated there for closed manifolds.

PL Remark 6.13. Let M be a compact, connected, piecewise linear manifold. The map that Anderson and Hsiang use to compute the lower homotopy groups of $|\mathcal{P}^b_\partial(M \times \mathbb{R}^n)|$ commutes with the inclusion map of the piecewise linear pseudoisotopy space to the topological one. So we get that the inclusion

$$|\mathcal{P}^{b,\mathcal{PL}}_\partial(M \times \mathbb{R}^n)| \to |\mathcal{P}^b_\partial(M \times \mathbb{R}^n)|$$

induces an isomorphism on π_i for $i < n$ by [AH77] and for $i \geq n$ by the ω-spectrum structure and [BL74, Theorem 6.2]. Thus for the same argument as in remark 5.5 it suffices to construct the topological version of a pseudoisotopy spectrum functor, because the same functor will also be a $\mathcal{PL}$ pseudoisotopy spectrum functor. We will again only construct the topological version and define the $\mathcal{PL}$ pseudoisotopy spectrum functor to be the same as the topological one.

Definition 6.14. Let M, N be families of manifolds over X. Define the *space of spectrum-level choices* $\mathbf{Ch}^{\mathbb{P}}(M, N)$ of maps between $\mathbb{P}(M)$ and $\mathbb{P}(N)$ as the subspace

$$\mathbf{Ch}^{\mathbb{P}}(M, N) \subseteq \mathbf{Ch}^{\mathcal{P}}(M, N) \times \mathrm{map}\big([0,1], \mathbf{Ch}^{\mathcal{P}}(M \times J, N \times J)\big)$$

of those $(c, f) \in \mathbf{Ch}^{\mathcal{P}}(M, N) \times \mathrm{map}\big([0,1], \mathbf{Ch}^{\mathcal{P}}(M \times J, N \times J)\big)$ such that

1. $f(0) \circ s_M = s_N \circ c$ with $s_M \in \mathbf{Ch}^{\mathcal{P}}(M, M \times J)$, $s_N \in \mathbf{Ch}^{\mathcal{P}}(N, N \times J)$ the stable stabilization choices defined in 6.9,

2. $f(1) = c \times \mathrm{id}_J$, where crossing with id_J is defined as in definition 6.6.

Composition of choices $(c, f) \in \mathbf{Ch}^{\mathbb{P}}(M, N)$, $(c', f') \in \mathbf{Ch}^{\mathbb{P}}(N, K)$ is defined via the composition of stable choices as

$$(c', f') \circ (c, f) := (c' \circ c, f'(-) \circ f(-)) \in \mathbf{Ch}^{\mathbb{P}}(M, K).$$

In particular, associativity of the composition follows directly from proposition 4.27.

Definition 6.15. Let M, N be families of manifolds over X. The realization map

$$r : \mathbf{Ch}^{\mathbb{P}}(M, N) \to \mathrm{map}(\mathbb{P}(M), \mathbb{P}(N))$$

is defined inductively. On the zeroth spectrum level it is given by sending $(c, f) \in \mathbf{Ch}^{\mathbb{P}}(M, N)$ to

$$r_0(c, f) := r(f(1)) \in \mathrm{map}(|\mathcal{P}_\partial(M \times J)_{red}|, |\mathcal{P}_\partial(N \times J)_{red}|) = \mathrm{map}(\mathbb{P}(M)_0, \mathbb{P}(N)_0).$$

The map $r_{n+1}(c, f)$ is then defined by gluing together the maps

$$\begin{aligned} \mathrm{id}_{S^1} \wedge r_n(c, f) :&\ S^1 \wedge \mathbb{P}(M)_n \to S^1 \wedge \mathbb{P}(N)_n \\ r(f(0) \times \mathrm{id}_{\mathbb{R}^{n+1}}) :&\ |\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1} \times J)_{red}| \to |\mathcal{P}^b_\partial(N \times \mathbb{R}^{n+1} \times J)_{red}| \end{aligned}$$

along the map

$$\begin{aligned} S^1 \wedge |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \wedge [0,1]_+ &\to S^1 \wedge |\mathcal{P}^b_\partial(N \times \mathbb{R}^n \times J)_{red}| \wedge [0,1]_+ \\ [s, x, t] &\mapsto [s, r(f(t) \times \mathrm{id}_{\mathbb{R}^n})(x), t] \end{aligned}$$

in the homotopy pushout.

