Renormalisation in discrete elasticity

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
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aus
Dortmund

Bonn 2019

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 28.08.2019
Erscheinungsjahr: 2019

Summary

This thesis deals with the statistical mechanics of lattice models. It has two main contributions. On the one hand we implement a general framework for a rigorous renormalisation group approach to gradient models. This approach relies on work by Bauerschmidt, Brydges, and Slade and extends earlier results for gradient interface models by Adams, Kotecký and Müller. On the other hand we use those results to analyse microscopic models for discrete elasticity at small positive temperature and in particular prove convexity properties of the free energy.

The first Chapter is introductory and discusses the necessary mathematical background and the physical motivation for this thesis.

Chapters 2 to 4 then contain a complete and almost self contained implementation of the renormalisation group approach for gradient models.

Chapter 2 is concerned with a new construction of a finite range decomposition with improved regularity. Finite range decompositions are an important ingredient in the renormalisation group approach but also appear at various other places. The new finite range decomposition helps to avoid a loss of regularity and several technical problems that were present in the earlier applications of the renormalisation group technique to gradient models.

In the third Chapter we analyse generalized gradient models and discrete models for elasticity and we state our main results: At low temperatures the surface tension is locally uniformly convex and the scaling limit is Gaussian. Moreover, we show that those statements can be reduced to a general statement about perturbations of massless Gaussian measures using suitable null Lagrangians. This is a first step towards a mathematical understanding of elastic behaviour of crystalline solids at positive temperatures starting from microscopic models.

The fourth Chapter contains the renormalisation group analysis of gradient models. The main result is a bound for certain perturbations of Gaussian gradient measures that implies the results of the previous chapters. This generalizes earlier results for scalar nearest neighbour models to vector-valued finite range interactions. We also require a much weaker growth assumption for the perturbation. This is possible because we introduce a new solution to the large field problem based on an alternative construction of the weight functions using Gaussian calculus.

The last Chapter has a slightly different focus. We investigate gradient interface models for a specific class of non-convex potentials for which phase transitions occur in dimension two. The analysis of these potentials is based on the relation to a random conductance model. We study properties of this random conductance model and in particular prove correlation inequalities and reprove the phase transition result relying on planar duality instead of reflection positivity.

Acknowledgments

First and foremost, I would like to thank my advisor Professor Stefan Müller for his invaluable advice and constant support.

I thank Professor Massimiliano Gubinelli, Professor Herbert Koch, and Professor Robert Glaum for being members of my thesis committee.

The Institute of Applied Mathematics provided a great working environment and I would like to thank all my colleagues from the analysis groups. In particular, I am grateful to Susanne and Florian for several stimulating discussions. In addition I am grateful to Florian for proofreading parts of this thesis.

I gratefully acknowledge financial support from the Bonn International Graduate School in Mathematics (BIGS). Parts of this work were done during stays at the Courant Institute and the Isaac Newton Institute and I thank those institutions for their hospitality.

Finally, I would like to thank Annika for her constant support.

Contents

Chapter 1

Introduction

1.1 Outline

The main focus of this thesis is the derivation of properties of the free energy and Gibbs measures of realistic atomistic models for crystalline elastic materials using renormalisation techniques. The purpose of this chapter is to explain the concepts that appear in the previous sentence and provide the necessary background, in particular regarding statistical mechanics, elasticity, and renormalisation. Let us now briefly outline the structure of this introduction.

In Section 1.2 we introduce statistical mechanics which is the mathematical and physical concept underlying this work. Statistical mechanics is the equilibrium theory of physical systems with a large number of degrees of freedom. We also introduce spin systems as an important class of models in statistical mechanics. This class contains systems with a fixed spatial ordering of particles which arise in the modelling of, e.g., magnetism but also elasticity, the focus of this work.

Then we discuss our main results and their implications which go in two directions. The most important results are about realistic microscopic discrete models for elasticity. They are presented along with the necessary background on mathematical elasticity theory, in particular the Cauchy-Born rule in Section 1.3. Our results do not only apply to discrete models for elasticity but to the more general class of finite range gradient models. The prototypical example are gradient interface models which have caught considerable attention in the literature. In Section 1.4 we discuss our results in the context of gradient interface models.

The next section is devoted to our method: The renormalisation group approach. This is a powerful technique to investigate the large (or small) scale behaviour of physical systems that has been employed at various places in physics and mathematics. We will briefly discuss the historic background as well as the recent advances in the mathematical rigorous theory made by Bauerschmidt, Brydges, and Slade and we give a brief sketch of the method in the context of gradient models. Finally, in the last Section 1.6 of this chapter we will briefly introduce phase transitions in the context of gradient interface models. This introduction does not contain original material even though we do not provide line by line references.

1.2 Statistical mechanics and spin systems

1.2.1 Statistical mechanics

Statistical mechanics deals with the study of macroscopic physical systems at equilibrium. This is an area of mathematics and physics research with a long history and we will only give a very brief introduction of the most important notions and refer to the literature (e.g., [56, 114, 117, 133]) for a more detailed overview. Macroscopic physical systems consist of a huge number of particles which is typically of the order 10^{23}. For a system we denote the state space of possible configurations by Ω which is then a high dimensional manifold. Classical systems can be parametrized by the position and the momenta of the constituents. For quantum systems the state is given by the wave function which is an element of a suitable Hilbert space. It is possible to describe the time evolution of the system using the Hamilton equations for classical mechanics or the Schrödinger equation for quantum systems. However, from a practical viewpoint this approach is completely infeasible: We can neither observe the exact state of the systems, nor can we calculate its evolution, and moreover the exact state is not very helpful to understand the macroscopic properties of the system arising from the collective behaviour of the particles. Therefore the approach of statistical mechanics is to consider instead of a microstate $\omega \in \Omega$ an equilibrium measure on Ω. This measure depends on a (canonical) a priori measure λ on Ω (typically the Lebesgue measure in the classical case) and a Hamiltonian, i.e., an energy function $H : \Omega \to \mathbb{R}$. Then the Gibbs measure (canonical ensemble) for a system that is in thermal equilibrium with a heat bath at inverse temperature $\beta = T^{-1}$ is given by

$$P(\mathrm{d}\omega) = \frac{e^{-\beta H(\omega)}\,\lambda(\mathrm{d}\omega)}{Z} \tag{1.2.1}$$

where $Z = Z(\beta)$ denotes the partition function that is the normalisation of the measure, i.e.,

$$Z = \int_\Omega e^{-\beta H(\omega)}\,\lambda(\mathrm{d}\omega). \tag{1.2.2}$$

The exponential weight $e^{-\beta H}$ is often called Boltzmann weight because Boltzmann gave the first justification for this distribution in [135]. Note that there is a competition between energy, i.e., states with low energy are more probable and entropy, i.e., the phase space volume contributes to the distribution. For zero temperature the measure is concentrated on the minimizers of the energy. It is also possible to describe systems which exchange particles with their environment within this formalism using the grand canonical ensemble. In this case there is the number operator $N : \Omega \to \mathbb{N}$ specifying the numbers of particles in a given state. Then the grand canonical ensemble for inverse temperature β and chemical potential μ is the probability distribution given by

$$P(\mathrm{d}\omega) = \frac{e^{-\beta H(\omega)+\mu N(\omega)}}{Z} \tag{1.2.3}$$

where $Z = Z(\beta, \mu)$ denotes the grand-canonical partition function that normalizes the measure.

1.2.2 Spin systems and Gibbs measures

Spin systems are a class of statistical mechanics systems that can be used to describe systems which are arranged in a fixed spatial configuration that have some internal degree of freedom (spin) at each point. Here we follow [86, 92] where a detailed exposition is given. Let S be a finite or countable set. In this thesis we will almost exclusively consider the case of the hypercubic lattices $S = \mathbb{Z}^d$ or $S = (\mathbb{Z}/L\mathbb{Z})^d$. Let $(E, \mathcal{E})$ be a measurable space which will be the single spin space. We will mostly consider $E = \mathbb{R}^m$. We investigate fields which are maps $\sigma : S \to E$ or equivalently $\sigma \in E^S = \Omega$. Corresponding to those equivalent viewpoints it is convenient to use both, $\sigma(x)$ and σ_x, for the value of the field at $x \in S$ and we will switch between the two expressions in the following. A random field or spin system is a probability measure on the space $(E^S, \mathcal{E}^S)$. We will denote the product σ-algebra by $\mathcal{F} = \mathcal{E}^S$ and the probability measures on $(\Omega, \mathcal{F})$ by $\mathcal{P}(\Omega, \mathcal{F})$. Random fields can be used to model, e.g., magnetism (this is where the name spin system arises), lattice gases, configuration models (height functions for dimer models), interfaces between phases, and displacements in crystals.

Example 1.2.1. *An important class of random fields are Gaussian fields. Let $C \in \mathbb{R}^{S\times S}$ be a positive definite matrix indexed by S. Then the centred (i.e., mean zero) Gaussian random field with covariance C is the probability measure μ_C characterized by its Laplace transform*

$$\int_{R^S} e^{(f,\varphi)_S}\, \mu_C(\mathrm{d}\varphi) = e^{\frac{1}{2}(f,Cf)} \tag{1.2.4}$$

for any $f \in \mathbb{R}^S$ where $(\cdot,\cdot)$ denotes the standard scalar product on $\mathbb{R}^S$. Gaussian fields will play a central role in this work.

While it is straightforward to consider Gibbs measures as introduced in the previous paragraph for finite S, the definition of Gibbs measures for infinite S is a major challenge because the energy is typically infinite in this case. Dobrushin, Lanford, and Ruelle proposed to define Gibbs measures in infinite volume by the condition that every finite subsystem of the infinite system is in thermal equilibrium [76, 133]. We will make this idea precise in the following. We denote by $\mathcal{F}_\Lambda$ for $\Lambda \subset S$ the pullback of the product σ-algebra $\mathcal{E}^\Lambda$ on E^Λ along the canonical projection. We write $\mathcal{F} = \mathcal{F}_S = \mathcal{E}^S$. Recall that a probability kernel from a measurable space $(X, \mathcal{X})$ to another measurable space $(Y, \mathcal{Y})$ is a map $\pi : \mathcal{Y} \times X \to \mathbb{R}_+$ such that

1. $\pi(\cdot, x)$ is a measure on $(X, \mathcal{X})$ for every $x \in X$.
2. $\pi(A, \cdot)$ is a $\mathcal{X}$ measurable function for $A \in \mathcal{Y}$.
3. $\pi(Y, x) = 1$ for all $x \in X$.

Let $\mathcal{X}'$ be a sub σ-algebra of $\mathcal{X}$. A probability kernel from $(X, \mathcal{X}')$ to $(X, \mathcal{X})$ is proper if

$$\pi(X', \cdot) = \mathbb{1}_{X'}, \text{for } X' \in \mathcal{X}'. \tag{1.2.5}$$

Definition 1.2.2. *A specification is a family of proper probability kernels γ_Λ from $(E^S, \mathcal{F}_{\Lambda^c})$ to $(E^S, \mathcal{F})$ indexed by finite subsets $\Lambda \subset S$ such that $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$ for $\Lambda \subset \Lambda'$.*

We define the set of Gibbs measures for this specification by

$$\mathcal{G}(\gamma) = \{\mu \in \mathcal{P}(\Omega, \mathcal{F}) : \mu(A \mid \mathcal{F}_{\Lambda^c})(\cdot) = \gamma_\Lambda(A, \cdot)\ \mu \text{ a.s. for } A \in \mathcal{F} \text{ and } \Lambda \subset S \text{ finite}\}. \tag{1.2.6}$$

We will now describe the specifications we are interested in. We restrict ourselves to the case $S = \mathbb{Z}^d$ and translation invariant potentials. Consider a finite set $A \subset \mathbb{Z}^d$ and let $U : E^A \to \mathbb{R}$ be a local interaction energy. For $x \in Z^d$ we introduce the shift τ_x acting on fields φ by $(\tau_x\varphi)_y = \varphi_{y-x}$, on sets $\Lambda \subset \mathbb{Z}^d$ by $\tau_x A = A + x$ and on events $A \in \mathcal{F}$ by $\tau_x(A) = \{\varphi : \tau_x\varphi \in A\}$. We define the energy $H_\Lambda : E^{\mathbb{Z}^d} \to \mathbb{R}$ for $\Lambda \subset \mathbb{Z}^d$ finite by

$$H_\Lambda(\varphi) = \sum_{x\in\mathbb{Z}^d \,:\, \tau_x(A)\cap\Lambda\neq\emptyset} U(\varphi_{\tau_x(A)}). \tag{1.2.7}$$

Here $\varphi_{\tau_x(A)}$ denotes the restriction of φ to $\tau_x(A)$. We follow the convention used in [116] although formally one should write $U((\tau_x\varphi)_A)$. We write $A_0 = \{0, e_1, \dots, e_d\}$.

Example 1.2.3. *The Ising model has single spin space* $E = \{-1, 1\}$ *and energy* $U(\sigma_{A_0}) = \sum_{i=1}^{d} J\sigma_0\sigma_{e_i} + h\sigma_0$ *where* $J, h \in \mathbb{R}$ *denote the coupling constant and magnetic field strength and* e_i *the standard unit vectors. The study of this model was the starting point of statistical mechanics and the Ising model is still an active topic of research.*

Example 1.2.4. *The discrete Gaussian free field has single spin space* $E = \mathbb{R}$ *and its energy is given by* $U(\varphi_{A_0}) = \sum_{i=1}^{d} \frac{1}{2}(\varphi_{e_i} - \varphi_0)^2 = \frac{1}{2}|\nabla\varphi_0|^2$ *where* $(\nabla\varphi_x)_i = \varphi_{x+e_i} - \varphi_x$ *denotes the discrete gradient. This thesis is mostly concerned with generalisation of this example.*

Example 1.2.5. *The single spin space for scalar* φ^4 *theory is* $E = \mathbb{R}$ *and the energy is given by* $U(\varphi_{A_0}) = \frac{1}{2}|\nabla\varphi_0|^2 + \frac{1}{2}\nu\varphi_0^2 + \frac{1}{4}g\varphi_0^4$. *This is the simplest and most studied example in quantum field theory. The renormalisation method we use was initially developed for this model. In the literature the gradient term is usually written with a Laplacian using a summation by parts.*

We assume that there is an a priori measure λ on E which typically is the counting measure for finite or countable E and the Lebesgue measure for $\mathbb{R}^m$. The Gibbs specification $\gamma_{\Lambda,\beta}^U$ for the energy U at inverse temperature β is defined by

$$\gamma_\Lambda^U(A, \psi) = \frac{1}{Z_{\Lambda,\psi,\beta}} \int_A e^{-\beta H_\Lambda(\varphi)} \prod_{x\in\Lambda} \lambda(\mathrm{d}\varphi_x) \prod_{x\notin\Lambda} \delta_{\psi_x}(\mathrm{d}\varphi_x) \tag{1.2.8}$$

where $Z_{\Lambda,\psi,\beta}$ denotes the normalisation. A simple calculation shows that this defines a specification.

We define for $a \in \mathbb{R}^m$ the shift t_a of a field $\varphi \in (\mathbb{R}^m)^{\mathbb{Z}^d}$ by $(t_a\varphi)_x = \varphi_x + a$. This should not be confused with the positional shift τ_a. An interaction U is shift invariant if $U(\varphi_A) = U((t_a\varphi)_A)$ for all $a \in \mathbb{R}^m$ and all $\varphi \in (\mathbb{R}^m)^{\mathbb{Z}^d}$. We refer to shift invariant interactions $U : (\mathbb{R}^m)^A \to \mathbb{R}$ as gradient models because $U(\varphi_A)$ can be expressed in terms of the field differences $\varphi_x - \varphi_y$ for $x, y \in A$. The discrete Gaussian free field introduced in Example 1.2.4 is the most prominent gradient model while φ^4-theory is not a gradient model. In this thesis we will restrict our attention to gradient models. Gradient models are sometimes called massless using the interpretation in quantum field theory. Other models are similarly called massive.

The question of existence and uniqueness of Gibbs measures are two important recurring questions in statistical mechanics. The uniqueness of Gibbs measures is often a subtle issue and we will discuss it in slightly more detail in Section 1.6. If more than one Gibbs measure exists one says that a phase transition occurs [92]. It was famously shown by Peierls in [130] that phase transitions occur for the Ising model without magnetic field in dimension $d \geq 2$. This shows that the approach is powerful enough to model materials where different phases (identified with the different Gibbs states) exist.

The existence question for Gibbs measures is in general simpler and answered positively in many situations. E.g., for finite spin space and bounded interaction it is rather easy to prove the existence of a Gibbs measure (see Theorem 4.23 in [92]) using compactness arguments. For unbounded state spaces the existence problem is non-trivial. We discuss this briefly in the context of $E = \mathbb{R}^m$ and gradient models. For gradient models existence of a Gibbs measure depends on the dimension. It can be shown that an infinite volume Gaussian free field as defined in Example 1.2.4 exists in dimension $d \geq 3$ but does not exist in $d \leq 2$. This is the case because the variance blows up when the size of Λ is increased. Indeed, in dimension $d = 1$ the central limit theorem indicates that $\mathbb{E}(\varphi_0^2) \approx \sqrt{N}$ for the Gaussian free field on $\Lambda = [-N, N]$ with zero boundary condition.

Therefore one often considers gradient Gibbs measures. Roughly speaking this means that we consider the gradient of a field instead of the field itself, i.e. one restricts attention to the

σ-algebra generated by the gradient variables

$$\eta_{xy} = \varphi_y - \varphi_x \quad \text{for } |x-y| = 1. \tag{1.2.9}$$

In particular, the law of $\nabla\varphi$ is a gradient Gibbs measure if φ is distributed according to a Gibbs measure. The formal definition of gradient Gibbs measure is a bit technical and we refer to Chapter 5 and the literature [89, 136]. Note that for shift invariant interaction U is measurable with respect to the gradient variables. Infinite volume gradient Gibbs measures often exist when Gibbs measures do not exist, e.g., there is a gradient Gibbs measure in $d = 1$ and $d = 2$ for the quadratic potential of the Gaussian free field.

In the context of models for solids or interfaces it is natural to impose tilted boundary conditions, i.e., one considers $\psi = \psi_F$ with $\psi_F(x) = Fx$ in (1.2.8). This corresponds to a deformed state or a tilted surface and we refer to F as the tilt vector (or deformation matrix for vector valued fields). In this setting no translation invariant Gibbs measures exist, however, in general translation invariant gradient Gibbs measures exist. For a translation invariant gradient Gibbs measure we define its tilt by

$$F = \mathbb{E}\left(\nabla\varphi(0)\right). \tag{1.2.10}$$

An important quantity in statistical mechanics is the free energy density that is defined by

$$W(\psi, \beta) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{-\ln(Z_{\Lambda,\psi,\beta})}{\beta|\Lambda|} \tag{1.2.11}$$

if this limit exists. Note that for E finite and bounded finite range interactions U the free energy W does not depend on the boundary conditions. If $E = \mathbb{R}^m$ we are mostly interested in the affine boundary conditions ψ_F and we write

$$W(F, \beta) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{-\ln(Z_{\Lambda,\psi_F,\beta})}{\beta|\Lambda|}. \tag{1.2.12}$$

Let us remark that the existence of the limit is non-trivial and we will briefly discuss this in Chapter 3.

1.3 Mathematical theory of nonlinear elasticity

A body is elastic if it returns to its original shape when an applied force is removed. This is an extremely common phenomenon observed in a wide range of materials ranging from rubber to metals. Therefore a sound theoretical understanding of elasticity is very important, e.g., for engineering applications. Starting with the work of Ball the macroscopic theory of elasticity has seem tremendous progress. However, it remains mostly open how the macroscopic theory can be justified from microscopic models. This section explains the difficulties of this problem, gives a brief overview of the physical background, and explains how microscopic models for elasticity fit in the context provided in the previous section. We also present a simplified version of the main results of this thesis. Parts of the following are close to [15] and [115] where a more detailed exposition is given.

1.3.1 Macroscopic theory of nonlinear elasticity

Since we do not actually work with the macroscopic theory of nonlinear elasticity we will only give a brief overview of the theory that motivates our results and refer to the extensive literature for a more detailed treatment (see, e.g., [8, 58, 59, 100, 124]). In continuum mechanics one considers a reference configuration $\Omega \subset \mathbb{R}^3$ that is deformed by applied forces and boundary conditions. This deformation is described by a deformation map $\varphi : \Omega \to \mathbb{R}^3$. A hyperelastic (or Greens elastic) material is characterized by a stored energy function $W_{SE} : \mathbb{R}^{3\times 3} \times \mathbb{R}_+ \to \mathbb{R}$ that gives the energy density as a function of a local deformation gradient (strain tensor) and the temperature. Note that the usual definition of an elastic material is more general than the case of hyperelasticity considered here (see [100, 61] for a thorough discussion and a justification that hyperelasticity is a natural condition for conservative elastic materials). The total energy assigned to a deformation φ at inverse temperature $\beta = T^{-1}$ is

$$I(\varphi) = \int_\Omega W_{SE}(D\varphi(x), \beta)\, \mathrm{d}x. \tag{1.3.1}$$

We mostly assume that the temperature is constant throughout the sample and drop it from the notation. The equilibrium deformations are given by minimisers of this functional subject to certain boundary conditions for $\varphi_{\partial\Omega}$. Indeed, the Euler-Lagrange equation is equivalent to the balance of forces for all subsets of Ω.

Let us remark that from a mathematical viewpoint convexity properties of W_{SE} are essential for the existence of minimisers of the functional I. It is well known that (strict) convexity and minor growth assumptions for W_{SE} are sufficient for the existence and uniqueness of minimizers of I and there are non-convex functions W_{SE} where minimizers do not exist. However, it is easy to see that realistic stored energy densities are not convex. Indeed it is natural to assume that $W_{SE}(\mathbb{1}_{3\times 3}) = \min W_{SE}$, i.e., the undeformed configuration has minimal energy. Moreover, frame indifference suggests that $W_{SE}(QA) = W_{SE}(A)$ for all $Q \in \mathrm{SO}(3)$, in particular W_{SE} is minimized on SO(3). This implies that W_{SE} is not convex. To address this problem new notions of convexity like polyconvexity and quasiconvexity were introduced as sufficient conditions for the existence of minimisers [14]. Under such convexity assumptions on the stored energy density, nonlinear elasticity theory has been intensively studied and applied to several physically interesting setting (microstructures, rods, beams, . . .).

In practice the shape of W_{SE} can be determined from measurements. For a deeper theoretical understanding it would be interesting to relate the macroscopic theory to an underlying microscopic theory of matter. Let us mention two key challenges. First, the samples of suitably rescaled microscopic models should concentrate on the minimisers of (1.3.1) for some stored energy density W_{SE}. Moreover, W_{SE} should be given by the free energy density of the underlying model and it might be possible to derive (convexity) properties of W_{SE} from the microscopic theory. Let us remark that elastic behaviour of different materials arises for different physical reasons even though their macroscopic behaviour can be modeled with the same approach. On the one hand, for rubber-like materials that consist of connected polymer chains the free energy is dominated by the entropy contributions and elastic behaviour arises from the stretching of the polymer chains (see [60] for a recent approach to microscopic models of rubber elasticity). On the other hand, for crystalline solids elastic behaviour arises from the change of energy that is required to deform the lattice ordering. Note that the class of crystalline solids contains many materials, in particular all metals. In the following we restrict our attention to crystals and we will briefly review the results for microscopic models.

1.3.2 Cauchy-Born rule

In this section we consider atomistic models for crystalline matter at zero temperature and review their relation to the continuum theory. For a proper understanding of the interaction of

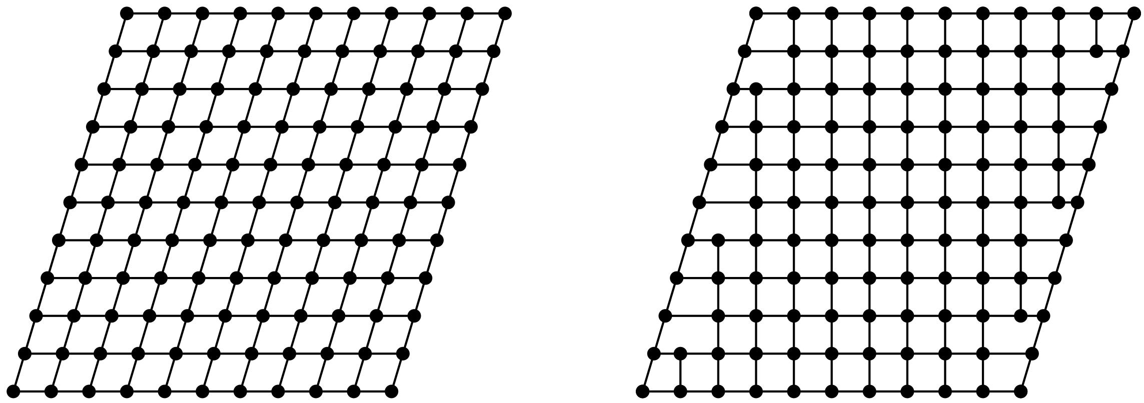

Figure 1.1: (left) The affine deformation following the Cauchy-Born rule. (right) A deformation with lower bulk energy.

atoms in crystals it is necessary to consider the (quantum mechanical) interactions of the electron hulls using, e.g., Thomas-Fermi theory (see [120]). Here we restrict our description to theories with an effective interaction between atoms that are considered as classical point particles. For ease of notation we only consider pairwise interactions of atoms at positions $x, y \in \mathbb{R}^d$ given by $V(|x-y|)$ for some potential $V : \mathbb{R}^+ \to \mathbb{R}$. A typical example of a potential that models the interaction of (free) atoms is the Lenard-Jones potential $V(r) = Ar^{-12} - Br^{-6}$ for $A, B > 0$. Here the first term models the strong short scale repulsion and the second term models the attraction due to van der Waals forces. The general shape of the Lenard-Jones potential that includes short distance repulsion and long distance attraction also applies to interaction potentials for bound atoms. We remark, however, that for a realistic model of crystalline solids it is not sufficient to consider only pairwise interactions. For n atoms with positions $x_i \in \mathbb{R}^d$ the associated energy is

$$E_n(x_1, \ldots, x_n) = \sum_{i<j} V(|x_i - x_j|). \tag{1.3.2}$$

For zero temperature the system configuration is concentrated on minimizers of the energy E_n. Empirically it can be observed that in several materials, namely crystals, the atoms are arranged in regular periodic patterns in the solid phase. Even the simple question, when minimizers of the energy 1.3.2 exhibit a periodic structure is not solved for general potentials V. This question is well known under the name crystallisation conjecture (see [34] for a recent review). We now consider a potential such that the minimizers of the energy arrange in the shape of some lattice $\Lambda \subset \mathbb{R}^3$. Then the connection between the free energy density W introduced in the previous subsection and the microscopic model is typically made using the Cauchy-Born hypothesis. This refers to the assumption that the lattice follows an applied linear deformation A of the boundary in an affine way, i.e., the lattice Λ is mapped under the linear boundary condition Ax to the lattice $A\Lambda$. Then the stored energy density $W_{SE}(A)$ of some linear map A in the reference configuration is given by the energy density of the transformed lattice $A\Lambda$. For a cubic lattice this amounts to (see [80])

$$W_{SE}(A) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} V(|Ax|). \tag{1.3.3}$$

In the literature the consequences and validity of this rule have been widely discussed, see [81, 153] for an overview, but several questions remain open. Let us present some of the results that are related to our analysis.

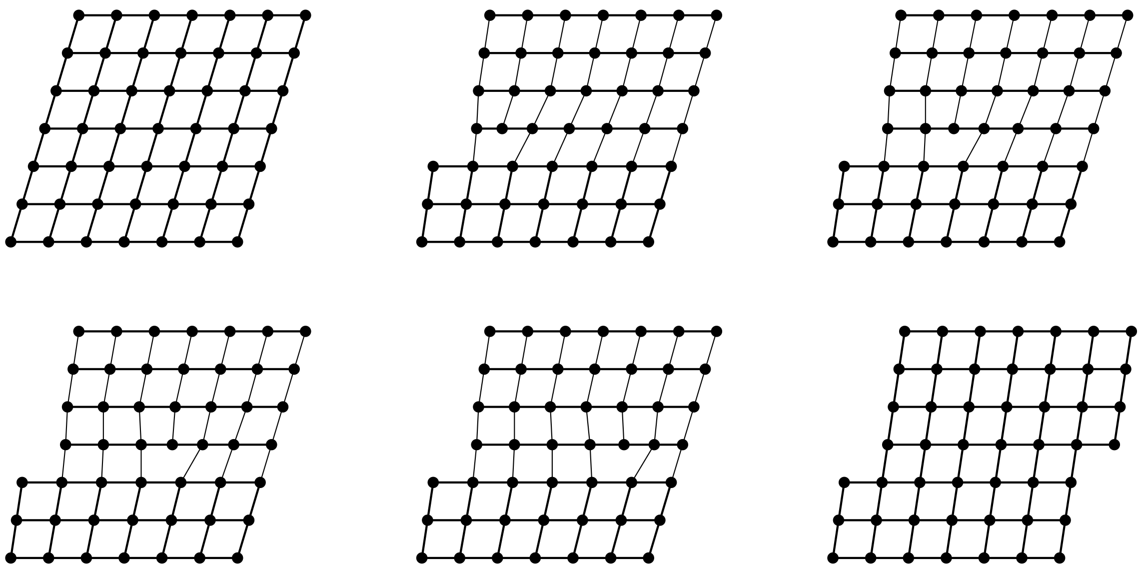

Figure 1.2: The creation of a slip plane through the motion of a dislocation from top left to bottom right.

For certain potentials it has been established by E and Ming in [80] that for affine boundary conditions close to the identity the affine transformation of the lattice is indeed a local minimum of the energy, thus justifying the Cauchy-Born rule. Let us emphasize one important observation: The affine transformation of the lattice is in general only a local minimum, not the global minimum of the energy [80]. The reason is that for volume preserving transformations (e.g., shears) large scale reordering allows to transform the bulk of the sample to a translation of the energy minimizing lattice Λ at the prize of certain boundary contributions. This is illustrated in Figure 1.1. We conclude that elasticity is not an equilibrium phenomenon in the strict sense. However, elastic deformations of solids persist on geological time scales and are therefore metastable.

The reason for this stability is that a reduction of the energy requires large scale reordering of the lattice (in particular through the formation of slip planes by the motion of dislocations) as illustrated in Figure 1.2. For small loads this is prevented by the large energy barrier. Note that for large loads such non-reversible deformations occur frequently and this is studied in the theory of plasticity.

Since large scale reordering is rare for small deformations, it is of interest to consider models that exclude any reordering by fixing the neighbourhood relations of the atoms in the crystal. Note that fixing the neighbourhood relation excludes all types of defects (point defects, line defects, etc.) that appear in solids and influence their behaviour [107].

This type of model is sometimes called mass and spring model because it can be thought of as atoms arranged in a fixed pattern and neighbouring atoms are connected by a spring modelling their interaction and preventing the atom to leave its position in the lattice (see Figure 1.3). Friesecke and Theil use a model where only nearest and next to nearest neighbours interact through potentials V_1 and V_2. They assume that V_1 and V_2 are quadratic, i.e., the atoms interact as if ideal springs were attached between nearest and next to nearest neighbours. In [87] they establish that the Cauchy-Born rule holds for an open range of spring constants for the potentials V_1 and V_2 in dimension $d = 2$ but there are also parameters for which the Cauchy Born rule does not hold.

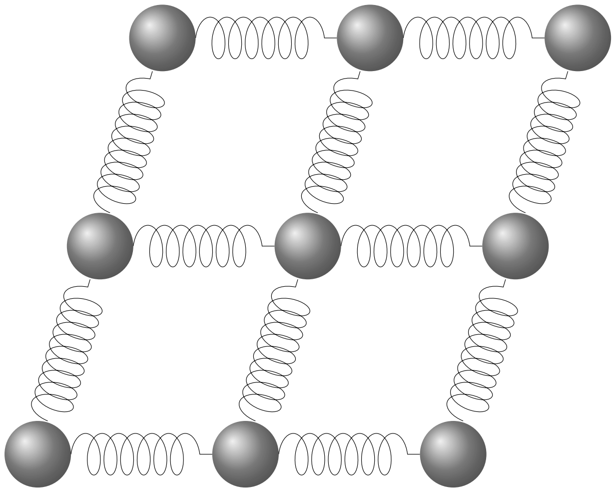

Figure 1.3: Schematic illustration of a nearest-neighbour mass and spring model

Their result was generalized to more general potentials and arbitrary dimensions by Conti, Dolzmann, Kirchheim, and Müller in [62]. They consider a general finite range interaction. It depends on a finite set $A \subset \mathbb{Z}^d$ that contains the unit cell $\{0,1\}^d$. To a deformation $\psi : A \to \mathbb{R}^d$ we assign an energy $U : (\mathbb{R}^d)^A \to \mathbb{R}$. Recall that for $a \in \mathbb{R}^d$ we defined the map $t_a : (\mathbb{R}^d)^A \to (\mathbb{R}^d)^A$ by $((t_a\psi)_x)_{x\in A} = (\psi_x + a)_{x\in A}$ and similarly we define for $F \in \mathbb{R}^{d\times d}$ the map $\psi \mapsto F\psi$ by $(F\psi)_x = F\psi_x$ for $x \in A$. Moreover we write $(\mathbb{R}^d)_0^A$ for the subspace of all ψ such that $\sum_{x\in A} \psi_x = 0$. Let us state assumptions on U:

i) For all $a \in \mathbb{R}^d$ and $Q \in \mathrm{SO}(d)$ we have $U(t_a(Q\psi)) = U(\psi)$.

ii) The energy satisfies $U \geq 0$ and $U(\psi) = 0$ if and only if ψ is a rigid body rotation, i.e., $\psi_x = Qx + a$ for all $x \in A$ and some $Q \in \mathrm{SO}(n)$ and $a \in \mathbb{R}^d$.

iii) We have $U \in C^2$ and $D^2U(\mathbb{1})$ is strictly positive on the orthogonal complement of the subspace spanned by all t_a and infinitesimal rotations $W \in \mathbb{R}^{d\times d}_{\text{skew}}$. Here $\mathbb{1}$ denotes the identity deformation given by $\mathbb{1}_x = x$ for $x \in A$.

iv) We assume

$$\liminf_{\psi\in(\mathbb{R}^d)_0^A,\psi\to\infty} \frac{U(\psi)}{|\psi|^d} > 0. \tag{1.3.4}$$

Under these conditions they establish the following theorem.

Theorem 1.3.1 (Theorem 5.1 in [62]). *Assume that U satisfies the assumptions i)-iv). Then the Cauchy-Born rule holds for all $A \in \mathbb{R}^{d\times d}$ close to $\mathrm{SO}(n)$, i.e., for every $\Lambda \subset \mathbb{Z}^d$ finite the unique minimizer of*

$$H_\Lambda(\psi) = \sum_{x\in\mathbb{Z}^d,\tau_x(A)\cap\Lambda\neq\emptyset} U(\psi_{\tau_x(A)}) \tag{1.3.5}$$

with affine boundary condition $\psi_x = Ax$ for $x \notin \Lambda$ is given by the affine function $\psi_x = Ax$ for all $x \in \Lambda$.

The key ingredient in the proof is the construction of a discrete null Lagrangian N such that $U + \mathsf{N}$ is locally strictly convex and lies above a strictly convex function. This observation will also be crucial in Chapter 3. Note that the conditions i)-iii) for the energy are physical. Only the finite range of interaction and the condition iv) that prevents the reordering are unphysical simplifications.

Let us mention that recently several generalisations of the previous results were found [39, 40, 79, 129]. Those works study, e.g. the dynamics of this model under a change of boundary conditions and the relation between minimisers of (1.3.1) and local minimisers of the atomistic models.

1.3.3 Solids at positive temperature

In the previous section we reviewed the validity of the Cauchy-Born rule on the level of energy minimisers which corresponds to temperature $T = 0$. This approach, however, completely neglects thermal fluctuations which are essential for a proper understanding of matter. Let us assume that we have particles in a heat and particle bath with inverse temperature β and chemical potential μ that are interacting through a two-body potential V. Then using (1.2.3) and the energy (1.3.2) we obtain for $n \geq 0$

$$\mathbb{P}_{\beta,\mu}(N = n, \mathrm{d}x_1, \ldots, \mathrm{d}x_n) = \frac{1}{Z_{\beta,z}} \frac{e^{-\beta E_n(x_1,\ldots,x_n)+\mu n}}{n!} \mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{1.3.6}$$

where $Z_{\beta,z}$ is a normalising constant and $n!$ accounts for the fact that the particles are indistinguishable (Gibbs paradox). Note that it is not necessary to include the momenta of the particles in the model because their integral just contributes a constant factor per particle which can be included in the chemical potential. One would expect that this model shows the different states of matter. In particular, the phase transition between gas and liquid/solid which would be visible as an abrupt change of the density (particle number) of the system. Again, rigorous results are completely missing for such detailed models of matter which shows that the properties of matter are still poorly understood from a mathematical point of view. Only for toy models such as the Widom-Rowlinson model [57] or under unrealistic mean field assumption [119] a discontinuity of the density has been shown. But even if a rigorous analysis of this equilibrium model was possible this cannot be applied to elasticity because equilibrium statistical mechanics and completely physical models are incompatible as discussed in the previous subsection. Indeed, the equilibrium measure will be concentrated on configurations involving plastic deformations. Hence, we restrict our analysis of elastic materials at non-zero temperature to models with a fixed neighbourhood relation, i.e., we consider atoms arranged in a regular (hypercubic) pattern and each atom interacts with a finite number of neighbours. The associated Gibbs measure in finite volume Λ is given by

$$\gamma(\mathrm{d}\psi, \varphi) = e^{-\beta H_\Lambda^U(\psi)} \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\notin\Lambda} \delta_{\varphi_x}(\mathrm{d}\psi_x) \tag{1.3.7}$$

where the field φ prescribes a boundary condition. Let us list some questions of interest that were already briefly discussed in Section 1.3.1 in the context of discrete finite temperature models for elasticity:

1) In what sense can the macroscopic stored energy density W_{SE} and the free energy density W of the microscopic model be related? What is the relation between typical microscopic configurations $\varphi : \Lambda \to \mathbb{R}^d$ and the macroscopic deformation $u : \Omega \to \mathbb{R}^d$ that minimizes (1.3.1)?

2) What are (convexity) properties of the free energy density W?

3) What properties do the Gibbs measures satisfy? In particular, how do the correlations decay for large distances?

4) Is there a unique infinite volume Gibbs measure for a given tilt?

There are very few results concerning this class of models. Most results are restricted to scalar fields and they will be discussed in Section 1.4. Let us already emphasize here that a major difficulty in the analysis of the Gibbs measure is the non-convexity of U. Some general results concerning, e.g., the structure of the set of Gibbs measures can be found in [92, 136]. Several very interesting results concerning the questions above can be found in the work [116] by Kotecký and Luckhaus. They work under rather mild conditions on the potential U: U has to be invariant under rigid body motions (this is condition i) above) and needs to satisfy a certain lower and upper bound and some bound on the oscillations. These assumptions are neither weaker nor stronger than the assumptions we consider. They show in particular that the free energy $F \mapsto W(F)$ (defined with slightly different boundary conditions) exists and is a quasiconvex function of the external deformation F. Moreover, they establish a large deviation principle for the distribution of the Gibbs measure with decreasing mesh-size $\Lambda_\epsilon = D \cap (\varepsilon\mathbb{Z})^d$ with $D \subset \mathbb{R}^d$. The rate function of the large deviation principle is given by the functional (1.3.1) with $W_{SE}(\cdot, \beta) = W(\cdot, \beta)$. Moreover, they show that the distribution of the field locally converges to an infinite volume gradient Gibbs measure in the sense of gradient Young-Gibbs measures. In particular, this gives a partial answer to the first two questions. Their very weak assumptions on the potential U do not allow them to address smoothness properties of the free energy $F \mapsto W(F)$. This is one of the questions that will be addressed here.

1.3.4 Results for discrete elasticity

We will now give a sketch of the two results that we obtain for the model introduced above at sufficiently low temperature. Our first main result is a convexity result for the free energy.

> Assume that U satisfies the assumptions i)-iv) and in addition $U \in C^r$ for some $r \geq 5$ and the derivatives satisfy a mild growth assumption at ∞. Then for $\beta > \beta_0$ and $\delta > 0$ sufficiently small the function
>
> $$W_\beta : B_\delta(\mathbb{1}_{d\times d}) \to \mathbb{R}, \tag{1.3.8}$$
>
> defined in (1.2.12) is C^{r-4} and $D^2 W_\beta(\mathbb{1})$ is strictly positive on the subspace orthogonal to infinitesimal rotations (skew-symmetric matrices).

The precise statement of this theorem is Theorem 3.3.1 in Chapter 3. Let us emphasize that for the sake of giving an overview we neglected several details in the statement. Heuristically the strict convexity of the free energy close to the identity follows from the strict convexity of the potential at the identity and a bound for the entropy that is small for low temperatures. Our second main result concerns the scaling limit of the model. For this result we work on the torus $T_N = (\mathbb{Z}/L^N\mathbb{Z})^d$ where $L > 1$ is an odd integer. We denote by $\gamma^F_{N,\beta}$ the finite volume Gibbs measure for the energy H_{T_N} and tilt F (cf. Chapter 3 for an explanation how the tilt is implemented in a periodic setting and note that $\gamma^F_{N,\beta}$ really has tilt F in the sense of definition (1.2.10)). It is consistent to denote the Gibbs measure with the same letter as the specification because no boundary conditions are required on the torus.

Assume that U satisfies the assumptions i)-iv) and in addition $U \in C^3$ and the derivatives satisfy a mild growth condition at ∞. Then there are δ, β_0, and $L_0 > 0$ such that for $F \in B_\delta(\mathbb{1}_{d\times d})$, $\beta \geq \beta_0$, and $L \geq L_0$ there is an operator C and a subsequence $N_\ell \to \infty$ such that for $f \in \mathbb{C}^\infty((\mathbb{R}/\mathbb{Z})^d, \mathbb{R}^d)$ and $f_N : T_N \to \mathbb{R}^d$ given by $f_N(x) = L^{-N\frac{d+2}{2}} f(L^{-Nx})$

$$\lim_{N_\ell \to \infty} \mathbb{E}_{\gamma^F_{N,\beta}} e^{(f_{N_\ell}, \varphi)} = e^{\frac{1}{2\beta}(f, Cf)}. \tag{1.3.9}$$

In particular the scaling limit is Gaussian.

The precise statement can be found in Theorem 3.3.2 in Chapter 3.

Those two results provide a partial answer to the second and third question above. There are several open questions concerning the nature of the Gibbs measures for this model and we will discuss those questions briefly in the context of gradient interface models below, but they equally apply to the models of discrete elasticity discussed here.

1.4 Gradient interface models

In this section we will discuss our results in the context of gradient interface models. Gradient interface models are spin systems as defined in Section 1.2 with state space $E = \mathbb{R}$, i.e., scalar real valued fields. Their local interaction energy U is of the form $U(\varphi_A) = \sum_{i=1}^d V(\nabla_i \varphi(0))$ where $A = \{0, e_1, \ldots, e_d\}$ and $V : \mathbb{R} \to R$ is a nearest neighbour potential. The most prominent example is the Gaussian free field introduced in Example 1.2.4 where the potential $V(x) = x^2/2$ is quadratic. Gradient interface models are a class of models that can be used as an effective model for a continuous interface or for interfaces in spin systems, e.g., the interface between the plus and the minus phase of an Ising model. Let us remark that the free energy is often called *s*urface tension in the context of gradient interface models.

Our main results are derived in a general framework that includes models for discrete elasticity and gradient interface models as special cases. We will present our results also in the context of gradient interface models because they have been a very active theme of research in the last years and this allows us to compare our approach to techniques used earlier. Here we just sketch some of the known results and we refer to the reviews [88, 93, 136] for a complete overview.

1.4.1 Gradient interface models with convex potentials

Many results about gradient interface models are restricted to convex and even interaction potentials V, i.e., potentials $V \in C^2(\mathbb{R})$ that satisfy $V(-x) = V(x)$ and

$$C_1 \leq V'' \leq C_2 \tag{1.4.1}$$

for two constants $0 < C_1 < C_2$. The prototypical example for this class is the Gaussian free field where the Gaussian structure and the Markov property of the field simplify the analysis. Very fine results are known for the discrete Gaussian free field and there has been a recent surge of results, e.g., about the maximum of the field, level sets of the field, entropic repulsion, level set percolation. The guiding principle is that most of the results for discrete Gaussian free fields can be generalised to strictly convex potentials. Let us highlight some of the results. The important work [89] by Funaki and Spohn established the strict convexity of the free energy surface tension, uniqueness and existence of the Gibbs measure for a given tilt $u \in \mathbb{R}^d$, and the derivation of the hydrodynamic limit. Estimates for the decay of covariances were shown in [67, 128]. It was shown that the scaling limit of the model is Gaussian (see [128] for zero tilt using homogenisation and [94] for the general case). See also [125] for a scaling limit in a bounded domain and some interesting coupling results. Large deviations results were shown in [69, 136]. Recently also finer results have been shown concerning, e.g., the scaling of the maximum [27, 150]. Several tools have been developed to study gradient interface models. Let us briefly sketch some of the most important ideas that are connected in various ways.

1. *Langevin-dynamics.* The Gibbs measure for gradient interface models is the equilibrium distribution of an associated Langevin dynamics $(\varphi_t(x))_{(t,x)\in\mathbb{R}^+\times\mathbb{Z}^d}$ given by

$$\mathrm{d}\varphi_t(x) = -\sum_{y\sim x} V'(\varphi_t(x)-\varphi_t(y))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t(x) \tag{1.4.2}$$

where $\{B_t(x),\, x\in\mathbb{Z}^d\}$ is a family of independent Brownian motions. This dynamics and a coupling argument was used in [89] to show uniqueness of the Gibbs measure.

2. *Helffer-Sjöstrand random walk representation and Brascamp-Lieb inequality.* The Hellfer-Sjöstrand random walk representation is a technique that was introduced by Helffer and Sjöstrand in [104] and first applied to gradient interface models by Naddaf and Spencer in [128]. The basic observation underlying this approach is that the gradient interface model can be related to a time dependent random walk in a random environment which is a generalisation of the relation between the simple random walk and the Gaussian free field. Very roughly the key relation can be expressed for a gradient Gibbs measure μ and observables $F(\varphi)$ and $G(\varphi)$ as

$$\mathrm{Cov}_\mu(F,G) = \sum_{x\in\mathbb{Z}^d}\int_0^\infty \mathbb{E}\left(\frac{\partial F(\varphi_0)}{\partial\varphi_0(x)}\frac{\partial G(\varphi_t)}{\partial\varphi_t(X_t)}\right)\mathrm{d}t \tag{1.4.3}$$

where φ_t evolves according to the dynamics (1.4.2) with φ_0 distributed according to μ and X_t is a random walk on $\mathbb{Z}^d$ with time dependent transition rates $q_{t,x,y} = V''(\varphi_t(x)-\varphi_t(y))$ and $X_0 = x$. This representation can be used to derive correlation inequalities like the FKG-correlation inequality and the Brascamp-Lieb inequality (see [38]) that states that variances for a Gibbs measure for the potential V satisfying (1.4.1) can be bounded by the variances of the rescaled discrete Gaussian free field with potential $x\to\frac{1}{2}C_1x^2$. The Helffer-Sjöstrand representation can also be applied to understand the large scale behaviour of the model. Here the Kipnis Varadhan approach [113] is an important ingredient.

3. *Cluster swapping.* Cluster swapping is a technique introduced by Sheffield in [136]. The basic idea is similar to the Swendsen-Wang algorithm that can be used for fast Monte-Carlo simulations of (near) critical Ising models. The key observation is that if $\varphi_x < \psi_x$ and $\varphi_y < \psi_y$ then strict convexity of the potential implies that

$$V(\varphi(x)-\varphi(y)) + V(\psi(x)-\psi(y)) < V(\varphi(x)-\psi(y)) + V(\psi(x)-\varphi(y)). \tag{1.4.4}$$

This implies in particular that

$$H_\Lambda(\min(\varphi,\psi)) + H_\Lambda(\max(\varphi,\psi)) < H_\Lambda(\varphi) + H_\Lambda(\psi). \tag{1.4.5}$$

Based on this observation he introduces for a coupling of two Gibbs measures a cluster swap map that maps very roughly $(\varphi,\psi)\mapsto(\min(\varphi,\psi),\max(\varphi,\psi))$ on infinite clusters of the set $\varphi>\psi$ (the fields are defined using a carefully chosen offset $\varphi_0-\psi_0=h$). This map can be used to show that the surface tension is strictly convex and the gradient Gibbs measure for a given tilt u is unique.

4. *Stochastic homogenisation.* Already in one of the early works by Naddaf and Spencer [128] stochastic homogenisation was used to investigate the large scale behaviour of the random walk in the Helffer-Sjöstrand representation. In the past years (quantitative) stochastic homogenisation became a highly active field of research and we refer in particular to the early works by Gloria, Otto, and Neukamm [97, 96] and to the works by Armstrong, Kuusi, and Mourrat [9, 10]. Dario recently presented a new approach to understand fluctuations and the partition function of the gradient interface model without using the Hellfer-Sjöstrand technique [65]. Instead his approach relies on stochastic homogenisation techniques similar to the approach in [9] and the extension of estimates for elliptic partial differential equations to gradient interface models.

However, all those techniques use the convexity of the potential in an essential way and it is not clear how to obtain any results for potentials that are not convex. Indeed, the Langevin dynamics is contracting when the same Brownian motion is used for two solutions if V is convex, the random walk only exists when $V'' \geq 0$, the inequality (1.4.4) is not true for non-convex V, and for V non-convex it is not expected that the elliptic theory can be applied. Therefore it remains a major challenge to find more robust techniques that apply to (some) non-convex potentials.

1.4.2 Gradient interface models with non-convex potentials

In this section we review the known results for gradient interface models with non-convex potentials. For non-convex potentials results are rather scarce because the techniques for convex potentials do not apply. Some general results about the surface tension can be found in [136]. Let us again (cf. Section 1.3.3) emphasize the work [116] where existence and convexity of the surface tension and large deviation principles were shown under very general assumptions. For high temperatures several results are known for potentials of the form $V + g$ where V is strictly convex and g satisfies some bound on the second derivative ($g'' \in L^p$ for some $p \geq 1$ is sufficient). Then for β sufficiently small the surface tension is strictly convex, the Gibbs measure is unique, covariances decay, the scaling limit is Gaussian, and the hydrodynamic limit can be derived [64, 63, 70]. All those results rely on the fact that the interaction potential becomes strictly convex after a one step integration procedure so that this case is effectively reduced to the convex case discussed above. One possibility to regain convexity is to integrate out the field on the even sublattice of $\mathbb{Z}^d$.

For small temperatures it is not expected that convexity can be restored using a one step integration procedure. Moreover, it is known that the behaviour for general non-convex potentials can be very different from the convex case as phase transitions (non-uniqueness) of the gradient Gibbs measure can occur (see [32] and Section 1.6 below). Nevertheless we do not expect this type of behaviour at low temperatures. When rescaling a potential $V \in C^2(\mathbb{R})$ with unique minimum in zero by $V \mapsto V_\beta = \beta V(\cdot/\sqrt{\beta})$ (this corresponds to a change of temperature and a rescaling of the field such that its fluctuations remain of constant size) the potential locally approaches a quadratic function as $\beta \to \infty$ (see Figure 1.4) and the non-convexity becomes energetically less favourable. This suggests that the interface model behaves similarly to the case where V is strictly convex. It is difficult to make this idea rigorous because the event that $\nabla_i \varphi(x)$ is in the non-convex region of V_β has positive probability for any β so it occurs with a positive density. A further aspect that makes the analysis of gradient models difficult is that they are critical. This means that the correlation of the gradient fields are expected to decay only polynomially as

$$\mathbb{E}(\nabla\varphi(x)\nabla\varphi(y)) \approx |x-y|^{-d}. \tag{1.4.6}$$

The slow decay of correlations implies that naive approaches that involve a sum over the covariance fail since the covariance is not integrable. Therefore a careful multiscale analysis seems to be necessary to obtain results for the low temperature case. To our knowledge the only results in this case seem to be the convexity of the surface tension [116] and the strict convexity of the surface tension for small tilts. Strict convexity was shown by Adams, Kotecký, and Müller using a renormalisation group approach in [4].

1.4.3 Results for gradient interface models and outlook

Most of this thesis is devoted to a generalisation of the results in [4]. The main improvement is that we are now able to handle interactions relevant for microscopic elasticity as discussed in the previous section, but we also derive some new results for gradient interface models which we discuss here. Again, we only provide a sketch of the result and refer to Chapter 3 for a precise statement. The results are very similar to the ones discussed in the previous section, however the assumptions are slightly different.

Assume that $V \in C^r$ satisfies $V''(0) > \varepsilon$ and

$$V(x) \geq V'(0)x + V(0) + \varepsilon|x|^2 \tag{1.4.7}$$

for some $\varepsilon > 0$ and that all derivatives up to order r of V grow at most polynomially. Then there is $\beta_0 > 0$ and $\delta > 0$ such that for $\beta \geq \beta_0$ the surface tension $W : B_\delta(0) \to \mathbb{R}$ is strictly convex and in C^{r-4}.

The precise statement of this statement in the more general context of generalized gradient models can be found in Proposition 3.2.4 and Theorem 3.2.3 in Chapter 3. We can also derive the scaling limit. We again work on the torus $T_N = (\mathbb{Z}/L^N\mathbb{Z})^d$ where $L > 1$ is an odd integer and denote by $\gamma^u_{N,\beta}$ the finite volume Gibbs measure for the energy H_{T_N} and tilt u.

Assume that $V \in C^3$ satisfies $V''(0) > \varepsilon$ and

$$V(x) \geq V'(0)x + V(0) + \varepsilon|x|^2 \tag{1.4.8}$$

for some $\varepsilon > 0$ and that all derivatives up to order 3 of V grow at most polynomially. Then there are β_0, $\delta > 0$, and L_0 such that for any $\beta \geq \beta_0$, $|u| \leq \delta$ and $L \geq L_0$ there is an operator C and a subsequence $N_\ell \to \infty$ such that for $f \in \mathbb{C}^\infty((\mathbb{R}/\mathbb{Z})^d, \mathbb{R})$ and $f_N : T_N \to \mathbb{R}^d$ given by $f_N(x) = L^{-N\frac{d+2}{2}} f(L^{-N}x)$

$$\lim_{N_\ell \to \infty} \mathbb{E}_{\gamma^u_{N,\beta}} e^{(f_{N_\ell}, \varphi)} = e^{\frac{1}{2\beta}(f, Cf)}. \tag{1.4.9}$$

In particular the scaling limit is Gaussian.

The precise statement can be found in Proposition 3.2.4 and Theorem 3.2.6 in Chapter 3. The first result was derived under more restrictive growth and smoothness assumptions in [4] (see Theorem 2.1 and Proposition 2.2 there). Indeed, they require that

$$V(x) - V'(0)x - V(0) \geq \tfrac{1}{2}(V''(0) - \varepsilon)x^2 \tag{1.4.10}$$

for a small $\varepsilon > 0$. The second result concerning the scaling limit was first shown in [105] in the setting of [4]. Here we adapt the proof to our setting. We extend the method used in [4] in several directions and we will discuss those changes in the next Section 1.5.

To understand finer properties of the model a natural question addresses the behaviour of the gradient-gradient correlations. This is answered in the Ph.D. thesis of Hilger [106] where the same decay of correlations as for an anisotropic Gaussian free field is shown. The major open question is whether uniqueness of the Gibbs measure can be established under the assumptions stated above. This might require new techniques because our analysis is confined to the periodic setting which cannot be sufficient to conclude uniqueness of the Gibbs measure (i.e., asymptotic independence of the boundary condition). This would be the basis of an analysis of the hydrodynamic limit but is also of interest for the existence of microstructures in materials. Note that our smoothness result for the surface tension is a hint towards uniqueness of the Gibbs measure. Indeed, for models with finite spin space and bounded interactions it was rigorously shown that coexistence of Gibbs measures and non-differentiability of the free energy are closely related (see [92, 132, 138]).

1.5 Renormalisation group approach to statistical mechanics

In this section we describe the renormalisation group approach. This is the method that will be used to prove the main results stated in the previous sections. We start with a brief historical overview and refer to, e.g., [110] for details.

1.5.1 The renormalisation group for critical phenomena

The term critical phenomena refers to the behaviour of systems at the critical point where a phase transition occurs. For many classical systems in statistical mechanics (e.g., Ising model, percolation, φ^4 theory, self-avoiding random walks) it is observed that there is a phase transition between a subcritical and a supercritical phase. The subcritical and supercritical phase are typically easier to understand because correlations decay exponentially. However, as the critical point is approached the correlation length diverges giving rise to many interesting phenomena. In particular the system becomes self similar at the critical point. Often universality is observed at the critical point, i.e., on large scales the system is independent of microscopic details of the interaction, e.g., the lattice structure, and only depends on the dimension. Various critical exponents are used to describe the large distance behaviour at and near the critical point. Often there is a critical dimension d_c for each model such that for $d > d_c$ the model exhibits mean field behaviour (i.e., the lattice can be replaced by a tree and the critical exponents agree in both cases) while for $d = d_c$ there are typically logarithmic corrections to the mean field behaviour. The case $d \gg d_c$ and sometimes also $d > d_c$ is well understood for many models using the lace expansion [50, 141]. Moreover, in dimension $d = 2$ special techniques relying on conformal invariance apply and a lot of progress has been made recently (see [118]). For $2 < d < d_c$ critical behaviour is poorly understood. Gradient models are special and do not fit in this framework because they are always critical in the sense that their correlations only decay polynomially for all temperatures and all dimensions.

The renormalisation group method was introduced as a tool to understand the large scale behaviour of critical systems. A first important step was the block spin approach developed by Kardanoff in [109]. He studied the Ising model and introduced the block spin as the average of the Ising spins on a square of sidelength L. Then he argued that the block spins interact approximately similar to an Ising model with different (renormalized) coupling constants. Later this procedure was made rigorous by Gawędzki and Kupiainen in [90, 91]. Kardanoff's ideas already contained two important ingredients that exploit the approximate self similarity: Coarse graining and rescaling. Coarse graining is used to average out the microscopic degrees of freedom and the rescaling is used to restore approximately the original structure of the model. Today one refers to the combination of a coarse graining step and a rescaling step as a renormalisation group transformation. This transformation gives rise to a dynamical system whose long time behaviour describes the large scale behaviour of the model.

Based on, amongst other ideas, Kardanoff's block spin approach and the removal of divergences in quantum field theory Wilson introduced the renormalisation group to study second-order phase transitions [146, 147] in 1971 and the Kondo problem [149] in 1974. For his contributions he was awarded the Nobel price in 1982. A review of his work can be found in his Nobel lecture [148]. Since then renormalisation group techniques have become a standard tool in theoretical physics. The rigorous implementation, however, is a major challenge since it is very difficult to control the error terms in the perturbative expansions used in physics. Nevertheless renormalisation techniques have been widely applied in mathematics, e.g., to quantum field theory [1, 2, 11, 12, 28, 95], the Coulomb gas [74, 75], the Fermi liquid [85], and the Bose gas [13]. Brydges and Yau introduced many important ideas in the seminal paper [52] about the dipole gas and those ideas heavily influenced many of the works mentioned before. Based on the approach used in [52], Bauerschmidt, Brydges, and Slade started a long programme to build a general framework for rigorous renormalisation group techniques. The general method is developed in a series of papers [44, 45, 20, 46, 47]. They apply their method to find the critical exponents and its logarithmic corrections for the φ^4-theory (see Example 1.2.5) [22] and the weakly self avoiding random walk [21] in the critical dimension $d_c = 4$. See also [25, 26, 140, 121] for some further results in this direction. Their approach has been generalised to the analysis of the Coulomb gas by Falco in [82, 83]. The technique was also applied to gradient models in [4]. Here we extend the results for gradient models.

Let us remark that recently there was another new development in the theory of renormalisation for the analysis of nonlinear stochastic partial differential equation introduced by Martin Hairer [102]. In this setting renormalisation is required to remove ultraviolet (i.e., short scale) divergences from a priori ill-posed equations in contrast to the infrared problems faced in statistical physics. However, understanding the long term behaviour of stochastic partial differential equation provides information about their equlibrium distribution which might relate the two approaches. See [126] for first results concerning the long term behaviour. Let us also mention [17] for a recent new approach using variational techniques to show existence of the equilibrium measure of φ^4-theory, providing another approach to renormalisation that heavily uses techniques from singular stochastic partial differential equations.

1.5.2 The renormalisation group approach for gradient models

We now provide a very brief sketch of the renormalisation method in the spirit of Bauerschmidt, Brydges, and Slade in the setting of gradient models. We refer to Section 4.2 for a slightly more detailed overview of our method and to the lecture notes [42], the recent book [24], and [19] for a description of the general method for φ^4-theory. The basic problem is to control perturbation of Gaussian integrals in the limit $\Lambda \to \mathbb{Z}^d$,

$$Z_\Lambda = \int P(\varphi)\, \mu(\mathrm{d}\varphi). \tag{1.5.1}$$

Here μ denotes a Gaussian measure and $P(\varphi) = \prod_{x\in\Lambda}(1 + \mathcal{K}(x,\varphi))$ denotes a perturbation. We assume that $\mathcal{K}(x,\varphi)$ is identical for each point x ($\mathcal{K}(\tau_y x, \tau_y \varphi) = \mathcal{K}(x,\varphi)$), local in the sense that

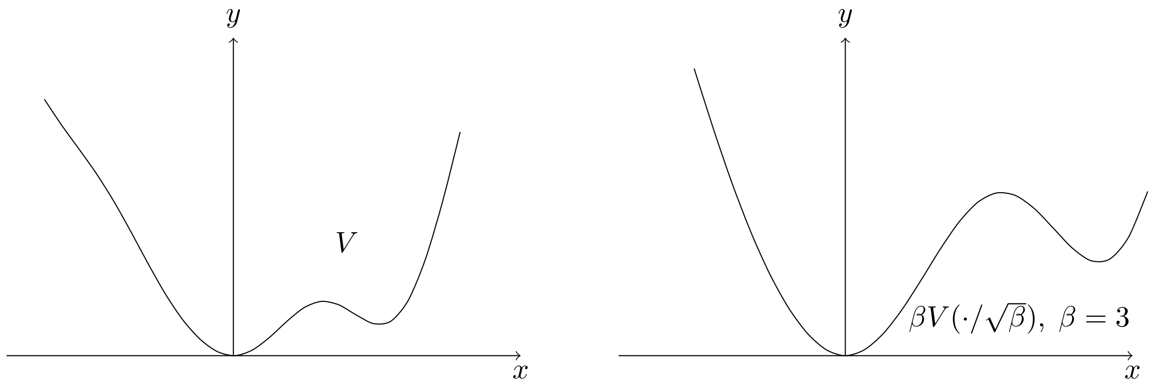

Figure 1.4: (left) Typical example of an admissible potential: V is convex at the global minimum satisfies a quadratic lower bound and may be non-convex away from the minimum. (right) The change of temperature decreases the influence of the non-convexity and drives the potential closer a the quadratic potential.

$\mathcal{K}(x, \varphi)$ only depends on the values of φ in a neighbourhood of x, of gradient type, i.e., shift invariant in the sense that $\mathcal{K}(x, t_a\varphi) = \mathcal{K}(x, \varphi)$, and small in a sense that will be made precise in Chapter 3. It is also of interest to understand finer properties in particular expectation values of the measure $P(\varphi)\,\mu(\mathrm{d}\varphi)$. Since this requires similar techniques, we restrict to the analysis of the partition function Z_Λ here. The basic structure appears in all the problems mentioned above, i.e., in the analysis of the φ^4 model, the weakly self-avoiding random walk, and the Coulomb gas. Let us very briefly explain how this structure arises in the analysis of gradient interface models. In this case the partition function is given by

$$Z_N = \int e^{-\beta \sum_{x\in\Lambda_N} \sum_{i=1}^{d} V(\nabla_i \varphi(x))}\,\lambda(\mathrm{d}\varphi) \tag{1.5.2}$$

where for concreteness we work on the torus $\Lambda_N = (\mathbb{Z}/(L^N\mathbb{Z}))^d$ where L is an integer and λ denotes the Hausdorff-measure on the subspace of $\mathbb{R}^{\Lambda_N}$ with average 0. Note that we can assume that $V(0) = V'(0) = 0$ because constants contribute $-\beta\,|\Lambda_N|\,dV(0)$ to the energy and the linear term disappears since $\sum_{x\in\Lambda_N} \nabla_i\varphi(x) = 0$. Moreover we assume by rescaling φ that $V''(0) = 1$. We write

$$V(x) = \frac{x^2}{2} + g(x) \tag{1.5.3}$$

where $g(0) = g'(0) = g''(0) = 0$. Using this decomposition and the coordinate change $\varphi' = \varphi\sqrt{\beta}$ we obtain

$$Z_N = \int e^{-\beta \sum_{x\in\Lambda_N} \sum_{i=1}^{d} g\left(\frac{\nabla_i\varphi'(x)}{\sqrt{\beta}}\right)} e^{-\frac{1}{2}\sum_{x\in\Lambda_N} |\nabla\varphi(x)|^2}\,\lambda(\mathrm{d}\varphi'). \tag{1.5.4}$$

Note that the expression $e^{-\frac{1}{2}\sum_{x\in\Lambda_N}|\nabla\varphi'(x)|^2}$ is proportional to the density of the discrete Gaussian free field on the torus. Moreover, as $\beta \to \infty$ the function $x \to \beta g(x/\sqrt{\beta})$ converges to 0 pointwise for $g \in C^3$, as a Taylor expansion at 0 shows.

Thus we have shown that Z_N equals up to a normalisation factor an expression as in (1.5.1) where $\mathcal{K}(x, \varphi) = e^{\sum_{i=1}^{d} \beta g(\nabla_i\varphi(x)/\sqrt{\beta})} - 1$. For a general version of this calculation we refer to Proposition 3.2.4 in Chapter 3. The basic approach is to use a coarse graining procedure to estimate the integral (1.5.1). This is implemented using a decomposition of the Gaussian measure $\mu = \mu_f * \mu_l$ where μ_f denotes the fluctuation field that captures the local strong fluctuations of the measure μ and μ_l captures the remaining long distance fluctuations. Then a renormalisation step consists first of the integration with respect to the fluctuation field that eliminates the strong fluctuations of the local degrees of freedom. The rescaling step is difficult to implement due to the lattice structure. Instead the rescaling is replaced by a change of the norms which require typical fields to be smooth on the next length scale. This procedure is then iterated. The concrete implementation is slightly different in that we use a fixed decomposition of the measure in $N+1$ terms that is fixed beforehand. Let us now describe the implementation in our setting in a bit more detail. For reasons that will become clear later we introduce for $\boldsymbol{q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ with $|\boldsymbol{q}| < \frac{1}{2}$ the family of Gaussian measures

$$\mu^{(\boldsymbol{q})} = \frac{e^{-\frac{1}{2}\sum_{x\in\Lambda_N}\sum_{i,j=0}^{d}\nabla_i\varphi(x)(\delta_{ij}+\boldsymbol{q}_{ij})\nabla_j\varphi(x)}}{Z^{(\boldsymbol{q})}}\lambda \tag{1.5.5}$$

where $Z^{(\boldsymbol{q})}$ denotes the normalisation constant. We denote the covariance of $\mu^{(\boldsymbol{q})}$ by $\mathcal{C}^{(\boldsymbol{q})} = (\nabla^*(\mathbb{1}+\boldsymbol{q})\nabla)^{-1}$ where ∇^* denotes the backward discrete derivative which is the adjoint of the discrete forward derivative. We use so called finite range decompositions which is a decomposition of $\mu^{(\boldsymbol{q})} = \mu_1^{(\boldsymbol{q})} * \ldots \mu_{N+1}^{(\boldsymbol{q})}$ in a series of Gaussian measures with covariances $\mathcal{C}_k^{(\boldsymbol{q})}$. We will discuss finite range decompositions in a bit more detail below in Section 1.5.4 and state here only there defining properties. The first property is the finite range property that states

$$\mathbb{E}_{\mu_k^{(\boldsymbol{q})}}\left(\nabla_i\varphi(x)\nabla_j\varphi(x)\right) = 0 \qquad \text{for } |x-y|_\infty \geq \frac{L^k}{2}. \tag{1.5.6}$$

The second property is the regularity of the covariance which ensures that $\mu_k^{(\boldsymbol{q})}$ captures the fluctuations on the length scale κ

$$|\nabla_x^\alpha\nabla_y^\beta\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x,y)| \leq CL^{-(d-2+|\alpha|+|\beta|)k} \tag{1.5.7}$$

where we denote by $\mathcal{C}_{k+1}^{(\boldsymbol{q})}$ the kernels of the covariance and α and β denote multiindices and $|\cdot|$ their degree. Using this decomposition we can rewrite the initial integral as a series of integrations and obtain

$$Z_N = \int P^{(\boldsymbol{q})}\Big(\sum_{k=1}^{N+1}\varphi_k\Big)\,\mu_1^{(\boldsymbol{q})}(\mathrm{d}\varphi_1)\ldots\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi_{N+1}) \tag{1.5.8}$$

where it will be useful to allow us to renormalise the measure using the matrix $\boldsymbol{q}$. If we define inductively $P_{k+1}(\varphi) = \boldsymbol{T}_K(P_k,\boldsymbol{q})(\varphi) = \int P_k(\varphi+\varphi_{k+1})\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi_{k+1})$ with $P_0 = P^{(\boldsymbol{q})}$ we can view this as a (scale dependent) dynamical system and $Z_N = P_{N+1}(0)$. Note that at this point it is not clear how this might be helpful because the space of P_k is infinite dimensional. The key observation that makes this dynamical system tractable is that only a finite number of explicit terms (coordinate directions) grow under this systems while all other directions are contracting. Let us provide a heuristic explanation of this fact. More detailed explanations can be found in the references given above and in Section 4.4.2. For a monomial $\nabla^\alpha\varphi(x)$ the variance under $\mu_{k+1}^{(\boldsymbol{q})}$ is bounded by

$$\mathbb{E}_{\mu_{k+1}^{(\boldsymbol{q})}}\left(|\nabla^\alpha\varphi_{k+1}(x)|^2\right) = \nabla^\alpha(\nabla^*)^\alpha\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x,x) \leq CL^{-k(2|\alpha|+(d-2))}. \tag{1.5.9}$$

Therefore the size of the field $\nabla^\alpha \varphi_{k+1}(x)$ is typically of the order of the square root of the variance which is $L^{-k(|\alpha|+\frac{(d-2)}{2})}$. We define the dimension of the field φ by $[\varphi] = (d-2)/2$ so that the size of a monomial $\nabla^\alpha \varphi(x)$ can be expressed as $L^{-k([\varphi]+|\alpha|)}$. Since we do not rescale the system the natural length scale at the k-th step is L^k. Therefore the natural size associated to the field $\nabla^\alpha \varphi_{k+1}$ is the sum over a block B of sidelength L^k

$$\sum_{x\in B} |\nabla^\alpha \varphi_{k+1}(x)| \approx L^{k(d-[\varphi]-|\alpha|)}. \tag{1.5.10}$$

We obtain a similar expression for the product of monomials

$$\sum_{x\in B} \left| \prod_{l=1}^{s} \nabla^{\alpha_l} \varphi_{k+1}(x) \right| \approx L^{k(d-s[\varphi]-\sum_{l=1}^{s}|\alpha_l|)}. \tag{1.5.11}$$

Only terms that satisfy $d - s[\varphi] - \sum_{l=1}^{s} |\alpha_l| \geq 0$ do not decrease as k becomes large. In renormalisation those terms are usually referred to as relevant or marginal (if $d - s[\varphi] - \sum_{l=1}^{s} |\alpha_l| = 0$). All other terms are labelled irrelevant because they contract under the renormalisation. Since we consider gradient models all the functionals P_k are measurable with respect to gradients of the fields so we only need to consider terms with $|\alpha_l| \geq 1$. Then one easily sees that the relevant and marginal terms amount to constants, the linear terms $\nabla^\alpha \varphi(x)$ with $|\alpha| \leq d/2 + 1$ and the quadratic terms $\nabla_i \varphi(x) \nabla_j \varphi(x)$. Note that in principle we also have to consider non-local products like $\nabla_i \varphi(x) \nabla_j \varphi(y)$. However, one easily sees that they can be rewritten as $\nabla_i \varphi(x) \nabla_j \varphi(x)$ plus irrelevant terms with additional discrete gradients.

It turns out that it is possible to rewrite $P_k = K_k \circ e^{-H_k}$ where the formal definition of the $\circ$ operation can be found in Section 4.4.2. Here it is sufficient to view this representation as a change of coordinates. The functional K_k is an element of a suitable infinite dimensional Banach space that collects the irrelevant directions on scale k and H_k is an element of a finite dimensional space that is spanned by the finite number of relevant directions. The expansion that is used for the representation of H_k and K_k is related to polymer expansions and cluster expansions that are a standard tool in statistical mechanics and we refer to [41] for details. We can define the renormalisation map $\boldsymbol{T}_k$ for the new coordinates so that $\boldsymbol{T}_k(K_k, H_k, q) = (K_{k+1}, H_{k+1})$ satisfies

$$(K_{k+1} \circ e^{-H_{k+1}})(\varphi) = \int (K_k \circ e^{-H_k})(\varphi + \psi)\, \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\psi). \tag{1.5.12}$$

The sequence of maps $\boldsymbol{T}_k$ defines a scale dependent dynamical system and the point $(0, 0, \boldsymbol{q})$ corresponds to a Gaussian fixed point for any $\boldsymbol{q}$. Moreover it turns out that $\boldsymbol{T}_k$ can be defined such that this fixed point is hyperbolic where the directions K_k are contracting and H_k are expanding. This requires a careful definition of $\boldsymbol{T}_k$ because even though K_k only contains the irrelevant directions the convolution with $\mu_{k+1}^{(\boldsymbol{q})}$ creates relevant terms. The structure of a dynamical system can be exploited in the vicinity of the fixed point using a stable manifold theorem and a fine tuning of $\boldsymbol{q}$. It can be shown that the scaling limit of the model agrees with the long distance behaviour of the Gaussian measure $\mu^{(\boldsymbol{q})}$ so that this fine tuning is physically meaningful. We also show that all operations are smooth as a function of the perturbation $\mathcal{K}(x, \varphi)$. The key challenges in this approach are the proper setup and proof of the hyperbolicity of the fixed point and smoothness of the maps $\boldsymbol{T}_k$ close to the fixed point which are the statements of Theorem 4.4.7 and 4.4.8.

1.5.3 Differences to earlier work

Let us here highlight the main differences of our work for gradient models to the implementation of the renormalisation group approach for φ^4-theory by Bauerschmidt, Brydges, and Slade.

At the end of this section we will also discuss the main differences to the earlier work for gradient models by Adams, Kotecký, and Müller.

There are two main difficulties in the analysis of φ^4-theory that do not appear for gradient models. First, the definition of the renormalisation map $\boldsymbol{T}_k$ for the φ^4-model is substantially more difficult because a second order perturbation theory is required. This is related to the fact that the φ^4 term is marginal and cannot be absorbed into the Gaussian measure. In this setting the fixed point of the dynamical system in non-hyperbolic and they need a finer analysis of finite dimensional dynamical systems with a non hyperbolic fixed point (cf. [23]). For gradient models first order perturbation theory is in general sufficient, but second order perturbation theory is necessary to handle observables that can be used, e.g., to estimate the decay of covariances [106].

A second difference to the work by Bauerschmidt, Brydges, and Slade is that they also deal with Grassmann variables and supersymmetric theories because the weakly self avoiding random walk corresponds to a supersymmetric version of the φ^4 theory. Therefore, they need to introduce very general abstract results and proofs while we restrict to special cases adapted to our settings and more simple minded proofs.

While the analysis of gradient models is much simpler with respect to the two aspects described above there are also new challenges. An additional difficulty compared to the analysis of φ^4-theory is the missing symmetry. While the invariance of all terms under all isometries of the lattice reduces the effective number of relevant terms for φ^4-theory to three we consider an anisotropic setting with far more relevant parameters. This makes the fine tuning of the matrix $\boldsymbol{q}$ in the stable manifold theorem slightly more subtle and we need smoothness of the renormalisation map $\boldsymbol{T}_k$ with respect to $\boldsymbol{q}$. In φ^4-theory only continuity is required at this point. To obtain the smoothness of $\boldsymbol{T}_k$ with respect to $\boldsymbol{q}$ we need additional regularity for the family of finite range decomposition $\mathcal{C}_k^{(\boldsymbol{q})}$. This will be discussed in the next Section.

A second difference of this work compared to the techniques used by Bauerschmidt, Brydges, and Slade is a new solution of the large field problem. The *large field problem* refers to the fact that the contracting error coordinate K_k is of higher order and therefore small for small fields φ but it is in general unbounded and increases for large or rough fields φ. Heuristically this contribution is suppressed by the Gaussian measure that has only little mass on large fields but rigorous implementation of this intuition has caused substantial difficulties. The first solution to this problem was found by Gawędzki and Kupiainen who treat small and large fields differently and explicitely show that the large field contribution is negligible [90, 91]. This approach is still widely used see [13, 72, 73]. Later, Brydges and Yau use suitable weight functions that control the growth of $K_k(\varphi)$ as the field increases [52]. This idea was adopted in the approach by Bauerschmidt, Brydges, and Slade [47] and also in the adaption to gradient models [4]. In all those works the weight functions are given by carefully chosen explicit functions. One important improvement in the present work is that we instead use function implicitly defined by Gaussian integration which are almost optimal (cf. Section 4.5). Let us remark that the large field problem seems to be slightly more delicate in the context of gradient models because we want to require minimal growth assumptions on the potential V at infinity. In contrast, large fields are highly suppressed due to the φ^4-term in φ^4-theory.

Let us now also list the most important changes of our work compared to [4]. We extend their results in several directions. We consider general finite range interactions for vector valued fields and we are able to consider $d \geq 2$ instead of $d = 2$ and $d = 3$. This requires mostly notational changes but can otherwise be easily integrated in the framework. The two main differences address the two challenges discussed above. We construct new large field regulators and we show smoothness of the renormalisation map $\boldsymbol{T}_k$ with respect to $\boldsymbol{q}$. This allows us to handle potentials with weaker growth assumptions than in [4] and to avoid several technical difficulties that are present in [4].

1.5.4 Finite range decompositions

The use of finite range decompositions as defined above is an important ingredient in the renormalisation analysis. They have the useful property that contributions from well separated regions with distance bigger than L^k factorize when integrated against the measure $\mu_k^{(\boldsymbol{q})}$, i.e.,

$$\mathbb{E}_{\mu_k^{(\boldsymbol{q})}}\left(F(\nabla\varphi(x))G(\nabla\varphi(y))\right) = \mathbb{E}_{\mu_k^{(\boldsymbol{q})}}\left(F(\nabla\varphi(x))\right)\mathbb{E}_{\mu_k^{(\boldsymbol{q})}}\left(G(\nabla\varphi(y))\right) \tag{1.5.13}$$

for any functionals F, G if $|x-y| \geq L^k$. This keeps the combinatorics of the polymer expansions much simpler than for decompositions whose covariance only decays exponentially in the distance. Such decompositions that have also been used in statistical mechanics [90]. On the other hand it is much harder to obtain finite range decompositions because the finite range constraint and the condition that $\mathcal{C}_k^{(\boldsymbol{q})}$ is the kernel of a positive operator are difficult to satisfy simultaneously. Finite range decompositions were constructed in [3, 43, 51] using the Poisson kernel. A very general construction was found by Bauerschmidt in [18] where he exploited the finite speed of propagation of the wave equation.

A major technical difficulty in the analysis of gradient models in [4] is that a loss of regularity occurs for the renormalisation map $\boldsymbol{T}_k$ which is not differentiable in $\boldsymbol{q}$. The origin of this loss of regularity is, roughly speaking, that the map

$$\boldsymbol{q} \to \mathbb{E}_{\mu_{k+1}^{(\boldsymbol{q})}}\left(F(\varphi)\right) \tag{1.5.14}$$

is not differentiable in $\boldsymbol{q}$ for bounded functionals F. Instead one is forced to estimate the derivative in terms of D^2F, thereby losing regularity.

The basic problem is already present in the one dimensional setting where a simple calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}C}\int_{\mathbb{R}} f(x)\frac{e^{-\frac{x^2}{2C}}}{\sqrt{2\pi C}}\,\mathrm{d}x = \frac{1}{2C}\int_{\mathbb{R}} f(x)\left(\frac{x^2}{C}-1\right)\frac{e^{-\frac{x^2}{2C}}}{\sqrt{2\pi C}}\,\mathrm{d}x. \tag{1.5.15}$$

This implies that the derivative can be bounded uniformly for $f \in L^\infty(\mathbb{R})$ only if C is bounded away from zero. Similarly bounds for (1.5.14) require lower bounds on the eigenvalues of the covariances of the finite range decomposition. Those can be obtained using a slight modification of earlier construction of finite range decompositions. Initially this was shown in the author's Master's thesis [55] based on the decomposition from [3]. The generalisation to finite range instead of nearest neighbour interactions based on the decomposition constructed in [18] was done as a part of the author's Ph.D. work and can be found in Chapter 2. This work was also published [54]. This decomposition is then employed in Chapter 4 to avoid the loss of regularity.

1.6 Phase transitions for gradient interface models

This last section of the introduction deals with a slightly different line of research that is presented in Chapter 5. Recall that we defined a phase transition as the coexistence of at least two Gibbs measures. Phase transitions are a huge topic of research in physics but also in statistical mechanics and we refer to the literature for an overview of the existing mathematical results [92]. Here we are concerned with phase transitions for gradient interface models. Our motivation is that results in this direction might be a small first step towards a understanding of phase transitions in crystalline materials. Indeed, the presence of different phases, e.g., martensite and austenite and the occurance of microstructures has been long studied using different effective macroscopic models [16, 127]. It is of considerable interest to relate the macroscopic models to a underlying microscopic model and relate the coexistence of phases of the two models.

As discussed in Section 1.4 robust techniques for gradient models only work for very small or high temperature. For example, the rigorous renormalisation group analysis discussed above requires initial conditions very close to the Gaussian fixed point. In the case of gradient models this restricts the results to very small temperatures. Since no robust techniques for the analysis of gradient models at intermediate temperatures are known the goal of studying phase transition of models for crystalline matter seems to be currently out of reach.

A more modest goal is to understand the range of possible phenomena that occur at intermediate temperatures using suitable simpler toy models. For simplicity we restrict our attention to scalar nearest neighbour interactions, i.e., gradient interface models. A promising class of toy models is given by potentials V that can be represented as a mixture of Gaussians, i.e., V can be written as

$$e^{-V(x)} = \int_{\mathbb{R}+} \rho(\mathrm{d}\kappa)\, e^{-\frac{\kappa x^2}{2}} \tag{1.6.1}$$

where ρ is a (positive) Borel measure. This class of potentials was introduced in [32] and has caught some attention.

One important result is that phase transitions may occur for this type of potential as shown in [32] by Biskup and Kotecký, i.e., the authors construct two distinct ergodic gradient Gibbs measures of zero tilt. In particular, this result shows that new phenomena occur for gradient interface models with non-convex potentials compared to convex potentials where the Gibbs measure is unique for every tilt. Their proof relies on the use of reflection positivity which is a powerful technique that has been widely used to show the existence of phase transitions and we refer to the review [31] for an overview. It is, however, not robust under perturbations and it can only be applied on the torus. A further result concerning this model was shown in [33] by Biskup and Spohn where they establish that the scaling limit of all ergodic zero tilt gradient Gibbs measures for V as in (1.6.1) is a Gaussian free field if ρ has compact support in $(0, \infty)$. Recently, this result was extended to the class of potentials $V(x) = (1 + x^2)^\alpha$ with $0 < \alpha < 1/2$ in [151]. This is a class of potential with subquadratic growth and therefore the corresponding measure ρ is not compactly supported in $(0, \infty)$. All those results rely on the observation that the structure of the potential allows to raise κ to a degree of freedom, enabling us to express Gibbs measures as a mixture of non-uniform Gaussian fields.

In Chapter 5 we are going to study the properties of the κ-marginal corresponding to gradient Gibbs measures with zero tilt for this type of potential in more detail. This gives rise to a random conductance model whose properties we analyse. In particular we establish correlation inequalities. Mostly we restrict our analysis to the case where

$$\rho = p\delta_q + (1-p)\delta_1 \tag{1.6.2}$$

for two parameters $p \in [0, 1]$ and $q \geq 1$. In this case it turns out that the random conductance model has several similarities to the random cluster model and we adapt some of the techniques that have been introduced in this context see [77]. In particular we use planar duality to reprove the phase coexistence result proved in [32] using reflection positivity, i.e., we show the following.

For $d = 2$ and $q > 1$ sufficiently large and p satisfying $q = p^4/(1-p)^4$ there are two translation invariant gradient Gibbs measures with zero tilt for the potential V given by

$$e^{-V(x)} = pe^{-\frac{qx^2}{2}} + (1-p)e^{-\frac{x^2}{2}}. \tag{1.6.3}$$

The precise statement is Theorem 5.2.4. Moreover, we establish uniqueness of the gradient Gibbs measure for almost all values of p and q and show that the Dobrushin criterion can be applied to prove uniqueness for a small range of parameters in $d \geq 4$. All the results for potentials as in (1.6.1) are restricted to zero tilt. We will briefly discuss some extensions and ideas for non-zero tilt at the end of Chapter 5.

Chapter 2

Finite range decompositions of Gaussian measures with improved regularity

This chapter is based on the author's master's thesis [55]. Here we extend and simplify the results. In particular, we no longer use the construction introduced in [3] and instead rely on a different finite range decomposition constructed in [18]. This allows us to consider general finite range interactions while our earlier results were restricted to nearest neighbour interactions. Moreover, we obtain optimal L dependence for the bounds of the discrete derivatives of the finite range decomposition. These extensions are necessary for the results in Chapter 4. The results of this chapter have been published as a research paper

> Simon Buchholz. Finite range decomposition for Gaussian measures with improved regularity. *J. Funct. Anal.*, 275(7):1674–1711, 2018.
> URL: https://doi.org/10.1016/j.jfa.2018.02.018.

The text of this chapter agrees with the publication up to minor editorial changes.

2.1 Introduction

In this paper we consider finite range decompositions for families of translation invariant Gaussian fields on a torus $T_N = (\mathbb{Z}/L^N\mathbb{Z})^d$. A Gaussian process ξ indexed by T_N has range M if $\mathbb{E}(\xi(x)\xi(y)) = 0$ for any x, y such that $|x-y| \geq M$. A finite range decomposition of ξ is a decomposition $\xi = \sum_k \xi_k$ such that the ξ_k are independent processes with range $\sim L^k$. Equivalently, if $\mathcal{C}(x, y)$ is the covariance of ξ then a finite range decomposition is possible if there are covariances $\mathcal{C}_k(x, y)$ such that $\mathcal{C} = \sum_k \mathcal{C}_k$, $\mathcal{C}_k(x, y) = 0$ for $|x-y| \gtrsim L^k$, and $\mathcal{C}_k$ is positive semi-definite.

Here we consider vector valued Gaussian fields ξ_A whose covariance is the Greens function of a constant coefficient, anisotropic, elliptic, discrete difference operator $\mathcal{A} = \nabla^* A \nabla$ (plus higher order terms). Our main object of interest is the corresponding gradient Gaussian field $\nabla\xi_A$, i.e., we consider the σ-algebra generated by the gradients. They are referred to as massless field in the language of quantum field theory. Gradient fields appear naturally in discrete elasticity where the energy only depends on the distance between the atoms. The analysis of gradient Gaussian fields is difficult because they exhibit long range correlations only decaying critically

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as $\mathbb{E}(\nabla_i \xi^r(x) \nabla_j \xi^s(y)) \propto |x-y|^{-d}$. Finite range decompositions of gradient Gaussian fields are the basis of a multi-scale approach to control the correlation structure of the fields and avoid logarithmic divergences that appear in naive approaches.

Finite range decompositions of quadratic forms have appeared in different places in mathematics. Hainzl and Seiringer obtained decompositions of radially symmetric functions into weighted integrals over tent functions [101]. The first decomposition for a setting without radial symmetry was obtained for the discrete Laplacian by Brydges, Guadagni, and Mitter in [43]. Their results are based on averaging the Poisson kernel. Brydges and Talarczyk in [51] generalised this result to quite general elliptic operators on $\mathbb{R}^m$ that can be written as $\mathcal{A} = \mathcal{B}^*\mathcal{B}$. Adams, Kotecký, and Müller adapted this work in [3] to the discrete anisotropic setting. Their decomposition has the property that the kernels $\mathcal{C}_{A,k}$ are analytic function of the operator A. Later, Bauerschmidt gave a very general construction based on the finite propagation speed of the wave equation and functional calculus [18].

The goal of this work is to improve the regularity of the previous constructions. We show lower bounds for the previous decomposition and modify the construction such that we can control the decay behaviour of the kernels in Fourier space from above and below. This implies that the integration map $F \to \mathbb{E}(F(\cdot + \xi_{A,k}))$ is differentiable with respect to the matrix A uniformly in the size N of the torus. Our results hold for vector valued fields and we allow for higher order terms in the elliptic operator which corresponds to general quadratic finite range interaction. This allows us to handle, e.g., realistic models for discrete elasticity where next to nearest neighbour interactions are included. The construction is based on the Bauerschmidt decomposition in [18] but in a previous version of this project [53] we started from the construction in [3].

The main application of finite range decompositions is the renormalisation group approach to problems in statistical mechanics. Renormalisation was introduced by Wilson in the analysis of phase transitions [148]. Brydges and Yau [52] adapted Wilson's ideas to the statistical mechanics setting and initiated a long stream of developments. Recently Bauerschmidt, Brydges, and Slade introduced a general framework and investigate the ϕ^4 model [22] and the weakly self avoiding random walk [21]. Their approach allows one to analyse functional integrals $\mathbb{E}(K)$ where K is a non-linear functional depending on a field on a large lattice and the expectation is with respect to a (gradient) Gaussian field with long ranged correlation. A key step is that this integral can be rewritten as a series of integrations using a finite range decomposition of the Gaussian field. Then one can analyse the correlation structure scale by scale.

Adams, Kotecký, and Müller extend this method to the anisotropic setting where $\mathcal{A}$ is not necessarily a multiple of the Laplacian and they show strict convexity of the surface tension for non-convex potentials for small tilt and low temperature [4]. However, they face substantial technical difficulties because the integration map $F \to \mathbb{E}(F(\cdot + \xi_{A,k}))$ is not differentiable with respect to A for their finite range decomposition and regularity is lost. The results of this paper allow to avoid this loss of regularity and therefore simplify their analysis.

This paper is structured as follows: In Section 2.2 we introduce the setting, state the main result, and give a brief motivation for the bounds of the new finite range decomposition. Then, in Section 2.3 we prove the main result based on the finite range decomposition from [3]. Finally, in Section 2.4 we show the smoothness of the integration map. For the convenience of the reader the appendix states the precise results of [3] when applied to our setting.

Notation: In this paper we always understand inequalities of the form

$$A \geq B \tag{2.1.1}$$

for $A, B \in \mathbb{C}^{n\times n}_{\text{her}}$ in the sense of Hermitian matrices,i.e., $A - B$ is semi-positive definite. Moreover we use the inclusion $\mathbb{R} \ni t \to t \cdot \mathrm{Id} \in \mathbb{C}^{n\times n}_{\text{her}}$ without reflecting this in the notationt.

2.2 Setting and main result

Fix an odd integer $L \geq 3$, a dimension $d \geq 2$, and the number of components $m \geq 1$ for the rest of this paper. Let $T_N = (\mathbb{Z}/(L^N\mathbb{Z}))^d$ be the d dimensional discrete torus of side length L^N. We equip T_N with the quotient distances d (or $|\cdot|$) and d_∞ (or $|\cdot|_\infty$) induced by the Euclidean and maximum norm respectively. We are interested in gradient fields. The condition of being a gradient is, however, complicated in dimension $d \geq 2$. Therefore we work with usual fields modulo a constant which are in one-to-one correspondence to gradient fields. Define the space of m-component fields as

$$\mathcal{V}_N = \{\varphi : T_N \to \mathbb{R}^m\} = (\mathbb{R}^m)^{T_N}. \tag{2.2.1}$$

Since we take the quotient modulo constant fields we can restrict our fields to have average zero, hence we set

$$\mathcal{X}_N = \left\{ \varphi \in \mathcal{V}_N : \sum_{x \in T_N} \varphi(x) = 0 \right\}. \tag{2.2.2}$$

Let the dot denote the standard scalar product on $\mathbb{R}^m$ which is later extended to $\mathbb{C}^m$. For $\psi, \varphi \in \mathcal{X}_N$ the expression

$$\langle \varphi, \psi \rangle = \sum_{x \in T_N} \varphi(x) \cdot \psi(x) \tag{2.2.3}$$

defines a scalar product on $\mathcal{X}_N$ and this turns $\mathcal{X}_N$ into a Hilbert space $\mathcal{H}$. The discrete forward and backward derivatives are defined by

$$\begin{aligned} (\nabla_j \varphi)_r(x) &= \varphi_r(x + e_j) - \varphi_r(x) \qquad r \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, d\}, \\ (\nabla_j^* \varphi)_r(x) &= \varphi_r(x - e_j) - \varphi_r(x) \qquad r \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, d\}. \end{aligned} \tag{2.2.4}$$

Here e_j are the standard unit vectors in $\mathbb{Z}^d$. Forward and backward derivatives are adjoints of each other.

Next we introduce the set of operators for which we discuss finite range decompositions. We fix some necessary notation. Let $\mathcal{I} \subset \mathbb{N}_0^d \setminus \{0, \ldots, 0\}$ be a finite set of multiindices which is fixed for the rest of this work. We assume that $\mathcal{I}$ contains all multiindices of order 1, i.e., the gradient. We define $R = \max_{\alpha \in \mathcal{I}} |\alpha|_\infty$. With $\mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}}$ we denote the space of discrete derivatives with $\alpha \in \mathcal{I}$ and for any field φ we denote $D\varphi(x) \in \mathcal{G}$ for the vector $(\nabla^\alpha \varphi(x))_{\alpha \in \mathcal{I}}$. We equip $\mathcal{G}$ with the standard scalar product

$$(D\varphi(x), D\psi(x)) = \sum_{\alpha \in \mathcal{I}} (\nabla^\alpha \varphi(x), \nabla^\alpha \psi(x))_{\mathbb{R}^m}. \tag{2.2.5}$$

For any $z \in \mathcal{G}$ we write z^∇ for the restriction of z to the gradient components $\alpha = e_1, \ldots, e_d$. We consider non-negative quadratic forms $Q : \mathcal{G} \to \mathbb{R}$ that satisfy

$$Q(z) \geq \omega_0 |z^\nabla|^2 \tag{2.2.6}$$

for some $\omega_0 > 0$. This condition for general finite range interactions already appeared in [128]. To keep the notation consistent with [3] we denote the corresponding symmetric generator by $A : \mathcal{G} \to \mathcal{G}$. By definition A satisfies $(z, Az) = Q(z)$. The matrix elements of A are denoted by $A_{\alpha\beta} \in \mathbb{R}^{m \times m}$, i.e., $Q(D\varphi(x), D\psi(x)) = (AD\varphi(x), D\psi(x)) = \sum_{\alpha,\beta \in \mathcal{G}} \nabla^\alpha \varphi(x) A_{\alpha\beta} \nabla^\beta \psi(x)$.

Usually we consider the set of generators A whose operator norm with respect to the standard scalar product on $\mathcal{G}$ satisfies

$$\|A\| \leq \Omega_0 \tag{2.2.7}$$

for some fixed $\Omega_0 > 0$. We denote the set of symmetric operators A such that (2.2.6) and (2.2.7) hold by $\mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$. We think of $\mathcal{I}$, ω_0, and Ω_0 as fixed and in the following and all constants depend on d, m, R, ω_0, and Ω_0 in the following. From the operator A we obtain a corresponding elliptic finite difference operator

$$\mathcal{A} = \sum_{\alpha,\beta\in\mathcal{G}} (\nabla^\alpha)^* A_{\alpha\beta} \nabla^\beta. \tag{2.2.8}$$

The operator $\mathcal{A}$ defines a Gaussian measure μ_A on $\mathcal{X}_N$ that is given by

$$\mu_A(\mathrm{d}\varphi) = \frac{e^{-\frac{1}{2}\langle\varphi,\mathcal{A}\varphi\rangle}}{\sqrt{\det\left(2\pi\mathcal{A}^{-1}\right)}} \lambda(\mathrm{d}\varphi) \tag{2.2.9}$$

where λ denotes the $m(L^{Nd}-1)$ dimensional Hausdorff-measure on the affine space $\mathcal{X}_N$.

In the following we discuss finite range decompositions for the Greens functions of operators $\mathcal{A} : \mathcal{X}_N \to \mathcal{X}_N$ with $A \in \mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$. In [18] and [3] only the case where $\mathcal{I} = \{e_1, \ldots, e_d\}$ was discussed. By assumption $\langle\varphi, \mathcal{A}\varphi\rangle \geq \omega_0\langle\nabla\varphi, \nabla\varphi\rangle$ which implies positivity of the operator. Hence the operator $\mathcal{A}$ is invertible and we call its inverse operator $\mathcal{C} = \mathcal{A}^{-1}$. The operator $\mathcal{C}$ is the covariance operator of the Gaussian measure μ_A. Since $\mathcal{A}$ is translation invariant (i.e., $[\tau_x, \mathcal{A}] = 0$ where $\tau_x : \mathcal{X}_N \to \mathcal{X}_N$ denotes the translation operator $(\tau_x\phi)(y) = \phi(y-x)$ and $[A, B] = AB - BA$ is the commutator) the same is true for $\mathcal{C}$. Translation invariance implies that the operator $\mathcal{C}$ has a unique kernel $\mathcal{C} : T_N \to \mathbb{R}^{m\times m}$ (cf. Lemma 3.5 in [3]) such that

$$(\mathcal{C}\varphi)(x) = \sum_{y\in T_N} \mathcal{C}(x-y)\varphi(y) \tag{2.2.10}$$

and $\mathcal{C} \in \mathcal{M}_N$ where $\mathcal{M}_N$ is the space of $m \times m$ matrix valued functions on T_N with average zero, i.e., $\mathcal{C}_{ij} \in \mathcal{X}_N$ for all $1 \leq i, j \leq m$ (this condition can always be satisfied because constant kernels generate the zero-operator on $\mathcal{X}_N$).

Remark 2.2.1. *Note that the kernel $\mathcal{C}$ defines an extension $\overline{\mathcal{C}}$ of the operator $\mathcal{C}$ from the space $\mathcal{X}_N$ to the space $\mathcal{V}_N$ that annihilates constant fields. Since the space of constant fields is the orthogonal complement of $\mathcal{X}_N$ in $\mathcal{V}_N$ the operator $\overline{\mathcal{C}}$ is a positive semi-definite operator on an euclidean space. Hence it is the covariance of a Gaussian measure on $\mathcal{V}_N$ and this measure is concentrated on $\mathcal{X}_N$ and its restriction to $\mathcal{X}_N$ agrees with μ_A given by equation* (2.2.9). *This implies by general Gaussian calculus that*

$$\mathbb{E}_{\mu_A}\left(\varphi(x)\varphi(y)\right) = \mathcal{C}(x-y). \tag{2.2.11}$$

Let us define the term *finite range.*

Lemma and Definition 2.2.2. *Let $\mathcal{C} : \mathcal{X}_N \to \mathcal{X}_N$ be a translation invariant operator with kernel $\mathcal{C} \in \mathcal{M}_N$. We say that $\mathcal{C}$ has range at most l for $2l + 3 < L^N$ if the following three equivalent statements hold.*

1. $\langle\varphi, \mathcal{C}\psi\rangle = 0$ *for all* $\varphi, \psi \in \mathcal{X}_N$ *with* $dist_\infty(supp\,\varphi, supp\,\psi) > l$.

2. *There is* $M \in Mat_{m,m}(\mathbb{R})$ *such that* $\mathcal{C}(x) = M$ *for* $\mathrm{d}_\infty(x,0) > l$.
3. $\operatorname{supp} \mathcal{C}\varphi \subset \operatorname{supp}\varphi + \{-l,\ldots,l\}^d$ *for all* $\varphi \in \mathcal{X}_N$.

For $2l+3 \geq L^N$ *property* 2 *shall be the defining property.*

Proof. This is Lemma 3.6 in [3]. The implication $(ii) \Rightarrow (iii) \Rightarrow (i)$ is always true. □

Note that under the condition of Definition 2.2.2 for a positive operator $\mathcal{C}$ the gradient field with covariance $\mathcal{C}$ has finite range of correlations. Indeed, from (2.2.11) we conclude that for $x, y \in T_N$ with $|x-y|_\infty > l+1$

$$\mathbb{E}_{\mathcal{C}}(\nabla_i\varphi(x)\nabla_j\varphi(y)) = \nabla_j^*\nabla_i \mathcal{C}(x-y) = 0. \tag{2.2.12}$$

Observe that the operator $\mathcal{A}$ introduced before has range at most R. We seek a decomposition of $\mathcal{C} = \mathcal{A}^{-1}$ into translation invariant and positive operators $\mathcal{C}_k$ with

$$\mathcal{C} = \sum_{k=1}^{N+1} \mathcal{C}_k \tag{2.2.13}$$

such that the range of $\mathcal{C}_k$ is smaller than $L^k/2$ and the $\mathcal{C}_k$ satisfy certain bounds. This implies that $\mathcal{C}_k$ is the covariance operator of a Gaussian measure on $\mathcal{X}_N$ and the Gaussian variables $\nabla_i\varphi(x)$ and $\nabla_j\varphi(y)$ (where φ is distributed according to this measure) are uncorrelated and therefore independent for $|x-y|_\infty \geq L^k/2$. A decomposition satisfying the finite range property and translation invariance will be called finite range decomposition. Note, however, that they are only useful in the presence of strong bounds because, e.g., the trivial decomposition $\mathcal{C}_{N+1} = \mathcal{C}$ and $\mathcal{C}_k = 0$ for $k \leq N$ has the finite range property.

Translation invariant operators are diagonal in Fourier space which will be introduced briefly because the strongest bounds of the kernel $\mathcal{C}_k$ are the bounds for its Fourier transform. Define the dual torus

$$\widehat{T}_N = \left\{ \frac{(-L^N+1)\pi}{L^N}, \frac{(-L^N+3)\pi}{L^N}, \ldots, \frac{(L^N-1)\pi}{L^N} \right\}^d. \tag{2.2.14}$$

For $p \in \widehat{T}_N$ the exponentials $f_p : T_N \to \mathbb{C}$ with $f_p(x) = e^{ip\cdot x}$ are well defined since $e^{ip\cdot x}$ is a $(L^N\mathbb{Z})^d$ periodic function on $\mathbb{Z}^d$. Here and in the following we use the immediate generalisations to complex valued fields. The Fourier transform $\widehat{\psi} : \widehat{T}_N \to \mathbb{C}$ of a scalar field $\psi : T_N \to \mathbb{C}$ is defined by

$$\widehat{\psi}(p) = \sum_{x\in T_N} e^{-ip\cdot x}\psi(x) = \sum_{x\in T_N} f_p(-x)\psi(x) \tag{2.2.15}$$

and the inverse transform is given by

$$\psi(x) = \frac{1}{L^{Nd}} \sum_{p\in\widehat{T}_N} e^{ip\cdot x}\widehat{\psi}(p). \tag{2.2.16}$$

The Fourier transform maps the space $\mathcal{X}_N$ bijectively on the subspace $\{\widehat{\psi} : \widehat{T}_N \to \mathbb{C} | \widehat{\psi}(0) = 0\}$. Clearly $\widehat{f_p}(q) = L^{Nd}\delta_{pq}$ for $p, q \in \widehat{T}_N$. For matrix- or vector-valued functions we define the Fourier transform component-wise. The Fourier transform satisfies

$$\langle \psi, \varphi \rangle = \frac{1}{L^{Nd}} \sum_{p\in\widehat{T}_N} \widehat{\psi}(p) \cdot \widehat{\varphi}(p) \tag{2.2.17}$$

where we extended the scalar product anti-linearly in the first component, this means $v \cdot w = \sum_{i=1}^m \overline{v}_i w_i$. Hence the functions $L^{-\frac{Nd}{2}} f_p e_i$ for $p \neq 0$ and $e_i \in \mathbb{R}^m$ a standard unit vector form an orthonormal basis of $\mathcal{X}_N$ (complexified). Let $\mathcal{A}, \mathcal{B} \in \mathcal{M}_N$ be two matrix-valued functions. Define the convolution by

$$\mathcal{A} * \mathcal{B}(x) = \sum_{y \in T_N} \mathcal{A}(x-y)\mathcal{B}(y). \tag{2.2.18}$$

Note that by (2.2.10) the kernel of the composition $\mathscr{A}\mathscr{B}$ of the operators $\mathscr{A}$ and $\mathscr{B}$ with kernels $\mathcal{A}$ and $\mathcal{B}$ is given by $\mathcal{A} * \mathcal{B}$.

Consider a translation invariant operator $\mathscr{K}$ with kernel $\mathcal{K} \in \mathcal{M}_N$. As in the continuous setting the Fourier transform of a convolution is the product of the Fourier transforms, i.e.,

$$\widehat{\mathscr{K}\psi}(p) = \widehat{\mathcal{K} * \psi}(p) = \widehat{\mathcal{K}}(p)\widehat{\psi}(p). \tag{2.2.19}$$

Hence translation invariant operators are indeed (block) diagonal in Fourier space with eigenvalues given by the Fourier transform of the kernel.

Next we calculate the Fourier modes of the kernel of the operator $\mathcal{A}$. A simple calculation shows that $\widehat{\nabla_i \varphi}(p) = q_i(p)\widehat{\varphi}(p)$ and $\widehat{\nabla_i^* \varphi}(p) = \overline{q}_i(p)\widehat{\varphi}(p)$ where $q_i(p) = e^{ip_i} - 1$ and $\overline{q}_i(p) = e^{-ip_i} - 1$ denotes the complex conjugate. This implies

$$\widehat{\mathcal{A}}(p) = \sum_{\alpha,\beta \in \mathcal{I}} \overline{q}(p)^\alpha A_{\alpha\beta} q(p)^\beta \tag{2.2.20}$$

where $q(p)^\alpha = \prod_{i=1}^d q_i(p)^{\alpha_i}$. Note again the formal similarity to the continuum setting. The estimate $\frac{4}{\pi^2} t^2 \leq |e^{it} - 1|^2 \leq t^2$ for $t \in [-\pi, \pi]$ immediately implies that $\frac{4}{\pi^2}|p|^2 \leq |q(p)|^2 \leq |p|^2$ for any $p \in \widehat{T}_N$. Hence we find using $|p| < \sqrt{d}\pi$ and $|\alpha|_1 \leq dR$ for $\alpha \in \mathcal{I}$

$$\|\widehat{\mathcal{A}}(p)\| = \left\| \sum_{\alpha,\beta \in \mathcal{I}} \overline{q}(p)^\alpha q(p)^\beta A_{\alpha\beta} \right\| \leq \|A\| \left| (q(p)^\alpha)_{\alpha \in \mathcal{I}} \right|_{\mathcal{G}}^2 \leq \Omega_0 |p|^2 \cdot |\mathcal{I}| (d\pi^2)^{dR} \tag{2.2.21}$$

On the other hand we also find a lower bound for the Fourier modes and $a \in \mathbb{R}^m$ using the assumption (2.2.6) for $z = (aq(p)^\alpha)_{\alpha \in \mathcal{I}}$

$$a \cdot \widehat{\mathcal{A}}(p) a \geq \omega_0 |q(p)|^2 |a|^2 \geq \frac{4\omega_0}{\pi^2} |p|^2 |a|^2. \tag{2.2.22}$$

Together this yields the important bound

$$\|\widehat{\mathcal{A}}(p)\| \leq \Omega |p|^2 \quad \text{and} \quad \widehat{\mathcal{A}}(p) \geq \omega |p|^2. \tag{2.2.23}$$

Finally we note that by the Fourier inversion formula (2.2.16) for any multi-index α

$$\nabla^\alpha \mathcal{C}_k(x) = \frac{1}{L^{Nd}} \sum_{p \in \widehat{T}_N} \widehat{\mathcal{C}}_k(p) q(p)^\alpha f_p(x). \tag{2.2.24}$$

The $L^\infty - L^1$ bounds for the Fourier transform also hold in the discrete case

$$\|\nabla^\alpha \mathcal{C}_k(x)\| \leq \frac{1}{L^{Nd}} \sum_{p \in \widehat{T}_N} \|\widehat{\mathcal{C}}_k(p)\| |p|^{|\alpha|}. \tag{2.2.25}$$

Finally we introduce a dyadic partition of the dual torus for $j = 1, \ldots, N$

$$\boldsymbol{A}_j = \{p \in \widehat{T}_N : L^{-j-1} < |p| \le L^{-j}\} \tag{2.2.26}$$

$$\boldsymbol{A}_0 = \{p \in \widehat{T}_N : L^{-1} < |p|\}. \tag{2.2.27}$$

When the size of the torus $\widehat{T}_N$ is not clear from the context we write $\boldsymbol{A}_j^N = \boldsymbol{A}_j$ for clarity. Observe that

$$|\boldsymbol{A}_j| \le \kappa(d) L^{(N-j)d} \tag{2.2.28}$$

for some constant $\kappa(d) > 0$.

Let us now state the main result of [18] adapted to our setting.

Theorem 2.2.3. *Assume that $d \ge 2$ and $A \in \bigcup_{\omega_0 > 0} \mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$. Then there is a decomposition $\mathcal{C}_A = \sum_{k=1}^{N+1} \mathcal{C}_{A,k}$ where $\mathcal{C}_{A,k}$ are positive, translation invariant operators and the map $A \to \mathcal{C}_{A,k}(x)$ only depends on R and Ω_0 but not on ω_0. The $\mathcal{C}_{A,k}$ are polynomials in A for $1 \le k \le N$ and real analytic for $k = N+1$. This decomposition has the finite range property*

1. *$\mathcal{C}_{A,k}(x) = -M_k$ for $1 \le k \le N$ and $|x|_\infty \ge L^k/2$ where $M_k \in \mathbb{R}^{m \times m}$ are positive semi-definite matrices independent of A (for Ω_0 and $\mathcal{G}$ fixed). In particular $\mathcal{C}_{A,k}$ has range smaller $L^k/2$.*

Moreover we have the following bounds for $\omega_0 > 0$ and $A \in \mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$

2. *In Fourier space the following bounds hold for any positive integers ℓ and $\tilde{n}$ and symmetric $\dot{A} \in \mathcal{L}(\mathcal{G})$*

$$\|D_A^\ell \widehat{\mathcal{C}}_{A,k}(p)\| \le \begin{cases} C_{\ell,\tilde{n}} |p|^{-2} (|p| L^{(k-1)})^{-\tilde{n}} & \text{for } |p| > L^{-k} \ (p \in \boldsymbol{A}_j,\ j \le k-1), \\ C_\ell L^{2k} \|\dot{A}\|^\ell & \text{for } p \le L^{-k} \ (p \in \boldsymbol{A}_j,\ j \ge k). \end{cases} \tag{2.2.29}$$

Here C_ℓ and $C_{\ell,\tilde{n}}$ are constants that do not depend on L, N, or k. The corresponding lower bound reads

$$\widehat{\mathcal{C}}_{A,k}(p) \ge \begin{cases} c \min\left(|p|^{-2}, L^2\right) & \text{for } \quad k = 1, \\ c L^{2k} & \text{for } |p| < L^{-k} \ (p \in \boldsymbol{A}_j,\ j \ge k). \end{cases} \tag{2.2.30}$$

for some constant $c > 0$ depending on the same quantities as C_ℓ.

3. *In particular, we have*

$$\sup_{x \in T_N} \left\| D_A^\ell \nabla^\alpha \mathcal{C}_{A,k}(x) \right\| \le \begin{cases} C(\alpha, \ell) L^{-(k-1)(d-2+|\alpha|)} & \text{for } d + |\alpha| > 2 \\ C(\alpha, \ell) \ln(L) L^{-(k-1)(d-2+|\alpha|)} & \text{for } d + |\alpha| = 2. \end{cases} \tag{2.2.31}$$

Here $C(\alpha, \ell)$ denotes a constant that does not depend on L, N, and k.

Remark 2.2.4. *We usually work on a torus with fixed side-length in this paper. In applications one is often interested in the thermodynamic limit $N \to \infty$. Hence, it is also interesting to investigate the dependence of the finite range decomposition on the size L^N of the torus. In [20] the dependence of a finite range decomposition on the size of the torus and the relation to decompositions of the corresponding operator on $\mathbb{Z}^d$ are discussed in more detail and this is used to extend the renormalisation analysis from discrete tori to $\mathbb{Z}^d$ in [47]. We state the corresponding result for* *the dependence on the size of the torus adapted to our setting. To compare decompositions for different sizes of the torus let us choose* $\Lambda_N = \{-(L^N-1)/2, -(L^N-3)/2, \ldots, (L^N-1)/2\}^d \subset \mathbb{Z}^d$ *as the underlying set of* T_N*. Note that* $\Lambda_N \subset \Lambda_{N'}$ *for* $N \le N'$*. Let us denote by* $\mathcal{C}_k^N$ *the kernels of the decomposition depending on the torus size* L^N*. It can be shown (cf. the proof of Theorem 2.2.3 in Appendix 2.A) that for* $k \le N \le N'$ *and* $x \in \Lambda_N$ *the decomposition from Theorem 2.2.3 satisfies*

$$\mathcal{C}_k^N(x) - \mathcal{C}_k^{N'}(x) = -(M_k^N - M_k^{N'}), \tag{2.2.32}$$

hence the kernels agree up to a constant shift locally and they are constant for $|x|_\infty \ge L^k/2$*. We define* $\Lambda_N' = \{x \in \mathbb{Z}^d : |x|_\infty < (L^N-1)/4\}$*. Then we have* $x - y \in \Lambda_N$ *for* $x, y \in \Lambda_N' \subset \mathbb{Z}^d$*. Now* (2.2.32) *implies that for* $x, y \in \Lambda_N'$ *such that* $x + e_i, y + e_j \in \Lambda_N'$

$$\mathbb{E}_{\mu_k^N} \nabla_i \varphi(x) \nabla_j \varphi(y) = \nabla_j^* \nabla_i \mathcal{C}_k^N(x-y) = \nabla_j^* \nabla_i \mathcal{C}_k^{N'}(x-y) = \mathbb{E}_{\mu_k^{N'}} \nabla_i \varphi(x) \nabla_j \varphi(y). \tag{2.2.33}$$

This means that the covariance structures of μ_k^N *and* $\mu_k^{N'}$ *agree locally. In particular we can conclude that for any set* $X \subset \Lambda_N'$ *satisfying* $X + e_i \subset \Lambda_N'$ *for* $1 \le i \le d$*, any* $1 \le k \le N$*, and any measurable functional* $F : (\mathbb{R}^m)^X \to \mathbb{R}$

$$\int_{\mathcal{X}_N} F(\nabla\varphi\restriction_X)\, \mu_k^N(\mathrm{d}\varphi) = \int_{\mathcal{X}_{N'}} F(\nabla\varphi\restriction_X)\, \mu_k^{N'}(\mathrm{d}\varphi). \tag{2.2.34}$$

Hence the influence of the finite size of the torus is restricted to the last term of the decomposition.

Theorem 2.2.3 is a special case of the much more general Theorem 1.2 in [18] except for the simple lower bound. For the convenience of the reader we include all relevant calculations for the concrete setting in the appendix. The estimates are similar to the ones that appeared in [20].

Our main result is an extension of this result which additionally gives controlled decay of the kernels in Fourier space. In particular the operators $\mathscr{C}_{k,A}$ and $\mathscr{C}_{k,A'}$ are comparable for the new construction. The main application of the following theorem is the regularity of the renormalisation map which is stated in Proposition 2.4.1 in Section 2.4.

Theorem 2.2.5. *Let* $L > 3$ *odd,* $N \ge 1$ *as before and let* $\tilde{n} > n$ *be two integers. Fix* $\Omega_0 > \omega_0 > 0$ *and consider the family of symmetric, positive operators* $A \in \mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$*. Then there exists a family of finite range decomposition* $\mathscr{C}_{A,k}$ *of* $\mathscr{C}_A$ *such that the map* $A \to \mathcal{C}_{A,k}(x)$ *only depends on* R*,* ω_0*, and* Ω_0*. The map is a polynomial in* A *for* $1 \le k \le N$ *and real analytic for* $k = N+1$*. The operators* $\mathscr{C}_{A,k}$ *satisfy*

$$\begin{aligned} &\mathscr{C}_A = \sum_{k=1}^{N+1} \mathscr{C}_{A,k}, \\ &\mathscr{C}_{A,k}(x) = M_k \text{ for } 1 \le k \le N, \text{ and } |x|_\infty \ge \frac{L^k}{2}, \end{aligned} \tag{2.2.35}$$

where $M_k \le 0$ *and* M_k *is independent of* A*. The* α*-th discrete derivative for all* α *with* $|\alpha|_1 \le n$ *is bounded by*

$$\sup_{x \in T_N} \sup_{\|\dot{A}\| \le 1} \left\| \nabla^\alpha D_A^\ell \mathcal{C}_{A,k}(x)(\dot{A}, \ldots, \dot{A}) \right\| \le \begin{cases} 3C(\alpha,\ell) L^{-(k-1)(d-2+|\alpha|)} & \text{for } d + |\alpha| > 2 \\ 3C(\alpha,\ell) \ln(L) L^{-(k-1)(d-2+|\alpha|)} & \text{for } d + |\alpha| = 2. \end{cases} \tag{2.2.36}$$

where the constants are the same as in Theorem 2.2.3. We have the following lower bounds in Fourier space for some $c = c(\tilde{n}) > 0$

$$\widehat{\mathcal{C}}_{A,k}(p) \geq \begin{cases} cL^{-2(d+\tilde{n})-1}L^{2j}L^{(k-j)(-d+1-n)} & \text{for } p \in \boldsymbol{A}_j \text{ and } j < k \\ cL^{-2(d+\tilde{n})-1}L^{2k} & \text{for } p \in \boldsymbol{A}_j \text{ and } j \geq k. \end{cases} \tag{2.2.37}$$

Similar upper bounds hold with some constant $C = C(\tilde{n})$

$$\|\widehat{\mathcal{C}}_{A,k}(p)\| \leq \begin{cases} CL^{2(d+\tilde{n})+1}L^{2j}L^{(k-j)(-d+1-n)} & \text{for } p \in \boldsymbol{A}_j \text{ and } j < k \\ CL^{2k} & \text{for } p \in \boldsymbol{A}_j \text{ and } j \geq k. \end{cases} \tag{2.2.38}$$

For the derivatives of the kernels and for $\|\dot{A}\| \leq 1$, $\ell \geq 1$, $p \in A_j$ we have the following stronger bounds in Fourier space

$$\left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} \widehat{\mathcal{C}}_{A+s\dot{A},k}(p) \right\| \leq \begin{cases} CL^{2(d+\tilde{n})+1}L^{2j}L^{(k-j)(-d+1-\tilde{n})} & \text{for } p \in \boldsymbol{A}_j \text{ and } j < k \\ CL^{2k} & \text{for } p \in \boldsymbol{A}_j \text{ and } j \geq k, \end{cases} \tag{2.2.39}$$

i.e., the decay of the derivative of the Fourier modes for large p is governed by $\tilde{n}$ and not by n as in (2.2.38)*. The lower and upper bound can be combined to give for $\ell \geq 1$ and $p \in A_j$*

$$\left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} \widehat{\mathcal{C}}_{A+s\dot{A},k}(p) \right\| \cdot \left\| \widehat{\mathcal{C}}_{A,k}(p)^{-1} \right\| \leq \begin{cases} \Xi L^{4(d+\tilde{n})+2}L^{(k-j)(n-\tilde{n})} & \text{for } p \in \boldsymbol{A}_j \text{ and } j < k \\ \Xi L^{2(d+\tilde{n})+1} & \text{for } p \in \boldsymbol{A}_j \text{ and } j \geq k. \end{cases} \tag{2.2.40}$$

The constants $\Xi = \Xi(\tilde{n}, \ell)$ do not depend on N, k, or L.

The proof of this theorem can be found in Section 2.3.

Remark 2.2.6. *For the calculations it is advantageous to express the bounds mostly in terms of the single quantity L. To get a better feeling for the bounds it is useful to write them in terms of $|p|$ and L. The definition of $\boldsymbol{A}_j$ implies that $|p| \approx L^{-j}$ for $p \in \boldsymbol{A}_j$. Hence up to constants that also depend on L we find $\widehat{\mathcal{C}}_k(p) \approx L^{2k}$ for $|p| \lesssim L^{-k}$ and $\widehat{\mathcal{C}}_k(p) \approx |p|^{-2}(L^k|p|)^{-(d-1+n)}$ otherwise. For $\ell \geq 1$, however, we find $\|\frac{\mathrm{d}^\ell}{\mathrm{d}A^\ell}\widehat{\mathcal{C}}_{A,k}(p)\| \lesssim |p|^{-2}(L^k|p|)^{-d+1-\tilde{n}}$ for $|p| \gtrsim L^{-k}$ and $\|\frac{\mathrm{d}^\ell}{\mathrm{d}A^\ell}\widehat{\mathcal{C}}_{A,k}(p)\| \lesssim L^{2k}$ otherwise. In particular the quotient of the derivative and the kernel itself is bounded by a constant for $|p| \lesssim L^{-k}$ and behaves as $(|p|L^k)^{n-\tilde{n}}$ for $p \gtrsim L^{-k}$, i.e., decays as fast as we like if we choose $\tilde{n} \gg n$.*

Remark 2.2.7. *From the proof it is clear that Remark 2.2.4 also applies to the finite range decomposition in Theorem 2.2.5.*

Remark 2.2.8. *It is possible to obtain similar results using the decomposition based on averaging the Poisson kernel over cubes that appeared in [3]. Some steps can be found in the earlier version [53] of this work. To prove Theorem 2.2.5, however, technical modifications of the construction in [3] must be implemented in order to handle the generalisation to higher order operators and to get rid of some L dependent constants in the bound* (2.2.31)*.*

Since the theorem is rather technical we briefly motivate the need for lower bounds and the specific structure of the bounds. As pointed out before we are interested in bounds for $\partial_A \mathbb{E}_{\mathcal{C}_{k,A}}(F(\cdot + \xi))$. By Theorem 2.2.3 the derivatives of the covariance $\mathcal{C}_{k,A}$ with respect to A are controlled. By the chain rule we need to bound the derivatives of expectations of Gaussian random variables with respect to their covariance. Let us briefly discuss this problem in a general setting. Consider a smooth map $M : \mathbb{R} \to \mathbb{R}^{s\times s}_{\mathrm{sym},+}$ mapping to the $s \times s$ dimensional, symmetric, and positive matrices. We denote the Gaussian measure on $\mathbb{R}^s$ with covariance $M(t)$ by

$$\mu_{M(t)}(\mathrm{d}x) = \varphi_{M(t)}(x)\,\mathrm{d}x = \frac{e^{-\frac{1}{2}\langle x, M(t)^{-1}x\rangle}}{\sqrt{\det(2\pi M(t))}}\,\mathrm{d}x. \tag{2.2.41}$$

Let $F \in C_b(\mathbb{R}^s, \mathbb{R})$ be a bounded and continuous function. We are interested in a bound for the expression

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^s} F(x)\,\mu_{M(t)}(\mathrm{d}x)\Big|_{t=0} \right|. \tag{2.2.42}$$

In principle this seems easy because the Gaussian integral acts as a heat semi-group which is infinitely smoothing. However this is only true as long as the eigenvalues do not approach zero (think of the delta distribution which is a (degenerate) Gaussian measure). Therefore lower bounds on the eigenvalues of $M(t)$ control the smoothing behaviour of the semigroup. Now we discuss the necessary bounds in a bit more detail. An elementary calculation, with the abbreviations $M = M(0)$ and $\dot{M} = \dot{M}(0)$, shows

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^s} F(x)\,\mu_{M(t)}(\mathrm{d}x)\Big|_{t=0} \right| = \left| \frac{1}{2} \int_{\mathbb{R}^s} F(x) \left(\langle x, M^{-1}\dot{M}M^{-1}x\rangle - \operatorname{Tr} M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}} \right) \mu_M(\mathrm{d}x) \right|. \tag{2.2.43}$$

The trace term arises as the derivative of the determinant. To bound this expression one needs bounds on M^{-1}, i.e., lower bounds on the spectrum of M are required. With the help of the Cauchy-Schwarz inequality it can be shown that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^s} F(x)\,\mu_{M(t)}(\mathrm{d}x)\Big|_{t=0} \right| \le \|F\|_{L^2(\mathbb{R}^s, \mu_{M(t)})} \|M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\|_{\mathrm{HS}}. \tag{2.2.44}$$

To bound the right hand side of this equation we need lower bounds on the finite range decomposition and the derivatives of the kernels with respect to A have to decay faster than the kernels itself. Denote $\dot{\mathcal{C}}_{k,A} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{k,A+t\dot{A}}$ where $\dot{A}$ is a fixed symmetric operator. Then the Hilbert Schmidt norm from the right hand side of (2.2.44) corresponds for scalar fields ($m = 1$) to the expression

$$\|\mathcal{C}_{N,A}^{-\frac{1}{2}}\dot{\mathcal{C}}_{N,A}\mathcal{C}_{N,A}^{-\frac{1}{2}}\|_{\mathrm{HS}}^2 = \sum_{p \in \widehat{T}_N \setminus \{0\}} \left(\frac{\widehat{\dot{\mathcal{C}}}_{N,A}(p)}{\widehat{\mathcal{C}}_{N,A}(p)} \right)^2. \tag{2.2.45}$$

In other words this expression shows that the derivative of the expectation is not controlled by the change of the covariance but rather by the relative change of the covariance. Moreover it is more difficult to bound the Hilbert-Schmidt norm for bigger number of degrees of freedoms, i.e., increasing torus size. We observe that since we have no lower bound for all $\varphi \in \widehat{T}_N$ in Theorem 2.2.3 we cannot bound the expression (2.2.45). Moreover this is not an issue of missing bounds. This can be seen easily for the decomposition constructed in [3]. Their decomposition has the property that $\mathcal{C}_{k,tA} = t^{-1}\mathcal{C}_{k,A}$. This implies $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{k,A-tA}\big|_{t=0} = \mathcal{C}_{k,A}$, i.e., each summand in (2.2.46) is 1. Hence the entire sum diverges like $L^{Nd} - 1$. Therefore we have to modify the construction such that we obtain better lower bounds for the kernels while their derivatives continue to have good decay properties. With the decomposition from Theorem 2.2.5 the bounds (2.2.23) and

(2.2.40) the expression (2.2.44) can be bounded uniformly in N for $\tilde{n} - n > d/2$ in the scalar case as follows

$$\begin{aligned}\|\mathcal{C}_{N,A}^{-\frac{1}{2}}\dot{\mathcal{C}}_{N,A}\mathcal{C}_{N,A}^{-\frac{1}{2}}\|_{\mathrm{HS}}^2 = \sum_{p\in\widehat{T}_N\setminus\{0\}} \left(\frac{\widehat{\dot{\mathcal{C}}}_{N,A}(p)}{\widehat{\mathcal{C}}_{N,A}(p)}\right)^2 &\leq C\sum_{j=0}^{N}|\boldsymbol{A}_j|L^{2(N-j)(n-\tilde{n})} \\ &\leq C\sum_{j=0}^{N}L^{(N-j)(2(n-\tilde{n})+d)} \leq C.\end{aligned} \tag{2.2.46}$$

This means that we can bound the derivative of expectation values of $F(\cdot + \xi_{N,A})$ with respect to A uniformly in N for the new decomposition. For $k < N$ one obtains similarly the bound

$$\|\mathcal{C}_{k,A}^{-\frac{1}{2}}\dot{\mathcal{C}}_{k,A}\mathcal{C}_{k,A}^{-\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq CL^{(N-k)d}. \tag{2.2.47}$$

This already indicates that for $k < N$ the derivative is bounded only for certain functionals F. For details and the general case $k < N$ and $m > 1$ cf. Section 2.4.

Note that (2.2.43) can be also bounded using the following observation

$$\left(\langle x, M^{-1}\dot{M}M^{-1}x\rangle - \operatorname{Tr} M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\right)\mu_M(\mathrm{d}x) = \sum_{i,j=1}^{s}(\dot{M}_{i,j}\partial_{x_i}\partial_{x_j}\varphi_M(x))\,\mathrm{d}x. \tag{2.2.48}$$

Integration by parts then implies

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^s}F(x)\,\mu_{M(t)}(\mathrm{d}x)\bigg|_{t=0}\right| = \left|\frac{1}{2}\int_{\mathbb{R}^s}\sum_{i,j=1}^{s}\dot{M}_{i,j}\left(\partial_{x_i}\partial_{x_j}F(x)\right)\mu_M(\mathrm{d}x)\right| \tag{2.2.49}$$

The bounds on M^{-1} are no longer needed. Now, however, we bound an integral over F by an integral over the second derivative of F, i.e., we lose two orders of regularity. This loss of regularity causes substantial difficulties in the renormalisation analysis in [4] which can be avoided by using the decomposition from Theorem 2.2.5.

2.3 Construction of the finite range decomposition

The lower bound for the Fourier transform of the kernel of the finite range decompositions $\mathcal{C}_k$ for $|p| \lesssim L^{-k}$ allows one to construct a new finite range decomposition which satisfies a global bound from below. The key idea is to use suitable linear combinations of the original decomposition, i.e., we use the ansatz $\mathcal{D}_k = \sum_{j=1}^{k}\lambda_{k,j}\mathcal{C}_j$. By construction of the $\mathcal{C}_j$ the range of $\mathcal{D}_k$ is not larger than $L^k/2$. The discrete derivatives of $\mathcal{D}_{k,A}$ shall be bounded as in (2.2.31) for all $|\alpha| \leq n$ for some integer $n > 0$. Thus we need for $|\alpha| \leq n$ the estimate $\lambda_{k,j}|\nabla^{\alpha}\mathcal{C}_j(x)| \leq \lambda_{k,j}L^{-(j-1)(d-2+|\alpha|)} \leq L^{-(k-1)(d-2+|\alpha|)}$ which is satisfied if $\lambda_{k,j} \leq L^{-(k-j)(d-2+n)}$. These bounds on $\lambda_{k,j}$ are the largest possible (later we will add 1 in the exponent so that the sum over j is still uniformly bounded). Then for $p \approx L^{-j}$ with $j \leq k$ we find, using $\widehat{\mathcal{C}}_j(p) \gtrsim L^{2j} \approx |p|^{-2}$, a lower bound

$$\widehat{\mathcal{D}}_k(p) \geq \lambda_{k,j}\widehat{\mathcal{C}}_j(p) \gtrsim L^{-(k-j)(d-2+n)}|p|^{-2} \approx |p|^{-2}(L^k p)^{-(d-2+n)}. \tag{2.3.1}$$

This decays much slower in Fourier space than the decomposition from Theorem 2.2.3 and therefore it is helpful to bound the expression in (2.2.44). The construction above yields a decomposition with good lower and upper global bounds on the Fourier modes of the finite range decomposition. The following proposition states the precise result.

Proposition 2.3.1. *Let $n > 0$ be an integer. Then the family $\mathcal{C}_A$ of operators with $A \in \mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$ has a finite range decomposition into operators $\mathcal{D}_{A,k}$ such that $\mathcal{C}_A = \sum_{k=0}^{N+1} \mathcal{D}_{A,k}$ and*

$$\mathcal{D}_{A,k}(x) = M_k \ \text{if} \ |x|_\infty \geq \frac{L^k}{2} \tag{2.3.2}$$

where $M_k \leq 0$ is a constant matrix. Furthermore for any multi-index α with $|\alpha| \leq n$ and the constants $C(\alpha, \ell)$ from Theorem 2.2.3 the discrete derivative in x and the directional derivative in A satisfy

$$\sup_{\|\dot{A}\| \leq 1} |\nabla^\alpha D_A^\ell \mathcal{D}_{A,k}(x)(\dot{A}, \cdots, \dot{A})| \leq 2C(\alpha, d) L^{-(k-1)(d-2+|\alpha|)}. \tag{2.3.3}$$

Moreover we also have a lower bound on $\mathcal{D}_{A,k}$ in Fourier space

$$\widehat{\mathcal{D}}_{A,k}(p) \geq \begin{cases} cL^{2j} L^{(k-j)(-d+1-n)} & \text{for } p \in \boldsymbol{A}_j \text{ and } j < k \\ cL^{2k} & \text{for } p \in \boldsymbol{A}_j \text{ and } j \geq k. \end{cases} \tag{2.3.4}$$

The upper bound in Fourier space reads

$$\|D_A^\ell \widehat{\mathcal{D}}_{A,k}(p)\| \leq \begin{cases} CL^{2(n+d)+1} L^{2j} L^{(k-j)(-d+1-n)} & \text{for } p \in \boldsymbol{A}_j \text{ and } j < k \\ CL^{2k} & \text{for } p \in \boldsymbol{A}_j \text{ and } j \geq k. \end{cases} \tag{2.3.5}$$

In particular the quotient between the lower bound (2.3.4) and the upper bound (2.3.5) is bounded by a constant for all $p \in \widehat{T}_N$ and $A, A' \in \mathcal{L}(\mathcal{G}, \omega_0, \Omega_0)$, i.e.,

$$KL^{2(d+n)+1} \widehat{\mathcal{D}}_{A,k}(p) \geq \|D_{A'}^\ell \widehat{\mathcal{D}}_{A',k}(p)\| \tag{2.3.6}$$

for some constants $K = K(n, \ell)$.

Remark 2.3.2. *The finite range decomposition $\mathcal{D}_{A,k}$ has the property*

$$\mathcal{D}_{A,k+1} \geq L^{-d+1-n} \mathcal{D}_{A,k}$$

which can be seen easily from the construction below. One easily sees that L^{-d+1-n} can be replaced by ηL^{-d+2-n} for any $\eta < 1$. This bound seems to be optimal under the condition that the discrete derivatives up to order n are bounded as in (2.3.3) because the bound for $\|\nabla^\alpha \mathcal{D}_{k,A}\|_\infty$ strengthens by a factor of L^{d-2+n} in each step if $|\alpha| = n$. Lower bounds of this type might be useful for a new approach to the definition of the norms for the renormalisation group approach.

