Statistical mechanics of gradient models

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Abstract

In this thesis, we consider gradient models on the lattice $\mathbb{Z}^d$. These models serve as effective models for interfaces and are also known as *continuous Ising models*. The height of the interface is modelled by a random field $\varphi : \Lambda \to \mathbb{R}$, where Λ is a finite subset of $\mathbb{Z}^d$. The energy of a configuration φ is given by a Hamiltonian $H_\Lambda(\varphi) = \sum_{x\in\Lambda}\sum_{i=1}^d W(\nabla_i\varphi(x))$ with a potential W and finite difference quotient $\nabla_i\varphi(x) = \varphi(x+e_i) - \varphi(x)$. We impose a tilt $u \in \mathbb{R}^d$ on the interface by equipping Λ with periodic boundary conditions and considering the Hamiltonian H_Λ^u with shifted potential $W(\cdot + u_i)$. We are interested in the Gibbs measure at tilt u and inverse temperature β of this model,

$$\gamma_{\Lambda,\beta}^u(\mathrm{d}\varphi) = \frac{1}{Z_{\Lambda,\beta}^u} e^{-\beta H_\Lambda^u(\varphi)} \lambda_\Lambda(\mathrm{d}\varphi), \quad \text{where } Z_{\Lambda,\beta}^u = \int_{\mathbb{R}/_{\{\text{constants}\}}} e^{-\beta H_\Lambda^u(\varphi)} \lambda_\Lambda(\mathrm{d}\varphi),$$

and $\lambda_\Lambda(\mathrm{d}\varphi)$ is an a priori measure on $\mathbb{R}/_{\{\text{constants}\}}$. For the potential W being a small non-convex perturbation of the quadratic interaction we prove scaling of the model to the Gaussian free field, strict convexity of the surface tension and algebraic decay of the covariance. The method of the proof is a rigorous implementation of the renormalisation group method.

Contents

1 Introduction

In the following introduction we will outline the motivation and goal for this thesis and briefly explain the mathematical tools and concepts we utilize. No originality is claimed and, to give an informative exposition, we sketch out a number of ideas from some references mentioned, but without explicit reference to the origin of each single idea.

1.1 Statistical mechanics

Motivation Statistical mechanics attempts to explain the macroscopic behaviour of large systems in equilibrium on the basis of their microscopic structure. The starting point for the development of statistical mechanics by Maxwell, Boltzmann, and Gibbs was the idea of a microscopic justification of thermodynamic laws.

Thermodynamics is the study of bulk matter. The state of a system, for example a gas, is specified by a few macroscopic quantities, for example by pressure and volume. A system is in thermal equilibrium if the state does not change with time. Thermodynamic laws and equations relating the thermodynamic quantities are viewed as hypotheses.

Statistical mechanics aims to derive these thermodynamic quantities and their relations, starting with the microscopic system of many interacting particles on the basis of microscopic forces between the components of the system. The microscopic state could be described by the equations of motion as done in classical mechanics. However, the microscopic structure is enormously complex, and any measurement of microscopic quantities is subject to statistical fluctuations.

The difficulty one has to overcome for providing a connection between macroscopic and microscopic levels is the contrast between an experimentally based description of a few quantities on the one hand and precise, but not amenable information of the behaviour of many variables on the other. The basic idea of statistical mechanics is to replace a perfect description at microscopic scale by a statistical description, i.e. by a probability measure on the state space. Of course the behaviour of the system of particles is not random, but it may be sufficiently complex that it is reasonable to view it as such. By this statistical approach the microscopic complexity may be overcome and the macroscopic determinism then may be regarded as a consequence of a suitable law of large numbers. A nice historical introduction into statistical mechanics can be found in the book by Thompson [Tho72].

Gibbs distribution Although the foundations of statistical mechanics were already laid in the nineteenth century, the mathematically rigorous study of systems only began in the late 1960s with the work of Dobrushin, Lanford and Ruelle who introduced the basic concept of a Gibbs measure.

The mathematical idealisation of the equilibrium distribution of a system is pos-

tulated to be given by the *Gibbs distribution* or *Gibbs measure*. Let S be a large (but finite) set which labels the components of the system (the particles), the *set space*. The possible states of each component are described by elements in a set E. A particular state of the system is specified by an element (a field) $\varphi = (\varphi(x))_{x\in S}$ of the product space $\Omega = E^S$, the *configuration space*. Let H be a Hamiltonian which assigns to each possible configuration a potential energy.
For a given boundary condition ψ, the finite-volume Gibbs distribution is given by the Boltzmann weight $e^{-\beta H_S^\psi}$ (the parameter β is physically associated to the inverse temperature) times an a priori measure λ_S on the configuration space Ω,

$$\gamma_{\beta,S}^{\psi}(\mathrm{d}\varphi) = \frac{1}{Z_{\beta,S}^{\psi}} e^{-\beta H_S^{\psi}(\varphi)} \lambda_S(\mathrm{d}\varphi),$$

where $Z_{\beta,S}^{\psi}$ is the normalisation of the measure, called *partition function*. The partition function is of major importance because all physically interesting macroscopic quantities can be expressed in terms of the partition function, usually in the form of logarithmic derivatives.

Since the number of particles in many-particle systems is extremely large, the intrinsic properties of the system can be made manifest by performing suitable limiting procedures. It is therefore a common practice in statistical mechanics to pass to the thermodynamic limit $|S| \to \infty$. Unfortunately, the Gibbs measure does not admit a direct extension to infinite systems. To overcome this obstacle, one characterises the Gibbs distribution by a property which can be formulated also on the infinite lattice. This property is given in terms of DLR-equations (named after Dobrushin, Lanford and Ruelle). We refer to the book by Georgii [Geo11] for a detailed introduction to Gibbs measures.

Questions The major aim is to determine the behaviour of the system at non-vanishing temperature in the thermodynamic limit.
From the probabilistic viewpoint the question of *universality* arises: Which Hamiltonians lead to similar behaviour? On appropriate limits, very different systems can have essentially identical properties.
More precisely, one can ask the following questions.

- Does the infinite-volume Gibbs measure exist, and, if yes, is it unique? Non-uniqueness of the Gibbs measure characterises the physical phenomenon of a phase transition (abrupt change in the physical properties of a system).

- What is the long-distance structure of the model, the *scaling limit*? We can study the measure in terms of a transform, for example its Laplace transform

$$\int e^{(f,\varphi)} \nu(\mathrm{d}\varphi).$$

  To analyse the behaviour of the model when looking at it from further and further away, the class of test functions f should be insensitive to fluctuations at short distances. The scaling limit would be determined by increasingly smooth f^ϵ given by $f^\epsilon(x) = \epsilon^\alpha f(\epsilon x)$, $x \in \mathbb{Z}^d$, for some exponent $\alpha \neq 0$ and $f \in C_c^\infty(\mathbb{R}^d)$, in the limit $\epsilon \to 0$. If a limiting distribution exists, the result leads to a central limit theorem for strongly correlated fields.

- Define the *free energy* or *surface tension* by

$$\sigma_\beta(\psi) = \lim_{|S|\to\infty} -\frac{1}{\beta|S|} \ln Z^\psi_{\beta,S}.$$

Which smoothness properties are satisfied by σ in dependence on the parameter β and the boundary condition ψ? This question is also related to the existence of phase transitions.

- How does the covariance

$$\begin{aligned}&\operatorname{Cov}(\varphi(a), \varphi(b))\\ &= \lim_{|S|\to\infty} \left(\int \varphi(a)\varphi(b)\gamma^\psi_{\beta,S}(\mathrm{d}\varphi) - \int \varphi(a)\gamma^\psi_{\beta,S}(\mathrm{d}\varphi) \int \varphi(b)\gamma^\psi_{\beta,S}(\mathrm{d}\varphi)\right)\end{aligned}$$

fall off with distance? The covariance expresses how strongly the fluctuations in the values of the fields are correlated.

1.2 Gradient models

Gradient models serve as effective models for interfaces (based on the idealisation that the interface can be described microscopically by a function, i.e., there are no overhangs or bubbles). We give a short introduction to the setting as it is used in this thesis. A more detailed description can be found in Section 2.1. Subsequently we give an overview of mathematical contributions to this model.

Setting Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset of the lattice. We consider fields $\varphi : \Lambda \to \mathbb{R}$ which can be interpreted as height variables of the interface. The Hamiltonian is given by a potential $W : \mathbb{R} \to \mathbb{R}$ that only depends on discrete gradients of the field,

$$H_\Lambda(\varphi) = \sum_{x\in\Lambda} \sum_{i=1}^d W(\nabla_i \varphi(x)),$$

where $\nabla_i \varphi(x) = \varphi(x + e_i) - \varphi(x)$ is the finite difference quotient on the lattice. We impose tilted boundary conditions, namely

$$\varphi(x) = \psi^u(x) \quad \text{for } x \in \partial\Lambda, \quad \psi^u(x) = u \cdot x \quad \text{for } u \in \mathbb{R}^d.$$

The finite-volume Gibbs measure with boundary condition ψ^u at inverse temperature $\beta > 0$ is then

$$\gamma^{\psi^u}_{\beta,\Lambda}(\mathrm{d}\varphi) = \frac{1}{Z^{\psi^u}_{\beta,\Lambda}} e^{-\beta H_\Lambda(\varphi)} \prod_{x\in\Lambda} \mathrm{d}\varphi(x) \prod_{x\in\partial\Lambda} \delta_{\psi^u(x)}(\mathrm{d}\varphi(x)),$$

where

$$Z^{\psi^u}_{\beta,\Lambda} = \int_{\mathbb{R}^\Lambda} e^{-\beta H_\Lambda(\varphi)} \prod_{x\in\Lambda} \mathrm{d}\varphi(x) \prod_{x\in\partial\Lambda} \delta_{\psi^u(x)}(\mathrm{d}\varphi(x))$$

is the partition function which normalises the measure.
The goal is to construct the infinite-volume limit $\Lambda \uparrow \mathbb{Z}^d$ of the Gibbs distribution. To realise this limiting procedure one has to find a way to define both an infinite-volume Gibbs state for a formal sum in the Hamiltonian and an infinite-dimensional a priori measure. This can be achieved by using the characterisation of finite-volume Gibbs measures in terms of DLR-equations which allow an extension to infinite volume. One is then particularly interested in shift-invariant, ergodic infinite-volume gradient Gibbs measures with mean u. In the case of strictly convex potentials W Funaki and Spohn ([FS97]) observed that these properties are generated by considering fields on a torus and a shifted potential $W(\cdot\ +\ u_i)$.

Let $\Lambda \subset \mathbb{Z}^d$ be a box, equip it with periodic boundary conditions, and consider the shifted Hamiltonian

$$H^u_\Lambda(\varphi) = \sum_{x\in\Lambda}\sum_{i=1}^{d} W(\nabla_i\varphi(x) + u_i).$$

In order to obtain a well-defined partition function in any dimension d we restrict the configuration space to $\mathbb{R}^\Lambda/_{\{\text{constants}\}}$. Let $\lambda_\Lambda(\mathrm{d}\varphi)$ be the unique (up to scalar multipliers) translation invariant measure on $\mathbb{R}^\Lambda/_{\{\text{constants}\}}$. We analyse the following finite-volume gradient Gibbs measure:

$$\gamma^u_{\beta,\Lambda}(\mathrm{d}\varphi) = \frac{1}{Z^u_{\beta,\Lambda}}\, e^{-\beta H^u_\Lambda(\varphi)}\lambda_\Lambda(\mathrm{d}\varphi)$$

with partition function

$$Z^u_{\beta,\Lambda} = \int_{\mathbb{R}^\Lambda/_{\{\text{constants}\}}} e^{-\beta H^u_\Lambda(\varphi)}\lambda_\Lambda(\mathrm{d}\varphi).$$

The surface tension is

$$\sigma_\beta(u) = \lim_{|\Lambda|\to\infty} -\frac{1}{\beta|\Lambda|}\ln Z^u_{\beta,\Lambda}.$$

The smoothness property of interest here is strict convexity of σ_β in u, since this is connected to the question of phase transition: In the region where entropy wins, the free energy is strictly convex. The opposite is true in the region where energy wins (strict convexity of the free energy rules out phase coexistence which corresponds to flat parts in the free energy).
A nice and detailed introduction into gradient models and gradient Gibbs measures can be found in [Fun05].

Known results For the case of the quadratic potential, $W(s) = \frac{1}{2}s^2$ (the so-called *massless free field*), the Gibbs measure is the Gaussian free field on the lattice, allowing many of the desired characteristics to be computed explicitly (see, e.g., [Fun05]):

- For any tilt u there is a unique infinite-volume gradient Gibbs measure which coincides with the Gaussian free field with mean u and covariance given by $\mathcal{C}_{\mathbb{Z}^d} = (-\Delta_{\mathbb{Z}^d})^{-1}$ (with kernel $C_{\mathbb{Z}^d}$), where $-\Delta_{\mathbb{Z}^d} = \sum_{i,j=1}^d \delta_{ij} \nabla_j^* \nabla_i$ is the discrete Laplacian on $\mathbb{Z}^d$.

- The scaling limit (when the lattice spacing tends to zero) of the model is the Gaussian free field on the continuum torus $\mathbb{T}^d$ with covariance $\mathcal{C}_{\mathbb{T}^d} = (-\Delta_{\mathbb{T}^d})^{-1}$, where $-\Delta_{\mathbb{T}^d} = -\sum_{i,j=1}^d \delta_{ij} \partial_i \partial_j$ is the Laplacian on the continuum torus.

- The surface tension is $\sigma(u) = \frac{1}{2}|u|^2 + \sigma(0)$ and thus strictly convex in u.

- The gradient-gradient covariance decays algebraically, namely

$$|\mathrm{Cov}\,(\nabla_i \varphi(a), \nabla_j \varphi(b))| = |\nabla_j^* \nabla_i C_{\mathbb{Z}^d}(a,b)| \leq C \frac{1}{|a-b|^d}.$$

From the viewpoint of probability the challenge is to develop an equivalent understanding for non-quadratic W's. How far can we enlarge the class of Hamiltonians such that the model behaves similar to the Gaussian free field? In the case of strictly convex potentials the picture is quite satisfactory:

- In [FS97] it is shown that for any tilt u there is an infinite-volume gradient Gibbs measure which is tempered, ergodic and shift invariant.

- The scaling limit is the Gaussian free field on the torus with covariance $\mathcal{C}$ where $\mathcal{C}^{-1} = -\sum_{i,j=1}^d a_{ij} \partial_i \partial_j$ for a constant positive definite matrix a, see [NS97] for $u = 0$ and in [GOS01] for arbitrary tilt u.

- The surface tension is strictly convex in u, see [FS97].

- In [DD05] it is shown that the covariance decays algebraically,

$$|\mathrm{Cov}\,(\nabla_i \varphi(a), \nabla_j \varphi(b))| \leq C \frac{1}{|a-b|^d}.$$

In summary, we observe similar behaviour to the case of quadratic potentials.

The proofs of the above results rely heavily on the strict convexity of the potentials. What about the non-convex case? Only partial results are available.

A special class of gradient fields with non-convex potentials (log-mixture of centered Gaussians) is considered in [BK07]. At tilt $u = 0$, a phase transition is shown to happen at some critical value of the inverse temperature β_c. This result demonstrates that one can expect neither the uniqueness of gradient Gibbs measures corresponding to a fixed tilt u nor strict convexity of the surface tension $\sigma(u)$. However, the scaling limit in this case is still the Gaussian free field, as shown in [BS11].

For a class of gradient models where the potential is a small non-convex perturbation of a strictly convex one, [CDM09] shows strict convexity of the surface tension at high temperature. For the same class in the same temperature regime in [CD12] it is shown that for any u there exists a unique ergodic, shift-invariant gradient Gibbs measure. Moreover, the measure scales to the Gaussian free field and the decay of the covariance is algebraic as above.

The complementary temperature regime is considered in [AKM16]. The authors consider potentials which are small perturbations of the quadratic one, the perturbation chosen such that it does not disturb the convexity at the minimum of the potential. For small tilt u and large inverse temperature β they prove strict convexity of the surface tension obtained as a limit of a subsequence of $(N_l)_{l\in\mathbb{N}}$, where L^N is the side length of the box Λ, and relying on a quite restrictive lower bound on W, namely

$$W(s) \geq (1-\epsilon)s^2$$

for a small ϵ.
In the same setting the paper [Hil16] shows that there is $q \in \mathbb{R}^{d\times d}_{\text{sym}}$ small, such that the scaling limit is the Gaussian free field on $\mathbb{T}^d$ with covariance $\mathcal{C}^q_{\mathbb{T}^d}$, where

$$\left(\mathcal{C}^q_{\mathbb{T}^d}\right)^{-1} = -\sum_{i,j=1}^{d} \left(\delta_{ij} + q_{ij}\right) \partial_i \partial_j,$$

and that a "smoothed" covariance decays algebraically. The convergences are on a subsequence.

In the PhD thesis of Simon Buchholz [Buc19] the class of potentials is widened to such which satisfy less restrictive bounds on the potential, namely

$$W(s) \geq \epsilon s^2,$$

and to vector-valued fields and finite-range instead of only nearest-neighbour interaction. The last two improvements are of interest for the application in nonlinear elasticity, see the motivation in Subsection 1.4. The authors show that the surface tension is strictly convex and that the scaling limit is the Gaussian free field on the torus. Unfortunately, all convergences are still on a subsequence.

New results The setting in this thesis is similar to the one from [Buc19]: We restrict to small tilts and large inverse temperature and use the same smallness condition on the potential. For the sake of simplicity we formulate our results and proofs for scalar-valued fields and nearest-neighbour interaction. We show that the necessity for the subsequence in the statements about the surface tension and the scaling limit can be removed. Moreover, refined covariance estimates are shown, namely

$$|\operatorname{Cov}(\nabla_i \varphi(a), \nabla_j \varphi(b))| \leq C \frac{1}{|a-b|^d}.$$

More precisely, it is shown that to first order in $|a-b|$ the Gaussian covariance $C^q_{\mathbb{Z}^d}$ appears, where $C^q_{\mathbb{Z}^d}$ is the kernel of $\mathcal{C}^q_{\mathbb{Z}^d}$ with $\left(\mathcal{C}^q_{\mathbb{Z}^d}\right)^{-1} = \sum_{i,j=1}^d (\delta_{ij} + q_{ij}) \nabla_j^* \nabla_i$:

$$\operatorname{Cov}(\nabla_i \varphi(a), \nabla_j \varphi(b)) = \nabla_j^* \nabla_i C^q(a,b) + R_{ab}, \quad |R_{ab}| \le C \frac{1}{|a-b|^{d+\nu}}, \quad \nu > 0.$$

The proof builds on a rigorous renormalisation group approach for the partition function. This approach is developed for the model at hand in [AKM16] and improved in [Buc19]. We augment the technique in two directions. On the one hand, we extend the finite-volume flow apparent in the renormalisation group method to infinite volume. This enables us to get rid of the restriction on the subsequence. On the other hand, the renormalisation group analysis is enlarged from the bulk flow (which determines the partition function) to observables. This allows us to prove fine estimates for the covariance.

1.3 Renormalisation group method

We give a rough motivation for the renormalisation group method used in the proofs of our results.

For massless Gaussian models the gradient-gradient covariance decay like the second derivative of the Greens function for the discrete Laplacian, i.e., like $|x|^{-d}$ in d dimensions. As the decay is not absolutely integrable, the models are outside the range of powerful techniques such as, for instance, the cluster expansion (at least in its original form).

The renormalisation group (RG) method is an elaborate technique originally invented to understand critical phenomena in quantum field theory and statistical physics. It has led to an understanding of universality of models in the critical regime. The method has provided a non-perturbative calculational framework as well as the basis for a rigorous mathematical understanding of these theories. However, even outside the realm of critical phenomena, the philosophy is useful and applicable, when other methods, like cluster expansion, fail.

The basic idea of renormalisation is to study the large-distance behaviour of a model by reducing the degrees of freedom. This is achieved by a version of *coarse graining*, i.e., by disregarding information about the behaviour at small distances.
The fundamental hypothesis of the renormalisation idea is that, after coarse graining and rescaling, the model should be similar to the original model with modified parameters. The combination of the two operations is called a renormalisation group transformation. The RG transformation can be viewed as a discrete, infinite-dimensional dynamical system and the model is identified with a point on its finite-dimensional stable manifold.

The philosophy of the RG method plays a key role in several rigorous investigations. The implementation in this thesis is based on Wilson's formulation of the RG (see [WK74]). A rigorous version of the method has been developed by Bauerschmidt, Brydges and Slade in a series of papers ([BS15a],[BS15b], [BBS15b], [BS15c], [BS15d]). Adams, Kotecký and Müller [AKM16] adapted the method to the setting of gradient models with non-convex interaction.

The method applies to measures which are a perturbation of a Gaussian measure and relies on the fact that when the Gaussian measure is chosen correctly, then the perturbation vanishes in some limit. By decomposing the Gaussian measure one can integrate out the degrees of freedom at different length scales present in the covariance of the Gaussian measure. In each step, expanding and contracting directions are carefully separated so that one obtains a discrete dynamical system where the stable manifold theorem can be applied.

More precisely (but still very sketchy), the problem is to study measures of the form

$$\gamma(\mathrm{d}\varphi) = F(\varphi)\mu(\mathrm{d}\varphi),$$

where μ is a Gaussian measure and F is local and satisfies $F \approx 1$ in some sense. In principle, the measure $F\mu$ can be studied in terms of $\int OF\mathrm{d}\mu$ for field functionals O which we call *observables*. For instance, with $O = 1$ it expresses the partition function, with $O(\varphi) = \nabla_i\varphi(a)\nabla_j\varphi(b)$ it gives the gradient-gradient covariance. A measure can also be studied in terms of transformations, e.g., its Laplace transform

$$Z(f) = \int e^{-(\varphi,f)}F(\varphi)\mu(\mathrm{d}\varphi).$$

The starting point is to decompose φ as $\varphi = \sum_k \varphi_k$ such that the fields φ_k are Gaussian, independent, and live on increasing scales but become smoother and smaller as k increases.
The decomposition of the fields corresponds to a decomposition $\mu = \mu_1 * \mu_2 * \mu_3 * \dots$ of the Gaussian measure, which allows to rewrite Z as a series of integrations,

$$\begin{aligned}\int F(\varphi)\mu(\mathrm{d}\varphi) &= \int F(\varphi_1+\varphi_2+\varphi_3+\dots)\mu_1(\mathrm{d}\varphi_1)\mu_2(\mathrm{d}\varphi_2)\mu_3(\mathrm{d}\varphi_3)\dots\\ &= \int F_1(\varphi_2+\varphi_3+\dots)\mu_2(\mathrm{d}\varphi_2)\mu_3(\mathrm{d}\varphi_3)\dots\\ &= \dots,\\ \text{where } F_k(\phi) &= \int F_{k-1}(\varphi_k+\phi)\mu_k(\mathrm{d}\varphi_k).\end{aligned}$$

Then

$$\int F(\varphi)\mu(\mathrm{d}\varphi) = F_\infty(0) = \lim_{k\to\infty} F_k(0),$$

if the limit exists, and (C being the covariance of the Gaussian measure μ),

$$Z(f) = e^{(f,Cf)}\int F(\varphi + Cf)\mu(\mathrm{d}\varphi) = F_\infty(Cf).$$

The map $F_{k-1} \mapsto F_k$ (the *RG-flow*) is a very complicated dynamical system. The hope is to find appropriate coordinates on a finite-dimensional space which determine the behaviour of the transformation.
At each step, local parts H_k are collected in e^{H_k} and traced explicitly, contributing to the *relevant* behaviour of the RG flow. On a perturbative level, this is enough. In the rigorous version we implement we also keep track of the error part K_k which is shown to be contractive (*irrelevant*). The flow (H_k, K_k) is then considered as a discrete dynamical system with an expanding and a contracting direction and with the fixed point $\bar{F} = 1$ corresponding to $\bar{H} = \bar{K} = 0$. If the initial data of the flow are chosen as elements in the stable manifold of the dynamical system, the sequence converges to its fixed point. The method is explained in detail in [AKM16], [Buc19] and in Section 3 below.

1.4 Statistical mechanics of elastic materials

Since the work [Buc19] intends to describe statistical mechanics of elastic materials, we give a short introduction to this application.

A unifying feature of elasticity is that the materials withstand (small) shear force. A basic hypothesis used in the mathematical formulation of solid mechanics is the so-called *Cauchy-Born rule*: The energy minimizer under affine boundary conditions is the affine function itself. The challenge is to obtain a microscopic justification of the Cauchy-Born rule.

Statistical mechanics is one possible tool to provide microscopic verification of macroscopic hypotheses. It applies to systems in equilibrium position. However, elasticity is not an equilibrium phenomenon – no material can withstand shear forever. Elastically deformed states are in general only local minimizers of the energy, not global ones. Fractured states, where the lattice is reordered, have less energy than the elastically deformed states.

Nevertheless, one can try to construct a lattice model where the equilibrium state is mimicking a metastable state at a short time scale. In the gradient model with vector valued fields $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ (fields of displacement), local neighbourhood relations are fixed to rule out complete reordering.

In [Buc19] it is shown that the Cauchy-Born rule holds at large inverse temperature β and small tilt u.

1.5 Structure of the thesis and notations

Structure of the thesis In Section 2, gradient models are introduced and the main results concerning the scaling limit (Theorem 2.1), strict convexity of the surface tension (Theorem 2.2) and a fine estimate on the covariance (Theorem 2.3) are stated. Furthermore, two technical theorems on which the proofs of these results are based are formulated (Theorems 2.7 and 2.11). They contain representations of

the generating partition function and provide straightforward proofs of the main results.
Section 3 contains the proof of the first technical result, Theorem 2.7. The proof is by RG analysis which closely follows [Buc19]. To improve the convergence results in [Buc19], the method is extended from finite-volume to infinite-volume flows. This extension is explained in [BS15d] for the φ^4-model and adapted to gradient models in this thesis.
Section 4 deals with the proof of the second technical result, Theorem 2.11. We extend the RG analysis from Section 3 to observables, as developed in [BS15a, BS15b, BBS15b, BS15c, BS15d] for the φ^4-model ([BBS15a]). We adapt the method to the setting of gradient models.
Finally, in Section 5, details for certain extensions and intermediate steps are provided. The presentation follows closely the one in [Buc19] in order to facilitate the understanding of the extensions. Proofs are only provided if they differ from the ones in [Buc19].

Notations Throughout the whole thesis we will use the following notations.

- C_c^∞ will denote the set of smooth, compactly supported functions.
- Partial derivatives will be denoted by ∂_s instead of $\frac{\partial}{\partial s}$.
- The symbol ∂_i will be used for usual derivatives, in contrast to ∇_i for discrete finite differences.
- C^r denotes the set of r-times differential functions.
- $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the set of $d \times d$ symmetric matrices.
- The Kronecker-delta δ_{ij} is 1 if $i = j$ and 0 else.
- The indicator function $\mathbb{1}_z$ is given by $\mathbb{1}_z = 1$ if condition z is satisfied and $\mathbb{1}_z = 0$ otherwise.
- We use the *big O notation* $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ to describe the limiting behaviour of the function f in terms of the function g. It means that for all sufficiently large values of x, the absolute value of $f(x)$ is at most a positive constant multiple of $g(x)$.
- For $x \in \mathbb{R}$ let $(x)_+$ be x if $x \geq 0$ and 0 else.
- For $x, y \in \mathbb{R}$ let $(x \wedge y)$ denote the minimum of x and y.
- The symbol C will mostly denote a positive constant whose value is allowed to change in a chain of inequalities from line to line.

2 Setting and results

We start by describing gradient models and their finite-volume Gibbs distributions and stating the main results, namely the scaling limit of the measure in Theorem 2.1, strict convexity of the surface tension in Theorem 2.2, and decay of correlations in Theorem 2.3.
Then we state two technical key theorems (Theorem 2.7 and Theorem 2.11), which are the main components of the proofs of the main results. They contain powerful representations of the normalisation constant of the Gibbs measure. From these representations the proofs of the main results can be deduced straightforwardly.

2.1 Gradient models

Fix an odd integer $L \geq 3$ and a dimension $d \geq 2$. Let $\mathbb{T}_N = \left(\mathbb{Z}/L^N\mathbb{Z}\right)^d$ be the d-dimensional discrete torus of side length L^N where N is a positive integer. We equip $\mathbb{T}_N$ with the quotient distances $|\cdot|$ and $|\cdot|_\infty$ induced by the Euclidean and maximum norm respectively. The torus can be represented by the cube

$$\Lambda_N = \left\{x \in \mathbb{Z}^d : |x|_\infty \leq \frac{1}{2}\left(L^N - 1\right)\right\}$$

of side length L^N once it is equipped with the metric

$$|x-y|_{\mathrm{per}} = \inf\left\{|x-y+k|_\infty : k \in \left(L^N\mathbb{Z}\right)^d\right\}.$$

Define the space of fields on Λ_N as

$$\mathcal{V}_N = \{\varphi : \Lambda_N \to \mathbb{R}\} = \mathbb{R}^{\Lambda_N}.$$

Since we will consider shift invariant energies, we are only interested in gradient fields on $\mathcal{V}_N$. Gradient fields can be described by elements in $\mathcal{V}_N/_{\{\mathrm{constants}\}}$, or, equivalently, by usual fields with vanishing average

$$\chi_N = \left\{\varphi \in \mathcal{V}_N : \sum_{x \in \Lambda_N} \varphi(x) = 0\right\}.$$

We equip χ_N with a scalar product via

$$(\varphi, \psi) = \sum_{x \in \Lambda_N} \varphi(x)\psi(x).$$

Let λ_N be the $\left(L^{Nd} - 1\right)$-dimensional Hausdorff measure on χ_N. Let e_i, $i = 1, \ldots, d$, be the standard unit vectors in $\mathbb{Z}^d$. Then the discrete forward and backward derivatives are defined by

$$\nabla_i \varphi(x) = \varphi(x + e_i) - \varphi(x), \quad i \in \{1, \ldots, d\},$$
$$\nabla_i^* \varphi(x) = \varphi(x - e_i) - \varphi(x), \quad i \in \{1, \ldots, d\}.$$

Let $W : \mathbb{R} \to \mathbb{R}$ be a potential which is a perturbation of a quadratic potential,

$$W(s) = \frac{1}{2}s^2 + V(s), \quad V : \mathbb{R} \to \mathbb{R}.$$

We study a class of random gradient fields defined in terms of a Hamiltonian

$$H_N(\varphi) = \sum_{x\in\Lambda_N} \sum_{i=1}^{d} W(\nabla_i\varphi(x)) = \sum_{x\in\Lambda_N} \sum_{i=1}^{d} \left(\frac{1}{2}|\nabla_i\varphi(x)|^2 + V(\nabla_i\varphi(x))\right).$$

We equip the space χ_N with the σ-algebra $\mathfrak{B}_{\chi_N}$ induced by the Borel-σ-algebra with respect to the product topology, and use $\mathcal{M}_1(\chi_N) = \mathcal{M}_1(\chi_N, \mathfrak{B}_{\chi_N})$ to denote the set of probability measures on χ_N.
The finite-volume gradient Gibbs measure $\gamma_{N,\beta} \in \mathcal{M}_1(\chi_N)$ at inverse temperature β is defined as

$$\gamma_{N,\beta}(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}} e^{-\beta H_N(\varphi)} \lambda_N(\mathrm{d}\varphi)$$

with partition function

$$Z_{N,\beta} = \int_{\chi_N} e^{-\beta H_N(\varphi)} \lambda_N(\mathrm{d}\varphi).$$

The model describes the behaviour of a random microscopic interface. A microscopic tilt applied to the discrete interface can be implemented by the Funaki-Spohn trick introduced in [FS97]. Given $u \in \mathbb{R}^d$, we define the Hamiltonian H_N^u on the torus $\mathbb{T}_N$ with tilt u by

$$H_N^u(\varphi) = \sum_{x\in\Lambda_N} \sum_{i=1}^{d} W(\nabla_i\varphi(x) + u_i).$$

Consequently, the finite-volume gradient Gibbs measure $\gamma_{N,\beta}^u$ with tilt u is defined as

$$\gamma_{N,\beta}^u(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}(u)} e^{-\beta H_N^u(\varphi)} \lambda_N(\mathrm{d}\varphi),$$

where $Z_{N,\beta}(u)$ is the normalisation constant. A useful generalisation of the partition function with a source term $f \in \mathcal{V}_N$ is given by the generating functional

$$Z_{N,\beta}(u, f) = \int_{\chi_N} e^{-\beta H_N^u(\varphi) + (f,\varphi)} \lambda_N(\mathrm{d}\varphi). \tag{1}$$

2.2 Main results

On the one hand we give improved versions of Theorem 3.2.9 in [Buc19] (which was firstly proven in [Hil16] with stronger assumptions on the potential W) and Theorem 3.2.6 in [Buc19]. The improvement consists in the removal of the need for a subsequence $(N_l)_l$.

On the other hand we assert an asymptotic expression for the gradient-gradient covariance of the Gibbs measure.

We impose the following assumptions on the potential W:

$$\begin{cases} \text{Let } r_0 \geq 3, r_1 \geq 2,\ V \in C^{r_0+r_1}, V'(0) = V''(0) = 0. \\ \text{Let } 0 < \omega < \frac{1}{16} \text{ and suppose that } \sum_{i=1}^d W(z_i) \geq \omega |z|^2 \text{ and} \\ \lim_{t\to\infty} t^{-2} \ln \Psi(t) = 0 \\ \qquad \text{where } \Psi(t) = \sup_{|z|\leq t} \sum_{3\leq|\alpha|\leq r_0+r_1} \frac{1}{\alpha!} |\partial^\alpha \sum_{i=1}^d W(z_i)|. \end{cases} \tag{$\star$}$$

Let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ be the continuum torus, $q \in \mathbb{R}^{d\times d}_{\text{sym}}$ and $\mathcal{C}^q_{\mathbb{T}^d}$ be the inverse of the elliptic partial differential operator $\mathcal{A}^q_{\mathbb{T}^d}$,

$$\mathcal{C}^q_{\mathbb{T}^d} = \left(\mathcal{A}^q_{\mathbb{T}^d}\right)^{-1}, \quad \mathcal{A}^q_{\mathbb{T}^d} = -\sum_{i,j=1}^d \left(\delta_{ij} + q_{ij}\right) \partial_j \partial_i,$$

which acts on the space of all functions $f \in W^{1,2}(\mathbb{T}^d)$ with mean zero.
The following theorem states that the Laplace transform of $\gamma^u_{N,\beta}$ converges to the Laplace transform of the Gaussian free field $\mu_{\mathcal{C}^q_{\mathbb{T}^d}}$ on the continuum torus with covariance $\mathcal{C}^q_{\mathbb{T}^d}$ as the lattice spacing tends to zero in a suitably scaled way.

Theorem 2.1 (Scaling limit)**.** *Let W satisfy $(\star)$. Then there is L_0 such that for all odd integers $L \geq L_0$ there is $\delta > 0$ and $\beta_0 > 0$ with the following property. For all $u \in B_\delta(0)$ and $\beta \geq \beta_0$ there is $q = q(u, \beta, V) \in \mathbb{R}^{d\times d}_{\text{sym}}$ such that for any $f \in C^\infty_c\left(\mathbb{T}^d\right)$ satisfying $\int f = 0$ and $f_N(x) = L^{-N\frac{d-2}{2}} f\left(L^{-N}x\right)$ for $x \in \Lambda_N$,*

$$\lim_{N\to\infty} \mathbb{E}_{\gamma^u_{N,\beta}}(e^{(f_N,\cdot)}) = \lim_{N\to\infty} \frac{Z_{N,\beta}(u, f_N)}{Z_{N,\beta}(u,0)} = e^{\frac{1}{2\beta}\left(f,\mathcal{C}^q_{\mathbb{T}^d} f\right)}.$$

Let us denote

$$\sigma_{N,\beta}(u) = -\frac{1}{\beta L^{Nd}} \ln Z_{N,\beta}(u,0). \tag{2}$$

The *free energy* or *surface tension* can be written as

$$\sigma_\beta(u) = \lim_{N\to\infty} \sigma_{N,\beta}(u). \tag{3}$$

The next theorem is concerned with smoothness properties of the free energy.

Theorem 2.2 (Strict convexity of surface tension)**.** *Let W satisfy $(\star)$. Then there is L_0 such that for all odd integers $L \geq L_0$ there is $\delta > 0$ and β_0 with the following property. For all $u \in B_\delta(0)$ and $\beta \geq \beta_0$ there is $q = q(u, \beta, V) \in \mathbb{R}^{d\times d}_{sym}$ such that for any N the free energy $\sigma_{N,\beta} : B_\delta(0) \to \mathbb{R}$ is in C^{r_1} and uniformly convex. Moreover, the limit $\sigma_\beta(u)$ is uniformly convex in $B_\delta(0)$.*

Furthermore, we give a formula for the gradient-gradient covariance. Given $a, b \in \Lambda_N$ and directions $m_a, m_b \in \{1, \dots, d\}$, define

$$\begin{aligned}&\operatorname{Cov}_{\gamma^u_{N,\beta}}\left(\nabla_{m_a}\varphi(a), \nabla_{m_b}\varphi(b)\right)\\&= \int_{\chi_N} \nabla_{m_a}\varphi(a)\nabla_{m_b}\varphi(b)\gamma^u_{N,\beta}(\mathrm{d}\varphi) - \int_{\chi_N} \nabla_{m_a}\varphi(a)\gamma^u_{N,\beta}(\mathrm{d}\varphi)\int_{\chi_N} \nabla_{m_b}\varphi(b)\gamma^u_{N,\beta}(\mathrm{d}\varphi).\end{aligned}$$

For $q \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, let $\mathcal{C}^q_{\mathbb{Z}^d}$ be the inverse of the differential operator on gradient fields on $\mathbb{Z}^d$,

$$\mathcal{C}^q_{\mathbb{Z}^d} = \left(\mathcal{A}^q_{\mathbb{Z}^d}\right)^{-1}, \quad \mathcal{A}^q_{\mathbb{Z}^d} = \sum_{i,j=1}^{d} \left(\delta_{ij} + q_{ij}\right) \nabla_j^* \nabla_i$$

(see [Fun05] for details on gradient fields on $\mathbb{Z}^d$ and existence of $\mathcal{C}^q_{\mathbb{Z}^d}$). Let $C^q_{\mathbb{Z}^d}$ be the kernel corresponding to the operator $\mathcal{C}^q_{\mathbb{Z}^d}$.

The following theorem states that in the thermodynamic limit $\Lambda_N \to \mathbb{Z}^d$ the gradient-gradient covariance is dominated by the covariance $\mathcal{C}^q_{\mathbb{Z}^d}$ of the discrete Gaussian free field on $\mathbb{Z}^d$.

Theorem 2.3 (Decay of the covariance)**.** *Let W satisfy $(\star)$. There is L_1 such that for all odd integers $L \geq L_1$ there is $\delta > 0$ and β_0 with the following property. For all $u \in B_\delta(0)$ and $\beta \geq \beta_0$ there is $q = q(u, \beta, V) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that*

$$\lim_{N\to\infty} \operatorname{Cov}_{\gamma^u_{N,\beta}}\left(\nabla_{m_a}\varphi(a), \nabla_{m_b}\varphi(b)\right) = \frac{1}{\beta}\left(\nabla^*_{m_b}\nabla_{m_a} C^q_{\mathbb{Z}^d}(a, b) + R_{ab}\right).$$

Here, R_{ab} can be estimated as follows. There is $\nu > 0$ and a constant $C_1 = C_1(L)$ such that for $a \neq b$

$$|R_{ab}| \leq C_1 \frac{1}{|a-b|^{d+\nu}}.$$

Remark 2.4. *Theorem 2.1 and Theorem 2.2 both follow from the same representation of the generating functional $Z_{N,\beta}(u, f)$ in Theorem 2.7. There and in Lemma 2.9 the parameters L_0, δ and β_0 are fixed, and the existence of q is stated. Therefore, these paramters coincide in Theorem 2.1 and 2.2.*
Theorem 2.3 follows from an extended representation in Theorem 2.11. The parameter L_1 has to be chosen larger than L_0 in Theorem 2.7. Accordingly, δ is smaller and β_0 larger. Aside from that, q is the same as before.

Let us mention a straightforward consequence of Theorem 2.3.

Corollary 2.5 (Algebraic decay of the covariance)**.** *Under the assumptions of Theorem 2.3 there is a constant C such that the following estimate holds:*

$$\left|\lim_{N\to\infty} \operatorname{Cov}_{\gamma^u_{N,\beta}}\left(\nabla_{m_a}\varphi(a), \nabla_{m_b}\varphi(b)\right)\right| \leq C \frac{1}{|a-b|^d}.$$

Proof. We apply Theorem 2.3 and use that the Gaussian covariance $C^q_{\mathbb{Z}^d}$ satisfies

$$\left|\nabla^\alpha C^q_{\mathbb{Z}^d}(x)\right| \le C \frac{1}{|x|^{d-2+|\alpha|}},$$

see, e.g., Proposition 2.6.14 in [Run14] for a proof of this estimate. □

Remark 2.6. *1. One can state the assumptions ($\star$) on the potential W in a more general form allowing a bigger class of perturbations V. We will comment on this again in the next section, see Lemma 2.9 and Remark 2.10. For the sake of simplicity we decided to state the main results with assumptions ($\star$).*

2. *Theorems 2.1, 2.2 and 2.3 can also be formulated for m-component fields on $\mathbb{T}_N$,*

$$\varphi : \Lambda \to \mathbb{R}^m.$$

Discrete derivatives are understood component-wise,

$$(\nabla_i \varphi)_s(x) = \varphi_s(x + e_i) - \varphi_s(x), \quad s \in \{1, \dots, m\}, i \in \{1, \dots, d\}.$$

The potential W and the perturbation V are maps from $\mathbb{R}^m$ to $\mathbb{R}$ and the tilted boundary condition $u \in \mathbb{R}^d$ is replaced by a deformation $F \in \mathbb{R}^{m \times d}$. See [Buc19] for more details on the set-up. This extension shows up in the notation but does not change the arguments in the proofs.