Proposition 6.16. *The realization map as defined in 6.15 is well-defined and compatible with the composition maps, i.e. we have $r((c', f') \circ (c, f)) = r(c', f') \circ r(c, f)$ for choices $(c, f) \in \mathbf{Ch}^{\mathbb{P}}(M, N)$ and $(c', f') \in \mathbf{Ch}^{\mathbb{P}}(N, K)$.*

Proof. To prove that the realization map is well defined, we need to check that for $(c, f) \in \mathbf{Ch}^{\mathbb{P}}(M, N)$ we can in fact glue the three maps in definition 6.15 together. That we can glue along the map $\mathrm{id}_{S^1} \wedge r_n(c, f) : S^1 \wedge \mathbb{P}(M)_n \to S^1 \wedge \mathbb{P}(N)_n$ follows by induction on n. For the gluing along the map $r(f(0) \times \mathrm{id}_{\mathbb{R}^{n+1}})$ to be well-defined, we have to check that the outer square in the following diagram commutes:

$$\begin{tikzcd}
S^1 \wedge |\mathcal{P}^b_\partial(M \times \mathbb{R}^n \times J)_{red}| \arrow[r, "\mathrm{id}_{S^1} \wedge r(f(1) \times \mathrm{id}_{\mathbb{R}^n})"] \arrow[d, "\theta"] & S^1 \wedge |\mathcal{P}^b_\partial(N \times \mathbb{R}^n \times J)_{red}| \arrow[d, "\theta"] \\
|\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1})_{red}| \arrow[r, "r(c \times \mathrm{id}_{\mathbb{R}^{n+1}})"] \arrow[d, "r(s \times \mathrm{id}_{\mathbb{R}^{n+1}})"] & |\mathcal{P}^b_\partial(N \times \mathbb{R}^{n+1})_{red}| \arrow[d, "r(s \times \mathrm{id}_{\mathbb{R}^{n+1}})"] \\
|\mathcal{P}^b_\partial(M \times \mathbb{R}^{n+1} \times J)_{red}| \arrow[r, "r(f(0) \times \mathrm{id}_{\mathbb{R}^{n+1}})"] & |\mathcal{P}^b_\partial(N \times \mathbb{R}^{n+1} \times J)_{red}|
\end{tikzcd}$$

First note that crossing choices with $\mathrm{id}_{\mathbb{R}^n}$ is compatible with composition: For $c_1 \in \mathbf{Ch}^{\mathcal{P}}(M,N)$ and $c_2 \in \mathbf{Ch}^{\mathcal{P}}(N,K)$ we have

$$(c_2 \circ c_1) \times \mathrm{id}_{\mathbb{R}^n} = (c_2 \times \mathrm{id}_{\mathbb{R}^n}) \circ (c_1 \times \mathrm{id}_{\mathbb{R}^n}) \in \mathbf{Ch}^{\mathcal{P}}(M \times \mathbb{R}^n, K \times \mathbb{R}^n),$$

since on both sides we crossed each transfer map and bending map of $c_2 \circ c_1$ with $\mathrm{id}_{\mathbb{R}^n}$. So we have for the lower square

$$(f(0) \times \mathrm{id}_{\mathbb{R}^{n+1}}) \circ (s \times \mathrm{id}_{\mathbb{R}^{n+1}}) = (f(0) \circ s) \times \mathrm{id}_{\mathbb{R}^{n+1}} = (s \circ c) \times \mathrm{id}_{\mathbb{R}^{n+1}} = (s \times \mathrm{id}_{\mathbb{R}^{n+1}}) \circ (c \times \mathrm{id}_{\mathbb{R}^{n+1}})$$

and thus the lower square commutes already before realization.

For the upper square we have $f(1) \times \mathrm{id}_{\mathbb{R}^n} = c \times \mathrm{id}_{\mathbb{R}^n} \times \mathrm{id}_J$. Now we use that by definition 6.6 the geometric transfer maps and the bending maps of $c \times \mathrm{id}_{\mathbb{R}^n} \times \mathrm{id}_J$ equal the restriction of the transfer and bending maps of $c \times \mathrm{id}_{\mathbb{R}^{n+1}}$ from $M \times \mathbb{R}^{n+1}$ to $M \times \mathbb{R}^n \times J$. In other words, for a pseudoisotopy relative boundary on $M \times \mathbb{R}^n \times J \times I$ it does not matter whether we first embed $M \times \mathbb{R}^n \times J$ into $M \times \mathbb{R}^{n+1}$ and then apply $r(c \times \mathrm{id}_{\mathbb{R}^{n+1}})$ or first apply $r(c \times \mathrm{id}_{\mathbb{R}^n} \times \mathrm{id}_J)$ and then embed $N \times \mathbb{R}^n \times J$ into $N \times \mathbb{R}^{n+1}$, since outside of the interval $J \subseteq \mathbb{R}$ the pseudoisotopy will be the identity in both cases. Because the S^1-coordinate only controls where we embed J into $\mathbb{R}$, the commutativity of the upper square follows.

The compatibility of the realization map with composition follows from

$$r(f'(t) \times \mathrm{id}_{\mathbb{R}^n}) \circ r(f(t) \times \mathrm{id}_{\mathbb{R}^n}) = r(f'(t) \times \mathrm{id}_{\mathbb{R}^n} \circ f(t) \times \mathrm{id}_{\mathbb{R}^n}) = r((f'(t) \circ f(t)) \times \mathrm{id}_{\mathbb{R}^n})$$

for $t \in [0,1]$, using proposition 4.27, remark 6.5 and the fact that crossing choices with $\mathrm{id}_{\mathbb{R}^n}$ is compatible with composition. □

Remark 6.17. For a family of manifolds M over X we have an identity choice $(c_{\mathrm{id}}, f_{\mathrm{id}}) \in \mathbf{Ch}^{\mathbb{P}}(M,M)$ with c_{id} the identity choice in $\mathbf{Ch}^{\mathcal{P}}(M,M)$ and $f_{\mathrm{id}} : [0,1] \to \mathbf{Ch}^{\mathcal{P}}(M \times J, M \times J)$ the constant map with value the identity choice in $\mathbf{Ch}^{\mathcal{P}}(M \times J, M \times J)$. As the name implies, the realization $r((c_{\mathrm{id}}, f_{\mathrm{id}})) : \mathbb{P}(M) \to \mathbb{P}(M)$ is the identity map on $\mathbb{P}(M)$.