Remark 2.3.3. *The construction is slightly more flexible when we start from the continuous decomposition in [18] (cf. Appendix A). Then we define*

$$\mathcal{D}_{A,k} = \int_{\mathbb{R}_+} \Phi_k(t) \cdot t W_t(\mathcal{A}) \, \mathrm{d}t \tag{2.3.7}$$

where $\Phi_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a family of positive functions that are a decomposition of unity with $\mathrm{supp}(\Phi_k) \subset (0, L^k/(2R))$ and it behaves as $\Phi_k(t) \approx (tL^{-k})^{d-1+n}$ for $t < L^k/(2R)$. Since this is more technical we used to the simpler construction below based on the discrete decomposition in Theorem 2.2.3.

Remark 2.3.4. *Note that the only relevant difference between the result of this Proposition and Theorem 2.2.5 is the decay of $D_A^\ell \widehat{\mathcal{D}}_{A,k}(p)$ for $\ell > 0$ which is here given by $L^{(k-j)(-d+1-n)}$. In Theorem 2.2.5 the decay is improved to $L^{(k-j)(-d+1-\tilde{n})}$ where $\tilde{n} > n$ is any integer. However, this small change helps a lot with the sum in equation* (2.2.46) *which cannot be bounded by using Proposition 2.3.1.*

Proof. We sometimes omit A from the notation. Let $\mathcal{C}_k$ be a finite range decomposition as in Theorem 2.2.3. Define for $1 \leq k \leq N$

$$\mathcal{D}_k = \sum_{j=1}^{k} \lambda_{k,j} \mathcal{C}_j \tag{2.3.8}$$

where the coefficients $\lambda_{k,j} = \lambda_{k-j}$ are given by

$$\begin{aligned} \lambda_{k,j} &= \lambda_{k-j} = L^{(k-j)(-d+2-n-1)} \quad \text{for } j < k, \\ \lambda_{k,k} &= 1 - \sum_{j=k+1}^{\infty} \lambda_{j,k} = 1 - \sum_{i=1}^{\infty} \lambda_i. \end{aligned} \tag{2.3.9}$$

Since $\lambda_{j,k}$ is a geometric series for $j > k$ we find that $1 \geq \lambda_{k,k} > 1/2$. We define the last term of the decomposition by

$$\mathcal{D}_{N+1} = \sum_{j=1}^{N+1} \left(\sum_{k=N+1}^{\infty} \lambda_{k,j} \right) \mathcal{C}_j \tag{2.3.10}$$

This definition implies that $\sum_{k=1}^{N+1} \mathcal{D}_k = \mathcal{C}$. The operators $\mathcal{D}_k$ clearly have the correct range. Since λ_i is a geometric series we can estimate $\lambda_{N+1,j} \leq \sum_{k=N+1}^{\infty} \lambda_{k,j} \leq 2\lambda_{N+1,j}$. Therefore we do not explicitly mention the case $k = N+1$ in the following. The discrete derivatives can be estimated easily for $|\alpha| \leq n$ using (2.2.31)

$$\begin{aligned} |\nabla^\alpha D_A^\ell \mathcal{D}_{A,k}(x)(\dot{A}, \ldots, \dot{A})| &\leq \sum_{j=1}^{k} \lambda_{k,j} |\nabla^\alpha D_A^\ell \mathcal{C}_{A,j}(x)(\dot{A}, \ldots, \dot{A})| \\ &\leq \sum_{j=1}^{k} C(\alpha, \ell) L^{(k-j)(-d+2-n-1)-(j-1)(d-2+|\alpha|)} \\ &\leq C(\alpha, \ell) L^{-(k-1)(d-2+|\alpha|)} \sum_{j=1}^{k} L^{(j-k)(n-|\alpha|+1)} \\ &\leq 2C(\alpha, \ell) L^{-(k-1)(d-2+|\alpha|)}. \end{aligned} \tag{2.3.11}$$

In the last step we used $n - |\alpha| \geq 0$ and we estimated the geometric series by 2. It remains to prove the bounds in Fourier space. For $j \geq k$ and $p \in \boldsymbol{A}_j$ we use (2.2.30) and $\lambda_{k,k} \geq \frac{1}{2}$

$$\widehat{\mathcal{D}}_k(p) \geq \frac{1}{2} \widehat{\mathcal{C}}_k(p) \geq \frac{c}{2} L^{2k}. \tag{2.3.12}$$

For $1 \leq j < k$ we employ again (2.2.30) on the $\mathcal{C}_j$ summand of $\mathcal{D}_k$

$$\widehat{\mathcal{D}}_k(p) \geq \lambda_{k,j} \widehat{\mathcal{C}}_j(p) \geq c L^{(k-j)(-d+2-n-1)} L^{2j} \geq c L^{(k-j)(-d+1-n)} L^{2j}. \tag{2.3.13}$$

Finally, for $j = 0$ equation (2.2.30) applied to the $\mathcal{C}_1$ term yields

$$\widehat{\mathcal{D}}_k(p) \geq \lambda_{k,1}\hat{\mathcal{C}}_1(p) \geq L^{(k-1)(-d+1-n)}c|p|^{-2} \geq cL^{k(-d+1-n)}L^{d+n-1}L^{-2} \geq cL^{(k-j)(-d+1-n)}L^{2j}. \tag{2.3.14}$$

The proof of the upper bound (2.3.5) is straightforward but technical. For $j \geq k$ and $p \in \boldsymbol{A}_j$ the bound is immediate from the second estimate of (2.2.29) because

$$\|\widehat{\mathcal{D}}_k(p)\| \leq \sum_{k'=1}^{k} \lambda_{k,k'}\|\widehat{\mathcal{C}}_{k'}(p)\| \leq C\sum_{k'=1}^{k} L^{2k'}\lambda_{k,k'} \leq 2CL^{2k}. \tag{2.3.15}$$

On the other hand for $j < k$ and $p \in \boldsymbol{A}_j$ we find with (2.2.29) for any $\bar{n} > 0$

$$\begin{aligned} \|\widehat{\mathcal{D}}_k(p)\| &\leq \sum_{k'=1}^{k} \lambda_{k,k'}\|\widehat{\mathcal{C}}_{k'}(p)\| \leq \\ &\leq \sum_{k'=1}^{j} L^{(k-k')(-d+1-n)}CL^{2k'} + \sum_{k'=j+1}^{k} L^{(k-k')(-d+1-n)}C_{\bar{n}}|p|^{-2}(|p|L^{k'-1})^{-\bar{n}}. \end{aligned} \tag{2.3.16}$$

The first summand is a geometric sum bounded by twice the largest term

$$\sum_{k'=1}^{j} L^{(k-k')(-d+1-n)}CL^{2k'} \leq 2CL^{(k-j)(-d+1-n)}L^{2j}. \tag{2.3.17}$$

The second summand in (2.3.16) is also a geometric series and for $\bar{n} = d + n$ it can be bounded similarly

$$\begin{aligned} \sum_{k'=j+1}^{k} L^{(k-k')(-d+1-n)}C|p|^{-2}(|p|L^{k'-1})^{-\bar{n}} &\leq CL^{2(j+1)}\sum_{k'=j+1}^{k} L^{(k-k')(-d+1-n)}(L^{k'-j-2})^{-(d+n)} \\ &\leq CL^{2(d+n)+1}L^{2j}L^{(k-j)(-d+1-n)}. \end{aligned} \tag{2.3.18}$$

Now the estimates (2.3.17) and (2.3.18) plugged in (2.3.16) imply (2.3.5) for $\ell = 0$. This ends the proof for $\ell = 0$. For $\ell > 0$ only the constants in the upper bound change. □

So far we have constructed a finite range decomposition with n controlled discrete derivatives and matching lower and upper bounds on the Fourier coefficients. Finally we want to prove Theorem 2.2.5 where we stated the existence of a finite range decomposition for $\mathcal{C}_A$ where the derivatives with respect to A of the kernel decay better in Fourier space than the kernel itself. The key idea is to start with a decomposition as in Proposition 2.3.1 with many controlled derivatives, then subtract something constant such that the decomposition remains positive. Finally we add the subtracted part in a way that we get strong lower bounds. Then the derivatives with respect to A only hit the fast decaying first term. The main problem is to ensure that the operators remain positive. In the scalar case the fact that all operators are simultaneously diagonalised by the Fourier transform simplifies the analysis slightly and this would allow one to obtain slightly stronger results than the one stated in Theorem 2.2.5.

For the construction we have to fix an operator where for simplicity we choose $-\Delta$, where $-\Delta = \nabla^*\nabla$ denotes the lattice Laplacian. The linear map $A : \mathcal{G} \to \mathcal{G}$ corresponding to the discrete Laplacian acting on vectors in $\mathbb{R}^m$ has the matrix elements $A_{\alpha\alpha} = \mathrm{id}_{m\times m}$ for $|\alpha| = 1$ and 0 otherwise. With a slight abuse of notation we denote finite range decompositions of the Laplacian by $\mathcal{D}_{-\Delta,k}$.

Proof of Theorem 2.2.5. Let $\mathcal{D}^{\tilde n}_{A,k}$ and $\mathcal{D}^{\tilde n}_{-\Delta,k}$ be finite range decompositions as constructed in Proposition 2.3.1 with $\tilde n$ controlled derivatives. Let $\mathcal{D}^{n}_{-\Delta,k}$ be a finite range decomposition of $\mathcal{C}_{-\Delta}$ as constructed in Proposition 2.3.1 with derivatives up to order n bounded. Then we define

$$\mathcal{C}_{A,k} = \mathcal{D}^{\tilde n}_{A,k} - \frac{L^{-2(d+\tilde n)-1}}{K(\tilde n, \ell = 0)} \mathcal{D}^{\tilde n}_{-\Delta,k} + \frac{L^{-2(d+\tilde n)-1}}{K(\tilde n, \ell = 0)} \mathcal{D}^{n}_{-\Delta,k} \tag{2.3.19}$$

where $K \geq 1$ is the constant from the inequality (2.3.6). Clearly we have

$$\mathcal{C}_A = \sum_{k=1}^{N+1} \mathcal{C}_{A,k}, \tag{2.3.20}$$

this decomposition has the correct finite range and it is translation invariant. Bounds on the discrete derivatives already hold for each term separately by (2.3.3). The same is true for the upper bounds in Fourier space. Moreover the stronger bound for the derivatives with respect to A follows from the fact that only the first term depends on A and the bounds given in equation (2.3.5) in Proposition 2.3.1. It remains to prove the lower bounds in Fourier space which also imply positivity of the decomposition. The third term satisfies the lower bound so it remains to proof that the first two terms together are positive. But this is a consequence of (2.3.6) which in particular gives

$$\widehat{\mathcal{D}}^{\tilde n}_{A,k}(p) \geq \frac{L^{-2(d+\tilde n)+1}}{K(\tilde n, \ell = 0)} \|\widehat{\mathcal{D}}^{\tilde n}_{-\Delta,k}(p)\|. \tag{2.3.21}$$

□

2.4 Smoothness of the renormalisation map

The goal of this section is to prove the following proposition stating the smoothness of the renormalisation map based on the finite range decomposition from Theorem 2.2.5. This is similar to the motivation given in Section 2.3. We first discuss the simplest smoothness statement as an illustration without technical problems. Then, in Proposition 2.4.5, we prove the more general smoothness statement that allows to avoid the loss of regularity in [4]. Both propositions rely on an interesting localisation property of finite range decompositions discussed in Lemma 2.4.3.

Proposition 2.4.1. *Let $B \subset T_N$ be a cube of side length L^k. Let $F : \mathcal{X}_N \to \mathbb{R}$ be a bounded functional that is measurable with respect to the σ-algebra generated by $\{\varphi(x) | x \in B\}$, i.e., F depends only on the restriction $\varphi\restriction_B$. Then the following bound holds*

$$\left| \partial_A \int_{\mathcal{X}_N} F(\varphi + \psi)\, \mu_{\mathcal{C}_{A,k+1}}(\mathrm{d}\varphi) \right| \leq C_L \|F\|_{L^2(\mathcal{X}_N, \mu_{\mathcal{C}_{A,k+1}})} \tag{2.4.1}$$

where $\mathcal{C}_{A,k+1}$ is a finite range decomposition as in Theorem 2.2.5 with $\tilde n - n > d/2$ and C_L is a constant that does not depend on N or k but in contrast to the previous sections it does depend on L.

Remark 2.4.2. *The condition that F depends on the values of the field in a cube of side length L^k appears naturally in the renormalisation group analysis, cf., e.g., [42].*

As stated in Proposition 2.4.1 we show a bound for derivatives of the renormalisation map which does not depend on N. In principle this could be difficult since the dimension of the space $\mathcal{X}_N$ over which we integrate increases with N. We have seen in equation (2.2.46) in Section 2.2 that the number of terms appearing in the bounds for the derivatives with respect to the covariance is proportional to the dimension of the space $\mathcal{X}_N$ and therefore that the naive estimate is only sufficient to bound the derivative for $k = N$. For $k < N$ (2.2.47) suggests that there is no uniform in N bound for general functionals F. This means we have to exploit the special structure of the functionals.

The following heuristics suggests that the specific structure of the functionals is indeed sufficient to bound the derivative uniformly in N. We integrate in the k-th integration step with respect to a measure of range L^k which satisfies bounds uniformly in N and the functionals $F_k(\varphi)$ that appear are local in the sense that they only depend on the values of φ on B. Hence in some sense the size of the torus should not be seen by this integration. This idea will be made rigorous in the sense that locally the distribution on a block B of a field ξ with distribution $\mu_{\mathcal{C}_{A,k}}$ is the same as the distribution of a field $\tilde{\xi}$ defined on a torus of size comparable to B.

A bit more quantitatively we motivate the $\mathcal{O}(1)$ bound as follows. From (2.2.47) we find the bound

$$\left|\partial_A \int_{\mathcal{X}_N} F(\varphi+\psi)\,\mu_{\mathcal{C}_{A,k+1}}(\mathrm{d}\varphi)\right| \le C\sqrt{L^{(N-k)d}}\|F\|_{L^2(\mathcal{X}_N,\mu_{A,k+1})} \tag{2.4.2}$$

for general functionals F. If F depends only on the values of φ in a block of side-length L^k it depends only on the fraction $L^{(k-N)d}$ of the degrees of freedom. We expect that each degree of freedom contributes equally which suggests the bound stated in (2.4.1).

Let us sketch a proof of Proposition 2.4.1 that makes the previous consideration precise. Below, we also give a longer proof based on Lemma 2.4.3 because the second proof can be used to establish Theorem 2.4.5. Consider the set T of translation operators given by

$$T = \{\tau_a : a = (a_1, \ldots, a_n) \in (3L^{k+1}\mathbb{Z})^d,\, 0 \le a_i \le L^N - 3L^{k+1}\}. \tag{2.4.3}$$

Then there is a constant such that $|T| \ge cL^{(N-k-1)d}$. Recall that F only depends on $\varphi\restriction_B$ hence the random variable $F(\tau_a\varphi)$ only depends on $\varphi\restriction_{B+a}$. The definition of T implies that for $\tau_{a_1}, \tau_{a_2} \in T$ with $\tau_{a_1} \ne \tau_{a_2}$ the sets $B + a_1$ and $B + a_2$ have distance at least L^{k+1} which implies that the random variables $F(\tau_{a_1}\varphi)$ and $F(\tau_{a_2}\varphi)$ are independent. Then we can estimate using translation invariance of $\mu_{A,k+1}$, independence of $\tau_a F$, and (2.4.2)

$$\begin{aligned}
\left|\partial_A \int_{\mathcal{X}_N} F(\varphi)\,\mu_{A,k+1}(\mathrm{d}\varphi)\right| &= \frac{1}{|T|}\left|\partial_A \sum_{\tau\in T}\int_{\mathcal{X}_N} F(\tau\varphi)\,\mu_{A,k+1}(\mathrm{d}\varphi)\right| \\
&\le \frac{C\sqrt{L^{(N-k-1)d}}}{|T|}\left\|\sum_{\tau\in T} F(\tau\varphi)\right\|_{L^2(\mathcal{X}_N,\mu_{A,k+1}} \\
&\le \frac{C\sqrt{L^{(N-k-1)d}}}{|T|}\left(\sum_{\tau\in T}\int_{\mathcal{X}_N} F^2(\tau\varphi)\,\mu_{A,k+1}\right)^{\frac{1}{2}} \\
&\le C\sqrt{\frac{L^{(N-k-1)d}}{|T|}}\|F\|_{L^2(\mathcal{X}_N,\mu_{A,k+1})} \\
&\le C\|F\|_{L^2(\mathcal{X}_N,\mu_{A,k+1})}.
\end{aligned} \tag{2.4.4}$$

We introduce some notation necessary for the next lemma. Let $k \le \overline{N} \le N$ be positive integers. We denote by $\pi = \pi_{N,\overline{N}} : T_N \to T_{\overline{N}}$ the projection and group homomorphism of discrete tori. Recall that $\mathcal{V}_N$ was defined in (2.2.1) and denotes the set of fields on T_N. Let $\sigma : \mathcal{V}_{\overline{N}} \to \mathcal{V}_N$ with $\sigma = \pi^*$ be the pull-back of fields, i.e., for $\varphi \in \mathcal{V}_{\overline{N}}$ we define $(\sigma\varphi)(x) = \varphi(\pi x)$. Clearly $\sigma\varphi$ has average zero if φ has average zero hence σ also maps the subspace $\mathcal{X}_{\overline{N}}$ to $\mathcal{X}_N$. In other words, identifying functions on T_N with periodic functions on $\mathbb{Z}^d$, the function $\sigma\varphi$ is just the $(L^{\overline{N}}\mathbb{Z})^d$ periodic function φ understood as a $(L^N\mathbb{Z})^d$ periodic function. With this notation we can state the following lemma. Here $\mathcal{C} = \sum_{k=0}^{N+1} \mathcal{C}_k$ denotes a finite range decomposition such that $\mathcal{C}_k$ is a non-negative translation invariant, positive operator on $\mathcal{V}_N$ (see Remark 2.4.4 below) with kernels $\mathcal{C}_k$ satisfying $\mathcal{C}_k(x) = -M \le 0$ for $\mathrm{d}_\infty(x,0) \ge L^k/2$ where M is a positive semi-definite symmetric matrix.

Lemma 2.4.3. *Let $X \subset T_N$ and $D = \operatorname{diam}(X) = \sup_{x,y\in X} d_\infty(x,y)$. Choose $\overline{N} \in \mathbb{N}$ such that $L^{\overline{N}} > 2D \ge L^{\overline{N}-1}$ and assume $k \le \overline{N} \le N$. Define a Gaussian measure $\nu_{k,\overline{N}}$ on $\mathcal{V}_{\overline{N}}$ by its covariance operator $\mathcal{D}_{k,\overline{N}}$ given by the kernel $\mathcal{D}_{k,\overline{N}} : T_{\overline{N}} \to \mathbb{R}^{m\times m}_{\mathrm{sym}}$*

$$\mathcal{D}_{k,\overline{N}}(x) = (L^{(N-\overline{N})d} - 1)M + \frac{1}{L^{\overline{N}d}} \sum_{p\in\widehat{T}_{\overline{N}}} e^{ipx}\widehat{\mathcal{C}}_k(p) \tag{2.4.5}$$

where $\widehat{\mathcal{C}}_k(p)$ are the Fourier coefficients of the kernel $\mathcal{C}_k$ of the covariance operator $\mathcal{C}_k$. They are well defined because $\widehat{T}_{\overline{N}} \subset \widehat{T}_N$. Let $F_X : \mathcal{V}_N \to \mathbb{R}$ be a measurable functional that only depends on $\{\varphi(x)|\, x \in X\}$, i.e., F is measurable with respect to the σ-algebra generated by $\{\varphi(x)|\, x \in X\}$. Then the following identity holds

$$\int_{\mathcal{V}_N} F_X(\xi)\, \mu_k(\mathrm{d}\xi) = \int_{\mathcal{V}_{\overline{N}}} F_X(\sigma\psi)\, \nu_{k,\overline{N}}(\mathrm{d}\psi). \tag{2.4.6}$$

Remark 2.4.4. *Note that the measure μ_k appearing on the left hand side of equation* (2.4.6) *was defined on $\mathcal{X}_N \subset \mathcal{V}_N$. In Remark 2.2.1 we discussed that μ_k agrees with a degenerate Gaussian measure on $\mathcal{V}_N$ which implies that the left hand side is well defined.*

We postpone the proof of this Lemma to the end of this section. Instead we continue with the smoothness estimates.

Proof of Proposition 2.4.1. The proof relies on the bound (2.2.44) which we recall here. We consider a smooth map $M : (-\varepsilon, \varepsilon) \to \mathbb{R}^{s\times s}_{\mathrm{sym},+}$ and we denote $M = M(0)$ and $\dot{M} = \dot{M}(0)$. Then

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^s} F(x)\, \mu_{M(t)}(\mathrm{d}x)\Big|_{t=0} \right| \le \|F\|_{L^2(\mathbb{R}^s,\mu_{M(t)})} \|M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\|_{\mathrm{HS}}. \tag{2.4.7}$$

To see this we start from (2.2.43) and apply Cauchy-Schwarz

$$\begin{aligned} &\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^s} F(x)\, \mu_{M(t)}(\mathrm{d}x)\Big|_{t=0} \right| \\ &\quad \le \frac{1}{2} \left(\int_{\mathbb{R}^s} |F(x)|^2\, \mu_M(\mathrm{d}x) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^s} \left(\langle x, M^{-1}\dot{M}M^{-1}x\rangle - \operatorname{Tr} M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}} \right)^2 \mu_M(\mathrm{d}x) \right)^{\frac{1}{2}}. \end{aligned} \tag{2.4.8}$$

Then the change of variable $y = M^{-\frac{1}{2}}x$ and an orthogonal transformation yields

$$\int_{\mathbb{R}^s} \left(\langle x, M^{-1}\dot{M}M^{-1}x\rangle - \operatorname{Tr} M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}} \right)^2 \mu_M(\mathrm{d}x) = 2\|M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\|^2_{\mathrm{HS}}. \tag{2.4.9}$$

Here the norm on the right hand side is the Hilbert Schmidt norm given by $\|A\|_{\mathrm{HS}} = \sqrt{\operatorname{Tr} AA^T}$. Lemma 2.4.6 below states a much more general version of this estimate.

We apply (2.4.7) to the measures from Lemma 2.4.3. We set $M(t) = \mathcal{D}_{A+t\dot{A},k+1,k+1}$ and denote $\dot{\mathcal{D}}_{A,k+1,k+1} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}_{A+t\dot{A},k+1,k+1}\Big|_{t=0}$ and $\tilde{\mathcal{D}} = \mathcal{D}_{A,k+1,k+1}^{-\frac{1}{2}}\dot{\mathcal{D}}_{A,k+1,k+1}\mathcal{D}_{A,k+1,k+1}^{-\frac{1}{2}}$. Combining this with Lemma 2.4.3 (here we need F to be local) we get the estimate

$$\left| D_A \int\limits_{\mathcal{X}_N} F(\varphi)\,\mu_k^{(A)}(\mathrm{d}\varphi)(\dot{A})\right| = \left| D_A \int\limits_{\mathcal{V}_{k+1}} F(\sigma\psi)\,\nu_{k+1,k+1}^{(A)}(\mathrm{d}\psi)(\dot{A})\right| \leq \|F\|_{L^2(\mathcal{V}_{k+1},\nu_{k+1,k+1}^{(A)})}\|\tilde{\mathcal{D}}\|_{\mathrm{HS}} \tag{2.4.10}$$

where $\mu_{k+1}^{(A)} = \mu_{\mathcal{C}_{A,k+1}}$ and $\nu_{k+1,k+1}^{(A)} = \nu_{\mathcal{D}_{A,k+1,k+1}}$. Note that reading Lemma 2.4.3 backwards implies $\|F\|_{L^2(\mathcal{V}_{k+1},\nu_{k+1,k+1}^{(A)})} = \|F\|_{L^2(\mathcal{X}_N,\mu_{k+1}^{(A)})}$. The operators $\mathcal{D}_{A,k+1,k+1}$ are diagonal in Fourier space and satisfy by definition (see (2.4.5)) the equality $\widehat{\mathcal{D}}_{A,k+1,k+1}(p) = \widehat{\mathcal{C}}_{A,k+1}(p)$ for $p \in \widehat{T}_{k+1}\setminus\{0\}$ hence the Hilbert-Schmidt norm is given by

$$\|\tilde{\mathcal{D}}\|_{\mathrm{HS}} = \sum_{p\in\widehat{T}_{k+1}\setminus\{0\}} \|\widehat{\mathcal{C}}_{A,k+1}(p)^{-\frac{1}{2}}\widehat{\dot{\mathcal{C}}}_{A,k+1}(p)\widehat{\mathcal{C}}_{A,k+1}(p)^{-\frac{1}{2}}\|_{\mathrm{HS}}^2. \tag{2.4.11}$$

Note that the Fourier mode for $p = 0$ does not contribute because it does not depend on A. Indeed, we have $\mathcal{C}_{A,k}(x) = -M$ for $|x| > L^{k+1}/2$ independent of A and $\widehat{\mathcal{C}}_{A,k}(0) = 0$ for all A hence (2.4.5) implies that $\widehat{\mathcal{D}}_{A,k+1,k+1}(0)$ is independent of A. We bound the expression in (2.4.11) using (2.2.40) and (2.2.23) (denoting $\boldsymbol{A}_j = \boldsymbol{A}_j^{k+1} \subset \widehat{T}_{k+1}$)

$$\begin{aligned}
\sum_{p\in\widehat{T}_{k+1}\setminus\{0\}} &\|\widehat{\mathcal{C}}_{A,k+1}(p)^{-\frac{1}{2}}\widehat{\dot{\mathcal{C}}}_{A,k+1}(p)\widehat{\mathcal{C}}_{A,k+1}(p)^{-\frac{1}{2}}\|_{\mathrm{HS}}^2 \\
&\leq m \sum_{p\in\widehat{T}_{k+1}\setminus\{0\}} \Big(\|\widehat{\mathcal{C}}_{A,k+1}(p)^{-1}\|\|\widehat{\dot{\mathcal{C}}}_{A,k+1}(p)\|\Big)^2 \\
&\leq m \sum_{j=0}^{k+1}\sum_{p\in\boldsymbol{A}_j} \Xi^2 L^{8(\tilde{n}+d)+4} L^{2(k+1-j)(n-\tilde{n})} \\
&\leq m\Xi^2 L^{8(\tilde{n}+d)+4}\sum_{j=0}^{k+1} |\boldsymbol{A}_j| L^{(k+1-j)(2n-2\tilde{n})} \\
&\leq m\Xi^2 L^{8(\tilde{n}+d)+4}\sum_{j=0}^{k+1} \kappa(d) L^{(k+1-j)d} L^{(k+1-j)(2n-2\tilde{n})} \\
&\leq C
\end{aligned} \tag{2.4.12}$$

where we used $2\tilde{n} - 2n > d$ in the last step. □

We need a more general version of Proposition 2.4.1 to avoid the loss of regularity in [4]. Namely we must generalise the result in Proposition 2.4.1 to higher order derivatives and we have to replace the L^2 norm of F on the right hand side of (2.4.1) by a L^p norm for any $p > 1$. To understand the motivation for this lemma we refer to the description of the renormalisation approach in the aforementioned work [4].

Theorem 2.4.5. *Let $\mathcal{C}_{A,k+1}$ a finite range decomposition as in Theorem 2.2.5 with $\tilde{n} - n > d/2$ and $X \subset T_N$ be a subset with diameter $D = \operatorname{diam}_\infty(X) \geq L^k$. Let $F : \mathcal{V}_N \to \mathbb{R}$ be a functional that is measurable with respect to the σ-algebra generated by $\{\varphi(x) | x \in X\}$, i.e., F depends only on the values of the field φ in X. Then for $\ell \geq 1$ and $p > 1$ the following bound holds*

$$\left| \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} \int_{\mathcal{X}_N} F(\varphi) \, \mu_{A+tA_1,k+1}(\mathrm{d}\varphi)|_{t=0} \right| \leq C_{p,\ell,L} (DL^{-k})^{\frac{d\ell}{2}} \|A_1\|^\ell \|F\|_{L^p(\mathcal{X}_N, \mu_{A,k+1})}. \tag{2.4.13}$$

Proof. We use the notation from Section 2.2 and the proof of Proposition 2.4.1. We first give explicit calculations for $\ell = 1$ and $\ell = 2$ and indicate the general case in the end. Recall in particular the definition of $\varphi_{M(t)}(x)$ in (2.2.42) and note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \varphi_{M(t)}(x) \Big|_{t=0} &= \frac{1}{2} \left(\langle x, M^{-1} \dot{M} M^{-1} x \rangle - \operatorname{Tr} M^{-1} \dot{M} \right) \varphi_M(x) \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} \varphi_{M(t)}(x) \Big|_{t=0} &= \Bigg[\frac{1}{4} \left(\langle x, M^{-1} \dot{M} M^{-1} x \rangle - \operatorname{Tr} M^{-1} \dot{M} \right)^2 - \langle x, M^{-1} \dot{M} M^{-1} \dot{M} M^{-1} x \rangle + \\ &\quad + \frac{1}{2} \left(\langle x, M^{-1} \ddot{M} M^{-1} x \rangle - \operatorname{Tr} \left(M^{-1} \ddot{M} \right) \right) + \frac{1}{2} \operatorname{Tr} \left(M^{-1} \dot{M} M^{-1} \dot{M} \right) \Bigg] \varphi_M(x). \end{aligned} \tag{2.4.14}$$

We need the following general lemma from [145].

Lemma 2.4.6. *Let X be a vector of n independent standard normal variables and $A \in \mathbb{R}^{n \times n}$ a matrix. Then for any real $s_0 \geq 2$ there is a constant $C(s_0)$ such that for $1 \leq s \leq s_0$ the estimate*

$$\mathbb{E} |\langle x, Ax \rangle - \operatorname{Tr} A|^s \leq C(s_0) \|A\|_{HS}^s \tag{2.4.15}$$

holds.

Proof. This is a special case of Theorem 2 in [145]. The extension from $s \geq 2$ to $s \geq 1$ is a direct consequence of Hölder's inequality. □

Using this lemma and the Hölder inequality with exponents p and p' we bound

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^s} F(x) \, \mu_{M(t)}(\mathrm{d}x) \Big|_{t=0} \right| &\leq \frac{1}{2} \|F\|_{L^p(\mu_M)} \| \langle x, M^{-\frac{1}{2}} \dot{M} M^{-\frac{1}{2}} x \rangle - \operatorname{Tr} M^{-\frac{1}{2}} \dot{M} M^{-\frac{1}{2}} \|_{L^{p'}(\mu_{\mathrm{id}})} \\ &\leq C_p \|F\|_{L^p(\mu_M)} \| M^{-\frac{1}{2}} \dot{M} M^{-\frac{1}{2}} \|_{\mathrm{HS}} \end{aligned} \tag{2.4.16}$$

For the second derivative we find similarly the bound

$$\begin{aligned} &\left| \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbb{R}^s} F(x) \, \mu_{M(t)}(\mathrm{d}x) \Big|_{t=0} \right| \leq \\ &\|F\|_{L^p(\mu_M)} \Bigg(\frac{1}{2} | \operatorname{Tr} M^{-\frac{1}{2}} \dot{M} M^{-1} \dot{M} M^{-\frac{1}{2}} | + \\ &\quad + \frac{1}{4} \| (\langle x, M^{-\frac{1}{2}} \dot{M} M^{-\frac{1}{2}} x \rangle - \operatorname{Tr} M^{-\frac{1}{2}} \dot{M} M^{-\frac{1}{2}})^2 \|_{L^{p'}(\mu_{\mathrm{id}})} + \\ &\quad + \frac{1}{2} \| \langle x, M^{-\frac{1}{2}} \ddot{M} M^{-\frac{1}{2}} x \rangle - \operatorname{Tr} M^{-\frac{1}{2}} \ddot{M} M^{-\frac{1}{2}} \|_{L^{p'}(\mu_{\mathrm{id}})} + \\ &\quad + \| (\langle x, M^{-\frac{1}{2}} \dot{M} M^{-1} \dot{M} M^{-\frac{1}{2}} x \rangle - \operatorname{Tr} M^{-\frac{1}{2}} \dot{M} M^{-1} \dot{M} M^{-\frac{1}{2}} \|_{L^{p'}(\mu_{\mathrm{id}})} \Bigg) \end{aligned} \tag{2.4.17}$$

where p' denotes the Hölder conjugate of p. The first trace term in (2.4.17) can be bounded by Hölder inequality for Schatten norms (cf. Theorem 2.8 in [139]) which yields

$$|\operatorname{Tr} M^{-\frac{1}{2}}\dot{M}M^{-1}\dot{M}M^{-\frac{1}{2}}| \leq \|M^{-\frac{1}{2}}\dot{M}M^{-1}\dot{M}M^{-\frac{1}{2}}\|_{\mathrm{Tr}} \leq \|M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\|_{\mathrm{HS}}^2. \tag{2.4.18}$$

The trace norm is defined by $\|A\|_{\mathrm{Tr}} = \operatorname{Tr}\sqrt{AA^*}$. With the bound (2.4.18), Lemma 2.4.6, and the estimate $\|AB\|_{\mathrm{HS}} \leq \|AB\|_{\mathrm{Tr}} \leq \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}$ we conclude from (2.4.17)

$$\begin{aligned} &\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{R}^s} F(x)\,\mu_{M(t)}(\mathrm{d}x)\big|_{t=0}\right| \leq \\ &\quad\leq C_p\|F\|_{L^p(\mu_M)}\left(\|M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\|_{\mathrm{HS}} + \|M^{-\frac{1}{2}}\ddot{M}M^{-\frac{1}{2}}\|_{\mathrm{HS}} + \|M^{-\frac{1}{2}}\dot{M}M^{-\frac{1}{2}}\|_{\mathrm{HS}}^2\right). \end{aligned} \tag{2.4.19}$$

For $\ell > 2$ the estimates are similar but more involved. We introduce some additional notation. Let us state the general structure of $\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}\varphi_{M(t)}(x)\Big|_{t=t_0}$. We write $\frac{\mathrm{d}^{\delta_i}}{\mathrm{d}t^{\delta_i}}M(t)\Big|_{t=t_0} = M^{\delta_i}(t_0)$ for any positive integer δ_i. In the following we consider multiindices $\boldsymbol{\delta} = (\delta_1,\ldots,\delta_r)$ such that $r \geq 1$ and $\delta_i > 0$ are integers. The length of an index is denoted by $\mathcal{L}(\boldsymbol{\delta}) = r$, we call $|\boldsymbol{\delta}|_1 = \sum_{i=1}^r \delta_i$ the degree of an index. With I_ℓ we denote the set of such multiindices $\boldsymbol{\delta}$ such that $|\boldsymbol{\delta}|_1 \leq \ell$ and $r \geq 2$. Moreover $\overline{I}_\ell$ is defined similarly with $r \geq 2$ replaced by $r \geq 1$. This distinction accounts for a cancellation of the $r = 1$ terms. We define for any $\boldsymbol{\delta} = (\delta_1,\ldots,\delta_r)$ and $r \geq 1$

$$\boldsymbol{M}_{\boldsymbol{\delta}}(t) = \prod_{i=1}^r M^{-\frac{1}{2}}(t)M^{\delta_i}(t)M^{-\frac{1}{2}}(t). \tag{2.4.20}$$

In the following three types of terms appear:

$$\begin{aligned} Q_{\boldsymbol{\delta}}(x,t) = \langle M(t)^{-\frac{1}{2}}x, \boldsymbol{M}_{\boldsymbol{\delta}}(t)M(t)^{-\frac{1}{2}}x\rangle, \quad R_{\boldsymbol{\delta}}(t) = \operatorname{Tr}\boldsymbol{M}_{\boldsymbol{\delta}}(t) \quad \text{with } \boldsymbol{\delta}\in I_\ell \text{ and} \\ S_d(x,t) = \langle x, M(t)^{-1}M^d(t)M(t)^{-1}x\rangle - \operatorname{Tr} M(t)^{-1}M^d(t) \quad \text{with } 1 \leq d \leq \ell. \end{aligned} \tag{2.4.21}$$

The general expression can be written as

$$\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}\varphi_{M(t)}(x)\Big|_{t=t_0} = P_\ell(x,t_0)\varphi_{M(t_0)}(x) \tag{2.4.22}$$

where $P_\ell(x,t)$ is a linear combination of terms

$$\prod_{i=1}^{k_1} Q_{\boldsymbol{\delta}_i}(x,t)\prod_{i=1}^{k_2} R_{\boldsymbol{\epsilon}_i}(t)\prod_{i=1}^{k_3} S_{d_i}(x,t). \tag{2.4.23}$$

Here $\boldsymbol{\delta}_i, \boldsymbol{\epsilon}_i \in I_\ell$ and $1 \leq d_i \leq \ell$ such that $\sum_{i=1}^{k_1}|\boldsymbol{\delta}_i|_1 + \sum_{i=1}^{k_2}|\boldsymbol{\epsilon}_i|_1 + \sum_{i=1}^{k_3} d_i = \ell$, i.e. the total order of derivatives is ℓ. From now on we drop the time argument and assume $t = 0$. The explicit calculations for $\ell = 1$ and $\ell = 2$ showed that we need to bound the p'-norm of P_ℓ. Using Hölder's inequality with exponents $\frac{\ell}{|\boldsymbol{\delta}_i|_1}$, $\frac{\ell}{|\boldsymbol{\epsilon}_i|_1}$, and $\frac{\ell}{d_i}$ we estimate

$$\begin{aligned} \left\|\prod_{i=1}^{k_1} Q_{\boldsymbol{\delta}_i}(x)\prod_{i=1}^{k_2} R_{\boldsymbol{\epsilon}_i}\prod_{i=1}^{k_3} S_{d_i}(x)\right\|_{L^{p'}(\mu_M)} &\leq \prod_{i=1}^{k_1}\|Q_{\boldsymbol{\delta}_i}(x)\|_{\frac{p'\ell}{|\boldsymbol{\delta}_i|_1}}\prod_{i=1}^{k_2}\|R_{\boldsymbol{\epsilon}_i}\|_{\frac{p'\ell}{|\boldsymbol{\epsilon}_i|_1}}\prod_{i=1}^{k_3}\|S_{d_i}(x)\|_{\frac{p'\ell}{d_i}} \\ &\leq \max_{\boldsymbol{\delta}\in I_\ell}\|Q_{\boldsymbol{\delta}}(x)\|_{\frac{p'\ell}{|\boldsymbol{\delta}|_1}}^{\frac{\ell}{|\boldsymbol{\delta}|_1}} \vee \max_{\boldsymbol{\delta}\in I_\ell}|R_{\boldsymbol{\delta}}|^{\frac{\ell}{|\boldsymbol{\delta}|_1}} \vee \max_{1\leq d\leq\ell}\|S_d(x)\|_{\frac{p'\ell}{d}}^{\frac{\ell}{d_i}} \\ &\leq \max_{\boldsymbol{\delta}\in I_\ell}\|Q_{\boldsymbol{\delta}}(x)\|_{p'\ell}^{\frac{\ell}{2}} \vee \max_{\boldsymbol{\delta}\in I_\ell}|R_{\boldsymbol{\delta}}|^{\frac{\ell}{2}} \vee \max_{1\leq d\leq\ell}\|S_d(x)\|_{p'\ell}^{\ell} \vee 1 \end{aligned} \tag{2.4.24}$$

In the second step we used that $\prod x_i^{\alpha_i} \leq \max x_i$ if $\sum \alpha_i = 1$. Then, in the third step, we exploited that the L_p norms are monotone on probability spaces and $|\boldsymbol{\delta}|_1 \geq \mathcal{L}(\boldsymbol{\delta}) \geq 2$ for $\boldsymbol{\delta} \in I_\ell$.

We estimate the three terms with the help of Lemma 2.4.6. For $1 \leq d \leq \ell$

$$\begin{aligned}\int_{\mathbb{R}^s} |S_d(x)|^{p'\ell}\, \mu_M(\mathrm{d}x) &= \int_{\mathbb{R}^s} |\langle x, M^{-\frac{1}{2}} M^d M^{-\frac{1}{2}} x\rangle - \operatorname{Tr} M^{-1} M^d|^{p'\ell}\, \mu_{\mathrm{id}}(\mathrm{d}x) \\ &\leq C_{p'\ell} \|M^{-\frac{1}{2}} M^d M^{-\frac{1}{2}}\|_{\mathrm{HS}}^{p'\ell}.\end{aligned} \tag{2.4.25}$$

Similarly we estimate the Q terms

$$\begin{aligned}\int_{\mathbb{R}^s} |Q_{\boldsymbol{\delta}}(x)|^{p'\ell}\, \mu_M(\mathrm{d}x) &= \int_{\mathbb{R}^s} |\langle y, \boldsymbol{M}_{\boldsymbol{\delta}} y\rangle|^{p'\ell}\, \mu_{\mathrm{id}}(\mathrm{d}y) \\ &\leq \int_{\mathbb{R}^s} 2^{p'\ell} \left(|\langle y, \boldsymbol{M}_{\boldsymbol{\delta}} y\rangle - \operatorname{Tr} \boldsymbol{M}_{\boldsymbol{\delta}}|^{p'\ell} + |\operatorname{Tr} \boldsymbol{M}_{\boldsymbol{\delta}}|^{p'\ell} \right) \mu_{\mathrm{id}}(\mathrm{d}y) \\ &\leq 2^{p'\ell} \left(C_{p'\ell} \|\boldsymbol{M}_{\boldsymbol{\delta}}\|_{\mathrm{HS}}^{p'\ell} + |R_{\boldsymbol{\delta}}|^{p'\ell} \right).\end{aligned} \tag{2.4.26}$$

For $\boldsymbol{\delta} \in I_\ell$ we find $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \in \bar{I}_\ell$ such that $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)$. We bound the trace term for $\boldsymbol{\delta}$ using $\boldsymbol{M}_{\boldsymbol{\delta}} = \boldsymbol{M}_{\boldsymbol{\delta}_1} \boldsymbol{M}_{\boldsymbol{\delta}_2}$ and the Hölder inequality for Schatten norms by

$$|R_{\boldsymbol{\delta}}| = |\operatorname{Tr} \boldsymbol{M}_{\boldsymbol{\delta}}| \leq \|\boldsymbol{M}_{\boldsymbol{\delta}_1}\|_{\mathrm{HS}} \|\boldsymbol{M}_{\boldsymbol{\delta}_2}\|_{\mathrm{HS}} \leq \|\boldsymbol{M}_{\boldsymbol{\delta}_1}\|_{\mathrm{HS}}^2 \vee \|\boldsymbol{M}_{\boldsymbol{\delta}_2}\|_{\mathrm{HS}}^2. \tag{2.4.27}$$

The estimates (2.4.25), (2.4.26), and (2.4.27) imply that there is a constant $\bar{C}_{\ell,p}$ such that the following estimate holds

$$\begin{aligned}\left| \frac{\mathrm{d}^\ell}{\mathrm{d}t^l} \int_{\mathbb{R}^s} F(x)\, \mu_{M(t)}(\mathrm{d}x) \Big|_{t=0} \right| &\leq \|F\|_{L^p(\mu_M)} \left(\int_{\mathbb{R}^s} |P_\ell(x)|^{p'}\, \mu_M(\mathrm{d}x) \right)^{\frac{1}{p'}} \\ &\leq \bar{C}_{\ell,p} \|F\|_{L^p(\mu_M)} \left(\max_{\boldsymbol{\delta} \in \bar{I}_\ell} \|\boldsymbol{M}_{\boldsymbol{\delta}}\|_{\mathrm{HS}}^\ell \vee 1 \right).\end{aligned} \tag{2.4.28}$$

After these technical estimates the rest of the proof is similar to the proof of Proposition 2.4.1. We consider $M(t) = \mathcal{D}_{A+t\dot{A},k+1,\overline{N}}$ where $\overline{N}$ is the smallest integer such that $L^{\overline{N}} \geq 2D$ (or $\overline{N} = N$ if $D \geq L^N/2$). Moreover we again denote $\mathcal{D}^r_{A,k+1,k+1} = \frac{\mathrm{d}^r}{\mathrm{d}t^r} \mathcal{D}_{A+t\dot{A},k+1,\overline{N}}\Big|_{t=0}$ and for $\boldsymbol{\delta} \in I_\ell$

$$\mathcal{M}_{\boldsymbol{\delta}} = \prod_{i=1}^{\mathcal{L}(\boldsymbol{\delta})} \mathcal{D}^{-\frac{1}{2}}_{A,k+1,\overline{N}} \mathcal{D}^{\delta_i}_{A,k+1,\overline{N}} \mathcal{D}^{-\frac{1}{2}}_{A,k+1,\overline{N}}. \tag{2.4.29}$$

Using Lemma 2.4.3 and (2.4.28) we bound

$$\begin{aligned}\left| D_A^\ell \int_{\mathcal{X}_N} F(\varphi) \mu^{(A)}_{k+1}(\mathrm{d}\varphi)(\dot{A}, \ldots, \dot{A}) \right| &= \left| D_A^\ell \int_{\mathcal{V}_{\overline{N}}} F(\sigma\psi) \nu^{(A)}_{k+1,\overline{N}}(\mathrm{d}\psi)(\dot{A}, \ldots, \dot{A}) \right| \\ &\leq C \|F\|_{L^p(\mathcal{V}_{\overline{N}}, \nu^{(A)}_{k+1,\overline{N}})} \left(\max_{\boldsymbol{\delta} \in I_\ell} \|\mathcal{M}_\delta\|_{\mathrm{HS}}^\ell \vee 1 \right) \\ &= C \|F\|_{L^p(\mathcal{X}_N, \mu^{(A)}_{k+1})} \left(\max_{\boldsymbol{\delta} \in I_\ell} \|\mathcal{M}_\delta\|_{\mathrm{HS}}^\ell \vee 1 \right)\end{aligned} \tag{2.4.30}$$

where $\mu^{(A)}_{k+1} = \mu_{\mathcal{C}_{A,k+1}}$ and $\nu^{(A)}_{k+1,\overline{N}} = \nu_{\mathcal{D}_{A,k+1,\overline{N}}}$. For the last identity we used Lemma 2.4.3 backwards.

It remains to estimate the Hilbert-Schmidt norm of the operators $\mathcal{M}_{\boldsymbol{\delta}}$. The operators $\mathcal{M}_{\boldsymbol{\delta}}$ are diagonal in Fourier space and by construction we have the equality $\widehat{\mathcal{D}}_{A,k+1,\overline{N}}(p) = \widehat{\mathcal{C}}_{A,k+1}(p)$ for $p \in \widehat{T}_{\overline{N}}$ and all A hence we can bound the operator norm of the Fourier modes of $\mathcal{M}_{\boldsymbol{\delta}}$ for $\|\dot{A}\| \leq 1$ using (2.2.40) for $p \in \boldsymbol{A}_j$ and $j < k$ as follows

$$\begin{aligned} \|\widehat{\mathcal{M}}_{\boldsymbol{\delta}}(p)\| &\leq \prod_{i=1}^{\mathcal{L}(\boldsymbol{\delta})} \|\widehat{\mathcal{D}}_{k,\bar{N}}^{-1}(p)\| \cdot \|\widehat{\mathcal{D}}_{k,\bar{N}}^{\delta_i}(p)\| \\ &\leq \prod_{i=1}^{\mathcal{L}(\boldsymbol{\delta})} \Xi(\delta_i) L^{4(\tilde{n}+d)+2} L^{(k-j)(n-\tilde{n})} \\ &\leq \bar{\Xi}(\boldsymbol{\delta})^2 L^{4\mathcal{L}(\boldsymbol{\delta})(\tilde{n}+d)+2\mathcal{L}(\boldsymbol{\delta})} L^{(k-j)(n-\tilde{n})}. \end{aligned} \tag{2.4.31}$$

Here we wrote $\bar{\Xi}(\boldsymbol{\delta}) = \prod_{j=1}^{\mathcal{L}(\boldsymbol{\delta})} \Xi(\delta_j)$ for the product of the constants. Similar, for $p \in A_j$ with $j \geq k$,

$$\|\widehat{\mathcal{M}}_{\delta}(p)\| \leq \bar{\Xi}(\boldsymbol{\delta}) L^{2\mathcal{L}(\boldsymbol{\delta})(\tilde{n}+d)+\mathcal{L}(\boldsymbol{\delta})}. \tag{2.4.32}$$

Then the Hilbert-Schmidt norm is bounded by (with $\boldsymbol{A}_j = \boldsymbol{A}_j^{\overline{N}} \subset \widehat{T}_{\overline{N}}$)

$$\begin{aligned} \|\mathcal{M}_{\boldsymbol{\delta}}\|_{\mathrm{HS}}^2 &= \sum_{p \in \widehat{T}_{\overline{N}} \setminus \{0\}} \|\widehat{\mathcal{M}}_{\boldsymbol{\delta}}(p)\|_{\mathrm{HS}}^2 \leq m \sum_{p \in \widehat{T}_{\overline{N}} \setminus \{0\}} \|\widehat{\mathcal{M}}_{\boldsymbol{\delta}}(p)\|^2 \\ &\leq m \sum_{j=0}^{k-1} \sum_{p \in \boldsymbol{A}_j} \bar{\Xi}^s 2 L^{8\ell(d+\tilde{n})+4} L^{2(k-j)(n-\tilde{n})} + m \sum_{j=k}^{\overline{N}} \sum_{p \in \boldsymbol{A}_j} \bar{\Xi}^2 L^{4\ell(d+\tilde{n})+2} \\ &\leq C L^{8\ell(d+\tilde{n})+4} \sum_{j=0}^{k-1} L^{(\overline{N}-j)d} L^{(k-j)(2n-2\tilde{n})} + C L^{4\ell(d+\tilde{n})+2} \sum_{j=k}^{\overline{N}} L^{(\overline{N}-j)d} \\ &\leq C L^{8\ell(d+\tilde{n})+4} L^{(\overline{N}-k)d}. \end{aligned} \tag{2.4.33}$$

where we used $2\tilde{n} - 2n > d$ in the last step. The bound (2.4.12) and $2D > L^{\overline{N}-1}$ imply that

$$\|\mathcal{M}_{\boldsymbol{\delta}}\|_{\mathrm{HS}}^{\ell} \leq C L^{\frac{\ell}{2}(\overline{N}-k)d} < C(DL^{-k})^{\frac{\ell d}{2}}. \tag{2.4.34}$$

Plugging this into (2.4.30) ends the proof.

□

Finally we prove Lemma 2.4.3.

Proof of Lemma 2.4.3. The proof of the lemma is divided into two steps.

Step 1: First we show that we can find another Gaussian measure on $\mathcal{V}_N$ such that the local covariance structure is the same but all Fourier coefficients of the kernel except the ones that satisfy $p \in \widehat{T}_{\overline{N}} \subset \widehat{T}_N$ vanish (see (2.4.38) below). Define the kernel of an operator $\mathcal{C}_{k,\overline{N}} : T_N \to \mathbb{R}_{\mathrm{sym}}^{m\times m}$ by

$$\mathcal{C}_{k,\overline{N}}(x) = \frac{1}{L^{d\overline{N}}} \sum_{p \in \widehat{T}_{\overline{N}}} e^{ipx} \widehat{\mathcal{C}}_k(p) + \lambda M \tag{2.4.35}$$

where here and in the following we write $\lambda = L^{(N-\overline{N})d} - 1$. Clearly the Fourier modes of $\mathcal{C}_{k,\overline{N}}$ are given by $\widehat{\mathcal{C}}_{k,\overline{N}}(p) = L^{(N-\overline{N})d}\widehat{\mathcal{C}}_k(p)$ for $p \in \widehat{T}_{\overline{N}} \setminus \{0\}$, $\widehat{\mathcal{C}}_{k,\overline{N}}(0) = L^{(N-\overline{N})d}\widehat{\mathcal{C}}_k(p) + L^{Nd}\lambda M$, and 0 otherwise. By assumption $\mathcal{C}_k$ is a non-negative operator on $\mathcal{V}_N$, i.e., all Fourier coefficients are non-negative and M is positive by assumption, hence $\mathcal{C}_{k,\overline{N}}$ also is a non-negative operator. Therefore it defines a (highly degenerate) Gaussian measure $\mu_{k,\overline{N}}$ on $\mathcal{X}_N$ with covariance operator $\mathcal{C}_{k,\overline{N}}$.

We are interested in this measure because of the following remarkable property: For $x \in T_N$ with $d_\infty(x,0) \leq (L^{\overline{N}} - 1)/2$ we have $\mathcal{C}_{k,\overline{N}}(x) = \mathcal{C}_k(x)$, i.e., locally the measures μ_k and $\mu_{k,\overline{N}}$ have the same covariance structure Let us prove this property. First we observe that for $p \in \widehat{T}_{\overline{N}}$ the exponential f_p is a well defined function on $T_{\overline{N}}$ and with slight abuse of notation (identifying the exponentials on T_N and $T_{\overline{N}}$) it satisfies $f_p(x) = f_p(\pi x)$ for $x \in T_N$. Hence we find for $x, y \in T_N$

$$\frac{1}{L^{\overline{N}d}} \sum_{p \in \widehat{T}_{\overline{N}}} e^{ip(x-y)} = \frac{1}{L^{\overline{N}d}} \sum_{p \in \widehat{T}_{\overline{N}}} e^{ip\cdot\pi(x-y)} = \begin{cases} 1 \text{ if } \ \pi(x) = \pi(y) \\ 0 \text{ else.} \end{cases} \tag{2.4.36}$$

Using this, we calculate

$$\begin{aligned} \mathcal{C}_{k,\overline{N}}(x) &= \frac{1}{L^{\overline{N}d}} \sum_{p \in \widehat{T}_{\overline{N}}} e^{ipx}\widehat{\mathcal{C}}_k(p) + \lambda M \\ &= \frac{1}{L^{\overline{N}d}} \sum_{p \in \widehat{T}_{\overline{N}}} e^{ipx} \left(\sum_{y \in T_N} e^{-ipy}\mathcal{C}_k(y) \right) + \lambda M \\ &= \sum_{y \in T_N} \mathcal{C}_k(y) \left(\frac{1}{L^{\overline{N}d}} \sum_{p \in \widehat{T}_{\overline{N}}} e^{ip(x-y)} \right) + \lambda M \\ &= \sum_{\substack{y \in T_N \\ \pi(y) = \pi(x)}} \mathcal{C}_k(y) + \lambda M. \end{aligned} \tag{2.4.37}$$

In the first step we used the definition of the kernel, in the second we used the definition of the Fourier transform and the third step interchanged the order of summation. Now for a given point $x \in T_N$ there is exactly one $y_0 \in T_N$ such that $\pi(x) = \pi(y_0)$ and $d_\infty(y_0, 0) \leq \frac{L^{\overline{N}}-1}{2}$. If $d_\infty(x,0) \leq \frac{L^{\overline{N}}-1}{2}$ we have $x = y_0$. Moreover, for $d_\infty(y,0) > \frac{L^{\overline{N}}-1}{2} \geq \frac{L^k-1}{2}$ we have $\mathcal{C}_k(y) = -M$ by assumption. Hence we have

$$\mathcal{C}_{k,\overline{N}}(x) = \mathcal{C}_k(x) - (L^{(N-\bar{N})d} - 1)M + \lambda M = \mathcal{C}_k(x) \tag{2.4.38}$$

for $d_\infty(x,0) \leq \frac{L^{\overline{N}}-1}{2}$ as claimed. Actually we have even shown that $\mathcal{C}_{k,\overline{N}}$ is the $(L^{\overline{N}}\mathbb{Z})^d$ periodic extension of $\mathcal{C}_k \restriction_{\left[-\frac{L^{\overline{N}}-1}{2},\dots,\frac{L^{\overline{N}}-1}{2}\right]^d}$.

Next we claim that if ξ is distributed according to μ_k and φ is distributed according to $\mu_{k,\overline{N}}$ then $\xi\restriction_X \overset{\text{Law}}{=} \varphi\restriction_X$. First we note that since the distribution of ξ and φ is Gaussian with mean zero the same holds for the restrictions $\xi\restriction_X$ and $\varphi\restriction_X$. Then it is enough to prove that all covariances agree because they determine the law. By the assumption $x, y \in X \Rightarrow d_\infty(x,y) \leq D \leq \frac{L^{\overline{N}}-1}{2}$. Hence $\mathcal{C}_k(x-y) = \mathcal{C}_{k,\overline{N}}(x-y)$ and therefore

$$\mathbb{E}\left(\xi^i(x)\xi^j(y)\right) = \mathcal{C}_k^{ij}(x-y) = \mathcal{C}_{k,\overline{N}}^{ij}(x-y) = \mathbb{E}\left(\varphi^i(x)\varphi^j(y)\right). \tag{2.4.39}$$

By assumption there exists a functional $\tilde{F}_X : (\mathbb{R}^m)^X \to \mathbb{R}$ such that $F_X(\xi) = \tilde{F}_X(\xi\restriction_X)$. Hence we find

$$\begin{aligned}\int_{\mathcal{V}_N} F_X(\xi)\,\mu_k(\mathrm{d}\xi) = \int_{\mathcal{V}_N} \tilde{F}_X(\xi\restriction_X)\,\mu_k(\mathrm{d}\xi) &= \int_{\mathcal{V}_N} \tilde{F}_X(\varphi\restriction_X)\,\mu_{k,\overline{N}}(\mathrm{d}\varphi) \\ &= \int_{\mathcal{V}_N} F_X(\varphi)\,\mu_{k,\overline{N}}(\mathrm{d}\varphi).\end{aligned} \tag{2.4.40}$$

Step 2: In the second step we show that the measure $\mu_{k,\overline{N}}$ is the push-forward of $\nu_{k,\overline{N}}$ along σ, i.e., $\mu_{k,\overline{N}} = \sigma_*\nu_{k,\overline{N}}$. The measure $\nu_{k,\overline{N}}$ was defined in the statement of the lemma by the kernel $\mathcal{D}_{k,\overline{N}} : T_{\overline{N}} \to \mathbb{R}$

$$\mathcal{D}_{k,\overline{N}}(x) = \frac{1}{L^{\overline{N}d}} \sum_{p\in\widehat{T}_{\overline{N}}} e^{ipx}\widehat{\mathcal{C}}_k(p) + \lambda M. \tag{2.4.41}$$

From this equation we can similarly to Step 1 extract the Fourier decomposition of this operator and see that this expression defines a non-negative operator and therefore the covariance of a Gaussian measure. Note that for $x \in T_N$

$$\mathcal{D}_{k,\overline{N}}(\pi(x)) = \mathcal{C}_{k,\overline{N}}(x) \tag{2.4.42}$$

which is again a consequence of $f_p(\pi x) = f_p(x)$ for $p \in \widehat{T}_{\overline{N}}$. In other words the kernel $\mathcal{C}_{k,\overline{N}}$ is already $(L^{\overline{N}}\mathbb{Z})^d$ periodic and hence also defines a function $T_{\overline{N}} \to \mathbb{R}$ which we call $\mathcal{D}_{k,\overline{N}}$. The previous definition has the advantage that it makes clear that this kernel defines a Gaussian measure. The proof of $\mu_{k,\overline{N}} = \sigma_*\nu_{k,\overline{N}}$ is standard. We prove that the characteristic functions for both measures agree. Let $v : T_N \to \mathbb{R}^m$ be a field. We have to show

$$\int_{\mathcal{V}_N} e^{i\langle v,\varphi\rangle}\,\mu_{k,\overline{N}}(\mathrm{d}\varphi) = \int_{\mathcal{V}_N} e^{i\langle v,\varphi\rangle}\,\sigma_*\nu_{k,\overline{N}}(\mathrm{d}\varphi). \tag{2.4.43}$$

The left hand side is the characteristic function of a Gaussian measure given by

$$\int_{\mathcal{V}_N} e^{i\langle v,\varphi\rangle}\,\mu_{k,\overline{N}}(\mathrm{d}\varphi) = \exp\left(-\frac{\langle v, \mathcal{C}_{k,\overline{N}}v\rangle}{2}\right) \tag{2.4.44}$$

as completion of the square shows. The right hand side is slightly more complicated. By a change of variable we find

$$\begin{aligned}\int_{\mathcal{V}_N} e^{i\langle v,\varphi\rangle}\,\sigma_*\nu_{k,\overline{N}}(\mathrm{d}\varphi) &= \int_{\mathcal{V}_{\overline{N}}} e^{i\langle v,\sigma\psi\rangle}\,\nu_{k,\overline{N}}(\mathrm{d}\psi) \\ &= \int_{\mathcal{V}_{\overline{N}}} e^{i\langle\sigma^* v,\psi\rangle}\,\nu_{k,\overline{N}}(\mathrm{d}\psi) \\ &= e^{-\frac{1}{2}\langle\sigma^* v,\mathcal{D}_{k,\overline{N}}\sigma^* v\rangle}.\end{aligned} \tag{2.4.45}$$

Here $\sigma^* : \mathcal{V}_N \to \mathcal{V}_{\overline{N}}$ is the adjoint of $\sigma : \mathcal{V}_{\overline{N}} \to \mathcal{V}_N$ with respect to the standard scalar product on both spaces, i.e., σ^* is characterised by $\langle\varphi,\sigma\xi\rangle = \langle\sigma^*\varphi,\xi\rangle$ for $\varphi \in \mathcal{V}_N$ and $\xi \in \mathcal{V}_{\overline{N}}$. It is easy to see that $\sigma^* v(\overline{x}) = \sum_{x \in T_N : \pi(x) = \overline{x}} v(x)$. Then we find

$$\begin{aligned}
\langle \sigma^* v, \mathcal{D}_{k,\overline{N}} \sigma^* v \rangle &= \sum_{\overline{x},\overline{y} \in T_{\overline{N}}} \sigma^* v(\overline{x}) \mathcal{D}_{k,\overline{N}}(\overline{x} - \overline{y}) \sigma^* v(\overline{y}) \\
&= \sum_{\overline{x},\overline{y} \in T_{\overline{N}}} \sum_{\substack{x \in T_N \\ \pi(x) = \overline{x}}} \sum_{\substack{y \in T_N \\ \pi(y) = \overline{y}}} v(x) \mathcal{D}_{k,\overline{N}}(\pi(x-y)) v(y) \\
&= \sum_{x,y \in T_N} v(x) \mathcal{C}_{k,\overline{N}}(x-y) v(y) \\
&= \langle v, \mathcal{C}_{k,\overline{N}} v \rangle.
\end{aligned} \tag{2.4.46}$$

Together with the equations (2.4.44) and (2.4.45) this shows the claim.

Conclusion: From equation (2.4.40) and Step 2 we conclude

$$\int_{\mathcal{V}_N} F_X(\xi)\, \mu_k(\mathrm{d}\xi) = \int_{\mathcal{V}_N} F_X(\varphi)\, \sigma_* \nu_{k,\overline{N}}(\mathrm{d}\varphi) = \int_{\mathcal{V}_{\overline{N}}} F_X(\sigma\psi)\, \nu_{k,\overline{N}}(\mathrm{d}\psi). \tag{2.4.47}$$

□

2.A Proof of Theorem 2.2.3

In this appendix we discuss those details of the proof of Theorem 2.2.3 that are not already contained in the much more general discussion in [18]. The key ingredient of the proof is the following lemma

Lemma 2.A.1 (Lemma 2.3. in [18])**.** *Let $B > 0$ be a constant. There is a smooth family of functions $W_t \in C^\infty(\mathbb{R})$ for $t > 0$ such that for $\lambda \in (0, B)$, $t > 0$*

$$\lambda^{-1} = \int_0^\infty t W_t(\lambda)\, \mathrm{d}t, \tag{2.A.1}$$

$$W_t(\lambda) \geq 0. \tag{2.A.2}$$

Moreover $W_t \restriction_{(0,B)}$ is the restriction of a polynomial in λ of degree at most t. For $t \leq 1$ we have the explicit formula

$$W_t(\lambda) = C/t \tag{2.A.3}$$

for some constant $C > 0$. For $t \geq 1$ and integers $\ell, n \geq 0$ the following estimate holds

$$(1 + t^2 \lambda)^n \lambda^\ell \left| \frac{\partial^\ell}{\partial \lambda^\ell} W_t(\lambda) \right| \leq C_{\ell,n}. \tag{2.A.4}$$

In addition we can choose W_t such that it satisfies

$$W_t(\lambda) \geq \epsilon \tag{2.A.5}$$

for some $\epsilon > 0$ and $\lambda \leq B \min(1, t^{-2})$.

Proof. This is Lemma 2.3. in [18] (with rescaled λ) except for the lower bound (2.A.5). The lower bound is however easily obtained from the construction in [18]. One possible choice for the function W_t is given by

$$W_t(\lambda) = \sum_{n\in\mathbb{Z}} \varphi\left(\arccos\left(1-\frac{\lambda}{2B}\right)t - 2\pi n t\right) \tag{2.A.6}$$

for $\lambda \in (0, B)$ where $\varphi : \mathbb{R} \to \mathbb{R}_0^+$ is any symmetric non-negative function such that $\widehat{\varphi}$ is supported in $[-1, 1]$ and smooth. More precisely, we let $\varphi = |\kappa|^2$ where $\widehat{\kappa}$ is even and supported in $[-\frac{1}{2}, \frac{1}{2}]$ and we can moreover choose $\widehat{\kappa}$ to be non-negative, which implies for $|x| < 2$ that

$$\kappa(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\kappa}(k)e^{ikx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{\kappa}(k)\cos(kx)\,\mathrm{d}x > \sqrt{\epsilon} \tag{2.A.7}$$

for some $\epsilon > 0$. The bound $\arccos(1-x) \le \frac{\pi}{2}\sqrt{2x}$ for $x \in [0, 2]$ implies that for $\lambda < B\min(t^{-2}, 1)$ the estimate $|t\arccos(1-\frac{\lambda}{2B})| < 2$ holds. Hence for those λ we bound

$$W_t(\lambda) \ge \varphi\left(\arccos\left(1-\frac{\lambda}{2B}\right)t\right) = \left|\kappa\left(\arccos\left(1-\frac{\lambda}{2B}\right)t\right)\right|^2 \ge \epsilon. \tag{2.A.8}$$

□

Proof of Theorem 2.2.3. We set $B = \pi^2 d\Omega \ge \max(\mathrm{spec}(\mathcal{A}))$ where Ω is the constant in (2.2.23). Based on the previous lemma we obtain a finite range decomposition by defining for $2 \le k \le N$

$$\begin{aligned} \mathcal{C}_k &= \int_{\frac{L^{k-1}}{2R}}^{\frac{L^k}{2R}} tW_t(\mathcal{A})\,\mathrm{d}t \\ \mathcal{C}_1 &= \int_0^{\frac{L}{2R}} tW_t(\mathcal{A})\,\mathrm{d}t \\ \mathcal{C}_{N+1} &= \int_{\frac{L^N}{2R}}^{\infty} tW_t(\mathcal{A})\,\mathrm{d}t \end{aligned} \tag{2.A.9}$$

This decomposition indeed satisfies $\sum_{k=1}^{N+1} \mathcal{C}_k = \mathcal{C}$ because $\mathrm{spec}(\mathcal{A}) \subset [0, B]$ by (2.2.23) and property (2.A.1). Since W_t is a polynomial of degree at most t and $\mathrm{supp}(\mathcal{A}\varphi) \subset \mathrm{supp}(\varphi) + [-R, \dots, R]^d$ the finite range property in the theorem holds.

Next, we want to show that the matrix $-M_k = \mathcal{C}_{A,k}(x)$ with $|x| \ge L^k/2$ is positive definite and independent of A and we want to show(2.2.32). The kernel $\mathcal{C}_{A,k}$ is uniquely characterised by the conditions that $\mathcal{C}_{A,k}\varphi = \mathcal{C}_{A,k} * \varphi$ for $\varphi \in \mathcal{X}_N$ and $\mathcal{C}_{A,k} \in \mathcal{M}_N$ (space of matrix valued kernels with average zero). By construction of $\mathcal{C}_{A,k} = \mathcal{C}_k(\mathcal{A})$ we know that for $1 \le k \le N$ there are coefficients $c_{k,l}$ such that $\mathcal{C}_k(\mathcal{A}) = \sum c_{k,l}\mathcal{A}^l$. Observe that the action of the finite difference operator $\mathcal{A}$ can be written as

$$\mathcal{A}\varphi(x) = \sum_{y\in[-R,R]^d} a_y\varphi(x+y) \tag{2.A.10}$$

where $a_y \in \mathbb{R}^{m\times m}$ are coefficients such that $\sum_y a_y = 0$. In particular the kernel $\mathcal{A} \in \mathcal{M}_N$ of $\mathcal{A}$ satisfies $\mathcal{A}(x) = 0$ for $d_\infty(x, 0) > R$. The same holds for powers $\mathcal{A}^l$ and $d_\infty(x, 0) > lR$ because the kernel of $\mathcal{A}^l$ is given by the l-fold convolution $\mathcal{A} * \ldots * \mathcal{A}$. Therefore only the constant term of the polynomial contributes to the kernel of $\mathcal{C}_{A,k}(x)$ for $|x| \ge L^k/2$ in particular it is independent of $\mathcal{A}$. Note that the kernel of $\mathcal{A}^0 = \mathrm{id}_{\mathcal{X}_N}$ is given by $(\delta_0 - L^{-Nd})\mathbb{1}_{m\times m}$ because this function has average zero and constant shifts do not change the operator so it generates the same operator as $\delta_0\mathbb{1}_{m\times m}$. In order to show $M_k \geq 0$ it is therefore sufficient to show $c_{k,0} \geq 0$ because $M_k = -c_{k,0}L^{-Nd}\mathrm{Id}$. The inequality $W_t(0) \geq 0$ implies that the constant term of the polynomial W_t is positive. Hence

$$c_{k,0} = \left(\int_{L^{k-1}/(2R)}^{L^k/(2R)} t \cdot W_t(0)\,\mathrm{d}t\right) > 0. \tag{2.A.11}$$

Using similar arguments we can show Remark 2.2.4. Note that for $l \leq (L^N - 1)/(2R)$ the operator $\mathcal{A}^l$ with kernel $\mathcal{A}^{*l}$ does not 'wrap around' the torus. In particular we have for $x \in \Lambda_N$ and $N \leq N'$

$$\mathcal{A}_N^{*l}(x) = \mathcal{A}_{N'}^{*l}(x) \tag{2.A.12}$$

where $\mathcal{A}_N$ denotes the kernel of $\mathcal{A}_N$ and N indicates the size of the torus. Since W_t is a fixed polynomial independent of N of degree at most t a simple calculation then implies (2.2.32).

It remains to establish the bounds. Here it is useful to rely on the estimate (2.2.23) for the spectrum of the Fourier coefficients instead of the bounds on the quadratic form Q. Since $\mathcal{A}$ is diagonal in Fourier space and this property carries over to polynomials in $\mathcal{A}$ the identity

$$\widehat{\mathcal{C}}_k(p) = \int_{\frac{L^{k-1}}{2R}}^{\frac{L^k}{2R}} t \cdot W_t(\widehat{\mathcal{A}}(p))\,\mathrm{d}t \tag{2.A.13}$$

holds for $2 \leq k \leq N$ and similar identities hold for $k = 1$ and $k = N+1$. Using this equation we can derive strong bounds for the Fourier modes of $\mathcal{C}_k$. Let us denote the eigenvalues of the symmetric and positive matrix $\widehat{\mathcal{A}}(p)$ by $\omega|p|^2 \leq \lambda_1 \leq \ldots \leq \lambda_m$ where we plugged in the lower bound (2.2.23). The key observation is that by estimate (2.A.4) for $t \geq 1$

$$\|W_t(\widehat{\mathcal{A}}(p))\| = \max_{1\leq i\leq m} |W_t(\lambda_i)| \leq \frac{C_n}{(\lambda_1 t^2)^n} \leq \frac{C_n}{(\omega|p|^2t^2)^n}. \tag{2.A.14}$$

The estimate (2.A.14) implies with $n \geq 2$ and $n = 0$ respectively

$$\|\widehat{\mathcal{C}}_k(p)\| \leq \int_{\frac{L^{k-1}}{2R}}^{\infty} \frac{C_n}{(\omega|p|^2t^2)^n} t\,\mathrm{d}t \leq \frac{C_n(2R)^{2n-2}}{(2n-1)\omega^n}|p|^{-2}(|p|L^{k-1})^{-(2n-2)}\ \forall 2 \leq k \leq N+1 \tag{2.A.15}$$

$$\|\widehat{\mathcal{C}}_k(p)\| \leq \int_0^1 \frac{C_0}{t} t\,\mathrm{d}t \int_1^{\frac{L^k}{2R}} C_0 t\,\mathrm{d}t \leq C_0 + \frac{C_0 L^{2k}}{8R^2}\ \forall 1 \leq k \leq N. \tag{2.A.16}$$

where we used $W_t(\lambda) \leq C_0/t$ for $t \leq 1$ in the second estimate. Note that the first bound does not hold for $k = 1$ and the last bound does not hold for $k = N+1$. Moreover there is the trivial bound

$$\|\widehat{\mathcal{C}}_k(p)\| \leq \|\widehat{\mathcal{C}}(p)\| = \|\widehat{\mathcal{A}}(p)^{-1}\| \leq \frac{|p|^{-2}}{\omega}. \tag{2.A.17}$$

The most useful combination of these bounds is

$$\|\widehat{\mathcal{C}}_k(p)\| \leq \begin{cases} C_{n'}|p|^{-2}(|p|L^{(k-1)})^{-n'} & \text{for } |p| \geq L^{-(k-1)},\ n' \geq 2, \\ C|p|^{-2} & \text{for } L^{-(k-1)} > |p| \geq L^{-k}, \\ CL^{2k} & \text{for } L^{-k} > |p|. \end{cases} \tag{2.A.18}$$

This bound also holds for $k = 1$ and $k = N+1$. Indeed we have $(|p|L^{k-1})^{-n'} \geq (\pi\sqrt{d})^{-n'}$ for $k = 1$ so (2.A.17) implies the first estimate and $\{p \in \widehat{T}_N : |p| < L^{-(N+1)}\} = \emptyset$. In particular (2.A.18) contains (2.2.29) for $\ell = 0$. For $|\alpha| + d > 2$ we get using (2.A.18) with $n' > d - 2 + |\alpha|$ and (2.2.25)

$$\begin{aligned}
\|\nabla^\alpha \mathcal{C}_k(x)\| &\leq \frac{C}{L^{Nd}}\Big(\sum_{\substack{p\in\widehat{T}_N\\ |p|\geq L^{-(k-1)}}} |p|^{|\alpha|-2}(|p|L^{(k-1)})^{-n'} + \sum_{\substack{p\in\widehat{T}_N\\ L^{-(k-1)}>|p|\geq L^{-k}}} |p|^{|\alpha|-2} + \sum_{\substack{p\in\widehat{T}_N\\ L^{-k}>|p|}} |p|^{|\alpha|}L^{2k}\Big) \\
&\leq C \int\limits_{L^{-(k-1)}\leq r} r^{d-3+|\alpha|}(rL^{(k-1)})^{-n'}\,\mathrm{d}r + C \int\limits_{L^{-k}<r<L^{-(k-1)}} r^{d-3+|\alpha|}\,\mathrm{d}r + C \int\limits_{r<L^{-k}} L^{2k} r^{|\alpha|+d-1}\,\mathrm{d}r \\
&\leq CL^{-(k-1)(d-2+|\alpha|)} + CL^{-k(d-2+|\alpha|)} \leq CL^{-(k-1)(d-2+|\alpha|)}
\end{aligned} \tag{2.A.19}$$

where we approximated the Riemann sums by their integrals possibly increasing the constant. This approximation can be justified using a dyadic decomposition for the sums. Note that the constant C does not depend on N or L. The condition $d + |\alpha| > 2$ was used to bound the second integral. For $d + |\alpha| = 2$ it behaves as $\int \frac{\mathrm{d}r}{r} \approx \ln(L)$ hence we get an additional logarithm in this case.

Next we consider derivatives with respect to the parameter matrix A. We need the following simple lemma for which we did not find an exact reference in the literature. Similar arguments can be found in [137].

Lemma 2.A.2. *Let $A, B \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$ be symmetric matrices and let $\lambda_1 \leq \ldots \leq \lambda_m$ be the eigenvalues of A counted with multiplicity. Let f be a holomorphic function. Then there is a combinatorial constant $C_{m,\ell}$ such that*

$$\left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} f(A+sB)\Big|_{s=0} \right\| \leq C_{m,\ell} \sup_{\lambda\in[\lambda_1,\lambda_m]} |f^{(\ell)}(\lambda)| \|B\|^\ell. \tag{2.A.20}$$

Proof. The proof is based on a representation of the matrix derivative using the Cauchy formula that appears e.g. in [111]. Note that the eigenvalues of $A + sB$ are continuous functions of $s \in \mathbb{R}$ for A and B symmetric [111]. Let C be a curve around all the eigenvalues of $A + sB$ for $s \in (-\epsilon, \epsilon)$ with winding number 1. By the Cauchy formula

$$f(A+sB) = \frac{1}{2\pi i}\int_C f(z)(z\mathrm{Id} - (A+sB))^{-1} \tag{2.A.21}$$

Differentiating ℓ times with respect to s or using the Neumann series for the matrix inverse gives

$$\frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} f(A+sB)\Big|_{s=0} = \frac{\ell!}{2\pi i}\int_C f(z)(z\mathrm{Id}-A)^{-1}B(z\mathrm{Id}-A)^{-1}B\ldots B(z\mathrm{Id}-A)^{-1}. \tag{2.A.22}$$

Now we write $A = \sum_{i=1}^m \lambda_i P_i$ as a sum of orthogonal projections such that $\sum_{i=1}^m P_i = \mathrm{Id}$. Then we find for $z \notin \mathrm{spec}(A)$ that $(z\mathrm{Id} - A)^{-1} = \sum_{i=1}^n (z-\lambda_i)^{-1}P_i$. Plugging this in (2.A.22) we bound

$$\begin{aligned}
\left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} f(A+sB)\Big|_{s=0} \right\| &= \left\| \frac{\ell!}{2\pi i} \sum_{i_1,\ldots,i_{\ell+1}=1}^m \int_C \frac{f(z)}{(z-\lambda_{i_1})\ldots(z-\lambda_{i_n})} P_{i_1}BP_{i_2}\ldots BP_{i_{\ell+1}} \right\| \\
&\leq \sum_{i_1,\ldots,i_{\ell+1}=1}^m \left| \frac{\ell!}{2\pi i}\int_C \frac{f(z)}{(z-\lambda_{i_1})\ldots(z-\lambda_{i_{\ell+1}})} \right| \|B\|^\ell
\end{aligned} \tag{2.A.23}$$

The term in absolute values is the sum of divided differences [35] and by the mean value theorem for finite difference (Proposition 43 in [35]) there is a $\xi \in (\min_{1\leq j\leq n+1} \lambda_{i_j}, \max_{1\leq j\leq n+1} \lambda_{i_j})$ such that

$$\frac{\ell!}{2\pi i}\int_C \frac{f(z)}{(z-\lambda_{i_1})\dots(z-\lambda_{i_\ell})} = f^{(\ell)}(\xi). \tag{2.A.24}$$

This implies the claim

$$\left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} f(A+sB)\Big|_{s=0} \right\| \leq m^{(\ell+1)} \sup_{\lambda\in[\lambda_1,\lambda_m]} |f^{(\ell)}(\lambda)| \|B\|^\ell. \tag{2.A.25}$$

□

We apply this lemma to the Fourier modes of the operators $\mathcal{A}_{A+sA_1}$ where $A_1 \in \mathcal{L}(\mathcal{G})$ is a linear and symmetric but not necessarily positive operator. Note that W_t can be extended to a holomorphic function in a neighbourhood of $(0, B)$ because this holds for the arccos function and φ is holomorphic since $\widehat{\varphi}$ has compact support. Then we find using linearity of the Fourier transform, Lemma 2.A.2, the bounds for the spectrum of the Fourier modes (2.2.23), and the estimate (2.A.4) for $t \geq 1$

$$\begin{aligned}
\left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} W_t(\widehat{\mathcal{A}}_{A+sA_1}(p)) \right\| &= \left\| \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} W_t(\widehat{\mathcal{A}}_{A}(p) + s\widehat{\mathcal{A}}_{A_1}(p)) \right\| \\
&\leq C_{m,\ell} \|\widehat{\mathcal{A}}_{A_1}(p)\|^\ell \sup_{\lambda\in \mathrm{Conv}(\mathrm{spec}\,\widehat{\mathcal{A}}_A(p))} |W_t^{(\ell)}(\lambda)| \\
&\leq C_{m,\ell}\Omega^\ell |p|^{2\ell} \|A_1\|^\ell \sup_{\lambda\in \mathrm{Conv}(\mathrm{spec}\,\widehat{\mathcal{A}}_A(p))} |W_t^{(\ell)}(\lambda)| \\
&\leq C_{m,\ell}\Omega^\ell \|A_1\|^\ell |p|^{2\ell} \sup_{\lambda\in \mathrm{Conv}(\mathrm{spec}\,\widehat{\mathcal{A}}_A(p))} \frac{C_{n,\ell}}{\lambda^\ell (1+\lambda t^2)^n} \\
&\leq C_{m,n,\ell} \left(\frac{\Omega}{\omega}\right)^\ell \|A_1\|^\ell \min(1, (\omega|p|^2t^2)^{-n}).
\end{aligned} \tag{2.A.26}$$

The bound extends to $t < 1$ because $W_t(\lambda) = C/t$ is constant in this case. We obtain up to a constant exactly the same bound we used before to find (2.A.15) (for $k \geq 2$) and (2.A.16) (for $k \leq N$). Let us check that also the bound $\|\widehat{\mathcal{C}}_k(p)\| < C|p|^{-2}$ generalises to $\ell \geq 1$. We bound for $L^{-k} < |p| < L^{-(k-1)}$ and $\|\dot{A}\| \leq 1$

$$\|D_A^\ell \widehat{\mathcal{C}}_k(p)(\dot{A},\dots,\dot{A})\| \leq \int_{\frac{L^{k-1}}{2R}}^{|p|^{-1}} C_{m,0,\ell}\cdot t\,\mathrm{d}t + \int_{|p|^{-1}}^{\frac{L^k}{2R}} \frac{C_{m,2,\ell}}{|p|^4t^4} t\,\mathrm{d}t \leq C|p|^{-2} \tag{2.A.27}$$

Hence we find a bound similar to (2.A.18)

$$\|D_A^\ell \widehat{\mathcal{C}}_k(p)(\dot{A},\dots,\dot{A})\| \leq \begin{cases} C_{n'}|p|^{-2}(|p|L^{(k-1)})^{-n'} & \text{for } |p| \geq L^{-(k-1)},\ n' \geq 2 \\ C|p|^{-2} & \text{for } L^{-(k-1)} > |p| \geq L^{-k} \\ CL^{2k} & \text{for } L^{-k} > |p| \end{cases} \tag{2.A.28}$$

This bound completes the proof of the upper bound in Fourier space (2.2.29). As in (2.A.19) this bound also implies (2.2.31) for $\ell \geq 1$.

Finally we consider the lower bound. For $|p| \leq t^{-1}$ we find $\|\widehat{\mathcal{A}}(p)\| \leq \Omega |p|^2 \leq Bt^{-2}$. Hence (2.A.5) implies $W_t(\widehat{\mathcal{A}}(p)) \geq \epsilon \mathbb{1}_{m\times m}$. Using this bound we find for $|p| \leq L^{-k}$ and $k \geq 2$

$$\widehat{\mathcal{C}}_k(p) = \int_{\frac{L^{k-1}}{2R}}^{\frac{L^k}{2R}} tW_t(\widehat{\mathcal{A}}(p))\,\mathrm{d}t \geq \frac{\epsilon L^{2k}}{16R^2}. \tag{2.A.29}$$

Using the positivity of W_t we get for $k = 1$

$$\widehat{\mathcal{C}}_1(p) = \int_0^{\frac{L}{2R}} tW_t(\widehat{\mathcal{A}}(p))\,\mathrm{d}t \tag{2.A.30}$$

$$\geq \epsilon \int_0^{\min(\frac{L}{2R}, |p|^{-1})} t\,\mathrm{d}t \geq \frac{\epsilon}{2} \min\left(|p|^{-2}, \frac{L^2}{4R^2}\right). \tag{2.A.31}$$

This completes the proof of Theorem 2.2.3. □

Chapter 3

Models for discrete elasticity at positive temperature

The results of this and the following chapter are joint work in progress with Stefan Adams, Roman Kotecký, and Stefan Müller.

3.1 Introduction

The purpose of this chapter is to state our main results for microscopic models of elasticity and gradient models and reduce them to statements about perturbations of Gaussian integrals that are then proved in Chapter 4. For a general background on gradient models, a motivation of our results, and references to the literature we refer to the introduction in Chapter 1.

Let us briefly recall the setting. We consider general finite range potentials given by a function $U : (\mathbb{R}^m)^A \to \mathbb{R}$ where A is a finite subset of $\mathbb{Z}^d$ that are invariant under shifts. The corresponding Gibbs measure describes the behaviour of, e.g., crystalline solids or discrete interfaces. We are mostly interested in properties of the free energy for this class of models. Recall that by (1.2.12) the free energy is given by

$$W(F, \beta) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{-\ln(Z_{\Lambda,\psi_F,\beta})}{\beta|\Lambda|} \tag{3.1.1}$$

where

$$Z_{\Lambda,\psi,\beta} = \int_A e^{-\beta \sum_{\tau_x(A)\cap\Lambda\neq\emptyset} U(\varphi_A)} \prod_{x\in\Lambda} \lambda(\mathrm{d}\varphi_x) \prod_{x\notin\Lambda} \delta_{\psi_x}(\mathrm{d}\varphi_x). \tag{3.1.2}$$

It is convenient to reformulate the finite range interaction as generalized gradient models, i.e., the interaction potential can equivalently be written as a function $\mathcal{U} : (\mathbb{R}^m)^{\mathcal{I}} \to \mathbb{R}$ acting on a set of discrete derivatives $(\nabla^\alpha \varphi)_{\alpha\in\mathcal{I}}$ indexed by $\mathcal{I}$. This contains gradient interface models as a special case. Our analysis is restricted to potentials that can be bounded below by a strictly convex function agreeing with the potential in the origin, i.e.,

$$\mathcal{U}(z) - D\mathcal{U}(0)z - \mathcal{U}(0) \geq \omega|z|^2 \tag{3.1.3}$$

where $\omega > 0$ is a positive constant.

Under this assumption it is possible to extract an explicit dominant contribution of the free energy and we obtain a perturbative contribution that can be expressed as a perturbation of a

Gaussian integral and that is formally of order β^{-1}. Our main results are that the assumption (3.1.3) together with a smoothness condition on $\mathcal{U}$ imply a good control on the perturbative part of the free energy. Then it is a simple consequence that the free energy is strictly convex in a neighbourhood of the origin for small temperatures and that the scaling limit is Gaussian for small external deformations and small temperatures. This extends earlier results for convex potentials by Funaki and Spohn [89] and Giacomin, Olla, and Spohn [94] to certain non convex potentials. For a detailed review of the literature we refer to Chapter 1.

Models for nonlinear discrete elasticity have the property that the energy is invariant under rotations and the minimum of the energy is obtained for the undeformed state, i.e., the identity deformation. This results in a degenerate minimum of the energy. In general the analysis of spontaneously broken symmetries in models with continuous spins is very difficult and no rigorous renormalisation group analysis for such models is known. In the context of elasticity one is usually interested in affine boundary conditions that break the symmetry. In fact, it is even possible to use discrete null Lagrangians to construct a physically equivalent energy that satisfies the growth assumption (3.1.3). This allows us to apply our results for generalised gradient models also to models for nonlinear elasticity.

This chapter is structured as follows. In Section 3.2 we properly define gradient interface models and state our main results Theorem 3.2.6 and Theorem 3.2.9. We reduce them to general statements about perturbations of Gaussian integrals by showing the smallness of the perturbations in Proposition 3.2.4. The proof of this proposition is the only longer proof in this section but nevertheless essentially an exercise in bounding derivatives using the product and chain rule. In Section 3.3 we generalise the results to statements about discrete elasticity in Theorem 3.3.1 and Theorem 3.3.2 using discrete null Lagrangians.

3.2 Setting and main results

3.2.1 General setup

Fix an odd integer $L \geq 3$ and a dimension $d \geq 2$. Let $T_N = (\mathbb{Z}/(L^N\mathbb{Z}))^d$ be the d-dimensional *discrete torus* of side length L^N where N is a positive integer. We equip T_N with the quotient distances $|\cdot|$ and $|\cdot|_\infty$ induced by the Euclidean and maximum norm respectively. Define the space of m-component fields on T_N as

$$\mathcal{V}_N = \{\varphi : T_N \to \mathbb{R}^m\} = (\mathbb{R}^m)^{T_N}. \tag{3.2.1}$$

Since the energies we consider are shift invariant we are only interested in gradient fields. However, the condition of being a gradient is not entirely straightforward in dimension $d \geq 2$; thus we rather work with usual fields modulo a constant or, equivalently, with fields with the vanishing average

$$\varphi \in \mathcal{X}_N = \Big\{\varphi \in \mathcal{V}_N : \sum_{x\in T_N} \varphi(x) = 0\Big\} \tag{3.2.2}$$

that are in one-to-one correspondence with *gradient fields*. Let the dot denote the standard scalar product on $\mathbb{R}^m$ which is later extended to $\mathbb{C}^m$. For $\psi, \varphi \in \mathcal{X}_N$ the expression

$$(\varphi, \psi) = \sum_{x\in T_N} \varphi(x) \cdot \psi(x) \tag{3.2.3}$$

defines a scalar product on $\mathcal{X}_N$ and this turns $\mathcal{X}_N$ into a Hilbert space. We use λ_N for the $m(L^{Nd} - 1)$-dimensional Hausdorff measure on $\mathcal{X}_N$, equip the space $\mathcal{X}_N$ with the σ-algebra

$\mathcal{B}_{\mathcal{X}_N}$ induced by the Borel σ-algebra with respect to the product topology, and use $\mathcal{M}_1(\mathcal{X}_N) = \mathcal{M}_1(\mathcal{X}_N, \mathcal{B}_{\mathcal{X}_N})$ to denote the set of probability measures on $\mathcal{X}_N$, referring to elements in $\mathcal{M}_1(\mathcal{X}_N)$ as *random gradient fields.*

The *discrete forward* and *backward derivatives* are defined by

$$\begin{aligned} (\nabla_i\varphi)_s(x) &= \varphi_s(x+e_i) - \varphi_s(x) \qquad s \in \{1,\dots,m\}, \quad i \in \{1,\dots,d\}, \\ (\nabla_i^*\varphi)_s(x) &= \varphi_s(x-e_i) - \varphi_s(x) \qquad s \in \{1,\dots,m\}, \quad i \in \{1,\dots,d\}. \end{aligned} \tag{3.2.4}$$

Here e_i are the standard unit vectors in $\mathbb{Z}^d$. Forward and backward derivatives are adjoints of each other. We use $\nabla\varphi(x)$ and $\nabla^*\varphi(x)$ for the corresponding $m \times d$ matrices.

In this article we study a class of random gradient fields defined (as Gibbs measures) in terms of Hamiltonians $H_N : \mathcal{X}_N \to \mathbb{R}$ that are in their turn given by a finite range potential $U : (\mathbb{R}^m)^A \to \mathbb{R}$. Here, $A \subset \mathbb{Z}^d$ is a finite set and we use R' to denote the range of the potential U, $R' = \mathrm{diam}_\infty A$. Anticipating that U is invariant with respect to translations in $\mathbb{R}^m$, $U(\psi) = U(t_a\psi)$ for any $\psi \in (\mathbb{R}^m)^A$ with $(t_a\psi)_s(x) = \psi_s(x) + a_s$, $a \in \mathbb{R}^m$, it depends on ψ only modulo translations by vectors from $\mathbb{R}^m$, or, equivalently, depends for connected sets A only on gradients $\nabla\psi(x)$, $x \in A$. For any $\varphi \in \mathcal{X}_N$ and any $B \subset T_N$, we use φ_B to denote the restriction of φ to B, and define

$$H_N(\varphi) = \sum_{x \in T_N} U(\varphi_{\tau_x(A)}) \tag{3.2.5}$$

where $\tau_x(A)$ denotes the set A translated by $x \in T_N$, $\tau_x(A) = A + x = \{y \colon y - x \in A\}$. The corresponding *gradient Gibbs measure* $\gamma_{N,\beta} \in \mathcal{M}_1(\mathcal{X}_N)$ *at inverse temperature* $\beta = T^{-1}$ is defined as

$$\gamma_{N,\beta}(\mathrm{d}\varphi) = \frac{\exp\{-\beta H_N(\varphi)\}}{Z_{N,\beta}} \lambda_N(\mathrm{d}\varphi) \tag{3.2.6}$$

with

$$Z_{N,\beta} = \int_{\mathcal{X}_N} \exp\{-\beta H_N(\varphi)\} \lambda_N(\mathrm{d}\varphi). \tag{3.2.7}$$

Given that the torus has a periodic structure, we implement suitable *boundary conditions* following the *Funaki-Spohn trick* as introduced in [89]. Given a linear map (deformation) $F : \mathbb{R}^d \to \mathbb{R}^m$, we define the *Hamiltonian* $H_N^F(\varphi)$ *on the torus* T_N *with the "external deformation"* F by

$$H_N^F(\varphi) = \sum_{x \in T_N} U((\varphi + F)_{\tau_x(A)}). \tag{3.2.8}$$

Here we identify F with the linear map $x \to Fx$ and $\varphi \in \mathcal{X}_N$ with a $(L^N\mathbb{Z})^d$ periodic function and the set T_N with $\mathbb{Z}^d \cap [-\frac{1}{2}(L^N - 1), \frac{1}{2}(L^N - 1)]^d$.

The *finite volume gradient Gibbs measure* $\gamma_{N,\beta}^F$ *under a deformation* F is then defined as

$$\gamma_{N,\beta}^F(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}(F,0)} \exp\left(-\beta H_N^F(\varphi)\right) \lambda_N(\mathrm{d}\varphi), \tag{3.2.9}$$

where $Z_{N,\beta}(F,0)$ is the normalizing *partition function.* A useful generalization of the partition function $Z_{N,\beta}(F,f)$ with a source term $f \in \mathcal{X}_N$ is defined by

$$Z_{N,\beta}(F,f) = \int_{\mathcal{X}_N} \exp\bigl(-\beta H_N^F(\varphi) + (f,\varphi)\bigr) \lambda_N(\mathrm{d}\varphi). \tag{3.2.10}$$

In particular, it characterizes the Gibbs measure $\gamma^F_{N,\beta}$ and will be used to analyse its scaling limit.

While our major long-term objective is the specification of the gradient Gibbs measures with a given deformation as it was done in [89] for the scalar case with convex interactions, in the present paper we will restrict our attention to the analysis of the partition function $Z_{N,\beta}(F,0)$ and the scaling limit of the partition function $Z_{N,\beta}(F,f)$. In particular, we investigate local convexity properties of the functions

$$W_{N,\beta}(F) = -\frac{\ln Z_{N,\beta}(F,0)}{\beta L^{Nd}}. \tag{3.2.11}$$

and of the free energy

$$W_\beta(F) = \lim_{N\to\infty} W_{N,\beta}(F) = -\lim_{N\to\infty} \frac{\ln Z_{N,\beta}(F,0)}{\beta L^{Nd}}. \tag{3.2.12}$$

For the scaling limit of the gradient Gibbs measure we will analyse the Laplace transform

$$\lim_{N\to\infty} \mathbb{E}_{\gamma^F_{N,\beta}} e^{(f_N,\varphi)} = \lim_{N\to\infty} \frac{Z_{N,\beta}(F,f_N)}{Z_{N,\beta}(F,0)} \tag{3.2.13}$$

where $f_N \in \mathcal{X}_N$ is the rescaled discretization $f_N(x) = L^{-N(\frac{d+2}{2})} f(L^{-N}x)$ of a smooth function $f : (\mathbb{R}/\mathbb{Z})^d \to \mathbb{R}^m$ with average zero. The function f_N is a slowly varying field that allows us to examine the long distance behaviour of the random field φ.

Let us remark that when $m = d$, this is the setting for microscopic model of nonlinear elasticity with F representing an affine deformation applied to a solid as will be discussed in detail in Sections 3.3.1 and 3.3.2. In the scalar case, $m = 1$, the model describes the behaviour of a random microscopic interface and the map F actually determines a vector—a macroscopic tilt applied to the discrete interface and the free energy $W_\beta(F)$ then corresponds to the interface free energy/surface tension with a given tilt.

3.2.2 Generalized gradient model

Up to now we considered finite range interactions with support A that is without loss of generality taken to be contained in a discrete cube of side R', $A \subset Q_{R'} = \{0,\dots,R'\}^d$. We introduce the m dimensional space of shifts $\mathcal{V}_{Q_{R'}} = \{(a,\dots,a) \in (\mathbb{R}^m)^{Q_{R'}} : a \in \mathbb{R}^m\}$ and its orthogonal complement $\mathcal{V}^\perp_{Q_{R'}}$ in $(\mathbb{R}^m)^{Q_{R'}}$. General interactions of range R' are thus functions on the $m((R'+1)^d - 1)$-dimensional space $\mathcal{V}^\perp_{Q_{R'}} \simeq (\mathbb{R}^m)^{Q_{R'}}/\sim$ of local field configurations where the equivalence relation $\sim$ identifies configurations that differ only by a constant field. However, for our analysis it is more convenient to use an equivalent formulation with a space of local deformations introduced in terms of higher order derivatives of the fields.

We consider sets of multiindices $\mathcal{I}$ satisfying

$$\{e_i \in \mathbb{R}^d : 1 \le i \le d\} \subset \mathcal{I} \subset \{\alpha \in \mathbb{N}_0^d \setminus \{(0,\dots,0)\} : |\alpha|_\infty \le R'\}. \tag{3.2.14}$$

Moreover we define the specific set $\mathcal{I}_{R'} = \{\alpha \in \mathbb{N}_0^d \setminus \{(0,\dots,0)\} : |\alpha|_\infty \le R'\}$. Note that the case $\mathcal{I} = \{e_1,\dots,e_d\}$ corresponds to nearest neighbour interactions.

We consider the vector space $\mathcal{G} = (R^m)^{\mathcal{I}}$ equipped with the standard scalar product

$$(z,z)_{\mathcal{G}} = \sum_{\alpha\in\mathcal{I}} z_\alpha \cdot z_\alpha, \; z = (z_\alpha)_{\alpha\in\mathcal{I}} \in \mathcal{G}. \tag{3.2.15}$$

We write $\mathcal{G}_{R'}$ if $\mathcal{I} = \mathcal{I}_{R'}$. For any $\varphi \in \mathcal{X}_N$ and any $x \in T_N$, we then use $D\varphi(x)$ to denote *the extended gradient*—the vector $(\nabla^\alpha \varphi(x))_{\alpha \in \mathcal{I}} \in \mathcal{G}$ with $\nabla^\alpha \varphi(x) = \prod_{j=1}^d \nabla_j^{\alpha(j)} \varphi(x)$.

Assuming that $L > R' + 1$ so that the definition of $D\varphi$ does not wrap around the torus, we have the following equivalence.

Lemma 3.2.1. *There exists an isomorphism $\Pi : \mathcal{G}_{R'} \to \mathcal{V}_{Q_{R'}}^\perp$ inducing a one-to-one correspondence between functions on $\mathcal{V}_{Q_{R'}}^\perp$ and those on $\mathcal{G}_{R'}$: for any $U : \mathcal{V}_{Q_{R'}}^\perp \to R'$, there is $\mathcal{U} : \mathcal{G}_{R'} \to \mathbb{R}$ such that $\mathcal{U}(D\psi(0)) = U(\psi)$ for any $\psi \in \mathcal{V}_{Q_{R'}}^\perp$.*

Proof. Both spaces $\mathcal{G}_{R'}$ and $\mathcal{V}_{Q_{R'}}^\perp$ have the same dimension $m((R'+1)^d - 1)$. The isomorphism between them can be explicitly given by the map $\mathcal{V}_{Q_{R'}}^\perp \ni \psi \mapsto D\psi(0) \in \mathcal{G}$. This map is linear and injective ($D\psi_1(0) = D\psi_2(0)$ implies $\psi_1 - \psi_2 \in \mathcal{V}_{Q_{R'}}$). We define Π to be its inverse.

For any $U : \mathcal{V}_{Q_{R'}}^\perp \to \mathbb{R}$, we define $\mathcal{U} : \mathcal{G}_{R'} \to \mathbb{R}$ by $\mathcal{U}(z) = U(\Pi(z))$. Given that Π is an isomorphism, we have $\mathcal{U}(D\psi(0)) = U(\psi)$ for any $\psi \in \mathcal{V}_{Q_{R'}}^\perp$. □

There are obvious generalisations of the previous lemma to index sets $\mathcal{I}$ with the property that if $\alpha \in \mathcal{I}$ and $0 \neq \beta \leq \alpha$ then $\beta \in \mathcal{I}$. In particular a similar statement holds for $A = \{0, e_1, \dots, e_d\}$ and $\mathcal{I} = \{e_1, \dots, e_d\}$.

Let $\mathcal{G}^\nabla$ and $\mathcal{G}^\perp$ be orthogonal subspaces of $\mathcal{G}$ given by $\mathcal{G}^\nabla = \{z \in \mathcal{G} : z_\alpha = 0 \text{ for } |\alpha|_1 \neq 1\}$ and $\mathcal{G}^\perp = \{z \in \mathcal{G} : z_\alpha = 0 \text{ for } |\alpha|_1 = 1\}$, respectively. For any $z \in \mathcal{G}$ let $z^\nabla \in \mathcal{G}^\nabla$ and $z^\perp \in \mathcal{G}^\perp$ be the corresponding projections. We refer to z^∇ as to the *gradient components* of z. Finally, let us observe that the vector space of linear maps $F : \mathbb{R}^d \to \mathbb{R}^m$ can be identified with the md-dimensional space $\mathcal{G}^\nabla$ employing the isomorphism $F \mapsto \overline{F} = DF(x)$ (for any $x \in \mathbb{R}^d$). On $\mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^m) \simeq \mathbb{R}^{m \times d}$ we define the usual Hilbert-Schmidt scalar product by

$$(F, G) = \sum_{i=1}^d Fe_i \cdot Ge_i = \sum_{i=1}^d \sum_{j=1}^m F_{i,s} G_{i,s}. \tag{3.2.16}$$

With this scalar product the isomorphism $F \mapsto \overline{F}$ becomes an isometry and we will often not distinguish between $|F|$ and $|\overline{F}|$.

With the function $\mathcal{U}$ on $\mathcal{G}_{R'}$ corresponding to U on $\mathcal{V}_{Q_{R'}}^\perp$, we get $U(\psi + F) = \mathcal{U}(D\psi(0) + \overline{F})$ for any $\psi \in \mathcal{V}_{Q_{R'}}^\perp$ leading to an alternative expression for the Hamiltonian $H_N^F(\varphi)$,

$$H_N^F(\varphi) = \sum_{x \in T_N} U(\varphi_{\tau_x(A)} + F) = \sum_{x \in T_N} \mathcal{U}(D\varphi(x) + \overline{F}). \tag{3.2.17}$$

Let us introduce the class of interactions $\mathcal{U}$, functions of the extended gradients $D\varphi$ of the fields, for which we will prove our claims that will be, eventually, used to prove Theorems 3.3.1 and 3.3.2.

First, let $\boldsymbol{Q} : \mathcal{G} \to \mathcal{G}$ be a symmetric positive linear operator and let $\mathcal{Q} : \mathcal{G} \to \mathbb{R}$ be the corresponding quadratic form $\mathcal{Q}(z) = (z, \boldsymbol{Q}z)$. For any $\varphi \in \mathcal{X}_N$ and any $x \in T_N$ we can now introduce the quadratic interaction $\mathcal{Q}(D\varphi(x))$ expressed explicitly in terms of the matrix $\big(\boldsymbol{Q}_{\alpha\beta}, \alpha, \beta \in \mathcal{I}\big)$ of the operator $\boldsymbol{Q}$ as

$$\mathcal{Q}(D\varphi(x)) = \sum_{\alpha, \beta \in \mathcal{I}} \nabla^\alpha \varphi(x) \cdot \boldsymbol{Q}_{\alpha\beta} \nabla^\beta \varphi(x). \tag{3.2.18}$$

For any twice differentiable function $\mathcal{U}$ on $\mathcal{G}$ we define the symmetric quadratic form $\mathcal{Q}_\mathcal{U}$ by

$$\mathcal{Q}_\mathcal{U}(z) := D^2\mathcal{U}(0)(z, z). \tag{3.2.19}$$

We will assume that the quadratic form $\mathcal{Q}_{\mathcal{U}}$ satisfies the bounds

$$\omega_0|z|^2 \le \mathcal{Q}_{\mathcal{U}}(z) \le \omega_0^{-1}|z|^2 \quad \text{for all} \quad z \in \mathcal{G} \tag{3.2.20}$$

for some $\omega_0 \in (0,1)$. We will see in Remark 3.3.12 in Section 3.3.2 that for the lower bound the condition

$$\omega_0|z^\nabla|^2 \le \mathcal{Q}_{\mathcal{U}}(z) \le \omega_0^{-1}|z|^2 \quad \text{for all} \quad z \in \mathcal{G} \tag{3.2.21}$$

is sufficient.

We begin the analysis of the general case by extracting the relevant leading low temperature contribution from the partition function and find a formula for the remainder that will be analysed in the following sections. Similarly to [4], we define the function $\overline{\mathcal{U}} : \mathcal{G} \times \mathbb{R}^{d\times m} \to \mathbb{R}$ by

$$\overline{\mathcal{U}}(z,F) = \mathcal{U}(z+\overline{F}) - \mathcal{U}(\overline{F}) - D\mathcal{U}(\overline{F})(z) - \frac{\mathcal{Q}_{\mathcal{U}}(z)}{2}. \tag{3.2.22}$$

It describes the remainder of the Taylor expansion of $\mathcal{U}(z+\overline{F})$ around $\overline{F}$ collecting all third order terms plus the difference $D^2\mathcal{U}(\overline{F})(z,z) - D^2\mathcal{U}(0)(z,z)$ since we want to keep only the quadratic term that does not depend on $\overline{F}$. Notice that the function $\mathcal{V}(z) = \overline{\mathcal{U}}(z,0) = \mathcal{U}(z) - \mathcal{U}(0) - D\mathcal{U}(0)z - \frac{\mathcal{Q}_{\mathcal{U}}(z)}{2}$ is the third order remainder of the Taylor expansion of $\mathcal{U}$ yielding $\mathcal{V}(0) = D\mathcal{V}(0) = D^2\mathcal{V}(0) = 0$.

In terms of the function $\overline{\mathcal{U}}$ the Hamiltonian can be expressed as

$$H_N^F(\varphi) = \sum_{x\in T_N} \mathcal{U}(D\varphi(x)+\overline{F}) = L^{Nd}\mathcal{U}(\overline{F}) + \sum_{x\in T_N}\Big(\overline{\mathcal{U}}(D\varphi(x),F) + \frac{\mathcal{Q}_{\mathcal{U}}(D\varphi(x))}{2}\Big), \tag{3.2.23}$$

where we used that the terms linear in $D\varphi(x)$ cancel because $\sum_{x\in T_N} D\varphi(x) = 0$ in the periodic setting. Using equation (3.2.23) we can rewrite the partition function (3.2.10) as

$$Z_{N,\beta}(F,f) = e^{-\beta L^{Nd}\mathcal{U}(\overline{F})} \int_{\mathcal{X}_N} e^{(f,\varphi)} e^{-\beta\sum_{x\in T_N}(\overline{\mathcal{U}}(D\varphi(x),F) + \frac{\mathcal{Q}_{\mathcal{U}}(D\varphi(x))}{2})} \lambda_N(\mathrm{d}\varphi). \tag{3.2.24}$$

The positive quadratic form $\beta\mathcal{Q}_{\mathcal{U}}$ defines the probabilistic Gaussian measure

$$\mu_\beta(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}}} \exp\big(-\tfrac{\beta}{2}\sum_{x\in T_N}\mathcal{Q}_{\mathcal{U}}(D\varphi(x))\big)\lambda_N(\mathrm{d}\varphi) \tag{3.2.25}$$

with an appropriate normalization factor $Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}}$.

Thus

$$Z_{N,\beta}(F,f) = e^{-\beta L^{Nd}\mathcal{U}(\overline{F})} Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}} \int_{\mathcal{X}_N} e^{(f,\varphi)} e^{-\beta\sum_{x\in T_N}\overline{\mathcal{U}}(D\varphi(x),F)}\, \mu_\beta(\mathrm{d}\varphi) \tag{3.2.26}$$

Finally, rescaling the field by $\sqrt{\beta}$, introducing the Mayer function corresponding to the remainder $\overline{\mathcal{U}}$,

$$\mathcal{K}_{F,\beta,\mathcal{U}}(z) = \exp\big(-\beta\overline{\mathcal{U}}(\tfrac{z}{\sqrt{\beta}},F)\big) - 1, \tag{3.2.27}$$

and using the shorthand $\mu = \mu_1$, we express the partition function $Z_{N,\beta}(F,f)$ in terms of the polymer expansion

$$\begin{aligned} Z_{N,\beta}(F,f) &= e^{-\beta L^{Nd}\mathcal{U}(\overline{F})} Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}} \int_{\mathcal{X}_N} e^{(f,\frac{\varphi}{\sqrt{\beta}})} e^{-\beta\sum_{x\in T_N}\overline{\mathcal{U}}(\frac{D\varphi(x)}{\sqrt{\beta}},F)}\,\mu(\mathrm{d}\varphi) \\ &= e^{-\beta L^{Nd}\mathcal{U}(\overline{F})} Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}} \int_{\mathcal{X}_N} e^{(\frac{f}{\sqrt{\beta}},\varphi)} \prod_{x\in T_N}\left(1+\mathcal{K}_{F,\beta,\mathcal{U}}(D\varphi(x))\right)\mu(\mathrm{d}\varphi) \\ &= e^{-\beta L^{Nd}\mathcal{U}(\overline{F})} Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}} \int_{\mathcal{X}_N} e^{(\frac{f}{\sqrt{\beta}},\varphi)} \sum_{X\subset T_N}\prod_{x\in X}\mathcal{K}_{F,\beta,\mathcal{U}}(D\varphi(x))\,\mu(\mathrm{d}\varphi). \end{aligned} \tag{3.2.28}$$

The integral in the last expression gives the perturbative contribution

$$\mathcal{Z}_{N,\beta}\left(F,\frac{f}{\sqrt{\beta}}\right) = \int_{\mathcal{X}_N} e^{(\frac{f}{\sqrt{\beta}},\varphi)} \sum_{X\subset T_N}\prod_{x\in X}\mathcal{K}_{F,\beta,\mathcal{U}}(D\varphi(x))\,\mu(\mathrm{d}\varphi). \tag{3.2.29}$$

Introducing the perturbative components of the free energy by

$$\mathcal{W}_{N,\beta}(F) = -\frac{\ln \mathcal{Z}_{N,\beta}(F,0)}{L^{dN}} \text{ and } \mathcal{W}_\beta(F) = -\lim_{N\to\infty}\frac{\ln \mathcal{Z}_{N,\beta}(F,0)}{L^{dN}} \tag{3.2.30}$$

we can rewrite the $W_{N,\beta}$ and the free energy W_β defined in (3.2.11) and (3.2.12) as

$$W_{N,\beta}(F) = \mathcal{U}(\overline{F}) + \frac{\mathcal{W}_{N,\beta}(F)}{\beta} - \frac{1}{\beta L^{Nd}} \ln Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}}, \tag{3.2.31}$$

$$W_\beta(F) = \mathcal{U}(\overline{F}) + \frac{\mathcal{W}_\beta(F)}{\beta} - \lim_{N\to\infty}\frac{1}{\beta L^{Nd}} \ln Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}}. \tag{3.2.32}$$

Here, in both expressions the last term is a constant independent of F.

It will be useful to generalise our formulation slightly and, instead of a particular $\mathcal{K}_{F,\beta,\mathcal{U}}$ above, to consider a general function $\mathcal{K}: \mathcal{G} \to \mathbb{R}^m$ and, instead of the quadratic form $\mathcal{Q}_{\mathcal{U}}$ depending on $\mathcal{U}$, to consider a general positive definite quadratic form $\mathcal{Q}$ and define the partition function

$$\mathcal{Z}_N(\mathcal{K},\mathcal{Q},f) = \int_{\mathcal{X}_N} e^{(f,\varphi)} \sum_{X\subset T_N}\prod_{x\in X}\mathcal{K}(D\varphi(x))\,\mu_{\mathcal{Q}}(\mathrm{d}\varphi). \tag{3.2.33}$$

with the Gaussian measure

$$\mu_{\mathcal{Q}}(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}^{\mathcal{Q}}} \exp\bigl(-\tfrac{1}{2}\sum_{x\in T_N}\mathcal{Q}(D\varphi(x))\bigr)\lambda_N(\mathrm{d}\varphi). \tag{3.2.34}$$

Introducing then the free energies

$$\mathcal{W}_N(\mathcal{K},\mathcal{Q}) = -\frac{\ln \mathcal{Z}_N(\mathcal{K},\mathcal{Q},0)}{L^{dN}} \tag{3.2.35}$$

and

$$\mathcal{W}(\mathcal{K},\mathcal{Q}) = -\lim_{N\to\infty}\frac{\ln \mathcal{Z}_N(\mathcal{K},\mathcal{Q},0)}{L^{dN}}, \tag{3.2.36}$$

we readily get

$$\mathcal{W}_{N,\beta}(F) = \mathcal{W}_N(\mathcal{K}_{F,\beta,\mathcal{U}},\mathcal{Q}_{\mathcal{U}}) \text{ and } \mathcal{W}_\beta(F) = \mathcal{W}(\mathcal{K}_{F,\beta,\mathcal{U}},\mathcal{Q}_{\mathcal{U}}). \tag{3.2.37}$$

The key result of this paper consists in a good control of the behaviour of the partition function $\mathcal{Z}_N(\mathcal{K}, \mathcal{Q}, f)$ and thus also $\mathcal{W}_N$ and $\mathcal{W}$ for a class of admissible perturbations $\mathcal{K}$. Introducing first an appropriate space for the functions $\mathcal{K}$, we will later formulate conditions on $\mathcal{U}$ that guarantee that $K_{F,\beta,\mathcal{U}}$ (accompanied with $\mathcal{Q} = \mathcal{Q}_{\mathcal{U}}$) belongs to this space.

Let $\mathcal{Q} : \mathcal{G} \to \mathbb{R}$ be a positive definite quadratic form and $\zeta \in (0,1)$. We define the Banach space $\boldsymbol{E}_{\zeta,\mathcal{Q}}$ consisting of functions $\mathcal{K} : \mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}} \to \mathbb{R}$ such that that the following norm is finite

$$\|\mathcal{K}\|_{\zeta,\mathcal{Q}} = \sup_{z\in\mathcal{G}} \sum_{|\alpha|\le r_0} \frac{1}{\alpha!} |\partial^\alpha \mathcal{K}(z)| e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}(z)}. \tag{3.2.38}$$

We will usually use the abbreviations

$$\boldsymbol{E} = \boldsymbol{E}_{\zeta,\mathcal{Q}}, \quad \|\cdot\|_\zeta = \|\cdot\|_{\zeta,\mathcal{Q}}. \tag{3.2.39}$$

The following theorem then provides bounds for the perturbative free energy.

Theorem 3.2.2. *Fix the spatial dimension d, the number of components m, the range of interaction R', the set of multiindices $\{e_1, \ldots, e_d\} \subset \mathcal{I} \subset \{\alpha \in \mathbb{N}_0^d \setminus \{(0,\ldots,0)\} : |\alpha|_\infty \le R'\}$, real constants $\omega_0 > 0$, $\zeta \in (0,1)$ and an integer $r_0 \ge 3$. For $\mathcal{K} \in \boldsymbol{E}$ let $\mathcal{W}_N(\mathcal{K}, \mathcal{Q})$ be defined by* (3.2.33) *and* (3.2.35).

Then there exist $L_0 \in \mathbb{N}$ such that for every odd integer $L \ge L_0$ there exists a constant $\varrho = \varrho(L) > 0$ with the following properties. For any integer $N \ge 1$ and any quadratic form $\mathcal{Q}$ on $\mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}}$ that satisfies the bounds

$$\omega_0 |z|^2 \le \mathcal{Q}(z) \le \omega_0^{-1} |z|^2 \quad \textit{for all} \quad z \in \mathcal{G}, \tag{3.2.40}$$

the map $\mathcal{K} \mapsto \overline{\mathcal{W}}_N(\mathcal{K})$ defined as $\overline{\mathcal{W}}_N(\mathcal{K}) = \mathcal{W}_N(\mathcal{K}, \mathcal{Q})$ is C^∞ in $B_\varrho(0) \subset \boldsymbol{E}_{\zeta,\mathcal{Q}}$ and its derivatives are bounded independently of N, i.e.,

$$\frac{1}{\ell!} \|D^\ell \overline{\mathcal{W}}_N(\mathcal{K})(\dot{\mathcal{K}}, \ldots, \dot{\mathcal{K}})\| \le C_\ell(L)\, \|\dot{\mathcal{K}}\|_{\zeta,\mathcal{Q}}^\ell \qquad \forall \mathcal{K} \in B_\varrho(0) \quad \forall \ell \in \mathbb{N}. \tag{3.2.41}$$

In particular there exist $\overline{\mathcal{W}} \in C^r(B_\varrho(0))$ and a subsequence $N_n \to \infty$ such that $\overline{\mathcal{W}}_{N_n}$ converges to $\overline{\mathcal{W}}$ for all $r \in \mathbb{N}$ and the derivatives of $\overline{\mathcal{W}}$ are bounded as in (3.2.41).

This is the main technical Theorem of the paper. The main steps of the proof will be summarised in Section 4.2 and it will be eventually proven in Section 4.9.

Its immediate consequence that we will use is the claim concerning smoothness of the function $F \mapsto \mathcal{W}_N(\mathcal{K}_F, \mathcal{Q})$ where $\mathbb{R}^{m\times d} \ni F \mapsto \mathcal{K}_F \in \boldsymbol{E}$ is a function that satisfies suitable conditions and $\mathcal{Q}$ is a fixed quadratic form.

Theorem 3.2.3. *Let d, m, R', $\mathcal{I}$, ω_0, ζ, r_0, $L \ge L_0$, $\varrho = \varrho(L)$ and a fixed $\mathcal{Q}$ be as in Theorem 3.2.2. Let $r_1 \ge 2$ be an integer. Then for each integer $N \ge 1$, each open set $\mathcal{O} \subset \mathbb{R}^{m\times d}$ and any map $\mathcal{O} \ni F \to \mathcal{K}_F \in \boldsymbol{E}_{\zeta,\mathcal{Q}}$ of class C^{r_1} that satisfies the bounds*

$$\sup_{F\in\mathcal{O}} \|\mathcal{K}_F\|_{\zeta,\mathcal{Q}} < \varrho, \tag{3.2.42}$$

$$\sup_{F\in\mathcal{O}} \sum_{|\gamma|\le r_1} \frac{1}{\gamma!} \|\partial_F^\gamma \mathcal{K}_F\|_{\zeta,\mathcal{Q}} < \infty, \tag{3.2.43}$$

the function $F \mapsto \mathcal{W}_N(F) := \mathcal{W}_N(\mathcal{K}_F, \mathcal{Q})$ is in $C^{r_1}(\mathcal{O})$ and the derivatives $|\partial_F^\alpha \mathcal{W}_N(F)|, |\alpha| \le r_1$ can be bounded in terms of L and $\Theta := \sup_{F\in\mathcal{O}} \sum_{|\gamma|\le r_1} \frac{1}{\gamma!} \|\partial_F^\gamma \mathcal{K}_F\|_{\zeta,\mathcal{Q}}$. In particular there exists $\mathcal{W} \in C^{r_1-1,1}(\mathcal{O})$ and a subsequence $N_n \to \infty$ such that $\mathcal{W}_{N_n} \to \mathcal{W}$ in C^{r_1-1}, and the derivatives of $\mathcal{W}$ up to order $r_1 - 1$ as well as the Lipschitz constant of the (r_1-1)-st derivative are bounded in terms of L and Θ.

Proof. The claim follows from Theorem 3.2.2 and the chain rule. □

Now we show that for potentials $\mathcal{U}$ from a reasonable class of functions, the assumptions of the previous theorem hold for the corresponding functions $\mathcal{K}_F = \mathcal{K}_{F,\beta,\mathcal{U}}$ defined as in (3.2.27),

$$\mathcal{K}_{F,\beta,\mathcal{U}}(z) = \exp\left(-\beta\overline{\mathcal{U}}(\frac{z}{\sqrt{\beta}}, F)\right) - 1, \tag{3.2.44}$$

with $\overline{\mathcal{U}}$ defined in terms of $\mathcal{U}$ in (3.2.22) and with $\mathcal{Q} = \mathcal{Q}_{\mathcal{U}}$ defined in (3.2.19).

Proposition 3.2.4. *Let $r_0 \geq 3$ and $r_1 \geq 0$ be integers and assume that*

$$\mathcal{U} \in C^{r_0+r_1}(\mathcal{G}) \tag{3.2.45}$$

Recall that $\mathcal{Q}_{\mathcal{U}}(z) = D^2\mathcal{U}(0)(z,z)$ and assume that

$$\omega_0|z|^2 \leq \mathcal{Q}_{\mathcal{U}}(z) \leq \omega_0^{-1}|z|^2 \tag{3.2.46}$$

for some $\omega_0 \in (0,1)$. Let $0 < \omega \leq \frac{\omega_0}{8}$ and suppose that $\mathcal{U} : \mathcal{G} \to \mathbb{R}$ satisfies the additional conditions

$$\mathcal{U}(z) - D\mathcal{U}(0)z - \mathcal{U}(0) \geq \omega|z|^2 \quad \forall z \in \mathcal{G}, \text{ and} \tag{3.2.47}$$

$$\lim_{t\to\infty} t^{-2}\ln\Psi(t) = 0 \quad \text{where} \quad \Psi(t) := \sup_{|z|\leq t} \sum_{3\leq|\alpha|\leq r_0+r_1} \frac{1}{\alpha!}|\partial^\alpha\mathcal{U}(z)|. \tag{3.2.48}$$

Then there exist $\tilde{\zeta}$ (depending on ω and ω_0), $\delta_0 > 0$ (depending on ω, ω_0 and $\Psi(1)$), C_1 (depending on ω, r_0 and the function Ψ) and Θ (depending on ω, r_0, r_1 and the function Ψ) such for all $\delta \in (0, \delta_0]$ and all $\beta \geq 1$ the map $B_\delta(0) \ni F \mapsto \mathcal{K}_F = \mathcal{K}_{F,\beta,\mathcal{U}} \in \boldsymbol{E} = \boldsymbol{E}_{\tilde{\zeta}}$ is C^{r_1} and satisfies

$$\|\mathcal{K}_F\|_{\tilde{\zeta},\mathcal{Q}_{\mathcal{U}}} \leq C_1(\delta + \beta^{-1/2}) \tag{3.2.49}$$

and

$$\sum_{|\gamma|\leq r_1} \frac{1}{\gamma!}\|\partial_F^\gamma\mathcal{K}_F\|_{\tilde{\zeta},\mathcal{Q}_{\mathcal{U}}} \leq \Theta. \tag{3.2.50}$$

In particular, given $\varrho > 0$ there exists $\delta > 0$ and $\beta_0 \geq 1$ (both depending on ω, ω_0, and the function Ψ) such that for all $\beta \geq \beta_0$ and all $F \in B_\delta$ we have (3.2.50) *and*

$$\|\mathcal{K}_F\|_{\tilde{\zeta},\mathcal{Q}_{\mathcal{U}}} \leq \varrho. \tag{3.2.51}$$

The proof is postponed to Section 3.2.3. It is shown there that we may take $\tilde{\zeta} = \frac{\omega\omega_0}{2}$. Explicit expressions for δ_0, C_1 and Θ are given in (3.2.89), (3.2.94) and (3.2.111), respectively. The proof also shows that dependence of δ_0 and C_1 on the number of derivatives of $\mathcal{U}$ can be slightly improved. If we set $\Psi_r(t) := \sup_{|z|\leq t}\sum_{3\leq|\alpha|\leq r}\frac{1}{\alpha!}|\partial^\alpha\mathcal{U}(z)|$ then δ_0 depends on ω, ω_0 and $\Psi_3(1)$ while C_1 depends on ω, r_0 and the function Ψ_{r_0}.

Remark 3.2.5. *Let us state some remarks concerning this result.*

1. *We may assume without loss of generality that $\mathcal{U}(0) = D\mathcal{U}(0) = 0$ since both the Mayer function $\overline{\mathcal{U}}$ and assumptions of the proposition are invariant under adding an affine function to $\mathcal{U}$. The lower growth assumption* (3.2.47) *is then much weaker than the corresponding condition in [4]. Assumption* (3.2.47) *only requires any quadratic bound from below while in [4] the condition $\mathcal{U}(z) \geq \frac{\mathcal{Q}(z)}{2} - \varepsilon|z|^2$ for some small $\varepsilon > 0$ was imposed, i.e., the potential was assumed to grow almost as fast as the quadratic approximation at* 0.

2. *Let us emphasize that we do not require that* 0 *is a minimum of the potential. In fact the theorem applies by a simple translation for all points* z_0 *such that*

$$\mathcal{U}(z) - D\mathcal{U}(z_0)(z - z_0) - \mathcal{U}(z_0) \geq \omega|z - z_0|^2 \tag{3.2.52}$$

for some $\omega \geq 0$. *In particular from Theorem 3.2.6 below we recover the known result that the surface tension is strictly convex everywhere for strictly convex potentials.*

3. *The proposition can be generalized to some singular potentials, e.g., it is possible to consider potentials* $\mathcal{U} + \mathcal{V}$ *where* $\mathcal{U}$ *is as before and* $\mathcal{V} : \mathcal{G} \to \mathbb{R} \cup \{\infty\}$ *satisfies* $\mathcal{V}(z) \geq 0$, $0 \notin \operatorname{supp} \mathcal{V}$, *and* $e^{-\mathcal{V}} \in C^{r_0+r_1}$. *The one dimensional potential* $\mathcal{V}(x) = \eta(x)|x-2|^{-1}$ *where* $\eta \in C_c^\infty((1,\infty))$ *satisfies* $e^{-\mathcal{V}(x)} \in C^\infty(\mathbb{R})$, *hence non-trivial examples for such potentials with singularities exist.*

   Let us briefly indicate the necessary extensions to prove this result. Suppose that $\varepsilon > 0$ *is chosen such that* $\operatorname{dist}(0, \operatorname{supp} \mathcal{V}) \geq \varepsilon$. *We choose* $\delta_0 \leq \varepsilon/2$. *On the complement of the support of* $\overline{\mathcal{V}}$ *we can argue as in the proof of Proposition 3.2.4 below. If* $(z/\sqrt{\beta}, F)$ *is in the support of* $\overline{\mathcal{V}}$ *and* $|F| \leq \delta_0$ *we conclude that* $|z| \geq \varepsilon\sqrt{\beta}/2$. *In this regime we use the estimate*

$$\left| e^{-\beta\overline{\mathcal{V}}(\frac{z}{\sqrt{\beta}},F)-\beta\overline{\mathcal{U}}(\frac{z}{\sqrt{\beta}},F)} - 1 \right|_{T_{z,F}} \leq \left| e^{-\beta\overline{\mathcal{V}}(\frac{z}{\sqrt{\beta}},F)} \right|_{T_{z,F}} \left| e^{-\beta\overline{\mathcal{U}}(\frac{z}{\sqrt{\beta}},F)} \right|_{T_{z,F}} + |1|_{T_{z,F}} \tag{3.2.53}$$

   where $|\cdot|_{T_{z,F}}$ *is defined in* (3.2.97) *below. Then the first term can be controlled by the assumption on* $\mathcal{V}$ *and the second term is bounded in* (3.2.110). *The condition* $|z| \geq \varepsilon\sqrt{\beta}/2$ *implies that when multiplied with the weight of the* $\|\cdot\|_{\boldsymbol{E}}$*-norm both summands are exponentially small in* β.

Theorem 3.2.6. *Under the assumptions of Proposition 3.2.4 with* $r_1 \geq 2$ *there is a* $\beta_0 > 0$ *and* $\delta_0 > 0$ *such that the free energies* $W_{N,\beta}|_{B_{\delta_0}(0)}$ *are* C^{r_1} *and uniformly convex for* $\beta \geq \beta_0$, *in particular* $D^2 W_{N,\beta}(F)(\dot{F},\dot{F}) \geq \frac{\omega_0}{4}|\dot{F}|^2$. *Also every limit* $W_\beta = \lim_{\ell\to\infty} W_{N_\ell,\beta}$ *is uniformly convex.*

Proof. Proposition 3.2.4 and Theorem 3.2.3 imply together that there are constants β_1 and δ_1 such that $\mathcal{W}_{N,\beta}|_{B_\delta(0)}$ is uniformly C^{r_1} for $\beta \geq \beta_1$ and $\delta \leq \delta_1$. This means in particular that there is a constant $\Xi > 0$ independent of β and δ such that $|D^2\mathcal{W}_{N,\beta}(\dot{F},\dot{F})| \leq \Xi|\dot{F}|^2$ in $B_\delta(0)$ for $\beta \geq \beta_1$ and $\delta \leq \delta_1$. The bound (3.2.48) on the third derivative of $\mathcal{U}$ implies that there is a $\delta_2 > 0$ such that for $\delta \leq \delta_2$ and $F \in B_\delta(0)$

$$|D^2\mathcal{U}(\overline{F})(z,z) - \mathcal{Q}_{\mathcal{U}}(z)| = |D^2\mathcal{U}(\overline{F})(z,z) - D^2\mathcal{U}(0)(z,z)| \leq \frac{\omega_0}{2}|z|^2 \tag{3.2.54}$$

and thus

$$D^2\mathcal{U}(\overline{F})(z,z) \geq \frac{\omega_0}{2}|z|^2. \tag{3.2.55}$$

Let $\beta_2 = 4\Xi/\omega_0$. Then for $\beta \geq \max(\beta_1,\beta_2)$, $\delta \leq \min(\delta_1,\delta_2)$ and $F \in B_\delta(0)$

$$D^2 W_{N,\beta}(F)(\dot{F},\dot{F}) = D^2\mathcal{U}(\overline{F})(\overline{\dot{F}},\overline{\dot{F}}) + \frac{D^2\mathcal{W}_\beta(F)}{\beta}(\dot{F},\dot{F}) \geq \frac{\omega_0}{2}|\overline{\dot{F}}|^2 - \frac{\Xi}{4\Xi/\omega_0}|\dot{F}|^2 \geq \frac{\omega_0}{4}|\dot{F}|^2 \tag{3.2.56}$$

The assertion for the limit $\mathcal{W}_\beta$ follows from the fact that the pointwise limit of uniformly convex functions is uniformly convex. □

Finally we address the scaling limit of the model. This is a statement about the Laplace transform of the measure with density $\sum_{X\subset T_N}\prod_{x\in X}\mathcal{K}(D\varphi(x))\,\mu_{\mathcal{Q}}(\mathrm{d}\varphi)/\mathcal{Z}(\mathcal{K},\mathcal{Q},0)$.

Theorem 3.2.7. *Fix the spatial dimension d, the number of components m, the range of interaction R', the set of multiindices $\{e_1,\ldots,e_d\}\subset\mathcal{I}\subset\{\alpha\in\mathbb{N}_0^d\setminus\{(0,\ldots,0)\}:\ |\alpha|_\infty\le R'\}$, real constants $\omega_0>0$, $\zeta\in(0,1)$ and an integer $r_0\ge 3$. Let $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$. For $f\in C^\infty(\mathbb{T}^d,\mathbb{R}^m)$ with $\int f=0$ we define $f_N\in\mathcal{V}_N$ by $f_N(x)=L^{-N\frac{d+2}{2}}f(L^{-N}x)$.*

For $\mathcal{K}\in\boldsymbol{E}$ let $\mathcal{Z}_N(\mathcal{K},\mathcal{Q},f_N)$ be defined by (3.2.33). Then there exist $L_0\in\mathbb{N}$ such that for every odd integer $L\ge L_0$ there exists a constant $\varrho=\varrho(L)>0$ with the following properties. For any quadratic form $\mathcal{Q}$ on $\mathcal{G}=(\mathbb{R}^m)^{\mathcal{I}}$ that satisfies the bounds

$$\omega_0|z|^2\le\mathcal{Q}(z)\le\omega_0^{-1}|z|^2\quad\textit{for all}\quad z\in\mathcal{G},\tag{3.2.57}$$

and any $\mathcal{K}\in B_\varrho(0)\subset\boldsymbol{E}_{\zeta,\mathcal{Q}}$ there is a subsequence $N_\ell\to\infty$ and a matrix $\boldsymbol{q}\in\mathbb{R}_{\mathrm{sym}}^{(m\times d)\times(m\times d)}$ such that for all $f\in C^\infty(\mathbb{T}^d,\mathbb{R}^m)$

$$\lim_{\ell\to\infty}\frac{\mathcal{Z}(\mathcal{K},\mathcal{Q},f_{N_\ell})}{\mathcal{Z}(\mathcal{K},\mathcal{Q},0)}=e^{\frac{1}{2}(f,\mathcal{C}_{\mathbb{T}^d}f)}\tag{3.2.58}$$

where $\mathcal{C}_{\mathbb{T}^d}$ is the inverse of the operator $\mathcal{A}_{\mathbb{T}^d}$ acting on $u\in H^1((\mathbb{R}/\mathbb{Z})^d,\mathbb{R}^m)$ with $\int u=0$ by

$$(\mathcal{A}_{\mathbb{T}^d}u)_s=-\sum_{t=1}^{m}\sum_{i,j=1}^{d}(\boldsymbol{Q}-\boldsymbol{q})_{i,j;s,t}\partial_i\partial_j u_t.\tag{3.2.59}$$

Here $\boldsymbol{Q}$ is the operator associated to the quadratic form $\mathcal{Q}$ via (3.2.18). The identity (3.2.59) states in particular that the operator $\mathcal{A}$ depends only on the restriction of $\boldsymbol{Q}$ to $\mathcal{G}^\nabla$ and for ease of notation we identify i with the multiindex e_i.

Remark 3.2.8.