3. *The statements in Theorems 2.1, 2.2 and 2.3 can also be extended to more general finite-range interaction (not only nearest-neighbour). Let $A \subset \mathbb{Z}^d$ be a finite set. Consider the potential*

$$W : (\mathbb{R}^m)^A \to \mathbb{R}.$$

Then one can define the Hamiltonian with finite-range interaction and external deformation $F \in \mathbb{R}^{d \times m}$ as

$$H_N^F(\varphi) = \sum_{x \in \mathbb{T}_N} W\left((\varphi + F)_{\tau_x(A)}\right),$$

where for any $\varphi \in \chi_N$ and $B \subset \mathbb{Z}^d$ we use φ_B to denote the restriction of φ to B, and $\tau_x(A)$ denotes the set A translated by x.

For $m = d$, this is the setting for microscopic models of nonlinear elasticity with F representing an affine deformation applied to a solid. See [Buc19] for more details on the set-up and the application to elasticity.

2.3 Two key theorems and proofs of the main results

The goal of this section is the formulation of two technical key theorems, which state powerful representations of the generating functional of the model. The proofs of these theorems are obtained by a subtle renormalisation group analysis which will be carefully introduced in Sections 3 and 4.

2.3.1 Reformulation of $Z_{N,\beta}(u,f)$

Let $\overline{V}(z,u)$ be the remainder of the linear Taylor expansion of $V(z+u)$ around u,

$$\overline{V}(z,u) = V(z+u) - V(u) - V'(u)z.$$

We can write the generating functional $Z_{N,\beta}(u,f)$ from (1) in the form

$$Z_{N,\beta}(u,f) = e^{-\beta L^{Nd}\left(\frac{|u|^2}{2}+\sum_{i=1}^d V(u_i)\right)} \times \int_{\chi_N} e^{(f,\varphi)} e^{-\beta\sum_{x\in\Lambda_N}\sum_{i=1}^d\left(\overline{V}(\nabla_i\varphi(x),u_i)+\frac{1}{2}|\nabla_i\varphi(x)|^2\right)}\lambda_N(\mathrm{d}\varphi).$$

Let

$$\mu_\beta(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}^{(0)}} e^{-\frac{\beta}{2}\sum_{x\in\Lambda_N}\sum_{i=1}^d|\nabla_i\varphi(x)|^2}\lambda_N(\mathrm{d}\varphi) \tag{4}$$

be the Gaussian measure at inverse temperature β with corresponding normalisation

$$Z_{N,\beta}^{(0)} = \int_{\chi_N} e^{-\frac{\beta}{2}\sum_{x\in\Lambda_N}\sum_{i=1}^d|\nabla_i\varphi(x)|^2}\lambda_N(\mathrm{d}\varphi). \tag{5}$$

Consequently,

$$Z_{N,\beta}(u,f) = e^{-\beta L^{Nd}\sum_{i=1}^d W(u_i)} Z_{N,\beta}^{(0)} \int_{\chi_N} e^{(f,\varphi)} e^{-\beta\sum_{x\in\Lambda_N}\sum_{i=1}^d\overline{V}(\nabla_i\varphi(x),u_i)}\mu_\beta(\mathrm{d}\varphi).$$

Now we rescale the field by $\sqrt{\beta}$ and introduce the *Mayer function* $\mathcal{K}_{u,\beta,V}:\mathbb{R}^d\to\mathbb{R}$,

$$\mathcal{K}_{u,\beta,V}(z) = e^{-\beta\sum_{i=1}^d\overline{V}(\frac{z_i}{\sqrt{\beta}},u_i)} - 1. \tag{6}$$

We can express the partition function $Z_{N,\beta}(u,f)$ in terms of the polymer expansion:

$$\begin{aligned}
Z_{N,\beta}(u,f) &= e^{-\beta L^{Nd}\sum_{i=1}^d W(u_i)} Z_{N,\beta}^{(0)} \int_{\chi_N} e^{\left(f,\frac{\varphi}{\sqrt{\beta}}\right)} e^{-\beta\sum_{x\in\Lambda_N}\sum_{i=1}^d\overline{V}\left(\frac{\nabla_i\varphi(x)}{\sqrt{\beta}},u_i\right)}\mu_1(\mathrm{d}\varphi)\\
&= e^{-\beta L^{Nd}\sum_{i=1}^d W(u_i)} Z_{N,\beta}^{(0)} \int_{\chi_N} e^{\left(f,\frac{\varphi}{\sqrt{\beta}}\right)} \prod_{x\in\Lambda_N}\left(1+\mathcal{K}_{u,\beta,V}(\nabla\varphi(x))\right)\mu_1(\mathrm{d}\varphi)\\
&= e^{-\beta L^{Nd}\sum_{i=1}^d W(u_i)} Z_{N,\beta}^{(0)} \int_{\chi_N} e^{\left(f,\frac{\varphi}{\sqrt{\beta}}\right)} \sum_{X\subset\Lambda_N}\prod_{x\in X}\mathcal{K}_{u,\beta,V}(\nabla\varphi(x))\mu_1(\mathrm{d}\varphi).
\end{aligned}$$

The integral in the last expression gives the perturbative contribution

$$\mathcal{Z}_{N,\beta}\left(u,\frac{f}{\sqrt{\beta}}\right) = \int_{\chi_N} e^{\left(\frac{f}{\sqrt{\beta}},\varphi\right)} \sum_{X\subset\Lambda_N}\prod_{x\in X}\mathcal{K}_{u,\beta,V}(\nabla\varphi(x))\mu_1(\mathrm{d}\varphi). \tag{7}$$

In summary, we obtain the representation

$$Z_{N,\beta}(u,f) = e^{-\beta L^{Nd}\sum_{i=1}^d W(u_i)} Z_{N,\beta}^{(0)}\,\mathcal{Z}_{N,\beta}\left(u,\frac{f}{\sqrt{\beta}}\right).$$

We introduce a space for the perturbation $\mathcal{K}_{u,\beta,V}$. Let $\zeta \in (0,1)$. For $r_0 \geq 3$ we define the Banach space $\mathbf{E}_\zeta$ consisting of functions $\mathcal{K} : \mathbb{R}^d \to \mathbb{R}$ such that the following norm is finite

$$\|\mathcal{K}\|_\zeta = \sup_{z\in\mathbb{R}^d} \sum_{|\alpha|\leq r_0} \frac{1}{\alpha!} |\partial^\alpha \mathcal{K}(z)| e^{-\frac{1}{2}(1-\zeta)|z|^2}.$$

Let us generalise the expression for the perturbative part to arbitrary $\mathcal{K} \in \mathbf{E}_\zeta$ from the rather explicit $\mathcal{K}_{u,\beta,V}$ in (6). Namely, let

$$\mathcal{Z}_N(\mathcal{K}, f) = \int_{\chi_N} e^{(f,\varphi)} \sum_{X\subset\Lambda_N} \prod_{x\in X} \mathcal{K}(\nabla\varphi(x)) \mu_1(\mathrm{d}\varphi). \tag{8}$$

Theorem 2.7 will give a useful representation of this perturbative part of the partition function.

2.3.2 Representations of $Z_{N,\beta}(u,f)$ and conclusions

Let us introduce $\mathcal{C}^q_{\Lambda_N} = \left(\mathcal{A}^q_{\Lambda_N}\right)^{-1}$ for $q \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, where

$$\mathcal{A}^q_{\Lambda_N} : \chi_N \to \chi_N, \quad \mathcal{A}^q_{\Lambda_N} = \sum_{i,j=1}^d (\delta_{ij} + q_{ij}) \nabla_j^* \nabla_i.$$

We use $\|q\|$ to denote the operator norm of q viewed as an operator on $\mathbb{R}^d$ equipped with the l_2 metric. If q is small, $\|q\| \leq \frac{1}{2}$, we can define a Gaussian measure $\mu_{\mathcal{C}^q_{\Lambda_N}}$ on χ_N with covariance $\mathcal{C}^q_{\Lambda_N}$,

$$\mu_{\mathcal{C}^q_{\Lambda_N}}(\mathrm{d}\varphi) = \frac{1}{Z_N^{(q)}} e^{-\frac{1}{2}\left(\varphi, \mathcal{A}^q_{\Lambda_N}\varphi\right)} \mathrm{d}\lambda_N(\varphi).$$

Observe that we changed notation from $Z^{(0)}_{N,\beta=1}$ in (5) to $Z^{(0)}_N$.

The following theorem states that the perturbative contribution $\mathcal{Z}_N(u,f)$ in (8) can be written as the product of a rather explicit term and a term which is almost 1, the error being exponentially decreasing in N if $\mathcal{K}$ is small enough. This result is the key ingredient for the proofs of Theorem 2.1 and Theorem 2.2 and also the basis for the extended version in Theorem 2.11. The proof is a subtle renormalisation group (RG) analysis established in [AKM16] and reviewed and extended in Section 3.

Theorem 2.7 (Representation of the partition function)**.** *Fix $\zeta, \eta \in (0,1)$. There is L_0 such that for all odd integers $L \geq L_0$ there is $\epsilon_0 > 0$ with the following properties. There exist smooth maps (with bounds on the derivatives which are independent of N)*

$$\lambda : B_{\epsilon_0}(0) \subset \mathbf{E}_\zeta \to \mathbb{R}, \quad q : B_{\epsilon_0}(0) \subset \mathbf{E}_\zeta \to \mathbb{R}^{d\times d}_{\mathrm{sym}},$$

and, for any $N \in \mathbb{N}$, a smooth map (with bounds on the derivatives which are independent of N) $Z_N^{\emptyset} : B_{\epsilon_0}(0) \times \chi_N \to \mathbb{R}$ such that for any $f \in \chi_N$ and $\mathcal{K} \in B_{\epsilon_0}(0)$ the following representation holds:

$$\mathcal{Z}_N(\mathcal{K}, f) = e^{\frac{1}{2}\left(f, \mathcal{C}_{\Lambda_N}^{q(\mathcal{K})} f\right)} \frac{Z_N^{(q(\mathcal{K}))}}{Z_N^{(0)}} e^{-L^{Nd}\lambda(\mathcal{K})} Z_N^{\emptyset}\left(\mathcal{K}, \mathcal{C}_{\Lambda_N}^{q(\mathcal{K})} f\right).$$

If $f(x) = g_N(x) - c_N$, $g_N(x) = L^{-N\frac{d+2}{2}} g(L^{-N}x)$ for $g \in C_c^\infty(\mathbb{T}^d)$ with $\int g = 0$, c_N such that $\sum_{x \in \mathbb{T}_N} f(x) = 0$, then there is a constant C which is independent of N such that the remainder $Z_N^{\emptyset}(\mathcal{K})$ satisfies the estimate

$$\left| Z_N^{\emptyset}\left(\mathcal{K}, \mathcal{C}_{\Lambda_N}^{q(\mathcal{K})} f\right) - 1 \right| \le C\eta^N.$$

Notice that the condition on f includes the case $f \equiv 0$.

Remark 2.8. *This statement is similar to Theorem 4.9.1 in [Buc19] with the key difference that in [Buc19] the quantities $\lambda(\mathcal{K})$ and $q(\mathcal{K})$ depend on the size of the torus, i.e., on N, and here they are independent of N. This improvement is obtained by introducing a global flow (see Section 3.2). As a consequence, there is no subsequence needed in Theorems 2.1 and 2.2.*

Proposition 3.2.4 in [Buc19] provides conditions on V such that $\mathcal{K} \in B_\rho(0) \subset \mathbf{E}_\zeta$ for any $\rho > 0$ is satisfied. We cite the proposition in the following lemma.

Lemma 2.9. *Let W satisfy $(\star)$. Then there exist $\tilde{\zeta}$, $\delta_0 > 0$, C_1 and $\Theta > 0$ such that for all $\delta \in (0, \delta_0]$ and for all $\beta \ge 1$ the map*

$$B_\delta(0) \ni u \mapsto \mathcal{K}_{u,\beta,V} \in \mathbf{E}_{\tilde{\zeta}}$$

is C^{r_1} and satisfies

$$\|\mathcal{K}_{u,\beta,V}\|_{\tilde{\zeta}} \le C_1\left(\delta + \beta^{-\frac{1}{2}}\right) \quad \text{and} \quad \sum_{|\gamma| \le r_1} \frac{1}{\gamma!} \|\partial_u^\gamma \mathcal{K}_{u,\beta,V}\|_{\tilde{\zeta}} \le \Theta. \tag{9}$$

In particular, given $\rho > 0$, there exist $\delta > 0$ and $\beta_0 \ge 1$ such that for all $\beta \ge \beta_0$ and all $u \in B_\delta(0)$ we have

$$\|\mathcal{K}_{u,\beta,V}\|_{\tilde{\zeta}} \le \rho$$

and the bound on the derivatives in (9) holds.

Remark 2.10. *As noted in the previous section we can state more general assumptions on the potential W than $(\star)$. Namely, it is enough to assume the smallness condition on the Mayer function $\mathcal{K}$, $\|\mathcal{K}_{u,\beta,V}\|_{\tilde{\zeta}} \le \rho$. Then the main theorems can be applied for every V such that its Mayer function satisfies the bound.*

The proofs of Theorems 2.1 and 2.2 are straightforward consequences of the representation of the partition function in Theorem 2.7.

Proof of Theorem 2.1. The proof may be handled in the very same way as in [Hil16] or [Buc19] but without the need for taking a subsequence. We review the main arguments.
Let $\tilde{\zeta}$ be the parameter from Lemma 2.9, and let L_0 and ϵ_0 the corresponding parameters from Theorem 2.7. Then, by Lemma 2.9, there is $\delta > 0$ and $\beta_0 \geq 1$ such that for all $\beta \geq \beta_0$ and $u \in B_\delta(0)$ we have $\mathcal{K}_{u,\beta,V} \in B_{\epsilon_0}(0) \subset \mathbf{E}_{\tilde{\zeta}}$. Fix $f \in \chi_N$. By Theorem 2.7, the function $\mathcal{Z}_{N,\beta}(u,f)$ in (7) can be written as an explicit term multiplied by a perturbation $Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V}, \mathcal{C}_{\Lambda_N}^{q(\mathcal{K}_{u,\beta,V})} f)$.
Let f_N be as in the assumptions of the theorem. Define

$$\tilde{f}_N = f_N - c_N, \quad c_N \text{ such that } \sum_{x \in \mathbb{T}_N} \tilde{f}_N(x) = 0.$$

Then $\tilde{f}_N \in \chi_N$. Since $(c_N, \varphi) = 0$ for all $\varphi \in \chi_N$,

$$\mathbb{E}_{\gamma^u_{N,\beta}} \left(e^{(f_N,\varphi)} \right) = \mathbb{E}_{\gamma^u_{N,\beta}} \left(e^{(\tilde{f}_N,\varphi)} \right),$$

and we can use Theorem 2.7 to rewrite, using $q = q(\mathcal{K}_{u,\beta,V})$,

$$\begin{aligned} \mathbb{E}_{\gamma^u_{N,\beta}} \left(e^{(f_N,\varphi)} \right) &= \frac{Z_{N,\beta}(u, \tilde{f}_N)}{Z_{N,\beta}(u,0)} = \frac{\mathcal{Z}_{N,\beta}\left(u, \frac{\tilde{f}_N}{\sqrt{\beta}}\right)}{\mathcal{Z}_{N,\beta}(u,0)} \\ &= e^{\frac{1}{2\beta}(\tilde{f}_N, \mathcal{C}^q \tilde{f}_N)} \frac{Z_N^{\emptyset}\left(\mathcal{K}_{u,\beta,V}, \mathcal{C}^q_{\Lambda_N} \frac{\tilde{f}_N}{\sqrt{\beta}}\right)}{Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V}, 0)}. \end{aligned}$$

A standard argument (see Proposition 4.7 in [Hil16] or the proof of Theorem 3.2.7 in [Buc19]) shows that

$$\left(\tilde{f}_N, \mathcal{C}^q_{\Lambda_N} \tilde{f}_N\right) \to \left(f, \mathcal{C}^q_{\mathbb{T}^d} f\right)_{L^2(\mathbb{T}^d)}, \quad \text{as } N \to \infty,$$

and from Theorem 2.7 it follows that

$$\left| Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V}, 0) - 1 \right|, \left| Z_N^{\emptyset}\left(\mathcal{K}_{u,\beta,V}, \mathcal{C}^q_{\Lambda_N} \frac{f_N}{\sqrt{\beta}}\right) - 1 \right| \to 0 \text{ as } N \to \infty.$$

This concludes the proof. □

Proof of Theorem 2.2. The proof is similar to the one in [Buc19] but without the need for taking a subsequence. We sketch the main steps here.
Let $\tilde{\zeta}$ be the parameter from Lemma 2.9, and let L_0 and ϵ_0 be as in Theorem 2.7. Then, by Lemma 2.9, there is $\delta_0 > 0$ and $\beta_0 \geq 1$ such that for all $\beta \geq \beta_0$ and $u \in B_{\delta_0}(0)$ we have $\mathcal{K}_{u,\beta,V} \in B_{\epsilon_0}(0) \subset \mathbf{E}_{\tilde{\zeta}}$. Hence we can apply the representation of the perturbative partition function (see (7)) in Theorem 2.7 and we can rewrite the

finite-volume surface tension as follows (using $\lambda = \lambda(\mathcal{K}_{u,\beta,V})$ and $q = q(\mathcal{K}_{u,\beta,V})$):

$$\begin{aligned}\sigma_{N,\beta}(u) &= -\frac{1}{\beta L^{Nd}} \ln Z_{N,\beta}(u,0)\\ &= \sum_{i=1}^{d} W(u_i) - \frac{1}{\beta L^{Nd}} \ln Z^{(0)}_{N,\beta} - \frac{1}{\beta L^{Nd}} \ln \mathcal{Z}_{N,\beta}(u,0)\\ &= \sum_{i=1}^{d} W(u_i) - \frac{1}{\beta L^{Nd}} \ln Z^{(0)}_{N,\beta} + \frac{\lambda}{\beta} - \frac{1}{\beta L^{Nd}} \ln \frac{Z_N^{(q)}}{Z_N^{(0)}} - \frac{1}{\beta L^{Nd}} \ln Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V},0).\end{aligned}$$

The assumptions $(\star)$ on the potential W in Theorem 2.2 imply that there is $\delta_1 > 0$ such that for $u \in B_{\delta_1}(0)$

$$D^2\left(\sum_{i=1}^{d} W(u_i)\right)(z,z) \geq \frac{\omega}{2}|z|^2.$$

The second term $\frac{1}{\beta L^{Nd}} \ln Z^{(0)}_{N,\beta}$ is independent of u.
Our next concern is to show that

$$\mathcal{W}_{N,\beta}(u) = \frac{\lambda(\mathcal{K})}{\beta} - \frac{1}{\beta L^{Nd}} \ln \frac{Z_N^{(q)}}{Z_N^{(0)}} - \frac{1}{\beta L^{Nd}} \ln Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V},0)$$

is C^{r_1} uniformly in N. The map $u \mapsto \lambda(\mathcal{K}_{u,\beta,V})$ is C^{r_1} uniformly in N by Theorem 2.7 and then chain rule. Similar arguments apply to the second term (see Lemma 4.9.2 in [Buc19]). The third term is C^{r_1} by smoothness of $Z_N^{\emptyset}(\mathcal{K})$ in $\mathcal{K}$ with uniform bounds in N as stated in Theorem 2.7. Thus there is a constant $\Xi > 0$ independent of β and δ such that

$$\left|D^2\mathcal{W}_{N,1}(u)(z,z)\right| \leq \Xi|z|^2.$$

In summary, with the choice $\beta_1 = \frac{4\Xi}{\omega}$ for $\beta \geq \max\{\beta_0,\beta_1\}$, $\delta \leq \min\{\delta_0,\delta_1\}$ and $u \in B_\delta(0)$, we get

$$\begin{aligned}D^2\sigma_{N,\beta}(u)(z,z) &= D^2\left(\sum_{i=1}^{d} W(u_i)\right)(z,z) + D^2\mathcal{W}_{N,\beta}(u)(z,z)\\ &\geq \frac{\omega}{2}|z|^2 - \frac{\Xi}{\beta}|z|^2 \geq \frac{\omega}{4}|z|^2.\end{aligned}$$

The uniform convexity of $\sigma_\beta(u)$ follows by using the fact that the pointwise limit of uniformly convex functions is uniformly convex. □

For the proof of Theorem 2.3 we want to proceed similarly. As is often the case in statistical mechanics we compute correlation functions as derivatives with respect to an external field, which we refer to as an *observable* field. Namely, we express the gradient-gradient covariance in terms of the perturbed generating partition function:

$$\begin{aligned}\operatorname{Cov}_{\gamma^u_{N,\beta}}\left(\nabla_{m_a}\varphi(a),\nabla_{m_b}\varphi(b)\right) &= \partial_s\partial_t\Big|_{s=t=0} \ln Z_{N,\beta}\left(u, f_{ab}(s,t)\right)\\ &= \partial_s\partial_t\Big|_{s=t=0} \ln \mathcal{Z}_{N,\beta}\left(u, \frac{f_{ab}(s,t)}{\sqrt{\beta}}\right), \qquad (10)\end{aligned}$$

where

$$f_{ab}(s,t) = s\nabla^*_{m_a}\mathbb{1}_a + t\nabla^*_{m_b}\mathbb{1}_b \tag{11}$$

is the observable. The observable fields s and t are constant external fields which couple to the field φ only at the points a and b due to the indicator functions. An external field is also employed to analyse the scaling limit, but there the macroscopic regularity of this test function is important. The application of the representation in Theorem 2.7 does not give a good estimate on $Z_N^{\emptyset}\left(\mathcal{K}, \mathcal{C}_{\Lambda_N}^{(q(\mathcal{K}))} f_{ab}\right)$ since f_{ab} is too rough. If we smooth out f_{ab}, we can get a decay for the "smoothed covariance" by exploiting the decay η^N. This is done in [Hil16].
Instead we use a finer analysis based on the RG method for the bulk flow but extended to observables and obtain a refined representation of the generating partition function in Theorem 2.11.

In view of (10), we are only interested in the behaviour of $\mathcal{Z}_{N,\beta}\left(u, \frac{f_{ab}(s,t)}{\sqrt{\beta}}\right)$ up to first order in s,t and st. To make this precise, one considers the quotient algebra in which two maps of s,t become equivalent if their formal power series in s,t agree to order $1, s, t, st$, see Section 4 for the details.

Theorem 2.11 (Representation of the extended partition function)**.** *Fix $a,b \in \Lambda_N$, $\zeta \in (0,1)$ and $\eta \in (0, \frac{1}{4})$. There is L_1 such that for all odd integers $L \geq L_1$ there is $\epsilon_1 > 0$ with the following properties. For any $N \in \mathbb{N}$ there is a smooth map $Z_N^{\mathrm{ext}} : B_{\epsilon_1}(0) \times \chi_N \to \mathbb{R}$ such that (up to first order in s and t)*

$$\mathcal{Z}_N(\mathcal{K}, f_{ab}) = \frac{Z_N^{(q(\mathcal{K}))}}{Z_N^{(0)}} e^{-L^{Nd}|\lambda(\mathcal{K})|} e^{stq_N^{ab} + s\lambda_N^a + t\lambda_N^b} Z_N^{\mathrm{ext}}(\mathcal{K}, 0), \tag{12}$$

where $\lambda(\mathcal{K})$ and $q(\mathcal{K})$ are given in Theorem 2.7. There is a constant $C_1 = C_1(L)$, such that

$$q_N^{ab} = \nabla^*_{m_b}\nabla_{m_a} C_{\Lambda_N}^{q(\mathcal{K})}(a,b) + R_{ab}, \quad |R_{ab}| \leq C_1 \frac{1}{|a-b|^{d+\nu}},$$

where $0 < \nu \leq -\frac{\ln(4\eta)}{\ln L}$, and λ_N^a and λ_N^b are uniformly bounded in N.
Moreover, the remainder $Z_N^{\mathrm{ext}}(\mathcal{K}, 0)$ can be expressed (up to first order in s and t) in terms of the error term $Z_N^{\emptyset}(\mathcal{K}, 0)$ from Theorem 2.7 and parts that are small in N:

$$\begin{aligned} Z_N^{\mathrm{ext}}(\mathcal{K}, 0) &= Z_N^{\emptyset}(\mathcal{K}, 0) + sK_N^a + tK_N^b + stK_N^{ab}, \\ K_N^a, K_N^b &= \mathcal{O}\left(2^{-N}\right), \quad K_N^{ab} = \mathcal{O}\left(\eta^N 4^{-N}\right). \end{aligned}$$

As before this representation can be used for a straightforward proof of Theorem 2.3.

Proof of Theorem 2.3. Let $\tilde{\zeta}$ be the parameter from Lemma 2.9, fix $\eta \in (0, 1/4)$ and let L_1 and ϵ_1 be the corresponding parameters from Theorem 2.7. Then, for β large enough and δ small enough, $\mathcal{K}_{u,\beta,V} \in B_{\epsilon_1}(0) \subset \mathbf{E}_{\tilde{\zeta}}$ is satisfied. Therefore we can apply the representation (12) from Theorem 2.11 with

$$f_{ab}(\tilde{s}, \tilde{t}) = f_{ab}\left(\frac{s}{\sqrt{\beta}}, \frac{t}{\sqrt{\beta}}\right)$$

in the computation of the correlations as follows:

$$\begin{aligned}
\mathrm{Cov}_{\gamma^u_{N,\beta}}\left(\nabla_{m_a}\varphi(a), \nabla_{m_b}\varphi(b)\right) &= \partial_s\partial_t\Big|_{s=t=0} \ln \mathcal{Z}_{N,\beta}\left(u, \frac{f_{ab}(s,t)}{\sqrt{\beta}}\right) \\
&= \partial_s\partial_t\Big|_{s=t=0} \ln \mathcal{Z}_{N,\beta}\left(u, f_{ab}(\tilde{s}, \tilde{t})\right) \\
&= \partial_s\partial_t\Big|_{s=t=0} \ln \left[e^{st\frac{q_N^{ab}}{\beta} + s\frac{\lambda_N^a}{\sqrt{\beta}} + t\frac{\lambda_N^b}{\sqrt{\beta}}} Z_N^{\mathrm{ext}}(\mathcal{K}_{u,\beta,V}, 0) \right] \\
&= \frac{1}{\beta} q_N^{ab} + \frac{K_N^{ab}}{\beta Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V}, 0)} - \frac{K_N^a K_N^b}{\beta Z_N^{\emptyset}(\mathcal{K}_{u,\beta,V}, 0)} \\
&= \frac{1}{\beta}\left(\nabla^*_{m_b}\nabla_{m_a} C^{q(\mathcal{K})}_{\Lambda_N}(a,b) + R_{ab} + \mathcal{O}\left(2^{-N}\right)\right).
\end{aligned}$$

Since $C^{q(\mathcal{K})}_{\Lambda_N} \to C^{q(\mathcal{K})}_{\mathbb{Z}^d}$ as $N \to \infty$ (see [Fun05] for details), the theorem is proven. □

3 RG analysis for the bulk flow

The proofs of Theorem 2.7 and Theorem 2.11 are carried out by renormalisation group analysis. This is an iterative averaging process over different scales. We will introduce the multiscale method in this section and prove Theorem 2.7, the *bulk* case. We start by motivating the idea of RG.

We aim to get an expression for

$$\mathcal{Z}_N(\mathcal{K}, f) = \int_{\chi_N} e^{(f,\varphi)} \sum_{X \subset \Lambda_N} \prod_{x \in X} \mathcal{K}(\nabla\varphi(x))\mu_1(\mathrm{d}\varphi),$$

where $f \in \chi_N$, $\mathcal{K} \in \mathbf{E}_\zeta$, and $\zeta \in (0,1)$ fixed. Remember that

$$\mathcal{C}^q_{\Lambda_N} = \left(\mathcal{A}^q_{\Lambda_N}\right)^{-1}, \quad \mathcal{A}^q_{\Lambda_N} = \sum_{i,j=1}^d (\delta_{ij} + q_{ij})\, \nabla_j^* \nabla_i,$$

is the covariance of the Gaussian free field on Λ_N. For ease of notation, we will drop the subscript Λ_N from now on.

To sketch the rough idea of the method, set $f = 0$ and let us denote

$$F(\varphi) = \sum_{X \subset \Lambda_N} \prod_{x \in X} \mathcal{K}(\nabla\varphi(x)).$$

The starting point is to put an additional parameter q into the measure,

$$\mathcal{Z}_N(\mathcal{K}, 0) = \int_{\chi_N} F(\varphi)\, \mu_1(\mathrm{d}\varphi) = \frac{Z^{(q)}}{Z^{(0)}} \int_{\chi_N} F^q(\varphi)\, \mu_{\mathcal{C}^q}(\mathrm{d}\varphi),$$
$$\text{where } F^q(\varphi) = e^{\frac{1}{2}\sum_{i,j=1}^d (\nabla_i\varphi, q_{ij}\nabla_j\varphi)} F(\varphi).$$

With the help of the implicit function theorem we "tune" q to find the "correct" Gaussian measure producing a useful formula for the partition function.

A *finite-range decomposition* of $\mu_{\mathcal{C}^q} = \mu_{\mathcal{C}_1} * \ldots * \mu_{\mathcal{C}_N}$ enables us to integrate out iteratively scale by scale,

$$\begin{aligned}
\int_{\chi_N} F^q(\varphi + \phi)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi) &= \int_{\chi_N} F^q(\xi_1 + \ldots + \xi_N + \phi)\mu_{\mathcal{C}_1}(\mathrm{d}\xi_1) \ldots \mu_{\mathcal{C}_N}(\mathrm{d}\xi_N) \\
&= \int_{\chi_N} F_1^q(\xi_2 + \ldots + \xi_N + \phi)\mu_{\mathcal{C}_2}(\mathrm{d}\xi_2) \ldots \mu_{\mathcal{C}_N}(\mathrm{d}\xi_N) \\
&= \ldots \\
&= \int_{\chi_N} F_{N-1}^q(\xi_N + \phi)\mu_{\mathcal{C}_N}(\mathrm{d}\xi_N) = F_N^q(\phi).
\end{aligned}$$

F^q can be written by polymer expansion as,

$$F^q = \sum_{X \subset \Lambda} e^{H_0(X)} K_0(\Lambda \setminus X) = \left(e^{H_0} \circ K_0\right)(\Lambda),$$

$$\text{where} \quad H_0(\varphi)(X) = \sum_{x \in X} \sum_{i,j=1}^{d} \nabla_i \varphi(x) q_{ij} \nabla_j \varphi(x),$$

$$\text{and} \quad K_0(\varphi)(Y) = e^{\sum_{x \in Y} \sum_{i,j=1}^{d} \nabla_i \varphi(x) q_{ij} \nabla_j \varphi(x)} \prod_{x \in Y} \mathcal{K}\left(\nabla \varphi(x)\right).$$

This decomposition can be maintained on each scale $k \in \{1, \ldots, N\}$, that is there are maps (H_k^q, K_k^q) such that $F_k^q = e^{H_k^q} \circ K_k^q$. This so-called *circ product* acts on scale k with polymers consisting of k*-blocks*, which are cubes of side length L^k (a precise definition can be found in (16) in Subsection 3.1.2). At the last scale N there is only one block left, namely the whole set Λ_N, and the circ product is just a sum of two terms, $\left(e^{H_N^q} + K_N^q\right)(\Lambda)$.

The maps H_k^q are the *relevant* (more precisely: relevant and marginal) directions which collect all increasing (and constant) parts in the procedure $F \mapsto \mu_{k+1} * F$ and they will live in finite dimensional spaces. The flow $(H, K) \mapsto H_+ = \mathbf{A}^q H + \mathbf{B}^q K$ will be defined in such a way that $(H, K) \mapsto K_+$ is a contraction (by a suitable choice of the map $\mathbf{B}^q$). Moreover, the linear part of H should remain relevant, so that H appears in K_+ to second order (by a suitable choice of the map $\mathbf{A}^q$). Then the implicit function theorem can be applied to the flow to find the stable manifold for the initial condition (H_0, K_0) so that the flow converges to its fixed point $(0, 0)$.

This method is described and performed in detail in [BS15a], [BS15b], [BBS15b], [BS15c] and [BS15d] and adapted to gradient models in [AKM16] and [Buc19]. For the convenience of the reader we review the relevant material from [Buc19] without proofs, see Subsection 3.1.

For the asserted improvement in Theorem 2.7, namely the N-independence of the maps $\lambda(\mathcal{K})$ and $q(\mathcal{K})$, we will need some additional properties which we will state explicitly as extensions from [Buc19]. These are the restriction property and $\mathbb{Z}^d$-property as stated in Propositions 3.7 and 3.9, an improved bound on the first derivative of the irrelevant part in Lemma 3.11, and the single step estimate in Proposition 3.14.

In Subsection 3.2 the flow in [Buc19] will be extended to an infinite-volume flow and the stable manifold theorem will be applied to this flow instead on the finite-volume flow as in [Buc19].

Finally, estimates on the finite-volume flow and the proof of Theorem 2.7 will be deduced (see Subsection 3.3).

3.1 Finite-volume bulk flow and single step estimates

We start by describing the *finite-range decomposition* of the measure $\mu_{\mathcal{C}^q}$. This decomposition is the starting point for the iterative procedure.

3.1.1 Finite-range decomposition

The operator $\mathcal{A}^q : \chi_N \to \chi_N$ commutes with translations and so does its inverse $\mathcal{C}^q$. Thus there exists a unique kernel $C^q : \Lambda_N \to \mathbb{R}$ with $\sum_{x\in\Lambda_N} C^q(x) = 0$ such that

$$\mathcal{C}^q\varphi(x) = \sum_{y\in\Lambda_N} C^q(x-y)\varphi(y).$$

The next proposition is Theorem 2.3 in [Buc18].

Proposition 3.1 (Finite-range decomposition). *Fix $q \in \mathbb{R}^{d\times d}_{sym}$ such that $\mathcal{C}^q$ is positive definite. Let $L > 3$ be an odd integer and $N \geq 1$. Then there exist positive, translation invariant operators $\mathcal{C}^q_k$ such that*

$$\mathcal{C}^q = \sum_{k=1}^{N+1} \mathcal{C}^q_k,$$

$$C^q_k(x) = -M_k \quad \textit{for} \quad |x|_\infty \geq \frac{L^k}{2}, \quad k \in \{1,\dots,N\},$$

where $M_k \geq 0$ is a constant that is independent of q. The following bounds hold for any positive integer l and any multiindex α:

$$\sup_{x\in\Lambda_N}\sup_{\|\dot q\|\leq\frac12}\left|\nabla^\alpha D^l_q C^q_k(x)(\dot q,\dots,\dot q)\right| \leq \begin{cases} C_{\alpha,l}L^{-(k-1)(d-2+|\alpha|)} & \textit{for } d+|\alpha|>2 \\ C_{\alpha,l}\ln(L)L^{-(k-1)(d-2+|\alpha|)} & \textit{for } d+|\alpha|=2. \end{cases}$$

Here, $C_{\alpha,l}$ denotes a constant that does not depend on L, N, and k.

In [Buc18] further bounds in Fourier space are stated. For the sake of simplicity they are omitted here.

In contrast to [Buc19] we combine the last two covariances to a single one:

$$\mathcal{C}^q_{N,N} = \mathcal{C}^q_N + \mathcal{C}^q_{N+1}. \tag{13}$$

We will use the following decomposition:

$$\mathcal{C}^q = \sum_{k=1}^{N-1} \mathcal{C}^q_k + \mathcal{C}^q_{N,N}, \tag{14}$$

where the last term is different from [Buc19]. The reason for this change is that we extend the [Buc19] flow to infinite volume. In order to have good estimates for the finite-volume covariance we have to perform the last step of integration in the RG flow instead of dealing with a remaining integral in $\int e^{H_N} + K_N \mathrm{d}\mu_{N+1}$ at the last step.
Let us denote by μ_k the Gaussian measure with covariance $\mathcal{C}^q_k$.
For the sake of completeness we state the following property of Gaussian measures. A proof can be found, e.g., in [Bry09].

Lemma 3.2. *Let C_k be a family of positive definite operators such that $C = \sum_k C_k$. Then a field φ which is distributed according to μ_C can be written as $\varphi = \sum_k \xi_k$ where ξ_k is distributed according to μ_{C_k}.*

Another property of the finite range decomposition is independence of N, which is stated in Remark 2.4 in [Buc18]. We need this property in order to expand the flow in [Buc19] to infinite volume.

Remark 3.3 (Independence of N)**.** *Let $N < N'$ and $\Lambda_N \subset \Lambda_{N'}$ be the corresponding tori. Let us denote by C_k^N and $C_k^{N'}$ the kernels of the decomposition depending on the torus size L^N, and M_k^N, $M_k^{N'}$ be the corresponding constants from Proposition 3.1. It can be shown that for $k < N \leq N'$ and $x \in \Lambda_N$ the decomposition satisfies*

$$C_k^N(x) - C_k^{N'}(x) = -\left(M_k^N - M_k^{N'}\right), \tag{15}$$

hence the kernels agree up to a constant shift locally, and they are constant for $|x|_\infty \geq L^k/2$. We define $\Lambda_N' = \{x \in \mathbb{Z}^d : |x|_\infty < (L^N - 1)/4\}$. Then we have $x - y \in \Lambda_N$ for $x, y \in \Lambda_N'$. Let $x, y \in \Lambda_N'$ such that $x + e_i, y + e_j \in \Lambda_N'$. Then (15) *implies that*

$$\begin{aligned} \mathbb{E}_{\mu_k^N} \nabla_i \varphi(x) \nabla_j \varphi(y) &= \nabla_j^* \nabla_i C_k^N(x-y) = \nabla_j^* \nabla_i C_k^{N'}(x-y) \\ &= \mathbb{E}_{\mu_k^{N'}} \nabla_i \varphi(x) \nabla_j \varphi(y). \end{aligned}$$

This means that the covariance structures of μ_k^N and $\mu_k^{N'}$ agree locally. In particular we can conclude that for any set $X \subset \Lambda_N'$ satisfying $X + e_i \subset \Lambda_N'$ for $1 \leq i \leq d$, any $1 \leq k \leq N$, and any measurable functional $F : \mathbb{R}^X \to \mathbb{R}$

$$\int_{\chi_N} F(\nabla\varphi|_X)\mu_k^N(\mathrm{d}\varphi) = \int_{\chi_{N'}} F(\nabla\varphi|_X)\mu_k^{N'}(\mathrm{d}\varphi).$$

3.1.2 Polymers, functionals and norms

As mentioned in the preface to Section 3, we apply an iterative averaging process over various scales. In this subsection, we discuss several key notions and introduce the setting of the scales and spaces for functionals. We follow closely the presentation in [Buc19].

At each scale k we pave the torus with blocks of side length L^k. These so-called *k-blocks* are translations by $(L^k\mathbb{Z})^d$ of the block $B_0 = \left\{z \in \mathbb{Z}^d : |z_i| \leq \frac{L^k-1}{2}\right\}$. Together, they form the set of k-blocks denoted by

$$\mathcal{B}_k = \{B : B \text{ is a } k\text{-block}\}.$$

Unions of blocks are called *polymers*. For $X \subset \Lambda$ let $\mathcal{P}_k(X)$ be the set of all k-polymers in X at scale k.
Furthermore we need the following notations:

- A polymer X is *connected* if for any $x, y \in X$ there is a path $x_1 = x, x_2, \ldots, x_n = y$ in X such that $|x_{i+1} - x_i|_\infty = 1$ for $i = 1, \ldots, n-1$. The set of all connected k-polymers in X is denoted by $\mathcal{P}_k^c(X)$. The set of connected components of a polymer X is denoted by $\mathcal{C}_k(X)$.
- Let $\mathcal{B}_k(X)$ be the set of k-blocks contained in X and $|X|_k = |\mathcal{B}_k(X)|$ be the number of k-blocks in X.
- The *closure* $\bar{X} \in \mathcal{P}_{k+1}$ of $X \in \mathcal{P}_k$ is the smallest $(k+1)$-polymer containing X.
- The set of *small polymers* $\mathcal{S}_k$ is given by all polymers $X \in \mathcal{P}_k^c$ such that $|X|_k \leq 2^d$. The other polymers in $\mathcal{P}_k \setminus \mathcal{S}_k$ are *large*.
- For any block $B \in \mathcal{B}_k$ let $\hat{B} \in \mathcal{P}_k$ be the cube of side length $(2^{d+1}+1)L^k$ centered at B.
- The *small set neighbourhood* $X^* \in \mathcal{P}_{k-1}$ of $X \in \mathcal{P}_k$ is defined by

$$X^* = \bigcup_{B \in \mathcal{B}_{k-1}(X)} \hat{B}.$$

- The *large neighbourhood* X^+ of $X \in \mathcal{P}_k$ is defined by

$$X^+ = \bigcup_{\substack{B \in \mathcal{B}_k: \\ B \text{ touches } X}} B \cup X.$$

Additionally, we introduce a class of functionals.

- Let $M(\mathcal{V}_N)$ be the set of measurable real functions on $\mathcal{V}_N$ with respect to the Borel-σ-algebra.
- Let $\mathcal{N}^\emptyset$ be the space of real-valued functions of φ which are in C^{r_0}.
- A map $F : \mathcal{P}_k \to \mathcal{N}^\emptyset$ is called *translation invariant* if for every $y \in (L^k\mathbb{Z})^d$ we have $F(\tau_y(X), \tau_y(\varphi)) = F(X, \varphi)$ where $\tau_y(B) = B + y$ and $\tau_y\varphi(x) = \varphi(x-y)$.
- A map $F : \mathcal{P}_k \to \mathcal{N}^\emptyset$ is called *local* if $\varphi\big|_{X^*} = \psi\big|_{X^*}$ implies $F(X, \varphi) = F(X, \psi)$.
- A map $F : \mathcal{P}_k \to \mathcal{N}^\emptyset$ is called *shift invariant* if $F(X, \varphi + \psi) = F(X, \varphi)$ for ψ such that $\psi(x) = c$, $x \in X^*$ on each connected component of X^*.

We set

$$M(\mathcal{P}_k, \mathcal{V}_N) = \{F : \mathcal{P}_k \to \mathcal{N}^\emptyset \big| F(X) \in M(\mathcal{V}_N), F \text{ translation inv., shift inv., local}\}.$$

Notice that we included C^{r_0}-smoothness in the definition of the space $M(\mathcal{P}_k, \mathcal{V}_N)$ which is not done in [Buc19].
Generalisations of $M(\mathcal{P}_k, \mathcal{V}_N)$ are given by $M(\mathcal{P}_k^c, \mathcal{V}_N)$, $M(\mathcal{S}_k, \mathcal{V}_N)$ and $M(\mathcal{B}_k, \mathcal{V}_N)$ where the first component is changed appropriately. We will write $M(\mathcal{P}_k)$, $M(\mathcal{P}_k^c)$, $M(\mathcal{S}_k)$ and $M(\mathcal{B}_k)$ for short.

