Open Topological Field Theories and 2-Segal Objects

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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aus
Santa Cruz, CA, USA

Bonn, 2019

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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2. Gutachter: Prof. Dr. Catharina Stroppel
Tag der Promotion: 27. September, 2019
Erscheinungsjahr: 2019

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Zusammenfassung

In this thesis we analyze 2-dimensional open topological field theories in both 1-categorical and ∞-categorical contexts. Making use of the formalism, introduced by Dyckerhoff and Kapranov, of graphs structured over a crossed simplicial group $\Delta\mathfrak{G}$, we give combinatorial models for 2-dimensional open cobordism categories with additional structure — orientations, N-spin structures, etc. We then use this model to effect a classification of the corresponding classes of 1-categorical topological field theories. This classification retrieves, in special cases, a number of results known in the literature, as well as providing new results.

We then turn to 2-dimensional open oriented topological field theories valued in an ∞-category $\mathrm{Span}(\mathcal{C})$ of spans in an ∞-category $\mathcal{C}$. Applying a theorem stated by Lurie in [33], such topological field theories are classified by Calabi-Yau algebras in $\mathrm{Span}(\mathcal{C})$. We define two 1-categories whose functors to $\mathcal{C}$ parameterize, respectively, associative algebras and Calabi-Yau algebras in $\mathrm{Span}(\mathcal{C})$. We prove that there is an equivalence of ∞-categories between associative algebras in $\mathrm{Span}(\mathcal{C})$ and 2-Segal simplicial objects in $\mathcal{C}$; and we prove an equivalence of ∞-categories between Calabi-Yau algebras in $\mathrm{Span}(\mathcal{C})$ and 2-Segal cyclic objects in $\mathcal{C}$. We discuss the invariants the resultant topological field theories assign to surfaces, and develop the example provided by cyclic structures on Čech nerves.

Acknowledgements

I have accrued a great many debts throughout the course of my doctoral studies. Thanks are due first and foremost to my advisor, Tobias Dyckerhoff, who has been patient, helpful, unstintingly generous with his time, and on the whole a far better mentor than I could have hoped for or deserved. Discussions with Nils Carqueville, Hans Jockers, and Chris Brav at various times enriched my studies, and I am quite grateful to them. Conversations with Lorant Szegedy about structured graphs and their relation to his PLCW decompositions have aided me as I revised my work on the former for inclusion here.

I have also had the good fortune to have had excellent officemates and coevals in my studies. Fernando has been quite patient in putting up with me while I prepared this document, and in helping me clarify my thinking and writing. Tashi seems at equal ease discussing go and math, and made both enjoyable. Thomas is no longer with us, but I remember with gratitude his good humor and helpful discussions and comments. The term 'balanced crossed simplicial group', which appears in this thesis, was his suggestion.

My friends and family have helped me immensely — without their support I could not have made it this far. Luigi, Raphael, and Jonas added color and variety to my life in Bonn, just as Vincent and Fernando have done in Hamburg. My parents, grandmother, and brother have been wonderfully supportive throughout my doctoral studies, and I cannot thank them enough.

I am also deeply grateful to the Max Planck Institute for Mathematics and Universität Hamburg for providing support during my studies.

Contents

Introduction

Topological field theories

The functorial formalization of the notion of a topological field theory first appears in [2], published by Atiyah in 1988. This paper furnishes the axioms which, in a slightly different form, are still used today as the definition of a topological field theory. In modern terminology, this definition amounts to saying that a topological field theory is given by a symmetric monoidal functor from a cobordism category, i.e. a symmetric monoidal category whose objects are closed $n-1$-manifold, whose morphisms $N_1 \to N_2$ are equivalence classes of n-manifolds M equipped with an isomorphism $\partial M \cong N_1 \amalg N_2$, and whose monoidal product is the disjoint union.

Though Atiyah provided the first formalization of topological field theory, some examples had already appeared in the literature. Notably, elements of Chern-Simons theory, which would later be expanded by Witten [43] and others (e.g. [22]), had already appeared in a classical form before 1988, see, e.g. [12]. Chern-Simons theory and the Seiberg-Witten invariants of [42], among many others, provide geometric examples of interesting topological field theories.

While topological field theory holds some interest for physicists as a toy model of quantum field theory, and as the result of the so-called 'topological twist' from string theory, mathematicians tend to approach it as a way of analyzing invariants of manifolds. To wit: the functoriality condition is something like the Seifert-van-Kampen theorem on steroids. Invariants which satisfy it can be built from the invariants assigned to a wide variety of allowed 'slicings' of the manifold.

Classification results in two dimensions

From a mathematical point of view, it is natural to ask what kind of manifold invariants can arise from topological field theories. In two dimensions, classification theorems for topological field theories defined on oriented cobordisms have been folkloric since the inception of topological field theory as a field of study. In the 'closed' case, where the objects of the bordism category are disjoint unions of circles, a formal proof first appeared in [1], showing an equivalence of categories between two-dimensional closed topological field theories and commutative Frobenius algebras.

A number of other variants of two-dimensional topological field theory have proven amenable to classifcation. Open-closed topological field theories (where objects can be disjoint unions of circles and intervals) have been shown to be equivalent to 'knowledgeable Frobenius algebras' in [29]. In [8], operadic methods are used to show that open unoriented topological field theories are equivalent to Frobenius algebras with trace-preserving anti-involution.

Extending: ∞-categorical topological field theories

In the seminal paper [3], Baez and Dolan argue that a 1-categorical viewpoint is insufficient to properly understand topological field theories, instead proposing a weak n-category whose objects are disjoint unions of points, 1-morphisms are 1-manifolds with boundary, 2-morphisms are 2-manifolds with corners, and so on up to n. In the paper, they propose what they call the *Extended TQFT Hypothesis*, namely, that such an 'extended' topological field theory is equivalent to its value on a point, which must satisfy dualizability conditions.

Lurie took up this idea in [33], expanding the notion of an extended topological field theory by including the mapping spaces between n-morphisms in his definition. There, he rechristens the Extended TQFT Hypothesis the *Baez-Dolan Cobordism Hypothesis* — the name under which it is now most commonly known — and sketches a proof in the language of (∞, n)-categories.

Much work has been done on constructions and classification results for higher-categorical topological field theories in the decades following Baez and Dolan's paper. Examples include Turaev-Viro theory ([27, 4, 5]) as a 3-2-1 topological field theory and Chern-Simons theory ([21]) as a fully extended topological field theory. Fully extended 2d theories were classified by Schommer-Pries in [39]. Also in two dimensions, Costello in [11] provided a classification theorem using a variant of the open cobordism category whose mapping spaces consisted of singular chains on the corresponding component of the moduli space of Riemann surfaces. This latter result, or rather its analogue in [33], is of particular interest for us, and classifies open 2d topological field theories in terms of Calabi-Yau algebra objects (see below).

Segal conditions

The classical Segal conditions (which, for reasons which will shortly become apparent, we will refer to as the 1-Segal condition), appear in the work of Segal on Γ-spaces [40]. The term 'Segal space' as it applies to simplicial spaces was introduced by Rezk in [37]. In a nutshell, the Segal conditions on a simplicial object $X : N(\Delta)^{op} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ say that the natural maps

$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

must be equivalences. The underlying algebraic meaning of this condition may be glimpsed by considering the span

$$X_1 \times_{X_0} X_1 \xleftarrow{\{0,1\},\{1,2\}} X_2 \xrightarrow{\{0,2\}} X_1, \tag{1}$$

which is well defined for any simplicial object. Under the 1-Segal condition, however, the left hand morphism has a homotopy inverse, defining a 'multiplication map'

$$X_1 \times_{X_0} X_1 \longrightarrow X_1 \tag{2}$$

up to contractible choice. The 1-Segal conditions also encode coherent associativity data, yielding, under appropriate conditions algebra objects in $\mathcal{C}$.

Particularly in cases where X_0 is a terminal object, the 1-Segal condition is widely used to encode homotopy coherent associative algebraic structures. It appears for instance, in the definitions of monoidal ∞-categories, ∞-operads, and monoid objects in ∞-categories, as well as (in a strict form) in the conditions necessary and sufficient for a simplicial set to be the nerve of a category.

Given its relation to homotopy associativity, it is not surprising that the 1-Segal condition is in some way related to the topology of the interval. Per [16], if we let $I[n]$ be the simplicial set

$$\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$

and define

$$(X, I[n]) := \lim_{\Delta^p \to I[n]} X_p \tag{3}$$

in $\mathcal{C}$, then X is 1-Segal if and only if, for every $n \geq 1$, the morphism $X_n \to (X, I[n])$ induced by the embedding $I[n] \to \Delta^n$ is an equivalence.

The 2-Segal conditions

The 2-Segal conditions, introduced by Dyckerhoff and Kapranov in [15], and independently by Gálvez-Carrillo, Kock, and Tonks in [23] (where 2-Segal spaces are called *decomposition spaces*), provide a generalization of the 1-Segal conditions that can be viewed from several perspectives.

Topologically, the 2-Segal conditions can be seen as a 2-dimensional generalization of (3). There, the space $(X, I[n])$ is given as a limit parameterized by a subdivision of the interval, expressed combinatorially via the simplicial set $I[n]$. In analogy to this, one may consider a convex $n+1$-gon P_{n+1} with vertices $\{0, 1, \ldots, n\}$, together with a triangulation $\mathcal{T}$ of P_{n+1} with vertices in $\{0, 1, \ldots n\}$. The triangles of $\mathcal{T}$ can be considered as 2-simplices of Δ^n, giving rise to a 2-dimensional simplicial subset $\Delta^{\mathcal{T}} \subset \Delta^n$ consisting of precisely these triangles. The 2-Segal conditions on a simplicial object $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ are then precisely that, for all $n \geq 2$, and every triangulation $\mathcal{T}$ of P_{n+1} the induced morphism

$$X_n \to \lim_{\Delta^k \to \Delta^{\mathcal{T}}} X_k$$

is an equivalence (see [16] for more details). By a similar construction, in the presence of a cyclic structure the 2-Segal condition can be read as a way to canonically assign invariants to a triangulated surface, as in [14]. Of natural interest, therefore, is the question of what – if any – functoriality these invariants are possessed of. Of particular interest for the purposes of this thesis are whether these invariants organize into some kind of topological field theory.

There is also another equivalent formulation for the 2-Segal conditions, one often more amenable to direct computation. In the reformulation, we require that for all $0 \leq i < j \leq n$, the diagram

$$\begin{array}{ccc} X([n]) & \longrightarrow & X(\{0,\dots,i,j,\dots,n\}) \\ \downarrow & & \downarrow \\ X(\{i,\dots,j\}) & \longrightarrow & X(\{i,j\}) \end{array}$$

is pullback in $\mathcal{C}$. This provides a more algebraic perspective on the 2-Segal conditions, acting as a generalization of the interpretation of the 1-Segal conditions in terms of the multiplication (2). Roughly speaking, the 2-Segal conditions retain the associativity data encoded in a 1-Segal object, while forgetting that the the left-hand morphism in the span (1) is a weak equivalence.

More precisely, we can think of the span

$$\overbrace{X_1 \times X_1 \times \cdots \times X_1}^{\times n} \xleftarrow{\{0,1\},\dots,\{n-1,n\}} X_n \xrightarrow{\{0,n\}} X_1$$

as a kind of n-ary operation in an ∞-category $\mathrm{Span}(\mathcal{C})$ whose morphisms are spans in $\mathcal{C}$, even absent any conditions on the left-hand morphism. Thinking of the pullback as a kind of composite, the 2-Segal condition then ensures that, up to equivalence, composing the n-ary operation defined by X with the m-ary operation, one obtains the $n+m-1$-ary operation. This gives rise to constructions associating to a 2-Segal object X an algebra object in an ∞-category whose morphisms are spans. Such constructions have appeared in numerous places in the literature, see e.g. [15] and [35].

One natural question to ask about such algebra objects is whether (or when) they are unital. The answer to this question — which was already present in [15] — gives rise to an additional condition, that for all $0 \leq i \leq n$ the square

$$\begin{array}{ccc} X(\{0,1,\dots,\hat{\imath},\dots,n\}) & \longrightarrow & X([n]) \\ \downarrow & & \downarrow \\ X(\{i\}) & \longrightarrow & X(\{i,i\}) \end{array}$$

is pullback in $\mathcal{C}$, where the two copies of i on the bottom right are considered as distinct. In the algebraic interpretation of 2-Segal spaces, this corresponds to saying that the span

$$* \longleftarrow X_0 \xrightarrow{\{0,0\}} X_1$$

acts as a unit for the multiplication. The formalism of decomposition spaces in [23] always includes the unitality condition. Throughout this thesis, the term 2-*Segal space* will always refer to what is elsewhere called a *unital* 2-*Segal space*. This abuse of terminology began as a means to ease reading, but acquired a *post-hoc* justification in the paper [17], which proves — rather surprisingly — that every 2-Segal object is, in fact, unital 2-Segal.

The algebraic view of 2-Segal objects — that they encode coherently associative multiplications — has been neatly encapsulated by Walde in [41]. There, using the dendroidal spaces formalism of Cisinski and Moerdijk from [9], he shows that there is an equivalence of ∞-categories between 2-Segal simplicial spaces and invertible ∞-operads.

There are further generalizations of the 1-Segal conditions to yet higher dimensions, already alluded to in [15]. A definition of these higher Segal conditions, as well as a generalization of the Waldhausen S-construction satisfying them, is given in [36].