Definition 6.18. Let $f : M \to N$ be a map between families of manifolds over X. Define the space $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ of spectrum-level choices *over* f of maps between $\mathbb{P}(M)$ and $\mathbb{P}(N)$ as

$$\mathbf{Ch}^{\mathbb{P}}_f(M,N) \coloneqq \mathbf{Ch}^{\mathbb{P}}(M,N) \cap \mathbf{Ch}^{\mathcal{P}}_f(M,N) \times \mathrm{map}([0,1], \mathbf{Ch}^{\mathcal{P}}_f(M \times J, N \times J))$$

Lemma 6.19. *Let $f : M \to N$ and $g : N \to K$ be maps between families of manifolds over X. Then the composition*

$$\mathbf{Ch}^{\mathbb{P}}_g(N,K) \times \mathbf{Ch}^{\mathbb{P}}_f(M,N) \to \mathbf{Ch}^{\mathbb{P}}(M,K),\ (\psi,\varphi) \mapsto \psi \circ \varphi$$

has image in $\mathbf{Ch}^{\mathbb{P}}_{g\circ f}(M,K)$.

Proof. This is a direct application of lemma 4.32. □

Proposition 6.20. *Let $f : M \to N$ be a map between families of compact manifolds over X with X a finite CW-complex and $f(M) \cap \partial N = \varnothing$. Then $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ is contractible.*

Proof. To show that $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ is not empty, let $g \in \mathbf{Ch}^{\mathcal{P}}_f(M \times J, N)$. Since $\mathbf{Ch}^{\mathcal{P}}_f(M \times J, N \times J)$ is contractible by corollary 4.33, there exists a homotopy

$$h : [0,1] \to \mathbf{Ch}^{\mathcal{P}}_f(M \times J, N \times J)$$

starting at $s \circ g$ and ending at $(g \circ s) \times \mathrm{id}_J$, where s always denotes the stable stabilization choice. Thus the tuple $(g \circ s, h)$ is an element in $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$.

To show that $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ is contractible, we will construct a contraction of the subspace of $\mathbf{Ch}^{\mathcal{P}}_f(M,N) \times \mathbf{Ch}^{\mathcal{P}}_f(M \times J, N \times J)$ of those tuples (c,c') that satisfy $s \circ c = c' \circ s$. Since for $(c,g) \in \mathbf{Ch}^{\mathbb{P}}_f(M,N)$ we always have $g(1) = c \times \mathrm{id}_J$, this defines the contraction on $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ restricted to $\mathbf{Ch}^{\mathcal{P}}_f(M,N) \times \mathrm{map}(\{0,1\}, \mathbf{Ch}^{\mathcal{P}}_f(M \times J, N \times J))$. Because $\mathbf{Ch}^{\mathcal{P}}_f(M \times J, N \times J)$ is contractible by corollary 4.33, we can extend this to a contraction of $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ which finishes the proof.

Let $c_{incl} \in \mathbf{Ch}^{\mathcal{P}}(M \times J, M)$ be the stable choice that is given component-wise by the identity choice defined in 4.28, i.e. $c_{incl} = (\varphi^{\mathrm{id}}_0, \varphi^{\mathrm{id}}_1, \ldots)$. Denote by

$$sh^M : [0,1] \to \mathbf{Ch}^{\mathcal{P}}(M,M)$$

the path from the identity choice $c_{\mathrm{id}} \in \mathbf{Ch}^{\mathcal{P}}(M,M)$ to $c_{incl} \circ s \in \mathbf{Ch}^{\mathcal{P}}(M,M)$ that we get by shifting each component $\varphi^{\mathrm{id}}_k : \Delta^1 \to |\mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, M)|$ by one step in the direction of the mapping telescope. Since sh^M does not change the unstable choices but only moves them around in the mapping telescope direction, it has the following properties:

- $sh^M(t) \in \mathbf{Ch}^{\mathcal{P}}_{\mathrm{id}}(M,M)$ for all $t \in [0,1]$,
- $sh^M(t) \circ s = s \circ sh^M(t) \in \mathbf{Ch}^{\mathcal{P}}(M, M \times J)$ for all $t \in [0,1]$.