The *circ product* of two functionals $F, G \in M(\mathcal{P}_k)$ is defined by

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}_k(X)} F(Y) G(X \setminus Y). \tag{16}$$

The space of *relevant Hamiltonians* $M_0(\mathcal{B}_k)$, a subspace of $M(\mathcal{B}_k)$, is given by all functionals of the form

$$H(B, \varphi) = \sum_{x \in B} \mathcal{H}(\{x\}, \varphi)$$

where $\mathcal{H}(\{x\}, \varphi)$ is a linear combination of the following *relevant monomials*:

- The constant monomial $M(\{x\})_\emptyset(\varphi) = 1$;
- the linear monomials $M(\{x\})_\beta(\varphi) = \nabla^\beta \varphi(x)$ for $1 \le |\beta| \le \lfloor \frac{d}{2} \rfloor + 1$;
- the quadratic monomials $M(\{x\})_{\beta,\gamma}(\varphi) = \nabla^\beta \varphi(x) \nabla^\gamma \varphi(x)$ for $1 = |\beta| = |\gamma|$.

Next we introduce norms on the space of functionals. Fix $r_0 \in \mathbb{N}$, $r_0 \ge 3$.

- Define

  $$\bigoplus_{r=0}^{\infty} \mathcal{V}_N^{\otimes r} = \left\{ g = \left(g^{(0)}, g^{(1)}, \dots \right) \,\middle|\, g^{(r)} \in \mathcal{V}_N^{(r)}, \text{ only finitely many non-zero elements} \right\}.$$

  The space of test function is given by

  $$\Phi = \Phi_{r_0} = \left\{ g \in \bigoplus_{r=0}^{\infty} \mathcal{V}_N^{\otimes r} : g^{(r)} = 0 \ \forall r \ge r_0 \right\}.$$

  A norm on Φ is given as follows: On $\mathcal{V}_N^{\otimes 0} = \mathbb{R}$ we take the usual absolute value on $\mathbb{R}$. For $\varphi \in \mathcal{V}_N$ we define

  $$|\varphi|_{j,X} = \sup_{x \in X^*} \sup_{1 \le |\alpha| \le p_\Phi} \mathfrak{w}_j(\alpha)^{-1} \left| \nabla^\alpha(\varphi)(x) \right|$$

  where $\mathfrak{w}_j(\alpha) = h_j L^{-j|\alpha|} L^{-j\frac{d-2}{2}}$, $h_j = 2^j h$ and $p_\Phi = \lfloor \frac{d}{2} \rfloor + 2$. For $g^{(r)} \in \mathcal{V}_N^{\otimes r}$ we define

  $$\left| g^{(r)} \right|_{j,X} = \sup_{x_1, \dots, x_r \in X^*} \sup_{1 \le |\alpha_1|, \dots, |\alpha_r| \le p_\Phi} \left(\prod_{l=1}^{r} \mathfrak{w}_j(\alpha_l)^{-1} \right) \nabla^{\alpha_1} \otimes \dots \otimes \nabla^{\alpha_r} g^{(r)}(x_1, \dots, x_r).$$

  Then set $|g|_{j,X} = \sup_{r \le r_0} \left| g^{(r)} \right|_{j,X}$.

- A homogeneos polynomial $P^{(r)}$ of degree r on $\mathcal{V}_N$ can be uniquely identified with a symmetric r-linear form and hence with an element $\overline{P^{(r)}}$ in the dual of $\mathcal{V}_N^{\otimes r}$. So we can define the pairing
$$\langle P, g\rangle = \sum_{r=0}^{\infty} \left\langle \overline{P^{(r)}}, g^{(r)} \right\rangle$$
and a norm
$$|P|_{j,X} = \sup \{\langle P, g\rangle : g \in \Phi, |g|_{j,X} \leq 1\}.$$
For $F \in C^{r_0}(\mathcal{V}_N) = \mathcal{N}^{\emptyset}$ the pairing is given by $\langle F, g\rangle_\varphi = \langle \mathrm{Tay}_\varphi F, g\rangle$ which defines a norm
$$|F|_{j,X,T_\varphi} = |\mathrm{Tay}_\varphi F|_{j,X} = \sup \{\langle F, g\rangle_\varphi : g \in \Phi, |g|_{j,X} \leq 1\}.$$
Here, $\mathrm{Tay}_\varphi F$ denotes the Taylor polynomial of order r_0 of F at φ.

- Let $F \in M(\mathcal{P}_k^c)$. In [Buc19] weights $W_k^X, w_k^X, w_{k:k+1}^X \in M(\mathcal{P}_k)$ are defined. Useful properties are summarized in Lemma 5.1. Weighted norms are given by
$$\begin{aligned} |||F(X)|||_{k,X} &= \sup_\varphi |F(X)|_{k,X,T_\varphi} W_k^X(\varphi)^{-1}, \\ \|F(X)\|_{k,X} &= \sup_\varphi |F(X)|_{k,X,T_\varphi} w_k^X(\varphi)^{-1}, \\ \|F(X)\|_{k:k+1,X} &= \sup_\varphi |F(X)|_{k,X,T_\varphi} w_{k:k+1}^X(\varphi)^{-1}. \end{aligned}$$
Observe that the last scale weight ($k = N$) is defined via a new covariance (see (13)) in contrast to [Buc19]. We will comment on this modification in Section 5.1.1.

- The global weak norm for $F \in M(\mathcal{P}_k^c)$ for $A \geq 1$ is given by
$$\|F\|_k^{(A)} = \sup_{X \in \mathcal{P}_k^c} \|F(X)\|_{k,X} A^{|X|_k}.$$

- A norm on relevant Hamiltonians is given as follows. For $H \in M_0(\mathcal{B}_k)$ we can write
$$H(B, \varphi) = \sum_{x \in B} \left(a_\emptyset + \sum_{\beta \in \mathfrak{v}_1} a_\beta \nabla^\beta \varphi(x) + \sum_{x \in B} \sum_{\beta,\gamma \in \mathfrak{v}_2} a_{\beta,\gamma} \nabla^\beta \varphi(x) \nabla^\gamma \varphi(x) \right).$$
Here
$$\begin{aligned} \mathfrak{v}_1 &= \left\{ \beta \in \mathbb{N}_0^{\mathcal{U}}, 1 \leq |\beta| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1 \right\}, \\ \mathfrak{v}_2 &= \left\{ (\beta, \gamma) \in \mathbb{N}_0^{\mathcal{U}} \times \mathbb{N}_0^{\mathcal{U}}, |\beta| = |\gamma| = 1, \beta < \gamma \right\}, \end{aligned}$$
where $\mathcal{U} = \{e_1, \ldots, e_d\}$ and the expression $\beta < \gamma$ refers to any ordering of $\{e_1, \ldots, e_d\}$. With these preparations we define a norm on $M_0(\mathcal{B}_k)$ as follows:
$$\|H\|_{k,0} = L^{dk} |a_\emptyset| + \sum_{\beta \in \mathfrak{v}_1} h_k L^{kd} L^{-k\frac{d-2}{2}} L^{-k|\beta|} |a_\beta| + \sum_{(\beta,\gamma) \in \mathfrak{v}_2} h_k^2 \left| a_{(\beta,\gamma)} \right|.$$

Remark 3.4. *Aside from the parameter L two parameters appear above in the definition of the norms: h and A.*
The parameter h is determined by desired properties for the weights $W_k, w_k, w_{k:k+1}$, see Theorem 4.5.1 in [Buc19] (cited here in Lemma 5.1), dependent on the choice of L. We will use the weights without explaining the construction and thus we will always choose h large enough as required, depending on L.
The parameter A (also dependent on L) will be fixed in Proposition 3.21. It will be chosen larger than in [Buc19].
Finally, there will be a small parameter $\kappa = \kappa(L)$. It constrains the parameter $q \in \mathbb{R}^{d\times d}_{sym}$ which determines the Gaussian covariance $\mathcal{C}^q$. The constraint will be that $q \in B_\kappa(0)$ for κ small. The parameter κ is determined by desired properties for the weights, $W_k, w_k, w_{k:k+1}$, see Theorem 4.5.1 in [Buc19] (cited here in Lemma 5.1).

3.1.3 The renormalisation map

We use the finite-range decomposition of $\mathcal{C}^q$ into covariances $\mathcal{C}^q_1, \dots, \mathcal{C}^q_{N-1}, \mathcal{C}^q_{N,N}$ defined in Subsection 3.1.1 (see (14)). The decomposition implies that a field φ distributed according to $\mu_{\mathcal{C}^{(q)}}$ can be decomposed into fields ξ_k distributed according to $\mu_{\mathcal{C}^q_k} =: \mu^q_k$,

$$\varphi \overset{\mathcal{D}}{=} \sum_{k=1}^N \xi_k,$$

and that $\mu_{\mathcal{C}^{(q)}} = \mu^q_1 * \dots * \mu^q_{N-1} * \mu^q_{N,N}$ (see Lemma 3.2).
Let us define the renormalisation map

$$\mathcal{R}_k F(\varphi) = \int_{\chi_N} F(\varphi + \xi)\mu_k(\mathrm{d}\xi).$$

Then

$$\int_{\chi_N} F(\varphi)\mu_{\mathcal{C}^{(q)}}(\mathrm{d}\varphi) = \mathcal{R}_{N,N}\mathcal{R}_{N-1}\dots\mathcal{R}_1(F)(0).$$

The flow under $\mathcal{R}_k$ will be described by two sequences of functionals $H_k \in M_0(\mathcal{B}_k)$ and $K_k \in M(\mathcal{P}^c_k)$. In the following we define those sequences and state properties as far as it is needed for our purpose of proving Theorem 2.7 and for the understanding of the extension to observables.

The flow is given by

$$\begin{aligned} \mathbf{T}_k : M_0(\mathcal{B}_k) \times M(\mathcal{P}^c_k) \times \mathbb{R}^{d\times d}_{\mathrm{sym}} &\to M_0(\mathcal{B}_{k+1}) \times M(\mathcal{P}^c_{k+1}), \\ (H, K, q) &\mapsto (H_+, K_+). \end{aligned}$$

Note that we sometimes omit the scale k from the notation; if doing so, the $+$ indicates the change of scale from k to $k+1$. The maps $H_+ \in M_0(\mathcal{B}_{k+1})$ and $K_+ \in M(\mathcal{P}_{k+1})$ are chosen such that

$$\mathcal{R}_+(e^H \circ K)(\Lambda_N) = (e^{H_+} \circ K_+)(\Lambda_N).$$

Let us introduce a projection $\Pi_2 : M(\mathcal{B}_k) \to M_0(\mathcal{B}_k)$ on the space of relevant Hamiltonians. For $F \in M(\mathcal{B}_k)$, $\Pi_2 F$ is attained as homogenisation of the second order Taylor expansion of $F(B)$ given by $\dot\varphi \mapsto F(B,0)+DF(B,0)\dot\varphi+\frac{1}{2}D^2F(B,0)(\dot\varphi,\dot\varphi)$. More precisely, $\Pi_2 F$ is the relevant Hamiltonian $F(B,0) + l(\dot\varphi) + Q(\dot\varphi,\dot\varphi)$ where l is the unique linear relevant Hamiltonian that satisfies $l(\dot\varphi) = DF(B,0)\dot\varphi$ for all $\dot\varphi$ who are polynomials of order $\lfloor \frac{d}{2} + 1 \rfloor$ on B^+, and Q is the unique quadratic relevant Hamiltonian that agrees with $\frac{1}{2}D^2F(B,0)(\dot\varphi,\dot\varphi)$ on all $\dot\varphi$ which are affine on B^+. These heuristics are made precise in [Buc19], Section 4.6.4.

The relevant part of the flow on the next scale, the map H_+, is defined as follows: For $B_+ \in \mathcal{B}_{k+1}$

$$\begin{aligned} H_+(B_+) &= \mathbf{A}_k^q H(B_+) + \mathbf{B}_k^q K(B_+) \\ &= \sum_{B\in\mathcal{B}_{k+1}(B_+)} \Pi_2\mathcal{R}_{k+1}H(B) + \sum_{B\in\mathcal{B}_{k+1}(B_+)} \Pi_2\mathcal{R}_{k+1}K(B). \end{aligned}$$

Remark 3.5. *We comment again on the motivation for the decomposition into H and K (see also at the beginning of this section). $\mathbf{A}_k^q H$ is a linear order perturbation which results in the fact that H appears to second order in K_+, see Proposition 3.14. Moreover, $\mathbf{B}_k^q K$ is defined in such a way that $(H,K) \mapsto K_+$ is a contraction, see Proposition 3.12.*

For the definition of the irrelevant part K_+ of the flow at the next scale, set

$$\widetilde{H}(B) = \Pi_2\mathcal{R}_{k+1}H(B) + \Pi_2\mathcal{R}_{k+1}K(B),$$

and for $X \in \mathcal{P}_k$ and $U \in \mathcal{P}_{k+1}$,

$$\begin{aligned} \chi(X,U) &= \mathbb{1}_{\pi(x)=U}, \quad \text{where} \\ \pi(X) &= \bigcup_{Y\in\mathcal{C}(X)} \tilde\pi(Y) \quad \text{and} \\ \tilde\pi(Y) &= \begin{cases} \bar X & \text{if } X \in \mathcal{P}^c \setminus \mathcal{S}, \\ B_+ & \text{where } B_+ \in \mathcal{B}_+ \text{ with } B_+ \cap X \neq \emptyset \text{ for } X \in \mathcal{S}\setminus\{\emptyset\}, \\ \emptyset & \text{if } X = \emptyset. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} K_+(U,\varphi) &= \mathbf{S}_k^q(H_+,K_+)(U,\varphi) \\ &= \sum_{X\in\mathcal{P}} \chi(X,U) \left(e^{\widetilde{H}(\varphi)}\right)^{U\setminus X} \left(e^{\widetilde{H}(\varphi)}\right)^{-X\setminus U} \\ &\quad \times \int \left[\left(1 - e^{\widetilde{H}(\varphi)}\right) \circ \left(e^{H(\varphi+\xi)} - 1\right) \circ K(\varphi+\xi)\right](X)\mu_+(\mathrm{d}\xi). \end{aligned} \tag{17}$$

If the dependence of $\mathbf{S}_k^q$ on q is not of direct importance we omit it from the notation.

We review the following properties of the map $(H,K) \mapsto K_+$ from Lemma 4.4.4 in [Buc19].

Lemma 3.6. *For $H \in M_0(\mathcal{B}_k)$ the functional K_+ defined above has the following properties.*

1. *If $K \in M(\mathcal{P}_k)$, then $K_+ \in M(\mathcal{P}_+)$.*
2. *If $K \in M(\mathcal{P}_k)$ factors on scale k, then K_+ factors on scale $k+1$.*

The construction on K_+ gives it a local dependence on K, as formulated in the next proposition.

Proposition 3.7. *The map $(H, K) \mapsto K_+$ satisfies the* restriction property*, that is for $U \in \mathcal{P}_{k+1}$ the value of $K_+(U)$ depends on U only via the restriction $K\big|_{U^*}$ of K to polymers in $\mathcal{P}(U^*)$.*

Proof. This follows from the definition of K_+ and from the fact that $\mathcal{R}_{k+1}$ preserves locality. □

For the construction of the infinite-volume flow later we consider the family $(K^\Lambda)_\Lambda$ in dependence on the torus Λ. More precisely, we consider tori Λ_N with increasing side length L^N, $N \in \mathbb{N}$. Let $\mathcal{P}_k(\mathbb{Z}^d)$ be the set of finite unions of k-blocks in $\mathbb{Z}^d$. We need the following compatibility condition.

Definition 3.8. *We say that a family of maps $(K^\Lambda)_\Lambda$ satisfies the $(\mathbb{Z}^d)$-property if for any $X \in \mathcal{P}_k(\mathbb{Z}^d)$ and for $\Lambda \subset \Lambda'$ satisfying $\operatorname{diam}(X) \leq \frac{1}{2}\operatorname{diam}(\Lambda)$ it holds that*

$$K^\Lambda(X) = K^{\Lambda'}(X).$$

Given (H, K^Λ), we note the dependence on Λ also in the map $\mathbf{S}^\Lambda$. By the definition of the map $(H, K) \mapsto \mathbf{S}_k^\Lambda(H, K)$ we directly get the following property.

Proposition 3.9. *Let $(K^\Lambda)_\Lambda$ satisfy the $(\mathbb{Z}^d)$-property and let $H \in M_0(\mathcal{B})$. Then $(\mathbf{S}^\Lambda(H, K, q))_\Lambda$ also satisfies the $(\mathbb{Z}^d)$-property.*

Proof. Let $U \in \mathcal{P}_+(\mathbb{Z}^d)$ such that $\operatorname{diam}(U) \leq \frac{1}{2}\operatorname{diam}(\Lambda)$. Let Λ' be a torus larger than Λ. Then

$$K_+^{\Lambda'}(U) = \mathbf{S}(H, K^{\Lambda'})(U).$$

We use the restriction property in Proposition 3.7 to see that $\mathbf{S}(H, K^{\Lambda'})(U)$ only depends on $K^{\Lambda'}$ through $K^{\Lambda'}\big|_{U^*}$. In fact, no polymers that are larger than U can appear in the formula for $\mathbf{S}_k$ due to the definition of $\chi(X, U)$. Thus for any $X \in \mathcal{P}(U^*)$ that appears in $\mathbf{S}$ it holds that $\operatorname{diam}(X) \leq \frac{1}{2}\operatorname{diam}(\Lambda)$, and we can apply the assumption that $(K^\Lambda)_\Lambda$ satisfies the $(\mathbb{Z}^d)$-property. □

3.1.4 Properties of the renormalisation map

Here we state important properties of the renormalisation map $\mathbf{T}_k$, namely smoothness of the irrelevant part (Proposition 3.10), an improved bound on the first derivative of the irrelevant part (Lemma 3.11), contractivity of the linearisation of the irrelevant part (Proposition 3.12), and a single step estimate (Proposition 3.14). Smoothness and contractivity are proven in [Buc19], but we add restriction and

($\mathbb{Z}^d$)-property in the statements which will be useful to perform the extension to infinite volume in the next section.
We explicitly analyse the dependence of $\mathbf{S}_k$ on q in the next statement, so we consider $\mathbf{S}_k$ as a map from $M_0(\mathcal{B}_k) \times M(\mathcal{P}_k^c) \times \mathbb{R}^{d\times d}_{\text{sym}}$ to $M(\mathcal{P}_{k+1})$. This proposition is an extension of Theorem 4.4.7 in [Buc19].

Proposition 3.10 (Smoothness of the bulk flow)**.** *Let*

$$U_{\rho,\kappa} = \{(H,K,q) \in M_0(\mathcal{B}_k) \times M(\mathcal{P}_k^c) \times \mathbb{R}^{d\times d}_{sym} : \|H\|_{k,0} < \rho, \|K\|_k^{(A)} < \rho, \|q\| < \kappa\}.$$

There is L_0 such that for all odd integers $L \geq L_0$ there are A_0, h_0 and κ with the following property. For all $A \geq A_0$ and $h \geq h_0$ there exists $\rho = \rho(A)$ such that for all $k \leq N$

$$\mathbf{S}_k \in C^\infty\left(U_{\rho,\kappa}, M(\mathcal{P}_{k+1}^c)\right).$$

For any $j_1, j_2, j_3 \in \mathbb{N}$ there are constants C_{j_1,j_2,j_3} independent of N such that for any $(H,K,q) \in U_{\rho,\kappa}$

$$\left\|D_1^{j_1} D_2^{j_2} D_3^{j_3} \mathbf{S}_k(H,K,q)(\dot{H}^{j_1}, \dot{K}^{j_2}, \dot{q}^{j_3})\right\|_{k+1}^{(A)} \leq C_{j_1,j_2,j_3} \|\dot{H}\|_{k,0}^{j_1} \left(\|\dot{K}\|_k^{(A)}\right)^{j_2} \|\dot{q}\|^{j_3}.$$

Moreover, $\mathbf{S}_k(H,K,q)(U)$ satisfies the restriction property and preserves the ($\mathbb{Z}^d$)-property.

Proof. The restriction property is stated in Proposition 3.7. The ($\mathbb{Z}^d$)-property is preserved by Proposition 3.9. The smoothness and bounds are part of Theorem 4.4.7 in [Buc19]. □

For the transfer of smoothness properties from the global flow back to the finite-volume flow in Proposition 3.21 we need the following improved bound on the first derivative of $\mathbf{S}_k$ on long polymers.

Lemma 3.11. *Assume that Proposition 3.10 holds. Let $\mathcal{P}^2_{k+1}(\Lambda)$ be the set of polymers $U \in \mathcal{P}_{k+1}(\Lambda)$ such that $\operatorname{diam}(U) > \frac{1}{2}\operatorname{diam}(\Lambda)$. Then, for any $x \in (0, 2\alpha)$, where $\alpha = \left[(1+2^d)(1+6^d)\right]^{-1}$, and for any $(H,K) \in U_\rho$,*

$$\left\|D_H D_K D_q \mathbf{S}_k(H,K,q)(\dot{H},\dot{K},\dot{q})\big|_{\mathcal{P}^2_{k+1}(\Lambda)}\right\|_{k+1}^{(A)} \leq C_1 A^{-\frac{x}{2}L^{N-(k+1)}} A^4 \|\dot{H}\|_{k,0} \|\dot{K}\|_k^{(A)} \|\dot{q}\|.$$

Proof. In Remark 5.23 we show that

$$A^{|U|_{k+1}} \left\|D_H D_K D_q \mathbf{S}_k(H,K,q)(\dot{H},\dot{K}\dot{q})(U)\right\|_{k+1,U} \leq C_1 A^{-x|U|_{k+1}} A^4 \|\dot{H}\|_{k,0} \|\dot{K}\|_k^{(A)} \|\dot{q}\|.$$

Since

$$\operatorname{diam}(\Lambda) = \sqrt{2}L^N \quad \text{and} \quad \operatorname{diam}(U) \leq |U|_k \sqrt{2} L^{k+1}$$

we get for $U \in \mathcal{P}^{c,2}_{k+1}(\Lambda)$

$$|U|_{k+1} > \frac{1}{2} L^{N-(k+1)}.$$

Thus the claim follows.

□

The following proposition is Theorem 4.4.8 in [Buc19] except for the minor difference that θ is kept arbitrarily instead of the choice $\frac{3}{4}\eta$.

Proposition 3.12 (Contractivity of the bulk flow)**.** *The first derivative of $\mathbf{T}_k$ at $H = 0$ and $K = 0$ has the triangular form*

$$D\mathbf{T}_k(0,0,q)\begin{pmatrix}\dot{H}\\ \dot{K}\end{pmatrix} = \begin{pmatrix}\mathbf{A}_k^q & \mathbf{B}_k^q\\ 0 & \mathbf{C}_k^q\end{pmatrix}\begin{pmatrix}\dot{H}\\ \dot{K}\end{pmatrix}$$

where

$$\mathbf{C}_k^q\dot{K}(U) = \sum_{B:\bar{B}=U}(1-\Pi_2)\mathcal{R}_+\dot{K}(B) + \sum_{\substack{X\in\mathcal{P}_k^c\setminus\mathcal{B}(X)\\ \pi(X)=U}}\mathcal{R}_+\dot{K}(X).$$

For any $\theta\in(0,1)$ there is L_0 such that for all odd integers $L\geq L_0$ there exist A_0, h_0 and κ with the following property. For all $A\geq A_0$, $h\geq h_0$ and for $\|q\|<\kappa$ the following bounds hold independent of k and N:

$$\|\mathbf{C}_k^q\|\leq\theta,\quad \|\left(\mathbf{A}_k^q\right)^{-1}\|\leq\frac{3}{4},\quad \|\mathbf{B}_k^q\|\leq\frac{1}{3}.$$

Moreover, the derivatives of the operators with respect to q are bounded.

Remark 3.13. *In [Buc19] θ is fixed to be $\frac{3}{4}\eta$, where η is the parameter that controls the contraction rate of the renormalisation flow. For the single step estimates in Proposition 3.14 and Proposition 4.10 we have to choose θ smaller than $\frac{3}{4}\eta$. Thus we formulated the Proposition with this additional flexibility. Inspection of the proof of the bound on $\|\mathbf{C}_k^q\|$ in [Buc19] shows that a smaller θ can be obtained by choosing larger L_0 and A_0.*

Proposition 3.10 and Proposition 3.12 can be combined to prove a single step estimate of the irrelevant part of the flow. This bound can not be found in [Buc19]. The estimate will help us deduce estimates on the finite-volume flow given the infinite-volume flow, see Proposition 3.21.

Let us introduce the space

$$\begin{aligned}&\mathbb{D}_k(\rho_0,\eta,\Lambda)\\ &\quad=\left\{(H,K)\in M_0(\mathcal{B}_k)\times M(\mathcal{P}_k(\Lambda)) : H\in B_{\rho_0\eta^k}(0), K\in B_{\rho_0\eta^{2k}}(0)\right\}.\end{aligned}\tag{18}$$

Proposition 3.14 (Single step estimate for the bulk flow)**.** *Fix $\eta\in(0,1)$. There is L_0 such that for all odd integers $L\geq L_0$ there are A_0,h_0,κ with the following property. For all $A\geq A_0, h\geq h_0$ and $q\in B_\kappa(0)$ there is $\rho_0^\emptyset>0$ such that if $(H,K)\in\mathbb{D}_k(\rho_0^\emptyset,\eta,\Lambda)$ then*

$$\|\mathbf{S}_k^q(H,K)\|_{k+1}^{(A)}\leq\rho_0^\emptyset\eta^{2(k+1)}.$$

As the proof will show this estimate reflects the fact that we use first order perturbation: Heuristically, up to first order in H,

$$\mathcal{R}_+(e^H) \approx e^{\mathcal{R}_+ H}$$

since $\mathcal{R}_+(e^H) \approx \mathcal{R}_+(1+H) = 1 + \mathcal{R}_+ H \approx e^{\mathcal{R}_+ H}$.

Proof. Fix $\theta < \eta^2$. Let L_0 be large enough such that Proposition 3.12 and Proposition 3.10 can be applied. Define

$$C_2 = \max\{C_{2,0,0}, C_{1,1,0}, C_{0,2,0}\}$$

where C_{j_1,j_2,j_3} are the constants from Proposition 3.10. Choose $\rho_0^{\emptyset}$ small enough that

$$\rho_0^{\emptyset} \leq \rho(A) \quad \text{and } \theta + 2C_2\rho_0^{\emptyset} \leq \eta^2.$$

Then $(H,K) \in \mathbb{D}_k(\rho_0^{\emptyset}, \eta, \Lambda)$ implies $(H,K) \in U_{\rho(A)}$ so we can apply Proposition 3.10 to estimate as follows.
We Taylor-expand $\mathbf{S}(H,K)$ up to first order with second order integral remainder around $(0,0)$:

$$\begin{aligned}\mathbf{S}(H,K) &= \mathbf{S}(0,0) + D\mathbf{S}(0,0)(H,K) + \int_0^1 D^2\mathbf{S}(tH,tK)(H,K)(H,K)(1-t)\mathrm{d}t \\ &= \mathbf{C}^q K + \int_0^1 D^2\mathbf{S}(tH,tK)(H,K)(H,K)(1-t)\mathrm{d}t.\end{aligned}$$

Then we estimate

$$\begin{aligned}\|\mathbf{S}(H,K)\|_{k+1}^{(A)} &\leq \|\mathbf{C}^q\|\|K\|_k^{(A)} + \frac{1}{2}C_2\left(\|H\|_{k,0}^2 + 2\|H\|_{k,0}\|K\|_k^{(A)} + \left(\|K\|_k^{(A)}\right)^2\right) \\ &\leq \rho_0^{\emptyset}\eta^{2k}\left(\theta + \frac{1}{2}C_2 4\rho_0^{\emptyset}\right) \leq \rho_0^{\emptyset}\eta^{2(k+1)}.\end{aligned}$$

The last inequality follows by the assumption on $\rho_0^{\emptyset}$. This finishes the proof. □

3.2 Infinite-volume flow: definition and existence

3.2.1 Definition of the infinite-volume flow

In our context, the renormalisation map T_k is most naturally defined to be a map in finite volume, since a defining property is that is should preserve the circ product under expectation. There is no analogue of this property for infinite volume. Nevertheless, there is a natural definition of a map $(H,K) \mapsto (H_+, K_+)$ which lives on $\mathbb{Z}^d$ rather than on a torus Λ, as an appropriate inductive limit of the corresponding maps on the family of all tori. The infinite-volume map has the advantage that it is defined for all scales $k \in \mathbb{N}$, with no restriction due to finite volume. In particular we can study the limit $k \to \infty$ which we use to apply an implicit function theorem to the dynamical system defined by the RG.

Let $\mathcal{B}_k(\mathbb{Z}^d)$ be the set of all k-blocks in $\mathbb{Z}^d$ and $\mathcal{P}_k(\mathbb{Z}^d)$ be the set of all finite unions of k-blocks. Since we are dealing with boxes Λ of varying side length L^N let us introduce the notation $N(\Lambda)$ for the exponent describing the side length of the box Λ.

A relevant functional $H \in M_0(\mathcal{B}_k)$ can easily be thought of as an element dependent on a block living in $\mathbb{Z}^d$ instead of Λ due to translation invariance. More precisely, given $H \in M_0(\mathcal{B}_k(\Lambda))$, we define $H^{\mathbb{Z}^d}$ on a block $B \in M_0(\mathbb{Z}^d)$ as $H(B)$ for a translation of B to the fundamental domain of Λ and suppress the index $\mathbb{Z}^d$ as well as the translation of the block in the notation.

The irrelevant part is extended as follows.

Definition 3.15. *Let $(K^\Lambda)_\Lambda$ be a family of maps which satisfy the $(\mathbb{Z}^d)$-property. For $X \in \mathcal{P}_k(\mathbb{Z}^d)$ choose Λ large enough such that $k < N(\Lambda)$ and* $\operatorname{diam}(X) \leq \frac{1}{2}\operatorname{diam}(\Lambda)$. *Then we define*

$$K^{\mathbb{Z}^d}(X) = K^\Lambda(X).$$

Here we use that $X \in \mathcal{P}_k(\mathbb{Z}^d)$ has a straight-forward analogon in $\mathcal{P}_k(\Lambda)$ if Λ is large enough which we do not record in the notation.
The definition does not depend on the choice of Λ owing to the $(\mathbb{Z}^d)$-property required for the family $(K^\Lambda)_\Lambda$.

Given $(H, K^{\mathbb{Z}^d})$ and the finite-volume maps $\left(\mathbf{S}^\Lambda\right)_\Lambda$, we define $K_+^{\mathbb{Z}^d}$ as follows.

Definition 3.16. *For $U \in \mathcal{P}_{k+1}(\mathbb{Z}^d)$ choose Λ large enough such that $k+1 < N(\Lambda)$ and* $\operatorname{diam}(U) \leq \frac{1}{2}\operatorname{diam}(\Lambda)$. *Then*

$$K_+^{\mathbb{Z}^d}\left(H, K^{\mathbb{Z}^d}\right)(U) = \mathbf{S}^\Lambda\left(H, K^\Lambda|_{U^*}\right).$$

As it is claimed in Proposition 3.10 the map $\mathbf{S}^\Lambda$ satisfies the restriction property and preserves the $(\mathbb{Z}^d)$-property. Moreover, the map $\mathbf{S}^\Lambda$ involves integration with respect to μ_{k+1} of functionals which again only depend on U^* and thus, referring to Remark 3.3, the covariance is also independent of the choice of Λ. So $K_+^{\mathbb{Z}^d}$ is well-defined.

Defining the relevant flow in infinite volume is straightforward: Fix $B \in \mathcal{B}_{k+1}(\mathbb{Z}^d)$ and $(H, K^{\mathbb{Z}^d})$. Define

$$H_+^{\mathbb{Z}^d}(B) = \mathbf{A}^q H(B) + \mathbf{B}^q K^{\mathbb{Z}^d}(B).$$

As before we can skip the index $\mathbb{Z}^d$ on H due to the following reasoning: Let $k < N(\Lambda)$ and $B \in \mathcal{B}_{k+1}(\mathbb{Z}^d)$. Then $B \in \mathcal{B}_{k+1}(\Lambda)$ and for all $b \in \mathcal{B}_k(B)$ it holds that $K^{\mathbb{Z}^d}(b) = K^\Lambda(b)$. Thus $H_{k+1}^{\mathbb{Z}^d}(B) = \sum_{b \in \mathcal{B}_k(b)} \mathcal{R}_+ H(b) + \Pi_2 \mathcal{R}_+ K^{\mathbb{Z}^d}(b) = \sum_{b \in \mathcal{B}_k(b)} \mathcal{R}_+ H(b) + \Pi_2 \mathcal{R}_+ K^\Lambda(b) = H_+^\Lambda(B)$.

We just defined the infinite-volume renormalisation map

$$\mathbf{T}_k^{\mathbb{Z}^d}(H_k, K_k^{\mathbb{Z}^d}, q) = (H_{k+1}, K_{k+1}^{\mathbb{Z}^d}).$$

Now we extend the norms.
There is no need to change the norm for the relevant variable since it does not depend at all on the size of the torus.
For the irrelevant variable let $X \in \mathcal{P}_k^c(\mathbb{Z}^d)$ and choose Λ large enough such that $\mathrm{diam}(X) \leq \frac{1}{2}\mathrm{diam}(\Lambda)$. Then $K^{\mathbb{Z}^d}(X) = K^\Lambda(X)$ and we can use the same definition as in [Buc19] for

$$\left\| K^{\mathbb{Z}^d}(X) \right\|_k = \left\| K^\Lambda(X) \right\|_k = \sup_{\varphi \in \mathcal{V}(X^*)} w_k^{-X}(\varphi) |K(X,\varphi)|_{k,X,T_\varphi}$$

(the weights w_k, $w_{k:k+1}$ and W_k do not depend on the size of the torus as long as X is small enough compared to the torus, see Remark 5.2).

3.2.2 Properties of the infinite-volume renormalisation map

Due to the definition the single step estimates for the map $(H, K^\Lambda) \mapsto (H_+, K_+^\Lambda)$ can be transferred to the infinite-volume flow.

Proposition 3.17 (Smoothness and contractivity in infinite volume)**.** *For any $\theta \in (0,1)$ there is L_0 such that for all odd integers $L \geq L_0$ (and corresponding A, h, κ) the following bounds hold independently of k and N for each $q \in B_\kappa(0)$:*

$$\|\mathbf{C}_k^q\| \leq \theta, \quad \left\| \left(\mathbf{A}_k^q\right)^{-1} \right\| \leq \frac{3}{4}, \quad \|\mathbf{B}_k^q\| \leq \frac{1}{3}.$$

The derivatives with respect to q are bounded. Moreover, there is $\rho(A)$ such that

$$\mathbf{S}_k \in C^\infty\left(U_{\rho,\kappa}, M(\mathcal{P}_{k+1}^c)\right)$$

and

$$\|D_1^{j_1} D_2^{j_2} D_3^{j_3} \mathbf{S}_k(H,K,q)(\dot{H}^{j_1}, \dot{K}^{j_2}, \dot{q}^{j_3})\|_{k+1}^{(A)} \leq C_{j_1,j_2,j_3} \|\dot{H}\|_{k,0}^{j_1} \left(\|\dot{K}\|_k^{(A)}\right)^{j_2} \|\dot{q}\|^{j_3}.$$

3.2.3 Global flow

Proposition 3.18 (Existence of the global flow)**.** *Fix $\zeta, \eta \in (0,1)$. There is L_0 such that for all odd integers $L \geq L_0$ there is A_0, h_0 and κ with the following property. Given $\epsilon > 0$ there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for each $(\mathcal{K}, \mathcal{H}, q) \in B_{\epsilon_1}(0) \times B_{\epsilon_2}(0) \times B_\kappa(0) \subset \mathbf{E} \times M_0(\mathcal{B}_0) \times \mathbb{R}_{sym}^{d\times d}$ there exists a unique global flow $\left(H_k, K_k^{\mathbb{Z}^d}\right)_{k\in\mathbb{N}}$ such that*

$$\|H_k\|_{k,0}, \left\|K_k^{\mathbb{Z}^d}\right\|_k^{(A)} \leq \epsilon\eta^k \quad \textit{for all } k \in \mathbb{N}_0,$$

with initial condition given by

$$K_0^{\mathbb{Z}^d}(X,\varphi) = e^{\mathcal{H}(X,\varphi)} \prod_{x\in X} \mathcal{K}(\nabla\varphi(x))$$

and

$$\left(H_{k+1}, K_{k+1}^{\mathbb{Z}^d}\right) = \mathbf{T}_k^{\mathbb{Z}^d}\left(H_k, K_k^{\mathbb{Z}^d}, q\right).$$

Moreover, the flow is smooth in $(\mathcal{K},\mathcal{H},q)$.

Proposition 3.18 implies that for any $(\mathcal{K},\mathcal{H}) \in B_{\epsilon_1} \times B_{\epsilon_2}$ there is $H_0(\mathcal{K},\mathcal{H})$ such that the flow (using the parameter $q(\mathcal{H})$ in the measures) converges to the fixed point of the RG. In our application we require the q-component of H_0 to correspond to the parameter $q(\mathcal{H})$ in the measure.

Proposition 3.19 (Global flow with renormalised initial condition)**.** *Let*

$$\left(H_k, K_k^{\mathbb{Z}^d}\right)_k = \left(H_k(\mathcal{K},\mathcal{H}), K_k^{\mathbb{Z}^d}(\mathcal{K},\mathcal{H})\right)_k$$

be the global flow from Proposition 3.18. There is $0 < \delta \le \epsilon_1$ *and a smooth map*

$$\hat{\mathcal{H}} : B_\delta(0) \subset \mathbf{E} \to B_{\epsilon_2}(0) \subset M_0(\mathcal{B}_0)$$

such that

$$H_0(\hat{\mathcal{H}}(\mathcal{K}),\mathcal{K}) = \hat{\mathcal{H}}(\mathcal{K})$$

and $q(\mathcal{H}\hat{(\mathcal{K})}) \subset B_\kappa(0)$ *for all* $\mathcal{K} \in B_\delta(0)$. *Moreover, the derivatives of* $\hat{\mathcal{H}}$ *can be bounded uniformly in* N.

In what follows we will prove Proposition 3.18 and Proposition 3.19. The proofs are very similar to the corresponding proofs in [Buc19]. In fact, here the arguments are slightly easier since we do not have to care about last scale maps due to the change of the finite-range decomposition, see (14). For the sake of completeness we review most of the steps.

The main ingredient is the application of the implicit function theorem. For the convenience of the reader, we state the implicit function theorem as we will use it in the following.

Theorem 3.20 (Implicit function theorem)**.** *Let* X, Y, Z *be Banachspaces, and for* $U \subset X, V \subset Y$ *open subsets, let* f *be a* C^p *Frechet differentiable map* $f : U \times V \to Z$. *If* $(x_0, y_0) \in U \times V$, $f(x_0, y_0) = 0$, *and* $y \mapsto D_2 f(x_0, y_0)y$ *isomorphism, then there exist a neighbourhood* U_0 *of* x_0 *in* U *and a Frechet differentiable* C^p *map* $g : U_0 \to V$ *such that* $g(x_0) = y_0$ *and* $f(x, g(x)) = f(x_0, y_0)$ *for all* $x \in U_0$.

We give definitions which prepare the proof of Proposition 3.18. Let us set

$$\mathcal{Z}_\infty = \Big\{ Z = (H_0, H_1, K_1, H_2, K_2, \ldots),\ H_k \in M_0(\mathcal{B}_k), K_k \in M(\mathcal{P}_k^c), \\ \|Z\|_{\mathcal{Z}_\infty} < \infty \Big\}$$

where

$$\|Z\|_{\mathcal{Z}_\infty} = \max\left(\sup_{k\geq 0} \frac{1}{\eta^k} \|H_k\|_{k,0},\ \sup_{k\geq 1} \frac{1}{\eta^k} \|K_k\|_k^{(A)} \right).$$

Clearly, $\|\cdot\|_{\mathcal{Z}_\infty}$ is a norm on $\mathcal{Z}_\infty$. We define a dynamical system on $\mathcal{Z}_\infty$ as follows:

$$\mathcal{T} : \mathbf{E} \times M(\mathcal{B}_0) \times \mathcal{Z}_\infty \to \mathcal{Z}_\infty, \quad \mathcal{T}(\mathcal{K}, \mathcal{H}, Z) = \tilde{Z},$$

where

$$\begin{aligned}
\tilde{H}_0(\mathcal{K}, \mathcal{H}, Z) &= \left(\mathbf{A}_0^{q(\mathcal{H})}\right)^{-1} \left(H_1 - \mathbf{B}_0^{q(\mathcal{H})} \hat{K}_0(\mathcal{K}, \mathcal{H})\right), \\
\tilde{H}_k(\mathcal{K}, \mathcal{H}, Z) &= \left(\mathbf{A}_k^{q(\mathcal{H})}\right)^{-1} \left(H_{k+1} - \mathbf{B}_k^{q(\mathcal{H})} K_k\right), \quad k \geq 1, \\
\tilde{K}_{k+1}(\mathcal{K}, \mathcal{H}, Z) &= \mathbf{S}_k\left(H_k, K_k, q(\mathcal{H})\right), \quad k \geq 1, \\
\tilde{K}_1(\mathcal{K}, \mathcal{H}, Z) &= \mathbf{S}_0\left(H_0, \hat{K}_0(\mathcal{K}, \mathcal{H}) q(\mathcal{H})\right),
\end{aligned}$$

with fixed initial condition

$$\hat{K}_0(\mathcal{K}, \mathcal{H})(X, \varphi) = e^{\mathcal{H}(X,\varphi)} \prod_{x\in X} \mathcal{K}(\nabla\varphi(x)),$$

and $q(\mathcal{H})$ is the projection on the coefficients of the quadratic part of $\mathcal{H}$.
One easily sees that

$$\mathcal{T}(\mathcal{K}, \mathcal{H}, Z) = Z$$

is satisfied if and only if

$$\mathbf{T}_k(H_k, K_k, q(\mathcal{H})) = (H_{k+1}, K_{k+1})$$

with $K_0 = \hat{K}_0(\mathcal{K}, \mathcal{H})$.
Proposition 3.18 is equivalent to the statement that for sufficiently small $(\mathcal{K}, \mathcal{H})$ there is a unique fixed point $\hat{Z}(\mathcal{K}, \mathcal{H})$ which depends smoothly on $(\mathcal{K}, \mathcal{H})$.