1. *Note that the rescaling $L^{-\frac{Nd}{2}}$ would correspond to a central limit law behaviour of the random field. Due to the strong correlations we need to use the stronger rescaling with $L^{-N(\frac{d+2}{2})}$. One easily sees that the scaling limit of the gradient field $\nabla\varphi$ involves the central limit scaling, cf. e.g. [33] and [128].*

2. *Note that the limiting covariance is dominated by the gradient-gradient contribution of the interaction while the higher order terms are not directly present, see also [128]. In other words, the limiting covariance $\mathcal{C}$ depends only on the action of $\boldsymbol{Q}$ on the subspace $\mathcal{G}^\nabla$, defined after Lemma 3.2.1, which can be identified with $\mathbb{R}^{m\times d}$. There might be an implicit dependence on the higher order terms through the matrix $\boldsymbol{q}$. This behaviour does not come as a surprise because it is already present in the Gaussian setting where $\mathcal{K}=0$. The higher order terms can change the local correlation structure. They have, however, little influence on the long distance correlation because roughly speaking their long wave Fourier modes are very small and decay with $|p|^{|\alpha|}$ with $|\alpha|\ge 3$ compared to $|p|^2$ for the gradient-gradient interaction.*

Again, the abstract Theorem 3.2.7 for the Laplace transform of perturbations of Gaussian measures has a concrete counterpart for the Gibbs measures of generalized gradient models. Recall that the Gibbs measure $\gamma_{N,\beta}^{F,\mathcal{U}}$ with tilt F was defined in (3.2.9) where the Hamiltonian is given by (3.2.17) in terms of $\mathcal{U}$.

Theorem 3.2.9. *Assume that $\mathcal{U}$ satisfies the assumptions of Proposition 3.2.4 with $r_0 = 3$ and $r_1 = 0$. Let $\beta \geq 1$ and $F \in B_\delta(0)$ be such that (3.2.51) holds with ϱ as in Theorem 3.2.2. Then there is a subsequence (N_ℓ) and a matrix $\boldsymbol{q} \in \mathbb{R}^{(m\times d)\times(m\times d)}_{\mathrm{sym}}$ such that for $f \in C^\infty(\mathbb{T}^d, \mathbb{R}^m)$ with $\int f = 0$ and $f_N(x) = L^{-N\frac{d+2}{2}} f(L^{-N}x)$*

$$\lim_{\ell\to\infty} \mathbb{E}_{\gamma^{F,\mathcal{U}}_{N_\ell,\beta}} e^{(f_{N_\ell},\varphi)} = e^{\frac{1}{2\beta}(f,\mathcal{C}_{\mathbb{T}^d} f)} \tag{3.2.60}$$

where $\mathcal{C}_{\mathbb{T}^d}$ is the inverse of the operator $\mathcal{A}_{\mathbb{T}^d}$ acting on $u \in H^1((\mathbb{R}/\mathbb{Z})^d, \mathbb{R}^d)$ with $\int u = 0$ by

$$(\mathcal{A}_{\mathbb{T}^d} u)_s = -\sum_{t=1}^{m}\sum_{i,j=1}^{d} (\boldsymbol{Q}_{\mathcal{U}} - \boldsymbol{q})_{i,j;s,t}\partial_i\partial_j u_t. \tag{3.2.61}$$

Proof. Combining (3.2.28), (3.2.29), and (3.2.33) we get

$$\mathbb{E}_{\gamma^{F,\mathcal{U}}_{N,\beta}} e^{(f_N,\varphi)} = \frac{Z_{N,\beta}(F, f_N)}{Z_{N,\beta}(F,0)} = \frac{\mathcal{Z}_{N,\beta}(F, \frac{f}{\sqrt{\beta}})}{\mathcal{Z}_{N,\beta}(F,0)} = \frac{\mathcal{Z}(\mathcal{K}_{F,\beta,\mathcal{U}}, \mathcal{Q}_{\mathcal{U}}, \frac{f_N}{\sqrt{\beta}})}{\mathcal{Z}(\mathcal{K}_{F,\beta,\mathcal{U}}, \mathcal{Q}_{\mathcal{U}}, 0)}. \tag{3.2.62}$$

The assumptions ensure that Theorem 3.2.7 can be applied which implies the claim. □

3.2.3 Embedding of the initial perturbation

Proof of Proposition 3.2.4. The main point is to obtain the additional factor $\beta^{-1/2}+\delta$ in (3.2.49) which can be made as small as desired by taking δ small and β large. This factor essentially comes from the third order Taylor expansion. We may assume that $\mathcal{U}(0) = D\mathcal{U}(0) = 0$ since the second and higher order derivatives of $\mathcal{U}$ (and thus also the function $\overline{\mathcal{U}}$) and the assumptions in Proposition 3.2.4 are invariant under addition of an affine function to $\mathcal{U}$. The rest of the argument is then essentially an exercise in estimating polynomials and their exponentials. Observe that for functions $f \in C^{r_0}(\mathcal{G})$ the norms $|f|_{T_z}$ introduced in Appendix A amount to

$$|f|_{T_z} = \sum_{|\alpha|\leq r_0} \frac{1}{\alpha!}\left|\partial_z^\alpha f(z)\right| \tag{3.2.63}$$

(see Example 4.A.8 and equation (4.A.43)).

The proof of Proposition 3.2.4 can be split into the following steps:

Step 1. For any $f \in C^{r_0}(\mathcal{G})$ we have

$$|e^f|_{T_z} \leq e^{f(z)}(1 + |f|_{T_z})^{r_0} \tag{3.2.64}$$

and

$$|e^f - 1|_{T_z} \leq \max(e^{f(z)}, 1)(1 + |f|_{T_z})^{r_0}\, |f|_{T_z}. \tag{3.2.65}$$

We first note that for $f_1, f_2 \in C^{r_0}(\mathcal{G})$ we have $|f_1 f_2|_{T_z} \leq |f_1|_{T_z}\, |f_2|_{T_z}$. This follows abstractly from Proposition 4.A.9 and Example 4.A.8 in the appendix. Alternatively one can easily verify this by a direct calculation using that the (truncated) product of Taylor polynomials is the Taylor polynomial of the product. To prove (3.2.64) we set $\tilde{f}(y) = f(y) - f(z)$. Then $e^{f(y)} = e^{f(z)}e^{\tilde{f}(y)}$. Since $\tilde{f}(z) = 0$ the r_0-th order Taylor polynomial of $e^{\tilde{f}}$ at z agrees with the Taylor polynomial of $\sum_{m=0}^{r_0} \frac{1}{m!}(\tilde{f})^m$. By the triangle inequality and the product property we get

$$|e^{\tilde{f}}|_{T_z} \leq \sum_{r=0}^{r_0} \frac{1}{r!}|\tilde{f}|^r_{T_z} \leq (1 + |\tilde{f}|_{T_z})^{r_0} \leq (1 + |f|_{T_z})^{r_0}. \tag{3.2.66}$$

This finishes the proof of (3.2.64). Now (3.2.65) follows from the identity

$$e^f - 1 = \int_0^1 e^{\tau f} f \,\mathrm{d}\tau. \tag{3.2.67}$$

Jensen's inequality and the product property.

We will now use the claims of Step 1 for

$$f(z) = \mathcal{U}_\beta(z, F) = \beta \overline{\mathcal{U}}(\frac{z}{\sqrt{\beta}}, F). \tag{3.2.68}$$

Step 2. For $\beta \geq 1$ and $|F| \leq \delta \leq 1$, we have

$$|\mathcal{U}_\beta(\cdot, F)|_{T_z} \leq (\beta^{-1/2} + \delta)\widetilde{\Psi}(|z|) \tag{3.2.69}$$

where

$$\widetilde{\Psi}(t) := 3(1+t)^3\, \Psi(t+1). \tag{3.2.70}$$

Actually, we show a slightly stronger bound,

$$|\mathcal{U}_\beta(\cdot, F)|_{T_z} \leq \Big[3(1 + |z|_\infty^2)|\overline{F}| + (1 + |z|_\infty)^3 \beta^{-1/2}\Big]\, \Psi\Big(\frac{|z|}{\sqrt{\beta}} + \delta\Big). \tag{3.2.71}$$

Let us remark that in this proof D refers as usual to total derivatives and ∂ to partial derivatives. Without reference to z or F, the derivatives $\partial\mathcal{U}$ (or $\partial_i\mathcal{U}$) and $D\mathcal{U}$ refer to the derivatives of the function $\mathcal{U}$ evaluated at the corresponding argument while $\partial_{z_i}\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})$ and $D_z\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})$ refer to the derivatives of the map $z \to \mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})$. Clearly $\partial_{z_i}\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F}) = \frac{1}{\sqrt{\beta}}\partial_i\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})$ and $\partial_{F_i}\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F}) = \partial_i\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})$.

For derivatives of the 3$^{\text{rd}}$ or higher order we use that $\partial_z^\alpha \mathcal{U}_\beta(z, F) = \beta^{1-\frac{|\alpha|}{2}}\, \partial^\alpha \mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})$ which yields

$$\sum_{3 \leq |\alpha| \leq r_0} \frac{1}{\alpha!}|\partial_z^\alpha \mathcal{U}_\beta(z, F)| \leq \beta^{-1/2}\, \Psi(\frac{|z|}{\sqrt{\beta}} + \delta). \tag{3.2.72}$$

To estimate the lower order terms we use the third order Taylor expansion in z. This yields

$$\begin{aligned}
\mathcal{U}_\beta(z, F) &= \frac{1}{2}D^2\mathcal{U}(\overline{F})(z, z) - \frac{1}{2}D^2\mathcal{U}(0)(z, z) + \beta^{-1/2}\int_0^1 \frac{(1-\tau)^2}{2} D^3\mathcal{U}(\overline{F} + \frac{\tau z}{\sqrt{\beta}})(z, z, z)\,\mathrm{d}\tau,\\
D_z\mathcal{U}_\beta(z, F)(\dot z) &= D^2\mathcal{U}(\overline{F})(z, \dot z) - D^2\mathcal{U}(0)(z, \dot z) + \beta^{-1/2}\int_0^1 (1-\tau) D^3\mathcal{U}(\overline{F} + \frac{\tau z}{\sqrt{\beta}})(z, z, \dot z)\,\mathrm{d}\tau,\\
D_z^2\mathcal{U}_\beta(z, F)(\dot z_1, \dot z_2) &= D^2\mathcal{U}(\overline{F})(\dot z_1, \dot z_2) - D^2\mathcal{U}(0)(\dot z_1, \dot z_2) + \beta^{-1/2}\int_0^1 D^3\mathcal{U}(\overline{F} + \frac{\tau z}{\sqrt{\beta}})(z, \dot z_1, \dot z_2)\,\mathrm{d}\tau.
\end{aligned} \tag{3.2.73}$$

Using further the bound

$$|D^2\mathcal{U}(\overline{F})(\dot z_1, \dot z_2) - D^2\mathcal{U}(0)(\dot z_1, \dot z_2)| \leq \int_0^1 D^3\mathcal{U}(\tau\overline{F})(\overline{F}, \dot z_1, \dot z_2)\,\mathrm{d}\tau \tag{3.2.74}$$

combined with

$$\frac{1}{3!}|D^3\mathcal{U}(\tau\overline{F})(\overline{F},z,z)| \le \sum_{|\alpha|=3}\frac{1}{\alpha!}\Big|\partial^\alpha\mathcal{U}(\tau\overline{F})\Big|\,|z|_\infty^2\,|\overline{F}|_\infty, \tag{3.2.75}$$

as well as

$$\begin{aligned}\frac{1}{3!}\Big|D^3\mathcal{U}(\overline{F}+\frac{\tau z}{\sqrt{\beta}})(z,z,z)\Big| &\le \frac{1}{3!}\sum_{i_1,i_2,i_3=1}^{\dim\mathcal{G}}\Big|\partial_{i_1}\partial_{i_2}\partial_{i_3}\mathcal{U}(\overline{F}+\frac{\tau z}{\sqrt{\beta}})\Big|\,|z|_\infty^3\\ &= \sum_{|\alpha|=3}\frac{1}{\alpha!}\Big|\partial^\alpha\mathcal{U}(\overline{F}+\frac{\tau z}{\sqrt{\beta}})\Big|\,|z|_\infty^3,\end{aligned} \tag{3.2.76}$$

with $\int_0^1\frac{(1-\tau)^2}{2}d\tau=\frac{1}{3!}$, we deduce that

$$|\mathcal{U}_\beta(z,F)| \le (3|z|_\infty^2\,|\overline{F}|_\infty+|z|_\infty^3\beta^{-1/2})\,\Psi(\frac{|z|}{\sqrt{\beta}}+\delta). \tag{3.2.77}$$

Reasoning similarly for the first and second derivatives of $\mathcal{U}_\beta$ we obtain (3.2.71). Since $|\overline{F}|_\infty \le |\overline{F}| = |F|$ we deduce (3.2.69).

Step 3. There exist $\delta_0>0$ such that

$$-\,\mathcal{U}_\beta(z,F) \le \frac{1}{2}\mathcal{Q}_{\mathcal{U}}(z)-\frac{\omega}{2}\,|z|^2 \qquad \forall\ \beta\ge 1,\ z\in\mathcal{G},\ F\in B_{\delta_0}(0). \tag{3.2.78}$$

Using the definitions (3.2.22) and (3.2.68), we need to show that

$$\beta\big(\mathcal{U}(\overline{F}+\frac{z}{\sqrt{\beta}})-\mathcal{U}(\overline{F})-D\mathcal{U}(\overline{F})(\frac{z}{\sqrt{\beta}})\big) \ge \frac{\omega}{2}\,|z|^2. \tag{3.2.79}$$

For $F=0$ this follows directly from the assumption (3.2.47),

$$\beta\big(\mathcal{U}(\frac{z}{\sqrt{\beta}})-\mathcal{U}(0)-D\mathcal{U}(0)(\frac{z}{\sqrt{\beta}})\big) \ge \beta\omega\,\left|\frac{z}{\sqrt{\beta}}\right|^2 = \omega\,|z|^2 \ge \frac{\omega}{2}\,|z|^2. \tag{3.2.80}$$

This can be extended to the case when F is small if compared with $z/\sqrt{\beta}$. On the other hand, if F is comparable or bigger than $z/\sqrt{\beta}$, we can rely on the third order Taylor expansion around 0.

Indeed, consider first the case when $\frac{z}{\sqrt{\beta}}$ is large. Let $\kappa:=\frac{9}{\omega\omega_0}\ge 9$ and assume that $\frac{|z|}{\sqrt{\beta}}\ge\kappa\delta$ and $|\overline{F}|=|F|\le\delta$. The estimate (3.2.47) with the assumption $\mathcal{U}(0)=D\mathcal{U}(0)=0$ implies

$$\beta\mathcal{U}(\overline{F}+\frac{z}{\sqrt{\beta}}) \ge \omega\beta\big|\overline{F}+\frac{z}{\sqrt{\beta}}\big|^2 \ge \omega\beta\big(\frac{|z|}{\sqrt{\beta}}-|\overline{F}|\big)^2 \ge \omega\big(1-\frac{1}{\kappa}\big)^2|z|^2. \tag{3.2.81}$$

For z and F as before and using $D\mathcal{U}(0)=0$, $D^2\mathcal{U}(0)=\mathcal{Q}_{\mathcal{U}}$, and the third order Taylor expansion, we bound

$$\beta\Big|D\mathcal{U}(\overline{F})(\frac{z}{\sqrt{\beta}})\Big| \le \beta\Big|D^2\mathcal{U}(0)(\overline{F},\frac{z}{\sqrt{\beta}})\Big|+\sup_{|\xi|\le|\overline{F}|}\frac{\beta}{2}\Big|D^3\mathcal{U}(\xi)(\overline{F},\overline{F},\frac{z}{\sqrt{\beta}})\Big| \tag{3.2.82}$$

Evaluating the first term as

$$\begin{aligned}\beta\Big|D^2\mathcal{U}(0)(\overline{F},\frac{z}{\sqrt{\beta}})\Big| &\le \beta\big|D^2\mathcal{U}(0)(\overline{F},\overline{F})\big|^{1/2}\Big|D^2\mathcal{U}(0)(\frac{z}{\sqrt{\beta}},\frac{z}{\sqrt{\beta}})\Big|^{1/2}\\ &\le \frac{\beta}{\omega_0}|F|\frac{|z|}{\sqrt{\beta}} \le \frac{\beta}{\kappa\omega_0}(\frac{|z|}{\sqrt{\beta}})^2 = \frac{|z|^2}{\kappa\omega_0}\end{aligned} \tag{3.2.83}$$

and the second term, assuming that $3\Psi(1)\delta \le 1$, as

$$\sup_{|\xi|\le|\overline{F}|}\frac{\beta}{2}\Big|D^3\mathcal{U}(\xi)(\overline{F},\overline{F},\frac{z}{\sqrt{\beta}})\Big| \le 3\beta\Psi(1)\delta\frac{1}{\kappa}\Big|\frac{z}{\sqrt{\beta}}\Big|^2 \le \frac{|z|^2}{\kappa}, \tag{3.2.84}$$

we get the bound

$$\beta\Big|D\mathcal{U}(\overline{F})(\frac{z}{\sqrt{\beta}})\Big| \le \big(1+\frac{1}{\omega_0}\big)\frac{|z|^2}{\kappa}. \tag{3.2.85}$$

Similarly, assuming again that $\delta \le \frac{1}{3\Psi(1)}$, we get

$$\beta|\mathcal{U}(\overline{F})| \le \beta\big|D^2\mathcal{U}(0)(\overline{F},\overline{F})\big| + \sup_{|\xi|\le|\overline{F}|}\frac{\beta}{2}\big|D^3\mathcal{U}(\xi)(\overline{F},\overline{F},\overline{F})\big| \le \beta\left(\frac{\delta^2}{\omega_0}+3\Psi(1)\delta^3\right) \le (1+\frac{1}{\omega_0})\frac{|z|^2}{\kappa^2}. \tag{3.2.86}$$

Combining the bounds (3.2.81), (3.2.85) and (3.2.86) imply (3.2.78) once

$$\big(1+\frac{1}{\omega_0}\big)\frac{1}{\kappa}(1+\frac{1}{\kappa}) \le \omega\big((1-\frac{1}{\kappa})^2-\frac{1}{2}\big). \tag{3.2.87}$$

For this to be true, it suffices when

$$2(1+\frac{1}{\kappa}) \le \kappa\omega_0\omega\big((1-\frac{1}{\kappa})^2-\frac{1}{2}\big). \tag{3.2.88}$$

Indeed, with the choice $\kappa = \frac{9}{\omega\omega_0} \ge 9$, the left hand side is bounded from above by $2(1+1/9) = 20/9$ while the right hand side from below by $9((8/9)^2-1/2) = 47/18 > 20/9$.

It remains to consider the case $|z|/\sqrt{\beta} < \kappa\delta$. We choose

$$\delta_0 := \min\Big(\frac{1}{1+\kappa},\frac{3\omega_0}{16\kappa\Psi(1)}\Big). \tag{3.2.89}$$

With $|z|/\sqrt{\beta} < \kappa\delta$ and $\delta \le \delta_0$, we get $\frac{|z|}{\sqrt{\beta}}+\delta \le (\kappa+1)\delta \le 1$. Hence, from (3.2.77) with $|z|_\infty \le |z|$, $\kappa \ge 9$, and assuming $\omega \le \frac{\omega_0}{2}$, we get

$$|\mathcal{U}_\beta(z,F)| \le (3+\kappa)\delta\Psi(1)|z|^2 \le \frac{4}{3}\kappa\delta\Psi(1)|z|^2 \le \frac{1}{4}\omega_0|z|^2 \le \frac{1}{2}(\omega_0-\omega)|z|^2 \le \frac{1}{2}\mathcal{Q}_{\mathcal{U}}(z)-\frac{1}{2}\omega|z|^2. \tag{3.2.90}$$

Thus (3.2.78) holds for this choice of δ_0 and $|z|/\sqrt{\beta} \le \kappa\delta$. Finally for $\delta \le \delta_0$ also the condition $\delta \le \frac{1}{3\Psi(1)}$ is satisfied and thus (3.2.78) holds for all z and all $F \in B_{\delta_0}(0)$.

Step 4. Let $0 < \delta < \delta_0$ with $\delta_0 \le 1$ given by (3.2.89). Then, with $\tilde{\zeta} = \frac{\omega\omega_0}{2}$, we have

$$\|e^{-\mathcal{U}_\beta(\cdot,F)}-1\|_{\tilde{\zeta},\mathcal{Q}_{\mathcal{U}}} \le C_1(\delta+\beta^{-1/2}) \qquad \forall\ \beta \ge 1,\ F \in B_\delta(0). \tag{3.2.91}$$

Combining (3.2.65), (3.2.69) and (3.2.78) and using that $\beta^{-1/2} + \delta \le 2$ we get

$$|e^{-\mathcal{U}_\beta(\cdot,F)} - 1|_{T_z} \le e^{\frac{1}{2}\mathcal{Q}_\mathcal{U}(z) - \frac{\omega}{2}|z|^2}(\beta_0^{-1/2} + \delta_0)\, \widetilde{\Psi}(|z|)\, (1 + 2\widetilde{\Psi}(|z|))^{r_0}. \tag{3.2.92}$$

Given that $\frac{1}{2}\frac{\omega\omega_0}{2}\mathcal{Q}_\mathcal{U}(z) \le \frac{1}{4}\omega|z|^2$ we have

$$e^{-\frac{1}{2}(1-\frac{\omega\omega_0}{2})\mathcal{Q}_\mathcal{U}(z)} \le e^{-\frac{1}{2}\mathcal{Q}_\mathcal{U}(z)}\, e^{\frac{1}{4}\omega|z|^2}. \tag{3.2.93}$$

Thus multiplying (3.2.92) by the weight $e^{-\frac{1}{2}(1-\frac{\omega\omega_0}{2})\mathcal{Q}_\mathcal{U}(z)}$ and setting

$$C_1 = \sup_{t\ge 0} e^{-\frac{\omega}{4}t^2}\, \widetilde{\Psi}(t)(1 + 2\widetilde{\Psi}(t))^{r_0} < \infty \quad \text{with} \quad \widetilde{\Psi}(t) = 3(1+t)^3\,\Psi(t+1), \tag{3.2.94}$$

we get (3.2.91), thus completing Step 4.

The estimates (3.2.92) and (3.2.93) imply that the assumptions of Lemma 3.2.10 below hold. This shows that $F \to \mathcal{K}_F$ is continuous. Together with (3.2.91) this ends the proof for $r_1 = 0$.

It remains to show the bound (3.2.50) for the derivatives with respect to F. Considering first the case $|\gamma| = 1$, we need to estimate

$$\frac{\partial}{\partial F_i} e^{-\mathcal{U}_\beta(z,F)} = -e^{-\mathcal{U}_\beta(z,F)}\, \frac{\partial}{\partial F_i}\mathcal{U}_\beta(z,F). \tag{3.2.95}$$

By the chain and product rules, the derivatives ∂_z^α of this expression exist for $|\alpha| \le r_0$. Moreover by (3.2.64) and the product property of the $|\cdot|_{T_z}$ norm,

$$\Big|\frac{\partial}{\partial F_i} e^{-\mathcal{U}_\beta(\cdot,F)}\Big|_{T_z} \le e^{-\mathcal{U}_\beta(z,F)}(1 + |\mathcal{U}_\beta(\cdot,F)|_{T_z})^{r_0}\, \Big|\frac{\partial}{\partial F_i}\mathcal{U}_\beta(z,F)\Big|_{T_z}. \tag{3.2.96}$$

Then it remains to bound $\Big|\frac{\partial}{\partial F_i}\mathcal{U}_\beta(z,F)\Big|_{T_z}$.

For the higher derivatives with respect to F the combinatorics becomes more complicated. Therefore, it is actually useful to introduce the norm $|\cdot|_{T_{z,F}}$ for Taylor polynomials in two variables (see Appendix 4.A.2),

$$|f|_{T_{z,F}} := \sum_{|\alpha|\le r_0}\sum_{|\gamma|\le r_1} \frac{1}{\alpha!}\frac{1}{\gamma!}\big|\partial_z^\alpha \partial_F^\gamma f(z,F)\big|. \tag{3.2.97}$$

Note that, in particular, the expression $\Big|\frac{\partial}{\partial F_i}\mathcal{U}_\beta(z,F)\Big|_{T_z}$ is controlled by this norm. As a preparation we show an estimate similar to the result of Step 2 for the $|\cdot|_{T_{z,F}}$ norm of $\mathcal{U}_\beta(z,F)$.

Step 5. For $\beta \ge 1$ and $|F| \le 1$ we have

$$|\mathcal{U}_\beta(z,F)|_{T_{z,F}} \le 2^{r_0+r_1+1}(1+|z|)^3\Psi(|z|+1). \tag{3.2.98}$$

To estimate the terms in the definition of $|\cdot|_{T_{z,F}}$ norm, we distinguish three cases depending on the order of derivatives.

For $|\gamma| = 0$ we have shown in Step 2 that for $\beta \ge 1$ and $|F| \le 1$,

$$\sum_{|\alpha|\le r_0} \frac{1}{\alpha!}\big|\partial_z^\alpha \mathcal{U}_\beta(z,F)\big| = |\mathcal{U}_\beta(\cdot,F)|_{T_z} \le 6(1+|z|)^3\Psi(1+|z|). \tag{3.2.99}$$

For $|\gamma| \geq 1$ and $|\alpha| \geq 2$ we use that $|\partial_z^\alpha \partial_F^\gamma \mathcal{U}_\beta(z,F)| = \beta^{1-|\alpha|/2} \big|\partial^{\alpha+\gamma}\mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})\big|$. The combinatorial identity

$$\sum_{\alpha+\gamma=\delta} \frac{1}{\alpha!}\frac{1}{\gamma!} = \frac{1}{\delta!}\sum_{\alpha+\gamma=\delta}\frac{\delta!}{\alpha!\gamma!} = \frac{1}{\delta!}2^{|\delta|} \tag{3.2.100}$$

then implies

$$\sum_{\substack{2\leq|\alpha|\leq r_0,\\ 1\leq|\gamma|\leq r_1}} \frac{1}{\alpha!}\frac{1}{\gamma!}|\partial_z^\alpha \partial_F^\gamma \mathcal{U}_\beta(z,F)| \leq 2^{r_0+r_1} \sum_{3\leq|\delta|\leq r_0+r_1} \frac{1}{\delta!}\Big|\partial^\delta \mathcal{U}(\frac{z}{\sqrt{\beta}} + \overline{F})\Big| \leq 2^{r_0+r_1}\Psi(|z|+1). \tag{3.2.101}$$

For the terms with $\alpha = 0$ and $|\gamma| \geq 1$, one differentiates with respect to F the second order Taylor expansion of $\mathcal{U}_\beta$ in the variable z,

$$\mathcal{U}_\beta(z,F) = \int_0^1 (1-\tau) D^2\mathcal{U}(\tau\frac{z}{\sqrt{\beta}} + \overline{F})(z,z)\, d\tau - \frac{D^2\mathcal{U}(0)(z,z)}{2}. \tag{3.2.102}$$

Using the identity

$$\sum_{|\gamma|=k} \frac{1}{\gamma!}|\partial^\gamma f(F)| = \frac{1}{k!}\sum_{i_1,\ldots i_k=1}^{\dim\mathcal{G}} |\partial^{i_1}\ldots\partial^{i_k} f(F)| \tag{3.2.103}$$

valid for any $f \in C^k(\mathcal{G})$, we get

$$\sum_{j_1,\ldots,j_\ell}\sum_{\alpha:|\alpha|=k} \frac{1}{\alpha!}|\partial_{j_1}\ldots\partial_{j_\ell}\partial^\alpha f(z)| = \frac{(k+\ell)!}{k!}\sum_{\overline{\alpha}:|\overline{\alpha}|=k+\ell}\frac{1}{\overline{\alpha}!}|\partial^{\overline{\alpha}} f(z)|. \tag{3.2.104}$$

Hence (3.2.102) implies

$$\sum_{1\leq|\gamma|\leq r_1} \frac{1}{\gamma!}|\partial_F^\gamma \mathcal{U}_\beta(z,F)| \leq \frac{(r_1+2)!}{2r_1!}|z|_\infty^2\Psi(|z|+1) \leq \frac{(r_1+2)^2}{2}|z|^2\Psi(|z|+1) \tag{3.2.105}$$

Similarly, the Taylor expansion for the derivative,

$$D_z\mathcal{U}_\beta(z,F)(\dot{z}) = \int_0^1 D^2\mathcal{U}(\tau\frac{z}{\sqrt{\beta}} + \overline{F})(z,\dot{z})\,\mathrm{d}\tau - D^2\mathcal{U}(0)(z,\dot{z}), \tag{3.2.106}$$

implies that

$$\sum_{|\alpha|=1}\sum_{1\leq|\gamma|\leq r_1} \frac{1}{\gamma!}|\partial_z^\alpha\partial_F^\gamma \mathcal{U}_\beta(z,F)| \leq \frac{(r_1+2)!}{r_1!}|z|_\infty\Psi(|z|+1) \leq (r_1+2)^2|z|\Psi(|z|+1). \tag{3.2.107}$$

Thus, combining (3.2.99), (3.2.101), (3.2.105), and (3.2.107) we get (3.2.98) since $(r_1+2)^2 \leq 4\cdot 2^{r_1} < 2^{r_0+r_1}$.

Step 6. Derivatives with respect to F.
Let δ_0 and $\tilde{\zeta}$ be like in Step 4. The map $B_{\delta_0}(0) \ni F \mapsto e^{-\mathcal{U}_\beta(\cdot,F)} \in \boldsymbol{E}$ is r_1 times continuously differentiable and

$$\sum_{|\gamma|\leq r_1} \frac{1}{\gamma!}\,\|\partial_F^\gamma \mathcal{K}_F\|_{\tilde{\zeta},\mathcal{Q}_\mathcal{U}} \leq \Theta. \tag{3.2.108}$$

with Θ depending on Ψ, ω, r_0, r_1, and R'.

By the chain and product rule it follows that the the derivatives $\partial_z^\alpha \partial_F^\gamma e^{\mathcal{U}_\beta(z,F)}$ exists for all $|\alpha| \le r_0$ and $|\gamma| \le r_1$ and are continuous in (z, F). To get a bound for $|\partial_F^\gamma e^{\mathcal{U}_\beta(z,F)}|_{T_z}$ and to prove higher differentiability of $F \mapsto e^{\mathcal{U}_\beta(\cdot,F)}$, we proceed similarly to Step 4. As shown in Appendix 4.A.2 the product property extends to the norm $|\cdot|_{T_{z,F}}$.

From the product property one deduces as in Step 1 that

$$\sum_{|\gamma|\le r_1} \frac{1}{\gamma!} |\partial_F^\gamma e^{f(\cdot,F)}|_{T_z} = |e^f|_{T_{z,F}} \le e^{f(z,F)} (1 + |f|_{T_{z,F}})^{r_0+r_1}. \tag{3.2.109}$$

For $f = -\mathcal{U}_\beta$ we find with the results of Step 3 and Step 5 that

$$\begin{aligned}\sum_{|\gamma|\le r_1} \frac{1}{\gamma!} |\partial_F^\gamma e^{-\mathcal{U}_\beta(\cdot,F)}|_{T_z} &\le e^{\frac{1}{2}\mathcal{Q}_\mathcal{U}(z) - \frac{\omega}{2}|z|^2} \left(1 + 2^{r_0+r_1+1}(1+|z|)^3 \Psi(|z|+1)\right)^{r_0+r_1} \\ &\le e^{\frac{1}{2}(1-\tilde{\zeta})\mathcal{Q}_\mathcal{U}(z)} e^{-\frac{1}{4}\omega|z|^2} \left(1 + 2^{r_0+r_1+1}(1+|z|)^3 \Psi(|z|+1)\right)^{r_0+r_1}.\end{aligned} \tag{3.2.110}$$

where we used (3.2.93) and the definition of $\tilde{\zeta}$ in the second step. Invoking Lemma 3.2.10 below, it follows by induction in $|\gamma|$ that the map $F \mapsto e^{-\mathcal{U}_\beta(\cdot,F)}$ is r_1 times continuously differentiable as a map from $B_{\delta_0}(0)$ to $\boldsymbol{E}$. Moreover (3.2.110) implies the estimate (3.2.108) for the higher derivatives with

$$\Theta = (|\mathcal{G}|+1)^{r_1} \sup_z e^{-\frac{1}{4}\omega|z|^2} \left(1 + 2^{r_0+r_1+1}(1+|z|)^3 \Psi(|z|+1)\right)^{r_0+r_1} \tag{3.2.111}$$

where $(|\mathcal{G}|+1)^{r_1} \ge |\{\gamma : |\gamma| \le r_1\}|$ counts the number of terms in the sum $\sum_{|\gamma|\le r_1}$ which arises because we interchange the sum with the supremum in the definition of the $\|\cdot\|_{\tilde{\zeta},\mathcal{Q}_\mathcal{U}}$ norm.

□

Lemma 3.2.10. *Let $\mathcal{O}$ be an open set in a finite dimensional space and $h : \mathcal{O} \times \mathcal{G} \to \mathbb{R}$ a map satisfying two conditions:*

(i) For each $(F, z) \in \mathcal{O} \times \mathcal{G}$ and each α with $|\alpha| \le r_0$ the partial derivatives $\partial_z^\alpha h(F, z)$ exist and are continuous in $\mathcal{O} \times \mathcal{G}$,

(ii) $\lim_{|z|\to\infty} e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}_\mathcal{U}(z)} \sup_{F\in\mathcal{O}} |h(F,\cdot)|_{T_z} = 0$.

Define the function $g : \mathcal{O} \to \boldsymbol{E}_\zeta$ by taking $(g(F))(z) = h(F, z)$. Then $g \in C^0(\mathcal{O}, \boldsymbol{E}_\zeta)$. Moreover, if the conditions (i) and (ii) hold for all partial derivatives $h_i(F, z) = \frac{\partial}{\partial F_i} h(f, z)$ then $g \in C^1(\mathcal{O}, \boldsymbol{E}_\zeta)$.

Proof. To prove that $F \to h(F, \cdot)$ is a continuous map from $\mathcal{O}$ to $\boldsymbol{E}_\zeta$ note that h is uniformly continuous on compact subsets of $\mathcal{O} \times \mathcal{G}$. Let $\delta > 0$. By assumption there exists an R such that $\sup_{F\in\mathcal{O}} e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}(z)} |h(F,\cdot)|_{T_z} \le \delta$ if $|z| > R$. Let $F_k \to F$. Then

$$\begin{aligned}&\limsup_{k\to\infty} \|h(F_k,\cdot) - h(F,\cdot)\|_\zeta \\ &= \limsup_{k\to\infty} \sup_{z\in\mathcal{G}} e^{\frac{1}{2}(1-\zeta)\mathcal{Q}_\mathcal{U}(z)} |h(F_k,\cdot) - h(F,\cdot)|_{T_z} \\ &\le 2\delta + \limsup_{k\to\infty} \sup_{|z|\le R} e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}_\mathcal{U}(z)} |h(F_k,\cdot) - h_i(F,\cdot)|_{T_z} = 2\delta\end{aligned}$$

by uniform continuity on compact sets. Since $\delta > 0$ was arbitrary this shows that $g \in C^0(\mathcal{O}, \boldsymbol{E}_\zeta)$.

Assume now that all partial derivatives $h_i = \frac{\partial}{\partial F_i} h$ satisfy (i) and (ii). The same reasoning as before implies that $F \mapsto h_i(F, \cdot)$ is a continuous map from $\mathcal{O}$ to $\boldsymbol{E}_\zeta$. Then we use that

$$h(F + \eta e_i, z) - h(F, z) - h_i(F, z)\eta = \int_0^1 [h_i(F + t\eta, z) - h_i(F, z)]\, \eta \, \mathrm{d}t, \tag{3.2.112}$$

divide by η, use Jensen's inequality for $|\cdot|_{T_z}$ and take the limit $\eta \to 0$ to show that the map $g : \mathcal{O} \to \boldsymbol{E}_\zeta$ has partial derivatives given by $h_i(F, \cdot)$. Moreover these partial derivatives are continuous. Since $\mathcal{O}$ is in a finite dimensional space this implies the assertion that $g \in C^1(\mathcal{O}, \boldsymbol{E}_\zeta)$. □

3.3 Discrete nonlinear elasticity

3.3.1 Main results for discrete elasticity

In this section we consider models of discrete elasticity and analyse local convexity properties of the free energy and the scaling limit of Gibbs measures. Indeed, the study of such models is a key motivation for the present work and it is the reason why we considered vector-valued fields and interactions beyond nearest neighbour interactions in the previous section. An additional difficulty in discrete nonlinear elasticity is that the invariance under rotations leads to a degeneracy of the quadratic form $\mathcal{Q}$ which we considered in the previous section. Thus condition (3.2.47) is violated and the results in the previous section cannot be applied directly. We will overcome this difficulty by adding a suitable discrete null Lagrangian, see Definition 3.3.3, equation (3.3.46) and Lemma 3.3.10 in the next subsection.

We consider the general setting of (3.2.5) with $m = d$. Thus let A be a finite subset of $\mathbb{Z}^d$ and let $U : (\mathbb{R}^d)^A \to \mathbb{R}$ be an interaction potential. For fields $\varphi : \Lambda_N \to \mathbb{R}^d$ we consider the Hamiltonian

$$H_N(\varphi) = \sum_{x \in T_N} U(\varphi_{\tau_x(A)}) \tag{3.3.1}$$

where $\tau_x(A)$ denotes the set A translated by $x \in T_N$, $\tau_x(A) = A + x = \{y \colon y - x \in A\}$. For simplicity (and without loss of generality), we suppose that the support set A of the potential U contains the unit cell of $\mathbb{Z}^d$, $\{0,1\}^d \subset A$.

For any $\psi, \psi' \in (\mathbb{R}^d)^A$ we introduce the scalar product

$$(\psi, \psi') = \sum_{x \in A} \psi(x) \cdot \psi'(x) \tag{3.3.2}$$

and the corresponding norm $|\psi|$. Then we can naturally split $(\mathbb{R}^d)^A = \mathcal{V}_A \times \mathcal{V}_A^\perp$, where $\mathcal{V}_A \sim \mathbb{R}^d$ is the d-dimensional subspace of shifts $\mathcal{V}_A = \{(a, \dots, a) \in (\mathbb{R}^d)^A : a \in \mathbb{R}^d\}$, and $\mathcal{V}_A^\perp$ is the $d(|A| - 1)$-dimensional orthogonal complement of $\mathcal{V}_A$.

For a linear map $F : \mathbb{R}^d \to \mathbb{R}^d$ we consider the extension to $(\mathbb{R}^d)^A$ given by $(F\psi)(x) = F(\psi(x))$. For ease of notation we will use the same symbol F for the original map and the extension to $(\mathbb{R}^d)^A$ and similarly for the extension to $(\mathbb{R}^d)^{\mathbb{Z}^d}$.

We assume that the potential $U : \left(\mathbb{R}^d\right)^A \to \mathbb{R}$ satisfies the following conditions.

(H1) *Invariance under rotations and shifts:* We have

$$U(\psi) = U(\mathbf{R}(t_a\psi)) \tag{3.3.3}$$

for any $\psi \in (\mathbb{R}^d)^A$ and any $\mathbf{R} \in SO(d)$, $a \in \mathbb{R}^d$, with $\mathbf{R}(t_a\psi)(x) = \mathbf{R}(\psi(x) + a)$.

(H2) *Ground state:* $U(\psi) \geq 0$ and $U(\psi) = 0$ if and only if ψ is a rigid body rotation, i.e., there exists $\mathbf{R} \in \mathrm{SO}(d)$ and $a \in \mathbb{R}^d$ such that $\psi(x) = \mathbf{R}x + a$ for any $x \in A$.

(H3) *Smoothness and convexity:* Let $\mathbb{1} \in (\mathbb{R}^d)^A$ denote the identity configuration $\mathbb{1}(x) = x$. Assume that U is a C^2 function and $D^2U(\mathbb{1})$ is positive definite on the subspace orthogonal to shifts and infinitesimal rotations given by skew-symmetric linear maps.

(H4) *Growth at infinity:*

$$\liminf_{\psi \in \mathcal{V}_A^{\perp}, |\psi| \to \infty} \frac{U(\psi)}{|\psi|^d} > 0. \tag{3.3.4}$$

(H5) *Additional smoothness and subgaussian bound:* $U \in C^{r_0+r_1}$ with $r_0 \geq 3$ and $r_1 \geq 0$ and

$$\lim_{|\psi| \to \infty} |\psi|^{-2} \ln \Big(\sum_{2 \leq |\alpha|_1 \leq r_0 + r_1} \frac{1}{\alpha!} |\partial_\psi^\alpha U(\psi)| \Big) = 0, \tag{3.3.5}$$

where $\partial_\psi^\alpha U(\psi) = \prod_{x \in A} \prod_{s=1}^d \frac{\partial^{|\alpha|}}{\partial_{\psi_s(x)}^{\alpha(x,s)}} U(\psi)$ for any multiindex $\alpha : A \times \{1, \ldots, d\} \to \mathbb{N}$.

The first four conditions are the same as in [62]. The last condition is a minor additional regularity assumption for the potential. It was stated as a separate item to make clear that it is only required in the renormalisation group analysis but not in the convexification argument in Section 3.3.2.

In [62] these assumptions are used to prove that the Cauchy-Born rules holds at zero temperature, in the sense that the energy minimiser subject to affine boundary conditions is affine. Here we use this result as a starting point for a study of the Gibbs distribution for the Hamiltonian H_N at low temperatures using the renormalisation group approach. The ground state in the setting of discrete elasticity corresponds to the affine deformation given by the identity. Therefore we consider deformations $F \in \mathbb{R}^{d \times d}$ for which $F - \mathbb{1}$ is small. For a linear function F, its restriction to A and to $\tau_x(A)$ differ by the constant vector $F(x) \in \mathbb{R}^d$ and thus $U(F|_A) = U(F|_{\tau_x(A)})$. Hence for linear maps F we simply write $U(F)$ instead of $U(F|_{\tau_x(A)})$. As in (3.2.8) we define

$$H_N^F(\varphi) = \sum_{x \in T_N} U((\varphi + F)_{\tau_x(A)}). \tag{3.3.6}$$

and we recall the definition of the corresponding partition function $Z_{N,\beta}(F, 0)$ and the function

$$W_{N,\beta}(F) = -\frac{\ln Z_{N,\beta}(F, 0)}{\beta L^{Nd}} \tag{3.3.7}$$

in (3.2.10) and (3.2.11), respectively.

Note that $W_{N,\beta}$ inherits the rotational invariance of U, i.e.

$$W_{N,\beta}(RF) = W_{N,\beta}(F) \quad \text{for all } R \in SO(d). \tag{3.3.8}$$

This follow immediately from the fact that the Hausdorff measure λ_N on the space $\mathcal{X}_N$ of L^N periodic fields with average zero is invariant under the map $\varphi \mapsto R\varphi$.

In analogy with (3.2.31) we define

$$\mathcal{W}_{N,\beta}(F) := \beta \left(W_{N,\beta}(F) - U(F) \right) + \frac{\ln Z_{N,\beta}^{Q_U}}{L^{dN}} \tag{3.3.9}$$

where $Z^{Q_U}_{N,\beta}$ is the partition function of the following Gaussian integral based on the quadratic form $\beta Q_U = \beta D^2U(\mathbb{1})$:

$$Z^{Q_U}_{N,\beta} := \int_{\mathcal{X}_N} \exp\Big(-\frac{\beta}{2}\sum_{x\in T_N} Q_U(\varphi_{\tau_x(A)})\Big)\,\lambda_N(\mathrm{d}\varphi). \tag{3.3.10}$$

Note the integral is well-defined since the quadratic form $\varphi \mapsto \sum_{x\in T_N} Q_U(\varphi_{\tau_x(A)})$ is positive definite on the finite-dimensional space $\mathcal{X}_N$ even though $D^2U(\mathbb{1})$ is only positive semidefinite. Indeed, if $\sum_{x\in T_N} Q_U(\varphi_{\tau_x(A)}) = 0$ for $\varphi \in \mathcal{X}_N$ then the assumption $\{0, e_1, \ldots, e_d\} \subset A$ and (H3) imply that $\nabla_i\varphi^j(x) = W_{ij}(x)$ where $W(x)$ is a skew-symmetric $d \times d$ matrix. Discrete Fourier transform shows that all Fourier modes of $\widehat{W}(p)$ are skew-symmetric rank one matrices and thus vanish. This implies $\varphi = 0$.

Rewriting (3.3.9) we get

$$W_{N,\beta}(F) = U(F) + \frac{\mathcal{W}_{N,\beta}(F)}{\beta} - \frac{1}{\beta}\frac{\ln Z^{Q_U}_{N,\beta}}{L^{dN}}. \tag{3.3.11}$$

Note that the last term on the right hand side is independent of F. It is easy to see, e.g. by discrete Fourier transform, that its limit for $N \to \infty$ exists.

Theorem 3.3.1. *Suppose the potential U satisfies the assumptions ($H1$) to ($H5$) with $r_0 = 3$ and $r_1 \geq 0$. Then for all sufficiently large odd L there exist a $\delta(L) > 0$ and $\beta_0(L) > 0$ such that, for any $\beta \geq \beta_0$ and any $N \geq 1$ the functions $\mathcal{W}_{N,\beta} : B_\delta(\mathbb{1}) \to \mathbb{R}$ are in C^{r_1}, with bounds on the C^{r_1} norm that are independent of N and β.*

In particular, for $r_1 \geq 2$, $D^2W_{N,\beta}(G)$ is positive definite on the subspace orthogonal to the tangent space at G of the orbit $SO(d)G$, for all $G \in B_\delta(\mathbb{1})$, uniformly in N.

Moreover there exists a subsequence (N_ℓ) such that $W_{N_\ell,\beta}$ converges in C^{r_1-1} to the free energy $W_\beta(F)$. For $r_1 \geq 3$ the second derivative $D^2W_\beta(\mathbb{1})$ is strictly positive on the subspace orthogonal to the skew-symmetric matrices.

The second part of the theorem asserts that $W_{N,\beta}$ is uniformly convex near $\mathbb{1}$ modulo rotational invariance. Equivalently this can be stated as follows. Since $W_{N,\beta}$ is rotational invariant there exists a smooth function $\hat{W}_{N,\beta}$, defined in a small neighbourhood of $\mathbb{1}$ such that $W_{N,\beta}(F) = \hat{W}_{N,\beta}(F^TF)$. Then $\hat{W}$ is uniformly convex in a neighbourhood of $\mathbb{1}$, uniformly in N.

The discussion in Section 3.3.2 below implies a variant of the convexity result. There exists a null Lagrangian N (actually a multiple of the determinant) such that $W_{N,\beta} + \mathsf{N}$ is uniformly convex in $B_\delta(\mathbb{1})$ and this property extends to $W_\beta + \mathsf{N}$ for all $r_1 \geq 2$.

We also get a result for the scaling limit of the Gibbs state.

We consider $\mathcal{Q}_U = D^2U(\mathbb{1})$. We define $\mathcal{Q}^\nabla_U$ as the restriction of $\mathcal{Q}_U$ to linear maps. More precisely consider a linear map $F : \mathbb{R}^d \to \mathbb{R}^d$ and recall that F_A denotes the restriction of F to the discrete set A. Now we set

$$\mathcal{Q}^\nabla_U(F) := \mathcal{Q}_U(F_A) = D^2U(\mathbb{1})(F_A, F_A) \tag{3.3.12}$$

We identify the space of linear maps with the space $\mathbb{R}^{d\times d}$ of $d \times d$ matrices. Using the Hilbert-Schmidt scalar product $(F, G) = \sum_{i,s=1}^d F_{i,s}G_{i,s}$ on $\mathbb{R}^{d\times d}$ there is a unique symmetric operator $\boldsymbol{Q}^\nabla_U$ such that $\mathcal{Q}^\nabla_U(F) = (\boldsymbol{Q}^\nabla_U F, F)$ and we denote the components of the associated matrix by $(\boldsymbol{Q}^\nabla_U)_{i,j;s,t}$.

Theorem 3.3.2. *Under the assumptions of Theorem 3.3.1, there is a subsequence* (N_ℓ) *and a matrix* $\boldsymbol{q}(F) \in \mathbb{R}_{\mathrm{sym}}^{(d\times d)\times(d\times d)}$ *such that for* $f \in C^\infty((\mathbb{R}/\mathbb{Z})^d, \mathbb{R}^d)$ *with* $\int f = 0$ *and* $f_N(x) = L^{-N\frac{d+2}{2}} f(L^{-N}x)$

$$\lim_{\ell\to\infty} \mathbb{E}_{\gamma^F_{N_\ell,\beta}} e^{(f_{N_\ell},\varphi)} = e^{\frac{1}{2\beta}(f,\mathcal{C}_{\mathbb{T}^d} f)}. \tag{3.3.13}$$

Here, $\mathcal{C}_{\mathbb{T}^d}$ *is the inverse of the operator* $\mathcal{A}_{\mathbb{T}^d}$ *acting on functions* $u \in H^1((\mathbb{R}/\mathbb{Z})^d, \mathbb{R}^d)$ *with* $\int u = 0$ *by*

$$(\mathcal{A}_{\mathbb{T}^d} u)_s = -\sum_{t=1}^{d}\sum_{i,j=1}^{d} (\boldsymbol{Q}_U^\nabla - \boldsymbol{q})_{i,j;s,t}\partial_i\partial_j u_t. \tag{3.3.14}$$

For a discussion why only the restriction $\mathcal{Q}_U^\nabla$ and not the full quadratic form $\mathcal{Q}_U$ appears in the limiting covariance see Remark 3.2.8. The operators $\boldsymbol{Q}^\nabla$ and $\boldsymbol{Q}^\nabla - \boldsymbol{q}$ are not positively definite on the set of all matrices because skew-symmetric matrices are in their null space. They are, however, positive definite on symmetric matrices. By Korn's inequality this implies that $\mathcal{A}$ is an elliptic operator and that its inverse $\mathcal{C}$ is well-behaved. Actually we will see in the proof of Theorem 3.3.2 that the operator $\mathcal{A}$ can be also written in terms of $\boldsymbol{Q}^\nabla_{U+\mathsf{N}}$ such that $\boldsymbol{Q}^\nabla_{U+\mathsf{N}}$ and $\boldsymbol{Q}^\nabla_{U+\mathsf{N}} - \boldsymbol{q}$ are positive definite.

Along the lines of Section 1.8.3 in [47] one can show that in both theorems convergence holds not only for a subsequence but for the full sequence and that the convergence of $W_{N,\beta}$ holds in C^{r_1} and not just C^{r_1-1}, see [106]. In a slightly different situation the existence of the thermodynamic limit $\lim_{N\to\infty} W_{N,\beta}(F)$ was established in [116] under very weak conditions on the interaction U.

3.3.2 Reformulation of discrete elasticity as generalized gradient models

We saw in (3.2.17) that the Hamiltonian H_N^F can be formulated in terms of a potential U with finite range support A as well as in terms of the generalized gradient potential $\mathcal{U}$. However, the potential U and thus also $\mathcal{U}$ has a degenerate minimum and we cannot directly apply the results stated in the previous section. Instead we first need to gain local coercivity. This can be done with the help of an addition of a discrete null Lagrangian.

Let us first introduce the concept of discrete null Lagrangians.

Definition 3.3.3. *A function* $\mathsf{N} : \left(\mathbb{R}^d\right)^A \to \mathbb{R}$ *is called a discrete null Lagrangian if for any finite set* $\Lambda \subset \mathbb{Z}^d$ *and any* $\varphi, \tilde\varphi \in (\mathbb{R}^d)^{\mathbb{Z}^d}$ *such that* $\varphi(x) = \tilde\varphi(x)$ *for all* $x \notin \Lambda$ *we have the following identity*

$$\sum_{x\in\Lambda_A} \mathsf{N}(\varphi_{\tau_x(A)}) = \sum_{x\in\Lambda_A} \mathsf{N}(\tilde\varphi_{\tau_x(A)}) \quad \textit{where } \Lambda_A := \{x \in \mathbb{Z}^d : \tau_x(A)\cap\Lambda \neq \emptyset\}. \tag{3.3.15}$$

If N is a discrete null Lagrangian and $\varphi(x) = F(x)$ for $x \notin \Lambda$ then, in particular,

$$\sum_{x\in\Lambda_A} \mathsf{N}(\varphi_{\tau_x(A)}) = \sum_{x\in\Lambda_A} \mathsf{N}(F_{\tau_x(A)}). \tag{3.3.16}$$

It is useful to note that (3.3.15) holds if and only if

$$\sum_{x\in\Lambda'} \mathsf{N}(\varphi_{\tau_x(A)}) = \sum_{x\in\Lambda'} \mathsf{N}(\tilde\varphi_{\tau_x(A)}) \quad \text{for some finite } \Lambda' \text{ with } \Lambda_A \subset \Lambda' \subset \mathbb{Z}^d. \tag{3.3.17}$$

This follows immediately from the observation that $x \in \Lambda'\setminus\Lambda_A$ implies that $\tau_x(A) \subset \mathbb{Z}^d\setminus\Lambda$ and hence $\varphi_{\tau_x(A)} = \tilde\varphi_{\tau_x(A)}$.

Example 3.3.4. *Let* $A = \{0, y\}$ *where* $y \in \mathbb{Z}^d$ *and* $\mathsf{N}(\varphi) = \varphi(y) - \varphi(0)$. *Then* N *is a discrete null Lagrangian. To see this we use the criterion* (3.3.17). *Consider a cube* Λ' *which is so large that*

$$\Lambda_A \subset \Lambda', \quad \Lambda \cap \big((y + \Lambda') \setminus \Lambda'\big) = \emptyset \quad \text{and} \quad \Lambda \cap \big(\Lambda' \setminus (y + \Lambda')\big) = \emptyset. \tag{3.3.18}$$

Now

$$\sum_{x \in \Lambda'} \mathsf{N}(\varphi_{\tau_x(A)}) = \sum_{x \in (y+\Lambda')\setminus\Lambda'} \varphi(x) - \sum_{x \in \Lambda'\setminus(y+\Lambda')} \varphi(x). \tag{3.3.19}$$

Thus the assertion follows from (3.3.18) *since* $\varphi = \tilde{\varphi}$ *in* $\mathbb{Z}^d \setminus \Lambda$.
It follows that all the maps $\varphi \mapsto \nabla^\alpha \varphi(0)$ *with* $\alpha \neq 0$ *are discrete null Lagrangians.*

Example 3.3.5. *An important example of a nonlinear discrete null Lagrangian is given by the discrete determinant. For* $d = 2$ *and a map* $\psi : \{0,1\}^2 \to \mathbb{R}^2$ *one defines the discrete determinant as the oriented area of the polygon generated by the points* $\psi(0)$, $\psi(e_1)$, $\psi(e_1 + e_2)$, $\psi(e_2)$. *Thus*

$$\mathsf{N}(\psi) := \frac{1}{2}\psi(e_1) \times (\psi(e_1 + e_2) - \psi(e_1)) - \frac{1}{2}\psi(e_2) \times (\psi(e_1 + e_2) - \psi(e_2)) \tag{3.3.20}$$

where $a \times b = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1, b_1 \\ a_2, b_2 \end{vmatrix}$ *denotes the vector product. This implies that for the square* $Q_\ell = \{0, 1, \ldots, \ell\}^2$ *with the oriented boundary*

$$\vec{P}_\ell = \big((0,0), (1,0), .., (\ell, 0), (\ell, 1), .., (\ell, \ell), (\ell - 1, \ell), .., (0, \ell), (0, \ell - 1), .., (0, 0)\big), \tag{3.3.21}$$

the sum $\sum_{x \in Q_\ell} \mathsf{N}(\varphi_{\tau_x(\{0,1\}^2)})$ *is the oriented area of the oriented polygon* $\varphi(\vec{P}_\ell)$. *Thus it follows from the criterion* (3.3.17) *that* N *is a discrete null Lagrangian (given* Λ, *take* $\Lambda' = Q_\ell - \lfloor \frac{\ell}{2} \rfloor$ *with sufficiently large* ℓ*).*

To generalise the discrete determinant to higher dimensions, it is useful to reformulate first the case $d = 2$ *and to express* N *with the help of the continuous null Lagrangian* $\det \nabla\psi$ *for* $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. *For* $\varphi : \mathbb{Z}^2 \to \mathbb{R}^2$ *define* $I\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *as the multilinear interpolation, i.e., for* $x \in \mathbb{Z}^2$ *the map* $I\varphi_{|x+[0,1]^2}(y)$ *is the unique map which is affine in each coordinate direction* y_i *and agrees with* φ *on* $x + \{0,1\}^2$. *Note that* $I\varphi$ *is defined consistently along the lines* $x_i \in \mathbb{Z}$ *and is continuous on* $\mathbb{R}^2$. *Note also that* $I\varphi(\vec{P}_1)$ *is the boundary of the polygon generated by he points* $\varphi(0)$, $\varphi(e_1)$, $\varphi(e_1 + e_2)$, $\varphi(e_2)$. *Thus*

$$\mathsf{N}(\varphi) = \int_{(0,1)^2} \det \nabla I\varphi \, \mathrm{d}x \tag{3.3.22}$$

and for Q_ℓ,

$$\sum_{x \in Q_\ell} \mathsf{N}(\varphi_{|\tau_x(\{0,1\}^2)}) = \int_{(0,\ell)^2} \det \nabla I\varphi \, \mathrm{d}x. \tag{3.3.23}$$

The integral on the right hand side depends only on $I\varphi_{|\partial(0,\ell)^2}$ *and thus only on* $\varphi_{|\vec{P}_\ell}$. *This gives another proof that* N *is a discrete null Lagrangian.*

For $d \geq 3$ *and* $\varphi : \mathbb{Z}^d \to \mathbb{R}^d$ *we define the multilinear interpolation in the same way. We then define the discrete determinant* $\mathsf{N}_{\det} : (\mathbb{R}^d)^{\{0,1\}^d} \to \mathbb{R}$ *by*

$$\mathsf{N}_{\det}(\psi) = \int_{(0,1)^d} \det \nabla I\psi \, \mathrm{d}x. \tag{3.3.24}$$

The same reasoning as above shows that $\mathsf{N}_{\det}$ *is a discrete null Lagrangian. Note that for each* $y \in (0,1)^d$ *the expression* $\nabla I\varphi(y)$ *is a linear combination of the values* $\varphi(x)$ *for* $x \in \{0,1\}^d$. *Thus* $\mathsf{N}_{\det}$ *is a homogeneous polynomial of degree d on* $(\mathbb{R}^d)^{\{0,1\}^d}$. *If* $F : \mathbb{R}^d \to \mathbb{R}^d$ *is linear and* $\varphi_F(y) = Fy$ *for all* $y \in \{0,1\}^d$ *then* $I\varphi(y) = Fy$ *and thus*

$$\mathsf{N}_{\det}(F) = \det F. \tag{3.3.25}$$

Discrete null Lagrangians are defined using Dirichlet boundary conditions on $\mathbb{Z}^d$. One can extend the null Lagrangian property to periodic perturbations. We will only need the following result.

Lemma 3.3.6. *Assume that* $\mathsf{N} : (\mathbb{R}^d)^A \to \mathbb{R}$ *is a discrete shift-invariant null Lagrangian and assume that* N *is bounded on bounded sets. Let* $F : \mathbb{R}^d \to \mathbb{R}^d$ *be a linear map. Assume that* $|A|_\infty := \sup\{|y|_\infty : y \in A\} \le \frac{1}{8}L^N$. *Then for all periodic functions* $\varphi : T_N = \mathbb{Z}^d/L^N\mathbb{Z}^d \to \mathbb{R}^d$

$$\sum_{x\in T_N} \mathsf{N}((F+\varphi)_{\tau_x(A)}) = \sum_{x\in T_N} \mathsf{N}(F_{\tau_x(A)}) = L^{dN}\mathsf{N}(F_A). \tag{3.3.26}$$

Proof. The proof is standard, but we include it for the convenience of the reader. Fix F and φ. Note that by shift-invariance $\mathsf{N}(F_{\tau_x(A)}) = \mathsf{N}(F_A)$. We use the set $[\frac{-L^N-1}{2}, \frac{L^N-1}{2}]^d$ as the fundamental domain of T_N. Extend φ to an L^N-periodic function on $\mathbb{Z}^d$. Let M be a large odd integer and consider a cut-off function $\eta : \mathbb{Z}^d \to [0,1]$ such that

$$\eta(x) = 1 \quad \text{if } |x|_\infty \le (M-2)\tfrac{L^N-1}{2} + |A|_\infty, \qquad \eta(x) = 0 \quad \text{if } |x|_\infty \ge M\tfrac{L^N-1}{2} - 2|A|_\infty, \tag{3.3.27}$$

Apply the criterion (3.3.17) with the set $\Lambda' = \Lambda_M := [-\frac{ML^N-1}{2}, \frac{ML^N-1}{2}]^d$, $\tilde{\varphi} = 0$ and $\eta\varphi$ in place of φ. This gives

$$\sum_{x\in\Lambda_M} \mathsf{N}((F+\eta\varphi)_{\tau_x(A)}) = \sum_{x\in\Lambda_M} \mathsf{N}(F_{\tau_x(A)}) = M^d L^{dN}\mathsf{N}(F_A). \tag{3.3.28}$$

Now

$$\sum_{x\in\Lambda_{M-2}} \mathsf{N}((F+\eta\varphi)_{\tau_x(A)}) = (M-2)^d \sum_{x\in T_N} \mathsf{N}((F+\varphi)_{\tau_x(A)}). \tag{3.3.29}$$

By shift invariance we have $\mathsf{N}((F+\eta\varphi)_{\tau_x(A)}) = \mathsf{N}(F_A + (\eta\varphi)_{\tau_x(A)})$. Since φ is bounded on $\mathbb{Z}^d$ we get $|F_A + \eta\varphi| \le C$. Using the assumption that N is bounded on bounded sets we get $|\mathsf{N}((F+\eta\varphi)_{\tau_x(A)})| \le C'$ and

$$\sum_{x\in\Lambda_M\setminus\Lambda_{M-2}} |\mathsf{N}((F+\eta\varphi)_{\tau_x(A)})| \le C'(M^d - (M-2)^d)L^{dN}. \tag{3.3.30}$$

Dividing (3.3.28) by M^d and passing to the limit $M \to \infty$ we get (3.3.26). □

Using the discrete determinant one can show the following result.

Theorem 3.3.7. *[Theorem 5.1 in [62]] Under the assumptions (H1)-(H4) there is a shift invariant discrete null Lagrangian* $\mathsf{N} \in C^\infty\big((\mathbb{R}^d)^A, \mathbb{R}\big)$ *and a shift invariant function* $E \in C^2\big((\mathbb{R}^d)^A, \mathbb{R}\big)$ *such that:*

(i) E *is uniformly convex on the subspace* $\mathcal{V}_A^\perp$ *orthogonal to the shifts;*

(ii) For all $\psi \in \left(\mathbb{R}^d\right)^A$

$$U(\psi) + \mathsf{N}(\psi) \geq E(\psi); \tag{3.3.31}$$

(iii) For $\psi \in \left(\mathbb{R}^d\right)^A$ *that are close to rotations* $\psi_{\mathbf{R}}(x) = \mathbf{R}x$ *with* $\mathbf{R} \in \mathrm{SO}(d)$*,*

$$U(\psi) + \mathsf{N}(\psi) = E(\psi). \tag{3.3.32}$$

In fact one can take $\mathsf{N} = \alpha \mathsf{N}_{\det}$ *for some* $\alpha \in \mathbb{R}$ *where* $\mathsf{N}_{\det}$ *is the discrete determinant defined in* (3.3.24)*. Hence* N *is polynomial of degree* d *and in particular smooth. Moreover* N *depends only on the values of the deformation in one unit cell whose corners are contained in* A *and for affine maps* $F : \mathbb{Z}^d \to \mathbb{R}^d$*, restricted to* A *it yields*

$$\mathsf{N}(F_A) = \alpha \det F. \tag{3.3.33}$$

Remark 3.3.8. *The heart of the matter is to show that for small* $\alpha > 0$ *the quadratic form* $D^2(U + \alpha N_{\det})(z)$ *is positive definite on* $\mathcal{V}_A^\perp$ *for* $z = \mathbb{1}$ *(and hence for* z *in a small neighbourhood of* $\mathbb{1}$*). This is easy. Indeed* $D^2U(\mathbb{1})$ *is positive semidefinite on* $\mathcal{V}_A^\perp$ *since* $\mathbb{1}$ *is a minimum of* U *and by assumption positive definite on the complement of the space* $\mathcal{S} \subset \mathcal{V}_A^\perp$ *of skew symmetric linear maps. It thus suffices to show that* $D^2 N_{\det}$ *is positive definite on* $\mathcal{S}$*. For* $F \in \mathcal{S}$ *we have* $N_{\det}(F) = \det F$*. Moreover* e^{tF} *is a rotation and hence* $\det e^{tF} = 1$*. Computing the second derivative at* $t = 0$ *we get*

$$0 = D\det(\mathbb{1})(F^2) + D^2\det(\mathbb{1})(F, F) = \operatorname{Tr} F^2 + D^2\det(\mathbb{1})(F, F) = -|F|^2 + D^2\det(\mathbb{1})(F, F). \tag{3.3.34}$$

Here we used that $\operatorname{Tr} F^2 = (F^T, F) = (-F, F)$*. Thus* $D^2\det(\mathbb{1})(F, F) = |F|^2$ *for all* $F \in \mathcal{S}$*.*

In the following we want to rephrase the model given by the Hamiltonian (3.2.17) in the setting introduced in Section 3.2.1. The key idea is to consider the energy given by $U + \mathsf{N}$ instead of U. The function $U + \mathsf{N}$ is bigger than a strictly convex function and agrees with it in a neighbourhood of the identity. In particular the second derivative at the identity is strictly positive (modulo shift invariance) so it almost falls in the class of energies satisfying the assumptions of Proposition 3.2.4 (up to a trivial shift from 0 to $\mathbb{1}_{(R')^d}$). One minor issue is that we restricted the passage from finite range interaction U to generalized gradient interactions $\mathfrak{U}$ to cubes $Q_{R'}$ and $\mathcal{G}_{R'}$ and the interactions need to satisfy the lower bound (3.2.20). Since the interaction term U only depends on the field in A its second derivative will never satisfy (3.2.20) when $A \subsetneq Q_{R'}$. The addition of another null Lagrangian, however, gives us an energy that has a strictly positive Hessian at the identity.

Recall the definition (3.3.2) of the norm on $(\mathbb{R}^d)^A$ and note that the assumption $\{0,1\}^d \subset A$ implies for $\psi \in (\mathbb{R}^d)^A$

$$|\nabla\psi(0)|^2 \leq 2\sum_{i=1}^d (|\psi(0)|^2 + |\psi(e_i)|^2) \leq 2d|\psi|^2. \tag{3.3.35}$$

Uniform convexity of E orthogonal to shifts and shift invariance imply that there is a constant $\mu > 0$ such that for $\psi \in \mathcal{V}_A^\perp$

$$E(\mathbb{1}_A + \psi) \geq E(\mathbb{1}_A) + DE(\mathbb{1}_A)(\psi) + \mu|\psi|^2 \geq E(\mathbb{1}_A) + DE(\mathbb{1}_A)(\psi) + \frac{\mu}{2d}|\nabla\psi(0)|^2. \tag{3.3.36}$$

Since the first and the last expression are shift invariant we conclude that we have for all $\psi \in (\mathbb{R}^d)^A$

$$E(\mathbb{1}_A + \psi) \geq E(\mathbb{1}_A) + DE(\mathbb{1}_A)(\psi) + \frac{\mu}{2d}|\nabla\psi(0)|^2. \tag{3.3.37}$$

Hence, the growth of E is controlled from below by the gradient in one point. We now introduce a null Lagrangian that allows us to redistribute the gradient lower bound to gain coercivity on $(R^d)^{Q_{R'}}$.

Lemma 3.3.9. *Define* $\mathsf{N}_0 : (\mathbb{R}^d)^{Q_{R'}} \to \mathbb{R}$ *by*

$$\mathsf{N}_0(\psi) = -\sum_{i=1}^{d}|\nabla_i\psi(0) - e_i|^2 + \frac{1}{R'(R'+1)^{d-1}}\sum_{i=1}^{d}\sum_{y,y+e_i\in Q_{R'}}|\nabla_i\psi(y) - e_i|^2. \tag{3.3.38}$$

Then the function N_0 *is a null Lagrangian and* $\mathsf{N}_0(\psi) = 0$ *if* ψ *is the restriction of an affine map.*

Proof. This is similar to Example 3.3.4. Note that $\#\{y \in Q_{R'} : y + e_i \in Q_{R'}\} = R'(R'+1)^{d-1}$. Thus

$$\mathsf{N}_0(\psi) = \frac{1}{R'(R'+1)^{d-1}}\sum_{i=1}^{d}\sum_{y,y+e_i\in Q_{R'}}\mathsf{N}_{y,i}(\psi) \tag{3.3.39}$$

$$\text{where}\quad \mathsf{N}_{y,i}(\psi) = |\nabla_i\psi(y) - e_i|^2 - |\nabla_i\psi(0) - e_i|^2. \tag{3.3.40}$$

Thus it suffices to show that for all $y \in Q_{R'}$ with $y + e_i \in Q_{R'}$ the map $\mathsf{N}_{y,i} : (\mathbb{R}^d)^{Q_{R'}} \to \mathbb{R}$ is a null Lagrangian. We use the criterion (3.3.17). Assume that $\tilde{\varphi} = \varphi$ in $\mathbb{Z}^d \setminus \Lambda$. Take Λ' so large that $\Lambda_{Q_{R'}} \subset \Lambda'$ and $\big((y+\Lambda')\Delta\Lambda'\big) \cap \Lambda_{Q_{R'}} = \emptyset$. Here Δ denotes the symmetric set difference. Since

$$\sum_{x\in\Lambda'}\mathsf{N}_{y,i}(\varphi_{\tau_x(Q_{R'})}) = \sum_{x\in(y+\Lambda')\setminus\Lambda}|\nabla_i\varphi(x) - e_i|^2 - \sum_{x\in\Lambda\setminus(y+\Lambda')}|\nabla_i\varphi(x) - e_i|^2 \tag{3.3.41}$$

and $\big((y+\Lambda')\Delta\Lambda'\big) \cap \Lambda_{Q_{R'}} = \emptyset$ we conclude that $\varphi_{\tau_x(Q_{R'})} = \tilde{\varphi}_{\tau_x(Q_{R'})}$ for all $x \in \big((y+\Lambda')\Delta\Lambda'\big)$ and in particular $\nabla_i\varphi(x) = \nabla_i\tilde{\varphi}(x)$. Thus $\sum_{x\in\Lambda'}\mathsf{N}_{y,i}(\varphi_{\tau_x(Q_{R'})}) = \sum_{x\in\Lambda'}\mathsf{N}_{y,i}(\tilde{\varphi}_{\tau_x(Q_{R'})})$. This shows that $\mathsf{N}_{y,i}$ is a null Lagrangian.

Finally, if ψ is the restriction of an affine map then $\nabla_i\psi(y) = \nabla_i\psi(0)$ and hence $\mathsf{N}_0(\psi) = 0$. □

We define the energies $\tilde{U}, \tilde{\mathsf{N}}, \tilde{E} : (\mathbb{R}^d)^{Q_{R'}} \to \mathbb{R}$ for $\psi \in (\mathbb{R}^d)^{Q_{R'}}$ by

$$\tilde{U}(\psi) = U(\psi|_A), \tag{3.3.42}$$

$$\tilde{\mathsf{N}}(\psi) = \mathsf{N}(\psi|_A) + \frac{\mu}{2d}\mathsf{N}_0(\psi), \tag{3.3.43}$$

$$\tilde{E}(\psi) = E(\psi|_A) + \frac{\mu}{2d}\mathsf{N}_0(\psi). \tag{3.3.44}$$

Those functionals inherit the properties $\tilde{U} + \tilde{\mathsf{N}} \geq \tilde{E}$ with equality in a neighbourhood of rotations (restrictions of rotations are still rotations) and from (3.3.37) we infer that for any $\psi \in (\mathbb{R}^d)^{Q_{R'}}$

$$\begin{aligned}\tilde{E}(\mathbb{1}_{Q_{R'}} + \psi) &= E(\mathbb{1}_A + \psi|_A) + \frac{\mu}{2d}\frac{1}{R'(R'+1)^{d-1}}\sum_{i=1}^{d}\sum_{x,x+e_i\in Q_{R'}}|\nabla_i\psi(x)|^2 \\ &\geq \tilde{E}(\mathbb{1}_{Q_{R'}}) + D\tilde{E}(\mathbb{1}_{Q_{R'}})(\psi) + \frac{\mu}{2d}\frac{1}{R'(R'+1)^{d-1}}\sum_{i=1}^{d}\sum_{x,x+e_i\in Q_{R'}}|\nabla_i\psi(x)|^2\end{aligned} \tag{3.3.45}$$

where we used the $D\mathsf{N}_0(\mathbb{1}_{Q_{R'}}) = 0$ and $\mathsf{N}_0(\mathbb{1}_{Q_{R'}}) = 0$.

Recalling Lemma 3.2.1, we use the isomorphism $\Pi : \mathcal{G}_{R'} \to \mathcal{V}^{\perp}_{Q_{R'}}$ to define the functions $\mathcal{U}, \mathcal{E}, \mathcal{N} : \mathcal{G}_{R'} \to \mathbb{R}$ by

$$\mathcal{U}(z) = \tilde{U}(\Pi z + \mathbb{1}_{Q_{R'}}), \quad \mathcal{E}(z) = \tilde{E}(\Pi z + \mathbb{1}_{Q_{R'}}), \quad \text{and} \quad \mathcal{N}(z) = \tilde{\mathsf{N}}(\Pi z + \mathbb{1}_{Q_{R'}}). \tag{3.3.46}$$

Because of (3.3.31) and (3.3.32) they satisfy

$$\mathcal{U}(z) + \mathcal{N}(z) \geq \mathcal{E}(z), \tag{3.3.47}$$
$$\mathcal{U}(z) + \mathcal{N}(z) = \mathcal{E}(z) \quad \text{for } z \text{ close to } 0. \tag{3.3.48}$$

Moreover, their definition implies that

$$U((\mathbb{1} + F + \varphi)_{\tau_x(A)}) = \mathcal{U}(\overline{F} + D\varphi(x)). \tag{3.3.49}$$

Hence the Hamiltonian for the discrete elasticity model defined in (3.3.6) can be written as

$$H_N^{\mathbb{1}+F}(\varphi) = \sum_{x \in T_N} \mathcal{U}(\overline{F} + D\varphi(x)). \tag{3.3.50}$$

The functionals $\mathcal{U}$, $\mathcal{N}$, and $\mathcal{E}$ are differentiable since they are a composition of a differentiable and a linear map. Moreover (3.3.47), (3.3.48), and the bound (3.3.45) imply that there is $\omega_1 > 0$ such that for all $z \in \mathcal{G}_{R'}$

$$\begin{aligned} \mathcal{U}(z) + \mathcal{N}(z) &\geq \mathcal{E}(z) \geq \mathcal{E}(0) + D\mathcal{E}(0)(z) + \omega_1 |z|^2 \\ &= (\mathcal{U} + \mathcal{N})(0) + D(\mathcal{U} + \mathcal{N})(0)z + \omega_1 |z|^2. \end{aligned} \tag{3.3.51}$$

where we used that $\sum_{i=1}^d \sum_{y, y+e_i \in Q_{R'}} |\nabla_i \psi(y)|^2$ defines a norm on $\mathcal{V}^{\perp}_{Q_{R'}} \simeq \mathcal{G}_{R'}$ and all norms on a finite dimensional space are equivalent.

We now show that under the assumptions (H1) to (H5) the potential $\mathcal{U} + \mathcal{N}$ satisfies the conditions in Proposition 3.2.4 and that the generalised gradient model with the potential $\mathcal{U} + \mathcal{N}$ is equivalent to the discrete elasticity model with the potential U (see Lemma 3.3.10 below). Once this is done we can easily deduce our main result for discrete elasticity, Theorem 3.3.1 and Theorem 3.3.2, from the corresponding results for generalized gradient models, Theorem 3.2.3 and Theorem 3.2.7.

As in (3.2.19) and (3.2.22), we define the quadratic part

$$\mathcal{Q}_{\mathcal{U}+\mathcal{N}}(z) := D^2(\mathcal{U} + \mathcal{N})(0)(z, z) \tag{3.3.52}$$

and the function

$$\overline{(\mathcal{U} + \mathcal{N})}(z, F) = (\mathcal{U} + \mathcal{N})(z + \overline{F}) - (\mathcal{U} + \mathcal{N})(\overline{F}) - D(\mathcal{U} + \mathcal{N})(\overline{F})(z) - \frac{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}(z)}{2}. \tag{3.3.53}$$

Note that (3.3.51) implies

$$\mathcal{Q}_{\mathcal{U}+\mathcal{N}}(z) \geq 2\omega_1 |z|^2 \tag{3.3.54}$$

Since U and N hence $\mathcal{U}$ and $\mathcal{N}$ are C^2 we also have

$$\mathcal{Q}_{\mathcal{U}+\mathcal{N}}(z) \leq \frac{1}{\omega_2} |z|^2 \tag{3.3.55}$$

for some $\omega_2 > 0$.

Lemma 3.3.6 implies that

$$L^{Nd}\mathcal{N}(\overline{F}) = \sum_{x \in T_N} \mathcal{N}(D\varphi(x) + \overline{F}). \tag{3.3.56}$$

From (3.3.56) and (3.3.53), we find

$$\begin{aligned} H_N^{\mathbb{1}+F}(\varphi) &= -L^{Nd}\mathcal{N}(\overline{F}) + \sum_{x \in T_N} (\mathcal{U}+\mathcal{N})(D\varphi(x) + \overline{F}) \\ &= L^{Nd}\mathcal{U}(\overline{F}) + \sum_{x \in T_N} \overline{(\mathcal{U}+\mathcal{N})}(D\varphi(x), F) + \sum_{x \in T_N} \Big(D(\mathcal{U}+\mathcal{N})(\overline{F})(D\varphi(x)) + \frac{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}(D\varphi(x))}{2} \Big) \\ &= L^{Nd}\mathcal{U}(\overline{F}) + \sum_{x \in T_N} \Big(\overline{(\mathcal{U}+\mathcal{N})}(D\varphi(x), F) + \frac{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}(D\varphi(x))}{2} \Big). \end{aligned} \tag{3.3.57}$$

In the last equality we used the equation $\sum_{x \in T_N} D\varphi(x) = 0$. As a result, the partition function for the discrete elasticity model defined in (3.2.10) can be expressed as

$$Z_{N,\beta}^U(\mathbb{1} + F, f_N)) = e^{-\beta L^{Nd}\mathcal{U}(\overline{F})} Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}} \mathcal{Z}_{N,\beta}^{\mathcal{U}+\mathcal{N}}\Big(F, \frac{f_N}{\sqrt{\beta}}\Big) \tag{3.3.58}$$

where

$$\mathcal{Z}_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F, f) := \int_{\mathcal{X}_N} e^{(f,\varphi)} \sum_{X \subset T_N} \prod_{x \in X} K_{F,\beta,\mathcal{U}+\mathcal{N}}(D\varphi(x))\, \mu(\mathrm{d}\varphi), \tag{3.3.59}$$

with $K_{F,\beta,\mathcal{U}+\mathcal{N}}$ defined by replacing $\mathcal{U}$ by $\mathcal{U}+\mathcal{N}$ and $\overline{\mathcal{U}}$ by $\overline{\mathcal{U}+\mathcal{N}}$ in (3.2.27), (3.2.22) and (3.2.19). The calculations so far can be summarised as follows.

Lemma 3.3.10. *Let $Z_{N,\beta}^U(F, 0)$ denote partition function of the discrete elasticity model with interaction U and deformation F, let $\gamma_{N,\beta}^{F,U}$ denote the corresponding finite volume Gibbs measure, let*

$$W_{N,\beta}^U(F) = -\frac{\ln Z_{N,\beta}(F,0)}{\beta L^{Nd}}, \tag{3.3.60}$$

and let

$$\mathcal{W}_{N,\beta}^U(F) = \beta(W_{N,\beta}^U(F) - U(F)) + \frac{\ln Z_{N,\beta}^{Q_U}}{L^{dN}} \tag{3.3.61}$$

be the quantity defined in (3.3.9)*. Let $Z_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F,0)$ denote the partition function of the generalised gradient model with interaction $\mathcal{U}+\mathcal{N}$ and deformation F, let $\gamma_{N,\beta}^{F,\mathcal{U}+\mathcal{N}}$ be the corresponding Gibbs measure and let*

$$\mathcal{W}_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F) = -\frac{\ln \mathcal{Z}_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F,0)}{L^{dN}}. \tag{3.3.62}$$

be the quantity in (3.2.30) *Then*

$$Z_{N,\beta}^U(\mathbb{1} + F, 0) = e^{\beta L^{dN}\mathcal{N}(F)} Z_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F, 0), \tag{3.3.63}$$

$$\mathcal{W}_{N,\beta}^U(\mathbb{1} + F) = \mathcal{W}_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F), \tag{3.3.64}$$

$$\mathbb{E}_{\gamma_{N,\beta}^{\mathbb{1}+F,U}} e^{(f,\varphi)} = \mathbb{E}_{\gamma_{N,\beta}^{F,\mathcal{U}+\mathcal{N}}} e^{(f,\varphi)}. \tag{3.3.65}$$

Proof. Equation (3.2.28), applied to $\mathcal{U}+\mathcal{N}$ instead of $\mathcal{U}$ gives

$$\begin{aligned} Z_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F,f) &= e^{-\beta L^{Nd}(\mathcal{U}(\overline{F})+\mathcal{N}(\overline{F}))} Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}} \mathcal{Z}_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F,\frac{f}{\sqrt{\beta}}) \\ &\underset{(3.3.58)}{=} e^{-\beta L^{Nd}\mathcal{N}(\overline{F})} Z_{N,\beta}^{U}(\mathbb{1}+F,f) \end{aligned} \tag{3.3.66}$$

Taking $f=0$ we get (3.3.63). Dividing both sides by the corresponding expression for $f=0$ we get (3.3.65). It follows from (3.3.58) that

$$\begin{aligned} L^{dN}\mathcal{W}_{N,\beta}^{\mathcal{U}+\mathcal{N}}(F) &= -\ln Z_{N,\beta}^{U}(\mathbb{1}+F,0) - \beta L^{Nd}\mathcal{U}(\overline{F}) + \ln Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}} \\ &= L^{dN}\mathcal{W}_{N,\beta}^{U}(\mathbb{1}+F) + \ln Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}} - \ln Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}} \end{aligned} \tag{3.3.67}$$

Thus (3.3.64) follows if we can show that

$$Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}+\mathcal{N}}} = Z_{N,\beta}^{\mathcal{Q}_{\mathcal{U}}}. \tag{3.3.68}$$

To prove (3.3.68) note that Lemma 3.3.6 implies that

$$\sum_{x\in T_N} \mathcal{N}(sD\varphi(x)+tD\varphi(x)) = \sum_{x\in T_N} \mathcal{N}(0) = 0. \tag{3.3.69}$$

Taking the derivative with respect to s and t at $s=t=0$ we get $\sum_{x\in T_N} \mathcal{Q}_{\mathcal{N}}(D\varphi(x)) = 0$. This yields (3.3.68). □

We now prove that the potential $\mathcal{U}+\mathcal{N}$ satisfies the conditions in Proposition 3.2.4 so that we can apply the results from the previous section.

Lemma 3.3.11. *Under the hypotheses (H1), (H2), (H3), (H4), and (H5) the function* $\mathcal{U}+\mathcal{N}$ *satisfies the assumptions of Proposition 3.2.4, i.e.,*

$$\mathcal{U}+\mathcal{N} \in C^{r_0+r_1}(\mathcal{G}_{R'}), \tag{3.3.70}$$

$$\omega_0|z|^2 \le \mathcal{Q}_{\mathcal{U}+\mathcal{N}}(z) \le \omega_0^{-1}|z|^2 \tag{3.3.71}$$

$$\mathcal{U}(z)+\mathcal{N}(z) - \big(\mathcal{U}(0)-\mathcal{N}(0)\big) - D\big(\mathcal{U}(0)+\mathcal{N}(0)\big)(z) \ge \omega_1|z|^2, \textit{ and} \tag{3.3.72}$$

$$\lim_{t\to\infty} t^{-2}\ln\Psi(t) = 0 \quad \textit{where} \quad \Psi(t) := \sup_{|z|\le t} \sum_{3\le|\alpha|\le r_0+r_1} \frac{1}{\alpha!}\big|\partial^\alpha\big(\mathcal{U}(z)+\mathcal{N}(z)\big)\big|. \tag{3.3.73}$$

with $\omega_0 = \min(2\omega_1,\omega_2)$*, where* ω_1 *and* ω_2 *are the constants in* (3.3.51) *and* (3.3.55)*, respectively.*

Proof. The first condition is a consequence of the smoothness of $\mathcal{Q}$, $\mathcal{N}$, and $\mathcal{U}$ which follows by the chain rule from the smoothness of U postulated in (H5) and the smoothness of the polynomial N. The second condition follows from (3.3.54) and (3.3.55). The third condition follows from (3.3.51). The last condition follows from the fact that the $\mathcal{U}$-term is controlled by (H5) and the chain rule and that $\mathcal{N}$ is a polynomial. □

Finally we show how to deduce the results for the discrete elasticity model from those for the generalised gradient models.

Proof of Theorem 3.3.1. By Lemma 3.3.11 the potential $\mathcal{U}+\mathcal{N}$ satisfies the assumption of Proposition 3.2.4 (with $\omega := \frac{\omega_0}{8}$). Thus Theorem 3.3.1 follows from Theorem 3.2.3 and (3.3.64). □

Proof of Theorem 3.3.2. By Lemma 3.3.11 the potential $\mathcal{U} + \mathcal{N}$ satisfies the assumption of Proposition 3.2.4 (with $\omega := \frac{\omega_0}{8}$) and thus Theorem 3.2.9 can be applied to the generalised gradient model. Together with (3.3.65) this gives

$$\lim_{\ell\to\infty} \mathbb{E}_{\gamma^{\mathbb{1}+F}_{N_\ell,\beta}} e^{(f_{N_\ell},\varphi)} = e^{\frac{1}{2\beta}(f,\mathcal{C}_{\mathbb{T}_d} f)} \tag{3.3.74}$$

where $\mathcal{C}_{\mathbb{T}_d}$ is the inverse of the operator $\mathcal{A}_{\mathbb{T}_d}$ given by

$$(\mathcal{A}_{\mathbb{T}_d} u)_s = -\sum_{t=1}^{d}\sum_{i,j=1}^{d} (\boldsymbol{Q}_{\mathcal{U}+\mathcal{N}} - \boldsymbol{q})_{i,j;s,t}\partial_i\partial_j u_t. \tag{3.3.75}$$

In particular the operator $\mathcal{A}_{\mathbb{T}_d}$ depends only on the action of $\boldsymbol{Q}_{\mathcal{U}+\mathcal{N}}$ on the subspace $\mathcal{G}^{\nabla}_{R'}$. Now each $z \in \mathcal{G}^{\nabla}_{R'}$ is of the form $z = \overline{F} = DF$ where $F : \mathbb{R}^d \to \mathbb{R}^d$ is linear. By the definition of $\mathcal{U}$ and $\mathcal{N}$, see (3.3.42)–(3.3.44) and (3.3.46), we have

$$(\mathcal{U} + \mathcal{N})(\overline{F}) = (U + \mathsf{N})(\mathbb{1}_A + F_A) + \frac{\mu}{2d}\mathsf{N}_0(\mathbb{1}_{Q_{R'}} + F_{Q_{R'}}) = (U + \mathsf{N})(\mathbb{1}_A + F_A) \tag{3.3.76}$$

since N_0 vanishes on linear maps. Thus

$$\boldsymbol{Q}_{\mathcal{U}+\mathcal{N}}(\overline{F}) = D^2(U + \mathsf{N})(\mathbb{1})(F_A, F_A) = \boldsymbol{Q}^{\nabla}_{U+\mathsf{N}}(F) \tag{3.3.77}$$

where we used the definition (3.3.12) of $\boldsymbol{Q}^{\nabla}_{U+\mathsf{N}}$ for the second identity. It follows that the operator $\mathcal{A}_{\mathbb{T}_d}$ can be written as

$$(\mathcal{A}_{\mathbb{T}_d} u)_s = -\sum_{t=1}^{d}\sum_{i,j=1}^{d} (\boldsymbol{Q}^{\nabla}_{U+\mathsf{N}} - \boldsymbol{q})_{i,j;s,t}\partial_i\partial_j u_t. \tag{3.3.78}$$

Now $\boldsymbol{Q}^{\nabla}_{U+\mathsf{N}} = \boldsymbol{Q}^{\nabla}_{U} + \boldsymbol{Q}^{\nabla}_{\mathsf{N}}$ and it only remains to show that $\boldsymbol{Q}^{\nabla}_{\mathsf{N}}$ generates the zero operator. Multiplying by a test function $g \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, denoting the scalar product on $\mathbb{R}^d$ by $\cdot$, recalling that $\mathsf{N}(F_A) = \alpha \det F$ and using that det is a null Lagrangian on maps defined on $\mathbb{T}^d$, i.e. $\int_{\mathbb{T}^d}(\det(\mathbb{1} + \nabla h) - \det \mathbb{1})\,\mathrm{d}x = 0$ for all $h \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, we get

$$-\int_{\mathbb{T}^d} g \cdot \sum_{i,j=1}^{d} (\boldsymbol{Q}^{\nabla}_{\mathsf{N}})_{i,j}\partial_i\partial_j f\,\mathrm{d}x = \int_{\mathbb{T}^d} \sum_{i,j=1}^{d} \partial_i g \cdot (\boldsymbol{Q}^{\nabla}_{\mathsf{N}})_{ij}\partial_j f \tag{3.3.79}$$

$$= \int_{\mathbb{T}^d} \alpha D^2 \det(\mathbb{1})(\nabla f, \nabla g) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}_{|s=t=0} \int_{\mathbb{T}^d} \alpha \det(\mathbb{1} + s\nabla f + t\nabla g) = 0. \tag{3.3.80}$$

Thus in (3.3.78) we may replace $\boldsymbol{Q}^{\nabla}_{U+\mathsf{N}}$ by $\boldsymbol{Q}^{\nabla}_{U}$ and this finishes the proof of Theorem 3.3.2. □

Remark 3.3.12. *Completely independent from the analysis of discrete elasticity the null Lagrangian* N_0 *introduced in Lemma 3.3.9 can be used to gain coercivity in generalized gradient models with* $\mathbb{R}^m$ *valued fields. Indeed, the same arguments as used in this section show that the requirement in* (3.2.20) *can be replaced by* (3.2.21).

Chapter 4

Renormalisation group analysis of gradient models

The results of this chapter are joint work in progress with Stefan Adams, Roman Kotecký, and Stefan Müller. A sketch how the loss of regularity can be avoided based on a new finite range decomposition (see Chapter 2) already appeared in the author's master's thesis [55]. More precisely, Lemma 4.6.4 is similar to Lemma 6.6 in [55].

4.1 Introduction

This chapter contains our renormalisation group analysis of gradient models based on the approach by Bauerschmidt, Brydges, and Slade [44, 45, 20, 46, 47] and extending the paper [4] by Adams, Kotecký, and Müller. For a general background on the approach and references to the literature we refer to Chapter 1. Recall that the goal of the renormalisation analysis is to control perturbations of Gaussian integrals of the form

$$\mathcal{Z}_N(\mathcal{K}, \mathcal{Q}, f) = \int_{\mathcal{X}_N} e^{(f,\varphi)} \sum_{X \subset T_N} \prod_{x \in X} \mathcal{K}(D\varphi(x))\, \mu_{\mathcal{Q}}(\mathrm{d}\varphi). \tag{4.1.1}$$

for small perturbations $\mathcal{K}$. In this chapter we control this expression using a careful multiscale analysis obtaining the representation formula in Theorem 4.9.1 as our main result. We will outline the general strategy in the next section. Here we just recall very briefly the differences to the earlier works and explain the outline of this chapter.

The main difference to the analysis of the φ^4-theory are, on the one hand, that only first order perturbation theory is necessary and we obtain a dynamical system with a hyperbolic fixed point. On the other hand, we have less symmetry and therefore more relevant terms. Moreover, the large field problem, i.e., the control of the perturbations $\mathcal{K}$ for large fields φ is more subtle in our setting.

We extend the results by Adams, Kotecky, and Müller in several directions. We consider general finite range interactions for vector valued fields. Since the method is rather robust in this respect this requires mostly notational changes. We use a new finite range decomposition as constructed in Chapter 2 that avoids a loss of regularity thus simplifying several arguments concerning the smoothness of the renormalisation maps (see Section 4.7). We treat norms in a more systematic way which allows us to consider all dimensions $d \geq 2$ instead of $d \in \{2, 3\}$ (see Section 4.6 and Appendix 4.A). The main improvement is the construction of new weight functions that allow us to handle potentials with much weaker growth assumptions, in particular we can deal with realistic interactions for discrete elasticity as discussed in Chapter 3. This provides a new solution of the large field problem in our setting. The construction of the weights is mostly independent of the remaining parts of this chapter and can be found in Section 4.5. Finally, there are several smaller changes regarding, e.g., the combinatorics.

Let us now outline the structure of this chapter. The main result we prove is the representation formula in Theorem 4.9.1. From this result it is rather straightforward to conclude the main results of the previous Chapter, Theorem 3.2.2 and Theorem 3.2.7. Since the proof is rather involved we provide the reader with an overview of the general strategy and the main steps of the proof in the next Section 4.2. Moreover, in Section 4.3, we describe the most important parameters used in the argument in order to facilitate the understanding of the interaction of the various parts of the following arguments and as a future reference for the various restrictions on their choice. In Section 4.4 we discuss the general setup of the multiscale analysis. Then, in Section 4.5 we construct the new weight functions. Section 4.6 contains important submultiplicativity estimates for our norms and the definition of the projection on the relevant directions which corresponds to the operator *loc* in the language of Bauerschmidt, Brydges, and Slade. The following two Sections 4.7 and 4.8 contain the proofs of the key results that the renormalisation map is smooth and has a hyperbolic fixed point. Then Section 4.9 contains the proof of the main results based on the representation theorem. The latter theorem is proved in 4.10 using a suitable stable manifold theorem.

4.2 Explanation of the method

In this section we outline our general approach. It follows closely the programme for the rigorous renormalisation group analysis of functional integrals which has been systematically developed by Brydges, Slade and coworkers over the last decades, see [42, 48, 24] for surveys and additional references to earlier and related work.

4.2.1 Set-up

We focus on an outline of the strategy to prove Theorem 3.2.2, the proof of Theorem 3.2.7 is very similar. We want to study the integral

$$\mathcal{Z} := \int_{\mathcal{X}_N} \sum_{X\subset\Lambda} K(X,\varphi)\,\mu^{(0)}(\mathrm{d}\varphi) \tag{4.2.1}$$

where

$$K(X,\varphi) = \prod_{x\in X} \mathcal{K}(D\varphi(x)) \tag{4.2.2}$$

and $\mu^{(0)}$ is the Gaussian measure given by

$$\mu^{(0)}(d\varphi) = \frac{1}{Z^{(0)}} e^{-\frac{1}{2}\sum_{x\in\Lambda_N} \mathcal{Q}(D\varphi(x))}\lambda(\mathrm{d}\varphi). \tag{4.2.3}$$

It turns out that it is convenient to embed this problem into a more general family of problems of the form

$$\mathcal{Z}(H_0, K_0, \boldsymbol{q}) := \int_{\mathcal{X}_N} (e^{-H_0} \circ K_0)(\Lambda_N, \varphi)\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi). \tag{4.2.4}$$

Here $\boldsymbol{q}$ is a small symmetric $md \times md$ matrix and $\mu^{(\boldsymbol{q})}$ is the Gaussian measure given by

$$\mu^{(\boldsymbol{q})}(d\varphi) = \frac{1}{Z^{(\boldsymbol{q})}} e^{-\frac{1}{2}\sum_{x\in\Lambda_N} \mathcal{Q}(D\varphi(x)) - (\boldsymbol{q}\nabla\varphi(x),\nabla p(x))} \lambda(\mathrm{d}\varphi). \tag{4.2.5}$$

The circ product $\circ$ of maps F, G defined the subsets of Λ_N is given by

$$F \circ G(X) = \sum_{Y \subset X} F(Y) G(X \setminus Y) \tag{4.2.6}$$

for $X \subset \Lambda_N$. The sum includes the empty set and we set $F(\emptyset) = G(\emptyset) = 1$. The definition of the circ product is motivated by the following property. If F and G factor, i.e. if $F(X) = \prod_{x\in X} F(\{x\})$ and $G(X) = \prod_{x\in X} G(\{x\})$ then

$$F \circ G(X) = \prod_{x\in X} (F+G)(\{x\}). \tag{4.2.7}$$

The term H_0 plays a special role which will be further discussed below. It only contains so called relevant terms, namely constants and certain linear and quadratic expressions in φ. More specifically we assume that

$$H_0(X,\varphi) = \sum_{x\in X} H_0(\{x\},\varphi) \tag{4.2.8}$$

$$H_0(\{x\},p) = a_\emptyset + \sum_{1\le|\alpha|\le\lfloor d/2\rfloor+1} \sum_{i=1}^{m} a_{i,\alpha}\nabla^\alpha\varphi_i(x) + \frac{1}{2}(\boldsymbol{a}\nabla\varphi(x),\nabla\varphi(x)) \tag{4.2.9}$$

where $\boldsymbol{a}$ is symmetric $md \times md$ matrix.

The original problem corresponds to the choices $\boldsymbol{q} = 0$, $H_0(X,\varphi) = 0$ and $K_0(X,\varphi) = \prod_{x\in X} \mathcal{K}(D\varphi(x))$.

4.2.2 Finite range decomposition

The first idea is to replace the integration against the Gaussian measure $\mu^{(\boldsymbol{q})}$, by a sequence of integration against Gaussian measures $\mu_k^{(\boldsymbol{q})}$, $k = 1, \dots N+1$ such that the measure $\mu_k^{(\boldsymbol{q})}$ essentially detects the behaviour of the fields φ on the spatial scales between L^{k-1} and L^k.

More precisely, we express the translation-invariant covariance operator $\mathcal{C}^{(\boldsymbol{q})}$ of the Gaussian measure $\mu^{(\boldsymbol{q})}$ as a sum of translation-invariant covariance operators with finite range, i.e.,

$$\mathcal{C}^{(\boldsymbol{q})} = \sum_{k=1}^{N+1} \mathcal{C}_k^{(\boldsymbol{q})}, \quad \text{and the corresponding kernels satisfy} \quad \mathcal{C}_k^{(\boldsymbol{q})} = -C_k \text{ for } |x|_\infty \ge \frac{L^k}{2}. \tag{4.2.10}$$

Moreover the kernel $\mathcal{C}_k^{(\boldsymbol{q})}$ behaves like the Green's function of the discrete Laplace operator on scale L^{k-1}, i.e.,

$$\left|\nabla^\alpha \mathcal{C}_k^{(\boldsymbol{q})}(x)\right| \le \begin{cases} C_\alpha L^{-(k-1)(d-2+|\alpha|)} \text{ for } d+|\alpha|>2 \\ C_\alpha \ln(L) L^{-(k-1)(d-2+|\alpha|)} \text{ for } d+|\alpha|=2. \end{cases} \tag{4.2.11}$$

Then $\mu^{(\boldsymbol{q})} = \mu_{N+1}^{(\boldsymbol{q})} * \dots * \mu_1^{(\boldsymbol{q})}$ and thus the quantity $\mathcal{Z}(H_0, K_0, \boldsymbol{q})$ can be expressed as an $N+1$ fold integral. Alternatively we can define the convolution operator $\boldsymbol{R}_k^{(\boldsymbol{q})}$ by

$$(\boldsymbol{R}_k^{(\boldsymbol{q})} F)(\psi) = \int_{\mathcal{X}_N} F(\psi+\varphi)\mu_k^{(\boldsymbol{q})}(\mathrm{d}\varphi). \tag{4.2.12}$$

Then the integral we are interested in can be written as

$$\mathcal{Z}(H_0, K_0, \boldsymbol{q}) = \big(\boldsymbol{R}^{(\boldsymbol{q})}_{N+1}\boldsymbol{R}^{(\boldsymbol{q})}_{N} \dots \boldsymbol{R}^{(\boldsymbol{q})}_{1}(e^{-H_0} \circ K_0)\big)(\Lambda_N, 0). \tag{4.2.13}$$

4.2.3 The renormalisation map

In view of (4.2.13) the key idea is to define a map $\boldsymbol{T}_k : (H_k, K_k, \boldsymbol{q}) \mapsto (H_{k+1}, K_{k+1})$ such that

$$e^{-H_{k+1}} \circ K_{k+1}(\Lambda_N) = \boldsymbol{R}^{(\boldsymbol{q})}_{k+1}(e^{-H_k} \circ K_k(\Lambda_N)). \tag{4.2.14}$$

Then

$$\mathcal{Z}(H_0, K_0, \boldsymbol{q}) = \big(\boldsymbol{R}^{(\boldsymbol{q})}_{N+1}(e^{-H_N} \circ K_N)\big)(\Lambda_N, 0) = \int_{\mathcal{X}_N} (e^{-H_N} \circ K_N)(\Lambda_N, \varphi)\, \mu^{(\boldsymbol{q})}_{N+1}(\mathrm{d}\varphi). \tag{4.2.15}$$

Of course the property (4.2.14) does not determine $\boldsymbol{T}_k$ uniquely. Indeed, for any $\tilde{H}$ such that $\tilde{H}$ satisfies (4.2.8) we can write using (4.2.7)

$$e^{-H} \circ K(X) = (e^{-\tilde{H}} + e^{-H} - e^{-\tilde{H}}) \circ K(X) = \Big(e^{-\tilde{H}} \circ (e^{-H} - e^{-\tilde{H}}) \circ K\Big)(X) = e^{-\tilde{H}} \circ \tilde{K}(X) \tag{4.2.16}$$

where $\tilde{K} = (e^{-H} - e^{-\tilde{H}}) \circ K$.

The guiding principle for the definition of $\boldsymbol{T}_k$ is that we want $\boldsymbol{T}_k(0, 0, \boldsymbol{q}) = (0, 0)$ and that the derivative of $\boldsymbol{T}_k$ at the origin is contracting in K_k and expanding in H_k. This will allow us to apply the stable manifold theorem to show that the term on the right hand side of (4.2.15) is 1 up to an exponentially small correction provided that we chose H_0 suitably in dependence of K_0, see the next subsection. Indeed, the special form of relevant Hamiltonians given in (4.2.9) stems from the fact that exactly monomials of this form do not lead to a contraction under application of $\boldsymbol{R}^{(\boldsymbol{q})}$ if we equip the space of functionals with natural scale dependent norms. See the text after (4.4.42) for further discussion on relevant vs. irrelevant monomials. The definition of the map $\boldsymbol{T}_k$ thus involves three key steps

- Integration against $\mu^{(\boldsymbol{q})}_{k+1}$, i.e., application of $\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}$
- Extraction of the relevant terms, see (4.4.66) where $\boldsymbol{R}' = \boldsymbol{R}^{(\boldsymbol{q})}_{k+1}$
- Coarse-graining to maps defined on disjoint blocks of size L^{k+1} ($k+1$-blocks) and their union ($k+1$-polymers) rather than single points and subsets of Λ_N, see (4.4.65) and (4.4.67)

The motivation for the coarse graining is that a field φ which is typical under the next-scale measure $\mu^{(\boldsymbol{q})}_{k+2}$ varies only slowly on scale L^{k+1}. The circ product is adjusted to the coarse graining: for two maps F, G on k-polymers the circ product is defined as

$$F \circ G(X) = \sum_{Y k\text{-polymer}, Y \subset X} F(Y) G(X \setminus Y). \tag{4.2.17}$$

In particular for $k = N$ there are only two polymers, the whole torus Λ_N and the empty set. Thus the right hand side of (4.2.15) simplifies further since $e^{-H_N} \circ K_N(\Lambda_N) = e^{-H_N}(\Lambda_N) + K_N(\Lambda_N)$.

The key results about the maps $\boldsymbol{T}_k$ are contained in Theorems 4.4.7 and 4.4.8 below: they are smooth in a small neighbourhood (uniformly in k and N) and the derivatives at the origin are given by

$$D\boldsymbol{T}_k(0)\begin{pmatrix}\dot{H}\\ \dot{K}\end{pmatrix} = \begin{pmatrix}\boldsymbol{A}_k & \boldsymbol{B}_k\\ 0 & \boldsymbol{C}_k\end{pmatrix}\begin{pmatrix}\dot{H}\\ \dot{K}\end{pmatrix} \tag{4.2.18}$$

where

$$\|\boldsymbol{A}_k^{-1}\| < c \le 1, \quad \|\boldsymbol{C}_k\| < c \le 1. \tag{4.2.19}$$

These estimate give a precise formulation of the idea that the flow is contracting in the K variable and expanding in the H variable.

4.2.4 Application of the stable manifold theorem and fine tuning

The uniform smoothness of the maps $\boldsymbol{T}_k$ and the contraction estimates (4.2.19) allow us to apply a discrete version of the stable manifold theorem. This guarantees that there exists a smooth function $\hat{H}_0$ such that for each sufficiently small K_0 the flow starting with $(\hat{H}_0(K_0, \boldsymbol{q}), K_0)$ satisfies $H_N = 0$ and $\|K_N\| \le C\eta^N$ for a suitable $\eta < 1$. This is described in full detail in Section 4.10 below (for a slightly modified situation).

The basic idea is very simple. One considers the vector $Z = (H_0, \dots, H_{N_1}, K_1, \dots K_N)$ and a weighted norm $\|Z\| = \max(\max_{0\le k\le N-1} \eta^{-k}\|H_k\|, \max_{1\le k\le N} \eta^{-k}\|K_k\|)$. The space of vectors with finite norm is denoted by $\mathcal{Z}$. Then one reformulates the conditions that $(H_{k+1}, K_{k+1}) = \boldsymbol{T}_k(H_k, K_k, \boldsymbol{q})$ and $H_N = 0$ as a fixed point condition. More precisely one defines a map $\widetilde{\mathcal{T}}$ on $\mathcal{Z}$ which has K_0 and $\boldsymbol{q}$ as an additional parameters such every Z which satisfies $\widetilde{\mathcal{T}}(\boldsymbol{q}, K_0, Z) = Z$ also satisfies $(H_{k+1}, K_{k+1}) = \boldsymbol{T}_k(H_k, K_k, \boldsymbol{q})$ for $k \le N-2$ and $\boldsymbol{T}_{N-1}(H_{k-1}, K_{k-1}) = (0, K_N)$.

The contraction estimates (4.2.19) will imply that that map $\widetilde{\mathcal{T}}(\boldsymbol{q}, K_0, \cdot)$ does indeed have a fixed point $Z = \hat{Z}(\boldsymbol{q}, K_0)$ for every small K_0. Then the map $\hat{H}_0$ is obtained by taking the H_0 component of $\hat{Z}$. Thus we get for each small K_0

$$\mathcal{Z}(\hat{H}_0(K_0, \boldsymbol{q}), K_0, \boldsymbol{q}) = \int_{\mathcal{X}_N} (1 + K_N(\Lambda_N, \varphi))\, \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi) \tag{4.2.20}$$

where K_N is exponentially small (and depends smoothly on K_0 and $\boldsymbol{q}$). If $\hat{H}_0(K_0, 0) = 0$ holds by chance we have solved our original problem. In general, there is, however, no reason why this should be true.

In the final step we will thus use the freedom to tune the free parameter $\boldsymbol{q}$ so that the effects of $\boldsymbol{q}$ in the Gaussian measure $\mu^{(\boldsymbol{q})}$ and the effect of $\hat{H}_0(K_0, \boldsymbol{q})$ cancel exactly up to a constant term which can be pulled out of the integral. Thus the final dependence our original partition function $Z(\mathcal{K})$ on $\mathcal{K}$ is encoded in this constant term, up to an exponentially small term which comes from K_N. This allows us to conclude easily.

The details of this fine-tuning procedure are explained in Section 4.10. It is actually convenient to write the enlarged family of problems in a slightly different way. Instead of working only with $\boldsymbol{q}$ as the main free parameter we use a full relevant Hamiltonian (see (4.2.9)) as the free parameter and identify $\boldsymbol{q}$ with the quadratic part $\boldsymbol{a}$ of the relevant Hamiltonian. Denoting the relevant Hamiltonian by $\mathcal{H}$ and the quadratic part by $\boldsymbol{q}(\mathcal{H})$ we are thus lead to study the family of problems

$$\int_{X_N} \left(e^{-H_0} \circ \hat{K}_0(\mathcal{H}, \mathcal{K})\right)(\Lambda_N)\, \mu^{(\boldsymbol{q}(\mathcal{H}))}(\mathrm{d}\varphi) \quad \text{with} \quad \hat{K}_0(\mathcal{H}, \mathcal{K}) = e^{-\mathcal{H}}\mathcal{K}, \tag{4.2.21}$$

see Section 4.10. We then show as above that there exists a function $\hat{H}_0$ such that the choice $H_0 = \hat{H}_0(\mathcal{H}, \mathcal{K})$ leads to $H_N = 0$ and an exponentially small K_N. In fact we can use exactly the argument given above in connection with the observation that the map $(\mathcal{H}, \mathcal{K}) \to e^{-\mathcal{H}}\mathcal{K}$ is smooth. It is then easy to see that there exists a map $\hat{\mathcal{H}}$ such that $\hat{H}_0(\hat{\mathcal{H}}(\mathcal{K}), \mathcal{K}) = \hat{\mathcal{H}}(\mathcal{K})$ and that the integral for $\mathcal{H} = \hat{\mathcal{H}}(\mathcal{K})$ and $H_0 = \hat{H}_0(\mathcal{H}, \mathcal{K})$ agrees with our original integral up to a scalar factor.

4.2.5 A glimpse at the implementation of the strategy

Our main objects are relevant Hamiltonians H_k and perturbations K_k. The relevant Hamiltonians are described by the parameters $a_\emptyset$ for the constant part, $a_{\alpha,i}$ for the linear part and $\boldsymbol{a}$ for the quadratic part. The K_k are functions depending on a k-polymer X and the field φ. One key ingredient is to design the RG maps $\boldsymbol{T}_k$ so that at each step the relevant terms are correctly extracted. This can already by guessed at the level of the linearised problem. Another key ingredient is to design norms for H_k and K_k which allow us to prove uniform smoothness and contraction estimates. The construction of such norms will be described in detail in Section 4.4. Here we just mention three guiding principles

- The norms at scale k for the fields φ should be such that a field which is 'typical' under the measure $\mu_k^{(\boldsymbol{q})}$ has norm of approximately order 1;
- For a fixed k polymer the norm on the functional $\varphi \mapsto K(X, \varphi)$ should be dual to the field norm. For linear functionals it is clear what duality means. Homogeneous polynomials or degree r can be viewed as linear functionals on the r-fold tensor product of the space of fields and there is a natural way to design norms which behave well under tensorisation (see Appendix 4.A);
- Our starting perturbation factors, i.e. $K_0(X) = \prod_{x \in X} K_0(\{x\})$. This suggests that for small K the size of $K(X)$ should decrease exponentially in the number of blocks in X. The property that K factors is lost in the iteration. To keep the idea that the contribution from large polymers is exponentially small, a weight $A^{|X|_k}$ where $|X|_k$ is the number of k-blocks in the polymer X is introduced in the definition of the norm of K_k.

Two further points turn out to be important. First, while the factorisation property is in general lost, the finite range condition (4.2.10) on the covariance in the finite range decomposition ensures that factorisation still holds between polymers that are separated by one block. Here we use the fact that we work on fields with zero average. Thus the the action of the kernel $\mathcal{C}_k^{(\boldsymbol{q})}$ on fields by discrete convolution does not change if add a constant to the kernel. Hence the condition condition $\mathcal{C}_k^{(\boldsymbol{q})} = -C_k$ for $|x|_\infty \geq \frac{L^k}{2}$ is equivalent to assuming that $\mathcal{C}_k^{(\boldsymbol{q})}$ is supported in $\{x : |x|_\infty < \frac{L^k}{2}\}$.

This factorisation property for polymers that are separated by one block allows us to track only $K_k(X, \cdot)$ for *connected* polymers X. The functional for general polymers is then obtained by multiplying over the connected components.

The second point is the so called large field problem. With exponential small probability very large values of the field $\nabla\varphi(x)$ may arise. Since a typical perturbation $\mathcal{K}(D\varphi(x)) = e^{-\overline{\mathcal{U}}(D\varphi(x))} - 1$ contains also exponential terms care has to be taken that the integrals in each step are well-defined. This problem is well known in rigorous renormalisation theory and handled by the introduction of carefully chosen weights, or large-field regulators, in the norms of K_k. In Section 4.5 we present a new construction of weights which leads to almost optimal weights.

4.3 Choice of parameters

The precise implementation of the renormalisation construction involves a number of parameters which help to fine tune the properties of the renormalisation map and to ensure the key smoothness and contraction estimates in Theorems 4.4.7 and 4.4.8 from which the main results Theorem 3.2.2 and 3.2.7 can be deduced.

The purpose of this subsection is to give an overview over these parameters and to explain how they are chosen. Detailed descriptions are given in the following subsections. Here we focus on a bird's eye view to emphasise the idea the parameters can be chosen in such a way that all the restriction which arise in the following sections can be satisfied simultaneously.

Actually most of the parameters can be chosen once and for all (in dependence on the dimension d of the model and the maximal order R_0 of discrete derivatives in a coordinate direction). We will refer to these as 'fixed parameters' and we will not track how the various constants depend on these parameters. A list of these fixed parameters is given in Subsection 4.3.2 below. We first discuss the free parameters which we will adjust to obtain the desired smoothness and contraction estimates.

4.3.1 The free parameters L, h, and A

There are three free parameters, namely

- $L \in \mathbb{N}$: The size of a basic block
- $h \gg 1$: A scaling factor in the norm for the fields; the field norm on level k involves a term h_k^{-1} with $h_k = 2^k h$, see (4.4.74) and (4.4.76). A field which is typical on scale k (i.e., under the measure μ_{k+1}) has norm of order h_k^{-1}. Since the norms on functionals are defined by duality the standard Hamiltonian $H(\varphi) = \sum_{x\in B} |\nabla\varphi|^2$ for a block B on scale k has norm h_k^2. In our earlier work [4] we used a scaling factor h which was independent of k. The reason we now need scaling factors h_k which grow sufficiently rapidly in k is related to the new choice of nearly optimal weights (see Section 4.5). Among others, we want to bound the field norm by the increase in the logarithm of the weights as we go from scale k to $k+1$ (see (4.5.25)). This essential requires that $\sum_{k<N} h_k^{-2}$ can be bounded independent of N. A similar issue arises for the estimates (4.5.23) and (4.5.24). The choice of exponentially growing scaling factors is mostly for convenience. We can not allow for faster than exponential growth because factors of h_{k+1}/h_k appear in the proof of the change of scale estimate in Lemma 4.6.1 and Lemma 4.6.9.
- $A \gg 1$: A parameter which penalises the contributions of functionals defined on long polymers. The norm on functionals involves a supremum over all k-polymers X of $A^{|X|}\|K(X)\|_k$ where $|X|$ denotes the number of k-blocks in X.

Our goal is to show that there exists a number L_0 and functions $L \mapsto h_0(L)$ and $L \mapsto A_0(L)$ such that the renormalisation maps $\boldsymbol{T}_k = \boldsymbol{T}_k^{(\boldsymbol{q})}$ have good properties (in a suitable small neighbourhood of 0 and for sufficiently small $\boldsymbol{q}$) if

$$L \geq L_0, \quad h \geq h_0(L), \quad \text{and} \quad A \geq A_0(L). \tag{4.3.1}$$

In the following we first review the choice for the fixed parameters. Then we describe the key steps in the proof and discuss which restrictions on the free parameters L, h, and A arise in each step.

4.3.2 Fixed parameters

The following parameters are fixed once and for all and dependence on them is usually not indicated in the following

- d: Spatial dimension.
- m: Number of components of φ.
- R_0: A nonzero integer which determines the maximal number of discrete (forward) derivatives through the set $\{e_1, \ldots, e_d\} \subset \mathcal{I} \subset \{\alpha \in \mathbb{N}_0^d \setminus \{(0, \ldots, 0)\} : \ |\alpha|_\infty \leq R_0\}$.
- $r_0 \geq 3$: An integer which measures smoothness of the functionals in the field. Loosely speaking, the restriction $r_0 \geq 3$ arises from the fact that third order terms are always irrelevant, but quadratic terms are not. More precisely, the condition $r_0 \geq 3$ is crucial for the two-norm estimate (4.6.2). This estimate in particular allows us to deduce the crucial contraction estimate for $\boldsymbol{C}^{(\boldsymbol{q})}$ from a contraction estimate for the action of the extraction operator $1 - \Pi_2$ on Taylor polynomials at zero. See Lemma 4.6.9 and Lemma 4.8.3 in connection with (4.8.2) for further details. We will take

$$r_0 = 3. \tag{4.3.2}$$

- $r_1 \geq 2$: An integer which measures smoothness with respect to external parameters (e.g., the deformation F)
- p_Φ: Number of discrete derivatives in the definition of the field norm $|\phi|_{j,X}$. We need $p_\Phi \geq \lfloor d/2 \rfloor + 2$ to get the right decay in L in the Poincaré type estimate in Lemma 4.6.10 which is the main ingredient in the proof of the contraction estimate for $1 - \Pi_2$ (see Lemma 4.6.9 and we will take

$$p_\Phi = \lfloor d/2 \rfloor + 2. \tag{4.3.3}$$

- M: Number of discrete derivatives in the definition of the quadratic form $\boldsymbol{M}_k^X$ in (4.5.2). We need $M \geq p_\phi + \lfloor d/2 \rfloor + 1$ to be able to apply the discrete Sobolev embedding and to get control of p_Φ discrete derivatives in the supremum norm. We will take

$$M = p_\Phi + \lfloor d/2 \rfloor + 1 = 2\lfloor d/2 \rfloor + 3. \tag{4.3.4}$$

- R: A geometric parameter which is used to define a neighbourhood around blocks (see (4.4.34)). It determines the allowed range of dependence of the functionals on the first scale, e.g., $\mathcal{K}(\{x\}, \varphi)$, $\mathcal{H}_0(\{x\}, \varphi)$, and $\boldsymbol{M}_0^{\{x\}}$ (see (4.2.2), (4.2.8), and (4.5.2)) may only depend on $\varphi\restriction_{x+[-R,R]^d}$. This implies that we need that $R \geq \max(R_0, M, p_\Phi) = \max(R_0, M)$ and we will take

$$R = \max(R_0, M) = \max(R_0, 2\lfloor d/2 \rfloor + 3). \tag{4.3.5}$$

- n: The number of discrete derivatives controlled in the finite range decomposition (see Theorem 4.4.1). We need $n \geq 2M$ to control the integral of the weights against the Gaussian measures obtained by the finite range decomposition (see Theorem 4.5.1x) and its proof in Lemma 4.5.7) and we will take

$$n = 2M = 4\lfloor d/2 \rfloor + 6. \tag{4.3.6}$$

- $\tilde{n}$: A secondary parameter in the finite range decomposition (see Theorem 4.4.1) which relates to the decay of the derivative of the Fourier symbols with respect to the quadratic form we decompose. We need $\tilde{n} \geq n + \lfloor d/2 \rfloor + 1$ to bound the derivative of the maps $\boldsymbol{R}_k^{(\boldsymbol{q})}$ with respect to $\boldsymbol{q}$ (see Theorem 4.4.2) and we will take

$$\tilde{n} = 2M = n + \lfloor d/2 \rfloor + 1 = 5\lfloor d/2 \rfloor + 7. \tag{4.3.7}$$

- $\omega_0 > 0$: A parameter which controls the coerciveness and boundedness of the quadratic form $\mathcal{Q}$. We require (see (3.2.40)

$$\omega_0 |z|^2 \leq \mathcal{Q}(z) \leq \omega_0^{-1} |z|^2 \quad \text{for all} \quad z \in \mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}}. \tag{4.3.8}$$

- $\zeta \in (0,1)$: This parameter controls the exponential weight in the norm $\|\cdot\|_\zeta$ which is defined in (3.2.38) and measures the allowed growth of the perturbation $\mathcal{K}(z)$ as $z \to \infty$.

- $\overline{\zeta} \in (0, \frac{1}{4})$: This parameter analogously controls the growth of the weights, see (4.5.10). To make the norms of the perturbation $\mathcal{K}$ and the corresponding functional K consistent we choose $\overline{\zeta} = \frac{1}{4}\zeta$, see (4.5.8) as well as (4.10.46), (4.10.26) and Lemma 4.10.3.

- $\eta \in (0, \frac{2}{3}]$: This parameter controls the rate of convergence of $\|H_k\|$ and $\|K_k\|$. More precisely it appears in the definition of the norm of the vector $(H_0, \ldots, H_{N-1}, K_1, \ldots K_N)$. Vectors with norm ≤ 1 satisfy $\|H_k\| \leq \eta^k$ and $\|K_k\| \leq \eta^k$, see (4.10.2). For the purpose of the current paper we could take $\eta = \frac{2}{3}$, but other applications require smaller values of η.

4.3.3 Choice of the free parameters in the key steps of the proof

The key technical results are the uniform smoothness and contraction estimates for the renormalisation maps $\boldsymbol{T}_k$ (see Theorem 4.4.7 and Theorem 4.4.8). From those the assertions follow by standard abstract results as outlined in Section 4.10 a discrete stable manifold theorem (Theorem 4.10.1), a second fixed point theorem (Lemma 4.10.6) which implies a representation formula for the partition function (Theorem 4.9.1). From this formula the desired results follow easily, see Section 4.9.2 and Section 4.9.3.

The key steps in the proof of the smoothness and contraction estimates are:

- construction of a familiy of finite range decompositions;
- definition of the renormalisation map and factorisation properties;
- construction of weights;
- submultiplicativity of the norms;
- estimates for the extraction map Π_2 and for $(1 - \Pi_2)$ (with change of scales);
- smoothness and uniform estimates on the derivatives of the renormalisation map (Theorem 4.4.7);
- contraction estimates for the linearised operator (Theorem 4.4.8).

We now review the role of the free parameters in the key steps.

Family of finite range decompositions. In Theorem 4.4.1 we obtain a finite range decomposition for all quadratic forms with $\omega_0/2 \leq \mathcal{Q} \leq 2\omega_0^{-1}$. Dependence on of the estimates on L is expressed explicitly and there is no restriction on L. The parameters h and A do not appear. A key property is that the convolution operators $\boldsymbol{R}_k^{(\boldsymbol{q})}$ which correspond to the finite range composition for the quadratic $\mathcal{Q}^{(\boldsymbol{q})}(z) = \mathcal{Q}(z) - (\boldsymbol{q}z^\nabla, z^\nabla)$ depend smoothly on $\boldsymbol{q}$, with bounds independent of N, see Theorem 4.4.2.

Definition of the renormalisation map: locality, factorisation, geometric properties. To make the combinatorics or coarse-graining and the properties of the finite range decompostion interact nicely we define various neighbourhoods of a polymer and locality conditions on the functionals $\varphi \mapsto K(X, \varphi)$, see Section 4.4.2.

Consistency of these definitions requires $L \geq 2^d + R$. The construction involves a map π which assigns to a polymer X at scale k (a union of blocks of size L^k) a polymer $\pi(X)$ at scale $k+1$. In general X is not contained in $\pi(X)$, but the condition $L \geq 2^d + R$ guarantees that the corresponding small scale neighbourhoods, defined in (4.4.34), satisfy $X^* \subset \pi(X)^*$. To ensure that the renormalisation map preserves the factorization property we need the stronger relation

$$L \geq 2^{d+2} + 4R, \tag{4.3.9}$$

see Proposition 4.4.6.

Weights. To deal with the large field problem we introduce families of weak weights $w_k^X(\varphi)$ and $w_{k:k+1}(\varphi)$ as well as strong weights $W_k^X(\varphi)$ which depend on the field φ and a polymer X. These weights need to satisfy certain natural supermultiplicativity properties and to be consistent with application of the integration map $\boldsymbol{R}_{k+1}^{\boldsymbol{q}}$. These properties are summarised in Theorem 4.5.1. They hold provided that the following constraints are satisfied

$$L \geq 2^{d+3} + 16R, \tag{4.3.10}$$

$$h \geq C\delta^{-1/2}(L), \tag{4.3.11}$$

Here $\delta(L)$ is a parameter that appears in the construction of the weights, see (4.5.10) and (4.5.57). It measures how much the weights can be perturbed using the terms $\boldsymbol{M}_k$ defined in (4.5.4). The free parameter A does not appear in the construction of the weights.

Submultiplicativity of the norms. The map $\boldsymbol{T}_k$ can be written as composition of linear maps, the harmless map $H \mapsto e^{-H}$ for relevant Hamiltonians and a number of polynomial maps which arise from the combinatorics of the circ product and the coarse-graining procedure. The key difficulty is that the degree of the polynomials is not bounded independent of N. Hence an important idea is to work with norms which are submultiplicative so that polynomials (and their derivatives) can be easily estimated. The submultiplicativity of the relevant norms, defined in (4.4.86)-(4.4.88), essentially follows from general facts about tensor product norms on (Taylor) polynomials (see Appendix 4.A) and the supermultiplicativity of the weights. The details are described in Section 4.6. The submultiplicativity estimates require only that the weights satisfy the properties stated in Theorem 4.5.1. Thus the conditions (4.3.10) and (4.3.11) are sufficient as discussed above. The parameter A does not appear.

Estimates for Π_2 and $(1 - \Pi_2)$. A key step in the definition of the map $\boldsymbol{T}_k$ is the extraction of relevant terms. We need that this leads to a bounded map from K_k to H_{k+1} and, more importantly that due to the extraction of the relevant terms the linearisation $\boldsymbol{C}_k^{(\boldsymbol{q})}$ of the map $K_k \mapsto K_{k+1}$ is a strong contraction.

As we will discuss below the main step is to analyse the extraction at the level of Taylor polynomials at zero. This leads to the definition of the projection Π_2 and the remainder map $1 - \Pi_2$, see Section 4.6.4. The key properties of these maps are stated in Lemma 4.6.7 and Lemma 4.6.9. The L dependence is handled explicitly in these lemmas and L only needs to satisfy a mild geometric condition: $L \geq 2^d + R$. The estimates only rely on the definition of the field norms in (4.4.74) and the dependence on h (or $h_k = 2^k h$) cancels exactly. The free parameter A does not appear.

Uniform smoothness estimates. The restrictions on L and h are of the form

$$L \geq L_0, \quad h \geq h_0(L) \tag{4.3.12}$$

and come from Theorem 4.5.1 through Lemma 4.6.3 (submultiplicativity of the norms) and Lemma 4.6.4 (smoothness of the integration map).

The restrictions on A take the form

$$A \geq A_0(L) \tag{4.3.13}$$

and arise from smoothness estimates for the polynomial maps, in particular P_1. An explicit choice of $A_0(L)$ is given in (4.7.91)

Otherwise the dependence of constants L, h and A is tracked explicitly in Section 4.7 and we get explicit bound for the final neighbourhood $U_{\rho,\kappa}$ on which $\boldsymbol{S}$ is smooth. We can take κ as in Theorem 4.5.1 and ρ can be taken of the form $\rho = cA^{-2}$ where c is given explicitly in terms of a constant in the finite range decomposition and the bound for the map Π_2, see Section 4.7.8. The bounds for the derivatives do not depend on h.

Contraction estimates. The contraction estimates impose conditions on all three parameters. To show that the contributions from single blocks are contracting we need $L \geq L_0$ in order to exploit the good L-dependence of $(1 - \Pi_2)$. Here L_0 depends on the constant $A_{\mathcal{B}}$ from Theorem 4.5.1 (this is the only non-geometric condition on L). An explicit choice is given in (4.8.8). The parameter A is used to cancel the combinatorial explosion and can be chosen as $A \geq A_0$ where A_0 depends on $A_{\mathcal{P}}$ from Theorem 4.5.1. An explicit choice of A_0 can be found in (4.8.9). Moreover, Lemma 4.8.6 imposes the minor additional condition (4.8.38) on A_0. For h we obtain the condition $h \geq h_0$ where h_0 must satisfy the condition in Theorem 4.5.1 and, in addition, $h_0 \geq \sqrt{C_{2,0}}$ (see (4.8.29)) where $C_{2,0}$ is the constant in the estimate (4.4.13) for the finite range decomposition.

4.4 Description of the multiscale analysis

In this section we introduce the key elements of the multiscale analysis. We recall the results on finite range decomposition from Chapter 2 and we define function spaces and norms. We continue to work on the discrete torus $T_N = (\mathbb{Z}/(L^N\mathbb{Z}))^d$.

4.4.1 Finite range decompositions

Recall that $\mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}}$ where $\{e_1, \ldots, e_d\} \subset \mathcal{I} \subset \{\alpha \in \mathbb{N}_0^d \setminus \{0, \ldots, 0\} : |\alpha|_\infty \le R_0\}$ and the extended gradient is the vector $D\varphi(x) = (\nabla^\alpha \varphi(x))_{\alpha \in \mathcal{I}} \in \mathcal{G}$. For a positive definite quadratic form $\mathcal{Q}$ on $\mathcal{G}$ the expression

$$\frac{e^{-\frac{1}{2}\sum_{x \in T_N} \mathcal{Q}(D\varphi(x))}}{Z} \lambda_N(\mathrm{d}\varphi) \tag{4.4.1}$$

defines a Gaussian measure on $\mathcal{X}_N$. In this section we recall the existence result of a finite range decomposition for the covariance of such Gaussian measures that we obtained in Chapter 2. Using this decomposition we can rewrite our initial functional integral as a series of integrations. For the convenience of the reader we also repeat some definitions that were already given in Chapter 2.

We denote the generator of $\mathcal{Q}$ by $\boldsymbol{Q} : \mathcal{G} \to \mathcal{G}$ and we get a corresponding elliptic finite difference operator $\mathcal{A}$ on $\mathcal{X}_N$

$$\mathcal{A}_{\boldsymbol{Q}}\varphi = \sum_{\alpha, \beta \in \mathcal{G}} (\nabla^\alpha)^* \boldsymbol{Q}_{\alpha\beta} \nabla^\beta \varphi. \tag{4.4.2}$$

We use $\mathcal{A}_{\boldsymbol{Q}}$ to denote the covariance of the Gaussian measure generated by $\mathcal{C}_{\boldsymbol{Q}} = \mathcal{A}_{\boldsymbol{Q}}^{-1}$.

The operator $\mathcal{A}_{\boldsymbol{Q}} : \mathcal{X}_N \to \mathcal{X}_N$ commutes with translations, hence its inverse $\mathcal{C}_{\boldsymbol{Q}}$ also commutes with translations. Thus there exists a unique kernel $\mathcal{C}_{\boldsymbol{Q}} : T_N \to \mathbb{R}^{m \times m}$ that satisfies $\sum_{x \in T_N} \mathcal{C}_{\boldsymbol{Q}}(x) = 0$ and

$$(\mathcal{C}_{\boldsymbol{Q}}\varphi)(x) = \sum_{y \in T_N} \mathcal{C}_{\boldsymbol{Q}}(x - y)\varphi(y). \tag{4.4.3}$$

Recall that $L \ge 3$ is odd. The dual torus is given by

$$\widehat{T}_N = \left\{ -\frac{(L^N - 1)\pi}{L^N}, -\frac{(L^N - 3)\pi}{L^N}, \ldots, \frac{(L^N - 1)\pi}{L^N} \right\}^d \tag{4.4.4}$$

For $p \in \widehat{T}_N$, we define the functions $f_p : T_N \to \mathbb{C}$ by $f_p(x) = e^{i\langle p, x\rangle}$. Then the Fourier transform $\widehat{\psi} : \widehat{T}_N \to \mathbb{C}$ of a function $\psi : T_N \to \mathbb{C}$ is defined by

$$\widehat{\psi}(p) = \sum_{x \in T_N} f_p(-x)\psi(x). \tag{4.4.5}$$

For vector and matrix valued functions the Fourier transform is defined component-wise. In particular, the Fourier transform diagonalises translation invariant operators

$$\widehat{\mathcal{C}_{\boldsymbol{Q}}\varphi}(p) = \widehat{\mathcal{C}}_{\boldsymbol{Q}}(p)\widehat{\varphi}(p). \tag{4.4.6}$$

We will also use the Plancherel identity

$$(\varphi, \psi)_{T_N} = \frac{1}{L^{Nd}} \sum_{p \in \widehat{T}_N} \widehat{\varphi}(p)\widehat{\psi}(p). \tag{4.4.7}$$

The discrete derivatives satisfy

$$\widehat{\nabla\varphi}(p) = q(p)\widehat{\varphi}(p) \tag{4.4.8}$$

with $q_j(p) = e^{ip_j} - 1$ for $1 \le j \le d$. For $p \in \widehat{T}_N$ we have $\frac{|p|}{2} \le |q(p)| \le |p|$. The Fourier transform of the kernel $\mathcal{A}_{\boldsymbol{Q}}$ of the operator $\mathcal{A}_{\boldsymbol{Q}}$ is therefore given by

$$\widehat{\mathcal{A}}_{\boldsymbol{Q}}(p) = \sum_{\alpha,\beta\in\mathcal{M}} \overline{q}(p)^\alpha \boldsymbol{Q}_{\alpha\beta} q(p)^\beta. \tag{4.4.9}$$

and $\widehat{\mathcal{C}}_{\boldsymbol{Q}}(p) = \left(\widehat{\mathcal{A}}_{\boldsymbol{Q}}(p)\right)^{-1}$.

We consider the set of all quadratic forms $\mathcal{Q}$ that satisfy,

$$\omega_0 |z^\nabla|^2 \le \mathcal{Q}(z) \le \frac{1}{\omega_0}|z|^2 \tag{4.4.10}$$

for some constant $\omega_0 \in (0,1)$ which is a slightly weaker condition than (3.2.40). Note that (4.4.9) then implies that there exists a constant ω such that

$$\begin{aligned} \omega|p|^2 \le \widehat{\mathcal{A}}_{\boldsymbol{Q}}(p) \le \frac{1}{\omega}|p|^2, \\ \frac{\omega}{|p|^2} \le \widehat{\mathcal{C}}_{\boldsymbol{Q}}(p) \le \frac{1}{\omega|p|^2}, \end{aligned} \tag{4.4.11}$$

where ω only depends on ω_0, R_0, and d. Let us also restate the main results from Chapter 2.