Cyclic objects

The topological interpretation of 2-Segal objects as providing invariants of polygons that can be computed from triangulations has a natural extension to invariants of surfaces. When working with simplicial objects one is obliged to fix a linear order on the vertices of the polygon P_{n+1}, and *a priori*, there is not a way of relating two such choices. If one works instead with Connes's cyclic category Λ (defined in [10]), and considers cyclic objects $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$ the underlying simplicial objects of which satisfy the 2-Segal condition, then the construction no longer depends on this linear order, but rather on a cyclic order on the vertices, i.e. an orientation on P_{n+1}. Given a stable, oriented marked surface (S, M), and a 2-Segal cyclic object $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$, one can then define an invariant of (S, M) valued in $\mathcal{C}$ by triangulating S with respect to M, and taking a limit of X over the category of simplices of this triangulation.

As shown in [14], this construction is one example of a more general phenomenon. The correspondence between certain of the *crossed simplicial groups* of [19] and [28] (crossed simplicial groups are called *skew-*

simplicial groups in the latter) and the planar Lie groups establishes a mechanism whereby an object $X : \Delta\mathfrak{G}^{\mathrm{op}} \to \mathcal{C}$ over a planar crossed simplicial group $\Delta\mathfrak{G}$ gives rise to an invariant of marked surfaces equipped with reduction of structure group to the corresponding planar Lie group G. In the special case of the crossed simplicial group Λ, the corresponding planar Lie group is $\mathrm{GL}^+(2, \mathbb{R})$, and the invariant retrieved is precisely the aforementioned invariant, which we will denote by $X(S, M)$, of oriented marked surfaces.

Walde's equivalence between ∞-operads and 2-Segal spaces also has a cyclic avatar. Namely, it is proved in [41] that there is an equivalence of ∞-categories between 2-Segal cyclic spaces and invertible cyclic ∞-operads, using the definition of the latter given in the same paper.

Topological field theories revisited

Since 2-Segal objects in $\mathcal{C}$ give, on the one hand, associative algebraic structures in $\mathrm{Span}(\mathcal{C})$ and, on the other hand, invariants of surfaces, it is natural to ask whether the latter association may, with the help of the former, be extended to give topological field theories valued in $\mathrm{Span}(\mathcal{C})$. The primary goal of this thesis is to answer this question in the affirmative — in fact, to provide a stronger result, namely that *every* topological field theory in spans comes from a 2-Segal cyclic object.

Calabi-Yau algebras

To achieve this goal, we make use of [33, Thm. 4.2.14]:

Theorem. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the following types of data are equivalent:*

1. *Open, oriented topological field theories valued in $\mathcal{C}$.*

2. *Calabi-Yau Algebras in $\mathcal{C}$.*

This equivalence is implemented by carrying a topological field theory Z to the Calabi-Yau algebra $Z([0, 1])$.

Lurie states this theorem without proof, instead referring to the work [11] by Costello, which proves a variant of the theorem in a chain complex-enriched context.

This is a higher-categorical analogue of the 1-categorical classification of open topological field theories. Calabi-Yau algebras are an ∞-categorical analogue of Frobenius algebras — in a loose sense, they are associative algebra objects in $\mathcal{C}$ equipped with a homotopically non-degenerate, cyclically symmetric trace.

To the end of relating 2-Segal objects and open topological field theories in $\mathrm{Span}(\mathcal{C})$, we first prove

Theorem. *There is an equivalence of ∞-categories*

$$\left\{\begin{matrix}\text{2-}\textit{Segal simplicial}\\ \textit{objects in } \mathrm{Span}(\mathcal{C})\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{associative algebras}\\ \textit{in } \mathrm{Span}(\mathcal{C})\end{matrix}\right\}$$

This appears as Theorem 3.1.29 in the text. It is worth noting at this juncture that the functoriality of this result is different than one might expect. The ∞-category appearing on the right does *not* have generic natural transformations in spans as it's morphisms, but rather only those for which the left-hand morphism in each span is invertible. This strengthens previous results which construct an associative algebra object associated to a given 2-Segal object.

The addition of a cyclic structure then gives the desired correspondence between 2-Segal objects and topological field theories via

Theorem. *There is an equivalence of ∞-categories*

$$\left\{\begin{matrix}\text{2-}\textit{Segal cyclic}\\ \textit{objects in } \mathrm{Span}(\mathcal{C})\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{Calabi-Yau algebras}\\ \textit{in } \mathrm{Span}(\mathcal{C})\end{matrix}\right\}$$

This appears as Theorem 3.2.31 in the text, and has a functoriality similar to that of the previous theorem.

Outline of the thesis

As expanded on above, the confluence of algebraic and topological intuition present in the 2-Segal condition suggests its suitability for application to two-dimensional topological field theories. In particular, the state-sum construction of surface invariants from [14] and the connection between 2-Segal objects and algebras from [15] are suggestive of a correspondence between cyclic 2-Segal objects and two-dimension open topological field theories valued in and ∞-category of spans. A proof of this correspondence forms the main thrust of the present thesis.

Chapter 1

The first chapter is devoted to providing background and preliminaries for the proofs that follow. The chapter is divided into three sections, which present background in roughly the order it will become necessary in later chapters. More precisely, section 1.1 presents general background material, sections 1.2 and 1.3 present background for chapter 2, and section 1.4 presents background relevant to chapter 3.

The first of the sections recalls information about crossed simplicial groups, orders, and structured sets from [14, 19, 28]. It also covers the correspondence between a class of crossed simplicial groups (called planar crossed simplicial groups) and the planar Lie groups. An original definition, that of a balanced crossed simplicial group, which axiomatizes properties used in Chapter 2, is introduced, and its relation to planar crossed simplicial groups explored.

The second section briefly recapitulates the formalisms of structured graphs and structured surfaces from [14], as well as the 2-Segal condition from [15]. The final section of the chapter recalls and extends constructions of monoidal ∞-categories of spans from [15], as well as relevant background material from [31]. It also provides the definition of a Calabi-Yau algebra in a symmetric monoidal infinity category, and some lemmata characterizing the same.

Chapter 2

The second chapter makes use of the formalism of structured graphs and structured surfaces to prove a classification result for two-dimensional closed topological field theories whose cobordisms carry a reduction of structure group to a planar Lie group G. The end result is an equivalence of categories from the category of such topological field theories to the category of $\Delta\mathfrak{G}$-Frobenius algebras — a generalization of the usual notion of a Frobenius algebra. This classification retrieves the known results for oriented and unoriented topological field theories.

The chapter begins by defining the necessary cobordism categories, denoted respectively Cob^G and $\mathfrak{G}$-Bord. The latter has morphisms given by equivalence classes of augmented $\Delta\mathfrak{G}$-structured graphs, as recollected in Chapter 1. It then proceeds through a proof that these cobordism categories are equivalent, before classifying symmetric monoidal functors out of the latter in terms of $\Delta\mathfrak{G}$-Frobenius algebras. A final section explores the particular example of equivariant topological field theories.

Chapter 3

The third and final chapter is devoted to proving Theorem 3.1.29 and 3.2.31, as well as teasing out their implications for topological field theories. The first section proves the equivalence between associative algebras in $\mathrm{Span}(\mathcal{C})$ and 2-Segal simplicial objects in $\mathcal{C}$, and the second proves the equivalence between Calabi-Yau algebras in $\mathrm{Span}(\mathcal{C})$ and 2-Segal cyclic objects in $\mathcal{C}$. These sections have parallel structure, beginning with a 1-category parameterizing the desired algebra objects, localizing it, and then identifying the simplex category and the cyclic category, respectively, with subcategories of the localization.

The final section of the chapter is devoted to the topological field theories arising from this equivalence. In it, we first identify the invariants of surfaces defined by these field theories with the surface invariants $X(S, M)$ of [14]. We then provide a cyclic analogue of the Čech nerve of a morphism of spaces, and explore the examples of topological field theories arising from it. In particular, we explore a special, 1-categorical case in which spaces of local systems arise as the invariants assigned by the corresponding topological field theory.

1 Preliminaries

We begin by providing the necessary background for the constructions and proofs which will appear in the following chapters. Much of the material presented here will follow [14] and [19] (for crossed simplicial groups and related concepts); [15] (for ∞-categories of spans and 2-Segal objects); and [31], [32], and [30] (for general ∞-categorical notions). The internal organization of the chapter is such that, as much as possible, background topics will appear in the order in which they will become necessary in the sequel, i.e. section 1.1 is general background, sections 1.2 and 1.3 correspond to chapter 2, and section 1.4 corresponds to chapter 3.

1.1 Basic definitions

Definition 1.1.1. We denote by $\mathbb{\Delta}$ the *enlarged simplex category*, whose objects are finite nonempty linearly ordered sets, and whose morphisms are maps of sets preserving the linear order. We denote by Δ the *simplex category*, i.e. the skeletal subcategory on the standard linearly ordered sets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$. We will denote by Δ_+ and $\mathbb{\Delta}_+$ the *augmented simplex categories* which include the empty set.

We denote by $\mathbb{\nabla}_+$ the *enlarged augmented interval category*, the subcategory of $\mathbb{\Delta}$ with objects $S \in \mathbb{\Delta}$, and morphisms preserving maximal and minimal elements. We denote the skeletal subcategory on $[n]$ for $n \geq 0$ by ∇_+. We denote by $\mathbb{\nabla}$ and ∇ the *interval categories*, i.e. the full subcategories on objects of cardinality ≥ 2. ▮

Definition 1.1.2. The category of the standard finite sets $\underline{n} := \{1, 2, \ldots, n\}$ for $n \geq 0$ will be denoted Fin. The category of the standard finite pointed sets $\langle n \rangle := \underline{n} \amalg \{*\}$ will be denoted Fin_*. The category of all finite sets (resp. the category of all finite pointed sets) will be denoted by $\mathbb{F}\mathrm{in}$ (resp.

by $\mathbb{F}\mathrm{in}_*$). When convenient, we will denote by $\mathbb{\Gamma}$ (resp. by Γ) the opposites of the categories $\mathbb{F}\mathrm{in}_*$ (resp. Fin_*). Given a pointed set $S \in \mathbb{F}\mathrm{in}_*$, we denote by S° the set $S \setminus \{*\}$, where $*$ denotes the basepoint of S.

We additionally denote by $\mathcal{A}\mathrm{ss}$ the *associative operad*, i.e. the category whose objects are objects of $\mathbb{F}\mathrm{in}_*$, and whose morphisms $\phi : S \to T$ are morphisms in $\mathbb{F}\mathrm{in}_*$ equipped with a chosen linear order on the fiber $\phi^{-1}(i)$ for each $i \in T^\circ$. Composition is defined by composition in $\mathbb{F}\mathrm{in}_*$, together with the lexicographic orders. Note that there is a forgetful functor $\mathcal{A}\mathrm{ss} \to \mathbb{F}\mathrm{in}_*$, which equips $N(\mathcal{A}\mathrm{ss})$ with the structure of an ∞-operad in the sense of [31]. ∎

Construction 1.1.3 (Linear interstices)**.** Given a linearly ordered set $S \in \mathbb{\Delta}$ we define an *inner interstice* of S to be an ordered pair $(k, k+1) \in S \times S$, where $k+1$ denotes the successor to k. The set of inner interstices of S is, itself, a linearly ordered set, with the order

$$(k, k+1) \leq (j, j+1) \Leftrightarrow k \leq j$$

We will denote the linearly ordered set of inner interstices of S by $\mathbb{I}(S)$. Note that $\mathbb{I}([0]) = \emptyset$.

Given a linearly ordered set $S \in \mathbb{\Delta}_+$, let $\hat{S}$ be the set $\{a\} \amalg S \amalg \{b\}$, where b is taken to be maximal and a minimal. We define an *outer interstice* of S to be an inner interstice of $\hat{S}$. We will denote the linearly ordered set of outer interstices of S by $\mathbb{O}(S)$. Note that $\mathbb{O}(\emptyset) = \{(a, b)\}$.

We define functors

$$\mathbb{O} : \mathbb{\Delta}_+^{\mathrm{op}} \to \mathbb{\nabla}_+; \quad S \mapsto \mathbb{O}(S)$$

and

$$\mathbb{I} : \mathbb{\nabla}_+^{\mathrm{op}} \to \mathbb{\Delta}; \quad S \mapsto \mathbb{I}(S)$$

as follows (we will define $\mathbb{O}$ explicitly, the definition of $\mathbb{I}$ is similar). Given a morphism $f : S \to T$ in $\mathbb{\Delta}_+$, we define a morphism $\mathbb{O}(f) : \mathbb{O}(S) \to \mathbb{O}(T)$ by setting

$$\mathbb{O}(f)(j, j+1) = \begin{cases} (k, k+1) & f(k) \leq j \leq j+1 \leq f(k+1) \\ (a, a+1) & j \leq f(k)\ \forall k \in S \\ (b-1, b) & j \geq f(k)\ \forall k \in S. \end{cases}$$

Pictorially, we can represent the morphism $\mathbb{O}(f)$ as a forest as in Fig. 1.1, thinking leaves $j \in \mathbb{O}(T)$ as being attached to the root $k \in \mathbb{O}(S)$ if $\mathbb{O}(f)(j) = k$.