Denote by $U \subseteq \mathbf{Ch}_f^{\mathcal{P}}(M,N) \times \mathbf{Ch}_f^{\mathcal{P}}(M \times J, N \times J)$ the subspace of those tuples (c,c') that satisfy $s \circ c = c' \circ s$. Let

$$\begin{aligned} h : U \times [0,1] &\to U, \\ (c,c',t) &\mapsto (sh^N(t) \circ c \circ sh^M(t), sh^N(t) \circ c' \circ sh^M(t)). \end{aligned}$$

Note that h is only well defined, because $sh^M(t)$ and $sh^N(t)$ commute with the stable stabilization map s for all $t \in [0,1]$. Now h is a homotopy from the identity on U and with

$$\begin{aligned} h(c,c',1) =& (c_{incl} \circ s \circ c \circ c_{incl} \circ s, c_{incl} \circ s \circ c' \circ c_{incl} \circ s) \\ =& (c_{incl} \circ s \circ c \circ c_{incl} \circ s, s \circ c_{incl} \circ c' \circ c_{incl} \circ s) \end{aligned}$$

where we used $c_{incl} \circ s = s \circ c_{incl} \in \mathbf{Ch}_{\mathrm{id}_N}^{\mathcal{P}}(N \times J, N \times J)$. That means, we have contracted U to a subspace that is contained in the image of the map

$$\alpha : \mathbf{Ch}_f^{\mathcal{P}}(M \times J, N) \to U, \ \ \alpha(c) = (c \circ s, s \circ c).$$

Now we need the following lemma:

Lemma 6.21. *The maps*

$$\begin{aligned} s \circ (-) :& \mathbf{Ch}_f^{\mathcal{P}}(M \times J, N) \to \mathbf{Ch}_f^{\mathcal{P}}(M \times J, N \times J) \\ (-) \circ s :& \mathbf{Ch}_f^{\mathcal{P}}(M \times J, N) \to \mathbf{Ch}_f^{\mathcal{P}}(M, N) \end{aligned}$$

are homeomorphisms onto their respective images.

Proof. The unstable compositions with the unstable stabilization map

$$\begin{aligned} s \circ (-) : \mathrm{Ch}^P(M \times J^k, N \times J^l) &\to \mathrm{Ch}^P(M \times J^k, N \times J^{l+1}) \\ (-) \circ s : \mathrm{Ch}^P(M \times J^{k+1}, N \times J^{l+1}) &\to \mathrm{Ch}^P(M \times J^k, N \times J^{l+1}) \end{aligned}$$

are by definition of the unstable composition injective simplicial maps for all $k, l \in \mathbb{N}$. Thus we get injective simplicial maps

$$\begin{aligned} s \circ (-) : \mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, N) &\to \mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, N \times J) \\ (-) \circ s : \mathrm{Ch}^{P,\mathcal{P}}(M \times J^{k+1}, N) &\to \mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, N) \end{aligned}$$

where $s \circ (-)$ is defined by sending $(\sigma, \alpha) \in \mathrm{Ch}^{P,\mathcal{P}}(M \times J^k, N)$ to $(s \circ \sigma, \alpha)$. The geometric realizations of the two maps are

$$|s \circ (-)| : |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^k, N)| \to |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^k, N \times J)|$$
$$(-) \circ |s| : |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^{k+1}, N)| \to |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^k, N)|$$

where one can easily check that $|s \circ (-)|$ is the same as composition with the stable stabilization map $s \circ (-)$. Both maps are now homeomorphisms onto their images since they are inclusions of CW-subcomplexes. Thus the maps

$$\begin{gathered} \operatorname{map}(\Delta^1, |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^k, N)|) \to \operatorname{map}(\Delta^1, |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^k, N \times J)|) \\ \varphi \mapsto s \circ \varphi(-) \end{gathered}$$

and

$$\begin{gathered} \operatorname{map}(\Delta^1, |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^{k+1}, N)|) \to \operatorname{map}(\Delta^1, |\operatorname{Ch}^{P,\mathcal{P}}(M \times J^k, N)|) \\ \varphi \mapsto \varphi(-) \circ |s| \end{gathered}$$

are also homeomorphisms onto their images. By taking products of these maps and restricting to subspaces (both operations again yield homeomorphisms onto their images), we get to the stable compositions and the lemma follows. □

From the lemma we conclude that we only need a contraction of $\mathbf{Ch}_f^{\mathcal{P}}(M \times J, N)$ to finish our contraction of U, since α is a homeomorphism onto its image. But $\mathbf{Ch}_f^{\mathcal{P}}(M \times J, N)$ is contractible by corollary 4.33 and thus we have finished the proof of proposition 6.20. □

Theorem 6.22. *Let* Spectra *be the category of sequential topological (pre-)spectra. There exists a coherent diagram*

$$\mathbb{P}^{coh} \in \operatorname{Coh}(\operatorname{Mfd}_c, \operatorname{Spectra})$$

such that $\mathbb{P}^{coh}(M) = \mathbb{P}(M)$ *for each* $M \in \operatorname{Mfd}_c$ *and for each map* $f \in \operatorname{map}_{\operatorname{Mfd}_c}(M, N)$ *there exists a map* $g : M \to N$ *homotopic to* f *and a choice* $c \in \mathbf{Ch}_g^{\mathbb{P}}(M, N)$ *with* $\mathbb{P}^{coh}(f) = r(c)$.