Theorem 4.4.1 (Theorem 2.2.5). *Fix $\overline{\omega}_0 > 0$. Consider the family of symmetric, positive operators $\boldsymbol{Q} : \mathcal{G} \to \mathcal{G}$ corresponding to quadratic forms $\mathcal{Q}$ that satisfy (4.4.10) with $\overline{\omega}_0$. Let $L > 3$ be odd, $N \ge 1$ as before and let $\tilde{n} > n$ be two integers. Then there exists a family of finite range decomposition $\mathcal{C}_{\boldsymbol{Q},k}$, $k = 1, 2, \dots, N+1$, of the operator $\mathcal{C}_{\boldsymbol{Q}}$ such that*

$$\begin{aligned} \mathcal{C}_{\boldsymbol{Q}} &= \sum_{k=1}^{N+1} \mathcal{C}_{\boldsymbol{Q},k}, \ \textit{with} \\ \mathcal{C}_{\boldsymbol{Q},k}(x) &= -C_k \ \textit{for} \ |x|_\infty \ge \frac{L^k}{2}, \end{aligned} \tag{4.4.12}$$

where $C_k \ge 0$ is a constant, positive semi-definite matrix that is independent of $\boldsymbol{Q}$. The family $\mathcal{C}_{\boldsymbol{Q},k}$ satisfies the following bounds where all constants may depend on R, d, m, $\overline{\omega}_0$, n, and $\tilde{n}$. The α-th discrete derivative for all α with $|\alpha| \le n$ is bounded by

$$\sup_{|\dot{\boldsymbol{Q}}|\le 1} \left| \nabla^\alpha D^\ell_{\boldsymbol{Q}} \mathcal{C}_{\boldsymbol{Q},k}(x)(\dot{\boldsymbol{Q}},\dots,\dot{\boldsymbol{Q}}) \right| \le \begin{cases} C_{\alpha,\ell} L^{-(k-1)(d-2+|\alpha|)} \ \textit{for}\ d + |\alpha| > 2 \\ C_{\alpha,\ell} \ln(L) L^{-(k-1)(d-2+|\alpha|)} \ \textit{for}\ d + |\alpha| = 2. \end{cases} \tag{4.4.13}$$

Further, for kernels in Fourier space we have the following lower bounds with a constant $c > 0$,

$$\widehat{\mathcal{C}}_{\boldsymbol{Q},k}(p) \ge \begin{cases} cL^{-2(d+\tilde{n})-1} L^{2j} L^{(k-j)(-d+1-n)} & \textit{for}\ L^{-j-1} < |p| \le L^{-j} \ \textit{and}\ j < k \\ cL^{-2(d+\tilde{n})-1} L^{2k} & \textit{for}\ |p| \le L^{-k-1}, \end{cases} \tag{4.4.14}$$

and similar upper bounds with a constant C,

$$\left| \widehat{\mathcal{C}}_{\boldsymbol{Q},k}(p) \right| \le \begin{cases} CL^{2(d+\tilde{n})+1} L^{2j} L^{(k-j)(-d+1-n)} & \textit{for}\ L^{-j-1} < |p| \le L^{-j} \ \textit{and}\ j < k \\ CL^{2k} & \textit{for}\ |p| \le L^{-k-1}. \end{cases} \tag{4.4.15}$$

For the derivatives of the kernels with $|\dot{\boldsymbol{Q}}| \leq 1$ and $\ell \geq 1$ we finally have the following stronger bounds

$$\left|\frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell}\widehat{\mathcal{C}}_{\boldsymbol{Q}+s\dot{\boldsymbol{Q}},k}(p)\right| \leq \begin{cases} C_\ell L^{(d+\tilde{n})+1}L^{2j}L^{(k-j)(-d+1-\tilde{n})} & \text{for } L^{-j-1} < |p| \leq L^{-j} \text{ and } j < k, \\ C_\ell L^{2k} & \text{for } |p| \leq L^{-k-1}. \end{cases} \tag{4.4.16}$$

The lower and upper bound can be combined to give, for $\ell \geq 1$ and $\boldsymbol{Q}$, $\widetilde{\boldsymbol{Q}}$ satisfying 3.2.20

$$\left|\frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell}\widehat{\mathcal{C}}_{\boldsymbol{Q}+s\dot{\boldsymbol{Q}},k}(p)\right| \cdot \left|\widehat{\mathcal{C}}_{\widetilde{\boldsymbol{Q}},k}(p)^{-1}\right| \leq \begin{cases} K_\ell L^{4(d+\tilde{n})+2}L^{(k-j)(n-\tilde{n})} & \text{for } L^{-j-1} < |p| \leq L^{-j} \text{ and } j < k, \\ K_\ell L^{2(d+\tilde{n})+1} & \text{for } |p| \leq L^{-k-1}, \end{cases} \tag{4.4.17}$$

where the constants K_ℓ do not depend on N or k.

Moreover we recall Theorem 2.4.5 that states that expectations with respect to $\mu_{\mathcal{C}^{\boldsymbol{Q}}_{k+1}}$ are differentiable in $\boldsymbol{Q}$. This will be a key ingredient in the proof of the smoothness of our renormalisation map.

Theorem 4.4.2 (Theorem 2.4.5)**.** *Let $\mathcal{C}_{\boldsymbol{Q},k+1}$ a finite range decomposition as in Theorem 4.4.1 with $\tilde{n} - n > d/2$ and $X \subset T_N$ be a subset with diameter $D = \mathrm{diam}_\infty(X) \geq L^k$. Let $F : \mathcal{V}_N \to \mathbb{R}$ be a functional that is measurable with respect to the σ-algebra generated by $\{\varphi(x) | \, x \in X\}$, i.e., F depends only on the values of the field φ in X. Then for $\ell \geq 1$ and $p > 1$ the following bound holds*

$$\left|\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}\int_{\mathcal{X}_N} F(\varphi)\, \mu_{\boldsymbol{Q}+t\boldsymbol{Q}_1,k+1}(\mathrm{d}\varphi)\Big|_{t=0}\right| \leq C_{\ell,p}(L)(DL^{-k})^{\frac{d\ell}{2}}\, |\boldsymbol{Q}_1|^\ell \, \|F\|_{L^p(\mathcal{X}_N,\mu_{\boldsymbol{Q},k+1})}. \tag{4.4.18}$$

The constant depends in addition on K_ℓ from (4.4.17) and therefore on ω_0, d, m, n, $\tilde{n}$, and R_0.

We already explained in Section 4.2 that in order to prove Theorem 3.2.3 it is not sufficient to decompose the Gaussian measure generated by $\mathcal{Q}$ but we have to consider small perturbations of this quadratic form. However, it is sufficient to consider only perturbations of the gradient-gradient term of the quadratic form. They are parametrized by symmetric maps $\boldsymbol{q} : \mathbb{R}^{d\times m} \to \mathbb{R}^{d\times m}$ and we denote with $|\boldsymbol{q}|$ its operator norm with respect to the standard scalar product on $\mathbb{R}^{d\times m}$. We consider the family of quadratic forms $\mathcal{Q}^{(\boldsymbol{q})}$ given by

$$\mathcal{Q}^{(\boldsymbol{q})}(z) = \mathcal{Q}(z) - z^\nabla \cdot \boldsymbol{q} z^\nabla \tag{4.4.19}$$

and the corresponding family of operators

$$\mathcal{A}^{(\boldsymbol{q})} = \sum_{\alpha,\beta\in\mathcal{G}}\sum_{i,j=1}^m (\nabla^\alpha)^*\boldsymbol{Q}_{(\alpha,i),(\beta,j)}\nabla^\beta - \sum_{|\alpha|=|\beta|=1}\sum_{i,j=1}^m \boldsymbol{q}_{(\alpha,i),(\beta,j)}(\nabla^{(\alpha,i)})^*\nabla^{(\beta,j)} \tag{4.4.20}$$

where $\nabla^{(\alpha,i)}\varphi(x) = \nabla^\alpha\varphi_i(x)$ and $\boldsymbol{Q}$ denotes the generator of $\mathcal{Q}$. The partition function of the Gaussian measure generated by $\mathcal{A}^{(\boldsymbol{q})}$ will be denoted by

$$Z^{(\boldsymbol{q})} = \int_{\mathcal{X}_N} e^{-\frac{1}{2}(\varphi,\mathcal{A}^{(\boldsymbol{q})}\varphi)}\, \mathrm{d}\varphi. \tag{4.4.21}$$

In the following we will always assume that $\boldsymbol{q} \in B_\kappa = B_\kappa(0) \coloneqq \{\boldsymbol{q} \in \mathbb{R}^{(d\times m)\times(d\times m))}_{\mathrm{sym}} : |\boldsymbol{q}| \leq \kappa\}$ for some κ with $\kappa \leq \frac{\omega_0}{2}$. Later we will impose additional conditions on κ.

Note that the family $\mathcal{Q}^{(\boldsymbol{q})}$ satisfies the condition (4.4.10) with $\overline{\omega}_0 = \omega_0/2$ for $\boldsymbol{q} \in B_\kappa$.

To obtain our main results we fix a finite range decomposition as in Theorem 4.4.1 with parameters $\overline{\omega}_0 = \omega_0/2$ and

$$n = 2M = 4\left\lfloor \frac{d}{2} \right\rfloor + 6, \quad \tilde{n} = n + \left\lfloor \frac{d}{2} \right\rfloor + 1 = 5\left\lfloor \frac{d}{2} \right\rfloor + 7. \tag{4.4.22}$$

The choice is related to the choice of the norms and a Sobolev embedding as we will see later. In particular we obtain a finite range decomposition $\mathcal{C}_k^{(\boldsymbol{q})}$ with kernels $\mathcal{C}_k^{(\boldsymbol{q})}$ with $1 \le k \le N+1$ for $\boldsymbol{q} \in B_\kappa$ of the covariances $\mathcal{C}^{(\boldsymbol{q})} = \left(\mathcal{A}^{(\boldsymbol{q})}\right)^{-1}$. To state the result in Theorem 4.5.1 in slightly bigger generality we consider more general choices of parameters there.

The key property of these decompositions is their finite range which implies for a random Gaussian field φ with covariance $\mathcal{C}_k^{(\boldsymbol{q})}$ that $\mathbb{E}(\nabla_i\varphi(x)\nabla_j\varphi(y)) = \nabla_j^*\nabla_i\mathcal{C}_k^{(\boldsymbol{q})}(x-y) = 0$ if $|x-y| \ge L^k/2$, rendering the gradient variables $\nabla_i\varphi(x)$ and $\nabla_j\varphi(y)$ to be independent. In particular, this implies

$$\mathbb{E}(F_1(\nabla\varphi\restriction_X)F_2(\nabla\varphi\restriction_Y)) = \mathbb{E}(F_1(\nabla\varphi\restriction_X))\mathbb{E}(F_2(\nabla\varphi\restriction_Y)) \tag{4.4.23}$$

for sets X and Y such that $\operatorname{dist}(X,Y) \ge L^k/2$. In analytic terms this means

$$\int_{\mathcal{X}_N} F_1(\nabla\varphi\restriction_X)F_2(\nabla\varphi\restriction_Y)\,\mu_{\mathcal{C}_k^{(\boldsymbol{q})}} = \int_{\mathcal{X}_N} F_1(\nabla\varphi\restriction_X)\,\mu_{\mathcal{C}_k^{(\boldsymbol{q})}} \int_{\mathcal{X}_N} F_2(\nabla\varphi\restriction_Y)\,\mu_{\mathcal{C}_k^{(\boldsymbol{q})}} \tag{4.4.24}$$

We will use this *factorization property* frequently in the following. Also, we will often use the shorthand $\mu_k^{(\boldsymbol{q})} = \mu_{\mathcal{C}_k^{(\boldsymbol{q})}}$, dropping occasionally $\boldsymbol{q}$ from the notation.

If φ is distributed according to μ and the fields φ_k are independent and distributed according to μ_k, the finite range decomposition amounts, in probabilistic language, to the claim that

$$\varphi \stackrel{\mathcal{D}}{=} \sum_{k=1}^{N+1} \varphi_k \tag{4.4.25}$$

in distribution. Or, from the analytic viewpoint, it is formulated in terms of the convolution of measures,

$$\mu = \mu_1 * \ldots * \mu_{N+1}. \tag{4.4.26}$$

The renormalisation maps are then defined by sequential integrations,

$$(\boldsymbol{R}_k^{(\boldsymbol{q})}F)(\varphi) = \int_{\mathcal{X}_N} F(\varphi+\xi)\,\mu_k^{(\boldsymbol{q})}(\mathrm{d}\xi) = F * \mu_k^{(\boldsymbol{q})}(\varphi) \tag{4.4.27}$$

for $1 \le k \le N+1$. Later we will define Banach spaces of functionals that will guarantee that this map is well-defined and continuous. For F integrable with respect to $\mu^{(\boldsymbol{q})}$ this definition implies

$$\begin{aligned}\int_{\mathcal{X}_N} F(\varphi)\,\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) &= \int_{\mathcal{X}_N\times\ldots\times\mathcal{X}_N} F\left(\sum_{i=1}^{N+1}\varphi_i\right)\mu_1^{(\boldsymbol{q})}(\mathrm{d}\varphi_1)\ldots\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi_{N+1}) \\ &= (\boldsymbol{R}_{N+1}^{(\boldsymbol{q})}\ldots\boldsymbol{R}_1^{(\boldsymbol{q})})(F)(0)\end{aligned} \tag{4.4.28}$$

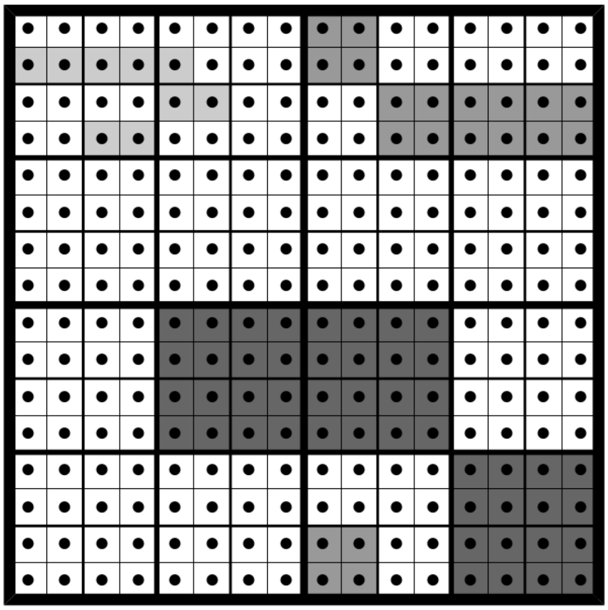

Figure 4.1: Torus $(\mathbb{Z}/(L\mathbb{Z}))^d$ with $d = 2$, $L = 2$, and $N = 4$. The light, middle, and dark gray regions are connected 0,1, and 2-polymers, respectively. The image is taken from [55] and similar to Figure 1 in [83].

4.4.2 Polymers and relevant Hamiltonians

In this section we define certain subsets of the torus that will be used to organize the multiscale analysis. To keep these definitions simple we introduce the constant

$$R = \max(R_0, 2\lfloor d/2 \rfloor + 3) = \max(R_0, M) \tag{4.4.29}$$

depending only on the range of the interaction R_0 and the dimension d.

We use Λ_N referring to the set underlying the torus $T_N = (\mathbb{Z}/(L^N\mathbb{Z}))^d$, and identify it sometimes with the set $\Lambda_N = \{z \in \mathbb{Z}^d : |z_i| \leq \frac{L^N-1}{2}\}$. For every $1 \leq k \leq N$ we pave the torus with blocks of side length L^k which are translates by $(L^k\mathbb{Z})^d$ of the block $B_0 = \{z \in \mathbb{Z}^d : |z_i| \leq \frac{L^k-1}{2}\}$. We refer to these blocks as k-blocks on Λ_N and denote their set by

$$\mathcal{B}_k = \{B : B \text{ is a } k\text{-block}\}. \tag{4.4.30}$$

Next, we summarise a notation for particular unions of blocks:

- A union of k-blocks is called a *k-polymer* and $\mathcal{P}_k$ will be the set of all k-polymers. Note that this definition of polymers differs from the definitions inspired directly by physics, in particular polymers need not be connected.

- A set $X \subset T_N$ is *connected* if for all $x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_m = y$ with $x_i \in X$ for $0 \leq i \leq m$ such that $|x_i - x_{i+1}|_\infty = 1$ for $0 \leq i \leq m-1$ (see also Figure 4.1). This notion corresponds to graph connectedness in the graph with vertices Λ_N and edges between $x, y \in \Lambda_N$ if $|x - y|_\infty = 1$. We say that $X, Y \subset T_N$ are *touching* if $X \cup Y$ is connected.

- Sets A and B are *strictly disjoint* if their union is not connected. An important property is that for $X, Y \in \mathcal{P}_k$ such that X and Y are strictly disjoint we have $\operatorname{dist}(X,Y) > L^k$. If ξ_k is distributed according to μ_k this implies that the gradient fields $\nabla\xi_k$ restricted to X and Y are independent by the finite range property.

- We use $\mathcal{P}_k^{\mathrm{c}}$ to denote the set of *connected k-polymers* and $\mathcal{C}(X)$ to denote the set of connected components of a polymer X.

- $\mathcal{B}_k(X)$ is the *set of all k-blocks* contained in a polymer X with $|X|_k$ denoting their *number*.

- The *closure* $\overline{X} \in \mathcal{P}_{k+1}$ of a k-polymer $X \in \mathcal{P}_k$ is the smallest $(k+1)$-polymer containing X.

- We say that a connected polymer $X \in \mathcal{P}_k^{\mathrm{c}}$ is *small* if $|X|_k \le 2^d$. We use $\mathcal{S}_k$ to denote all small k-polymers. All other polymers in $\mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k$ will be called *large*. Small polymers are introduced because they need a special treatment in the renormalisation procedure. The reason boils down to the fact that for large polymers $X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k$ the closure satisfies $|\overline{X}|_{k+1} \le \alpha(d)|X|_k$ for some $\alpha(d) < 1$. For $X \in \mathcal{S}_k$, however, it is possible that $|\overline{X}|_{k+1} = |X|_k$.

- For any block $B \in \mathcal{B}_k$ and $k \ge 1$ let $\widehat{B} \in \mathcal{P}_k$ be the cube of side length $(2^{d+1}+1)L^k$ centred at B. Note that this is similar to the definition of the small set neighbourhood in [4] but the side length is slightly bigger. For $B \in \mathcal{B}_0$ let $\widehat{B} \in \mathcal{P}_0$ denote the cube centred at B of side length $(2^{d+1} + 2R + 1)$ where R denotes the range of the interaction as defined in (4.4.29).

- For any polymer $X \in \mathcal{P}_k$ and $k \ge 1$ we define the *small neighbourhood* $X^* \in \mathcal{P}_{k-1}$ of X by
$$X^* = \bigcup_{B \in \mathcal{B}_{k-1}(X)} \widehat{B}. \tag{4.4.31}$$
For $k = 0$ we define $X^* = X + [-R,R]^d \cap T_N \in \mathcal{P}_0$. Note that we view $*$ as a map from $\mathcal{P}_k$ to $\mathcal{P}_{k-1}$ for $k \ge 1$. In particular, $X^{**} = (X^*)^* \in \mathcal{P}_{k-2}$ for $X \in \mathcal{P}_k$ and $k \ge 2$. If the scale of the considered polymer is not clear from the context it will be indicated explicitly. The definition of $\widehat{B}$ implies that for $X \in \mathcal{P}_k$, $k \ge 1$, and $x \in X^*$,
$$\operatorname{dist}_\infty(x, X) \le (2^d + R)L^{k-1}. \tag{4.4.32}$$

- Finally, for any $X \in \mathcal{P}_k$ we define the *large neighbourhood*
$$X^+ = \bigcup_{B \in \mathcal{B}_k \text{ is touching } X} B \quad \text{for } k \ge 1 \text{ and } X^+ = X^* \text{ for } k = 0. \tag{4.4.33}$$

For future reference we recapitulate the definitions of neighbourhoods:
$$\begin{gathered}
\widehat{B} = \begin{cases} B + [-2^d - R, 2^d + R]^d \cap T_N & \text{for } B \in \mathcal{B}_0 \\ B + [-2^d L^k, 2^d L^k]^d \cap T_N & \text{for } B \in \mathcal{B}_k,\ k \ge 1 \end{cases} \\
X^* = \begin{cases} X + [-R,R]^d \cap T_N & \text{for } X \in \mathcal{P}_0 \\ X + [-2^d - R, 2^d + R]^d \cap T_N & \text{for } X \in \mathcal{P}_1 \\ X + [-2^d L^{k-1}, 2^d L^{k-1}]^d \cap T_N & \text{for } X \in \mathcal{P}_k,\ k \ge 2 \end{cases} \\
X^+ = \begin{cases} X + [-R,R]^d \cap T_N & \text{for } X \in \mathcal{P}_0 \\ X + [-L^k, L^k]^d \cap T_N & \text{for } X \in \mathcal{P}_k,\ k \ge 1. \end{cases}
\end{gathered} \tag{4.4.34}$$

Let us also collect several obvious geometric consequences of the definitions.

For strictly disjoint $U_1, U_2 \in \mathcal{P}_{k+1}$ and $L \geq 2^{d+2} + 4R$ we have

$$\operatorname{dist}(U_1^*, U_2^*) \geq L^{k+1} - 2(2^d L^k + R) \geq \frac{L^{k+1}}{2}. \tag{4.4.35}$$

For $L \geq 2^d + R$ and $X \in \mathcal{P}_k$ we have

$$X^* \subset X^+. \tag{4.4.36}$$

Indeed, for $k = 0$ it holds as the equality and for $k \geq 1$ the inclusion follows from (4.4.32). Moreover, for $L \geq 2^d + R$ and $k \geq 0$

$$X^* \subset X^+ \subset Y^* \text{ for } X \in \mathcal{S}_k \text{ and } Y \in \mathcal{P}_{k+1} \text{ such that } X \cap Y \neq \emptyset. \tag{4.4.37}$$

To verify the second inclusion, let $B \in \mathcal{B}_k(X \cap Y)$. We will show that then $X^+ \subset \widehat{B}$ and thus $X^+ \subset Y^*$. Indeed, given that X is small, it is contained in a cube of side length $(2^{d+1} - 1)L^k$ centred at B. For $k \geq 1$ this implies that X^+ is contained in a cube of side length $(2^{d+1} + 1)L^k$ centred at B, while for $k = 0$ in a cube of side length $2^{d+1} + 2R + 1$ centred at B. In both cases it implies that $X^+ \subset \widehat{B}$.

Now we introduce the class of functionals we are going to work with. We set

$$M(\mathcal{P}_k, \mathcal{V}_N) = \{F : \mathcal{P}_k \times \mathcal{V}_N \to \mathbb{R} |\, F(X, \cdot) \in M(\mathcal{V}_N),\ F \text{ local, translation and shift invariant}\}. \tag{4.4.38}$$

Here, $M(\mathcal{V}_N)$ is the set of measurable real functions on $\mathcal{V}_N$ with respect to the Borel σ-algebra. Locality of F is defined by assuming that $F(X, \varphi)$ depends only on the value of the field φ on X^*, that is, assuming the equality $F(X, \varphi) = F(X, \psi)$ to be valid whenever $\varphi\restriction_{X^*} = \psi\restriction_{X^*}$. The translation invariance of F means that for any $a \in (L^k\mathbb{Z})^d$ we have $F(\tau_a(X), \tau_a(\varphi)) = F(X, \varphi)$, where $\tau_a(B) = B + a$ and $\tau_a\varphi(x) = \varphi(x - a)$. Finally, for a local functional F and a connected polymer X, the shift invariance means that $F(X, \varphi + \psi) = F(X, \varphi)$, where ψ is a constant function, $\psi(x) = c$ for $x \in X^*$. For general polymers X we define the shift invariance by assuming that $F(X, \varphi + \psi) = F(X, \varphi)$ whenever ψ is a step function—a constant on each nearest neighbour graph-connected component of X^*. Here nearest neighbour graph-connectedness refers to the usual nearest neighbour graph structure on Λ_N (defining the set $E(\Lambda_N)$ of edges in Λ_N as $E(\Lambda_N) = \{\{x, y\}, x, y \in \Lambda_N \text{ such that } |x - y|_2 = 1\}$ in contrast to the relation $|x - y|_\infty = 1$ used when defining connectedness of polymers). Note that for $k \geq 1$ and $X \in \mathcal{P}_k$ the graph-connected components of X^* agree with the connected components we defined before.

It is convenient to define the functionals on $\mathcal{V}_N$ instead of $\mathcal{X}_N$ the space of fields with average zero which are in one-to-one correspondence with gradient fields. Nevertheless, all the measures $\mu_k^{(\boldsymbol{q})}$ appearing in the following are supported on $\mathcal{X}_N$ which implies that the functionals are only evaluated for $\varphi \in \mathcal{X}_N$. Moreover the measures $\mu_k^{(\boldsymbol{q})}$ are absolutely continuous with respect to the Hausdorff measure on $\mathcal{X}_N$. Note that for $F \in M(\mathcal{V}_N)$ such that $F(\varphi + c) = F(\varphi)$ for any $\varphi \in \mathcal{V}_N$ and any constant field $c \in \mathcal{V}_N$, the restriction $F\restriction_{\mathcal{X}_N}$ is measurable with respect to the Borel σ-algebra on $\mathcal{X}_N$. Indeed, the condition $F(\varphi + c) = F(\varphi)$ implies that for any Borel $O \subset \mathbb{R}$ with $A = (F\restriction_{\mathcal{X}_N})^{-1}(O)$, we have $F^{-1}(O) = A \times \mathcal{X}_N^\perp \subset \mathcal{X}_N \oplus \mathcal{X}_N^\perp = \mathcal{V}_N$.

Let us formulate an equivalent characterisation of shift invariance. For any subset $X \subset \Lambda_N$ we introduce the set of edges $E(X) = \{\{x, y\} \in E(\mathbb{Z}^d), x, y \in X\}$ and the set of directed edges $\vec{E}(X) = \{(x, y), (y, x), \{x, y\} \in E\}$. For $\varphi \in \mathcal{V}_N$ we can view $\nabla\varphi$ as a function from $\vec{E}(\Lambda_N)$ to $\mathbb{R}^m$ by taking $\nabla\varphi((x, x + e_i)) = \nabla_i\varphi(x) = \varphi(x + e_i) - \varphi(x)$ and $\nabla\varphi((x + e_i, x)) = \nabla_i^*\varphi(x + e_i) = \varphi(x) - \varphi(x + e_i)$.

Lemma 4.4.3. *A functional $F : \mathcal{P}_k \times \mathcal{V}_N \to \mathbb{R}$ is local and shift invariant iff for each $X \in \mathcal{P}_k$ there is a functional $\widetilde{F}_X : \vec{E}(X^*) \to \mathbb{R}$ such that $F(X,\varphi) = \widetilde{F}_X(\nabla\varphi\restriction_{\vec{E}(X^*)})$ for any $\varphi \in \mathcal{V}_N$, i. e., $F(X,\cdot)$ is measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(X^*)}$.*

Proof. We first observe that for a graph connected set Y, a fixed $y \in Y$, and $\eta : \vec{E}(Y) \to \mathbb{R}^m$, there is at most one function $\widetilde{\varphi} : Y \to \mathbb{R}^m$ such that $\widetilde{\varphi}(y) = 0$ and $\nabla\widetilde{\varphi} = \eta$. Note that a necessary condition is that $\eta((x,y)) = -\eta((y,x))$ for any $(x,y) \in \vec{E}(Y)$.) Indeed, if there were two such functions $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ and a point $z \in Y$ such that $\widetilde{\varphi}_1(z) \neq \widetilde{\varphi}_2(z)$, we would get a contradiction since $\widetilde{\varphi}_1(z) - \widetilde{\varphi}_2(z) = \sum_{i=0}^n \eta_{(x_i,x_{i+1})} - \sum_{i=0}^n \eta_{(x_i,x_{i+1})} = 0$ for any path $x_0 = y, x_1, \dots, x_n = z$. Such a path exists since the graph $(Y, E(Y))$ is connected.

Now, let F be shift invariant and local, $Y_1, Y_2, \dots, Y_n$ be the graph connected components of X^*, and let $y_i \in Y_i, i = 1, 2, \dots, n$. Note that the argument above implies that for $\eta : \vec{E}(X^*) \to \mathbb{R}^m$ there is at most one $\varphi \in \mathcal{V}_N$ such that $\varphi(y_i) = 0$ and $\nabla\varphi\restriction_{\vec{E}(X^*)} = \eta$. Then we define $\widetilde{F}_X(\eta) = F(\varphi)$ if such a φ exists and $\widetilde{F}_X(\eta) = 0$ otherwise. For $\varphi \in \mathcal{V}_N$ define $\widetilde{\varphi}(x) = \varphi(x) - \sum_i \varphi(y_i)\mathbf{1}_{Y_i}(x)$. Then by shift invariance we have

$$F(X,\varphi) = F(X,\widetilde{\varphi}) = \widetilde{F}_X(\nabla\widetilde{\varphi}\restriction_{\vec{E}(X^*)}) = \widetilde{F}_X(\nabla\varphi\restriction_{\vec{E}(X^*)}). \tag{4.4.39}$$

The opposite implication is obvious. □

In addition to the set $M(\mathcal{P}_k, \mathcal{V}_N)$ of functionals we consider its obvious generalizations $M(\mathcal{P}_k^{\mathrm{c}}, \mathcal{V}_N)$, $M(\mathcal{S}_k, \mathcal{V}_N)$ and $M(\mathcal{B}_k, \mathcal{V}_N)$. We use the shorthand $M(\mathcal{P}_k), M(\mathcal{P}_k^{\mathrm{c}}), M(\mathcal{S}_k)$, and $M(\mathcal{B}_k)$. There are two canonical inclusions $\iota_1 : M(\mathcal{B}_k) \to M(\mathcal{P}_k^{\mathrm{c}})$ and $\iota_2 : M(\mathcal{P}_k^{\mathrm{c}}) \to M(\mathcal{P}_k)$ given by $(\iota_1 F)(X,\varphi) = \prod_{B\in\mathcal{B}_k(X)} F(B,\varphi)$ and $(\iota_2 F)(X,\varphi) = \prod_{Y\in\mathcal{C}(X)} F(Y,\varphi)$, respectively. In the following we will usually drop ι from the notation and write $F(X,\varphi) = F^X(\varphi)$ for $F \in M(\mathcal{B}_k)$ and $F(X,\varphi) = \prod_{Y\in\mathcal{C}(X)} F(Y,\varphi)$ for $F \in M(\mathcal{P}_k)$. The set $M(\mathcal{P}_k)$ can be endowed by an associative and commutative product $\circ$,

$$(F_1 \circ F_2)(X,\varphi) = \sum_{Y\in\mathcal{P}_k(X)} F_1(Y,\varphi)F_2(X\setminus Y,\varphi), \quad F_1, F_2 \in M(\mathcal{P}_k) \tag{4.4.40}$$

that is useful to streamline the notation. For example, it serves as a shorthand for the expansion of the product

$$(F_1 + F_2)^X(\varphi) = (F_1 \circ F_2)(X,\varphi) \tag{4.4.41}$$

with $F_1, F_2 \in M(\mathcal{B}_k)$.

Finally, we introduce the space of relevant Hamiltonians $M_0(\mathcal{B}_k) \subset M(\mathcal{B}_k)$ given by all functionals of the form

$$H(B,\varphi) = \sum_{x\in B} \mathcal{H}(\{x\},\varphi) \tag{4.4.42}$$

where $\mathcal{H}(\{x\})(\varphi)$ is a linear combination of the following *relevant monomials*:

- The constant monomial $\mathcal{M}_\emptyset(\{x\})(\varphi) \equiv 1$;
- the linear monomials $\mathcal{M}_{i,\alpha}(\{x\})(\varphi) := \nabla^{i,\alpha}\varphi(x) := \nabla^\alpha\varphi_i(x)$ with $1 \le |\alpha| \le \lfloor d/2 \rfloor + 1$;
- the quadratic monomials $\mathcal{M}_{(i,\alpha),(j,\beta)}(\{x\})(\varphi) = \nabla^\alpha\varphi_i(x)\,\nabla^\beta\varphi_j(x)$ with $|\alpha| = |\beta| = 1$.

The rationale for declaring exactly these monomials as relevant is based on the following heuristic argument concerning the decay of their expectations under the measures μ_k: Let us assign the *scaling dimension* $[\varphi] = \frac{d-2}{2}$ to the field φ, and the scaling dimension $[\mathcal{M}_m] = r[\varphi] + \sum_{i=1}^r |\alpha_i|$ to a general monomial $\mathcal{M}_m(\{x\})(\varphi) = \nabla^{\alpha_1}\varphi_{i_1}(x)\cdots\nabla^{\alpha_r}\varphi_{i_r}(x)$ (with $\alpha_i \neq 0$). The relevance of the scaling dimension follows from the asymptotics $\mathbb{E}_{\mu_k}|\mathcal{M}_m(\{x\})|^2 \sim L^{-2k[\mathcal{M}_m]}$ and the fact that, by the smoothness properties of correlations of μ_k, we expect that the fields $\varphi(x)$ and $\varphi(y)$ are correlated only if $|x-y| \leq cL^d$. As a result, for a k-block B we get $\mathbb{E}_{\mu_k}\big(\sum_{x\in B}\mathcal{M}_m(\{x\})^2\big) \sim L^{-2k[\mathcal{M}_m]+2kd}$. Hence the relevant monomials are exactly those for which the expectation of $|\sum_{x\in B}\mathcal{M}_m(\{x\})|$ under μ_k is not expected to decay for large k. One often calls the monomials for which this quantity grows with k *relevant*, those for which it remains of order 1 *marginal* and those for which it decays as *irrelevant*. To avoid clumsy notation such as 'not irrelevant' or 'relevant or marginal' we include marginal monomials into our list of relevant polynomials.

Any $H \in M_0(\mathcal{B}_k)$ is clearly shift invariant and local (the fact that $B + [-R,R] \cap T_N \subset B^+$ once $R \geq \lfloor d/2 \rfloor + 1$ implies that $H \in M_0(\mathcal{B}_k)$ and thus $M_0(\mathcal{B}_k) \subset M(\mathcal{B}_k)$).

4.4.3 Definition of the renormalisation map

In this section we define the flow of the functionals under the renormalisation maps (4.4.27). Specifically the flow will be described by two sequences of functionals H_k and K_k. The coordinate $H_k \in M_0(\mathcal{B}_k)$ stems from the finite dimensional space of relevant Hamiltonians and collects the relevant and marginal directions whereas the perturbation $K_k \in M(\mathcal{P}_k^c)$ is an element of an infinite dimensional space that collects all remaining irrelevant directions of the model. In this section we introduce the map $\boldsymbol{T}_k$ that maps the operators H_k and K_k to the next scale operators H_{k+1} and K_{k+1}. Formally it is given by a map

$$\boldsymbol{T}_k : M_0(\mathcal{B}_k) \times M(\mathcal{P}_k^c) \times \mathbb{R}^{(d\times m)\times(d\times m)} \to M_0(\mathcal{B}_{k+1}) \times M(\mathcal{P}_{k+1}^c), \tag{4.4.43}$$

where we reflected the fact that it also depends on the *a priori* tuning matrix $\boldsymbol{q}$ which is mostly suppressed in the notation in this section. In the following we fix a scale k and write $(H,K) = (H_k, K_k)$ and $(H',K') = (H_{k+1}, K_{k+1})$. Using $\boldsymbol{R}_k^{(\boldsymbol{q})}$ or a shorthand $\boldsymbol{R}_k$ for $\boldsymbol{T}_k$ with a fixed $\boldsymbol{q}$, the key requirement for the renormalisation transformation is the identity

$$\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}(e^{-H}\circ K)(\Lambda_N,\varphi) = (e^{-H'}\circ K')(\Lambda_N,\varphi). \tag{4.4.44}$$

Moreover it must be chosen in such a way that the map $K \mapsto K'$ is contracting. For most polymers this will follow from the definition of the norms and the fact that typically the number of blocks decreases when the scale is changed, i.e., $|\overline{X}|_{k+1} < |X|_k$. However, for k-blocks $X \in \mathcal{B}_k$, and in general also for $X \in \mathcal{S}_k$, this is not true, $|\overline{X}|_{k+1} = |X|_k$. As a result, we have to subtract the dominant part of their contribution and include it in the Hamiltonian H'. The process of selection of the relevant part that is to be included to the space of relevant Hamiltonians determines a projection

$$\Pi_2 : M(\mathcal{B}_k) \to M_0(\mathcal{B}_k). \tag{4.4.45}$$

Existence, boundedness and further properties of this projection are discussed in Subsection 4.6.4 below. Slightly informally, $\Pi_2 F$ is defined as a "homogenization" of the second order Taylor expansion T_2 around zero. Namely, considering the second order Taylor expansion of $F(B)$ given by $\dot\varphi \mapsto F(B)(0) + DF(B)(0)(\dot\varphi) + \frac{1}{2}D^2F(0)(\dot\varphi,\dot\varphi)$, we define $\Pi_2 F$ as the ideal Hamiltonian $F(B)(0) + \ell(\dot\varphi) + Q(\dot\varphi,\dot\varphi)$ where ℓ is the unique linear relevant Hamiltonian that satisfies the condition $\ell(\dot\varphi) = DF(B,0)(\dot\varphi)$ for all $\dot\varphi$ whose restriction to B^+ is a polynomial of degree $\lfloor d/2 \rfloor + 1$

and similarly $Q(\dot\varphi,\dot\varphi)$ is the unique quadratic relevant Hamiltonian that agrees with $D^2F(B,0)$ on all functions whose restriction to B^+ is affine. Note that B^+ does not wrap around the torus for $k \le N-1$ and $L \ge 5$ and, as a consequence, the condition that φ restricted to B^+ is a polynomial is well-defined.

We defer the definition of $\boldsymbol{T}_k$ and first motivate its definition with a sequence of manipulations starting with the left hand side of (4.4.44). We define the relevant Hamiltonian on the next scale by

$$H'(B',\varphi) = \sum_{B\in\mathcal{B}_k(B')} \widetilde{H}(B,\varphi) \tag{4.4.46}$$

where $\widetilde{H}(B,\varphi)$ is defined by

$$\widetilde{H}(B,\varphi) = \Pi_2 \boldsymbol{R}' H(B,\varphi) - \Pi_2 \boldsymbol{R}' K(B,\varphi). \tag{4.4.47}$$

Note that we need to subtract only the contributions that stem from a single block. In the following we write

$$I(B,\varphi+\xi) = \exp(-H(B,\varphi+\xi)), \quad \tilde{I}(B,\varphi) = \exp(-\widetilde{H}(B,\varphi)), \quad \text{and} \quad \tilde{J} = 1-\tilde{I}. \tag{4.4.48}$$

Using repeatedly the identities (4.4.41), we rewrite the initial integral in (4.4.44) in terms of the next scale Hamiltonian,

$$\begin{aligned}\int_{\mathcal{X}_N} I(\varphi+\xi)\circ K(\varphi+\xi)\,\mu_{k+1}(\mathrm{d}\xi) &= \int_{\mathcal{X}_N} \tilde{I}(\varphi)\circ(I-\tilde{I})(\varphi+\xi)\circ K(\varphi+\xi)\,\mu_{k+1}(\mathrm{d}\xi)\\ &= \int_{\mathcal{X}_N} \tilde{I}(\varphi)\circ(1-\tilde{I})(\varphi)\circ(I-1)(\varphi+\xi)\circ K(\varphi+\xi)\,\mu_{k+1}(\mathrm{d}\xi)\\ &= \tilde{I}(\varphi)\circ\left(\int_{\mathcal{X}_N} \tilde{J}(\varphi)\circ(I-1)(\varphi+\xi)\circ K(\varphi+\xi)\,\mu_{k+1}(\mathrm{d}\xi)\right).\end{aligned} \tag{4.4.49}$$

This allows to to introduce an intermediate perturbation functional $\widetilde{K}: \mathcal{P}\times\mathcal{V}_N\times\mathcal{V}_N \to \mathbb{R}$ by

$$\widetilde{K}(X,\varphi,\xi) = (\tilde{J}(\varphi)\circ(I(\varphi+\xi)-1)\circ K(\varphi+\xi))(X). \tag{4.4.50}$$

The initial integral then becomes

$$\boldsymbol{R}'(I\circ K)(\Lambda_N,\varphi) = \sum_{X\in\mathcal{P}(\Lambda_N)} \tilde{I}^{\Lambda_N\setminus X}(\varphi)\int_{\mathcal{X}_N} \widetilde{K}(X,\varphi,\xi)\,\mu_{k+1}(\mathrm{d}\xi). \tag{4.4.51}$$

In the next step we regroup the terms in a such a way that we obtain an expression in the form $e^{-H'}\circ K'$ with $H' \in M_0(\mathcal{B}_{k+1})$ and $K' \in M(\mathcal{P}^{\mathrm{c}}_{k+1})$. For $X \in \mathcal{P}_k \setminus \mathcal{S}_k$ we just include the contribution of the integral of $\widetilde{K}(X)$ to the terms labelled by $U = \overline{X}$ in K'. Introducing, on the spaces $M(\mathcal{P}_k)$, the norms with the weight $A^{|X|_k}$ we will prove the contractivity of the linearisation of the map T_k. For $A > 1$ and $X \in \mathcal{P}_k \setminus \mathcal{S}_k$ for which we can show that $|X|_{k+1} < |X|_k$, this is based on the suppression factor $A^{|X|_{k+1}-|X|_k}$. However, for $X \in \mathcal{S}_k$ this strategy does not work since we might have $|X|_{k+1} = |X|_k$. In this case, as explained above, we have to include the dominant part of their contribution into the Hamiltonian H' as anticipated in (4.4.47).

In addition, for $X \in \mathcal{S} = \mathcal{S}_k$, we have to determine to which of the blocks $B' \in \mathcal{B}' = \mathcal{B}_{k+1}$, among those that intersect X, we attribute the corresponding contribution. This is achieved in the following claim. There exists a map $\widetilde{\pi} : \mathcal{P}^{\mathrm{c}} \to \mathcal{P}^{\mathrm{c}\prime}$ that is translation invariant, i.e., $\widetilde{\pi}(\tau_a X) = \tau_a \widetilde{\pi}(X)$ for $a \in (L^{k+1}\mathbb{Z})^d$ and, for connected polymers, satisfies

$$\widetilde{\pi}(X) = \begin{cases} \overline{X} & \text{if } X \in \mathcal{P}^{\mathrm{c}} \setminus \mathcal{S}, \\ B' & \text{where } B' \in \mathcal{B}' \text{ with } B' \cap X \neq \emptyset \quad \text{for } X \in \mathcal{S} \setminus \{\emptyset\}, \\ \emptyset & \text{if } X = \emptyset. \end{cases} \tag{4.4.52}$$

We then extend $\widetilde{\pi}$ to a map $\pi : \mathcal{P} \to \mathcal{P}'$ defined on all polymers by

$$\pi(X) = \bigcup_{Y \in \mathcal{C}(X)} \widetilde{\pi}(Y) \quad \forall\, X \in \mathcal{P}. \tag{4.4.53}$$

To show the existence of a map $\widetilde{\pi}$, it suffices to define the image $\widetilde{\pi}(X)$ for any $X \in \mathcal{S}$ and show that the resulting $\widetilde{\pi}$ is translation invariant. "Unwrapping" the torus T_N, it can be viewed as a projection $P : \mathbb{Z}^d \to T_N$ with the preimage of any point $x \in T_N$ being the set $P^{-1}(\{x\}) = \{\tau_a x, a \in (L^N\mathbb{Z})^d\}$. The preimage of any $X \in \mathcal{S}$ is a collection of sets $\{\tau_a \widehat{X}, a \in (L^N\mathbb{Z})^d\}$, where $\widehat{X} \subset \mathbb{Z}^d$ can be chosen as a connected set $\widehat{X} \subset \mathbb{Z}^d$ (recall that any $X \in \mathcal{S}$ is connected) for which $X = P(\widehat{X})$. For any $X \in \mathcal{S}$ consider the k-block $B(X) \in \mathcal{B}(X)$ such that the preimage of its centre in $\widehat{X}$ is the first one in the lexicographic order in $\mathbb{Z}^d$ among the preimages in $\widehat{X}$ of centres of k-blocks in $\mathcal{B}(X)$. We determine the image $\widetilde{\pi}(X)$ as the $(k+1)$-block $B' = \overline{B(X)}$. Translation invariance of the map $\widetilde{\pi}$ follows immediately from the fact that $B(\tau_a X) = \tau_a B(X)$ for any $a \in (L^k\mathbb{Z})^d$.

We claim that for $X \in \mathcal{P}_k$ and $L \geq 2^d + R$,

$$\mathcal{P}_{k-1} \ni X^* \subset \pi(X)^* \in \mathcal{P}_k. \tag{4.4.54}$$

By (4.4.53) it is sufficient to show this for X connected. For connected polymers X that are not small this is clear by (4.4.52). For $X \in \mathcal{S}_k$ this is a consequence of (4.4.37) applied with $Y = \pi(X)$.

We define the function $\chi : \mathcal{P} \times \mathcal{P}' \to \mathbb{R}$ by

$$\chi(X, U) = \mathbb{1}_{\pi(X) = U}. \tag{4.4.55}$$

This definition ensures $\sum_{U \in \mathcal{P}'} \chi(X, U) = 1$. Using the relation $(\Lambda_N \setminus X) \cup (X \setminus U) = (\Lambda_N \setminus U) \cup (U \setminus X)$ we rearrange the right hand side of (4.4.51)

$$\boldsymbol{R}'(I \circ K)(\Lambda_N, \varphi) = \sum_{U \in \mathcal{P}'} \tilde{I}^{\Lambda_N \setminus U}(\varphi) \left[\sum_{X \in \mathcal{P}} \chi(X, U) \tilde{I}^{U \setminus X}(\varphi) \tilde{I}^{-(X \setminus U)}(\varphi) \int_{\mathcal{X}_N} \widetilde{K}(X, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) \right] \tag{4.4.56}$$

where the shorthand expression $I^{-X} = (I^X)^{-1}$ was used. Therefore we define

$$K'(U, \varphi) = \sum_{X \in \mathcal{P}} \chi(X, U) \tilde{I}^{U \setminus X}(\varphi) \tilde{I}^{-(X \setminus U)}(\varphi) \int_{\mathcal{X}_N} \widetilde{K}(X, \varphi, \xi)\, \mu(\mathrm{d}\xi) \tag{4.4.57}$$

for any $U \in \mathcal{P}'$.

Lemma 4.4.4. *For $H, \widetilde{H} \in M_0(\mathcal{B}_k)$, I, $\tilde{I}$, and J as in* (4.4.48)*, and $L \geq 2^{d+2} + 4R$ the functional K' defined in* (4.4.57) *has the following properties* *i) If K is translation invariant on scale k, i.e., $K(X,\varphi) = K(\tau_a X, \tau_a \varphi)$ for $a \in (L^k\mathbb{Z})^d$ then K' is translation invariant on scale $k+1$.*

ii) If K is local, i.e., $K(X,\varphi)$ only depends on the values of φ in X^* then K' is local.

iii) If K is invariant under shifts then K' is also shift invariant.

iv) If $K \in M(\mathcal{P})$ then $K' \in M(\mathcal{P}')$.

v) If $K \in M(\mathcal{P})$ factors on the scale k, i.e.,

$$K(X_1 \cup X_2, \varphi) = K(X_1,\varphi)K(X_2,\varphi) \quad \text{for strictly disjoint } X_1, X_2 \in \mathcal{P}, \tag{4.4.58}$$

then K' factors on scale $k+1$, i.e.,

$$K'(U_1 \cup U_2, \varphi) = K'(U_1,\varphi)K'(U_2,\varphi) \quad \text{for strictly disjoint } U_1, U_2 \in \mathcal{P}'. \tag{4.4.59}$$

Proof. The first claim is a consequence of the translation invariance of K, H, and π.

For the second claim we observe that $I(X,\varphi)$, $J(X,\varphi)$, $\tilde{I}(X,\varphi)$, and $K(X,\varphi)$ only depend on the values of $\varphi\restriction X^*$. Moreover $\chi(X,U) = 1$ implies by (4.4.54) for $L \geq 2^d + R$ that $X^* \subset U^*$. Since the $*$ operation is monotone and the renormalisation map $\boldsymbol{R}$ preserves locality, the functionals $K'(U,\varphi)$ only depend on $\varphi\restriction_{U^*}$.

To prove the shift invariance of K', we first notice that I, $\tilde{I}$, and J are shift invariant because they are compositions of $H, \widetilde{H} \in M_0(\mathcal{B}_k)$ with the exponential. Thus, by Lemma 4.4.3, the functionals $K(X,\varphi)$, $I(X,\varphi)$, $\tilde{I}(X,\varphi)$, and $J(X,\varphi)$ can be rewritten as functionals of $\nabla\varphi\restriction_{\vec{E}(X^*)}$. As in the locality argument this implies that K' can be written as $K'(U,\varphi) = \widetilde{K}_U(\nabla\varphi\restriction_{\vec{E}(U^*)})$ and is thus shift invariant.

The claim iv) is a consequence of the first three claims.

To prove the last claim, we observe that functionals $F, G \in M(\mathcal{P})$ that factor on the scale k, satisfy the equality $(F \circ G)(X \cup Y) = (F \circ G)(X)(F \circ G)(Y)$ whenever the polymers X and Y are strictly disjoint. Indeed,

$$\begin{aligned}
(F \circ G)(X \cup Y) &= \sum_{Z \subset X \cup Y} F(Z)G((X \cup Y) \setminus Z) \\
&= \sum_{\substack{Z_1 \subset X \\ Z_2 \subset Y}} F(Z_1 \cup Z_2) G(X \cup Y \setminus (Z_1 \cup Z_2)) \\
&= \sum_{Z_1 \subset X} \sum_{Z_2 \subset Y} F(Z_1)F(Z_2)G(X \setminus Z_1)G(Y \setminus Z_2) = (F \circ G)(X)(F \circ G)(Y).
\end{aligned} \tag{4.4.60}$$

Given that the $\circ$-product is associative, we can extend this to three functionals: the product $F \circ G \circ H$ factors if F, G, and H factor. In particular, the functional $\widetilde{K}$ factors on the scale k.

Let $U_1, U_2 \in \mathcal{P}_{k+1}$ be strictly disjoint polymers and let $X \in \mathcal{P}_k$ be a polymer such that we have $\bigcup_{Y \in \mathcal{C}(X)} \pi(Y) = \pi(X) = U = U_1 \cup U_2$. We claim that there is a unique decomposition $X = X_1 \cup X_2$ such that X_1 and X_2 are strictly disjoint and satisfy $\pi(X_i) = U_i$.

For the existence, consider $X_1 = U_1^* \cap X$, $X_2 = U_2^* \cap X$. Clearly, X_1 and X_2 are strictly disjoint and $X_1 \cup X_2 = X$ since by (4.4.54) we know that $X \subset (U_1 \cup U_2)^* = U_1^* \cup U_2^*$. The inclusions $\pi(X_i) \subset U_i^+$ together with $U_1^+ \cap U_2 = U_1 \cap U_2^+ = \emptyset$ and $U = \pi(X) = \pi(X_1) \cup \pi(X_2)$ imply that $\pi(X_i) = U_i$.

Uniqueness follows from the observation that $\pi(\widetilde{X}_i) = U_i$ implies by (4.4.54) that $\widetilde{X}_i \subset U_i^*$, and thus $\widetilde{X}_i \subset X_i$.

Assuming $L \geq 2^{d+2} + 4R$ and using (4.4.32) and (4.4.35), we conclude that the distance between X_1^* and X_2^* is bigger than the range of μ_{k+1},

$$\operatorname{dist}(X_1^*, X_2^*) \geq \operatorname{dist}(U_1^*, U_2^*) \geq \frac{L^{k+1}}{2}. \tag{4.4.61}$$

Thus, using that $\widetilde{K}$ factors on scale k, we get

$$\begin{aligned}\int_{\mathcal{X}_N} \widetilde{K}(X_1 \cup X_2, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) &= \int_{\mathcal{X}_N} \widetilde{K}(X_1, \varphi, \xi)\widetilde{K}(X_2, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) \\ &= \int_{\mathcal{X}_N} \widetilde{K}(X_1, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) \int_{\mathcal{X}_N} \widetilde{K}(X_2, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi).\end{aligned} \tag{4.4.62}$$

Finally, we observe that

$$\begin{aligned}(X_1 \cup X_2) \setminus (U_1 \cup U_2) &= (X_1 \setminus U_1) \cup (X_2 \setminus U_2) \\ (U_1 \cup U_2) \setminus (X_1 \cup X_2) &= (U_1 \setminus X_1) \cup (U_2 \setminus X_2).\end{aligned} \tag{4.4.63}$$

The inclusion '$\subset$' holds in general, the other inclusion follows from $X_1 \cap U_2 = X_2 \cap U_1 = \emptyset$.

As a result, using manipulations similar to (4.4.60) for strictly disjoint $U_1, U_2 \in \mathcal{P}'$, these facts imply

$$\begin{aligned}K'(U_1 \cup U_2, \varphi) &= \sum_{X \in \mathcal{P}} \chi(X, U_1 \cup U_2) \tilde{I}^{(U_1 \cup U_2) \setminus X}(\varphi) \tilde{I}^{-(X \setminus (U_1 \cup U_2))}(\varphi) \int_{\mathcal{X}_N} \widetilde{K}(X, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) \\ &= \sum_{X \in \mathcal{P}} \mathbb{1}_{\pi(X) = U_1 \cup U_2} \tilde{I}^{(U_1 \cup U_2) \setminus X}(\varphi) \tilde{I}^{-(X \setminus (U_1 \cup U_2))}(\varphi) \int_{\mathcal{X}_N} \widetilde{K}(X, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) \\ &= \sum_{X_1, X_2 \in \mathcal{P}} \mathbb{1}_{\pi(X_1) = U_1} \mathbb{1}_{\pi(X_2) = U_2} \frac{\tilde{I}^{(U_1 \cup U_2) \setminus (X_1 \cup X_2)}(\varphi)}{\tilde{I}^{(X_1 \cup X_2) \setminus (U_1 \cup U_2)}(\varphi)} \int_{\mathcal{X}_N} \widetilde{K}(X_1 \cup X_2, \varphi, \xi)\, \mu_{k+1}(\mathrm{d}\xi) \\ &= \sum_{X_1, X_2 \in \mathcal{P}} \mathbb{1}_{\pi(X_1) = U_1} \mathbb{1}_{\pi(X_2) = U_2} \frac{\tilde{I}^{U_1 \setminus X_1}(\varphi)}{\tilde{I}^{X_1 \setminus U_1}(\varphi)} \frac{\tilde{I}^{U_2 \setminus X_2}(\varphi)}{\tilde{I}^{X_2 \setminus U_2}(\varphi)} \int_{\mathcal{X}_N} \widetilde{K}(X_1, \varphi, \xi) \mu_{k+1}(\mathrm{d}\xi) \int_{\mathcal{X}_N} \widetilde{K}(X_2, \varphi, \xi) \mu_{k+1}(\mathrm{d}\xi) \\ &= K'(U_1, \varphi) K'(U_2, \varphi)\end{aligned} \tag{4.4.64}$$

□

For future reference we state a concise definition of $\boldsymbol{T}_k$. Recall that we defined for $0 \leq k \leq N-1$ and $H_k \in M_0(\mathcal{B}_k)$ the next scale Hamiltonian $H_{k+1} \in M_0(\mathcal{B}_{k+1})$ by

$$H_{k+1}(B', \varphi) = \sum_{B \in \mathcal{B}_k(B')} \widetilde{H}_k(B, \varphi) \tag{4.4.65}$$

where

$$\widetilde{H}_k(B, \varphi) = \Pi_2 \boldsymbol{R}' H(B, \varphi) - \Pi_2 \boldsymbol{R}' K(B, \varphi). \tag{4.4.66}$$

For $K_k \in M(\mathcal{P}_k^c)$ we denote $\widetilde{K}_k(\varphi,\xi) = \Big(1 - e^{-\widetilde{H}_k(\varphi)}\Big) \circ \big(e^{-H_k(\varphi+\xi)} - 1\big) \circ K_k(\varphi+\xi)$ and we define $K_{k+1} \in M(\mathcal{P}_{k+1}^c)$ for $U \in \mathcal{P}_{k+1}^c$ by

$$K_{k+1}(U,\varphi) = \sum_{X\in\mathcal{P}} \chi(X,U) \exp\Big(- \sum_{B\in\mathcal{B}_k(U\setminus X)} \widetilde{H}_k(B,\varphi) + \sum_{B\in\mathcal{B}_k(X\setminus U)} \widetilde{H}_k(B,\varphi)\Big) \int_{\mathcal{X}_N} \widetilde{K}(X,\varphi,\xi)\, \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi) \tag{4.4.67}$$

where $\chi(X,U) = \mathbb{1}_{\pi(X)=U}$ and $\pi : \mathcal{P}_k \to \mathcal{P}_{k+1}$ was defined in (4.4.53).

Definition 4.4.5. *Let* $0 \le k \le$ *the renormalisation transformation*

$$\boldsymbol{T}_k : M_0(\mathcal{B}_k) \times M(\mathcal{P}_k^c) \times \mathbb{R}_{\mathrm{sym}}^{(d\times m)\times(d\times m)} \to M_0(\mathcal{B}_{k+1}) \times M(\mathcal{P}_{k+1}^c) \tag{4.4.68}$$

is defined by

$$\boldsymbol{T}_k(H_k, K_k, \boldsymbol{q}) = (H_{k+1}, K_{k+1}) \tag{4.4.69}$$

where H_{k+1} *and* K_{k+1} *are given by* (4.4.65) *and* (4.4.67) *respectively.*

We have the following result for $\boldsymbol{T}_k$.

Proposition 4.4.6. *For* $L \ge 2^{d+2} + 4R$ *and* $0 \le k \le N-1$ *the renormalisation transformation* $\boldsymbol{T}_k$ *is well-defined and satisfies for* $H_k \in M_0(\mathcal{B}_k)$*,* $K_k \in M(\mathcal{P}_k^c)$*,* $H_{k+1} \in M_0(\mathcal{B}_{k+1})$*, and* $K_{k+1} \in M(\mathcal{P}_{k+1}^c)$ *with* $\boldsymbol{T}_k(H_k, K_k, \boldsymbol{q}) = (H_{k+1}, K_{k+1})$ *the identity*

$$\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}(e^{-H_k} \circ K_k)(\Lambda_N, \varphi) = (e^{-H_{k+1}} \circ K_{k+1})(\Lambda_N, \varphi). \tag{4.4.70}$$

Proof. Lemma 4.4.4 iv) implies that $K_{k+1} \in M(\mathcal{P}_{k+1}^c)$. From Lemma 4.4.4v) we conclude that $\iota_2 K_{k+1}(U)$ equals the expression on the right hand side of (4.4.67) for all $U \in \mathcal{P}_{k+1}$. Now (4.4.70) follows from (4.4.56). □

Of course the condition (4.4.70) is not sufficient for our analysis. In addition we need smoothness and boundedness results for the map $\boldsymbol{T}_k$. This requires to equip the spaces $M(\mathcal{P}_k^c)$ with a norm. In the next section we will define the relevant norms which will allow us to establish the smoothness result and to prove contraction properties of $\boldsymbol{T}_k$.

4.4.4 Norms

Next we introduce suitable norms on the space $M(\mathcal{P}_k, \mathcal{V}_N)$ of local functionals (see (4.4.38)). For any $F \in M(\mathcal{P}_k^c, \mathcal{V}_N)$ and any $X \in \mathcal{P}_k^c$ we define $F(X) \in M(\mathcal{V}_N)$ by $F(X)(\varphi) = F(X,\varphi)$. Fixing now $r_0 \in \mathbb{N}$ with $r_0 \ge 3$, we introduce a norm $\|F(X)\|_{k,T_\varphi}$ based on a norm of the r_0-th order Taylor polynomial of the functional $F(X)$ at φ as well as the norm $\|F(X)\|_{k,X} = \sup_\varphi w^{-X}(\varphi)\|F(X)\|_{k,T_\varphi}$, where $w^{-X}(\varphi) = \frac{1}{w^X(\varphi)}$ and w^X is an appropriately chosen weight function. The main difference in comparison with [4] (which was based on earlier work of Brydges et al., cf. e.g. [44] and [42]) is in the choice of these weights. The current choice allows us to relax substantially the growth condition for the potential. An additional difference with respect to [4] is that we use a different norm on polynomials (essentially the projective instead of the injective norm on the dual tensor product, see Appendix 4.A). This is not crucial but it puts our approach in line with the much more general framework developed in [44, 45].

The main observation for the definition of the norms on Taylor polynomials is that the action of polynomials can be linearised by looking at their action on (direct sums) of tensor products.

More precisely a homogeneous polynomial $P^{(r)}$ of degree r on the space of fields $\mathcal{X}$ can be uniquely identified with a symmetric r-linear form and hence with an element $\overline{P^{(r)}}$ in the dual of $\mathcal{X}^{\otimes r}$ (see Lemma 4.A.1).

To define a linear action of a general polynomial P we recall that $\oplus_{r=0}^{\infty}\mathcal{X}^{\otimes r}$ is the space of sequences $g = (g^{(0)}, g^{(1)}, \ldots)$ with $g^{(r)} \in \mathcal{X}^{\otimes r}$ and with only finitely many non-vanishing terms. Then we define the dual pairing

$$\langle P, g\rangle = \sum_{r=0}^{\infty} \langle \overline{P^{(r)}}, g^{(r)}\rangle \tag{4.4.71}$$

with the space of test functions

$$\Phi := \Phi_{r_0} := \{g \in \oplus_{r=0}^{\infty}\mathcal{X}^{\otimes r} : g^{(r)} = 0 \;\text{ for all }\; r > r_0\}. \tag{4.4.72}$$

The restriction to the space Φ_{r_0} means that the linear maps P correspond to polynomials of order at most r_0.

In the following we take $\mathcal{X} = \mathcal{V}_N$ as the space of fields with norms defined on Φ as follows. On $\mathcal{V}_N^{\otimes 0} = \mathbb{R}$ we take the usual absolute value on $\mathbb{R}$. Let

$$X \in \mathcal{P}_k \;\text{ and }\; j \in \{k, k+1\}. \tag{4.4.73}$$

For $\varphi \in \mathcal{V}_N$ and $x \in \Lambda$ we define $\nabla_x^{i,\alpha}\varphi = (\nabla^\alpha \varphi_i)(x)$ and consider the norms

$$\begin{aligned} |\varphi|_{j,X} &= \sup_{x\in X^*} \sup_{1\le i\le m} \sup_{1\le|\alpha|\le p_\Phi} w_j(\alpha)^{-1} \left|\nabla_x^{i,\alpha}\varphi\right| \\ &= \sup_{x\in X^*} \sup_{1\le i\le m} \sup_{1\le|\alpha|\le p_\Phi} w_j(\alpha)^{-1} \left|\nabla^{\alpha}\varphi_i(x)\right|. \end{aligned} \tag{4.4.74}$$

where

$$p_\Phi = \lfloor d/2 \rfloor + 2 \tag{4.4.75}$$

and the weights $w_j(\alpha)$ are given by

$$w_j(\alpha) = h_j\, L^{-j|\alpha|}\, L^{-j\frac{d-2}{2}} \quad \text{with} \quad h_j = 2^j h. \tag{4.4.76}$$

The $|\cdot|_{j,X}$-norm for the fields depends on a k-polymer X and a scale $j \in \{k, k+1\}$ and it measures the size of the field in a weighted maximum-norm in a neighbourhood of this polymer. The weights are chosen so that a typical value of the field ξ distributed according to $\mu_{j+1}^{(\boldsymbol{q})}$ has norm of order h_j^{-1} (cf. (4.4.13)). The parameters h_j allow to control the scaling of the field norms $|\cdot|_{j,X}$ and since norms are defined by duality the parameter h_j also appears in the norm for Hamiltonians $H \in M_0(\mathcal{B})$. See Section 4.3.1 for further discussion why we choose scaling factor h_j which grow with j.

Viewing homogeneous terms $g^{(r)} \in \mathcal{V}_N^{\otimes r}$ as maps (or more precisely equivalence classes of maps modulo tensor products involving constant fields, see Section 4.A.4 in the appendix) from Λ^r to $(\mathbb{R}^p)^{\otimes r}$ with ∇^{α_j} acting on the j-th argument of $g^{(r)}_{i_1\ldots i_r}$, we introduce the norm

$$\begin{aligned} |g^{(r)}|_{j,X} &= \sup_{x_1,\ldots,x_r\in X^*} \sup_{\boldsymbol{m}\in\mathfrak{m}_{p_\Phi,r}} w_j(\boldsymbol{m})^{-1}\, \nabla^{\boldsymbol{m}_1} \otimes \ldots \otimes \nabla^{\boldsymbol{m}_r} g^{(r)}(x_1,\ldots,x_r) \\ &= \sup_{x_1,\ldots,x_r\in X^*} \sup_{\boldsymbol{m}\in\mathfrak{m}_{p_\Phi,r}} w_j(\boldsymbol{m})^{-1}\, \nabla^{\alpha_1} \otimes \ldots \otimes \nabla^{\alpha_r} g^{(r)}_{i_1\ldots i_r}(x_1,\ldots,x_r) \end{aligned} \tag{4.4.77}$$

where $\mathfrak{m}_{p_\Phi,r}$ is the set of r-tuples $m = (m_1, \dots, m_r)$ with $m_\ell = (i_\ell, \alpha_\ell)$ and $1 \le |\alpha_\ell| \le p_\Phi$ and

$$w_j(m) = \prod_{\ell=1}^{r} w_j(\alpha_\ell). \tag{4.4.78}$$

The norm defined above is actually the injective tensor norm on $(\mathcal{V}_N, |\cdot|_{j,X})^{\otimes r}$, see (4.A.71), implying, in particular, that

$$|\varphi^{(1)} \otimes \dots \otimes \varphi^{(r)}|_{j,X} = |\varphi^{(1)}|_{j,X} \dots |\varphi^{(r)}|_{j,X} \quad \text{for any} \quad \varphi^{(1)}, \dots, \varphi^{(r)} \in \mathcal{X}. \tag{4.4.79}$$

We now define a norm on the space Φ of test functions by

$$|g|_{j,X} = \sup_{r \in \mathbb{N}_0} |g^{(r)}|_{j,X} = \sup_{r \le r_0} |g^{(r)}|_{j,X}. \tag{4.4.80}$$

and a dual norm on polynomials by

$$|P|_{j,X} := \sup\{\langle P, g\rangle : g \in \Phi,\ |g|_{j,X} \le 1\}. \tag{4.4.81}$$

Assume that $F \in C^{r_0}(\mathcal{V}_N)$ satisfies the locality condition with respect to a polymer $X \in \mathcal{P}^{\mathrm{c}}$,

$$F(\varphi + \psi) = F(\varphi) \quad \text{if} \quad \psi|_{X^*} = 0. \tag{4.4.82}$$

We define the pairing

$$\langle F, g\rangle_\varphi := \langle \mathrm{Tay}_\varphi F, g\rangle. \tag{4.4.83}$$

and the norm

$$|F|_{j,X,T_\varphi} = |\mathrm{Tay}_\varphi F|_{j,X} = \sup\{\langle F, g\rangle_\varphi : g \in \Phi,\ |g|_{j,X} \le 1\}. \tag{4.4.84}$$

Here $\mathrm{Tay}_\varphi F$ denotes the Taylor polynomial of order r_0 of F at φ.

We remark in passing that the right hand side of (4.4.84) may be infinite since $|\cdot|_{j,X}$ is only a seminorm, but this will not occur in the cases we are interested in, namely when F is local and shift invariant in the sense described in the paragraph following (4.4.38). More precisely the right hand side of (4.4.84) is finite if and only if $\mathrm{Tay}_\varphi F(\dot\varphi + \dot\psi) = \mathrm{Tay}_\varphi F(\dot\varphi)$ for all $\dot\varphi \in \mathcal{V}_N$ and all $\dot\psi \in \mathcal{V}_N$ with $|\dot\psi|_{j,X} = 0$ (to see this one uses the fact $\mathcal{V}_N$ is finite dimensional and the zero norm elements of $\mathcal{V}_N^{\otimes r}$ are linear combinations of tensor products $\xi_1 \otimes \dots \otimes \xi_r$ where at least one of the ξ_i has zero norm). Note that $|\dot\psi|_{k,X} = 0$ implies that $\dot\psi$ is constant on each graph-connected component of X^* and therefore by the definition of shift invariance $F(\varphi + \dot\psi) = F(\varphi)$ for all $\varphi \in \mathcal{V}_N$.

The final norms for the functional F are weighted sup-norms over φ of the norm $|F|_{k,X,T_\varphi}$. Dividing the norm $|F|_{k,X,\varphi}$ by a regulator $w_k(\varphi)$, we allow the functional to grow for large fields. A way to think about these regulators is that $|F(\varphi)| \le \|F(X)\|\, w_k(\varphi)$. This bound must behave well with respect to integration against μ_{k+1} and satisfy certain submultiplicativity properties. The exact definition of the regulator is slightly involved and will be given in the next section.

Now, we define a norm on the class of functionals $M(\mathcal{P}_k^{\mathrm{c}}) = M(\mathcal{P}_k^{\mathrm{c}}, \mathcal{V}_N)$ defined in (4.4.38). Writing $F(X)(\varphi) = F(X, \varphi)$ for any $F \in M(\mathcal{P}_k^{\mathrm{c}}, \mathcal{V}_N)$, we sometimes use the abbreviation

$$|F(X)|_{k,T_\varphi} := |F(X)|_{k,X,T_\varphi}. \tag{4.4.85}$$

Let $W_k^X, w_k^X, w_{k:k+1}^X \in M(\mathcal{P}_k)$ be weight functions that will be defined in the next section. Let us denote $W_k^{-X} = (W_k^X)^{-1}$ and similarly for w. The strong and weak norms are defined, respectively, by

$$|||F(X)|||_{k,X} = \sup_{\varphi} |F(X)|_{k,T_\varphi} W_k^{-X}(\varphi), \tag{4.4.86}$$

$$\|F(X)\|_{k,X} = \sup_{\varphi} |F(X)|_{k,T_\varphi} w_k^{-X}(\varphi), \tag{4.4.87}$$

$$\|F(X)\|_{k:k+1,X} = \sup_{\varphi} |F(X)|_{k,T_\varphi} w_{k:k+1}^{-X}(\varphi). \tag{4.4.88}$$

The last norm is a version of the weak norm which lies between the weak norms of scales k and $k+1$. In fact we will use the strong norm only for functionals in $M(\mathcal{B})$ which already factor over single blocks. We write $|||F|||_k = |||F(B)|||_{k,B}$ where the right hand side is independent of B by translation invariance.

Finally, for any $A \geq 1$ we define the global weak norm for $F \in M(\mathcal{P}_k^c)$ given by a weighted maximum of the weak norms over the connected polymers

$$\|F\|_k^{(A)} = \sup_{X \in \mathcal{P}_k^c} \|F(X)\|_{k,X} A^{|X|_k}. \tag{4.4.89}$$

For polymers X that are not connected we will usually estimate the norm of $F(X,\cdot)$ by the product of the norms of $F(Y_i,\cdot)$ where $Y_1, Y_2, \ldots$ are the connected components of X. In Lemma 4.6.3 we will state submultiplicativity properties of the norms needed for these estimates. With the norm (4.4.89) we also consider the version where we replace the weak k norm by the in-between $k : k+1$ norm.

We finally introduce another norm on the space of relevant Hamiltonians (at scale k). Recall that we defined these to be functionals of the form

$$H(B,\varphi) = L^{dk} a_\emptyset + \sum_{x\in B} \sum_{(i,\alpha)\in\mathfrak{v}_1} a_{i,\alpha} \nabla^\alpha \varphi_i(x) + \sum_{x\in B} \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} a_{(i,\alpha),(j,\beta)} \nabla^\alpha\varphi_i(x) \nabla^\beta \varphi_j(x). \tag{4.4.90}$$

Here B is a k-block and the index sets $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are given by

$$\mathfrak{v}_1 := \{(i,\alpha) : 1 \leq i \leq m,\, \alpha \in \mathbb{N}_0^{\mathcal{U}}, 1 \leq |\alpha| \leq \lfloor d/2 \rfloor + 1\}, \tag{4.4.91}$$

$$\mathfrak{v}_2 := \{(i,\alpha),(j,\beta) : 1 \leq i,j \leq m,\, \alpha,\beta \in \mathbb{N}_0^{\mathcal{U}},\, |\alpha| = |\beta| = 1,\, (i,\alpha) \leq (j,\beta)\}, \tag{4.4.92}$$

where $\mathcal{U} = \{e_1, \ldots, e_d\}$. The expression $(i,\alpha) \leq (j,\beta)$ refers to any ordering on $\{1,\ldots,m\} \times \{e_1,\ldots,e_d\}$, e.g. lexicographic ordering. We use ordered indices to avoid double counting since $\nabla^\alpha\varphi_i(x)\nabla^\beta\varphi_j(x) = \nabla^\beta\varphi_j(x)\nabla^\alpha\varphi_i(x)$. We now introduce a norm for relevant Hamiltonians which is expressed directly in terms of the coefficients a_m and given by

$$\|H\|_{k,0} = L^{kd}\,|a_\emptyset| + \sum_{(i,\alpha)\in\mathfrak{v}_1} h_k L^{kd} L^{-k\frac{d-2}{2}} L^{-k|\alpha|}\,|a_{i,\alpha}| + \sum_{m\in\mathfrak{v}_2} h_k^2\,|a_m|. \tag{4.4.93}$$

The weights in front of the coefficients are chosen in such a way that the norm $\|\cdot\|_{k,0}$ is equivalent (uniformly in k and N) to the strong norm $|||\cdot|||$ (see Lemma 4.6.7 and Lemma 4.6.8 below). Intuitively the weight in L can also be understood by recalling that the typical value of $|\nabla^\alpha\varphi_i(x)|$ under μ_{k+1} is of order $L^{-k|\alpha|}L^{-k\frac{d-2}{2}}$.

Note that the norms depend on the constants h_k, A and also on L that will be chosen later. We will need one additional norm because the renormalisation map $\boldsymbol{R}_{k+1}$ does not preserve factorisation on scale k so that we cannot rely on submultiplicativity. This norm will only be required in the smoothness result in Section 4.7 and we postpone the definition of the last norm to that section.

4.4.5 Properties of the renormalisation map

We are interested in results for fixed values of

$$d, m, R_0, \omega_0, \zeta \tag{4.4.94}$$

and therefore we usually do not keep track of the dependence of constants on these parameters.

Our definition of the renormalisation transformation $\boldsymbol{T}_k$ in Definition 4.4.5 satisfies the condition (4.4.44). A second requirement for the map $\boldsymbol{T}_k$ is that it separates relevant and irrelevant contributions properly. Observe that the origin $(0,0)$ is a fixed point of the transformation for every $\boldsymbol{q}$. The separation of relevant and irrelevant contributions can be made precise by showing that the linearisation of $\boldsymbol{T}_k$ at the origin defines a hyperbolic dynamical system. A close look at the definition of $\boldsymbol{T}_k$ reveals that H' is in fact a linear function of K and H, i.e., we can write

$$\boldsymbol{T}_k(H, K, \boldsymbol{q}) = (\boldsymbol{A}_k^{(\boldsymbol{q})} H + \boldsymbol{B}_k^{(\boldsymbol{q})} K, \boldsymbol{S}_k(H, K, \boldsymbol{q})) \tag{4.4.95}$$

where $\boldsymbol{A}_k^{(\boldsymbol{q})}$ and $\boldsymbol{B}_k^{(\boldsymbol{q})}$ are linear operators. We need two theorems concerning the renormalisation transformation $\boldsymbol{T}_k$. The first theorem states local smoothness of the map $\boldsymbol{S}$ which is required to apply an implicit function theorem. Let us denote with $\mathcal{U}_{\rho,\kappa} \subset M_0(\mathcal{B}_k) \times M(\mathcal{P}_k^{\mathrm{c}}) \times \mathbb{R}_{\mathrm{sym}}^{(d\times m)\times(d\times m)}$ the subset

$$\mathcal{U}_{\rho,\kappa} = \{(H; K, \boldsymbol{q}) \in M_0(\mathcal{B}_k) \times M(\mathcal{P}_k) \times \mathbb{R}_{\mathrm{sym}}^{(d\times m)\times(d\times m)} : \|H\|_{k,0} < \rho,\ \|K\|_k^{(A)} < \rho,\ |\boldsymbol{q}| < \kappa\} \tag{4.4.96}$$

Theorem 4.4.7. *Let $L_0 = \max(2^{d+3} + 16R, 4d(2^d + R))$. For every $L \geq L_0$ there are $h_0(L)$, $A_0(L)$, and $\kappa(L)$ such that for $h \geq h_0(L)$ and $A \geq A_0(L)$ there exists $\rho = \rho(A)$ such that the map $\boldsymbol{S}_k$ satisfies*

$$\boldsymbol{S}_k \in C^\infty\left(\mathcal{U}_{\rho,\kappa}, (M(\mathcal{P}_{k+1}^{\mathrm{c}}), \|\cdot\|_{k+1}^{(A)})\right). \tag{4.4.97}$$

Moreover there are constants $C = C_{j_1,j_2,j_3}(A, L)$ such that

$$\|D_1^{j_1} D_2^{j_2} D_3^{j_3} \boldsymbol{S}_k(H, K, \boldsymbol{q})(\dot{H}^{j_1}, \dot{K}^{j_2}, \dot{\boldsymbol{q}}^{j_3})\|_{k+1}^{(A)} \leq C \|\dot{H}\|_0^{j_1} \left(\|\dot{K}\|_k^{(A)}\right)^{j_2} \|\dot{\boldsymbol{q}}\|^{j_3} \tag{4.4.98}$$

for any $(H, K, \boldsymbol{q}) \in \mathcal{U}_{\rho,\kappa}$ and any $j_1, j_2, j_3 \geq 0$.

The proof of this theorem can be found in Section 4.7. The second theorem concerns the hyperbolicity of the linearisation of the renormalisation transformation. Recall that $\eta \in \left(0, \frac{2}{3}\right]$ is a fixed parameter that controls the contraction rate of the renormalisation flow.

Theorem 4.4.8. *The first derivative of $\boldsymbol{T}_k$ at $H = 0$ and $K = 0$ has the triangular form*

$$D\boldsymbol{T}_k(0, 0, \boldsymbol{q}) \begin{pmatrix} \dot{H} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_k^{(\boldsymbol{q})} & \boldsymbol{B}_k^{(\boldsymbol{q})} \\ 0 & \boldsymbol{C}_k^{(\boldsymbol{q})} \end{pmatrix} \begin{pmatrix} \dot{H} \\ \dot{K} \end{pmatrix} \tag{4.4.99}$$

where

$$(\boldsymbol{A}_k^{(\boldsymbol{q})}\dot{H})(B',\varphi) = \sum_{B\in\mathcal{B}(B')} \dot{H}(B,\varphi) + L^{(k+1)d} \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} a_{(i,\alpha),(j,\beta)}\,(\nabla^\beta)^*\nabla^\alpha \mathcal{C}_{k+1,ij}^{(\boldsymbol{q})}(0) \quad (4.4.100)$$

$$(\boldsymbol{B}_k^{(\boldsymbol{q})}\dot{K})(B',\varphi) = -\sum_{B\in\mathcal{B}(B')} \Pi_2\left(\int_{\mathcal{X}_N} \dot{K}(B,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\right) \quad (4.4.101)$$

$$\begin{aligned}(\boldsymbol{C}_k^{(\boldsymbol{q})}\dot{K})(U,\varphi) = &\sum_{B:\,\overline{B}=U} (1-\Pi_2)\int_{\mathcal{X}_N} \dot{K}(B,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi) \\ &+ \sum_{\substack{X\in\mathcal{P}_k^{\mathrm{c}}\setminus\mathcal{B}(X)\\ \pi(X)=U}} \int_{\mathcal{X}_N} \dot{K}(X,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi).\end{aligned} \quad (4.4.102)$$

There exists a constant L_0 such that there are constants $h_0 = h_0(L)$, $A_0 = A_0(L)$, and $\kappa(L) > 0$ such that for any $L \geq L_0$, $A \geq A_0(L)$, $h \geq h_0(L)$ and for $|\boldsymbol{q}| < \kappa(L)$ the following bounds hold independent of k and N

$$\|\boldsymbol{C}_k^{(\boldsymbol{q})}\| \leq \frac{3}{4}\eta, \quad \|(\boldsymbol{A}_k^{(\boldsymbol{q})})^{-1}\| \leq \frac{3}{4}, \text{ and } \|\boldsymbol{B}_k^{(\boldsymbol{q})}\| \leq \frac{1}{3}. \quad (4.4.103)$$

Here the norms denote the operator norms for maps $(M(\mathcal{P}_k^{\mathrm{c}}), \|\cdot\|_k^{(A)}) \to (M(\mathcal{P}_{k+1}^{\mathrm{c}}), \|\cdot\|_{k+1}^{(A)})$, $(M_0(\mathcal{B}_{k+1}), \|\cdot\|_{k+1,0}) \to (M_0(\mathcal{B}_k), \|\cdot\|_{k,0})$, and $(M(\mathcal{P}_k^{\mathrm{c}}), \|\cdot\|_k^{(A)}) \to (M_0(\mathcal{B}_{k+1}),, \|\cdot\|_{k+1,0})$. In addition the derivatives of the operators with respect to $\boldsymbol{q}$ are bounded:

$$\|\partial_{\boldsymbol{q}}^\ell \boldsymbol{A}_k^{(\boldsymbol{q})}\dot{H}\|_0 \leq C\|\dot{H}\|_0, \ \|\partial_{\boldsymbol{q}}^\ell \boldsymbol{B}_k^{(\boldsymbol{q})}\dot{K}\|_0 \leq C\|\dot{K}\|, \ \|\partial_{\boldsymbol{q}}^\ell \boldsymbol{C}_k^{(\boldsymbol{q})}\dot{K}\| \leq C\|\dot{K}\| \quad (4.4.104)$$

for some constant $C = C_\ell(A, L)$. The proof shows that L_0 only depends on d, m, R_0, and on ζ and ω_0 through $A_{\mathcal{B}}$ where $A_{\mathcal{B}}$ comes from Theorem 4.5.1.

Proof. Here we only show the validity of the expressions for the operators $\boldsymbol{A}_k^{(\mathbf{q})}$, $\boldsymbol{B}_k^{(\boldsymbol{q})}$, and $\boldsymbol{C}_k^{(\boldsymbol{q})}$ and the bound (4.4.104). The bounds for the operator norms will be shown in Section 4.8 in Lemma 4.8.5, Lemma 4.8.6 and Lemma 4.8.1. The proof of the bounds (4.4.104) can be found in Corollary 4.7.9 for the operators $\boldsymbol{A}_k^{(\boldsymbol{q})}$ and $\boldsymbol{B}_k^{(\boldsymbol{q})}$. For $\boldsymbol{C}_k^{(\boldsymbol{q})}$ it follows from Theorem 4.4.7 and the identity

$$\partial_{\boldsymbol{q}}^\ell \boldsymbol{C}_k^{(\boldsymbol{q})} = \partial_{\boldsymbol{q}}^\ell \partial_K \boldsymbol{S}_k(0,0,\boldsymbol{q}). \quad (4.4.105)$$

To obtain the formula for $\boldsymbol{A}_k^{(\boldsymbol{q})}$ we recall that by (4.4.65) and (4.4.66)

$$(\boldsymbol{A}_k^{(\boldsymbol{q})}\dot{H})(B',\varphi) = \sum_{B\in\mathcal{B}_k(B')} \Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})}\dot{H}(B,\varphi). \quad (4.4.106)$$

We write the Hamiltonian $\dot{H}$ as a sum of constant, linear and quadratic terms, $\dot{H}(\varphi) = L^{dk}a_\emptyset + \ell(\varphi) + Q(\xi)$. Then

$$\dot{H}(B,\varphi+\xi) = \dot{H}(B,\varphi) + Q(\xi) + \text{terms linear in } \xi \quad (4.4.107)$$

where in view of (4.4.90)

$$Q(\xi) = \sum_{x\in B} \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} a_{(i,\alpha),(j,\beta)} \nabla^\alpha \xi_i(x) \nabla^\beta \xi_j(x).$$

Linear terms vanish when integrated against $\mu_{k+1}(\mathrm{d}\xi)$. Observe that the projection Π_2 preserves relevant Hamiltonians, i.e., $\Pi_2 H = H$ for $H \in M_0(\mathcal{B}_k)$. It remains to evaluate the integral of the quadratic form $Q(\xi,\xi)$. Since the covariance of μ_{k+1} is translation invariant we have for $\mathbb{E} = \mathbb{E}_{\mu_{k+1}}$

$$\mathbb{E}(\nabla^\alpha \xi^i(x)\nabla^\beta \xi^j(y)) = \mathbb{E}((\nabla^\beta)^*\nabla^\alpha \xi^i(x)\xi^j(y)) = \big((\nabla^\beta)^*\nabla^\alpha \mathcal{C}_{ij}\big)(x-y). \tag{4.4.108}$$

This implies that

$$\int_{\mathcal{X}_N} Q(\xi)\,\mu^{(\boldsymbol{q})}_{k+1}(\mathrm{d}\xi) = \sum_{x\in B}\sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} a_{(i,\alpha),(j,\beta)}\,(\nabla^\beta)^*\nabla^\alpha \mathcal{C}^{(\boldsymbol{q})}_{k+1,ij}(0). \tag{4.4.109}$$

Summing over $B \in \mathcal{B}_k(B')$ we get the formula (4.4.100) for $\boldsymbol{A}^{(\boldsymbol{q})}$ using that $|B| = L^{dk}$ and $|\mathcal{B}_k(B')| = L^d$ to obtain the prefactor $L^{(k+1)d}$ of the constant term.

The formula for $\boldsymbol{B}^{(\boldsymbol{q})}_k$ is a direct consequence of the definitions (4.4.65) and (4.4.66).

We now derive the formula for $\boldsymbol{C}^{(\boldsymbol{q})}_k$. Recall that we defined $\widetilde{K}(K,H)(\varphi,\xi) = (1-e^{-\widetilde{H}_k(\varphi)}))\circ (e^{-H_k(\varphi+\xi)}-1)\circ K_k(\varphi+\xi)$. We calculate the derivative at 0 in direction $\dot{K}$, hence we set $H_k = 0$ and

$$\widetilde{H}_k(B,\varphi) = -\Pi_2 \boldsymbol{R}^{(\boldsymbol{q})}_{k+1} K_k(B,\varphi). \tag{4.4.110}$$

This implies for the derivative of $\widetilde{K}$ at zero

$$D_K\widetilde{K}(0)(\dot{K})(X,\varphi,\xi) = \begin{cases} \dot{K}(X,\varphi+\xi) - \Pi_2\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}\dot{K}(X,\varphi) \text{ if } X\in\mathcal{B}_k,\\ \dot{K}(X,\varphi+\xi) \text{ if } X\in\mathcal{P}^{\mathrm{c}}_k\setminus\mathcal{B}_k,\\ 0 \text{ if } X\in\mathcal{P}_k\setminus\mathcal{P}^{\mathrm{c}}_k.\end{cases} \tag{4.4.111}$$

The derivative vanishes for non-connected polymers because K factors on scale k. Now the definition (4.4.67) implies (4.4.102).

Finally we show that the derivative of K_{k+1} with respect to H_k vanishes. To this end we notice that

$$D_H\widetilde{K}(0)(\dot{H})(X,\varphi,\xi) = \begin{cases} \dot{H}(X,\varphi+\xi) - \Pi_2\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}\dot{H}(X,\varphi) \ \text{ for } X\in\mathcal{B}_k,\\ 0 \ \text{ otherwise.}\end{cases} \tag{4.4.112}$$

Thus (4.4.67) implies that the derivative vanishes for $U\notin\mathcal{B}_{k+1}$ and we infer that for $B'\in\mathcal{B}_{k+1}$

$$\begin{aligned} D_H K_{k+1}(\dot{H})(B',\varphi) &= \sum_{B\in\mathcal{B}_k(B')}\int_{\mathcal{X}_N} \dot{H}(B,\varphi+\xi) - (\Pi_2\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}\dot{H})(B,\varphi)\,\mu^{(\boldsymbol{q})}_{k+1}(\mathrm{d}\xi)\\ &= \sum_{B\in\mathcal{B}_k(B')} (\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}\dot{H})(B,\varphi) - (\Pi_2\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}\dot{H})(B,\varphi) = 0 \end{aligned} \tag{4.4.113}$$

where we used that $\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}$ maps relevant Hamiltonians to relevant Hamiltonians as shown above and Π_2 is the identity on relevant Hamiltonians. □

4.5 A new large field regulator

In this section we construct a new large field regulator. It allows for substantially rougher initial perturbations than the previous regulator in [4] or [42]. Previously explicit estimates for carefully chosen Gaussian integrals were used to construct the regulators. In the new approach we define the weights implicitly based on the abstract formula for Gaussian integrals.

Recall that we defined the constant

$$M = M(d) = p_\Phi + \lfloor d/2 \rfloor + 1 = 2\lfloor d/2 \rfloor + 3 \tag{4.5.1}$$

that is related to the discrete Sobolev embedding (note that compared to [4] we changed M). For any k-polymer X we define the linear operator $\boldsymbol{M}_k^X : \mathcal{X}_N \to \mathcal{X}_N$ by

$$\boldsymbol{M}_k^X = \sum_{1 \le |\alpha| \le M} L^{2k(|\alpha|-1)} (\nabla^*)^\alpha \chi_X \nabla^\alpha \tag{4.5.2}$$

where $\chi_X : T_N \to \mathbb{R}$ is defined by

$$\chi_X(x) = \sum_{B \in \mathcal{B}_k(X)} \mathbf{1}_{B^+}(x) = \big|\{B \in \mathcal{B}_k(X) : x \in B^+\}\big|. \tag{4.5.3}$$

Here $\mathbf{1}$ denotes the indicator function. Recall that $B^+ = (B + [-L^k, L^k]) \cap T_N^d$ for $k \ge 1$ and $B^+ = (B + [-R, R]^d) \cap T_N$ for $k = 0$. Note that here and in the following we sometimes use the natural inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^{m \times m}$ given by $\lambda \to \lambda \, \mathrm{Id}$ without reflecting this in the notation. Let us also introduce the operator

$$\boldsymbol{M}_k = \sum_{1 \le |\alpha| \le M} L^{2k(|\alpha|-1)} (\nabla^*)^\alpha \nabla^\alpha. \tag{4.5.4}$$

The operators $\boldsymbol{M}_k^{\Lambda_N}$ and $\boldsymbol{M}_k$ are related by

$$\boldsymbol{M}_k^{\Lambda_N} = \Xi_k \sum_{1 \le |\alpha| \le M} L^{2k(|\alpha|-1)} (\nabla^*)^\alpha \nabla^\alpha = \Xi_k \boldsymbol{M}_k \tag{4.5.5}$$

where $\Xi_k = |B^+|_k$, $B \in \mathcal{B}_k$ accounts for the sum over $\mathbf{1}_{B^+}$. From the definition of B^+ we find $\Xi_0 = (2R+1)^d$, $\Xi_N = 1$, and $\Xi_k = 3^d$ for $1 \le k < N$ and therefore in particular

$$\Xi_k \le \Xi_{\max} = (2R+1)^d. \tag{4.5.6}$$

Note that $\boldsymbol{M}_k$ is translation invariant and therefore diagonal in Fourier space.

Recall that we consider the space $\mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}}$ where $\mathcal{I}$ satisfies $\{e_1, \ldots, e_n\} \subset \mathcal{I} \subset\subset \{\alpha \in \mathbb{N}_0^d \setminus \{0, \ldots, 0\} : |\alpha|_\infty \le R_0\}$. We assume that $\mathcal{Q}$ is a quadratic form on $\mathcal{G}$ that satisfies (4.4.10). From now on we use the shorthand notation $\mathcal{A} = \mathcal{A}_{\boldsymbol{Q}} = \mathcal{A}^{(0)}$ for the operator generated by $\mathcal{Q}$ on $\mathcal{X}_N$ (cf. (4.4.2) and (4.4.20)),

$$(\varphi, \mathcal{A}\varphi) = \sum_{x \in T_N} \mathcal{Q}(D\varphi(x)). \tag{4.5.7}$$

Let $\overline{\zeta} \in (0, \frac{1}{4})$ be a parameter. We will later set

$$\overline{\zeta} = \zeta/4 \tag{4.5.8}$$

where $\zeta \in (0, 1/2)$ is the parameter in the norm on $\boldsymbol{E}$ that appears in Theorem 3.2.3. Let $\delta_j = 4^{-j}\delta > 0$ be a sequence of real numbers with δ to be specified later. We define large field regulators $w_k^X, w_{k:k+1}^X$ for the weak norm for $0 \le k \le N$ by

$$w_k^X(\varphi) = e^{\frac{1}{2}(\boldsymbol{A}_k^X \varphi, \varphi)}, \quad w_{k:k+1}^X(\varphi) = e^{\frac{1}{2}(\boldsymbol{A}_{k:k+1}^X \varphi, \varphi)} \tag{4.5.9}$$

where $\boldsymbol{A}_k^X$ and $\boldsymbol{A}_{k:k+1}^X$ are linear symmetric operators on $\mathcal{X}_N$ that are defined iteratively by

$$\begin{aligned}
(\varphi, \boldsymbol{A}_0^X \varphi) &= (1-4\bar{\zeta}) \sum_{x\in X} \mathcal{Q}(D\varphi(x)) + \delta_0(\varphi, \boldsymbol{M}_0^X \varphi) && \text{for } X \in \mathcal{P}_0, \\
\boldsymbol{A}_{k:k+1}^X &= \left((\boldsymbol{A}_k^X)^{-1} - (1+\bar{\zeta})\mathcal{C}_{k+1}\right)^{-1} && \text{for } X \in \mathcal{P}_k \text{ and } 0 \le k \le N, \\
\boldsymbol{A}_{k+1}^X &= \boldsymbol{A}_{k:k+1}^{X^*} + \delta_{k+1}\boldsymbol{M}_{k+1}^X && \text{for } X \in \mathcal{P}_{k+1} \text{ and } 0 \le k \le N-1.
\end{aligned} \tag{4.5.10}$$

Here $\mathcal{C}_{k+1}$ is a finite range decomposition for the operator $\mathcal{A} = \mathcal{A}^{(0)}$ as in Theorem 4.4.1. The definition of $\boldsymbol{A}_{k:k+1}^X$ is a bit sloppy because $\boldsymbol{A}_k^X$ is in general not invertible, however the definition makes sense on the space $\ker(\boldsymbol{A}_k^X)^\perp$ and then $\boldsymbol{A}_{k:k+1}^X$ is the extension by zero of this operator; see the beginning of the Subsection 4.5.1 and Lemma 4.5.5 i) below. We use the neighbourhood X^* in the definition of $\boldsymbol{A}_{k+1}^X$ to account for the fact that in the reblocking step we also add contributions to X that come from polymers that are not contained in X but only in X^*.

We define the strong norm weight functions almost as in [4] by

$$W_k^X(\varphi) = e^{\frac{1}{2}(\boldsymbol{G}_k^X\varphi,\varphi)} \text{ with } (\varphi, \boldsymbol{G}_k^X\varphi) = \frac{1}{h_k^2} \sum_{1\le|\alpha|\le\lfloor\frac{d}{2}\rfloor+1} L^{2k(|\alpha|-1)}(\nabla^\alpha\varphi, \mathbf{1}_X\nabla^\alpha\varphi) \tag{4.5.11}$$

where as before $h_k = 2^k h$ with $h = h(L)$ to be chosen later.

To motivate the definition of the weight functions, we add several observations. In the evaluation of functional integrals $\int F(\varphi)\,\mu(\mathrm{d}\varphi)$ where μ is a Gaussian measure it is a well known problem that the functional F is in general unbounded for large fields φ. This is the *large field problem* that makes the construction of good norms for F difficult. A more detailed discussion can be found in [42]. In our approach we defined the norms for F in (4.4.87) by $\|F\|_{k,X} = \sup_\varphi |F(\varphi)|_{k,X,T_\varphi}(w_k^X(\varphi))^{-1}$ where w_k^X are the weight functions. They regulate the allowed growth at infinity. The larger the weight function the weaker the norm. So the results get stronger, i.e., the class of admissible potentials is bigger, if we can choose w_k bigger. The growth assumptions for the potential V in our theorems are weaker then those in [4] due to the larger weights that we construct in this section.

The first key requirement for the norm is that the renormalisation map, i.e., convolution with the partial measures μ_{k+1} is bounded. This yields the condition

$$\begin{aligned}
\left\| \int_{\mathcal{X}_N} F(\varphi+\cdot)\,\mu_{k+1}(\mathrm{d}\varphi) \right\| &= \sup_\psi w_{k+1}^{-X}(\psi) \left| \int_{\mathcal{X}_N} F(\varphi+\psi)\,\mu_{k+1}(\mathrm{d}\varphi) \right| \\
&\le \|F\| \sup_\psi w_{k+1}^{-X}(\psi) \int_{\mathcal{X}_N} w_k^X(\varphi+\psi)\,\mu_{k+1}(\mathrm{d}\varphi).
\end{aligned} \tag{4.5.12}$$

In other words the renormalisation map is bounded if and only if $w_k^X * \mu_{k+1} \lesssim C w_{k+1}^X$. Therefore the optimal choice is $w_{k+1}^X \propto w_k^X * \mu_{k+1}$. In general this is a very implicit definition that is not very useful. If, however, $w_k(\varphi) = e^{\frac{1}{2}(\varphi, \boldsymbol{A}_k^X\varphi)}$ is an exponential of a quadratic form, the convolution can be carried out explicitly and then the next weight has the same structure, i.e., it is again the exponential of a quadratic form. Indeed, by general Gaussian calculus the following identity holds for a given linear symmetric positive operator A on a finite dimensional vector space V and a covariance operator C

$$\begin{aligned}
\int_V e^{\frac{1}{2}(A(\varphi+\psi),\varphi+\psi)}\,\mu_C(\mathrm{d}\psi) &= \left(\frac{\det(C^{-1}-A)}{\det C^{-1}}\right)^{-\frac{1}{2}} e^{\frac{1}{2}((A^{-1}-C)^{-1}\varphi,\varphi)} \\
&= \det\left(\mathbb{1} - C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1}{2}} e^{\frac{1}{2}((A^{-1}-C)^{-1}\varphi,\varphi)}
\end{aligned} \tag{4.5.13}$$

under the assumption that $A < C^{-1}$. This implies that the next scale quadratic form is essentially given by the expression for $\boldsymbol{A}^X_{k:k+1}$ in (4.5.10).

The second key requirement for the norms of the functionals is that they are submultiplicative for distant polymers, i.e., $\|F^X F^Y\| \le \|F^X\| \cdot \|F^Y\|$ if X and Y are strictly disjoint polymers. This condition is necessary to regroup the terms and estimate products. Since the maximum norm is sub-multiplicative we find the condition $w_k^X w_k^Y \ge w_k^{X\cup Y}$ for the weights. At first sight this might appear problematic because we have no explicit expression for w_k^X. But it turns out that the finite range property of μ_{k+1} ensures that the weight functions factor for strictly disjoint polymers if we choose $w_{k+1}^X \propto w_k^X * \mu_{k+1}$. To show this we note that $w_0^X(\varphi)$ only depends on the values of φ in a neighbourhood of X. The same is true for $w_k^X(\varphi)$ because it is a convolution of w_0 with some measure. Then the factorisation follows by induction from the finite range property

$$w_{k+1}^{X\cup Y} = w_k^{X\cup Y} * \mu_{k+1} = (w_k^X \cdot w_k^Y) * \mu_{k+1} = (w_k^X * \mu_{k+1})(w_k^Y * \mu_{k+1}) = w_{k+1}^X w_{k+1}^Y. \quad (4.5.14)$$

Finally, let us briefly mention why we need the second set of weights $w_{k:k+1}$ that includes the operator $\boldsymbol{M}_k^X$. The reason is twofold. On the one hand, in every step we also need to control contribution from the Hamiltonian terms on blocks that are bounded in the strong norm but the blocks are not separated from the considered polymer. Therefore sub-multiplicativity does not hold in this case. Instead we add the operator $\boldsymbol{M}_k^X$ that allows us to bound the terms from the Hamiltonian. Secondly, the field norm $|\varphi|_{k,X}$ must be controlled by the weight function w_k^X. This is also guaranteed by the addition of the term $\boldsymbol{M}_k^X$. It turns out, however, that this changes the weight functions only slightly for sufficiently small prefactor δ (see Lemma 4.5.3 below).

4.5.1 Properties of the weight functions

Here and in the following we consider the extensions of the quadratic forms $\boldsymbol{G}_k^X$, $\boldsymbol{M}_k^X$, $\boldsymbol{A}_k^X$, and $\boldsymbol{A}^X_{k:k+1}$ to $\mathcal{V}_N$ by $\boldsymbol{G}_k^X\varphi = 0$, for $\varphi \in \mathcal{X}_N^\perp = \{\text{constant fields}\}$ and similarly for the other forms. Then we can also extend the weight functions w_k^X, $w_{k:k+1}^X$, and W_k^X to $\mathcal{V}_n$ using their definition (4.5.9) and (4.5.11). This extension has the property that $w_k^X(\varphi+\psi) = w_k(\varphi)$ if ψ is a constant field.

In the following theorem we collect the properties of the weight functions w_k^X, $w_{k:k+1}^X$, and W_k^X. The claims of the theorem will be reformulated and proven directly in terms of the operators $\boldsymbol{A}_k^X$, $\boldsymbol{A}^X_{k:k+1}$, and $\boldsymbol{G}_k^X$ in the following subsections. We state our results for general values of p_Φ, M, n, and $\tilde{n}$ but we will later only use the weights for the parameters chosen as indicated before. Recall our convention that we do not indicate dependence on the fixed parameters ω_0, $\overline{\zeta}$, d, m, R_0, M, n, and $\tilde{n}$

Theorem 4.5.1. *Consider $\mathcal{G}$ as above and let $\mathcal{Q}$ be a quadratic form on $\mathcal{G}$ satisfying*

$$\omega_0 |z^\nabla|^2 \le \mathcal{Q}(z) \le \omega_0^{-1}|z|^2 \quad (4.5.15)$$

with a constant $\omega_0 \in (0,1)$ and let $\overline{\zeta} \in (0, \frac{1}{4})$. Let $M \ge p_\Phi + \lfloor \frac{d}{2} \rfloor + 1$ and let $\mathcal{C}_k^{(\boldsymbol{q})}$ be a family of finite range decompositions for the quadratic forms $z \mapsto \mathcal{Q}(z) - (\boldsymbol{q}z^\nabla, z^\nabla)$, with $n \ge 2M$ and $\tilde{n} > n$. Then, for every

$$L \ge 2^{d+3} + 16R, \quad (4.5.16)$$

there are constants $\lambda > 0$, $\delta(L) > 0$, $\kappa(L)$ (specified in (4.5.55), (4.5.57), *and* (4.5.81)*) and $h_0(L)$ given by*

$$h_0(L) = \delta(L)^{-\frac{1}{2}} \max(8^{\frac{1}{2}}, c_d) \quad (4.5.17)$$

such that the weight functions defined in (4.5.9) *and* (4.5.11) *are well-defined and satisfy:*

i) For any $Y \subset X \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$w_k^Y(\varphi) \le w_k^X(\varphi) \quad \text{and} \quad w_{k:k+1}^Y(\varphi) \le w_{k:k+1}^X(\varphi). \tag{4.5.18}$$

ii) The estimate

$$w_k^X(\varphi) \le \exp\left(\frac{(\varphi, \boldsymbol{M}_k\varphi)}{2\lambda}\right) \quad \text{and} \quad w_{k:k+1}^X(\varphi) \le \exp\left(\frac{(\varphi, \boldsymbol{M}_k\varphi)}{2\lambda}\right) \tag{4.5.19}$$

holds for $0 \le k \le N$, $X \in \mathcal{P}_k$, *and* $\varphi \in \mathcal{V}_N$.

iii) For any strictly disjoint polymers $X, Y \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$w_k^{X\cup Y}(\varphi) = w_k^X(\varphi) w_k^Y(\varphi). \tag{4.5.20}$$

iv) For any polymers $X, Y \in \mathcal{P}_k$ *such that* $\operatorname{dist}(X,Y) \ge \frac{3}{4}L^{k+1}$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$w_{k:k+1}^{X\cup Y}(\varphi) = w_{k:k+1}^X(\varphi) w_{k:k+1}^Y(\varphi). \tag{4.5.21}$$

v) For any disjoint polymers $X, Y \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$W_k^{X\cup Y}(\varphi) = W_k^X(\varphi) W_k^Y(\varphi). \tag{4.5.22}$$

vi) For $h \ge h_0(L)$, *disjoint polymers* $X, Y \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$w_k^{X\cup Y}(\varphi) \ge w_k^X(\varphi) W_k^Y(\varphi). \tag{4.5.23}$$

vii) For $h \ge h_0(L)$, $X \in \mathcal{P}_k$ *and* $U = \pi(X) \in \mathcal{P}_{k+1}$, $0 \le k \le N-1$, *and* $\varphi \in \mathcal{V}_N$,

$$w_{k+1}^U(\varphi) \ge w_{k:k+1}^X(\varphi)\left(W_k^{U^+}(\varphi)\right)^2. \tag{4.5.24}$$

viii) For any $h \ge h_0(L)$ *and all* $0 \le k \le N-1$, $X \in \mathcal{P}_{k+1}$ *and* $\varphi \in \mathcal{V}_N$,

$$e^{\frac{|\varphi|^2_{k+1,X}}{2}} w_{k:k+1}^X(\varphi) \le w_{k+1}^X(\varphi). \tag{4.5.25}$$

ix) There is a constant $A_\mathcal{P} = A_\mathcal{P}(L)$ *such that for* $\boldsymbol{q} \in B_\kappa$, $\rho = (1+\overline{\zeta})^{1/3} - 1$, $X \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$\left(\int_{\mathcal{X}_N} \left(w_k^X(\varphi+\xi)\right)^{1+\rho} \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\right)^{\frac{1}{1+\rho}} \le \left(\frac{A_\mathcal{P}}{2}\right)^{|X|_k} w_{k:k+1}^X(\varphi). \tag{4.5.26}$$

x) There is a constant $A_\mathcal{B}$ *independent of* L *such that for* $\boldsymbol{q} \in B_\kappa$, $\rho = (1+\overline{\zeta})^{1/3} - 1$, $X \in \mathcal{P}_k$, $0 \le k \le N$, *and* $\varphi \in \mathcal{V}_N$,

$$\left(\int_{\mathcal{X}_N} \left(w_k^B(\varphi+\xi)\right)^{1+\rho} \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\right)^{\frac{1}{1+\rho}} \le \frac{A_\mathcal{B}}{2} w_{k:k+1}^B(\varphi). \tag{4.5.27}$$

Proof. The theorem follows from a sequence of lemmas in the following sections. Lemma 4.5.5 establishes basic properties of the operators $\boldsymbol{A}_k^X$ and $\boldsymbol{A}_{k:k+1}^X$ that imply i) and ii). Lemma 4.5.6 concerns factorisation properties of the operators $\boldsymbol{A}_k^X$ and $\boldsymbol{A}_{k:k+1}^X$ that allow us to conclude iii)-vii). Lemma 4.5.7 gives a bound on a particular determinant that implies ix) and x). Finally, Lemma 4.5.8 bounds the field norm $|\cdot|_{k,X}$ in terms of the weights. This easily yields property viii). □

4.5.2 The main technical matrix estimate

In this subsection we prove a crucial technical estimate which shows that the iterative procedure (4.5.10) introducing the operators $A_k \to A_{k:k+1} \to A_{k+1}$ is well-defined.

We first recall some standard facts about monotone matrix functions. We say that two Hermitian matrices A and B satisfy $A \le B$ if $(Ax,x) \le (Bx,x)$ for all x. We say that a map f from a subset U of the Hermitian matrices to the Hermitian matrices is matrix monotone if $A \le B$ implies $f(A) \le f(B)$ for all $A, B \in U$.

Lemma 4.5.2. *i) The map $A \mapsto -A^{-1}$ is matrix-monotone on the set of positive definite Hermitian matrices.*

ii) Let C be Hermitian and positive definite. For positive definite Hermitian matrices A with $A < C^{-1}$ define

$$f(A) := (A^{-1} - C)^{-1}. \tag{4.5.28}$$

Then f is matrix monotone.

iii) If we extend f to Hermitian matrices A with $0 \le A < C^{-1}$ by

$$f(A) = \begin{cases} ((A_{\ker A^\perp})^{-1} - (P_{\ker A^\perp} C P_{\ker A^\perp}))^{-1} & \text{on } \ker A^\perp, \\ 0 & \text{on } \ker A. \end{cases} \tag{4.5.29}$$

then the extended function is still matrix monotone.

iv) If $0 \le A < C^{-1}$ then $A^{1/2} C A^{1/2} < \mathbb{1}$ and the extended function f satisfies

$$f(A) = A^{1/2}(\mathbb{1} - A^{1/2} C A^{1/2})^{-1} A^{1/2}. \tag{4.5.30}$$

There is the following absolutely convergent series representation for f and $0 \le A < C^{-1}$

$$f(A) = \sum_{i=0}^{\infty} A(CA)^i. \tag{4.5.31}$$

Proof. The assertions are classical. We include a proof for the convenience of the reader.

The first assertion follows from Löwner's theorem ([122], see also [103]) since the imaginary part of the map $\mathbb{C}\setminus\{0\} \ni z \mapsto -z^{-1} = \frac{-\bar{z}}{|z|^2}$ is non-negative in the upper half-plane. Alternatively, it can be proved elementary as follows. First, monotonicity is clear for $B = \mathbb{1}$ since for a positive definite symmetric matrix A the condition $A \le \mathbb{1}$ is equivalent to $\operatorname{spec}(A) \subset (0,1]$ while the condition $A \ge 1$ is equivalent to $\operatorname{spec}(A) \subset [1,\infty)$. To prove the result for a general B assume $A \le B$ and note that this implies $\overline{F}^T AF \le \overline{F}^T BF$ for all matrices F. Taking $F = B^{-1/2}$ we get $B^{-1/2}AB^{-1/2} \le \mathbb{1}$ and thus $B^{1/2}A^{-1}B^{1/2} \ge \mathbb{1}$ which implies that $A^{-1} \ge B^{-1/2}\mathbb{1}B^{-1/2} = B^{-1}$.

The second assertion follows by applying the monotonicity of the inversion map twice.

The third assertion follows since the right hand side is the limit $\lim_{\varepsilon\downarrow 0} f(A + \varepsilon\mathbb{1})$.

The fourth assertion is clear for $0 < A < C^{-1}$. Fix A with $0 \le A < C^{-1}$. Then there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $A_\varepsilon := A + \varepsilon\mathbb{1} \le (1-\delta)C^{-1}$. Thus $C \le (1-\delta)A_\varepsilon^{-1}$ and hence $A_\varepsilon^{1/2} C A_\varepsilon^{1/2} \le (1-\delta)\mathbb{1}$. Passing to the limit $\varepsilon \downarrow 0$ we get $A^{1/2}CA^{1/2} \le (1-\delta)\mathbb{1}$ and the validity of (4.5.30). Equation (4.5.31) follows from (4.5.30) by expanding the the Neumann series. □

We now show the crucial technical lemma that allows us to find suitable bounds for the operators $\boldsymbol{A}_k$. Basically this lemma shows that for sufficiently small δ the $\boldsymbol{M}_k^X$ terms are just a small perturbation of the operators.

Lemma 4.5.3. *Suppose that* $\mathcal{Q}$, M, n, $\tilde{n}$, *and* $\mathcal{C}_k = \mathcal{C}_k^{(0)}$ *satisfy the assumptions of Theorem 4.5.1. Then the following holds. For all* $\lambda \in (0, 1/4)$ *and* $L \geq 3$ *odd, there is a constant* $\mu(\lambda, L) \geq 1$ *such that for any* $\varepsilon \in (0,1)$, $0 \leq \delta < \frac{1+\varepsilon}{\mu}$ *and for all* $0 \leq k \leq N-1$, *the bound*

$$\left(\lambda \boldsymbol{M}_k^{-1} + (1+\varepsilon)\sum_{j=k+2}^{N+1} \mathcal{C}_j\right)^{-1} + \delta \boldsymbol{M}_{k+1} \leq \left(\lambda \boldsymbol{M}_{k+1}^{-1} + (1+\varepsilon-\mu\delta)\sum_{j=k+2}^{N+1} \mathcal{C}_j\right)^{-1} \tag{4.5.32}$$

holds in the sense of Hermitian operators on $\mathcal{X}_N$.

Remark 4.5.4. *The proof is quite technical and not very insightful. Therefore we first give a brief heuristic argument. All operators are diagonal in the Fourier space. Thus it is sufficient to show the bound for all Fourier modes* $p \in \widehat{T}_N \setminus \{0\}$ *of the kernels of the operators that actually are* $m \times m$ *matrices. Note that* $\boldsymbol{M}_k$ *acts diagonally with respect to the* m *components and thus its Fourier modes are multiples of the identity. We use* $\widehat{\boldsymbol{M}_k}(p) \in \mathbb{R}$ *to denote the coefficient of the Fourier mode and use the embedding into* $\mathbb{R}^{m\times m}$ *when necessary. Let* $q(p)_j = e^{ip_j} - 1$ *and note that the Fourier multiplier of* ∇ *is the vector* $q(p)$ *whose norm is of the order* $|p|$: $|p|/2 \leq |q(p)| \leq |p|$ *for* $p \in \widehat{T}_N$ *(cf. (4.4.8)). Therefore we can write*

$$\widehat{\boldsymbol{M}_k}(p) = \sum_{1\leq|\alpha|\leq M} L^{2k(|\alpha|-1)} |q(p)^{2\alpha}| \tag{4.5.33}$$

To shorten the notation we introduce the notation

$$\mathcal{C}_{k+1}^{N+1} = \sum_{j=k+1}^{N+1} \mathcal{C}_j. \tag{4.5.34}$$

There are two regimes $|p| \leq L^{-k}$ *and* $|p| \geq L^{-k}$ *requiring different treatment.*

Using (4.5.33), for $|p| < L^{-k}$ *we find that* $\widehat{\boldsymbol{M}}_k(p) \approx |p|^2$. *Indeed, since, roughly speaking,* $\widehat{\mathcal{C}}_j(p) \approx |p|^{-2}$ *for* $|p| \approx L^{-j}$, *we observe that* $|p|^{-2} \approx \widehat{\mathcal{C}}_{k+1}^{N+1}(p)$ *for* $|p| \leq L^{-k}$. *Hence, after factoring out the term* $|p|^2$, *we are left to show an inequality of the type* $\alpha^{-1} + \delta \leq (\alpha - \mu\delta)^{-1}$ *for given real numbers* α *and* δ. *This is true for some large* μ *if* α *is uniformly bounded above and below and* $\delta > 0$ *is bounded above.*

For $|p| \geq L^{-k}$ *the asymptotic behaviour is* $\widehat{\boldsymbol{M}}_k(p) \approx |p|^{2M} L^{(2M-2)k}$ *and* $\widehat{\mathcal{C}}_k^{N+1}(p) \approx 0$. *Then the bound is implied by* $\widehat{\boldsymbol{M}}_k(p) \ll \widehat{\boldsymbol{M}}_{k+1}(p)$.

Proof. Here, we implement rigorously the heuristics described above. The first step is to compare the operators $\boldsymbol{M}_k$ and $\boldsymbol{M}_{k+1}$. For $|p| \geq L^{-k}$ and $L \geq 8$ we observe using $|p|/2 \leq |q(p)| \leq |p|$ that

$$\begin{aligned} 4|q(p)|^2 \leq 16|q(p)|^4 L^{2k} \leq \frac{32}{L^2} \sum_{|\alpha|=2} L^{2(k+1)(|\alpha|-1)} |q(p)^{2\alpha}| &\leq \frac{1}{2} \sum_{|\alpha|=2} L^{2(k+1)(|\alpha|-1)} |q(p)^{2\alpha}| \\ &\leq \frac{1}{2} \widehat{\boldsymbol{M}}_{k+1}(p). \end{aligned} \tag{4.5.35}$$

Hence, for $|p| \geq L^{-k}$ and $L \geq 8$, we have

$$4\widehat{\boldsymbol{M}}_k(p) = 4 \sum_{1\leq|\alpha|\leq M} L^{2k(|\alpha|-1)} |q(p)^{2\alpha}| \leq 4|q(p)|^2 + \frac{4}{L^2} \sum_{2\leq|\alpha|\leq M} L^{2(k+1)(|\alpha|-1)} |q(p)|^{2|\alpha|} \leq \widehat{\boldsymbol{M}}_{k+1}(p). \tag{4.5.36}$$

We claim that there is a constant $k_0 = k_0(\lambda)$ independent of k and N such that

$$2\,\widehat{\mathcal{C}}_{k+2}^{N+1}(p) = 2\sum_{k'=k+2}^{N+1}\widehat{\mathcal{C}}_{k'}(p) \le \lambda\widehat{\boldsymbol{M}}_{k+1}(p)^{-1} \tag{4.5.37}$$

for $|p| \ge L^{-k+k_0}$. To prove this we observe that for $L^{-j-1} < |p| \le L^{-j}$ and $j < k-k_0$ the sum on the left hand side is by (4.4.15) dominated by a geometric series which implies

$$\left|\sum_{k'=k+2}^{N+1}\widehat{\mathcal{C}}_{k'}(p)\right| \le C_1 L^{2(d+\tilde{n})+1}L^{2j}L^{-(k+2-j)(d-1+n)} = C_1 L^{d+2\tilde{n}+2-n}L^{2j}L^{-(k+1-j)(d-1+n)}. \tag{4.5.38}$$

Note that

$$\sum_{|\alpha|=l}|q(p)^{2\alpha}| \le |q(p)|^{2l} \le |p|^{2l} \tag{4.5.39}$$

This implies that, for $j \le k$ and $|p| \le L^{-j}$, the right hand side of (4.5.37) satisfies

$$\widehat{\boldsymbol{M}}_{k+1}(p) \le \sum_{l=1}^{M} L^{2(l-1)(k+1)}L^{-2lj} \le 2L^{-2j}L^{2(M-1)(k+1-j)}. \tag{4.5.40}$$

Therefore we find that

$$\lambda^{-1}\widehat{\boldsymbol{M}}_{k+1}(p)\left|\sum_{k'=k+2}^{N+1}\widehat{\mathcal{C}}_{k'}(p)\right| \le \frac{2C_1}{\lambda}L^{d+2\tilde{n}+2-n}L^{(k+1-j)(2M-1-d-n)} \le \frac{1}{2} \tag{4.5.41}$$

for $k-j > k_0$ with $k_0 = k_0(\lambda) = \lceil \log_3(4C_1/\lambda)\rceil + d + 2\tilde{n} + 2 - n$ where we used that $L \ge 3$ and $n \ge 2M$ and thus $2M-1-d-n \le -1$. Note that the constant C_1 from (4.4.15) does not depend on L. Hence, in particular, k_0 is independent of L. This proves (4.5.37). The bounds (4.5.36) and (4.5.37) thus, for $|p| \ge L^{-k+k_0}$, $\delta < \frac{1}{4\lambda}$, and $\varepsilon < 1$, jointly imply

$$\begin{aligned}
\Big(\lambda\widehat{\boldsymbol{M}}_k^{-1}(p) + (1+\varepsilon)\widehat{\mathcal{C}}_{k+2}^{N+1}(p)\Big)^{-1} + \delta\widehat{\boldsymbol{M}}_{k+1}(p) &\le \frac{1}{\lambda}\widehat{\boldsymbol{M}}_k(p) + \delta\widehat{\boldsymbol{M}}_{k+1}(p)\\
&\le \frac{1}{4\lambda}\widehat{\boldsymbol{M}}_{k+1}(p) + \frac{1}{4\lambda}\widehat{\boldsymbol{M}}_{k+1}(p)\\
&\le \Big(2\lambda\widehat{\boldsymbol{M}}_{k+1}^{-1}(p)\Big)^{-1}\\
&\le \Big(\lambda\widehat{\boldsymbol{M}}_{k+1}^{-1}(p) + (1+\varepsilon)\widehat{\mathcal{C}}_{k+2}^{N+1}(p)\Big)^{-1}.
\end{aligned} \tag{4.5.42}$$

In the first and the last step we used the fact that the inversion of a Hermitian positive definite matrix is a monotone operation (see Lemma 4.5.2). This ends the proof for $p \in \widehat{T}_N$ with $|p| \ge L^{-k+k_0}$.

For $p \in \widehat{T}_N$ such that $|p| < L^{-k+k_0}$ we note that there are constants $\omega_1, \omega_2, \Omega_1, \Omega_2$ depending on L, $k_0(\lambda)$, and λ such that

$$\omega_1|p|^{-2} \le \widehat{\mathcal{C}}_{k+2}^{N+1}(p) \le (1+\varepsilon)\widehat{\mathcal{C}}_{k+2}^{N+1}(p) \le \Omega_1|p|^{-2} \text{ and} \tag{4.5.43}$$

$$\omega_2|p|^{-2} \le \lambda\widehat{\boldsymbol{M}}_{k+1}^{-1}(p) \le \Omega_2|p|^{-2}. \tag{4.5.44}$$

Indeed, the upper bounds are trivial and even hold uniformly in k_0 and N for all p because $\widehat{\mathcal{C}}(p) \leq \Omega_1 |p|^{-2}$ for some constant Ω_1 by (4.4.11) and $\boldsymbol{M}_{k+1} \geq -\Delta$. The first lower bound follows from (4.4.14) which implies the bound

$$\widehat{\mathcal{C}}^{N+1}_{k+2}(p) \geq \widehat{\mathcal{C}}_j(p) \geq cL^{-2(d+\tilde{n})-1} L^{2j} \geq cL^{-2(d+\tilde{n})-3} |p|^{-2} \tag{4.5.45}$$

for $L^{-j-1} < |p| < L^{-j}$ and $j \geq k+2$. For $L^{-j-1} < |p| < L^{-j}$ and $k - k_0 \leq j < k+2$ we use

$$\widehat{\mathcal{C}}^{N+1}_{k+2}(p) \geq \widehat{\mathcal{C}}_{k+2}(p) \geq cL^{-2(d+\tilde{n})-1} L^{(k+2-j)(-d+1-n)} L^{2j} \geq cL^{-2(d+\tilde{n})-3} L^{(k_0+2)(-d+1-n)} |p|^{-2}. \tag{4.5.46}$$

Therefore the lower bound in (4.5.43) holds with $\omega_1 = cL^{-2(d+\tilde{n})-3+(k_0+2)(-d+1-n)}$. The second lower bound is a consequence of (4.5.39) which implies

$$\widehat{\boldsymbol{M}}_{k+1}(p) \leq \sum_{l=1}^{M} L^{2(l-1)(k+1)} |p|^{2l} \leq \sum_{l=1}^{M} L^{2(l-1)(k+1)} L^{2(l-1)(-k+k_0)} |p|^2 \leq 2L^{2(M-1)(k_0+1)} |p|^2. \tag{4.5.47}$$

if $|p| < L^{-k+k_0}$. So the lower bound in (4.5.44) holds with $\omega_2 = \lambda(2L^{2(M-1)(k_0+1)})^{-1}$.

Observe that $\widehat{\mathcal{C}}^{N+1}_{k+2}(p)$ and $\widehat{\boldsymbol{M}}_{k+1}(p)$ are Hermitian and they commute because $\widehat{\boldsymbol{M}}_{k+1}(p)$ is a multiple of the identity. Therefore we can work in basis where both matrices are diagonal which reduces the estimates to the scalar case $m = 1$. Then the bound we want to show is essentially the estimate $(a-x)^{-1} - a^{-1} > x/a^2$ for $a > x > 0$. In more detail, using (4.5.43) and the trivial estimate $\widehat{\boldsymbol{M}}_k(p) \leq \widehat{\boldsymbol{M}}_{k+1}(p)$, we find for $|p| < L^{-k+k_0}$, $m = 1$, and $0 < \delta < (1+\varepsilon)/\mu$,

$$\begin{aligned}
&\Big(\lambda \widehat{\boldsymbol{M}}^{-1}_{k+1}(p) + (1+\varepsilon-\mu\delta)\widehat{\mathcal{C}}^{N+1}_{k+2}(p)\Big)^{-1} - \Big(\lambda \widehat{\boldsymbol{M}}^{-1}_{k}(p) + (1+\varepsilon)\widehat{\mathcal{C}}^{N+1}_{k+2}(p)\Big)^{-1} \\
&\geq \Big(\lambda \widehat{\boldsymbol{M}}^{-1}_{k+1}(p) + (1+\varepsilon-\mu\delta)\widehat{\mathcal{C}}^{N+1}_{k+2}(p)\Big)^{-1} - \Big(\lambda \widehat{\boldsymbol{M}}^{-1}_{k+1}(p) + (1+\varepsilon)\widehat{\mathcal{C}}^{N+1}_{k+2}(p)\Big)^{-1} \\
&\geq \frac{\mu\delta \widehat{\mathcal{C}}^{N+1}_{k+2}(p)}{\Big(\lambda \widehat{\boldsymbol{M}}^{-1}_{k+1}(p) + (1+\varepsilon-\mu\delta)\widehat{\mathcal{C}}^{N+1}_{k+2}(p)\Big)\Big(\lambda \widehat{\boldsymbol{M}}^{-1}_{k+1}(p) + (1+\varepsilon)\widehat{\mathcal{C}}^{N+1}_{k+2}(p)\Big)} \\
&\geq \frac{\mu\delta\omega_1 |p|^{-2}}{(\Omega_1+\Omega_2)^2 |p|^{-4}} \\
&\geq \delta \widehat{\boldsymbol{M}}_{k+1}(p) \frac{\mu\omega_1\omega_2}{\lambda(\Omega_1+\Omega_2)^2}.
\end{aligned} \tag{4.5.48}$$

Then for

$$\mu \geq \lambda \frac{(\Omega_1+\Omega_2)^2}{\omega_1\omega_2} \tag{4.5.49}$$

(where ω_1, ω_2, Ω_1, and Ω_2 where introduced in (4.5.43) and (4.5.44)) the inequality (4.5.32) follows. For $m > 1$ the claim follows by applying (4.5.48) to each diagonal entry of the diagonalised matrices. The estimates (4.5.42) and (4.5.48) imply the claim. □

4.5.3 Basic properties of the operators $\boldsymbol{A}^X_{k:k+1}$ and $\boldsymbol{A}^X_k$

Recall that $\overline{\zeta} \in (0, \frac{1}{4})$ is a fixed parameter and $\Xi_{\max}$ was defined in (4.5.6). For given values of h, δ, and μ (that will be specified later) we define sequences

$$h_j = 2^j h, \quad \delta_j = 4^{-j}\delta, \quad \overline{\zeta}_j = 2\overline{\zeta} - \sum_{i=0}^{j} \mu \Xi_{\max} \delta_i. \tag{4.5.50}$$

The following lemma proves the claims i) and ii) from Theorem 4.5.1.

Lemma 4.5.5. *Under the assumptions of Theorem 4.5.1, for every $L \geq 2^{d+3} + 16R$, there are constants $\lambda > 0$, $\mu(L) > 1$, and $\delta(L) \in \left(0, \frac{\overline{\zeta}}{2\mu\Xi_{\max}}\right)$ such that $\overline{\zeta}_j \geq \overline{\zeta}$ for all $j = 0, \ldots, N$, and for all $0 \leq k \leq N$:*

i) The operators $\boldsymbol{A}^X_{k:k+1}$ and $\boldsymbol{A}^X_k$ are well-defined, symmetric and non-negative operators on $\mathcal{X}_N$ for any $X \in \mathcal{P}_k$.

ii) Translation invariance: For any translation $\tau_a\varphi(x) = \varphi(x-a)$ with $a \in (L^k\mathbb{Z})^d/(L^N\mathbb{Z})^d$ the equalities $(\varphi, \boldsymbol{A}^X_k\varphi) = (\tau_a\varphi, \tau_{-a}\boldsymbol{A}^{X+a}_k\tau_a\varphi)$ and $(\varphi, \boldsymbol{A}^X_{k:k+1}\varphi) = (\tau_a\varphi, \boldsymbol{A}^{X+a}_{k:k+1}\tau_a\varphi)$ hold.

iii) Locality: The operators $\boldsymbol{A}^X_k$ and $\boldsymbol{A}^X_{k:k+1}$ only depend on the values of φ in X^{++} and the are shift invariant, i.e., they are measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(X^{++})}$.

iv) Monotonicity: For $Y \subset X$ the inequalities $\boldsymbol{A}^Y_k \leq \boldsymbol{A}^X_k$ and $\boldsymbol{A}^Y_{k:k+1} \leq \boldsymbol{A}^X_{k:k+1}$ hold in the sense of operators.

v) Bounds: The weight functions are bounded from above as follows

$$\boldsymbol{A}^X_k \leq \left(\lambda\boldsymbol{M}_k^{-1} + (1+\overline{\zeta}_k)\sum_{j=k+1}^{N+1}\mathcal{C}_j\right)^{-1} \tag{4.5.51}$$

$$\boldsymbol{A}^X_{k:k+1} \leq \left(\lambda\boldsymbol{M}_k^{-1} + (1+\overline{\zeta}_k)\sum_{j=k+2}^{N+1}\mathcal{C}_j\right)^{-1}. \tag{4.5.52}$$

Proof. Note first that the estimate $\zeta_j \geq \zeta$ is an immediate consequence of the definition of δ.

The proof is by induction on k. First, for $k = 0$, the properties i), iii), and iv) are obvious. Indeed, $\mathcal{Q}$ has range at most R, is positive, and $D\varphi(x)$ can be expressed as a function of $\nabla\varphi\restriction_{\vec{E}(x+[0,R]^d)}$. Similarly $\boldsymbol{M}^X_0$ is non-negative, symmetric, and monotone in X and $\boldsymbol{M}^X_0\varphi$ only depends on the values of $\nabla\varphi$ restricted to the bonds $\vec{E}((X^+ + [-M,M]^d)\cap T_N)$ and $(X^+ + [-M,M]^d)\cap T_N \subset X^{++}$ since $R \geq M$ by (4.4.29) and (4.5.1).

Translation invariance for $k = 0$ follows from the facts that the discrete derivatives commute with translations and $\tau_{-a}\mathbf{1}_{X+a}\tau_a = \mathbf{1}_X$ where $\mathbf{1}_X$ denotes the multiplication operator with the indicator function of X which implies translation invariance of the operators $\boldsymbol{M}_k$ in the set variable that is $(\varphi, \boldsymbol{M}^X_k\varphi) = (\tau_a\varphi, \boldsymbol{M}^{X+a}_k\tau_a\varphi)$. A similar statement holds for $\mathcal{Q}$. Finally, we establish the bound (4.5.51). First we note that there exist two constants $\Omega, \omega > 0$ independent of L, such that the operator $\mathcal{A}$ (see (4.5.7)) satisfies the bounds

$$\omega\mathcal{A} \leq \boldsymbol{M}_0 \leq \Omega\mathcal{A}. \tag{4.5.53}$$

This is a consequence of the fact that both operators have the Fourier modes bounded uniformly by $|p|^2$ from above and below. For $\mathcal{A}$ the bounds follow from (4.4.11) and for $\boldsymbol{M}_0$ the lower bound follows from $|q(p)|^2 \geq \frac{|p|^2}{4}$ while the upper bound follows from $|q(p)| \leq |p|$ and the fact that the dual torus is bounded. Then, for $\delta_0 \leq \overline{\zeta}/\Omega$, we estimate

$$\boldsymbol{A}^X_0 \leq \boldsymbol{A}^{\Lambda_N}_0 = (1-4\overline{\zeta})\mathcal{A} + \delta_0\boldsymbol{M}_0 \leq (1-3\overline{\zeta})\mathcal{A} \leq \frac{1}{(1+3\overline{\zeta})\mathcal{A}^{-1}} \leq \frac{1}{\overline{\zeta}\omega\boldsymbol{M}_0^{-1} + (1+2\overline{\zeta})\mathcal{C}}. \tag{4.5.54}$$

Hence (4.5.51) holds for $k = 0$ and $\lambda \le \overline{\zeta}\omega$. We now fix

$$\lambda = \min\left(\overline{\zeta}\omega, \frac{1}{4}\right) \tag{4.5.55}$$

where ω was introduced in (4.5.53) and

$$\mu = \mu(\lambda, L) > 1 \tag{4.5.56}$$

as in Lemma 4.5.3. We then set

$$\delta(L) = \min\left(\frac{\overline{\zeta}}{\Omega}, \frac{\overline{\zeta}}{2\mu\Xi_{\max}}\right), \tag{4.5.57}$$

where $\Xi_{\max}$ was introduced in (4.5.6)

Now we perform the induction step from $\boldsymbol{A}^X_k$ to $\boldsymbol{A}^X_{k:k+1}$.

First we show that $\boldsymbol{A}^X_{k:k+1}$ is well defined. By the induction hypothesis, the operator $\boldsymbol{A}^X_k$ is non-negative and symmetric and the bound (4.5.51) implies

$$\boldsymbol{A}^X_k \le \left((1+\overline{\zeta}_k)\mathcal{C}_{k+1}\right)^{-1}. \tag{4.5.58}$$

Since $\mathcal{C}_{k+1}$ is also symmetric, Lemma 4.5.2 implies that $\boldsymbol{A}^X_{k:k+1}$ is well-defined using the extension defined in (4.5.29) and it can be expressed as follows

$$\boldsymbol{A}^X_{k:k+1} = \left(\boldsymbol{A}^X_k\right)^{\frac{1}{2}} \left(1 - \left(\boldsymbol{A}^X_k\right)^{\frac{1}{2}} \mathcal{C}_{k+1} \left(\boldsymbol{A}^X_k\right)^{\frac{1}{2}}\right)^{-1} \left(\boldsymbol{A}^X_k\right)^{\frac{1}{2}}. \tag{4.5.59}$$

This expression shows that the operator $\boldsymbol{A}^X_{k:k+1}$ is symmetric and, again by Lemma 4.5.2, also non-negative. Moreover, the matrix monotonicity stated in Lemma 4.5.2 implies that the monotonicity $\boldsymbol{A}^Y_{k:k+1} \le \boldsymbol{A}^X_{k:k+1}$ for $Y \subset X$ follows from the induction hypothesis $\boldsymbol{A}^Y_k \le \boldsymbol{A}^X_k$.