Proof of Proposition 3.18. Let L_0 (and A_0, h_0, κ) and $\rho(A)$ be as in Proposition 3.17. Let $f : \mathbf{E} \times M(\mathcal{B}_0) \times \mathcal{Z}_\infty \to \mathcal{Z}_\infty$ be the map

$$f(\mathcal{K}, \mathcal{H}, Z) = \mathcal{T}(\mathcal{K}, \mathcal{H}, Z) - Z.$$

We apply the implicit function theorem on f. The required assumptions on f are checked below.
It holds that $f(0,0,0) = 0$. To show that f is smooth we have to check that $\mathcal{T}$ is smooth.
Claim: For every triple (L, h, A) which satisfies $L \geq L_0, h \geq h_0(L), A \geq A_0(L)$ there exist constants $\rho_1 > 0, \rho_2 > 0$ such that $\mathcal{T}$ is smooth in

$$B_{\rho_1}(0) \times B_{\rho_2}(0) \times B_{\rho(A)}(0) \subset \mathbf{E} \times M(\mathcal{B}_0) \times \mathcal{Z}_\infty,$$

i.e., for all $(\mathcal{K},\mathcal{H},Z)\in B_{\rho_1}(0)\times B_{\rho_2}(0)\times B_{\rho(A)}(0)$,

$$\frac{1}{j_1!j_2!j_3!}\|D_{\mathcal{K}}^{j_1}D_{\mathcal{H}}^{j_2}D_Z^{j_3}\mathcal{T}(\mathcal{K},\mathcal{H},Z)(\dot{\mathcal{K}},\dots,\dot{\mathcal{H}},\dots,\dot{Z})\|_{\mathcal{Z}_\infty} \leq C_{j_1,j_2,j_3}(L,h,A)\|\dot{\mathcal{K}}\|_\zeta^{j_1}\|\dot{\mathcal{H}}\|_{0,0}^{j_2}\|\dot{Z}\|_{\mathcal{Z}_\infty}^{j_3}.$$

Furthermore $q(\mathcal{H})\in B_\kappa(0)$ for all $\mathcal{H}\in B_{\rho_2}(0)$.
Proof of the claim: We establish smoothness of the coordinate maps for $\tilde{H}_k$ and $\tilde{K}_k$ in a neighbourhood of the origin. Let $Z\in B_{\rho(A)}(0)$.

- Since $\tilde{K}_{k+1}(\mathcal{K},\mathcal{H},Z)=\mathbf{S}_k(H_k,K_k,q(\mathcal{H}))$ for $k\geq 1$, smoothness follows from the smoothness of $\mathbf{S}_k$ in Proposition 3.17. The proposition can be applied if

$$(H_k,K_k,q(\mathcal{H}))\in U_{\rho(A),\kappa}.$$

 Since $Z\in B_{\rho(A)}(0)$ is assumed, $(H_k,K_k)\in U_{\rho(A)}$ is satisfied. Moreover, the map $\mathcal{H}\mapsto q(\mathcal{H})$ is linear and satisfies

$$|q(\mathcal{H})|\leq \frac{C}{h^2}\|\mathcal{H}\|_{0,0}.$$

 For ρ_2 small enough we thus have $q(\mathcal{H})\in B_\kappa$. Bounds on the derivatives of $\tilde{K}_{k+1}$ are obtained as follows. Note that for $k\geq 1$ the function $\tilde{K}_{k+1}$ does not depend on $\mathcal{K}$.

$$\begin{aligned}\frac{1}{j_2!j_3!}\frac{1}{\eta^{k+1}}\|D_{\mathcal{H}}^{j_2}D_Z^{j_3}\tilde{K}_{k+1}(\mathcal{K},\mathcal{H},Z)(\dot{\mathcal{H}},\dots,\dot{Z})\|_{k+1}^{(A)} &\leq C_{j_2,j_3}\frac{1}{\eta^{k+1}}\left(\|\dot{H}_k\|_{k,0}+\|\dot{K}_k\|_k^{(A)}\right)^{j_3}C_{j_2}\|\dot{\mathcal{H}}\|_{0,0}^{j_2}\\ &\leq C_{j_2,j_3}\frac{1}{\eta}\|\dot{Z}\|_k^{j_3}C_{j_2}\|\dot{\mathcal{H}}\|_{0,0}^{j_2}.\end{aligned}$$

- The smoothness of $\tilde{H}_k$ follows similarly with the help of Proposition 3.17.
- The smoothness of the map $\tilde{K}_1(\mathcal{K},\mathcal{H},Z)=\mathbf{S}_0(\hat{K}_0(\mathcal{K},\mathcal{H}),H_0,q(\mathcal{H}))$ and bounds on the derivatives are done in detail in [Buc19]. Smoothness for $\hat{K}_0$ is proven in Lemma 4.10.2 in [Buc19], and then we apply Proposition 3.17 and chain rule.

Now we show that $Z\mapsto D_Zf(0,0)Z$ is an isomorphism. Since

$$D_Zf(0,0)Z=D_Z\mathcal{T}(0,0,0)Z-Z$$

one needs $Z\mapsto\mathcal{T}(\mathcal{K},\mathcal{H},Z)$ to be a contraction at the origin. From the definition of the maps $\tilde{H}_k$ and $\tilde{K}_k$ and from Proposition 3.12 it follows that

$$\begin{aligned}\frac{\mathrm{d}\tilde{H}_k}{\mathrm{d}H_{k+1}} &= \left(\mathbf{A}_k^0\right)^{-1} \quad \text{for } k\geq 0,\\ \frac{\mathrm{d}\tilde{H}_k}{\mathrm{d}K_k} &= -\left(\mathbf{A}_k^0\right)^{-1}\mathbf{B}_k^0 \quad \text{for } k\geq 1,\\ \frac{\mathrm{d}\tilde{K}_{k+1}}{\mathrm{d}K_k} &= \mathbf{C}_k^0 \quad \text{for } k\geq 1,\end{aligned}$$

and all other derivatives vanish. Let $Z \in \mathcal{Z}_\infty$ satisfy $\|Z\|_{\mathcal{Z}_\infty} \leq 1$. Let us denote by

$$Z' = \frac{\partial \mathcal{T}(0,0,Z)}{\partial Z}\bigg|_{Z=0} Z,$$

and denote the coordinates of Z' by H'_k and K'_k. The bounds on the operators $\left(\mathbf{A}_k^q\right)^{-1}$, $\mathbf{B}_k^q$ and $\mathbf{C}_k^q$ from Proposition 3.12 and $\|Z\|_{\mathcal{Z}_\infty}$ imply that

$$\begin{aligned}
&\|H'_0\|_{0,0} \leq \left\|\left(\mathbf{A}_0^0\right)^{-1}\right\| \eta \leq \frac{3}{4}\eta,\\
&\eta^{-k}\|H'_k\|_{k,0} \leq \eta^{-k}\left\|\left(\mathbf{A}_k^0\right)^{-1}\right\| \eta^{k+1} + \eta^{-k}\left\|\left(\mathbf{A}_k^0\right)^{-1}\right\| \|\mathbf{B}_k^0\|\eta^k \leq \frac{3}{4}\left(\eta + \frac{1}{3}\right),\ 1 \leq k,\\
&\eta^{-1}\|K'_1\| = 0,\\
&\eta^{-k}\|K'_k\| \leq \eta^{-k}\|\mathbf{C}_{k-1}^0\|\eta^{k-1} \leq \frac{\theta}{\eta},\ k \geq 2.
\end{aligned}$$

For $\eta < 1$ this implies that

$$\left\|\frac{\partial \mathcal{T}(0,0,Z)}{\partial Z}\bigg|_{Z=0}\right\| \leq \varrho < 1.$$

Thus we can apply the implicit function theorem. It follows that there exist ϵ_1 and ϵ_2 and a smooth function $\hat{Z} : B_{\epsilon_1}(\mathbf{E}) \times B_{\epsilon_2}(M_0(\mathcal{B}_0)) \to B_{\rho(A)}(\mathcal{Z}_\infty)$ such that $\hat{Z}(0,0) = 0$ and $\mathcal{T}(\mathcal{K},\mathcal{H},\hat{Z}(\mathcal{K},\mathcal{H})) = \hat{Z}(\mathcal{K},\mathcal{H})$ for all $(\mathcal{K},\mathcal{H}) \in B_{\epsilon_1}(0) \times B_{\epsilon_2}(0)$.

It remains to show that the bounds mentioned in Proposition 3.18 are satisfied. The fixed point map satisfies

$$\|\hat{Z}(\mathcal{K},\mathcal{H})\|_{\mathcal{Z}_\infty} \leq \rho(A)$$

uniformly in $(\mathcal{K},\mathcal{H}) \in B_{\epsilon_1}(0) \times B_{\epsilon_2}(0)$. The connections between the parameters ϵ_1, ϵ_2 and ϵ is clearly explained in [Buc19], Theorem 4.10.1.
From this it follows that

$$\|\hat{H}_k\|_{k,0} \text{ and } \|\hat{K}_k\|_k^{(A)} \leq \epsilon\eta^k.$$

□

Proof of Proposition 3.19. Let $\hat{Z} : B_{\epsilon_1}(0) \times B_{\epsilon_2}(0) \to B_\epsilon(0)$ be the fixed point map from Proposition 3.18. Denote by $\Pi_{H_0} : \mathcal{Z}_\infty \to M_0(\mathcal{B}_0)$ the bounded linear map that extracts the coordinate H_0 form Z.
Define

$$f(\mathcal{K},\mathcal{H}) = \Pi_{H_0}\hat{Z}(\mathcal{K},\mathcal{H}) - \mathcal{H}$$

as a map from $B_{\epsilon_1}(0) \times B_{\epsilon_2}(0) \to M_0(B_0)$. f is surely smooth. The equality

$$f(0,0) = \Pi_{H_0}\hat{Z}(0,0) = 0$$

holds since $\hat{Z}(0,0)=0$. Our next concern is to show that

$$D_2 f(0,0)\mathcal{H} = -\mathcal{H}.$$

By definition, $\mathcal{T}(0,\mathcal{H},0)=0$ for all $\mathcal{H}\in B_{\epsilon_2}(0)$. Due to the uniqueness of the fixed point, $\hat{Z}(0,\mathcal{H})=0$ for all $\mathcal{H}\in B_{\epsilon_2}(0)$. It follows that $D_2\hat{Z}(0,0)=0$ and thus $D_2\Pi_{H_0}\hat{Z}(0,0)\mathcal{H}=0$ for all $\mathcal{H}\in B_{\epsilon_2}(0)$.
In summary we obtain that $D_2 f(0,0)\mathcal{H}$ is an isomorphism. By the implicit function theorem it follows that there is δ and a smooth function $\hat{\mathcal{H}}: B_\delta(0)\subset \mathbf{E}\to B_{\epsilon_2}(0)\subset M_0(\mathcal{B}_0)$ such that $\Pi_{H_0}\hat{Z}(\mathcal{K},\hat{\mathcal{H}}(\mathcal{K}))=\hat{\mathcal{H}}(\mathcal{K})$. □

3.3 Back to finite volume and proof of Theorem 2.7

In the last section we constructed the global flow $(H_k, K_k^{\mathbb{Z}^d})_{k\in\mathbb{N}}$ and proved useful estimates. Now we transfer the properties to the finite-volume flow and deduce the proof of Theorem 2.7.
The relevant part of the flow is the same in finite and infinite volume,

$$H_k^{\mathbb{Z}^d} = H_k^\Lambda \quad \text{for } k\le N(\Lambda),$$

so the estimates of the global flow are also valid in finite volume. The irrelevant parts coincide only for polymers X with $\operatorname{diam}(X)\le \frac{1}{2}\operatorname{diam}(\Lambda)$. However, we can use the improved bound on $D\mathbf{S}_k$ in Lemma 3.11 and the single step estimate in Proposition 3.14 to prove inductively that K_k^Λ also satisfies the desired estimates.

Proposition 3.21 (Existence of the finite-volume bulk flow)**.** *Fix $\zeta,\eta\in(0,1)$. There is L_0 such that for all odd integers $L\ge L_0$ there is A_0, h_0, κ with the following property. There is $\bar{\delta}$ and $\bar{\epsilon}$ such that for a fixed Λ the finite-volume flow*

$$(H_k, K_k^\Lambda)\mapsto (H_{k+1}, K_{k+1}^\Lambda)$$

exists for all $k<N(\Lambda)$, is smooth in $\mathcal{K}\in B_{\bar{\delta}}(0)$ with bounds on the derivatives which are uniform in $N(\Lambda)$ and satisfies $(H_k, K_k^\Lambda)\in\mathbb{D}_k(\bar{\epsilon},\eta,\Lambda)$ for all $k\le N(\Lambda)$. Moreover,

$$\Pi_2(H_0(\mathcal{K})) = q(\mathcal{K})$$

and

$$K_0(\varphi,X) = K_0(\mathcal{K},H_0)(\varphi,X) = e^{H_0(\varphi,X)}\prod_{x\in X}\mathcal{K}(\nabla\varphi(x)).$$

Proof. Let (L_0, A_0, h_0, κ) be as in Proposition 3.18 and let $(H_k, K_k^{\mathbb{Z}^d})_{k\in\mathbb{N}}$ be the global flow with renormalised initial condition from Proposition 3.19. Let $\bar{\epsilon}=\min\{\rho_0^\emptyset,\epsilon\}$, where $\rho_0^\emptyset$ is the quantity from Proposition 3.14 and ϵ is as in Proposition 3.18. From the infinite-volume flow we already know that $\|H_k\|_{k,0}\in B_{\epsilon\eta^k}(0)$ for any $k\le N$ where ϵ can be made arbitrarily small by decreasing ϵ_1, in particular we can presume that $\|H_k\|_{k,0}\in B_{\bar{\epsilon}\eta^k}(0)$ for $\mathcal{K}\in B_{\bar{\delta}}(0)$ for sufficiently small $\bar{\delta}$. Thus we just have to show that $K_k^\Lambda\in B_{\bar{\epsilon}\eta^{2k}}(0)$ for $\bar{\delta}$ small enough, for any $k\le N(\Lambda)$.
We proceed by induction. For $k=0$ it holds by definition that $K_0^\Lambda = K_0^{\mathbb{Z}^d}$ and thus $K_0^\Lambda\in B_{\bar{\epsilon}}(0)$ is satisfied.

Now let $K_k^\Lambda \in B_{\bar{\epsilon}\eta^{2k}}(0)$ for $\mathcal{K} \in B_{\bar{\delta}}(0)$. To advance the induction, we apply Proposition 3.14 and obtain that also K_{k+1}^Λ satisfies the desired estimate.

Smoothness in $\mathcal{K}$ with bounds on the derivatives which are uniform in N can be proven as follows. Let $\mathcal{P}_k^1(\Lambda)$ be the set of polymers $X \in \mathcal{P}_k(\Lambda)$ such that $\operatorname{diam}(X) \leq \frac{1}{2}\operatorname{diam}(\Lambda)$. Since $K_k^\Lambda = K_k^{\mathbb{Z}^d}$ on $\mathcal{P}_k^1(\Lambda)$for any $k \leq N(\Lambda)$ and since the global flow $(H_k, K_k^{\mathbb{Z}^d})_{k\in\mathbb{N}}$ is smooth in $\mathcal{K}$, we know that for all $r \in \mathbb{N}$ there is $\tilde{C}_r > 0$ such that for all $k \leq N(\Lambda)$

$$\left\| D_{\mathcal{K}}^r H_k(\dot{\mathcal{K}}, \ldots, \dot{\mathcal{K}}) \right\|_{k,0} \leq \tilde{C}_r \|\dot{\mathcal{K}}\|_\zeta^r, \tag{19}$$

$$\left\| D_{\mathcal{K}}^r K_k^\Lambda|_{\mathcal{P}_k^1(\Lambda)}(\dot{\mathcal{K}}, \ldots, \dot{\mathcal{K}}) \right\|_{k,0} \leq \tilde{C}_r \|\dot{\mathcal{K}}\|_\zeta^r. \tag{20}$$

We will prove inductively (induction on k and r) that also $D_{\mathcal{K}}^r K_k^\Lambda|_{\mathcal{P}_k^2(\Lambda)}$ satisfies a bound with a constant $\bar{C}_r$ that is uniform in k and N,

$$\left\| D_{\mathcal{K}}^r K_k^\Lambda|_{\mathcal{P}_k^2(\Lambda)}(\dot{\mathcal{K}}, \ldots, \dot{\mathcal{K}}) \right\|_{k,0} \leq \bar{C}_r \|\dot{\mathcal{K}}\|_\zeta^r. \tag{21}$$

For all scales $k \in \{0, 1, \ldots, N-2$ we can use Lemma 3.11 to prevent accumulation of large constants. Then only two scales remain, where large constants are allowed to appear.

By Lemma 3.11 it holds

$$\left\| D_H D_K D_q \mathbf{S}_k|_{\mathcal{P}_{k+1}^2(\Lambda)}(\dot{H}, \dot{K}) \right\|_{k+1}^{(A)} \leq C_1 A^{4-\frac{x}{2}L^{N-(k+1)}} \|\dot{H}\|_{k,0} \|\dot{K}\|_k^{(A)} \|\dot{q}\|.$$

Fix $\vartheta \in (0,1)$. Choose L_0 large enough such that

$$4 - \frac{x}{2}L \leq -\vartheta.$$

Then, for all $k \leq N-2$,

$$4 - \frac{x}{2}L^{N-(k+1)} \leq 4 - \frac{x}{2}L \leq -\vartheta.$$

Now fix $\varrho \in (0,1)$ and choose A_0 large enough such that

$$C_1 A^{4-\frac{x}{2}L^{N-(k+1)}} \leq C_1 A^{-\vartheta} \leq \varrho < 1.$$

Then

$$\left\| D_H D_K D_q \mathbf{S}_k(H, K, q)(\dot{H}, \dot{K})|_{\mathcal{P}_{k+1}^2(\Lambda)} \right\|_{k+1}^{(A)} \leq \varrho \|\dot{H}\|_{k,0} \|\dot{K}\|_k^{(A)} \|\dot{q}\| \tag{22}$$

This estimate will be the main point in the argument to advance the induction. For the remaining scales $k = N-1$ and $k = N$ we will use

$$\left\| D_H D_K D_q \mathbf{S}_k(H, K, q)(\dot{H}, \dot{K})|_{\mathcal{P}_{k+1}^2(\Lambda)} \right\|_{k+1}^{(A)} \leq C_1 \|\dot{H}\|_{k,0} \|\dot{K}\|_k^{(A)} \|\dot{q}\| \tag{23}$$

where C_1 is the constant that appears in Proposition 3.10.

We start with the case $r = 1$. We use induction on k until scale $k = N - 2$. Choose

$$\bar{C}_1 \geq \max\left\{\tilde{C}_1, \frac{\varrho}{(1-\varrho)} 3\tilde{C}_1\right\}.$$

For $k = 0$ nothing is to show since both H_0 and K_0^Λ coincide with the corresponding maps in the global flow. To advance the induction, let us assume that

$$\left\| D_{\mathcal{K}} K_k^\Lambda|_{\mathcal{P}_k^2(\Lambda)} \dot{\mathcal{K}} \right\|_k^{(A)} \leq \bar{C}_1 \|\dot{\mathcal{K}}\|_\zeta.$$

Then, as long as $k < N - 2$, by (22), (19), (20) and induction hypothesis,

$$\begin{aligned}
\left\| D_{\mathcal{K}} K_{k+1}^\Lambda|_{\mathcal{P}_{k+1}^2(\Lambda)} \dot{\mathcal{K}} \right\|_{k+1}^{(A)} &= \left\| D\mathbf{S}_k(H_k, K_k, q)|_{\mathcal{P}_{k+1}^2(\Lambda)} D_{\mathcal{K}}(H_k, K_k, q) \dot{\mathcal{K}} \right\|_{k+1}^{(A)} \\
&\leq \varrho \left(\left\| D_{\mathcal{K}} H_k \dot{\mathcal{K}} \right\|_{k,0} + \left\| D_{\mathcal{K}} K_k|_{\mathcal{P}_k^1(\Lambda)} \dot{\mathcal{K}} \right\|_k^{(A)} + \left\| D_{\mathcal{K}} K_k|_{\mathcal{P}_k^2(\Lambda)} \dot{\mathcal{K}} \right\|_k^{(A)} + \left\| D_{\mathcal{K}} q \dot{\mathcal{K}} \right\| \right) \\
&\leq \varrho \left(\tilde{C}_1 + \tilde{C}_1 + \bar{C}_1 + \tilde{C}_1 \right) \|\dot{\mathcal{K}}\|_\zeta.
\end{aligned}$$

Our choice of $\bar{C}_1$ and $\varrho < 1$ implies that $\varrho\left(\tilde{C}_1 + \tilde{C}_1 + \bar{C}_1\right) \leq \bar{C}_1$ and the induction step is proven.
If $k = N - 1$ and $k = N$, then we accept accumulation of constants, and we get by (23)

$$\left\| D_{\mathcal{K}} K_{N-1}^\Lambda|_{\mathcal{P}_{N-1}^2(\Lambda)} \dot{\mathcal{K}} \right\|_{N-1}^{(A)} \leq C_1 \left(3\tilde{C}_1 + \bar{C}_1\right) \|\dot{K}\|_\zeta,$$

and

$$\left\| D_{\mathcal{K}} K_N^\Lambda|_{\mathcal{P}_N^2(\Lambda)} \dot{\mathcal{K}} \right\|_N^{(A)} \leq C_1 \left(3\tilde{C}_1 + C_1\left(3\tilde{C}_1 + \bar{C}_1\right)\right) \|\dot{K}\|_\zeta.$$

Next we consider the case $r = 2$. Again we use induction on k until scale $k = N - 2$ to show

$$\left\| D_{\mathcal{K}}^2 K_k^\Lambda|_{\mathcal{P}_k^2(\Lambda)} \dot{\mathcal{K}}^2 \right\|_k^{(A)} \leq \bar{C}_2 \|\dot{\mathcal{K}}\|_\zeta^2,$$

whith

$$\bar{C}_2 \geq \max\left\{\tilde{C}_2, \frac{1}{1-\varrho}\left(C_2(3\tilde{C}_1 + \bar{C}_1)^2 + \varrho 3\tilde{C}_2\right)\right\},$$

where C_2 is the constant which appears in the estimate $\left\|D^2\mathbf{S}_k(H, K)\right\| \leq C_2$ in Proposition 3.10.
For $k = 0$ nothing is to show.
Let us assume that the bound holds for $k < N - 2$. By chain rule we have

$$\begin{aligned}
&D_{\mathcal{K}}^2 K_{k+1}^\Lambda|_{\mathcal{P}_{k+1}^2(\Lambda)} \left(\dot{\mathcal{K}}, \dot{\mathcal{K}}\right) \\
&\quad = D^2\mathbf{S}_k(H_k, K_k, q)|_{\mathcal{P}_{k+1}^2(\Lambda)} \left(D_{\mathcal{K}}(H_k, K_k, q)\dot{\mathcal{K}}\right)^2 \\
&\quad\quad + D\mathbf{S}_k(H_k, K_k, q)|_{\mathcal{P}_{k+1}^2(\Lambda)} D_{\mathcal{K}}^2(H_k, K_k, q)\left(\dot{\mathcal{K}}, \dot{\mathcal{K}}\right)
\end{aligned}$$

and thus we can estimate with (22), (19), (20) and induction hypothesis,

$$\begin{aligned}&\left\| D^2_{\mathcal{K}} K^{\Lambda}_{k+1}|_{\mathcal{P}^2_{k+1}(\Lambda)} \left(\dot{\mathcal{K}},\dot{\mathcal{K}}\right)\right\|^{(A)}_{k+1}\\ &\qquad \le C_2\left(\tilde{C}_1+\tilde{C}_1+\bar{C}_1+\tilde{C}_1\right)^2+\varrho\left(\tilde{C}_2+\tilde{C}_2+\bar{C}_2+\tilde{C}_2\right)\|\dot{\mathcal{K}}\|^2_{\zeta}.\end{aligned}$$

The desired bound is satisfied by our choice of $\bar{C}_2$ and since $\varrho < 1$. The key point here is, that the "dangerous" bound $\bar{C}_2$ (the application of the induction hypothesis) comes with the occurrence of ϱ.
As before the scales $k = N-1$ and $k = N$ can be handled by allowing the constants to accumulate.

By a second induction in r we show that (21) holds for any r.
From the chain rule we deduce inductively that

$$D^r_{\mathcal{K}} K^{\Lambda}_{k+1}|_{\mathcal{P}^2_{k+1}(\Lambda)}\left(\dot{\mathcal{K}},\dots,\dot{\mathcal{K}}\right)$$

is a linear combination of terms

$$\left(D^i\mathbf{S}_k(H_k,K_k,q)\right)\left(D^{j_1}_{\mathcal{K}}(H_k,K_k,q)\dot{\mathcal{K}}^{j_1}\right)\dots\left(D^{j_k}_{\mathcal{K}}(H_k,K_k,q)\dot{\mathcal{K}}^{j_k}\right),$$

where $1 \le i \le r$, $j_s \ge 1$ and $\sum_{s=1}^{i} j_s = r$. For $i > 1$ this term is estimated as follows:

$$\begin{aligned}&\left\|\left(D^i\mathbf{S}_k(H_k,K_k,q)\right)\left(D^{j_1}_{\mathcal{K}}(H_k,K_k,q)\dot{\mathcal{K}}^{j_1}\right)\dots\left(D^{j_s}_{\mathcal{K}}(H_k,K_k,q)\dot{\mathcal{K}}^{j_s}\right)\right\|^{(A)}_{k+1}\\ &\qquad\le C_i\prod_{s=1}^{i}\left(3\tilde{C}_{j_s}+\bar{C}_{j_s}\right)\|\dot{\mathcal{K}}\|^{j_s}_{\zeta},\end{aligned}$$

where we used that $\|D^i\mathbf{S}_k(H,K)\| \le C_i$, (19), (20) and induction hypothesis. Note that for $i > 1$ it holds that $j_s < r$ so that only constants $\bar{C}_l$ for $l < r$ appear. The term with $i = 1$ is

$$\left(D\mathbf{S}_k(H_k,K_k,q)\right)D^r_{\mathcal{K}}(H_k,K_k,q)\dot{\mathcal{K}}^r,$$

which can be bounded for scales $k \le N-2$ with the help of (22) by

$$\left\|\left(D\mathbf{S}_k(H_k,K_k,q)\right)D^r_{\mathcal{K}}(H_k,K_k,q)\dot{\mathcal{K}}^r\right\|^{(A)}_{k+1} \le \varrho\left(3\tilde{C}_r+\bar{C}_r\right).$$

Again the "dangerous" term $\bar{C}_r$ appears with ϱ in front, so that in summary we get

$$\left\|D^r_{\mathcal{K}} K^{\Lambda}_{k+1}|_{\mathcal{P}^2_{k+1}(\Lambda)}\dot{\mathcal{K}}^r\right\|^{(A)}_{k+1} \le D+\varrho\left(3\tilde{C}_r+\bar{C}_r\right)$$

for a constant D which depends on C_i for $1 < i \le r$ and $\tilde{C}_{j_s}$ for $1 \le j_s < r$. By the choice

$$\bar{C}_r \ge \frac{1}{1-\varrho}\left(D+\varrho 3\tilde{C}_r\right)$$

we obtain (21).
Constants are allowed to accumulate for scales $k = N-1$ and $k = N$.

This finishes the proof of smoothness of the finite-volume flow in $\mathcal{K}$.

□

Proof of Theorem 2.7. Let L_0 and $\epsilon_0 = \bar{\delta}$ be as in Proposition 3.21. Let $f \in \chi_N$. The starting point is the identity

$$\mathcal{Z}_N(\mathcal{K}, f) = \int e^{(f,\varphi)} \sum_{X \subset \mathbb{T}^N} \prod_{x \in X} \mathcal{K}\left(\nabla\varphi(x)\right) \mu_1(\mathrm{d}\varphi).$$

Let us denote

$$F(\Lambda, \varphi) = \sum_{X \subset \mathbb{T}^N} \prod_{x \in X} \mathcal{K}\left(\nabla\varphi(x)\right).$$

For $\mathcal{K} \in B_{\epsilon_0}(0)$ let $q = q(\mathcal{K})$ be the quadratic part in $H_0(\mathcal{K})$ from Proposition 3.21. Then

$$\begin{aligned}\mathcal{Z}_N(\mathcal{K}, f) = \int e^{(f,\varphi)} F(\Lambda,\varphi)\mu_1(\mathrm{d}\varphi) &= \frac{Z_N^{(q)}}{Z_N^{(0)}} \int e^{(f,\varphi)} F^q(\Lambda,\varphi)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi) \\ &= \frac{Z_N^{(q)}}{Z_N^{(0)}} e^{\frac{1}{2}(f,\mathcal{C}^q f)} \int F^q(\Lambda, \varphi + \mathcal{C}^q f)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi)\end{aligned}$$

with

$$F^q(\Lambda,\varphi) = e^{\frac{1}{2}\sum_{i,j=1}^d (\nabla_i\varphi, q_{ij}\nabla_j\varphi)} F(\Lambda,\varphi).$$

Now let $\lambda = \lambda(\mathcal{K})$ be the constant part and $l(\mathcal{K})(\varphi)$ the linear part of $H_0(\mathcal{K})(\varphi)$. Since $\sum_{x\in\Lambda} l(\mathcal{K})(\varphi)(x) = 0$, and since K_0 satisfies the correct initial data, it holds that

$$F^q(\Lambda,\varphi) = e^{-\lambda L^{Nd}} e^{H_0} \circ K_0(\Lambda,\varphi),$$

and thus, by Proposition 3.21,

$$\begin{aligned}\mathcal{Z}_N(\mathcal{K}, f) &= \frac{Z_N^{(q)}}{Z_N^{(0)}} e^{\frac{1}{2}(f,\mathcal{C}^q f)} e^{-\lambda L^{Nd}} \int \left(e^{H_0} \circ K_0\right)(\Lambda, \varphi + \mathcal{C}^q f)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi) \\ &= \frac{Z_N^{(q)}}{Z_N^{(0)}} e^{\frac{1}{2}(f,\mathcal{C}^q f)} e^{-\lambda L^{Nd}} \left(e^{H_N} + K_N\right)(\Lambda, \mathcal{C}^q f).\end{aligned}$$

Let

$$Z_N^{\emptyset}(\mathcal{K}, \mathcal{C}^q f) = \left(e^{H_N(\mathcal{K})} + K_N(\mathcal{K})\right)(\mathcal{C}^q f).$$

By Proposition 3.21 the map $Z_N^{\emptyset}$ is smooth in $\mathcal{K}$. We shall have established the proof of the theorem if we show that there is a constant C such that $Z_N^{\emptyset}(\mathcal{K}, \mathcal{C}^q f)$ satisfies the estimate $\left|Z_N^{\emptyset}(\mathcal{K}, \mathcal{C}^q f) - 1\right| \le C\eta^N$ for special choices of f. First we get

$$\begin{aligned}\left|Z_N^{\emptyset}(\mathcal{K}, \mathcal{C}^q f) - 1\right| &\le \left|e^{H_N(\mathcal{C}^q f)} - 1\right| + |K_N(\mathcal{C}^q f)| \\ &\le \|K_N\|_N^{(A)} w_N^{\Lambda_N}(\mathcal{C}^q f) A^{-1} + \left|\!\left|\!\left| e^{H_N} - 1 \right|\!\right|\!\right|_N W_N^{\Lambda_N}(\mathcal{C}^q f).\end{aligned}$$

For $f = g_N - c_N$ as given in the assumptions of the theorem it holds that $f \in \chi_N$. Then one can show (see Lemma 5.1 in [Hil16] or the proof of Theorem 3.2.7 in [Buc19]) that

$$w_N^{\Lambda_N}(\mathcal{C}^q f), W_N^{\Lambda_N}(\mathcal{C}^q f) \leq C$$

for a constant which is independent of N. Moreover, by Lemma 4.7.3 in [Buc19], one can estimate

$$\left|\left|\left| e^{H_N} - 1 \right|\right|\right|_N \leq 8 \|H_N\|_{N,0}$$

and since $(H_N, K_N) \in \mathbb{D}_N(\bar{\epsilon}, \eta, \Lambda)$ by Proposition 3.21 we finally get

$$\left| Z_N^{\emptyset}(\mathcal{K}, \mathcal{C}^q f) - 1 \right| \leq C \eta^N$$

for a constant C which is independent of N. □

4 RG analysis for the observable flow

This section is dedicated to the proof of Theorem 2.11. The theorem contains a representation of the partition function with inserted observables $s\nabla_{m_a}\varphi(a)$ and $t\nabla_{m_b}\varphi(b)$. In order to work with such a singular external field we extend the analysis of Section 3. This will truly be an extension in the sense that the bulk flow needs no modification. We will show how observables can be incorporated into the analysis to obtain the pointwise asymptotic formula in Theorem 2.11.
We will follow the flow of these observables in detail and study the corresponding properties. First we extend spaces and norms in Subsection 4.1. In Subsection 4.2 the RG map is defined. We have to provide a good definition for the flow such that we can extract the Gaussian covariance $\mathcal{C}^q$. This is achieved by using second order perturbation in the map $\mathbf{A}$ instead of a first order expansion as before.
The proof of Theorem 2.11 consists of two steps. A first estimate on the covariance is proven in Subsection 4.3, a refined one in Subsection 4.4. The proof of Theorem 2.11 is then immediate from these estimates (see Subsection 4.5).

Remember that we aim to obtain a representation of

$$\mathcal{Z}_N(\mathcal{K}, f_{ab}) \text{ from (8)}, \quad \text{where} \quad f_{ab} = s\nabla^*_{m_a}\mathbb{1}_a + t\nabla^*_{m_b}\mathbb{1}_b.$$

Let (H_k, K_k) be the bulk flow of the last section. We can rewrite $\mathcal{Z}_N(\mathcal{K}, f_{ab})$ as follows:

$$\begin{aligned}\mathcal{Z}_N(\mathcal{K}, f_{ab}) &= \int e^{(\varphi, f_{ab})} \sum_{X\subset\Lambda_N} \prod_{x\in X} \mathcal{K}(\nabla\varphi(x))\mu_1(\mathrm{d}\varphi)\\ &= \frac{Z_N^{(q(\mathcal{K}))}}{Z_N^{(0)}} e^{-L^{Nd}\lambda(\mathcal{K})} \int e^{(\varphi, f_{ab})} \left(e^{H_0} \circ K_0\right)(\Lambda_N, \varphi)\mu_{\mathcal{C}^{q(\mathcal{K})}}(\mathrm{d}\varphi).\end{aligned}$$

We include (φ, f_{ab}) into the circ product and extend the maps H_0 and K_0 to

$$\begin{aligned} H_0^{\mathrm{ext}}(\varphi) &= H_0(\varphi) + s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b,\\ K_0^{\mathrm{ext}}(\varphi) &= K_0(\varphi)e^{s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b}.\end{aligned}$$

Then

$$\mathcal{Z}_N(\mathcal{K}, f_{ab}) = \frac{Z_N^{(q(\mathcal{K}))}}{Z_N^{(0)}} e^{-L^{Nd}\lambda(\mathcal{K})} \int e^{H_0^{\mathrm{ext}}} \circ K_0^{\mathrm{ext}}(\Lambda_N, \varphi)\mu_{\mathcal{C}^{q(\mathcal{K})}}(\mathrm{d}\varphi).$$

We want to follow the relevant observable flow explicitly in order to extract the Gaussian covariance $C^{q(\mathcal{K})}(a,b)$. For this purpose we extend the space of functionals of the bulk flow to these observables. We introduce extended norms, where the observable part is weighted by a carefully chosen weight $l_{\mathrm{obs},k}$, see Definition 4.1 and the motivation in Remark 4.11. In order to gain the factor $\nabla^*\nabla C^{q(\mathcal{K})}(a,b)$ in every step we define the flow

$$(H^{\mathrm{ext}}, K^{\mathrm{ext}}) \mapsto H_+^{\mathrm{ext}} = \mathbf{A}H^{\mathrm{ext}} + \mathbf{B}K^{\mathrm{ext}}$$

such that second order perturbation is reflected in the observable part of the map $\mathbf{A}$. Then the observable part of H^{ext} appears in K_+^{ext} only to third order (see Proposition 4.6) which leads to a refined single step estimate (Proposition 4.10). For the contractivity property of the extended map $(H^{\text{ext}}, K^{\text{ext}}) \mapsto K_+^{\text{ext}}$ in Proposition 4.8 the operator $\mathbf{B}$ also has to be adjusted.

Roughly speaking, the flow then satisfies estimates which result in a leading term

$$(1+S^a)(1+S^b)\nabla^*_{m_b}\nabla_{m_a}C^{q(\mathcal{K})}(a,b)$$

in the covariance, see Proposition 4.13.
In order to show that S^a, S^b do not contribute to the leading order but only at order $\frac{1}{|a-b|^{d+\nu}}$ we will have to perform an additional step: we consider the flow with just one observable in infinite volume and compare a smoothed version to the result on the scaling limit (Proposition 4.14). Finally Proposition 4.14 together with Proposition 4.13 will result in the proof of Theorem 2.11.

From this point on we use the following **change of notation**: quantities which belong to the bulk flow will get an superscript $\emptyset$. Consequently, the bulk flow becomes $\left(H_k^\emptyset, K_k^\emptyset\right)$. The superscript "ext" which was used in the motivation above will disappear in most cases, so (H_k, K_k) will denote the extended flow.

4.1 Extension of functionals, spaces and norms

4.1.1 Extended spaces

As before, let $\mathcal{N}^\emptyset = C^{r_0}(\chi_N, \mathbb{R})$ be the space of real-valued functions of fields having at least r_0 continuous derivatives. We are interested in functions not only of $\varphi \in \chi_N$ but also of s and t, but only in the dependence up to terms of the form $1, s, t, st$. We formalise this via the introduction of a quotient space, in which two functions of φ, s, t become equivalent if their formal power series in the observable fields agree to order $1, s, t, st$, as follows.
Let $\tilde{\mathcal{N}}$ be the space of real-valued functions of φ, s, t which are C^{r_0} in φ and C^∞ in s, t. Consider the elements of $\tilde{\mathcal{N}}$ whose formal power series expansion to second-order in the external fields s, t is zero. These elements form an ideal $\mathcal{I}$ in $\tilde{\mathcal{N}}$, and the quotient algebra $\mathcal{N} = \tilde{\mathcal{N}}/\mathcal{I}$ has a direct sum decomposition

$$\mathcal{N} = \mathcal{N}^\emptyset \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}.$$

The elements of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ are given by elements of $\mathcal{N}^\emptyset$ multiplied by s, by t and by st respectively. As functions of the observable field, elements of $\mathcal{N}$ are then identified with polynomials of degree at most 2. For example, we identify $e^{s\nabla\varphi(a)+t\nabla\varphi(b)}$ and $1 + s\nabla\varphi(a) + t\nabla\varphi(b) + st\nabla\varphi(a)\nabla\varphi(b)$, as both are elements of the same equivalence class in the quotient space. An element $F \in \mathcal{N}$ can be written as

$$F = F^\emptyset + sF^a + tF^b + stF^{ab},$$

where $F^\alpha \in \mathcal{N}^\emptyset$ for each $\alpha \in \{\emptyset, a, b, ab\}$. We define projections $\pi^\alpha : \mathcal{N} \to \mathcal{N}^\alpha$ by $\pi^\emptyset F = F^\emptyset$, $\pi^a F = sF^a$, $\pi^b F = tF^b$ and $\pi^{ab} F = stF^{ab}$.
Furthermore, let $\pi^* F = \pi^a F + \pi^b F + \pi^{ab} F$ be the projection to the observable part.

The class of functionals we are going to work with is

$$M^{\text{ext}}(\mathcal{P}_k, \mathcal{V}_N) = \Big\{F : \mathcal{P}_k \to \mathcal{N} \,\Big|\, F^\alpha(X) \in M(\mathcal{V}_N) \text{ for all } X \in \mathcal{P}_k \text{ and } \alpha \in \{\emptyset, a, b, ab\}, \\ \pi^\emptyset F \in M(\mathcal{P}_k), \pi^* F \text{ shift invariant and local}\Big\}.$$

Note that $\pi^* F$ is not required to be translation invariant.
As in the case of bulk functionals we have immediate generalisations to $M^{\text{ext}}(\mathcal{P}_k^c)$, $M^{\text{ext}}(\mathcal{S}_k)$ and $M^{\text{ext}}(\mathcal{B}_k)$.

We define the *coalescence scale*

$$j_{ab} = \Big\lfloor \log_L(2|a-b|) \Big\rfloor. \tag{24}$$

Since by definition

$$\frac{L^k}{2} \le |a-b| \quad \text{for all } k \le j_{ab},$$

it holds that

$$\nabla_j^* \nabla_i C_k(a,b) = 0 \quad \text{for all } k \le j_{ab}, \quad i, j \in \{1, \dots, d\}, \tag{25}$$

due to the finite-range property of the covariance decomposition.