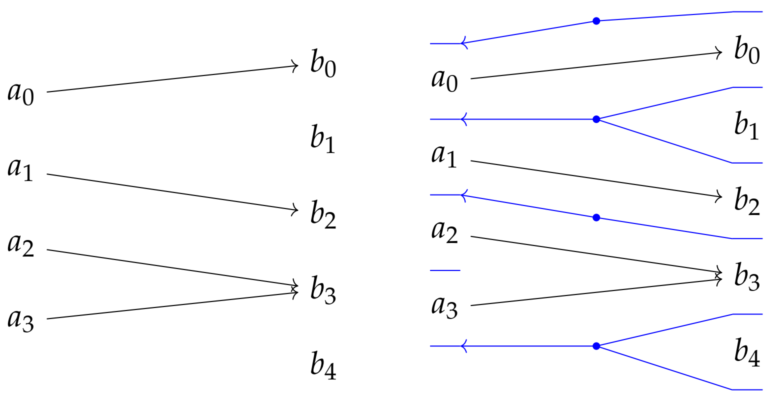


Figure 1.1: Left: a morphism f of linearly ordered sets. Right: the morphism $\mathbb{O}(f)$, visualized as a forest (blue).

Note that the functors $\mathbb{I}$ and $\mathbb{O}$ define an equivalence of categories. Since Δ_+ (resp. ∇_+) is the skeletal version of $\mathbb{\Delta}_+$ (resp. $\mathbb{\nabla}_+$), all isomorphisms in these categories are identities, we see that we get an induced *isomorphism* of categories

$$O : \Delta_+^{\mathrm{op}} \overset{\cong}{\longleftrightarrow} \nabla_+ : I$$

Moreover, we can define a functor $\mathbb{\nabla}_+ \to \mathrm{Fin}_*$ by

$$S \mapsto (S \amalg \{*\})_{/\max(S) \sim \min(s) \sim *}$$

We then find that the induced functor

$$\Delta_+^{\mathrm{op}} \hookrightarrow \Delta_+^{\mathrm{op}} \overset{O}{\to} \nabla_+ \to \mathrm{Fin}_*$$

is precisely the functor $\mathrm{cut} : \Delta^{\mathrm{op}} \to \mathrm{Fin}_*$ defined in [31, p. 4.1.2.9]. ∎

Definition 1.1.4. Given two linearly ordered sets $S, T \in \mathbb{\Delta}_+$ define the *ordinal sum* $S \oplus T$ to be the set $S \amalg T$ equipped with the linear order defined by the orders on S and T and the proscription that for all $s \in S$ and $t \in T$, $s \leq t$. The ordinal sum defines a monoidal structure on $\mathbb{\Delta}_+$.

Given two linearly ordered sets $S, T \in \mathbb{\nabla}_+$, with b the maximum of S and a the minimum of T, define the *imbrication* $S \star T$ to be the linearly ordered set $(S \oplus T)_{/a \sim b}$ (note that since a is the successor to b in $S \oplus T$, there is a canonical linear order on $S \star T$ compatible with the quotient map). ∎

Lemma 1.1.5. *The functor* $\mathbb{O}$ *is a monoidal functor sending the ordinal sum to the imbrication.*

Proof. Follows immediately from the definitions. □

Definition 1.1.6. A *cyclic order* on a finite, non-empty set S is a simply transitive action of $\mathbb{Z}/|S|$ on S. We will sometimes denote the action of $1 \in \mathbb{Z}$ on an element $s \in S$ as $s+1$. ▮

Construction 1.1.7. Given a finite non-empty set S with a cyclic order, and a collection $\{T_s\}_{s\in S}$ of elements $T_s \in \mathbb{\Delta}_+$ with at least one $T_s \neq \emptyset$, we define the *lexicographic cyclic order* on the set $\bigcup_{s\in S} T_s$ to be defined by the $\mathbb{Z}$-action

$$t+1 = \begin{cases} t+1 \in T_s & t \in T_s \text{ not maximal} \\ \min(T_{s+k_s}) & t \in T_s \text{ maximal} \end{cases}$$

where k_s is the smallest positive element in $\mathbb{Z}$ such that $T_{s+k_s} \neq \emptyset$ We will sometimes denote the set $\bigcup_{s\in S} T_s$ together with its lexicographic cyclic order as $\bigcup^S T_s$. ▮

Definition 1.1.8. A *morphism* of cyclically ordered sets $T \to S$ consists of a morphism $f : T \to S$ of underlying sets together with a choice of linear order on each fiber $f^{-1}(t)$ such that the induced lexicographic order on S agrees with the given cyclic order on S.

The *enlarged cyclic category* $\mathbb{\Lambda}$ is the category whose objects are finite non-empty sets with a cyclic order, and whose morphisms are morphisms of cyclically ordered sets. The *cyclic category* Λ is the skeletal subcategory on the standard cyclically ordered sets $\langle n\rangle := \{0, 1, \dots, n\}$. ▮

Construction 1.1.9 (Cyclic Duality)**.** In analogy to the construction of the linear interstice functors, we define a duality

$$\mathbb{D} : \mathbb{\Lambda}^{\mathrm{op}} \to \mathbb{\Lambda}$$

on the cyclic category. Let $S \in \mathbb{\Lambda}$ be a cyclicly ordered set. We define a *cyclic interstice* of S to be an ordered pair $(a, a+1) \in S \times S$, where $a+1$ denotes the successor of a under the cyclic order. We denote the set of cyclic interstices of S by $\mathbb{D}(S)$. The set $\mathbb{D}(S)$ inherits a canonical cyclic order from S, which can be visualized as in Fig. 1.2. The functor $\mathbb{D}$ is specified on morphisms by an analogue of Construction 1.1.3, namely, for $f : S \to T$ in $\mathbb{\Lambda}$, we set

$$\mathbb{D}(f)(j, j+1) := k \quad \text{where}$$

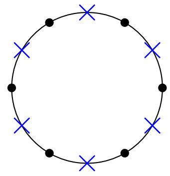


Figure 1.2: A cyclic set with its cyclic order visualized via an embedding into the oriented circle (black), together with its set of cyclic interstices (blue crosses).

This functor is an equivalence of categories.Since Λ is the skeletal version of $\mathbb{\Lambda}$, $\mathbb{D}$ descends to an equivalence $D : \Lambda^{\mathrm{op}} \to \Lambda$ ∎

Construction 1.1.10 (Cyclic closures)**.** We define a functor $\mathbb{K} : \mathbb{\Delta} \to \mathbb{\Lambda}$ in the following way. Given a linearly ordered set S of cardinality $n+1$, there is a unique order-preserving bijection $\phi : S \to [n]$. We define a bijection

$$S \to r(n); \quad j \mapsto \exp\left(\frac{2\pi i \phi(j)}{n+1}\right)$$

to the n^{th} roots of unity in S^1. The orientation on S^1 then yields a canonical cyclic order on S. Passing to skeletal versions yields the well-known functor $\kappa : \Delta \to \Lambda$.

Via the equivalences $\mathbb{O}$ and $\mathbb{D}$ we can then define a functor $\mathbb{C} : \mathbb{\nabla} \to \mathbb{\Lambda}$ such that the diagram

$$\begin{array}{ccc} \mathbb{\Delta}^{\mathrm{op}} & \xrightarrow{\mathbb{O}} & \mathbb{\nabla} \\ {\scriptstyle \mathbb{K}}\downarrow & & \downarrow{\scriptstyle \mathbb{C}} \\ \mathbb{\Lambda}^{\mathrm{op}} & \xrightarrow[\mathbb{D}]{} & \mathbb{\Lambda} \end{array}$$

commutes up to natural isomorphism. The functor $\mathbb{C}$ admits the following explicit description on objects. Let $S \in \mathbb{\nabla}$ with maximal element b and minimal element a. Then $\mathbb{C}(S)$ can be identified with with quotient of $\mathbb{K}(S)$ by the identification $a \sim b$. Once again, we have that $\mathbb{C}$ descends to a functor $C : \nabla \to \Lambda$. ∎

Definition 1.1.11. Given an object $S \in \mathbb{\Lambda}$, a *linear order on S compatible with the cyclic order* consists of a pair $([n], \phi)$ consisting of an object $[n] \in \Delta$, and an isomorphism $\phi : \mathbb{K}([n]) \cong S$.

We introduce one more equivalent variant of $\mathbb{\Lambda}$, which we will denote $\boldsymbol{\Lambda}$. The objects of $\boldsymbol{\Lambda}$ consist of pairs (S, ϕ) where $S \in \mathbb{\Lambda}$, and $\phi : \mathbb{K}([n]) \cong S$ is a compatible linear order on S. The morphisms of S are simply the morphisms of $\mathbb{\Lambda}$. It is clear that the forgetful functor $\mathbf{\Lambda} \to \mathbb{\Lambda}$ is an equivalence. ∎

Construction 1.1.12. The functor $\mathbb{K}$ clearly extends to a functor $\mathbf{K} : \mathbb{\Delta} \to \mathbf{\Lambda}$ by choosing the identity as the compatible linear order. We can then define functors $\mathbf{D} : \mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$ and $\mathbf{C} : \mathbb{\nabla} \to \mathbf{\Lambda}$ such that the diagram

$$\begin{array}{ccc} \mathbb{\Delta}^{\mathrm{op}} & \xrightarrow{\mathbf{O}} & \mathbb{\nabla} \\ {\scriptstyle \mathbb{K}}\downarrow & & \downarrow{\scriptstyle \mathbf{C}} \\ \mathbf{\Lambda}^{\mathrm{op}} & \xrightarrow[\mathbf{D}]{} & \mathbf{\Lambda} \end{array}$$

commutes *strictly*. ∎

Lemma 1.1.13. *Given $S \in \mathbb{\Lambda}$, a set $\{[n_i]\}_{i \in S}$ of elements in Δ_+, and a compatible linear order $\phi : \mathbb{K}([m]) \cong S$, there is a canonical isomorphism*

$$\mathbb{K}\left(\bigoplus_{i \in [m]} [n_{\phi(i)}]\right) \cong \bigcup^{S} [n_i]$$

which acts as the identity on underlying sets.

Proof. We compare the $\mathbb{Z}/n$-actions. When $j \in \bigoplus_{i \in [m]} [n_i]$ is not maximal, the successor function for the ordinal sum agrees with the $\mathbb{Z}/n$-action on $\bigcup^S [n_i]$. If j is maximal, we have that the action on the left sends j to $0 \in n_{\phi(0)}$, which agrees with the definition of the cyclic order on the right. □

1.2 Crossed simplicial groups

In this section, we will quickly introduce the theory of *crossed simplicial groups*, introduced by Krasauskas in [28] and Fiedorowicz and Loday in [19] in '87 and '91 respectively. We will then give an exposition of the correspondence between planar Lie groups and crossed simplicial groups, as well as the theories of *structured graphs* and *structured surfaces* from [14] which lie at the core of the classification of 2d open structured topological field theories.

Definition 1.2.1. A *crossed simplicial group* is a category $\Delta\mathfrak{G}$ equipped with an embedding $i : \Delta \to \Delta\mathfrak{G}$ which is bijective on objects, such that there is a bijection

$$\mathrm{CF}_{\Delta\mathfrak{G}} : \mathrm{Hom}_{\Delta\mathfrak{G}}(i[m], i[n]) \stackrel{\cong}{\to} \mathrm{Hom}_{\Delta}([m],[n]) \times \mathrm{Aut}_{\Delta\mathfrak{G}}([m])$$

inverse to composition. We call $\mathrm{CF}_{\Delta\mathfrak{G}}$ the *canonical factorization* of the crossed simplicial group. We will denote by $\mathfrak{G}_n$ the automorphism group $\mathrm{Aut}_{\Delta\mathfrak{G}}([n])$. ∎

Examples 1.2.2.

1. It is immediate the that simplex category itself is the unique crossed simplicial group whose objects have trivial automorphism groups.

2. The cyclic category Λ with the embedding $K : \Delta \to \Lambda$ is a crossed simplicial group. This is in some sense the prototypical example of the crossed simplicial groups we wish to study.

3. The category Fin is a crossed simplicial group with automorphism group of $i[n]$ given by the symmetric group S_{n+1}. When considering Fin as a crossed simplicial group, we will refer to it as the *symmetric crossed simplicial group*

4. The *braid crossed simplicial group* $\Delta\mathfrak{B}$. Objects are once again the standard ordinals, and morphisms are given by "generalized braids". The automorphism groups of $\Delta\mathfrak{B}$ are the braid groups B_n. See [28, Ex. 1.7 (iii)] or [19, p. 3.7] for more detailed accounts.

5. Following [14, Section I.2], we define a *signed linear order* on a finite set S to consist of a linear order on S together with a map of sets $\varepsilon_S : S \to \mathbb{Z}/2\mathbb{Z}$. Given a set T with a signed linear order, and a collection $\{S_t\}_{t\in T}$ of sets with signed linear orders, we form the *lexicographic signed linear order* on $\bigcup_{t\in T} S_t$ as follows. The linear order is simply the lexicographic linear order. We define

$$\varepsilon : \bigcup_{t\in T} S_t \to \mathbb{Z}/2\mathbb{Z}$$

by setting, for $s \in S_t$

$$\varepsilon(s) = \varepsilon_{S_t}(s) + \varepsilon_T(t).$$

The *Weyl crossed simplicial group* $\Delta\mathfrak{W}$ has morphisms $[n] \to [m]$ consisting of maps of sets $f : \{0, 1 \ldots, n\} \to \{0, 1 \ldots, m\}$, together with a choice of signed linear order on each fiber. Composition in $\Delta\mathfrak{W}$ is given by composition of maps of sets, together with the formation of lexicographic signed linear orders. The automorphism groups in the Weyl crossed simplicial group are given by the wreath products $\mathfrak{W}_n := \mathbb{Z}/2\mathbb{Z} \wr S_{n+1}$. See also [28, Ex. 1.2], [19, Thm 3.3], and [14, Prop. I.6].