To prove the claim ii) for $\boldsymbol{A}^X_{k:k+1}$, we use the induction hypothesis, the series representation (4.5.31) for $\boldsymbol{A}^X_{k:k+1}$, and the translation invariance of the kernel $\mathcal{C}_{k+1}$, $[\tau_a, \mathcal{C}_{k+1}] = 0$. The easiest way to show the locality of $\boldsymbol{A}^X_{k:k+1}$ stated in iii) is based on the observation that, by Gaussian integration (4.5.13), we get the identity

$$\begin{aligned}\int_{\mathcal{X}_N} e^{\frac{1}{2}(\varphi+\xi, \boldsymbol{A}^X_k(\varphi+\xi))} \mu_{(1+\overline{\zeta})\mathcal{C}_{k+1}}(\mathrm{d}\xi) &= \frac{e^{\frac{1}{2}(\varphi, ((\boldsymbol{A}^X_k)^{-1} - (1+\overline{\zeta})\mathcal{C}_{k+1})^{-1}\varphi)}}{\det\left(\mathbb{1} - \left((1+\overline{\zeta})\mathcal{C}_{k+1}\right)^{\frac{1}{2}} \boldsymbol{A}^X_k \left((1+\overline{\zeta})\mathcal{C}_{k+1}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}} \\ &= \frac{e^{\frac{1}{2}(\varphi, \boldsymbol{A}^X_{k:k+1}\varphi)}}{\det\left(\mathbb{1} - \left((1+\overline{\zeta})\mathcal{C}_{k+1}\right)^{\frac{1}{2}} \boldsymbol{A}^X_k \left((1+\overline{\zeta})\mathcal{C}_{k+1}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}}.\end{aligned} \tag{4.5.60}$$

By the induction hypothesis the left hand side is measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(X^{++})}$, hence the same is true for the right hand side.

For the proof of v) for $\boldsymbol{A}^X_{k:k+1}$ we first note that, by the monotonicity iv), it is sufficient to prove the bound for $X = \Lambda_N$. This is an immediate consequence of the bound for $\boldsymbol{A}^{\Lambda_N}_k$, Lemma 4.5.2 iii), and the inequality $\overline{\zeta}_k \geq \overline{\zeta}$ which implies

$$\begin{aligned} \boldsymbol{A}_{k:k+1}^{\Lambda_N} = ((\boldsymbol{A}_k^{\Lambda_N})^{-1} - (1+\overline{\zeta})\mathcal{C}_{k+1})^{-1} &\leq \left(\lambda \boldsymbol{M}_k^{-1} + (1+\overline{\zeta}_k)\sum_{j=k+1}^{N+1} \mathcal{C}_j - (1+\overline{\zeta})\mathcal{C}_{k+1}\right)^{-1} \\ &\leq \left(\lambda \boldsymbol{M}_k^{-1} + (1+\overline{\zeta}_k)\sum_{j=k+2}^{N+1} \mathcal{C}_j\right)^{-1}. \end{aligned} \tag{4.5.61}$$

It remains to show the induction step from $\boldsymbol{A}_{k:k+1}^X$ to $\boldsymbol{A}_{k+1}^X$. We begin with the observation that the operators $\boldsymbol{M}_{k+1}^X$ are well-defined, symmetric, non-negative, monotone in X, and translation invariant. Moreover $\boldsymbol{M}_{k+1}^X$ only depends on $\nabla\varphi\restriction_{\vec{E}(X^+ + [-M,M]^d)}$ and $X^+ + [-M,M]^d \subset X^{++}$ once $L \geq M$, the inequality that follows from $M \leq R \leq L$.

Now, the points i) and ii) follow from the induction hypothesis applied to $\boldsymbol{A}_{k:k+1}^{X^*}$ and the previous observation. The claim iv) follows from the induction hypothesis for $\boldsymbol{A}_{k:k+1}$ applied to $X^* \subset Y^*$ for $X \subset Y$ and the monotonicity of $\boldsymbol{M}_{k+1}^X$. To show iii), it remains to check that $\boldsymbol{A}_{k:k+1}^{X^*}$ is measurable with respect to $\nabla\varphi\restriction_{\vec{E}(X^{++})}$. Using the induction hypothesis we are left to show the inclusion $(X^*)^{++} \subset X^{++}$. Note that by (4.4.34),

$$(X^*)^{++} = \begin{cases} X + [-2^d - 3R, 2^d + 3R]^d & \text{for } X \in \mathcal{P}_1, \\ X + [-(2^d+2)L^{k-1}, (2^d+2)L^{k-1}]^d & \text{for } X \in \mathcal{P}_k,\ k \geq 2. \end{cases} \tag{4.5.62}$$

Therefore $(X^*)^{++} \subset X^{++}$ holds for $X \in \mathcal{P}_k$, $k \geq 1$, and $L \geq 2^d + 3R$.

Finally, the bound for $\boldsymbol{A}_{k+1}^X$ is a direct consequence of Lemma 4.5.3 and our choice for δ. Indeed, recall that $\delta\Xi_{\max} \leq \frac{\overline{\zeta}}{2\mu} \leq \frac{1}{\mu}$ and $\delta_{k+1} \leq \delta$, hence Lemma 4.5.3 and the induction hypothesis imply

$$\begin{aligned} \boldsymbol{A}_{k+1}^X \leq \boldsymbol{A}_{k+1}^{\Lambda_N} = \boldsymbol{A}_{k:k+1}^{\Lambda_N} + \delta_{k+1}\boldsymbol{M}_{k+1}^{\Lambda_N} &\leq \left(\lambda \boldsymbol{M}_k^{-1} + (1+\overline{\zeta}_k)\sum_{j=k+2}^{N+1} \mathcal{C}_j\right)^{-1} + \delta_{k+1}\Xi_{\max}\boldsymbol{M}_{k+1} \leq \\ &\leq \left(\lambda \boldsymbol{M}_{k+1}^{-1} + (1+\overline{\zeta}_k - \mu\delta_{k+1}\Xi_{\max})\sum_{j=k+2}^{N+1} \mathcal{C}_j\right)^{-1}. \end{aligned} \tag{4.5.63}$$

The claim follows from $\overline{\zeta}_k - \overline{\zeta}_{k+1} = \mu\Xi_{\max}\delta_{k+1}$.

□

4.5.4 Subadditivity properties of the operators $\boldsymbol{A}_k^X$ and $\boldsymbol{A}_{k:k+1}^X$

In this section we prove that the weight operators satisfy additivity properties that directly imply the statements iii)-vii) in Theorem 4.5.1. In Section 4.6 we will also prove that they imply that the norms we defined in Section 4.4.4 are sub-multiplicative.

Lemma 4.5.6. *The weight operators $\boldsymbol{A}_k^X$ and $\boldsymbol{A}_{k:k+1}^X$ satisfy for $0 \leq k \leq N-1$, under the same assumptions as in Theorem 4.5.1 with δ and λ as in Lemma 4.5.5, the following (sub)additivity properties:*

i) Additivity: For any strictly disjoint $X, Y \in \mathcal{P}_k$*, the equality*

$$\boldsymbol{A}_k^{X\cup Y} = \boldsymbol{A}_k^X + \boldsymbol{A}_k^Y \tag{4.5.64}$$

holds. For any $X, Y \in \mathcal{P}_k$ *such that* $\operatorname{dist}(X,Y) \geq \frac{3}{4}L^{k+1}$*, we have*

$$\boldsymbol{A}_{k:k+1}^{X\cup Y} = \boldsymbol{A}_{k:k+1}^X + \boldsymbol{A}_{k:k+1}^Y. \tag{4.5.65}$$

ii) Subadditivity: For any disjoint k-polymers $X, Y \in \mathcal{P}_k$*, the inequality*

$$\boldsymbol{A}_k^X + \boldsymbol{G}_k^Y \leq \boldsymbol{A}_k^{X\cup Y} \tag{4.5.66}$$

holds if $h^{-2} < \delta$*. For any* $(k+1)$*-polymer* $U \in \mathcal{P}_{k+1}$ *and a k-polymer* $X \in \mathcal{P}_k$ *such that* $\pi(X) = U$*, the inequality*

$$\boldsymbol{A}_{k:k+1}^X + 2\boldsymbol{G}_k^{U+} \leq \boldsymbol{A}_{k+1}^U \tag{4.5.67}$$

holds if $8h^{-2} < \delta$*.*

Proof. We first prove (4.5.64) and proceed by induction. Note that for all $k \geq 0$ and any disjoint $X, Y \in \mathcal{P}_k$ we have

$$\boldsymbol{M}_k^{X\cup Y} = \boldsymbol{M}_k^X + \boldsymbol{M}_k^Y \tag{4.5.68}$$

since a block $B \in \mathcal{B}_k$ is contained in $X \cup Y$ if and only if either $B \subset X$ or $B \subset Y$. From (4.5.68) with $k = 0$ and (4.5.10), it follows that (4.5.64) holds for $k = 0$. Hence it suffices to show that $(4.5.64)_k \Rightarrow (4.5.65)_k$ for $k \geq 0$ and $(4.5.65)_k \Rightarrow (4.5.64)_{k+1}$ for $k \geq 0$.

To prove the second statement, $(4.5.65)_k \Rightarrow (4.5.64)_{k+1}$, we consider strictly disjoint $X, Y \in \mathcal{P}_{k+1}$. Then $\operatorname{dist}(X,Y) \geq L^{k+1}$ and, by (4.4.32), $X^*, Y^* \in \mathcal{P}_k$ satisfy

$$\operatorname{dist}(X^*, Y^*) \geq L^{k+1} - 2(2^d + R)L^k \geq \frac{3}{4}L^{k+1} \tag{4.5.69}$$

for $L \geq 2^{d+3} + 8R$. Then

$$\boldsymbol{A}_{k:k+1}^{X^*\cup Y^*} = \boldsymbol{A}_{k:k+1}^{X^*} + \boldsymbol{A}_{k:k+1}^{Y^*} \tag{4.5.70}$$

by $(4.5.65)_k$. Together with (4.5.68) this implies $\boldsymbol{A}_{k+1}^{X\cup Y} = \boldsymbol{A}_{k+1}^X + \boldsymbol{A}_{k+1}^Y$.

To prove the statement $(4.5.64)_k \Rightarrow (4.5.65)_k$, we observe that by property iii) in Lemma 4.5.5, the operator $\boldsymbol{A}_k^X$ is, for a k-polymer $X \in \mathcal{P}_k$, measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(X^{++})}$ and similarly $\boldsymbol{A}_k^Y\varphi$ is measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(Y^{++})}$. Let $X, Y \in \mathcal{P}_k$ be polymers such that $\operatorname{dist}(X,Y) \geq \frac{3}{4}L^{k+1}$. Note that $\operatorname{dist}(X^{++}, Y^{++}) \geq \operatorname{dist}(X,Y) - 4L^k > L^{k+1}/2$ for $L > 16$ and $k \geq 1$ and thus $\operatorname{dist}(X^{++}, Y^{++}) \geq \operatorname{dist}(X,Y) - 4R > L/2$ for $k = 0$ and $L \geq 16R$. This implies that $\nabla\xi_{k+1}\restriction_{\vec{E}(X^{++})}$ and $\nabla\xi_{k+1}\restriction_{\vec{E}(Y^{++})}$ are independent under μ_{k+1} and therefore also under the measure $\mu_{(1+\overline{\zeta}_{k+1})\mathcal{C}_{k+1}}$. Hence the random variables $(\varphi + \xi_{k+1}, \boldsymbol{A}_k^X(\varphi + \xi_{k+1}))$ and $(\varphi + \xi_{k+1}, \boldsymbol{A}_k^Y(\varphi + \xi_{k+1}))$ are independent under the same measure for ξ_{k+1} and any φ. To simplify the notation we denote $C = (1 + \overline{\zeta}_{k+1})\mathcal{C}_{k+1}$. Independence and the formula (4.5.13) for Gaussian integration shows that there exist positive constants c_X, c_Y, and $c_{X\cup Y}$ such that

$$\begin{aligned}\frac{e^{\frac{1}{2}(\boldsymbol{A}_{k:k+1}^{X\cup Y}\varphi,\varphi)}}{c_{X\cup Y}} &= \int_{\mathcal{X}_N} e^{\frac{1}{2}(\boldsymbol{A}_k^{X\cup Y}(\varphi+\xi),\varphi+\xi)}\,\mu_C(\mathrm{d}\xi)\\ &= \int_{\mathcal{X}_N} e^{\frac{1}{2}((\boldsymbol{A}_k^{X}+\boldsymbol{A}_k^{Y})(\varphi+\xi),\varphi+\xi)}\,\mu_C(\mathrm{d}\xi)\\ &= \int_{\mathcal{X}_N} e^{\frac{1}{2}(\boldsymbol{A}_k^{X}(\varphi+\xi),\varphi+\xi)}\,\mu_C(\mathrm{d}\xi)\int_{\mathcal{X}_N} e^{\frac{1}{2}(\boldsymbol{A}_k^{Y}(\varphi+\xi),\varphi+\xi)}\,\mu_C(d\xi)\\ &= \frac{e^{\frac{1}{2}(\boldsymbol{A}_{k:k+1}^{X}\varphi,\varphi)}}{c_X}\frac{e^{\frac{1}{2}(\boldsymbol{A}_{k:k+1}^{Y}\varphi,\varphi)}}{c_Y},\end{aligned} \tag{4.5.71}$$

where, in the second step, we used the induction hypothesis. In the third step we used that the integral factors by the finite range property of $\mathcal{C}_{k+1}$. Evaluation for $\varphi\equiv 0$ shows that the constants must satisfy $c_{X\cup Y}=c_Xc_Y$ which implies $\boldsymbol{A}_{k:k+1}^{X\cup Y}=\boldsymbol{A}_{k:k+1}^{X}+\boldsymbol{A}_{k:k+1}^{Y}$. The equality $c_{X\cup Y}=c_Xc_Y$ can also checked explicitly. Using (4.5.13) we can rewrite

$$\begin{aligned}c_X^2c_Y^2 &= \det(\mathbb{1}-C^{\frac{1}{2}}\boldsymbol{A}_k^XC^{\frac{1}{2}})\det(1-C^{\frac{1}{2}}\boldsymbol{A}_k^YC^{\frac{1}{2}})\\ &= \det(\mathbb{1}-C^{\frac{1}{2}}(\boldsymbol{A}_k^X+\boldsymbol{A}_k^Y)C^{\frac{1}{2}}+C^{\frac{1}{2}}\boldsymbol{A}_k^XC\boldsymbol{A}_k^YC^{\frac{1}{2}})\\ &= \det(\mathbb{1}-C^{\frac{1}{2}}\boldsymbol{A}_k^{X\cup Y}C^{\frac{1}{2}})=c_{X\cup Y}^2.\end{aligned} \tag{4.5.72}$$

Here we used the induction hypothesis for the linear term. The quadratic term vanishes because $\boldsymbol{A}_k^XC\boldsymbol{A}_k^Y=0$ which we now show. Note that $\operatorname{supp}(\boldsymbol{A}_k^Y\varphi)\subset X^{++}$. Indeed, the symmetry of $\boldsymbol{A}_k^Y$ and the locality property iii) in Lemma 4.5.5 imply that $(\psi,\boldsymbol{A}_k^Y\varphi)=0$ for any ψ with $\operatorname{supp}\psi\cap Y^{++}=\emptyset$. Since the kernel $\mathcal{C}(x)$ of C is constant for $|x|_\infty\geq L^{k+1}/2$ we find that $C\varphi(x)=c$ for some constant c for $x\notin B_{L^{k+1}/2}(\operatorname{supp}\varphi)$. Using $\operatorname{dist}(X^{++},Y^{++})\geq L^{k+1}/2$ we conclude that $\nabla C\boldsymbol{A}_k^Y\varphi\restriction_{\vec{E}(X^{++})}=0$ for all $\varphi\in\mathcal{X}_N$ and therefore $\boldsymbol{A}_k^XC\boldsymbol{A}_k^Y\varphi=0$.

Now we prove ii). We first observe that the following operator inequality is true for $h^{-2}<\delta$

$$\delta_k\boldsymbol{M}_k^X\geq\delta_k\sum_{1\leq|\alpha|\leq M}L^{2k(|\alpha|-1)}(\nabla^*)^\alpha\mathbf{1}_{X^+}\nabla^\alpha\geq\delta_kh_k^2\boldsymbol{G}_k^{X^+}\geq\boldsymbol{G}_k^X. \tag{4.5.73}$$

This implies (4.5.66) for $k=0$. For $k\geq 1$ the monotonicity of $\boldsymbol{A}_{k-1:k}^X$ in X, the positivity of $\boldsymbol{M}_k^X$, and the additivity property (4.5.68) of $\boldsymbol{M}_k^X$ imply

$$\boldsymbol{A}_k^{X\cup Y}=\boldsymbol{A}_{k-1:k}^{(X\cup Y)^*}+\delta_k\boldsymbol{M}_k^{X\cup Y}\geq\boldsymbol{A}_{k-1:k}^{X^*}+\delta_k\boldsymbol{M}_k^X+\delta_k\boldsymbol{M}_k^Y\geq\boldsymbol{A}_k^X+\boldsymbol{G}_k^Y. \tag{4.5.74}$$

It remains to prove (4.5.67). Note that $\delta_{k+1}h_{k+1}^2=\delta h^2\geq 8$. Similar to (4.5.73) we conclude that for $U\in\mathcal{P}_{k+1}$

$$\delta_{k+1}\boldsymbol{M}_{k+1}^U\geq\delta_{k+1}\sum_{1\leq|\alpha|\leq M}L^{2(k+1)(|\alpha|-1)}(\nabla^*)^\alpha\mathbf{1}_{U^+}\nabla^\alpha\geq\delta_{k+1}h_{k+1}^2\boldsymbol{G}_{k+1}^{U^+}\geq 8\boldsymbol{G}_{k+1}^{U^+}. \tag{4.5.75}$$

Recall that by (4.4.54) we have $X\subset X^*\subset U^*$ if $U=\pi(X)$. Together with (4.5.75) this implies

$$\boldsymbol{A}_{k+1}^U=\boldsymbol{A}_{k:k+1}^{U^*}+\delta_{k+1}\boldsymbol{M}_{k+1}^U\geq\boldsymbol{A}_{k:k+1}^X+8\boldsymbol{G}_{k+1}^{U^+}\geq\boldsymbol{A}_{k:k+1}^X+2\boldsymbol{G}_k^{U^+} \tag{4.5.76}$$

where we used in the last step that $h_{k+1}^2=4h_k^2$ and therefore $4\boldsymbol{G}_{k+1}^X\geq\boldsymbol{G}_k^X$. Note that in the last expression the operation U^+ in $\boldsymbol{G}_k^{U^+}$ is still on scale $k+1$. □

4.5.5 Consistency of the weights under $\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}$

In this subsection we prove the necessary bounds that imply the integration property of the weights ix) and x) in Theorem 4.5.1. They follow from a Gaussian integration stated in (4.5.13) with the operators $\boldsymbol{A}_k^X$ and the covariances $\mathcal{C}_{k+1}^{(\boldsymbol{q})}$.

Lemma 4.5.7. *Under the same assumptions as in Theorem 4.5.1 with δ and λ as in Lemma 4.5.5 the operators $\boldsymbol{A}_k^X$ satisfy the following additional properties:*

i) Let $\rho = (1+\bar{\zeta})^{\frac{1}{3}} - 1$. There is a constant $A_{\mathcal{P}}$ depending on $\bar{\zeta}$, and in addition on L if $d = 2$, and a constant $\kappa = \kappa(L)$ such that for any k-polymer X and $\boldsymbol{q} \in B_\kappa = \{\boldsymbol{q} \in \mathbb{R}_{\text{sym}}^{(d\times m)\times(d\times m)} \mid |\boldsymbol{q}| < \kappa\}$ the following estimate holds

$$\det\left(\mathbb{1} - (1+\rho)\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{1/2} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{1/2}\right)^{-1/2} \leq \left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k}. \tag{4.5.77}$$

For blocks $X \in \mathcal{B}_k$ the same estimate holds for a constant $A_{\mathcal{B}}$ which does not depend on L.

ii) Integration property: Let $A_{\mathcal{P}}$ and ρ be the constants from i). Then

$$\int_{\mathcal{X}_N} e^{\frac{1+\rho}{2}(\boldsymbol{A}_k^X(\varphi+\xi),\varphi+\xi)}\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi) \leq \left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k} e^{\frac{1+\rho}{2}(\boldsymbol{A}_{k:k+1}^X\varphi,\varphi)} \tag{4.5.78}$$

for any polymer X and the same bound with $A_{\mathcal{P}}$ replaced by $A_{\mathcal{B}}$ holds for any block $X \in \mathcal{B}_k$.

Proof. The statement i) can be proved similarly to Lemma 5.3 in [4]. We rely on the abstract Gaussian calculus sketched at the beginning of this section. One difficulty is the fact that we need to renormalise the covariance. Hence we later need the integration property ii) not only for $\mu_{k+1}^{(0)}$ but also for $\boldsymbol{q}$ in a small neighbourhood $B_\kappa(0)$. As in Section 4.4.1 we impose the condition $\kappa \leq \omega_0/2$. This condition ensures that the finite range decomposition of the covariance $\mathcal{C}^{(q)}$ is defined for $\boldsymbol{q} \in B_\kappa$ under the assumption (4.4.10) on $\mathcal{Q}$.

The first step is to bound the spectrum of the operator $(\mathcal{C}_{k+1}^{(\boldsymbol{q})})^{\frac{1}{2}} \boldsymbol{A}_k^X (\mathcal{C}_{k+1}^{(\boldsymbol{q})})^{\frac{1}{2}}$. This is a necessary condition for the convergence of the integral in (4.5.78) and it is also needed to bound the determinant. We show that the covariance operators $\mathcal{C}_{k+1}^{(\boldsymbol{q})}$ and $\mathcal{C}_{k+1}^{(0)}$ are comparable for small $\boldsymbol{q}$. Namely, for a sufficiently small neighbourhood B_κ of the origin, the inequality $\mathcal{C}_{k+1}^{(\boldsymbol{q})} \leq (1+\rho)\mathcal{C}_{k+1}^{(0)}$ holds for $\boldsymbol{q} \in B_\kappa$. Since both operators are block-diagoinal in the Fourier space, it is sufficient to show the estimate for all Fourier modes. Indeed, we observe that for $p \in \widehat{T}_N$ and $\boldsymbol{q}$ satisfying $|\boldsymbol{q}| < \omega_0/2$, the bound (4.4.17) with $\ell = 1$ implies

$$\left|\widehat{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}(p) - \widehat{\mathcal{C}}_{k+1}^{(0)}(p)\right| \leq \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{C}}_{k+1}^{(t\boldsymbol{q})}(p)\right| \mathrm{d}t \leq |\boldsymbol{q}| K_1 L^{4(d+\tilde{n})+2} \frac{1}{\left|(\widehat{\mathcal{C}}_{k+1}^{(0)}(p))^{-1}\right|}. \tag{4.5.79}$$

From this and the bound $\mathrm{Id}/|A^{-1}| \leq A$, we infer that

$$\widehat{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}(p) - \widehat{\mathcal{C}}_{k+1}^{(0)}(p) \leq |\boldsymbol{q}| K_1 L^{4(d+\tilde{n})+2} \widehat{\mathcal{C}}_{k+1}^{(0)}(p). \tag{4.5.80}$$

The claim now follows for $\boldsymbol{q} \in B_\kappa$ where

$$\kappa = \min(\rho L^{-4(d+\tilde{n})-2}/K_1, \omega_0/2). \tag{4.5.81}$$

Note that here, the lower bounds for the finite range decomposition are essential. We can rewrite $\mathcal{C}_{k+1}^{(\boldsymbol{q})} \leq (1+\rho)\mathcal{C}_{k+1}^{(0)}$ equivalently as

$$(\mathcal{C}_{k+1}^{(\boldsymbol{q})})^{\frac{1}{2}}(\mathcal{C}_{k+1}^{(0)})^{-1}(\mathcal{C}_{k+1}^{(\boldsymbol{q})})^{\frac{1}{2}} \leq (1+\rho). \tag{4.5.82}$$

The constants $\overline{\zeta}_k$ that appear in Lemma 4.5.5 satisfy the inequality $\overline{\zeta}_k \geq \overline{\zeta}$, and we assumed $\overline{\zeta} \in (0,1/4)$. Thus we have $\rho \in (0,1/4)$. Using this, the bounds (4.5.51) and (4.5.82), for $X \in \mathcal{P}_k$ we estimate

$$\begin{aligned} (1+\rho)\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} &\leq \frac{(1+\rho)}{1+\overline{\zeta}_k}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}}\left(\mathcal{C}_{k+1}^{(0)}\right)^{-1}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}}(p) \\ &\leq \frac{(1+\rho)^2}{(1+\rho)^3} < 1-\frac{\rho}{2}. \end{aligned} \tag{4.5.83}$$

Therefore we have shown that the determinant in (4.5.77) is non-vanishing.

To complete the proof of (4.5.77), we bound the trace of $(\mathcal{C}_{k+1}^{(\boldsymbol{q})})^{\frac{1}{2}}\boldsymbol{A}_k^X(\mathcal{C}_{k+1}^{(\boldsymbol{q})})^{\frac{1}{2}}$. Recall that the operators $\mathcal{C}_{k+1}^{(\boldsymbol{q})}$ and $\boldsymbol{A}_k^X$ can be extended to $\mathcal{V}_N$ so that they annihilate constant fields. This extension does not change the trace. Let $\eta_X : T_N \to \mathbb{R}$ be a cut-off function such that $\eta_X\restriction_{X^{++}} = 1$, $\operatorname{supp}(\eta) \subset X^{+++}$, and η_X satisfies the smoothness estimate

$$|\nabla^l \eta_X| \leq \Theta L^{-lk} \tag{4.5.84}$$

for $l \leq 2M$ where Θ does not depend on L or X. We use m_{η_X} to denote the operator of multiplication by η_X. First we note that

$$m_{\eta_X}\boldsymbol{A}_k^X m_{\eta_X} = \boldsymbol{A}_k^X \tag{4.5.85}$$

because $\boldsymbol{A}_k^X$ is self adjoint and depends only on $\varphi(x)$ for $x \in X^{++}$. We observe that, for symmetric operators, the inequality $A \geq B$ implies that $\operatorname{Tr} A \geq \operatorname{Tr} B$ which by (4.5.51) yields

$$\begin{aligned} \operatorname{Tr}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} &= \operatorname{Tr}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} m_{\eta_X}\boldsymbol{A}_k^X m_{\eta_X}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\lambda}\operatorname{Tr}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} m_{\eta_X}\boldsymbol{M}_k m_{\eta_X}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \\ &= \frac{1}{\lambda}\operatorname{Tr} m_{\eta_X}\boldsymbol{M}_k m_{\eta_X}\mathcal{C}_{k+1}^{(\boldsymbol{q})}. \end{aligned} \tag{4.5.86}$$

Recall that λ from (4.5.55) does not depend on L.

The remaining part of the proof is, up to minor details, the same as in [4, Lemma 5.3]. For the trace calculation we will use the orthonormal basis $e_x^i(y) = \delta_x^y e^i$ of $\mathcal{V}_N$, where $e^i \in \mathbb{R}^m$ is a standard basis vector. Note that

$$\left(m_{\eta_X}\mathcal{C}_{k+1}^{(\boldsymbol{q})}e_{x_0}^i\right)(x) = \eta_X(x)\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x-x_0)e^i. \tag{4.5.87}$$

For the evaluation of the operator $\boldsymbol{M}_k$, we need the product rule for discrete derivatives that reads

$$\nabla_i(fg) = \nabla_i f S_i g + S_i f \nabla_i f \tag{4.5.88}$$

where

$$(S_i f)(x) := \frac{1}{2}f(x) + \frac{1}{2}f(x+e_i). \tag{4.5.89}$$

The operators S_i commute with discrete derivatives and we use the usual multiindex notation S^α for $\alpha \in \mathbb{N}_0^d$. This implies that

$$\begin{aligned}&\boldsymbol{M}_k m_{\eta X} \mathcal{C}_{k+1}^{(\boldsymbol{q})} e_{x_0}^i(\cdot) = \\ &= \sum_{|\alpha|\le M} L^{2k(|\alpha|-1)} \sum_{\substack{\beta_1+\beta_2=\alpha\\ \gamma_1+\gamma_2=\alpha}} K_{\gamma_1,\gamma_2}^{\beta_1,\beta_2} (S^{\beta_2})^* S^{\gamma_2} (\nabla^{\beta_1})^* \nabla^{\gamma_1} \eta(\cdot) (S^{\beta_1})^* S^{\gamma_1} (\nabla^{\beta_2})^* \nabla^{\gamma_2} \mathcal{C}_k^{(\boldsymbol{q})}(\cdot - x_0) e^i\end{aligned} \tag{4.5.90}$$

where $K_{\gamma_1,\gamma_2}^{\beta_1,\beta_2}$ is a combinatorial constant. Note that $\|S_i\| = 1$ where, only here, we use $\|\cdot\|$ to denote the operator norm with respect to the maximum norm $\|\cdot\|_\infty$ on $\mathcal{V}_N$. The bound (4.4.13) for the discrete derivatives of $\mathcal{C}_{k+1}^{(\boldsymbol{q})}$ and the choice of $n \ge 2M$ for the regularity parameter of the finite range decomposition, jointly imply that there is a constant $C_M = C_M(L) > 0$ such that

$$\sup_{x \in T_N} |\nabla^\alpha \mathcal{C}_{k+1}^{(q)}(x)| \le C_M L^{-k(d-2+|\alpha|)} \qquad \text{for all } |\alpha| \le 2M, \tag{4.5.91}$$

where C_M is independent of L for $d > 2$, but $C_M \propto \ln(L)$ for $d = 2$. Using this combined with (4.5.84) and (4.5.91), we get

$$\begin{aligned}&\|\boldsymbol{M}_k m_{\eta X} \mathcal{C}_{k+1}^{(\boldsymbol{q})} e_{x_0}^i\|_\infty \\ &\le \sum_{|\alpha|\le M} L^{2k(|\alpha|-1)} \sum_{\substack{\beta_1+\beta_2=\alpha\\ \gamma_1+\gamma_2=\alpha}} K_{\gamma_1,\gamma_2}^{\beta_1,\beta_2} \|(S^{\beta_2})^* S^{\gamma_2} (\nabla^{\beta_1})^* \nabla^{\gamma_1} \eta\|_\infty \|(S^{\beta_1})^* S^{\gamma_1} (\nabla^{\beta_2})^* \nabla^{\gamma_2} \mathcal{C}_{k+1}^{(\boldsymbol{q})}(\cdot - x_0) e^i\|_\infty \\ &\le \sum_{|\alpha|\le M} L^{2k(|\alpha|-1)} \sum_{\substack{\beta_1+\beta_2=\alpha\\ \gamma_1+\gamma_2=\alpha}} K_{\gamma_1,\gamma_2}^{\beta_1,\beta_2} \|\nabla^{\beta_1+\gamma_1}\eta\|_\infty \|\nabla^{\beta_2+\gamma_2} \mathcal{C}_{k+1}^{(\boldsymbol{q})} e^i\|_\infty \\ &\le \sum_{|\alpha|\le M} L^{2k(|\alpha|-1)} \sum_{\substack{\beta_1+\beta_2=\alpha\\ \gamma_1+\gamma_2=\alpha}} K_{\gamma_1,\gamma_2}^{\beta_1,\beta_2} \Theta L^{-k(|\beta_1|+|\gamma_1|)} C_M L^{-k(d-2+|\beta_2|+|\gamma_2|)} \\ &\le C_M \Theta K \sum_{l=1}^{M} L^{2k(l-1)} L^{-k(d-2+2l)} \le C_M \Theta K M L^{-kd} = \Omega L^{-kd}.\end{aligned} \tag{4.5.92}$$

Here K is a purely combinatorial constant depending on the $K_{\gamma_1,\gamma_2}^{\beta_1,\beta_2}$.

Using (4.5.86) and (4.5.92), this implies

$$\begin{aligned}&\operatorname{Tr}\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac12} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac12} \le \frac{1}{\lambda} \sum_{x\in\Lambda_N} \sum_{i=1}^{m} (e_x^i, m_{\eta X} \boldsymbol{M}_k^{\Lambda_N} m_{\eta X} \mathcal{C}_{k+1}^{(\boldsymbol{q})} e_x^i) \\ &\le \frac{1}{\lambda} \sum_{x\in \operatorname{supp}(\eta_X)} \sum_{i=1}^{m} \|\boldsymbol{M}_k^{\Lambda_N} m_{\eta X} \mathcal{C}_{k+1}^{(\boldsymbol{q})} e_x^i\|_\infty \le \frac{\Omega m |X^{+++}| L^{-kd}}{\lambda} \le \frac{\Omega m (7R+1)^d}{\lambda} |X|_k = \Theta_1 |X|_k.\end{aligned} \tag{4.5.93}$$

where Θ_1 depends on L if $d = 2$. The factor $(7R+1)^d$ arises because $X^{+++} = (X + [-3R, 3R]^d) \cap T_N$ for $X \in \mathcal{P}_0$. It could be strengthened to 7^d for $k \ge 1$.

The appearance of an L-dependent term seems to be only an artefact of the use of a cutoff function. Let us show how we can get rid of the L-dependence if X is a single block. This shows the second part of the first statement. First, let us consider $0 \leq k \leq N-1$. Note that by (4.4.13) there is a constant C'_M independent of L such that

$$\sup_{x \in T_N} |\nabla^\alpha \mathcal{C}_{k+1}(x)| \leq C'_M L^{-k(d-2+|\alpha|)} \qquad \text{for all } 1 \leq |\alpha| \leq 2M. \tag{4.5.94}$$

Consider the set

$$T = \{0, 2L^k, 4L^k \cdots, (L^{(N-k)} - 3)L^k\}^d. \tag{4.5.95}$$

Then the blocks $\tau_a(B)$ and $\tau_b(B)$ with $a, b \in T, a \neq b$, have distance at least L^k for $B \in \mathcal{B}_k$. Therefore we find using properties ii), iv), and v) from Lemma 4.5.5 and i) from Lemma 4.5.6

$$\operatorname{Tr} \mathcal{C}_{k+1}^{\frac{1}{2}} \boldsymbol{A}_k^B \mathcal{C}_{k+1}^{\frac{1}{2}} = \frac{1}{|T|} \sum_{a \in T} \operatorname{Tr} \mathcal{C}_{k+1}^{\frac{1}{2}} \boldsymbol{A}_k^{\tau_a(B)} \mathcal{C}_{k+1}^{\frac{1}{2}} \leq \frac{1}{|T|} \operatorname{Tr} \mathcal{C}_{k+1}^{\frac{1}{2}} \boldsymbol{A}_k^{\Lambda_N} \mathcal{C}_{k+1}^{\frac{1}{2}} \leq \frac{1}{\lambda |T|} \operatorname{Tr} \mathcal{C}_{k+1}^{\frac{1}{2}} \boldsymbol{M}_k \mathcal{C}_{k+1}^{\frac{1}{2}} \tag{4.5.96}$$

This trace is estimated similarly to (4.5.92) using (4.5.94) as follows

$$\begin{aligned} \frac{1}{\lambda |T|} \operatorname{Tr} \mathcal{C}_{k+1}^{\frac{1}{2}} \boldsymbol{M}_k \mathcal{C}_{k+1}^{\frac{1}{2}} &= \frac{1}{\lambda |T|} \sum_{x \in T_N} \sum_{i=1}^m (e_x^i, \boldsymbol{M}_k \mathcal{C}_{k+1} e_x^i) \leq \frac{m L^{Nd}}{\lambda |T|} \|\boldsymbol{M}_k \mathcal{C}_{k+1}\|_\infty \\ &\leq \frac{K m L^{Nd}}{\lambda |T|} \sum_{l=1}^{2M} L^{2l(k-1)} C'_M L^{-k(d-2+2l)} \leq \frac{C'_M K m L^{Nd}}{\lambda |T|} L^{-kd}, \end{aligned} \tag{4.5.97}$$

where K denotes again a combinatorial constant and none of the constants depends on L. Using the bound $|T| = (L^{N-k} - 1)^d / 2^d \geq 4^{-d} L^{(N-k)d}$ we find that there is a constant Θ_2 independent of L such that for all blocks $B \in \mathcal{B}_k$

$$\operatorname{Tr} \mathcal{C}_{k+1}^{\frac{1}{2}} \boldsymbol{A}_k^B \mathcal{C}_{k+1}^{\frac{1}{2}} \leq \Theta_2 |B|_k = \Theta_2. \tag{4.5.98}$$

Note that for $k = N$ there is only one block and we can use the same argument with $T = \{0\}$.

The estimate for the determinant is now standard. We denote the eigenvalues of the operator $(1+\rho)\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}}$ by λ_i. Recall that $\rho = \rho(\bar{\zeta}) < 1/4$ is a constant and the bound (4.5.83) on the spectrum implies that $\lambda_i \in [0, 1-\rho/2]$. Concavity of the logarithm implies that $\ln(1-x) \geq \frac{\ln(\rho/2)}{1-\rho/2} x = -\frac{\ln(2/\rho)}{1-\rho/2} x$ for $x \in [0, 1-\rho/2]$. Using this we obtain the bound

$$\begin{aligned} \ln \det \left(\mathbb{1} - (1+\rho) \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \right) &= \sum_i \ln(1-\lambda_i) \geq \frac{\ln(\rho/2)}{1-\rho/2} \sum_i \lambda_i \\ &= -\frac{\ln(2/\rho)}{1-\rho/2} (1+\rho) \operatorname{Tr} \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}}. \end{aligned} \tag{4.5.99}$$

From (4.5.93) we conclude that, for $A_{\mathcal{P}} \geq 2 \exp(\Theta_1 (1+\rho) \ln(2/\rho) / (2(1-\rho/2)))$,

$$\begin{aligned} \ln \det \left(\mathbb{1} - (1+\rho) \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \right) &\geq -\frac{\ln(2/\rho)}{1-\rho/2} (1+\rho) \operatorname{Tr} \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \\ &\geq -2|X|_k \ln\left(\frac{A_{\mathcal{P}}}{2}\right) \end{aligned} \tag{4.5.100}$$

which implies the claim (4.5.77). Similarly we find the same statement for blocks $B \in \mathcal{B}_k$ for the constant $A_{\mathcal{B}} \geq 2\exp(\Theta_2(1+\rho)\ln(2/\rho)/(2(1-\rho/2))$ which does not depend on L.

The integration property ii) follows directly from Gaussian calculus (which is justified because of (4.5.83)) and the previous point i),

$$\begin{aligned}
&\int_{\mathcal{X}_N} e^{\frac{1+\rho}{2}(\boldsymbol{A}_k^X(\varphi+\xi),\varphi+\xi)}\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi) \\
&\quad = \left(\det \mathbb{1} - (1+\rho)\left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}} \boldsymbol{A}_k^X \left(\mathcal{C}_{k+1}^{(\boldsymbol{q})}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\left(\varphi, (((1+\rho)\boldsymbol{A}_k^X)^{-1} - \mathcal{C}_{k+1}^{(\boldsymbol{q})})^{-1}\varphi\right)\right) \\
&\quad \leq \left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k} \exp\left(\frac{1+\rho}{2}(\varphi, ((\boldsymbol{A}_k^X)^{-1} - (1+\rho)^2\mathcal{C}_{k+1}^{(0)})^{-1}\varphi)\right)\left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k} e^{\frac{1+\rho}{2}(\varphi,\boldsymbol{A}_{k:k+1}^X\varphi)},
\end{aligned} \tag{4.5.101}$$

where we again used the monotonicity of the inversion combined with the bound $(1+\rho)\mathcal{C}_{k+1}^{(\boldsymbol{q})} \leq (1+\rho)^2\mathcal{C}_{k+1}^{(0)} \leq (1+\bar{\zeta})\mathcal{C}_{k+1}^{(0)}$ for $\boldsymbol{q} \in B_\kappa$. If X is a block we can replace $A_{\mathcal{P}}$ by $A_{\mathcal{B}}$. □

Finally, we prove the property Theorem 4.5.1viii).

Lemma 4.5.8. *Under the same assumptions as in Theorem 4.5.1 and with δ and λ as in Lemma 4.5.5, the norm for the fields can be bounded in terms of the weights as follows*

i) Interaction with the field norm: For any polymer $X \in \mathcal{P}_{k+1}$ the bound

$$|\varphi|_{k+1,X}^2 \leq (\varphi, \boldsymbol{A}_{k+1}^X\varphi) - (\varphi, \boldsymbol{A}_{k:k+1}^X\varphi) \tag{4.5.102}$$

holds if $h \geq h_0 = (M'3^{2M'}S/\delta)^{1/2} \coloneqq c_d\delta^{-1/2}$ where $M' = \lfloor \frac{d}{2} \rfloor + 1$, $S = S(d)$ is the Sobolev constant in Lemma 4.5.9, and δ is the constant from Lemma 4.5.5.

Proof. This property follows from the discrete Sobolev inequality stated in the next lemma.

Lemma 4.5.9. *Let $B_\ell = [0,\ell]^d \cap \mathbb{Z}^d$ and $M' = \lfloor \frac{d}{2} \rfloor + 1$. For $f : B_\ell \to \mathbb{R}$ we define the norm*

$$\|f\|_{B_\ell,2} = \left(\sum_{x \in B_\ell} |f(x)|^2\right)^{\frac{1}{2}}. \tag{4.5.103}$$

Then the following Sobolev inequality holds for some constant $S(d) > 0$

$$\max_{x \in B_\ell} |f(x)| \leq S(d)\ell^{-\frac{d}{2}} \sum_{0 \leq |\alpha| \leq M'} \|(\ell\nabla)^\alpha f\|_2 \tag{4.5.104}$$

where we assume that f is defined in a neighbourhood of B_ℓ such that all discrete derivatives exist.

Proof. Sobolev already considered such inequalities on lattices, see [142] for a similar statement. Also, a similar claim with d derivatives appeared in [43][Proposition B2] and [44][Lemma 6.6]. For the statement above a proof can be found, e.g., in [4, Appendix A]. □

We apply this lemma to the function $\nabla^\alpha \varphi_i$ for $1 \le |\alpha| \le p_\Phi = \lfloor d/2 \rfloor + 2$ and the set B^* for $B \in \mathcal{B}_{k+1}$. Using that $B \subset B^* \subset B^+$ for $B \in \mathcal{B}_{k+1}$ we obtain that the side-length of B^* is contained in $[L^{k+1}, 3L^{k+1}]$ and therefore

$$\begin{aligned} \max_{x \in B^*} |\nabla^\alpha \varphi_i(x)|^2 &\le M' S(d) (L^{k+1})^{-d} \sum_{0 \le |\beta| \le M'} \|(3L^{k+1}\nabla)^\beta \nabla^\alpha \varphi_i\|^2_{B^*,2} \\ &\le M' 3^{2M'} S(d) L^{-(k+1)d} \sum_{1 \le |\gamma| \le M} L^{2(|\gamma|-|\alpha|)(k+1)} \left(\nabla^\gamma \varphi_i, \mathbf{1}_{B^*} \nabla^\gamma \varphi_i\right) \\ &\le M' 3^{2M'} S(d) L^{-(k+1)(d+2|\alpha|-2)} (\varphi, \boldsymbol{M}^B_{k+1}\varphi). \end{aligned} \tag{4.5.105}$$

Here we used that $M = M' + p_\Phi = 2\lfloor d/2 \rfloor + 3$ by (4.5.1) and the definition of $\boldsymbol{M}^X_k$ in (4.5.2). Note that the definition of $\boldsymbol{M}^B_k$ involves the term $\mathbf{1}_{B^+} \ge \mathbf{1}_{B^*}$. Using the definition (4.4.74) of the primal norm we deduce

$$\begin{aligned} |\varphi|^2_{k+1,B} &= \frac{1}{h^2_{k+1}} \max_{x \in B^*} \max_{1 \le i \le m} \max_{1 \le |\alpha| \le p_\Phi} L^{(k+1)(d-2+2j)} |\nabla^\alpha \varphi(x)|^2 \\ &\le \frac{M' 3^{2M'} \Xi(d)}{h^2_{k+1}} (\varphi, \boldsymbol{M}^B_{k+1}\varphi) \\ &\le \delta_{k+1} (\varphi, \boldsymbol{M}^B_{k+1}\varphi) \end{aligned} \tag{4.5.106}$$

provided that $h^2 \ge M' 3^{2M'} S(d)/\delta$. We can now easily conclude for a general polymer $X \in \mathcal{P}_{k+1}$,

$$\begin{aligned} |\varphi|^2_{k+1,X} &= \max_{B \in \mathcal{B}(X)} |\varphi|^2_{k+1,B} \\ &\le \sum_{B \in \mathcal{B}(X)} \delta_{k+1} (\varphi, \boldsymbol{M}^B_{k+1}\varphi) \\ &= \delta_{k+1} (\varphi, \boldsymbol{M}^X_{k+1}\varphi) = (\varphi, \boldsymbol{A}^X_{k+1}\varphi) - (\varphi, \boldsymbol{A}^X_{k:k+1}\varphi). \end{aligned} \tag{4.5.107}$$

□

4.6 Properties of the norms

In this section we collect some bounds for the norms we defined before. In particular we establish submultiplicativity of the norms as well as bounds for the renormalisation maps $\boldsymbol{R}_k$ and the projection Π_2. Here and in the following section we assume that for any given L our norms are defined using weights as in Theorem 4.5.1 with $\overline{\zeta}$, M, p_Φ, n, and $\tilde{n}$ as indicated in Section 4.3.2.

4.6.1 Pointwise properties of the norms

Specialising the general properties of norms on Taylor polynomials described in Appendix 4.A to the (injective) tensor norms defined in (4.4.77) and the dual norm in (4.4.84) we obtain the following result.

Recall that $\vec{E}(X^*)$ denotes the set of directed edges in the small neighbourhood X^* of a polymer $X \in \mathcal{P}_k$. Lemma 4.4.3 states that functional $F : \mathcal{P}_k \times \mathcal{V}_N \to \mathbb{R}$ is local and shift invariant if and only if for each $X \in \mathcal{P}_k$ the map $\varphi \mapsto F(X, \varphi)$ is measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(X^*)}$. Recall also that we always assume $r_0 \ge 3$.

Lemma 4.6.1. *Let $X \in \mathcal{P}_k$, $F, G \in C^{r_0}(\mathcal{V}_N)$ and assume that F and G are measurable with respect to the σ-algebra generated by $\nabla\varphi\restriction_{\vec{E}(X^*)}$. Then*

$$|FG|_{k,X,T_\varphi} \le |F|_{k,X,T_\varphi}\, |G|_{k,X,T_\varphi} \tag{4.6.1}$$

and

$$|F|_{k+1,X,T_\varphi} \le (1 + |\varphi|_{k+1,X})^3 \big(|F|_{k+1,X,T_0} + 16L^{-\frac{3}{2}d} \sup_{0\le t\le 1} |F|_{k,X,T_{t\varphi}}\big). \tag{4.6.2}$$

Proof. The first inequality will follow from Proposition 4.A.9 applied to a certain quotient space. In the following we will define this quotient space and show that it is a Banach space on which the Taylor polynomials of F and G act. We first note that $|\psi|_{k,X} = 0$ implies that $\nabla\psi\restriction_{\vec{E}(X^*)} = 0$ and therefore by assumption $F(\varphi + \psi) = F(\varphi)$ for $\varphi \in \mathcal{V}_N$. Hence, F and G have the property that $(\mathrm{Tay}_\varphi F)(\dot\varphi + \dot\psi) = (\mathrm{Tay}_\varphi F)(\dot\varphi)$ and $(\mathrm{Tay}_\varphi G)(\dot\varphi + \dot\psi) = (\mathrm{Tay}_\varphi G)(\dot\varphi)$ for all $\dot\varphi \in \mathcal{V}_N$ and all $\dot\psi \in \mathcal{V}_N$ with $|\dot\psi|_{k,X} = 0$

This implies that the norms in (4.6.1) are finite (see the remark after (4.4.84)) and that the Taylor polynomials act on the quotient space $\mathcal{V}_N/\!\sim$ and on $\oplus_{r=0}^{r_0} (\mathcal{V}_N/\!\sim)^{\otimes r}$ where $\varphi \sim \xi$ if and only if $|\xi - \varphi|_{k,X} = 0$. Moreover $|\cdot|_{k,X}$ is a norm on this quotient space. Thus the assertion follows from Proposition 4.A.9.

Similarly for the second inequality we again use that F acts on the quotient space $\mathcal{V}_N/\!\sim$ where $\varphi \sim \xi$ if $|\varphi - \xi|_{k,X} = 0$. Since $|\varphi|_{k,X} = 0 \Leftrightarrow |\varphi|_{k+1,X} = 0$ both $|\cdot|_{k,X}$ and $|\cdot|_{k+1,X}$ define norms on $\mathcal{V}_N/\!\sim$. We may thus apply the two norm estimate (4.A.51) in Proposition 4.A.11 with the norms $|g|_{k,X}$ and $|g|_{k+1,X}$ and $\overline{r} = 2$. It follows directly from the definition of the norms $|g|_{j,X}$ in (4.4.76), (4.4.77) and (4.4.78) (and the fact that $|\alpha_i| \ge 1$) that

$$|g^{(r)}|_{k,X} \le 2^r L^{-r\frac{d}{2}}\, |g^{(r)}|_{k+1,X} \quad \forall g^{(r)} \in \mathcal{V}_N^{\otimes r}. \tag{4.6.3}$$

Here we used in particular that $h_{k+1}/h_k = 2$. Thus the quantity $\rho^{(3)}$ in Proposition (4.A.11) satisfies

$$\rho^{(3)} \le 16L^{-\frac{3}{2}d}. \tag{4.6.4}$$

Therefore the two norm estimate (4.A.51) with $\overline{r} = 2$ implies (4.6.2). $\square$

Lemma 4.6.2. *Let $\varphi \in \mathcal{V}_N$. Then*

1. *for any $F_1, F_2 \in M(\mathcal{P}_k)$ and any (not necessarily disjoint) $X_1, X_2 \in \mathcal{P}_k$, we have*

$$|F_1(X_1)F_2(X_2)|_{k,X_1\cup X_2,T_\varphi} \le |F_1(X_1)|_{k,X_1,T_\varphi}\, |F_2(X_2)|_{k,X_2,T_\varphi}; \tag{4.6.5}$$

2. *for any $F \in M(\mathcal{P}_k)$ and any polymer $X \in \mathcal{P}_k$ the bound*

$$|F(X)|_{k+1,\pi(X),T_\varphi} \le |F(X)|_{k,X\cup\pi(X),T_\varphi} \le |F(X)|_{k,X,T_\varphi} \tag{4.6.6}$$

holds if $L \ge 2^d + R$.

Proof. In view of (4.6.1) (applied with $X = X_1 \cup X_2$, $F = F_1$ and $G = F_2$) the first inequality follows from the bound

$$|F(X)|_{k,X\cup Y,T_\varphi} \le |F(X)|_{k,X,T_\varphi} \tag{4.6.7}$$

which itself is a consequence of the estimate $|\varphi|_{k,X} \le |\varphi|_{k,X\cup Y}$.

The second inequality in (4.6.6) follows from (4.6.7). To prove the first inequality in (4.6.6) it is sufficient to show that for any polymer $X \in \mathcal{P}_k$ and any $\varphi \in \mathcal{V}_N$ the primal norms satisfy the estimate

$$|\varphi|_{k,X\cup\pi(X)} \le |\varphi|_{k+1,\pi(X)} \tag{4.6.8}$$

for $L \ge 2^d + R$. Note that by (4.4.54) the condition $L \ge 2^d + R$ implies that $X^* \subset \pi(X)^*$. This fact and the bound

$$h_{k+1}^{-1} L^{(k+1)(\frac{d-2}{2}+|\alpha|)} \ge h_k^{-1} L^{k(\frac{d-2}{2}+|\alpha|)} \frac{L}{2} \ge h_k^{-1} L^{k(\frac{d-2}{2}+|\alpha|)} \tag{4.6.9}$$

for $|\alpha| \ge 1$ imply (4.6.8). □

4.6.2 Submultiplicativity of the norms

Lemma 4.6.3. *Assume that $L \ge 2^{d+3} + 16R$ odd, and $h \ge h_0(L)$ where $h_0(L)$ is specified in (4.5.17) in Theorem 4.5.1. Let $K \in M(\mathcal{P}_k)$ factor at scale $0 \le k \le N-1$ and let $F, F_1, F_2, F_3 \in M(\mathcal{B}_k)$. Then the following bounds hold:*

i) For every $X \in \mathcal{P}_k$

$$\|K(X)\|_{k,X} \le \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_{k,Y} \tag{4.6.10}$$

$$\|K(X)\|_{k:k+1,X} \le \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_{k:k+1,Y}. \tag{4.6.11}$$

More generally the same bounds hold for any decomposition $X = \bigcup_i Y_i$ such that the Y_i are strictly disjoint.

ii) For every $X, Y \in \mathcal{P}_k$ with X and Y disjoint

$$\|K(Y)F^X\|_{k,X\cup Y} \le \|K(Y)\|_{k,Y} |||F|||_k^{|X|_k}. \tag{4.6.12}$$

iii) For any polymers $X, Y, Z_1, Z_2 \in \mathcal{P}_k$ such that $X \cap Y = \emptyset$, $Z_1 \cap Z_2 = \emptyset$, and $Z_1, Z_2 \subset \pi(X\cup Y) \cup X \cup Y$

$$\|F_1^{Z_1} F_2^{Z_2} F_3^X K(Y)\|_{k+1,\pi(X\cup Y)} \le \|K(Y)\|_{k:k+1,Y} |||F_1|||_k^{|Z_1|_k} |||F_2|||_k^{|Z_2|_k} |||F_3|||_k^{|X|_k}. \tag{4.6.13}$$

iv) For $B \in \mathcal{B}_k$

$$|||\mathbb{1}(B)|||_{k,B} = 1. \tag{4.6.14}$$

Proof. The proof is the same as in [4] with the difference that the definition of the weight functions changed. The submultiplicativity from Lemma 4.6.2 reduces the proof to the factorisation of the weight functions stated in Theorem 4.5.1 iii). Indeed, for i) we observe that

$$\begin{aligned} \|K(X)\|_{k,X} &= \sup_{\varphi\in\mathcal{V}_N} \frac{|K(X)|_{k,X,T_\varphi}}{w_k^X(\varphi)} = \sup_{\varphi\in\mathcal{V}_N} \frac{\left|\prod_{Y\in\mathcal{C}(X)} K(Y)\right|_{k,X,T_\varphi}}{\prod_{Y\in\mathcal{C}(X)} w_k^Y(\varphi)} \\ &\le \sup_{\varphi\in\mathcal{V}_N} \prod_{Y\in\mathcal{C}(X)} \frac{|K(Y)|_{k,Y,T_\varphi}}{w_k^Y(\varphi)} \le \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_{k,Y}. \end{aligned} \tag{4.6.15}$$

The same proof applies for a general decomposition $X = \bigcup_i Y_i$ into strictly disjoint sets Y_i. To prove the estimate for the $\|\cdot\|_{k:k+1,X}$ norm it suffices to use property iv) in Theorem 4.5.1 instead of property iii). The proof of ii) relies on Theorem 4.5.1 vi) which together with (4.6.5) implies

$$\begin{aligned} \|K(Y)F^X\|_{k,X\cup Y} &\leq \sup_{\varphi\in\mathcal{V}_N} \frac{|K(Y)|_{k,Y,T_\varphi} \prod_{B\in\mathcal{B}_k(X)} |F(B)|_{k,B,T_\varphi}}{w_k^Y(\varphi)W_k^X(\varphi)} \\ &\leq \sup_{\varphi\in\mathcal{V}_N} \frac{|K(Y)|_{k,Y,T_\varphi}}{w_k^X(\varphi)} \prod_{B\in\mathcal{B}_k(X)} \frac{|F(B)|_{k,B,T_\varphi}}{W_k^B(\varphi)} \\ &\leq \|K\|_{k,X} |||F|||_k^{|X|_k}. \end{aligned} \tag{4.6.16}$$

To prove iii) we use property vii) in Theorem 4.5.1 and estimate (4.6.6) to get

$$\|F_1^{Z_1}F_2^{Z_2}F_3^X K(Y)\|_{k+1,\pi(X\cup Y),T_\varphi} \leq \sup_{\varphi\in\mathcal{V}_N} \frac{|F_1^{Z_1}F_2^{Z_2}F_3^X K(Y)|_{k,X\cup Y\cup\pi(X\cup Y),T_\varphi}}{w_{k:k+1}^{X\cup Y}(\varphi)\left(W_k^{\pi(X\cup Y)^+}(\varphi)\right)^2}. \tag{4.6.17}$$

where for $U \in \mathcal{P}_{k+1}$ the neighbourhood U^+ is given by $U^+ = U + [-L^{k+1}, L^{k+1}]^d \cap T_N$, see (4.4.34). Now Theorem 4.5.1 i) implies that $w_{k:k+1}^{X\cup Y} \geq w_{k:k+1}^{Y}$. Moreover we have $X \subset \pi(X \cup Y) \cup X \cup Y$, $Z_1 \cup Z_2 \subset \pi(X \cup Y) \cup X \cup Y$ and $Z_1 \cap Z_2 = \emptyset$. Thus the factorisation property (4.5.22) of the strong weight function yields

$$\left(W_k^{\pi(X\cup Y)^+}(\varphi)\right)^2 \geq W_k^{Z_1\cup Z_2}(\varphi)W_k^X(\varphi) = W_k^{Z_1}(\varphi)W_k^{Z_2}(\varphi)W_k^X(\varphi). \tag{4.6.18}$$

Together with (4.6.5) we get

$$\begin{aligned} &\|F_1^{Z_1}F_2^{Z_2}F_3^X K(Y)\|_{k+1,\pi(X\cup Y),T_\varphi} \\ &\leq \sup_{\varphi\in\mathcal{V}_N} \frac{\prod_{B\in\mathcal{B}_k(Z_1)} |F_1(B)|_{k,B,T_\varphi} \prod_{B\in\mathcal{B}_k(Z_2)} |F_2(B)|_{k,B,T_\varphi} \prod_{B\in\mathcal{B}_k(X)} |F_3(B)|_{k,B,T_\varphi} |K(Y)|_{k,Y,T_\varphi}}{W_k^{Z_1}(\varphi)W_k^{Z_2}(\varphi)W_k^X(\varphi)w_{k:k+1}^Y(\varphi)}. \\ &\leq |||F_1|||^{|Z_1|_k} |||F_2|||^{|Z_2|_k} |||F_3|||^{|X|_k} \|K(Y)\|_{k:k+1,Y} \end{aligned} \tag{4.6.19}$$

where we used the definition of the norms in the last inequality. The last property is clear from the definitions. □

4.6.3 Regularity of the integration map

The next lemma gives the bound for the renormalisation maps $\boldsymbol{R}_k^{(\boldsymbol{q})}$. Moreover it states regularity of the renormalisation map with respect to the parameter $\boldsymbol{q}$. This is one of the major differences compared to [4] where the authors have to deal with a loss of regularity for the $\boldsymbol{q}$ derivatives. The regularity we obtain here is a consequence of the new finite range decomposition from Theorem 4.4.1 that was constructed in [54].

Lemma 4.6.4. *Assume that $L \geq 2^{d+3} + 16R$ and let $A_\mathcal{P} = A_\mathcal{P}(L) \geq 1$, $\kappa = \kappa(L) > 0$ be the constants from from Theorem 4.5.1. Then for $\boldsymbol{q} \in B_\kappa$ and $X \in \mathcal{P}_k$*

$$\|(\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}K)(X)\|_{k:k+1,X} \leq {A_\mathcal{P}}^{|X|_k}\|K(X)\|_{k,X}. \tag{4.6.20}$$

Let $X \in \mathcal{P}_k$ be a polymer such that $\pi(X) \in \mathcal{P}_{k+1}^{\mathrm{c}}$. Then for $\ell \geq 1$ and $\boldsymbol{q} \in B_\kappa$

$$\sup_{|\dot{\boldsymbol{q}}|\le 1} \|\partial^\ell_{\boldsymbol{q}}(\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}K)(X)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k:k+1,X} \le C_\ell(L) A_{\mathcal{P}}{}^{|X|_k}\|K(X)\|_{k,X}. \tag{4.6.21}$$

The same bounds hold with $A_{\mathcal{P}}$ replaced by $A_{\mathcal{B}}$ if $X \in \mathcal{B}_k$ is a single block.

Proof. We first consider $\ell = 0$. Here we argue similar to [4]. Since Taylor expansion commutes with convolution we have

$$|(\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}K)(X)|_{k,X,T_\varphi} \le \int_{\mathcal{X}_N} |K(X)|_{k,X,T_{\varphi+\xi}}\ \mu^{(\boldsymbol{q})}_{k+1}(\mathrm{d}\xi). \tag{4.6.22}$$

It follows that

$$\begin{aligned}\|(\boldsymbol{R}^{(\boldsymbol{q})}_{k+1}K)(X)\|_{k:k+1,X} &\le \sup_\varphi w^{-X}_{k:k+1}(\varphi)\int |K(X)|_{k,X,T_{\varphi+\xi}}\ \mu^{(\boldsymbol{q})}_{k+1}(\mathrm{d}\xi)\\ &\le \sup_\varphi w^{-X}_{k:k+1}(\varphi)\int \|K(X)\|_{k,X}\ w^X_k(\varphi+\xi)\,\mu^{(\boldsymbol{q})}_{k+1}(\mathrm{d}\xi)\\ &\le \left(\frac{A_{\mathcal{P}}}{2}\right)^{|X|_k}\|K(X)\|_{k,X}\end{aligned} \tag{4.6.23}$$

where we used Theorem 4.5.1 ix) in the last step. Using Theorem 4.5.1 x) $A_{\mathcal{P}}$ can be replaced by $A_{\mathcal{B}}$ for single blocks.

For the derivatives we argue similarly. First we bound the diameter of X. Note that we have $B\cap X \neq \emptyset$ for any block $B \in \mathcal{B}_{k+1}(\pi(X))$ by definition of π. This implies $|\pi(X)|_{k+1} \le |X|_k$. By (4.4.54) we have $X^* \subset \pi(X)^*$. We get using (4.4.32)

$$\operatorname{diam}(X^*) \le \operatorname{diam}(\pi(X)^*) \le L^{k+1}|\pi(X)|_{k+1} + 2(2^d+R)L^k \le 2L^{k+1}|X|_k. \tag{4.6.24}$$

Next we claim that for $\ell \ge 1$, $p > 1$ and $D = \operatorname{diam}(X^*)$

$$\sup_{|\dot{\boldsymbol{q}}|\le 1}\left|\frac{d^\ell}{dt^\ell}_{|t=0}\boldsymbol{R}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})}K(X)\right|_{k,X,T_\varphi} \le C_{p,\ell}(L)(DL^{-k})^{\frac{d\ell}{2}}\left(\int_{\mathcal{X}_N}|K(X)|^p_{k,X,T_{\varphi+\xi}}\ \mu^{(\boldsymbol{q})}_{k+1}(d\xi)\right)^{1/p} \tag{4.6.25}$$

Indeed we have

$$\left\langle \mathrm{Tay}_\varphi \frac{d^\ell}{dt^\ell}_{|t=0}(\boldsymbol{R}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})}K)(X,\varphi), g\right\rangle = \frac{d^\ell}{dt^\ell}_{|t=0}\int_{\mathcal{X}_N}\langle \mathrm{Tay}_{\varphi+\xi}K(X), g\rangle\ \mu^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})}_{k+1}(\mathrm{d}\xi). \tag{4.6.26}$$

Denote the integrand in (4.6.26) by $F(\xi) = F_{\varphi,g}(\xi)$ and observe that we have the bound $|F(\xi)| \le |K(X)|_{k,X,T_{\varphi+\xi}}\,|g|_{k,X}$. Passing to absolute values and using Theorem 4.4.2 with $\boldsymbol{Q}_1(z)$ being the generator of the quadratic form $z \mapsto -(\dot{\boldsymbol{q}}z^\nabla, z^\nabla)$ we get

$$\begin{aligned}&\left|\left\langle \mathrm{Tay}_\varphi \tfrac{d^\ell}{dt^\ell}_{|t=0}\boldsymbol{R}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})}K(X), g\right\rangle\right|^p\\ &\le C^p_{\ell,p}(L)(DL^{-k})^{\frac{d\ell p}{2}}\,\|F\|^p_{L^p(\mathcal{X},\mu^{(\boldsymbol{q})}_{k+1})}\,\|\dot{\boldsymbol{q}}\|^{p\ell}\\ &\le C^p_{\ell,p}(L)(DL^{-k})^{\frac{d\ell p}{2}}\int_{\mathcal{X}_N}|K(X)|^p_{k,X,T_{\varphi+\xi}}\ \mu^{(\boldsymbol{q})}_{k+1}(d\xi)\,|g|^p_{k,X}\,\|\dot{\boldsymbol{q}}\|^{p\ell}.\end{aligned}$$

Taking the supremum over g with $|g|_{k,X} \le 1$ and over $\dot{\boldsymbol{q}}$ with $\|\dot{\boldsymbol{q}}\| \le 1$ we get (4.6.25).

Now (4.6.24) implies that $DL^{-k} \le 2L|X|_k$. Using that $x^{d\ell/2}2^{-x}$ is bounded and (4.6.25) we see that there is another constant $C'_{\ell,p}(L)$ such that

$$\sup_{\|\dot{\boldsymbol{q}}\|\le 1} \left| \frac{d^\ell}{dt^\ell}_{|t=0} \boldsymbol{R}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})} K(X) \right|_{k,X,T_\varphi} \le C'_{\ell,p}(L) 2^{|X|_k} \|K(X)\|_{k,X} \left(\int |w_k^X(\varphi+\xi)|^p \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi) \right)^{\frac{1}{p}}. \tag{4.6.27}$$

Now we set $p = 1+\rho$ where $\rho = (1+\overline{\zeta})^{1/3} - 1$. Then Theorem 4.5.1 ix) implies

$$\sup_{\|\dot{\boldsymbol{q}}\|\le 1} \left| \frac{d^\ell}{dt^\ell}_{|t=0} \boldsymbol{R}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})} K(X) \right|_{k,X,T_\varphi} \le C_\ell(L)\, A_{\mathcal{P}}^{|X|_k} \|K(X)\|_{k,X}\, w_{k:k+1}^X(\varphi) \tag{4.6.28}$$

The conclusion follows by multiplying with $w_{k:k+1}^{-X}(\varphi)$ and then taking the supremum over φ. Again, using Theorem 4.5.1 x) we can replace $A_{\mathcal{P}}$ by $A_{\mathcal{B}}$ for single blocks. □

4.6.4 The projection Π_2 to relevant Hamiltonians

In this subsection we introduce the projection Π_2 to relevant Hamiltonians and prove its key properties. The argument is based on a natural duality between relevant monomials in the fields and monomials on $\mathbb{Z}^d$. The projection Π_2 is a very special case of the operator loc (in fact loc_B) introduced by Brydges and Slade [45], except that we do not need to symmetrise between forward and backward derivatives. Since our situation is much simpler than the general case considered in [45] we give a self-contained exposition, which follows the strategy in [45], for the convenience of the reader. For $d \le 3$ a more simple-minded proof of the boundedness and contraction properties of Π_2 was given in Lemma 6.2 and Lemma 7.3 of [4]. This argument can be extended to the case $d > 3$, but we prefer to follow the more elegant approach of [45]. As pointed out in [45], related questions are discussed in the paper [36] by de Boor and Ron.

Regarding dependencies on the various parameters we recall our convention that we do not indicate dependence on the fixed parameters described in Section 4.3. The parameter A does not enter at all, so we only indicate dependence on L and h.For the contraction estimate which involves norms on scales k and $k+1$ we use that the ratio h_{k+1}/h_k is bounded, in fact with our choice $h_{k+1}/h_k = 2$. Inspection of the proofs shows that the constants which appear in the rest of this subsection depend only on the spatial dimension d the number of components m and the parameter $R = \max(R_0, M)$ where R_0 is the range of the interaction and $M = p_\Phi + \lfloor d/2 \rfloor + 1 = 2\lfloor d/2 \rfloor + 3$.

We follow closely the notation of [45], with the following exception. Since we only deal with forward derivatives we set

$$\mathcal{U} = \{e_1, \ldots, e_d\} \simeq \{1, \ldots, d\}. \tag{4.6.29}$$

In contrast, $\mathcal{U}$ is the set $\{\pm e_1, \ldots, \pm e_d\}$ in [45]. We also drop various subscripts $+$ which refer to forward derivatives.

Relevant monomials in the fields. Recall that we declared the following monomials to be relevant.

- The constant monomial $\mathcal{M}_\emptyset(\{x\})(\varphi) \equiv 1$;
- the linear monomials $\mathcal{M}_{i,\alpha}(\{x\})(\varphi) := \nabla^{i,\alpha}\phi(x) := \nabla^\alpha \phi_i(x) \quad$ for $1 \le |\alpha| \le \lfloor d/2 \rfloor + 1$;

- the quadratic monomials $\mathcal{M}_{(i,\alpha),(j,\beta)}(\{x\})(\varphi) = \nabla^\alpha \varphi_i(x)\, \nabla^\beta \varphi_j(x) \quad$ for $|\alpha| = |\beta| = 1$.

We introduced the corresponding index sets (recall that $\mathcal{U} = \{e_1, \dots, e_d\} \simeq \{1, \dots, d\}$)

$$\mathfrak{v}_0 := \{\emptyset\}, \quad \mathfrak{v}_1 := \{(i, \alpha) : 1 \le i \le m,\, \alpha \in \mathbb{N}_0^{\mathcal{U}}, 1 \le |\alpha| \le \lfloor d/2 \rfloor + 1\}, \tag{4.6.30}$$

$$\mathfrak{v}_2 := \{\big((i, \alpha), (j, \beta)\big) : 1 \le i, j \le m,\, \alpha, \beta \in \mathbb{N}_0^{\mathcal{U}},\, |\alpha| = |\beta| = 1,\, (i, \alpha) \le (j, \beta)\}. \tag{4.6.31}$$

and $\mathfrak{v} = \mathfrak{v}_0 \cup \mathfrak{v}_1 \cup \mathfrak{v}_2$. Here $(i, \alpha) \le (j, \beta)$ refers to any ordering on $\{1, \dots, m\} \times \{e_1, \dots, e_d\}$, e.g. lexicographic. We use ordered indices to avoid double counting since $\mathcal{M}_{(i,\alpha),(j,\beta)}(\{x\})(\varphi) = \mathcal{M}_{(j,\beta),(i,\alpha)}(\{x\})(\varphi)$.

In the following we will always consider levels k with

$$0 \le k \le N - 1. \tag{4.6.32}$$

For a k-block B and $m \in \mathfrak{v}$ we define

$$\mathcal{M}_m(B) = \sum_{x \in B} \mathcal{M}_m(\{x\}). \tag{4.6.33}$$

We denote by $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2$ the space of relevant Hamiltonians, with

$$\mathcal{V}_0 = \mathbb{R}, \quad \mathcal{V}_1 = \operatorname{span}\{M_m(B) : m \in \mathfrak{v}_1\}, \quad \mathcal{V}_2 = \operatorname{span}\{M_m(B) : m \in \mathfrak{v}_2\}. \tag{4.6.34}$$

Given a local functional $K(B)$ we want to extract a 'relevant' part $H = \Pi_2 K(B) \in \mathcal{V}$ in such a way that the functional $K(B) - \Pi_2 K(B)$ measured in the next scale norm $\| \cdot \|_{k+1,B}$ is much smaller than $K(B)$ measured in the $\| \cdot \|_{k,B}$ norm, see Lemma 4.6.9 below. This is not true without extraction as can be seen by considering the constant functional. In fact we need to gain a factor which is small compared to L^{-d} (to compensate the effect of reblocking which combines L^d blocks on the scale k to a single block on the scale $k+1$) and for this we need to extract exactly the elements of $\mathcal{V}$.

We will show that $H = \Pi_2 K(B)$ can be characterised as follows. Let $K^{(0)} + K^{(1)} + K^{(2)}$ denote the second order Taylor polynomial of K at 0 written as a sum of the constant, linear and quadratic part. We will show that there exist unique $H^{(i)} \in \mathcal{V}_i$ such that

$$H^{(0)} = K^{(0)}; \tag{4.6.35}$$

$$H^{(1)}(\varphi) = K^{(1)}(\varphi) \quad \text{for all } \varphi \text{ such that } \varphi\restriction_{B^+} \text{ is a polynomial of degree} \le \lfloor d/2 \rfloor + 1; \tag{4.6.36}$$

$$H^{(2)}(\varphi) = K^{(2)}(\varphi) \quad \text{for all } \varphi \text{ such that } \varphi\restriction_{B^+} \text{ is a linear map.} \tag{4.6.37}$$

Here the large set neighbourhood B^+ was defined in (4.4.33). We then define $H = \Pi_2 K$ by $H = H^{(0)} + H^{(1)} + H^{(2)}$.

We can write this in a more concise notation by using the dual pairing $\langle K, g \rangle_0$ introduced in (4.4.71) and (4.4.83). Before we do so we note that both $H(\varphi)$ and $K(B)(\varphi)$ depend only on values of the field on the set B^+ if $L \ge 2^d + R$ (see Section 4.4.2).

Since $k \le N - 1$ the enlarged block B^{++} does not wrap around the torus T_N for $L \ge 7$ and we can view B^{++} as a subset of $\mathbb{Z}^d$ rather than of T_N. Note that $\nabla^\alpha \varphi_i(x)$ for $|\alpha| \le p_\Phi$ and $x \in B^*$ only depends on $\varphi\restriction_{B^{++}}$ for $L \ge 2^d + R$ since by (4.4.36) $B^* \subset B^+$.

We will thus consider in this subsection the space of fields

$$\mathcal{X} = (\mathbb{R}^m)^{B^{++}} / N_{k,B} \tag{4.6.38}$$

equipped with the norm

$$|\varphi|_{k,B} = \frac{1}{h_k} \sup_{x\in B^*} \sup_{1\le|\alpha|\le p_\Phi} \sup_{1\le i\le m} L^{k|\alpha|} L^{k\frac{d-2}{2}} |\nabla^\alpha \varphi_i(x)| \tag{4.6.39}$$

where

$$N_{k,B} = \{\varphi \in (\mathbb{R}^m)^{B^+} : |\varphi|_{k,B} = 0\}. \tag{4.6.40}$$

Note that $N_{k,B}$ contains in particular the constant functions.

Polynomials on $\mathbb{Z}^d$. We introduce a convenient basis for polynomials on $\mathbb{Z}^d$ as follows. For $t \in \mathbb{Z}$ and $k \in \mathbb{N}$ we define the polynomial

$$t \mapsto \binom{t}{k} := \frac{t(t-1)\dots(t-k+1)}{k!} \tag{4.6.41}$$

and we extend this by $\binom{t}{0} = 1$ and $\binom{t}{k} = 0$ if $k \in \mathbb{Z}\setminus\mathbb{N}_0$. Then $\nabla\binom{t}{k} = \binom{t}{k-1}$ where ∇ denotes the one dimensional forward difference operator. For a multiindex $\alpha \in \mathbb{N}_0^{\{1,\dots,d\}}$ and $z \in \mathbb{Z}^d$ define

$$b_\alpha(z) = \binom{z_1}{\alpha_1}\dots\binom{z_d}{\alpha_d}. \tag{4.6.42}$$

Then

$$\nabla^\beta b_\alpha = b_{\alpha-\beta}. \tag{4.6.43}$$

This relation leads to a natural duality between monomials in ∇ and polynomials on $\mathbb{Z}^d$. Finally we set

$$b_{(i,\alpha)}(z) = b_\alpha(z) e_i, \tag{4.6.44}$$

where $e_1, \dots e_m$ is the standard basis of $\mathbb{R}^m$, and

$$b_m = b_{i,\alpha} \otimes b_{j,\beta} \quad \text{for } m = ((i,\alpha),(j,\beta)). \tag{4.6.45}$$

We also define the normalised symmetrised tensor products

$$f_m = N_m b_m = N_m \frac{1}{2}\big(b_{i,\alpha}\otimes b_{j,\beta} + b_{j,\beta}\otimes b_{i,\alpha}\big) \quad \text{for } m = ((i,\alpha),(j,\beta)). \tag{4.6.46}$$

where

$$N_{(i,\alpha),(j,\beta)} = \begin{cases} 1 & \text{if } (i,\alpha) = (j,\beta), \\ 2 & \text{if } (i,\alpha) \neq (j,\beta). \end{cases} \tag{4.6.47}$$

This agrees with the much more general definition $N_m = \frac{|\vec{\Sigma}(m)|}{|\vec{\Sigma}_0(m)|}$. in (3.9) of [45]. There $\vec{\Sigma}(m)$ denotes the group of permutation that fix the species and $\vec{\Sigma}_0$ is the subgroup that fixes $m = (m_1, m_2)$. In our case there is only one species so that $\vec{\Sigma}(m)$ is simply the group of permutations of two elements and $\vec{\Sigma}_0(m) = \vec{\Sigma}(m)$ if $m_1 = m_2$ and $\vec{\Sigma}_0 = \{\text{id}\}$ otherwise.

We now define the subspaces $\mathcal{P}_k \subset \mathcal{X}^{\otimes k}$ of (equivalence classes of) functions by

$$\mathcal{P}_0 := \mathbb{R}, \quad \mathcal{P}_1 = \text{span}\{b_{(i,\alpha)} : (i,\alpha) \in \mathfrak{v}_1\}, \quad \mathcal{P}_2 := \text{span}\{f_m : m \in \mathfrak{v}_2\}. \tag{4.6.48}$$

and we set $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2$.

Definition and properties of the projection Π_2.

Lemma 4.6.5. *Let $K \in M(\mathcal{P}_k^c)$ and let B be a k-block. Then there exist one and only one $H \in \mathcal{V}$ such that*

$$\langle H, g\rangle_0 = \langle K(B), g\rangle_0 \quad \forall g \in \mathcal{P}. \tag{4.6.49}$$

We remark in passing that (4.6.49) is equivalent to (4.6.35)–(4.6.37). For $H^{(0)}$ we simply evaluate at $\varphi = 0$, for $H^{(1)}$ we use test functions φ such that φ_{B^+} is a polynomial of degree $\lfloor d/2 \rfloor + 1$. For $H^{(2)}$ the implication (4.6.49) $\Longrightarrow$ (4.6.37) follows by taking $g = \varphi \otimes \varphi$ for a linear function φ. For the converse implication one can use polarisation, i.e., the identity $\frac{d}{ds}\frac{d}{dt}_{|s=t=0}(H^{(2)} - K(B))(sb_{i,\alpha} + tb_{j,\beta}) = 0$.

Definition 4.6.6. *We define $\Pi_2 K(B) = H$ where H is given by Lemma 4.6.5.*

We now state the main properties of Π_2: the maps Π_2 is bounded on a fixed scale and $1 - \Pi_2$ is a contraction under change of scale.

Recall that on relevant Hamiltonians $H = \sum_{m \in v} a_m \mathcal{M}_m(B)$ we defined in (4.4.93) the norm

$$\|H\|_{k,0} = L^{kd}|a_\emptyset| + \sum_{(i,\alpha)\in\mathfrak{v}_1} h_k L^{kd} L^{-k\frac{d-2}{2}} L^{-k|\alpha|}|a_{i,\alpha}| + \sum_{m\in\mathfrak{v}_2} h_k^2 |a_m|. \tag{4.6.50}$$

Lemma 4.6.7 (Boundedness of Π_2)**.** *There exists a constant C such that for $L \geq 2^d + R$ and $0 \leq k \leq N-1$*

$$\|\Pi_2 K(B)\|_{k,0} \leq C|K(B)|_{k,B,T_0}. \tag{4.6.51}$$

Since $\Pi_2 H = H$ for $H \in \mathcal{V}$, Lemma 4.6.7 shows in particular that $\|H\|_{k,0} \leq C|H|_{k,T_0} \leq C|||H|||_{k,B}$. We can also prove the converse estimate, in fact a slightly stronger result which will be useful to bound e^H (see Lemma 4.7.3 below). Define

$$|\varphi|_{k,\ell_2(B)}^2 := \frac{1}{h_k^2}\sup_{(i,\alpha)\in\mathfrak{v}_1} \frac{1}{L^{kd}} \sum_{x\in B} L^{2k|\alpha|} L^{k(d-2)} |\nabla^\alpha \varphi_i(x)|^2 = \frac{1}{h_k^2}\sup_{(i,\alpha)\in\mathfrak{v}_1} \sum_{x\in B} L^{2k(|\alpha|-1)} |\nabla^\alpha \varphi_i(x)|^2. \tag{4.6.52}$$

Then it follows directly from the definition of $|\varphi|_{k,B}$ in (4.4.74) that

$$|\varphi|_{k,\ell_2(B)} \leq |\varphi|_{k,B}. \tag{4.6.53}$$

Lemma 4.6.8. *For $H \in M_0(\mathcal{B}_k)$, $L \geq 3$, and $0 \leq k \leq N$ we have*

$$|H|_{T_\varphi} \leq \left(1 + |\varphi|_{k,\ell_2(B)}\right)^2 \|H\|_{k,0} \leq 2(1 + |\varphi|_{k,\ell_2(B)}^2)\, \|H\|_{k,0} \tag{4.6.54}$$

and in particular

$$|||H|||_{k,B} \leq 4\|H\|_{k,0}. \tag{4.6.55}$$

Lemma 4.6.9 (Contraction estimate)**.** *There exists a constant C such that for all $L \geq 2^d + R$*

$$|(1 - \Pi_2)K(B)|_{k+1,B,T_0} \leq C L^{-(d/2 + \lfloor d/2 \rfloor + 1)} |K(B)|_{k,B,T_0}. \tag{4.6.56}$$

Proofs.

Proof of Lemma 4.6.5 (existence and uniqueness of Π_2). Clearly $H^{(0)} = K^{(0)} = K(0)$.

Step 1: There exist one and only one $H^{(2)} \in \mathcal{V}_2$ such that

$$\langle H^{(2)}, g\rangle_0 = \langle K(B), g\rangle_0 \quad \forall g \in \mathcal{P}_2. \tag{4.6.57}$$

Indeed each $H^{(2)} \in \mathcal{V}_2$ is of the form $H^{(2)} = \sum_{m\in\mathfrak{v}_2} a_m \mathcal{M}_m$. Now $\mathcal{M}_{(i,\alpha),(j,\beta)}(B)$ defines a unique symmetric element of $(\mathcal{X}\otimes\mathcal{X})'$ via (see Lemma 4.A.1)

$$\langle \mathcal{M}_{(i,\alpha),(j,\beta)}(B), \varphi\otimes\psi\rangle = \frac{1}{2}\sum_{x\in B} \nabla^\alpha\varphi_i(x)\nabla^\beta\psi_j(x) + \nabla^\beta\varphi_j(x)\nabla^\alpha\psi_i(x). \tag{4.6.58}$$

Thus in view of (4.6.43), (4.6.46) and (4.6.47) we get

$$\langle \mathcal{M}_m(B), f_{m'}\rangle_0 = L^{kd}\delta_{mm'} \quad \forall m, m' \in \mathfrak{v}_2. \tag{4.6.59}$$

It follows that there is one and only one $H^{(2)}$ which satisfies (4.6.57) and the coefficients are given by

$$a_m = L^{-dk}\langle K(B), f_m\rangle_0 = L^{-kd}\langle K^{(2)}, f_m\rangle_0 \quad \forall m \in \mathfrak{v}_2. \tag{4.6.60}$$

Step 2: There exist one and only one $H^{(1)} \in \mathcal{V}_1$ such that

$$\langle H^{(1)}, \varphi\rangle_0 = \langle K(B), \varphi\rangle_0 \quad \forall \varphi \in \mathcal{P}_1. \tag{4.6.61}$$

Writing $H^{(1)} = \sum_{(i,\alpha)\in\mathfrak{v}_1} a_{i,\alpha}\,\mathcal{M}_{i,\alpha}(B)$ and testing against the basis $\{b_{i',\alpha'} : (i',\alpha') \in \mathfrak{v}_1\}$ of $\mathcal{P}_1$ we see that the condition for $H^{(1)}$ is equivalent to

$$\sum_{m\in\mathfrak{v}_1} B_{m'm}\, a_m = \langle K(B), b_{m'}\rangle_0 \quad \forall m' \in \mathfrak{v}_1 \tag{4.6.62}$$

where

$$B_{m'm} = \sum_{x\in B} \nabla^\alpha b_{\alpha'}(x)\,\delta_{ii'} = \sum_{x\in B} b_{\alpha'-\alpha}(x)\,\delta_{ii'} \quad \text{for } m = (i,\alpha),\ m' = (i',\alpha'). \tag{4.6.63}$$

In particular

$$B_{mm} = L^{dk} \qquad \text{and} \qquad B_{m'm} = 0 \quad \text{if } |\alpha| > |\alpha'|. \tag{4.6.64}$$

Thus if we order the indices (i,α) in such a way that $(i,\alpha) < (i,\alpha')$ if $|\alpha| < |\alpha'|$ then B is a triangular matrix with entries L^{dk} on the diagonal. Therefore B is invertible and hence the coefficients of $H^{(1)}$ are uniquely determined. □

Proof of Lemma 4.6.7 (boundedness of Π_2). We have

$$L^{kd}|a_\emptyset| = |H^{(0)}| = |K(0)|. \tag{4.6.65}$$

Since $L \geq 2^d + R$ we can again view B^{++} as a subset of $\mathbb{Z}^d$. Moreover, since the space of polynomials of a certain degree is invariant by translation we assume without loss of generality that $0 \in B$. This implies that

$$|b_{i,\alpha}|_{k,B} = \frac{1}{h_k}L^{k\frac{d}{2}} \quad \text{if } |\alpha| = 1 \quad \text{and thus} \quad |f_m|_{k,B} \leq 2\frac{1}{h_k^2}L^{kd} \quad \forall m \in \mathfrak{v}_2. \tag{4.6.66}$$

Then (4.6.60) implies that

$$|a_m| \le L^{-dk} |K^{(2)}|_{k,B,T_0} \, |f_m|_{k,B} \le 2\frac{1}{h_k^2} |K^{(2)}|_{k,B,T_0}$$

and therefore

$$\sum_{m\in\mathfrak{v}_2} h_k^2 |a_m| \le 2\,\#\mathfrak{v}_2\, |K^{(2)}|_{k,B,T_0}. \tag{4.6.67}$$

To estimate the coefficients of $H^{(1)}$ we note that the system (4.6.63) for the coefficients $a_{(i,\alpha)}$ decouples for different i since $B_{(i,\alpha)(i',\alpha')} = C_{\alpha\alpha'}\delta_{ii'}$. Hence it is sufficient to prove the estimate for the scalar case $m = 1$. It convenient to work in a rescaled basis. Using again that $0 \in B$ we get for $|\alpha'| \ge |\alpha|$

$$\sup_{x\in B^*} |\nabla^\alpha b_{\alpha'}(x)| = \sup_{x\in B^*} |b_{\alpha'-\alpha}(x)| \le (\mathrm{diam}_\infty B^*)^{|\alpha'|-|\alpha|} \tag{4.6.68}$$

and the left hand side vanishes for $|\alpha'| < |\alpha|$. Thus

$$|b_{\alpha'}|_{k,B} \le \sup_{1\le|\alpha|\le|\alpha'|} \frac{1}{h_k} L^{k\frac{d-2}{2}} L^{k|\alpha|} (\mathrm{diam}_\infty B^*)^{|\alpha'|-|\alpha|} \le C' \frac{1}{h_k} L^{k\frac{d-2}{2}} L^{k|\alpha'|} \tag{4.6.69}$$

where

$$C' := (L^{-k}\mathrm{diam}_\infty B^*)^{p_\Phi - 1} = (L^{-k}\mathrm{diam}_\infty B^*)^{\lfloor d/2\rfloor + 1} \tag{4.6.70}$$

depends only on d and R (the dependence from R arises from the fact that for $k = 0$ we have $\mathrm{diam}_\infty B^* = 2R + 1$).

Now we use the basis of test functions given by

$$\widetilde{b}_{\alpha'} = h_k L^{-k\frac{d-2}{2}} L^{-k|\alpha'|} b_{\alpha'}. \tag{4.6.71}$$

Then

$$|\widetilde{b}_{\alpha'}|_{k,B} \le C'. \tag{4.6.72}$$

We define rescaled coefficients

$$\widetilde{a}_\alpha = h_k L^{dk} L^{-k\frac{d-2}{2}} L^{-k|\alpha|} a_\alpha \tag{4.6.73}$$

In these new quantities (4.6.62) can be rewritten as

$$\sum_{\alpha\in\mathfrak{v}_1} A_{\alpha'\alpha}\, \widetilde{a}_\alpha = \langle K, \widetilde{b}_{\alpha'}\rangle. \tag{4.6.74}$$

with

$$A_{\alpha',\alpha} = h_k^{-1} L^{-dk} L^{k\frac{d-2}{d}} L^{k|\alpha|} \quad h_k L^{-k\frac{d-2}{2}} L^{-k|\alpha'|} \quad B_{\alpha'\alpha} \underset{(4.6.63)}{=} L^{-dk} L^{k(|\alpha|-|\alpha'|)} \sum_{x\in B} b_{\alpha'-\alpha}(x).$$

Hence

$$A_{\alpha'\alpha} = \delta_{\alpha'\alpha} \quad \text{if } |\alpha'| = |\alpha|, \quad |A_{\alpha'\alpha}| \le \frac{1}{(\alpha'-\alpha)!} \quad \text{if } \alpha' - \alpha \in \mathbb{N}_0^{\{1,\dots,d\}} \setminus \{0\} \tag{4.6.75}$$

and $A_{\alpha'\alpha} = 0$ if $\alpha'_i < \alpha_i$ for some $i \in \{1, \dots, d\}$. This implies that $(A - \mathbb{1})^{\lfloor d/2 \rfloor + 1} = 0$. Indeed, let $V_\ell := \operatorname{span}(e_\alpha : |\alpha| \le \ell)$. Then $A^T - \mathbb{1}$ acts on $V_{\lfloor d/2 \rfloor+1}$ and we have $(A - \mathbb{1})^T V_l \subset V_{l_1}$ and $(A^T - \mathbb{1})V_1 = \{0\}$. Thus

$$A^{-1} = (\mathbb{1} + (A - \mathbb{1}))^{-1} = \mathbb{1} + \sum_{r=1}^{\lfloor d/2 \rfloor} (A - \mathbb{1})^r. \tag{4.6.76}$$

Since the matrix elements of $A - \mathbb{1}$ are bounded this implies that

$$|\tilde{a}_\alpha| \le C \sup_{\alpha' \in \mathfrak{v}_1} \langle K(B), \tilde{b}_{\alpha'} \rangle_0 \underset{(4.6.72)}{\le} CC' |K^{(1)}(B)|_{k,B,T_0}. \tag{4.6.77}$$

Here C is a combinatorial constant which depends only on the dimension d. Thus

$$\|H^{(1)}\|_{k,0} = \sum_{\alpha \in \mathfrak{v}_1} |\tilde{a}_\alpha| \le CC' \, \#\mathfrak{v}_1 \, |K^{(1)}(B)|_{k,B,T_0} \tag{4.6.78}$$

in the scalar case $m = 1$. For $m > 1$ the equations for the different components i decouple and thus the estimate holds with an additional factor m. Combining this with (4.6.65) and (4.6.67) we get $\|H\|_{k,0} \le C \sum_{r=0}^{2} |K^{(r)}(B)|_{k,B,T_0} \le C|K(B)|_{k,B,T_0}$. □

Proof of Lemma 4.6.8. The assertion (4.6.55) follows from (4.6.54), the definition of the strong norm in (4.4.86) and (4.5.11) as well as the estimates $|\varphi|^2_{k,\ell_2(B)} \le \boldsymbol{G}_k^B(\varphi)$ and $(1 + t) \le 2e^{t/2}$ for $t \ge 0$.

To prove (4.6.54) we use that $|M_\emptyset(\{x\})|_{k,T_0} = 1$ and that by (4.A.74) and (4.A.70) we have

$$|\mathcal{M}_{i,\alpha}(\{x\})|_{k,B,T_0} \le h_k L^{-k|\alpha|} L^{-k\frac{d-2}{2}} \quad \text{and} \quad |\mathcal{M}_m(\{x\})|_{k,B,T_0} \le h_k^2 L^{-kd} \quad \forall m \in \mathfrak{v}_2. \tag{4.6.79}$$

Now for $\varphi = 0$ the estimate (4.6.54) follows directly by summing (4.6.79) over $x \in B$. For $\varphi \ne 0$ we use that for the decomposition of $H = H_0 + H_1 + H_2$ in constant, linear and quadratic terms we get

$$\operatorname{Tay}_\varphi H = \operatorname{Tay}_0 H + (H_1(\varphi) + H_2(\varphi)) + L_\varphi \tag{4.6.80}$$

where $H_1(\varphi) + H_2(\varphi)$ is a constant term and L_φ is the linear functional defined by $L_\varphi(\psi) = 2\overline{H}_2(\varphi \otimes \psi)$ or explicitly by

$$L_\varphi(\psi) = \sum_{x \in B} \sum_{(i,\alpha) \le (j,\beta), |\alpha| = |\beta| = 1} a_{(i,\alpha),(j,\beta)} \big(\nabla^\alpha \varphi_i(x) \nabla^\beta \psi_j(x) + \nabla^\beta \varphi_j(x) \nabla^\alpha \psi_i(x)\big). \tag{4.6.81}$$

Since $\nabla^\alpha \psi_i(x) = \mathcal{M}_{i,\alpha}(\{x\})(\psi)$ we get from (4.6.79) (with $|\alpha| = 1$) and the Cauchy-Schwarz inequality for $\sum_{x \in B}$

$$\begin{aligned} |L_\varphi|_{k,T_0} \le & \sum_{x \in B} \sum_{(i,\alpha) \le (j,\beta), |\alpha| = |\beta| = 1} |a_{(i,\alpha),(j,\beta)}| \big(|\nabla^\alpha \varphi_i(x)| + |\nabla^\beta \varphi_j(x)|\big) \; h_k L^{-k\frac{d}{2}} \\ \le & 2 \sup_{(i,\alpha), |\alpha| = 1} \frac{1}{h_k} \Big(\sum_{x \in B} |\nabla^\alpha \varphi_i(x)|^2 \Big)^{1/2} \sum_{m \in \mathfrak{v}_2} h_k^2 |a_m| \end{aligned} \tag{4.6.82}$$

It follows directly from the definition of H_2 and the inequality $|ab| \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ applied to $\nabla^\alpha \varphi_i \nabla^\beta \varphi_j$ that

$$|H_2(\varphi)| \le \sup_{(i,\alpha), |\alpha| = 1} \frac{1}{h_k^2} \Big(\sum_{x \in B} |\nabla^\alpha \varphi_i(x)|^2 \Big) \sum_{m \in \mathfrak{v}_2} h_k^2 |a_m|. \tag{4.6.83}$$

Finally the Cauchy-Schwarz inequality for $\sum_{x\in B}$ gives

$$|H_1(\varphi)| \le \sup_{(i,\alpha)\in\mathfrak{v}_1} \frac{1}{h_k}\Big(\sum_{x\in B} L^{2k(|\alpha|-1)}\,|\nabla^\alpha\varphi_i(x)|^2\Big)^{1/2} \sum_{(i,\alpha)\in\mathfrak{v}_1} h_k L^{k\frac{d}{2}} L^{-k(|\alpha|-1)}\,|a_{(i,\alpha)}| \qquad (4.6.84)$$

Now (4.6.82)–(4.6.84) imply that

$$|\mathrm{Tay}_\varphi H - \mathrm{Tay}_0 H|_{k,T_0} \le \big(2\|\varphi\|_{k,\ell_2(B)} + \|\varphi\|^2_{k,\ell_2(B)}\big)\,\|H\|_{k,0}. \qquad (4.6.85)$$

Together with the estimate for $\varphi = 0$, i.e., $|H|_{k,T_0} \le \|H\|_{k,0}$, this concludes the proof. □

Proof of Lemma 4.6.9 (contraction estimate). This will easily follow from a duality argument given below and the following result. □

Lemma 4.6.10. *There exists a constant C such that for all $L \ge 2^d + R$*

$$\min_{P\in\mathcal{P}_1} |\varphi - P|_{k,B} \le CL^{-(d/2+\lfloor d/2\rfloor+1)}|\varphi|_{k+1,B} \quad \forall\varphi\in\mathcal{X} \qquad (4.6.86)$$

and

$$\min_{P\in\mathcal{P}_2} |Sg - P|_{k,B} \le CL^{-(d+1)}|g|_{k+1,B} \quad \forall g\in\mathcal{X}\otimes\mathcal{X}. \qquad (4.6.87)$$

Here S is the symmetrisation operator, defined by $S(\varphi\otimes\psi) = \frac{1}{2}(\varphi\otimes\psi) + \frac{1}{2}(\psi\otimes\varphi)$ and linear extension.

Proof. Since $L \ge 2^d + R$ we can view B^{++} as a subset of $\mathbb{Z}^d$. We first show (4.6.86). It suffices to consider the scalar case $m = 1$ since the estimate can be done component by component. The small set neighbourhood B^* can be written as

$$B^* = a + [0,\rho]^d \quad \text{with} \quad L^{-k}\rho \le C \qquad (4.6.88)$$

where $C = \max(2R+1, 3)$. We will apply Lemma 4.B.1 for the estimate of the remainder term in the Taylor expansion with

$$s := \lfloor d/2\rfloor + 1 = p_\Phi - 1 \qquad (4.6.89)$$

and

$$M_s := M_{s,\rho} = \sup\{|\nabla^\alpha\varphi(x)| : |\alpha| = s+1,\ x\in\mathbb{Z}^d\cap\big(a+[0,\rho]^d\big)\}. \qquad (4.6.90)$$

Then it follows from the definition of the field norm $|\varphi|_{k+1,B}$ that

$$M_s \le h_{k+1}L^{-(k+1)(s+1)}L^{-(k+1)\frac{d-2}{2}}\,|\varphi|_{k+1,B}. \qquad (4.6.91)$$

Let $P = \mathrm{Tay}^s_a\,\varphi$ be the discrete Taylor polynomial of order s of φ at a. Then by Lemma 4.B.1 we have for $t = |\beta| \le s$ and all $x\in\mathbb{Z}^d\cap\big(a+[0,\rho]^d\big)$

$$\Big|\nabla^\beta[\varphi(x) - P(x)]\Big| \le M_s\binom{|x-a|_1}{s-t+1} \le M_s(d\rho)^{s+1-t} \le M_s C L^{k(s+1-t)}. \qquad (4.6.92)$$

Here C depends only on d and R and we used that $|x-a|_1 \le d\rho$. Taking into account that for $|\beta| = s+1$ we have $\nabla^\beta(\varphi - P) = \nabla^\beta \varphi$ and using (4.6.92), the definition of M_s and the fact that $h_{k+1}/h_k = 2$ we get

$$\begin{aligned} |\varphi - P|_{k,B} \le C\frac{1}{h_k} L^{k\frac{d-2}{2}} L^{k(s+1)} M_s \underset{(4.6.91)}{\le} & CL^{-\frac{d-2}{2}} L^{-(s+1)} |\varphi|_{k+1,B} \\ &= CL^{-(s+\frac{d}{2})}|\varphi|_{k+1,B}. \end{aligned} \tag{4.6.93}$$

This finishes the proof of (4.6.86).

The proof of the second estimate is similar. We consider the space

$$\widetilde{\mathcal{P}}_2 := \operatorname{span}\{b_{i,\alpha} \otimes b_{j,\beta} : |\alpha| = |\beta| = 1\}$$

Thus $\widetilde{\mathcal{P}}_2$ is the non symmetrised counterpart of $\mathcal{P}_2$. In particular $S\widetilde{\mathcal{P}}_2 = \mathcal{P}_2$ where S is the symmetrisation operator. For $\widetilde{P} \in \widetilde{\mathcal{P}}_2$ and $|\alpha| + |\beta| \ge 3$ we have $(\nabla^{i,\alpha} \otimes \nabla^{j,\beta})\widetilde{P} = 0$. Using again that $h_{k+1}/h_k = 2$ we deduce that

$$\begin{aligned} & h_k^{-2} L^{k(|\alpha|+|\beta|)} L^{k(d-2)} |(\nabla^{i,\alpha} \otimes \nabla^{j,\beta})(g - \widetilde{P})(x,y)| \\ & \le 4L^{-(|\alpha|+|\beta|+d-2)} |g|_{k+1,B} \le 4L^{-(d+1)}|g|_{k+1,B} \qquad \text{if } |\alpha| + |\beta| \ge 3. \end{aligned} \tag{4.6.94}$$

To prove (4.6.87) it only remains to estimate $\nabla^{i,\alpha} \otimes \nabla^{j,\beta}(g - \widetilde{P})$ for $|\alpha| = |\beta| = 1$. We define $\widetilde{P} \in \widetilde{\mathcal{P}}_2$ by

$$\widetilde{P} = \sum_{(i',\alpha'),(j',\beta'),|\alpha'|=|\beta'|=1} (\nabla^{i',\alpha'} \otimes \nabla^{j',\beta'} g)(a,a) \;\; b_{i',\alpha'} \otimes b_{j',\beta'}. \tag{4.6.95}$$

Then $\nabla^{i,\alpha} \otimes \nabla^{j,\beta}\widetilde{P} = \text{const} = (\nabla^{i,\alpha} \otimes \nabla^{j,\beta} g)(a,a)$ for $|\alpha| = |\beta| = 1$.

We now define

$$M_2 := \sup\{|(\nabla^{i,\alpha} \otimes \nabla^{j,\beta} g)(x,y)| : |\alpha| \ge 1,\ |\beta| \ge 1,\ |\alpha| + |\beta| = 3,\ x,y \in a + [0,\rho]^d\}. \tag{4.6.96}$$

Then

$$M_2 \le h_{k+1}^2 L^{-3(k+1)} L^{-(k+1)(d-2)} |g|_{k+1,B} \tag{4.6.97}$$

We claim that for

$$\begin{aligned} |(\nabla^{i,\alpha} \otimes \nabla^{j,\beta} g)(x,y) - \underbrace{(\nabla^{i,\alpha} \otimes \nabla^{j,\beta} g)(a,a)}_{=\nabla^{i,\alpha} \otimes \nabla^{j,\beta}\widetilde{P}}| & \le M_2(|x-a|_1 + |y-a|_1) \\ & \le 2d\rho M_2 \quad \text{for } |\alpha| = |\beta| = 1. \end{aligned} \tag{4.6.98}$$

This estimate is a special case of the Taylor remainder estimate in Lemma 3.5. of [45], but it can also be easily verified as follows. For $h : \mathbb{R}^{B^{++}} \times \mathbb{R}^{B^{++}} \to \mathbb{R}$ the difference $h(x,y) - h(a,a)$ can be estimated in $B^* \times B^*$ by the maximum of the first order forward derivatives of h in B^* times $|x-a|_1 + |y-a|_1$. Now apply this with $h = \nabla^{i,\alpha} \otimes \nabla^{j,\beta} g$.

Since $\rho \le CL^k$ the estimates (4.6.98), (4.6.97), and (4.6.94) jointly imply that $|g - \widetilde{P}|_{k,B} \le CL^{-(d+1)}|g|_{k+1,B}$. Application of the symmetrisation operator S does not increase the norm (see Lemma 4.A.5) and thus $|Sg - S\widetilde{P}|_{k,B} \le CL^{-(d+1)}$. Since $P := S\widetilde{P} \subset \mathcal{P}_2$ the assertion (4.6.87) follows. □

Proof of Lemma 4.6.9 (continued). It follows from the definition of the norm $|g|_{j,B}$ for $j \in \{k, k+1\}$ and $g \in \mathcal{X}^{\otimes r}$ in (4.4.77) and the fact that $h_{k+1}/h_k = 2$ that

$$|g|_{k,B} \leq 8L^{-\frac{3}{2}d}|g|_{k+1,B} \quad \forall g \in \mathcal{X}^{\otimes r} \quad \forall r \geq 3. \tag{4.6.99}$$

Since $\Pi_2 K(B)$ depends only on the second order Taylor polynomial of K we get the estimate

$$\begin{aligned} |\langle(\mathbb{1} - \Pi_2)K(B), g\rangle_0| &= |\langle K(B), g\rangle_0| \leq |K(B)|_{k,B,T_0}\, |g|_{k,B} \\ &\leq 8L^{-\frac{3}{2}d}|K(B)|_{k,B,T_0}\, |g|_{k+1,B} \qquad \forall g \in \mathcal{X}^{\otimes r} \quad \forall r \geq 3. \end{aligned} \tag{4.6.100}$$

Now for $\varphi \in \mathcal{X}$ we have by the definition of Π_2, the boundedness of Π_2 and Lemma 4.6.10

$$\begin{aligned} |\langle(\mathbb{1} - \Pi_2)K(B), \varphi\rangle_0| &= \min_{P\in\mathcal{P}_1} |\langle(\mathbb{1} - \Pi_2)K(B), \varphi - P\rangle_0| \\ &\leq\ |(\mathbb{1} - \Pi_2)K(B)|_{k,B,T_0} \ \min_{P\in\mathcal{P}_1} |\varphi - P|_{k,B} \\ &\leq\ C\,|K(B)|_{k,B,T_0}\, L^{-(d/2+\lfloor d/2\rfloor+1)}\, |\varphi|_{k+1,B}. \end{aligned} \tag{4.6.101}$$

Since the pairing $\langle(\mathbb{1} - \Pi_2)K(B), g\rangle_0$ depends only on Sg we get similarly for $g \in \mathcal{X} \otimes \mathcal{X}$

$$\begin{aligned} |\langle(\mathbb{1} - \Pi_2)K(B), g\rangle_0| &= \min_{P\in\mathcal{P}_2} |\langle(\mathbb{1} - \Pi_2)K(B), Sg - P\rangle_0| \\ &\leq\ |(\mathbb{1} - \Pi_2)K(B)|_{k,B,T_0} \ \min_{P\in\mathcal{P}_2} |Sg - P|_{k,B} \\ &\leq\ C\,|K(B)|_{k,B,T_0}\, L^{-(d+1)}\, |g|_{k+1,B}. \end{aligned} \tag{4.6.102}$$

The desired assertion follows from (4.6.100)– (4.6.102) and the definition (4.4.84) of the norm $|K(B)|_{k+1,B,T_0}$. □

4.7 Smoothness of the renormalisation map

In this section we prove Theorem 4.4.7. The strategy is to write the renormalisation map $\boldsymbol{S}$ as a composition of simpler maps and to show smoothness for those maps. For this section we fix a scale k. No index will in the following denote quantities on scale k while a prime will denote quantities on the next scale $k+1$.

4.7.1 Decomposition of the renormalisation map

Recall from Section 4.4.2 that the space of functionals $K \in M(\mathcal{P}_k)$ which factorise over connected components can be identified with the space $M(\mathcal{P}_k^c)$ via the map $\iota_2 : M(\mathcal{P}_k^c) \to M(\mathcal{P}_k)$ given by $(\iota_2 K)(X) = \prod_{Y\in\mathcal{C}(X)} K(Y)$. We often do not distinguish between K and $\iota_2 K$. Similarly the space of functionals F which factorise over k-blocks can be identified with the elements of $M(\mathcal{B}_k)$ via $F^X := (\iota_1 F)(X) := \prod_{B\in\mathcal{B}_k(X)} F(B)$.

To simplify the notation we introduce the following abbreviations from [4] for the Banach spaces involved in the decomposition of the map S:

$$\begin{aligned} \boldsymbol{M}^{(A)} &= (M(\mathcal{P}_k^c), \|\cdot\|_k^{(A)}), \\ \boldsymbol{M}'^{(A)} &= (M(\mathcal{P}_{k+1}^c), \|\cdot\|_{k+1}^{(A)}), \\ \boldsymbol{M}_0 &= (M(\mathcal{B}_k), \|\cdot\|_{k,0}), \\ \boldsymbol{M}_{|||} &= (M(\mathcal{B}_k), |||\cdot|||_k), \\ B_\kappa &= \left\{\boldsymbol{q} \in \mathbb{R}_{\text{sym}}^{(d\times m)\times(d\times m)} :\ |\boldsymbol{q}|_{\text{op}} < \kappa\right\}. \end{aligned} \tag{4.7.1}$$

Here it is understood that $\boldsymbol{M}^{(A)}$ consists of those elements of $M(\mathcal{P}_k^c)$ for which the norm $\|\cdot\|_k^{(A)}$ is finite and similarly for the other spaces. The abbreviations $\boldsymbol{M}^{(A)}$ etc. should not be confused with the notation for the quadratic forms that appeared in Section 4.5.

We also need a slight modification of the spaces $\boldsymbol{M}^{(A)}$ because the renormalisation map does not preserve factorisation on scale k, i.e., in general for $K \in M(\mathcal{P}_k^c)$

$$\boldsymbol{R}K(X,\varphi) \neq \prod_{Y\in\mathcal{C}(X)} \boldsymbol{R}K(Y,\varphi) \tag{4.7.2}$$

(here we identified K in $\iota_2 K$). In other word $\boldsymbol{R}K$ cannot be identified with an element of $M(\mathcal{P}_k^c)$. In [4] this problem is solved by the use of the embedding $M(\mathcal{P}^c) \to M(\mathcal{P})$ and the submultiplicativity estimates from Lemma 4.6.3. In the current setting, however, it is not possible to estimate the derivative with respect to $\boldsymbol{q}$ of the renormalisation map $\boldsymbol{R}^{(\boldsymbol{q})}$ on arbitrary polymers (cf. Lemma 4.6.4).

To overcome this difficulty we introduce the space of functionals that live on scale k but factor only on scale $k+1$.

More precisely we use the following definition. Recall the definition of the map $\pi : \mathcal{P}_k \to \mathcal{P}_{k+1}$ in (4.4.52) and (4.4.53).

Definition 4.7.1. *We say that $X \in \mathcal{P}_k \setminus \emptyset$ is a cluster, $X \in \mathcal{P}_k^{\mathrm{cl}}$, if $\pi(X) \in \mathcal{P}_{k+1}^{\mathrm{c}}$. For $X \in \mathcal{P}_k$, $Y \subset X$ is a cluster of X if there is $U \in \mathcal{C}_{k+1}(\pi(X))$ such that*

$$Y = \bigcup_{Z\in\mathcal{C}(X):\pi(Z)\subset U} Z \tag{4.7.3}$$

We use $\mathcal{C}^{\mathrm{cl}}(X)$ to denote the set of all clusters of X.

Lemma 4.7.2. *Assume that $L \geq 2^{d+2} + 4R$. Let $X \in \mathcal{P}_k \setminus \emptyset$. Then*

(i) For any $U \in \mathcal{C}_{k+1}(\pi(X))$, there is a cluster Y of X, $Y \in \mathcal{C}^{\mathrm{cl}}(X)$, such that $\pi(Y) = U$.

(ii) $X = \bigcup_{Y\in\mathcal{C}^{\mathrm{cl}}(X)} Y$;

(iii) Two clusters of $Y_1, Y_2 \in \mathcal{C}^{\mathrm{cl}}(X)$ are either identical or strictly disjoint on scale k;

(iv) $\sum_{Y\in\mathcal{C}^{\mathrm{cl}}(X)} |\mathcal{C}(Y)| = |\mathcal{C}(X)|$;

(v) If $K \in M(\mathcal{P}_k)$ factors over connected components on the scale k then

$$(\boldsymbol{R}_{k+1}K)(X,\varphi) = \prod_{Y\in\mathcal{C}^{\mathrm{cl}}(X)} (\boldsymbol{R}_{k+1}K)(Y,\varphi). \tag{4.7.4}$$

Proof. Let $X \in \mathcal{P}_k$ and $U = \pi(X)$. By definition (4.4.53) of π we have

$$U = \bigcup_{Z\in\mathcal{C}(X)} \pi(Z). \tag{4.7.5}$$

Note first that a component of X cannot be shared between two components of U:

$$Z \in \mathcal{P}_k^{\mathrm{c}} \text{ implies that } \pi(Z) \in \mathcal{P}_{k+1}^{\mathrm{c}}. \tag{4.7.6}$$

Indeed, if $Z \in \mathcal{S}_k$ then $\pi(Z)$ is a single block and hence connected. If $Z \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k$ then $\pi(Z) = \overline{Z}$ and, in particular, $Z \subset \pi(Z)$ and every block $B \in \mathcal{B}_{k+1}(\pi(Z))$ contains at least one point from

Z. For any two points $x, y \in \pi(Z)$ consider $x' \in Z \cap B_x$, where $B_x \in \mathcal{B}_{k+1}(\pi(Z))$ is the block that contains the point x and similarly $y' \in Z \cap B_y$. Given that Z as well as any block are connected, there exist a path joining x with y via x' and y'.

Thus, in view of (4.7.5) and the fact that a connected set cannot be contained in a union of two nonempty disjoint sets, we get

$$Z \in \mathcal{C}(X) \text{ implies that } \pi(Z) \text{ is contained in one component of } U \text{ (on scale } k+1) \tag{4.7.7}$$

For a connected component $U_1 \in \mathcal{C}(U)$ we consider the corresponding cluster Y_1 defined by (4.7.3), i.e.,

$$Y_1 = \bigcup_{Z \in \mathcal{C}(X):\, \pi(Z) \subset U_1} Z. \tag{4.7.8}$$

Then (4.7.5) and (4.7.7) jointly imply $\pi(Y_1) = U_1$ thus proving the first claim. Moreover, (4.7.7) also implies the second claim. To prove the third claim let U_1 and U_2 be two different components of $\pi(X)$. Again by (4.7.7). the corresponding clusters Y_1 and Y_2 defined by (4.7.3) are disjoint. Since Y_1 and Y_2 are unions of k-components of X they must by strictly disjoint on scale k.

The fourth claim follows now from the fact that clusters are union of distinct elements of $\mathcal{C}(X)$.

To prove the last claim, it is sufficient to show that for different components U_1 and U_2 of $U = \pi(X)$ with the corresponding clusters $Y_1 \subset U_1$ and $Y_2 \subset U_2$, the fields $\nabla\xi_{k+1}\restriction_{Y_1^*}$ and $\nabla\xi_{k+1}\restriction_{Y_2^*}$ are independent if ξ_{k+1} is distributed according to μ_{k+1}. Note that by (4.4.54) $Y_i^* \subset U_i{}^*$ and by (4.4.35) $\operatorname{dist}(U_1^*, U_2^*) \geq \frac{L^{k+1}}{2}$ for $L \geq 2^{d+2} + 4R$ which implies the independence of the gradient fields. Therefore we find for any polymer $X \in \mathcal{P}_k^{\mathrm{c}}$ and $K \in M(\mathcal{P}_k^{\mathrm{cl}})$ the identity (4.7.4). □

The space of functionals which factorise over clusters can again be identified with the space $M(\mathcal{P}_k^{\mathrm{cl}})$. Now we need to equip this space with a norm.It turns out that we need norms that involve in addition to the parameter A that regulates the growth depending on the number of blocks another parameter B that regulates the growth depending on the number of connected components of the polymer. For $K \in M(\mathcal{P}_k^{\mathrm{cl}})$ and $A, B > 1$, we define

$$\|K\|_k^{(A,B)} = \sup_{X \in \mathcal{P}_k^{\mathrm{cl}}} A^{|X|_k} B^{|\mathcal{C}(X)|} \|K(X)\|_{k,X}. \tag{4.7.9}$$

We also consider the norm $\|\cdot\|_{k:k+1}^{(A,B)}$ obtained by replacing, in the right hand side above, the norm $\|\cdot\|_{k,X}$ by the norm $\|\cdot\|_{k:k+1,X}$.

Again we introduce abbreviations for the corresponding normed spaces

$$\begin{aligned} \widehat{\boldsymbol{M}}^{(A,B)} &= \{M(\mathcal{P}_k^{\mathrm{cl}}), \|\cdot\|_k^{(A,B)}\}, \\ \widehat{\boldsymbol{M}}_{:}^{(A,B)} &= \{M(\mathcal{P}_k^{\mathrm{cl}}), \|\cdot\|_{k:k+1}^{(A,B)}\}. \end{aligned} \tag{4.7.10}$$

Recall the definition of $K' = \boldsymbol{S}(H, K, \boldsymbol{q})$ in (4.4.67): for $U \in \mathcal{P}'$ we have

$$K'(U, \varphi) = \sum_{X \in \mathcal{P}_k} \chi(X, U) \tilde{I}^{U \setminus X}(\varphi) \tilde{I}^{-X \setminus U}(\varphi) \int_{\mathcal{X}_N} (\tilde{J}(\varphi) \circ P(\varphi + \xi))(X)\, \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi), \tag{4.7.11}$$

where $\tilde{I} = e^{-\tilde{H}}$, $\tilde{J} = 1 - \tilde{I}$, $P = (I-1) \circ K$, $I = e^{-H}$, and $\tilde{H}(B, \varphi) = (\Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})} H)(B, \varphi) - (\Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K)(B, \varphi)$.

Using first the definition of the $\circ$ product and then the factorisation property (4.7.4) (and (4.4.60) for $P_2 = (I-1)\circ K$ to verify its assumption) we get

$$\begin{aligned}K'(U,\varphi) &= \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1\cup X_2,U)\tilde{I}^{U\setminus(X_1\cup X_2)}\tilde{I}^{-X_1\cup X_2\setminus U}(\varphi)\tilde{J}^{X_1}(\varphi)\int_{\mathcal{X}_N} P(X_2,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\\ &= \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1\cup X_2,U)\tilde{I}^{U\setminus(X_1\cup X_2)}\tilde{I}^{-X_1\cup X_2\setminus U}(\varphi)\tilde{J}^{X_1}(\varphi)\prod_{Y\in\mathcal{C}^{\mathrm{cl}}(X_2)} (\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}P)(Y,\varphi)\end{aligned} \tag{4.7.12}$$

It is now easy to see that the map $\boldsymbol{S}$ can be rewritten as a composition of the following maps. The exponential map

$$E : \boldsymbol{M}_0 \to \boldsymbol{M}_{|||}, \quad E(H) = \exp(H), \tag{4.7.13}$$

three polynomial maps

$$\begin{aligned}&P_1 : \boldsymbol{M}_{|||}\times\boldsymbol{M}_{|||}\times\boldsymbol{M}_{|||}\times\widehat{\boldsymbol{M}}_{:}^{(A/(2A_{\mathcal{P}}),B)} \to \boldsymbol{M}'^{(A)},\\ &P_1(I_1,I_2,J,K)(U,\varphi) = \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1\cup X_2,U)I_1^{U\setminus(X_1\cup X_2)}(\varphi)I_2^{(X_1\cup X_2)\setminus U}J^{X_1}(\varphi)\prod_{Y\in\mathcal{C}^{\mathrm{cl}}(X_2)} K(Y,\varphi),\\ &P_2 : \boldsymbol{M}_{|||}\times\boldsymbol{M}^{(A)} \to \boldsymbol{M}^{(A/2)}, \quad P_2(I,K) = (I-1)\circ K,\\ &P_3 : \boldsymbol{M}^{(A/2)} \to \widehat{\boldsymbol{M}}^{(A/2,B)}, \quad P_3K(X,\varphi) = \prod_{Y\in\mathcal{C}(X)} K(Y,\varphi),\end{aligned} \tag{4.7.14}$$

and, finally, two maps which include an integration with respect to $\mu_{k+1}^{(\boldsymbol{q})}$. This is the point where regularity is lost for derivatives in $\boldsymbol{q}$ direction if the original finite range decomposition from [3] is used. These maps are given by

$$\begin{aligned}&R_1 : \widehat{\boldsymbol{M}}^{(A/2,B)}\times B_\kappa \to \widehat{\boldsymbol{M}}_{:}^{(A/(2A_{\mathcal{P}}),B)},\\ &R_1(P,\boldsymbol{q})(X,\varphi) = (\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}P)(X,\varphi) = \int_{\mathcal{X}_N} P(X,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\end{aligned} \tag{4.7.15}$$

and

$$\begin{aligned}&R_2 : \boldsymbol{M}_0\times\boldsymbol{M}^{(A)}\times B_\kappa \to \boldsymbol{M}_0,\\ &R_2(H,K,\boldsymbol{q})(B,\varphi) = \Pi_2\Big((\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}H)(B,\varphi) - (\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}K)(B,\varphi)\Big).\end{aligned} \tag{4.7.16}$$

In terms of these maps the map $\boldsymbol{S}$ can be expressed as

$$\begin{aligned}&\boldsymbol{S}(H,K,\boldsymbol{q}) =\\ &P_1\Big(E\big(-R_2(H,K,\boldsymbol{q})\big), E\big(R_2(H,K,\boldsymbol{q})\big), 1-E\big(-R_2(H,K,\boldsymbol{q})\big), R_1\big(P_3\big(P_2(E(-H),K)\big),\boldsymbol{q}\big)\Big).\end{aligned} \tag{4.7.17}$$

Note that when we insert in the arguments I_1 and I_2 of P_1 we find $I_1 = I_2^{-1}$. Since the inversion is not continuous for the strong norm we have to introduce the two terms as different arguments of P_1. They are, however, equal to $E(H)$ and $E(-H)$ for some H and we clearly have $\|H\|_{k,0} = \|-H\|_{k,0}$.

Compared to [4] the smoothness estimates for R_1 and R_2 change. Actually they become much simpler because the bulk of the work has been done in [54]. The estimate for P_1 changes slightly because of the slight changes in the combinatorics. The proof for the smoothness of E has been simplified. The remaining smoothness estimates are very similar.

To control the polynomial maps P_2 and P_3 we will use the assumptions on L and h in Lemma 4.6.3, i.e.,

$$L \geq 2^{d+3} + 16R, \quad h \geq h_0(L), \tag{4.7.18}$$

where $h_0(L)$ is as in Lemma 4.6.3. For P_1 we need a slightly stronger assumption for L

$$L \geq \max(2^{d+3} + 16R, 4d(2^d + R)), \quad h \geq h_0(L). \tag{4.7.19}$$

For the maps R_1 and R_2 we use the assumption

$$L \geq 2^{d+3} + 16R, \tag{4.7.20}$$

in Lemma 4.6.4. Finally, for the map E we use the weaker condition

$$L \geq 3. \tag{4.7.21}$$

4.7.2 The immersion E

Lemma 4.7.3. *Assume* (4.7.21). *Then the map*

$$E : B_{\frac{1}{8}}(M_0(\mathcal{B}_k), \|\cdot\|_{k,0}) \to (M(\mathcal{B}_k), |||\cdot|||_{k,B}) \qquad \text{defined by } E(H) = e^H \tag{4.7.22}$$

is smooth and the r-th derivative (viewed as a map from $B_{\frac{1}{8}}(M_0(\mathcal{B}_k))$ to the set of r-multilinear forms on $M_0(\mathcal{B}_k)$ with values in $M(\mathcal{B}_k)$) is uniformly bounded. More precisely if we set

$$\|D^r E(H)\| := \sup\left\{ \left|\left|\left| D^r E(H)(\dot{H}_1, \ldots, \dot{H}_r) \right|\right|\right|_{k,B} : \|\dot{H}_i\|_{k,0} \leq 1 \text{ for } i = 1, \ldots, r \right\} \tag{4.7.23}$$

and

$$C_r := 2^r e^{\frac{1}{4}} \max_{t \geq 0} e^{-\frac{t}{4}} (1+t)^r, \tag{4.7.24}$$

then

$$D^r E(H)(\dot{H}_1, \ldots, \dot{H}_r) = e^H \dot{H}_1 \ldots \dot{H}_r \tag{4.7.25}$$

and

$$\|D^r E(H)\| \leq C_r \text{ for any } H \in B_{\frac{1}{8}}(M_0, \|\cdot\|_{k,0}). \tag{4.7.26}$$

Moreover,

$$\left|\left|\left| e^H - 1 \right|\right|\right|_{k,B} \leq 8\|H\|_{k,0} \text{ for any } H \in B_{\frac{1}{8}}(M_0, \|\cdot\|_{k,0}). \tag{4.7.27}$$

Proof. We first recall some notation. In (4.6.52) we defined the (semi)norm on fields

$$|\varphi|^2_{k,\ell_2(B)} = \frac{1}{h_k^2} \sup_{(i,\alpha) \in \mathfrak{v}_1} \sum_{x \in B} L^{2k(|\alpha|-1)} |\nabla^\alpha \varphi_i(x)|^2. \tag{4.7.28}$$

Since (4.7.21) holds we can apply Lemma 4.6.8 guaranteeing that

$$|H|_{k,B,T_\varphi} \le 2(1+|\varphi|^2_{k,\ell_2(B)})\,\|H\|_{k,0} \text{ for all } H \in M_0(\mathcal{B}_k). \tag{4.7.29}$$

The strong norm $|\!|\!|\cdot|\!|\!|_{k,B}$ is defined using the weight $W_k^B = e^{\frac{1}{2}(\varphi,\boldsymbol{G}_k^B\varphi)}$ where

$$(\varphi,\boldsymbol{G}_k^B\varphi) \underset{(4.5.11)}{=} \frac{1}{h_k^2} \sum_{1\le|\alpha|\le\lfloor d/2\rfloor+1} L^{2k(|\alpha|-1)}(\nabla^\alpha\varphi, \mathbb{1}_B\nabla^\alpha\varphi) \ge |\varphi|^2_{k,\ell_2(B)}. \tag{4.7.30}$$

Thus

$$|\!|\!|F|\!|\!|_{k,B} \underset{(4.4.86),(4.5.11)}{=} \sup_\varphi e^{-\frac{1}{2}(\varphi,G_k^B\varphi)}|F|_{k,B,T_\varphi} \le \sup_\varphi e^{-\frac{1}{2}|\varphi|^2_{k,\ell_2(B)}}|F|_{k,B,T_\varphi}. \tag{4.7.31}$$

To prove the differentiability we argue by induction. The main point is to show that

$$\lim_{\dot H\to 0}\frac{1}{\|\dot H\|_{k,0}} \sup_{\|\dot H_i\|_{k,0}\le 1} \Big|\!\Big|\!\Big|\underbrace{(e^{H+\dot H}-e^H-e^H\dot H)\dot H_1\dots\dot H_r}_{=:f(\dot H)}\Big|\!\Big|\!\Big|_{k,B} = 0. \tag{4.7.32}$$

We have

$$f(\dot H) = e^H(e^{\dot H}-1-\dot H)\dot H_1\dots\dot H_r \tag{4.7.33}$$

In the following we assume, without loss of generality, that

$$\|\dot H\|_{k,0} \le \frac{1}{16}. \tag{4.7.34}$$

Combining the equality

$$e^{\dot H}-1-\dot H = \sum_{m=2}^\infty \frac{1}{m!}\dot H^m = \dot H^2\sum_{m=0}^\infty\frac{1}{(m+2)!}\dot H^m, \tag{4.7.35}$$

with the product property of the T_φ norm, the estimate $\sum_{m=0}^\infty \frac{1}{(m+2)!}x^m \le e^x$ valid for $x\ge 0$, and (4.7.29), we infer that

$$|e^{\dot H}-1-\dot H|_{k,T_\varphi} \le |\dot H|^2_{k,T_\varphi}e^{|\dot H|_{k,T_\varphi}} \le \|\dot H\|^2_{k,0}\,4(1+|\varphi|^2_{k,\ell_2(B)})^2\,e^{\frac{1}{8}(1+|\varphi|^2_{k,\ell_2(B)})}. \tag{4.7.36}$$

Thus, using again the product property, the assumptions $\|H\|_{k,0}\le\frac{1}{8}$ and $\|\dot H_i\|_{k,0}\le 1$, as well as (4.7.29), we get

$$\begin{aligned}
|f(\dot H)|_{k,T_\varphi} &\le e^{|H|_{k,T_\varphi}}\ \|\dot H\|^2_{k,0}\,4(1+|\varphi|^2_{k,\ell_2(B)})^2\,e^{\frac{1}{8}(1+|\varphi|^2_{k,\ell_2(B)})}\ \prod_{j=1}^r|\dot H_j|_{k,T_\varphi}\\
&\le e^{\frac{1}{4}(1+|\varphi|^2_{k,\ell_2(B)})}\ \|\dot H\|^2_{k,0}\,4(1+|\varphi|^2_{k,\ell_2(B)})^2\,e^{\frac{1}{8}(1+|\varphi|^2_{k,\ell_2(B)})}\ 2^r((1+|\varphi|^2_{k,\ell_2(B)})^r\\
&\le \|\dot H\|^2_{k,0}\,2^{r+2}(1+|\varphi|^2_{k,\ell_2(B)})^{(r+2)}e^{\frac{3}{8}}e^{\frac{3}{8}|\varphi|^2_{k,\ell_2(B)}}\\
&\le 2^{r+2}C'_r e^{\frac{3}{8}}\|\dot H\|^2_{k,0}\,e^{\frac{1}{2}|\varphi|^2_{k,\ell_2(B)}},
\end{aligned}$$

where $C'_r = \sup_{t\ge 0} e^{-\frac{t}{8}}(1+t)^{r+2}$. Using (4.7.31) we get $|\!|\!|f(\dot H)|\!|\!|_{k,B} \le 2^{r+2}C'_r e^{\frac{3}{8}}\|\dot H\|^2_{k,0}$ and the assertion (4.7.32) follows.

To prove the bound (4.7.26) we use the product property of the T_φ norm to deduce that

$$\begin{aligned}
&|D^r E(H)(\dot H_1,\dots,\dot H_r)|_{k,T_\varphi}\\
&\le e^{|H|_{k,T_\varphi}}\,|\dot H_1|_{k,T_\varphi}\dots|\dot H_r|_{k,T_\varphi}\\
&\le e^{\frac14} e^{\frac14|\varphi|^2_{k,\ell_2(B)}}\,\|\dot H_1\|_{k,0}\dots\|\dot H_r\|_{k,0}\,2^r(1+|\varphi|^2_{k,\ell_2(B)})^r,\\
&\le C_r e^{\frac12|\varphi|^2_{k,\ell_2(B)}}\,\|\dot H_1\|_{k,0}\dots\|\dot H_r\|_{k,0}.
\end{aligned}$$

Dividing both sides by $e^{\frac12|\varphi|^2_{k,\ell_2(B)}}$, taking the supremum over φ, and using the definition (4.7.31), we get (4.7.26). Finally, the bound (4.7.27) follows from (4.7.26) with $r=1$ since $e^H-1=\int_0^1 DE(tH)(H)\,dt$ and $C_1=2e^{1/4}\max_{t\ge0}e^{-\frac{t}{4}}(1+t)=8e^{-1/2}$. □

4.7.3 The map P_2

We next consider the map

$$P_2(I,K)=(I-1)\circ K. \tag{4.7.37}$$

Lemma 4.7.4. *Let L and h satisfy the lower bounds* (4.7.18). *Then the map P_2 restricted to $B_{\rho_1}(1)\times B_{\rho_2}\subset \boldsymbol{M}_{|||}\times\boldsymbol{M}^{(A)}$ with $\rho_1<(2A)^{-1}$ and $\rho_2<\frac12$ is smooth for any $A\ge2$ and satisfies the bounds*

$$\frac{1}{j_1!j_2!}\|(D_I^{j_1}D_K^{j_2}P_2)(I,K)(\dot I,\dots,\dot I,\dot K,\dots,\dot K)\|_k^{(A/2)}\le\Big(2A|||\dot I|||_k\Big)^{j_1}\Big(2\|\dot K\|_k^{(A)}\Big)^{j_2}. \tag{4.7.38}$$

In particular, for $I\in B_{\rho_1}(1)$ and $K\in B_{\rho_2}$ this implies

$$\|P_2(I,K)\|_k^{(A/2)}\le 2A|||I-1|||_k+2\|K\|_k^A. \tag{4.7.39}$$

On right hand side of (4.7.38) we used the convention

$$a^0=1 \tag{4.7.40}$$

that we will use also in the rest of this section.

Proof. We have

$$\begin{aligned}
P_2(I,K)(X)&=((I-1)\circ K)(X)=\sum_{Y\in\mathcal{P}(X)}(I-1)^{X\setminus Y}K(Y)\\
&=\sum_{Y\in\mathcal{P}(X)}\prod_{B\in\mathcal{B}_k(X\setminus Y)}(I(B)-1)\prod_{Z\in\mathcal{C}(Y)}K(Y).
\end{aligned}\tag{4.7.41}$$

Using i) and ii) of Lemma 4.6.3 and $\Gamma_{k,A}(Y)=A^{|Y|_k}$ we get

$$\|P_2(I,K)(X)\|_{k,X}\le\sum_{Y\in\mathcal{P}(X)}|||I-1|||_k^{|X\setminus Y|}\Big(\underbrace{\|K\|_k^{(A)}}_{\le1}\Big)^{|\mathcal{C}(Y)|}A^{-|Y|}\le\Big(\frac{1}{2A}+\frac1A\Big)^{|X|}\le1 \tag{4.7.42}$$

where we used that $\sum_{Y\in\mathcal{P}(X)}a^{|X\setminus Y|}b^{|Y|}=(a+b)^{|X|}$ and $A\ge2$.

The derivatives of P_2 are given by

$$\frac{1}{j_1!j_2!}(D_1^{j_1}D_2^{j_2}P_2(I,K)(\dot I,\dots,\dot I,\dot K,\dots,\dot K))(X) = \sum_{\substack{Y\in\mathcal{P}(X),Y_1\in\mathcal{P}(X\setminus Y),|Y_1|=j_1\\ \mathcal{J}\subset\mathcal{C}(Y),|\mathcal{J}|=j_2}} (I-1)^{X\setminus(Y\cup Y_1)}\dot I^{Y_1}\prod_{Z\in\mathcal{C}(Y)\setminus\mathcal{J}}K(Z)\prod_{Z\in\mathcal{J}}\dot K(Z). \tag{4.7.43}$$

Using the bound $\binom{n}{j}\le 2^n$, we can estimate the norm of the expression above similarly as in (4.7.42),

$$\begin{aligned}\frac{1}{j_1!j_2!}&\|(D_1^{j_1}D_2^{j_2}P_2(I,K)(\dot I,\dots,\dot I,\dot K,\dots,\dot K))(X)\|_{k,X}\\ &\le \sum_{Y\in\mathcal{P}(X)}\binom{|X\setminus Y|}{j_1}|||I-1|||_k^{|X\setminus Y|-j_1}|||\dot I|||_k^{j_1}\binom{|\mathcal{C}(Y)|}{j_2}\left(\|K\|_k^{(A)}\right)^{|\mathcal{C}(Y)|-j_2}\left(\|\dot K\|_k^{(A)}\right)^{j_2}A^{-|Y|}\\ &\le \sum_{Y\in\mathcal{P}(X)}2^{|X\setminus Y|}(2A)^{-|X\setminus Y|+j_1}|||\dot I|||_k^{j_1}2^{|\mathcal{C}(Y)|}2^{-|\mathcal{C}(Y)|+j_2}\left(\|\dot K\|_k^{(A)}\right)^{j_2}A^{-|Y|}\\ &\le \left(\frac{A}{2}\right)^{-|X|}\left(2A|||\dot I|||_k\right)^{j_1}\left(2\|\dot K\|_k^{(A)}\right)^{j_2}.\end{aligned} \tag{4.7.44}$$

Equation (4.7.39) follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}P_2(1+t(I-1),tK) = D_1P_2(1+t(I-1),tK)(I-1)+D_2P_2(1+t(I-1),tK)K \tag{4.7.45}$$

using that $P_2(1,0)=0$.