The extended space of relevant Hamiltonians $M_0^{\text{ext}}(\mathcal{B}_k) \subset M^{\text{ext}}(\mathcal{B}_k)$ consists of all functionals of the form

$$H(B,\varphi) = H^\emptyset(B,\varphi) + sH^a(B,\varphi) + tH^b(B,\varphi) + stH^{ab}(B,\varphi)$$

where

$$H^\alpha(B,\varphi) = \mathbb{1}_{\alpha \in B}\left(\lambda^\alpha + \sum_{i=1}^d n_i^\alpha \nabla_i \varphi(\alpha)\right), \quad \lambda^\alpha \in \mathbb{R}, n^\alpha \in \mathbb{R}^d, \quad \alpha \in \{a,b\},$$
$$H^{ab}(B,\varphi) = \mathbb{1}_{a,b \in B}\, q^{ab}, \quad q^{ab} \in \mathbb{R}.$$

We also define a subspace where no constants appear in the observable part: Let

$$\mathcal{V}_k^{(0)} = \{H \in M_0^{\text{ext}}(\mathcal{B}_k) : \lambda^a = \lambda^b = q^{ab} = 0\},$$

so $H \in \mathcal{V}_k^{(0)}$ is of the form

$$H(\varphi) = H^\emptyset(\varphi) + sn^a \nabla\varphi(a)\mathbb{1}_a + tn^b \nabla\varphi(b)\mathbb{1}_b, \quad n^a, n^b \in \mathbb{R}^d.$$

Here the scalar product on $\mathbb{R}^d$ is hidden in the notation,

$$n^\alpha \nabla\varphi(\alpha) = \sum_{i=1}^d n_i^\alpha \nabla_i \varphi(\alpha).$$

4.1.2 Extended norms

Definition 4.1. *Let $h_k = 2^k h$ and $l_k = L^{-\frac{d}{2}k} h_k$. For a fixed $\eta \in (0,1)$ set $g_k = \eta^k$. Fix $\rho_0 > 0$. We define the observable weight $l_{\mathrm{obs},k}$ by*

$$l_{\mathrm{obs},k} = \rho_0 g_k 2^{-k} 4^{(k-j_{ab})_+} L^{\frac{d}{2}(k \wedge j_{ab})}.$$

The parameter ρ_0 will be determined a-posteriori in Proposition 4.12.

In the following we provide a brief motivation for the choice of $l_{\mathrm{obs},k}$. A more detailed discussion can be found in Remark 4.11.

- The sequence h_k is a scaling factor in the norm for the fields, see Subsection 3.1.2. It has the effect that in norm $s\nabla\varphi(a) \approx l_{\mathrm{obs},k} l_k$, where the growing factor 2^k appears on the right hand side in l_k. This term is eliminated by 2^{-k} in $l_{\mathrm{obs},k}$.
- $4^{(k-j_{ab})_+}$ makes a sum converging at the end of the analysis;
- $L^{\frac{d}{2}(k\wedge j_{ab})}$ gives the desired decay since $\left(L^{\frac{d}{2}j_{ab}}\right)^2 = \left(L^{j_{ab}}\right)^d \approx \frac{1}{|a-b|^d}$;
- g_k makes sure that the observables live in decreasing balls.

Note that

$$\frac{l_{\mathrm{obs},k+1}}{l_{\mathrm{obs},k}} = \begin{cases} \frac{\eta}{2} L^{d/2} & \text{if } k \le j_{ab} - 1, \\ 2\eta & \text{else .} \end{cases} \tag{26}$$

We set, for $F \in M^{\mathrm{ext}}(\mathcal{P}_k)$,

$$\left|F(X,\varphi)\right|^{\mathrm{ext}}_{k,X,T_\varphi} = \sum_{\alpha \in \{\emptyset, a, b, ab\}} \left|F^\alpha(X,\varphi)\right|_{k,X,T_\varphi} l^{|\alpha|}_{\mathrm{obs},k}$$

where, with a slight abuse of notation, $|\emptyset| = 0$, $|a| = |b| = 1$ and $|ab| = 2$. The norms $\|\cdot\|^{\mathrm{ext}}_{k,X}$, $\|\cdot\|^{\mathrm{ext}}_{k:k+1,X}$, $|||\cdot|||^{\mathrm{ext}}_{k,X}$ and $\|\cdot\|^{(A),\mathrm{ext}}_k$ on functionals $F \in M^{\mathrm{ext}}(\mathcal{P}^c_k)$ are defined as before in Section 3.1.2.

The norm on $M_0(\mathcal{B}_k)$ is extended to $M_0^{\mathrm{ext}}(\mathcal{B}_k)$ as follows. Recall that we defined elements of $M_0^{\mathrm{ext}}(\mathcal{B}_k)$ to be functionals of the form

$$H(\varphi) = H^\emptyset(\varphi) + s\mathbb{1}_a\Big(\lambda^a + \sum_i n_i^a \nabla_i\varphi(a)\Big) + t\mathbb{1}_b\Big(\lambda^b + \sum_i n_i^b \nabla_i \varphi(b)\Big) + st\mathbb{1}_{a,b}\, q^{ab}.$$

Then

$$\|H\|^{\mathrm{ext}}_{k,0} = \left\|H^\emptyset\right\|_{k,0} + l_{\mathrm{obs},k}\left(|\lambda^a| + l_k \sum_{i=1}^d |n_i^a| + \left|\lambda^b\right| + l_k \sum_{i=1}^d \left|n_i^b\right|\right) + l^2_{\mathrm{obs},k}\left|q^{ab}\right|.$$

We will use the following notation:

$$\|H^\alpha\|^\alpha_{k,0} = l_{\mathrm{obs},k}\left(|\lambda^\alpha| + l_k \sum_{i=1}^d |n_i^\alpha|\right) \quad \text{for } \alpha \in \{a,b\},$$

$$\left\|H^{ab}\right\|^{ab}_{k,0} = l^2_{\mathrm{obs},k}\left|q^{ab}\right|.$$

4.2 Extension of the renormalisation map

4.2.1 Definition of the extended map

The goal of this section is the definition and preliminary study of the extended renormalisation map

$$\begin{aligned} \mathbf{T}_k^{\mathrm{ext}} : \mathbb{R}^3 \times \mathcal{V}_k^{(0)} \times M^{\mathrm{ext}}(\mathcal{P}_k^c) &\to \mathbb{R}^3 \times \mathcal{V}_{k+1}^{(0)} \times M^{\mathrm{ext}}(\mathcal{P}_{k+1}^c), \\ (\lambda^a, \lambda^b, q^{ab}, H, K) &\mapsto (\lambda_+^a, \lambda_+^b, q_+^{ab}, H_+, K_+). \end{aligned}$$

Initially, we extend the operator $\mathbf{B}_k$:

$$\mathbf{B}_k : M^{\mathrm{ext}}\left(\mathcal{P}_k^c\right) \to M_0^{\mathrm{ext}}\left(\mathcal{B}_{k+1}\right), \quad \mathbf{B}_k K(B_+) = \sum_{B \in \mathcal{B}_k(B_+)} \Pi_k \mathcal{R}_{k+1} K(B)$$

where Π_k is the scale-dependent localisation operator

$$\Pi_k : M^{\mathrm{ext}}(\mathcal{B}_k) \to M_0^{\mathrm{ext}}(\mathcal{B}_k), \quad \Pi_k F = \Pi_2 F^{\emptyset} + \mathbb{1}_a \Pi_k^a F^a + \mathbb{1}_b \Pi_k^b F^b + \mathbb{1}_{ab} \Pi_0 F^{ab},$$

Π_k^α defined explicitly below in Section 5.1.5. Roughly speaking, for $\alpha \in \{a, b\}$,

$$\Pi_k^\alpha = \begin{cases} \Pi_1 & \text{if } k < j_{ab}, \\ \Pi_0 & \text{if } k \geq j_{ab}. \end{cases}$$

Similar to the definition of Π_2 in the bulk flow case (see Section 3.1.3),

$$\Pi_0 F(\varphi) = F(0), \quad \text{and} \quad \Pi_1^\alpha F(\varphi) = F(0) + l^\alpha(\varphi)$$

where $l^\alpha(\varphi)$ is the unique map of the form $l^\alpha(\varphi) = \sum_j n_j^\alpha \nabla_j \varphi(\alpha)$ which coincides with $DF(0)(\varphi)$ for all functions φ which are on $(B_\alpha^*)^*$ of the form

$$\varphi(x) = \sum_i m_i (x_i - \alpha_i), \quad m \in \mathbb{R}^d.$$

This implies that in $(\mathbf{B}_k K)^{ab}$ only the zeroth order polynomial remains after projection whereas in the a- and b-part of $\mathbf{B}_k K$ we follow the linear flow up to the scale j_{ab} but not further.
Note that $\mathbf{B}_k$ is a linear operator, so $(\mathbf{B}_k K)^\alpha = \mathbf{B}_k\left(K^\alpha\right)$.
Let us introduce the following notation: For $\alpha \in \{a, b\}$, we denote the constant and linear coefficients of $\mathbf{B}_k K^\alpha$ by

$$\mathbf{B}_k K^\alpha = \left(\mathbf{B}_k K^\alpha\right)^0 + \sum_{i=1}^d \left(\mathbf{B}_k K^\alpha\right)_i^1 \nabla_i \varphi(\alpha).$$

Now we can give a definition of the map

$$\mathbf{T}_k^{\mathrm{ext}} : (\lambda^a, \lambda^b, q^{ab}, H, K) \mapsto (\lambda_+^a, \lambda_+^b, q_+^{ab}, H_+, K_+).$$

Namely,

$$
\begin{aligned}
&\lambda_+^\alpha = \lambda^\alpha + (\mathbf{B}_k K^\alpha)^0, \quad \alpha \in \{a,b\},\\
&q_+^{ab} = q^{ab} + \mathbf{B}_k K^{ab} + \int H^a H^b \mathrm{d}\mu_{k+1},\\
&(H_+)^\emptyset = \left(H^\emptyset\right)_+, \quad H_+^\alpha = H^\alpha + (\mathbf{B}_k K^\alpha)^1 \nabla\varphi(\alpha), \quad \alpha \in \{a,b\},
\end{aligned}
$$

and the irrelevant K_+ is defined by

$$
K_+ = e^{-s(\mathbf{B}_k K^a)^0 - t\left(\mathbf{B}_k K^b\right)^0 - st\left(\int H^a H^b \mathrm{d}\mu_{k+1} + \mathbf{B}_k K^{ab}\right)} \mathbf{S}_k(H,K),
$$

where $\mathbf{S}_k$ is the map from the bulk flow, defined in (17). Let us denote

$$
\mathbf{S}_k^{\mathrm{ext}}(H,K) = e^{-s(\mathbf{B}_k K^a)^0 - t\left(\mathbf{B}_k K^b)\right)^0 - st\left(\int H^a H^b \mathrm{d}\mu_{k+1} + \mathbf{B}_k K^{ab}\right)} \mathbf{S}_k(H,K).
$$

Moreover, let us combine the definitions above into the map $\mathbf{A}_k$,

$$
\begin{aligned}
&\mathbf{A}_k : \mathcal{V}_k^{(0)} \to M_0^{\mathrm{ext}}(\mathcal{B}_{k+1}), \quad \mathbf{A}_k H = \mathbf{A}_k H^\emptyset + \mathbf{A}_k H^{\mathrm{obs}},\\
&\mathbf{A}_k H^\emptyset(B_+) = \sum_{B \in \mathcal{B}_k(B_+)} \Pi_2 \mathcal{R}_{k+1} H^\emptyset(B),\\
&\mathbf{A}_k H^{\mathrm{obs}} = sH^a + tH^b + st \int H^a H^b \mathrm{d}\mu_{k+1}.
\end{aligned}
$$

Remark 4.2. *We are no longer interested in the dependence of the maps on the parameter q since we will fix the bulk flow obtained in the last section - with the caveat that the choice of κ in $q \in B_\kappa(0)$ depends on the choice of L which will be chosen larger than in [Buc19].*

In the next lemma we show that the map $\mathbf{T}_k$ is well-defined, and we state first properties. A motivation for the definition of $\mathbf{T}_k$ follows afterwards in Remark 4.4.

Let $K \in M^{\mathrm{ext}}(\mathcal{P}_k)$ satisfy *field locality* if for $\alpha \in \{a,b,ab\}$ and for any $X \in \mathcal{P}_k$, $K^\alpha(X) = 0$ unless $\alpha \in X$. Here we use the notation $ab \in X$ which means $a \in X$ and $b \in X$.

Lemma 4.3. *Fix $(\lambda^a, \lambda^b, q^{ab}, H, K) \in \mathbb{R}^3 \times \mathcal{V}_k^{(0)} \times M^{\mathrm{ext}}(\mathcal{P}_k^c)$. Then the map $\mathbf{T}_k^{\mathrm{ext}}$ defined above satisfies the following properties.*

1. *$K_+ \in M^{\mathrm{ext}}(\mathcal{P}_{k+1}^c)$, and the map $\mathbf{S}_k^{\mathrm{ext}}$ satisfies the restriction property and preserves the $(\mathbb{Z}^d)$-property as well as field locality.*
2. *If K satisfies field locality, then $H_+ \in \mathcal{V}_{k+1}^{(0)}$, i.e., there are $n_+^a, n_+^b \in \mathbb{R}^d$ such that*
$$
H_+(\varphi) = H_+^\emptyset(\varphi) + sn_+^a \nabla\varphi(a) \mathbb{1}_a + tn_+^b \nabla\varphi(b) \mathbb{1}_b.
$$
3. *Let us denote $\zeta = s\lambda^a + t\lambda^b + stq^{ab}$ and $\zeta_+ s\lambda_+^a + t\lambda_+^b + stq_+^{ab}$. Then*
$$
e^{\zeta} \mathcal{R}_{k+1}\left(e^H \circ K\right) = e^{\zeta_+}\left(e^{H_+} \circ K_+\right). \tag{27}
$$

4. *If K satisfies field locality, then H_+^a is independent of H^b, K^b and K^{ab}, and the same holds for a, b interchanged.*

5. *The observable flow leaves the bulk flow unchanged, i.e.,*

$$(H_+)^\emptyset = \left(H^\emptyset\right)_+, \quad (K_+)^\emptyset = \mathbf{S}(H^\emptyset, K^\emptyset).$$

Proof. 1. The definition immediately implies that $K_+ \in M^{\mathrm{ext}}(\mathcal{P}_{k+1}^c)$ and that $\mathbf{S}^{\mathrm{ext}}$ satisfies the restriction property and preserves the $(\mathbb{Z}^d)$-property, since the map $\mathbf{S}$ fulfils the desired properties. The preservation of field locality can be verified by inspection of the definition.

2. Since K satisfies field locality, it holds that $\mathbf{B}_k K^\alpha = \mathbf{B}_k K^\alpha \mathbb{1}_\alpha$. Thus we can set

$$n_+^\alpha = n^\alpha + (\mathbf{B}_k K^\alpha)^1$$

and so $H_+ \in \mathcal{V}_{k+1}^{(0)}$.

3. The definition of the map $\mathbf{S}^{\mathrm{ext}}$ is specifically designed so that this integration property holds. Namely, use that in the bulk flow case the maps $\mathbf{A}_k, \mathbf{B}_k$ and $\mathbf{S}_k$ are made such that

$$e^{(\mathbf{A}_k H + \mathbf{B}_k K)} \circ \mathbf{S}_k(H, K) = \mathcal{R}_{k+1}(e^H \circ K).$$

Then

$$\begin{aligned}
&e^\zeta \mathcal{R}_{k+1}(e^H \circ K) \\
&\quad = e^\zeta \left[e^{(\mathbf{A}_k + \mathbf{B}_k)} \circ \mathbf{S}_k(H, K)\right] \\
&\quad = e^{\zeta + s(\mathbf{B}_k K^a)^0 + t\left(\mathbf{B}_k K^b\right)^0 + st\left(\int H^a H^b \mathrm{d}\mu_{k+1} + \mathbf{B}_k K^{ab}\right)} \\
&\qquad\quad \times \left[e^{H_+} \circ \left(e^{-s(\mathbf{B}_k K^a)^0 - t\left(\mathbf{B}_k K^b\right)^0 - st\left(\int H^a H^b \mathrm{d}\mu_{k+1} + \mathbf{B}_k K^{ab}\right)} \mathbf{S}_k(H, K)\right)\right] \\
&\quad = e^{\zeta_+}\left[e^{H_+} \circ \mathbf{S}_k^{\mathrm{ext}}(H, K)\right].
\end{aligned}$$

4. Since $H_+^a = H^a + (\mathbf{B} K^a)^1 \nabla\varphi(a)$ the statement follows straightforwardly by field locality.

5. Due to the definition of $\mathbf{A}_k$ and $\mathbf{B}_k$, for $H = H^\emptyset + \pi^* H$ and $K = K^\emptyset + \pi^* K$, it holds that $H_+^\emptyset = \mathbf{A}_k H^\emptyset + \mathbf{B}_k K^\emptyset$.

□

Remark 4.4. *We try to motivate the definition of the map $\mathbf{T}_k^{\mathrm{ext}}$.*
In principle we want to define $H_+ = \mathbf{A}_k H + \mathbf{B}_k K$ as before in the bulk flow case through extended maps $\mathbf{A}_k$ and $\mathbf{B}_k$. We perform some changes in the definition of $\mathbf{A}_k$ and $\mathbf{B}_k$.

On the one hand, we want to extract not only to linear but also to quadratic order in H, so that we can observe the Gaussian covariance. Heuristically, up to second order in H,

$$\mathcal{R}_+\left(e^H\right) \approx 1 + \mathcal{R}_+ H + \frac{1}{2}\mathcal{R}_+\left(H^2\right)$$

since

$$\mathcal{R}_+\left(e^H\right) \approx \mathcal{R}_+\left(1 + H + \frac{1}{2}H^2\right) = 1 + \mathcal{R}_+ H + \frac{1}{2}\mathcal{R}_+\left(H^2\right)$$

and

$$\begin{aligned} e^{\mathcal{R}_+ H + \frac{1}{2}\mathcal{R}_+(H^2) - \frac{1}{2}(\mathcal{R}_+ H)^2} &\approx 1 + \mathcal{R}_+ H + \frac{1}{2}\mathcal{R}_+\left(H^2\right) - \frac{1}{2}\left(\mathcal{R}_+ H\right)^2 + \frac{1}{2}\left(\mathcal{R}_+ H\right)^2 \\ &= 1 + \mathcal{R}_+ H + \frac{1}{2}\mathcal{R}_+\left(H^2\right). \end{aligned}$$

Given $H \in \mathcal{V}_k^{(0)}$ with

$$\begin{aligned} &H^{\mathrm{obs}} = sH^a + tH^b, \\ &H^a(\varphi) = n^a \nabla\varphi(a)\mathbb{1}_a, \quad H^b(\varphi) = n^b \nabla\varphi(b)\mathbb{1}_b, \quad n^a, n^b \in \mathbb{R}^d, \end{aligned}$$

then, up to first order in s, t and st,

$$\begin{aligned} &\mathcal{R}_{k+1} H^{\mathrm{obs}} + \frac{1}{2}\mathcal{R}_{k+1}\left((H^{\mathrm{obs}})^2\right) - \frac{1}{2}\left(\mathcal{R}_{k+1}(H^{\mathrm{obs}})\right)^2 \\ &\quad = sH^a + tH^b + st\int H^a H^b \mathrm{d}\mu_{k+1}. \end{aligned}$$

Since

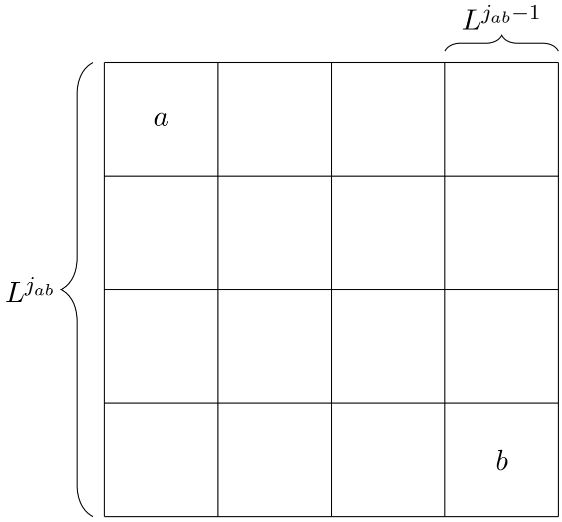
$$\int H^a H^b \mathrm{d}\mu_{k+1} = n^a n^b \nabla^* \nabla C_{k+1}(a,b),$$

we explicitly observe a part of the Gaussian covariance. This motivates the definition of the map $\mathbf{A}_k$ given above. Note that the map is no longer linear, unlike in the bulk flow case.
On the other hand, the map $\mathbf{B}_k$ extracts as much from $\mathcal{R}_+ K$ as is needed in order to have a contraction in the irrelevant part. In the case of observables it is enough to extract the linear order up to coalescence scale j_{ab} and only the constant order above.
In a last step in the definition of the map $(H, K) \mapsto H_+$ we extract constant observable parts which arise by the application of the maps $\mathbf{A}_k$ and $\mathbf{B}_k$. We put them out of the circ product into ζ_+.
The irrelevant part K_+ is defined such that (27) holds.

Let us denote by $B_a \in \mathcal{B}_k$ and $B_b \in \mathcal{B}_k$ the block at scale k which contains a and b, respectively. By definition of the coalescence scale j_{ab},

$$\frac{L^{j_{ab}-1}}{2} < \frac{L^{j_{ab}}}{2} \leq |a-b| < \frac{L^{j_{ab}+1}}{2}.$$

For simplicity let us assume that there is $B \in \mathcal{B}_{j_{ab}}$ such that $a, b \in B$, but $B_a, B_b \in \mathcal{B}_{j_{ab}-1}$ are disjoint as in the following picture. All other cases can be done similarly.



Lemma 4.5. *For initial coupling constants $\lambda_0^a = \lambda_0^b = q^{ab} = 0$, $n_0^a, n_0^b \in \mathbb{R}^d$ we obtain the following formulas for the coupling constants:*

1. $\lambda_k^\alpha = \sum_{l=0}^{k-1} (\mathbf{B}_l K_l^\alpha)^0$,
2. $q_k^{ab} = 0$ *for* $k \leq j_{ab}$ *and*

$$q_k^{ab} = \sum_{l=j_{ab}}^{k-1} \left(\mathbf{B}_l K_l^{ab} + \int H_l^a H_l^b \mathrm{d}\mu_{l+1} \right), \quad \textit{for } k > j_{ab},$$

3. $n_k^\alpha = n_0^\alpha + \sum_{l=0}^{(k-1)\wedge(j_{ab}-1)} (\mathbf{B}_l K_l^a)^1$.

Proof. These formulas follow iteratively by definition of the flow and Lemma 4.3. □

In the next statement we will deliver a precise formulation of what was described heuristically in Remark 4.4 when we motivated the definition of the map $\mathbf{A}_k$, namely that the relevant flow absorbs the irrelevant part up to second order.

Proposition 4.6. *The st-part of the second derivative in direction H of $\mathbf{S}^{\text{ext}}$ is zero:*

$$\left[D_H^2 \mathbf{S}^{\text{ext}}(0,0)(\dot{H}, \dot{H}) \right]^{ab} = 0.$$

The proof can be found in Lemma 5.29.

At this point, we have obtained that

$$\int e^{H_0} \circ K_0 \, \mathrm{d}\mu_{\mathcal{C}^q} = e^{\zeta_N} \left(e^{H_N(\varphi=0)} + K_N(\varphi = 0) \right), \quad \zeta_N = stq_N^{ab} + s\lambda_N^a + t\lambda_N^b.$$

Since

$$\sum_{k=j_{ab}}^{N} C_k(a,b) = \sum_{k=0}^{N} C_k(a,b) = C^q(a,b),$$

it holds that

$$q_N^{ab} = \left(n_0^a + S_{j_{ab}}^a\right)\left(n_0^b + S_{j_{ab}}^b\right)\nabla_{m_b}^*\nabla_{m_a} C^q(a,b) + \tilde{R}_{ab},$$
$$S_{j_{ab}}^\alpha = \sum_{l=0}^{j_{ab}-1} \left(\mathbf{B}_l K_l^\alpha\right)^1, \quad \tilde{R}_{ab} = \sum_{l=j_{ab}}^{N-1} \mathbf{B}_l K_l^{ab},$$
$$\lambda_N^\alpha = \sum_{l=0}^{N-1} \left(\mathbf{B}_l K_l^\alpha\right)^0.$$

In the following section we develop estimates on the involved quantities which lead to a first bound on the covariance in Proposition 4.13. In order to get rid of the $S_{j_{ab}}^\alpha$ in the leading term, an additional argument is needed. We implement this by considering the flow of a single observable. The refined bound on the covariance can be found in Proposition 4.14.

4.2.2 Estimates on the extended map

The separation of the bulk flow into relevant and irrelevant directions with corresponding estimates can be extended to the observable flow.
Let $U_\rho \subset \mathcal{V}_k^{(0)} \times M^{\mathrm{ext}}(\mathcal{P}_k^c)$ be the subset

$$U_\rho = \{(H,K) \in \mathcal{V}_k^{(0)} \times M^{\mathrm{ext}}(\mathcal{P}_k^c) : \|H\|_{k,0}^{\mathrm{ext}} < \rho, \|K\|_k^{(A),\mathrm{ext}} < \rho\}.$$

Proposition 4.7 (Smoothness of the extended flow)**.** *There exists a constant L_0 such that for all odd integers $L \geq L_0$ there is A_0 and h_0 with the following property. For all $A \geq A_0$ and $h \geq h_0$ there exists $\rho^* = \rho^*(A)$ such that the map $\mathbf{S}_k^{\mathrm{ext}}$ satisfies*

$$\mathbf{S}_k^{\mathrm{ext}} \in C^\infty\left(U_{\rho^*}, M^{\mathrm{ext}}(\mathcal{P}_{k+1}^c)\right).$$

For any $j_1, j_2 \in \mathbb{N}$ there is a constant $C_{j_1,j_2}^* = C_{j_1,j_2}^*(L,h,A)$ such that for any $(H,K) \in U_{\rho^*}$

$$\left\|D_H^{j_1} D_K^{j_2} \mathbf{S}_k^{\mathrm{ext}}(H,K)(\dot{H}^{j_1}, \dot{K}^{j_2})\right\|_{k+1}^{(A),\mathrm{ext}} \leq C_{j_1,j_2}^* \left(\|\dot{H}\|_{k,0}^{\mathrm{ext}}\right)^{j_1} \left(\|\dot{K}\|_k^{(A),\mathrm{ext}}\right)^{j_2}.$$

The proof of this proposition can be found in Section 5.2.

The extended flow also satisfies contraction estimates for the derivative of $\mathbf{S}_k^{\mathrm{ext}}$ at zero.

Proposition 4.8 (Contractivity of the extended flow)**.** *The first derivative of $\mathbf{S}_k^{\mathrm{ext}}$ at $(H,K) = (0,0)$ satisfies*

$$D\mathbf{S}_k^{\mathrm{ext}}(0,0)(\dot{H},\dot{K}) = \mathbf{C}_k \dot{K},$$

where

$$\mathbf{C}_k \dot{K}(U,\varphi) = \sum_{\substack{B\in\mathcal{B}_k:\\ \bar{B}=U}} (1-\Pi)\mathcal{R}_{k+1}\dot{K}(B,\varphi) + \sum_{\substack{X\in\mathcal{P}_k^c\setminus\mathcal{B}_k\\ \pi(X)=U}} \mathcal{R}_{k+1}\dot{K}(X,\varphi).$$

For any $\theta \in (0,1)$ there is L_0 such that for all odd integers $L \geq L_0$ there is A_0 and h_0 with the following property. For all $A \geq A_0$, $h \geq h_0$ the following estimate holds independent of k and N,

$$\|\mathbf{C}_k\| \leq \theta.$$

The norm on the left hand side denotes the operator norm for the map

$$\left(M^{\text{ext}}(\mathcal{P}_k^c), \|\cdot\|_k^{(A),\text{ext}}\right) \to \left(M^{\text{ext}}(\mathcal{P}_{k+1}^c), \|\cdot\|_{k+1}^{(A),\text{ext}}\right).$$

Proof. Here we only show the validity of the expression for $\mathbf{C}_k$. The contractivity is shown in Section 5.3.1, see Lemma 5.24.
We claim that

$$D\mathbf{S}_k^{\text{ext}}(0,0)(\dot{H},\dot{K}) = D\mathbf{S}_k(0,0)(\dot{H},\dot{K}).$$

Then the expression for $\mathbf{C}_k$ follows just as in the case of the bulk flow, see Proposition 3.12. The above equation holds with product rule since $\mathbf{S}_k(0,0) = 0$:

$$\begin{aligned}
&D\mathbf{S}^{\text{ext}}(0,0)(\dot{H},\dot{K})\\
&= D_H\mathbf{S}(0,0)\dot{H} + D_H\left(e^{-s(\mathbf{B}K^a)^0 - t(\mathbf{B}K^b)^0 - st(\int H^a H^b \mathrm{d}\mu_+ + \mathbf{B}K^{ab})}\right)\dot{H}\Big|_{H=K=0}\mathbf{S}(0,0)\\
&\quad + D_K\left(e^{-s(\mathbf{B}K^a)^0 - t(\mathbf{B}K^b)^0 - st(\int H^a H^b \mathrm{d}\mu_+ + \mathbf{B}K^{ab})}\right)\dot{K}\Big|_{H=K=0}\mathbf{S}(0,0)\\
&\quad + e^0 D_K\mathbf{S}(0,0)\dot{K}.
\end{aligned}$$

□

We also state bounds on the map $\mathbf{B}_k$. They are proven in Lemma 5.27.

Proposition 4.9 (Bounds on $\mathbf{B}_k$)**.** *The following bounds on the observable part of the map $\mathbf{B}_k$ hold:*

$$\begin{aligned}
\left|(\mathbf{B}_k K_k^\alpha)^1\right| &\leq l_k^{-1} l_{\text{obs},k}^{-1} \frac{A_{\mathcal{B}}}{2} \|K_k\|_k^{(A),\text{ext}}, \quad \alpha \in \{a,b\}\\
\left|(\mathbf{B}_k K_k^\alpha)^0\right| &\leq l_{\text{obs},k}^{-1} \frac{A_{\mathcal{B}}}{2} \|K_k\|_k^{(A),\text{ext}}, \quad \alpha \in \{a,b\}\\
\left|\mathbf{B}_k K_k^{ab}\right| &\leq l_{\text{obs},k}^{-2} \frac{A_{\mathcal{B}}}{2} \|K_k^{ab}\|_k^{(A),\text{ext}}.
\end{aligned}$$

As in the case of the bulk flow (see Proposition 3.14) we can combine Proposition 4.7 and 4.8 and additionally Proposition 4.6 to get a refined single step estimate.

To state it, we extend the space $\mathbb{D}_k(\rho_0, \eta, \Lambda)$ (defined in (18)) to observables. In the following definition, $C_{\mathcal{D}}$ is fixed, determined a posteriori in the proof of Proposition 4.12. Let

$$\mathbb{D}_k^{\mathrm{ext}}(\rho_0, g_k, \Lambda) \\ = \left\{ (H, K) \in \mathcal{V}_k^{(0)} \times M^{\mathrm{ext}}(\mathcal{P}_k^c)(\Lambda) : H \in B_{C_{\mathcal{D}}\rho_0 g_k}, K \in B_{\rho_0 g_k^2}, K^{ab} \in B_{\rho_0 g_k^3} \right\}.$$

Proposition 4.10 (Single step estimate for the extended flow)**.** *Fix $\eta \in (0,1)$ and $C_{\mathcal{D}} > 1$. There is L_0 such that for all odd integers $L \geq L_0$ there are A_0 and h_0 with the following property. For $A \geq A_0$ and $h \geq h_0$ there is $\rho_0 > 0$ such that if $(H, K) \in \mathbb{D}_k^{\mathrm{ext}}(\rho_0, g_k, \Lambda)$ then*

$$\|\mathbf{S}^{\mathrm{ext}}(H, K, q)\|_{k+1}^{(A),\mathrm{ext}} \leq \rho_0 g_{k+1}^2 \quad \text{and} \quad K_{k+1}^{ab} \in B_{\rho_0 g_{k+1}^3}.$$

Proof. Fix $\theta < \eta^3$. Let L_0 be large enough such that Proposition 4.7 and 4.8 can be applied. Define $C_2^* = \max(C_{2,0}^*, C_{1,1}^*, C_{0,2}^*)$ where C_{j_1,j_2}^* are the constants from Proposition 4.7. Choose ρ_0 small enough that

$$C_{\mathcal{D}}\rho_0 \leq \rho^*(A) \quad \text{and} \quad \theta + \frac{1}{2}C_2^*\rho_0 \left(C_{\mathcal{D}} + 1\right)^2 \leq \eta^2.$$

Then $(H, K) \in \mathbb{D}_k^{\mathrm{ext}}(\rho_0, g_k, \Lambda)$ implies $(H, K) \in U_{\rho^*(A)}$ so we can apply Proposition 4.7 to estimate as follows.
As in the proof of Proposition 3.14 we expand $\mathbf{S}^{\mathrm{ext}}$ around $(0,0)$ up to linear order,

$$\mathbf{S}^{\mathrm{ext}}(H, K) = \mathbf{C}K + \int_0^1 D^2\mathbf{S}^{\mathrm{ext}}(tH, tK)(H, K)^2(1-t)\mathrm{d}t.$$

Then

$$\begin{aligned} &\|\mathbf{S}^{\mathrm{ext}}(H, K)\|_{k+1}^{(A),\mathrm{ext}} \\ &\quad \leq \theta\|K\|_k^{(A),\mathrm{ext}} + \frac{1}{2}C_2^* \left(\left(\|H\|_{k,0}^{\mathrm{ext}}\right)^2 + 2\|H\|_{k,0}^{\mathrm{ext}}\|K\|_k^{(A),\mathrm{ext}} + \left(\|K\|_k^{(A),\mathrm{ext}}\right)^2 \right) \\ &\quad \leq \rho_0 g_{k+1}^2 \frac{1}{\eta^2} \left(\theta + \frac{1}{2}C_2^*\rho_0(C_{\mathcal{D}} + 1)^2 \right) \leq \rho_0 g_{k+1}^2. \end{aligned}$$

The last inequality follows by the assumption on ρ_0.

For the improved estimate on the ab-part we expand $\mathbf{S}^{\mathrm{ext}}$ up to second order and exploit the fact that we used second order perturbation in the observable flow. With Lemma 4.6 we obtain

$$\begin{aligned} K_+^{ab} = \mathbf{C}K^{ab} &+ 2\left[D_H D_K \mathbf{S}^{\mathrm{ext}}(0,0)(H, K)\right]^{ab} + \left[D_K^2 \mathbf{S}^{\mathrm{ext}}(0,0)K^2\right]^{ab} \\ &+ \left[\frac{1}{2}\int_0^1 D^3\mathbf{S}^{\mathrm{ext}}(tH, tK)(H, K)^3(1-t)^2 dt\right]^{ab}. \end{aligned}$$

Now let $C_3^* = \max(C_{3,0}^*, C_{2,1}^*, C_{1,2}^*, C_{0,3}^*)$ and choose ρ_0 such that additionally

$$\theta + C_2^* \rho_0 (2C_{\mathcal{D}} + 1) + \frac{1}{6} C_3^* \rho_0^2 (C_{\mathcal{D}} + 1)^3 \le \eta^3$$

is satisfied. Then

$$\begin{aligned}
\|K_+^{ab}\|_{k+1}^{(A),\mathrm{ext}} &\le \theta \|K^{ab}\|_k^{(A),\mathrm{ext}} + 2C_2^* \|H\|_{k,0}^{\mathrm{ext}} \|K\|_k^{(A),\mathrm{ext}} + C_2^* \left(\|K\|_k^{(A),\mathrm{ext}}\right)^2 \\
&\quad + \frac{1}{2}\frac{1}{3} C_3^* \Big(\left(\|H\|_{k,0}^{\mathrm{ext}}\right)^3 + 3\left(\|H\|_{k,0}^{\mathrm{ext}}\right)^2 \|K\|_k^{(A),\mathrm{ext}} \\
&\quad + 3\|H\|_{k,0}^{\mathrm{ext}} \left(\|K\|_k^{(A),\mathrm{ext}}\right)^2 + \left(\|K\|_k^{(A),\mathrm{ext}}\right)^3 \Big) \\
&\le \rho_0 g_{k+1}^3 \frac{1}{\eta^3} \left(\theta + C_2^* \rho_0 (2C_{\mathcal{D}} + 1) + \frac{1}{6} C_3^* \rho_0^2 (C_{\mathcal{D}} + 1)^3 \right) \le \rho_0 g_{k+1}^3
\end{aligned}$$

and the proof is finished. □

Remark 4.11. *Here we give some motivation for the choice of the weight for the extended norms and the choice of the extended localisation operator.*
The relevant part of the flow at scale $k = 0$ is

$$H_0(\varphi) = H_0^{\emptyset}(\varphi) + s n_0^a \nabla\varphi(a) \mathbb{1}_a + t n_0^b \nabla\varphi(b) \mathbb{1}_b.$$

So at least on that scale one has a linear part in the observable flow. The norm of the linear part creates the factor $l_{\mathrm{obs},k} l_l$ which has to satisfy $l_{\mathrm{obs},k} l_k \le \rho^*(A)$ for the smoothness statement on $\mathbf{S}^{\mathrm{ext}}$ and $l_{\mathrm{obs},k} l_k \le \rho_0 \eta^k$ for the single step estimate. Thus $l_{\mathrm{obs},k}$ has to include $\rho_0 \eta^k$ for ρ_0 small enough.
To get a contraction we have to put at least the constant part of the integrated irrelevant flow into the relevant flow. We aim to get an estimate

$$\sum_{k=j_{ab}}^{N} \mathbf{B} K_k^{ab} \le C \frac{1}{|a-b|^{d+\nu}}$$

Since

$$\sum_{k=j_{ab}}^{N} |\mathbf{B} K_k^{ab}| \le \sum_{k=j_{ab}}^{N} l_{\mathrm{obs},k}^{-2} \|K_k^{ab}\|_k^{(A),\mathrm{ext}} \le \sum_{k=j_{ab}}^{N} l_{\mathrm{obs},k}^{-2} \rho_0 \eta^{3k}$$

we need $L^{\frac{d}{2} j_{ab}}$ in $l_{\mathrm{obs},k}$ for $k \ge j_{ab}$.
We cannot just put the constant $L^{\frac{d}{2} j_{ab}}$ in each $l_{\mathrm{obs},k}$ for any k since then $l_{\mathrm{obs},k} l_k \le \rho^*(A)$ cannot be satisfied for the scales where the linear part exists (at least at scale 0). So we insert $L^{\frac{d}{2}(k \wedge j_{ab})}$ into the weight, until scale j_{ab}. Then we have to extract the linear part out of the irrelevant flow until coalescence to get a contraction since $\frac{l_{\mathrm{obs},k+1}}{l_{\mathrm{obs},k}}$ contains $L^{d/2}$ up to scale j_{ab} which has to be extinguished for contraction by pulling out the linear part.
Another consequence of the inserted factor $L^{\frac{d}{2} k}$ into the weight is, that now we have to kill the growing sequence h_k in l_k so that the factor 2^{-k} appears in the weight.

4.3 A first estimate on the covariance

Propositions 4.8, 4.9 and 4.10 provide us with the following intermediate result: If $(H_k, K_k) \in \mathbb{D}_k^{\mathrm{ext}}(\rho_0, g_k, \Lambda)$, then we have good control of the differences $q_+^{ab} - q^{ab}$, $\lambda_+^\alpha - \lambda^\alpha$, $n_+^\alpha - n^\alpha$ and also of the observable part of K_+ (whose bulk part had been controlled along with the bulk coupling constants already in Proposition 3.21). The following proposition links scales together via an inductive argument to conclude that (H_k, K_k) remains in $\mathbb{D}_k^{\mathrm{ext}}$ for all $k \le N$. It establishes a choice for the parameters ρ_0 and $C_{\mathcal{D}}$ as we had indicated above Proposition 4.10.

Proposition 4.12 (Existence of the observable flow)**.** *Fix $\eta \in (0,1)$. There is L_0 such that for all odd integers $L \ge L_0$ there are A_0, h_0 with the following property. For all $A \ge A_0$ and $h \ge h_0$ there is $\tilde\epsilon$ and ρ_0 (and $C_{\mathcal{D}}$) such that the flow $(\zeta_k, H_k, K_k)_{k\le N}$ satisfies*

$$(H_k, K_k) \in \mathbb{D}_k^{\mathrm{ext}}(\rho_0, g_k, \Lambda) \tag{28}$$

for any $k \le N$.

Proof. Let L_0 be large enough such that Propositions 4.7, 4.8 and 4.10 hold. The proof of (28) is by induction on k with the induction hypothesis

$$(IH)_k: \text{ for all } l \le k,\ (H_l, K_l) \in \mathbb{D}_l^{\mathrm{ext}}(\rho_0, g_l, \Lambda).$$

Note that by Proposition 3.21 the bulk flow satisfies

$$\left(H_k^\emptyset, K_k^\emptyset\right) \in B_{\bar\epsilon\eta^k}(0) \times B_{\bar\epsilon\eta^{2k}}(0)$$

if $\mathcal{K} \in B_{\bar\delta}$. Furthermore, $\bar\epsilon$ can be made arbitrarily small by decreasing $\bar\delta$.

- Base clause $k = 0$: We show that $H_0 \in B_{C_{\mathcal{D}}\rho_0}$ and $K_0 \in B_{\rho_0}$. First, we have that, for $\alpha \in \{a, b\}$,

  $$\|H_0^\alpha\|_{0,0} = l_{\mathrm{obs},0} l_0 |n_0^\alpha| = \rho_0 h$$

  and thus

  $$\|H_0\|_{0,0}^{\mathrm{ext}} = \|H_0^\emptyset\|_{0,0} + \|H_0^a\|_{0,0} + \|H_0^b\|_{0,0} \le \|H_0^\emptyset\|_{0,0} + 2\rho_0 h.$$

  Choose $\tilde\epsilon$ sufficiently small such that $\mathcal{K} \in B_{\tilde\epsilon}(0)$ implies $H_0^\emptyset \in B_{\rho_0}(0)$. Let $C_{\mathcal{D}} \ge 1 + 2h$. Then

  $$\|H_0\|_{0,0}^{\mathrm{ext}} \le C_{\mathcal{D}}\rho_0.$$

  To estimate K_0, note that

  $$\begin{aligned} K_0(\varphi) &= e^{s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b} K_0^\emptyset(\varphi) = e^{s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b} e^{H_0^\emptyset(\varphi)}\mathcal{K}(\varphi) \\ &= e^{H_0^\emptyset + s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b}\mathcal{K} \\ &= K_0^\emptyset\left(\mathcal{K}, \mathcal{H} + s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b\right). \end{aligned}$$

Choose $\tilde{\epsilon}$ small enough such that $\mathcal{K} \in B_{\tilde{\epsilon}}(0)$ implies that $\mathcal{H} + s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b$ in turn is small enough such that

$$K_0^{\emptyset}(\mathcal{K}, \mathcal{H} + s\nabla_{m_a}\varphi(a)\mathbb{1}_a + t\nabla_{m_b}\varphi(b)\mathbb{1}_b) \in B_{\rho_0}(0)$$

(use Lemma 4.10.2 in [Buc19] for verification).