Proposition 1.2.3. *Let $\Delta\mathfrak{G}$ be a crossed simplicial group. Then the assignment*

$$\lambda : \Delta\mathfrak{G} \to \mathrm{Fin}, \quad [n] \mapsto \mathrm{Hom}_{\Delta\mathfrak{G}}([0], [n])/\mathfrak{G}_0$$

defines a functor. In particular, for every $n \geq 0$, there is a canonical homomorphism of groups

$$\lambda_n : \mathfrak{G}_n \to S_{n+1}.$$

Proof. This is [14, Prop. I.5]. □

Construction 1.2.4. The canonical factorization for a crossed simplicial group $\Delta\mathfrak{G}$ gives the collection of automorphism groups $\mathfrak{G}_\bullet$ the structure of a simplicial set in the following canonical way. Let $g \in \mathfrak{G}_n$ and $\phi : \mathrm{Hom}_\Delta([m], [n])$, then by canonical factorization, there is a unique $\phi^*(g) \in \mathfrak{G}_m$ and a unique $g^*(\phi) \in \mathrm{Hom}_\Delta([m], [n])$ such that the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\phi} & [n] \\ {\scriptstyle \phi^*(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [m] & \xrightarrow[g^*(\phi)]{} & [n] \end{array}$$

commutes. ∎

Proposition 1.2.5. *A crossed simplicial group $\Delta\mathfrak{G}$ is a simplicial set $\mathfrak{G}_\bullet$ such that each $\mathfrak{G}_n$ is a group, together with a group homomorphism $\lambda_n : \mathfrak{G}_n \to S_{n+1}$ such that*

1. $d_i(gg') = d_i(g) d_{g^{-1}(i)}(g')$
2. $s_i(g, g') = s_i(g) s_{g^{-1}(i)}(g')$

3. *The diagrams*

$$\begin{array}{ccc} [n-1] & \xrightarrow{\delta_{g^{-1}(i)}} & [n] \\ {\scriptstyle d_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n-1] & \xrightarrow[\delta_i]{} & [n] \end{array} \quad \text{and} \quad \begin{array}{ccc} [n+1] & \xrightarrow{\sigma_{g^{-1}(i)}} & [n] \\ {\scriptstyle s_i(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n+1] & \xrightarrow{\sigma_i} & [n] \end{array}$$

commute in Fin*.*

Proof. This is [19, Prop. 1.7] □

Remark 1.2.6. In particular, every simplicial group is a crossed simplicial group such that the action of $\mathfrak{G}_n$ on $\mathrm{Hom}_\Delta([m],[n])$ is trivial. A particularly simple class of crossed simplicial groups which will play an outsized role in a number of constructions are the constant simplicial groups. ∎

Remark 1.2.7. In particular, Proposition 1.2.5 implies that given a crossed simplicial group $\Delta\mathfrak{G}$ and sequence of subgroups $\mathfrak{H}_n \subset \mathfrak{G}_n$ stable under the action of morphisms in Δ, there is a crossed simplicial subgroup $\Delta\mathfrak{H} \subset \Delta\mathfrak{G}$ with automorphism groups $\mathfrak{H}_n$.

In the special case of the Weyl crossed simplicial group $\Delta\mathfrak{W}$ of Examples 1.2.2 (5), this leads to a number of further examples:

1. The dihedral groups $D_{n+1} \subset \mathfrak{W}_n$ are stable under the action of morphisms in Δ. Therefore, we have the *dihedral crossed simplicial group* $\Xi \subset \Delta\mathfrak{W}$.

2. There are subgroups $\mathbb{Z}/2\mathbb{Z} \subset \mathfrak{W}_n$ generated by the automorphism

$$\zeta_n : [n] \to [n]$$

whose underlying map of sets sends $i \mapsto n-i$, and such that the sign on the fiber $\{i\}$ sends $i \mapsto 1 \in \{0,1\} = \mathbb{Z}/2\mathbb{Z}$. These subgroups are stable under the action of morphisms in Δ, and so define a crossed simplicial subgroup $\Delta\mathbb{Z}/2\mathbb{Z} \subset \Delta\mathfrak{W}$, called the *reflexive crossed simplicial group* . (note that this is *not* the constant simplicial group on $\mathbb{Z}/2\mathbb{Z}$).

3. The symmetric groups $S_{n+1} \subset \mathfrak{W}_n$ are stable under the action of morphisms in Δ, showing that $\mathrm{Fin} \subset \Delta\mathfrak{W}$.

4. The subgroup $\mathbb{Z}/2\mathbb{Z} \ltimes S_{n+1} \subset \Delta\mathfrak{W}$ generated by S_{n+1} and $\zeta_n$is similarly stable under Δ. This defines the *reflexosymmetric crossed simplicial group* $\Delta\mathfrak{R} \subset \Delta\mathfrak{W}$.

∎

1.2.0.1 Classification of crossed simplicial groups

Definition 1.2.8. Given two crossed simplicial groups $\Delta\mathfrak{G}$ and $\Delta\mathfrak{H}$, a *morphism of crossed simplicial groups* $F : \Delta\mathfrak{G} \to \Delta\mathfrak{H}$ is a functor of underlying categories commuting with the inclusions of Δ.

An *extension of crossed simplicial groups* is a pair of morphisms of crossed simplicial groups

$$\Delta\mathfrak{H} \xrightarrow{L} \Delta\mathfrak{G} \xrightarrow{F} \Delta\mathfrak{K}$$

such that, for every $n \geq 0$,

$$\mathfrak{H}_n \xrightarrow{L_n} \mathfrak{G}_n \xrightarrow{F_n} \Delta\mathfrak{K}$$

is a short exact sequence of groups. ∎

Lemma 1.2.9. *A morphism $F : \Delta\mathfrak{H} \to \Delta\mathfrak{G}$ of crossed simplicial groups is equivalently a map of simplicial sets $f : \mathfrak{H}_\bullet \to \mathfrak{G}_\bullet$ such that*

1. *each f_n is a group homomorphism, and*

2. *for all $n \geq 0$ the diagram*

$$\begin{array}{ccc} \mathfrak{H}_\bullet & \xrightarrow{f_n} & \mathfrak{G}_\bullet \\ & {\scriptstyle\lambda_n}\searrow \quad \swarrow{\scriptstyle\lambda_n} & \\ & S_{n+1} & \end{array}$$

commutes.

Proof. Immediate from the characterization in Proposition 1.2.5. □

Definition 1.2.10. We call a morphism $F : \Delta\mathfrak{G} \to \Delta\mathfrak{H}$ of crossed simplicial groups *surjective* if the corresponding map of simplicial sets is surjective in each degree. ∎

Proposition 1.2.11. *Let $\Delta\mathfrak{G}$ be a crossed simplicial group.*

1. *There is a canonical morphism of crossed simplicial groups*

$$F_{\mathfrak{G}} : \Delta\mathfrak{G} \to \Delta\mathfrak{W}.$$

which factors through a surjective morphism to a crossed simplicial subgroup $\Delta\mathfrak{G}'' \subseteq \Delta\mathfrak{W}$.

2. *There is an extension of crossed simplicial groups*

$$\Delta\mathfrak{G}' \longrightarrow \Delta\mathfrak{G} \xrightarrow{F_{\mathfrak{G}}} \Delta\mathfrak{G}''$$

where $\Delta\mathfrak{G}$ *is a simplicial group.*

Proof. This is [19, Thm 3.6], recapitulated in [14, Thm. I.7]. □

Proposition 1.2.12. *The crossed simplicial subgroups of* $\Delta\mathfrak{W}$ *are: the trivial crossed simplicial group* Δ, *the crossed simplicial group* $\Delta\mathbb{Z}/2\mathbb{Z}$, *the cyclic category* Λ, *the dihedral category* Ξ, *the symmetric crossed simplicial group* Fin, *the reflexosymmetric crossed simplicial group* $\Delta\mathfrak{R}$, *and* $\Delta\mathfrak{W}$ *itself*

Proof. See [28, Prop. 1.5] or [19, Prop. 1.5]. □

Definition 1.2.13. Similarly, the unique morphism $\omega_n \in \mathrm{Hom}_\Delta([n],[0])$, gives rise to a group homomorphism

$$\omega_n^* : \mathfrak{G}_0 \to \mathfrak{G}_n$$

induced by canonical factorization. We will call $\Delta\mathfrak{G}$ *semi-constant* if these homomorphisms are isomorphisms for every n. ▮

Remark 1.2.14. Of particular note in Proposition 1.2.12 are the rates at which the automorphism groups of the subgroups grow. These growth rates, listed in Fig. 1.3 place restrictions on the growth rates of crossed simplicial groups via the extensions from Proposition 1.2.11. ▮

1.2.0.2 $\Delta\mathfrak{G}$-orders

Definition 1.2.15. Let $\Delta\mathfrak{G}$ be a crossed simplicial group. A $\Delta\mathfrak{G}$-*order* on a finite, non-empty set S with $|S| = n+1$ consists of a right $\mathfrak{G}_n$-torsor $\mathcal{O}(S)$, together with a λ_n-equivariant map

$$\rho_S : \mathcal{O}(S) \to \mathrm{Isom}_{\mathbb{F}\mathrm{in}}([n], S).$$

▮

Subgroup $\Delta\mathfrak{G} \subset \Delta\mathfrak{W}$	$\lvert\mathfrak{G}_n/\omega_n^*(\mathfrak{G}_0)$
Δ	1
$\Delta\mathbb{Z}/2\mathbb{Z}$	1
Λ	$n+1$
Ξ	$n+1$
Fin	$(n+1)!$
$\Delta\mathfrak{R}$	$(n+1)!$
$\Delta\mathfrak{W}$	$2^n \cdot (n+1)!$

Figure 1.3: The growth rates of automorphism groups for the crossed simplicial subgroups of $\Delta\mathfrak{W}$.

Example 1.2.16. The triple $([n], \mathfrak{G}_n, \lambda_n)$ defines a $\Delta\mathfrak{G}$-order on the set $[n]$.

Examples 1.2.17.

1. A Δ-order on a set S specifies a single isomorphism $[n] \cong S$, i.e., a linear order on S.

2. A Λ-order on a set S consists of a collection of linear orders ϕ_i : $\{0, \ldots, n\} \cong S$ related by cyclic permutations of $\{0, \ldots, n\}$. For $s \in S$, and any i such that s is not maximal in the linear order ϕ_i, we can define $s + 1 := \phi_i(\phi_i^{-1}(s) + 1)$. This yields a simply transitive $\mathbb{Z}/(n + 1)$-action on S, i.e. a cyclic order.

Remark 1.2.18. The elements of $\mathcal{O}(S)$ can be thought of as trivializations of $\mathcal{O}(S)$ as a $\mathfrak{G}_n$-torsor. From this point of view, we wish to think of each $x \in \mathcal{O}(S)$ as providing an isomorphism from $[n] \in \Delta\mathfrak{G}$ to $(S, \mathcal{O}(S))$. A family of such isomorphisms under the $\mathfrak{G}_n$-action is a then a $\Delta\mathfrak{G}$-order. ▮

Definition 1.2.19. A *morphism of* $\Delta\mathfrak{G}$*-ordered sets* $\psi : (S, \mathcal{O}(S)) \to (T, \mathcal{O}(T))$ consists of a collection

$$\{\psi_{f,f'} \in \mathrm{Hom}_{\Delta\mathfrak{G}}([n], [m]) | f \in \mathcal{O}(T), f' \in \mathcal{O}(S)\}$$

where $|S| = n + 1$ and $|T| = m + 1$ such that, for every $g \in \mathfrak{G}_n$ and every $h \in \mathfrak{G}_m$,

$$\psi_{fh,f'g} = g^{-1} \circ \psi_{f,f'} \circ h. \tag{1.1}$$

Composition

$$(S, \mathcal{O}(S)) \xrightarrow{\psi} (S, \mathcal{O}(S)) \xrightarrow{\phi} (U, \mathcal{O}(U))$$

is given by the formula

$$(\phi \circ \psi)_{f,f'} := \phi_{f,f''} \circ \psi_{f'',f'}$$

for any $f'' \in \mathcal{O}(S)$.

We denote the category of $\Delta\mathfrak{G}$-ordered sets by $\mathcal{G}$. ∎

Remark 1.2.20. Since $\mathcal{O}(S)$ and $\mathcal{O}(T)$ are torsors, a morphism

$$\psi = \{\psi_{f,f'}\} : (S, \mathcal{O}(S)) \to (T, \mathcal{O}(T))$$

is uniquely determined by one of the $\psi_{f,f'}$ together with (1.1). ∎

Construction 1.2.21. It is clear that the assignment

$$[n] \mapsto ([n], \mathfrak{G}_n)$$

defines a functor $\epsilon_{\mathfrak{G}} : \Delta\mathfrak{G} \to \mathcal{G}$. We can define a weak inverse $\pi_{\mathcal{G}} : \mathcal{G} \to \Delta\mathfrak{G}$ by choosing, for every $\Delta\mathfrak{G}$-ordered set $(S, \mathcal{O}(S))$ a trivialization $f_S \in \mathcal{O}(S)$. The assignment

$$(S, \mathcal{O}(S)) \mapsto [|S| - 1]$$
$$\psi : (S, \mathcal{O}(S)) \to (T, \mathcal{O}(T)) \mapsto \psi_{f_T, f_S}$$

then defines the functor $\pi_{\mathcal{G}}$.