□

4.7.4 The map P_3

The smoothness of the maps P_3 is implied by similar estimates, but simpler, as those for P_2.

Lemma 4.7.5. *Let L and h satisfy the lower bounds* (4.7.18) *and let $A\ge 2$, $B\ge 1$. Consider the map $P_3:\boldsymbol{M}^{(A/2)}\to\widehat{\boldsymbol{M}}^{(A/2,B)}$ given by*

$$P_3K(X)=\prod_{Y\in\mathcal{C}(X)}K(Y). \tag{4.7.46}$$

Its restriction to $B_\rho=\{K\in\boldsymbol{M}^{(A/2)}:\ \|K\|_k^{(A/2)}\le\rho\}$ is smooth for any ρ satisfying

$$\rho\le(2B)^{-1}. \tag{4.7.47}$$

Moreover the following estimate holds for all $j\ge 0$

$$\frac{1}{j!}\|(D^jP_3K)(\dot K,\dots,\dot K)\|_k^{(A/2,B)}\le\left(2B\|\dot K\|_{k,r}^{(A/2)}\right)^j. \tag{4.7.48}$$

Proof. We note that

$$\frac{1}{j!}D^jP_3(K)(X)(\dot K,\dots,\dot K)=\sum_{\substack{\mathcal{J}\subset\mathcal{C}(X)\\ \mathcal{J}=j}}\prod_{Z\in\mathcal{C}(X)\setminus\mathcal{J}}K(Z)\prod_{Z\in\mathcal{J}}\dot K(Z). \tag{4.7.49}$$

Using the bound $\binom{|\mathcal{C}(X)|}{j} \leq 2^{|\mathcal{C}(X)|}$ and i) from Lemma 4.6.3 for $K \in B_\rho$ we get

$$B^{|\mathcal{C}(X)|}\Big(\frac{A}{2}\Big)^{|X|}\frac{1}{j!}\|(D^jP_3K)(\dot K,\dots,\dot K)(X)\|_{k,X} \leq (2B)^{|\mathcal{C}(X)|}\big(\|K\|_k^{(A/2)}\big)^{|\mathcal{C}(X)|-j}\big(\|\dot K\|_{k,r}^{(A/2)}\big)^j$$
$$\leq \big(2B\|\dot K\|_{k,r}^{(A/2)}\big)^j. \tag{4.7.50}$$

□

4.7.5 The map P_1

Next we show smoothness of the outermost map P_1 given by

$$P_1(I_1,I_2,J,K)(U,\varphi) = \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1\cup X_2,U) I_1^{U\setminus(X_1\cup X_2)}(\varphi) I_2^{(X_1\cup X_2)\setminus U} J^{X_1}(\varphi) \prod_{Y\in\mathcal{C}^{\mathrm{cl}}(X_2)} K(Y,\varphi) \tag{4.7.51}$$

Lemma 4.7.6. *Let L and h satisfy the lower bounds* (4.7.19) *and*

$$A_0(L) = (48A_{\mathcal{P}})^{\frac{L^d}{\alpha}} \tag{4.7.52}$$

where $A_{\mathcal{P}}$ was introduced in Theorem 4.5.1 ix) and $\alpha(d) = (1+2^d)^{-1}(1+6^d)^{-1}$. Further, let $A \geq A_0(L)$, $B = A$ and

$$\rho_1 = \rho_2 \leq \frac{1}{2}, \quad \rho_3 \leq A^{-2}, \quad \rho_4 \leq 1. \tag{4.7.53}$$

Then the map P_1 restricted to the neighbourhood

$$U = B_{\rho_1}(1)\times B_{\rho_2}(1)\times B_{\rho_3}(0)\times B_{\rho_4}(0) \subset \boldsymbol{M}_{|||}\times\boldsymbol{M}_{|||}\times\boldsymbol{M}_{|||}\times\widehat{\boldsymbol{M}}_{:}^{(A/(2A_{\mathcal{P}}),B)} \tag{4.7.54}$$

is smooth with the bound on derivatives,

$$\frac{1}{i_1!i_2!j_1!j_2!}\|D_{I_1}^{i_1}D_{I_2}^{i_2}D_J^{j_1}D_K^{j_2}P_1(I_1,I_2,J,K)(\dot I_1,\dots,\dot I_1,\dot I_2,\dots,\dot I_2,\dot J,\dots,\dot J,\dot K,\dots,\dot K)\|_{k+1,r}^{(A)}$$
$$\leq |||\dot I_1|||^{i_1}|||\dot I_2|||^{i_2}(A^2|||\dot J|||)^{j_1}(\|\dot K\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)})^{j_2}. \tag{4.7.55}$$

Proof. We first note some simple inequalities for polymers. Recall from Lemma 4.7.2 that

$$\sum_{Y\in\mathcal{C}^{\mathrm{cl}}(X)} |\mathcal{C}(Y)| = |\mathcal{C}(X)|. \tag{4.7.56}$$

Next let $X \in \mathcal{P}_k$ and $U = \pi(X)$. Then by (4.4.54) we have $X \subset U^*$ and hence

$$|X\setminus U|_k + |U\setminus X|_k \leq |U^*|_k, \quad |X|_k \leq |U^*|_k. \tag{4.7.57}$$

We also have

$$|U^*|_k \leq 2|U|_k \text{ if } L \geq 4d(2^d+R). \tag{4.7.58}$$

Indeed for $B' \in \mathcal{B}_{k+1}$ and $k \geq 1$

$$|B'^*|_k \leq (L+2^{d+1})^d \leq L^d\Big(1+\frac{1}{2d}\Big)^d \leq L^d e^{\frac{1}{2}}, \tag{4.7.59}$$

while for $k = 0$,

$$|B'^*|_0 \le (L + 2^{d+1} + 2R)^d \le L^d\Big(1 + \frac{1}{2d}\Big)^d \le L^d e^{\frac{1}{2}}.$$

Finally, for $X_2, X \in \mathcal{P}_k$ with $X_2 \subset X$ we use the identity

$$|\mathcal{C}(X_2)| = \sum_{Y\in\mathcal{C}(X)} |\mathcal{C}(X_2 \cap Y)|. \tag{4.7.60}$$

It suffices to show that each connected component of X_2 is a connected component of $X_2 \cap Y$ for some $Y \in \mathcal{C}(X)$ (with $Y \cap X_2 \ne \emptyset$) and vice versa. Now if $Z \in \mathcal{C}(X_2)$ then Z is a connected subset of X and hence contained in exactly one component Y of X. Thus Z is a connected subset of $X_2 \cap Y$. In fact $Z \in \mathcal{C}(X_2 \cap Y)$ because $\mathrm{dist}_\infty(Z, (X_2 \cap Y) \setminus Z) \ge \mathrm{dist}_\infty(Z, X_2 \setminus Z) \ge L^k$ as $Z \in \mathcal{C}(X_2)$.

Conversely consider $Y \in \mathcal{C}(X)$ with $X_2 \cap Y \ne \emptyset$ and $Z \in \mathcal{C}(X_2 \cap Y)$. Then Z is a connected subset of X_2. Moreover $\mathrm{dist}_\infty(Z, (X_2 \setminus Y) \setminus Z) \ge \mathrm{dist}_\infty(Y, X \setminus Y) \ge L^k$ and $\mathrm{dist}_\infty(Z, (X_2 \cap Y) \setminus Z) \ge L^k$. Thus $\mathrm{dist}(Z, X_2 \setminus Z) \ge L^k$ and therefore $Z \in \mathcal{C}(X_2)$. This concludes the proof of (4.7.60).

Now let $U \in \mathcal{P}^c_{k+1}$ be a connected polymer. Lemma 4.6.3 implies that

$$\begin{aligned}
&\|P_1(I_1, I_2, J, K)\|_{k+1,U,r} \\
&\le \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1 \cup X_2, U)\, |||I_2|||_k^{|(X_1\cup X_2)\setminus U|}\, |||I_1|||_k^{|U\setminus(X_1\cup X_2)|}\, |||J|||_k^{|X_1|}\, \Big\| \prod_{Y\in\mathcal{C}^{\mathrm{cl}}(X_2)} K(Y)\Big\|_{k:k+1,X_2} \\
&\underset{(4.7.56)}{\le} \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1 \cup X_2, U)\, 2^{|(X_1\cup X_2)\setminus U|}\, 2^{|U\setminus(X_1\cup X_2)|}\, A^{-2|X_1|} \times \\
&\qquad \times \Big(\frac{A}{2A_{\mathcal{P}}}\Big)^{-|X_2|} B^{-|\mathcal{C}(X_2)|} \big(\|K\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)}\big)^{|\mathcal{C}'(X_2)|} \\
&\underset{(4.7.57)}{\le} 2^{2|U^*|_k} (A_{\mathcal{P}})^{|U^*|_k} \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1 \cup X_2, U)\, A^{-2|X_1|-|X_2|} B^{-|\mathcal{C}(X_2)|} \\
&\underset{(4.7.58),(4.7.60)}{\le} (4A_{\mathcal{P}})^{2|U|_k} \sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}} \chi(X_1 \cup X_2, U) \prod_{Y\in\mathcal{C}(X_1\cup X_2)} A^{-2|X_1\cap Y|-|X_2\cap Y|-|\mathcal{C}(X_2\cap Y)|},
\end{aligned} \tag{4.7.61}$$

where we used $B = A$ to get the last inequality.

Now we use the crucial fact that, for connected polymers X, their closure typically satisfies the bound $|\overline{X}|_{k+1} < c|X|_k$ for some $c < 1$. For the precise formulation, we record this standard inequality (4.C.1) in Lemma 4.C.1 (Appendix C). It is stating that for connected polymers $X \in \mathcal{P}^c_k \setminus \mathcal{S}_k$, we have

$$|X|_k \ge (1 + 2\alpha(d))|\overline{X}|_{k+1}, \tag{4.7.62}$$

where $0 < \alpha(d) = ((1+2^d)(1+6^d))^{-1} < 1$ is a positive constant. This implies, for $Y \in \mathcal{C}(X_1 \cup X_2)$ and $Y \notin \mathcal{S}_k$, that

$$2|X_1 \cap Y| + |X_2 \cap Y| + |\mathcal{C}(X_2 \cap Y)| \ge |Y| \ge (1 + 2\alpha(d))|\pi(Y)|_{k+1}, \tag{4.7.63}$$

where we used that $\pi(Y) = \overline{Y}$ since Y is not small. If $Y \in \mathcal{S}_k$ we note that either $|X_1 \cap Y| \geq 1$ or $|X_2 \cap Y| \geq 1$. In either case we get

$$2|X_1 \cap Y| + |X_2 \cap Y| + |\mathcal{C}(X_2 \cap Y)| \geq |Y| + 1 \geq 2|\pi(Y)|_{k+1} \geq (1 + 2\alpha(d))|\pi(Y)|_{k+1}. \quad (4.7.64)$$

Inserting (4.7.63) and (4.7.64) into (4.7.61), we get

$$\begin{aligned} \|P_1(I_1, I_2, J, K)\|_{k+1,U} &\leq (4A_{\mathcal{P}})^{2|U|_k} \sum_{\substack{X_1, X_2 \in \mathcal{P}_k \\ X_1 \cap X_2 = \emptyset}} \chi(X_1 \cup X_2, U) \prod_{Y \in \mathcal{C}(X_1 \cup X_2)} A^{-(1+2\alpha)|\pi(Y)|_{k+1}} \\ &\leq (4A_{\mathcal{P}})^{2|U|_k} 3^{|U^*|_k} A^{-(1+2\alpha)|U|_{k+1}} \underset{(4.7.58)}{\leq} (12A_{\mathcal{P}})^{2|U|_k} A^{-(1+2\alpha)|U|_{k+1}} \\ &\leq \Big(\frac{(12A_{\mathcal{P}})^{2L^d}}{A^{2\alpha}} \Big)^{|U|_{k+1}} A^{-|U|_{k+1}}. \end{aligned} \quad (4.7.65)$$

For the second inequality we used that $X_1 \cup X_2 \subset U^*$ if $\chi(X_1 \cup X_2, U) \neq 0$ and that there are $3^{|U^*|_k}$ possibilities for partitions of U^* into three disjoint sets X_1, X_2 and $X_3 = U^* \setminus (X_1 \cup X_2)$. We also used that by the definition of π we have $\pi(X) = \cup_{Y \in \mathcal{C}(X)} \pi(Y)$ and thus $|\pi(X)|_{k+1} \leq \sum_{Y \in \mathcal{C}(X)} |\pi(Y)|_{k+1}$.

Thus we get for $A \geq (12A_{\mathcal{P}})^{\frac{L^d}{\alpha}}$

$$\|P_1(I_1, I_2, J, K)\|_{k+1}^{(A)} \leq 1. \quad (4.7.66)$$

Let us now proceed to the bounds for derivatives.Similarly to the derivatives of P_3 in Lemma 4.7.4, we get

$$\frac{1}{j!} D^j \Big(\prod_{Y \in \mathcal{C}^{\mathrm{cl}}(X)} K(Y) \Big)(\dot{K}, \ldots, \dot{K}) = \sum_{\substack{\mathcal{J} \subset \mathcal{C}^{\mathrm{cl}}(X) \\ |\mathcal{J}| = j}} \prod_{Y \in \mathcal{J}} \dot{K}(Y) \prod_{Y \in \mathcal{C}^{\mathrm{cl}}(X) \setminus \mathcal{J}} K(Y). \quad (4.7.67)$$

For $\|K\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} \leq 1$ we use Lemma 4.6.3 to get,

$$\begin{aligned} \frac{1}{j!} \|D^j \Big(\prod_{Y \in \mathcal{C}^{\mathrm{cl}}(X)} K(Y) \Big)(\dot{K}, \ldots, \dot{K})\|_{k:k+1,X} &\leq \sum_{\substack{\mathcal{J} \subset \mathcal{C}'(X) \\ |\mathcal{J}| = j}} \prod_{Y \in \mathcal{J}} \|\dot{K}(Y)\|_{k:k+1,Y} \prod_{Y \in \mathcal{C}'(X) \setminus \mathcal{J}} \|K(Y)\|_{k:k+1,Y} \\ &\underset{(4.7.56)}{\leq} \binom{|\mathcal{C}'(X)|}{j} \Big(\frac{A}{2A_{\mathcal{P}}} \Big)^{-|X|} B^{-|\mathcal{C}(X)|} \big(\|\dot{K}\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} \big)^j. \end{aligned} \quad (4.7.68)$$

A similar bound holds for the factors of I_1, I_2, and J. Therefore, similarly to (4.7.61), we bound

$$\begin{aligned} &\frac{1}{i_1! i_2! j_1! j_2!} \|D_{i_1}^{I_1} D_{i_2}^{I_2} D_{j_1}^{J} D_{j_2}^{K} P_1(I_1, I_2, J, K)\|_{k+1,U} \\ &\leq \sum_{\substack{X_1, X_2 \in \mathcal{P}_k \\ X_1 \cap X_2 = \emptyset}} \chi(X_1 \cup X_2, U) \binom{|(X_1 \cup X_2) \setminus U|}{i_2} |||I_2|||_k^{|(X_1 \cup X_2) \setminus U| - i_2} |||\dot{I}_2|||_k^{i_2} \times \\ &\quad \times \binom{|U \setminus (X_1 \cup X_2)|}{i_1} |||I_1|||_k^{|U \setminus (X_1 \cup X_2)| - i_1} |||\dot{I}_1|||_k^{i_1} \binom{|X_1|}{j_1} |||J|||_k^{|X_1| - j_1} |||\dot{J}|||_k^{j_1} \times \\ &\quad \times \binom{\mathcal{C}'(X_2)}{j_2} \Big(\frac{A}{2A_{\mathcal{P}}} \Big)^{-|X_2|} B^{-|\mathcal{C}(X_2)|} \big(\|\dot{K}\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} \big)^{j_2}. \end{aligned} \quad (4.7.69)$$

Assume that $\chi(U, X_1\cup X_2) = 1$. Then $X_1 \cup X_2 \subset U^*$. and we can bound the combinatorial factor by

$$\begin{aligned}\tbinom{|(X_1\cup X_2)\setminus U|}{i_2}\tbinom{|U\setminus(X_1\cup X_2)|}{i_1}\tbinom{|X_1|}{j_1}\tbinom{|\mathcal{C}'(X_2)|}{j_2} &\le 2^{|(X_1\cup X_2)\setminus U|+|U\setminus(X_1\cup X_2)|+|X_1|+|X_2|}\\ &\underset{(4.7.57)}{\le} 2^{2|U^*|_k} \underset{(4.7.58)}{\le} 4^{2|U|_k}.\end{aligned} \tag{4.7.70}$$

Then we bound, exactly as in (4.7.61),

$$\begin{aligned}&\frac{1}{i_1!i_2!j_1!j_2!}\|D_{i_1}^{I_1}D_{i_2}^{I_2}D_{j_1}^{J}D_{j_2}^{K}P_1(I_1,I_2,J,K)\|_{k+1,U}\\ &\le (16A_{\mathcal{P}})^{2|U|_k}\sum_{\substack{X_1,X_2\in\mathcal{P}_k\\ X_1\cap X_2=\emptyset}}\chi(X_1\cup X_2,U)\prod_{Y\in\mathcal{C}(X_1\cup X_2)}A^{-2|X_1\cap Y|-|X_2\cap Y|-|\mathcal{C}(X_2\cap Y)|}\times\\ &\qquad\times\big(\tfrac{1}{2}|||\dot I_1|||_k\big)^{i_1}\big(\tfrac{1}{2}|||\dot I_2|||_k\big)^{i_2}\big(A^2|||\dot J|||_k\big)^{j_1}\big(\|\dot K\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)}\big)^{j_2}.\end{aligned} \tag{4.7.71}$$

Now, we can conclude as in (4.7.65) that

$$\begin{aligned}&\frac{1}{i_1!i_2!j_1!j_2!}\|D_{i_1}^{I_1}D_{i_2}^{I_2}D_{j_1}^{J}D_{j_2}^{K}P_1(I_1,I_2,J,K)\|_{k+1,U,r}\\ &\qquad\le\Big(\frac{(48A_{\mathcal{P}})^{2L^d}}{A^{2\alpha}}\Big)^{|U|_{k+1}}A^{-|U|_{k+1}}|||\dot I_2|||_k^{i_2}|||\dot I_1|||_k^{i_1}\big(A|||\dot J|||_k\big)^{j_1}\big(\|\dot K\|_{k:k+1,r}^{(A/(2A'),B)}\big)^{j_2}\\ &\qquad\le A^{-|U|_{k+1}}|||\dot I_1|||_k^{i_1}|||\dot I_2|||_k^{i_2}\big(A^2|||\dot J|||_k\big)^{j_1}\big(\|\dot K\|_{k:k+1,r}^{(A/(2A'),B)}\big)^{j_2}\end{aligned} \tag{4.7.72}$$

once $A > (48A_{\mathcal{P}})^{\frac{L^d}{\alpha}}$. This implies the claim (4.7.55). □

4.7.6 The map R_1

Next we discuss the smoothness of the maps R_1 and R_2 which depend explicitly on $\boldsymbol{q}$. The proofs are similar to those in [4] however we do not have to deal with the $\boldsymbol{q}$ derivatives explicitly because we already controlled them in Lemma 4.6.4. Let us begin with the map R_1 which is defined by

$$R_1(P,\boldsymbol{q})(X,\varphi) = (\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}P)(X,\varphi) = \int_{\mathcal{X}_N} P(X,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi). \tag{4.7.73}$$

Lemma 4.7.7. *Let L and h satisfy the lower bound (4.7.20) and let $\kappa = \kappa(L)$ be the constant introduced in Theorem 4.5.1 and specified in (4.5.81). For $B \ge 1$ and any $A \ge 4A_{\mathcal{P}}$ the map R_1 restricted to $\widehat{\boldsymbol{M}}_k^{(A/2,B)} \times \mathcal{U}_\kappa$ is smooth and satisfies*

$$\|D_P^j R_1(P,\boldsymbol{q})(X,\cdot)(\dot P,\ldots\dot P)\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} \le (\|\dot P\|_k^{(A/2)})^j(\|P\|_k^{(A/2)})^{1-j}. \tag{4.7.74}$$

and

$$\|D_{\boldsymbol{q}}^{\ell}D_P^j R_1(P,\boldsymbol{q})(X,\cdot)(\dot{\boldsymbol{q}},\ldots,\dot{\boldsymbol{q}},\dot P,\ldots\dot P)\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} \le C_\ell(L)\|\dot{\boldsymbol{q}}\|^{\ell}(\|\dot P\|_k^{(A/2)})^j(\|P\|_k^{(A/2)})^{1-j}. \tag{4.7.75}$$

for $\ell \ge 1$ and $0 \le j \le 1$. The constants $C_\ell(L)$ do not depend on h or A. The derivatives vanish for $j > 1$.

Proof. Note first that the map R_1 is linear in P. Therefore the statement for the derivative in P direction is trivial and we only need to consider the $\boldsymbol{q}$ derivative. Note that $X \in \mathcal{P}_k^{\mathrm{cl}}$ is equivalent to the condition that $\pi(X)$ is connected. Therefore we can apply Lemma 4.6.4. From (4.6.20) we get

$$\|(\boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K)(X)\|_{k:k+1,X} \le A_{\mathcal{P}}{}^{|X|_k} \|K(X)\|_{k,X}. \tag{4.7.76}$$

and hence

$$\begin{aligned}
\|\boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} &= \sup_{X \in \mathcal{P}_k^{\mathrm{cl}}} B^{|\mathcal{C}(X)|} \Big(\frac{A}{2A_{\mathcal{P}}}\Big)^{|X|_k} \|\boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K(X)\|_{k:k+1,X} \\
&\le \sup_{X \in \mathcal{P}_k^{\mathrm{cl}}} B^{|\mathcal{C}(X)|} \Big(\frac{A}{2A_{\mathcal{P}}}\Big)^{|X|_k} A_{\mathcal{P}}{}^{|X|_k} \|K(X)\|_{k,X} \\
&= \|K(X)\|_{k:k+1}^{(A/2,B)}.
\end{aligned} \tag{4.7.77}$$

Similarly, for $\ell \ge 1$, we get

$$\begin{aligned}
\|D_{\boldsymbol{q}}^{\ell} \boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K\|_{k:k+1}^{(A/(2A_{\mathcal{P}}),B)} &= \sup_{X \in \mathcal{P}_k^{\mathrm{cl}}} B^{|\mathcal{C}(X)|} \Big(\frac{A}{2A_{\mathcal{P}}}\Big)^{|X|_k} \|D_{\boldsymbol{q}}^{\ell} \boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K(X)\|_{k:k+1,X} \\
&\le C_\ell(L) \sup_{X \in \mathcal{P}_k^{\mathrm{cl}}} B^{|\mathcal{C}(X)|} \Big(\frac{A}{2A_{\mathcal{P}}}\Big)^{|X|_k} A_{\mathcal{P}}{}^{|X|_k} \|K(X)\|_{k,X} \\
&= C_\ell(L) \|K(X)\|_{k:k+1}^{(A/2,B)}.
\end{aligned} \tag{4.7.78}$$

□

4.7.7 The map R_2

Lemma 4.7.8. *Let L and h satisfy the lower bound* (4.7.20)*. For any $h \ge 1$ and $A \ge 1$ the map R_2 defined in* (4.7.16) *is smooth. Moreover there exist a constant C_0 (which is independent of L, h and A) and for each $\ell \ge 1$ there exist a constant $C_\ell(L)$ (which is independent of h an A) such that*

$$\|D_H^{j_1} D_K^{j_2} D_{\boldsymbol{q}}^{\ell} R_2(H,K,\boldsymbol{q})(\dot H, \dot K, \dot{\boldsymbol{q}})\|_{k,0} \le C_\ell(L) \|\dot{\boldsymbol{q}}\|^{\ell} \begin{cases} \|H\|_{k,0} + \|K\|_k^{(A)} & \text{if } j_1 = j_2 = 0 \\ \|\dot H\|_{k,0} & \text{if } j_1 = 1, j_2 = 0 \\ \|\dot K\|_k^{(A)} & \text{if } j_1 = 0, j_2 = 1, \end{cases} \tag{4.7.79}$$

and

$$D_H^{j_1} D_K^{j_2} D_{\boldsymbol{q}}^{\ell} R_2(H,K,\boldsymbol{q})(\dot H, \ldots, \dot H, \dot K, \ldots, \dot K, \dot{\boldsymbol{q}}, \ldots, \dot{\boldsymbol{q}}) = 0 \quad \text{if } j_1 + j_2 \ge 2. \tag{4.7.80}$$

Proof. First we observe that $R_2(H,K,\boldsymbol{q}) = R_{2,a}^{(\boldsymbol{q})} H - R_{2,b}^{(\boldsymbol{q})} K$ where both $R_{2,a}^{(\boldsymbol{q})}$ and $R_{2,b}^{(\boldsymbol{q})}$ are linear maps given by

$$R_{2,a}^{(\boldsymbol{q})} H = \Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})} H, \qquad R_{2,b}^{(\boldsymbol{q})} K = \Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})} K. \tag{4.7.81}$$

This implies (4.7.80). Due to the linearity with respect to H and K the bounds for the derivatives with respect to H and K follow from the case without derivatives in H or K direction. We consider the two operators separately.

The estimate for the operator $R_{2,a}^{(\boldsymbol{q})}$ is simple because its action on Hamiltonians can be calculated explicitly. It only changes the constant part

$$a_\emptyset \mapsto a_\emptyset + \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} a_{(i,\alpha),(j,\beta)} (\nabla^\beta)^* \nabla^\alpha \mathcal{C}_{k+1,ij}^{(\boldsymbol{q})}(0), \tag{4.7.82}$$

see Proposition 4.4.8. Using the bound (4.4.13) and the definition (4.4.93), we get

$$\|D_{\boldsymbol{q}}^\ell R_{2,a}^{(\boldsymbol{q})} H\|_{k,0} \le \|H\|_{k,0} + c_{2,\ell} h_k^{-2} \|H\|_{k,0} \le (1 + c_{2,\ell(4.4.13)}) \|H\|_{k,0} \tag{4.7.83}$$

if $h \ge 1$.

Further, let us consider the map $R_{2,b}^{(\boldsymbol{q})}$. From the linearity of Π_2 and Lemma 4.6.7 we get

$$\begin{aligned} \|D_{\boldsymbol{q}}^\ell \Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} K(B,\cdot)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k,0} &\le \|\Pi_2 (D_{\boldsymbol{q}}^\ell \boldsymbol{R}^{(\boldsymbol{q})} K(B,\cdot))(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k,0} \\ &\le C_{(4.6.51)} |D_{\boldsymbol{q}}^\ell \boldsymbol{R}^{(\boldsymbol{q})} K(B,0)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})|_{k,B,T_0} \\ &\le C_{(4.6.51)} \|D_{\boldsymbol{q}}^\ell \boldsymbol{R}^{(\boldsymbol{q})} K(B)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k:k+1,B}. \end{aligned} \tag{4.7.84}$$

In the last step we used that by definition (4.4.88),

$$\|F(B)\|_{k:k+1,B} = \sup_\varphi w_{k:k+1}^{-B}(\varphi) \, |F(B)|_{k,B,T_0} \ge |F(B)|_{k,B,T_0} \tag{4.7.85}$$

since $w_{k:k+1}^{-B}(0) = 1$. Now, Lemma 4.6.4 for $\ell \ge 1$ yields

$$\begin{aligned} \|D_{\boldsymbol{q}}^\ell \Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} K(B,\cdot)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k,0} &\le C_{(4.6.51)} \|D_{\boldsymbol{q}}^\ell \boldsymbol{R}^{(\boldsymbol{q})} K(B)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k:k+1,B} \\ &\le C_{(4.6.51)} C_{\ell,(4.6.21)}(L) A_{\mathcal{B}} \|\dot{\boldsymbol{q}}\|^\ell \, \|K(B)\|_{k,B} \\ &\le \frac{C_{(4.6.51)} C_{\ell,(4.6.21)}(L) A_{\mathcal{B}} \|\dot{\boldsymbol{q}}\|^\ell}{A} \|K\|_k^{(A)}. \end{aligned} \tag{4.7.86}$$

This implies that

$$\|D_{\boldsymbol{q}}^\ell \boldsymbol{R}_{2,b}^{(\boldsymbol{q})} K\|_{k,0} \le C_\ell(L) \|\dot{\boldsymbol{q}}\|^\ell \|K\|_k^{(A)} \tag{4.7.87}$$

for $\ell \ge 1$. The bounds (4.7.83) and (4.7.87) jointly yield the desired estimate for $\ell \ge 1$. For $\ell = 0$ we get, instead of (4.7.86), a slightly sharper estimate,

$$\|\Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} K(B,\cdot)\|_{k,0} \le C_{(4.6.51)} \|\boldsymbol{R}^{(\boldsymbol{q})} K(B)\|_{k:k+1,B} \le \frac{C_{(4.6.51)} A_{\mathcal{B}}}{A} \|K\|_k^{(A)}. \tag{4.7.88}$$

Together with (4.7.83) and the assumption $A \ge 1$ this implies the desired estimate for $\ell = 0$ with

$$C_0 = 1 + c_{2,\ell,(4.4.13)} + C_{(4.6.51)} \frac{A_{\mathcal{B}}}{A}. \tag{4.7.89}$$

□

Corollary 4.7.9. *The operators* $\boldsymbol{A}_k^{(\boldsymbol{q})}$ *and* $\boldsymbol{B}_k^{(\boldsymbol{q})}$ *satisfy the estimate* (4.4.104).

Proof. The operators $\boldsymbol{A}_k^{(\boldsymbol{q})}$ and $\boldsymbol{B}_k^{(\boldsymbol{q})}$ satisfy the identities

$$\begin{aligned} \boldsymbol{A}_k^{(\boldsymbol{q})} H(B',\varphi) &= \sum_{B\in\mathcal{B}(B')} R_{2,a}^{(\boldsymbol{q})} H(B,\varphi) \\ \boldsymbol{B}_k^{(\boldsymbol{q})} K(B',\varphi) &= -\sum_{B\in\mathcal{B}(B')} R_{2,b}^{(\boldsymbol{q})} K(B,\varphi). \end{aligned} \tag{4.7.90}$$

Hence, the claim follows from bounds (4.7.83) and (4.7.87). □

4.7.8 Proof of Theorem 4.4.7

Proof of Theorem 4.4.7. The assertion follows from the smoothness of the individual maps E, P_1, P_2, P_3, R_1, and R_2 and the chain rule. To get an estimate for a neighbourhood $U_{\rho,\kappa} \ni 0$ on which the map $\boldsymbol{S}$ is smooth and to see on which parameters the constants ρ and κ depend, we sequentially trace the dependence back to the neighbourhoods on which the individual maps and their compositions are smooth.

First, we fix a constant $A \geq A_0(L)$, where

$$A_0(L) = (48 A_{\mathcal{P}}(L))^{\frac{L^d}{\alpha}} \quad \text{with} \quad \alpha(d) = (1+2^d)^{-1}(1+6^d)^{-1} \tag{4.7.91}$$

is as in Lemma 4.7.6 and set

$$B = A. \tag{4.7.92}$$

Thus, by Lemma 4.7.6, the map P_1 is smooth in a neighbourhood $O_1 = B_{\rho_1}(1) \times B_{\rho_2}(1) \times B_{\rho_3}(0) \times B_{\rho_4}(0)$ with

$$\rho_1 = \rho_2 = \tfrac{1}{2}, \ \rho_3 = A^{-2}, \ \text{and } \rho_4 = 1. \tag{4.7.93}$$

Using Lemma 4.7.7, we find a neighbourhood $O_2 = B_{\rho_5} \times B_\kappa$ of the origin such that R_1 is smooth on O_2 and $R_1(O_2) \subset B_{\rho_4}$. Indeed, we may take

$$\kappa = \kappa(L) \text{ to be the constant } \kappa \text{ defined in (4.5.81)} \tag{4.7.94}$$

and

$$\rho_5 = \rho_4 = 1. \tag{4.7.95}$$

Similarly, by Lemma 4.7.3, there exists a neighbourhood $O_3 = B_{\rho_6}(0)$ such that E is smooth on O_3 and $E(O_3) \subset B_{\rho_1}(1) \cap B_{\rho_2}(1) \cap B_{\rho_3}(1)$. Indeed, since $A \geq A_0(L) \geq 2$, it suffices to take

$$\rho_6 = \tfrac{1}{8}\min(1, \rho_1, \rho_2, \rho_3) = \tfrac{1}{8}A^{-2}. \tag{4.7.96}$$

In view of Lemma 4.7.8, there exists a neighbourhood $O_4 = B_{\rho_7}(0) \times B_{\rho_8}(0) \times B_\kappa$ such that $R_2(O_4) \subset B_{\rho_6}$. Indeed, we may take

$$\rho_7 = \rho_8 = \frac{\rho_6}{C_{0,(4.7.79)}} = \frac{1}{8A^2\, C_{0,(4.7.79)}}. \tag{4.7.97}$$

This defines the first restriction on the final neighbourhood $U_{\rho,\kappa}$, namely,

$$U_{\rho,\kappa} \subset B_{\rho_7}(0) \times B_{\rho_8}(0) \times B_\kappa(0). \tag{4.7.98}$$

The second restriction comes from the condition

$$P_3(P_2(E(-H), K)) \in B_{\rho_5}(0). \tag{4.7.99}$$

To satisfy this condition, we note that by Lemma 4.7.5 there exists a neighbourhood $O_5 = B_{\rho_9}(0)$ such that $P_3(O_5) \subset B_{\rho_5}$. It suffices to take

$$\rho_9 = \frac{1}{2B}\min(\rho_5, 1) = \frac{1}{2A}. \tag{4.7.100}$$

By Lemma 4.7.4, there exists a neighbourhood $O_6 = B_{\rho_{10}}(1) \times B_{\rho_{11}}(0)$ with $P_2(O_6) \subset B_{\rho_9}(0)$. Taking into account the bound (4.7.39) and the fact that $\rho_9 \leq 1$, we may take

$$\rho_{10} = \frac{\rho_9}{4A} = \frac{1}{8A^2}, \quad \rho_{11} = \frac{\rho_9}{4} = \frac{1}{8A}. \tag{4.7.101}$$

Using once more Lemma 4.7.3, we see that the condition (4.7.99) holds if

$$(H, K) \in B_{\rho_{12}}(0) \times B_{\rho_{11}}(0) \quad \text{with} \quad \rho_{12} = \tfrac{1}{8}\rho_{10} = \frac{1}{64A^2}. \tag{4.7.102}$$

Combining this with (4.7.98), we see that the map $\boldsymbol{S}$ is C^∞ in the set

$$U_{\rho,\kappa} = B_\rho(0) \times B_\rho(0) \times B_\kappa(0) \tag{4.7.103}$$

once

$$\rho = \min(\rho_7, \rho_8, \rho_{11}, \rho_{12}) = \frac{1}{8A^2} \min\Big(\frac{1}{C_{0,(4.7.79)}}, \frac{1}{8}\Big) \tag{4.7.104}$$

and the constant $\kappa = \kappa(L)$ is as above. Since $A \geq A_0 \geq A_{\mathcal{P}} \geq A_{\mathcal{B}}$ we deduce from (4.7.89) that

$$C_{0,(4.7.79)} \leq 1 + c_{2,\ell,(4.4.13)} + C_{(4.6.51)}. \tag{4.7.105}$$

Thus we may take

$$\rho = \frac{1}{8A^2} \min\Big(\frac{1}{1 + c_{2,\ell,(4.4.13)} + C_{(4.6.51)}(d, m, R)}, \tfrac{1}{8}\Big). \tag{4.7.106}$$

Finally, the chain rule implies the estimate (4.4.98). □

4.8 Linearisation of the renormalisation map

In this section we prove the bounds for the operator norms stated in Theorem 4.4.8. These contraction estimates make precise the idea that H_k and K_k collect the relevant and the irrelevant terms, respectively. Throughout this section we assume that

$$\boldsymbol{q} \in B_\kappa \quad \text{where } B_\kappa = B_\kappa(0) \text{ and } \kappa = \kappa(L) \text{ is introduced in Theorem 4.5.1.} \tag{4.8.1}$$

4.8.1 Bounds for the operator $\boldsymbol{C}^{(\boldsymbol{q})}$

By (4.4.102) we have, for $K \in M(\mathcal{P}_k^{\mathrm{c}})$,

$$(\boldsymbol{C}^{(\boldsymbol{q})} K)(U, \varphi) = F(U, \varphi) + G(U, \varphi) \tag{4.8.2}$$

where $F \in M(\mathcal{P}_{k+1}^{\mathrm{c}})$ is defined by

$$F(U, \varphi) = \sum_{\substack{X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{B} \\ \pi(X) = U}} \int_{\mathcal{X}_N} K(X, \varphi + \xi) \, \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi). \tag{4.8.3}$$

and

$$G(B', \varphi) = \sum_{B \in \mathcal{B}_k(B')} G(B)(\varphi) \tag{4.8.4}$$

with

$$G(B)(\varphi) := (1 - \Pi_2)\boldsymbol{R}^{(\boldsymbol{q})}_{k+1} K(B, \varphi). \tag{4.8.5}$$

if B' is a $k+1$ block while

$$G(U, \varphi) = 0 \quad \forall U \in \mathcal{P}^{\mathrm{c}}_{k+1} \setminus \mathcal{B}_{k+1}. \tag{4.8.6}$$

For ease of reading we restate the key bound from Theorem 4.4.8 as Lemma 4.8.1 below. Recall the definition of R in (4.4.29) and let $A_{\mathcal{B}}$ and $A_{\mathcal{P}}(L)$ denote the constants which appear in the integration estimates in Theorem 4.5.1 ix) and x). Recall also that $\eta \in (0, \frac{2}{3}]$ is a fixed parameter. This parameter actually controls the contraction rate of the flow.

Lemma 4.8.1. *There exists an L_0 such that for each $L \geq L_0$ there exists an $A_0(L)$ and a $h_0(L)$ with the property that for all $A \geq A_0(L)$ and all $h \geq h_0(L)$*

$$\|\boldsymbol{C}^{(\boldsymbol{q})}\|^{(A)} = \sup_{\|K\|^{(A)}_k \leq 1} \|\boldsymbol{C}^{(\boldsymbol{q})} K\|^{(A)}_{k+1} \leq \frac{3}{4}\eta \qquad \text{for all } \boldsymbol{q} \in B_\kappa. \tag{4.8.7}$$

We may take

$$L_0 = \max\big((4\eta^{-1} C' A_{\mathcal{B}} C_1)^{\frac{1}{d'-d}}, (32\eta^{-1} C' A_{\mathcal{B}}(C_2+1))^{\frac{2}{d}}, 2^{d+3} + 16R\big), \tag{4.8.8}$$

$$A_0(L) = \max\Big(\frac{8}{\eta} {A_{\mathcal{P}}}^2 L^d (2^{d+1}+1)^{d2^d}, \big(\frac{8A_{\mathcal{P}}}{\eta\delta}\big)^{\frac{1+2\alpha}{2\alpha}}\Big) \tag{4.8.9}$$

and $h_0(L)$ as in (4.5.17) *in Theorem 4.5.1. Here C_1 is the constant in the estimate $|(1 - \Pi_2)K(B)|_{k+1,B,0} \leq C_1 L^{-d'} |K(B)|_{k,B,0}$ in Lemma 4.6.9 and C_2 is the constant in the estimate $|\Pi_2 K(B)|_{k+1,B,0} \leq C_2 |K(B)|_{k,B,0}$ in Lemma 4.6.7. Moreover $d' = \lfloor d/2 \rfloor + d/2 + 1$, $C' = \max_{x\geq 0}(1+x)^5 e^{-\frac{1}{2}x^2}$, and α and δ are the constants from Lemma 4.C.1 and Lemma 4.C.2, respectively.*

There are two mechanisms that ensure contractivity of the map $\boldsymbol{C}^{(\boldsymbol{q})}$. For the operator F defined in (4.8.3) we use that the operation π reduces the number of blocks, i.e., $|\pi(X)|_{k+1} < |X|_k$. The definition of the norm ensures that we gain a factor of $A^{|X|_k - |\pi(X)|_{k+1}}$ which can be used to cancel the combinatorial explosion of the number of terms. For the operator G, i.e., the contributions of single blocks this is not possible. For single block we use instead that $(1-\Pi_2)K$ measured at scale $k+1$ is much smaller than K measured at scale k (see Lemma 4.6.9).

We first consider the simpler large polymer term F.

Lemma 4.8.2. *Let $L \geq 2^{d+3} + 16R$ and define*

$$A_0(L) := \max\Big(\frac{8}{\eta} {A_{\mathcal{P}}}^2 L^d (2^{d+1}+1)^{d2^d}, \big(\frac{8A_{\mathcal{P}}}{\eta\delta}\big)^{\frac{1+2\alpha}{2\alpha}}\Big) \tag{4.8.10}$$

where $A_{\mathcal{P}}$ is the constant from Theorem 4.5.1 ix) and α and δ are the constants from Lemma 4.C.1 and Lemma 4.C.2, respectively. Then for all $A \geq A_0(L)$

$$\|F\|^{(A)}_{k+1} \leq \tfrac{1}{4}\eta \|K\|^{(A)}_k. \tag{4.8.11}$$

Proof. Lemma 4.6.2 states that for $U = \pi(X)$

$$|\boldsymbol{R}^{(\boldsymbol{q})}_{k+1} K(X, \varphi)|_{k+1,U,T_\varphi} \leq |\boldsymbol{R}^{(\boldsymbol{q})}_{k+1} K(X, \varphi)|_{k,X,T_\varphi}. \tag{4.8.12}$$

The inequality (4.5.24) in Theorem 4.5.1 vii) implies that

$$w^X_{k:k+1}(\varphi) \le w^U_{k+1}(\varphi). \tag{4.8.13}$$

We conclude that

$$\begin{aligned} \|\boldsymbol{R}_{k+1}K(X,\varphi)\|_{k+1,U} = \sup_{\varphi\in\mathcal{X}_N} \frac{|\boldsymbol{R}_{k+1}K(X,\varphi)|_{k+1,U,T_\varphi}}{w^U_{k+1}(\varphi)} &\le \sup_{\varphi\in\mathcal{X}_N} \frac{|\boldsymbol{R}_{k+1}K(X,\varphi)|_{k,X,T_\varphi}}{w^X_{k:k+1}(\varphi)} \\ &= \|\boldsymbol{R}_{k+1}K(X,\varphi)\|_{k:k+1,X}. \end{aligned} \tag{4.8.14}$$

Using this bound we can estimate

$$A^{|U|_{k+1}}\|F(U)\|_{k+1,U} \le A^{|U|_{k+1}}\Big(\sum_{\substack{X\in\mathcal{P}^c_k\setminus\mathcal{S}_k\\ \pi(X)=U}} \|\boldsymbol{R}_{k+1}K(X)\|_{k:k+1,X} + \sum_{\substack{X\in\mathcal{S}_k\setminus\mathcal{B}_k\\ \pi(X)=U}} \|\boldsymbol{R}_{k+1}K(X)\|_{k:k+1,X}\Big). \tag{4.8.15}$$

We bound the two summands separately. For the first term we use that $\pi(X)=U$ implies $\overline{X}=U$ for large polymers X so that we can use the bound $|\overline{X}|_{k+1}\le \frac{1}{1+2\alpha}|X|_k$ in Lemma 4.C.1 and Lemma 4.C.2. Bounding in addition the map $\boldsymbol{R}_{k+1}$ using Lemma 4.6.4 we infer that

$$\begin{aligned} A^{|U|_{k+1}} \sum_{\substack{X\in\mathcal{P}^c_k\setminus\mathcal{S}_k\\ \pi(X)=U}} \|\boldsymbol{R}_{k+1}K(X)\|_{k:k+1,X} &\le A^{|U|_{k+1}} \sum_{\substack{X\in\mathcal{P}^c_k\setminus\mathcal{S}_k\\ \overline{X}=U}} {A_\mathcal{P}}^{|X|_k}\|K(X)\|_{k,X} \\ \le A^{|U|_{k+1}} \sum_{\substack{X\in\mathcal{P}^c_k\setminus\mathcal{S}_k\\ \overline{X}=U}} \|K\|^{(A)}_k \Big(\frac{A_\mathcal{P}}{A}\Big)^{|X|_k} &\le \|K\|^{(A)}_k \sum_{\substack{X\in\mathcal{P}^c_k\setminus\mathcal{S}_k\\ \overline{X}=U}} \big(A_\mathcal{P}A^{-\frac{2\alpha}{1+2\alpha}}\big)^{|X|_k} \le \tfrac{1}{8}\eta\|K\|^{(A)}_k \end{aligned} \tag{4.8.16}$$

for $A\ge \big(\frac{8A_\mathcal{P}}{\eta\delta}\big)^{\frac{1+2\alpha}{2\alpha}}$. For the second contribution we observe that $\pi(X)$ is a single block for $X\in\mathcal{S}_k$, i.e., the second summand in (4.8.15) is only non-zero if $U\in\mathcal{B}_{k+1}$. Moreover we can bound the number of small polymers X that intersect a block $B'\in\mathcal{B}_{k+1}$ by $L^d(2^{d+1}+1)^{d2^d}$. Indeed there are L^d possibilities to pick the first block B of X and then all further blocks are contained in a cube of side-length $(2^{d+1}+1)L^k$ centred at B and there are at most 2^d of them.

This implies for $U\in\mathcal{B}_{k+1}$ and $A\ge A_\mathcal{P}$

$$\begin{aligned} A^{|U|_{k+1}} \sum_{\substack{X\in\mathcal{S}_k\setminus\mathcal{B}_k\\ \pi(X)=U}} \|\boldsymbol{R}_{k+1}K(X)\|_{k:k+1,X} &\le A \sum_{\substack{X\in\mathcal{S}_k\setminus\mathcal{B}_k\\ \pi(X)=U}} {A_\mathcal{P}}^{|X|_k}\|K(X)\|_{k,X} \le A\|K\|^{(A)}_k \sum_{\substack{X\in\mathcal{S}_k\setminus\mathcal{B}_k\\ \pi(X)=U}} \Big(\frac{A_\mathcal{P}}{A}\Big)^{|X|_k} \\ &\le A\|K\|^{(A)}_k L^d(2^{d+1}+1)^{d2^d}\frac{{A_\mathcal{P}}^2}{A^2} \le \tfrac{1}{8}\eta\|K\|^{(A)}_k \end{aligned} \tag{4.8.17}$$

for $A\ge 8\eta^{-1}{A_\mathcal{P}}^2L^d(2^{d+1}+1)^{d2^d}$. □

Next we consider the contribution from single blocks. Recall from (4.8.5) that for a k-block B we defined $G(B)(\varphi)=(1-\Pi_2)\boldsymbol{R}_{k+1}K(B,\varphi)$.

Lemma 4.8.3. *Assume that $L\ge 2^{d+3}+16R$. Then we have*

$$|G(B)|_{k+1,B,T_\varphi} \le A_\mathcal{B}(1+|\varphi|_{k+1,B})^5\left(C_1L^{-d'}+8(C_2+1)L^{-\frac{3}{2}d}w^B_{k:k+1}(\varphi)\right)\|K\|_{k,B}. \tag{4.8.18}$$

where

$$d' = d/2 + \lfloor d/2 \rfloor + 1 > d \tag{4.8.19}$$

and where $A_{\mathcal{B}}$ is the constant which appears in the integration estimate for the weights in Theorem 4.5.1 x). The constant C_1 is the constant in the estimate $|(1-\Pi_2)K(B)|_{k+1,B,0} \leq C_1 L^{-d'}|K(B)|_{k,B,0}$ in Lemma 4.6.9 while the constant C_2 is the constant in the estimate $|\Pi_2 K(B)|_{k+1,B,0} \leq C_2 |K(B)|_{k,B,0}$ in Lemma 4.6.7.

Proof. From the two norm estimate (4.6.2) and the contraction estimate (4.6.56) we get

$$\begin{aligned} &|G(B)|_{k+1,B,T_\varphi} \\ &\leq (1+|\varphi|_{k+1,B})^3 \left(|(1-\Pi_2)\boldsymbol{R}_{k+1}K(B)|_{k+1,B,T_0} + 8L^{-\frac{3}{2}d} \sup_{0\leq t\leq 1} |(1-\Pi_2)\boldsymbol{R}_{k+1}K(B)|_{k,B,T_{t\varphi}}\right) \\ &\leq (1+|\varphi|_{k+1,B})^3 \left(C_1 L^{-d'}|\boldsymbol{R}_{k+1}K(B)|_{k,B,T_0} + 8L^{-\frac{3}{2}d} \sup_{0\leq t\leq 1} |(1-\Pi_2)\boldsymbol{R}_{k+1}K(B)|_{k,B,T_{t\varphi}}\right) \end{aligned} \tag{4.8.20}$$

Now by Jensen's inequality and the estimate (4.5.27) in Theorem 4.5.1 x) with $\varphi = 0$

$$\begin{aligned} |\boldsymbol{R}_{k+1}K(B)|_{k,B,T_0} &\leq \int_{\mathcal{X}_N} |K(B)|_{k,B,T_\xi}\, \mu_{k+1}(\mathrm{d}\xi) \\ &\leq \int_{\mathcal{X}_N} \|K\|_{k,B}\, w_k^B(\xi)\; \mu_{k+1}(\mathrm{d}\xi) \leq A_{\mathcal{B}}\, \|K\|_{k,B}. \end{aligned} \tag{4.8.21}$$

The second term is bounded similarly. By (4.6.54), (4.6.53), Lemma 4.6.7 and (4.8.21) we get, for all $t \in [0,1]$,

$$\begin{aligned} &|\Pi_2 \boldsymbol{R}_{k+1}K(B)|_{k,B,T_{t\varphi}} \leq (1+|\varphi|_{k,B})^2 \|\Pi_2 \boldsymbol{R}_{k+1}K(B)\|_{k,0} \\ &\leq C_2(1+|\varphi|_{k,B})^2 |\boldsymbol{R}_{k+1}K(B)|_{k,B,T_0} \leq C_2(1+|\varphi|_{k,B})^2 A_{\mathcal{B}}\, \|K\|_{k,B}. \end{aligned} \tag{4.8.22}$$

Using the monotonicity of $t \mapsto w_{k:k+1}(t\varphi)$ we get from (4.6.20) in Lemma 4.6.4

$$|\boldsymbol{R}_{k+1}K(B)|_{k,B,T_{t\varphi}} \leq w_{k:k+1}^B(\varphi)\, \|\boldsymbol{R}_{k+1}K(B)\|_{k:k+1,B} \leq A_{\mathcal{B}}\, w_{k:k+1}^B(\varphi)\, \|K(B)\|_{k,B}. \tag{4.8.23}$$

Since $|\varphi|_{k,B} \leq |\varphi|_{k+1,B}$ the estimate (4.8.18) now follows from (4.8.20), (4.8.21), (4.8.22) and (4.8.23). □

Lemma 4.8.4. *Assume that $L \geq 2^{d+3} + 16R$ and that $h \geq h_0(L)$ where $h_0(L)$ satisfies (4.5.17). Let $B' \in \mathcal{B}_{k+1}$ be a $k+1$ block and recall that $G(B') = \sum_{B\in\mathcal{B}_k(B')} G(B)$. Then*

$$|G(B)|_{k+1,B',T_\varphi} \leq C' A_{\mathcal{B}}\big(C_1 L^{-d'} + 8(C_2+1)L^{-\frac{3}{2}d}\big)\, w_{k+1}^{B'}(\varphi)\, \|K\|_{k,B}. \tag{4.8.24}$$

and

$$\|G(B')\|_{k+1,B'} \leq C' A_{\mathcal{B}}\big(C_1 L^{d-d'} + 8(C_2+1)L^{-\frac{1}{2}d}\big)\, \|K\|_{k,B}. \tag{4.8.25}$$

where

$$C' = \max_{x\geq 0}(1+x)^5 e^{-\frac{1}{2}x^2}.$$

In particular there exists an L_0 such that for $L \geq L_0$ and $h \geq h_0(L)$

$$\|G\|_{k+1}^{(A)} \leq \tfrac{1}{2}\eta \|K\|_k^{(A)} \quad \forall A \geq 1. \tag{4.8.26}$$

We may take

$$L_0 = \max\big((4\eta^{-1}C' A_{\mathcal{B}} C_1)^{\frac{1}{d'-d}}, (32\eta^{-1}C' A_{\mathcal{B}}(C_2+1))^{\frac{2}{d}}, 2^{d+3} + 16R\big). \tag{4.8.27}$$

Proof. Indeed by (4.5.25) in Theorem 4.5.1 viii) and the definition of C' we have

$$(1+|\varphi|_{k+1,B'})^5 \le (1+|\varphi|_{k+1,B'})^5\, w^{B'}_{k:k+1}(\varphi) \le C' w^{B'}_{k+1}(\varphi).$$

Since $|\varphi|_{k+1,B} \le |\varphi|_{k+1,B'}$ and $w^{B}_{k:k+1} \le w^{B'}_{k:k+1}$ the estimate (4.8.24) follows from Lemma 4.8.3. Now (4.8.25) follows from (4.8.24) after summing over B, dividing by $w_{k+1,B'}(\varphi)$ and taking the supremum over φ. Finally (4.8.26) holds if we take L_0 so large that

$$C' A_{\mathcal{B}} C_1 L_0^{d-d'} \le \frac{1}{4}\eta \quad \text{and} \quad 8C' A_{\mathcal{B}}(C_2+1) L_0^{-d/2} \le \frac{1}{4}\eta. \tag{4.8.28}$$

Clearly both conditions are satisfied if L satisfies $L \ge L_0$ and L_0 is the number in (4.8.27). □

Proof of Lemma 4.8.1. This follows from (4.8.2), Lemma 4.8.2 and Lemma 4.8.4. □

4.8.2 Bound for the operator $\big(\boldsymbol{A}^{(\boldsymbol{q})}\big)^{-1}$

Lemma 4.8.5. *Let* $C_{2,0}$ *be the constant in* (4.4.13) *for* $\ell = 0$*. Then for*

$$h \ge \sqrt{C_{2,0}} \tag{4.8.29}$$

and $h_k = 2^k h$ *the operator* $\boldsymbol{A}^{(\boldsymbol{q})} : (M_0(\mathcal{B}_k), \|\cdot\|_{k,0}) \to (M_0(\mathcal{B}_{k+1}), \|\cdot\|_{k+1})$ *satisfies*

$$\|\big(\boldsymbol{A}^{(\boldsymbol{q})}\big)^{-1}\| \le \frac{3}{4}. \tag{4.8.30}$$

Proof. Let $H' = \boldsymbol{A}^{(\boldsymbol{q})} H$. As before we denote the coefficients of the expansion of H and H' in monomials by a_m and a'_m, respectively. Here $m \in \mathfrak{v}$. By (4.4.100) we have $a'_m = a_m$ for $m \ne \emptyset$ and

$$a'_\emptyset = a_\emptyset + \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} a_{(i,\alpha),(j,\beta)}\, (\nabla^\beta)^* \nabla^\alpha \mathcal{C}^{(\boldsymbol{q})}_{k+1,ij}(0). \tag{4.8.31}$$

Thus $\boldsymbol{A} := \boldsymbol{A}^{(\boldsymbol{q})}$ is invertible and by the definition (4.4.93) of the $\|\cdot\|_{k,0}$ norm in connection with the relations $h_{k+1} \ge 2h_k$ and $L \ge 2$ we get

$$\begin{aligned}
\|H\|_{k,0} &= L^{kd}\,|a_\emptyset| + \sum_{(i,\alpha)\in\mathfrak{v}_1} h_k L^{kd} L^{-k\frac{d-2}{2}} L^{-k|\alpha|}\,|a_{i,\alpha}| + \sum_{m\in\mathfrak{v}_2} h_k^2\,|a_m| \\
&\le L^{kd}\,|a'_\emptyset| + \sum_{(i,\alpha)\in\mathfrak{v}_1} h_k L^{kd} L^{-k\frac{d-2}{2}} L^{-k|\alpha|}\,|a'_{i,\alpha}| + \sum_{m\in\mathfrak{v}_2} h_k^2\,|a'_m| \\
&\quad + L^{kd} \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} |a_{(i,\alpha),(j,\beta)}|\,|(\nabla^\beta)^*\nabla^\alpha \mathcal{C}^{(\boldsymbol{q})}_{k+1,ij}(0)| \\
&\le \tfrac{1}{2}\|H'\|_{k+1,0} + L^{kd} \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} |a_{(i,\alpha),(j,\beta)}|\,|(\nabla^\beta)^*\nabla^\alpha \mathcal{C}^{(\boldsymbol{q})}_{k+1,ij}(0)|
\end{aligned} \tag{4.8.32}$$

The bound (4.4.13) implies that for $((i,\alpha),(j,\beta)) \in \mathfrak{v}_2$

$$\Big|(\nabla^\beta)^*\nabla^\alpha \mathcal{C}^{(\boldsymbol{q})}_{k+1,ij}(0)\Big| \le C_{2,0}\, L^{-kd}. \tag{4.8.33}$$

Using in addition that

$$\sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} |a_{(i,\alpha),(j,\beta)}| = \sum_{(i,\alpha),(j,\beta)\in\mathfrak{v}_2} |a'_{(i,\alpha),(j,\beta)}| \le \frac{\|H'\|_{k+1,0}}{h_{k+1}^2} \tag{4.8.34}$$

and $h_{k+1} = 2^{k+1}h \geq 2h$ we conclude that

$$\|\boldsymbol{A}^{-1}H'\|_{k,0} \leq \frac{1}{2}\|H'\|_{k+1,0} + \frac{C_{2,0}\|H'\|_{k+1,0}}{h_{k+1}^2} \leq \frac{3}{4}\|H'\|_{k+1,0} \tag{4.8.35}$$

provided that $h^2 \geq C_{2,0}$. □

4.8.3 Bound for the operator $\boldsymbol{B}^{(\boldsymbol{q})}$

Recall from (4.4.101) that $\boldsymbol{B}_k^{(\boldsymbol{q})} : (M(\mathcal{P}_k^{\mathrm{c}}), \|\cdot\|_k^{(A)}) \to (M_0(\mathcal{B}_{k+1}), \|\cdot\|_{k+1,0})$ is defined by

$$(\boldsymbol{B}_k^{(\boldsymbol{q})}K)(B', \varphi) = \sum_{B \in \mathcal{B}_k(B')} \Pi_2\Big(\int_{\mathcal{X}_N} K(B, \varphi + \xi)\, \mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\Big) \tag{4.8.36}$$

Lemma 4.8.6. *Assume that*

$$L \geq 2^{d+3} + 16R, \tag{4.8.37}$$

and

$$A \geq A_0 := 3C_2 A_{\mathcal{B}} L^d \tag{4.8.38}$$

where C_2 is the constant in Lemma 4.6.7 and $A_{\mathcal{B}}$ is the constant in Theorem 4.5.1 x). Then the operator norm of $\boldsymbol{B}^{(\boldsymbol{q})}$ satisfies

$$\|\boldsymbol{B}^{(\boldsymbol{q})}\| \leq \frac{C_2 A_{\mathcal{B}} L^d}{A} \leq \frac{1}{3}. \tag{4.8.39}$$

Proof. Set $H'(B') = -(B_k^{(\boldsymbol{q})}K)(B')$. For a $B \in \mathcal{B}_k(B')$ set $H(B) = -\Pi_2 \boldsymbol{R}^{(\boldsymbol{q})}K(B)$. Then $H(B)$ can be written as

$$H(B) = \sum_{x \in B} \sum_{m \in \mathfrak{v}} a_m \,\mathcal{M}_m(\{x\}).$$

By translation invariance $H'(B')$ can be written as

$$H'(B') = \sum_{x \in B'} \sum_{m \in \mathfrak{v}} a_m \,\mathcal{M}_m(\{x\})$$

with the *same* coefficients a_m. Thus it follows from the definition (4.4.93) of the norm $\|\cdot\|_{k,0}$ on relevant Hamiltonians and the relation $h_{k+1} = 2h_k$ that

$$\|\boldsymbol{B}^{(\boldsymbol{q})}K\|_{k+1,0} \leq \max(L^d, 2L^{d/2}, 4)\,\|\Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})}K(B)\|_{k,0} \leq L^d\,\|\Pi_2 \boldsymbol{R}_{k+1}^{(\boldsymbol{q})}K(B)\|_{k,0}. \tag{4.8.40}$$

Lemma 4.6.7 and (4.8.21) (which is a consequence of (4.5.27)) imply that

$$\|\Pi_2(\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}K)(B)\|_{k,0} \leq C_2 |\boldsymbol{R}_{k+1}^{(\boldsymbol{q})}K(B)|_{k,B,T_0} \leq C_2 A_{\mathcal{B}} \|K(B)\|_{k,B}. \tag{4.8.41}$$

Since $\|K(B)\|_{k,B} \leq A^{-1}\|K\|_k^{(A)}$ the desired assertion follows. □

4.9 Proofs of the main results

4.9.1 Main result of the renormalisation analysis

We fix $\zeta \in (0,1)$ and we recall from (3.2.38) that the Banach space $\boldsymbol{E}$ consists of functions $\mathcal{K} : \mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}} \to \mathbb{R}$ such that that the following norm is finite

$$\|\mathcal{K}\|_\zeta = \sup_{z \in \mathcal{G}} \sum_{|\alpha| \le r_0} \frac{1}{\alpha!} |\partial^\alpha \mathcal{K}(z)| e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}(z)}. \tag{4.9.1}$$

Recall that $\eta \in (0, \frac{2}{3}]$ is a parameter controlling the rate of contraction of the renormalisation flow.

Theorem 4.9.1. *Let $\kappa = \kappa(L)$ be as in Theorem 4.5.1. Moreover, let L_0, $h_0(L)$, $A_0(L)$, $\rho(A)$, $C_{j_1,j_2,j_3}(L,A)$, and $C_\ell(L,A)$ be such that the conclusions of Theorem 4.4.7 and Theorem 4.4.8 hold for every triple (L,h,A) with $L \ge L_0$, $h \ge h_0(L)$, $A \ge A_0(L)$. Assume also that*

$$h_0(L) \ge \delta(L)^{-\frac{1}{2}} \tag{4.9.2}$$

where $\delta(L)$ is the constant introduced in (4.5.57) *in Lemma 4.5.5.*

Then for every triple (L,h,A) with $L \ge L_0$, $h \ge h_0(L)$, and $A \ge A_0(L)$ there exists a $\varrho = \varrho(L,h,A) > 0$ such that for each $N \ge 1$ there are C^∞ maps $\widehat{e}_N : B_\varrho(0) \subset \boldsymbol{E} \to \mathbb{R}$, $\widehat{\boldsymbol{q}}_N : B_\varrho(0) \subset \boldsymbol{E} \to B_\kappa(0) \subset \mathbb{R}^{(d\times m)\times(d\times m)}_{sym}$ and $\widehat{K}_N : B_\varrho(0) \subset \boldsymbol{E} \to \boldsymbol{M}_N^{(A)}$ (defined in (4.7.1)*)with the following properties. For each $\mathcal{K} \in B_\varrho(0) \subset E$*

$$\int_{\mathcal{X}_N} \sum_{X \subset T_N} \prod_{x \in X} \mathcal{K}(D\varphi(x))\, \mu^{(0)}(\mathrm{d}\varphi) = \frac{Z_N^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))} e^{L^{Nd}\widehat{e}_N(\mathcal{K})}}{Z_N^{(0)}} \int_{\mathcal{X}_N} \Big(1 + \widehat{K}_N(\mathcal{K})(\Lambda_N, \varphi)\Big)\, \mu_{N+1}^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))}(\mathrm{d}\varphi) \tag{4.9.3}$$

where $Z_N^{(\boldsymbol{q})}$ denotes the normalisation introduced in (4.4.21)*. The derivatives of these maps satisfy bounds that are uniform in N and the map $\widehat{K}_N$ is contracting in the sense that there is a constant $C > 0$ such that for all $\ell \ge 0$*

$$\frac{1}{\ell!} \|\partial_{\mathcal{K}}^\ell \widehat{K}_N(\mathcal{K})(\dot{\mathcal{K}}, \ldots, \dot{\mathcal{K}})\|_N^{(A)} \le C_\ell(L,h,A)\, \eta^N\, \|\dot{\mathcal{K}}\|_\zeta^\ell \tag{4.9.4}$$

Moreover

$$\int_{\mathcal{X}_N} |\widehat{K}_N(\mathcal{K})(\Lambda_N, \varphi)|\, \mu_{N+1}^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))}(d\varphi) \le \frac{1}{2}. \tag{4.9.5}$$

More generally the following identity holds for $f_N \in \mathcal{X}_N$ and $\mathcal{K} \in B_\varrho(0)$

$$\begin{aligned} &\int_{\mathcal{X}_N} e^{(f_N,\varphi)} \sum_{X \subset T_N} \prod_{x \in X} \mathcal{K}(D\varphi(x))\, \mu^{(0)}(\mathrm{d}\varphi) \\ &= e^{\frac{1}{2}(f_N, \mathcal{C}^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))} f_N)} \frac{Z_N^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))} e^{L^{Nd}\widehat{e}_N(\mathcal{K})}}{Z_N^{(0)}} \int_{\mathcal{X}_N} \Big(1 + \widehat{K}_N(\mathcal{K})(\Lambda_N, \varphi + \mathcal{C}^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))} f_N)\Big)\, \mu_{N+1}^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))}(\mathrm{d}\varphi). \end{aligned} \tag{4.9.6}$$

We may take ϱ as the minimum of the radius $\tilde{\varrho}$ in Lemma 4.10.6 and $\frac{A}{2A_{\mathcal{B}}C_{1,(4.9.4)}}$.

Actually the proof shows that we may take ϱ as the minimum $\tilde{\varrho}$ in Lemma 4.10.6 and $\frac{A}{2A_{\mathcal{B}}C_{1,(4.9.4)}}\eta^{-N}$. Thus for $N \ge N_0(L,h,A)$ we may take ϱ simply as in Lemma 4.10.6.

We will prove this theorem at the end of Section 4.10. In the remainder of the current section we show how Theorem 4.9.1 implies the main results in Section 3.2.

4.9.2 Proof of the main theorem

Proof of Theorem 3.2.2. Choose the parameter ϱ in the statement of Theorem 3.2.3 as the number $\varrho(L, h_0(L), A_0(L))$ in Theorem 4.9.1. We apply first (3.2.35) and then (3.2.33) and (4.9.3) from Theorem 4.9.1 and obtain that the perturbative free energy can be expressed as

$$\begin{aligned}\overline{\mathcal{W}}_N(\mathcal{K}) &= -\frac{1}{L^{Nd}} \ln \mathcal{Z}_N(\mathcal{K}, \mathcal{Q}, 0)\\ &= -\widehat{e}_N(\mathcal{K}) - \frac{1}{L^{Nd}} \ln \left(\frac{Z_N^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))}}{Z_N^{(0)}} \right) - \frac{1}{L^{Nd}} \ln \left(\int_{\mathcal{X}_N} \left(1 + \widehat{K}_N(\mathcal{K})(\Lambda_N, \varphi)\right) \mu_{N+1}^{(\widehat{\boldsymbol{q}}_N(\mathcal{K}))}(\mathrm{d}\varphi) \right).\end{aligned} \tag{4.9.7}$$

The first term is C^∞ uniformly in N by Theorem 4.9.1. Similarly the second term is C^∞ uniformly in N by Theorem 4.9.1, Lemma 4.9.2 below and the chain rule.

To address the last term we introduce the notation

$$G(K_N, \boldsymbol{q}) = \int_{\mathcal{X}_N} K_N(X, \varphi)\, \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \boldsymbol{R}_{N+1}^{(\boldsymbol{q})} K_N(\Lambda_N, 0). \tag{4.9.8}$$

Then the last term is given by $L^{-Nd} \ln(1 + G(\widehat{K}_N(\mathcal{K}), \widehat{\boldsymbol{q}}_N(\mathcal{K})))$. Note that for any positive function G the derivative $D^k \ln(1+G)$ is given by a polynomial in derivatives of G divided by $(1+G)^k$. It follows from (4.9.5) that $1 + G \geq \frac{1}{2}$. By the chain rule it is sufficient to show that $G : B_\kappa(0) \times \boldsymbol{M}_N^{(A)} \to \mathbb{R}$ is smooth because $\widehat{\boldsymbol{q}}(\mathcal{K})$ and $\widehat{K}_N(\mathcal{K})$ are smooth functions. For the derivatives with respect to $\boldsymbol{q}$ we use (4.6.21) from Lemma 4.6.4 to estimate

$$\begin{aligned}|\partial_{\boldsymbol{q}}^\ell G(K_N, \boldsymbol{q}))| &\leq \|\partial_{\boldsymbol{q}}^\ell \boldsymbol{R}_{N+1}^{(\boldsymbol{q})} K_N(\Lambda_N)\|_{N:N+1,\Lambda_N}\\ &\leq C_\ell(L)\, \frac{A_{\mathcal{B}}}{A} \|\widehat{K}_N\|_N^{(A)}\end{aligned} \tag{4.9.9}$$

We have thus established that $\bar{\mathcal{W}}_N$ is C^∞ with uniform bounds. □

To show smoothness of the second term on the right hand side of (4.9.7) we used the following result.

Lemma 4.9.2. *Let $f_N(\boldsymbol{q}) = \frac{1}{L^{Nd}} \ln \left(\frac{Z^{(\boldsymbol{q})}}{Z^{(0)}} \right)$. Then $f_N \in C^\infty(B_{\omega_0/2}(0))$ and the derivatives of f_N can be bounded uniformly in N.*

Proof. To emphasise dependence on N we denote the operator $\mathcal{A}^{(\boldsymbol{q})}$ on $L^2(\mathcal{X}_N)$ defined in (4.4.20) temporarily by $\mathcal{A}_N^{(\boldsymbol{q})}$. Fourier transform diagonalises this operator in the scalar case $m = 1$ and block-diagonalises it for general m (with $m \times m$ blocks). By (4.4.9) the Fourier transform is given by

$$\widehat{\mathcal{A}_N^{(\boldsymbol{q})}}(p) = \sum_{\alpha,\beta \in \mathcal{I}} \overline{q}(p)^\alpha \boldsymbol{Q}_{\alpha\beta} q(p)^\beta + \sum_{|\alpha|=|\beta|=1} \overline{q}(p)^\alpha \boldsymbol{q}_{\alpha\beta} q(p)^\beta \tag{4.9.10}$$

where $\boldsymbol{q}_{\alpha\beta}$ denotes the $m \times m$ matrix with entries $\boldsymbol{q}_{(\alpha,i)(\beta,j)}$ and the j-th component of $q(p)$ is given by $q_j(p) = e^{ip_j} - 1$. Since $\boldsymbol{q} \in B_{\omega/2}(0)$ it follows from (4.4.11) that $\mathcal{A}_N^{(\boldsymbol{q})}$ is positive definite and Gaussian calculus gives

$$\begin{aligned}f_N(\boldsymbol{q}) &= \frac{1}{L^{Nd}} \frac{1}{2} \ln \frac{\det \mathcal{A}_N^{(\boldsymbol{q})}}{\det \mathcal{A}_N^{(0)}} = \frac{1}{L^{Nd}} \frac{1}{2} \sum_{p \in \widehat{T}_N \setminus \{0\}} \ln \det \left(\widehat{\mathcal{A}_N^{(\boldsymbol{q})}}(p) \right) - \ln \det \left(\widehat{\mathcal{A}_N^{(0)}}(p) \right)\\ &= \frac{1}{L^{Nd}} \frac{1}{2} \sum_{p \in \widehat{T}_N \setminus \{0\}} \ln \det \left(\frac{1}{|p|^2} \widehat{\mathcal{A}_N^{(\boldsymbol{q})}}(p) \right) - \ln \det \left(\frac{1}{|p|^2} \widehat{\mathcal{A}_N^{(0)}}(p) \right)\end{aligned} \tag{4.9.11}$$

Now it follows from (4.9.10) and (4.4.11) that $\boldsymbol{q} \mapsto \frac{1}{|p|^2}\widehat{\mathcal{A}^{(\boldsymbol{q})}}(p)$ is linear in $\boldsymbol{q}$ and both $\frac{1}{|p|^2}\widehat{\mathcal{A}^{(\boldsymbol{q})}}(p)$ and its inverse are bounded uniformly in N and $p \in \widehat{T}_N \setminus \{0\}$. In particular $\det\left(\frac{1}{|p|^2}\widehat{\mathcal{A}_N^{(\boldsymbol{q})}}(p)\right)$ lies in a fixed compact subset of $(0,\infty)$. The determinant is smooth and the logarithm is smooth away from 0. Since the sum contains $L^{dN}-1$ terms it follows that the function f_N is smooth and the derivatives are bounded uniformly in N. □

4.9.3 Proof of the scaling limit

In the setting of [4] the scaling limit was derived by Hilger [105]. Here we follow a similar strategy. In this section $\mathcal{K}$ is fixed and we use the abbreviations

$$e_N = \widehat{e}_N(\mathcal{K}), \quad \boldsymbol{q}_N = \widehat{\boldsymbol{q}}_N(\mathcal{K}), \quad K_N = \widehat{K}_N(\mathcal{K}). \tag{4.9.12}$$

Proof of Theorem 3.2.7. Recall that we consider $f \in C^\infty(\mathbb{T}^d;\mathbb{R}^m)$ where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and define rescaled functions on $\Lambda_N = \mathbb{Z}^d/(L^N\mathbb{Z}^d)$ by

$$f_N(x) = L^{-N\frac{d+2}{2}} f(L^{-N}x) - c_N \tag{4.9.13}$$

where the constant c_N is chosen so that

$$\sum_{x\in T_N} f_N(x) = 0. \tag{4.9.14}$$

Note that in the statement of Theorem 3.2.7 we did not subtract to constant from f_N. Since $(c_N,\varphi)=0$ for all $\varphi\in\mathcal{X}_N$ subtracting the constant does, however, not affect the statement of Theorem 3.2.7.

We rewrite the right hand side of equation (3.2.58) using the definition (3.2.33) and (4.9.6) from Theorem 4.9.1

$$\frac{\mathcal{Z}(\mathcal{K},\mathcal{Q},f_N)}{\mathcal{Z}(\mathcal{K},\mathcal{Q},0)} = e^{\frac{1}{2}(f_N,\mathcal{C}_N^{(\boldsymbol{q}_N)}f_N)} \frac{\int_{\mathcal{X}_N}\left(1+K_N(\Lambda_N,\varphi+\mathcal{C}^{(\boldsymbol{q}_N)}f_N)\right)\mu_{N+1}^{(\boldsymbol{q}_N)}(\mathrm{d}\varphi)}{\int_{\mathcal{X}_N}\left(1+K_N(\Lambda_N,\varphi)\right)\mu_{N+1}^{(\boldsymbol{q}_N)}(\mathrm{d}\varphi)}. \tag{4.9.15}$$

Here we used that the contribution of the term $\frac{Z^{(\boldsymbol{q}_N)}e^{L^{Nd}e_N}}{Z^{(0)}}$ in (4.9.6) cancels in the quotient because that term does not depend on f_N. The matrix $\boldsymbol{q}_N$ depends on N. Since $\boldsymbol{q}_N$ is bounded independently of N we find a subsequence $N_\ell\to\infty$ such $\boldsymbol{q}_{N_\ell}$ converges to $\boldsymbol{q}$. In the following we only consider this subsequence, but for ease of notation we still write $\boldsymbol{q}_N$. One can actually show convergence of the whole sequence [106] using the techniques from [47].

We consider the two terms on the right hand side of (4.9.15) in two steps. First we show that the second term converges to 1 by showing that this holds for the numerator and the denominator. In fact it suffices to show convergence for the numerator since the denominator corresponds to the special case $f_N=0$. Theorem 4.5.1 x) and (4.9.4) imply that

$$\begin{aligned}\left|\int_{\mathcal{X}_N} K_N(\Lambda_N,\varphi+\mathcal{C}^{(\boldsymbol{q}_N)}f_N)\,\mu_{N+1}^{(\boldsymbol{q}_N)}(\mathrm{d}\varphi)\right| &\le \frac{\|K_N\|_N^{(A)}}{A}\int_{\mathcal{X}_N} w_N^{\Lambda_N}(\varphi+\mathcal{C}^{(\boldsymbol{q}_N)}f_N)\,\mu_{N+1}^{(\boldsymbol{q}_N)}(\mathrm{d}\varphi)\\ &\le C\eta^N\frac{1}{A}A_{\mathcal{B}}\,w_{N:N+1}^{\Lambda_N}(\mathcal{C}^{(\boldsymbol{q}_N)}f_N).\end{aligned} \tag{4.9.16}$$

The weight function can be bounded using Theorem 4.5.1 ii)

$$\ln(w_{N:N+1}^{\Lambda_N}(\mathcal{C}^{(\boldsymbol{q}_N)}f_N)) \le \frac{1}{2\lambda}\left(\mathcal{C}_N^{(\boldsymbol{q}_N)}f_N, \boldsymbol{M}_N\mathcal{C}_N^{(\boldsymbol{q}_N)}f_N\right) \tag{4.9.17}$$

By (4.4.11) the Fourier modes of the kernel of $\mathcal{C}^{(\boldsymbol{q}_N)}$ satisfy $|\widehat{\mathcal{C}}^{(\boldsymbol{q}_N)}(p)| \leq C|p|^{-2} \leq CL^{2N}$. Recall that $q_i(p) = e^{ip_i} - 1$ and $q(p)^\alpha = \prod_{i=1}^d q_i(p)^{\alpha_i}$ for any multiindex $\alpha \in \mathbb{N}^d$. Using the Plancherel identity (4.4.7) we get

$$\begin{aligned}\left(\mathcal{C}^{(\boldsymbol{q}_N)} f_N, \boldsymbol{M}_N \mathcal{C}^{(\boldsymbol{q}_N)} f_N\right) &= L^{-Nd} \sum_{1\leq|\alpha|\leq M} \sum_{p\in\widehat{T}_N} (\widehat{\mathcal{C}}^{(\boldsymbol{q}_N)}(p)\widehat{f}_N(p), L^{2N(|\alpha|-1)}|q(p)^{2\alpha}|\widehat{\mathcal{C}}^{(\boldsymbol{q}_N)}(p)\widehat{f}_N(p)) \\ &= L^{-Nd-2N} \sum_{1\leq|\alpha|\leq M} \sum_{p\in\widehat{T}_N} L^{2N|\alpha|}|q(p)^{2\alpha}|\,|\widehat{\mathcal{C}}^{(\boldsymbol{q}_N)}(p)\widehat{f}_N(p)|^2 \\ &\leq CL^{-Nd+2N} \sum_{1\leq|\alpha|\leq M} \sum_{p\in\widehat{T}_N} L^{2N|\alpha|}|q(p)^{2\alpha}||\widehat{f}_N(p)|^2 \\ &= CL^{2N} \sum_{1\leq|\alpha|\leq M} L^{2N|\alpha|}(\nabla^\alpha f_N, \nabla^\alpha f_N).\end{aligned} \tag{4.9.18}$$

To estimate the discrete derivatives at x we apply a Taylor expansion of f of order r. This gives

$$\begin{aligned}f_N(x+a) &= L^{-N\frac{d+2}{2}} f(L^{-N}x + L^{-N}a) \\ &= L^{-N\frac{d+2}{2}} \left(\sum_{0\leq\beta\leq r} \frac{(L^{-N}a)^\beta}{\beta!} \partial^\beta f(L^{-N}x) + R_r \right)\end{aligned} \tag{4.9.19}$$

where R_r denotes the remainder that can be bounded by $C_{r+1}|\nabla^{r+1}f|_\infty |L^{-N}a|^{r+1}$. Since the discrete derivative of order $|\alpha|$ annihilates polynomials up to order $|\alpha|-1$ and since the discrete derivative is a bounded operator the identity (4.9.19) implies that

$$|\nabla^\alpha f_N(x)| \leq C_{|\alpha|} L^{-N\frac{d+2}{2}} |\nabla^{|\alpha|} f|_\infty L^{-N|\alpha|} \tag{4.9.20}$$

and thus

$$L^{2N} \sum_{1\leq|\alpha|\leq M} L^{2N|\alpha|}(\nabla^\alpha f_N, \nabla^\alpha f_N) \leq C \sum_{r=0}^{M} |\nabla^r f|_\infty. \tag{4.9.21}$$

Combining (4.9.16), (4.9.17), (4.9.18), and (4.9.21) we conclude that

$$\left| \int_{\mathcal{X}_N} K_N(\Lambda_N, \varphi + \mathcal{C}^{(\boldsymbol{q}_N)} f_N)\, \mu_{N+1}^{(\boldsymbol{q}_N)}(\mathrm{d}\varphi) \right| \leq C\eta^N \frac{A_{\mathcal{B}}}{A} \exp\left(C \sum_{r=0}^{M} |\nabla^r f|_\infty \right) \to 0 \tag{4.9.22}$$

as $N \to \infty$. This implies that the numerator on the right hand side of (4.9.15) converges to 1.

The second step is to prove the convergence of the the prefactor

$$\frac{1}{2}(f_N, \mathcal{C}_N^{(\boldsymbol{q}_N)}\ f_N) \to \frac{1}{2}(f, \mathcal{C}_{\mathbb{T}^d} f). \tag{4.9.23}$$

To show this we change the scaling of the system. Usually we think that the system size grows with N while the distance between the atoms remains fixed , but now it is more convenient to fix the system size and to let the distance between the atoms go to zero.

We define the rescaled torus T_N' and the corresponding dual torus $\widehat{T}_N'$ in Fourier space by

$$T_N' = L^{-N} T_N, \quad \widehat{T}_N' = L^N \widehat{T}_N. \tag{4.9.24}$$

Recall from (4.4.4) that

$$\widehat{T}'_N = \{\xi \in (2\pi\mathbb{Z})^d : |\xi|_\infty \le (L^N - 1)\pi\} \tag{4.9.25}$$

(here we use that L is odd and hence $L^N - 1$ is even and we identify the dual torus with its fundamental domain). To make the notation clearer we will write x and z for coordinates in T_N and T'_N, respectively and similarly p and ξ for coordinates in $\widehat{T}_N$ and $\widehat{T}'_N$, respectively. Note that there is an inclusion $T'_N \to (\mathbb{R}/\mathbb{Z})^d = \mathbb{T}^d$. For a function $g : T'_N \to \mathbb{C}$ we define the discrete Fourier transform by

$$\hat{g}(\xi) := L^{-dN} \sum_{z \in T'_N} g(z) e^{-i\xi\cdot z} \quad \forall \xi \in \widehat{T}'_N. \tag{4.9.26}$$

The prefactor L^{-dN} is chosen so that for $g \in C^0(\mathbb{T}^d)$ the sum is the Riemann sum which corresponds to the integral for the coefficient in the Fourier series of g. For brevity we write for the rest of this subsection

$$\mathcal{C}_N = \mathcal{C}^{(\boldsymbol{q}_N)}. \tag{4.9.27}$$

This quantity should not be confused with the finite range decomposition at scale N. We define the rescaled functions $f'_N : T'_N \to \mathbb{R}^m$,

$$\begin{aligned} f'_N(z) &= L^{N\frac{(d+2)}{2}} f_N(L^N z) = f(z) \\ \mathcal{C}'_N(z) &= L^{N(d-2)} \mathcal{C}_N(L^N z). \end{aligned} \tag{4.9.28}$$

Note that the rescaling of $\mathcal{C}_N$ reflects the expected behaviour of the Green's function of the Laplacian, namely $\mathcal{C}_N(x) \sim |x|^{2-d}$. Then the corresponding Fourier transforms $\widehat{f'_N} : \widehat{T}'_N \to \mathbb{C}^m$, and $\widehat{\mathcal{C}'_N} : \widehat{T}'_N \to \mathbb{C}^{m\times m}_{\text{her}}$ satisfy

$$\begin{aligned} \widehat{f'_N}(\xi) &= L^{-Nd} \sum_{z \in T'_N} f(z) e^{-iz\xi} = L^{-N\frac{d-2}{2}} \sum_{x \in T_N} f_N(x) e^{-iL^{-N}\xi x} = L^{-N\frac{d-2}{2}} \widehat{f}_N(L^{-N}\xi) \\ \widehat{\mathcal{C}'_N}(\xi) &= L^{-2N} \widehat{\mathcal{C}}_N(L^{-N}\xi). \end{aligned} \tag{4.9.29}$$

Using this rescaling, Plancherel and the zero-average condition (4.9.14) we find that

$$(f_N, \mathcal{C}_N f_N) = \frac{1}{L^{Nd}} \sum_{p \in \widehat{T}_N \setminus \{0\}} (\widehat{f}_N(p), \widehat{\mathcal{C}}_N(p) \widehat{f}_N(p)) = \sum_{\xi \in \widehat{T}'_N \setminus \{0\}} (\widehat{f'_N}(\xi), \widehat{\mathcal{C}'_N}(\xi) \widehat{f'_N}(\xi)) \tag{4.9.30}$$

On the other hand the Plancherel identity and the fact that f has average 0 yield

$$(f, \mathcal{C} f) = \sum_{\xi \in (2\pi\mathbb{Z})^d \setminus \{0\}} (\widehat{f}(\xi), \widehat{\mathcal{C}}_{\mathbb{T}^d}(\xi) \widehat{f}(\xi)) \tag{4.9.31}$$

where the Fourier modes are given by

$$\begin{aligned} \widehat{f}(\xi) &= \int_{\mathbb{T}^d} f(z) e^{-i\xi z}, \\ \widehat{\mathcal{C}}_{\mathbb{T}^d}(\xi) &= \left(\sum_{i,j=1}^d \xi_i \xi_j (\boldsymbol{Q} + \boldsymbol{q})_{ij} \right)^{-1}. \end{aligned} \tag{4.9.32}$$

The last expression is well defined because $\boldsymbol{Q} + \boldsymbol{q}$ is positive definite. Now we show the pointwise convergence

$$\lim_{N\to\infty} (\widehat{f'_N}(\xi), \widehat{\mathcal{C}'_N}(\xi) \widehat{f'_N}(\xi)) = (\widehat{f}(\xi), \widehat{\mathcal{C}}_{\mathbb{T}^d}(\xi) \widehat{f}(\xi)) \tag{4.9.33}$$

for $\xi \in (2\pi\mathbb{Z})^d \setminus \{0\}$. First note that $\widehat{f}'_N(\xi) \to \widehat{f}(\xi)$ for all $\xi \in (2\pi\mathbb{Z})^d \setminus \{0\}$ because $\widehat{f}'_N(\xi)$ is a Riemann sum approximation of the integral for $\widehat{f}(\xi)$. For the covariance we observe that by (4.9.10)

$$\begin{aligned}\widehat{\mathcal{C}}'_N(\xi) &= \widehat{\mathcal{C}}_N(L^{-N}\xi)L^{-2N}\\ &= \left(\sum_{\alpha,\beta\in\mathcal{I}} L^N\overline{q}(L^{-N}\xi)^\alpha \boldsymbol{Q}_{\alpha\beta}L^N q(L^{-N}\xi)^\beta + \sum_{|\alpha|=|\beta|=1}^{d} L^N\overline{q}(L^{-N}\xi)^\alpha \boldsymbol{q}_{\alpha\beta}L^N q(L^{-N}\xi)^\beta\right)^{-1}.\end{aligned} \tag{4.9.34}$$

We have $L^N q(L^{-N}\xi)^\alpha = L^N(e^{iL^{-N}\xi_j} - 1) \to i\xi_j$ as $N \to \infty$ for $\alpha = e_j$ and $L^N q(L^{-N}\xi)^\alpha \to 0$ as $N \to \infty$ for $|\alpha| \geq 2$. Then the assumption $\boldsymbol{q}_N \to \boldsymbol{q}$ along the subsequence we consider and the fact that the inversion of matrices is continuous yield

$$\widehat{\mathcal{C}}'_N(\xi) \to \widehat{\mathcal{C}}_{\mathbb{T}^d}(\xi) \quad \text{as } N \to \infty \tag{4.9.35}$$

This establishes (4.9.33).

Next we show that the Fourier modes are uniformly bounded from above. Note that $|\widehat{\mathcal{C}}'_N(\xi)| = L^{-2N}|\widehat{\mathcal{C}}_N(L^{-N}\xi)| \leq C|\xi|^{-2}$ by (4.4.11). The definition of $q(p)$ and discrete integration by parts yield

$$|q(p)|^{2r}\widehat{f}_N(p) = \sum_{x\in T_N} f_N(x)\Delta^r e^{-ipx} = \sum_{x\in T_N} \Delta^r f_N(x)e^{-ipx}. \tag{4.9.36}$$

The estimate $|p| \leq 2|q(p)|$ the rescaling (4.9.29), and (4.9.20) imply for $\xi \in \widehat{T}'_N$ and $p = L^{-N}\xi$

$$\begin{aligned}|\xi|^{2r}|\widehat{f}'_N(\xi)| &= L^{2rN}|p|^{2r}L^{-N\frac{d-2}{2}}|\widehat{f}_N(p)|\\ &\leq C_r\, L^{2rN}L^{-N\frac{d-2}{2}}\sum_{x\in T_N}|\Delta^r f_N(x)|\\ &\leq C_r\, L^{2rN}L^{-N\frac{d-2}{2}}\sum_{x\in T_N}L^{-N\frac{d+2}{2}}\,|\nabla^{2r}f|_\infty\, L^{-2rN} \leq C_r|\nabla^{2r}f|_\infty.\end{aligned} \tag{4.9.37}$$

Note that by (4.4.11) and (4.9.29) we have $|\widehat{\mathcal{C}}'_N(\xi)| \leq CL^{-2N}L^{2N}|\xi|^{-2} \leq C|\xi|^{-2}$. We hence deduce that

$$(\widehat{f}'_N(\xi), \widehat{\mathcal{C}}'_N(\xi)\widehat{f}'_N(\xi)) \leq C_r|\xi|^{-2r-2}\,|\nabla^{2r}f|_\infty \qquad \forall \xi \in \widehat{T}'_N \setminus \{0\}. \tag{4.9.38}$$

For $r \geq \lfloor \frac{d}{2} \rfloor$ the right hand side is summable over $\xi \in (2\pi\mathbb{Z})^d \setminus \{0\}$ and the dominated convergence theorem and the pointwise convergence (4.9.33) imply that

$$\begin{aligned}\sum_{\xi\in\widehat{T}'_N\setminus\{0\}} (\widehat{f}'_N(\xi), \widehat{\mathcal{C}}'_N(\xi)\widehat{f}'_N(\xi)) &\underset{(4.9.25)}{=} \sum_{\xi\in(2\pi\mathbb{Z})^d\setminus\{0\}} 1_{|\xi|_\infty\leq(L^N-1)\pi}\,(\widehat{f}'_N(\xi), \widehat{\mathcal{C}}'_N(\xi)\widehat{f}'_N(\xi))\\ &\to \sum_{\xi\in(2\pi\mathbb{Z})^d\setminus\{0\}} (\widehat{f}(\xi), \widehat{\mathcal{C}}_{\mathbb{T}^d}(\xi)\widehat{f}(\xi)).\end{aligned} \tag{4.9.39}$$

Now (4.9.30) and (4.9.31) show that $(f_N, \mathcal{C}_N f_N) \to (f, \mathcal{C}_{\mathbb{T}^d} f)$ (along the subsequence considered).

□

4.10 Fine tuning of the initial condition

In this section we prove Theorem 4.9.1 by the use of a stable manifold theorem and an additional application of the implicit function theorem to determine the renormalised Hamiltonian. The setting for the stable manifold theorem is very similar to the situation in Theorem 2.16 of [42] but for completeness and for the convenience of the reader we provide a detailed proof.

The stable manifold theorem boils down to an application of the implicit function theorem to the whole trajectory of relevant and irrelevant interactions (H_k, K_k). We define the Banach space

$$\mathcal{Z} = \{Z = (H_0, H_1, \ldots, H_{N-1}, K_1, \ldots, K_N) : H_k \in M_0(\mathcal{B}_k), K_k \in M(\mathcal{P}_k^c)\} \tag{4.10.1}$$

equipped with the norm

$$\|Z\|_{\mathcal{Z}} = \max\left(\max_{0\le k\le N-1} \frac{1}{\eta^k}\|H_k\|_{k,0}, \max_{1\le k\le N} \frac{1}{\eta^k}\|K_k\|_k^{(A)}\right) \tag{4.10.2}$$

where

$$\eta \in \left(0, \tfrac{2}{3}\right]. \tag{4.10.3}$$

is a fixed parameter. Note that a bound on $\|Z\|_{\mathcal{Z}}$ implies exponential decay of the norms of H_k and K_k in k. The functionals H_N and K_0 do not appear in $\mathcal{Z}$ because we want to achieve the final condition $H_N = 0$ and we treat K_0 as a fixed initial condition, see (4.10.10) below.

We define a dynamical system $\mathcal{T}$ on $\mathcal{Z}$. The map $\mathcal{T}$ depends in addition on two parameters, a relevant Hamiltonian $\mathcal{H} \in M(\mathcal{B}_0)$ and the interaction $\mathcal{K} \in \boldsymbol{E}$. Here we fix $\zeta \in (0,1)$ and we recall from (3.2.38) that the Banach space $\boldsymbol{E}$ consists of functions $K : \mathcal{G} = (\mathbb{R}^m)^{\mathcal{I}} \to \mathbb{R}$ so that the following norm is finite

$$\|K\|_\zeta = \sup_{z\in\mathcal{G}} \sum_{|\alpha|\le r_0} \frac{1}{\alpha!}|\partial^\alpha K(z)|e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}_0(z)}. \tag{4.10.4}$$

The Hamiltonian $\mathcal{H}$ will eventually allow us to extract the correct Gaussian part in the measure (the renormalized covariance).

More precisely we consider a map $\mathcal{T} : \boldsymbol{E} \times M(\mathcal{B}_0) \times \mathcal{Z} \to \mathcal{Z}$ defined by $\mathcal{T}(\mathcal{K}, \mathcal{H}, Z) = \tilde{Z}$ where the coordinates of $\tilde{Z}$ are given by

$$\tilde{H}_0(\mathcal{K}, \mathcal{H}, Z) = (\boldsymbol{A}_0^{(\boldsymbol{q}(\mathcal{H}))})^{-1}\left(H_1 - \boldsymbol{B}_0^{(\boldsymbol{q}(\mathcal{H}))}\widehat{K}_0(\mathcal{K}, \mathcal{H})\right), \tag{4.10.5}$$

$$\tilde{H}_k(\mathcal{K}, \mathcal{H}, Z) = (\boldsymbol{A}_k^{(\boldsymbol{q}(\mathcal{H}))})^{-1}(H_{k+1} - \boldsymbol{B}_k^{(\boldsymbol{q}(\mathcal{H}))}K_k), \quad \text{for} \quad 1 \le k \le N-2, \tag{4.10.6}$$

$$\tilde{H}_{N-1}(\mathcal{K}, \mathcal{H}, Z) = -(\boldsymbol{A}_{N-1}^{(\boldsymbol{q}(\mathcal{H}))})^{-1}\boldsymbol{B}_{N-1}^{(\boldsymbol{q}(\mathcal{H}))}K_{N-1}, \tag{4.10.7}$$

$$\tilde{K}_{k+1}(\mathcal{K}, \mathcal{H}, Z) = \boldsymbol{S}_k(H_k, K_k, \boldsymbol{q}), \quad \text{for} \quad 1 \le k \le N-1, \tag{4.10.8}$$

$$\tilde{K}_1(\mathcal{K}, \mathcal{H}, Z) = \boldsymbol{S}_0(H_0, \widehat{K}_0(\mathcal{K}, \mathcal{H}), \boldsymbol{q}(\mathcal{H})). \tag{4.10.9}$$

Here the map $\widehat{K}_0$ is defined by

$$\widehat{K}_0(\mathcal{K}, \mathcal{H})(X, \varphi) = \exp\left(-\mathcal{H}(X, \varphi)\right) \prod_{x\in X} \mathcal{K}(D\varphi(x)). \tag{4.10.10}$$

and $\boldsymbol{q}(\mathcal{H})$ is the projection on the coefficients of the quadratic part of $\mathcal{H}$, i.e., $\boldsymbol{q}_{(i,\alpha)(j,\beta)} = \frac{1}{2}a_{(i,\alpha),(j,\beta)}$ for $(i,\alpha) < (j,\beta)$, $\boldsymbol{q}_{(i,\alpha)(j,\beta)} = \frac{1}{2}a_{(j,\beta),(i,\alpha)}$ for $(i,\alpha) > (j,\beta)$, and $\boldsymbol{q}_{(i,\alpha)(j,\beta)} = a_{(i,\alpha),(j,\beta)}$

for $(i,\alpha) = (j,\beta)$ where $a_{(i,\alpha),(j,\beta)}$ denotes the coefficients of the quadratic term of $\mathcal{H}$. The factor $\frac{1}{2}$ arises because $\boldsymbol{q}$ is symmetric. Note that the definition (4.10.7) of H_{N-1} reflects the final condition $H_N = 0$.

One easily sees that

$$\mathcal{T}(\mathcal{K},\mathcal{H},Z) = Z \quad \textit{if and only if}$$
$$\boldsymbol{T}_k(H_k, K_k, \boldsymbol{q}(\mathcal{H})) = (H_{k+1}, K_{k+1}) \quad \forall 0 \le k \le N-1 \quad \text{with } H_N = 0 \text{ and } K_0 = \widehat{K}_0(\mathcal{K},\mathcal{H}). \tag{4.10.11}$$

Here $\boldsymbol{T}_k$ is the renormalisation group map defined in Definition 4.4.5. Proposition 4.4.6 then implies that a fixed point of $\mathcal{T}$ satisfies

$$\int_{\mathcal{X}_N} (e^{-H_0} \circ \widehat{K_0}(\mathcal{K},\mathcal{H}))(\Lambda_N, \varphi+\psi)\, \mu^{(\boldsymbol{q}(\mathcal{H}))}(\mathrm{d}\varphi) = \int_{\mathcal{X}_N} \big(1 + K_N(\Lambda_N, \varphi+\psi)\big)\, \mu_{N+1}^{(\boldsymbol{q}(\mathcal{H}))}(\mathrm{d}\varphi). \tag{4.10.12}$$

4.10.1 Existence of a fixed point of the map $\mathcal{T}(\mathcal{K},\mathcal{H},\cdot)$

Theorem 4.10.1 below states that for sufficiently small $\mathcal{H}$ and $\mathcal{K}$ there is a unique fixed point $\hat{Z}(\mathcal{K},\mathcal{H})$ which depends smoothly on $\mathcal{K}$ and $\mathcal{H}$. In particular (4.10.12) holds with $H_0 = \Pi_{H_0}\widehat{Z}(\mathcal{K},\mathcal{H})$ and $K_N = \Pi_{K_N}\widehat{Z}(\mathcal{K},\mathcal{H})$ where $\Pi_{H_0} Z$ and $\Pi_{K_N} Z$ denote the projection onto the H_0 component and the K_N component, respectively. Now the right hand side of (4.10.12) deviates from 1 only by an error of order $O(\eta^N)$ and the left hand side of (4.10.12) looks very similar to the functional

$$\int_{\mathcal{X}} \sum_{X\subset\Lambda_N} \prod_{x\in X} \mathcal{K}(D\varphi(x))\, \mu^{(0)}(d\varphi) \tag{4.10.13}$$

which we want to study, but is in general not identical to it due to the presence of the terms $\Pi_{H_0}\widehat{Z}(\mathcal{K},\mathcal{H})$ and $\boldsymbol{q}(\mathcal{H})$. Another application of the implicit function theorem then leads to Lemma 4.10.6 below which shows that there exist an $\mathcal{H} = \widehat{\mathcal{H}}(\mathcal{K})$ such that $\Pi_{H_0}\widehat{Z}(\mathcal{K},\mathcal{H}) = \mathcal{H}$. Then short calculation shows that left hand side of (4.10.12) agrees with the expression (4.10.13) up to an explicit scalar factor which involves the constant term in $\mathcal{H}$ and the ratio $Z^{(\boldsymbol{q}(\mathcal{H}))}/Z^{(0)}$ of the Gaussian partition functions, see (4.10.79) below. From this representation we will easily deduce the main theorem of the previous section, Theorem 4.9.1.

Recall the convention that, say, $C_{(4.10.50)}$ denotes the constant which appears in equation (4.10.50).

Theorem 4.10.1. *Let $\kappa = \kappa(L)$ be as in Theorem 4.5.1. Moreover, let L_0, $h_0(L)$, $A_0(L)$, $\rho(A)$, $C_{j_1,j_2,j_3}(L,A)$, and $C_\ell(L,A)$ be such that the conclusions of Theorem 4.4.7 and Theorem 4.4.8 hold for every triple (L,h,A) with $L \ge L_0$, $h \ge h_0(L)$, $A \ge A_0(L)$. Assume also that*

$$h_0(L) \ge \max(\delta(L)^{-1/2}, 1) \tag{4.10.14}$$

where $\delta(L)$ is the constant introduced in (4.5.57). Then for every triple (L,h,A) that satisfies $L \ge L_0$, $h \ge h_0(L)$, $A \ge A_0(L)$ there exist constants $\rho_1 = \rho_1(h,A) > 0$, $\rho_2 = \rho_2(L) > 0$ and $\overline{C}_{j_1,j_2,j_3}$ such that $\mathcal{T}$ is smooth in $B_{\rho_1}(0) \times B_{\rho_2}(0) \times B_{\rho(A)}(0) \subset (M(\mathcal{B}_0); \|\cdot\|_{0,0}) \times \boldsymbol{E} \times \mathcal{Z}$,

$$\frac{1}{j_1!j_2!j_3!}\|D_{\mathcal{K}}^{j_1} D_{\mathcal{H}}^{j_2} D_Z^{j_3} \mathcal{T}(\mathcal{K},\mathcal{H},Z)(\dot{\mathcal{K}},\ldots\dot{\mathcal{H}},\ldots,\dot{Z})\|_{\mathcal{Z}} \le \overline{C}_{j_1,j_2,j_3}(L,A)\, \|\dot{\mathcal{K}}\|_\zeta^{j_1}\, \|\dot{\mathcal{H}}\|_{0,0}^{j_2}\, \|\dot{Z}\|_{\mathcal{Z}}^{j_3}$$
$$\forall\ (\mathcal{K},\mathcal{H},Z) \in B_{\rho_1}(0) \times B_{\rho_2}(0) \times B_{\rho(A)}(0) \tag{4.10.15}$$

and

$$\boldsymbol{q}(\mathcal{H}) \in B_\kappa(0) \qquad \forall \mathcal{H} \in B_{\rho_2}(0). \tag{4.10.16}$$

We may take

$$\rho_1(h,A) = \frac{\rho(A)}{2^{R_0 d r_0+3} h^{r_0} A}, \quad \rho_2(L) = \min\left(\frac{1}{8}, \frac{\kappa(L)}{C_{(4.10.50)}(m,d)}\right) \tag{4.10.17}$$

Moreover there exist $\epsilon = \epsilon(L,h,A) > 0$, $\epsilon_1 = \epsilon_1(L,h,A) > 0$, $\epsilon_2 = \epsilon_2(L,h,A) > 0$, and $C_{j_1,j_2}(L,A) > 0$ such that for each $(\mathcal{K},\mathcal{H}) \in B_{\epsilon_1}(0) \times B_{\epsilon_2}(0)$ there exists a unique $Z = \widehat{Z}(\mathcal{K},\mathcal{H})$ in $B_\epsilon(0)$ that satisfies

$$\mathcal{T}(\mathcal{K},\mathcal{H},\hat{Z}(\mathcal{K},\mathcal{H})) = \hat{Z}(\mathcal{K},\mathcal{H}). \tag{4.10.18}$$

The map $\hat{Z}$ is smooth in $B_{\epsilon_1}(0) \times B_{\epsilon_2}(0)$ and satisfies the bounds

$$\begin{aligned}\frac{1}{j_1! j_2!} \|D_{\mathcal{K}}^{j_1} D_{\mathcal{H}}^{j_2} \hat{Z}(\mathcal{K},\mathcal{H})(\dot{\mathcal{K}},\ldots,\dot{\mathcal{H}})\|_{\mathcal{Z}} \le C_{j_1,j_2}(L,h,A)\, \|\dot{\mathcal{K}}\|_\zeta^{j_1}\, \|\dot{\mathcal{H}}\|_{0,0}^{j_2} \\ \forall\ (\mathcal{K},\mathcal{H}) \in B_{\epsilon_1}(0) \times B_{\epsilon_2}(0).\end{aligned} \tag{4.10.19}$$

The parameters ϵ, ϵ_1 and ϵ_2 can be bounded from below by $\rho_1, \rho_2, \rho(A)$ and bounds on the first and second derivatives of $\mathcal{T}$. We may take

$$\epsilon = \min\left(\frac{1}{48\overline{C}_{0,0,2}}, \frac{\rho(A)}{2}\right), \quad \epsilon_1 = \min\left(\frac{1}{24\overline{C}_{1,0,1}}, \frac{\epsilon}{8\overline{C}_{1,0,0}}, \rho_1\right), \quad \epsilon_2 = \min\left(\frac{1}{24\overline{C}_{0,1,1}}, \rho_2\right) \tag{4.10.20}$$

where $\overline{C}_{j_1,j_2,j_3}$ are the constants in (4.10.15).

The condition (4.10.14) is implied by the conditions we use to prove Theorem 4.4.7 and Theorem 4.4.8. We added it here since in principle the conclusions of these theorems might hold under weaker conditions on L and h. Condition (4.10.14) is used in Lemma 4.10.2 which ensures smoothness of the map $(\mathcal{K},\mathcal{H}) \mapsto K_0$.

Proof of Theorem 4.10.1, Set-up. The proof is mostly along the lines of the proof of Proposition 8.1 in [4]. The situation here is, however, much simpler then in [4] because no loss of regularity occurs when we take derivative with respect to $\boldsymbol{q}$ (or $\mathcal{H}$). Thus we can use the usual implicit function theorem in Banach spaces which can be found, e.g., in Theorem 4.E. [154]. To apply the implicit function theorem we verify its assumptions.

Here Theorem 4.4.7 and Theorem 4.4.8 are the key ingredients. The first result gives smoothness of the maps $\tilde{K}_k$ (except for $k=1$) while Theorem 4.4.8 will be used to show that the derivatives of $\mathcal{T}$ are small. Then we can apply the implicit function theorem to the map $\mathcal{T} - \pi_3$ where π_3 is the projection on the third component. The main remaining point in showing smoothness of the map $\mathcal{T}$ is to show smoothness of the maps $(\mathcal{K},\mathcal{H}) \mapsto K_0$. We first state and prove this result. Then we will continue the proof of Theorem 4.10.1. □

Lemma 4.10.2. *Assume that $L \ge 5$ and*

$$h \ge \max(\delta(L)^{-1/2}, 1) \tag{4.10.21}$$

where $\delta(L)$ *is the constant defined in* (4.5.57). *Set*

$$\rho_1 = (2^{R_0 d r_0 + 3} h^{r_0} A)^{-1}, \quad \rho_2 = \frac{1}{8}. \tag{4.10.22}$$

Then the map $\widehat{K}_0 : (\boldsymbol{E}, \|\cdot\|_\zeta) \times (M_0(\mathcal{B}_0), \|\cdot\|_{0,0}) \to (M(\mathcal{P}_0^c), \|\cdot\|_0)$ *defined in* (4.10.10) *is smooth on* $B_{\rho_1}(0) \times B_{\rho_2}(0)$ *and there exist numerical constants* C_{j_2} *and* C_{j_1,j_2} *such that*

$$\frac{1}{j_2!}\|D^{j_2}_{\mathcal{H}} \widehat{K}_0(\mathcal{K},\mathcal{H})(\dot{\mathcal{H}},\dots,\dot{\mathcal{H}})\|_0^{(A)} \le C_{j_2}\, 2^{R_0 d r_0 + 2} h^{r_0} A \,\|\mathcal{K}\|_\zeta \;\|\dot{\mathcal{H}}\|^{j_2} \quad \forall (\mathcal{K},\mathcal{H}) \in B_{\rho_1}(0) \times B_{\rho_2}(0), \tag{4.10.23}$$

with

$$C_0 = 1 \tag{4.10.24}$$

and, for $j_1 \ge 1$,

$$\|D^{j_1}_{\mathcal{K}} D^{j_2}_{\mathcal{H}} \widehat{K}_0(\mathcal{K},\mathcal{H})(\dot{\mathcal{K}},\dots \dot{\mathcal{H}})\|_0^{(A)} \le C_{j_1,j_2} (2^{R_0 d r_0 + 3} h^{r_0} A)^{j_1} \,\|\dot{\mathcal{K}}\|_\zeta^{j_1} \;\|\dot{\mathcal{H}}\|^{j_2} \quad \forall (\mathcal{K},\mathcal{H}) \in B_{\rho_1}(0) \times B_{\rho_2}(0). \tag{4.10.25}$$

To prove this lemma we decompose K_0 in a series of maps and show smoothness for each of them. Then the chain rule implies the claim. It is convenient to introduce the weight function

$$w^X_{-1:0}(\varphi) = \exp\left(\frac{1}{2}(1-\zeta)\sum_{x\in X} \mathcal{Q}(D\varphi(x))\right) \tag{4.10.26}$$

and to define $\|\cdot\|_{-1:0}^{(4A)}$ as in (4.4.89) and (4.4.88). We can write $K_0(\mathcal{K},\mathcal{H}) = P_4(I(\mathcal{K}), E(\mathcal{H}))$, where E is the exponential defined in (4.7.13) and where the inclusion map I and the product map P_4 are given by

$$I : (\boldsymbol{E}, \|\cdot\|_\zeta) \to (M(\mathcal{P}_0^c), \|\cdot\|_{-1:0}^{(4A)}),\; I(\mathcal{K})(X,\varphi) = \prod_{x\in X} \mathcal{K}(D\varphi(x)) \tag{4.10.27}$$

$$P_4 : (M(\mathcal{P}_0^c), \|\cdot\|_{-1:0}^{(4A)}) \times (M(\mathcal{B}_0), |||\cdot|||_0) \to (M(\mathcal{P}_0^c),\, \|\cdot\|_0^{(A)})),\; P_4(K,F)(X,\varphi) = K(X,\varphi) F^X(\varphi). \tag{4.10.28}$$

Smoothness of E was established in Lemma 4.7.3. We will now show smoothness of I and of P_4 in Lemma 4.10.3 and Lemma 4.10.5, respectively, and then conclude the proof of Lemma 4.10.2.

Lemma 4.10.3. *Let* I *be the map defined in* (4.10.27). *Assume that*

$$\rho_1 \le (2^{R_0 d r_0 + 3} h^{r_0} A)^{-1} \quad \textit{and} \quad h \ge 1. \tag{4.10.29}$$

Then I *is a smooth on* $B_\rho(0) \subset \boldsymbol{E}$ *and, for all* $\mathcal{K} \in B_\rho(0)$

$$\|I(\mathcal{K})\|_{-1:0}^{(4A)} \le 2^{R_0 d r_0 + 2} h^{r_0} A \,\|\mathcal{K}\|_\zeta \tag{4.10.30}$$

$$\frac{1}{j!}\|D^j I(\mathcal{K})(\dot{\mathcal{K}},\dots,\dot{\mathcal{K}})\|_{-1:0}^{(4A)} \le (2^{R_0 d r_0 + 3}\, h^{r_0} A)^j \,\|\dot{\mathcal{K}}\|_\zeta^j. \tag{4.10.31}$$

.

Remark 4.10.4. *We could avoid h-dependence of the constants and neighbourhoods here and in all other statements in this section as well as in Theorem 4.9.1 if we work with the norm*

$$\|\mathcal{K}\|_{\zeta,h} := \sup_{z\in\mathcal{G}} \sum_{|\alpha|\le r_0} \frac{1}{\alpha!} h^{|\alpha|} |\partial^\alpha K(z)| e^{-\frac{1}{2}(1-\zeta)\mathcal{Q}_0(z)}, \tag{4.10.32}$$

This gives slightly better results, because, roughly speaking our current setting leads to conditions of the type '$h^{r_0}\|\mathcal{K}\|_\zeta$ small' while it suffices that $\|\mathcal{K}\|_{\zeta,h}$ is small which is a weaker condition on the low derivatives of $\mathcal{K}$. We prefer, however, to keep the notation in Section 3.2 simple and not to introduce a more complicated norm with another parameter.

Proof. Note that the functional $I(\mathcal{K})$ is translation invariant, shift invariant and local. Thus $I(\mathcal{K})$ is an element of $M(\mathcal{P}_0^c)$. We first estimate $|I(\mathcal{K})(\{x\})|_{0,\{x\},T_\varphi}$. Let us introduce the set $\mathcal{I}_m = \mathcal{I}\times\{1,\dots,m\}$ where we recall that $\mathcal{I}\subset\{0,\dots,R_0\}^d\setminus\{0,\dots,0\}$. We consider multiindices $\gamma\in\mathbb{N}_0^{\mathcal{I}_m}$. Recall that for $m=(\alpha,i)\in\mathcal{I}_m$ we defined the monomials

$$\mathcal{M}_m(\{x\})(\dot\psi) := \nabla^m\dot\psi(x) := \nabla^\alpha\dot\psi_i(x). \tag{4.10.33}$$

The Taylor expansion of order r_0 of $I(\mathcal{K})(\{x\})$ is given by

$$\mathrm{Tay}_\varphi I(\mathcal{K})(\{x\})(\dot\psi) = \sum_{|\gamma|\le r_0} \frac{1}{\gamma!}\partial^\gamma\mathcal{K}(D\varphi(x)) \prod_{m\in\mathcal{M}_m} (\nabla^m\dot\psi(x))^{\gamma_m}. \tag{4.10.34}$$

Hence we have

$$\mathrm{Tay}_\varphi I(\mathcal{K})(\{x\}) = \sum_{|\gamma|\le r_0} \frac{1}{\gamma!}\partial^\gamma\mathcal{K}(D\varphi(x)) \prod_{m\in\mathcal{M}_m} (\mathcal{M}_m(\{x\}))^{\gamma_m}. \tag{4.10.35}$$

The triangle inequality and the product property in Lemma 4.6.1 imply

$$\begin{aligned} |I(\mathcal{K})(\{x\})|_{0,\{x\},T_\varphi} &\le \sum_{|\gamma|\le r_0} \frac{1}{\gamma!}\,|\partial^\gamma\mathcal{K}(D\varphi(x))| \left|\prod_{m\in\mathcal{M}_m} (\mathcal{M}_m(\{x\}))^{\gamma_m}\right|_{0,\{x\},T_0} \\ &\le \sum_{|\gamma|\le r_0} \frac{1}{\gamma!}\,|\partial^\gamma\mathcal{K}(D\varphi(x))| \prod_{m\in\mathcal{M}_m} \left|\mathcal{M}_m(\{x\})\right|^{\gamma_m}_{0,\{x\},T_0} \end{aligned} \tag{4.10.36}$$

Next we give a crude estimate for $\left|M_m(\{x\})\right|_{0,\{x\},T_0}$. The definition of $\{x\}^* = \{x\}+[-R,R]^d$ ensures that the reiterated difference quotient $\nabla^\alpha\dot\varphi(x)$ for $\alpha\in\mathcal{I}$ can be written as a linear combination of values $\nabla_i\varphi(y)$ with $y\in\{x\}^*$ involving at most $2^{|\alpha|-1}$ terms. Using an induction argument we easily see that for $\alpha\in\mathcal{I}$

$$|\nabla_i^\alpha\dot\psi(x)| \le 2^{|\alpha|-1} \sup_{y\in\{x\}^*} |\nabla_i\dot\psi(y)| \le 2^{R_0 d}\, h\, |\dot\psi|_{0,\{x\}} \tag{4.10.37}$$

where we used the definition (4.4.74) of $|\cdot|_{0,X}$ and the fact that for $j=0$ the weights in (4.4.76) reduce to $w_0(i,\alpha)=h$. Now (4.10.37) and the definition of the $|\cdot|_{0,\{x\},T_0}$ norm by duality give the estimate

$$\left|\mathcal{M}_m(\{x\})\right|_{0,\{x\},T_0} \le 2^{R_0 d}\, h. \tag{4.10.38}$$

From (4.10.36), (4.10.37), the condition $h \geq 1$ and the definition (3.2.38) we infer that

$$\frac{|I(\mathcal{K})(\{x\})|_{0,\{x\},T_\varphi}}{w_{-1:0}^{\{x\}}(\varphi)} \leq 2^{R_0 d r_0}\, h^{r_0}\, \|\mathcal{K}\|_\zeta \frac{e^{\frac{1}{2}(1-\zeta)\mathcal{Q}_0(D\varphi(x))}}{e^{\frac{1}{2}(1-\zeta)\mathcal{Q}_0(D\varphi(x))}} = 2^{R_0 d r_0}\, h^{r_0} \|\mathcal{K}\|_\zeta \tag{4.10.39}$$

Since $w_{-1,0}^X$ factors over any polymer the submultiplicativity estimate (4.6.1) combined with the trivial estimate $|\cdot|_{0,X,T_\varphi} \leq |\cdot|_{0,x,T_\varphi}$ whenever $x \in X$ implies that

$$\frac{|I(\mathcal{K})(X)|_{0,X,T_\varphi}}{w_{-1:0}^X(\varphi)} \leq (2^{R_0 d r_0}\, h^{r_0})^{|X|}\, \|\mathcal{K}\|_\zeta^{|X|} \tag{4.10.40}$$

Thus we get for $\rho_1 \leq (2^{R_0 d r_0}\, h^{r_0} 4A)^{-1}$.

$$\|I(\mathcal{K})(X)\|_{-1:0}^{(4A)} \leq \sup_{X \in \mathcal{P}_0^c} \left(2^{R_0 d r_0}\, h^{r_0} 4A\, \|\mathcal{K}\|_\zeta\right)^{|X|} \leq 2^{R_0 d r_0}\, h^{r_0} 4A\, \|\mathcal{K}\|_\zeta \leq 1. \tag{4.10.41}$$

This proves (4.10.30).

As in Lemma 4.7.5 the derivatives are estimated similarly. For $\rho_1 \leq (2^{R_0 d r_0 + 1}\, h^{r_0} 4A)^{-1}$ we get

$$\begin{aligned}(4A)^{|X|}\, \frac{1}{j!} \|D^j I(\mathcal{K})(\dot{\mathcal{K}}, \dots, \dot{\mathcal{K}})(X)\|_{0,X} &\leq \binom{|X|}{j} (2^{R_0 d r_0}\, h^{r_0} 4A)^{|X|}\ \|\mathcal{K}\|_\zeta^{|X|-j}\ \|\dot{\mathcal{K}}\|_\zeta^j \\ \leq & (2^{R_0 d r_0 + 1}\, h^{r_0} 4A)^{|X|}\ \|\mathcal{K}\|_\zeta^{|X|-j}\ \|\dot{\mathcal{K}}\|_\zeta^j \\ \leq & \left(2^{R_0 d r_0 + 1}\, h^{r_0} 4A \|\dot{\mathcal{K}}\|_\zeta\right)^j.\end{aligned} \tag{4.10.42}$$

This shows that I is smooth on $B_{\rho_1}(0)$ and the estimate (4.10.31) holds. □

Lemma 4.10.5. *Assume that*

$$h \geq \max(\delta(L)^{-1/2}, 1) \tag{4.10.43}$$

where $\delta(L)$ *was introduced in* (4.5.57). *Then the map* P_4 *defined in* (4.10.28) *is smooth on the set* $\boldsymbol{M}_{-1:0}^{(4A)} \times B_1(1)$ *with* $B_1(1) \subset (M(\mathcal{B}_0), |||\cdot|||_0)$. *Moreover on that set we have*

$$\|P_4(K, F)\|_0^{(A)} \leq \|K\|_{-1:0}^{(4A)}, \tag{4.10.44}$$

$$\frac{1}{j_2!} \|D_K^{j_1} D_F^{j_2} P_4(K, F)\|_0^{(A)} \leq \left(\|K\|_{-1:0}^{(4A)}\right)^{1-j_1} \left(\|\dot{K}\|_{-1:0}^{(4A)}\right)^{j_1} \left|\left|\left|\dot{F}\right|\right|\right|_0^j. \tag{4.10.45}$$

for $j_1 \in \{0, 1\}$ *while the left hand side vanishes for* $j_1 \geq 2$.

Proof. For brevity we write δ instead of $\delta(L)$. It follows from the definitions of the quadratic forms $\boldsymbol{M}_0^X$ and $\boldsymbol{G}_0^X$ in (4.5.2) and (4.5.11) and assumption (4.10.43) that $\delta \boldsymbol{M}_0^X \geq \boldsymbol{G}_0^X$. Taking into account that in (4.5.10) we have $\delta_0 = \delta$ and $4\overline{\zeta} = \zeta$ (see (4.5.8)) we deduce from (4.5.10) that

$$(\varphi, \boldsymbol{A}_0^X \varphi) \geq (1 - \zeta) \sum_{x \in X} \mathcal{Q}(D\varphi(x)) + (\boldsymbol{G}_0^X \varphi, \varphi). \tag{4.10.46}$$

Since $w_0^X(\varphi) = e^{\frac{1}{2}(\boldsymbol{A}_0^X\varphi,\varphi)}$ and $W_0^X(\varphi) = e^{\frac{1}{2}(\boldsymbol{G}_0^X\varphi,\varphi)}$ the definition of $w_{-1:0}^X$ in (4.10.26) implies that

$$w_0^X \geq w_{-1:0}^X W_0^X = w_{-1:0}^X \prod_{B\in\mathcal{B}_0(X)} W_0^B.$$

Together with Lemma 4.6.1 we get

$$\begin{aligned}\|P_4(K,F)(X)\|_{0,X} &= \sup_\varphi \frac{|F^X(\varphi)K(X,\varphi)|_{0,X,T_\varphi}}{w_0^X(\varphi)}\\ &\leq \sup_\varphi \frac{|K(X,\varphi)|_{0,X,T_\varphi}}{w_{-1:0}^X(\varphi)} \prod_{B\in\mathcal{B}_0(X)} \sup_\varphi \frac{|F(B,\varphi)|_{0,B,T_\varphi}}{W_0^B(\varphi)}\\ &\leq \|K\|_{-1:0}^{(4A)}(4A)^{-|X|}\, |||F|||_0^{|X|} \leq \|K\|_{-1:0}^{(4A)}(2A)^{-|X|}\end{aligned} \tag{4.10.47}$$

where we used that $F \in B_1(1) \subset B_2(0) \subset (M(\mathcal{B}_0), |||\cdot|||_0)$. Multiplying by $A^{|X|}$ and taking the supremum over X we get (4.10.44).

To estimate the derivatives we observe that P_4 is linear in K. Therefore it is sufficient to note that

$$\frac{1}{j!}\|D_F^j P_4(K,F)(\dot{F},\ldots,\dot{F})(X)\|_{0,X} \leq \|K\|_{-1:0}^{(4A)}(2A)^{-|X|}2^{|X|}\, |||\dot{F}|||_0^j \tag{4.10.48}$$

where the additional factor $2^{|X|}$ is again the combinatorial factor of the derivatives. Hence

$$\frac{1}{j!}\|D_F^j P_4(K,F)(\dot{F},\ldots,\dot{F})\|_0^{(A)} \leq \|K\|_{-1:0}^{(4A)}\, |||\dot{F}|||_0^j. \tag{4.10.49}$$

□

Proof of Lemma 4.10.2. To see that the $\widehat{K}_0$ is smooth on $B_{\rho_1}\times B_{\rho_2}$ for the given values of ρ_1 and ρ_2 it suffices to note that I is smooth on B_{ρ_1} and E maps $B_{\frac{1}{8}}(0)$ to $B_1(1)$ (see (4.7.27)). Then the assertion follows from the fact that P_4 is smooth on $\boldsymbol{M}_{-1:0}^{(4A)} \times B_1(1)$. The bound (4.10.23) for $j_2 = 0$ with $C_0 = 1$ follows from (4.10.44) and (4.10.30). The other bounds follow from the bounds in Lemma 4.7.3, Lemma 4.10.3 and Lemma 4.10.5 in connection with the chain rule. □

Proof of Theorem 4.10.1, conclusion. We first note that the map $\mathcal{H} \mapsto \boldsymbol{q}(\mathcal{H})$ is linear and satisfies

$$|\boldsymbol{q}(\mathcal{H})| \leq \frac{C(m,d)}{h^2}\|\mathcal{H}\|_{0,0} \leq C\,\|\mathcal{H}\|_{0,0}. \tag{4.10.50}$$

This follows from the definition of the norm $\|\cdot\|_{0,0}$ in (4.4.93) and the fact that all norms on $\mathbb{R}_{\text{sym}}^{(d\times m)\times(d\times m)}$ are equivalent.

Next we establish smoothness of the coordinate maps for $\tilde{H}_k$ and $\tilde{K}_k$ in a neighbourhood of the origin. We first consider the maps $\tilde{K}_{k+1}$ with $k\geq 1$. Then $\tilde{K}_{k+1}(\mathcal{K},\mathcal{H},Z) = \boldsymbol{S}_k(H_k,K_k,\boldsymbol{q}(\mathcal{H}))$ and in particular $\tilde{K}_{k+1}$ does not depend on $\mathcal{K}$. Smoothness of $\tilde{K}_{k+1}$ follows from the smoothness of $\boldsymbol{S}_k$ (see Theorem 4.4.7) and (4.10.50) as long as

$$\rho_2 \leq \frac{\kappa(L)}{C_{(4.10.50)}} \tag{4.10.51}$$

Regarding the bounds on the derivatives of $\tilde{K}_{k+1}$ we have

$$\begin{aligned}
&\frac{1}{j_2! j_3!}\left\| \frac{1}{\eta^{k+1}} D_{\mathcal{H}}^{j_2} D_Z^{j_3} \tilde{K}_{k+1}(\mathcal{K},\mathcal{H},Z)(\dot{\mathcal{H}},\ldots,\dot{\mathcal{Z}})\right\|_{k+1}^{(A)} \\
\leq & C_{j_2,j_3}(L,A)\frac{1}{\eta}\frac{1}{\eta^k}\left(\|\dot{K}_k\|_k^{(A)} + \|\dot{H}_k\|_{k,0}\right)^{j_3} C_{(4.10.50)}^{j_2} \|\dot{\mathcal{H}}\|_{0,0}^{j_2} \\
\leq & C_{j_2,j_3}(L,A)\, \|\dot{Z}\|_{\mathcal{Z}}^{j_3}\ \|\dot{\mathcal{H}}\|_{0,0}^{j_2}.
\end{aligned} \tag{4.10.52}$$

He we used the convention that we denote indicate the dependence of constants on fixed parameters like η. Similarly smoothness of $\tilde{H}_k$ and the bounds on the derivatives follow from (4.10.50), (4.4.103) and (4.4.104).

The main point is to show smoothness of the map $\tilde{K}_1(\mathcal{K},\mathcal{H},Z) = \boldsymbol{S}_0(\widehat{K}_0(\mathcal{K},\mathcal{H}), H_0, \boldsymbol{q}(\mathcal{H}))$ and to bound the derivatives of $\tilde{K}_1$. We first note that for ρ_1 and ρ_2 given by (4.10.17)

$$\widehat{K}_0(B_{\rho_1}(0)\times B_{\rho_2}(0)) \subset B_{\rho(A)}(0). \tag{4.10.53}$$

Indeed this follows directly from (4.10.23) with $j_2 = 0$. Now the desired properties of $\tilde{K}_1$ follow from Lemma 4.10.2, Theorem 4.4.7 and the chain rule.

Next we show that at the origin the differential of the map $Z \mapsto \mathcal{T}(\mathcal{K},\mathcal{H},Z)$ is contraction. It follows from the definition of the maps $\tilde{K}_k$ and $\tilde{H}_k$ in (4.10.5)–(4.10.8) in combination with (4.4.99) in Theorem 4.4.8 that

$$D_{H_{k+1}}\tilde{H}_k(0,0,0) = (\boldsymbol{A}_k^{(0)})^{-1} \quad \text{for} \quad 0 \leq k \leq N-2, \tag{4.10.54}$$

$$D_{K_k}\tilde{H}_k(0,0,0) = -(\boldsymbol{A}_k^{(0)})^{-1}\boldsymbol{B}_k^{(0)}, \quad \text{for} \quad 1 \leq k \leq N-1, \tag{4.10.55}$$

$$D_{K_k}\tilde{K}_{k+1}(0,0,0) = \boldsymbol{C}_k^{(0)} \quad \text{for} \quad 1 \leq k \leq N-1 \tag{4.10.56}$$

and all other derivatives vanish. To estimate the operator norm of $D_Z\mathcal{T}(0,0,0)$ let $\dot{Z} \in \mathcal{Z}$ with $\|\dot{Z}\|_{\mathcal{Z}} \leq 1$ and set

$$Z' = D_Z\mathcal{T}(0,0,0)\dot{Z} \tag{4.10.57}$$

We denote the coordinates of $\dot{Z}$ by $\dot{H}_k$ and $\dot{K}_k$ and the coordinates of Z' by H'_k and K'_k. The definition of the norm on $\mathcal{Z}$ implies that $\|\dot{H}_k\|_{k,0} \leq \eta^k$ and $\|\dot{K}_k\|_k \leq \eta^k$. The bounds from Theorem 4.4.8 imply that

$$\begin{aligned}
&\|H'_0\|_{0,0} \leq \|(\boldsymbol{A}_0^{(0)})^{-1}\|\eta \leq \frac{3}{4}\eta, \\
&\eta^{-k}\|H'_k\|_{k,0} \leq \eta^{-k}\|(\boldsymbol{A}_k^{(0)})^{-1}\|\eta^{k+1} + \eta^{-k}\|(\boldsymbol{A}_k^{(0)})^{-1}\|\ \|\boldsymbol{B}_k^{(0)}\|\eta^k \leq \frac{3}{4}(\eta + \frac{1}{3}), \quad 1 \leq k \leq N-2, \\
&\eta^{-(N-1)}\|H'_{N-1}\|_{N-1,0} \leq \eta^{-(N-1)}\|(\boldsymbol{A}_{N-1}^{(0)})^{-1}\|\ \|\boldsymbol{B}_{N-1}^{(0)}\|\eta^{N-1} \leq \frac{3}{4}\cdot\frac{1}{3}, \\
&\eta^{-1}\|K'_1\| = 0, \\
&\eta^{-k}\|K'_k\| \leq \eta^{-k}\|\boldsymbol{C}_{k-1}^{(0)}\|\eta^{k-1} \leq \frac{1}{\eta}\frac{3}{4}\eta = \frac{3}{4}, \quad 2 \leq k \leq N.
\end{aligned} \tag{4.10.58}$$

Since $\eta \leq \frac{2}{3}$ this implies that

$$\|D_Z\mathcal{T}(0,0,0)\| \leq \frac{3}{4}. \tag{4.10.59}$$

Thus the assumptions of the implicit function theorem are satisfied for the map $\mathcal{T}-\pi_3$ and since $\mathcal{T}$ is smooth (with bounds on the derivatives that are independent of N) the implicit function $\hat{Z}$ is defined in a neighbourhood $B_{\epsilon_1}\times B_{\epsilon_2}\subset \boldsymbol{E}\times M(\mathcal{B}_0)$ with ϵ_1 and ϵ_2 independent of N and the derivatives of $\widehat{Z}$ can be bounded independent of N. □

To show that the specific choice of ϵ_1, ϵ_2 and ϵ given Theorem 4.10.1 is sufficient assume that

$$\epsilon_1\le\rho_1,\quad \epsilon_2\le\rho_2,\quad \epsilon\le\rho(A)/2 \tag{4.10.60}$$

and that

$$2\overline{C}_{0,0,2}\,\epsilon+\overline{C}_{1,0,1}\,\epsilon_1+\overline{C}_{0,1,1}\,\epsilon_2\le\frac{1}{8}. \tag{4.10.61}$$

Then for $(\mathcal{K},\mathcal{H},Z)\in B_{\epsilon_1}\times B_{\epsilon_2}\times\overline{B_\epsilon}$ we have

$$\|D_Z\mathcal{T}(\mathcal{K},\mathcal{H},Z)\|\le\frac{3}{4}+\|D_Z\mathcal{T}(\mathcal{K},\mathcal{H},Z)-D_Z\mathcal{T}(0,0,0)\|\le\frac{7}{8}. \tag{4.10.62}$$

Note that the definition of $\mathcal{T}$ implies that $\mathcal{T}(0,\mathcal{H},0)=0$ for all $\mathcal{H}\in B_{\epsilon_2}(0)\subset B_{\rho_2}(0)$. Thus if in addition

$$\overline{C}_{1,0,0}\,\epsilon_1\le\frac{1}{8}\epsilon \tag{4.10.63}$$

then we have

$$\|\mathcal{T}(\mathcal{K},\mathcal{H},0)\|<\frac{1}{8}\epsilon\qquad\forall\,(\mathcal{H},\mathcal{K})\in B_{\epsilon_1}(0)\times B_{\epsilon_2}(0). \tag{4.10.64}$$

It follows from (4.10.62) and (4.10.64) that for all $(\mathcal{K},\mathcal{H})\in B_{\epsilon_1}(0)\times B_{\epsilon_2}(0)$ the map $Z\mapsto\mathcal{T}(\mathcal{K},\mathcal{H},Z)$ is a contraction and maps the closed ball $\overline{B_\epsilon}$ to itself. Thus by the Banach fixed point theorem there is a unique $\widehat{Z}(\mathcal{K},\mathcal{H})\in\overline{B_\epsilon}$ such that

$$\mathcal{T}(\mathcal{K},\mathcal{H},\widehat{Z}(\mathcal{K},\mathcal{H}))=\widehat{Z}(\mathcal{K},\mathcal{H}). \tag{4.10.65}$$

Moreover $\|\widehat{Z}(\mathcal{K},\mathcal{H})\|\le 8\|\mathcal{T}(\mathcal{K},\mathcal{H},0)\|<\epsilon$ so that $\widehat{Z}(\mathcal{K},\mathcal{H})\in B_\epsilon(0)$. It follows from the implicit function theorem applied at the point $(\mathcal{K},\mathcal{H},\widehat{Z}(\mathcal{H},\mathcal{K}))$ that the function $\hat{Z}$ is locally smooth. By uniqueness $\hat{Z}$ is smooth in $B_{\epsilon_1}(0)\times B_{\epsilon_2}(0)$. Finally one easily sees that the choices in (4.10.20) imply (4.10.60), (4.10.61) and (4.10.63).