Proof. The proof is analogous to the proof of theorem 5.3: We define categories $\mathfrak{C}$ and $\mathbf{Ch}^{\mathbb{P}}$ with objects the same as Mfd_c and morphism spaces

$$\begin{aligned} \operatorname{map}_{\mathbf{Ch}^{\mathbb{P}}}(M, N) &= \mathbf{Ch}^{\mathbb{P}}(M, N) \\ \operatorname{map}_{\mathfrak{C}}(M, N) &= \mathbf{Ch}^{\mathbb{P}}(M, N) \times \operatorname{map}([0,1], \operatorname{map}(M, N)), \end{aligned}$$

then construct a coherent diagram on $\mathfrak{C}$ and then use the forgetful functor $\mathfrak{C} \to \mathbf{Ch}^{\mathbb{P}}$ and the realization functor $r : \mathbf{Ch}^{\mathbb{P}} \to \operatorname{Spectra}$ to get a coherent diagram with the desired properties.

We use the same isotopies $\alpha_M : [0,1] \to \operatorname{map}_{\operatorname{Mfd}_c}(M, M)$ as in the proof of 5.3 but replace the corresponding homotopies of choices $c_M : [0,1] \to \mathbf{Ch}^{\mathcal{P}}(M,M)$ with

$$c_M^{\mathbb{P}} : [0,1] \to \mathbf{Ch}^{\mathbb{P}}(M,M), \;\; c_M^{\mathbb{P}}(t) = (c_M(t), f_M(t))$$

with $f_M(t) : [0,1] \to \mathbf{Ch}^{\mathcal{P}}(M \times J, M \times J)$, $f_M(t)(r) = c_M(t) \times \mathrm{id}_J$. These choices again have the property $c_M^{\mathbb{P}}(t) \in \mathbf{Ch}^{\mathbb{P}}_{\alpha_M(t)}(M,M)$.

The rest of the proof is the same as in the proof of 5.3, we just have to replace the stable choice spaces with the spectrum choice spaces, c_M with $c_M^{\mathbb{P}}$ and use proposition 6.20 instead of corollary 4.33 for contractibility of choice spaces. □

Proposition 6.23. *Let $f \simeq g : M \to N$ be homotopic maps of families of compact manifolds over X with X a finite CW-complex and let $c \in \mathbf{Ch}^{\mathbb{P}}_f(M,N)$, $c' \in \mathbf{Ch}^{\mathbb{P}}_g(M,N)$. Then the two maps $r(c), r(c') : \mathbb{P}(M) \to \mathbb{P}(N)$ are homotopic.*

Proof. As in the proof of theorem 6.22, we can push away from the boundary of N using composition with choices coming from codimension-zero-embeddings. So without loss of generality we can assume that $f(M) \cap \partial N = g(M) \cap \partial N = \varnothing$. Let $h : M \times [0,1] \to N \times [0,1]$ be a homotopy from f to g with $h(M \times [0,1]) \cap (\partial N) \times [0,1] = \varnothing$. Then an element in $\mathbf{Ch}^{\mathbb{P}}_h(M \times [0,1], N \times [0,1])$, with $M \times [0,1]$ and $N \times [0,1]$ considered as families of manifolds over $X \times [0,1]$, yields a path from an element in $\mathbf{Ch}^{\mathbb{P}}_f(M,N)$ to an element in $\mathbf{Ch}^{\mathbb{P}}_g(M,N)$. Since all three choice spaces are contractible by proposition 6.20, there exists a path from c to c' in $\mathbf{Ch}^{\mathbb{P}}(M,N)$. So after applying realization the claim follows. □

Theorem 6.24. *There exists a functor*

$$\mathbb{P}^{strict} : \mathfrak{Top} \to \mathrm{Spectra}$$

together with maps $\tau_M : \mathbb{P}(M) \to \mathbb{P}^{strict}(M)$ for each $M \in \mathrm{Mfd}_c$ such that all τ_M are homotopy equivalences and for each map $f \in \mathrm{map}_{\mathrm{Mfd}_c}(M,N)$ with corresponding choice $c \in \mathbf{Ch}^{\mathbb{P}}_f(M,N)$ the diagram

$$\begin{tikzcd} \mathbb{P}(M) \arrow[r, "r(c)"] \arrow[d, "\tau_M"] & \mathbb{P}(N) \arrow[d, "\tau_N"] \\ \mathbb{P}^{strict}(M) \arrow[r, "\mathbb{P}^{strict}(f)"] & \mathbb{P}^{strict}(N) \end{tikzcd}$$

commutes up to homotopy. Furthermore, $\mathbb{P}^{strict}$ is again functorial up to homotopy in the sense that for $f, g \in \mathrm{map}_{\mathfrak{Top}}(X,Y)$, with $f \simeq g$ the maps $\mathbb{P}^{strict}(f)$ and $\mathbb{P}^{strict}(g)$ are also homotopic.