Moreover, given a morphism $\psi = \{\psi_{f,f'}\} : (S, \mathcal{O}(S)) \to (T, \mathcal{O}(T))$, we get an induced morphism of sets

$$\psi_* : S \xrightarrow{\rho_S(f')^{-1}} [n] \xrightarrow{\lambda(\psi_{f,f'})} [m] \xrightarrow{\rho_T(f)} T$$

which is independent of the trivializations f and f'. ∎

Proposition 1.2.22. *For any crossed simplicial group $\Delta\mathfrak{G}$ and any object $(S, \mathcal{O}(S)) \in \mathcal{G}$,*

1. *There is a canonical isomorphism of $\mathfrak{G}_n$-torsors*

$$\mathcal{O}(S) \cong \mathrm{Isom}_{\mathcal{G}}(\epsilon_{\mathfrak{G}}[n], (S, \mathcal{O}(S))).$$

2. *There is a canonical bijection*

$$S \cong \mathrm{Hom}_{\mathcal{G}}(\epsilon_{\mathfrak{G}}[0], (S, \mathcal{O}(S)))/\mathfrak{G}_0.$$

In particular, there is a functor $\lambda_{\mathcal{G}} : \mathcal{G} \to \mathbb{F}\mathrm{in}$, $(S, \mathcal{O}(S)) \mapsto S$, *such that the diagram*

$$\begin{array}{ccc} \Delta\mathfrak{G} & \xrightarrow{\epsilon_{\mathfrak{G}}} & \mathcal{G} \\ & {\scriptstyle \lambda} \searrow \quad \swarrow {\scriptstyle \lambda_{\mathcal{G}}} & \\ & \mathbb{F}\mathrm{in} & \end{array}$$

commutes.

Proof. This is [14, Prop. II.3]. □

1.2.1 Balanced and planar crossed simplicial groups

The collection of *planar crossed simplicial groups* is marked by its relation to Lie groups covering $O(2)$. It is precisely these planar crossed simplicial groups which will provide the interface between our combinatorial formalism and the topology and geometry of surfaces. However, the proofs that will be presented in the sequel hold in greater generality than simply for planar crossed simplicial groups, and some of the non-planar cases themselves provide interesting types of topological field theories. This being the case, we begin with an axiomatization of precisely the properties necessary for our combinatorial construction of field theories – the *balanced crossed simplicial groups*.

1.2.1.1 Balanced Crossed simplicial groups

Construction 1.2.23. Given a crossed simplicial group $\Delta\mathfrak{G}$, the morphisms

$$i_n : [0] \to [n], \quad 0 \mapsto i$$

in Δ define maps (*a priori* only of sets)

$$i_n^* : \mathfrak{G}_n \to \mathfrak{G}_0$$

via the canonical factorization. If we define the i^{th} *stabilizer subgroup*

$$\mathrm{Stab}_n(i) \subset \mathfrak{G}_n$$

to be the set of $g \in \mathcal{G}_n$ such that $\lambda_n(g)(i) = i$, we see that i_n^* defines a homomorphism

$$i_n^* : \mathrm{Stab}_n(i) \to \mathfrak{G}_0.$$

of groups. ∎

Remark 1.2.24. If a semi-constant crossed simplicial group $\Delta\mathfrak{G}$ is, in particular, a simplicial group, then for every $\phi \in \mathrm{Hom}_\Delta([n],[m])$, we can write the commutative diagram

$$\begin{array}{ccc} [n] & \xrightarrow{\ \phi\ } & [m] \\ & {\scriptstyle \omega_n} \searrow \quad \swarrow {\scriptstyle \omega_m} & \\ & [0] & \end{array}$$

in Δ. The 2-out-of-3 property then implies that ϕ^* is an isomorphism. Moreover, identifying $\mathfrak{G}_n$ with $\mathfrak{G}_0$ via ω_n^* makes ϕ^* the identity. Therefore, we see that $\Delta\mathfrak{G}$ is a *constant* simplicial group. ∎

Definition 1.2.25. A crossed simplicial group $\Delta\mathfrak{G}$ is called *balanced* if:

1. $\Delta\mathfrak{G}$ admits a duality

   $$D_{\mathfrak{G}} : \Delta\mathfrak{G} \xrightarrow{\cong} \Delta\mathfrak{G}^{\mathrm{op}}$$

   satisfying the conditions:

   a) $D_{\mathfrak{G}}$ is the identity on objects

   b) denoting by $\{i,j\}_n \in \mathrm{Hom}_\Delta([1],[n])$ the morphism which sends $0 \mapsto i$ and $1 \mapsto j$, then

   $$\lambda(D_{\mathfrak{G}}(\{i-1,i\}_n)) = \psi_i$$

   and

   $$\lambda(D_{\mathfrak{G}}(\{0,n\}_n) = \phi$$

   where $\psi_i^{-1}(1) = \{i\}$ and $\phi^{-1}(0) = \{0\}$.

2. The maps

   $$i_n^* : \mathrm{Stab}_n(i) \to \mathfrak{G}_0$$

   are isomorphisms.

3. $1_1^* = 0_1^*$ on $\mathrm{Stab}_1(1) = \mathrm{Stab}_1(0)$.

Note that the duality $D_{\mathfrak{G}}$ need not be unique. ∎

Example 1.2.26. The cyclic category Λ is a balanced crossed simplicial group. The cyclic duality of Construction 1.1.9 provides the necessary duality functor D_Λ. Conditions Items 2 and 3 are immediate from that fact that both the stabilizer groups and $\mathfrak{G}_0$ are trivial. The cyclic category is the prototypical example which the condition of balancedness is supposed to capture.

Definition 1.2.27. In the special case of $\mathfrak{G}_1$, we will use the notation $\mathfrak{G}_1^0 := \mathrm{Stab}_1(1) = \mathrm{Stab}_1(0)$. ▮

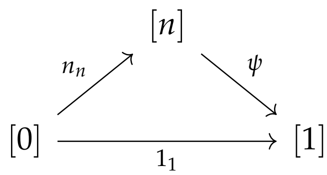
Lemma 1.2.28. *Let $\Delta\mathfrak{G}$ be a balanced crossed simplicial group, and let $\phi : [n] \to [1]$ be a morphism in $\Delta\mathfrak{G}$ such that $\lambda(\phi)^{-1}(1) = \{i\}$. Then ϕ induces an isomorphism*

$$\phi^* : \mathfrak{G}_1^0 \xrightarrow{\cong} \mathrm{Stab}_n(i)$$

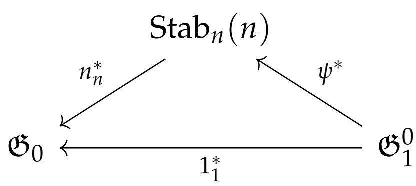
of groups.

Proof. By canonical factorization, we can write $\phi = \psi \circ g$, where $\psi \in \Delta$, $\psi^{-1}(1) = \{n\}$, and $\lambda(g)(i) = n$. The morphism ψ then induces a group homomorphism $\psi^* : \mathfrak{G}_1^0 \to \mathrm{Stab}_n(n)$, and, and conjugation by g induces an isomorphism $\mathrm{Stab}_n(n) \cong \mathrm{Stab}_n(i)$. The homomorphism ϕ^* is defined to be the composite of these two.

It therefore suffices to show that $\psi^* : \mathfrak{G}_1^0 \to \mathrm{Stab}_n(n)$ is an isomorphism. The diagram



commutes in Δ, and so we get a commutative diagram



of group homomorphisms. Since $\Delta\mathfrak{G}$ is balanced, 1_1^* and n_n^* are isomorphisms, so the lemma follows from the 2-out-of-3 property. □

Remark 1.2.29. The lemma also holds (with effectively the same argument) if we require that $\lambda(\phi)^{-1}(0) = \{i\}$ instead of requiring that $\lambda(\phi)^{-1}(1) = \{i\}$. ∎

Examples 1.2.30. We give some *non-examples* of balanced crossed simplicial groups.

1. The simplex category Δ is not a balanced crossed simplicial group, since, for $n \geq 1$

$$|\mathrm{Hom}_\Delta([n],[0])| \neq |\mathrm{Hom}_\Delta([0],[n])|,$$

   precluding the existence of a duality.

2. The braid crossed simplicial group $\Delta\mathfrak{B}$ is not balanced, since $\mathfrak{B}_0 \cong \{1\}$ is trivial, but $\mathrm{Stab}_n(i)$ is typically infinite.

Construction 1.2.31. Let H be a discrete group, and let $\Delta\mathfrak{G}$ be a balanced crossed simplicial group. The product $\Delta\mathfrak{G}\mathfrak{H} := \Delta\mathfrak{G} \times BH$ is a crossed simplicial group with

$$\mathfrak{G}\mathfrak{H}_n := \mathfrak{G}_n \times H.$$

The product of the duality $D_\mathfrak{G}$ with the canonical isomorphism of categories $BH \cong BH^{\mathrm{op}}$ yields a duality $D_{\mathfrak{G}\mathfrak{H}} : \Delta\mathfrak{G}\mathfrak{H} \to \Delta\mathfrak{G}\mathfrak{H}^{\mathrm{op}}$ which satisfies the conditions from Definition 1.2.25 (1). Similarly, since the simplicial maps act trivially on H, conditions (2) and (3) follow immediately for $\Delta\mathfrak{G}\mathfrak{H}$ from their counterparts for $\Delta\mathfrak{G}$. Consequently $\Delta\mathfrak{G}\mathfrak{H}$ is a balanced crossed simplicial group. ∎

Definition 1.2.32. Given a balanced crossed simplicial group $\Delta\mathfrak{G}$ and a morphism $\phi \in \mathrm{Hom}_{\Delta\mathfrak{G}}([n],[m])$, we will employ the notation $\phi^\vee := D_\mathfrak{G}(\phi)$. ∎

Proposition 1.2.33. *Every balanced simplicial group $\Delta\mathfrak{G}$ is an extension of Λ or Ξ by a constant simplicial group.*

Proof. Before starting the proof, we fix the notation

$$\Delta\mathfrak{G}' \xrightarrow{L} \Delta\mathfrak{G} \xrightarrow{F} \Delta\mathfrak{G}''$$

for the sequence from Proposition 1.2.11.

We first show that $\Delta\mathfrak{G}'$ is constant. Note that, since $\Delta\mathfrak{G}'$ is a simplicial group, the action $\mathfrak{G}_n \to S_{n+1}$ is trivial. Therefore, we have that $\mathrm{Stab}_n(i) =$

$\mathfrak{G}'_n$ for any $0 \leq i \leq n$, and hence, by Definition 1.2.25 (2), i_n induces an isomorphism

$$i_n^* : \mathfrak{G}'_n \cong \mathfrak{G}'_0.$$

Since $\omega_n \circ i_n = \mathrm{id}_{[0]}$, this implies that, for all n, ω_n^* is an isomorphism. By Remark 1.2.6, $\Delta\mathfrak{G}'$ is thus a constant simplicial group.

This also implies a condition on $\Delta\mathfrak{G}''$. If $\Delta\mathfrak{G}''$ is semi-constant then so is $\Delta\mathfrak{G}$, which precludes the existence of a duality on $\Delta\mathfrak{G}$. Therefore, $\Delta\mathfrak{G}''$ cannot be the trivial or reflexive crossed simplicial group.

We now prove a condition restricting the growth of the groups $\mathfrak{G}_n$. The maps $\omega_n : [n] \to [0]$ induce group homomorphisms

$$\omega_n^* : \mathfrak{G}_0 \to \mathcal{G}_n.$$

Let $k, h \in \mathfrak{G}_n$. Then we have an equality of cosets

$$(\omega_n^*(\mathfrak{G}_0))h = (\omega_n^*(\mathfrak{G}_0))k$$

if and only if, for all $g \in \mathfrak{G}_0$ there exists an $m \in \mathfrak{G}_0$ such that

$$(\omega_n^*(g))h = (\omega_n^*(m))k,$$

i.e. if and only if $g \circ \omega_n \circ k = m \circ \omega_n \circ h$.

By canonical factorization, we can uniquely write $\omega_n^\vee = \gamma \circ a$, with $\gamma \in \mathrm{Hom}_\Delta([0], [n])$ and $a \in \mathfrak{G}_0$. Passing through the duality $D_\mathfrak{G}$, we find that h and k define the same coset if and only if for all $g \in \mathfrak{G}_0$ there exists an $m \in \mathfrak{G}_0$ such that

$$k^\vee \circ \gamma \circ a \circ g^\vee = h^\vee \circ \gamma \circ a \circ m^\vee.$$

Taking canonical factorizations of $k^\vee \circ \gamma$ and $h^\vee \circ \gamma$, we can rewrite this condition as

$$(k^\vee)^*(\gamma) \circ \gamma^*(k^\vee) \circ a \circ g^\vee = (h^\vee)^*(\gamma) \circ \gamma^*(h^\vee) \circ a \circ m^\vee.$$

Consequently, we see that h and k will define the same coset if and only if $(k^\vee)^*(\gamma) = (h^\vee)^*(\gamma)$.