4.10.2 Existence of a fixed point of the map $\Pi_{H_0}\hat{Z}(\mathcal{K},\cdot)$

Theorem 4.10.1 and the definition of the $\|\cdot\|_{\mathcal{Z}}$ norm show the existence of sequence of maps H_k and K_k such that $\boldsymbol{R}^{(q)}_{k+1}\left(e^{H_k}\circ K_k\right)(\varphi)=\left(e^{H_{k+1}}\circ K_{k+1}\right)(\varphi)$, the coordinate K_k is exponentially decreasing and $H_N=0$. In particular equation (4.10.12) holds. But this sequence will in general not satisfy the correct initial condition because the H_0 coordinate of the fixed point is only given implicitly by the fixed point equation. We can, however, use the artificially inserted coordinate $\mathcal{H}$ and apply the implicit function theorem once more to show that we can choose $\mathcal{H}$ such that $H_0=\mathcal{H}$. Then a simple calculation shows that this fixed point satisfies the correct initial condition up to an explicit scalar factor,see (4.10.79) and (4.10.82) below.

We use the same notation as in Theorem 4.10.1. In particular $\widehat{Z}:B_{\epsilon_1}(0)\times B_{\epsilon_2}(0)\to\mathcal{Z}$ denotes the fixed point map. We denote by $\Pi_{H_0}:\mathcal{Z}\to M_0(\mathcal{B}_0)$ the bounded linear map that extracts the coordinate H_0 from Z.

Lemma 4.10.6. *Under the assumptions of Theorem 4.10.1 there is a constant* $\tilde{\varrho} > 0$ *which can be chosen independently of* N *and a map* $\widehat{\mathcal{H}} : B_{\tilde{\varrho}}(0) \subset \boldsymbol{E} \to B_{\epsilon_2}(0) \subset M_0(\mathcal{B}_0)$ *such that*

$$\Pi_{H_0} \hat{Z}(\mathcal{K}, \widehat{\mathcal{H}}(\mathcal{K})) = \widehat{\mathcal{H}}(\mathcal{K}) \quad \text{and} \quad \boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K})) \subset B_\kappa(0) \qquad \text{for all } \mathcal{K} \in B_{\tilde{\varrho}}(0). \tag{4.10.66}$$

Moreover $\widehat{\mathcal{H}}$ *is smooth in* $B_{\tilde{\varrho}}(0)$ *and the derivatives can be bounded uniformly in* N. *We may take*

$$\tilde{\varrho} = \min\left(\frac{1}{4C_{1,1}}, \frac{\rho'}{2C_{1,0}}, \epsilon_1\right) \quad \text{where} \quad \rho' = \min\left(\frac{1}{8C_{0,2}}, \frac{\epsilon_2}{2}\right) \tag{4.10.67}$$

and where C_{j_1,j_2} *are the constants in the estimate* (4.10.19) *for the derivatives of* $\widehat{Z}$.

Note that the condition $\widehat{\mathcal{H}}(B_{\tilde{\varrho}}(0)) \subset B_{\epsilon_2}(0)$ and (4.10.16) imply that

$$\boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K})) \in B_\kappa(0) \quad \text{for all } \mathcal{K} \in B_\rho(0) \tag{4.10.68}$$

since $\epsilon_2 \le \rho_2$.

Proof. We first note that $\mathcal{T}(0, \mathcal{H}, 0) = 0$. Hence by uniqueness of the fixed point we get

$$\widehat{Z}(0, \mathcal{H}) = 0 \qquad \text{for all } \mathcal{H} \in B_{\epsilon_2}(0) \tag{4.10.69}$$

and in particular

$$D_{\mathcal{H}} \hat{Z}(0,0) = 0. \tag{4.10.70}$$

We now consider the function

$$f = \Pi_{H_0} \circ \widehat{Z} - \pi_2 : B_{\epsilon_1} \times B_{\epsilon_2} \subset \boldsymbol{E} \times M_0(\mathcal{B}_0) \to M_0(\mathcal{B}_0) \tag{4.10.71}$$

where $\pi_2(\mathcal{K}, \mathcal{H}) := \mathcal{H}$. Condition (4.10.70) implies that $D_{\mathcal{H}} f(0,0)\dot{\mathcal{H}} = -\dot{\mathcal{H}}$. Hence we can apply the implicit function theorem to f and find a $\tilde{\varrho} > 0$ and a smooth function $\widehat{\mathcal{H}} : B_{\tilde{\varrho}}(0) \subset \boldsymbol{E} \to M_0(\mathcal{B}_0)$ such that $f(\Pi_{H_0} \hat{Z}(\mathcal{K}, \widehat{\mathcal{H}}(\mathcal{K})), \widehat{\mathcal{H}}(\mathcal{K})) = 0$, i.e.,

$$\Pi_{H_0} \hat{Z}(\mathcal{K}, \widehat{\mathcal{H}}(\mathcal{K})) = \widehat{\mathcal{H}}(\mathcal{K}). \tag{4.10.72}$$

We can choose $\tilde{\varrho}$ independent of N because the derivatives of $\hat{Z}$ are bounded uniformly in N.

It only remains to show that the choice (4.10.67) for $\tilde{\varrho}$ is admissible and $\widehat{\mathcal{H}}(B_\rho(0)) \subset B_{\epsilon_2}(0)$. To see this we argue exactly as in the proof of Theorem 4.10.1. First assume that $\rho' \le \epsilon_2/2$ and

$$2C_{0,2}\, \rho' + C_{1,1}\, \tilde{\varrho} \le \frac{1}{2} \tag{4.10.73}$$

Then $D_{\mathcal{H}} \widehat{Z}(0,0) = 0$ implies that

$$\|D_{\mathcal{H}}(\Pi_{H_0} \circ \hat{Z})\| \le \|D_{\mathcal{H}} \widehat{Z}\| \le \frac{1}{2} \quad \text{in } B_{\tilde{\varrho}}(0) \times B_{\rho'}(0). \tag{4.10.74}$$

If in addition

$$C_{1,0}\, \tilde{\varrho} \le \frac{1}{2}\rho' \tag{4.10.75}$$

then $\|(\Pi_{H_0} \circ \hat{Z})(\mathcal{K}, 0)\| \le \frac{1}{2}\rho'$ for $\mathcal{K} \in B_\rho(0)$. Thus for such $\mathcal{K}$ the map $\mathcal{H} \mapsto (\Pi_{H_0} \circ \widehat{Z})(\mathcal{K}, \mathcal{H})$ is a contraction and maps $\overline{B_{\rho'}(0)}$ to itself. Hence this map has a unique fixed point $\widehat{\mathcal{H}}(\mathcal{K}) \in \overline{B_{\rho'}(0)} \subset B_{\epsilon_2}(0)$. Smoothness of $\widehat{\mathcal{H}}$ follows from the implicit function theorem. □

4.10.3 Proof of Theorem 4.9.1

Proof. The heart of the matter is the identity (4.10.79) below. In combination with Lemma 4.10.6 and the identity (4.10.12) it yields immediately the representation (4.9.3). The further assertions in Theorem 4.9.1 then follow from the properties of the map $\widehat{\mathcal{H}}$ in Lemma 4.10.6. To simplify the notation we write $\hat{e}$ and $\widehat{\boldsymbol{q}}$ instead of $\hat{e}_N$ and $\widehat{\boldsymbol{q}}_N$ for the maps whose existence is asserted in Theorem 4.9.1.

Recall that for an ideal Hamiltonian $\mathcal{H}$ we denote the matrix which defines the quadratic part by $\boldsymbol{q}(\mathcal{H})$. We denote the constant part by $e(\mathcal{H})$. Then

$$\sum_{x\in\Lambda_N} \mathcal{H}(\mathcal{K})(x,\varphi) = e(\mathcal{H})L^{Nd} + \frac{1}{2}\sum_{x\in\Lambda_N} \langle \boldsymbol{q}(\mathcal{H})\nabla\varphi(x), \nabla\varphi(x)\rangle \tag{4.10.76}$$

where we used that the sum over the linear terms in the field vanishes because $\sum_{x\in\Lambda_N} \nabla^\alpha\varphi_i(x) = 0$ for any $\varphi\in\mathcal{X}_N$ and any multiindex α and $1\le i\le m$, due to the periodic boundary conditions. Recall that λ is the Hausdorff measure on $\mathcal{X}_N$. The definition of the partition function $Z^{(\boldsymbol{q})}$ implies that

$$\begin{aligned} e^{\frac{1}{2}\sum_{x\in\Lambda_N}\langle \boldsymbol{q}\nabla\varphi(x),\nabla\varphi(x)\rangle}\,\mu^{(0)}(\mathrm{d}\varphi) =& \frac{Z^{(\boldsymbol{q})} e^{-\frac{1}{2}\sum_{x\in\Lambda_N}\mathcal{Q}(D\varphi(x)) - \langle \boldsymbol{q}\nabla\varphi(x),\nabla\varphi(x)\rangle}\,\lambda(\mathrm{d}\varphi)}{Z^{(\boldsymbol{q})}Z^{(0)}} \\ =& \frac{Z^{(\boldsymbol{q})}}{Z^{(0)}}\,\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) \end{aligned} \tag{4.10.77}$$

Recall also that $\mathcal{K}(X,\varphi) = \prod_{x\in X}\mathcal{K}(D\varphi(x))$. Thus by the definition (4.10.10) of $\widehat{K}_0(\mathcal{K},\mathcal{H})$

$$\begin{aligned} &(\widehat{K}_0(\mathcal{K},\mathcal{H})\circ e^{-\mathcal{H}})(\Lambda_N,\varphi) = (\mathcal{K}e^{-\mathcal{H}}\circ e^{-\mathcal{H}})(\Lambda_N,\varphi) \\ =& \sum_{X\subset\Lambda_N} \mathcal{K}(X,\varphi)e^{-\mathcal{H}}(X,\varphi)\,e^{-\mathcal{H}}(\Lambda_N\setminus X,\varphi) \\ =& \sum_{X\subset\Lambda_N} \mathcal{K}(X,\varphi)\; e^{-\sum_{x\in\Lambda_N}\mathcal{H}(x,\varphi)}. \end{aligned} \tag{4.10.78}$$

Using the identities (4.10.76)–(4.10.78) we get

$$\begin{aligned} &\int_{\mathcal{X}_N}\sum_{X\subset\Lambda_N}\mathcal{K}(X,\varphi)\,\mu^{(0)}(\mathrm{d}\varphi) \\ &\quad= \int_{\mathcal{X}_N}\left(\widehat{K}_0(\mathcal{K},\mathcal{H})\circ e^{-\mathcal{H}}\right)(\Lambda_N,\varphi)\cdot e^{\sum_{x\in\Lambda_N}\mathcal{H}(x,\varphi)}\,\mu^{(0)}(\mathrm{d}\varphi) \\ &\quad= \frac{Z^{(\boldsymbol{q}(\mathcal{H}))}}{Z^{(0)}} e^{L^{Nd}e(\mathcal{H})}\int_{\mathcal{X}_N}\left(\widehat{K}_0(\mathcal{K},\mathcal{H})\circ e^{-\mathcal{H}}\right)(\Lambda_N,\varphi)\,\mu^{(\boldsymbol{q}(\mathcal{H}))}(\mathrm{d}\varphi) \end{aligned} \tag{4.10.79}$$

Now let $0<\varrho<\tilde{\varrho}$ with $\tilde{\varrho}$ as in Lemma 4.10.6 and define the following maps on $B_\varrho(0)\subset\boldsymbol{E}$

$$\widehat{\boldsymbol{q}}(\mathcal{K}) := \boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K})),\quad \widehat{e}(\mathcal{K}) := e(\widehat{\mathcal{H}}(\mathcal{K})),\quad \widehat{K}_N(\mathcal{K}) := \Pi_{K_N}\widehat{Z}(\mathcal{K},\widehat{\mathcal{H}}(\mathcal{K})). \tag{4.10.80}$$

Here Π_{K_N} denotes the projection from Z to the K_N coordinate of Z. By Lemma 4.10.6 we have

$$\Pi_{H_0}\widehat{Z}(\mathcal{K},\widehat{\mathcal{H}}(\mathcal{K})) = \widehat{\mathcal{H}}(\mathcal{K}). \tag{4.10.81}$$

Using the abbreviation $H_0 = \Pi_{H_0}\widehat{Z}(\mathcal{K},\widehat{\mathcal{H}}(\mathcal{K}))$ we get

$$\begin{aligned}\int_{\mathcal{X}_N} &\left(\widehat{K}_0(\mathcal{K},\widehat{\mathcal{H}}(\mathcal{K}))\circ e^{-\widehat{\mathcal{H}}(\mathcal{K})}\right)(\Lambda_N,\varphi)\,\mu^{(\boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K}))}(\mathrm{d}\varphi)\\ &= \int_{\mathcal{X}_N}\left(\widehat{K}_0(\mathcal{K},\widehat{\mathcal{H}}(\mathcal{K}))\circ e^{-H_0}\right)(\Lambda_N,\varphi)\,\mu^{(\boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K}))}(\mathrm{d}\varphi) \\ &\underset{(4.10.12)}{=} \int_{\mathcal{X}_N}\left(1+\widehat{K}_N(\mathcal{K})(\Lambda_N,\varphi)\right)\mu_{N+1}^{(\boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K}))}(\mathrm{d}\varphi)\end{aligned} \tag{4.10.82}$$

Taking $\mathcal{H} = \widehat{\mathcal{H}}(\mathcal{K})$ in (4.10.79) and using that $\boldsymbol{q}(\widehat{\mathcal{H}}(\mathcal{K})) = \widehat{\boldsymbol{q}}(\mathcal{K})$ and $e(\widehat{\mathcal{H}}(\mathcal{K})) = \widehat{e}(\mathcal{K})$ we obtain the desired representation (4.9.3)

Smoothness of maps $\widehat{\boldsymbol{q}}$, $\widehat{e}$ and $\widehat{K}_N$ as well as bounds on the derivatives which are independent on N follow from the same property for $\widehat{\mathcal{H}}$ and $\widehat{Z}$ as well the linearity and uniform boundedness of the projections $\mathcal{H}\mapsto\boldsymbol{q}(\mathcal{H})$, $\mathcal{H}\mapsto e(\mathcal{H})$ and $Z\mapsto K_N$. In particular uniform bounds on the derivatives of $\mathcal{K}\mapsto\widehat{Z}(\mathcal{K},\widehat{\mathcal{H}}(\mathcal{K}))$ and the definition $\|\cdot\|_{\mathcal{Z}}$ imply that

$$\frac{1}{\eta^N}\frac{1}{l!}\|D^\ell_{\mathcal{K}}\widehat{K}_N(\mathcal{K})(\dot{\mathcal{K}},\dots,\dot{\mathcal{K}})\|_N^{(A)} \le C_\ell(L,h,A)\,\|\dot{\mathcal{K}}\|_\zeta^\ell. \tag{4.10.83}$$

This proves (4.9.4). To show (4.9.5) we note that the definition of $\|\cdot\|_N^{(A)}$ and Theorem 4.5.1 x) yield

$$\begin{aligned}\int_{\mathcal{X}}\widehat{K}_N(\Lambda_N,\varphi)\,\mu_{N+1}^{(\boldsymbol{q})}(d\varphi) &\le \int_{\mathcal{X}}\frac{1}{A}\|\widehat{K}_N\|_N^{(A)}w_N(\varphi)\,\mu_{N+1}^{(\boldsymbol{q})}(d\varphi)\\ &\le \frac{1}{A}\|\widehat{K}_N\|_N^{(A)}A_{\mathcal{B}}\,w_{N:N+1}(0) = \frac{A_{\mathcal{B}}}{A}\|\widehat{K}_N\|_N^{(A)}.\end{aligned} \tag{4.10.84}$$

Since $\widehat{K}_N(0) = 0$ it follows from (4.9.4) with $\ell = 1$ that $\|\widehat{K}_N\|_N^{(A)} \le C_{1,(4.9.4)}\eta^N\|\mathcal{K}\|_\zeta$. Thus (4.9.5) holds if ϱ satisfies in addition

$$\frac{A_{\mathcal{B}}}{A}C_{1,(4.9.4)}\eta^N\varrho \le \frac{1}{2}. \tag{4.10.85}$$

Finally the representation (4.9.6) can be derived arguing as in (4.10.79) and (4.10.82) and using Gaussian calculus. More precisely we use that for every positive quadratic form $\mathcal{C}$

$$(f_N,\varphi+\mathcal{C}f_N) - \frac{1}{2}(\mathcal{C}^{-1}(\varphi+\mathcal{C}f_N),\varphi+\mathcal{C}f_N) = \frac{1}{2}(f_N,\mathcal{C}f_N) - \frac{1}{2}(\mathcal{C}^{-1}\varphi,\varphi). \tag{4.10.86}$$

Since the Hausdorff measure λ on $\mathcal{X}_N$ is translation invariant this implies that

$$\int_{\mathcal{X}_N} e^{(f_N,\varphi)}G(\varphi)\,\mu^{(\boldsymbol{q})}(d\varphi) = e^{\frac{1}{2}(f_N,\mathcal{C}^{(\boldsymbol{q})}f_N)}\int_{\mathcal{X}_N}G(\varphi+\mathcal{C}^{(\boldsymbol{q})}f_N)\,\mu^{(\boldsymbol{q})}(d\varphi). \tag{4.10.87}$$

Using now first (4.10.78) as in (4.10.79) and then (4.10.87) we get

$$\begin{aligned}&\int_{\mathcal{X}_N} e^{(f_N,\varphi)}\sum_{X\subset\Lambda_N}\mathcal{K}(X,\varphi)\,\mu^{(0)}(\mathrm{d}\varphi)\\ &= \frac{Z^{(\boldsymbol{q}(\mathcal{H}))}}{Z^{(0)}}e^{L^{Nd}e(\mathcal{H})}\int_{\mathcal{X}_N}e^{(f_N,\varphi)}\left(\widehat{K}_0(\mathcal{K},\mathcal{H})\circ e^{-\mathcal{H}}\right)(\Lambda_N,\varphi)\,\mu^{(\boldsymbol{q}(\mathcal{H}))}(\mathrm{d}\varphi)\\ &= e^{\frac{1}{2}(f_N,\mathcal{C}^{(\boldsymbol{q}(\mathcal{H}))}f_N)}\frac{Z^{(\boldsymbol{q}(\mathcal{H}))}}{Z^{(0)}}e^{L^{Nd}e(\mathcal{H})}\int_{\mathcal{X}_N}\left(\widehat{K}_0(\mathcal{K},\mathcal{H})\circ e^{-\mathcal{H}}\right)(\Lambda_N,\varphi+\mathcal{C}^{(\boldsymbol{q}(\mathcal{H}))}f_N)\,\mu^{(\boldsymbol{q}(\mathcal{H}))}(\mathrm{d}\varphi)\end{aligned} \tag{4.10.88}$$

Taking as before $\mathcal{H} = \widehat{\mathcal{H}}(\mathcal{K})$, using the abbreviation $H_0 = \Pi_{H_0}(\widehat{Z}(\mathcal{K}, \widehat{\mathcal{H}}(\mathcal{K})) = \mathcal{H}$, the relations $\boldsymbol{q}(\mathcal{H}) = \widehat{\boldsymbol{q}}(\mathcal{K})$ and $e(\mathcal{H}) = \widehat{e}(\mathcal{K})$ and finally (4.10.12) we see that the right hand side of (4.10.88) equals

$$e^{\frac{1}{2}(f_N, \mathcal{C}^{(\widehat{\boldsymbol{q}}(\mathcal{K}))} f_N)} \frac{Z^{(\widehat{\boldsymbol{q}}(\mathcal{K}))}}{Z^{(0)}} e^{L^{Nd} \widehat{e}(\mathcal{K})} \int_{\mathcal{X}_N} \Big(1 + \widehat{K}_N(\mathcal{K})\Big) (\Lambda_N, \varphi + \mathcal{C}^{(\widehat{\boldsymbol{q}}(\mathcal{K}) f_N)})) \, \mu_{N+1}^{(\widehat{\boldsymbol{q}}(\mathcal{K}))}(\mathrm{d}\varphi). \quad (4.10.89)$$

This concludes the proof of (4.9.6) and thus of Theorem 4.9.1. □

4.A Norms on Taylor polynomials

The following material is essentially contained in [44]. We include it for the convenience of the reader because the notation is simpler than in [44] (since we do not have to deal with fermions) and because we would like to emphasise that the basic results (product property, polynomial property and two-norm estimate) follow from general features of tensor products and are not dependent on the special choice of the norm in (4.A.68).

Before we start on the details let us put this appendix more precisely into context. The uniform smoothness estimates for the polynomial maps and the exponential map in Section 4.7 rely heavily on the submultiplicativity of the norms on the functionals $K(X, \varphi)$. This submultiplicativity in turn is based on two ingredients: submultiplicativity of the weights (see Theorem 4.5.1 iii) and Theorem 4.5.1 iv)) and the choice of a submultiplicative norm on Taylor polynomials which we address in this appendix.

For smooth functions on $\mathbb{R}^p$ one can easily check that a suitable ℓ_1 type norm on the Taylor coefficients (see (4.A.43) below) is submultiplicative. We deal with smooth maps on the space $\mathcal{X}_N$ of fields and, more importantly, we want the norm on Taylor polynomials to reflect the typical behaviour of the field on different scales k, i.e., under the measure $\mu_{k+1}^{(\boldsymbol{q})}$. In this setting a more systematic approach to the construction of the norms is useful.

The main idea is to view a homogeneous polynomial of degree r on a finite dimensional space $\mathcal{X}$ as a linear functional on the tensor product $\mathcal{X}^{\otimes r}$. A norm on $\mathcal{X}$ induces in a natural way norms on the tensor products (see Definition (4.A.43)) and by duality on polynomials (see (4.A.24), (4.A.25) and (4.A.27)). This norm automatically satisfies submultiplicativity (see Propositions 4.A.6 and 4.A.9) and in addition we get useful properties such as the polynomial property in Proposition 4.A.10 and the two-norm estimate in Proposition 4.A.11.

4.A.1 Norms on polynomials

Let $\mathcal{X}$ be a finite dimensional space vector space. For definiteness we consider only vector spaces over $\mathbb{R}$, but the arguments apply also to vector spaces over $\mathbb{C}$. The main idea is to linearise the action of polynomials on $\mathcal{X}$. We say that $P : \mathcal{X} \to \mathbb{R}$ is a polynomial if in some (and hence in any) basis P is a polynomial in the coordinate with respect to that basis. For r-homogeneous polynomials we can use the following representation (alternatively this representation can be used as a coordinate-free definition of an r-homogeneous polynomial).

Lemma 4.A.1. *Let P be an r-homogeneous polynomial on $\mathcal{X}$. Then there exist a unique symmetric element $\overline{P}$ of the dual space $(\mathcal{X}^{\otimes r})'$ such that $P(\xi) = \langle \overline{P}, \xi \otimes \ldots \otimes \xi \rangle$.*

Here we write $\langle \cdot, \cdot \rangle$ to denote the dual pairing of $(\mathcal{X}^{\otimes r})'$ and $\mathcal{X}^{\otimes r}$. We say that $g \in \mathcal{X}^{\otimes r}$ is symmetric if $Sg = g$ where the symmetrisation operator S is defined in (4.A.20).

Proof. Existence: define $\langle \overline{P}, \xi_1 \otimes \ldots \otimes \xi_r \rangle = \frac{1}{r!}\frac{d}{dt_r} \cdots \frac{d}{dt_1}_{|t_i=0} P(\xi(t))$ where $\xi(t) = \sum_{i=1}^r t_i \xi_i$ and where the ξ_i run through a basis. Then extend $\overline{P}$ by linearity. Homogeneity implies that

$$\overline{P}(\xi \otimes \ldots \otimes \xi) = \frac{1}{r!}\frac{d}{dt_r} \cdots \frac{d}{dt_1}_{|t_i=0} (t_1 + \ldots + t_r)^r P(\xi) = P(\xi). \tag{4.A.1}$$

Uniqueness: if $\overline{P}, \overline{Q} \in (\mathcal{X}^{\otimes r})'$ are symmetric and $\langle \overline{P} - \overline{Q}, \xi(t) \otimes \ldots \otimes \xi(t) \rangle = 0$ then applying $\frac{d}{dt_r} \cdots \frac{d}{dt_1}_{|t_i=0}$ we deduce that $\overline{P} - \overline{Q} = 0$. □

We denote by $\bigoplus_{r=0}^{\infty} \mathcal{X}^{\otimes r}$ the space of sequences $(g^{(0)}, g^{(1)}, \ldots)$ with $g^{(r)} \in \mathcal{X}^{\otimes r}$ for which only finitely many of the $g^{(r)}$ are non-zero. By writing a general polynomial P as a sum of homogeneous polynomials we can associate to P a linear map on $\bigoplus_{r=0}^{\infty} \mathcal{X}^{\otimes r}$ via[1]

$$\langle \overline{P}, g \rangle = \sum_{r=0}^{\infty} \langle \overline{P^{(r)}}, g^{(r)} \rangle. \tag{4.A.2}$$

Here $\mathcal{X}^{\otimes 0} := \mathbb{R}$ and $P^{(0)}$ is the constant term $P(0)$. We will define a norm on $\bigoplus_{r=0}^{\infty} \mathcal{X}^{\otimes r}$. This induces a norm on P by duality. The point is to define the norm on $\bigoplus_{r=0}^{\infty} \mathcal{X}^{\otimes r}$ in such a way that the norm on P enjoys the product property: $\|PQ\| \le \|P\| \, \|Q\|$.

Here we consider only finite dimensional spaces $\mathcal{X}_i$. The study of tensor products of (infinite dimensional) Banach spaces has been a very active field of research beginning with Grothendieck's seminal work [99], see, e.g., [66, 134].

Let $\mathcal{X}_i$ be finite dimensional normed vector spaces over $\mathbb{R}$ and with dual spaces $\mathcal{X}_i'$. We say that an element of $\xi \in \mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_r$ is *simple* if

$$\xi = \xi_i \otimes \ldots \otimes \xi_r \quad \text{with } \xi_i \in \mathcal{X}_i \quad \text{and we define} \quad \|\xi_1 \otimes \ldots \otimes \xi_r\| = \|\xi_1\| \ldots \|\xi_r\|. \tag{4.A.3}$$

Note that by definition of the tensor product every element of $\mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_r$ can be written as a finite combination of simple elements. We recall the definition of two standard norms on tensor products.

Definition 4.A.2. *The projective norm (or largest reasonable norm) on $\mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_r$ is given by*

$$\|g\|_\wedge = \inf \left\{ \sum_i \|\xi_i\| : g = \sum_i \xi_i \textit{ with } \xi_i \textit{ simple} \right\} \tag{4.A.4}$$

Here the infimum is taken over finite sums. The injective norm (or smallest reasonable norm) is given by

$$\|g\|_\vee = \sup \left\{ \langle \xi_1' \otimes \ldots \otimes \xi_r', g \rangle : \|\xi_i'\|_{\mathcal{X}_i'} \le 1 \textit{ for all } i = 1, \ldots, r \right\}. \tag{4.A.5}$$

There is a third important norm based on the Hilbertian structure, but we will not use this here.

One easily sees that

$$\|g\|_\vee \le \|g\|_\wedge \quad \text{and} \quad \|\xi_1 \otimes \ldots \otimes \xi_r\|_\vee = \|\xi_1 \otimes \ldots \otimes \xi_r\|_\wedge = \|\xi_1\| \ldots \|\xi_r\|. \tag{4.A.6}$$

Therefore for simple elements we write simple write $\|g\|$ instead of $\|g\|_\vee$ or $\|g\|_\wedge$.

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[1]Actually polynomials act even more naturally on the space of symmetric tensor products $\oplus_{m=0}^{\infty} \odot_m \mathcal{X}$, see Chapters 1.9 and 1.10 in [84], but the easier duality with $\oplus_{r=0}^{\infty} \mathcal{X}^{\otimes r}$ is good enough of us.

Example 4.A.3. *We show that the injective norm on $(\mathbb{R}^p, |\cdot|_\infty)^{\otimes r}$ is the ℓ_∞ norm and the projective norm on $(\mathbb{R}^p, |\cdot|_1)^{\otimes r}$ is the ℓ_1 norm.*

Let $e_1, \ldots, e_p$ be the standard basis of $\mathbb{R}^p$. For $\varphi = \sum_{i=1}^p \varphi_i e_i$ set $|\varphi|_\infty = \max_{1\le i\le p} |\varphi_i|$ and consider $\mathcal{X} = (\mathbb{R}^p, |\cdot|_\infty)$. Denote the dual basis by e_i', i.e. $e_i'(\varphi) = \varphi_i$. Then the dual space consists of functionals of the form $\ell = \sum_{i=1}^p a_i e_i'$ and the dual norm is given by $|\ell|_{\mathcal{X}'} = |a|_1 = \sum_{i=1}^p |a_i|$. Thus $\mathcal{X}'$ is isometrically isomorphic to $(\mathbb{R}^p, |\cdot|_1)$.

Let $E = \{1, \ldots, p\}$. Then an element $g \in \mathcal{X}^\otimes$ can be identified with an element of $\mathbb{R}^{E^r}$ via $g = \sum_{(i_1,\ldots,i_r)\in E^r} g_{i_1\ldots i_r}\, e_{i_1} \otimes \ldots \otimes e_{i_r}$. Similarly $L \in (\mathcal{X}')^{\otimes r}$ can be uniquely expressed as $L = \sum_{(i_1,\ldots,i_r)\in E^r} a_{i_1\ldots i_r}\, e_{i_1}' \otimes \ldots \otimes e_{i_r}'$. We claim that

$$\|g\|_\vee = |g|_\infty := \max_{(i_1,\ldots,i_r)\in E^r} |g_{i_1\ldots i_r}|, \tag{4.A.7}$$

$$\|L\|_\wedge = |L|_1 := \sum_{(i_1,\ldots,i_r)\in E^r} |a_{i_1\ldots i_r}|. \tag{4.A.8}$$

Indeed $\|L\|_\wedge \le |L|_1$ since $e_{i_1}' \otimes \ldots \otimes e_{i_r}'$ is simple. On the other hand for every simple $L = l_1 \otimes \ldots \otimes l_r$ with $l_j = \sum_{i_j=1}^p a_{i_j}^{(j)} e_{i_j}'$ we have

$$|L|_1 = \sum_{(i_1,\ldots,i_r)\in E^r} \left| \prod_{j=1}^r \sum_{i_j=1}^p a_{i_j}^{(j)} \right| \le \prod_{j=1}^r \sum_{i_j=1}^p |a_{i_j}^{(j)}| = \prod_{j=1}^r |\ell_j|_{\mathcal{X}'} = \|L\|_\wedge. \tag{4.A.9}$$

Thus $|L|_1 \le \|L\|_\wedge$ for all simple L and by definition of $\|\cdot\|_\wedge$ this implies $|L|_1 \le \|L\|_\wedge$ for all L.

To prove (4.A.7) we first note that

$$\pm g_{i_1\ldots i_r} = \langle \pm e_{i_1}' \otimes \ldots \otimes e_{i_r}', g\rangle \le \|g\|_\vee \tag{4.A.10}$$

and hence $|g|_\infty \le \|g\|_\vee$. To prove the converse inequality we note that for $l_j \in \mathcal{X}'$ as above the estimate (4.A.9) implies that

$$\langle \ell_1 \otimes \ldots \otimes \ell_r, g\rangle = \sum_{(i_1,\ldots,i_r)\in E^r} \prod_{j=1}^r \sum_{i_j=1}^p a_{i_j}^{(j)}\, g_{i_1\ldots i_r} \le \prod_{j=1}^r \|\ell_j\|_{\mathcal{X}'}\, |g|_\infty. \tag{4.A.11}$$

Thus $\|g\|_\vee \le |g|_\infty$.

Define dual norms on $(\otimes_{i=1}^r \mathcal{X}_i)'$ by

$$\|L\|_\vee' := \sup\{\langle L, g\rangle : g \in \otimes_{i=1}^r \mathcal{X}_i,\ \|g\|_\vee \le 1\}, \qquad \|L\|_\wedge' := \sup\{\langle L, g\rangle : g \in \otimes_{i=1}^r \mathcal{X}_i,\ \|g\|_\wedge \le 1\}. \tag{4.A.12}$$

The dual space $(\otimes_{i=1}^r \mathcal{X}_i)'$ can be identified with $(\otimes_{i=1}^r \mathcal{X}_i')$. Indeed, let $\xi_i' \in \mathcal{X}_i'$, let ξ_i run through a basis of $\mathcal{X}_i$ and define

$$\iota(\xi_1' \otimes \ldots \otimes \xi_r')(\xi_1 \otimes \ldots \otimes \xi_r) = \prod_{i=1}^r \langle \xi_i', \xi_i\rangle. \tag{4.A.13}$$

By linearity $\iota(\xi_1' \otimes \ldots \otimes \xi_r')$ can be extended to a linear functional on $\otimes_{i=1}^r \mathcal{X}_i$, i.e., to an element of $(\otimes_{i=1}^r \mathcal{X}_i)'$. Now let ξ_i' run through a basis of $\mathcal{X}_i'$. Then ι can be extended to a unique linear map from $(\otimes_{i=1}^r \mathcal{X}_i')$ to $(\otimes_{i=1}^r \mathcal{X}_i)'$ and one easily checks that ι is injective and hence bijective since both spaces have the same dimension. With this identification and using the fact that the closed unit ball in the projective norm is the convex hull $C = \text{conv}(\{\xi : \xi \text{ simple}, \|\xi\| \le 1\})$, the Hahn-Banach separation theorem and the fact that for finite dimensional spaces $\mathcal{X}'' = \mathcal{X}$ one easily verifies that

$$\|L\|'_\wedge = \|L\|_\vee \quad \text{and} \quad \|L\|'_\vee = \|L\|_\wedge. \tag{4.A.14}$$

One can also easily check that the projective and the injective norm are associative with respect to iterated tensorisation.

Lemma 4.A.4. *The following properties hold*

1. *(Tensorisation estimate) Assume that $\Box = \vee$ or $\Box = \wedge$. Then for $g \in \mathcal{X}^{\otimes r}$, $h \in \mathcal{X}^{\otimes s}$ and $L \in (\mathcal{X}^{\otimes r})'$, $M \in (\mathcal{X}^{\otimes r})'$*

$$\|g \otimes h\|_\Box \le \|g\|_\Box \|h\|_\Box \quad \textit{and} \quad \|L \otimes M\|'_\Box \le \|L\|'_\Box \, \|M\|'_\Box. \tag{4.A.15}$$

2. *(Contraction estimate) Assume that $\Box = \vee$ or $\Box = \wedge$. For $L \in (\mathcal{X}^{\otimes(r+s)})'$ and $h \in \mathcal{X}^{\otimes s}$ define $M \in (\mathcal{X}^{\otimes r})'$ by $\langle M, g\rangle = \langle L, (g \otimes h)\rangle$. Then*

$$\|M\|'_\Box \le \|L\|'_\Box \, \|h\|_\Box \tag{4.A.16}$$

Proof. To prove the first estimate in (4.A.15) for $\Box = \vee$ assume that $\|\xi'_i\| \le 1$ for $i \in \{1, \dots, r+s\}$. Then

$$\langle \xi'_1 \otimes \ldots \otimes \xi'_{r+s}, g \otimes h\rangle = \langle \xi'_1 \otimes \ldots \otimes \xi'_r, g\rangle \, \langle \xi'_{r+1} \otimes \ldots \otimes \xi'_{r+s}, h\rangle \le \|g\|_\vee \, \|h\|_\vee. \tag{4.A.17}$$

Next we consider $\Box = \wedge$. For each $\delta > 0$ there exist g_i, h_j simple such that

$$\sum_i \|g_i\| \le (1+\delta)\|g\|_\wedge, \quad \sum_j \|h_j\| \le (1+\delta)\|h\|_\wedge. \tag{4.A.18}$$

Now $g_i \otimes h_j$ is simple and thus $\|g_i \otimes h_j\|_\wedge = \|g_i\| \, \|h_j\|$. The assertion follows from the triangle inequality and fact that

$$\sum_i \sum_j \|g_i\| \, \|h_j\| = \sum_i \|g_i\| \ \sum_j \|h_j\| \le (1+\delta)^2 \|g\|_\vee \, \|h\|_\vee. \tag{4.A.19}$$

The second estimate in (4.A.15) follows from the first (applied to $\mathcal{X}'$ instead of $\mathcal{X}$) and (4.A.14). Finally (4.A.16) follows from (4.A.15) and the definition of the dual norm. □

On $\mathcal{X}^{\otimes r}$ we define the symmetrisation operator by

$$S(\xi_1 \otimes \ldots \otimes \xi_r) = \frac{1}{r!} \sum_{\pi \in S_r} \xi_{\pi(1)} \otimes \ldots \otimes \xi_{\pi(r)}, \tag{4.A.20}$$

where the sum runs over all permutation of the set $\{1, \dots, r\}$, and extension by linearity. Similarly we can define S on $\mathcal{X}'^{\otimes r} = (\mathcal{X}^{\otimes r})'$. Then

$$\langle SL, g\rangle = \langle L, Sg\rangle. \tag{4.A.21}$$

Indeed the identity holds if g is simple and hence by linearity for all g.

Lemma 4.A.5. *For $\Box = \vee$ or $\Box = \wedge$ we have*

$$\|Sg\|_\Box \le \|g\|_\Box \quad \forall g \in \mathcal{X}^{\otimes r} \quad \textit{and} \quad \|SL\|'_\Box \le \|L\|'_\Box \quad \forall L \in \mathcal{X}^{\otimes r}. \tag{4.A.22}$$

Proof. The second assertion follows from the first and (4.A.21). To prove the first assertion for $\Box = \wedge$ it suffices to note that S maps a simple element of norm 1 to a convex combination of simple elements of norm 1. For $\Box = \vee$ we use (4.A.21) to get $\langle \xi_1' \otimes \ldots \xi_r', Sg\rangle = \langle S(\xi_1' \otimes \ldots \xi_r'), g\rangle$. Now we use again that S maps a simple element of norm 1 to a convex combination of simple elements of norm 1. $\Box$

We now define a norm on $\oplus_{r=0}^{\infty} \mathcal{X}^{\otimes r}$ by

$$\|g\|_{\mathcal{X},\Box} := \sup_r \|g^{(r)}\|_{\mathcal{X},\Box} \tag{4.A.23}$$

Here $\|g^{(0)}\| = |g^{(0)}|_{\mathbb{R}}$ where $|\cdot|_{\mathbb{R}}$ is the absolute value on $\mathbb{R}$. For a polynomial $P = \sum_r P^{(r)}$ written as a sum of homogeneous polynomials of degree r the norm is defined by

$$\|P\|'_{\mathcal{X},\Box} = \sup\{\langle \overline{P}, g\rangle : \|g\|_{\mathcal{X},\Box} \le 1\} \tag{4.A.24}$$

where $\langle \overline{P}, g\rangle$ was defined in (4.A.2). We have

$$\|P\|'_{\mathcal{X},\Box} = \sum_{r=0}^{\infty} \|P^{(r)}\|'_{\mathcal{X},\Box} = \sum_{r_0}^{\infty} \|\overline{P^{(r)}}\|'_{\Box}. \tag{4.A.25}$$

Similarly we can define a seminorm by considering only test functions g in the space

$$\Phi_{r_0} := \{g \in \oplus_{r=0}^{\infty} \mathcal{X}^{\otimes r} : g^{(r)} = 0 \quad \forall r > r_0\}. \tag{4.A.26}$$

Then

$$\|P\|'_{r_0,\mathcal{X},\Box} := \sup\{\langle P, g\rangle : g \in \Phi_{r_0},\ \|g\| \le 1\} = \sum_{r=0}^{r_0} \|P^{(r)}\|'_{\mathcal{X},\Box}. \tag{4.A.27}$$

This defines is a seminorm on the space of all polynomials and a norm on polynomials of degree $\le r_0$. When r_0 and $\mathcal{X}$ and $\Box$ are clear we simply write $\|P\| = \|P\|'_{r_0,\mathcal{X},\Box}$.

Proposition 4.A.6 (Product property)**.** *Let $r_0 \in \mathbb{N}_0 \cup \{\infty\}$. Assume that $\Box \in \{\vee, \wedge\}$. Let P and Q be polynomials on $\mathcal{X}$. Then*

$$\|PQ\| \le \|P\|\,\|Q\|. \tag{4.A.28}$$

Proof. We first show the assertion for an r-homogeneous polynomial P and a $k-r$-homogeneous polynomial Q. If $r = 0$ or $k - r = 0$ the assertion is clear. We hence assume $r \ge 1$ and $k - r \ge 1$. We first note that $\overline{PQ} = S(\overline{P} \otimes \overline{Q})$ where S is the symmetrisation operator introduced above. Indeed both sides are symmetric elements of $\mathcal{X}^{\otimes k}$ and they agree on $\xi \otimes \ldots \otimes \xi$. Thus the desired identity follows from the uniqueness statement in Lemma 4.A.1. Now it follows from the second estimate in (4.A.15) and (4.A.22) that $\|PQ\| \le \|\overline{P}\|'_{\Box} \|\overline{Q}\|'_{\Box} = \|P\|\,\|Q\|$. This finishes the proof for homogeneous polynomials.

Finally consider general P, Q and their decompositions into homogeneous polynomials $P = \sum_r P^{(r)}$, $Q = \sum_s Q^{(s)}$. Then it follows from (4.A.27) and the triangle inequality that

$$\|PQ\| \le \sum_{k=0}^{r_0} \sum_{r=0}^{k} \|P^{(r)} Q^{(k-r)}\| \le \sum_{k=0}^{r_0} \|P^{(r)}\| \|Q^{(k-r)}\| \le \|P\|\,\|Q\|. \tag{4.A.29}$$

$\Box$

4.A.2 Norms on polynomials in several variables

The product property for polynomials can be easily extended to polynomials in several variables. To simplify the notation we illustrate this for the case of two variables. A polynomial $P(x, y)$ on $\mathcal{X} \times \mathcal{Y}$ which is r-homogeneous in x and s-homogeneous in y can be identified with an element $\overline{P}$ of $\mathcal{X}^{\otimes r} \otimes \mathcal{Y}^{\otimes s}$ which is symmetric in the sense that

$$\overline{P}(\xi_{\pi(1)} \otimes \ldots \otimes \xi_{\pi(r)} \otimes \eta_{\pi'(1)} \otimes \ldots \otimes \eta_{\pi'(s)}) = \overline{P}(\xi_1 \otimes \ldots \otimes \xi_r \otimes \eta_1 \otimes \ldots \otimes \eta_s) \tag{4.A.30}$$

for all permutations π and π'. We define a space of test functions

$$\Phi_{r_0,s_0} := \{g \in \oplus_{r,s\in\mathbb{N}_0} \mathcal{X}^{\otimes r} \otimes \mathcal{Y}^{\otimes s} : g^{(r,s)} = 0 \quad \text{if } r > r_0 \text{ or } s > s_0\} \tag{4.A.31}$$

with the norm

$$\|g\|_\square := \sup_{r,s\in\mathbb{N}} \|g^{(r,s)}\|_{\mathcal{X},\mathcal{Y},\square}. \tag{4.A.32}$$

Decomposing a general polynomial in homogeneous pieces $P^{(r,s)}$ we define the pairing

$$\langle P, g\rangle = \sum_{r,s\in\mathbb{N}_0} \langle \overline{P^{(r,s)}}, g^{(r,s)}\rangle \tag{4.A.33}$$

and the dual norm

$$\|P\|'_\square = \|P\|'_{r_0,s_0,\mathcal{X},\mathcal{Y},\square} = \sup\{\langle P, g\rangle : g \in \Phi_{r_0,s_0},\ \|g\|_\square\}. \tag{4.A.34}$$

Then

$$\|P\|'_\square = \sum_{r=0}^{r_0} \sum_{s=0}^{s_0} \|P^{(r,s)}\|'_\square = \sum_{r=0}^{r_0} \sum_{s=0}^{s_0} \|\overline{P^{(r,s)}}\|'_\square \tag{4.A.35}$$

where $P^{(r,s)}$ are the (r, s)-homogeneous pieces of P.

For $M \in (\mathcal{X}^{\otimes r_1} \otimes \mathcal{Y}^{\otimes s_1})'$ and $L \in (\mathcal{X}^{\otimes r_2} \otimes \mathcal{Y}^{\otimes s_2})'$ we define $M \otimes L$ in $(\mathcal{X}^{\otimes r_1+r_2} \otimes \mathcal{Y}^{\otimes s_1+s_2})'$ by

$$\langle M \otimes L, \xi_1 \otimes \ldots \otimes \xi_{r_1+r_2} \otimes \eta_1 \otimes \ldots \otimes \eta_{s_1+s_2}\rangle \tag{4.A.36}$$
$$= \langle M, \xi_1 \otimes \ldots \otimes \xi_{r_1} \otimes \eta_1 \otimes \ldots \otimes \eta_{s_1}\rangle\ \langle L, \xi_{r_1+1} \otimes \ldots \otimes \xi_{r_1+r_2} \otimes \eta_{s_1+1} \otimes \ldots \otimes \eta_{s_1+s_2}\rangle. \tag{4.A.37}$$

Then the same argument as before shows that

$$\|M \otimes L\|_\square \le \|M\|_\square\, \|L\|_\square \quad \text{for } \square \in \{\vee, \wedge\}. \tag{4.A.38}$$

We also define a symmetrisation operator $S_{\mathcal{X},\mathcal{Y}}$ which symmetrises separately in the variables on $\mathcal{X}$ and the ones in $\mathcal{Y}$, i.e.,

$$S(\xi_1 \otimes \ldots \otimes \xi_r \otimes \eta_1 \otimes \ldots \otimes \eta_s) := \frac{1}{r!}\frac{1}{s!} \sum_\pi \sum_{\pi'} (\xi_{\pi(1)} \otimes \ldots \otimes \xi_{\pi(r)} \otimes \eta_{\pi'(1)} \otimes \ldots \otimes \eta_{\pi'(s)}). \tag{4.A.39}$$

Again it is easy to see that S has norm 1. Thus for two homogeneous polynomials P and Q one sees as before

$$\|PQ\| = \|S(\overline{P} \otimes \overline{Q})\| \le \|\overline{P}\|\, \|\overline{Q}\| = \|P\|\, \|Q\|. \tag{4.A.40}$$

Now the product property for polynomials is obtained as before by decomposing P and Q in (r, s)-homogeneous polynomials.

4.A.3 Norms on Taylor polynomials

Definition 4.A.7. *Let $p_0 \in \mathbb{N}_0$, let $U \subset \mathcal{X}$ be open and let $F \in C^{r_0}(U)$. For $\varphi \in U$ denote the Taylor polynomial of F at φ by* $\mathrm{Tay}_\varphi F$ *and define*

$$\|F\|_{T_\varphi} = \|\mathrm{Tay}_\varphi F\|'_{r_0,\mathcal{X},\square}. \tag{4.A.41}$$

where $\square$ refers to the norm used for the tensor products.

When the norm on the tensor products is clear we often drop $\square$.

Example 4.A.8. *Let $\mathcal{X} = (\mathbb{R}^p, |\cdot|_\infty)$ and set $E = \{1, \dots, p\}$. In* (4.A.7) *we have seen that the injective norm of $g \in \mathcal{X}^{\otimes r}$ is given by $\|g\|_\vee = \sup_{(i_1,\dots i_r)\in E^r} |g_{i_1\dots i_r}| = |g|_\infty$. Let $F \in C^{r_0}(\mathcal{X})$. The Taylor polynomial of order r_0 at zero can be written as*

$$P(\varphi) = \sum_{r=0}^{r_0} \frac{1}{r!} \sum_{i_1,\dots,i_r=1}^{p} \frac{\partial^r F}{\partial\varphi_{i_1}\dots\partial\varphi_{i_r}}(0) \prod_{j=1}^{r} \varphi_{i_j} = \sum_{|\gamma|_1\le r_0} \frac{1}{\gamma!}\partial^\gamma F(0)\ \varphi^\gamma \tag{4.A.42}$$

where the sum in the term on the right hand side runs over multiindices $\gamma \in \mathbb{N}_0^E$ and $|\gamma| := \sum_{i\in E}\gamma(i)$. The term corresponding to $r = 0$ is defined as $F(0)$. We claim that

$$\|F\|_{T_0} = \sum_{r=0}^{r_0} \frac{1}{r!} \sum_{i_1,\dots,i_r=1}^{p} \left|\frac{\partial^r F}{\partial\varphi_{i_1}\dots\partial\varphi_{i_r}}(0)\right| = \sum_{|\gamma|\le r_0} \frac{1}{\gamma!}\left|\partial^\gamma F(0)\right|. \tag{4.A.43}$$

Indeed it suffices to verify the first identity, the second follows by the usual combinatorics. Denote the middle term in (4.A.43) *by M. Since we use the ℓ_∞ norm on $\mathcal{X}^{\otimes r} = \mathbb{R}^{E^r}$ we get for all $g \in \Phi_{r_0}$*

$$\langle F, g\rangle_0 = \sum_{r=0}^{r_0} \frac{1}{r!} \sum_{i_1,\dots,i_r=1}^{\varphi} \frac{\partial^r F}{\partial\varphi_{i_1}\dots\partial\varphi_{i_r}}(0)\, g_{i_1\dots i_r} \le M \sup_{0\le r\le r_0} |g^{(r)}|_\infty \le M\|g\|_{\mathcal{X},\vee} \tag{4.A.44}$$

The inequality becomes sharp if we take $g_{i_1\dots i_r} = \operatorname{sgn} \frac{\partial^r F}{\partial\varphi_{i_1}\dots\partial\varphi_{i_r}}(0)$. This proves (4.A.43).

Proposition 4.A.9 (Product property, see [44], Proposition 3.7). *Let $U \subset \mathcal{X}$ be open and let $F \in C^{r_0}(U)$. Then*

$$\|FG\|_{T_\varphi} \le \|F\|_{T_\varphi}\, \|G\|_{T_\varphi}. \tag{4.A.45}$$

Proof. This follows from Proposition 4.A.6 and the fact that the Taylor polynomial of the product is the product of the Taylor polynomials, truncated at degree r_0. □

By the considerations in Section 4.A.2 the product property also holds for polynomials in several variables.

Proposition 4.A.10 (Polynomial estimate, see [44], Proposition 3.10). *Assume $\square \in \{\vee, \wedge\}$. Let F be a polynomial of degree $\overline{r} \le r_0$. Then*

$$\|F\|_{T_\varphi} \le (1 + \|\varphi\|)^{\overline{r}}\|F\|_{T_0}. \tag{4.A.46}$$

Proof. Let F be a polynomial of degree $\overline{r}$ with homogeneous pieces F_r. Then we can write $F(\varphi) = \sum_{r=0}^{\overline{r}}\langle\overline{F_r}, \varphi\otimes\dots\otimes\varphi\rangle$. Set $G(\xi) = F(\varphi + \xi)$. For $r > k$ define $B_{k,r} \in (\mathcal{X}^{\otimes r})'$ by

$$\langle B_{k,r}, g\rangle = \langle\overline{F_r}, g\otimes\varphi\otimes\dots\otimes\varphi\rangle. \tag{4.A.47}$$

Set $B_{k,k} = \overline{F_k}$. Since the $\overline{F_r}$ are symmetric we get

$$G(\xi) = \sum_{k=0}^{\overline{r}} \langle B_k, \xi \otimes \ldots \otimes \xi \rangle \quad \text{where } B_k = \textstyle\sum_{r=k}^{\overline{r}} \binom{r}{k} B_{k,r}. \tag{4.A.48}$$

Now by the contraction estimate (4.A.16) we have $\|B_{k,r}\|'_{\Box} \le \|\overline{F_r}\|'_{\Box} \, \|\varphi\|_{\mathcal{X}}^{r-k}$. Thus

$$\|G\|_{T_0} \le \sum_{k=0}^{\overline{r}} \sum_{r=k}^{\overline{r}} \binom{r}{k} \|\overline{F_r}|'_{\Box} \, \|\varphi\|_{\mathcal{X}}^{r-k} \, 1^k \le \sum_{r=0}^{\overline{r}} (1 + \|\varphi\|_{\mathcal{X}})^r \, \|\overline{F_r}\|'_{\Box} \le (1 + \|\varphi\|_{\mathcal{X}})^{\overline{r}} \, \|F\|_{T_0}. \tag{4.A.49}$$

Since $\|F\|_{T_\varphi} = \|G\|_{T_0}$ this concludes the proof. $\Box$

Proposition 4.A.11 (Two norm estimate, see [44], Proposition 3.11)**.** *Let $F \in C^{r_0}(\mathcal{X})$. Assume that $\Box \in \{\vee, \wedge\}$. Let $\|\cdot\|_{\mathcal{X},\Box}$ and $\|\cdot\|_{\tilde{\mathcal{X}},\Box}$ denote norms on the tensor products $\mathcal{X}^{\otimes r}$ based on norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\tilde{\mathcal{X}}}$. Denote the corresponding norms of the Taylor polynomials of F by $\|F\|_{T_\varphi}$ and $\|F\|_{\tilde{T}_\varphi}$ Define*

$$\rho^{(n)} := 2 \sup\{\|g\|_{\mathcal{X},\Box} : g \in \mathcal{X}^{\otimes r}, \, \|g\|_{\tilde{\mathcal{X}},\Box} \le 1, \, n \le r \le r_0\}. \tag{4.A.50}$$

Then, for any $\overline{r} < r_0$,

$$\|F\|_{\tilde{T}_\varphi} \le (1 + \|\varphi\|_{\tilde{\mathcal{X}}})^{\overline{r}+1} \left(\|F\|_{\tilde{T}_0} + \rho^{(\overline{r}+1)} \sup_{0 \le t \le 1} \|F\|_{T_{t\varphi}} \right). \tag{4.A.51}$$

Proof. Let P denote the Taylor polynomial of order $\overline{r}$ of F computed at 0. By Proposition 4.A.10 and the trivial estimate $\|P\|_{\tilde{T}_0} \le \|F\|_{\tilde{T}_0}$ we have

$$\|P\|_{\tilde{T}_\varphi} \le (1 + \|\varphi\|_{\tilde{\mathcal{X}}})^{\overline{r}} \, \|P\|_{\tilde{T}_0} \le (1 + \|\varphi\|_{\tilde{\mathcal{X}}})^{\overline{r}+1} \, \|F\|_{\tilde{T}_0}. \tag{4.A.52}$$

Let $R = F - P$. It thus suffices to show that

$$\|R\|_{\tilde{T}_\varphi} \le (1 + \|\varphi\|_{\tilde{\mathcal{X}}})^{\overline{r}+1} \, \rho^{(\overline{r}+1)} \sup_{0 \le t \le 1} \|F\|_{T_{t\varphi}}. \tag{4.A.53}$$

To abbreviate set

$$M := \sup_{0 \le t \le 1} \|F\|_{T_{t\varphi}} = \sup_{0 \le t \le 1} \sum_{k=0}^{r_0} \frac{1}{k!} \|D^k F(t\varphi)\|'_{\mathcal{X},\Box}. \tag{4.A.54}$$

Here we view $D^k F(\varphi)$ as an element of $(\mathcal{X}^{\otimes r})'$. For $k \ge \overline{r} + 1$ we have $D^k R = D^k F$ and

$$\langle D^k R(\varphi), g \rangle = \langle D^k F(\varphi), g \rangle \le \|D^k F\|'_{\mathcal{X},\Box} \, \|g\|_{\mathcal{X},\Box} \le \|D^k F\|'_{\mathcal{X},\Box} \, \frac{1}{2} \, \rho^{(\overline{r}+1)} \|g\|_{\tilde{\mathcal{X}},\Box} \tag{4.A.55}$$

and thus

$$\sum_{k=\overline{r}+1}^{r_0} \frac{1}{k!} \|D^k R(\varphi)\|'_{\tilde{\mathcal{X}},\Box} \le \frac{1}{2} \, \rho^{(\overline{r}+1)} M. \tag{4.A.56}$$

For $k \le \overline{r}$ we apply the Taylor formula with remainder term in integral form to $\langle D^k R, g\rangle$ and get

$$
\begin{aligned}
&|\langle D^k R(\varphi), g\rangle| \qquad (4.A.57)\\
\le &\int_0^1 \frac{1}{(\overline{r}-k)!}(1-t)^{\overline{r}-k}\,|\langle D^{\overline{r}+1}F(t\varphi), g\otimes\varphi\otimes\ldots\otimes\varphi\rangle|\,dt\\
\le &M\frac{(\overline{r}+1)!}{(\overline{r}+1-k)!}\;\|g\otimes\varphi\otimes\ldots\otimes\varphi\|'_{\mathcal{X},\Box}\\
\le &\frac{1}{2}\rho^{(\overline{r}+1)}M\frac{(\overline{r}+1)!}{(\overline{r}+1-k)!}\;\|g\otimes\varphi\otimes\ldots\otimes\varphi\|'_{\tilde{\mathcal{X}},\Box}\\
\le &\frac{1}{2}\rho^{(\overline{r}+1)}M\frac{(\overline{r}+1)!}{(\overline{r}+1-k)!}\;\|g\|'_{\tilde{\mathcal{X}},\Box}\,\|\varphi\|_{\tilde{\mathcal{X}}}^{\overline{r}+1-k}.
\end{aligned}
$$

Thus

$$\frac{1}{k!}\|D^k R(\varphi)\|'_{\tilde{\mathcal{X}},\Box} \le \frac{1}{2}\rho^{(\overline{r}+1)}M\binom{\overline{r}+1}{k}\|\varphi\|_{\tilde{\mathcal{X}}}^{\overline{r}+1-k}\;1^k. \tag{4.A.58}$$

Summing this from $k=0$ to $\overline{r}$ we get

$$\sum_{k=0}^{\overline{r}}\frac{1}{k!}\|D^k R(\varphi)\|'_{\tilde{\mathcal{X}},\Box} \le \frac{1}{2}\rho^{(\overline{r}+1)}M(1+\|\varphi\|_{\tilde{\mathcal{X}}})^{\overline{r}+1}. \tag{4.A.59}$$

Together with (4.A.56) this concludes the proof of (4.A.53) □

4.A.4 Examples with a more general injective norm on $\mathcal{X}^{\otimes r}$

We will be mostly interested in the case that the norm on $\mathcal{X}$ is defined by a specific family of linear functionals on $\mathcal{X}$ (abstractly one can always define the norm in this way since for finite dimensional space $\mathcal{X}'' = \mathcal{X}$). Then the injective norm on $\mathcal{X}^{\otimes r}$ is defined by the tensor products of these functionals (see Proposition 4.A.13 below).

Let E be a finite set. On $\mathbb{R}^E$ consider a finite family $\mathcal{B}$ of linear functionals $\ell : \mathbb{R}^E \to \mathbb{R}$. Let

$$N_{\mathcal{B}} := \{\varphi \in \mathbb{R}^E : \ell(\varphi) = 0\ \forall \ell \in \mathcal{B}\}. \tag{4.A.60}$$

Then the linear functionals induce a norm on $\mathcal{X} := \mathbb{R}^E/N_{\mathcal{B}}$, namely

$$\|\varphi\|_{\mathcal{X}} := \sup\{|\ell(\varphi)| : \ell \in \mathcal{B}\}. \tag{4.A.61}$$

Proposition 4.A.12. *The dual space of $\mathcal{X}$ is given by $\mathcal{X}' := \operatorname{span}\{\ell : \ell \in \mathcal{B}\}$ and the norm on $\mathcal{X}'$ is given by*

$$\|\xi'\|_{\mathcal{X}'} = \inf\Big\{\sum_i |\lambda_i| : \xi' = \sum_i \lambda_i \ell_i,\ \ell_i \in \mathcal{B},\ \lambda_i \in \mathbb{R}\Big\}. \tag{4.A.62}$$

In particular $\|\ell\|_{\mathcal{X}'} \le 1$ for all $\ell \in \mathcal{B}$.

Proof. Let C denote the closed convex hull of $\mathcal{B} \cup -\mathcal{B}$. It follows from the definition of norm on $\mathcal{X}$ that $C \subset \overline{B_1(\mathcal{X}')}$. For the reverse inclusion one uses that points $\xi' \notin C$ can be separated by a linear functional, i.e., that there exist a $g \in \mathcal{X}$ such that $\xi'(g) > 1$ and $\tilde{\xi}'(g) \le 1 \quad \forall \tilde{\xi}' \in C$. This implies $\|g\| \le 1$ and hence $\|\xi'\| > 1$. □

Proposition 4.A.13. *The injective norm on $\mathcal{X}^{\otimes r}$ can be characterized by*

$$\|g\|_{\vee} = \sup\{|\langle \ell_1 \otimes \ldots \otimes \ell_r, g\rangle| : \ell_i \in \mathcal{B}\}. \tag{4.A.63}$$

Note that in the special case $E = \{1, \dots, p\}$ and $\mathcal{B} = \{e'_1, \dots, e'_p\}$ we recover (4.A.7).

Proof. Denote the right hand side of (4.A.63) by m. Since $\|\ell\|_{\mathcal{X}'} \le 1$ for all $\ell \in \mathcal{B}$ we get $m \le \|g\|_\vee$. To prove the reverse inequality let $\delta > 0$ and assume that $\|\xi'_i\| \le 1$. Then by (4.A.62) there exist $\lambda_{i,j} \in \mathbb{R}$ and $\ell_{i,j} \in \mathcal{B}$ such that $\xi'_i = \sum_j \lambda_{i,j}\ell_{i.j}$ and $\sum_j |\lambda_{i,j}| \le 1 + \delta$. Thus $|\langle \xi'_1 \otimes \dots \otimes \xi'_r, g\rangle| \le (1+\delta)^r m$ and hence $\|g\|_\vee \le (1+\delta)^r m$. Since $\delta > 0$ was arbitrary we conclude that $\|g\|_\vee \le m$. $\square$

4.A.5 Main example

We now come to our main example. Consider the torus $\Lambda = \mathbb{Z}^d/L^N\mathbb{Z}^d$ and set $\mathbf{\Lambda} = \{1, \dots, m\} \times \Lambda$. The elements of $\mathbb{R}^{\mathbf{\Lambda}} = \mathbb{R}^m \otimes \mathbb{R}^\Lambda$ can be viewed as maps from $\mathbf{\Lambda}$ to $\mathbb{R}$ or as maps from $\Lambda \to \mathbb{R}^m$. We will use both viewpoints interchangeably.

We are interested in linear functionals $\mathbb{R}^{\mathbf{\Lambda}}$ which are based on discrete derivatives. More precisely let $e_1, \dots, e_d$ denote the standard unit vectors in $\mathbb{Z}^d$ and set

$$\mathcal{U} = \{e_1, \dots, e_d\}. \tag{4.A.64}$$

We remark in passing that here our notation differs from [45]. There they also consider backward derivatives and $\mathcal{U}$ denotes the set $\{\pm e_1, \dots, \pm e_d\}$. For $e \in \mathcal{U}$ and $f : \Lambda \to \mathbb{R}$ the forward difference operator is given by

$$\nabla^e f(x) = f(x+e) - f(x). \tag{4.A.65}$$

For a multiindex $\alpha \in \mathbb{N}_0^{\mathcal{U}}$ we write

$$\nabla^\alpha = \prod_{e\in\mathcal{U}} (\nabla^e)^{\alpha(e)}, \qquad \nabla^0 = \mathrm{Id}\,. \tag{4.A.66}$$

For a pair $(i, \alpha) \in \{1, \dots, m\} \times \mathbb{N}_0^{\mathcal{U}}$ and $x \in \Lambda$ we define

$$\nabla_x^{i,\alpha}\varphi = \nabla^\alpha \varphi_i(x). \tag{4.A.67}$$

We set $N_\Lambda = \{\varphi : \Lambda \to \mathbb{R}^p : \varphi \text{ constant}\}$. Given weights $w(i,\alpha) > 0$ we define a norm on $\mathcal{X} = \mathbb{R}^{\mathbf{\Lambda}}/N_\Lambda$ by

$$\|\varphi\|_{\mathcal{X}} = \sup_{x\in\Lambda} \sup_{1\le i\le m} \sup_{1\le|\alpha|\le p_\Phi} w(i,\alpha)^{-1}\nabla_x^{i,\alpha}\varphi \tag{4.A.68}$$

Here and in the following we always use the ℓ_1 norm for multiindices

$$|\alpha| = |\alpha|_1 = \sum_{i\in\mathcal{U}} \alpha_i. \tag{4.A.69}$$

On the scale k we will usually use the weight

$$w_k(i,\alpha) = L^{-k|\alpha|}\mathfrak{h}_k, \quad \mathfrak{h}_k = h_k L^{-k\frac{d-2}{2}}, \quad h_k = 2^k h. \tag{4.A.70}$$

Note that for an element $\varphi \in \mathcal{X}$ we cannot define a pointwise value $\varphi(x)$ but the derivative $\nabla^\alpha\varphi(x)$ are well defined if $\alpha \ne 0$. Indeed φ is uniquely determined by the derivatives with $|\alpha|_1 = 1$. We can choose a unique representative $\widetilde{\varphi}$ of the equivalence class $\varphi + N$ by requiring $\sum_{x\in\Lambda}\widetilde{\varphi}(x) = 0$ and we sometimes identify the space $\mathcal{X} = \mathbb{R}^{\mathbf{\Lambda}}/N$ with the space $\{\psi \in \mathbb{R}^{\mathbf{\Lambda}} : \sum_{x\in\Lambda}\psi(x) = 0\}$.

The tensor product $\mathcal{X}\otimes\mathcal{X}$ is the quotient of $\mathbb{R}^{\mathbf{\Lambda}} \otimes \mathbb{R}^{\mathbf{\Lambda}}$ by span$\{\text{constants} \otimes \varphi, \varphi \otimes \text{constants} : \varphi \in \mathbb{R}^{\mathbf{\Lambda}}\}$. Again an element $g^{(2)} \in \mathcal{X}\otimes\mathcal{X}$ does not have pointwise values $g_{ij}(x,y)$ but the derivatives $\nabla^{i,\alpha} \otimes \nabla^{j,\beta} g^{(2)}(x,y) = \nabla^\alpha \otimes \nabla^\beta g_{ij}(x,y)$ are well defined (for $\alpha \neq 0$ and $\beta \neq 0$) and the derivatives with $|\alpha|_1 = |\beta|_1 = 1$ determine $g^{(2)}$ uniquely. Here $\nabla^{i,\alpha}$ acts on the first argument of $g^{(2)}$ and $\nabla^{j,\beta}$ on the second argument. Similar reasoning applies to $\mathcal{X}^{\otimes r}$ and by Proposition 4.A.13 the injective norm on $\mathcal{X}^{\otimes r}$ is given by

$$\|g^{(r)}\|_{\mathcal{X},\vee} = \sup_{x_1,\ldots,x_r \in \Lambda} \sup_{m \in \mathfrak{m}_{p_\Phi,r}} w(m)^{-1} \nabla^{m_1} \otimes \ldots \otimes \nabla^{m_r} g^{(r)}(x_1,\ldots,x_r). \tag{4.A.71}$$

Here

$$w(m) = \prod_{j=1}^{r} w(m_j) \tag{4.A.72}$$

and $\mathfrak{m}_{p_\Phi,r}$ is the set of r-tupels $m = (m_1,\ldots,m_r)$ with $m_j = (i_j,\alpha_j)$ and $1 \leq |\alpha_j| \leq p_\Phi$. Note that here each α_j is a multiindex, i.e., an element of $\mathbb{N}_0^{\mathcal{U}}$, not a number. For $m \in \mathfrak{m}_{p_\Phi,r}$ consider the monomial

$$\mathcal{M}_m(\{x\})(\varphi) := \prod_{j=1}^{r} \nabla^{m_j} \varphi(x). \tag{4.A.73}$$

Then the element $\overline{\mathcal{M}_m(\{x\})} \in (\mathcal{X}^{\otimes r})'$ which corresponds to $\mathcal{M}_m(\{x\})$ is given by the symmetrisation $S(\nabla_x^{m_1} \otimes \ldots \otimes \nabla_x^{m_r})$. Thus in view of (4.A.71) and (4.A.22) we get

$$\|\mathcal{M}_m(\{x\})\|_{T_0} = \|\overline{\mathcal{M}_m(\{x\})}\|'_{\mathcal{X},\vee} \leq w(m). \tag{4.A.74}$$

We consider functionals F localised near a polymer $X \subset \Lambda$, i.e. $F(\varphi) = F(\psi)$ if $\varphi = \psi$ in X^* where X^* is the small set neighbourhood of X, see (4.4.34). Then it is natural to work with field norms which are also localised. There are different ways to do that. We follow the approach in [4] and define

$$\|\varphi\|_{\mathcal{X},X} := \sup_{x \in X^*} \sup_{1 \leq i \leq p} \sup_{1 \leq |\alpha| \leq p_\Phi} w(i,\alpha)^{-1} |\nabla^{i,\alpha}\varphi(x)|, \tag{4.A.75}$$

see (4.4.77). Let us remark that Brydges and Slade take a more abstract, but similar, approach and define

$$\|\varphi\|_{\widetilde{\mathcal{X}},X} = \inf\{\|\varphi - \xi\|_{\mathcal{X}} : \xi\restriction_{\{1,\ldots,p\}\times X} = 0\}, \tag{4.A.76}$$

see eqns. (3.37)–(3.39) in [44].

4.B Estimates for Taylor polynomials in $\mathbb{Z}^d$

Here we give a proof of the remainder estimate which was the key ingredient in proving the contraction estimate for the linearised operator $\boldsymbol{C}^{(\boldsymbol{q})}$. Recall that for $f : \mathbb{Z}^d \to \mathbb{R}$ the discrete s-th order Taylor polynomial at a is given by

$$\mathrm{Tay}_a^s f(z) := \sum_{|\alpha| \leq s} \nabla^\alpha f(a)\; b_\alpha(z-a) \tag{4.B.1}$$

where

$$b_\alpha(z) = \prod_{i=1}^{d} \binom{z_j}{\alpha_j} \quad \text{and} \quad \binom{z_j}{\alpha_j} = \frac{z_j \ldots (z_j - \alpha_j + 1)}{\alpha_j!}. \tag{4.B.2}$$

It is easy to see that $\nabla^\beta b_\alpha = b_{\alpha-\beta}$ with the conventions $b_0 \equiv 1$ and $b_{\alpha-\beta} = 0$ if $\alpha - \beta \notin \mathbb{N}_0^{\{1,\ldots,d\}}$. Recall that $\mathcal{U} = \{e_1,\ldots,e_d\}$.

Lemma 4.B.1. *Let $s \in \mathbb{N}_0$, $\rho \in \mathbb{N}$ and define*

$$M_{\rho,s} = \sup\{|\nabla^\alpha f(z)| : |\alpha| = s+1,\ z \in \mathbb{Z}^d \cap \big(a + [0,\rho]^d\big)\}. \tag{4.B.3}$$

Then for all $\beta \in \mathbb{N}_0^{\mathcal{U}}$ with $t = |\beta| \le s$

$$\Big|\nabla^\beta[f(z) - \mathrm{Tay}_a^s f(z)]\Big| \le M_{\rho,s} \binom{|z-a|_1}{s-t+1} \quad \forall z \in \mathbb{Z}^d \cap \big(a + [0,\rho]^d\big). \tag{4.B.4}$$

The estimate is sharp for $a = 0$ and $t = 0$ since the function $f(z) = \binom{z_1+\ldots+z_d}{t+1}$ satisfies $\nabla^\alpha f = 1$ for all α with $|\alpha| = t+1$ (see proof).

Proof. This result in classical and is a (very) special case of Lemma 3.5 in [45]. Since the notation here is simpler we include the short proof along the lines of [45] for the convenience of the reader. We may assume that $a = 0$. It suffices to show (4.B.4) for $t = 0$. Indeed if the result is known for $t = 0$ we can use that $\nabla\beta\, \mathrm{Tay}_0^s f = \mathrm{Tay}_0^{s-t} \nabla^\beta f$ and deduce that $|\nabla^\beta f(z) - \mathrm{Tay}_0^{s-t} \nabla^\beta f(z)| \le M_{\rho,s}\binom{|z|_1}{s-t+1}$. Here we used that $M_{\rho,s-t}(\nabla^\beta f) \le M_{\rho,s}(f)$.

The proof for $t = 0$ is by induction over the dimension d. We first note that for $z \in \mathbb{N}_0^d$

$$\binom{|z|_1}{s+1} = b_{s+1}(z_1 + \ldots + z_d) = \sum_{|\alpha|=s+1} b_\alpha(z). \tag{4.B.5}$$

Indeed the first identity follows immediately from the definition of b_{s+1} (as a polynomial on $\mathbb{Z}$) since $z_i \ge 0$. To prove the second identity we show that both side have the same discrete derivatives at $z = 0$. Indeed the discrete derivative ∇^β of the left hand side evaluated at zero is given by $b_{s+1-|\beta|}(0)$. This equals 1 for $|\beta| = s+1$ and 0 if $|\beta| \ne s+1$. The same assertion is true for the right hand side.

Thus it suffices to show that

$$|f(z) - \mathrm{Tay}_0^s f(z)| \le M_{\rho,s} \sum_{|\alpha|=s+1} b_\alpha(s). \tag{4.B.6}$$

Note that if $z_j \in \mathbb{Z}$ and $0 < z_j < \alpha_j$ for some j then $b_\alpha(z) = 0$. Thus

$$b_\alpha(z) \ge 0 \quad \forall z \in \mathbb{Z}^d \cap [0,\rho]^d. \tag{4.B.7}$$

For $d = 1$ we use the discrete Taylor formula with remainder

$$f(z) = \sum_{r=0}^{s} \nabla^r f(0)\, b_r(z) + \sum_{z'=0}^{z-1} b_s(z-1-z')\, \nabla^{s+1} f(z'). \tag{4.B.8}$$

This formula is easily proved using induction over s and the summation by parts formula

$$\sum_{z'=0}^{z-1} b_s(z-1-z')\, g(z') = b_{s+1}(z) g(0) + \sum_{z''=0}^{z-1} b_{s+1}(z-1-z'')(g(z''+1) - g(z'')). \tag{4.B.9}$$

Since $\nabla b_{s+1} = b_s$ we have

$$\sum_{z'=0}^{z-1} |b_s(z-1-z')| = \sum_{z'=0}^{z-1} b_s(z-1-z') = \sum_{z'=0}^{z-1} b_s(z') = b_{s+1}(z) \tag{4.B.10}$$

and thus the Taylor formula with remainder implies (4.B.6) for $d = 1$.

Now assume that (4.B.6) holds for $d-1$. Set $\alpha' = (\alpha_1, \dots, \alpha_{d-1})$ and $\alpha = (\alpha', \alpha_d)$ and similarly $z = (z', z_d)$. Then the induction hypothesis gives (for $z_j \geq 0$)

$$\Big| f(z', z_d) - \sum_{|\alpha'| \leq s} \nabla^{\alpha'} f(0, z_d)\, b_{\alpha'}(z') \Big| \leq M_{\rho,s} \sum_{|\alpha'| = s+1} b_{\alpha',0}(z). \tag{4.B.11}$$

Now by the result for $d = 1$ applied to the z_d direction

$$\Big| \nabla^{\alpha'} f(0, z_d) - \sum_{\alpha_d \leq s - |\alpha'|} \nabla^{(\alpha', \alpha_d)} f(0)\, b_{\alpha_d}(z_d) \Big| \leq M_{\rho,s}\, b_{s+1-|\alpha'|}(z_d). \tag{4.B.12}$$

Since $b_{\alpha'}(z')\, b_{\alpha_d}(z_d) = b_\alpha(z)$ it follows that

$$\begin{aligned} &\Big| \sum_{|\alpha'| \leq s} \nabla^{\alpha'} f(0, z_d)\, b_{\alpha'}(z') - \sum_{|\alpha| \leq s} \nabla^{\alpha} f(0)\, b_\alpha(z) \Big| \\ &\leq M_{\rho,s} \sum_{|\alpha'| \leq s} b_{\alpha'}(z')\, b_{s+1-|\alpha'|}(z_d) = M_{\rho,s} \sum_{|\alpha| = s+1, |\alpha'| \leq s} b_\alpha(z). \end{aligned} \tag{4.B.13}$$

Combining (4.B.11) and (4.B.13) we see that (4.B.6) holds for d.

□

4.C Combinatorial lemmas

In this appendix we state two lemmas that are used in the reblocking step.

Lemma 4.C.1. *Let $X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k$ and $\alpha(d)) = (1 + 2^d)^{-1}(1 + 6^d)^{-1}$. Then*

$$|X|_k \geq (1 + 2\alpha(d))|\overline{X}|_{k+1}. \tag{4.C.1}$$

For any $X \in \mathcal{P}_k$ we have

$$|X|_k \geq (1 + \alpha(d))|\overline{X}|_{k+1} - (1 + \alpha(d))2^{d+1}|\mathcal{C}(X)|. \tag{4.C.2}$$

Proof. This is Lemma 6.15 in [42]. □

Lemma 4.C.2. *There exists $\delta(d, L) < 1$ such that*

$$\sum_{\substack{X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k \\ \overline{X} = U}} \delta^{|X|_k} \leq 1 \tag{4.C.3}$$

for any $k \in \mathbb{N}$ and $U \in \mathcal{P}_{k+1}^{\mathrm{c}}$.

Proof. This is Lemma 6.16 in [42]. □

Chapter 5

Phase transitions for a class of gradient fields

5.1 Introduction

In this chapter we are going to study gradient interface models for a class of non-convex potentials. We refer to the introduction in Chapter 1 for an overview of the results for gradient interface models. Let us just recall two results for convex interaction potentials. Funaki and Spohn showed in [89] that for every tilt vector u there exists a unique translation invariant gradient Gibbs measure. Moreover, the scaling limit of the model is a massless Gaussian field as shown by Naddaf and Spencer [128] for zero tilt and generalised to arbitrary tilt by Giacomin, Olla, and Spohn [94]. For non-convex potentials in general far less is known with two exceptions. First, in the high temperature phase for certain potentials essentially the same results as for convex potentials can be shown ([64, 63, 70]). On the other hand, for small temperatures some results in particular the existence of the scaling limit can be proved for periodic boundary conditions as discussed in Chapter 3. For intermediate temperatures that correspond to very non-convex potentials no robust techniques are known. All results to date are restricted to the special class of potentials introduced by Biskup and Kotecky in [32] that can be represented as

$$e^{-V(x)} = \int_{\mathbb{R}_+} e^{-\frac{\kappa x^2}{2}}\, \rho(\mathrm{d}\kappa) \tag{5.1.1}$$

where ρ is a non-negative Borel measure on the positive real line. Biskup and Kotecky mostly considered the simplest nontrivial case, denoting the Dirac measure at $x \in \mathbb{R}$ by δ_x,

$$\rho = p\delta_q + (1-p)\delta_1 \tag{5.1.2}$$

where $p \in [0,1]$ and $q \geq 1$. They show that in dimension $d = 2$ and for $q > 1$ sufficiently large there exist two ergodic zero-tilt gradient Gibbs measures. Later Biskup and Spohn showed in [33] that nevertheless the scaling limit of every zero-tilt gradient Gibbs measure is Gaussian if the measure ρ is compactly supported in $(0,\infty)$. In [151] their result was recently extended by Ye to potentials of the form $V(s) = (1+s^2)^\alpha$ with $0 < \alpha < \frac{1}{2}$. Those potentials can be expressed as in (5.1.1) but ρ has unbounded support so that the results from [33] do not directly apply.

The main reason to study this class of potentials is that they are much more tractable because using the representation (5.1.1) the variable κ can be considered as an additional degree of freedom. This leads to extended gradient Gibbs measures which are given by the joint law of $(\eta_e, \kappa_e)_{e\in\mathbf{E}(\mathbb{Z}^d)}$. These extended gradient Gibbs measures can be represented as a mixture of non-homogeneous Gaussian fields with bond potential $\kappa_e \eta^2/2$ for every edge $e \in \mathbf{E}(\mathbb{Z}^d)$ and $\kappa_e \in \mathbb{R}_+$. This implies that for a given κ the distribution of the gradient field is Gaussian with covariance given by the inverse of the operator Δ_κ where

$$\Delta_\kappa f(x) = \sum_{y \sim x} \kappa_{\{x,y\}} (f(x) - f(y)). \tag{5.1.3}$$

where $x \sim y$ denotes the neighbourhood relation in a graph. In all the works mentioned before this structure is frequently used, e.g. in [33] it is proved that the resulting κ-marginal of the extended gradient Gibbs measure is ergodic so that well known homogenization results for random walks in ergodic environments can be applied. The main purpose of this chapter is to investigate the properties of the κ-marginal of extended gradient Gibbs measures in a bit more detail. The starting point is the observation that the κ-marginal of an extended gradient Gibbs measures with zero tilt is itself the Gibbs measure for a certain specification. This specification arises as the infinite volume of an infinite range random conductance model defined on finite graphs. On the other hand, we show that starting from Gibbs measure for the random conductance model we can construct a zero tilt gradient Gibbs measure thus showing a one to one relation between the two notions of Gibbs measures. In particular, we can lift results about the random conductance model to results about gradient Gibbs measures. Note that one major drawback is the restriction to zero tilt that applies here and to all earlier results for this model. Let us mention that massive $\mathbb{R}$-valued random fields have been earlier connected to discrete percolation models to analyse the existence of phase transitions [152]. For gradient models the setting is slightly different because we consider a random conductance model on the bonds while for massive models one typically considers some type of site percolation.

The main motivation for our analysis is that it provides a first step to the completion of the phase diagram for this potential and zero tilt and a better understanding of the two coexisting Gibbs states. Moreover, the random conductance model appears to be interesting in its own right. We could define the random conductance model and prove several of the results for arbitrary ρ but we mostly restrict our analysis to the simplest case where ρ is as in (5.1.2) and the potential is of the form

$$e^{-V_{p,q}(x)} = pe^{-\frac{qx^2}{2}} + (1-p)e^{-\frac{x^2}{2}}. \tag{5.1.4}$$

We prove several results about the random conductance model in particular correlation inequalities (that extend to arbitrary ρ). One helpful observation is that the random conductance model is closely related to determinantal processes because its definition involves a determinant weight. This simplifies several of the proofs because all correlation inequalities can be immediately led back to similar results for the uniform spanning tree. Using the correlation inequalities it is possible to show uniqueness of its Gibbs measure in certain regimes.

It was already observed in [32] that the gradient interface model with potential $V_{p,q}$ exhibits a duality property when defined on the torus. Moreover, there is a self dual point $p_{\mathrm{sd}} = p_{\mathrm{sd}}(q) \in (0,1)$ where the model agrees with its own dual. The self dual point satisfies the equation

$$\left(\frac{p_{\mathrm{sd}}}{1-p_{\mathrm{sd}}}\right)^4 = q. \tag{5.1.5}$$

In [32] it is shown that the location of the phase transition in $d = 2$ must be the self dual point.

We extend the duality to the random conductance model and arbitrary planar graphs. Using the fact that $\mathbb{Z}^2$ as a graph is self-dual we can use the duality to prove non-uniqueness of the Gibbs measure therefore reproving the result from [32] without the use of reflection positivity.

Many of our techniques and results for the random conductance model originated in the study of the random cluster model and we conjecture further similarities.

This chapter is structured as follows. In Section 5.2 we give a precise definition of gradient Gibbs measures and state our main results. Then, in Section 5.3 we introduce and motivate the random conductance model and its relation to extended gradient Gibbs measures. We prove properties of the random conductance model in Sections 5.4 and 5.5. In Section 5.6 we use the duality of the model to reprove the phase transition result. Finally, in Section 5.7 we discuss some possible future directions. Two technical proofs and some results about regularity properties of discrete elliptic equations are delegated to appendices.

5.2 Model and main results

Specifications. Let us briefly recall the definition of a specification because the concept will be needed in full generality for the random conductance model (see Section 5.4). We consider a countable set S (mostly $\mathbb{Z}^d$ or the edges of $\mathbb{Z}^d$) and a measurable state space $(F, \mathcal{F})$ (mostly either $|F| = 2$ or $(F, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$). Random fields are probability measures on $(F^S, \mathcal{F}^S)$ where $\mathcal{F}^S$ denotes the product σ-algebra. The set of probability measures on a measurable space $(X, \mathcal{X})$ will be denoted by $\mathcal{P}(X, \mathcal{X})$. For any $\Lambda \subset S$ we denote by $\pi_\Lambda : F^S \to F^\Lambda$ the canonical projection. We often consider the σ-algebra $\mathcal{F}_\Lambda = \pi_\Lambda^{-1}(\mathcal{F}^\Lambda)$ of events depending on the set Λ. Recall that a probability kernel γ from $(X, \mathcal{B})$ to $(X, \mathcal{X})$, where $\mathcal{B} \subset \mathcal{X}$ is a sub-σ-algebra, is called proper if $\gamma(B, \cdot) = \mathbb{1}_B$ for $B \in \mathcal{B}$.

Definition 5.2.1. *A specification is a family of proper probability kernels γ_Λ from $\mathcal{F}_{\Lambda^c}$ to $\mathcal{F}_S$ indexed by finite subsets $\Lambda \subset S$ such that $\gamma_{\Lambda_1}\gamma_{\Lambda_2} = \gamma_{\Lambda_1}$ if $\Lambda_2 \subset \Lambda_1$. We define the set of random fields admitted to γ by*

$$\mathcal{G}(\gamma) = \{\mu \in \mathcal{P}(F^S, \mathcal{F}_S) : \mu(A|\mathcal{F}_{\Lambda^c})(\cdot) = \gamma_\Lambda(A|\cdot)\ \mu\text{-a.s. for all } A \in \mathcal{F}_S \text{ and } \Lambda \subset S \text{ finite}\}. \tag{5.2.1}$$

Remark 5.2.2. *1. We use the convention that we call $\mu \in \mathcal{G}(\gamma)$ a Gibbs measure for any specification, not only for Gibbsian specifications (see [92] for a definition of Gibbsian specifications).*

2. There is a well known equivalent definition of Gibbs measures. A cofinal set I is a subset of subsets of S with the property that for any finite set $\Lambda_0 \subset S$ there is $\Lambda \in I$ such that $\Lambda_0 \subset \Lambda$. Then $\mu \in \mathcal{G}(\gamma)$ if and only if $\mu\gamma_\Lambda = \mu$ for $\Lambda \in I$ where I is a cofinal subset of subsets of S. See Remark 1.24 in [92] for a proof.

Gradient Gibbs measures. We introduce the relevant notation and the definition of Gibbs and gradient Gibbs measures to state our results. For a broader discussion see [92, 136]. In this paragraph we consider real valued random fields indexed by a lattice $\Lambda \subset \mathbb{Z}^d$. We will denote the set of nearest neighbour bonds of $\mathbb{Z}^d$ by $\mathbf{E}(\mathbb{Z}^d)$. More generally, we will write $\mathbf{E}(G)$ and $\mathbf{V}(G)$ for the edges and vertices of a graph G. To consider gradient fields it is useful to choose on orientation of the edges. We orient the edges $e = \{x, y\} \in \mathbf{E}(\mathbb{Z}^d)$ from x to y iff $x \le y$ (coordinate-wise), i.e., we can view the graph $(\mathbb{Z}^d, \mathbf{E}(\mathbb{Z}^d))$ as a directed graph but mostly we work with the undirected graph.

To any random field $\varphi : \mathbb{Z}^d \to \mathbb{R}$ we associate the gradient field $\eta = \nabla\varphi \in \mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)}$ given by $\eta_e = \varphi_y - \varphi_x$ if $\{x, y\} \in \mathbf{E}(\mathbb{Z}^d)$ are nearest neighbours and $x \le y$. We formally write

$\eta_{x,y} = \eta_e = \varphi_y - \varphi_x$ and $\eta_{y,x} = -\eta_e = \varphi_x - \varphi_y$. The gradient field η satisfies the plaquette condition

$$\eta_{x_1,x_2} + \eta_{x_2,x_3} + \eta_{x_3,x_4} + \eta_{x_4,x_1} = 0 \tag{5.2.2}$$

for every plaquette, i.e., nearest neighbours x_1, x_2, x_3, x_4, x_1. Vice versa, given a field $\eta \in \mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)}$ that satisfies the plaquette condition there is a up to constant shifts a unique field φ such that $\eta = \nabla\varphi$ (the antisymmetry of the gradient field is contained in our definition). We will refer to those fields as gradient fields and denote them by $\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$. To simplify the notation we write φ_Λ for $\Lambda \subset \mathbb{Z}^d$ and η_E for $E \subset \mathbf{E}(\mathbb{Z}^d)$ for the the restriction of fields and gradient fields. We usually identify a subset $\Lambda \subset \mathbb{Z}^d$ with the graph generated by it and as before we write $\mathbf{E}(\Lambda)$ for the bonds with both endpoints in Λ.

For a subgraph $H \subset G$ we write ∂H for the (inner) boundary of H consisting of all points $x \in \mathbf{V}(H)$ such that there is an edge $e = \{x, y\} \in \mathbf{E}(G) \setminus \mathbf{E}(H)$. In the case of a graph generated by $\Lambda \subset G$ we have $x \in \partial\Lambda$ if there is $y \in \Lambda^c$ such that $\{x, y\} \in \mathbf{E}(G)$. We define $\mathring{\Lambda} = \Lambda \setminus \partial\Lambda$. For a finite subset $\Lambda \subset \mathbb{Z}^d$ we denote by $\mathrm{d}\varphi_\Lambda$ the Lebesgue measure on $\mathbb{R}^\Lambda$. We define for $\omega \in \mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$ and Λ finite and simply connected (i.e., Λ^c connected) the following a priori measure on gradient configurations

$$\nu_\Lambda^{\omega_{\mathbf{E}(\Lambda)^c}}(\mathrm{d}\eta) = \nabla_* \left(\prod_{x \in \mathring{\Lambda}^c} \delta_{\tilde{\varphi}(x)}(\cdot)\, \mathrm{d}\varphi_{\mathring{\Lambda}} \right) \tag{5.2.3}$$

where $\tilde{\varphi}$ is a configuration such that $\nabla\tilde{\varphi} = \omega$, $\mathrm{d}\varphi_{\mathring{\Lambda}}$ denotes the Lebesgue measure on $\mathbb{R}^{\mathring{\Lambda}}$, and ∇_* the push-forward of this measure along the gradient map $\nabla : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$. The shift invariance of the Lebesgue measure implies that this definition is independent of the choice of $\tilde{\varphi}$ and it only depends on the restriction $\omega_{\mathbf{E}(\Lambda)^c}$ since Λ is simply connected. For a potential $V : \mathbb{R} \to \mathbb{R}$ satisfying some growth condition we define the specification γ_Λ

$$\gamma_\Lambda(\mathrm{d}\eta, \omega_{\mathbf{E}(\Lambda)^c}) = \frac{\exp\left(-\sum_{e \in \mathbf{E}(\Lambda)} V(\eta_e)\right)}{Z_\Lambda(\omega_{\mathbf{E}(\Lambda)^c})} \nu_\Lambda^{\omega_{\mathbf{E}(\Lambda)^c}}(\mathrm{d}\eta) \tag{5.2.4}$$

where the constant $Z_\Lambda(\omega_{\mathbf{E}(\Lambda)^c})$ ensures the normalization of the measure. We introduce the notation $\mathcal{E}_E = \pi_E^{-1}(\mathcal{B}(\mathbb{R})^E)$ for $E \subset \mathbf{E}(\mathbb{Z}^d)$ for the σ-algebra of events depending only on E. Measures that are admitted to the specification γ, i.e., measures μ that satisfy for simply connected $\Lambda \subset \mathbb{Z}^d$

$$\mu(A \mid \mathcal{E}_{\mathbf{E}(\Lambda)^c})(\cdot) = \gamma_\Lambda(A, \cdot) \qquad \mu \text{ a.s.} \tag{5.2.5}$$

will be called gradient Gibbs measures for the potential V.

For $a \in \mathbb{Z}^d$ we consider the shift $\tau_a : \mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)} \to \mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)}$ that is defined by

$$(\tau_a \eta)_{x,y} = \eta_{x+a,y+a}. \tag{5.2.6}$$

A measure is translation invariant if $\mu(\tau_a^{-1}(A)) = \mu(A)$ for all a and $A \in \mathcal{B}(\mathbb{R})^{\mathbf{E}(\mathbb{Z}^d)}$. An event is translation invariant if $\tau_a(A) = A$ for all $a \in \mathbb{Z}^d$. A gradient measure is ergodic if $\mu(A) \in \{0, 1\}$ for all translation invariant A.

Main results. Our first main result is the following almost always uniqueness result for the gradient Gibbs measures for potentials as in (5.1.4).

Theorem 5.2.3. *For every q and $d \geq 2$ there is an at most countable set $N(q,d) \subset [0,1]$ such that for any $p \in [0,1] \setminus N(q,d)$ there is a unique shift invariant ergodic gradient Gibbs measure μ with zero tilt for the potential $V_{p,q}$.*

This theorem is proved in Section 5.5 below the proof of Theorem 5.5.1. Moreover, we reprove the non-uniqueness result originally shown in [32] for this type of potential.

Theorem 5.2.4. *There is $q_0 \geq 1$ such that for $d = 2$, $q \geq q_0$, and $p = p_{\mathrm{sd}}(q)$ the solution of* (5.1.5)*, there are at least two shift invariant gradient Gibbs measures with 0 tilt.*

The proof of this theorem is given at the end of Section 5.6. Moreover we prove uniqueness for 'high temperatures' and dimension $d \geq 4$, i.e., in the regime where the Dobrushin condition holds.

Theorem 5.2.5. *Let $d \geq 4$. For any $q \geq 1$ there exists $p_0 = p_0(q,d) > 0$ such that for all $p \in [0,p_0) \cup (1-p_0,1]$ there is a unique shift invariant ergodic gradient Gibbs measure with zero tilt for the potential $V_{p,q}$. Moreover, there exists $q_0 = q_0(d) > 1$ such that for any $q \in [1,q_0]$ and any $p \in [0,1]$ there is a unique shift invariant ergodic gradient Gibbs measure with zero tilt for the potential $V_{p,q}$.*

The proof of this Theorem is given in Section 5.5 below the proof of Theorem 5.5.6.

The main tool in the proofs of these theorems is the fact that the structure of the potentials V in (5.1.1) allows us to consider κ as a further degree of freedom and we consider the joint distribution of the gradient field η and κ. We show that the law of the κ-marginal can be related to a random conductance model. The analysis of this model then translates back into the theorems stated before. We will make those statements precise in the next section. Let us end this section with some remarks.

Remark 5.2.6.

1. *For spin systems with finite state space and bounded interactions there are general results that show that phase transitions, i.e., non-uniqueness of the Gibbs measure are rare, see, e.g., [92]. Theorem 5.2.3 establishes a similar result for a specific class of potentials for a unbounded spin space. As discussed in more detail at the end of Section 5.5 we expect that for every $q \geq 1$ the Gibbs measure is unique for all $p \in [0,1]$ except for $p = p_c$ for some critical value $p_c = p_c(q)$. Hence, Theorem 5.2.3 is far from optimal but we hope that the results provided in this chapter prove useful to establish stronger results.*

2. *Let us compare the results to earlier results in the literature. For $p/(1-p) < 1/q$ the potential $V_{p,q}$ is strictly convex so that uniqueness of the Gibbs measure is well known and holds for every tilt. The two step integration used by Cotar and Deuschel extends the uniqueness result to the regime $p/(1-p) < C/\sqrt{q}$ (see Section 3.2 in [63]). In particular the case $p \in [0,p_0)$ in Theorem 5.2.5 is included in earlier results. However, the potential becomes very non-convex (has a very negative second derivative at some points) for p close to 1 and the uniqueness result for $p \in (1-p_0,1]$ and $d \geq 4$ appears to be new. In this regime the only known result seems to be convexity of the surface tension as a function of the tilt which was shown in [4] (see in particular Proposition 2.4 there). Their results apply to p very close to one, $q-1$ very small, and $d \leq 3$. The results from in Chapter 3 extend this result to any dimension $d \geq 2$ and to arbitrary q for p sufficiently close to 1 depending on q.*

3. *The restriction to dimension $d \geq 4$ arises from the fact that the Green's function for inhomogeneous elliptic operators in divergence form decays slower than in the homogeneous case.*

5.3 Extended gradient Gibbs measures and random conductance model

Extended gradient Gibbs measure. In this work we restrict to potentials of the form introduced in (5.1.1). As already discussed in more detail in [32] and [33] it is possible to use the special structure of V to raise κ to a degree of freedom. Let μ be a gradient Gibbs measure for V. For a finite set $E \subset \mathbf{E}(\mathbb{Z}^d)$ and Borel sets $\mathbf{A} \subset \mathbb{R}^E$ and $\mathbf{B} \subset \mathbb{R}_+^E$ we define the extended gradient Gibbs measure

$$\tilde{\mu}((\eta_b, \kappa_b)_{b\in E} \in \mathbf{A} \times \mathbf{B}) = \int_{\mathbf{B}} \rho_E(\mathrm{d}\kappa)\mathbb{E}_\mu \left(\mathbb{1}_{\mathbf{A}} \prod_{e\in E} e^{-\frac{1}{2}\kappa_e \eta_e^2 + V(\eta_e)} \right). \tag{5.3.1}$$

It can be checked that this is a consistent family of measures and thus we can extend $\tilde{\mu}$ to a measure on $(\mathbb{R} \times \mathbb{R}_+)^{\mathbf{E}(\mathbb{Z}^d)}$. It was explained in [32] that $\tilde{\mu}$ is itself a Gibbs measure for the specification $\tilde{\gamma}_\Lambda$ defined by

$$\tilde{\gamma}_\Lambda((\mathrm{d}\bar{\eta}, \mathrm{d}\bar{\kappa}), (\eta, \kappa)) = \frac{\exp\left(-\frac{1}{2}\sum_{e\in\mathbf{E}(\Lambda)} \bar{\kappa}_e \bar{\eta}_e^2\right)}{Z_\Lambda(\eta_{\mathbf{E}(\Lambda)^c})} \nu_\Lambda^{\eta_{\mathbf{E}(\Lambda)^c}}(\mathrm{d}\bar{\eta}) \prod_{e\in\mathbf{E}(\Lambda)} \rho(\mathrm{d}\bar{\kappa}_e) \prod_{e\in\mathbf{E}(\Lambda)^c} \delta_{\kappa_e}(\mathrm{d}\bar{\kappa}_e). \tag{5.3.2}$$

Note that the distribution $(\mathrm{d}\bar{\eta}, \mathrm{d}\bar{\kappa})_{\mathbf{E}(\Lambda)}$ actually only depends on $\eta_{\mathbf{E}(\Lambda)^c}$ and is independent of κ. Let us add one remark concerning the notation. In this work we essentially consider three strongly related viewpoints of one model. The first viewpoint are gradient Gibbs measures that are measures on $\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$. Thy will be denoted by μ and the corresponding specification is denoted by γ. Then there are extended gradient Gibbs measures for a specification $\tilde{\gamma}$. They are measures on $\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)} \times \mathbb{R}_+^{\mathbf{E}(\mathbb{Z}^d)}$ and will be denoted by $\tilde{\mu}$. The η-marginal of $\tilde{\mu}$ is a gradient Gibbs measure μ. Finally there is also the κ-marginal of $\tilde{\mu}$ which is a measure on $\mathbb{R}_+^{\mathbf{E}(\mathbb{Z}^d)}$ and will be denoted by $\bar{\mu}$. An important result here is that $\bar{\mu}$ is a Gibbs measure for a specification $\bar{\gamma}$ if ρ is a measure as in (5.1.2). In this case $\bar{\mu}$ is a measure on the discrete space $\{1, q\}^{\mathbf{E}(\mathbb{Z}^d)}$. We expect that this result can be extended to far more general measures ρ but we do not pursue this matter here. To keep the notation consistent we denote objects with single spin space $\mathbb{R}$, e.g., gradient Gibbs measures without symbol modifier, objects with single spin space $\{1, q\}$, e.g., the κ-marginal with a bar, and objects with single spin space $\{1, q\} \times \mathbb{R}$, e.g., extended Gibbs with a tilde. Let us also fix a notation for the corresponding relevant σ-algebras. We write as before $\mathcal{E}_E$ for the σ-algebra on $\mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)}$ generated by $(\eta_e)_{e\in E}$ and we define $\mathcal{E} = \mathcal{E}_{\mathbf{E}(\mathbb{Z}^d)}$. For the κ-marginal we similarly consider the σ-algebra $\mathcal{F}_E$ on $\{1, q\}^{\mathbf{E}(\mathbb{Z}^d)}$ generated by $(\kappa_e)_{e\in E}$ and we write again $\mathcal{F} = \mathcal{F}_{\mathbf{E}(\mathbb{Z}^d)}$. For the extended space $\mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)} \times \{1, q\}^{\mathbf{E}(\mathbb{Z}^d)}$ we use the product σ-algebra $\mathcal{A}_E = \pi_1^{-1}(\mathcal{E}_E) \otimes \pi_2^{-1}(\mathcal{F}_E)$.

It was already remarked in [32] that this setting resembles the situation for the Potts model that can be coupled to the random cluster model via the Edwards-Sokal coupling measure.

The random conductance model. As explained before our strategy is to analyse the κ-marginal of extended gradient Gibbs measures and use the results to deduce properties of the gradient Gibbs measures for V. The key observation is that the κ-marginal of extended gradient

Gibbs measures is given by the infinite volume limit of a strongly coupled random conductance model. To motivate the definition of the random conductance model we consider the κ-marginal of the extended specification $\tilde{\gamma}$ defined in (5.3.2). For zero boundary value $\bar{0} \in \mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$ with $\bar{0}_e = 0$ and $\lambda \in \{1, q\}^{\mathbf{E}(\mathbb{Z}^d)}$ we obtain

$$\tilde{\gamma}_\Lambda\big(\kappa_{\mathbf{E}(\Lambda)} = \lambda_{\mathbf{E}(\Lambda)}\,,\, \bar{0}\big) = \frac{1}{Z} \int \prod_{e \in \mathbf{E}(\Lambda)} p^{\mathbb{1}_{\lambda_e = q}} (1-p)^{\mathbb{1}_{\lambda_e = 1}} e^{-\frac{1}{2}\lambda_e^2 \omega_e^2}\, \nu_\Lambda^{\bar{0}_{\mathbf{E}(\Lambda)^c}}(\mathrm{d}\omega). \tag{5.3.3}$$

We write $\Lambda^w = \bar{\Lambda}/\partial\Lambda$ for the graph where the entire boundary is collapsed to a single point (this is called wired boundary conditions and we will discuss this below in more detail). We denote the lattice Laplacian with conductances λ and zero boundary condition outside of $\mathring{\Lambda}$ by $\tilde{\Delta}_\lambda^{\Lambda^w}$, i.e., $\tilde{\Delta}_\lambda^{\Lambda^w}$ acts on functions $f : \mathring{\Lambda} \to \mathbb{R}$ by $\tilde{\Delta}_\lambda^{\Lambda^w} f(x) = \sum_{y \sim x} \lambda_{\{x,y\}}(f(x) - f(y))$ where we set $f(y) = 0$ for $y \notin \mathring{\Lambda}$. The definition (5.3.2) and an integration by parts followed by Gaussian calculus imply then

$$\begin{aligned} \tilde{\gamma}_\Lambda\big(\kappa_{\mathbf{E}(\Lambda)} = \lambda_{\mathbf{E}(\Lambda)}\,,\, \bar{0}\big) &= \frac{1}{Z} p^{|\{e \in \mathbf{E}(\Lambda)\,:\,\lambda_e = q\}|} (1-p)^{|\{e \in \mathbf{E}(\Lambda)\,:\,\lambda_e = 1\}|} \int e^{-\frac{1}{2}(\varphi, \tilde{\Delta}_\lambda^{\Lambda^w}\varphi)}\, \mathrm{d}\varphi_{\mathring{\Lambda}} \\ &= \frac{1}{Z} \frac{p^{|\{e \in \mathbf{E}(\Lambda)\,:\,\lambda_e = q\}|} (1-p)^{|\{e \in \mathbf{E}(\Lambda)\,:\,\lambda_e = 1\}|}}{\sqrt{\det (2\pi)^{-1} \tilde{\Delta}_\lambda^{\Lambda^w}}}. \end{aligned} \tag{5.3.4}$$

It simplifies the presentation to introduce the random conductance model of interest in a slightly more general setting. We consider a finite and connected graph $G = (V, E)$. The combinatorial graph Laplacian Δ_c associated to set of conductances $c : E \to \mathbb{R}_+$ is defined by

$$\Delta_c f(x) = \sum_{y \sim x} c_{\{x,y\}}(f(x) - f(y)) \tag{5.3.5}$$

for any function $f : V \to \mathbb{R}$. Note that we defined the graph Laplacian as a non-negative operator which is convenient for our purposes and common in the context of graph theory. In the following we view the Laplacian Δ_c as a linear map on the space $H_0 = \{f : V \to \mathbb{R} : \sum_{x \in V} f(x) = 0\}$ of functions with vanishing average. We define $\det \Delta_c$ as the determinant of this linear map. By the maximum principle the Laplacian is injective on H_0, hence $\det \Delta_c > 0$. Sometimes we clarify the underlying graph by writing Δ_c^G.

Remark 5.3.1. *In the general setting it is more natural to let the Laplacian act on H_0 instead of fixing a point to 0 as in the definition of $\tilde{\Delta}_\lambda^{\Lambda^w}$ above where this corresponds to Dirichlet boundary conditions. It would also be possible to fix a point $x \in \mathbf{V}(G)$ and consider $\tilde{\Delta}_c^G$ acting on functions $f : \mathbf{V}(G) \setminus \{x_0\} \to \mathbb{R}$ defined by $(\tilde{\Delta}_c^G f)(x) = \sum_{y \sim x} c_{\{x,y\}} f(x) - f(y)$ for $x \in \mathbf{V}(G) \setminus \{x_0\}$ where we set $f(x_0) = 0$. It is easy to see using, e.g., Gaussian calculus and a change of measure that the determinant of $\tilde{\Delta}_c^G$ is independent of x_0 and*

$$|G| \det \tilde{\Delta}_c^G = \det \Delta_c^G. \tag{5.3.6}$$

Motivated by (5.3.4) we fix a real number $q \geq 1$ and consider the following probability measure on $\{1, q\}^E$

$$\mathbb{P}^{G,p}(\kappa) = \frac{1}{Z} \frac{p^{|\{e \in E : \kappa_e = q\}|} (1-p)^{|\{e \in E : \kappa_e = 1\}|}}{\sqrt{\det \Delta_\kappa}} \tag{5.3.7}$$

where $Z = Z^{G,p}$ denotes a normalisation constant such that $\mathbb{P}^{G,p}$ is a probability measure. In the following we will often drop G and p from the notation and we will always suppress q. We restrict our attention to $q \geq 1$ because by scaling the model with conductances $\{1, q\}$ has the same distribution as a model with conductances $\{\alpha, \alpha q\}$ for $\alpha > 0$ so that we can set the smaller conductance to 1. Let us state a remark concerning the relation to the random cluster model.

Remark 5.3.2. *1. We chose the notation such that the similarity to the random cluster model is apparent. Both models have the parameter p as a priori distribution of the bonds that is then correlated by a complicated infinite range interaction depending on q. They reduce to Bernoulli percolation for $q = 1$. At the end of Section 5.5 we state a couple of conjectures about the behaviour of this model that show that we expect similarities with the random cluster model in many more aspects.*

2. *While there are several close similarities to the random cluster model there is also one important difference that seems to pose additional difficulties in the analysis of this model. The conditional distribution in a finite set depends on the entire configuration of the conductances outside the finite set (not just a partition of the boundary as in the random cluster model). In particular the often used argument that the conditional distribution of a random cluster model in a set given that all boundary edges are closed is the free boundary random cluster distribution has no analogue in our setting.*

3. *We refer to the model as a random conductance model since we will (not very surprisingly) use tools from the theory of electrical networks. Note that in the definition of the potential V the parameters correspond to different (random) stiffness of the bonds.*

5.4 Basic properties of the random conductance model

Preliminaries. As before we consider a connected graph $G = (V, E)$. To simplify the notation we introduce for $E' \subset E$ and $\kappa \in \{1, q\}^E$ the notation

$$h(\kappa, E') = |\{e \in E' : \kappa_e = q\}| \tag{5.4.1}$$

$$s(\kappa, E') = |\{e \in E' : \kappa_e = 1\}| \tag{5.4.2}$$

for the number of hard and soft edges respectively and we define $h(\kappa) = h(\kappa, E)$ and $s(\kappa) = s(\kappa, E)$. Let us introduce the weight of a subset of edges $t \subset E$ by defining

$$w(\kappa, t) = \prod_{e \in t} \kappa_e. \tag{5.4.3}$$

We will denote the set of all spanning trees of a graph by $\mathrm{ST}(G)$. We will identify spanning trees with their edge sets. In the following, we will frequently use the Kirchhoff formula

$$\det \Delta_c = |G| \sum_{t \in \mathrm{ST}(G)} w(c, t). \tag{5.4.4}$$

for the determinant of a weighted graph Laplacian (cf. [144] for a proof). Let us remark that the Kirchhoff formula is frequently used in statistical mechanics and has also been used in the context of gradient interface models for some potentials as in (5.1.1) in [49].

Remark 5.4.1. *Note that equation* (5.4.4) *remains true for graphs with multi-edges and loops. Indeed, loops have no contribution on both sides and multi-edges can be replaced by a single edge with the sum of the conductances as conductance.*

Correlation inequalities We will now show correlation inequalities for the measures $\mathbb{P} = \mathbb{P}^{G,p}$. We start by recalling several of the well known correlation inequalities. To state our results we introduce some notation. Let E be a finite or countable infinite set. Let $\Omega = \{1, q\}^E$ and $\mathcal{F}$ the σ-algebra generated by cylinder events. We consider the usual partial order on Ω given by $\omega^1 \leq \omega^2$ iff $\omega^1_e \leq \omega^2_e$ for all $e \in E$. A function $X : \Omega \to \mathbb{R}$ is increasing if $X(\omega_1) \leq X(\omega_2)$ for $\omega_1 \leq \omega_2$ and decreasing if $-X$ is increasing. An event $A \subset \Omega$ is increasing if its indicator function is increasing. We write $\bar{\mu}_1 \succsim \bar{\mu}_2$ if $\bar{\mu}_1$ stochastically dominates $\bar{\mu}_2$ which is by Strassen's Theorem equivalent to the existence of a coupling (ω_1, ω_2) such that $\omega^1 \sim \bar{\mu}_1$ and $\omega^2 \sim \bar{\mu}_2$ and $\omega^1 \geq \omega^2$ (see [143]). We introduce the minimum $\omega^1 \wedge \omega^2$ and the maximum $\omega^1 \vee \omega^2$ of two configurations given by $(\omega^1 \wedge \omega^2)_e = \min(\omega^1_e, \omega^2_e)$ and $(\omega^1 \vee \omega^2)_e = \max(\omega^1_e, \omega^2_e)$ for any $e \in E$. We call a measure $\bar{\mu}$ on Ω strictly positive if $\bar{\mu}(\omega) > 0$ for all $\omega \in \Omega$. Finally we introduce for $f, g \in E$ and $\omega \in \Omega$ the notation $\omega^{\pm\pm}_{fg} \in \Omega$ for the configuration given by $(\omega^{\pm\pm}_{fg})_e = \omega_e$ for $e \notin \{f, g\}$ and $(\omega^{\pm *}_{fg})_f = 1 + (q-1)_\pm$, $(\omega^{*\pm}_{fg})_g = 1 + (q-1)_\pm$. We define $\omega^\pm_f$ similarly. We sometimes drop the edges f, g from the notation. We write $\bar{\mu}(\omega) = \bar{\mu}(\{\omega\})$ for $\omega \in \Omega$ and $\bar{\mu}(X) = \int_\Omega X \,\mathrm{d}\bar{\mu}$ for $X : \Omega \to \mathbb{R}$.

Theorem 5.4.2 (Holley inequality)**.** *Let $\Omega = \{1, q\}^E$ be finite and $\bar{\mu}_1$, $\bar{\mu}_2$ strictly positive measures on Ω that satisfy the Holley inequality*

$$\bar{\mu}_2(\omega_1 \vee \omega_2)\bar{\mu}_1(\omega_1 \wedge \omega_2) \geq \bar{\mu}_1(\omega_1)\bar{\mu}_2(\omega_2) \quad \textit{for } \omega_1, \omega_2 \in \Omega. \tag{5.4.5}$$

Then $\bar{\mu}_1 \precsim \bar{\mu}_2$.

Proof. The original proof appeared in [108], a simpler proof can be found , e.g., in [98, Theorem 2.1]. □

A strictly positive measure is called *strongly positively associated* if it satisfies the FKG lattice condition

$$\bar{\mu}(\omega_1 \vee \omega_2)\bar{\mu}(\omega_1 \wedge \omega_2) \geq \bar{\mu}(\omega_1)\bar{\mu}(\omega_2) \qquad \text{for } \omega_1, \omega_2 \in \Omega. \tag{5.4.6}$$

Theorem 5.4.3. *A strongly positively associated measure $\bar{\mu}$ satisfies the FKG inequality, i.e., for increasing functions $X, Y : \Omega \to \mathbb{R}$*

$$\bar{\mu}(XY) \geq \bar{\mu}(X)\bar{\mu}(Y). \tag{5.4.7}$$

Proof. A proof can be found in [98, Theorem 2.16]. □

The next theorem provides a simple way to verify the assumptions of Theorem 5.4.2 and Theorem 5.4.3. Basically it states that it is sufficient to check the conditions when varying at most two edges.

Theorem 5.4.4. *Let $\Omega = \{1, q\}^E$ be finite and $\bar{\mu}_1$, $\bar{\mu}_2$ strictly positive measures on Ω. Then $\bar{\mu}_1$ and $\bar{\mu}_2$ satisfy* (5.4.5) *iff the following two inequalities hold*

$$\bar{\mu}_2(\omega^+_f)\bar{\mu}_1(\omega^-_f) \geq \bar{\mu}_1(\omega^+_f)\bar{\mu}_2(\omega^-_f), \quad \textit{for } \omega \in \Omega,\ f \in E, \tag{5.4.8}$$

$$\bar{\mu}_2(\omega^{++}_{fg})\bar{\mu}_1(\omega^{--}_{fg}) \geq \bar{\mu}_1(\omega^{+-}_{fg})\bar{\mu}_2(\omega^{-+}_{fg}), \quad \textit{for } \omega \in \Omega,\ f, g \in E.. \tag{5.4.9}$$

In particular, (5.4.8) *and* (5.4.9) *together imply $\bar{\mu}_1 \precsim \bar{\mu}_2$.*

Proof. See [98, Theorem 2.3]. □

We state one simple corollary of the previous results.

Corollary 5.4.5. *Let $\bar\mu_1$, $\bar\mu_2$ be strictly positive measures on $\Omega = \{1,q\}^E$ such that one of the measure $\bar\mu_1$, $\bar\mu_2$ is strongly positively associated. Then*

$$\bar\mu_2(\omega_f^+)\bar\mu_1(\omega_f^-) \geq \bar\mu_1(\omega_f^+)\bar\mu_2(\omega_f^-), \quad \text{for } \omega \in \Omega,\ f \in E \tag{5.4.10}$$

implies $\bar\mu_1 \precsim \bar\mu_2$.

Proof. Assuming that $\bar\mu_1$ is strongly positively associated we find using first the assumption (5.4.10) and then (5.4.6)

$$\bar\mu_2(\omega_{fg}^{++})\bar\mu_1(\omega_{fg}^{--}) \geq \frac{\bar\mu_1(\omega_{fg}^{++})\bar\mu_2(\omega_{fg}^{-+})}{\bar\mu_1(\omega_{fg}^{-+})}\bar\mu_1(\omega_{fg}^{--}) \geq \bar\mu_2(\omega_{fg}^{-+})\bar\mu_1(\omega_{fg}^{+-}). \tag{5.4.11}$$

Now Theorem 5.4.4 implies the claim. The proof if $\bar\mu_2$ is strictly positively associated is similar. □

It is convenient to derive the following correlation results for the measures $\mathbb{P}^{G,p}$ from corresponding results for the weighted spanning tree measure. The weighted spanning tree measure on a connected weighted graph (G,κ) is a measure on $\mathrm{ST}(G)$ with distribution

$$\mathbb{Q}_\kappa^G(t) = \frac{w(\kappa,t)}{\sum_{t'\in \mathrm{ST}(G)} w(\kappa,t')}. \tag{5.4.12}$$

This model has been studied extensively, see [29] for a survey. An important special case is the uniform spanning tree corresponding to constant conductances κ that assigns equal probability to every spanning tree.

The following lemma provides the basic estimate to check the condition (5.4.9) for the measures $\mathbb{P}^{G,p}$. Recall the notation $\kappa_{fg}^{\pm\pm}$ introduced before Theorem 5.4.2 and also the shorthand $\kappa^{\pm\pm}$.

Lemma 5.4.6. *For a finite graph G and $\kappa \in \{1,q\}^E$ as above*

$$\det \Delta_{\kappa^{++}} \det \Delta_{\kappa^{--}} \leq \det \Delta_{\kappa^{+-}} \det \Delta_{\kappa^{-+}}. \tag{5.4.13}$$

Remark 5.4.7. *The proof in fact extends to any $\kappa \in \mathbb{R}_+^E$ and $(\kappa_{fg}^{\pm\pm})_f = c_f^\pm$, $(\kappa_{fg}^{\pm\pm})_g = c_g^\pm$ with $c_f^- \leq c_f^+$ and $c_g^- \leq c_g^+$.*

Proof. The lemma can be derived from the fact that the weighted spanning tree has negative correlations. It is well known (see, e.g., [29]) that for all positive weights κ on a finite graph G the measure $\mathbb{Q}_\kappa^G$ has negative edge correlations

$$\mathbb{Q}_\kappa^G(e\in t | f \in t) \leq \mathbb{Q}_\kappa^G(e \in t). \tag{5.4.14}$$

Simple algebraic manipulations show that this is equivalent to

$$\mathbb{Q}_\kappa^G(e\in t, f\in t)\mathbb{Q}_\kappa^G(e\notin t, f\notin t) \leq \mathbb{Q}_\kappa^G(e\in t, f\notin t)\mathbb{Q}_\kappa^G(e\notin t, f\in t). \tag{5.4.15}$$

We introduce the following sums

$$\begin{aligned} A_{fg} &= \sum_{t\in\mathrm{ST}(G),\, f,g\in t} w(\kappa,t), \qquad & A_f &= \sum_{t\in\mathrm{ST}(G),\, f\in t,\, g\notin t} w(\kappa,t),\\ A_g &= \sum_{t\in\mathrm{ST}(G),\, g\in t,\, f\notin t} w(\kappa,t), \qquad & A &= \sum_{t\in\mathrm{ST}(G),\, f,g\notin t} w(\kappa,t). \end{aligned} \tag{5.4.16}$$

With this notation multiplication by $(A_{fg} + A_f + A_g + A)^2$ shows that (5.4.15) is equivalent to

$$A_{fg}A \leq A_f A_g. \tag{5.4.17}$$

It remains to show that the statement in the lemma can be deduced from (5.4.17) (actually the statements are equivalent). Clearly we can assume $\kappa = \kappa^{--}$, i.e., $\kappa_f = \kappa_g = 1$. Using the Kirchhoff formula (5.4.4) we find the following expression

$$|G|^{-1} \det \Delta_{\kappa^{\pm\pm}} = \sum_{t \in \mathrm{ST}(G)} w(\kappa^{\pm\pm}, t) = (\kappa^{\pm\pm})_f (\kappa^{\pm\pm})_g A_{fg} + (\kappa^{\pm\pm})_f A_f + (\kappa^{\pm\pm})_g A_g + A. \tag{5.4.18}$$

Hence we obtain

$$\begin{aligned} |G|^{-2} \det \Delta_{\kappa^{+-}} \ \det \Delta_{\kappa^{-+}} &= (qA_{fg} + qA_f + A_g + A)\,(qA_{fg} + A_f + qA_g + A)\,, \\ |G|^{-2} \det \Delta_{\kappa^{++}} \ \det \Delta_{\kappa^{--}} &= \left(q^2 A_{fg} + qA_f + qA_g + A\right)(A_{fg} + A_f + A_g + A)\,. \end{aligned} \tag{5.4.19}$$

Subtracting those two identities we find that only the cross-terms between A_f, A_g and between A_{fg}, A do not cancel and we get

$$\begin{aligned} |G|^{-2} \left(\det \Delta_{\kappa^{+-}} \ \det \Delta_{\kappa^{-+}} - \det \Delta_{\kappa^{++}} \ \det \Delta_{\kappa^{--}}\right) &= (q^2 + 1 - 2q)(A_f A_g - A_{fg}A) \\ &= (q-1)^2 (A_f A_g - A_{fg}A). \end{aligned} \tag{5.4.20}$$

We can conclude using (5.4.17). □

The previous lemma directly implies that the measures $\mathbb{P}^{G,p}$ are strongly positively associated.

Corollary 5.4.8. *The measure $\mathbb{P}^{G,p}$ satisfies the FKG lattice condition for any $\kappa_1, \kappa_2 \in \{1, q\}^E$*

$$\mathbb{P}^{G,p}(\kappa_1 \wedge \kappa_2)\mathbb{P}^{G,p}(\kappa_1 \vee \kappa_2) \geq \mathbb{P}^{G,p}(\kappa_1)\mathbb{P}^{G,p}(\kappa_2) \tag{5.4.21}$$

and the FKG inequality

$$\mathbb{E}^{G,p}(XY) \geq \mathbb{E}^{G,p}(X)\,\mathbb{E}^{G,p}(Y) \tag{5.4.22}$$

for any increasing functions $X, Y : \{1, q\}^E \to \mathbb{R}$.

Proof. Lemma 5.4.6 and the trivial observation that $h(\kappa^{++}) + h(\kappa^{--}) = h(\kappa^{+-}) + h(\kappa^{-+})$ imply for any $\kappa \in \{1, q\}^E$ and $f, g \in E$ the lattice inequality

$$\mathbb{P}^{G,p}(\kappa^{++})\mathbb{P}^{G,p}(\kappa^{--}) \geq \mathbb{P}^{G,p}(\kappa^{+-})\mathbb{P}^{G,p}(\kappa^{-+}). \tag{5.4.23}$$

Then Theorem 5.4.4 applied to $\bar{\mu}_1 = \bar{\mu}_2 = \mathbb{P}^{G,p}$ implies that the FKG lattice condition (5.4.21) holds and therefore by Theorem 5.4.3 also the FKG-inequality (5.4.22).

□

Let us first state a trivial consequence of this corollary.

Lemma 5.4.9. *The measures $\mathbb{P}^{G,p}$ and $\mathbb{P}^{G,p'}$ satisfy for $p \leq p'$*

$$\mathbb{P}^{G,p'} \succsim \mathbb{P}^{G,p}. \tag{5.4.24}$$

Proof. Using Corollary 5.4.8 and Corollary 5.4.5 we only need to check whether (5.4.10) holds for $\bar{\mu}_1 = \mathbb{P}^{G,p}$ and $\bar{\mu}_2 = \mathbb{P}^{G,p'}$. This is clearly the case if $p \leq p'$. □

The next step is to show correlation inequalities with respect to the size of the graph. More specifically we show statements for subgraphs and contracted graphs. This will later easily imply the existence of infinite volume limits. Moreover, we can bound infinite volume states by finite volume measures in the sense of stochastic domination. Let $F \subset E$ be a set of edges. We define the contracted graph G/F by identifying for every edge $f \in F$ the endpoints of f. Similarly for a set $W \subset V$ of vertices we define the contracted graph G/W by identifying all vertices in W. The resulting graphs may have multi-edges. We also consider connected subgraphs $G' = (V', E')$ of G. Recall the notation $\kappa^{\pm} = \kappa_f^{\pm}$ for $f \in E$. We use the notation $\Delta_\kappa^{G'}$ for the graph Laplacian on G' where we restrict the conductances κ to E' and we denote by $\Delta_\kappa^{G/F}$ the graph Laplacian on G/F. The following lemma relates the determinants of the different graph Laplacians.

Lemma 5.4.10. *With the notation introduced above we have for* $\kappa \in \{1, q\}^E$

$$\frac{\det \Delta_{\kappa^+}^{G'}}{\det \Delta_{\kappa^-}^{G'}} \geq \frac{\det \Delta_{\kappa^+}^{G}}{\det \Delta_{\kappa^-}^{G}} \geq \frac{\det \Delta_{\kappa^+}^{G/F}}{\det \Delta_{\kappa^-}^{G/F}}. \tag{5.4.25}$$

Remark 5.4.11. *The lemma again extends to* $\kappa \in \mathbb{R}_+^E$ *and* $\kappa_f^{\pm}$ *with* $(\kappa_f^+)_f = c_+ > c_- = (\kappa_f^-)_f$.

Proof. The proof is similar to the proof of Lemma 5.4.6. We derive the statement from a property of the weighted spanning tree model. For graphs as above and $e \in E'$ the estimate

$$\mathbb{Q}_\kappa^{G'}(e \in t) \geq \mathbb{Q}_\kappa^{G}(e \in t) \geq \mathbb{Q}_\kappa^{G/F}(e \in t) \tag{5.4.26}$$

holds (see Corollary 4.3 in [29] for a proof). We can rewrite (assuming again $\kappa_f = 1$, i.e., $\kappa = \kappa^-$)

$$\frac{\det \Delta_{\kappa^+}^{G}}{\det \Delta_{\kappa^-}^{G}} = \frac{\sum_{t \in \mathrm{ST}(G), f \notin t} w(\kappa, t) + q \sum_{t \in \mathrm{ST}(G), f \in t} w(\kappa, t)}{\sum_{t \in \mathrm{ST}(G), f \notin t} w(\kappa, t) + \sum_{t \in \mathrm{ST}(G), f \in t} w(\kappa, t)}. \tag{5.4.27}$$

Note that

$$\frac{\sum_{t \in \mathrm{ST}(G), f \in t} w(\kappa, t)}{\sum_{t \in \mathrm{ST}(G), f \notin t} w(\kappa, t)} = \frac{\mathbb{Q}_\kappa^G(f \in t)}{\mathbb{Q}_\kappa^G(f \notin t)} \tag{5.4.28}$$

and therefore (using $\kappa = \kappa^-$)

$$\frac{\det \Delta_{\kappa^+}^{G}}{\det \Delta_{\kappa^-}^{G}} = \frac{1 + q \frac{\mathbb{Q}_{\kappa^-}^G(f \in t)}{\mathbb{Q}_{\kappa^-}^G(f \notin t)}}{1 + \frac{\mathbb{Q}_{\kappa^-}^G(f \in t)}{\mathbb{Q}_{\kappa^-}^G(f \notin t)}} = 1 + (q-1)\mathbb{Q}_{\kappa^-}^G(f \in t). \tag{5.4.29}$$

Similar statements hold for the graphs G/F and G'. Hence (5.4.26) implies (5.4.25). □

Let us remark that the probability $\mathbb{Q}_\kappa^G(f \in t)$ can also be expressed as a current in a certain electrical network. In order to avoid unnecessary notation at this point we kept the weighted spanning tree measure and we will only exploit this connection when necessary below.

Again, the previous estimates implies correlation inequalities for the measures $\mathbb{P}^{G,p}$. In the following we consider a fixed value of p but different graphs so that we drop only p from the notation but we keep the graph G. We introduce the distribution under boundary conditions for a connected subgraph $G' = (V', E')$ of G. For $\lambda \in \{1, q\}^E$ we define the measure $\mathbb{P}^{G,E',\lambda}$ on $\{1, q\}^{E'}$ by

$$\mathbb{P}^{G,E',\lambda}(\kappa) = \frac{1}{Z} \frac{p^{h(\kappa)}(1-p)^{s(\kappa)}}{\sqrt{\det \Delta_{(\lambda,\kappa)}^{G}}} \tag{5.4.30}$$

where $(\lambda,\kappa)\in\{1,q\}^E$ denotes the conductances given by κ on E' and by λ on $E\setminus E'$. This definition implies that we have the following domain Markov property for $\omega\in\{1,q\}^{E'}$

$$\mathbb{P}^G(\kappa_{E'}=\omega\mid\kappa_{E\setminus E'}=\lambda_{E\setminus E'})=\mathbb{P}^{G,E',\lambda}(\omega). \tag{5.4.31}$$

Since the measure $\mathbb{P}^G$ is strongly positively associated, (5.4.31) and Theorem 2.24 in [98] implies that the measure $\mathbb{P}^{G,E',\lambda}$ is strongly positively associated. We now state the consequences of Lemma 5.4.10 on stochastic ordering.

Corollary 5.4.12. *For a finite graph $G=(V,E)$, a connected subgraph $G'=(V',E')$, an edge subset $F\subset E$, and configurations $\lambda_1,\lambda_2\in\{1,q\}^E$ such that $\lambda_1\le\lambda_2$ the following holds*

$$\mathbb{P}^{G'}\precsim\mathbb{P}^{G,E',\lambda_1},\qquad \mathbb{P}^{G,E',\lambda_1}\precsim\mathbb{P}^{G,E',\lambda_2},\qquad P^{G,E\setminus F,\lambda_2}\precsim\mathbb{P}^{G/F}. \tag{5.4.32}$$

More generally, we have for $\lambda\in\{1,q\}^E$ and $E''\subset E'$ or $E''\cap F=\emptyset$ respectively

$$\mathbb{P}^{G',E'',\lambda_{E'}}\precsim\mathbb{P}^{G,E'',\lambda},\qquad \mathbb{P}^{G,E'',\lambda}\precsim\mathbb{P}^{G/F,E'',\lambda_{E\setminus F}}. \tag{5.4.33}$$

Proof. From Lemma 5.4.10 we obtain for $f\in E'$ and any $\kappa\in\{1,q\}^{E'}$

$$\frac{\mathbb{P}^{G,E',\lambda}(\kappa^+)}{\mathbb{P}^{G,E',\lambda}(\kappa^-)}\ge\frac{\mathbb{P}^{G'}(\kappa^+)}{\mathbb{P}^{G'}(\kappa^-)}. \tag{5.4.34}$$

Similarly, Lemma 5.4.10 implies for $f\in E\setminus F$ and $\kappa\in\{1,q\}^{E\setminus F}$

$$\frac{\mathbb{P}^{G/F}(\kappa^+)}{\mathbb{P}^{G/F}(\kappa^-)}\ge\frac{\mathbb{P}^{G,E\setminus F,\lambda}(\kappa^+)}{\mathbb{P}^{G,E\setminus F,\lambda}(\kappa^-)}. \tag{5.4.35}$$

Then the the strong positive association of $\mathbb{P}^G$ and Corollary 5.4.5 imply the first and the last stochastic ordering claimed in (5.4.32). The stochastic domination result in the middle of (5.4.32) follows from (5.4.31) and a general result for strictly positive associated measures (see [98, Theorem 2.24]). The proof of (5.4.33) is similar. □

Infinite volume measures. The definition of the measure $\mathbb{P}$ shows that it is a finite volume Gibbs measure for the energy $E(\kappa)=\ln(\det\Delta_\kappa)/2$ and a homogeneous Bernoulli a priori measure. We would like to define infinite volume limits for the measures $\mathbb{P}^G$ and define a notion of Gibbs measures in infinite volume. This requires some additional definitions. Recall the definition of the σ-algebras $\mathcal{F}_E$ for $E\subset\mathbf{E}(\mathbb{Z}^d)$ and note that there is a similar definition for general graphs which will be used in the following. An event $A\subset\mathcal{F}$ is called local if it measurable with respect to $\mathcal{F}_E$ for some finite set E, i.e., A depends only on finitely many edges. Similarly we define a local function as a function that is measurable with respect to $\mathcal{F}_E$ for a finite set E. We say that a sequence of measures μ_n on $\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ converges in the topology of local convergence to a measure μ if $\mu_n(A)\to\mu(A)$ for all local events A. For a background on the choice of topologies in the context of Gibbs measures we refer to [92]. The construction of the infinite volume states proceeds similarly to the construction for the random cluster model by defining a specification and introducing the notion of free and wired boundary conditions. For simplicity we restrict the analysis to $\mathbb{Z}^d$ but the generalisation to more general graphs is straightforward. First, we define infinite volume limits of the finite volume distributions with wired and free boundary conditions. Let us denote by $\Lambda_n=[-n,n]\cap\mathbb{Z}^d$ the ball with radius n in the maximum norm around the origin and we denote by $E_n = \mathbf{E}(\Lambda_n)$ the edges in Λ_n. We introduce the shorthand $\Lambda_n^w = \Lambda_n/\partial\Lambda_n$ for the box with wired boundary conditions. We define

$$\bar{\mu}^0_{n,p} = \mathbb{P}^{\Lambda_n,p}, \qquad \bar{\mu}^1_{n,p} = \mathbb{P}^{\Lambda_n^w,p} \tag{5.4.36}$$

for the measure $\mathbb{P}$ on Λ_n with free and wired boundary conditions respectively. From Corollary 5.4.12 and equation (5.4.31) we conclude that for any increasing event A depending only on edges in E_n

$$\bar{\mu}^0_{n+1}(A) = \mathbb{P}^{\Lambda_{n+1}}(A) = \mathbb{P}^{\Lambda_{n+1}}(\mathbb{P}^{\Lambda_{n+1},E_n,\kappa}(A)) \geq \mathbb{P}^{\Lambda_n}(A) = \bar{\mu}^0_n(A). \tag{5.4.37}$$

We conclude that for any increasing event A depending only on finitely many edges the limits $\lim_{n\to\infty}\bar{\mu}^0_{n,p}(A)$ and similarly $\lim_{n\to\infty}\bar{\mu}^1_{n,p}(A)$ exist. Using standard arguments we can write every local event A as a union and difference of increasing local events and we conclude that $\lim_{n\to\infty}\bar{\mu}^0_{n,p}(A)$ and $\lim_{n\to\infty}\bar{\mu}^1_{n,p}(A)$ exist. It is well known (see [30]) that this implies convergence of $\bar{\mu}^0_{n,p}$ and $\bar{\mu}^1_{n,p}$ to a measure on $\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ in the topology of local convergence. We denote the infinite volume measures by $\bar{\mu}^0_p$ and $\bar{\mu}^1_p$.

Lemma 5.4.13. *The measure $\bar{\mu}^0_p$ and $\bar{\mu}^1_p$ satisfy the FKG-inequality and for $0 \leq p \leq p' \leq 1$ the relations*

$$\bar{\mu}^0_p \precsim \bar{\mu}^1_p, \quad \bar{\mu}^0_p \precsim \bar{\mu}^0_{p'}, \quad \bar{\mu}^1_p \precsim \bar{\mu}^1_{p'}. \tag{5.4.38}$$

Moreover they are invariant under symmetries of the lattice and ergodic with respect to translations.