Proof. We can use theorem 5.2 to strictify the coherent diagram constructed in theorem 6.22 to get a functor $F : \mathrm{Mfd}_c \to \mathrm{Spectra}$, which satisfies all the desired properties including functoriality up to homotopy by proposition 6.23. To extend it to a functor from topological spaces, we define

$$\mathbb{P}^{strict} : \mathfrak{Top} \to \mathrm{Spectra}$$

as the homotopy left Kan extension of F along the inclusion $\mathrm{Mfd}_c \hookrightarrow \mathfrak{Top}$. Using the level model structure on prespectra, i.e. weak equivalences and fibrations are defined level-wise, we can construct the homotopy left Kan extension level-wise as a homotopy colimit over comma categories as in the proof of theorem 5.6. Thus we can also reuse the rest of the proof of theorem 5.6 applied to pointed topological spaces to show that $\mathbb{P}^{strict}$ satisfies the desired properties on objects and morphisms already contained in Mfd_c and that $\mathbb{P}^{strict}$ is again functorial up to homotopy. For the latter note that [Dug08, Theorem 6.9] (and the proof given there) also holds for the category of pointed topological spaces and that proposition 6.23 also yields functoriality up to homotopy for the individual levels of the pseudoisotopy spectrum. $\square$

CHAPTER 7

Appendix

7.1 Existence of Piecewise Linear Parallel Transports

Proposition 7.1. *Let $p: E \to M$ be a piecewise linear bundle map of $\mathcal{PL}$ families of manifolds over X. Then there exists a piecewise linear parallel transport ν over p.*

Proof. Equip M with a piecewise linear metric $d: M \times M \to [0,\infty)$. On the path space $M^{[0,1]}$ we use the induced supremum metric, i.e. $d(\omega_1, \omega_2) = \sup_{t \in [0,1]} d(\omega_1(t), \omega_2(t))$ for $\omega_1, \omega_2 \in M^{[0,1]}$. For each path $\omega \in M^{[0,1]}$ one can choose an $\varepsilon(\omega) > 0$ such that for each $t \in [0,1]$ the bundle $p: E \to M$ is trivial over the ball of radius $\varepsilon(\omega)$ around $\omega(t)$. Since the set of piecewise linear paths is dense in $M^{[0,1]}$, we can choose a locally finite covering $\{U_i(\omega_i)\}_{i \in \mathcal{I}}$ of $M^{[0,1]}$ such that each ω_i is a piecewise linear path and $U_i(\omega_i) \subseteq M^{[0,1]}$ is the ball of radius $\varepsilon_i \le \varepsilon(\omega_i)/3$ around ω_i. Also define $u_i : M^{[0,1]} \to [0,\infty)$ by $u_i(\omega) = \max\{0, \varepsilon_i - d(\omega, \omega_i)\}$. Now $\{U_i, u_i\}_{i \in \mathcal{I}}$ is almost a partition of unity on $M^{[0,1]}$, except that the sum $\sum_{i \in \mathcal{I}} u_i(\omega)$ for $\omega \in M^{[0,1]}$ is a positive number but does not need to equal 1.

For each $i \in \mathcal{I}$ let $V_i = \{(m,t) \in M \times [0,1] : d(\omega_i(t), m) < 2\varepsilon_i\}$. By definition of the ε_i we can choose piecewise linear local trivializations $\varphi_i : F \times V_i \to E \times [0,1]$ of the bundle $p \times \mathrm{id}_{[0,1]} : E \times [0,1] \to M \times [0,1]$, where F denotes the fiber of the bundle $p: E \to M$.

For each $i \in \mathcal{I}$ we define a map

$$\sigma_i : \{(\omega, e, t) \in M^{[0,1]} \times E \times [0,1] : p(e) = \omega(t)\} \to \{(\omega, e, t) \in M^{[0,1]} \times E \times [0,1] : p(e) = \omega(t)\}$$

which is given for $d(\omega, \omega_i) < 2\varepsilon_i$ by the formula

$$(\omega, e, t) \mapsto \Big(\omega, \varphi_i\big(\operatorname{pr}_F \circ \varphi_i^{-1}(e,t), \underbrace{\omega(\min\{1, t + u_i(\omega)\})\big), \min\{1, t + u_i(\omega)\}}_{\in V_i \subseteq M \times [0,1]}\big)\Big)$$

and by the identity map for $d(\omega, \omega_i) \ge \varepsilon_i$. In words, for each ω the map $\sigma_i(\omega, -, -)$ uses the local trivialization given by (V_i, φ_i) to construct an isomorphism of the fiber over $\omega(t)$ in E to the fiber over $\omega(\min\{1, t + u_i(\omega)\})$ in E. Define for $\omega \in M^{[0,1]}$

$$\sigma_{(i,\omega)} := \sigma_i(\omega, -, -) : \{(e,t) \in E \times [0,1] : p(e) = \omega(t)\} \to \{(e,t) \in E \times [0,1] : p(e) = \omega(t)\}.$$