We thus have an injective map

$$\mathfrak{G}_n / (\omega_n^*(\mathfrak{G}_0)) \hookrightarrow \mathrm{Hom}_\Delta([0], [n]), \quad \omega_n^*(\mathfrak{G}_0)h \mapsto (h^\vee)^*(\gamma),$$

meaning that

$$\left| \frac{\mathfrak{G}_n}{\omega_n^*(\mathfrak{G}_0)} \right| \leq n + 1.$$

Now note that the functor F must be surjective, and that $F_n(\omega_n^*(\mathfrak{G}_0)) = \omega_n^*(\mathfrak{G}_0'')$. In particular, F induces surjective maps of sets

$$\mathfrak{G}_n/(\omega_n^*(\mathfrak{G}_0)) \to \mathfrak{G}_n''/(\omega_n^*(\mathfrak{G}_0'')).$$

consequently,

$$\left|\frac{\mathfrak{G}_n''}{\omega_n^*(\mathfrak{G}_0'')}\right| \leq n+1,$$

which in particular implies that $\Delta\mathfrak{G}$ can only be an extension of the trivial, reflexive, dihedral, or cyclic crossed simplicial groups. Since we have already ruled out the trivial and reflexive cases, this proves the proposition. □

1.2.1.2 Planar crossed simplicial groups

Proposition 1.2.34. *Given any crossed simplicial group $\Delta\mathfrak{G}$, the realization $G := |\mathfrak{G}_\bullet|$ is a topological group. Moreover, $|N(\Delta\mathfrak{G})| \simeq BG$.*

Proof. This is [19, Thm. 5.3 (i)] and [19, Thm. 5.13]. □

Remark 1.2.35. Proposition 1.2.34 generalizes the well-known result of Connes in [10] relating the cyclic category Λ to the circle S^1. Many of the related properties of cyclic objects can also be generalized. Of particular import for us is the underlying fact that the combinatorial structure of a crossed simplicial group carries significant topological content. For the 2-dimensional topological field theories we consider, cases related to the Lie group $O(2)$ will be of particular interest. ∎

Examples 1.2.36.

1. Per [10], the realization of $\Lambda_\bullet$ is $SO(2)$.
2. The realization of the simplicial set $\Xi_\bullet$ associated to the dihedral category is $O(2)$.

Remark 1.2.37. Examples 1.2.36 provide us with a topological interpretation of Proposition 1.2.33. Any balanced crossed simplicial group $\Delta\mathfrak{G}$ admits a surjective morphism $\Delta\mathfrak{G} \to \Lambda$ or $\Delta\mathfrak{G} \to \Xi$ of crossed simplicial groups. Passing to realizations, we get a surjective morphism $|\mathfrak{G}_\bullet| \to SO(2)$ or $|\mathfrak{G}_\bullet| \to O(2)$ of topological groups. In particular, this relates $|\mathbb{G}_\bullet|$ to the geometry of (oriented or unoriented) surfaces. ∎

Definition 1.2.38. A morphism of topological groups $p : \widetilde{G} \to G$ is called a *connective covering* if the following conditions are satisfied.

1. p is a covering of its image.
2. Let G_e be the connected component of the identity in G. Then $p^{-1}(G_e)$ is connected.

A connective covering is called a *proper covering* if it is, additionally, surjective. A connective covering $p : G \to O(2)$ is called a *planar Lie group*.

We denote by Con_G the category whose objects are connective coverings of G, and whose morphisms are morphisms of topological groups commuting with the projections to G. ▮

Proposition 1.2.39. *For each planar Lie group $p : G \to O(2)$, there is a crossed simplicial group $\Delta\mathfrak{G}$ such that the functor $F_{\mathfrak{G}}$ from Proposition 1.2.11 has target either Λ or Ξ and such that $|\mathfrak{G}| \cong G$.*

Proof. This is [14, Thm. I.33 (b1)]. □

Definition 1.2.40. We call the crossed simplicial groups corresponding to the planar Lie groups under Proposition 1.2.39 the *planar crossed simplicial groups*. ▮

Remark 1.2.41. There is a well-known classification of planar Lie groups which, coupled with the classification Proposition 1.2.11, allows us to list explicitly the correspondence between planar Lie groups and planar crossed simplicial groups. The correspondence is detailed in 1.4, and further exposed in [14, p. I.5.2]. ▮

Proposition 1.2.42. *Every planar crossed simplicial group is balanced.*

We defer the proof of Proposition 1.2.42 for a while, while we construct one further model for $\Delta\mathfrak{G}$-structures.

Proposition 1.2.43. *If $f : G \to H$ is a morphism of topological groups which is a homotopy equivalence on the underlying topological spaces, then pullback along f gives an equivalence of categories*

$$f^* : \mathrm{Con}_H \to \mathrm{Con}_G \,.$$

Proof. This is [14, Prop. I.30 (c)]. □

$\Delta\mathfrak{G}$	$\mathfrak{G}_n$	$G = \lvert\mathfrak{G}_\bullet\rvert$
Λ	$\mathbb{Z}/(n+1)$	$SO(2)$
Λ_N	$\mathbb{Z}/N(n+1)$	$\mathrm{Spin}_N(2)$
Λ_∞	$\mathbb{Z}$	$\mathbb{R} = \widetilde{SO(2)}$
Ξ	D_{n+1}	$O(2)$
Ξ_N	$D_{N(n+1)}$	$\mathrm{Pin}^+_N(2)$
Ξ_∞	D_∞	$\widetilde{O(2)}$
∇_M	$Q_{M(n+1)}$	$\mathrm{Pin}^-_{2M}(2)$

Figure 1.4: The planar Lie groups and corresponding planar crossed simplicial groups.

Definition 1.2.44. We define the *homeomorphism group of the circle* to be the topological group $\mathrm{Homeo}(S^1) := \mathrm{Aut}_{\mathrm{Top}}(S^1)$ with the compact-open topology. Proposition 1.2.43 implies that, given a planar Lie group $p : G \to O(2)$, we can pass along the homotopy equivalences

$$O(2) \to GL(2,\mathbb{R}) \to \mathrm{Homeo}(S^1)$$

to get a connective covering $p_{\mathrm{Homeo}} : \mathrm{Homeo}^G(S^1) \to \mathrm{Homeo}(S^1)$. We will call such coverings *planar homeomorphism groups*. ∎

Construction 1.2.45. Let C be a topological space homeomorphic to S^1 (which, in the sequel, we will simply refer to as a circle). Let $\rho : F \to \mathrm{Homeo}(S^1, C)$ be a reduction of structure group to G along p_{Homeo}. We will call the datum of C and ρ a *G-structured circle*.

Let C' and $\rho' : F' \to \mathrm{Homeo}(S^1, C')$ denote another such structured circle, and let $\mathrm{Homeo}^{\mathrm{Homeo}^G(S^1)}(F, F')$ denote the space of $\mathrm{Homeo}^G(S^1)$-equivariant homeomorphisms from F to F'. A *morphism of structured circles* $(C, \rho) \to (C', \rho')$ is then a pair $(\tilde{\phi}, \phi) \in \mathrm{Homeo}^{\mathrm{Homeo}^G(S^1)}(F, F') \times \mathrm{Homeo}(C, C')$ such that the diagram

$$\begin{array}{ccc} F, & \xrightarrow{\tilde{\phi}} & F' \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho'} \\ \mathrm{Homeo}(S^1, C) & \xrightarrow[\phi]{} & \mathrm{Homeo}(S^1, C') \end{array}$$

commutes. In a mild abuse of notation, we denote the space of such morphisms by $\mathrm{Homeo}^G(C, C')$.

Now let $J \subset C$ and $J' \subset C'$ be closed subsets homeomorphic to a disjoint union of copies of $[0,1]$. We will denote by $\mathrm{Homeo}^G((C,J),(C',J')) \subset \mathrm{Homeo}^G(C,C')$ the closed subspace of pairs $(\tilde{\phi},\phi)$ such that $\phi(J) \subset J'$. ∎

Definition 1.2.46. For a planar Lie group $G \to O(2)$, the *category of G-structured marked circles $\mathcal{C}_G$* has objects G-structured marked circles and hom-sets

$$\mathrm{Hom}_{\mathcal{C}_G}((C,J),(C,J')) := \pi_0\left(\mathrm{Homeo}^G((C,J),(C',J'))\right).$$

That is, it is the homotopy category of the topological category of marked structured circles. ∎

Proposition 1.2.47. *Let $p : G \to O(2)$ be a planar Lie group corresponding to a crossed simplicial group $\Delta\mathfrak{G}$. Then there is a functor*

$$\lambda_{\mathcal{C}_G} : \mathcal{C}_G \to \mathbb{F}\mathrm{in}, \quad (C,J) \mapsto \pi_0(J)$$

and an equivalence of categories $\pi : \mathcal{C}_G \to \mathcal{G}$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}_G & \xrightarrow{\ \pi\ } & \mathcal{G} \\ & {\scriptstyle \lambda_{\mathcal{C}_G}} \searrow \quad \swarrow {\scriptstyle \lambda_{\mathcal{G}}} & \\ & \mathbb{F}\mathrm{in} & \end{array}$$

commutes.

Proof. This is [14, Theorem II.13]. □

Remark 1.2.48. Though we will not reproduce the proof here, it will be of use to briefly write down the set and torsor associated to a structured marked circle (C,J), $F \to \mathrm{Homeo}(S^1,C)$. The set in question will simply be

$$I = \pi_0(J)$$

To define the 'torsor,' Let $\mathrm{Homeo}((S^1,[n]),(C,J))$ be the subspace of $\mathrm{Homeo}(S^1,C)$ which maps the standard set of $n+1$ marked points (roots of unity) into the intervals comprising J in such a way as to induce a bijection on connected components. Let F_J be the restriction of the bundle F to $\mathrm{Homeo}((S^1,[n]),(C,J))$. Then $\pi_0(F_J)$ can be given a canonical $\mathfrak{G}_n$-torsor structure such that the obvious map

$$\pi_0(F_J) \to \mathrm{Isom}_{\mathbf{FSet}}([n],I) = \pi_0\left(\mathrm{Homeo}\left((S^1,[n]),(C,J)\right)\right)$$

is equivariant. ∎

Construction 1.2.49 (Topological interstice duality)**.** Let G be a planar Lie group. Define a functor

$$D_{\mathcal{C}_G} : \mathcal{C}_G \to \mathcal{C}_G^{\mathrm{op}}$$

by sending $(C, J) \mapsto (C, \overline{C \setminus J})$ and $(\tilde{\phi}, \phi) \mapsto (\tilde{\phi}^{-1}, \phi^{-1})$. ∎

Proposition 1.2.50. *The functor $D_{\mathcal{C}_G}$ is an equivalence of categories.*

Proof. This is [14, Cor. II.17]. □

Corollary 1.2.51. *Any planar crossed simplicial group is balanced.*

Proof. Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group corresponding to the planar Lie group $p : G \to O(2)$. The properties of the duality from Definition 1.2.25 (1) follow from unwinding the definition of $D_{\mathcal{C}_G}$ in Construction 1.2.49.

To check condition (2), let (C, I, F) be a G-structured marked circle with $|\pi_0(I)| = 1$ and let (K, J, E) be a G-structured marked circle with $|\pi_0(J)| = n + 1$. Choose a morphism

$$(\tilde{\phi}, \phi) : (C, I) \to (K, J)$$

of structured circles sending I into a connected component $A \subset J$. We can identify this morphism with i_n in Δ by choosing connected components $x \in \pi_0(F_I)$ and $y \in \pi_0(E_A)$ trivializing the respective torsors.

An automorphism $(\gamma, \tilde{\gamma})$ of (K, J, E) which fixes $A \subset J$ can be uniquely specified by the connected component of E_A to which $\tilde{\gamma}$ sends y. Similarly, we can specify an automorphism $(\delta, \tilde{\delta})$ of (C, I, F) by specifying the connected component of $F_I)$ to which $\tilde{\delta}$ sends x. Since $\tilde{\phi}$ is a bundle map, it induces an bijection

$$\pi_0(F_I) \to \pi_0(E_A)$$

This bijection is precisely the map i_n^*, proving condition (2). The condition (3) can be checked case-by-case. □

1.3 Graphs and surfaces

We now recall the formalism of structured graphs developed in [14, Ch IV], which will form the basis of our combinatorial approach to the bordism category in Chapter 2.

1.3.1 Structured graphs

Definition 1.3.1. A *graph* Γ consists of:

1. A set H of *half-edges* of Γ.
2. A set V of *vertices* of Γ.
3. A map $s : H \to V$.
4. An involution $\eta : H \to H$.

For $v \in V$ we denote by $H(v)$ the set $s^{-1}(v)$ of *half-edges incident to v*. We additionally denote by E the pullback

$$\begin{array}{ccc} E & \longrightarrow & H \\ \downarrow & & \downarrow \eta \\ H & \xrightarrow[\mathrm{id}]{} & H \end{array}$$

i.e., the set of *edges* of Γ. A fixed point h of η (or the associated pair $\{h, h\}$) will be referred to as an *external half-edge* of Γ. ∎

Remark 1.3.2. The symbol Γ does double duty — first as the opposite of the category Fin_*, and second as a generic symbol for a graph. Fortunately for us, the former usage is mainly confined to Chapter 3, and the latter to Chapter 2. ∎

Definition 1.3.3. We call a graph Γ *compact* if it has no external half-edges (i.e. if η is fixed-point free). We denote by $M(\Gamma)$ the maximal compact subgraph of Γ. ∎

Construction 1.3.4. Given a graph Γ, the *incidence category* $I(\Gamma)$ has as its set of objects $E \amalg V$, and, for every $h \in H(v)$, a morphism $v \to \{h, \eta(h)\}$.