Proof. This is a consequence of Corollary 5.4.8 and Corollary 5.4.12 and a limiting argument. See the proof of Theorem 4.17 and Corollary 4.23 in [98] for a detailed proof for the random cluster model which also applies to the model considered here. Ergodicity is proved by showing that the measures are even mixing. □

Infinite volume specifications. We now introduce the concept of infinite volume Gibbs measures for this model. We first consider the case of a finite connected graph G. For $E \subset \mathbf{E}(G)$ we consider the finite volume specifications $\bar{\gamma}^G_E : \mathcal{F} \times \{1,q\}^{\mathbf{E}(G)} \to \mathbb{R}$

$$\bar{\gamma}^G_E(A,\lambda) = \frac{1}{Z_\lambda}\sum_{\kappa\in A} \mathbb{1}_{\kappa_{E^c}=\lambda_{E^c}} \frac{p^{h(\kappa)}(1-p)^{s(\kappa)}}{\sqrt{\det \Delta_\kappa}} \tag{5.4.39}$$

where the normalisation Z_λ ensures that $\bar{\gamma}^G_E(\cdot,\lambda)$ is a probability measure. A simple calculation shows that $\bar{\gamma}^G$ is indeed a specification, i.e., $\bar{\gamma}^G_E$ are proper probability kernels that satisfy for $E \subset E'$

$$\bar{\gamma}^G_{E'}\bar{\gamma}^G_E = \bar{\gamma}^G_{E'}. \tag{5.4.40}$$

Since $\bar{\gamma}_E(\cdot,\lambda)$ is concentrated on a finite set it is helpful to use the notation $\bar{\gamma}_E(\kappa,\lambda) = \bar{\gamma}_E(\{\kappa\},\lambda)$. The measure $\mathbb{P}^G$ is a finite volume Gibbs measure, i.e., it satisfies

$$\mathbb{P}^G\bar{\gamma}^G_E = \mathbb{P}^G \tag{5.4.41}$$

or put differently for $\kappa,\lambda \in \{1,q\}^E$

$$\bar{\gamma}^G_E(\kappa,\lambda) = \mathbb{P}^{G,E,\lambda}(\kappa_E)\mathbb{1}_{\kappa_{\mathbf{E}(G)\setminus E}=\lambda_{\mathbf{E}(G)\setminus E}}. \tag{5.4.42}$$

We would like to call μ a Gibbs measure on $\{1,q\}^{\mathbb{Z}^d}$ for the random conductance model if

$$\bar{\mu}\bar{\gamma}_E^{\mathbb{Z}^d} = \bar{\mu} \tag{5.4.43}$$

holds for all $E \subset \mathbf{E}(\mathbb{Z}^d)$ finite. However, $\bar{\gamma}_E^G$ is a priori only well defined for finite graphs so that we use an approximation procedure for infinite graphs. Let G be an connected infinite graph. We are a bit sloppy with the notation and do not distinguish between $\bar{\gamma}_E^H$ for a subgraph H of G and its proper extension to $\mathcal{F} \times \{1,q\}^{\mathbf{E}(G)}$, i.e., we define for $\kappa, \lambda \in \{1,q\}^{\mathbf{E}(G)}$

$$\bar{\gamma}_E^H(\kappa,\lambda) = \mathbb{1}_{\kappa_{E^c}=\lambda_{E^c}} \bar{\gamma}_E^H(\kappa_{\mathbf{E}(H)}, \lambda_{\mathbf{E}(H)}). \tag{5.4.44}$$

We denote for $f \in \mathbf{E}(G)$ and $\kappa \in \{1,q\}^{\mathbf{E}(G)}$ by κ^+ and κ^- as before the configurations such that $\kappa_e^+ = \kappa_e^-$ for $e \neq f$ and $\kappa_f^- = 1$, $\kappa_f^+ = q$.

In the following we assume $p \in (0,1)$. For $p \in \{0,1\}$ the measures $\mathbb{P}^{G,p}$ agree with the Dirac measure on the constant 1 or constant q configuration. Since we assume that E is finite the specification $\bar{\gamma}_E^H$ is uniquely characterized by the fact that it is proper and it satisfies for $\kappa, \lambda \in \{1,q\}^{\mathbf{E}(H)}$ such that $\kappa_{E^c} = \lambda_{E^c}$

$$\frac{\bar{\gamma}_{E'}^H(\kappa^-,\lambda)}{\bar{\gamma}_{E'}^H(\kappa^+,\lambda)} = \frac{1-p}{p}\sqrt{\frac{\det \Delta_{\kappa^+}^H}{\det \Delta_{\kappa^-}^H}} = \frac{1-p}{p}\sqrt{1+(q-1)\mathbb{Q}_{\kappa^-}^H(f \in t)} \tag{5.4.45}$$

where we used (5.4.29) in the second step. We show that we can give meaning to this expression in infinite volume. For this we sketch the definition of spanning trees in infinite volume but we refer to the literature for details (see [29]). A monotone exhaustion of an infinite graph G is a sequence of subgraphs G_n such that $G_n \subset G_{n+1}$ and $G = \bigcup_{n\geq 1} G_n$. It can be shown that for any finite sets $E_1 \subset E_2 \subset \mathbf{E}G$ the limit $\lim_{n\to\infty} \mathbb{Q}_\kappa^{G_n}(t \cap E_2 = E_1)$ exists. In fact this is a consequence of (5.4.26) and the arguments we used for $\bar{\mu}_n^0$ above. Hence it is possible to define a measure $\mathbb{Q}_\kappa^{G,0}$ on $2^{\mathbf{E}(G)}$, the power set of $\mathbf{E}(G)$ which will be called the weighted free spanning forest on G (as the name suggest the measure is supported on forests but not necessarily on trees, i.e., on connected subsets of edges). Similarly, we can define the wired spanning forest $\mathbb{Q}_\kappa^{G,1}$ replacing the subgraphs G_n by the contracted graphs $G_n/\partial G_n$. By definition those measures satisfy

$$\lim_{n\to\infty} \mathbb{Q}_\kappa^{G_n}(f \in t) = \mathbb{Q}_\kappa^{G,0}(f \in t) \tag{5.4.46}$$

$$\lim_{n\to\infty} \mathbb{Q}_\kappa^{G_n/\partial G_n}(f \in t) = \mathbb{Q}_\kappa^{G,1}(f \in t) \tag{5.4.47}$$

for any $f \in E$. Then it is possible to define two families of proper probability kernels $\bar{\gamma}_E^{G,0}$ and $\bar{\gamma}_E^{G,1}$ for $E \subset (\mathbf{E}(G))$ finite by the property that for $f \in E$ and $\kappa, \lambda \in \{1,q\}^{\mathbf{E}(G)}$ such that $\kappa_{E^c} = \lambda_{E^c}$

$$\frac{\bar{\gamma}_E^{G,0}(\kappa^-,\lambda)}{\bar{\gamma}_E^{G,0}(\kappa^+,\lambda)} = \frac{1-p}{p}\sqrt{1+(q-1)\mathbb{Q}_{\kappa^-}^{G,0}(f \in t)} \tag{5.4.48}$$

$$\frac{\bar{\gamma}_E^{G,1}(\kappa^-,\lambda)}{\bar{\gamma}_E^{G,1}(\kappa^+,\lambda)} = \frac{1-p}{p}\sqrt{1+(q-1)\mathbb{Q}_{\kappa^-}^{G,1}(f \in t)}. \tag{5.4.49}$$

From and (5.4.45) and (5.4.45) we conclude that $\bar{\gamma}^{G,0}$ and $\bar{\gamma}^{G,1}$ are well defined. Moreover we obtain that this family of probability kernels satisfy for $\lambda, \kappa \in \{1,q\}^{\mathbf{E}(G)}$

$$\bar{\gamma}_E^{G,0}(\kappa,\lambda) = \lim_{n\to\infty} \bar{\gamma}_E^{G_n}(\kappa,\lambda), \tag{5.4.50}$$

$$\bar{\gamma}_E^{G,1}(\kappa,\lambda) = \lim_{n\to\infty} \bar{\gamma}_E^{G_n/\partial G_n}(\kappa,\lambda). \tag{5.4.51}$$

Note that the concatenation for $\bar{\gamma}^{G,0}$ for $E', E \subset \mathbf{E}(\mathbb{Z}^d)$ is given by

$$\bar{\gamma}_E^{G,0}\bar{\gamma}_{E'}^{G,0}(\kappa,\lambda) = \sum_{\sigma:\sigma_{E^c}=\lambda_{E^c}} \bar{\gamma}_E^{G,0}(\sigma,\lambda)\bar{\gamma}_{E'}^{G,0}(\kappa,\sigma), \tag{5.4.52}$$

in particular it only involves a finite sum in the case of a finite spin space. We conclude using (5.4.50) and (5.4.51) that $\bar{\gamma}_E^{G,0}$ and $\bar{\gamma}_E^{G,1}$ define two specifications on G.

Suppose the wired and the free uniform spanning forest on G agree. This implies that also the weighted wired and free spanning forest $\mathbb{Q}_\kappa^{G,0}$ and $\mathbb{Q}_\kappa^{G,1}$ on G agree if the conductances κ_e are contained in a compact subset of $(0,\infty)$ (see Theorem 7.3 and Theorem 7.7 in [29]). Thus $\bar{\gamma}_E^{G,1} = \bar{\gamma}_E^{G,0}$ in this case. In particular we obtain that $\bar{\gamma}_E^{\mathbb{Z}^d,0} = \bar{\gamma}_E^{\mathbb{Z}^d,1}$ because the free and the wired uniform spanning forest on $\mathbb{Z}^d$ agree (Corollary 6.3 in [29]). In the following we will denote this specification by $\bar{\gamma}_E$. To ensure consistency with the earlier definition of $\tilde{\gamma}$ we define for a connected subset $\Lambda \subset \mathbb{Z}^d$ that $\bar{\gamma}_\Lambda = \bar{\gamma}_{\mathbf{E}(\Lambda)}$. We can now give a formal definition of Gibbs measures for the random conductance model.

Definition 5.4.14. *A measure $\bar{\mu} \in \mathcal{P}(\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)})$ is a Gibbs measure if it is admitted to the specification $\bar{\gamma}_E$.*

As one would expect the infinite volume measures $\bar{\mu}_p^0$ and $\bar{\mu}_p^1$ are Gibbs measures.

Lemma 5.4.15. *The measures $\bar{\mu}_p^0$ and $\bar{\mu}_p^1$ are Gibbs measures as defined in Definition 5.4.14. Moreover any Gibbs measure $\bar{\mu}$ satisfies $\bar{\mu}_p^0 \precsim \bar{\mu} \precsim \bar{\mu}_p^1$.*

Proof. By equation (5.4.41) we have for $E \subset E_n$

$$\bar{\mu}_n^0 \bar{\gamma}_E^{\Lambda_n} = \bar{\mu}_n^0. \tag{5.4.53}$$

We show that both sides converge in the topology of local convergence as $n \to \infty$. Let A be an increasing event depending on a finite number of edges. We have seen in (5.4.37) that $\mu_n^0(A)$ is an increasing sequence and converges by definition to $\mu^0(A)$. We derive the convergence of the left hand side of equation (5.4.53) from the following three observations. First, we conclude from (5.4.32) and (5.4.42) that $\bar{\gamma}_E^{\Lambda_n}(A,\cdot)$ is an increasing function. Second, using (5.4.33) and (5.4.42) we obtain $\bar{\gamma}_E^{\Lambda_{n+1}}(A,\kappa) \geq \bar{\gamma}_E^{\Lambda_n}(A,\kappa_{E_n})$ for all $\kappa \in \{1,q\}^{E_{n+1}}$. The third observation is that (5.4.37) can also be applied to an increasing function instead of an increasing event. These three facts imply

$$\bar{\mu}_n^0(\bar{\gamma}_E^{\Lambda_n}(A,\cdot)) \leq \bar{\mu}^0(\bar{\gamma}_E^{\Lambda_n}(A,\cdot)) \leq \bar{\mu}^0(\bar{\gamma}_E(A,\cdot)). \tag{5.4.54}$$

On the other hand, we obtain for any $m \in \mathbb{N}$

$$\lim_{n\to\infty} \bar{\mu}_n^0(\bar{\gamma}_E^{\Lambda_n}(A,\cdot)) \geq \lim_{n\to\infty} \bar{\mu}_n^0(\bar{\gamma}_E^{\Lambda_m}(A,\cdot)) = \bar{\mu}^0(\bar{\gamma}_E^{\Lambda_m}(A,\cdot)). \tag{5.4.55}$$

Sending $m \to \infty$ we get

$$\lim_{n\to\infty} (\bar{\mu}_n^0 \bar{\gamma}_E^{\Lambda_n})(A) \geq (\bar{\mu}^0 \bar{\gamma}_E)(A). \tag{5.4.56}$$

Hence, we have shown that

$$\bar{\mu}^0 \bar{\gamma}_E(A) = \lim_{n\to\infty} \bar{\mu}_n^0 \bar{\gamma}_E^{\Lambda_n}(A) = \lim_{n\to\infty} \bar{\mu}_n^0(A) = \bar{\mu}^0(A) \tag{5.4.57}$$

holds for any increasing and local event A. Using standard arguments (5.4.57) holds for all local events. Therefore μ^0 is a Gibbs measure. The proof for μ^1 is the same up to a reverse of inequalities. Finally, a limiting argument and the comparison of boundary conditions show that $\bar{\mu}_p^0 \precsim \bar{\mu} \precsim \bar{\mu}_p^1$ for any Gibbs measure μ (see [98, Proposition 4.10]). □

Let us introduce the class of quasilocal specifications. A quasilocal function on a general state space is a function $X : F^S \to \mathbb{R}$ that can be approximated arbitrarily well by local functions, i.e.,

$$\inf_{Y \text{ local}} \sup_{\omega \in F^S} |X(\omega) - Y(\omega)| = 0. \tag{5.4.58}$$

A specification γ is called quasilocal if $\gamma_\Lambda X$ is a quasilocal function for every local function X. Quasilocality is a natural and useful condition for a specification (see [92]). We will show that the specification $\bar{\gamma}_E$ is quasilocal. This is direct consequence of the following result that shows uniform convergence of $\bar{\gamma}_E^{\Lambda_n^w}$ to $\bar{\gamma}_E$. This result will be of independent use later.

Lemma 5.4.16. *The specifications* $\bar{\gamma}_{E_n}$ *and* $\bar{\gamma}_{E_n}^{\Lambda_N^w}$ *satisfy*

$$\limsup_{N\to\infty} \sup_{\kappa,\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}} |\bar{\gamma}_{E_n}(\kappa,\lambda) - \bar{\gamma}_{E_n}^{\Lambda_N^w}(\kappa,\lambda)| = 0. \tag{5.4.59}$$

Proof. First, we claim that it is sufficient to show that

$$\limsup_{N\to\infty} \sup_{\kappa \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}} \sup_{f\in E_n} |\mathbb{Q}_\kappa^{\mathbb{Z}^d}(f\in t) - \mathbb{Q}_\kappa^{\Lambda_N^w}(f \in t)| = 0. \tag{5.4.60}$$

Indeed, using (5.4.60) in (5.4.45) we obtain

$$\limsup_{N\to\infty} \sup_{\substack{\kappa,\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)} \\ \kappa_{E_n^c} = \lambda_{E_n^c}}} \sup_{f\in E_n} \frac{\bar{\gamma}_{E_n}(\kappa_f^-,\lambda)}{\bar{\gamma}_{E_n}(\kappa_f^+,\lambda)} \Big/ \frac{\bar{\gamma}_{E_n}^{\Lambda_N^w}(\kappa_f^-,\lambda)}{\bar{\gamma}_{E_n}^{\Lambda_N^w}(\kappa_f^+,\lambda)} = 1. \tag{5.4.61}$$

Since E_n is finite this implies the claim.

It remains to prove (5.4.60). This is a consequence of the transfer current theorem (see Theorem 4.1 in [29]) that states in the special case of the occupation property that for $f = \{x,y\} \in \mathbf{E}(G)$

$$\mathbb{Q}_\kappa^G(f\in t) = I_f(f) = \kappa_f(\delta_x - \delta_y)(\Delta_\kappa^G)^{-1}(\delta_x - \delta_y) \tag{5.4.62}$$

where the expression $I_f(f)$ denotes the current through the edge f when 1 unit of current is induced respectively removed at the two ends of f. In the last step we used that $I_f(f)$ can be calculated by applying the inverse Laplacian to the sources to obtain the potential which can be used to calculate the current through f. Now (5.4.60) follows from the display (5.4.62) and Lemma 5.B.3. □

Relation to extended gradient Gibbs measures In this paragraph we state the results that relate the random conductance model to extended gradient Gibbs measure. This is finally the justification to consider this model. The proofs of the results in this paragraph are deferred to Section 5.A. The first Proposition establishes that the κ-marginal of extended gradient Gibbs measures are Gibbs states for the random conductance model.

Proposition 5.4.17. *Let* $\tilde{\mu}$ *be an extended gradient Gibbs measure associated to a translation invariant and ergodic gradient Gibbs measure* μ *with zero tilt. Then the* κ*-marginal* $\bar{\mu}$ *of* $\tilde{\mu}$ *is a Gibbs measure in the sense of definition 5.4.14.*

The second main result in this paragraph is a reverse of Proposition 5.4.17, namely that it is possible to obtain an extended Gibbs measure with zero tilt for the potential $V_{p,q}$, given a Gibbs measure $\bar{\mu}$ for the random conductance model with parameters p, q.

Proposition 5.4.18. *Let $\bar{\mu}$ be a Gibbs measure in the sense of Definition 5.4.14 for parameters p and q and $\kappa \sim \bar{\mu}$. Let φ^κ be the random field that for given κ is a Gaussian field with zero average, $\varphi^\kappa(0) = 0$, and covariance $(\Delta_\kappa)^{-1}$, i.e., φ^κ satisfies for $f : \mathbb{Z}^d \to \mathbb{R}$ with finite support and $\sum_x f(x) = 0$*

$$\operatorname{Var}\left((f, \varphi^\kappa)_{\mathbb{Z}^d}\right) = (f, (\Delta_\kappa)^{-1} f). \tag{5.4.63}$$

Let $\tilde{\mu}$ be the joint law of $(\kappa, \nabla\varphi^\kappa)$. Then $\tilde{\mu}$ is an extended Gibbs measure for the potential $V_{p,q}$ with zero tilt, in particular its η-marginal is a gradient Gibbs measure with zero tilt.

As a last result in this direction we state a very useful result from [33] that characterizes the law of φ given κ for extended gradient Gibbs measures if φ is distributed according to a gradient Gibbs measure.

Proposition 5.4.19. *Let μ be a translation invariant, ergodic gradient Gibbs measure with zero tilt and $\tilde{\mu}$ the corresponding extended gradient Gibbs measure. Then the conditional law of φ given κ is $\tilde{\mu}$-almost surely Gaussian. It is determined by its expectation*

$$\mathbb{E}\big(\varphi_x \mid \mathcal{F}\big)(\kappa) = 0 \tag{5.4.64}$$

and the covariance given by $(\Delta_\kappa)^{-1}$, i.e., for $f : \mathbb{Z}^d \to \mathbb{R}$ with finite support and $\sum_x f(x) = 0$

$$\operatorname{Var}_{\tilde{\mu}}\left((f, \varphi)_{\mathbb{Z}^d} \mid \mathcal{F}\right)(\kappa) = (f, (\Delta_\kappa)^{-1} f). \tag{5.4.65}$$

Proof. This is Lemma 3.4 in [33]. □

In particular those results establish the following. Assume that μ is an ergodic zero tilt gradient Gibbs measure. Let $\bar{\mu}$ be the κ-marginal of the corresponding extended gradient Gibbs measure $\tilde{\mu}$ (which by Proposition 5.4.17 is Gibbs for the random conductance model). We can use Proposition 5.4.18 to construct an extended gradient Gibbs measure $\tilde{\mu}'$. Using the definition of $\bar{\mu}'$ in Proposition 5.4.18 and Proposition 5.4.19 we conclude that we get back the extended gradient Gibbs measure we started from, i.e., $\tilde{\mu} = \tilde{\mu}'$.

5.5 Further properties of the random conductance model

In this section we state and prove more results about the random conductance model considered in this work and use the results from the previous section to derive corresponding results for the associated gradient interface model. We end this section with some conjectures and open questions. We start by proving $\bar{\mu}_p^0 = \bar{\mu}_p^1$ for almost all values of p which will in particular implies uniqueness of the Gibbs measure for those p.

Theorem 5.5.1. *For every $q \geq 1$ there are at most countably many $p \in [0, 1]$ such that $\bar{\mu}_p^1 \neq \bar{\mu}_p^0$.*

Proof. It is a standard consequence of the invariance under lattice symmetries and $\bar{\mu}_p^0 \precsim \bar{\mu}_p^1$ that $\bar{\mu}_p^1 = \bar{\mu}_p^0$ is equivalent to $\bar{\mu}_p^1(\kappa_e = q) = \bar{\mu}_p^0(\kappa_e = q)$ for one and therefore any $e \in \mathbf{E}(\mathbb{Z}^d)$ (see, e.g, Proposition 4.6 in [98]). Lemma 5.5.3 below implies for $e \in \mathbf{E}(\mathbb{Z}^d)$

$$\bar{\mu}_p^0(\kappa_e = q) \leq \bar{\mu}_p^1(\kappa_e = q) \leq \bar{\mu}_{p'}^0(\kappa_e = q) \tag{5.5.1}$$

for any $p' > p$. In particular, we can conclude that $\bar{\mu}_p^0 = \bar{\mu}_p^1$ holds for all points of continuity of the map $p \mapsto \bar{\mu}_p^0(\kappa_e = q)$. Since this map is increasing by Lemma 5.4.9 it has only countably many points of discontinuity. □

We are now in the position to prove Theorem 5.2.3.

Proof of Theorem 5.2.3. We note that a translation invariant zero tilt Gibbs measure exists for any p and q, e.g., as a limit of torus Gibbs states (see the proof of Theorem 2.2 in [32]). It remains to show uniqueness. Consider p such that $\bar{\mu}_p^1 = \bar{\mu}_p^0$ which is true for all but a countable number of $p \in [0,1]$ by Theorem 5.5.1 above. Let μ_1 and μ_2 be ergodic zero tilt gradient Gibbs measures for $V = V_{p,q}$. By Proposition 5.4.17 the corresponding κ-marginals $\bar{\mu}_1$ and $\bar{\mu}_2$ of the extended Gibbs measures $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are Gibbs measures in the sense of Definition 5.4.14 and therefore equal. Using Proposition 5.4.19 we conclude that since μ_1 and μ_2 are ergodic zero tilt gradient Gibbs measures their laws are determined by $\bar{\mu}_1$ and $\bar{\mu}_2$, hence $\mu_1 = \mu_2$. □

Remark 5.5.2. *Similar arguments for this model appeared already in the proof of Theorem 2.4 in [32] where they use the convexity of the pressure to show that the number of q-bonds on the torus is concentrated around its expectation in the thermodynamic limit. However, this is not sufficient to conclude uniqueness.*

The key ingredient in the proof of Theorem 5.5.1 is the following lemma that compares $\bar{\mu}_p^1(\kappa_e = q)$ with $\bar{\mu}_{p'}^0(\kappa_e = q)$ for $p < p'$. Intuitively the reason for this result is that a change of p is a bulk effect of order $|\Lambda|$ while the effect of the boundary conditions is of order $|\partial\Lambda|$.

Lemma 5.5.3. *For any $p < p'$ we have*

$$\mu_{p'}^0(\kappa_e = q) \geq \mu_p^1(\kappa_e = q). \tag{5.5.2}$$

Proof. The proof follows the proof of Theorem 1.12 in [77] for the random cluster model that shows the analogous result for the random cluster model. The only difference is that the comparison between free and wired boundary conditions is slightly less direct. We define $a = \bar{\mu}_{p'}^0(\kappa_e = q)$ and $b = \bar{\mu}_p^1(\kappa_e = q)$. Comparison between boundary condition implies $\bar{\mu}_{n,p'}^0(\kappa_e = q) \leq \bar{\mu}_{p'}^0(\kappa_e = q) = a$ for any $e \in E_n$. Recall that $h(\kappa) = |\{e \in \mathbf{E}(G) : \kappa_e = q\}|$ denotes the number of q-bonds and $s(\kappa)$ similarly the number of 1-bonds. The definition of a and b implies for $0 < \varepsilon < 1 - a$

$$\bar{\mu}_{n,p'}^0(h(\kappa)) \leq a|E_n| \ \Rightarrow \bar{\mu}_{n,p'}^0(h(\kappa) \leq (a+\varepsilon)|E_n|) \geq \varepsilon. \tag{5.5.3}$$

Similarly for $0 < \varepsilon < b$

$$\bar{\mu}_{n,p}^1(h(\kappa)) \geq b|E_n| \ \Rightarrow \bar{\mu}_{n,p}^1(h(\kappa) \geq (b-\varepsilon)|E_n|) \geq \varepsilon. \tag{5.5.4}$$

Our goal is to show that $b - \varepsilon \leq a + \varepsilon$. We denote by Δ^0 and Δ^1 the graph Laplacian on Λ_n with free and wired boundary conditions respectively. To compare the boundary conditions we denote by $T_1 = \mathrm{ST}(\Lambda_n^w)$ the set of wired spanning trees on Λ_n and by $T_0 = \mathrm{ST}(\Lambda_n)$ the set of spanning trees on Λ_n with free boundary conditions. There is a map $\Phi : T_0 \to T_1$ such that $\Phi(t)\restriction_{\Lambda_{n-1}} = t\restriction_{\Lambda_{n-1}}$. Indeed, removing all edges in $E_n \subset E_{n-1}$ from t we obtain an acyclic subtree of Λ_n^w, hence we can find a tree $\Phi(t)$ such that $t\restriction_{\Lambda_{n-1}} \subset \Phi(t) \subset t$. The observation $|t \setminus \Phi(t)| = |\partial\Lambda_n| - 1$ implies that $w(\kappa,t) \leq w(\kappa,\Phi(t))q^{|\partial\Lambda_n|-1}$. Since Φ does not change the edges in E_{n-1} each tree $t \in T_1$ has at most $2^{|E_n \setminus E_{n-1}|}$ preimages. We obtain that

$$\begin{aligned} |\Lambda_n|^{-1} \det \Delta_\kappa^0 &= \sum_{t\in T_0} w(\kappa,t) \leq \sum_{t \in T_0} w(\kappa,\Phi(t))q^{|\partial\Lambda_n|-1} \leq 2^{|E_n\setminus E_{n-1}|} q^{|\partial\Lambda_n|} \sum_{t\in T_1} w(\kappa,t) \\ &= 2^{|E_n\setminus E_{n-1}|} q^{|\partial\Lambda_n|} |\Lambda_n^w|^{-1} \det \Delta_\kappa^1. \end{aligned} \tag{5.5.5}$$

Similarly, there is an injective mapping $\Psi : T_1 \to T_0$ such that $t \subset \Psi(t)$. Indeed, we fix a tree t_b in the graph $(\Lambda_n \setminus \Lambda_{n-1}, \mathbf{E}(\Lambda_n \setminus \Lambda_{n-1})$ and define $\Psi(t) = t \cup t_b \in T_0$. We get

$$|\Lambda_n^w|^{-1} \det \Delta_\kappa^1 = \sum_{t\in T_1} w(\kappa, t) \le \sum_{t\in T_1} w(\kappa, \Psi(t)) \le \sum_{t\in T_0} w(\kappa, t) = |\Lambda_n|^{-1} \det \Delta_\kappa^0. \tag{5.5.6}$$

Inserting the bound $|E_n \setminus E_{n-1}| \le 2d|\partial\Lambda_n|$ we infer from the definition (5.3.7) for any $\kappa \in \{1, q\}^{E_n}$

$$\left(2^{2d}q\right)^{-|\partial\Lambda_n|/2} \bar{\mu}^0_{n,p}(\kappa) \le \bar{\mu}^1_{n,p}(\kappa) \le \left(2^{2d}q\right)^{|\partial\Lambda_n|/2} \bar{\mu}^0_{n,p}(\kappa). \tag{5.5.7}$$

We define the constant $\alpha = p'(1-p)/(p(1-p')) > 1$. Simple manipulation show that for any function $X : \{1, q\}^{E_n} \to \mathbb{R}$

$$\bar{\mu}^0_{\Lambda_n,p'}(X) = \frac{\bar{\mu}^0_{\Lambda_n,p}(\alpha^{h(\kappa)}X)}{\bar{\mu}^0_{\Lambda_n,p}(\alpha^{h(\kappa)})}. \tag{5.5.8}$$

Therefore we obtain

$$\begin{aligned}
\bar{\mu}^0_{\Lambda_n,p'}\left(h(\kappa) \le (a+\varepsilon)|E_n|\right) &= \frac{\bar{\mu}^0_{\Lambda_n,p}(\alpha^{h(\kappa)} \mathbb{1}_{h(\kappa)\le(a+\varepsilon)|E_n|})}{\mu^0_{\Lambda_n,p}(\alpha^{h(\kappa)})} \\
&\le \frac{\bar{\mu}^0_{\Lambda_n,p}(\alpha^{h(\kappa)} \mathbb{1}_{h(\kappa)\le(a+\varepsilon)|E_n|})}{\bar{\mu}^0_{\Lambda_n,p}(\alpha^{h(\kappa)} \mathbb{1}_{h(\kappa)\ge(b-\varepsilon)|E_n|})} \\
&\le \frac{\alpha^{(a+\varepsilon)|E_n|}}{(2^{2d}q)^{-|\partial\Lambda_n|/2}\, \alpha^{(b-\varepsilon)|E_n|} \bar{\mu}^1_{\Lambda_n,p}\left(h(\kappa) \ge (b-\varepsilon)|E_n|\right)}.
\end{aligned} \tag{5.5.9}$$

From (5.5.3) and (5.5.4) we conclude

$$\varepsilon^2 \le \left(2^{2d}q\right)^{|\partial\Lambda_n|/2} \alpha^{(a-b+2\varepsilon)|E_n|} \tag{5.5.10}$$

which implies $a - b + 2\varepsilon \ge 0$ as $n \to \infty$ since $\alpha > 1$ and $|E_n|/|\partial\Lambda_n| \to \infty$. The lemma follows as $\varepsilon \to 0$. □

The next result is a non-uniqueness result for the random conductance model.

Theorem 5.5.4. *In dimension $d = 2$ and for $q > 1$ sufficiently large there are two distinct Gibbs measures $\bar{\mu}^1_{p_{\mathrm{sd}}} \neq \bar{\mu}^0_{p_{\mathrm{sd}}}$ at the self-dual point defined by equation* (5.1.5).

The proof uses duality of the random conductance model and can be found in Section 5.6. This result easily implies Theorem 5.2.4.

Proof of Theorem 5.2.4. Using Proposition 5.4.18 we infer from Theorem 5.5.4 the existence of two translation invariant extended gradient Gibbs measures $\tilde{\mu}_0$ and $\tilde{\mu}_1$ constructed from $\bar{\mu}^0_{p_{\mathrm{sd}}} \neq \bar{\mu}^1_{p_{\mathrm{sd}}}$. Their η-marginals μ_0 and μ_1 are not equal since then the κ-marginals $\bar{\mu}_1$ and $\bar{\mu}_2$ would agree. They both have zero tilt by Proposition 5.4.18 and the definition of $\tilde{\mu}$ shows that $\tilde{\mu}$ is translation invariant if $\bar{\mu}$ is translation invariant. □

Remark 5.5.5. *A proof similar to Lemma 3.2 in [33] shows that ergodicity of $\bar{\mu}_1$ and $\bar{\mu}_2$ implies that μ_0 and μ_1 are itself ergodic. The only difference is that η given κ is not independent (which κ given η is). Instead one has to rely on the decay of correlations for Gaussian fields stated in Appendix 5.B.*

Theorem 5.5.6. *For $d \geq 4$ there is $q_0 > 1$ such that for $p \in [0,1]$ and $q \in [1, q_0)$ the Gibbs measure for the random conductance model is unique. Similarly, for $d \geq 4$ and $q \geq 1$ there is a $p_0 = p_0(q,d) > 0$ such that the Gibbs measure is unique for $p \in [0, p_0) \cup (1-p_0, 1]$.*

Proof. We are going to apply Dobrushin's criterion (see, e.g., [92, Theorem (8.7)]. The necessary estimate is basically a refined version of the proof of Lemma 5.4.6. Fix two edges $f, g \in \mathbf{E}(\mathbb{Z}^d)$. Recall the notation $\lambda^{\pm\pm} = \lambda^{\pm\pm}_{fg}$ and $\lambda^{\pm} = \lambda^{\pm}_f$ introduced above Theorem 5.4.2. We will write $\bar{\gamma}_f = \bar{\gamma}_{\{f\}}$ in the following. Note that (5.4.45) and $\bar{\gamma}_f(\lambda^+, \lambda) + \bar{\gamma}_f(\lambda^-, \lambda) = 1$ imply that

$$\bar{\gamma}_f(\lambda^+, \lambda) = \frac{\bar{\gamma}_f(\lambda^+, \lambda)}{\bar{\gamma}_f(\lambda^+, \lambda) + \bar{\gamma}_f(\lambda^-, \lambda)} = \frac{p}{p + (1-p)\sqrt{1 + (q-1)\mathbb{Q}_{\lambda^-}(f \in t)}} \tag{5.5.11}$$

We need to bound the entries of the Dobrushin matrix given by

$$\begin{aligned} C_{fg} &= \sup_{\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}} |\bar{\gamma}_f(\lambda^{++}_{fg}, \lambda^{++}_{fg}) - \bar{\gamma}_f(\lambda^{+-}_{fg}, \lambda^{+-}_{fg})| \\ &= \sup_{\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}} \left| \frac{p}{p + (1-p)\sqrt{1 + (q-1)\mathbb{Q}_{\lambda^{-+}}(f \in t)}} - \frac{p}{p + (1-p)\sqrt{1 + (q-1)\mathbb{Q}_{\lambda^{--}}(f \in t)}} \right|. \end{aligned} \tag{5.5.12}$$

Since the derivative of the map $x \mapsto p/(p + (1-p)\sqrt{x})$ is bounded by $p(1-p)$ for $x \geq 1$ we conclude that

$$\sup_{\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}} |\bar{\gamma}_f(\lambda^{++}, \lambda^{++}) - \bar{\gamma}_f(\lambda^{+-}, \lambda^{+-})| \leq p(1-p)(q-1)\, |\mathbb{Q}_{\lambda^{-+}}(f \in t) - \mathbb{Q}_{\lambda^{--}}(f \in t)|. \tag{5.5.13}$$

We can express $\mathbb{Q}_{\lambda^{-+}}(f \in t)$ through the measure $\mathbb{Q}_{\lambda^{--}}$ as follows

$$\mathbb{Q}_{\lambda^{-+}}(f \in t) = \frac{\mathbb{Q}_{\lambda^{--}}(f \in t, g \notin t) + q\mathbb{Q}_{\lambda^{--}}(f \in t, g \in t)}{q\mathbb{Q}_{\lambda^{--}}(g \in t) + \mathbb{Q}_{\lambda^{--}}(g \notin t)}. \tag{5.5.14}$$

A sequence of manipulations then shows that

$$\mathbb{Q}_{\lambda^{-+}}(f \in t) - \mathbb{Q}_{\lambda^{--}}(f \in t) = \frac{(q-1)\big(\mathbb{Q}_{\lambda^{--}}(f \in t, g \in t) - \mathbb{Q}_{\lambda^{--}}(f \in t)\mathbb{Q}_{\lambda^{--}}(g \in t)\big)}{q\mathbb{Q}_{\lambda^{--}}(g \in t) + \mathbb{Q}_{\lambda^{--}}(g \notin t)}. \tag{5.5.15}$$

The numerator can be rewritten using the transfer-current Theorem for two edges (see [29, Page 10] and equation below 4.3 in [123])

$$\mathbb{Q}_{\lambda^{--}}(f \in t, g \in t) - \mathbb{Q}_{\lambda^{--}}(f \in t)\mathbb{Q}_{\lambda^{--}}(g \in t) = -I^{\lambda^{--}}_f(g) I^{\lambda^{--}}_g(f). \tag{5.5.16}$$

where $I^{\kappa}_f(g)$ denotes the current through g in a resistor network with conductances κ when 1 unit of current is inserted (respectively removed) at the ends of f (using a fixed orientation of the edges here, e.g., lexicographic). All together we have shown that

$$C_{fg} \leq \sup_{\kappa \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}} p(1-p)(q-1)^2 I^{\kappa}_f(g) I^{\kappa}_g(f). \tag{5.5.17}$$

Using electrical network theory we can express for $f = (x, x+e_i)$ and $g = (y, y+e_j)$

$$I^{\kappa}_f(g) = \kappa_g \big(\delta_{y+e_i} - \delta_y, (\Delta_\kappa)^{-1}(\delta_{x+e_i} - \delta_x)\big)_{\mathbb{Z}^d} = \kappa_g \nabla_{x,i} \nabla_{y,j} G_\kappa(y, x) \tag{5.5.18}$$

where G_κ denotes the inverse of the operator Δ_κ which exists in dimension $d \geq 3$ and whose derivative exists in dimension $d \geq 2$. Combining the bound (5.B.6) in Lemma 5.B.2, (5.5.17), and (5.5.18) we conclude for $f \in \mathbf{E}(\mathbb{Z}^d)$ that

$$\sum_{g \in \mathbf{E}(\mathbb{Z}^d)} C_{fg} \leq C(q,d)p(1-p)(q-1)^2 \sum_{x \in \mathbb{Z}^d} (1+|x|)^{2(2-d-2\alpha)}. \tag{5.5.19}$$

In dimension $d \geq 4$ the sum is finite. Now, for fixed q, the sum becomes smaller 1 for p sufficiently close to 0 or 1. Therefore there is $p_0 = p_0(q,d)$ such that the Gibbs measure is unique for $p \in [0, p_0) \cup (1-p_0, 1]$. On the other hand, the constant $C(q,d)$ from Lemma 5.B.1 is decreasing in q. Therefore we can estimate for all $p \in [0,1]$ and $q \leq 2$

$$\sum_{g \in \mathbf{E}(\mathbb{Z}^d)} C_{fg} \leq \frac{C(2,d)}{4}(q-1)^2 \sum_{x \in \mathbb{Z}^d} (1+|x|)^{2(2-d-2\alpha)}. \tag{5.5.20}$$

Hence the Dobrushin criterion is satisfied for q sufficiently close to 1 and all $p \in [0,1]$. □

Remark 5.5.7. *1. Note that the gradient-gradient correlations in gradient models at best only decay critically with $|x|^{-d}$ (which is the decay rate for the discrete Gaussian free field). In particular, the sum of the covariance $\sum_{g \in \mathbf{E}(\mathbb{Z}^d)} Cov(\eta_f, \eta_g)$ diverges in this type of model. We use crucially in the previous theorem that the decay of correlations is better for the discrete model: They decay with the square of the gradient-gradient correlations.*

2. *The averaged (annealed) second order derivative of the Greens functions decays with the optimal decay rate $|x|^{-d}$ as shown in [67]. For the application of the Dobrushin criterion we, however need deterministic bounds which are weaker.*

3. *To extend the uniqueness result to $d = 3$ and $d = 2$ and q close to 1 one would need estimates for the optimal Hölder exponent α depending on the ellipticity contrast of discrete elliptic operators. Here the ellipticity contrast can be bounded by q. There do not seem to be any results in this direction in the discrete setting. In the continuum setting the problem is open in for $d \geq 3$, but has been solved for $d = 2$ in the continuum where $\alpha \to 1$ as the ellipticity contrast converges to 1 [131].*

Note that we can again lift the uniqueness result for the Gibbs measure of the random conductance model to a uniqueness result for the ergodic gradient Gibbs measures with zero tilt.

Proof of Theorem 5.2.5. The proof follows from the uniqueness of the Gibbs measure proven in Theorem 5.5.6 in the same way as the proof of Theorem 5.2.3. □

Open questions Let us end this section by stating one further result and two conjectures regarding the phase transitions of this model. They are most easily expressed in terms of percolation properties of the model even though the interpretation as open and closed bonds is somehow misleading in this context. We write $x \leftrightarrow y$ for $x, y \in \mathbb{Z}^d$ and κ if there is a path of q-bonds in κ connecting x and y and similarly for sets. Observe that the results of [78] can be applied to the model introduced here and we obtain the existence of a sharp phase transition.

Theorem 5.5.8. *For every q the model undergoes a sharp phase transition in p, i.e., there is $p_c(q,d)$ such that the following two properties hold. On the one hand there is a constant $c_1 > 0$ such that for $p > p_c$ sufficiently close to p_c*

$$\bar{\mu}^1(0 \leftrightarrow \infty) \geq c_1(p - p_c). \tag{5.5.21}$$

On the other hand, for $p < p_c$ there is a constant c_p such that

$$\bar{\mu}_n^1(0 \leftrightarrow \partial\Lambda_n) \leq e^{-c_p n}. \tag{5.5.22}$$

Proof. The proof of Theorem 1.2 in [78] for the random cluster model applies to this model. Indeed, it only relies on $\mu_{n,p}^1$ being strongly positively associated and a certain relation for the p derivative of events stated in Theorem 3.12 in [98] which is still true since the p-dependence is the same as for the random cluster model. □

Remark 5.5.9. *For $d = 2$ the self dual point defined in* (5.1.5) *and the critical point agree: $p_c = p_{\mathrm{sd}}$. This can be seen based on Theorem 1.5 and the arguments used in the proof of Theorem 1.4 in [78] for the random cluster model .*

In the random cluster model the most interesting phenomena happen for $p = p_c$ and the subcritical and supercritical phase are much simpler to understand (in particular in $d = 2$). Due to the differences explained in Remark 5.3.2 those questions seem to be harder for our random conductance model. Nevertheless we conjecture the following stronger version of Theorem 5.5.1 and Theorem 5.5.6

Conjecture 5.5.10. For $p \neq p_c$ there is a unique Gibbs measure.

Note that the sharpness result Theorem 5.5.8 shows that the probability of subcritical q-clusters to be large is exponentially small. Nevertheless it is not clear how this can be used to show uniqueness of the Gibbs measure in our setting. The behaviour at p_c is also very interesting. A phase transition is called continuous if $\mu_{p_c}^1(0 \leftrightarrow \infty) = 0$ and otherwise it is discontinuous. In the random cluster model the uniqueness of the Gibbs measure at p_c is equivalent to a continuous phase transition. We do not know whether the same is true for the random conductance model considered here. We state a second conjecture about the nature of the phase transition in terms of Gibbs measures.

Conjecture 5.5.11. There is a q_0 such that for $q > q_0$ there is non-uniqueness of Gibbs measures $\bar{\mu}_{p_c,q}^1 \neq \bar{\mu}_{p_c,q}^0$ at the critical point while for $q < q_0$ the Gibbs measures agree $\bar{\mu}_{p_c,q}^1 = \bar{\mu}_{p_c,q}^0$.

A partial result in the direction of this conjecture is Theorem 5.5.4 that states non-uniqueness for large q in dimension $d = 2$ and Theorem 5.5.6 that shows uniqueness for q close to 1 and $d \geq 4$.

5.6 Duality and coexistence of Gibbs measures

In this section we are going to prove that $\mu_{p_{\mathrm{sd}}}^0 \neq \mu_{p_{\mathrm{sd}}}^1$ for large q which implies the non-uniqueness of gradient Gibbs measures stated in Theorem 5.2.4. This is a new proof for the result in [32]. They consider conductances q_1, q_2 with $q_1 q_2 = 1$ which makes the presentation slightly more symmetric.

In contrast to their work we do not rely on reflection positivity but instead we exploit the planar duality that is already used in [32] to find the location of the phase transition. Therefore it is not possible to extend the argument given here to $d \geq 3$ while the proof using reflection positivity is in principle independent of the dimension (see also Section 5.7). In addition to planar duality we rely on the properties proved in Section 5.4, in particular on the Kirchhoff formula. Similar arguments were developed in the context of the random cluster model and we refer to [98, Section 6 and 7].

We proceed now by stating the duality property in our setting. For a planar graph $G = (V, E)$ we denote its dual graph by $G^* = (V^*, E^*)$. The dual graph has the faces of G as vertices and the vertices of G as faces and each edge has a corresponding dual edge. For a formal definition of the dual of a graph and the necessary background we refer to the literature, e.g., [144].

For any configuration $\kappa : E \to \{1, q\}$ we define its dual configuration $\kappa^* \in \{1, q\}^{E^*}$ by $\kappa^*_{e^*} = 1 + q - \kappa_e$ where $e^* \in E^*$ denotes the dual edge of an edge $e \in E$. More generally we denote for $E_1 \subset E$ by $E_1^* = \{e^* : e \in E_1\}$ the dual edges of the edges E_1. We also introduce the notation $E_1^{\mathrm{d}} = \{e^* \in E^* : e \notin E_1\} = (E_1^{\mathrm{c}})^*$ for $E_1 \subset E$ for the dual set of an edge subset. Note that E_1 is acyclic if and only if E_1^{d} is spanning, i.e., every two points $x^*, y^* \in V^*$ are connected by a path in E_1^{d}. In particular, $t \subset E$ is a spanning tree in G if and only if t^{d} is a spanning tree in G^* and the map $t \mapsto t^d$ is an involution and in particular bijective from $\mathrm{ST}(G)$ to $\mathrm{ST}(G^*)$.

Recall that $h(\kappa, t) = |\{e \in t : \kappa_e = q\}|$ denotes the number of q-bonds in the set $t \subset \mathbf{E}(G)$ of κ and the similar definition of $s(\kappa, t)$ for the number of soft 1-bonds in t. The definitions imply that

$$h(\kappa) = s(\kappa^*), \qquad s(\kappa) = h(\kappa^*), \tag{5.6.1}$$

$$h(\kappa, t) = s(\kappa^*) - s(\kappa^*, t^{\mathrm{d}}), \qquad s(\kappa, t) = h(\kappa^*) - h(\kappa^*, t^{\mathrm{d}}). \tag{5.6.2}$$

The last two identities follow from the observation that $s(\kappa^*, t^{\mathrm{d}}) = h(\kappa, E \setminus t)$ and similarly for s and h interchanged. We calculate the distribution of κ^* if κ is distributed according to $\mathbb{P}^{G,p}$

$$\mathbb{P}(\kappa^*) = \mathbb{P}(\kappa) \propto \frac{p^{h(\kappa)}(1-p)^{s(\kappa)}}{\sqrt{\sum_{t \in \mathrm{ST}(G)} q^{h(\kappa,t)}}} = \frac{p^{s(\kappa^*)}(1-p)^{h(\kappa^*)}}{\sqrt{\sum_{t^{\mathrm{d}} \in \mathrm{ST}(G^*)} q^{s(\kappa^*) - s(\kappa^*, t^{\mathrm{d}})}}} = \frac{\left(\frac{p}{\sqrt{q}}\right)^{s(\kappa^*)} (1-p)^{h(\kappa^*)}}{\sqrt{\sum_{t^{\mathrm{d}} \in \mathrm{ST}(G^*)} q^{h(\kappa^*, t^{\mathrm{d}}) - |t^{\mathrm{d}}|}}}. \tag{5.6.3}$$

This implies that if κ is distributed according to $\mathbb{P}^{G,p}$ the dual configuration κ^* is distributed according to $\mathbb{P}^{G^*,p^*}$ where $q^* = q$ and

$$\frac{p^*}{1-p^*} = \frac{(1-p)}{p/\sqrt{q}}. \tag{5.6.4}$$

Note that the self dual point p_{sd} defined by $p^*_{\mathrm{sd}} = p_{\mathrm{sd}}$ is given by the solution of

$$\frac{p^4}{(1-p)^4} = q. \tag{5.6.5}$$

We will now restrict our attention to $\mathbb{Z}^2$. Let us mention that detailed proofs of the topological statements we use can be found in [112].

We can identify the dual of the graph $(\mathbb{Z}^2, \mathbf{E}(\mathbb{Z}^2))$, which will be denoted by $((\mathbb{Z}^2)^*, \mathbf{E}(\mathbb{Z}^2)^*)$, with $\mathbb{Z}^2$ shifted by the vector $w = (\frac{1}{2}, \frac{1}{2})$. We also consider the set of directed bonds $\vec{\mathbf{E}}(\mathbb{Z}^2)$ and $\vec{\mathbf{E}}(\mathbb{Z}^2)^*$. For a directed bond $\vec{e} = (x, y) \in \mathbf{E}(\mathbb{Z}^2)$ we define its dual bond as the directed bond $\vec{e}^* = (\frac{1}{2}(x + y + (x-y)^\perp), \frac{1}{2}(x + y + (y-x)^\perp)$ where $\perp$ denotes counter-clockwise rotation by $90°$, i.e., the linear map that satisfies $e_1^\perp = e_2$, $e_2^\perp = -e_1$. In other words, the dual of a directed bond $\vec{e}$ is the bond whose orientation is rotated by $90°$ counter-clockwise and crosses $\vec{e}$.

Every point $x \in \mathbb{Z}^2$ determines a plaquette with corners $z_1, z_2, z_3, z_4 \in (\mathbb{Z}^2)^*$ where z_i are the four nearest neighbours of x in $(\mathbb{Z}^2)^*$ and the plaquette has faces $e_1^*, e_2^*, e_3^*, e_4^* \in \mathbf{E}(\mathbb{Z}^2)^*$ where e_i^* are the dual bonds of the four bonds e_i that are incident to x. Vice versa every point $z \in (\mathbb{Z}^2)^*$ determines a plaquette in $\mathbb{Z}^2$. We write $\mathbf{P}(\mathbb{Z}^2)$ for the set of plaquettes of $\mathbb{Z}^2$.

For a bond $e = \{x, y\}$ we define the shifted dual bond $e + w = \{x + w, y + w\}$. Similarly, we define $E + w = \{e + w \in \mathbf{E}(\mathbb{Z}^2)^* : e \in E\}$ for a set $E \subset \mathbf{E}(\mathbb{Z}^2)$. For a subgraph $G \subset \mathbb{Z}^2$ we

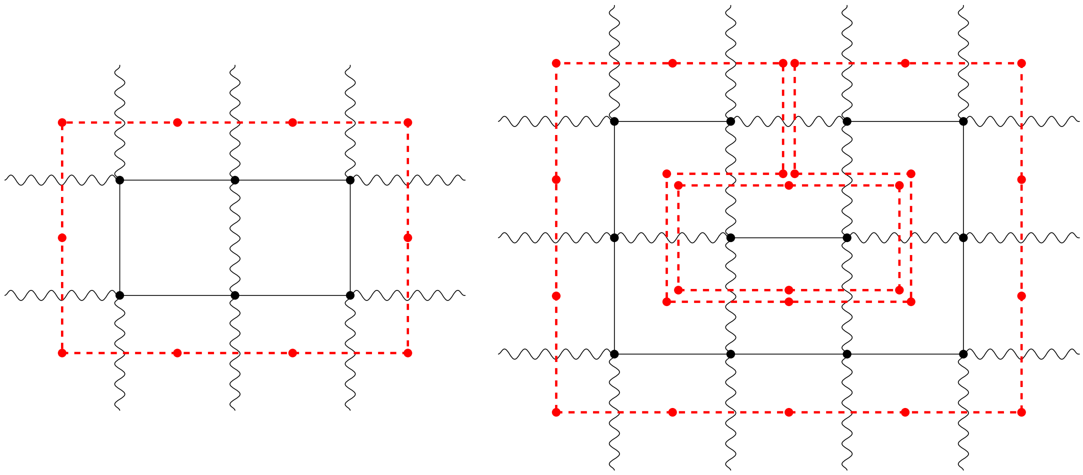

Figure 5.1: Examples of q-contours. In the second example there are two nested q-contours. Curly bonds indicate soft bonds with $\kappa_e = 1$ and straight bonds indicate hard bonds with $\kappa_e = q$. In red the dual bonds of the contours are shown. The horizontal curly bond in the top middle connects two point in $\operatorname{int}\gamma$.

denote by $\mathbf{P}(G) = \{P \in \mathbf{P}(\mathbb{Z}^2) : \text{all faces of } P \text{ are in } \mathbf{E}(G)\}$ the plaquettes of G. A subgraph $G \subset \mathbb{Z}^2$ is called simply connected if the union of all vertices $v \in \mathbf{V}(G)$, all edges $\{x, y\} \in \mathbf{E}(G)$ which are identified with the line segment from x to y in $\mathbb{R}^2$ and all plaquettes $\mathbf{P}(G)$ is a simply connected subset of $\mathbb{R}^2$.

An important tool in the analysis of planar models from statistical mechanics is the use of contours. Let us provide a notion of contours that is useful for our purposes. Our definition is slightly more complicated than the definition of contours for the random cluster model. We consider closed paths $\gamma = (x_1^*, \ldots, x_n^*, x_1^*)$ with $x_i^* \in (\mathbb{Z}^2)^*$ (not necessarily all distinct) along pairwise distinct directed dual bonds $\vec{b}_1^* = (x_1^*, x_2^*), \ldots, \vec{b}_n^* = (x_n^*, x_1^*)$. We denote the vertices in the contour by $\mathbf{V}(\gamma)^* = \{x_i^* : 1 \le i \le n\}$ and the bonds by $\vec{\mathbf{E}}(\gamma)^* = \{\vec{b}_i^* : 1 \le i \le n\}$. Similarly we write $\vec{\mathbf{E}}(\gamma) = \{\vec{b}_i : 1 \le i \le n\}$ for the corresponding primal bonds. We also consider the underlying sets of undirected bonds $\mathbf{E}(\gamma)$ and $\mathbf{E}(\gamma)^*$. Finally, we denote the heads and tails of $\vec{b}_i$ by y_i and z_i, i.e., $\vec{b}_i = (z_i, y_i)$.

Definition 5.6.1. *A contour γ is a closed path in the dual lattice without self-crossings in the sense that there is a bounded connected component $\operatorname{int}(\gamma)$ of the graph $(\mathbb{Z}^2, \mathbf{E}(\mathbb{Z}^2) \setminus \mathbf{E}(\gamma))$ such that $\partial(\operatorname{int}(\gamma)) = \{z_i : 1 \le i \le n\}$. We denote the union of the remaining connected components by $\operatorname{ext}(\gamma)$ and we define the length $|\gamma|$ of the contour as the number of (directed) bonds it contains, i.e., $|\gamma| = |\vec{\mathbf{E}}(\gamma)| = n$.*

Note that $\operatorname{ext}(\gamma)$ is not necessarily connected and that $\{x, y\} \in \mathbf{E}(\gamma)$ if $x \in \operatorname{int}(\gamma)$ and $y \in \operatorname{ext}(\gamma)$ (see Figure 5.1).

Contours are a suitable notion to define interfaces between hard and soft bonds.

Definition 5.6.2. *A contour γ is a q-contour for κ if the following two conditions hold. First, the primal bonds $b \in \mathbf{E}(\gamma)$ are soft, i.e., $\kappa_b = 1$. Moreover, for every plaquette with center $x^* \in \mathbf{V}(\gamma)^*$ all its faces b such that $b \in \mathbf{E}(\operatorname{int}(\gamma))$ are hard, i.e., satisfy $\kappa_b = q$.*

Our goal is to show that q-contours are unlikely for large values of q and $p \le p_{\mathrm{sd}}$. We now fix a contour γ and introduce some useful notation and helpful observations for the proof of the following theorem. We use the shorthand $G_i = \mathrm{int}(\gamma)$ and $E_i = \mathbf{E}(\mathrm{int}(\gamma))$. We observe that G_i is simply connected because γ is connected and without self-crossings. Therefore the faces of G_i consist of plaquettes in $\mathbb{Z}^2$ and one infinite face. We also consider the graph G with edges $E = E_i \cup \mathbf{E}(\gamma)$ and endpoints of edges as vertices. Let $\bar{1} \in \{1, q\}^E$ denote the configuration given by $\bar{1}_e = 1$ for all $e \in E$. We write $G^w = G/\partial G = G/\mathrm{ext}(\gamma)$ for the graph G with wired boundary conditions. Moreover we introduce the graph H^* with edges E_i^* and their endpoints as vertices. We claim that $H^*/\partial H^*$ agrees with the graph theoretic dual of G_i. To show this we need to prove that we identify all vertices that lie in the same face of G_i. First we note that every point in $\mathring{H}^* = H^* \setminus \partial H^*$ determines a plaquette in $\mathbf{P}(G_i)$ and this is a bijection. Then it remains to show that all vertices in ∂H^* lie in the infinite face of G_i. This follows from the observation

$$\partial H^* = \mathbf{V}(\gamma)^* \cap \mathbf{V}(H^*). \tag{5.6.6}$$

To show the observation we note that if $x^* \in \partial H^*$ then there are edges $e_1^* \notin \mathbf{E}(H^*)$ and $e_2^* \in \mathbf{E}(H^*)$ incident to x^*. This implies that there is a face $e' = \{z_1, z_2\}$ of the plaquette with center x^* such that $z_1 \in \mathbf{V}(G_i)$ but $e' \notin \mathbf{E}(G_i)$. Then $e \in \mathbf{E}(\gamma)$ and therefore $x^* \in \mathbf{V}(\gamma)^*$ because x^* is an endpoint of $e^* \in \mathbf{E}(\gamma)^*$. This ends the proof of the inclusion '$\subset$'. Now we note that if $x^* \in \mathbf{V}(H^*) \cap \mathbf{V}(\gamma)^*$ there is an edge $e^* \in \mathbf{E}(\gamma)^*$ incident to x^* which is not contained in $\mathbf{E}(H^*)$ and therefore $x^* \in \partial H^*$.

Finally we remark that if γ is a q-contour for κ then

$$\kappa^*_{e^*} = 1 + q - \kappa_e = 1 \qquad \text{if } e^* \in \mathbf{E}(H^*) \text{ is incident to } \partial H^*. \tag{5.6.7}$$

Indeed, we argued above that if $e^* \in \mathbf{E}(H^*)$ is incident to ∂H^* then $x^* \in \mathbf{V}(\gamma)^*$. Thus $e \in E_i$ is a face of the plaquette with center x^* so that the definition of q-contours implies that $\kappa_e = q$.

Theorem 5.6.3. *Let γ be a contour. The probability that γ is a q-contour under the measure $\mathbb{P}^{G^w, E_i, \bar{1}}$ for $p = p_{\mathrm{sd}}$ is bounded by*

$$\mathbb{P}^{G^w, E_i, \bar{1}}(\gamma \textit{ is a } q\textit{-contour}) \le \left(\frac{4}{q^{\frac{1}{8}}}\right)^{|\gamma|} q^{\frac{1}{2}}. \tag{5.6.8}$$

Remark 5.6.4. *The general idea of the proof is the same as when proving similar estimates for the Ising model. One tries to find a map from configurations where the contour is present to configurations where this is not the case and then estimates the corresponding probabilities. The more similar argument for the random cluster model can be found, e.g., in Theorem 6.35 in [98]. For an illustrated version see [77].*

Proof. We denote the set of all $\kappa \in \{1, q\}^E$ such that γ is a q-contour for κ by Ω_γ.

Step 1. We define a map $\Phi : \Omega_\gamma \to \{1, q\}^E$ with $\Phi(\kappa) = \kappa^\#$ as follows. Recall the definition of the dual configuration κ^* on $E^* \subset \mathbf{E}(\mathbb{Z}^2)^*$ and define for $e \in E$

$$\kappa_e^\# = \begin{cases} \kappa^*_{e-w} & \text{if } e - w \in {E_i}^*, \\ 1 & \text{otherwise.} \end{cases} \tag{5.6.9}$$

We claim that

$$\kappa_e^\# = 1 \qquad \text{if } e \in E \setminus E_i. \tag{5.6.10}$$

By definition of $\kappa^\#$, we only need to consider the case $e - w \in {E_i}^* = \mathbf{E}(H^*)$. We will show a slightly more general statement. Let us introduce the set $\tilde{E} = \mathbf{E}(H^*) + w = E_i^* + w \subset E$

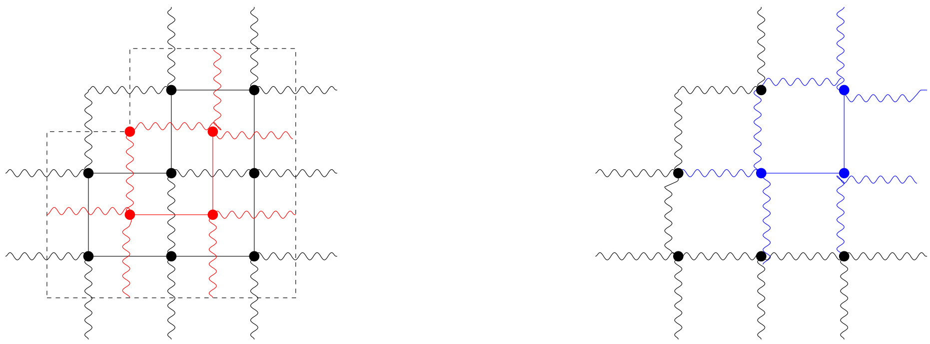

Figure 5.2: (left) The dashed line indicates a q-contour for the depicted configuration. In red the dual configuration on E^* for the edges E is shown. (right) The configuration $\kappa^\#$ for κ depicted on the left. The blue edges are the shifted dual edges forming the set $\tilde{E}$.

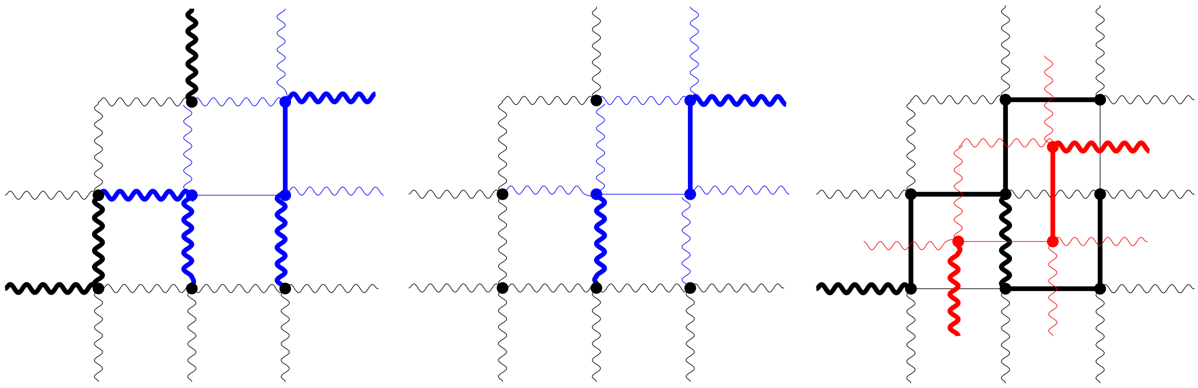

Figure 5.3: (left) An example of a wired tree $t^\#$ for $\kappa^\#$. (center) The subtree $\tilde{t}$. (right) The shifted tree $\tilde{t} - w$ and the dual tree $\Psi(t^\#)$.

and the graph $\tilde{G}$ consisting of the edges $\tilde{E}$ and their endpoints as vertices. See Figure 5.2 for an illustration of this construction. We remark that $\tilde{G}$ agrees with H^* shifted by w, which we denote by $\tilde{G} = H^* + w$. Equation (5.6.7) implies that

$$\kappa^\#_e = \kappa^*_{e-w} = 1 \qquad \text{for } e \in \tilde{G} \text{ incident to } \partial\tilde{G} \tag{5.6.11}$$

because then $e - w \in \mathbf{E}(H^*)$ is incident to ∂H^*. It remains to show that all edges $e \in E \cap \tilde{E} \setminus E_i$ are incident to $\partial\tilde{G}$. From $e \in E \setminus E_i$ we conclude that $e \in \mathbf{E}(\gamma)$. The edge $e - w$ has a common endpoint with $e^* \in \mathbf{E}(\gamma)^*$ and is therefore incident to $\mathbf{V}(\gamma)^*$ in this case. Using the observation (5.6.6) this implies that $e - w \in \mathbf{E}(H^*)$ is incident to ∂H^*.

Our goal is to compare the probabilities of $\mathbb{P}^{G^w,E_i,\bar{1}}(\kappa_{E_i})$ and $\mathbb{P}^{G^w,E_i,\bar{1}}(\kappa^\#_{E_i})$. To achieve this we use a strategy similar to the proof of Lemma 5.5.3.

Step 2. We define a map $\Psi : \mathrm{ST}(G^w) \to \mathrm{ST}(G^w)$ with $\Psi(t^\#) = t$ in the following steps

1. We choose deterministically a subset $\tilde{t} \subset t^\#\restriction_{\tilde{E}}$ such that $\tilde{t}$ is a spanning tree on $\tilde{G}/\partial\tilde{G}$ and all edges in $t^\#\restriction_{\tilde{E}} \setminus \tilde{t}$ are incident to $\partial\tilde{G}$.
2. We set $\Psi(t^\#)\restriction_{E_i} = \{e \in E_i \,:\, e^* \notin \tilde{t} - w\} = (\tilde{t} - w)^d$ (as a subset of E_i^*).
3. We consider a fixed $b \in \mathbf{E}(\gamma)$ that is incident to $\mathrm{int}(\gamma)$ and $\mathrm{ext}(\gamma)$ and we define $t = \Psi(t^\#) = \Psi(t^\#)\restriction_{E_i} \cup b$

See Figure 5.3 for an illustration of the construction. We have to show that this construction is possible, in particular that $t \in \mathrm{ST}(G^w)$.

We start with the first step. The relation $\tilde{G} \subset G$ implies

$$\partial G \cap \tilde{G} \subset \partial \tilde{G}. \tag{5.6.12}$$

Hence $\tilde{G}/\partial\tilde{G}$ agrees with $(G/\partial G)/(\tilde{G}^{\mathrm{c}} \cup \partial\tilde{G})$ up to self loops. This implies that $t^{\#}\restriction_{\tilde{E}}$ is spanning in $\tilde{G}/\partial\tilde{G}$ if $t^{\#} \in \mathrm{ST}(G^w)$. We consider the subset $t' \subset t^{\#}\restriction_{\tilde{E}}$ consisting of all edges $e \in t^{\#}\restriction_{\tilde{E}}$ that are not incident to $\partial\tilde{G}$. The set t' contains no cycles because $t^{\#} \in \mathrm{ST}(G^w)$ and no edge in t' is incident to ∂G by (5.6.12). Therefore we can select a spanning tree $\tilde{t}$ in $\tilde{G}/\partial\tilde{G}$ with $t' \subset \tilde{t} \subset t^{\#}\restriction_{\tilde{E}}$ deterministically, e.g., using Kruskal's algorithm.

We now argue that the second and third step yield a spanning tree in G^w. Clearly it is sufficient to show that $\Psi(t^{\#})\restriction_{E_i} \in \mathrm{ST}(G_i)$. We note that the relation between $\tilde{G}$ and H^* implies that $\tilde{t} - w$ is a spanning tree on $H^*/\partial H^*$. As shown before the theorem $H^*/\partial H^*$ agrees with the dual of G_i and thus $(\tilde{t} - w)^d \in \mathrm{ST}(G_i)$.

Step 3. The next step is to consider $\kappa^{\#} = \Phi(\kappa)$ and $t = \Psi(t^{\#})$ and compare the weights $w(\kappa^{\#}, t^{\#})$ and $w(\kappa, t)$. First we argue that

$$w(\kappa^{\#}, t^{\#}) = w(\kappa^{\#}, \tilde{t}). \tag{5.6.13}$$

Since $\tilde{t} \subset t^{\#}$ it is sufficient to show that $t^{\#} \setminus \tilde{t}$ contains only edges e such that $\kappa^{\#}_e = 1$. Indeed, let e be an edge in $t^{\#} \setminus \tilde{t}$. For $e \notin \tilde{E}$ we have $\kappa^{\#}_e = 1$ by definition. Let us now consider

$$e \in \tilde{E} \cap (t^{\#} \setminus \tilde{t}). \tag{5.6.14}$$

By construction of $\tilde{t}$ the edge e is incident to a vertex $v \in \partial\tilde{G}$. This implies that $e - w \in \mathbf{E}(H^*)$ is incident to $v - w \in \partial H^* \subset \mathbf{V}(\gamma)^*$. Using (5.6.7) we conclude that

$$\kappa^{\#}_e = \kappa^*_{e-w} = 1. \tag{5.6.15}$$

For the trees $\tilde{t}$ and $\Psi(t^{\#})\restriction_E$ we can apply the usual duality relations stated before. Using (5.6.2) and as before $\kappa^{\#} = \Phi(\kappa)$ and $t = \Psi(t^{\#})$ we obtain

$$h(\kappa^{\#}, t^{\#}) = h(\kappa^{\#}, \tilde{t}) = h(\kappa^*, \tilde{t} - w) = s(\kappa, E_i) - s(\kappa, E_i \cap t). \tag{5.6.16}$$

We compute

$$\frac{w(\kappa^{\#}, t^{\#})}{w(\kappa, t)} = \frac{q^{h(\kappa^{\#}, t^{\#})}}{q^{h(\kappa, t)}} = q^{s(\kappa, E_i) - s(\kappa, t\cap E_i) - h(\kappa, t\cap E_i)} = q^{s(\kappa, E_i) - |t\cap E_i|} = q^{s(\kappa, E_i) - |\mathbf{V}(G_i)| + 1}. \tag{5.6.17}$$

In the last step we used that $t \cap E_i$ is a free spanning tree on G_i and therefore has $|\mathbf{V}(G_i)| - 1$ edges.

Step 4. We bound the number of preimages of a tree t under Ψ. Note that Ψ factorizes into two maps $t^{\#} \to \tilde{t} \to t$. The second map is injective since we only pass to the dual tree which is an injective map and we add one additional edge. For the first map we observe that we only delete edges e incident to $\partial\tilde{G}$. However, for $x \in \partial\tilde{G}$ the point $x - w \in \partial H^*$ is contained in the contour by (5.6.6). Therefore there are at most $4|\gamma|$ such edges. We conclude that

$$|\{t^{\#} \in \mathrm{ST}(G^w) : \Psi(t^{\#}) = t\}| \leq 2^{4|\gamma|} \tag{5.6.18}$$

for every $t \in \mathrm{ST}(G^w)$. The displays (5.6.17) and (5.6.18) imply

$$\sum_{t^\# \in \mathrm{ST}(G^w)} w(\kappa^\#, t^\#) \le \sum_{t^\# \in \mathrm{ST}(G^w)} w(\kappa, \Psi(t^\#)) q^{s(\kappa,E_i)-|\mathbf{V}(G_i)|+1} \le q^{s(\kappa,E_i)-|\mathbf{V}(G_i)|+1} 2^{4|\gamma|} \sum_{t \in \mathrm{ST}(G^w)} w(\kappa, t). \tag{5.6.19}$$

Step 5. We can now estimate the probabilities of the patterns κ and $\kappa^\# = \Psi(\kappa)$ under $\mathbb{P}^{G^w,E_i,\bar{1}}$ using (5.6.19) and $\kappa^\#_e = 1 = \bar{1}_e$ for $e \in E \setminus E_i$

$$\begin{aligned}\frac{\mathbb{P}^{G^w,E_i,\bar{1}}_{p_{\mathrm{sd}}}(\kappa_{E_i})}{\mathbb{P}^{G^w,E_i,\bar{1}}_{p_{\mathrm{sd}}}(\kappa^\#_{E_i})} &= \frac{\mathbb{P}^{G^w}_{p_{\mathrm{sd}}}(\kappa)}{\mathbb{P}^{G^w}_{p_{\mathrm{sd}}}(\kappa^\#)} = \frac{p_{\mathrm{sd}}^{h(\kappa,E_i)}(1-p_{\mathrm{sd}})^{s(\kappa,E_i)}}{p_{\mathrm{sd}}^{h(\kappa^\#,E_i)}(1-p_{\mathrm{sd}})^{s(\kappa^\#,E_i)}} \sqrt{\frac{\sum_{t^\# \in \mathrm{ST}(G^w)} w(\kappa^\#, t^\#)}{\sum_{t \in \mathrm{ST}(G^w)} w(\kappa, t)}} \\ &\le \frac{\left(\frac{p_{\mathrm{sd}}}{1-p_{\mathrm{sd}}}\right)^{h(\kappa,E_i)}}{\left(\frac{p_{\mathrm{sd}}}{1-p_{\mathrm{sd}}}\right)^{h(\kappa^\#,E_i)}} q^{\frac{s(\kappa,E_i)-|\mathbf{V}(G_i)|+1}{2}} 2^{2|\gamma|} = 2^{2|\gamma|} q^{\frac{1}{4}(h(\kappa,E_i)-h(\kappa^\#,E_i))} q^{\frac{s(\kappa,E_i)-|\mathbf{V}(G)|+1}{2}}\end{aligned} \tag{5.6.20}$$

where we used equation (5.6.5) of p_{sd} in the last step. The definition of $\kappa^\#$ implies that $h(\kappa^\#, E_i) = h(\kappa^\#, \tilde{E}) = s(\kappa, E_i)$ and we get

$$\frac{\mathbb{P}^{G^w,E_i,\bar{1}}(\kappa_{E_i})}{\mathbb{P}^{G^w,E_i,\bar{1}}(\kappa^\#_{E_i})} \le 2^{2|\gamma|} q^{\frac{1}{4}(h(\kappa,E_i)+s(\kappa,E_i)-2|\mathbf{V}(G_i)|+2)} = 2^{2|\gamma|} q^{\frac{1}{4}(|E_i|-2|\mathbf{V}(G_i)|+2)} \tag{5.6.21}$$

Now we observe that $4|\mathbf{V}(G_i)| - 2|\mathbf{E}(G_i)| = |\vec{\mathbf{E}}(\gamma)\}| = |\gamma|$. We end up with the estimate

$$\frac{\mathbb{P}^{G^w,E_i,\bar{1}}(\kappa_{E_i})}{\mathbb{P}^{G^w,E_i,\bar{1}}(\kappa^\#_{E_i})} \le 2^{2|\gamma|} q^{-\frac{1}{8}|\gamma|} q^{\frac{1}{2}} = \left(4q^{-\frac{1}{8}}\right)^{|\gamma|} q^{\frac{1}{2}}. \tag{5.6.22}$$

Conclusion. Note that the map Φ is injective, hence

$$\mathbb{P}^{G^w,E_i,\bar{1}}(\gamma \text{ is a } q\text{-contour}) \le \frac{\sum_{\kappa \in \Omega_\gamma} \mathbb{P}^{G^w,E_i,\bar{1}}(\kappa_{E_i})}{\sum_{\kappa \in \Omega_\gamma} \mathbb{P}^{G^w,E_i,\bar{1}}(\Phi(\kappa)_{E_i})} \le \left(4q^{-\frac{1}{8}}\right)^{|\gamma|} q^{\frac{1}{2}}. \tag{5.6.23}$$

□

Using correlation inequalities we can derive the following stronger version of the previous theorem. For a simply connected subgraph $H \subset \mathbb{Z}^2$ we say that γ is contained in H if all faces of plaquettes with center x^* for $x^* \in \mathbf{V}(\gamma)^*$ are contained in $\mathbf{E}(H)$.

Corollary 5.6.5. *For any $p \le p_{\mathrm{sd}}$, any simply connected subgraph $H \subset \mathbb{Z}^2$, and a contour γ that is contained in H the probability that γ is a q-contour can be estimated by*

$$\mathbb{P}^H_p(\gamma \textit{ is a } q\textit{-contour}) \le \left(4q^{-\frac{1}{8}}\right)^{|\gamma|} q^{\frac{1}{2}}. \tag{5.6.24}$$

Proof. We estimate

$$\begin{aligned}\mathbb{P}^H_p(\gamma \text{ is a } q\text{-contour}) &= \mathbb{P}^H_p(\gamma \text{ is a } q\text{-contour}, \kappa_b = 1 \text{ for } b \in \mathbf{E}(\gamma)\ ) \\ &\le \mathbb{P}^H_p(\gamma \text{ is a } q\text{-contour} \mid \kappa_b = 1 \text{ for } b \in \mathbf{E}(\gamma)\ ) \\ &= \mathbb{P}^{H,\mathbf{E}(H)\setminus\mathbf{E}(\gamma),\bar{1}}_p(\gamma \text{ is } q\text{-contour}).\end{aligned} \tag{5.6.25}$$

For the measure $\mathbb{P}^{H,\mathbf{E}(H)\setminus\mathbf{V}(\gamma),\bar{1}}$ the bonds crossing the contour are fixed to the correct value. Hence the event that γ is a q-contour for κ is increasing under this event, such that the stochastic domination results proved in Lemma 5.4.9 and Corollary 5.4.12 imply that

$$\mathbb{P}_p^{H,\mathbf{E}(H)\setminus\mathbf{V}(\gamma),\bar{1}}(\gamma \text{ is a } q\text{-contour}) \leq \mathbb{P}_{p_{\text{sd}}}^{G^w,\mathbf{E}(G),\bar{1}}(\gamma \text{ is } q\text{-contour}) \tag{5.6.26}$$

where G denotes the graph corresponding to γ as introduced above Theorem 5.6.3. Theorem 5.6.3 implies the claim. □

We can now give a new proof for the coexistence result stated in Theorem 5.2.4.

Proof of Theorem 5.5.4. First we note that the duality between free and wired boundary conditions in finite volume implies that $\mu^0_{p_{\text{sd}}}$ and $\mu^1_{p_{\text{sd}}}$ are dual to each other in the sense that if $\kappa \sim \mu^0_{p_{\text{sd}}}$ then $\kappa^* \sim \mu^1_{p_{\text{sd}}}$ (on $(\mathbb{Z}^2)^*$)). The proof is the same as for the random cluster model, see, e.g., [98, Chapter 6]. Hence, it is sufficient to show that $\mu^0_{p_{\text{sd}}}(\kappa_e = q) < 1/2$ because then we can conclude that

$$\bar{\mu}^1_{p_{\text{sd}}}(\kappa_e = q) = \bar{\mu}^0_{p_{\text{sd}}}(\kappa_e = 1) > 1/2 \tag{5.6.27}$$

whence $\bar{\mu}^1_{p_{\text{sd}}} \neq \bar{\mu}^0_{p_{\text{sd}}}$.

Note that if $\kappa_e = q$ and there is any contour γ such that $e \in \mathbf{E}(\text{int}(\gamma))$ and $\kappa_b = 1$ for $b \in \mathbf{E}(\gamma)$ then there is a q-contour surrounding e. We can thus estimate for $e \in E_n$

$$\mathbb{P}^{\Lambda_{n+1},E_n,\bar{1}}(\kappa_e = q) \leq \mathbb{P}^{\Lambda_{n+1},E_n,\bar{1}}(\text{there is a } q\text{-contour around } e) \tag{5.6.28}$$

where as before $\bar{1}_e = 1$ for all e. The shortest contour γ that surrounds the edge e has length 6 so the bound in Corollary 5.6.5 implies that $\mathbb{P}^{\Lambda_{n+1},E_n,\bar{1}}(\gamma \text{ is a } q\text{-contour}) \leq C/q^{\frac{1}{4}}$ for any γ surrounding e. Using Corollary 5.4.12 we can compare boundary conditions to obtain the relation $\bar{\mu}^0_n \precsim \mathbb{P}^{\Lambda_{n+1},E_n,\bar{1}}$. This and a standard Peierls argument imply for q sufficiently large

$$\bar{\mu}^0_{n,p_{\text{sd}}}(\kappa_e = q) \leq \mathbb{P}^{\Lambda_{n+1},E_n,\bar{1}}(\kappa_e = q) \leq \frac{C}{q^{\frac{1}{4}}} \leq \frac{1}{4}. \tag{5.6.29}$$

Taking the limit $n \to \infty$ we obtain $\bar{\mu}^0_{p_{\text{sd}}}(\kappa_e = q) \leq \frac{1}{4}$. □

5.7 Further directions

In this section we give a brief overview about some further open questions and possible directions that might be of interested.

Spin wave calculation. The most important technical ingredient in the proof of the phase transition in [32] are spin wave calculations for the partition functions of certain periodic configurations (see Section 3.2 and Theorem 3.3 there). Here we show how the Kirchhoff formula (5.4.4) for the determinant of the graph Laplacian can be used to simplify those calculations substantially and we sketch how this makes the spin wave calculations in higher dimensions feasible. This basically extends the proof of the phase transition to dimension $d \geq 3$ but a detailed account of the Peierls argument and a proof of reflection positivity would still require some work.

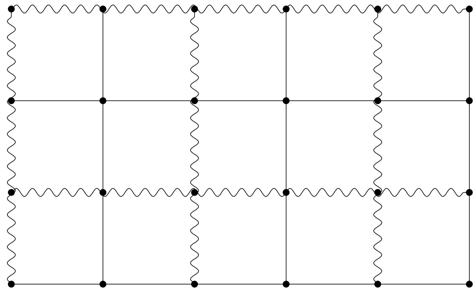

Figure 5.4: A small part of the pattern κ^m.

Let $T_L = (\mathbb{Z}/(L\mathbb{Z}))^d$ with L even denote the d-dimensional torus with side-length L. For $\kappa \in \{1, q\}^{\mathbf{E}(T_L)}$ we introduce the free energy

$$F_L(\kappa) = -L^{-d} \ln \left(\frac{p^{h(\kappa)}(1-p)^{s(\kappa)}}{(2\pi)^{\frac{1}{2}(L^d-1)}} \int e^{-\frac{1}{2}(\varphi, \tilde{\Delta}_\kappa \varphi)} \lambda(\mathrm{d}\varphi) \right) \tag{5.7.1}$$

where λ denotes the Lebesgue measure on $\mathbb{R}^{T_L}$ with one point pinned to 0. Using (5.3.6), (5.4.4), and Gaussian calculus we obtain

$$F_L(\kappa) = -L^{-d}\Big(h(\kappa)\ln(p) + s(\kappa)\ln(1-p) - \tfrac{1}{2}\ln\Big(\sum_{t \in \mathrm{ST}(T_L)} w(\kappa, t)\Big)\Big) \tag{5.7.2}$$

The key result of the spin wave calculations in [32] is that for large q the free energy of one of the homogeneous patterns is much lower than of certain mixed periodic pattern uniformly in $p \in [0, 1]$. Those mixed periodic patterns are obtained by considering any configuration on the edges of the unit cube $Q_d = \{0, 1\}^d$ and then we extend this configuration by repeated reflection along planes given by $\{x \in \mathbb{R}^d : e_i \cdot x = n\}$ for some $n \in \mathbb{Z}$ and $1 \le i \le d$. Since we assumed that L is even this defines a configuration in $\{1, q\}^{\mathbf{E}(T_L)}$.

To clarify the setting we consider a concrete example in dimension $d = 2$. We consider the patterns κ^1 with $\kappa^1_e = 1$ for every edge, κ^q with $\kappa^q_e = q$ for every edge and the mixed pattern κ^m sketched in Figure 5.4 which can be formally defined by $\kappa_e = 1$ iff and only if one of the endpoints of e has two even coordinates (this is well defined for L even). Since the number of trees on T_L is clearly bounded by 2^{2L^2} we obtain

$$\begin{aligned} F_L(\kappa^q) &\le -2\ln(p) + \frac{L^2-1}{2L^2}\ln(q) + \ln(2), \\ F_L(\kappa^1) &\le -2\ln(1-p) + \ln(2). \end{aligned} \tag{5.7.3}$$

To estimate the free energy of κ^m from below we construct a tree by choosing a spanning forest on the subgraph induced by the q-edges which can then be extended to a tree on T_L. For κ^m this construction yields a tree t containing $\frac{3}{4}L^2 - 1$ edges $e \in t$ with $\kappa_e = q$. Hence,

$$F_L(\kappa^m) \ge -\ln(p) - \ln(1-p) + \frac{\frac{3}{4}L^2 - 1}{2L^2}\ln(q). \tag{5.7.4}$$

Together these estimates imply

$$F_L(\kappa^m) - \min(F_L(\kappa^1), F_L(\kappa^q)) \ge F_L(\kappa^m) - \frac{1}{2}(F_L(\kappa^1) + F_L(\kappa^q)) \ge \frac{1}{8}\ln(q) - \frac{\ln(q)}{4L^2} - \ln(2). \tag{5.7.5}$$

In the thermodynamic limit the middle term vanishes and we see that as q becomes large there is an increasing energy gap between the free energy of the inhomogeneous pattern compared to the free energy of the optimal homogeneous pattern uniformly in p. This is the result of Theorem 3.3 in [32] for the pattern κ^m.

Let us sketch how to extend the result to a general pattern κ^m with periodic structure as described above. The generalisation of the free energies of the homogeneous patterns to $d \geq 2$ is straightforward

$$\begin{aligned} F_L(\kappa^q) &\leq -d\ln(p) + \frac{L^d-1}{2L^d}\ln(q) + \frac{d\ln(2)}{2}, \\ F_L(\kappa^1) &\leq -d\ln(1-p) + \frac{d\ln(2)}{2}, \end{aligned} \tag{5.7.6}$$

where we used that there are at most $2^{|\mathbf{E}(T_L)|} = 2^{dL^d}$ trees on T_L. Let N_V denote the number of vertices in Q_d that are endpoint of an edge $e \in \mathbf{E}(Q_d)$ with $\kappa_e^m = q$ (call them q-vertices) and N_E be the number of edges $e \in Q_d$ that satisfy $\kappa_e^m = q$. The subgraph H of T_L induced by the q-edges of κ^M and its endpoints has $L^d N_V/2^d$ vertices (every vertex is contained in 2^d cells (translates of Q_d)) and $L^d N_E/2^{d-1}$ edges (every edge is contained in 2^{d-1} cells). Moreover, due to the construction by repeated reflection of κ^m each connected component of H has at least L vertices. We denote the connected components of H by $\mathbf{C}(H)$. We can find an acyclic subgraph of H with

$$|\mathbf{V}(H)| - |\mathbf{C}(H)| = \frac{L^d N_V}{2^d} - \frac{1}{L}\frac{L^d N_V}{2^d} = \frac{L^d N_V}{2^d}(1-L^{-1}) \tag{5.7.7}$$

edges that can be extended to a tree on T_L. We estimate the sum in (5.7.2) by the contribution of this tree and obtain the estimate

$$F_L(\kappa^m) \geq -\frac{N_E}{2^{d-1}}\ln(p) - \frac{d2^{d-1}-N_E}{2^{d-1}}\ln(1-p) + \frac{L^d N_V(1-L^{-1})}{2^{d+1}L^d}\ln(q). \tag{5.7.8}$$

This implies for $\kappa^m \notin \{\kappa^1, \kappa^q\}$

$$\begin{aligned} F_L(\kappa^m) - \min(F_L(\kappa^1), F_L(\kappa^q)) &\geq F_L(\kappa^m) - \frac{d2^{d-1}-N_E}{d2^{d-1}}F_L(\kappa^1) - \frac{N_E}{d2^{d-1}}F_L(\kappa^q) \\ &\geq \left(\frac{N_V(1-L^{-1})}{2^{d+1}} - \frac{N_E}{d2^d}(1-L^{-d})\right)\ln(q) - \frac{d\ln(2)}{2} \\ &\geq \frac{1}{d2^{d+1}}(dN_V - 2N_E)\ln(q) - \frac{1}{2L}\ln(q) - \frac{d\ln(2)}{2} \\ &\geq \frac{1}{d2^d}\ln(q) - \frac{1}{2L}\ln(q) - \frac{d\ln(2)}{2}. \end{aligned} \tag{5.7.9}$$

In the second to last step we bounded $N_V \leq 2^d$. In the last step we used that $dN_V - 2N_E$ counts the number of edges $e \in Q_d$ (with multiplicity) such that $\kappa_e^m = 1$ that are going out of one of the N_V q-vertices, i.e., vertices that are incident to a q-bond. Since κ^m is not homogeneous this is at least 2 with equality if there is only one edge e in Q_d such that $\kappa_e^m = 1$. We conclude that there is an energy gap uniformly in κ^m and $p \in [0,1]$ that diverges as $q \to \infty$. Note that the prefactor of q is optimal and we recover the optimal scaling $1/8\ \ln(q)$ for $d = 2$ as in Theorem 3.3. in [32].

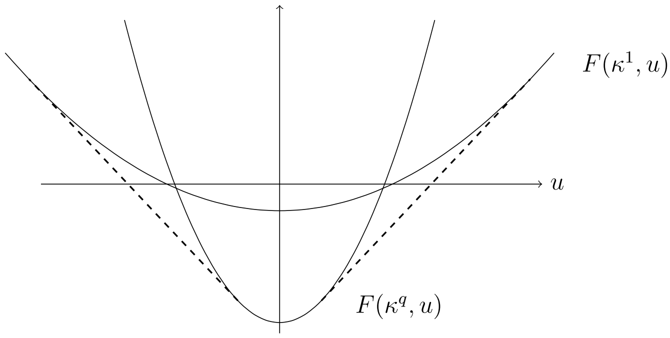

Figure 5.5: Sketch of the free energies of the homogenous patterns as a function of their tilt. The dashed line is a sketch of their lower convex envelope.

Extension of the scaling limit result. The correlation inequalities derived in Section 5.4 might be useful to extend the scaling limit result from [33] to certain measures ρ on $(0, \infty)$ with non-compact support. They can be used to derive a priori bounds on the κ-marginal for any Gibbs measure $\bar{\mu}$ of the random conductance model. Then one could use the recent sharp results for invariance principles and heat kernel estimates for random walks in random environment shown in [6, 7].

For a specific example this was done in [151]. Note, however, that the proof for the passage from extended gradient Gibbs measures to Gibbs measures for the random conductance model stated in Proposition 5.4.17 cannot be directly generalised to ρ with support that extends to 0 or ∞.

Models with disorder. It was suggested by Codina Cotar to consider models with disorder in this setting in dimension 2, i.e., in the simplest case $(p_e)_{e \in \mathbf{E}(\mathbb{Z}^2)} \in [0,1]^{\mathbf{E}(\mathbb{Z}^2)}$ is a set of i.i.d. random variables. Aizenman and Wehr showed that there is no phase transition in $d = 2$ for several models with disorder including the Ising model [5]. The basic heuristic given by Imry and Ma is that the energy fluctuations in a domain $|\Lambda|$ are of order $\sqrt{|\Lambda|}$ while the strength of the symmetry breaking is bounded by $C|\partial\Lambda|$. This lead to the prediction that symmetry breaking cannot persist in the presence of disorder in $d = 2$ where both terms are of the same order.

It might be possible to extend this result to our setting. Here the correlation inequalities are very helpful because they introduce an ordering on the phase space. This potentially simplifies the proof, e.g., for the Ising model a streamlined proof can be found in [37]. While the lower bound on the fluctuation of the disorder rely on general abstract arguments the bound for the strength of the symmetry breaking is more model dependent. Here one might bound this energy using techniques similar to the proof of Lemma 5.5.3.

Non-zero tilt The most interesting extension would be results about the model at non-zero tilt. This work and all earlier works heavily rely on the assumption of zero tilt. As discussed in [33] the main problem is that one needs to understand the behaviour of the corrector for non-zero tilt. This becomes also apparent in the proof of Proposition 5.4.17 (see (5.A.5)).

Let us provide a simple heuristic about the behaviour of the model with non-zero tilt if $\rho = p\delta_q + (1-p)\delta_1$ and q large. Note that for large q typically the measure is concentrated on almost homogenous configurations of κ for zero tilt. The thermodynamic limit of the free energy of the homogenous configurations can be easily evaluated (see [32]) and one obtains for some constant $c \in \mathbb{R}$

$$F(\kappa^q, u) = -2\ln(p) + \frac{\ln(q)}{2} + q\frac{|u|^2}{2} + c, \tag{5.7.10}$$

$$F(\kappa^1, u) = -2\ln(1-p) + \frac{|u|^2}{2} + c. \tag{5.7.11}$$

Those functions are plotted for $p \geq p_{\mathrm{sd}}$ in Figure 5.5. If the κ-marginal is mostly homogenous we expect the free energy to be approximately the lower convex envelope of those two functions. In particular there might be a linear piece of the surface tension. This would be in contrast to convex potentials or the setting considered in Chapter 3 where the surface tension is strictly convex. For tilts u in the linear region one could conjecture that there is no translation invariant and ergodic Gibbs state because the tilt might not concentrate in the thermodynamic limit. Instead every infinite volume Gibbs state might be a mixture of a state with mostly 1-bonds and tilt u_1 with $|u_1| > |u|$ and a state with mostly q-bonds and tilt u_q with $|u_q| < |u|$. This is, however probably very difficult to investigate because this is not a bulk phenomenon but one needs to understand, roughly speaking, the free energy up to surface order.

5.A Proofs of Proposition 5.4.17 and Proposition 5.4.18

In this section we pay the last remaining debt of proving two propositions from Section 5.2.

Proof of Proposition 5.4.17. For $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ and $E \subset \mathbf{E}(\mathbb{Z}^d)$ finite we define the cylinder event

$$\mathbf{A}(\lambda_E) = \{\kappa \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)} : \kappa_E = \lambda_E\} \in \mathcal{F}_E. \tag{5.A.1}$$

With a slight abuse of notation we drop the pullback from the notation when we consider the set $\pi_2^{-1}(\mathbf{A}(\lambda_E)) \subset \mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)} \times \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$. Since all local cylinder events in $\mathcal{F}$ can be written as a union of events of the form $\mathbf{A}(\lambda_{E_L})$ it is by Remark 5.2.2 sufficient to show

$$\bar{\mu}(\mathbf{A}(\lambda_{E_L})) = \bar{\mu}\bar{\gamma}_{E_n}(\mathbf{A}(\lambda_{E_L})) \tag{5.A.2}$$

for all $L, n \geq 0$ and all $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$. Using the quasilocality of $\bar{\gamma}$ and Remark 4.21 in [92] it is sufficient to consider $L = n$ and we will do this in the following. We are going to show this claim in a series of steps.

Step 1. We investigate the distribution of the κ-marginal conditioned on $\omega_{E_N^c}$.

Since $\tilde{\mu}$ is a gradient Gibbs measure we know by (5.3.2) that for $\omega \in \mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$ and $\kappa \in \{1,q\}^{\mathbf{E}(\Lambda)}$

$$\tilde{\mu}(\mathbf{A}(\kappa_{\mathbf{E}(\Lambda)}) \mid \mathcal{E}_{\mathbf{E}(\Lambda)^c})(\omega) = \frac{1}{Z}\int p^{h(\kappa)}(1-p)^{s(\kappa)} \prod_{e\in\mathbf{E}(\Lambda)} e^{-\kappa_e\eta_e^2}\, \nu_\Lambda^{\omega_{\mathbf{E}(\Lambda)^c}}(\mathrm{d}\eta) = \frac{Z(\kappa,\omega)}{Z} \tag{5.A.3}$$

where Z is the normalisation and

$$Z(\kappa,\omega) = \int p^{h(\kappa)}(1-p)^{s(\kappa)} \prod_{e\in\mathbf{E}(\Lambda)} e^{-\frac{1}{2}\kappa_e\eta_e^2}\, \nu_\Lambda^{\omega_{\mathbf{E}(\Lambda)^c}}(\mathrm{d}\eta) \tag{5.A.4}$$

denotes the partition function corresponding to the configuration κ. Let $\varphi \in \mathbb{R}^{\mathbb{Z}^d}$ be the configuration such that $\nabla\varphi = \omega$ and $\varphi(0) = 0$. We denote by χ_κ the corrector of κ, i.e., the solution of $\nabla^* \kappa \nabla \chi_\kappa = 0$ with boundary values φ_{Λ^c}. A shift of the integration variables and Gaussian calculus implies (see also (5.3.4))

$$Z(\kappa,\omega) = Z(\kappa,\bar{0})e^{-\frac{1}{2}(\nabla\chi_\kappa,\kappa\nabla\chi_\kappa)_{\mathbf{E}(\Lambda)}} = e^{-\frac{1}{2}(\nabla\chi_\kappa,\kappa\nabla\chi_\kappa)_{\mathbf{E}(\Lambda)}} \frac{p^{h(\kappa)}(1-p)^{s(\kappa)}}{\sqrt{\det 2\pi(\tilde{\Delta}_\kappa^{\Lambda^w})^{-1}}} \tag{5.A.5}$$

where $\bar{0}$ is the configuration with vanishing gradients, i.e., $\bar{0}_e = 0$ for $e \in \mathbf{E}(\mathbb{Z}^d)$. The necessary calculation to obtain (5.A.5) basically agrees with the calculation that shows that the discrete Gaussian free field can be decomposed in a zero boundary discrete Gaussian free field and a harmonic extension. We now restrict our attention to $\Lambda = \Lambda_N = [-N,N]^d \cap \mathbb{Z}^d$ for $N \in \mathbb{N}$. We introduce the law of the κ-marginal for wired non-constant boundary conditions for $\kappa \in \{1,q\}^{E_N}$ by

$$\bar{\mu}_N^{1,\omega}(\kappa) = \frac{Z(\kappa,\omega)}{Z}. \tag{5.A.6}$$

Note that $\bar{\mu}_N^{1,\bar{0}} = \bar{\mu}_N^1$ where $\bar{\mu}_N^1$ was defined in (5.4.36).

Step 2. In this step we are going to show that there is $N_0 \in \mathbb{N}$ depending on n such that for $N \geq N_0$ and uniformly in $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$

$$\left| \bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N \setminus E_n})\Big) - \bar{\mu}_N^1\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big)\right| \leq 4\varepsilon., \tag{5.A.7}$$

i.e., the boundary effect is negligible. We start by showing that typically the difference between the corrector energies for configurations κ and $\tilde{\kappa}$ that only differ in E_n will be small. This will allow us to estimate the difference between $\bar{\mu}_N^1$ and $\bar{\mu}_N^{1,\omega}$ conditioned to agree close to the boundary.

Recall that we consider the case that $\Lambda = \Lambda_N$ is a box. The Nash-Moser estimate stated in Lemma 5.B.1 combined with the maximum principle for the equation $\nabla^*\kappa\nabla\chi_\kappa = 0$ imply for $b \in E_n$ and some $\alpha = \alpha(q) > 0$

$$|\nabla \chi_\kappa(b)| \leq \frac{C\left(\max_{x\in\partial\Lambda_N}\varphi(x) - \min_{y\in\partial\Lambda_N}\varphi(y)\right)}{|N-n|^\alpha}. \tag{5.A.8}$$

We introduce the event $M(N) = \{\omega : \max_{x\in\partial\Lambda_N}\varphi(x) - \min_{y\in\partial\Lambda_N}\varphi(y) \leq (\ln N)^3\}$. Consider configurations $\kappa, \tilde{\kappa} \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ such that $\kappa_e = \tilde{\kappa}_e$ for $e \notin E_n$. Using the fact that the corrector is the minimizer of the quadratic form $(\nabla\chi_\kappa, \kappa\nabla\chi_\kappa)_{E_N}$ with given boundary condition we can estimate

$$(\nabla\chi_\kappa,\kappa\nabla\chi_\kappa)_{E_N} \leq (\nabla\chi_{\tilde{\kappa}},\kappa\nabla\chi_{\tilde{\kappa}})_{E_N} \leq (\nabla\chi_{\tilde{\kappa}},\tilde{\kappa}\nabla\chi_{\tilde{\kappa}})_{E_N} + |E_n| q \sup_{b\in E_n} |\nabla\chi_{\tilde{\kappa}}|^2. \tag{5.A.9}$$

From (5.A.8) we infer that for $N \geq 2n$ and $\varphi \in M(N)$

$$|(\nabla\chi_\kappa,\kappa\nabla\chi_\kappa)_{E_N} - (\nabla\chi_{\tilde{\kappa}},\tilde{\kappa}\nabla\chi_{\tilde{\kappa}})_{E_N}| \leq C|E_n| q \frac{(\ln N)^3}{N^\alpha}. \tag{5.A.10}$$

By choosing $N_1 \geq 2n$ sufficiently large we can ensure that for $N \geq N_1$, $\varphi \in M(N)$, and uniformly in $\kappa, \tilde{\kappa}$ as before

$$1-\varepsilon \leq e^{\frac{1}{2}(\nabla\chi_\kappa,\kappa\nabla\chi_\kappa)_{E_N} - \frac{1}{2}(\nabla\chi_{\tilde{\kappa}},\tilde{\kappa}\nabla\chi_{\tilde{\kappa}})_{E_N}} \leq 1+\varepsilon. \tag{5.A.11}$$

Using this in (5.A.5) we conclude that for $N \geq N_1 \vee 2n$, $\omega \in M(N)$, $\varepsilon < 1/3$, and $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$

$$\left|\bar{\mu}_N^{1,\omega}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big) - \bar{\mu}_N^1\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big)\right| \leq \frac{2\varepsilon}{1-\varepsilon} \leq 3\varepsilon. \tag{5.A.12}$$

This implies

$$\left|\tilde{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N - E_n}) \cap M(N)\Big) - \bar{\mu}_N^1\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big)\right| \leq 3\varepsilon. \tag{5.A.13}$$

From Lemma 5.A.1 below and Proposition 5.4.19 we infer that for an extended gradient Gibbs measure $\tilde{\mu}$ associated to an ergodic zero tilt Gibbs measure μ and any $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$

$$\tilde{\mu}\Big(M(N)^{\mathrm{c}} \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big) \leq \frac{C}{\ln(N)} \leq \varepsilon \tag{5.A.14}$$

for all $N \geq N_2$ and N_2 sufficiently large. We conclude that for $N \geq N_0 := N_1 \vee N_2 \vee 2n$

$$\begin{aligned}
&\left|\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big) - \bar{\mu}_N^1\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big)\right| \\
&\leq \tilde{\mu}\Big(M(N) \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big) \left|\tilde{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n}), M(N)\Big) - \bar{\mu}_N^1\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n}\Big)\right| \\
&+ \tilde{\mu}\Big(M(N)^{\mathrm{c}} \mid \mathbf{A}(\lambda_{E_N\setminus E_n})\Big) \left|\tilde{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n}), M(N)^{\mathrm{c}}\Big) - \bar{\mu}_N^1\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\lambda_{E_N\setminus E_n}\Big)\right| \\
&\leq 3\varepsilon + \varepsilon = 4\varepsilon.
\end{aligned} \tag{5.A.15}$$

Step 3. Using the previous results we can now finish the proof. We rewrite

$$\begin{aligned}
\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n})\Big) &= \sum_{\sigma' \in \{1,q\}^{E_N\setminus E_n}} \bar{\mu}\Big(\mathbf{A}(\sigma'_{E_N\setminus E_n})\Big)\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\sigma'_{E_N\setminus E_n})\Big) \\
&= \sum_{\sigma' \in \{1,q\}^{E_N\setminus E_n}} \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma'_{E_N\setminus E_n}) \cap \mathbf{A}(\sigma_{E_N})\Big)\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\sigma'_{E_N\setminus E_n})\Big) \\
&= \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\sigma_{E_N\setminus E_n})\Big).
\end{aligned} \tag{5.A.16}$$

The identity above and the fact that $\bar{\gamma}_{E_n}$ is proper imply adding and subtracting the same term

$$\begin{aligned}
&\left|\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n})\Big) - \bar{\mu}\bar{\gamma}_{E_n}\Big(\mathbf{A}(\lambda_{E_n})\Big)\right| \\
&= \left|\sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\sigma_{E_N\setminus E_n})\Big) - \int \bar{\mu}(\mathrm{d}\kappa)\mathbb{1}_{\mathbf{A}(\sigma_{E_N})}(\kappa)\, \bar{\gamma}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \kappa\Big)\right| \\
&\leq \left|\sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\sigma_{E_N\setminus E_n})\Big) - \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\gamma}_{E_n}^{\Lambda_N^w}\Big(\mathbf{A}(\lambda_{E_L}), \sigma_{E_N}\Big)\right| \\
&\quad + \left|\sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\gamma}_{E_n}^{\Lambda_N^w}\Big(\mathbf{A}(\lambda_{E_n}), \sigma_{E_N}\Big) - \int \bar{\mu}(\mathrm{d}\kappa)\mathbb{1}_{\mathbf{A}(\sigma_{E_N})}(\kappa)\, \bar{\gamma}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \kappa\Big)\right|.
\end{aligned} \tag{5.A.17}$$

We continue to estimate the right hand side of this expression. We start with the first term. Since $\bar{\mu}^1_{\Lambda_N}$ is a finite volume Gibbs measure (see (5.4.53)) we have for $\mathbf{A} \in \mathcal{F}_{E_N}$

$$\bar{\mu}^1_{\Lambda_N}\Big(\mathbf{A} \mid \mathbf{A}(\sigma_{E_N \setminus E_n})\Big) = \bar{\mu}^1_{\Lambda_N}\Big(\mathbf{A} \mid \mathcal{F}_{E_n^c}\Big)(\sigma_{E_N}) = \bar{\gamma}^{\Lambda_N^w}_{E_n}(\mathbf{A}, \sigma_{E_N}). \tag{5.A.18}$$

Using this and the bound (5.A.7) we obtain for $N \geq N_0$

$$\begin{aligned}
&\Bigg| \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n}) \mid \mathbf{A}(\sigma_{E_N \setminus E_n})\Big) - \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\gamma}^{\Lambda_N^w}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \sigma_{E_N}\Big)\Bigg| \\
&\quad \leq \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\Big|\bar{\mu}\Big(\mathbf{A}(\lambda_{E_L}) \mid \mathbf{A}(\sigma_{E_N \setminus E_n})\Big) - \bar{\mu}^1_{\Lambda_N}\Big(\mathbf{A}(\lambda_{E_N}) \mid \mathbf{A}(\sigma_{E_N \setminus E_n})\Big)\Big| \\
&\quad \leq 4\varepsilon \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big) \leq 4\varepsilon.
\end{aligned} \tag{5.A.19}$$

We now address the second term on the right hand side of (5.A.17). By Lemma 5.4.16 there is N_3 such that for $N \geq N_3$ and any $\lambda, \sigma \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$

$$|\bar{\gamma}_{E_n}(\sigma, \lambda) - \bar{\gamma}^{\Lambda_N^w}_{E_n}(\sigma, \lambda)| < \varepsilon. \tag{5.A.20}$$

This implies for $N \geq N_3$

$$\begin{aligned}
&\Bigg| \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big)\bar{\gamma}^{\Lambda_N^w}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \kappa\Big) - \int \bar{\mu}(\mathrm{d}\kappa)\mathbb{1}_{\mathbf{A}(\sigma_{E_N})}(\kappa)\, \bar{\gamma}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \kappa\Big)\Bigg| \\
&\quad \leq \sum_{\sigma \in \{1,q\}^{E_N}} \int \bar{\mu}(\mathrm{d}\kappa)\mathbb{1}_{\mathbf{A}(\sigma_{E_N})}(\kappa)\, \Big|\bar{\gamma}^{\Lambda_N^w}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \sigma_{E_N}\Big) - \bar{\gamma}_{E_n}\Big(\mathbf{A}(\lambda_{E_n}), \kappa\Big)\Big| \\
&\quad \leq \varepsilon \sum_{\sigma \in \{1,q\}^{E_N}} \bar{\mu}\Big(\mathbf{A}(\sigma_{E_N})\Big) \leq \varepsilon.
\end{aligned} \tag{5.A.21}$$

Using (5.A.17), (5.A.19), and (5.A.21) we conclude that for any $\varepsilon > 0$

$$\Big|\bar{\mu}\Big(\mathbf{A}(\lambda_{E_n})\Big) - \bar{\mu}\bar{\gamma}_{E_n}\Big(\mathbf{A}(\lambda_{E_n})\Big)\Big| \leq 5\varepsilon. \tag{5.A.22}$$

This ends the proof. □

The following simple Lemma was used in the proof of Proposition 5.4.17.

Lemma 5.A.1. *Let $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ and denote by φ^λ the centred Gaussian field on $\mathbb{Z}^d$ with $\varphi(0) = 0$ and covariance Δ_λ^{-1}. Then φ^λ satisfies*

$$\mathbb{P}\Big(\max_{x \in \partial\Lambda_N} \varphi^\lambda(x) - \min_{y \in \partial\Lambda_N} \varphi^\lambda(y) \geq (\ln N)^3\Big) \leq C(\ln N)^{-1}. \tag{5.A.23}$$

Proof. We use the notation $\bar{1} \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ for the configuration given by $\bar{1}_e = 1$ for $e \in \mathbf{E}(\mathbb{Z}^d)$. The Brascamp-Lieb inequality (see [38, Theorem 5.1]) implies for the centred Gaussian fields φ^λ and $\varphi^{\bar{1}}$ that

$$\mathbb{E}\left((\varphi^\lambda(x) - \varphi^\lambda(0))^2\right) \leq \mathbb{E}\left((\varphi^{\bar{1}}(x) - \varphi^{\bar{1}}(0))^2\right) \leq \begin{cases} C\ln(|x|) & \text{for } d = 2 \\ C & \text{for } d \geq 3. \end{cases} \tag{5.A.24}$$

It is well known that for a centred Gaussian random vector $X \in \mathcal{P}(\mathbb{R}^m)$ with $\mathbb{E}(X_i^2) \leq \sigma^2$ the expectation of the maximum is bounded by

$$\mathbb{E}(\max_i X_i) \leq \sigma\sqrt{2\ln m}. \tag{5.A.25}$$

We use this for the Gaussian field φ^λ and conclude that

$$\mathbb{E}\left(\max_{x\in\partial\Lambda_N} \varphi^\lambda(x) - \varphi^\lambda(0)\right) \leq \begin{cases} C\ln(N)^2 & \text{for } d=2 \\ C\ln(N^{d-1}) & \text{for } d\geq 3. \end{cases} \tag{5.A.26}$$

A simple Markov bound implies that there is $C = C(d) > 0$ such that

$$\mathbb{P}\left(\max_{x\in\partial\Lambda_N} \varphi^\lambda(x) - \min_{y\in\partial\Lambda_N} \varphi^\lambda(y) \geq (\ln N)^3\right) \leq C(\ln N)^{-1}. \tag{5.A.27}$$

□

It remains to provide a proof of Proposition 5.4.18. We will only sketch the argument.

Proof of Proposition 5.4.18. First we remark that the law of $(\kappa, \nabla\varphi^\kappa)$ is a Borel-measure on $\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)} \times \mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$. This follows from Carathéodory's extension theorem and the observation that for a local event $\mathbf{A} \in \mathcal{E}_E$ with $E \subset \mathbf{E}(\mathbb{Z}^d)$ finite the function $\kappa \mapsto \mu_{\varphi^\kappa}(\mathbf{A})$ is continuous (this can be shown using Lemma 5.B.3). By Remark 5.2.2 it is sufficient to prove that $\tilde{\mu}\tilde{\gamma}_{\Lambda_n} = \tilde{\mu}$ for all n. To prove this we use an approximation procedure. We fix n and define for $N > n$ a measure $\tilde{\mu}_N$ on $\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)} \times \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ as follows. The κ-marginal of $\tilde{\mu}_N$ is given by $\bar{\mu}_N = \bar{\mu}\bar{\gamma}_{E_n}^{\Lambda_N^w}$ where as before we extended $\bar{\gamma}_{E_n}^{\Lambda_N^w}$ to a proper probability kernel on $\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$. For given κ, let φ^κ be the centred Gaussian field with zero boundary data outside of $\mathring{\Lambda}_N$ and covariance $(\tilde{\Delta}_{\kappa_{E_N}}^{\Lambda_N^w})^{-1}$ where $\tilde{\Delta}_{\kappa_{E_N}}^{\Lambda_N^w}$ was defined in Section 5.3. The measure $\tilde{\mu}_N$ is the joint law of (κ, φ^κ) where κ has law $\bar{\mu}_N$. We claim that for $N > n$

$$\tilde{\mu}_N\tilde{\gamma}_{\Lambda_n} = \tilde{\mu}_N. \tag{5.A.28}$$

We prove this by showing the statement for the measures $\tilde{\mu}_N\big(\,\cdot\,|\mathbf{A}(\lambda_{E_N\setminus E_n})\big)$ for every configuration $\lambda \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$. To shorten the notation we write $\tilde{\mu}_N^\lambda = \tilde{\mu}_N\big(\,\cdot\,|\mathbf{A}(\lambda_{E_N\setminus E_n})\big)$. By definition of $\tilde{\mu}_N$ the φ-field conditioned on κ has density $\exp(-\frac{1}{2}(\varphi, \tilde{\Delta}_\kappa^{\Lambda_N^w}\varphi))/\sqrt{\det 2\pi(\tilde{\Delta}_\kappa^{\Lambda_N^w})^{-1}}\,\mathrm{d}\varphi_{\mathring{\Lambda}_N}$ where $\mathrm{d}\varphi_\Lambda = \prod_{x\in\Lambda} \mathrm{d}\varphi_x$ denotes the Lebesgue measure. This implies for $\mathbf{B} \in \mathcal{B}(\mathbb{R}^{\Lambda_N})$ and $\sigma \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ such that $\sigma_{E_N\setminus E_n} = \lambda_{E_N\setminus E_n}$

$$\tilde{\mu}_N^\lambda\Big(\varphi \in \mathbf{B}, \kappa \in \mathbf{A}(\sigma_{E_N})\Big) = \tilde{\mu}_N^\lambda\Big(\mathbf{A}(\sigma_{E_N})\Big) \int_{\mathbf{B}} \frac{\exp(-\frac{1}{2}(\varphi, \tilde{\Delta}_\sigma^{\Lambda_N^w}\varphi))}{\sqrt{\det 2\pi(\tilde{\Delta}_\sigma^{\Lambda_N^w})^{-1}}}\,\mathrm{d}\varphi_{\mathring{\Lambda}_N} \tag{5.A.29}$$

We use the definition of $\tilde{\mu}_N$ and the fact that specifications are proper to rewrite

$$\begin{aligned}\tilde{\mu}_N^\lambda\Big(\mathbf{A}(\sigma_{E_N})\Big) &= \frac{\bar{\mu}\bar{\gamma}_{E_n}^{\Lambda_N^w}\Big(\mathbf{A}(\sigma_{E_N}) \cap \mathbf{A}(\lambda_{E_N\setminus E_n})\Big)}{\bar{\mu}\bar{\gamma}_{E_n}^{\Lambda_N^w}\Big(\mathbf{A}(\lambda_{E_N\setminus E_n})\Big)} = \frac{\bar{\mu}\Big(\mathbb{1}_{\mathbf{A}(\lambda_{E_N\setminus E_n})}(\kappa)\bar{\gamma}_{E_n}^{\Lambda_N^w}(\mathbf{A}(\sigma_{E_N}, \kappa)\Big)}{\bar{\mu}\Big(\mathbf{A}(\lambda_{E_N\setminus E_n})\Big)} \\ &= \bar{\gamma}_{E_n}^{\Lambda_N^w}(\sigma_{E_N}, \lambda_{E_N}) = \mathbb{1}_{\sigma_{E_N\setminus E_n} = \lambda_{E_N\setminus E_n}} \frac{1}{Z_\lambda} \frac{p^{h(\sigma,E_N)}(1-p)^{s(\sigma,E_N)}}{\sqrt{\det \Delta_\sigma^{\Lambda_N^w}}}.\end{aligned} \tag{5.A.30}$$

Note that

$$p^{h(\sigma,\{e\})}(1-p)^{s(\sigma,\{e\})}e^{-\frac{\sigma_e\eta_e^2}{2}} = \int_{\{\sigma_e\}} \rho(\mathrm{d}\kappa_e)\, e^{-\frac{\kappa_e\eta_e^2}{2}}. \tag{5.A.31}$$

The last three displays, a summation by parts, and (5.3.6) lead us to

$$\begin{aligned}\tilde{\mu}_N^\lambda(\varphi\in\mathbf{B},\kappa\in\mathbf{A}(\sigma_{E_N})) &= \frac{1}{(2\pi)^{|\Lambda_N|}|\Lambda_N^w|Z_\lambda}\int_{\mathbf{B}} p^{h(\sigma,E_N)}(1-p)^{s(\sigma,E_N)}\prod_{e\in E_N} e^{-\frac{\sigma_e\eta_e^2}{2}}\,\mathrm{d}\varphi_{\mathring{\Lambda}_N}\\ &= \frac{1}{Z'_\lambda}\int_{\mathbf{B}}\mathrm{d}\varphi_{\mathring{\Lambda}_N}\int_{\mathbf{A}(\sigma_{E_N})}\prod_{e\in E_N}\rho(\mathrm{d}\kappa_e)\, e^{-\frac{\kappa_e\eta_e^2}{2}}.\end{aligned} \tag{5.A.32}$$

Combining this with the definition (5.3.2) we conclude that for $\sigma'\in\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ such that $\sigma'_{E_N\setminus E_n}=\lambda_{E_N\setminus E_n}$ and $\omega\in\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}$ such that $\omega_{E_N^c}=0$

$$\begin{aligned}&\tilde{\mu}_N^\lambda\Big(\mathbf{B}\times\mathbf{A}(\sigma_{E_N})\,|\mathcal{A}_{E_n^c}\Big)((\omega,\sigma'))\\ &\quad= \frac{1}{Z_{\lambda,\omega}}\int_{\mathbf{B}}\nu_{\Lambda_n}^{\omega_{E_n^c}}(\mathrm{d}\eta)\int_{\mathbf{A}(\sigma_{E_N})}\prod_{e\in E_n}\rho(\mathrm{d}\kappa_e)\prod_{e\notin E_n}\delta_{\sigma'_e}(\mathrm{d}\kappa_e)\prod_{e\in E_N}e^{-\frac{\kappa_e\eta_e^2}{2}}\\ &\quad= \tilde{\gamma}_{\Lambda_n}\Big(\mathbf{B}\times\mathbf{A}(\sigma_{E_N}),(\omega,\sigma')\Big).\end{aligned} \tag{5.A.33}$$

This implies $\tilde{\mu}_N^\lambda\tilde{\gamma}_{\Lambda_n}=\tilde{\mu}_N^\lambda$ and (5.A.28) follows directly.

It remains to pass to the limit in equation (5.A.28), i.e., we show that the right hand side converges in the topology of local convergence to $\tilde{\mu}$ and the left hand side to $\tilde{\mu}\tilde{\gamma}_{\Lambda_n}$ thus finishing the proof. We only sketch the argument. Since $\tilde{\gamma}(A,\cdot)$ is a measurable, local, and bounded function if A is a local event it is sufficient to show that $\tilde{\mu}_N$ converges to $\tilde{\mu}$ locally in total variation, that is for every $\Lambda\subset\subset\mathbb{Z}^d$

$$\lim_{N\to\infty}\sup_{A\in\mathcal{A}_{\mathbf{E}(\Lambda)}}|\tilde{\mu}_N(A)-\tilde{\mu}(A)|=0. \tag{5.A.34}$$

Where we used the σ-algebra $\mathcal{A}_E$ on $\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)}\times\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ defined in Section 5.3 as the product of the pullbacks of $\mathcal{E}_E$ and $\mathcal{F}_E$. We first consider the κ-marginals of $\tilde{\mu}_N$ and $\tilde{\mu}$. They are given by $\bar{\mu}_N=\bar{\mu}\bar{\gamma}_{E_n}^{\Lambda_N^w}$ and $\bar{\mu}=\bar{\mu}\bar{\gamma}_{E_n}$ where we use that $\bar{\mu}$ is a Gibbs measure. We can estimate the total variation of those two measures by

$$\begin{aligned}\|\bar{\mu}\bar{\gamma}_{E_n}^{\Lambda_N^w}-\bar{\mu}\bar{\gamma}_{E_n}\|_{\mathrm{TV}} &\le \sup_{\kappa\in\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}}\ \sup_{A\subset\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}}|\bar{\gamma}_{E_n}^{\Lambda_N^w}(A,\kappa)-\bar{\gamma}_{E_n}(A,\kappa)|\\ &\le 2^{|E_n|}\sup_{\sigma,\kappa\in\{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}}|\bar{\gamma}_{E_n}^{\Lambda_N^w}(\sigma,\kappa)-\bar{\gamma}_{E_n}(\sigma,\kappa)|.\end{aligned} \tag{5.A.35}$$

In the second step we used that the specifications are proper thus we can assume $A\subset\mathbf{A}(\kappa_{E_n^c})$ and use that $|\mathbf{A}(\kappa_{E_n^c})|\le 2^{|E_n|}$. Using Lemma 5.4.16 we conclude

$$\lim_{N\to\infty}\|\bar{\mu}_N-\bar{\mu}\|_{\mathrm{TV}}=\lim_{N\to\infty}\|\bar{\mu}\bar{\gamma}_{E_n}^{\Lambda_N^w}-\bar{\mu}\bar{\gamma}_{E_n}\|_{\mathrm{TV}}=0. \tag{5.A.36}$$

We address the η-marginals of the measures $\tilde{\mu}$ and $\tilde{\mu}_N$. We write $\tilde{\mu}(\cdot \mid \kappa), \tilde{\mu}_N(\cdot \mid \kappa) \in \mathcal{P}(\mathbb{R}_g^{\mathbf{E}(\mathbb{Z}^d)})$ for the conditional distribution of the η-field for a given $\kappa \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$. From the construction this is well defined for every κ. We define the centred Gaussian field φ^κ by $\varphi^\kappa(0) = 0$ and its covariance $(\Delta_\kappa)^{-1}$ and the centred fields φ_N^κ pinned to 0 outside of $\mathring{\Lambda}_N$ with covariance $(\tilde{\Delta}_\kappa^{\Lambda_N^w})^{-1}$ and we denote their gradients by $\eta^\kappa = \nabla\varphi^\kappa$ and $\eta_N^\kappa = \nabla\varphi_N^\kappa$. Note that by definition of $\tilde{\mu}$ and $\tilde{\mu}_N$ the law of η^κ and η_N^κ coincides with $\tilde{\mu}(\cdot \mid \kappa)$ and $\tilde{\mu}_N(\cdot \mid \kappa)$. Fix an integer L. We introduce the Gaussian vectors $X^\kappa = (\varphi^\kappa(x) - \varphi^\kappa(0))_{x\in\Lambda_L}$ and $X_N^\kappa = (\varphi_N^\kappa(x) - \varphi_N^\kappa(0))_{x\in\Lambda_L}$. Note that given X^κ, X_N^κ the gradient field $\eta^\kappa\restriction_{\mathbf{E}(\Lambda_L)}$ respectively $\eta_N^\kappa\restriction_{\mathbf{E}(\Lambda_L)}$ can be expressed as a function of X^κ and X_N^κ respectively. This implies that

$$\sup_{B\in\mathcal{E}_{\mathbf{E}(\Lambda_L)}} \left|\tilde{\mu}_N(B \mid \kappa) - \tilde{\mu}(B \mid \kappa)\right| \le \|X^\kappa - X_N^\kappa\|_{\mathrm{TV}}. \tag{5.A.37}$$

Theorem 1.1 in [71] states that the total variation distance between two centred Gaussian vectors Z_1, Z_2 with covariance matrices Σ_1 and Σ_2 can be bounded by $\frac{3}{2}|\Sigma_1^{-1}\Sigma_2 - \mathbb{1}|_F$ where $|\cdot|_F$ denotes the Frobenius norm. Using this theorem and the uniform convergence of the covariance of η_N^κ to the covariance of η^κ stated in Lemma 5.B.3 we conclude that

$$\lim_{N\to\infty} \sup_{B\in\mathcal{E}_{\mathbf{E}(\Lambda_L)}} \left|\tilde{\mu}_N(B \mid \kappa) - \tilde{\mu}(B \mid \kappa)\right| \le \lim_{N\to\infty} \|X^\kappa - X_N^\kappa\|_{\mathrm{TV}} = 0. \tag{5.A.38}$$

We denote for a set $A \in \mathcal{A}$ and $\kappa \in \{1,q\}^{\mathbf{E}(\mathbb{Z}^d)}$ by A_κ the intersection of A and the line through κ, i.e., $A_\kappa = \{\eta \in \mathbb{R}^{\mathbf{E}(\mathbb{Z}^d)} : (\eta,\kappa) \in A\}$. Using disintegration, (5.A.36), (5.A.38), and the dominated convergence theorem we estimate

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{A\in\mathcal{A}_{\mathbf{E}(\Lambda_L)}} |\tilde{\mu}_N(A) - \tilde{\mu}(A)| \\
&= \lim_{N\to\infty} \sup_{A\in\mathcal{A}_{\mathbf{E}(\Lambda_L)}} \left| \int \bar{\mu}(\mathrm{d}\kappa)\, \tilde{\mu}(A_\kappa \mid \kappa) - \int \bar{\mu}_N(\mathrm{d}\kappa)\, \tilde{\mu}_N(A_\kappa \mid \kappa)\right| \\
&\le \lim_{N\to\infty} \sup_{A\in\mathcal{A}_{\mathbf{E}(\Lambda_L)}} \int \bar{\mu}(\mathrm{d}\kappa) \left|\tilde{\mu}(A_\kappa \mid \kappa) - \tilde{\mu}_N(A_\kappa \mid \kappa)\right| \\
&\quad + \lim_{N\to\infty} \sup_{A\in\mathcal{A}_{\mathbf{E}(\Lambda_L)}} \left| \int \bar{\mu}(\mathrm{d}\kappa)\, \tilde{\mu}_N(A_\kappa \mid \kappa) - \int \bar{\mu}_N(\mathrm{d}\kappa)\, \tilde{\mu}_N(A_\kappa \mid \kappa)\right| \\
&\le \lim_{N\to\infty} \int \bar{\mu}(\mathrm{d}\kappa)\, \|X^\kappa - X_N^\kappa\|_{\mathrm{TV}} + \lim_{N\to\infty} \|\bar{\mu} - \bar{\mu}_N\|_{\mathrm{TV}} = 0.
\end{aligned} \tag{5.A.39}$$

We conclude that for any local event A

$$\tilde{\mu}(A) = \lim_{N\to\infty} \tilde{\mu}_N(A) = \lim_{N\to\infty} \tilde{\mu}_N \tilde{\gamma}_{\Lambda_n}(A) = \tilde{\mu}\tilde{\gamma}_{\Lambda_n}(A). \tag{5.A.40}$$

□

5.B Estimates for discrete elliptic equations

In this appendix we collect some regularity estimates for discrete elliptic equations. We consider as before uniformly elliptic $\kappa : \mathbf{E}(\mathbb{Z}^d) \to \mathbb{R}_+$ with $0 < c_- \le \kappa_e \le c_+ < \infty$ for all $e \in \mathbf{E}(\mathbb{Z}^d)$. We denote corresponding set of conductances by $M(c_-, c_+) = [c_-, c_+]^{\mathbf{E}(\mathbb{Z}^d)}$.

Next we state a discrete version of the well known Nash-Moser estimates for scalar elliptic partial differential equations with L^∞ coefficients.

Lemma 5.B.1. *Let $0 < c_- < c_+ < \infty$, $\Lambda \subset \mathbb{Z}^d$, and $\kappa \in M(c_-, c_+)$. Let $u : \Lambda \to \mathbb{R}$ be a solution of*

$$-\nabla^* \kappa \nabla u = 0 \quad \textit{in } \mathring{\Lambda} \tag{5.B.1}$$

Then there are constants $\alpha = \alpha(c_-, c_+, d)$ and $C = C(c_-, c_+, d)$ such that the following estimate holds for $x, y \in \Lambda$

$$|u(x) - u(y)| \leq C\|u\|_{L^\infty(\Lambda)} \left(\frac{|x-y|}{d(x, \partial\Lambda) \wedge d(y, \partial\Lambda)} \right)^\alpha. \tag{5.B.2}$$

Proof. This is Proposition 6.2 in [68]. □

Moreover we state some consequences for the Green's function of uniformly elliptic operators in divergence form. We define the Green's function $G_\kappa : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ as the inverse of Δ_κ, i.e., G_κ satisfies for $d \geq 3$

$$\Delta_\kappa G_\kappa(\cdot, y) = \delta_y, \quad \lim_{x \to \infty} G_\kappa(x, y) = 0. \tag{5.B.3}$$

It is well known that such a Green's function does not exist in dimension 2, however the derivative $\nabla_{x,i} G_\kappa$ does exist in dimension 2, in particular one can make sense of expressions as $G_\kappa(x_1, y) - G_\kappa(x_2, y)$. Formally one can define ∇G_κ by adding a mass m^2 to the Laplace operator, i.e., consider the Green's function of $\Delta_\kappa + m^2$ and then send $m^2 \to 0$. The following estimates hold for the Green's function.

Lemma 5.B.2. *For any $d \geq 3$ and $\kappa \in M(c_-, c_+)$ the estimate*

$$0 \leq G_\kappa(x, y) \leq \frac{C}{|x-y|^{d-2}} \tag{5.B.4}$$

holds where the constant C depends only on c_-, c_+, and d. Moreover there exist $\alpha > 0$ depending on c_+/c_-,and d and C depending on c_-, c_+, and d such that for $d \geq 2$

$$|\nabla_x G_\kappa(x, y)| \leq \frac{C}{|x-y|^{d-2+\alpha}}, \tag{5.B.5}$$

$$|\nabla_x \nabla_y G_\kappa(x, y)| \leq \frac{C}{|x-y|^{d-2+2\alpha}}. \tag{5.B.6}$$

Proof. These estimates are well known. Estimates for the corresponding parabolic Green's function are called Nash-Aronson estimates and they can be found, e.g., in Proposition B.3 in [94]. Integrating the bound for the parabolic Green's function implies (5.B.4). The estimates (5.B.5) and (5.B.6) follow for $d > 2$ from (5.B.4) and Lemma 5.B.1. For $d = 2$ one can bound the oscillation of the Green's function using Nash-Aronson estimates and the parabolic Nash-Moser estimate. In particular as shown, e.g., in [9, Chapter 8] there is a constant $C = C(c_-, c_+)$ such that for all $r > 0$

$$\sup_{x,y \in B_{2r}(0) \setminus B_r(0)} |G_\kappa(x, 0) - G_\kappa(y, 0)| \leq C. \tag{5.B.7}$$

Lemma 5.B.1 then implies (5.B.5) and (5.B.6). □

The previous results allow us to bound the difference between the Green's function in a set with Dirichlet boundary conditions and the Green's function on whole space. We define the Green's function $G_\kappa^{\Lambda^w}: \Lambda \times \Lambda \to \mathbb{R}$ with Dirichlet boundary values in finite volume by

$$\begin{aligned} \Delta_\kappa G_\kappa^{\Lambda^w}(\cdot, y) &= \delta_y \quad \text{in } \mathring{\Lambda}, \\ G_\kappa^{\Lambda^w}(x, y) &= 0 \quad \text{for } x \in \partial\Lambda. \end{aligned} \tag{5.B.8}$$

For clarity we write $G_\kappa^{\mathbb{Z}^d} = G_\kappa$ in the following.

Lemma 5.B.3. *Let $0 < c_- < c_+ < \infty$ and $R > 0$, then*

$$\lim_{n\to\infty} \sup_{\substack{x,y\in B_R(0)\\ 1\le i,j\le d}} \sup_{\kappa\in M(c_-,c_+)} \left| \left(\delta_{x+e_i} - \delta_x, G_\kappa^{\Lambda_n^w}(\delta_{y+e_j} - \delta_y)\right) - \left(\delta_{x+e_i} - \delta_x, G_\kappa^{\mathbb{Z}^d}(\delta_{y+e_j} - \delta_y)\right) \right| = 0. \tag{5.B.9}$$

Remark 5.B.4. *Note that the two scalar products can be rewritten as $\nabla_{x,i}\nabla_{y,j}G_\kappa^{\Lambda_n^w}(x, y)$ and $\nabla_{x,i}\nabla_{y,j}G_\kappa^{\mathbb{Z}^d}(x, y)$. This expression agrees with the gradient correlations of a Gaussian field:*

$$\mathbb{E}_{(\Delta_\kappa^{\Lambda_n^w})^{-1}} \left(\eta_{x,x+e_i}\eta_{y,y+e_j}\right) = \nabla_{x,i}\nabla_{y,j}G_\kappa^{\Lambda_n^w}(x, y). \tag{5.B.10}$$

A similar equation holds when Λ_n^w is replaced by $\mathbb{Z}^d$. Thus the lemma implies local uniform convergence of the covariance matrix of those two gradient Gaussian fields.

Proof. The difference of the Green's function in $d > 2$ can be expressed through the corrector $\varphi_{\kappa,n,y}: \Lambda_n \to \mathbb{R}$ that is defined by

$$G_\kappa^{\Lambda_n}(\cdot, y) = G_\kappa^{\mathbb{Z}^d}(\cdot, y) - \varphi_{\kappa,n,y}(\cdot). \tag{5.B.11}$$

Using the definition of the Green's function we obtain that the corrector satisfies

$$\begin{aligned} \Delta_\kappa \varphi_{\kappa,n,y} &= 0 \quad \text{in } \mathring{\Lambda}_n \\ \varphi_{\kappa,n,y}(x) &= G_\kappa^{\mathbb{Z}^d}(x, y) \quad \text{for } x \in \partial\Lambda_n. \end{aligned} \tag{5.B.12}$$

The estimate (5.B.4) in Lemma 5.B.2 now implies

$$|\varphi_{k,n,y}(z)| \le \frac{C}{|\mathrm{dist}(y, \partial\Lambda_n)|^{d-2}} \tag{5.B.13}$$

for $z \in \partial\Lambda_n$. By the maximum principle for Δ_κ the bound extends to all $z \in \Lambda_n$. The claim then follows from

$$\begin{aligned} &\left(\delta_{x+e_i} - \delta_x, G_\kappa^{\mathbb{Z}^d}(\delta_{y+e_j} - \delta_y)\right) - \left(\delta_{x+e_i} - \delta_x, G_\kappa^{\Lambda_n^w}(\delta_{y+e_j} - \delta_y)\right) \\ &= \varphi_{\kappa,n,y}(x) + \varphi_{\kappa,n,y+e_j}(x + e_i) - \varphi_{\kappa,n,y}(x + e_i) - \varphi_{\kappa,n,y+e_j}(x) \\ &= \nabla_i\varphi_{\kappa,n,y+e_j}(x) - \nabla_i\varphi_{\kappa,n,y}(x). \end{aligned} \tag{5.B.14}$$

The extension to dimension $d = 2$ is again slightly technical. We can define

$$\varphi_{\kappa,n,y}(\cdot) = \left(G_\kappa^{\mathbb{Z}^d}(\cdot, y) - G_\kappa^{\mathbb{Z}^d}(y, y)\right) - G_\kappa^{\Lambda_n}(\cdot, y). \tag{5.B.15}$$

The corrector satisfies

$$\begin{aligned} \Delta_\kappa \varphi_{\kappa,n,y} &= 0 \quad \text{in } \mathring{\Lambda}_n \\ \varphi_{\kappa,n,y}(x) &= G_\kappa^{\mathbb{Z}^d}(x,y) - G_\kappa^{\mathbb{Z}^d}(y,y) \quad \text{for } x \in \partial\Lambda_n. \end{aligned} \tag{5.B.16}$$

Using (5.B.7) we can bound for $y \in B_{n/2}(0)$

$$\max_{x\in\partial\Lambda_n} \varphi_{k,n,y}(x) - \min_{x\in\partial\Lambda_n} \varphi_{k,n,y}(x) \leq C. \tag{5.B.17}$$

Lemma 5.B.1 implies $\nabla\varphi_{\kappa,n,y}(x) \leq Cn^{-\alpha}$ for $x \in \Lambda_{n/2}$ and we can conclude using (5.B.14). □

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