Now choose a total ordering on the index set $\mathcal{I}$ and let $\omega \in M^{[0,1]}$. For each $i \in \mathcal{I}$ with $u_i(\omega) = 0$ the map $\sigma_{(i,\omega)}$ is the identity map, so we can define the infinite composition $\prod_{i\in\mathcal{I}} \sigma_{(i,\omega)}$ (with the composition order given by the chosen total ordering on $\mathcal{I}$) which is given by a finite composition

$$\sigma_{i_n} \circ \ldots \circ \sigma_{i_1} : \{(e,t) \in E \times [0,1] : p(e) = \omega(t)\} \to \{(e,t) \in E \times [\varepsilon,1] : p(e) = \omega(t)\}$$

for $\varepsilon = \min\{1, \sum_{i\in\mathcal{I}} u_i(\omega)\} > 0$ and some indices $i_1, \ldots, i_n \in I$. Iterating this map k times with $k \geq 1/\varepsilon$ yields an isomorphism from $p^{-1}(\omega(0))$ to $p^{-1}(\omega(1))$. Since $\prod_{i\in\mathcal{I}} \sigma_{(i,\omega)}$ is also the identity on $(p \times \mathrm{id}_{[0,1]})^{-1}(\omega(1), 1)$, we can define the k-fold composition as the infinite composition to get an isomorphism of fibers

$$\Big(\prod_{i\in\mathcal{I}} \sigma_{(i,\omega)}\Big)^{\infty} : p^{-1}(\omega(0)) \to p^{-1}(\omega(1)).$$

Since this is continuous in ω, it defines an isomorphism

$$\nu' : \{(\omega, e) \in M^{[0,1]} \times E : p(e) = \omega(0)\} \to \{(\omega, e) \in M^{[0,1]} \times E : p(e) = \omega(1)\}$$

out of which we can construct a parallel transport by the standard formula

$$\nu(\omega, e) := \nu'(\omega, \mathrm{pr}_E \circ (\nu')^{-1}(\mathrm{const}_{p(e)}, e)).$$

Now suppose that we have a piecewise linear map $f : \Delta^k \times [0,1] \to M$ for some $k \in \mathbb{N}$. Denote by $\omega_f : \Delta^k \to M^{[0,1]}$ the induced map that sends $x \in \Delta^k$ to the path defined by $t \in [0,1] \mapsto f(x,t)$. Since all ω_i are piecewise linear paths, the induced maps

$$\Delta^k \to [0, \infty),\ x \mapsto d_{M^{[0,1]}}(\omega_i, \omega_f(x))$$

are all piecewise linear. But that implies that the map induced by ν

$$\Delta^k \times p^{-1}(f(\Delta^k \times 0)) \to p^{-1}(f(\Delta^k \times 1)),\ \ (x,e) \mapsto \mathrm{pr}_E \circ \nu(f(x,(-)), e)$$

is given by locally finite compositions of piecewise linear maps and is thus also piecewise linear. Thus ν is a piecewise linear parallel transport. $\square$

7.2 On Families of Normal Microbundles

This section follows the proofs given in [KS77, Essay IV, Appendix A] very closely. We added some generalisation to it, but the proof idea is still the same as in [KS77].

Definition 7.2. Let M be a family of manifolds over X. A *family of microbundles* ν over M of dimension n consist of a space E over X and maps (commuting with the projection onto X)

$$M \xrightarrow{i} E \xrightarrow{r} M$$

such that $r \circ i = \mathrm{id}_M$ and r looks near $i(M)$ locally like a $\mathbb{R}^n$-bundle. That means for each point $x \in M$ there exists a neighborhood $U \subseteq E$ of $i(x)$ such that $U \cong K \times \mathbb{R}^n$ with $K = U \cap i(M)$ and r restricted to U corresponds to the projection onto K under this homeomorphism. We call $E = E_\nu$ the total space of the microbundle and n the dimension of the microbundle. $i = i_\nu$ and $r = r_\nu$ are called the inclusion and projection map of the microbundle.

Remark 7.3. If ν is a family of microbundles over M and η is a family of microbundles over E_ν, then we get a family of microbundles given by

$$M \xrightarrow{i_\eta \circ i_\nu} E_\eta \xrightarrow{r_\nu \circ r_\eta} M$$

called the *composed family of microbundles* $\nu \circ \eta$. An example for this is $\nu \circ \varepsilon^n$, where ε^n denotes the trivial n-dimensional family of microbundles given by

$$M = M \times \{0\} \to M \times \mathbb{R}^n \xrightarrow{\mathrm{pr}_M} M.$$

Definition 7.4. Let $M \subseteq N$ be a family of submanifolds over X. A normal microbundle of M in N is a $(\dim(N) - \dim(M))$-dimensional family of microbundles ν whose total space E_ν is a subspace of N and whose inclusion map i_ν is the inclusion map of M into N.

We call two normal microbundles ν, ν' *microbundle-isomorphic* if there exists a common subspace $M \subseteq V \subseteq E_\nu \cap E'_\nu$ such that ν and ν' restricted to V yield the same $(\dim(N) - \dim(M))$-dimensional microbundle

We call two normal microbundles ν, ν' *isotopic* if there exists a subspace $M \subseteq V \subseteq E_\nu$ and an isotopy of embeddings over X from the standard embedding of V to an embedding $h : V \to N$ such that the isotopy is constant on M and the microbundle given by

$$M \xrightarrow{h \circ i_\nu} h(V) \xrightarrow{r_\nu \circ h} M$$

is microbundle-isomorphic to ν'.