The incidence category comes equipped with a functor

$$I_\Gamma : I(\Gamma) \to \mathbb{F}\mathrm{in}$$

called the *incidence diagram of* Γ and defined as follows. To each $e = \{h, \eta(h)\} \in E$, we assign the set $\{h, \eta(h)\}$, and to each $v \in V$, we assign the set $H(v)$. To a morphism $v \to e$ corresponding to $h \in H(v)$, we assign the map $\phi : H(v) \to \{h, \eta(h)\}$ specified by requiring that $\phi^{-1}(h) = \{h\}$. ∎

Notation 1.3.5. We will denote by MI_Γ the restriction of I_Γ to $I(M(\Gamma))$. Note that this is *not* the same functor as $I_{M(\Gamma)}$, and that, if Γ is compact, MI_Γ is identical to I_Γ. ∎

Construction 1.3.6. Given a non-compact graph Γ, with $h \in H(v)$ an external half-edge, an *augmentation of the incidence diagram* $MI_\Gamma : I(M(\Gamma)) \to \mathbb{F}\mathrm{in}$ *at* h is a map $\phi_h : H(v) \to \{0,1\}$ such that

$$\phi_h^{-1}(\phi_h) = \{h\}.$$

A choice of augmentation at every external half-edge of Γ determines an *augmented incidence diagram* $AI_\Gamma : I(\Gamma) \to \mathbb{F}\mathrm{in}$. .

Note that there is not a unique choice of augmented incidence diagram AI_Γ, because there are two possible maps $\phi_h : H(v) \to \{0,1\}$ for each external half-edge h.

1. If $\phi_h(h) = 1$ we call the augmentation *incoming*.

2. If $\phi_n(h) = 0$ we call the augmentation *outgoing*.

Note that a choice of augmented incidence diagram AI_Γ determines a partition of the set of external half-edges of Γ into two sets, $\mathrm{Out}(\Gamma)$ and $\mathrm{In}(\Gamma)$. . Conversely there is a unique augmented incidence diagram corresponding to a specified partition. ∎

Definition 1.3.7. Let $\Delta\mathfrak{G}$ be a crossed simplicial group, with $\mathcal{G}$ the corresponding category of structured sets. Let Γ be a graph. A $\Delta\mathfrak{G}$*-structure* on Γ is is a choice of lift

$$\begin{array}{ccc} & \overset{\widetilde{MI}_\Gamma}{\nearrow} & \mathcal{G} \\ & & \downarrow{\scriptstyle \lambda_{\mathcal{G}}} \\ I(M(\Gamma)) & \xrightarrow[MI_\Gamma]{} & \mathbb{F}\mathrm{in} \end{array}$$

of the incidence diagram. In the case where Γ is compact, we will denote this lift by $\widetilde{I}_\Gamma$.

An *augmented* $\Delta\mathfrak{G}$*-structure on* Γ consists of:

1. A choice of augmented incidence diagram AI_Γ.

2. A choice of lift

$$\begin{array}{ccc} & & \mathcal{G} \\ & \overset{\widetilde{AI}_\Gamma}{\nearrow} & \downarrow{\scriptstyle \lambda_{\mathcal{G}}} \\ I(\Gamma) & \xrightarrow[AI_\Gamma]{} & \mathbb{F}\mathrm{in} \end{array}$$

   of AI_Γ.

Satisfying the condition that, for each external half-edge h of Γ, $\widetilde{AI}_\Gamma(\{h, h\}) = \epsilon_{\mathfrak{G}}[1] \in \mathcal{G}$. ∎

Notation 1.3.8. We will call a morphism $\phi : (S, \mathcal{O}(S)) \to \epsilon_{\mathfrak{G}}[1]$ in $\mathcal{G}$ an *augmentation map* (at s) if there is an $s \in S$ such that $\lambda_{\mathcal{G}}(\phi)^{-1}(\lambda_{\mathcal{G}}(\phi)(s)) = \{s\}$. We will call a morphism $\phi : [n] \to [1]$ in $\Delta\mathfrak{G}$ an augmentation map if $\epsilon_{\mathfrak{G}}(\phi)$ is an augmentation map in $\mathcal{G}$. ∎

Definition 1.3.9. Let Γ and Γ' be graphs. A *morphism of graphs* $\phi : \Gamma \to \Gamma'$ is a functor $\phi : I(\Gamma) \to I(\Gamma')$ which induces a bijection on the sets of external half-edges and such that, if v is a vertex in Γ then $\phi(v)$ is a vertex in Γ'.[1] A *weak equivalence of graphs* is a morphism $\phi : I(\Gamma) \to I(\Gamma')$ such that

$$|\phi| : |I(\Gamma)| \to |I(\Gamma')|$$

is a homotopy equivalence. ∎

Construction 1.3.10. Let Γ and Γ' be two graphs and $\phi : I(\Gamma) \to I(\Gamma)'$ a weak equivalence. We obtain a natural transformation

$$\mu : \phi^* MI_{\Gamma'} \Rightarrow MI_\Gamma$$

as follows. Let v be a vertex of Γ, and h a half-edge incident to $\phi(v)$. The edge $\{h, \eta(h)\}$ determines a path $\gamma : [0, 1] \to |I(\Gamma')|$. Let $\tilde{\gamma} : [0, 1] \to |I(\Gamma)|$ be a lift of γ starting at v. Then the germ of $\tilde{\gamma}$ at v determines a half-edge $\mu_v(h)$ incident to v.

Given an edge $e = \{h, \eta(h)\}$ in Γ there are two cases to consider:

1. $\phi(e)$ is an edge in Γ'. In this case, $\phi^{-1}(\phi(e)) = \{e\}$, and ϕ canonically induces an isomorphism $MI_{\Gamma'}(\phi(e)) \to MI_\Gamma(e)$.

[1]This is not quite identical to the corresponding definition [14, p. IV.10]. The additional condition that vertices be sent to vertices is added to simplify the cases in which 2-valent vertices appear.

2. $\phi(e)$ is a vertex in Γ'. In this case μ_e is uniquely determined by the value of μ on the two vertices adjacent to e.

Note that, since ϕ must preserve external half-edges, any augmentation on Γ (resp. Γ') induces an augmentation on Γ' (resp. Γ) such μ commutes with the augmentation maps. ∎

Definition 1.3.11. A *morphism of* $\Delta\mathfrak{G}$*-structured graphs* $\Gamma \to \Gamma'$ consists of

1. A weak equivalence of graphs $\psi : I(\Gamma) \to I(\Gamma')$.
2. A lift $\tilde{\mu} : \psi^* \widetilde{MI}_{\Gamma'} \Rightarrow \widetilde{MI}_{\Gamma}$ of the pullback morphism μ.

Given augmented $\Delta\mathfrak{G}$-structured graphs $\widetilde{AI}_{\Gamma}$ and $\widetilde{AI}_{\Gamma'}$, a *morphism of augmented* $\Delta\mathfrak{G}$*-structured graphs* is a morphism of the underlying $\Delta\mathfrak{G}$-structured graphs which commutes with the augmentation maps.

Given augmented $\Delta\mathfrak{G}$-structures $\widetilde{AI}_{\Gamma}$ and $\widetilde{AI}_{\Gamma'}$, a *weak morphism of augmented* $\Delta\mathfrak{G}$*-structured graphs* consists of

1. A morphism $\tilde{\mu}$ of the underlying $\Delta\mathfrak{G}$-structured graphs.
2. For every external half-edge h of Γ', an isomorphism $\xi_h : \epsilon_{\mathfrak{G}}[1] \xrightarrow{\cong} \epsilon_{\mathfrak{G}}[1]$

Satisfying the condition that for every external half-edge h of Γ attached to a vertex v of Γ, the diagram

$$\begin{array}{ccc} \epsilon_{\mathfrak{G}}[1] & \xleftarrow{\phi_{\psi(h)}} & \widetilde{AI}_{\Gamma'}(\psi(v)) \\ \downarrow{\scriptstyle \xi_h} & & \downarrow{\scriptstyle \tilde{\mu}} \\ \epsilon_{\mathfrak{G}}[1] & \xleftarrow[\phi_h]{} & \widetilde{AI}_{\Gamma'}(v) \end{array}$$

commutes. ∎

Notation 1.3.12. Let Γ be a graph. We denote by $\partial\Gamma$ the set of external half-edges of Γ. Let Γ be a compact graph. We denote by $\partial_1\Gamma$ the set of 1-valent vertices of Γ. ∎

Remark 1.3.13. Given a graph Γ and an internal edge $e = \{h, \eta(h)\}$ which is not a loop, there is a canonical graph Γ_e formed by contracting e. This gives rise to a canonical weak equivalence of graphs $\psi_e : \Gamma \to \Gamma_e$. We will reference to this operation as *edge contraction*. The induced morphism

$\mu : \psi_e^* MI_{\Gamma_e} \Rightarrow MI(\Gamma)$ exhibits the set $MI_{\Gamma(e)}(\phi(e))$ as a pullback over the diagram

$$H(v) \longrightarrow e \longleftarrow H(v')$$

in $\mathbb{F}$in.

Similarly, given a collection S of edges of Γ containing no closed walks, we can form a weak equivalence $\psi_S : \Gamma \to \Gamma_S$ contracting all the edges of S. Indeed, every weak equivalence of graphs is (isomorphic to) an edge contraction of the form ψ_S for some collection S of edges. ∎

Lemma 1.3.14. *Let $\Delta\mathfrak{G}$ be a crossed simplicial group. Then any diagram of the form*

$$\begin{array}{ccc} & & [m] \\ & & \downarrow{\scriptstyle \phi} \\ [n] & \xrightarrow[\psi]{} & [1] \end{array}$$

where ϕ is an outgoing augmentation map, ψ is an incoming augmentation map, and $[m]$ and $[n]$ are not both $[0]$ admits a pullback in $\Delta\mathfrak{G}$. The forgetful functor to $\mathbb{F}$in *preserves these pullback diagrams.*

Proof. Follows from direct computation. □

Corollary 1.3.15. *For any crossed simplicial group $\Delta\mathfrak{G}$, let Γ be a $\Delta\mathfrak{G}$-structured graph, and let S be a collection of internal edges of Γ containing no closed walks. Then there is a canonical morphism of $\Delta\mathfrak{G}$-structured graphs $\tilde{\mu} : \psi_S^* \widetilde{MI}_{\Gamma_S} \to \widetilde{MI}_{\Gamma}$ covering ψ_S.*

Remark 1.3.16. As in the unstructured case, any morphism between $\Delta\mathfrak{G}$-structured graphs $\Gamma \to \Gamma'$ can be expressed uniquely as a composite

$$\Gamma \xrightarrow{\psi_S} \Gamma_S \xrightarrow{\cong} \Gamma'.$$

In particular, every morphism of structured graphs is the composite of an isomorphism with a set of edge contractions. ∎

Definition 1.3.17. Let $\mathfrak{G}$-Graph be the category whose objects are compact $\Delta\mathfrak{G}$-structured graphs without 2-valent vertices, and whose morphisms are morphisms of structured graphs inducing bijections between the sets of 1-valent vertices.

Let $\mathfrak{G}$-Aug be the category with objects augmented $\Delta\mathfrak{G}$-structured graphs, and morphisms (strict) morphisms of $\Delta\mathfrak{G}$-structured graphs.

Finally, let $\mathfrak{G}$-WkAug be the category whose objects are augmented $\Delta\mathfrak{G}$-structured graphs, and whose morphisms are weak morphisms of $\Delta\mathfrak{G}$-structured graphs. ∎

1.3.2 Structured surfaces

Throughout this section, we fix a planar crossed simplicial group $\Delta\mathfrak{G}$ corresponding to connective coverings $p : G \to O(2)$ and $\overline{p} : \overline{G} \to \mathrm{GL}(2, \mathbb{R})$. There will, throughout, be two cases to consider: (1) connective coverings of $O(2)$ (and $GL(2, \mathbb{R})$), and (2) connective coverings of $SO(2)$ (and $GL_+(2, \mathbb{R})$.

Definition 1.3.18. Let S be a compact C^∞ surface (possibly with boundary ∂S). The frame bundle $\mathrm{Fr}_S \to S$ is a principal $GL(2, \mathbb{R})$ bundle over S. A *G-structure on S* is a reduction of structure group along $\overline{p}$, i.e. a principal $\overline{G}$ bundle $F \to S$ together with a $\overline{p}$-equivariant map $\rho : F \to \mathrm{Fr}_S$ over the identity. We will call the pair (S, F) a *G-structured surface*.

A *marked G-structured surface* consists of a surface S, a finite set $M \subset S$, and a G-structure F on $S \setminus (M \cap S^\circ)$. We will, for ease of notation, denote $M^\circ := M \cap S^\circ$. ∎

Definition 1.3.19. Let (S, M, F) and (S', M', F') be two given marked G-structured surfaces. A *structured diffeomorphism* $(S, M, F) \to (S', M', F')$ consists of a diffeomorphism $\phi : S \to S'$ taking M to M' and a diffeomorphism $\tilde{\phi} : F \to F'$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\phi}} & F' \\ \downarrow & & \downarrow \\ \mathrm{Fr}_{S\setminus M^\circ} & \xrightarrow{d\phi} & \mathrm{Fr}_{S'\setminus(M')^\circ} \\ \downarrow & & \downarrow \\ S \setminus M^\circ & \longrightarrow & S' \setminus (M')^\circ \end{array}$$

commutes.