- Induction hypothesis:

$$\forall\, 0 \leq l \leq k \quad (IH)_l \quad \text{holds.}$$

- Induction step:

For $\alpha \in \{a, b\}$, the following formula for the relevant observable flow holds:

$$H_{k+1}^{\alpha} = \sum_{i=1}^{d} \left(\delta_{m^\alpha}(i) + \sum_{l=0}^{k\wedge(j_{ab}-1)} (\mathbf{B}_l K_l^{\alpha})_i^1 \right) \nabla_i \varphi(\alpha).$$

We use Proposition 4.9 and the induction hypothesis to estimate

$$\begin{aligned}
\|H_{k+1}^{\alpha}\|_{k+1,0}^{\alpha} &\leq l_{\text{obs},k+1} l_{k+1} \sum_{i=1}^{d} \left(\delta_{m^\alpha}(i) + \sum_{l=0}^{k\wedge(j_{ab}-1)} \left|(BK_l^{\alpha})_i^1\right| \right) \\
&\leq \rho_0 g_{k+1} h \left(1 + \frac{A_{\mathcal{B}}}{2} d \sum_{l=0}^{k\wedge(j_{ab}-1)} l_{\text{obs},l}^{-1} l_l^{-1} \|K_l\|_l^{(A),\text{ext}} \right) \\
&\leq \rho_0 g_{k+1} h \left(1 + \frac{A_{\mathcal{B}}}{2} h^{-1} d \sum_{l=0}^{k\wedge(j_{ab}-1)} (\rho_0 g_l)^{-1} \rho_0 g_l^2 \right) \\
&\leq \rho_0 g_{k+1} h \left(1 + \frac{A_{\mathcal{B}}}{2} h^{-1} d \sum_{l=0}^{\infty} \eta^l \right).
\end{aligned}$$

Let $C_{\mathcal{D}} \geq 1 + 2h + A_{\mathcal{B}} d \frac{1}{1-\eta}$ and choose $\tilde{\epsilon}$ small enough such that $\mathcal{K} \in B_{\tilde{\epsilon}}(0)$ implies $H_k^{\emptyset} \in B_{\rho_0 \eta^k}$. Then

$$\begin{aligned}
\|H_{k+1}\|_{k+1,0}^{\text{ext}} &\leq \rho_0 \eta^{k+1} + 2\rho_0 g_{k+1} h \left(1 + \frac{A_{\mathcal{B}}}{2} d h^{-1} \frac{1}{1-\eta} \right) \\
&\leq \rho_0 g_{k+1} \left(1 + 2h + A_{\mathcal{B}} d \frac{1}{1-\eta} \right) \leq C_{\mathcal{D}} \rho_0 g_{k+1}.
\end{aligned}$$

For the estimate on K_{k+1} we use Proposition 4.10. We can apply it by induction hypothesis and we obtain exactly what we want.

□

From this result we can conclude a first estimate on the covariance.

Proposition 4.13. *Fix $\eta \in \left(0, \frac{1}{4}\right)$. Then there is L_1 such that for all odd integers $L \geq L_1$ and the corresponding A_0, h_0 there is $\tilde{\epsilon} > 0$ with the following property. For all $\mathcal{K} \in B_{\tilde{\epsilon}} \subset \mathbf{E}_\zeta$*

$$\int e^{H_0(\varphi)} \circ K_0(\varphi) \mu_{\mathcal{C}^q}(\mathrm{d}\varphi) = e^{\zeta_N} \left(e^{H_N(0)} + K_N(0)\right), \tag{29}$$

$$\textit{with} \quad \zeta_N = stq_N^{ab} + s\lambda_N^a + t\lambda_N^b$$

where (ζ_k, H_k, K_k) is the flow from Proposition 4.12. The term q_N^{ab} can be written as follows:

$$q_N^{ab} = \left(\delta_{m_a} + S_{j_{ab}}^a\right)\left(\delta_{m_b} + S_{j_{ab}}^b\right) \nabla^* \nabla C^q(a, b) + \tilde{R}_{ab}, \tag{30}$$

$$\textit{with} \quad S_{j_{ab}}^\alpha = \sum_{k=0}^{j_{ab}-1} \left(\mathbf{B} K_k^\alpha\right)^1, \tag{31}$$

and there is $\tilde{C}_1$ such that for $0 < \nu \leq -\frac{\ln(4\eta)}{\ln L}$

$$\left|\tilde{R}_{ab}\right| \leq \tilde{C}_1 \frac{1}{|a-b|^{d+\nu}}.$$

Moreover, λ_N^α is uniformly bounded in N.

Proof. The formulas (29), (30) and (31) follow from Proposition 4.12 and Lemma 4.5 with

$$\tilde{R_{ab}} = \sum_{k=j_{ab}}^{N-1} \int K_k^{ab}(\xi) \mu_{k+1}(\mathrm{d}\xi).$$

Fix $\eta < \frac{1}{4}$. Choose L_1 large enough such that $\theta < \eta^3$, and that Proposition 4.12 can be applied. Then there is $\tilde{\epsilon} > 0$ such that for all $\mathcal{K} \in B_{\tilde{\epsilon}}(0)$ we can estimate:

$$\begin{aligned}
\left|\tilde{R}_{ab}\right| &\leq \sum_{l=j_{ab}}^{N} \left|\int K_l^{ab} \mathrm{d}\mu_{l+1}\right| \leq \frac{A_{\mathcal{B}}}{2} \sum_{l=j_{ab}}^{N} l_{\mathrm{obs},l}^{-2} \left\|K_l^{ab}\right\|_l^{(A),\mathrm{ext}} \\
&\leq \frac{A_{\mathcal{B}}}{2} \rho_0^{-1} L^{-dj_{ab}} \sum_{l=j_{ab}}^{N} 4^{-2(l-j_{ab})} 4^l g_l \leq \frac{A_{\mathcal{B}}}{2} \rho_0^{-1} L^{-dj_{ab}} (4\eta)^{j_{ab}} \sum_{k=j_{ab}}^{N} 16^{-(l-j_{ab})} \\
&\leq \frac{A_{\mathcal{B}}}{2} \rho_0^{-1} L^{-dj_{ab}} (4\eta)^{j_{ab}} \sum_{k=0}^{\infty} 16^{-k} = \frac{A_{\mathcal{B}}}{2} \rho_0^{-1} L^{-dj_{ab}} (4\eta)^{j_{ab}} \frac{1}{1 - 1/16}.
\end{aligned}$$

If $\eta < \frac{1}{4}$ there is additional decay on terms of $|a-b|$ due to $(4\eta)^{j_{ab}}$:

$$(4\eta)^{j_{ab}} \leq (4\eta)^{\log_L(2|a-b|)} = (2|a-b|)^{\frac{\ln(4\eta)}{\ln L}}$$

and so

$$\left(L^{-d} 4\eta\right)^{j_{ab}} \leq (2|a-b|)^{-(d - \frac{\ln(4\eta)}{\ln L})} \leq (2|a-b|)^{-(d+\nu)}$$

for $0 < \nu \leq -\frac{\ln(4\eta)}{\ln L}$. This gives

$$\left|\tilde{R}_{ab}\right| \leq C \frac{1}{|a-b|^{d+\nu}}.$$

The uniform bound on λ_N^α follows similarly. □

4.4 A refined estimate on the covariance

Proposition 4.13 can be used to show that

$$\begin{aligned}\operatorname{Cov}_{\gamma^u_{N,\beta}}\left(\nabla_{m_a}\varphi(a),\nabla_{m_b}\varphi(b)\right) &= q_N^{ab} + \mathcal{O}\left(2^N\right) \\ &= \left(\delta_{m_a} + S^a_{j_{ab}}\right)\left(\delta_{m_b} + S^b_{j_{ab}}\right)\nabla^*\nabla C^q(a,b) + \tilde{R}_{ab} + \mathcal{O}\left(2^N\right).\end{aligned}$$

The goal of this subsection is to establish an improved formula for q_N^{ab}, namely

$$q_N^{ab} = \nabla^*_{m_b}\nabla_{m_a} C^q(a,b) + R_{ab}, \quad \text{with} \quad |R_{ab}| \leq C\frac{1}{|a-b|^{d+\nu}}.$$

This estimate follows from formula (30) if we can show that

$$\left|S^\alpha_{j_{ab}}\nabla^*\nabla C^q(a,b)\right| \leq C\frac{1}{|a-b|^{d+\nu}}.$$

We analyse the dependence of $S^\alpha_{j_{ab}}$ on j_{ab} as $j_{ab}\to\infty$ in order to obtain the desired bound. Precisely, we prove the following.

Proposition 4.14. *Under the assumptions of Proposition 4.13 there is a constant C which depends on $A_{\mathcal{B}}$, h, and η such that*

$$S^a_{j_{ab}}, S^b_{j_{ab}} \leq C\eta^{j_{ab}}.$$

We start by motivating the ideas of the proof in the following section. Afterwards, the rigorous proof follows.

4.4.1 Motivation for the proof of Proposition 4.14

Using the results in Subsection 4.3 we can construct sequences $(n_k^a, n_k^b)_{k\leq j_{ab}}$ and $(q_k^{ab})_{k\leq N}$ with a coalescence scale j_{ab} and

$$n_k^\alpha = n_0^\alpha + \sum_{l=0}^{k-1}\left(\mathbf{B}_l K_l^\alpha\right)^1 = n_0^\alpha + S_k^\alpha.$$

The goal is to analyse the dependence of $n^\alpha_{j_{ab}}$ on j_{ab} as $j_{ab}\to\infty$. The key steps in the proofs are:

- *Single observable flow:* From 4. in Lemma 4.3 we can deduce that n_k^a is independent of $(n_l^b)_{l\leq k}$. In particular we can choose $n_0^b = 0$ without changing the flow n_k^a. In this case we regard the observable at b as being absent, so the concept of coalescence becomes vacuous. We use the convention that in this case $j_{ab}=\infty$. If $n_0^b = 0$ then no b-term or ab-term arise in the flow. Nevertheless, the estimates on $\mathbf{B}K^a$ and K^a hold as before.
- *Extension to an infinite sequence:* We show that $(n_k^a)_{k\leq j_{ab}\wedge N}$ is independent of the size of the torus Λ. This allows us to extend the flow to an infinite sequence $n_k^{a,\mathbb{Z}^d}$ which can be written as

$$n_k^{a,\mathbb{Z}^d} = n_0^a + \sum_{l=0}^{k-1}\left(\mathbf{B}_l K_l^{a,\mathbb{Z}^d}\right)^1.$$

- *Convergence of the sequence:* A subtle argument shows that $n_k^{a,\mathbb{Z}^d} \to n_0^a$ and from this convergence we can deduce that

$$\sum_{k=0}^{\infty} \left(\mathbf{B}_l K_l^{a,\mathbb{Z}^d}\right)^1 = 0, \quad \text{and thus} \quad \sum_{k=0}^{m-1} \left(\mathbf{B}_l K_l^{a,\mathbb{Z}^d}\right)^1 = \mathcal{O}\left(\eta^m\right).$$

- *Back to finite volume:* If $\mathbf{B}_k K_k^{a,\mathbb{Z}^d} = \mathbf{B}_k K_k^{a,\Lambda}$ holds for any $k \leq j_{ab} - 1$, then

$$\sum_{k=0}^{j_{ab}-1} \left(\mathbf{B}_k K_k^{a,\Lambda}\right)^1 = \mathcal{O}\left(\eta^{j_{ab}}\right).$$

The computation of the limit of $n_k^{a,\mathbb{Z}^d}$ can be motivated as follows.
From the result on the scaling limit in Theorem 2.1 we know that the Gaussian covariance $\mathcal{C}^q$ arises without any correction term. We try to establish a connection to this result by smoothing the observable flow. Namely we will consider

$$\int n_0^a(\nabla(\varphi + \xi), g_N) F_0^{\emptyset}(\varphi + \xi) \mu_{\mathcal{C}^q}(\mathrm{d}\xi)$$

for a suitable chosen g_N (as in Theorem 2.1). Here, we denote $F_0^{\emptyset} = e^{H_0^{\emptyset}} \circ K_0^{\emptyset}$ the bulk flow.
On the one hand we can write this expression as

$$\sum_x g_N(x) \int n_0^a \nabla(\varphi + \xi)(x) F_0^{\emptyset}(\varphi + \xi) \mu_{\mathcal{C}^q}(\mathrm{d}\xi),$$

which can be related to the observable flow if we show that the flow of coefficients n_k^a is independent of the placing of the observable $a \in \Lambda$. Let us include the choice of a placing $a \in \Lambda$ in the notation as $Z_N(\varphi; a)$. Then

$$\begin{aligned}
\int & n_0^a(\nabla(\varphi + \xi), g_N) F_0^{\emptyset}(\varphi + \xi) \mu_{\mathcal{C}^q}(\mathrm{d}\xi) \\
&= \sum_x g_N(x) \partial_s\big|_{s=0} \ln \int e^{s n_0^a \nabla(\varphi+\xi)(x)} F_0^{\emptyset}(\varphi + \xi) \mu_{\mathcal{C}^q}(\mathrm{d}\xi) \\
&= \sum_x g_N(x) \partial_s\big|_{s=0} \ln Z_N(\varphi; x).
\end{aligned}$$

On the other hand we can relate the original expression to the bulk flow and the scaling limit as follows:

$$\begin{aligned}
\int & n_0^a(\nabla(\varphi + \xi), g_N) F_0^{\emptyset}(\varphi + \xi) \mu_{\mathcal{C}^q}(\mathrm{d}\xi) \\
&= n_0^a \partial_f \left[\int e^{(\varphi+\xi, f)} F_0^{\emptyset}(\varphi + \xi) \mu_{\mathcal{C}^q}(\mathrm{d}\xi)\right]_{f=0} (\nabla^* g_N).
\end{aligned}$$

4.4.2 Proof of Proposition 4.14

The procedure described above will be implemented here.

Single observable flow Let $(H_k, K_k)_{k\le N}$ be the flow from Section 4.3 with initial data $n_0^a = \delta_{m_a}$ and $n_0^b = \delta_{m_b}$ which satisfies $(H_k, K_k) \in \mathbb{D}_k^{\text{ext}}(\rho_0, g_k, \Lambda)$. Remember from Lemma 4.3 that n_+^a is independent of n^b, K^b and K^{ab}. Thus we can consider the initial datum $n_0^b = 0$ without changing the n_k^a-flow. Moreover, no b- and ab-term will ever arise. We summarize the properties in the following lemma.

Lemma 4.15. *1. Let $n_0^\alpha \in \{0, \delta_m\}$. For any $k \le j_{ab} \wedge N$, the term n_k^a is independent of $(n_l^b)_{l\le k}$.*

2. If $n_0^\alpha = 0$ then $H_k^\alpha = 0 = K_k^\alpha$ for all $k \le N$.

Proof. The claims follows inductively from 4. in Lemma 4.3. □

Since Propositions 4.6, 4.7, 4.8 and 4.9 hold as before also in the case $n_0^b = 0$, the following proposition can be proven by induction in the same way as Proposition 4.12.

Proposition 4.16. *Let $n_0^a = \delta_{m_a}$ and $n_0^b = 0$. By the same assumptions as in Proposition 4.13 the flow $(\zeta_k, H_k, K_k)_{k\le N}$ exists with*

$$\zeta_k = \lambda_k^a = \sum_{l=0}^{k-1} (\mathbf{B}_l K_l^a)^1 ,$$

$$H_k(\varphi) = H_k^\emptyset(\varphi) + s n_k^a \nabla\varphi(a)\mathbb{1}_a, \quad \text{where} \quad n_k^a = n_0^a + \sum_{l=0}^{k-1} (\mathbf{B}_l K_l^a)^1 ,$$

and

$$(H_k, K_k) \in \mathbb{D}_k^{\text{ext}}(\rho_0, g_k, \Lambda).$$

Extension to an infinite sequence Now we extend n_k^a to an infinite sequence. This is possible in view of the following independence property.

Lemma 4.17. *Let us denote the dependence on the torus Λ by writing $n_k^a = n_k^{a,\Lambda}$. Let Λ' be a larger torus. Then*

$$n_k^{a,\Lambda} = n_k^{a,\Lambda'} \text{ for all } k \le N(\Lambda).$$

Proof. From the N-independence of the map $\mathbf{B}$ and the $(\mathbb{Z}^d)$-property for K we can conclude that for $k < N$ and $B \in \mathcal{P}_{k+1}$

$$\mathbf{B}K_k^{a,\Lambda}(B) = \sum_{b\in\mathcal{B}_k(B)} \Pi_k^a \mathcal{R}_{k+1} K_k^{a,\Lambda}(b) = \sum_{b\in\mathcal{B}_k(B)} \Pi_k^a \mathcal{R}_{k+1} K_k^{a,\Lambda'}(b) = \mathbf{B}K_k^{a,\Lambda'}(B)$$

since for $b \in \mathcal{B}_k(B)$ and $k < N$ it holds that $\text{diam}(b) \le \frac{1}{2}\text{diam}(\Lambda)$. For $k \le N$ we thus get

$$n_k^{a,\Lambda} = n_0^a + \sum_{l=0}^{k-1} \left(\mathbf{B}K_l^{a,\Lambda}\right)^1 = n_0^a + \sum_{l=0}^{k-1} \left(\mathbf{B}K_l^{a,\Lambda'}\right)^1 = n_k^{a,\Lambda'}.$$

□

For $k \in \mathbb{N}$ define

$$n_k^{a,\mathbb{Z}^d} = n_k^{a,\Lambda}, \quad \Lambda \text{ large enough such that } k \leq N(\Lambda).$$

The sequence is well-defined by Lemma 4.17. By definition, it holds that

$$n_k^{a,\mathbb{Z}^d} = n_0^a + \sum_{l=0}^{k-1} \left(\mathbf{B} K_l^{a,\mathbb{Z}^d}\right)^1.$$

Convergence of the sequence First of all we need another generalisation. Namely, we start with an arbitrary position $x \in \Lambda$ of the observable instead of a fixed a.
Let

$$H_0(\varphi; x) = H_0^{\emptyset}(\varphi) + sn_0 \nabla\varphi(x)\mathbb{1}_x, \quad n_0 = \delta_m \text{ for } m \in 1, \ldots, d, \quad K_0 \ = e^{H_0} K_0^{\emptyset}.$$

Lemma 4.18. *The sequence $(n_k^x)_k$ is independent of the choice of the position x. More precisely, fix $x, a \in \Lambda$ and n_0 and consider two flows with initial condition $H_0^x(\varphi; x) = n_0 \nabla\varphi(x)\mathbb{1}_x$ and $H_0^a(\varphi; a) = n_0 \nabla\varphi(a)\mathbb{1}_a$ and the corresponding K_0. Then $n_k^x = n_k^a$ for all $k \leq N$.*

We can drop the superscript x from the notation by this property.

Proof. Fix $x, a \in \Lambda$. We need the following "translation property" of K^a:

$$\text{at any scale } k, \text{ for any } X \text{ and } \varphi,\ K^a(\varphi, X) = K^x(\tau_{x-a}\varphi, \tau_{x-a}X). \tag{32}$$

We will prove (32) subsequently. This property and translation invariance of the measure imply that the coefficients of $\mathbf{B}_k K_k^a$ equal the coefficients of $\mathbf{B}_k K_k^x$:

$$\begin{aligned}(\mathbf{B}K^a)^0 &= \int K^a(\varphi, B^a)\mu_+(\mathrm{d}\varphi) = \int K^x(\tau_{x-a}\varphi, B^x)\mu_+(\mathrm{d}\varphi) \\ &= \int K^x(\psi, B^x)\mu_+(\mathrm{d}\psi) = (\mathbf{B}K^x)^0.\end{aligned}$$

and, since (by (35))

$$(\mathbf{B}K^a)_i^1 = \langle \mathcal{R}_+ K^a, b_i^a \rangle_0 = D(\mathcal{R}_+ K^a)(0)(\varphi_i^a), \quad \varphi_i^a(x) = x_i - a_i,$$

we similarly get

$$\begin{aligned}(\mathbf{B}K^a)^1 &= \int DK^a(\varphi, B^a)\varphi_i^a \mu_+(\mathrm{d}\varphi) = \int D\left(K^x(\tau_{x-a}\varphi, \tau_{x-a}B^a)\right)\varphi_i^a \mu_+(\mathrm{d}\varphi) \\ &= \int DK^x(\tau_{x-a}\varphi, B^x)(\tau_{x-a}\varphi_i^a)\mu_+(\mathrm{d}\varphi) = \int DK^x(\psi, B^x)(\tau_{x-a}\varphi_i^a)\mu_+(\mathrm{d}\psi) \\ &= \int DK^x(\psi, B^x)(\varphi_i^x)\mu_+(\mathrm{d}\varphi) = (\mathbf{B}K^x)^1.\end{aligned}$$

By induction we verify that $n_k^a = n_k^x$ for any k.

It remains to prove (32). We again argue by induction. The induction hypothesis is

$$\text{For all } l \leq k \text{ and } X \in \mathcal{P}_l, \quad K_l^a(\varphi, X) = K_l^x(\tau_{x-a}\varphi, \tau_{x-a}X). \tag{33}$$

The case $k = 0$ is immediate:

$$\begin{aligned} K_0^x(\tau_{x-a}\varphi, \tau_{x-a}X) &= n_0 \nabla(\tau_{x-a}\varphi)(x) \mathbb{1}_x(\tau_{x-a}X) K_0^{\emptyset}(\tau_{x-a}\varphi, \tau_{x-a}X) \\ &= n_0 \nabla\varphi(a) \mathbb{1}_a(X) K_0^{\emptyset}(\varphi, X) = K_0^a(\varphi, X). \end{aligned}$$

For the induction step we have to show that for all $U \in \mathcal{P}_{k+1}$

$$\left[\mathbf{S}^{\text{ext}}(H, K)\right]^a(\varphi, U) = \left[\mathbf{S}^{\text{ext}}(H, K)\right]^x(\tau_{x-a}\varphi, \tau_{x-a}U).$$

From the definition of $\mathbf{S}^{\text{ext}}$ it holds that

$$\left[\mathbf{S}^{\text{ext}}(H, K)\right]^a(\varphi, U) = (\mathbf{B}K^a)^0 \mathbf{S}^{\emptyset}(H, K)(\varphi, U) + \left[\mathbf{S}(H, K)\right]^a(\varphi, U).$$

We already showed that $(\mathbf{B}K^a)^0 = (\mathbf{B}K^x)^0$ and that the bulk part satisfies translation invariance, so the first term becomes

$$(\mathbf{B}K^a)^0 \mathbf{S}^{\emptyset}(H, K)(\varphi, U) = (\mathbf{B}K^x)^0 \mathbf{S}^{\emptyset}(H, K)(\tau_{x-a}\varphi, \tau_{x-a}U).$$

For the second term, from the definition of $\mathbf{S}$, there is always one $a-$part falling on either $e^{\tilde{H}}(U \setminus X)$ or $e^{-\tilde{H}(X \setminus U)}$ or $(1 - e^{\tilde{H}})$ or $(e^H - 1)$ or K. The others form the bulk part. The bulk part always satisfies translation invariance, so we just have to check if the a-part translates correctly.
If the a-part falls on K, we use the induction hypothesis and translation invariance of the measure, and translate the sum over polymers $\sum_{X \in \mathcal{P}_k} \chi(X, U)$ into $\sum_{X \in \mathcal{P}_k} \chi(X, \tau_{x-a}U)$. The input field is then $\tau_{x-a}\varphi$.
If the a-part falls on $e^{\tilde{H}}$, we have

$$\tilde{H}^a(B_a)(\varphi) = H^a(B_a, \varphi) + \mathbf{B}K^a(B_a, \varphi) = H^x(B_x, \tau_{x-a}\varphi) + \mathbf{B}K^x(B_x, \tau_{x-a}\varphi).$$

□

Now we can prove the convergence result.

Proposition 4.19. *Given the assumptions of Proposition 4.13, the sequence*

$$\left(n_k^{a,\mathbb{Z}^d}\right)_{k \in \mathbb{N}}$$

converges to the limit $n_\infty = n_0^a$.

Proof. Convergence of the sequence is clear since by Proposition 4.9 and Proposition 4.16 we can bound the sum uniformly in N:

$$\sum_{l=0}^{k-1} \left| \left(\mathbf{B}K_l^{a,\mathbb{Z}^d}\right)^1 \right| \leq \sum_{l=0}^{k-1} \frac{A_{\mathcal{B}}}{2} h^{-1} \eta^k < \infty.$$

Let us denote the limit by n_∞.
We show $n_\infty = n_0^a$ by a limiting procedure involving the result on the scaling limit in Theorem 2.1. Let

$$Z_N(\varphi; x) = e^{\zeta_N}\left(e^{H_N(\varphi;x)} + K_N(\varphi)\right)$$

be the generating partition function at scale N, with one observable at position x. Let $g_N(x) = L^{-N\frac{d}{2}} g\left(L^{-N}x\right)$ for $g \in C_c^\infty(\mathbb{T}^d)$ satisfying $\int g = 0$ as in the assumptions of Theorem 2.1 and $h_N = \mathcal{C}^q \nabla_j g_N$, $h = \mathcal{C}^q \partial_j g$ for a fixed direction $j \in \{1, \ldots, d\}$. We will show that

$$D\left[\int n_0^a\left(\nabla(\varphi+\xi), g_N\right) F_0^\emptyset(\varphi+\xi)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi)\right]_{\varphi=0}(h_N) \xrightarrow{N\to\infty} n_\infty(\partial h, g)_{L^2(\mathbb{T}^d)} \quad \text{(A)}$$

by the statements on the observable flow. Here, the left hand side denotes the directional derivative of the term in brackets of φ in direction h_N.
On the other hand, by transforming the term into derivatives of the bulk partition function and using results there, we will show that

$$D\left[\int n_0^a\left(\nabla(\varphi+\xi), g_N\right) F_0^\emptyset(\varphi+\xi)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi)\right]_{\varphi=0}(h_N) \xrightarrow{N\to\infty} n_0^a(h, \partial^* g)_{L^2(\mathbb{T}^d)}. \quad \text{(B)}$$

By uniqueness of the limit we can conclude that $n_\infty = n_0^a$.

We start by proving (A). We can transform

$$\begin{aligned}
&D\left[\int n_0^a\left(\nabla(\varphi+\xi), g_N\right) F_0^\emptyset(\varphi+\xi)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi)\right]_{\varphi=0}(h_N)\\
&\quad= \sum_x g_N(x) D\left[\partial_s\big|_{s=0} \ln Z_N(\varphi;x)\right]_{\varphi=0}(h_N)\\
&\quad= n_N^a(\nabla h_N, g_N)\frac{e^{H_N^\emptyset(0)}}{Z_N^\emptyset(0)}\\
&\qquad\quad + \frac{1}{Z_N^\emptyset(0)}\sum_x g_N(x) D K_N^x(0)(h_N) + \frac{D Z_N^\emptyset(0)(h_N)}{\left(Z_N^\emptyset(0)\right)^2}\sum_x g_N(x) K_N^x(0).
\end{aligned}$$

By Lemma 5.16 we can estimate

$$\left|e^{H_N^\emptyset(0)} - 1\right| \leq C \left|\!\left|\!\left| e^{H_N^\emptyset(0)} - 1 \right|\!\right|\!\right|_N \leq C\|H_N^\emptyset\|_{N,0},$$

and, since $(H_N^\emptyset, K_N^\emptyset) \in \mathbb{D}_k(\rho_0, g_k, \Lambda)$, we conclude that

$$\left|e^{H_N^\emptyset(0)} - 1\right|, \left|Z_N^\emptyset(0) - 1\right| \to 0.$$

Together with the convergence result of Proposition 4.7 in [Hil16] we obtain

$$n_N^a(\nabla h_N, g_N)\frac{e^{H_N^\emptyset(0)}}{Z_N^\emptyset(0)} \quad\to\quad n_\infty(\partial h, g)_{L^2(\mathbb{T}^d)} \text{ as } N\to\infty.$$

Furthermore,

$$\left|\sum_x g_N(x)\right| = L^{Nd/2}L^{-Nd}\left|\sum_x g(L^{-N}x)\right| \le CL^{Nd/2}\left|\int g(x)\mathrm{d}x\right|,$$

but

$$|DK_N^a(0)(h_N)| \le l_{\text{obs},N}^{-1}\|K_N\|_N^{(A),\text{ext}}|\mathcal{C}^q\nabla^* g_N|_{N,\Lambda_N} \le CL^{-Nd/2}\eta^N 2^N$$

for a constant independent of N, such that

$$\left|\sum_x g_N(x)DK_N^x(0)(h_N)\right| \le C(2\eta)^N \to 0.$$

We estimate $DZ_N^\emptyset(0)(h_N)$ as in the proof of Theorem 2.7. Namely,

$$\begin{aligned}\left|DZ_N^\emptyset(0)(h_N)\right| &= \left|D\left(Z_N^\emptyset - 1\right)(0)(h_N)\right| \\ &\le \left|D\left(e^{H_N^\emptyset} - 1\right)(0)(h_N)\right| + \left|DK_N^\emptyset(0)(h_N)\right| \\ &\le C\left(\left|\left|\left|e^{H_N^\emptyset} - 1\right|\right|\right|_N |h_N|_{N,\Lambda_N} + \left\|K_N^\emptyset\right\|_N^{(A)}|h_N|_{N,\Lambda_N}\right).\end{aligned}$$

By Lemma 5.16 it holds that

$$\left|\left|\left|e^{H_N^\emptyset} - 1\right|\right|\right|_N \le 8\left\|H_N^\emptyset\right\|_{N,0}.$$

Moreover, similar to Lemma 5.2 from [Hil16] one can show that

$$|h_N|_{N,\Lambda_N} = |\mathcal{C}^q\nabla^* g_N|_{N,\Lambda_N} \le C$$

for a constant C which is independent of N. With $(H_N^\emptyset, K_N^\emptyset) \in \mathbb{D}(\rho_0, g_k, \Lambda)$ it follows that

$$\left|DZ_N^\emptyset(0)(h_N)\right| \le C\left(\|H_N^\emptyset\|_{N,0} + \|K_N^\emptyset\|_N^{(A)}\right) \le C\eta^N.$$

Thus

$$\frac{\left|DZ_N^\emptyset(0)(h_N)\right|}{\left|Z_N^\emptyset(0)\right|^2} \to 0,$$

and

$$\left|\sum_x g_N(x)K_N^x(0)\right| \le CL^{Nd/2}L^{-Nd/2}(2\eta)^N \to 0.$$

Now we prove (B). We start with the following transformations:

$$\begin{aligned}
&D\left[\int n_0^a\left(\nabla(\varphi+\xi), g_N\right) F_0^{\emptyset}(\varphi+\xi)\mu_{\mathcal{C}^q}(\mathrm{d}\varphi)\right]_{\varphi=0}(h_N)\\
&\quad= n_0^a\partial_f\left[D\left[\int e^{(\varphi+\xi,f)}F_0^{\emptyset}(\varphi+\xi)\mu_{\mathcal{C}^q}(\mathrm{d}\xi)\right]_{\varphi=0}(h_N)\right]_{f=0}(\nabla^* g_N)\\
&\quad= n_0^a\partial_f\left[e^{\frac{1}{2}(f,\mathcal{C}^q f)}D\left[e^{(\varphi,f)}Z_N^{\emptyset}(\mathcal{C}^q f+\varphi)\right]_{\varphi=0}(h_N)\right]_{f=0}(\nabla^* g_N)\\
&\quad= n_0^a\partial_f\left[e^{\frac{1}{2}(f,\mathcal{C}^q f)}(h_N,f)Z_N^{\emptyset}(\mathcal{C}^q f)+DZ_N^{\emptyset}(\mathcal{C}^q f)(h_N)\right]_{f=0}(\nabla^* g_N)\\
&\quad= n_0^a\left[(h_N,\nabla^* g_N)Z_N^{\emptyset}(0)+D^2Z_N^{\emptyset}(0)(h_N)(\mathcal{C}^q\nabla^* g_N)\right].
\end{aligned}$$

The first term converges

$$(h_N,\nabla^* g_N)Z_N^{\emptyset}(0)\to(h,\partial^* g)_{L^2(\mathbb{T}^d)}$$

as $N\to\infty$, due to $\left|Z_N^{\emptyset}(0)-1\right|\to 0$ and the convergence result of Proposition 4.7 from [Hil16]. The second term tends to zero by the following considerations which resemble the arguments in the proof of Theorem 2.7 and (A). It holds that

$$D^2Z_N^{\emptyset}(0)(h_N,\mathcal{C}^q\nabla^* g_N)=D^2\left(Z_N^{\emptyset}-1\right)(0)(h_N,\mathcal{C}^q\nabla^* g_N),$$

and thus

$$\begin{aligned}
&\left|D^2Z_N^{\emptyset}(0)(h_N,\mathcal{C}^q\nabla^* g_N)\right|\\
&\quad\le\left|D^2\left(e^{H_N^{\emptyset}}-1\right)(0)(h_N,\mathcal{C}^q\nabla^* g_N)\right|+\left|D^2K_N^{\emptyset}(0)(h_N,\mathcal{C}^q\nabla^* g_N)\right|\\
&\quad\le C\left(\left|\left|\left|e^{H_N^{\emptyset}}-1\right|\right|\right|_N+\left\|K_N^{\emptyset}\right\|_N^{(A)}\right)|h_N|_{N,\Lambda_N}|\mathcal{C}^q\nabla^* g_N|_{N,\Lambda_N}.
\end{aligned}$$

As before it holds that

$$|\mathcal{C}^q\nabla^* g_N|_{N,\Lambda_N},|h_N|_{N,\Lambda_N}\le C$$

for a constant C which is independent of N, and

$$\left|\left|\left|e^{H_N^{\emptyset}}-1\right|\right|\right|_N\le C\left\|H_N^{\emptyset}\right\|_{N,0}.$$

Together with $(H_N^{\emptyset},K_N^{\emptyset})\in\mathbb{D}_k(\rho_0,g_k,\Lambda)$ we conclude that

$$\left|D^2Z_N^{\emptyset}(0)(h_N,\mathcal{C}^q\nabla^* g_N)\right|\le C\eta^k\to 0.$$

This proves the claim.

□

Back to finite volume Now we can prove Proposition 4.14.

Proof of Proposition 4.14. We conclude from Proposition 4.19 and the construction of the flow that

$$n_\infty = n_0^a + \sum_{k=0}^{\infty} \left(\mathbf{B}_k K_k^{a,\mathbb{Z}^d}\right)^1 = n_0^a \quad \Rightarrow \quad \sum_{k=0}^{\infty} \left(\mathbf{B}_k K_k^{a,\mathbb{Z}^d}\right)^1 = 0.$$

Using Proposition 4.16 we can estimate

$$\left|\sum_{k=0}^{m-1} \left(\mathbf{B}_k K_k^{a,\mathbb{Z}^d}\right)^1\right| = \left|\sum_{k=m}^{\infty} \left(\mathbf{B}_k K_k^{a,\mathbb{Z}^d}\right)^1\right| \leq \sum_{k=m}^{\infty} \frac{A_{\mathcal{B}}}{2} h^{-1} \eta^k = \frac{A_{\mathcal{B}}}{2} h^{-1} \frac{1}{1-\eta} \eta^m.$$

By definition of the infinite sequence, the $(\mathbb{Z}^d)$-property and the local dependence of the relevant flow it holds that for all $k \leq j_{ab} - 1$

$$\mathbf{B}_k K_k^{a,\mathbb{Z}^d} = \mathbf{B}_k K_k^{a,\Lambda}.$$

Thus

$$S_{j_{ab}}^a = \sum_{k=0}^{j_{ab}-1} \left(\mathbf{B}_k K_k^a\right)^1 = \mathcal{O}\left(\eta^{j_{ab}}\right).$$

□

4.5 Proof of Theorem 2.11

The proof of Theorem 2.11 consists of two steps. By direct observation of the flow we get the estimate for q_N^{ab} in Proposition 4.13. In a second step Proposition 4.14 is used to get a refined leading term.

Proof of Theorem 2.11. Let L_1 and ϵ_1 be as in Proposition 4.12 with $\eta < \frac{1}{4}$. Then, by Proposition 4.13, (12) holds with the estimates on $Z_N^{\mathrm{ext}}(\mathcal{K}, 0)$ and on λ_N^α and with

$$q_N^{ab} = \left(\delta_{m_a} + S_{j_{ab}}^a\right)\left(\delta_{m_b} + S_{j_{ab}}^b\right) \nabla^* \nabla C^{q(\mathcal{K})}(a,b) + \tilde{R}_{ab}.$$

Proposition 4.14 gives the improved estimate as can be found in the statement of Theorem 2.11. □

5 Proofs of extensions and intermediate steps

Note that in this section any dependencies on q are omitted since $q \in B_\kappa(0)$ is fixed with κ depending on ζ, η and L in Proposition 3.21. As an exception we note the dependency explicitely in Lemma 5.1 since this is the place where the parameter κ is determined in dependence on L.
In this whole section R is a parameter which depends only on d.

5.1 Properties of the norms

In this subsection we follow closely the presentation in [Buc19], Section 4.6. Arguments from [Buc19] which can be applied without any change to the extended setting will be omitted in proofs.

5.1.1 Properties of the weights

For the sake of completeness we review Proposition 4.5.1 from [Buc19]. The last scale weights ($k = N$) differ from [Buc19] due to the modified definition of the last scale covariance (see (13)). However, this does not change the properties of the weights as stated in the following lemma.

Lemma 5.1. *Let $L \geq 2^{d+3} + 16R$. The weight functions w_k, $w_{k:k+1}$ and W_k are well-defined and satisfy the following properties:*

1. *For any $Y \subset X \in \mathcal{P}_k$, $0 \leq k \leq N$, and $\varphi \in \mathcal{V}_N$*
$$w_k^Y(\varphi) \leq w_k^X(\varphi) \quad \text{and} \quad w_{k:k+1}^Y(\varphi) \leq w_{k:k+1}^X(\varphi).$$
2. *For any strictly disjoint polymers $X, Y \in \mathcal{P}_k$, $0 \leq k \leq N$, and $\varphi \in \mathcal{V}_N$*
$$w_k^{X\cup Y}(\varphi) = w_k^X(\varphi) w_k^Y(\varphi).$$
3. *For any polymers $X, Y \in \mathcal{P}_k$ such that $\operatorname{dist}(X,Y) \geq \frac{3}{4}L^{k+1}$, $0 \leq k \leq N$, and $\varphi \in \mathcal{V}_N$*
$$w_{k:k+1}^{X\cup Y}(\varphi) = w_{k:k+1}^X(\varphi) w_{k:k+1}^Y(\varphi).$$
4. *For any disjoint polymers $X, Y \in \mathcal{P}_k$, $0 \leq k \leq N$, and $\varphi \in \mathcal{V}_N$*
$$W_k^{X\cup Y}(\varphi) = W_k^X(\varphi) W_k^Y(\varphi).$$

Moreover, there is a constant $h_0 = h_0(L,\zeta)$ such that for all $h \geq h_0$ the weight functions satisfy the following properties:

5. *For any disjoint polymers $X, Y \in \mathcal{P}_k$ and $U = \pi(X) \in \mathcal{P}_{k+1}$, $0 \leq k \leq N-1$, and $\varphi \in \mathcal{V}_N$*
$$w_{k+1}^U(\varphi) \geq w_{k:k+1}^X(\varphi)\left(W_k^{U^+}(\varphi)\right)^2.$$

6. *For all* $0 \le k \le N-1$, $X \in \mathcal{P}_{k+1}$ *and* $\varphi \in \mathcal{V}_N$,

$$e^{\frac{|\varphi|^2_{k+1,X}}{2}} w^X_{k:k+1}(\varphi) \le w^X_{k+1}(\varphi).$$

Lastly, there exists a constant $\kappa = \kappa(L,\zeta)$ *with the following properties:*

7. *There is a constant* $A_{\mathcal{P}}$ *such that for* $q \in B_\kappa(0)$, $\rho = (1+\frac{\zeta}{4})^{1/3} - 1$, $Y \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$

$$\left(\int_{\chi_N} \left(w_k^X(\varphi+\xi) \right)^{1+\rho} \mu_{k+1}(\mathrm{d}\xi) \right)^{\frac{1}{1+\rho}} \le \left(\frac{A_{\mathcal{P}}}{2} \right)^{|X|_k} w^X_{k:k+1}(\varphi).$$

8. *There is a constant* $A_{\mathcal{B}}$ *independent of* L *such that for* $q \in B_\kappa$, $\rho = (1+\frac{\zeta}{4})^{1/3} - 1$, $B \in \mathcal{B}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$

$$\left(\int_{\chi_N} \left(w_k^B(\varphi+\xi) \right)^{1+\rho} \mu_{k+1}(\mathrm{d}\xi) \right)^{\frac{1}{1+\rho}} \le \frac{A_{\mathcal{B}}}{2} w^B_{k:k+1}(\varphi).$$

Remark 5.2. *The weights* w_k *and* $w_{k:k+1}$ *are independent of the size of the torus as long as the input polymer is small enough. This can be seen when examining the construction of the weights. The weights essentially arise as follows: Take the local quadratic form from [AKM16], integrate against the covariance of the finite-range decomposition (which is independent of the size of the torus by Remark 3.3) and add explicit local perturbing terms. These steps are independent of the size of the torus as long as the input-polymer is small enough compared to the torus.*
The weights W_k *are given explicitly and obviously local.*

5.1.2 Pointwise properties of the norms

The following lemma is an extension to observables of Lemma 4.6.1 from [Buc19].