Example 7.5. For a family of manifolds M over X without boundary and with projection map $p : M \to X$ let

$$T(M) := \{(x, y) \in M \times M : p(x) = p(y)\}.$$

Then the (two) tangential microbundles τ_k for $k = 1, 2$ are given by

$$M \xrightarrow{\Delta} T(M) \xrightarrow{\mathrm{pr}_k} M$$

where Δ denotes the diagonal map given by $x \mapsto (x, x)$ and $\mathrm{pr}_k : T(M) \to M$ is the projection onto the k-th factor of $M \times M \supseteq T(M)$. Note that both microbundles are normal microbundles to the diagonal $\Delta(M) \subseteq T(M)$.

Lemma 7.6. *Let $i : M \to N$ be an embedding of families of manifolds over X, where X is a locally finite CW-complex and such that $i(M) \cap \partial N = \varnothing$. Denote $m = \dim(M)$ and $n = \dim(N)$. Then there exists a number $k > 0$ such that the embedding*

$$j := i \times 0 : M \times 0 \to N \times \mathbb{R}^k$$

is locally flat, that is for each $x \in M$ there exists a neighborhood $U \subseteq M$ of x such that there is a codimension-zero-embedding $e : U \times \mathbb{R}^{n-m+k} \to N \times \mathbb{R}^k$ such that e commutes with the projection onto X, we have $e(U \times \mathbb{R}^{n-m+k}) \cap j(M) = j(U)$ and for all $y \in U$ we have $e(y, 0) = j(y)$.

Proof. Choose $k := m$ and let $x \in M$. By restriction we can assume without loss of generality that

- $N = \mathbb{R}^n \times V$ for some open set $V \subseteq X$,
- M is a subset of $\mathbb{R}^m \times V$ that is either open in $\mathbb{R}^m \times V$ (for $x \notin \partial M$) or open in $[0, \infty) \times \mathbb{R}^{m-1} \times V$ (for $x \in \partial M$),
- $i(M) \subseteq N$ is a closed subset.

Because M is an absolute neighborhood retract, there exists a retraction $r : U \to M$ of the embedding

$$j = i \times 0 : M = M \times 0 \to N \times \mathbb{R}^m$$

which commutes with the projection onto V and where $U \subseteq N \times \mathbb{R}^m$ is some neighborhood of $j(M)$.

The projection $\mathrm{pr}_{\mathbb{R}^m \times V} : N \times \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^m \times V \to \mathbb{R}^m \times V$ yields a normal microbundle structure on $0 \times M \subseteq N \times \mathbb{R}^m$. Via the homeomorphism

$$\begin{aligned} \{(x, y, v) \in \mathbb{R}^n \times \mathbb{R}^m \times V : (y, v) \in M\} \to& \{(x, y, v) \in \mathbb{R}^n \times \mathbb{R}^m \times V : (y, v) \in M\} \\ (x, y, v) \mapsto& (x + \mathrm{pr}_{\mathbb{R}^n} \circ i(y, v), y, v) \end{aligned}$$

the normal microbundle of $0 \times M$ is sent to a normal microbundle of the diagonal embedding

$M \to N \times \mathbb{R}^m$, $(y, v) \mapsto (\mathrm{pr}_{\mathbb{R}^n} \circ i(y, v), y, v)$. Now the embedding

$$\begin{aligned}\{(x, y, v) \in \mathbb{R}^n \times \mathbb{R}^m \times V : (x, v) \in U \text{ and } (y, v) \in M\} \to &\mathbb{R}^n \times \mathbb{R}^m \times V \\ (x, y, v) \mapsto &(x, y - \mathrm{pr}_{\mathbb{R}^m} \circ r(x, v), v)\end{aligned}$$

sends the normal microbundle of the diagonal embedding to a normal microbundle of the embedding $j = i \times 0 : M \to N \times \mathbb{R}^m$. Thus the embedding j admits a normal microbundle, in particular this implies that j is locally flat. □

Theorem 7.7. *Let $i : M \to N$ be an embedding of families of manifolds over X with X a finite CW-complex and such that $i(M) \cap \partial N = \varnothing$. Then there exists an $n \in \mathbb{N}$ such that $i(M) \times 0 \subseteq N \times \mathbb{R}^n$ admits a normal disk bundle in $N \times \mathbb{R}^n$, i.e. there is a codimension-zero-submanifold $E \subseteq N \times \mathbb{R}^n$ and a disk bundle $p : E \to M$ over X such that i is a zero-section of the disk bundle p.*

Let $p_1, p_2 : E \to M$ be normal disk bundles to the embedding $i : M \to N$. Denote by $p := \mathrm{pr}_N : N \times J^n \to N$ the standard normal disk bundle of N in $N \times \mathbb{R}^n$ for some $n \in \mathbb{N}$. Then there exists an $n \in \mathbb{N}$ such that $p_1 \circ p$ and $p_2 \circ p$ are isotopic as normal disk bundles over X to the embedding $i \times 0 : M \times 0 \to N \times \mathbb{R}^n$.

By lemma 7.6 it suffices to prove the theorem for locally flat embeddings. This is done in [RS70, Corollary 5.5].

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