Lemma 5.3. *Assume that* $F, G \in \mathcal{N}$, $X \in \mathcal{P}_k$ *and* $F(\varphi)$ *and* $G(\varphi)$ *depend only on* $\varphi|_{X^*}$. *Assume furthermore that* $F(\varphi+\psi) = F(\varphi), G(\varphi+\psi) = G(\varphi)$ *if* $\psi|_{X^*}$ *is constant. Then*

$$|FG|^{\mathrm{ext}}_{k,X,T_\varphi} \le |F|^{\mathrm{ext}}_{k,X,T_\varphi} |G|^{\mathrm{ext}}_{k,X,T_\varphi}$$

and, for $X \in \mathcal{P}_k$ *and* $\alpha \in \{\emptyset, a, b, ab\}$,

$$|F^\alpha|_{k+1,X,T_\varphi} \le (1+|\varphi|_{k+1,X})^3 \left(|F^\alpha|_{k+1,X,T_0} + 16 L^{-\frac{3}{2}d} \sup_{0 \le t \le 1} |F^\alpha|_{k,X,T_{t\varphi}} \right).$$

Proof. We write the extended norm as sum $|K|^{\mathrm{ext}}_{k,X,T_\varphi} = \sum_\alpha l^{|\alpha|}_{\mathrm{obs},k} |K^\alpha|_{k,X,T_\varphi}$ and apply Lemma 4.6.1. from [Buc19] on each (bulk) norm $|F^\alpha G^\beta|_{k,X,T_\varphi}$. This yields that

$$l^{|\alpha|+|\beta|}_{\mathrm{obs},k} \left| F^\alpha G^\beta \right|_{k,X,T_\varphi} \le \left(l^{|\alpha|}_{\mathrm{obs},k} |F^\alpha|_{k,X,T_\varphi} \right) \left(l^{|\beta|}_{\mathrm{obs},k} |G^\beta|_{k,X,T_\varphi} \right).$$

Thus

$$|FG|_{k,X,T_\varphi} = \sum_\alpha l^\alpha_{\text{obs}} |(FG)^\alpha|_{k,X,T_\varphi} \leq \left(\sum_\alpha l^{|\alpha|}_{\text{obs},k} |F^\alpha|_{T_\varphi} \right) \left(\sum_\alpha l^{|\alpha|}_{\text{obs},k} |G^\alpha|_{T_\varphi} \right)$$

since

$$\begin{aligned} FG = F^\emptyset G^\emptyset &+ s \left(F^a G^\emptyset + F^\emptyset G^a \right) + t \left(F^b G^\emptyset + F^\emptyset G^b \right) \\ &+ st \left(F^\emptyset G^{ab} + F^{ab} G^\emptyset + F^a G^b + F^b G^a \right). \end{aligned}$$

This proves the first inequality. The second inequality is the same as in [Buc19]. □

The following statement is an extension to observables of Lemma 4.6.2 from [Buc19].

Lemma 5.4. *Let $\varphi \in \chi_N$. Then*

1. *for any $F_1, F_2 \in M^{\text{ext}}(\mathcal{P}_k)$ and any $X_1, X_2 \in \mathcal{P}_k$ we have*

$$|F_1(X_1) F_2(X_2)|^{\text{ext}}_{k,X_1 \cup X_2,T_\varphi} \leq |F_1(X_1)|^{\text{ext}}_{k,X_1,T_\varphi} |F_2(X_2)|^{\text{ext}}_{k,X_2,T_\varphi};$$

2. *for any $F \in M^{\text{ext}}(\mathcal{P}_k)$ and any polymer $X \in \mathcal{P}_k$ the bound*

$$\begin{aligned} |F(X)|_{k+1,\pi(X),T_\varphi} &\leq \max\left\{1, \frac{\eta^2}{4} L^d \right\} |F(X)|_{k,X \cup \pi(X),T_\varphi} \\ &\leq \max\left\{1, \frac{\eta^2}{4} L^d \right\} |F(X)|_{k,X,T_\varphi} \end{aligned}$$

   holds if $L \geq 2^d + R$.

In 2., the factor $\frac{\eta^2}{4} L^d$ is new in comparison to [Buc19].

Proof. The first inequality follows from Lemma 5.3 and the estimate

$$|F(X)|_{k,X \cup Y,T_\varphi} \leq |F(X)|_{k,X,T_\varphi}.$$

as in [Buc19].
For the second inequality note that due to the change of scale we have an additional factor

$$\frac{l^{|\alpha|}_{\text{obs},k+1}}{l^{|\alpha|}_{\text{obs},k}} \leq \frac{\eta^2}{4} L^d$$

for $|\alpha| = 1, 2$, which appears in the stated inequality. The remaining steps are as in [Buc19]. □

5.1.3 Submultiplicativity of the norms

The following claim is based on Lemma 4.6.3 in [Buc19], extended to observables.

Lemma 5.5. *Let $L \geq 2^{d+3} + 16R$ be an odd integer and $h \geq h_0(L)$, where h_0 is fixed in Lemma 5.1. For $k \in \{0, \dots, N-1\}$, let $K \in M^{\text{ext}}(\mathcal{P}_k)$ factor at scale k and let $F, F_1, F_2, F_3 \in M(\mathcal{B}_k)$. Then the following bounds hold:*

1. $\|K(X)\|^{\text{ext}}_{k,X} \leq \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|^{\text{ext}}_{k,Y}$ *and*
   $\|K(X)\|^{\text{ext}}_{k:k+1,X} \leq \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|^{\text{ext}}_{k:k+1,Y}$
   and more generally the same bounds hold for any decomposition $X = \bigcup Y_i$ such that the Y_i are strictly disjoint.
2. $\|F^X K(Y)\|_{k,X\cup Y} \leq \|K(Y)\|_{k,Y} |||F|||_k^{|X|_k}$ *for $X, Y \in \mathcal{P}_k$ with X and Y disjoint.*
3. *For any polymers $X, Y, Z_1, Z_2 \in \mathcal{P}_k$ such that $X \cap Y = \emptyset$, $Z_1 \cap Z_2 = \emptyset$, and $Z_1, Z_2 \subset \pi(X \cup Y) \cup X \cup Y$,*
   $$\|F_1^{Z_1} F_2^{Z_2} F_3^X K(Y)\|_{k+1,\pi(X\cup Y)} \leq \max\left\{1, \frac{\eta^2}{4} L^d\right\} \|K(Y)\|_{k:k+1,Y} |||F_1|||_k^{|Z_1|_k} |||F_2|||_k^{|Z_2|_k} |||F_1|||_k^{|X|_k}.$$
4. $|||\mathbb{1}(B)|||_{k,B} = 1$ *for $B \in \mathcal{B}_k$.*

In 3., the factor $\frac{\eta^2}{4} L^d$ is new in comparison to[Buc19].

Proof. Ingredients for the proof are the submultiplicativity of the T_φ-seminorm in Lemma 5.4 and properties of the weights. Since the submultiplicativity also holds for extended functionals the proof is exactly the same as in [Buc19]. The new factor $\frac{\eta^2}{4} L^d$ appears in the transition from one scale to the next one using (26). □

5.1.4 Regularity of the integration map

We extend Lemma 4.6.4 from [Buc19] to observables.

Lemma 5.6. *Let $L \geq 2^{d+3} + 16R$ and let $A_{\mathcal{P}}$ be the constant from Lemma 5.1. Then*

$$\|\mathcal{R}_{k+1} K(X)\|^{\text{ext}}_{k:k+1,X} \leq \left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k} \|K(X)\|^{\text{ext}}_{k,X}.$$

If X is a block the constant is $A_{\mathcal{B}}$ which is independent of L.

Proof. The proof in [Buc19] does not use any special property of the T_φ-seminorm, so it works exactly as in [Buc19]. □

For later reference we state the following inequality which appears in the proof of Lemma 4.6.4 from [Buc19].

Lemma 5.7. *Assume that Lemma 5.6 holds. Then*

$$|\mathcal{R}_{k+1}K(X)|^{\text{ext}}_{k,X,T_\varphi} \le \|K(X)\|^{\text{ext}}_{k,X}\left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k} w^X_{k:k+1}(\varphi). \tag{34}$$

If X is a block the constant is $A_{\mathcal{B}}$ which is independent of L.

5.1.5 The extended projection Π_k to relevant Hamiltonians

We extend the space of relevant Hamiltonians to observables.
Let $\mathcal{U} = \{e_1, \ldots, e_d\}$. The monomials which appear in [Buc19] are

$$\begin{aligned} M_\emptyset(\{x\})(\varphi) &= 1, \\ M_\beta(\{x\})(\varphi) &= \nabla^\beta \varphi(x), \\ M_{\beta,\gamma}(\{x\})(\varphi) &= \nabla^\beta \varphi(x) \nabla^\gamma \varphi(x). \end{aligned}$$

Then the corresponding index sets are

$$\begin{aligned} &\mathfrak{v}_0 = \{\emptyset\}, \\ &\mathfrak{v}_1 = \{\beta : \beta \in \mathbb{N}_0^{\mathcal{U}}, 1 \le |\beta| \le \lfloor d/2 \rfloor + 1\}, \\ &\mathfrak{v}_2 = \{(\beta,\gamma) : \beta,\gamma \in \mathbb{N}_0^{\mathcal{U}}, |\beta| = |\gamma| = 1, \beta < \gamma\}. \end{aligned}$$

Here, $\beta < \gamma$ refers to any ordering of $\mathcal{U}$. We additionally define

$$\begin{aligned} \mathfrak{v}_0^\alpha &= \{\emptyset\}, \quad \alpha \in \{a, b, ab\}, \\ \mathfrak{v}_1^\alpha &= \{\beta \in \mathbb{N}_0^{\mathcal{U}} : |\beta| = 1\}, \quad \alpha \in \{a, b\}. \end{aligned}$$

We set

$$\mathfrak{v}^{\text{ext}} = \mathfrak{v}_0 \cup \mathfrak{v}_1 \cup \mathfrak{v}_2 \cup \mathfrak{v}_0^a \cup \mathfrak{v}_1^a \cup \mathfrak{v}_0^b \cup \mathfrak{v}_1^b \cup \mathfrak{v}_0^{ab}.$$

A monomial on a block B for $\mathfrak{m} \in \mathfrak{v}^{\text{ext}}$ can then be written as

$$M_{\mathfrak{m}}(B)(\varphi) = \sum_{x \in B} M_{\mathfrak{m}}(\{x\})(\varphi).$$

The space of relevant Hamiltonians is given by

$$\mathcal{V}^{\text{ext}} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0^a \oplus \mathcal{V}_1^a \oplus \mathcal{V}_0^b \oplus \mathcal{V}_1^b \oplus \mathcal{V}_0^{ab}$$

where

$$\begin{aligned} \mathcal{V}_0 &= \mathbb{R}, \\ \mathcal{V}_1 &= \operatorname{span}\{M_{\mathfrak{m}}(B) : \mathfrak{m} \in \mathfrak{v}_1\}, \\ \mathcal{V}_2 &= \operatorname{span}\{M_{\mathfrak{m}}(B) : \mathfrak{m} \in \mathfrak{v}_2\}, \\ \mathcal{V}_0^\alpha &= \mathbb{R}, \quad \alpha \in \{a, b, ab\}, \\ \mathcal{V}_1^\alpha &= \operatorname{span}\{M_{\mathfrak{m}}(\{\alpha\}) : \mathfrak{m} \in \mathfrak{v}_1^\alpha\}, \quad \alpha \in \{a, b\}. \end{aligned}$$

As in [Buc19], we set

$$b_\beta(z) = \binom{z_1}{\beta_1}\cdots\binom{z_d}{\beta_d}, \quad z \in \mathbb{Z}^d, \quad \beta \in \mathbb{N}_0^{\{1,\dots,d\}}.$$

We extend the basis for polynomials on $\mathbb{Z}^d$ for $\alpha \in \{a, b\}$ by

$$b^\alpha_\beta(z) = \binom{z_1 - \alpha_1}{\beta_1}\cdots\binom{z_d - \alpha_d}{\beta_d}.$$

Using these functions we can extend the space $\mathcal{P}$ in [Buc19] to observables by defining

$$\mathcal{P}^\alpha_0 = \mathbb{R}, \quad \alpha \in \{a, b, ab\},$$
$$\mathcal{P}^\alpha_1 = \operatorname{span}\{b^\alpha_\beta : \beta \in \mathfrak{v}^\alpha_1\}, \quad \alpha \in \{a, b\},$$

and setting

$$\mathcal{P}^{\mathrm{ext}} = \mathcal{P} \oplus \mathcal{P}^a_0 \oplus \mathcal{P}^a_1 \oplus \mathcal{P}^b_0 \oplus \mathcal{P}^b_1 \oplus \mathcal{P}^{ab}_0.$$

Now we can formulate the extension of Lemma 4.6.5 from [Buc19]. The notation $\langle F, g\rangle_\varphi = \langle \mathrm{Tay}_\varphi F, g\rangle$ is used as in [Buc19].

Lemma 5.8. *Let $K \in M^{\mathrm{ext}}(\mathcal{P}^c_k, \chi_N)$ and let $B \in \mathcal{B}_k$. Then there exists one and only one $H \in \mathcal{V}^{\mathrm{ext}}$ such that*

$$\langle H, g\rangle_0 = \langle K(B), g\rangle_0 \quad \text{for all } g \in \mathcal{P}^{\mathrm{ext}}.$$

More precisely, for $\alpha \in \{a, b\}$,

$$H^\alpha(\varphi) = K^\alpha(0) + n^\alpha \nabla\varphi(\alpha),$$

where

$$n^\alpha_\gamma = \langle K^\alpha(B), b^\alpha_\gamma\rangle_0 \quad \text{for all } \gamma \in \mathfrak{v}^\alpha_1 \tag{35}$$

and

$$q^{ab} = K^{ab}(0).$$

Definition 5.9. *We define $\Pi K(B) = H$ where H is given by Lemma 5.8.*

Proof of Lemma 5.8. The bulk part of K is handled in [Buc19]. The constant observable part of $H \in \mathcal{V}^{\mathrm{ext}}$ is given by

$$\lambda^a = K^a(B, 0), \quad \lambda^b = K^b(B, 0), \quad q^{ab} = K^{ab}(B, 0).$$

We turn to the linear observable part of H. We claim that for $\alpha \in \{a, b\}$ there is a unique $H^{1,\alpha} \in \mathcal{V}^\alpha_1$ such that

$$\langle H^{1,\alpha}, g\rangle_0 = \langle K^\alpha(B), g\rangle_0 \quad \text{for all } g \in \mathcal{P}^\alpha_1.$$

An element $H^{1,\alpha} \in \mathcal{V}_1^\alpha$ is of the form $\sum_{\beta\in\mathfrak{v}_1^\alpha} n_\beta^\alpha M_\beta(\{\alpha\})$ for some n_β^α yet to be determined.
Testing against the basis $\{b_\beta^\alpha : \beta \in \mathfrak{v}_1^\alpha\}$ of $\mathcal{P}_1^\alpha$ we have to show that there is a family $\left(n_\beta^\alpha\right)_{\beta\in\mathfrak{v}_1^\alpha}$ such that

$$\sum_{\beta\in\mathfrak{v}_1^\alpha} n_\beta^\alpha \langle M_\beta(\{\alpha\}), b_\gamma^\alpha\rangle_0 = \langle K^\alpha(B), b_\beta^\alpha\rangle_0 \quad \text{for all } \gamma \in \mathfrak{v}_1^\alpha.$$

The last equality is equivalent to

$$\sum_{\beta\in\mathfrak{v}_1^\alpha} n_\beta^\alpha B_{\beta\gamma} = \langle K^\alpha(B), b_\beta^\alpha\rangle_0 \quad \text{for all } \gamma \in \mathfrak{v}_1^\alpha$$

with

$$B_{\beta\gamma} = \langle \nabla^\beta\varphi(\alpha), b_\gamma^\alpha\rangle_0 = \langle \mathrm{Tay}_0\nabla^\beta\varphi(\alpha), b_\gamma^\alpha\rangle = \nabla^\beta b_\gamma^\alpha(\alpha) = b_{\gamma-\beta}^\alpha(\alpha).$$

For $\beta, \gamma \in \mathfrak{v}_1^\alpha$ we get that $B_{\beta,\gamma} = \mathbb{1}_{\beta=\gamma}$ and thus

$$n_\gamma^\alpha = \langle K^\alpha(B), b_\gamma^\alpha\rangle_0 \quad \text{for all } \gamma \in \mathfrak{v}_1^\alpha.$$

□

The following statement is an extension to observables of Lemma 4.6.7 from [Buc19].

Lemma 5.10. *There exists a constant C such that for $L \geq 2^d + R$ and $0 \leq k \leq N-1$*

$$\|\Pi_k K(B)\|_{k,0}^{\text{ext}} \leq C|K(B)|_{k,B,T_0}^{\text{ext}}.$$

Proof. The bulk part of the estimate is done in [Buc19]. What remains to prove is

$$\|\Pi_k^\alpha K^\alpha(B)\|_{k,0}^\alpha \leq C l_{\text{obs},k}^{|\alpha|} |K^\alpha(B)|_{k,B,T_0}.$$

Since for the constant part of the projection we have $\lambda^\alpha = K^\alpha(B, 0)$ for $\alpha \in \{a, b\}$ and $q^{ab} = K^{ab}(B, 0)$ we just have to estimate the coefficients n^α of the linear part of the projection.
Since $n^\alpha = \langle K^\alpha(B), b^\alpha\rangle_0$ (see (35) in Lemma 5.8) we have to show that

$$l_{\text{obs},k} l_k |\langle K^\alpha(B), b^\alpha\rangle_0| \leq C l_{\text{obs},k} |K^\alpha(B)|_{k,B,T_0}.$$

However, this follows directly from the definition of the T_φ-seminorm and since $|b^\alpha|_{k,B} = l_k^{-1}$:

$$\langle K^\alpha(B), b^\alpha\rangle_0 \leq |b^\alpha|_{k,B} \sup_{|g|_{k,B}\leq 1} \langle K^\alpha(B), g\rangle_0 \leq l_k^{-1} |K^\alpha(B)|_{k,B,T_0}.$$

□

We extend Lemma 4.6.8 from [Buc19] to observables.

Lemma 5.11. *For $H \in M_0^{\mathrm{ext}}$, $L \geq 3$, and $0 \leq k \leq N$ we have*

$$|H|_{T_\varphi}^{\mathrm{ext}} \leq (1+|\varphi|_{k,B})^2 \|H\|_{k,0}^{\mathrm{ext}} \leq 2(1+|\varphi|_{k,B}^2)\|H\|_{k,0}^{\mathrm{ext}}.$$

Proof. The only difference to [Buc19] is that additional terms in $|H|_{T_\varphi}^{\mathrm{ext}}$ and $\|H\|_{k,0}^{\mathrm{ext}}$ exist:

$$\begin{aligned} |H|_{T_\varphi}^{\mathrm{ext}} &= |H^\emptyset|_{T_\varphi} + l_{\mathrm{obs},k}\left(|\lambda^a| + |n^a \nabla\varphi(a)\mathbb{1}_a|_{T_\varphi}\right) \\ &\qquad + l_{\mathrm{obs},k}\left(|\lambda^b| + |n^b \nabla\varphi(b)\mathbb{1}_b|_{T_\varphi}\right) + l_{\mathrm{obs},k}^2 |q^{ab}|, \\ \|H\|_{k,0}^{\mathrm{ext}} &= \|H^\emptyset\|_{k,0} + l_{\mathrm{obs},k}\left(|\lambda^a| + l_k|n^a|\right) + l_{\mathrm{obs},k}\left(|\lambda^b| + l_k|n^b|\right) + l_{\mathrm{obs},k}^2|q^{ab}|. \end{aligned}$$

Thus the proof is finished if we show that, for $\alpha \in \{a,b\}$,

$$l_{\mathrm{obs},k}|n^\alpha \nabla\varphi(\alpha)\mathbb{1}_\alpha(B)|_{T_\varphi} \leq (1+|\varphi|_{k,B})^2 l_{\mathrm{obs},k} l_k |n^\alpha|.$$

This follows straightforwardly since

$$|\nabla\varphi(\alpha)\mathbb{1}_\alpha(B)|_{T_\varphi} = (|\nabla\varphi(a)| + l_k)\mathbb{1}_\alpha(B) \leq l_k|\varphi|_{k,B} + l_k \leq l_k\left(1+|\varphi|_{k,B}^2\right).$$

□

The following lemma is an extension of Lemma 4.6.9 from [Buc19].

Lemma 5.12. *Let $A(\alpha,k) = 0$ when $k \geq j_{ab}$, $\alpha \in \{a,b,ab\}$, and $A(\alpha,k) = 1$ when $k < j_{ab}$, $\alpha \in \{a,b\}$. There exists a constant C such that for $L \geq 2^d + R$, for $\alpha \in \{a,b,ab\}$,*

$$|(1-\Pi_k^\alpha)K^\alpha(B)|_{k+1,B,T_0} \leq CL^{-(d/2+A(\alpha,k))}|K^\alpha|_{k,B,T_0}.$$

Proof. We start with $\alpha \in \{a,b,ab\}$ and $k \geq j_{ab}$, i.e. $\Pi_k^\alpha = \Pi_0$. Note that

$$|(1-\Pi_0)K^\alpha(B)|_{k+1,B,T_0} = \sup\left\{\langle(1-\Pi_0)K^\alpha, g\rangle_0 : g \in \Phi, |g|_{k+1,B} \leq 1\right\}.$$

For $g \in \chi^{\otimes r}$, $r \geq 1$, it holds that

$$\langle(1-\Pi_0)\,K^\alpha, g\rangle_0 = \langle K^\alpha, g\rangle_0$$

since $\Pi_0 K^\alpha$ depends only on the first order Taylor polynomial. For $g \in \chi^{\otimes r}$, $r \geq 1$, we can use the estimate

$$|g|_{k,B} \leq 8L^{-\frac{1}{2}d}|g|_{k+1,B}$$

as in [Buc19]. Thus

$$|\langle(1-\Pi_0)K^\alpha, g\rangle_0| \leq |K^\alpha|_{k,B,T_0}|g|_{k,B} \leq 8L^{-\frac{1}{2}d}|g|_{k+1,B}|K^\alpha|_{k,B,T_0}.$$

For $g \in \chi^{\otimes 0} = \mathbb{R} = \mathcal{P}_0^\alpha$ it holds that

$$\langle\Pi_0 K^\alpha, g\rangle_0 = \langle K^\alpha, g\rangle_0$$

and thus

$$\langle (1-\Pi_0)K^\alpha, g\rangle_0 = 0 \quad \text{for all } g \in \mathbb{R}.$$

This argument finishes the case $k \geq j_{ab}$.
Now let $\alpha \in \{a,b\}$ and $k < j_{ab}$, i.e., $\Pi_k^\alpha = \Pi_1$. As above we can use for all $g \in \chi^{\otimes r}$ and $r \geq 2$

$$|\langle (1-\Pi_1)K^\alpha, g\rangle_0| = |\langle K^\alpha, g\rangle_0| \leq 8L^{-\frac{1}{2}d}|K^\alpha|_{k,B,T_0}|g|_{k+1,B}.$$

Again,

$$\langle (1-\Pi_0)K^\alpha, g\rangle_0 = 0 \quad \text{for all } g \in \mathbb{R} = \mathcal{P}_0^\alpha.$$

Let $\varphi \in \chi$. For all $P \in \mathcal{P}_1^\alpha$ we have $\langle \Pi_1 K^\alpha, P\rangle_0 = \langle K^\alpha, P\rangle_0$. Using additionally boundedness of Π, we can estimate

$$\begin{aligned}|\langle (1-\Pi_1)K^\alpha, \varphi\rangle_0| &= \min_{P\in\mathcal{P}_1^\alpha} |\langle (1-\Pi_1)K^\alpha, \varphi - P\rangle_0| \\ &\leq |(1-\Pi_1)K^\alpha|_{k,B,T_0} \min_{P\in\mathcal{P}_1^\alpha} |\varphi - P|_{k,B} \\ &\leq C|K^\alpha|_{k,B,T_0} \min_{P\in\mathcal{P}_1^\alpha} |\varphi - P|_{k,B}.\end{aligned}$$

With Lemma 5.13 below the proof is finished. □

Lemma 5.13. *There exists a constant C such that for $L \geq 2^d + R$ and for all $\varphi \in \chi$*

$$\min_{P\in\mathcal{P}_1^\alpha} |\varphi - P|_{k,B} \leq CL^{-(\frac{d}{2}+1)}|\varphi|_{k+1,B}.$$

Proof. The statement is an extension of Lemma 4.6.10 from [Buc19]. The proof is as in [Buc19] with the only difference being the choice of parameter $s = 1$, which originally was $s = \lfloor \frac{d}{2} \rfloor + 1$. The reason for this change is that $\mathcal{P}_1^\alpha = \operatorname{span}\left\{b_\beta^\alpha : |\beta| = 1\right\}$, whereas in the bulk flow higher derivatives are also allowed. Then $P = \operatorname{Tay}_a^s \varphi$ provides the minimizer. □

5.2 Smoothness of the extended renormalisation map

In this section we prove Proposition 4.7 which claims that there is L_0 and corresponding A_0 and h_0 and a parameter $\rho^*(A)$ such that $\mathbf{S}_k^{\text{ext}} \in U_{\rho^*(A)}$ with bounds on derivatives which are uniformly in N.

Remember that

$$\mathbf{S}^{\text{ext}}(H,K) = e^{-s(\mathbf{B}K^a)^0 - t\left(\mathbf{B}K^b\right)^0 - st\left(\int H^a H^b \mathrm{d}\mu_+ + \mathbf{B}K^{ab}\right)}\mathbf{S}(H,K)$$

where we drop the subscript k and $k+1$ in the notation. To nevertheless note the change of scale, we abbreviate $k+1$ by $+$.
Let us denote

$$F = sF^a + tF^b + stF^{ab} := -s(\mathbf{B}K^a)^0 - t(\mathbf{B}K^b)^0 - st\left(\int H^a H^b \mathrm{d}\mu_+ + \mathbf{B}K^{ab}\right).$$

We divide the proof of Proposition 4.7 into two steps. The first step is the analysis of $\mathbf{S}$.

Lemma 5.14. *There is L_0 such that for all odd integers $L \geq L_0$ there is A_0, h_0 with the following property. For all $A \geq A_0$, $h \geq h_0$ there is $\rho^* = \rho^*(A)$ such that*

$$\mathbf{S} \in C^\infty\left(U_{\rho^*}, M^{\text{ext}}(\mathcal{P}_{k+1}^c)\right)$$

and for any $p, q \in \mathbb{N}$ there is a constant $C_{p,q} = C_{p,q}(L, h, A)$ such that for any $(H, K) \in U_{\rho^*}$

$$\left\| D_H^p D_K^q \mathbf{S}(H,K)(\dot{H}^p, \dot{K}^q) \right\|_{k+1}^{(A),\text{ext}} \leq C_{p,q} \left(\|\dot{H}\|_{k,0}^{\text{ext}} \right)^p \left(\|\dot{K}\|_k^{(A),\text{ext}} \right)^q.$$

The second step includes the analysis of the prefactor e^F.

Lemma 5.15. *Assume that Lemma 5.14 holds. Then*

$$\mathbf{S}^{ext} \in C^\infty\left(U_{\rho^*}, M^{\text{ext}}(\mathcal{P}_{k+1}^c)\right)$$

and for each $p, q \in \mathbb{N}$ there is a constant $C_{p,q}^*$ such that for any $(H, K) \in U_{\rho^*}$,

$$\left\| D_H^p D_K^q \mathbf{S}^{\text{ext}}(H,K)(\dot{H}^p, \dot{K}^q) \right\|_{k+1}^{(A),\text{ext}} \leq C_{p,q}^* \left(\|\dot{H}\|_{k,0}^{\text{ext}} \right)^p \left(\|\dot{K}\|_k^{(A),\text{ext}} \right)^q.$$

Proposition 4.7 follows from Lemma 5.15 with the assumptions of Lemma 5.14.

We first prove Lemma 5.15.

Proof of Lemma 5.15. We show smoothness via bounds on the derivatives. Since F is a constant in φ, we can estimate

$$\begin{aligned}
&\left\| D_H^p D_K^q \mathbf{S}^{\text{ext}}(H,K)(\dot{H}^p, \dot{K}^q) \right\|_{k+1}^{(A),\text{ext}} = \left\| D_H^p D_K^q \left[e^F \mathbf{S}(H,K) \right] (\dot{H}^p, \dot{K}^q) \right\|_{k+1}^{(A),\text{ext}} \\
&\quad \leq C_{p,q} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} \left\| D_H^{p_1} D_K^{q_1} \left[e^F \right] (\dot{H}^{p_1}, \dot{K}^{q_1}) D_H^{p_2} D_K^{q_2} \mathbf{S}(H,K)(\dot{H}^{p_2}, \dot{K}^{q_2}) \right\|_{k+1}^{(A),\text{ext}} \\
&\quad \leq C_{p,q} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} \sup_U \left\{ A^{|U|_{k+1}} \left| D_H^{p_1} D_K^{q_1} \left[e^{F(U)} \right] (\dot{H}^{p_1}, \dot{K}^{q_1}) \right|_{k+1,U,T_0}^{\text{ext}} \right. \\
&\qquad\qquad \left. \left\| D_H^{p_2} D_K^{q_2} \mathbf{S}(H,K)(U)(\dot{H}^{p_2}, \dot{K}^{q_2}) \right\|_{k+1,U}^{\text{ext}} \right\}.
\end{aligned}$$

By assumption $\mathbf{S}$ is smooth with the desired bounds, so it is enough to show that

$$\left| D_H^{p_1} D_K^{q_1} \left(e^{F(U)} \right) (\dot{H}^{p_1}, \dot{K}^{q_1}) \right|_{k+1,U,T_0}^{\text{ext}} \leq C \left(\|\dot{H}\|_{k,0}^{\text{ext}} \right)^{p_1} \left(\|\dot{K}\|_k^{(A),\text{ext}} \right)^{q_1}.$$

Note that if $a, b \notin U$ then $e^{F(U)} = 1$ such that any derivative $D_H^{p_1}$ or $D_K^{q_1}$ gives just zero which is not optimal for the supremum. Thus either $a, b \notin U$ and $p_1 = q_1 = 0$ or $\alpha \in U$ for $\alpha \in \{a, b, ab\}$. In the first case we are done – the constant we get is 1. In the second case we go through all possible cases. Let $(H, K) \in U_{\rho^*}$.

- $p_1 = 0$, $q_1 = 0$:

  We use Lemma 5.27, Lemma 5.28 and estimate (26) to get

$$\begin{aligned}\left|e^{F(U)}\right|^{\text{ext}}_{k+1,U,T_0} &= 1 + |F^a(U)| + |F^b(U)| + |F^{ab}(U)| + |F^a(U)F^b(U)| \\ &= 1 + l_{\text{obs},k+1}\left(\left|(\mathbf{B}K^a)^0\right| + \left|(\mathbf{B}K^b)^0\right|\right) \\ &\quad + l^2_{\text{obs},k+1}\left(\left|\mathbf{B}K^{ab}\right| + \left|\int H^a H^b \mathrm{d}\mu_+\right| + \left|(\mathbf{B}K^a)^0(\mathbf{B}K^b)^0\right|\right) \\ &\le 1 + \frac{A_{\mathcal{B}}}{2}L^{d/2}\frac{\eta}{2}\rho^* + L^d\frac{\eta^2}{4}\rho^*\left(\frac{A_{\mathcal{B}}}{2} + \frac{A^2_{\mathcal{B}}}{4}\rho^* + C_{FRD}h^{-2}\rho^*\right)\end{aligned}$$

  which is bounded by a constant.

- $p_1 = 0$, $q_1 = 1$: By Lemma 5.27 and estimate (26) we get

$$\begin{aligned}&\left|D_K e^{F(U)}\dot{K}\right|^{\text{ext}}_{k+1,U,T_0} \\ &= l_{\text{obs},k+1}\left(\left|(\mathbf{B}\dot{K}^a)^0\right| + \left|(\mathbf{B}\dot{K}^b)^0\right|\right) \\ &\quad + l^2_{\text{obs},k+1}\left(\left|(\mathbf{B}\dot{K}^{ab})^0\right| + |(\mathbf{B}K^a)^0|\left|(\mathbf{B}\dot{K}^b)^0\right| + |(\mathbf{B}K^b)^0|\left|(\mathbf{B}\dot{K}^a)^0\right|\right) \\ &\le l_{\text{obs},k+1}l^{-1}_{\text{obs},k}A_{\mathcal{B}}\|\dot{K}\|^{(A),\text{ext}}_k + l^2_{\text{obs},k+1}l^{-2}_{\text{obs},k}\left(\frac{A_{\mathcal{B}}}{2} + 2\left(\frac{A_{\mathcal{B}}}{2}\right)^2\rho^*\right)\|\dot{K}\|^{(A),\text{ext}}_k \\ &\le C\|\dot{K}\|^{(A),\text{ext}}_k.\end{aligned}$$

- $p_1 = 0$, $q_1 = 2$: By Lemma 5.27 and estimate (26) we get

$$\begin{aligned}\left|D^2_K\left(e^{F(U)}\right)(\dot{K},\dot{K})\right|^{\text{ext}}_{k+1,U,T_0} &= l^2_{\text{obs},k+1}2\left|(\mathbf{B}\dot{K}^a)^0\right|\left|(\mathbf{B}\dot{K}^b)^0\right| \\ &\le 2l^2_{\text{obs},k+1}l^{-2}_{\text{obs},k+1}\left(\frac{A_{\mathcal{B}}}{2}\right)^2\left(\|\dot{K}\|^{(A),\text{ext}}_k\right)^2 \\ &\le C\left(\|\dot{K}\|^{(A),\text{ext}}_k\right)^2.\end{aligned}$$

- $p_1 = 0$, $q_1 > 2$: The derivative is zero.

- $p_1 = 1$, $q_1 = 0$: By Lemma 5.28 we get

$$\begin{aligned}\left|D_H e^{F(U)}\dot{H}\right|^{\text{ext}}_{k+1,U,T_0} &= l^2_{\text{obs},k+1}\left|\int \dot{H}^a H^b \mathrm{d}\mu_+ + \int H^a \dot{H}^b \mathrm{d}\mu_+\right| \\ &\le 2C_{FRD}l^2_{\text{obs},k+1}l^{-2}_{\text{obs},k}h_k^{-2}\rho^*\|\dot{H}\|^{\text{ext}}_{k,0} \\ &\le C\|\dot{H}\|^{\text{ext}}_{k,0}.\end{aligned}$$

- $p_1 = 1$, $q_1 > 0$: The derivative is zero.

- $p_1 = 2$, $q_1 = 0$: By Lemma 5.28 we get

$$\left| D_H^2 e^F \dot{H}^2 \right|_{k+1,U,T_0}^{\text{ext}} = l_{\text{obs},k+1}^2 2 \left| \int \dot{H}^a \dot{H}^b \mathrm{d}\mu_+ \right| \\ \leq 2C_{FRD} l_{\text{obs},k+1}^2 l_{\text{obs},k}^{-2} h_k^{-2} \left(\|\dot{H}\|_{k,0}^{\text{ext}} \right)^2 \leq C \left(\|\dot{H}\|_{k,0}^{\text{ext}} \right)^2 .$$

- $p_1 = 2$, $q_1 > 0$: The derivative is zero.

In summary we get

$$\left\| D_H^p D_K^q \mathbf{S}^{\text{ext}}(H,K)(\dot{H}^p, \dot{K}^q) \right\|_{k+1}^{(A),\text{ext}} \\ \leq C_{p,q} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} \left(\|\dot{H}\|_{k,0}^{\text{ext}} \right)^{p_1} \left(\|\dot{K}\|_{k}^{(A),\text{ext}} \right)^{q_1} \left\| D_H^{p_2} D_K^{q_2} \mathbf{S}(H,K)(\dot{H}^{p_2}, \dot{K}^{q_2}) \right\|_{k+1}^{(A),\text{ext}} .$$

□

Now we turn to the analysis of $\mathbf{S}$ and the proof of Lemma 5.14.

As in [Buc19], the strategy is to write the map $\mathbf{S}^{\text{ext}}$ as a composition of simpler maps and show smoothness for those maps. We follow closely the presentation in [Buc19] and do not repeat arguments in proofs which can be applied without change to the extended setting here.
We consider the following spaces:

$$\mathbf{M}^{(A)} = \left(M^{\text{ext}}(\mathcal{P}_k^c), \|\cdot\|_k^{(A),\text{ext}} \right),$$
$$\mathbf{M}'^{(A)} = \left(M^{\text{ext}}(\mathcal{P}_{k+1}^c), \|\cdot\|_{k+1}^{(A),\text{ext}} \right),$$
$$\mathbf{M}_0 = \left(M^{\text{ext}}(\mathcal{B}_k), \|\cdot\|_{k,0}^{\text{ext}} \right),$$
$$\mathbf{M}_{|||} = \left(M^{\text{ext}}(\mathcal{B}_k), |||\cdot|||_k^{\text{ext}} \right).$$

We need a slight modification of $\mathbf{M}^{(A)}$. Define $\mathcal{P}_k^{c'} \subset \mathcal{P}_k$ as

$$\mathcal{P}_k^{c'} = \{X \in \mathcal{P}_k : \pi(X) \in \mathcal{P}_{k+1}^c\}.$$

The space $M^{\text{ext}}(\mathcal{P}_k^{c'})$ of functionals is defined similarly to $M^{\text{ext}}(\mathcal{P}_k^c)$ except that $\mathcal{P}_k^c$ is replaced by $\mathcal{P}_k^{c'}$ in the definition.
A norm on $M^{\text{ext}}(\mathcal{P}_k^{c'})$ with parameters $A, B > 1$ is given by

$$\|K\|_k^{(A,B),\text{ext}} = \sup_{X \in \mathcal{P}_k^{c'}} A^{|X|_k} B^{|\mathcal{C}(X)|} \|K(X)\|_{k,X}^{\text{ext}}.$$

We also use the norm $\|\cdot\|_{k:k+1}^{(A,B),\text{ext}}$ where we replace the $\|\cdot\|_{k,X}^{\text{ext}}$ norm by the norm $\|\cdot\|_{k:k+1,X}^{\text{ext}}$ on the right hand side.

As in [Buc19], we introduce short hand notations for the corresponding normed spaces

$$\widehat{\mathbf{M}}^{A,B} = \left\{ M(\mathcal{P}_k^{c'}), \|\cdot\|_k^{(A,B),\mathrm{ext}} \right\}, \quad \widehat{\mathbf{M}}_{:}^{A,B} = \left\{ M(\mathcal{P}_k^{c'}), \|\cdot\|_{k:k+1}^{(A,B),\mathrm{ext}} \right\}.$$

The map $\mathbf{S}$ is, as in [Buc19], rewritten in terms of the following maps. Observe the use of the subspace $\mathcal{V}_k^{(0)}$ of $\mathbf{M}_0$ here in the definition of R_2 in comparison to [Buc19]. However, on the bulk flow part, this subspace coincides with the whole space. Another difference to [Buc19] is the definition of the map R_2, since the second order perturbation in the observable part appears.

$$\begin{aligned}
&E : \mathbf{M}_0 \to \mathbf{M}_{|||}, \quad E(H) = e^H, \\
&P_1 : \mathbf{M}_{|||} \times \mathbf{M}_{|||} \times \mathbf{M}_{|||} \times \widehat{\mathbf{M}}_{:}^{(A/(2A_{\mathcal{P}}),B)} \to \mathbf{M}'^{(A)}, \\
&P_1(I_1, I_2, J, K)(U) = \sum_{\substack{X_1, X_2 \in \mathcal{P}_k \\ X_1 \cap X_2 = \emptyset}} \chi(X_1 \cup X_2, U) I_1^{U \setminus (X_1 \cup X_2)} I_2^{(X_1 \cup X_2) \setminus U} J^{X_1} K(X_2) \\
&P_2 : \mathbf{M}_{|||} \times \mathbf{M}^{(A)} \to \mathbf{M}^{(A/2)}, \quad P_2(I, K) = (I - 1) \circ K, \\
&P_3 : \mathbf{M}^{(A/2)} \to \widehat{\mathbf{M}}^{(A/2,B)}, \quad P_3 K(X, \varphi) = \prod_{y \in \mathcal{C}(X)} K(Y, \varphi), \\
&R_1 : \widehat{\mathbf{M}}^{(A/2,B)} \to \widehat{\mathbf{M}}_{:}^{(A/(2A_{\mathcal{P}}),B)}, \quad R_1(P) = \mathcal{R}_+ P, \\
&R_2 : \mathcal{V}_k^{(0)} \times \mathbf{M}^{(A)} \to \mathbf{M}_0, \quad R_2(H, K) = \mathcal{R}_+ H + st \int H^a H^b \mathrm{d}\mu_+ + \Pi \mathcal{R}_+ K.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{S}(H, K) = \\
&P_1 \left(E(R_2(H, K)), E(-R_2(H, K)), 1 - E(R_2(H, K)), R_1(P_3(P_2(E(H), K))) \right).
\end{aligned}$$

In the following we extend estimates on these maps to observables.

5.2.1 The immersion E

The following statement is an extension of Lemma 4.7.3 from [Buc19] to observables.

Lemma 5.16. *Let $L \geq 3$. The map*

$$E : B_{\frac{1}{8}}(0) \subset \mathbf{M}_0 \to \mathbf{M}_{|||}, \quad E(H) = e^H,$$

is smooth and for any $r \in \mathbb{N}$ there is a constant C_r (which is independent or A) such that for all $H \in B_{\frac{1}{8}}(0)$

$$\left|\left|\left| D^r E(H)(\dot{H}_1, \dots, \dot{H}_r) \right|\right|\right|_k^{\mathrm{ext}} = \left|\left|\left| e^H \dot{H}_1 \dots \dot{H}_r \right|\right|\right|_k^{\mathrm{ext}} \leq C_r \|\dot{H}_1\|_{k,0}^{\mathrm{ext}} \cdots \|\dot{H}_r\|_{k,0}^{\mathrm{ext}}.$$

Moreover, for all $H \in B_{\frac{1}{8}}(0)$,

$$\left|\left|\left| e^H - 1 \right|\right|\right|_k^{\mathrm{ext}} \leq 8 \|H\|_{k,0}^{\mathrm{ext}}.$$

Proof. The difference to [Buc19] is that $H \in \mathbf{M}_0$ is of the following form:

$$H = H^{\emptyset} + s\left(\lambda^a + \sum_i n_i^a \nabla_i \varphi(a)\right) \mathbb{1}_a + t\left(\lambda^b + \sum_i n_i^b \nabla_i \varphi(b)\right) \mathbb{1}_b + stq^{ab}.$$

In Lemma 5.11 it is shown that for the extended relevant variable $H \in \mathbf{M}_0$

$$|H|^{\mathrm{ext}}_{k,B,T_\varphi} \leq 2(1 + |\varphi|^2_{k,B})\|H\|^{\mathrm{ext}}_{k,0}.$$

This is the only ingredient for the proof where the observables play a role; for $\|H\|^{\mathrm{ext}}_{k,0} \leq \frac{1}{8}$ the remaining proof follows as in [Buc19].

□

5.2.2 The map P_2

We extend Lemma 4.7.4 from [Buc19] to the setting with observables. Here, $h_0(L)$ is fixed in Lemma 5.1.

Lemma 5.17. *Let $L \geq 2^{d+3} + 16R$ and $h \geq h_0(L)$. Consider the map*

$$P_2 : \mathbf{M}_{|||} \times \mathbf{M}^{(A)} \to \mathbf{M}^{(A/2)}, \quad P_2(I, K) = (I - 1) \circ K.$$

Restricted to $B_{\rho_1}(1) \times B_{\rho_2}(0)$ with $\rho_1 < (2A)^{-1}$ and $\rho_2 < \frac{1}{2}$, the map P_2 is smooth for any $A \geq 2$ and satisfies

$$\frac{1}{j_1! j_2!}\|(D_I^{j_1} D_K^{j_2} P_2)(I, K)(\dot{I}, \ldots \dot{I}, \dot{K}, \ldots, \dot{K})\|^{(A/2),\mathrm{ext}}_k$$
$$\leq \left(2A\left|\left|\left|\dot{I}\right|\right|\right|^{\mathrm{ext}}_k\right)^{j_1} \left(2\|\dot{K}\|^{(A),\mathrm{ext}}_k\right)^{j_2}.$$

This implies in particular for $I \in B_{\rho_1}(1)$ and $K \in B_{\rho_2}(0)$ that

$$\|P_2(I, K)\|^{(A/2),\mathrm{ext}}_k \leq 2A|||I - 1|||^{\mathrm{ext}}_k + 2\|K\|^{(A),\mathrm{ext}}_k.$$

Proof. Ingredients here are the norm estimates in Lemma 5.5 which also hold for the extended norms. Thus the claim follows as in [Buc19]. □

5.2.3 The map P_3

The following lemma is based on Lemma 4.7.5 in [Buc19] and extended to observables. Here, $h_0(L)$ is fixed in Lemma 5.1.

Lemma 5.18. *Assume $L \geq 2^{d+3} + 16R$ and $h \geq h_0(L)$. Let $A \geq 2$ and $B \geq 1$. Consider the map*

$$P_3 : \mathbf{M}^{(A/2)} \to \widehat{\mathbf{M}}^{(A/2,B)}, \quad P_3K(X) = \prod_{Y \in \mathcal{C}(X)} K(Y).$$

Its restriction to $B_\rho(0)$ is smooth for any ρ such that $\rho \leq (2B)^{-1}$ and it satisfies the following bound for $j \geq 0$,

$$\frac{1}{j!}\left\|(D^j P_3 K)(\dot{K}, \ldots, \dot{K})\right\|^{(A/2,B),\mathrm{ext}}_k \leq \left(2B\|\dot{K}\|^{(A/2),\mathrm{ext}}_{k,r}\right)^j.$$

Proof. The proof follows as in [Buc19] by using 1. from Lemma 5.5. □

5.2.4 The map R_2

The following statement is an extension of Lemma 4.7.8 in [Buc19]. The estimates look different from those in [Buc19] due to the second order perturbation in the observable flow.

Lemma 5.19. *Assume $L \geq 2^{d+3} + 16R$. Consider*

$$R_2 : \mathcal{V}_k^{(0)} \times \mathbf{M}^{(A)} \to \mathbf{M}_0, \quad R_2(H, K) = \mathcal{R}_+ H + st \int H^a H^b \mathrm{d}\mu_+ + \Pi \mathcal{R}_+ K.$$

For any $h \geq 1$ and $A \geq 1$ the map R_2 is smooth and there is a constant C which is independent of A such that

$$\begin{aligned} &\|D_H^{j_1} D_K^{j_2} R_2(H,K)(\dot{H},\ldots,\dot{H},\dot{K},\ldots,\dot{K})\|_{k,0}^{\mathrm{ext}} \\ &\quad \leq C \begin{cases} \|H\|_{k,0}^{\mathrm{ext}} + \|H^a\|_{k,0}^{a} \|H^b\|_{k,0}^{b} + \|K\|_k^{(A),\mathrm{ext}} & \text{if } j_1 = j_2 = 0 \\ \left(\|\dot{H}\|_{k,0}^{\mathrm{ext}} + \|\dot{H}^a\|_{k,0}^{a} \|H^b\|_{k,0}^{b} + \|H^a\|_{k,0}^{a} \|\dot{H}^b\|_{k,0}^{b} \right) & \text{if } j_1 = 1, j_2 = 0 \\ \|\dot{K}\|_k^{(A),\mathrm{ext}} & \text{if } j_1 = 0, j_2 = 1 \\ \|\dot{H}^a\|_{k,0}^{a} \|\dot{H}^b\|_{k,0}^{b} & \text{if } j_1 = 2, j_2 = 0 \end{cases} \end{aligned}$$

and $D_H^{j_1} D_k^{j_2} R_2(H,K)(\dot{H},\ldots,\dot{H},\dot{K},\ldots,\dot{K}) = 0$ else.

Proof. The extended norm consists of the following terms:

$$\begin{aligned} \|R_2(H,K)\|_{k,0}^{\mathrm{ext}} &= \sum_{\alpha \in \{\emptyset, a, b, ab\}} \|(R_2(H,K))^\alpha\|_{k,0}^\alpha \\ &= \|\mathcal{R}_+ H^\emptyset\|_{k,0} + \|H^a\|_{k,0}^a + \|H^b\|_{k,0}^b + \left\| \int H^a H^b \mathrm{d}\mu_+ \right\|_{k,0}^{ab} + \sum_{\alpha \in \{\emptyset, a, b, ab\}} \|\Pi^\alpha \mathcal{R}_+ K^\alpha\|_{k,0}^\alpha . \end{aligned}$$

The first four terms can be estimated, using Lemma 5.28, as follows:

$$\begin{aligned} \left\| \mathcal{R}_+ H^\emptyset \right\|_{k,0} + \|H^a\|_{k,0}^a + \|H^b\|_{k,0}^b + \left\| \int H^a H^b \mathrm{d}\mu_+ \right\|_{k,0}^{ab} \\ \leq C \|H\|_{k,0}^{\mathrm{ext}} + C_{FRD} h^{-1} \|H^a\|_{k,0}^a \|H^b\|_{k,0}^b . \end{aligned}$$

Derivatives with respect to H are bounded similarly since

$$\left[D_H R_2(H,K) \dot{H} \right]^{\mathrm{obs}} = s \dot{H}^a + t \dot{H}^b + st \left(\int \dot{H}^a H^b \mathrm{d}\mu_+ + \int H^a \dot{H}^b \mathrm{d}\mu_+ \right)$$

and

$$\left[D_H^2 R_2(H,K) (\dot{H})^2 \right]^{\mathrm{obs}} = 2st \int \dot{H}^a \dot{H}^b \mathrm{d}\mu_+ .$$

It remains to show that, for $\alpha \in \{a, b, ab\}$,

$$\|\Pi^\alpha \mathcal{R}_+ K^\alpha\|_{k,0}^\alpha \leq C \|K\|_k^{(A)} .$$

To show this inequality, we use Lemma 5.10 to obtain

$$\|\Pi^\alpha \mathcal{R}_+ K^\alpha\|_{k,0}^\alpha \leq C|\mathcal{R}_+ K|_{k,B,T_0}^{\text{ext}}.$$

For the extended seminorm it holds as in [Buc19] that

$$\|F(B)\|_{k:k+1,B}^{\text{ext}} = \sup_\varphi w_{k:k+1}^{-B}(\varphi)|F(B)|_{k,B,T_\varphi}^{\text{ext}} \geq |F(B)|_{k,B,T_0}^{\text{ext}}.$$

Thus

$$\|\Pi^\alpha \mathcal{R}_+ K^\alpha\|_{k,0}^\alpha \leq C\|\mathcal{R}_+ K(B)\|_{k:k+1,B}^{\text{ext}}.$$

Now we can proceed as in [Buc19], using Lemma 5.6.
Due to the linearity with respect to K the bounds for the derivatives with respect to K follow from the case without derivatives.

□

5.2.5 The map R_1

We extend Lemma 4.7.7 from [Buc19] to our setting.

Lemma 5.20. *Assume $L \geq 2^{d+3} + 16R$. Consider the map*

$$R_1 : \widehat{\mathbf{M}}^{(A/2,B)} \to \widehat{\mathbf{M}}_:^{(A/(2A_\mathcal{P}),B)}, \quad R_1(P) = \mathcal{R}_+ P.$$

For $B \geq 1$ and any $A \geq 4A_\mathcal{P}$ the map R_1 is smooth and satisfies

$$\|D_P^j R_1(P)(\dot{P}, \dots, \dot{P})\|_{k:k+1}^{(A/(2A_\mathcal{P}),B),\text{ext}} \leq \left(\|\dot{P}\|_k^{(A/2),\text{ext}}\right)^j \left(\|P\|_k^{(A/2),\text{ext}}\right)^{1-j}$$

for $j \in \{0,1\}$. The derivatives vanish for $j > 1$.

Proof. The statement for $j = 0$ follows directly from Lemma 5.6. Note that the map R_1 is linear in P so that the statement for $j > 0$ is trivial. □

5.2.6 The map P_1

In the following we extend Lemma 4.7.6 from [Buc19] to observables. Here, $h_0(L)$ is fixed in Lemma 5.1.

Lemma 5.21. *Assume $L \geq \max\left\{2^{d+3} + 16R, 4d(2^d + R)\right\}$, and $h \geq h_0(L)$. Consider the map*

$$\begin{aligned}
&P_1 : \mathbf{M}_{|||} \times \mathbf{M}_{|||} \times \mathbf{M}_{|||} \times \widehat{\mathbf{M}}_:^{(A/(2A_\mathcal{P}),B)} \to \mathbf{M}'^{(A)},\\
&P_1(I_1, I_2, J, K)(U) = \sum_{\substack{X_1, X_2 \in \mathcal{P}_k \\ X_1 \cap X_2 = \emptyset}} \chi(X_1 \cup X_2, U) I_1^{U \setminus (X_1 \cup X_2)} I_2^{(X_1 \cup X_2) \setminus U} J^{X_1} K(X_2).
\end{aligned}$$

Let $A_0(L,d) = (48A_\mathcal{P})^{\frac{L^d}{\alpha}}$ with $\alpha = (1 + 2^d)^{-1}(1 + 6^d)^{-1}$. If $A \geq A_0, B = A$ and if ρ_1, ρ_2, ρ_3 satisfy

$$\rho_1 \leq \frac{1}{2}, \quad \rho_2 \leq A^{-2}, \quad \rho_3 \leq 1,$$

then the map P_1 *restricted to* $U = B_{\rho_1}(1) \times B_{\rho_1}(1) \times B_{\rho_2}(0) \times B_{\rho_3}(0)$ *is smooth and satisfies*

$$\frac{1}{i_1!i_2!j_1!j_2!} \left\| D_{I_1}^{i_1} D_{I_2}^{i_2} D_J^{j_1} D_K^{j_2} P_1(I_1, I_2, J, K)(\dot I_1, \ldots, \dot I_1, \dot I_2, \ldots, \dot I_2, \dot J, \ldots, \dot J, \dot K, \ldots, \dot K) \right\|_{k+1,r}^{(A),\mathrm{ext}}$$
$$\leq \frac{\eta^2}{4} L^d \left(\left|\left|\left| \dot I_1 \right|\right|\right|^{\mathrm{ext}} \right)^{i_1} \left(\left|\left|\left| \dot I_2 \right|\right|\right|^{\mathrm{ext}} \right)^{i_2} \left(A^2 \left|\left|\left| \dot J \right|\right|\right|^{\mathrm{ext}} \right)^{j_1} \left(\left\| \dot K \right\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B),\mathrm{ext}} \right)^{j_2}.$$

Proof. The difference to [Buc19] is the additional factor $\frac{\eta^2}{4} L^d$ here which appears in Lemma 5.5. Apart from that the proof is the same as in [Buc19]. □

Remark 5.22. *Consider the case of the bulk flow, i.e., set* $s = t = 0$. *When inspecting the proof of Lemma 4.7.6 in [Buc19], we get*

$$A^{|U|_{k+1}} \left\| D_{I_1} D_{I_2} D_J D_K P_1(I_1, I_2, J, K)(U)(\dot I_1, \dot I_2, \dot J, \dot K) \right\|$$
$$\leq A^{-x|U|_{k+1}} A^2 \left|\left|\left| \dot I_1 \right|\right|\right| \left|\left|\left| \dot I_2 \right|\right|\right| \left|\left|\left| \dot J \right|\right|\right| \left\| \dot K \right\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)}$$

for $x \in (0, 2\alpha)$. *Namely, we have that*

$$A^{|U|_{k+1}} \left\| D_{I_1} D_{I_2} D_J D_K P_1(I_1, I_2, J, K)(U)(\dot I_1, D_{I_2}, D_J, D_K) \right\|$$
$$\leq \left(\frac{(48 A_{\mathcal{P}})^{2L^d}}{A^{2\alpha}} \right)^{|U|_{k+1}} A^2 \left|\left|\left| \dot I_1 \right|\right|\right| \left|\left|\left| \dot I_2 \right|\right|\right| \left|\left|\left| \dot J \right|\right|\right| \left\| \dot K \right\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)}$$
$$\leq A^{-x|U|_{k+1}} A^2 \left|\left|\left| \dot I_1 \right|\right|\right| \left|\left|\left| \dot I_2 \right|\right|\right| \left|\left|\left| \dot J \right|\right|\right| \left\| \dot K \right\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)}$$

if we choose

$$A \geq (48 A_{\mathcal{P}})^{\frac{2L^d}{2\alpha - x}}.$$

5.2.7 Proof of Lemma 5.14

For the sake of completeness we review the proof as it is done in [Buc19].

Proof of Lemma 5.14. The assertion follows from the smoothness of the individual maps E, P_1, P_2, P_3, R_1 and R_2 and the chain rule.
Let A_0 be as in Lemma 5.21 and set $B = A$. By Lemma 5.21 there exists a neighbourhood

$$O_1 = B_{\rho_1}(1) \times B_{\rho_1}(1) \times B_{\rho_2}(0) \times B_{\rho_3}(0)$$

such that P_1 is smooth in O_1. By Lemma 5.16 there is a neighbourhood

$$O_2 = B_{\rho_4}(0) \subset B_{\frac{1}{8}}(0)$$

such that E is smooth in O_2 and $E(O_2) \subset B_{\rho_1}(1)$ and $1 - E(O_2) \subset B_{\rho_2}(0)$. By Lemma 5.19 there is a neighbourhood

$$O_3 = B_{\rho_5}(0) \times B_{\rho_6}(0)$$

such that R_2 is smooth in O_3 and $R_2(O_3) \subset O_2$. This defines the first restriction on U_{ρ^*}, namely

$$U_{\rho^*} \subset B_{\rho_5}(0) \times B_{\rho_6}(0)$$

The second restriction comes from the condition

$$R_1\left(P_3\left(P_2\left(E(H), K\right)\right)\right) \in B_{\rho_3}(0).$$

By Lemma 5.20 there is a neighbourhood

$$O_4 = B_{\rho_7}(0)$$

such that R_1 is smooth in O_4 and $R_1(O_4) \subset B_{\rho_3}(0)$. By Lemma 5.18 there is a neighbourhood

$$O_5 \subset B_\rho(0)$$

such that P_3 is smooth in O_5 and $P_3(O_5) \subset O_4$. By Lemma 5.17 there is a neighbourhood

$$O_6 = B_{\rho_8}(1) \times B_{\rho_9}(0)$$

such that P_2 is smooth in O_6 and $P_2(O_6) \subset O_5$. Finally, by Lemma 5.16 there is a neighbourhood

$$O_7 = B_{\rho_{10}}(0) \subset B_{\rho_4}(0)$$

such that $E(O_7) \subset B_{\rho_8}(1)$. We obtain the second restriction:

$$U_{\rho_*} \subset B_{\rho_{10}}(0) \times B_{\rho_9}(0).$$

The combination of both constraints yields that $\mathbf{S}$ is C^∞ in the set

$$U_{\rho_*} \subset B_{\rho_{10} \wedge \rho_5}(0) \times B_{\rho_9 \wedge \rho_6}(0).$$

The chain rule implies the bounds on the derivatives. □

Remark 5.23. *Remark 5.22 and chain rule implies that in the case of the bulk flow there is a constant C_1 such that for any $x \in (0, 2\alpha)$ and $(H, K) \in U_\rho$*

$$A^{|U|_{k+1}} \left\| D_H D_K D_q \mathbf{S}_k(H, K, q)(\dot{H}, \dot{K}, \dot{q})(U) \right\|_{k+1,U}$$
$$\leq C_1 A^{-x|U|_{k+1}} A^4 \|\dot{H}\|_{k,0} \|\dot{K}\|_k^{(A)} \|\dot{q}\|,$$

where the factors A come from the estimates on $D_J P_1$, DP_3, and $D_I P_2$.

5.3 Derivatives of the extended renormalisation map at $(0, 0)$

In this section we prove the bounds on $\mathbf{C}$ stated in Proposition 4.8, the bounds on $\mathbf{B}$ stated in Proposition 4.9, a bound on the second order part in $\mathbf{A}$ as used in the proof of Lemma 5.15, and we compute the ab-part of the second derivative of $\mathbf{S}^{\text{ext}}$ at $(0, 0)$ as stated in Proposition 4.6.

5.3.1 Bound on the extended operator C

Let $K \in M^{\mathrm{ext}}(\mathcal{P}_k^c)$, $U \in \mathcal{P}_k^c$, and $\varphi \in \chi_N$. Then $\mathbf{C}K$ can be decomposed into two parts,

$$\mathbf{C}K(U,\varphi) = F(U,\varphi) + G(U,\varphi). \tag{36}$$

The large-polymer part $F \in M^{\mathrm{ext}}(\mathcal{P}_{k+1}^c)$ is defined by

$$F(U,\varphi) = \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{B}_k \\ \pi(X)=U}} \mathcal{R}_+ K(X,\varphi),$$

and G satisfies $G(U,\varphi) = 0$ for all $U \in \mathcal{P}_{k+1}^c \setminus \mathcal{B}_{k+1}$, otherwise, for $U = B_+ \in \mathcal{B}_{k+1}$,

$$G(B_+,\varphi) = \sum_{B \in \mathcal{B}_k(B_+)} G(B,\varphi) \quad \text{with} \quad G(B,\varphi) = (1-\Pi)\mathcal{R}_+ K(B,\varphi).$$

We restate the key bound from Proposition 4.8 as Lemma 5.24 below.

Lemma 5.24. *For any $\theta \in (0,1)$ there exists an L_0 such that for all odd integers $L \geq L_0$ there is A_0 and h_0 with the following property. For all $A \geq A_0$ and for all $h \geq h_0$,*

$$\|\mathbf{C}\|_{k+1}^{(A),\mathrm{ext}} \leq \theta$$

independently of k and N.

The proof is very similar to the proof in [Buc19]. For the argument of the large-polymer part F we have to deal with the additional factor $\frac{\eta^2}{4}L^d$ arising in the transformation of scales from the factor $\frac{l_{\mathrm{obs},k+1}^{|\alpha|}}{l_{\mathrm{obs},k}^{|\alpha|}}$, see 2. in Lemma 5.4.
The following lemma extends Lemma 4.8.2. from [Buc19] to observables.

Lemma 5.25. *Let $L \geq 2^{d+3} + 16R$. There is A_0 such that for all $A \geq A_0$*

$$\|F\|_{k+1}^{(A),\mathrm{ext}} \leq \frac{\theta}{2}\|K\|_k^{(A),\mathrm{ext}}.$$

Proof. Lemma 5.4 states that for $U = \pi(X)$

$$\Big|\mathcal{R}_+ K(X,\varphi)\Big|_{k+1,U,T_\varphi}^{\mathrm{ext}} \leq \frac{\eta^2}{4} L^d \Big|\mathcal{R}_+ K(X,\varphi)\Big|_{k,X,T_\varphi}^{\mathrm{ext}}.$$

By Lemma 5.1 it follows that

$$w_{k:k+1}^X(\varphi) \leq w_{k+1}^U(\varphi).$$

We conclude that

$$\|\mathcal{R}_+ K(X,\varphi)\|_{k+1,U}^{\mathrm{ext}} \leq \frac{\eta^2}{4} L^d \|\mathcal{R}_+ K(X,\varphi)\|_{k:k+1,X}^{\mathrm{ext}}.$$

By this inequality we can estimate

$$A^{|U|_{k+1}} \|F(U)\|^{\text{ext}}_{k+1,U} \\ \leq A^{|U|_{k+1}} \frac{\eta^2}{4} L^d \left(\sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \pi(X)=U}} \|\mathcal{R}_+ K(X)\|^{\text{ext}}_{k:k+1,X} + \sum_{\substack{X \in \mathcal{S}_k \setminus \mathcal{B}_k \\ \pi(X)=U}} \|\mathcal{R}_+ K(X)\|^{\text{ext}}_{k:k+1,X} \right). \tag{37}$$

We bound the two summands in (37) seperately. The first term can be estimated similar to [Buc19], with a change in the choice of A:

$$A^{|U|_{k+1}} \frac{\eta^2}{4} L^d \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \pi(X)=U}} \|\mathcal{R}_+ K(X)\|^{\text{ext}}_{k:k+1,X} \\ \leq \|K\|^{(A),\text{ext}}_k \frac{\eta^2}{4} L^d \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \bar{X}=U}} \left(A_{\mathcal{P}} A^{-\frac{2\alpha}{1+2\alpha}} \right)^{|X|_k},$$

where $\alpha = \left[(1+2^d)(1+6^d)\right]^{-1}$. Let

$$A \geq \left(\frac{A_{\mathcal{P}}}{\bar{\delta}} \frac{4}{\theta} \frac{\eta^2}{4} L^d \right)^{\frac{1+2\alpha}{2\alpha}}$$

where $\bar{\delta}$ is the constant from Lemma C.2 in [Buc19]. Then

$$\sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \pi(X)=U}} \|\mathcal{R}_+ K(X)\|^{\text{ext}}_{k:k+1,X} \leq \frac{\theta}{4} \|K\|^{(A),\text{ext}}_k.$$

For a bound on the second contribution in (37) we again follow closely the proof in [Buc19], with a change in the choice of A. For $U \in \mathcal{B}_{k+1}$ we have

$$A^{|U|_{k+1}} \frac{\eta^2}{4} L^d \sum_{\substack{X \in \mathcal{S}_k \setminus \mathcal{B}_k \\ \pi(X)=U}} \|\mathcal{R}_+ K(X)\|^{\text{ext}}_{k:k+1,X} \leq A \|K\|^{(A),\text{ext}}_k L^d (2^{d+1}+1)^{d2^d} \frac{A_{\mathcal{P}}^2}{A^2} \frac{\eta^2}{4} L^d.$$

If

$$A \geq \frac{4}{\theta} A_{\mathcal{P}}^2 L^d (2^{d+1}+1)^{d2^d} \frac{\eta^2}{4} L^d,$$

then

$$A^{|U|_{k+1}} \frac{\eta^2}{4} L^d \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \pi(X)=U}} \|\mathcal{R}_+ K(X)\|^{\text{ext}}_{k:k+1,X} \leq \frac{\theta}{4} \|K\|^{(A),\text{ext}}_k.$$

For A large enough this finishes the claim. □

Next we consider the contribution from single blocks. We extend Lemma 4.8.4 from [Buc19] to observables.

Lemma 5.26. *There is L_0 such that for all $L \geq L_0$, $h \geq h_0(L)$ and for all $A \geq 1$*

$$\|G\|_{k+1}^{(A),\mathrm{ext}} \leq \frac{\theta}{2}\|K\|_k^{(A),\mathrm{ext}}.$$

Proof. Remember that $G(U) = 0$ for $U \notin \mathcal{B}_{k+1}$ and

$$G(B_+) = \sum_{B\in\mathcal{B}_k(B_+)} G(B) = \sum_{B\in\mathcal{B}_k(B_+)} (1-\Pi)\mathcal{R}_+K(B)$$

for $B_+ \in \mathcal{B}_{k+1}$. Thus

$$\begin{aligned}
\|G\|_{k+1}^{(A),\mathrm{ext}} &\leq A \sup_\varphi w_{k+1}^{-B'}(\varphi) \sum_{B\in\mathcal{B}_k(B_+)} |G(B)|_{k+1,B,T_\varphi}^{\mathrm{ext}} \\
&\leq A \sup_\varphi w_{k+1}^{-B'}(\varphi) \sum_{B\in\mathcal{B}_k(B_+)} \sum_{\alpha\in\{\emptyset,a,b,ab\}} \mathbb{1}_{\alpha\in B} l_{\mathrm{obs},k+1}^{|\alpha|} |G^\alpha(B)|_{k+1,B,T_\varphi}.
\end{aligned}$$

Fix $\alpha \in \{a,b,ab\}$. We use the second inequality in Lemma 5.3 to get

$$\begin{aligned}
|G^\alpha(B)|_{k+1,B,T_\varphi} \leq (1+|\varphi|_{k+1,B})^3 \Big(&|(1-\Pi_k^\alpha)\mathcal{R}_+K^\alpha(B)|_{k+1,B,T_0} \\
&+ 16L^{-\frac{3}{2}d} \sup_{0\leq t\leq 1} |(1-\Pi_k^\alpha)\mathcal{R}_+K^\alpha(B)|_{k,B,T_{t\varphi}}\Big).
\end{aligned}$$

By Lemma 5.12 we proceed the estimate as follows

$$\begin{aligned}
|G^\alpha(B)|_{k+1,B,T_\varphi} \leq (1+|\varphi|_{k+1,B})^3 \Big(&CL^{-(d/2+A(\alpha,k))}|\mathcal{R}_+K^\alpha|_{k,B,T_0} \\
&+ 16L^{-\frac{3}{2}d} \sup_{0\leq t\leq 1} |(1-\Pi_k^\alpha)\mathcal{R}_+K^\alpha(B)|_{k,B,T_{t\varphi}}\Big).
\end{aligned}$$

We continue as in [Buc19] with the estimates

$$\begin{aligned}
|\mathcal{R}_+K^\alpha(B)|_{k,B,T_0} &\leq l_{\mathrm{obs},k}^{-|\alpha|} A_{\mathcal{B}} \|K\|_{k,B}, \\
|\Pi_k^\alpha \mathcal{R}_+K^\alpha(B)|_{k,B,T_{t\varphi}} &\leq C(1+|\varphi|_{k,B})^2 A_{\mathcal{B}} l_{\mathrm{obs},k}^{-|\alpha|}\|K\|_{k,B}, \quad \text{and} \\
|\mathcal{R}_+K^\alpha(B)|_{k,B,T_{t\varphi}} &\leq A_{\mathcal{B}} w_{k:k+1}^B(\varphi) l_{\mathrm{obs},k}^{-|\alpha|}\|K(B)\|_{k,B},
\end{aligned}$$

where we have the additional factor $l_{\mathrm{obs},k}^{-|\alpha|}$ on the right hand sides in contrast to [Buc19]. We obtain

$$\begin{aligned}
&|G^\alpha(B)|_{k+1,B,T_\varphi} \\
&\quad\leq l_{\mathrm{obs},k}^{-|\alpha|} (1+|\varphi|_{k+1,B})^3 \Big(CL^{-(d/2+A(\alpha,k))} A_{\mathcal{B}}\|K\|_{k,B} \\
&\qquad\quad + 16L^{-\frac{3}{2}d} A_{\mathcal{B}} w_{k:k+1}^B(\varphi)\|K\|_{k,B} + 16L^{-\frac{3}{2}d} C(1+|\varphi|_{k,B})^2 A_{\mathcal{B}}\|K\|_{k,B}\Big) \\
&\quad\leq A_{\mathcal{B}} C l_{\mathrm{obs},k}^{-|\alpha|} (1+|\varphi|_{k+1,B})^5 \|K\|_{k,B} \Big(L^{-(d/2+A(\alpha,k))} + L^{-\frac{3}{2}d} w_{k:k+1}^B(\varphi)\Big) \\
&\quad\leq C' l_{\mathrm{obs},k}^{-|\alpha|} w_{k+1}^{B'}(\varphi)\|K\|_{k,B} \Big(L^{-(d/2+A(\alpha,k))} + L^{-\frac{3}{2}d}\Big).
\end{aligned}$$

For $\alpha = \emptyset$ we use the result from [Buc19], namely that

$$|G^{\emptyset}(B)|_{k+1,B,T_\varphi} \le C' w_{k+1}^{B'}(\varphi)\|K\|_{k,B}\left(L^{-d'} + L^{-\frac{3}{2}d}\right)$$

with $d' = \frac{d}{2} + \lfloor d/2 \rfloor + 1 > d$.
Let $d'(\alpha,k) = d'$ for $\alpha = \emptyset$ and $d'(\alpha,k) = d/2 + A(\alpha,k)$ else. We combine the estimates obtained so far and obtain

$$\|G\|_{k+1}^{(A),\mathrm{ext}} \le C' \sum_{\alpha\in\{\emptyset,a,b,ab\}} \sum_{B\in\mathcal{B}_k(B_+)} \mathbb{1}_{\alpha\in B} l_{\mathrm{obs},k+1}^{|\alpha|} l_{\mathrm{obs},k}^{-|\alpha|} A^{|B|_k}\|K\|_{k,B}\left(L^{-d'(\alpha,k)} + L^{-\frac{3}{2}d}\right).$$

In the case $\alpha = \emptyset$, the sum over all $B \in \mathcal{B}_k(B_+)$ gives an additional factor L^d. In contrast, for $\alpha \in \{a,b,ab\}$, the sum reduces to one term so this factor does not arise. However, we have

$$\left(\frac{l_{\mathrm{obs},k+1}}{l_{\mathrm{obs},k}}\right)^{|\alpha|} = \begin{cases} (2\eta)^{|\alpha|} & \text{if } \alpha \in \{a,b,ab\},\ k \ge j_{ab}, \\ \left(\frac{\eta}{2}L^{d/2}\right)^{|\alpha|} & \text{if } \alpha \in \{a,b\},\ k < j_{ab} \end{cases}$$

which is canceled by $L^{-d'(\alpha,k)}$. In summary we thus get

$$\|G\|_{k+1}^{(A),\mathrm{ext}} \le C\|K\|_k^{(A),\mathrm{ext}}\left(L^{d-d'} + L^{-\frac{1}{2}d} + L^{-1} + L^{-d} + L^{-\frac{d}{2}} + L^{-\frac{3}{2}d}\right).$$

Now choose L large enough such that

$$\|G\|_{k+1}^{(A),\mathrm{ext}} \le \frac{\theta}{2}\|K\|_k^{(A),\mathrm{ext}}.$$

□

5.3.2 Bounds on the extended operator B

Here we prove Proposition 4.9. We restate the result in the following lemma.

Lemma 5.27. *For $\alpha \in \{a,b\}$, with the constant $A_{\mathcal{B}}$ from Lemma 5.1 which is independent of L, the following estimates hold:*

$$\left|(\mathbf{B}K_k^{\alpha})^1\right| \le l_k^{-1} l_{\mathrm{obs},k}^{-1}\frac{A_{\mathcal{B}}}{2}\|K_k\|_k^{(A),\mathrm{ext}},$$
$$\left|(\mathbf{B}K_k^{\alpha})^0\right| \le l_{\mathrm{obs},k}^{-1}\frac{A_{\mathcal{B}}}{2}\|K_k\|_k^{(A),\mathrm{ext}},$$
$$\left|\mathbf{B}K_k^{ab}\right| \le l_{\mathrm{obs},k}^{-2}\frac{A_{\mathcal{B}}}{2}\|K_k\|_k^{(A),\mathrm{ext}}.$$

Proof. The proof is similar to the one of Lemma 5.10. First, by Lemma 5.8,

$$\left|(\mathbf{B}K_k^{\alpha})^1\right| = \left|\langle \mathcal{R}_+K_l^{\alpha}, b^{\alpha}\rangle_0\right| \le |b^{\alpha}|_{k,B}\left|\mathcal{R}_+K_k^{\alpha}(B)\right|_{k,B,T_0} \le l_k^{-1} l_{\mathrm{obs},k}^{-1}\frac{A_{\mathcal{B}}}{2}\|K_k\|_k^{(A),\mathrm{ext}}.$$

Furthermore,

$$\left|(\mathbf{B}K_k^{\alpha})^0\right| \le \int \left|K_k^{\alpha}(B,\xi)\right|\mu_{k+1}(\mathrm{d}\xi) \le l_{\mathrm{obs},k}^{-1}\frac{A_{\mathcal{B}}}{2}\|K_k\|_k^{(A),\mathrm{ext}}$$

and similarly,

$$\left|\mathbf{B}K_k^{ab}\right| \le \int \left|K_k^{ab}(B,\xi)\right|\mu_{k+1}(\mathrm{d}\xi) \le l_{\mathrm{obs},k}^{-2}\frac{A_{\mathcal{B}}}{2}\|K_k\|_k^{(A),\mathrm{ext}}.$$

□

5.3.3 Bound on the extended operator A

Lemma 5.28. *Let $H^a = n^a \nabla\varphi(b)$, $H^b = n^b \nabla\varphi(b)$, $k \geq j_{ab}$. Then*

$$\Big| \int H^a H^b \mathrm{d}\mu_{k+1} \Big| \leq C_{FRD} l_{\mathrm{obs},k}^{-2} h_k^{-2} \|H^a\|_{k,0}^a \|H^b\|_{k,0}^b.$$

Proof. Note that

$$\int \nabla\varphi(a)\nabla\varphi(b)\mu_{k+1}(\mathrm{d}\varphi) = \nabla^*\nabla C_{k+1}(a,b)$$

and

$$|n^a| \leq l_{\mathrm{obs},k}^{-1} l_k^{-1} \|H^a\|_{k,0}^a.$$

By the properties of the finite-range decomposition the proof follows straightforwardly. □

5.3.4 Second derivative of $\mathbf{S}^{\mathrm{ext}}$ at $(0,0)$

Here we prove Proposition 4.6. We restate the result in the following lemma.

Lemma 5.29. *The st-part of the second derivative in direction H of $\mathbf{S}^{\mathrm{ext}}$ is zero:*

$$\Big[D_H^2 \mathbf{S}^{\mathrm{ext}}(0,0)(\dot{H},\dot{H}) \Big]^{ab} = 0.$$

Proof. Note that

$$D_H^2 \mathbf{S}^{\mathrm{ext}}(0,0)(\dot{H},\dot{H}) = D_H^2 \mathbf{S}(0,0)(\dot{H},\dot{H})$$

since $\mathbf{S}(0,0) = 0$ and

$$D_H \left(e^{-s(\mathbf{B}K^a)^0 - t(\mathbf{B}K^b)^0 - st\left(\int H^a H^b \mathrm{d}\mu_+ + \mathbf{B}K^{ab}\right)} \right) \Big|_{H=K=0} \dot{H} = 0.$$

By the product rule we get a sum of the following three terms:

$$\begin{aligned}
& D_H^2 \mathbf{S}^{\mathrm{ext}}(0,0)(\dot{H},\dot{H}) \\
&\quad = 2 \sum_{X \in \mathcal{P}_k} \chi(X,U) D_H \left(\left(e^{\tilde{H}} \right)^{U \setminus X} \right) \dot{H} \Big|_{H=K=0} \times \\
&\qquad\qquad \int D_H \left(\left(e^H - e^{\tilde{H}} \right)^X \right) \dot{H} \Big|_{H=K=0} \mathrm{d}\mu_+ \\
&\qquad + 2 \sum_{X \in \mathcal{P}_k} \chi(X,U) D_H \left(\left(e^{\tilde{H}} \right)^{-X \setminus U} \right) \dot{H} \Big|_{H=K=0} \times \\
&\qquad\qquad \int D_H \left(\left(e^H - e^{\tilde{H}} \right)^X \right) \dot{H} \Big|_{H=K=0} \mathrm{d}\mu_+ \\
&\qquad + \sum_{X \in \mathcal{P}_k} \chi(X,U) \int D_H^2 \left(\left(e^H - e^{\tilde{H}} \right)^X \right) (\dot{H},\dot{H}) \mathrm{d}\mu_+ .
\end{aligned}$$

Let us consider the second term in the right hand side above. We compute

$$D_H\left(\left(e^H-e^{\tilde{H}}\right)^X\right)\dot{H}\Big|_{H=K=0}=\mathbb{1}_{X=B}\left(\dot{H}(B)-D_H\tilde{H}(B)\dot{H}\big|_{H=K=0}\right).$$

The constraint $X=B$ for any $B\in\mathcal{B}_k$ implies that $X\setminus U=\emptyset$ for any U satisfying $\chi(X,U)\neq 0$. Thus the second term is zero.

The ab-part of the first term is zero as well. We compute

$$D_H\left(\left(e^{\tilde{H}}\right)^{U\setminus X}\right)\dot{H}\Big|_{H=K=0}=\sum_{B\in\mathcal{B}_k(U\setminus X)}\left(\tilde{A}\dot{H}^\emptyset+s\dot{H}^a+t\dot{H}^b\right)(B)$$

and

$$\begin{aligned}\int D_H\left(\left(e^H-e^{\tilde{H}}\right)^X\right)\dot{H}\Big|_{H=K=0}\mathrm{d}\mu_+ \\ &=\mathbb{1}_{X=B}\int\dot{H}^\emptyset(B,\varphi+\xi)+s\dot{H}^a(B,\varphi+\xi)+t\dot{H}^b(B,\varphi+\xi)\\ &\qquad-\tilde{A}\dot{H}^\emptyset(B,\varphi)-s\dot{H}^a(B,\varphi)-t\dot{H}^b(B,\varphi)\mathrm{d}\mu_+\\ &=\mathbb{1}_{X=B}\int\dot{H}^\emptyset(B,\varphi+\xi)-\tilde{A}\dot{H}^\emptyset(B,\varphi)\mathrm{d}\mu_+.\end{aligned}$$

The last equality holds since

$$\dot{H}^a(B,\varphi+\xi)=\dot{H}^a(B,\varphi)+\dot{H}^a(B,\xi)$$

and

$$\int\dot{H}^a(B,\xi)\mathrm{d}\mu_+ = 0$$

due to linearity. Thus the first term has bulk parts and a- and b-parts, but the projection to the ab-part is zero.

For the third term we distinguish the case that $X=B$ for $B\in\mathcal{B}_k$ and $X=B\cup B'$ for $B,B'\in\mathcal{B}_k$, $B\neq B'$. In the case $X=B$ we compute

$$\begin{aligned}\int D_H^2\left(\left(e^H-e^{\tilde{H}}\right)^B\right)(\dot{H},\dot{H})\mathrm{d}\mu_+ \\ &=\int\left(\dot{H}(B,\varphi+\xi)\right)^2-2st\int\dot{H}^a(B)\dot{H}^b(B)\mathrm{d}\mu_+\\ &\qquad-\left(\tilde{A}\dot{H}^\emptyset(B,\varphi)+s\dot{H}^a(B,\varphi)+t\dot{H}^b(B,\varphi)\right)^2\mathrm{d}\mu_+\\ &=2\int\dot{H}^a(B,\varphi+\xi)\dot{H}^b(B,\varphi+\xi)\mathrm{d}\mu_+-2\int\dot{H}^a(B,\xi)\dot{H}^b(B,\xi)\mathrm{d}\mu_+\\ &\qquad-2\int\dot{H}^a(B,\varphi)\dot{H}^b(B,\varphi)\mathrm{d}\mu_+=0.\end{aligned}$$

In the other case we compute

$$\int D_H\left(\left(e^H - e^{\tilde{H}}\right)^B\right)\dot{H} D_H\left(\left(e^H - e^{\tilde{H}}\right)^{B'}\right)\dot{H}\mathrm{d}\mu_+$$
$$= \int \left(\dot{H}(B,\varphi+\xi) - \tilde{A}\dot{H}^{\emptyset}(B,\varphi) - s\dot{H}^a(B,\varphi) - t\dot{H}^b(B,\varphi)\right)$$
$$\left(\dot{H}(B',\varphi+\xi) - \tilde{A}\dot{H}^{\emptyset}(B',\varphi) - s\dot{H}^a(B',\varphi) - t\dot{H}^b(B',\varphi)\right)\mathrm{d}\mu_+.$$

We project this term to the ab-part and obtain:

$$\int \left(\dot{H}^a(B,\varphi+\xi) - \dot{H}^a(B,\varphi)\right)\left(\dot{H}^b(B',\varphi+\xi) - \dot{H}^b(B',\varphi)\right)\mathrm{d}\mu_+$$
$$+ \int \left(\dot{H}^b(B,\varphi+\xi) - \dot{H}^b(B,\varphi)\right)\left(\dot{H}^a(B',\varphi+\xi) - \dot{H}^a(B',\varphi)\right)\mathrm{d}\mu_+$$
$$= \int \dot{H}^a(B,\xi)\dot{H}^b(B',\xi)\mathrm{d}\mu_+ + \int \dot{H}^b(B,\xi)\dot{H}^a(B',\xi)\mathrm{d}\mu_+.$$

Now we distinguish the scales $k \geq j_{ab}$ and the scales $k < j_{ab}$. If $k \geq j_{ab}$, then $a, b \in B_{ab} \in \mathcal{B}_k$, and either $B = B_{ab}$ and the B'-term is zero, or vice versa. If $k < j_{ab}$ only the choices $B \cup B' = B_a \cup B_b$ and $B \cup B' = B_b \cup B_a$ are relevant. Then we get

$$\int \dot{H}^a(B,\xi)\dot{H}^b(B',\xi)\mathrm{d}\mu_+ + \int \dot{H}^b(B,\xi)\dot{H}^a(B',\xi)\mathrm{d}\mu_+$$
$$= 2n_a n_b \int \nabla\varphi(a)\nabla\varphi(b)\mathrm{d}\mu_{k+1} = 2n_a n_b \nabla^*\nabla C_{k+1}(ab).$$

Due to the definition of the scale j_{ab} and the finite-range property of the covariances we have

$$\nabla^*\nabla C_{k+1}(a,b) = 0 \text{ for all } k < j_{ab}.$$

This finishes the claim. □

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