We denote by $\mathrm{Diff}^G(S, M)$ the topological group of G-structured self-diffeomorphisms of (S, M, F). We denote by

$$\mathrm{Mod}^G(S, M) := \pi_0 \left(\mathrm{Diff}^G(S, M) \right)$$

the *structured mapping class group* of (S, M, F). ∎

Definition 1.3.20. Let (S, M) be a marked C^∞ surface with boundary. The *Schottky double* $S^\#$ of S is obtained by taking the orientation cover $\widetilde{S^\circ}$ then compactifying it by gluing in a single copy of ∂S. This yields a two-sheeted covering $\pi : S^\# \to S$ ramified along the boundary. Additionally, We can equip $S^\#$ with the structure of a marked surface by taking $M^\# = \pi^{-1}(M)$. ∎

Definition 1.3.21. A marked surface (S, M) is called *stable* if

1. $M \neq \emptyset$ and M meets every boundary component of S.
2. $\chi(S^\# \setminus M^\#) < 0$

∎

Remark 1.3.22. To clarify this definition somewhat, we note that in the oriented case, the second condition amounts to requiring that (S, M) is not

- S^2 with $|M| \leq 2$
- D^2 with $|M| = 1$
- D^2 with $|M| = 2$ and $M \subset \partial S$.

In the unoriented case, we additionally prohibit the case of $\mathbb{R}P^2$ with $|M| = 1$. ∎

Definition 1.3.23. Let Γ be a graph. We denote by $|\Gamma|$ the topological space $|I(\Gamma)|$. If Γ is compact, we denote by $|\Gamma|^\circ$ the space $|\Gamma| \setminus \partial_1\Gamma$. If Γ is not compact, we denote by $|\Gamma|^\circ$ the space $|\Gamma| \setminus \partial\Gamma$. ∎

Definition 1.3.24. Let S be a C^∞ surface, and let Γ be a graph. An *embedding of* Γ *into* S is an injective, continuous map

$$\gamma : |I(\Gamma)| \to S$$

such that:

1. γ is smooth along every edge of $|I(\Gamma)|$.
2. For every object $x \in I(\Gamma)$, the tangent directions of half-edge germs leaving $\gamma(x)$ are distinct.

3. $\gamma(|\Gamma|^\circ) \subset S^\circ$.

We say that an embedding of a compact graph is *strict* if, in addition $\gamma(\partial_1\Gamma) \subset \partial S$.[2] We say and an embedding of an augmented graph is *strict* if $\gamma(\partial\Gamma) \subset \partial S$. ∎

Proposition 1.3.25. *Let G be a planar Lie group corresponding to the planar crossed simplicial group $\Delta\mathfrak{G}$. Let S be a surface with a G-structure. An embedding of a graph Γ into S endows Γ with a $\Delta\mathfrak{G}$-structure.*

Proof. This is [14, Prop. IV.8]. □

Proposition 1.3.26. *There is a homotopy equivalence*

$$|\mathfrak{G}\text{-Graph}^{\neq 2}| \simeq \coprod_{(S,M)} B\operatorname{Mod}^G(S,M)$$

where the coproduct is taken over all topological types of stable, marked, G-structured surfaces.

Proof. This is [14, Thm. IV.12]. □

1.3.3 Segal conditions

We fix an ∞-category $\mathcal{C}$ with limits.

Definition 1.3.27. Let $X : N(\Delta^{\mathrm{op}}) \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. We say that X is 1-*Segal* (often merely *Segal* in the literature) if, for every $n \geq 0$ and every decomposition $[n] = S \cup S'$ of $[n]$ into linearly ordered sets with $S \cap S' = \{s\}$, the induced diagrams

$$\begin{array}{ccc} X([n]) & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ X(S') & \longrightarrow & X(\{s\}) \end{array}$$

are pullback.

For $\Delta\mathfrak{G}$ a crossed simplicial group, we say that a $\Delta\mathfrak{G}$-object $X : N(\Delta\mathfrak{G}^{\mathrm{op}}) \to \mathcal{C}$ is 1-Segal if the underlying simplicial object

$$N(\Delta^{\mathrm{op}}) \xrightarrow{\iota} N(\Delta\mathfrak{G}^{\mathrm{op}}) \to \mathcal{C}$$

is 1-Segal. ∎

[2] Note that in [14], what we call strict embedded graphs are called embedded graphs. Similarly, what we denote $\partial_1\Gamma$ is there denoted by $\partial\Gamma$. Both of these changes are to ease our treatment of augmented graphs in Chapter 2.

Remark 1.3.28. The terminology *Segal space* was introduced by Rezk in [37], drawing on the work of Segal on Γ-spaces. In general terms, the 1-Segal conditions encode the existence of a homotopy associative algebraic structure on the object $X([1])$ of 1-simplices. In various manifestations, variants of the 1-Segal conditions occur in several models for the category of ∞-categories (see e.g. [37]), in the study of monoidal ∞-categories and algebra objects therein ([30]), and as the nerves of 1-categories. ∎

Proposition 1.3.29. *Let $\mathcal{C}$ be an ∞-category with limits and let $X : N(\Delta^{\mathrm{op}}) \to \mathcal{C}$ be a simplicial object. The following are equivalent*

1. *X is a 1-Segal object.*

2. *For every n, the natural map*

$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$ is an equivalence in $\mathcal{C}$.

Proof. See, e.g., [15, Prop. 2.1.3]. □

Definition 1.3.30. We call a simplicial object $X : N(\Delta^{\mathrm{op}}) \to \mathcal{C}$ 2-*Segal* if, for all $0 \leq i \leq j \leq n$, the diagram

$$\begin{array}{ccc} X([n]) & \longrightarrow & X(\{0,\ldots,i,j,\ldots,n\}) \\ \downarrow & & \downarrow \\ X(\{i,\ldots,j\}) & \longrightarrow & X(\{i,j\}) \end{array}$$

is pullback in $\mathcal{C}$. We here take the convention that, if $i = j$, the elements i, j in the right-hand column are taken to be distinct, while those in the left-hand column are not.

For a crossed simplicial group $\Delta\mathfrak{G}$, we call a $\Delta\mathfrak{G}$-object $X : \Delta\mathfrak{G}^{\mathrm{op}} \to \mathcal{C}$ 2-*Segal* if the underlying simplicial object is 2-Segal. ∎

Remark 1.3.31. The concept of 2-Segal objects was introduced independently in [15] and [23]. The terminology we use comes from the former, where spaces satisfying Definition 1.3.30 are called *unital 2-Segal spaces*. The adjective 'unital' corresponds to the inclusion of the degenerate case $i = j$ in the definition, and [15] use '2-Segal space' to refer to simplicial spaces satisfying only the non-degenerate conditions. However, recent work [17] has shown that every 2-Segal object is, in fact, unital 2-Segal, providing a post-hoc justification for our elision of the terminology. ∎

Proposition 1.3.32. *Every* 1*-Segal simplicial object* X *in an* ∞*-category* $\mathcal{C}$ *is a* 2*-Segal object.*

Proof. This is (the dual of) [13, Prop. 1.5]. □

Construction 1.3.33. Let Γ be a graph with an augmented $\Delta\mathfrak{G}$-structure

$$\widetilde{AI}_\Gamma : I(\Gamma) \to \mathcal{G}$$

on Γ. Further let $X : \Delta\mathfrak{G}^{\mathrm{op}} \to \mathcal{C}$ be a $\Delta\mathfrak{G}$-object in $\mathcal{C}$. We can factor $\widetilde{AI}_\Gamma$ through $\Delta\mathfrak{G}$ uniquely up to natural isomorphism, yielding a functor $\mathfrak{AI}_\Gamma : I(\Gamma) \to \Delta\mathfrak{G}$, which we will call a *reduced augmented* $\Delta\mathfrak{G}$*-structure on* Γ. Passing through the duality $D_\mathfrak{G}$ we obtain a composite:

$$X_\Gamma : I(\Gamma) \overset{\mathfrak{AI}_\Gamma}{\to} \Delta\mathfrak{G} \overset{D_\mathfrak{G}}{\to} \Delta\mathfrak{G}^{\mathrm{op}} \overset{X}{\to} \mathcal{C}.$$

We can then take the limit of X_Γ to obtain an element

$$X(\Gamma) := \lim_{I(\Gamma)} X_\Gamma$$

in $\mathcal{C}$. We will call $X(\Gamma)$ the *state sum of* X *over* Γ. ∎

Proposition 1.3.34. *Let* $X : N(\Lambda^{\mathrm{op}}) \to \mathcal{C}$ *be a cyclic object in* $\mathcal{C}$*.*

1. *The assignment* $X \mapsto X(\Gamma)$ *defines a functor* $\rho_X : \Lambda\text{-Graph} \to \mathcal{C}$*.*
2. *The cyclic object* X *is* 2*-Segal if and only if the functor* ρ_X *maps all edge contractions in* Λ-Graph *to equivalences.*
3. *If* Γ *represents a stable oriented marked surface* (S, M)*, then* $X(\Gamma)$ *comes equipped with a coherent action of the mapping class group* $\mathrm{Mod}(S, M)$*.*

Proof. This summarizes [13, Prop. 1.20, Prop. 1.23] and [14, Thm. IV.12]. □

Remark 1.3.35. Note that in [13], the state sum is constructed without first passing to a reduced cyclic structure on the graph. However, since the functors $\Delta\mathfrak{G} \to \mathcal{G}$ are equivalences of categories, doing so does not change the value of the invariants thus obtained. ∎

Construction 1.3.36. Let Γ be a graph with augmented cyclic structure $\widetilde{AI}_\Gamma$ and reduction $\mathfrak{AI}_\Gamma : I(\Gamma) \to \Lambda$. Let $X : \Lambda \to \mathcal{C}$ be a cyclic object in $\mathcal{C}$. Considering $\mathrm{Out}(\Gamma)$ and $\mathrm{In}(\Gamma)$ as discrete categories, we get canonical inclusions $\mathrm{Out}(\Gamma) \to I(\Gamma)$ and $\mathrm{In}(\Gamma) \to I(\Gamma)$, which, passing to state sums, yield a span

$$\prod_{\mathrm{In}(\Gamma)} X_1 \longleftarrow X(\Gamma) \longrightarrow \prod_{\mathrm{Out}(\Gamma)} X_1$$

in $\mathcal{C}$ induced by the limit cone of $X(\Gamma)$. ∎

1.4 Monoidal structures, operads, & spans

We now turn to a brief overview of the ∞-categorical preliminaries necessary for our constructions. Throughout, we will assume familiarity with the material from [32] and [31], presenting instead the specific constructions and variants we will need. This section will, for the most part, follow [15].

1.4.1 Cartesian monoidal structures and ∞-categories of spans

Throughout this thesis, we will model (symmetric) monoidal structures by Cartesian fibrations, rather than the coCartesian fibrations used in [31]. Throughout this section, $\mathcal{C}$ will denote an ∞-category which admits finite products.

1.4.1.1 Cartesian monoidal structures

Definition 1.4.1. The category $\Delta^{\amalg}$ has as its objects pairs $([n], \{i, j\})$, where $[n] \in \Delta$ and $i \leq j$ are elements in $[n]$. The morphisms $([n], \{i, j\}) \to ([m], \{k, \ell\})$ consist of a morphism $\phi : [n] \to [m]$ such that $\phi(i) \leq k \leq \ell \leq \phi(j)$. We will, in general, think of $\{i, j\}$ as an interval inside $[n]$, and denote by $\{i \leq j\}$ the linearly ordered set

$$\{i \leq j\} := \{i, i+1, \ldots, j\} \subset [n].$$

The category $\mathbb{F}\mathrm{in}_*^{\amalg}$ has as its objects pairs (S, T) where $S \in \mathbb{F}\mathrm{in}_*$ and $T \subset S^\circ$. A morphism $(S, T) \to (P, Q)$ consists of a morphism $\phi : S \to P$ in $\mathbb{F}\mathrm{in}_*$ such that $\phi(T) \subset Q$. We will sometimes denote by $\Gamma^{\amalg}$ the category $(\mathbb{F}\mathrm{in}_*^{\amalg})^{\mathrm{op}}$. ∎

Remark 1.4.2. We can provide an alternate characterization of $\Delta^{\amalg}$ and $\mathbb{F}\mathrm{in}_*^{\amalg}$. The functor $\Delta^{\amalg} \to \Delta$ is the coCartesian fibration defined as a Grothendieck construction of the functors

$$\Delta \to \mathrm{Cat}; \quad [n] \mapsto I_{[n]}^{\mathrm{op}}.$$

The functor $\mathbb{F}\mathrm{in}_*^{\amalg} \to \mathbb{F}\mathrm{in}$ is the Cartesian fibration defined as a Grothendieck construction of the (contravariant) power set functor

$$\mathbb{F}\mathrm{in}_*^{\mathrm{op}} \to \mathrm{Cat}; \quad S \mapsto \mathcal{P}(S^\circ).$$

Note that, as in [15, Remark 10.3.2], these constructions relate to the constructions $\Delta^\times \to \Delta$ and $\Gamma^\times \to \mathrm{Fin}_*$ from [30, Proposition 1.2.8] and [31, Proposition 2.4.1.5] respectively. In particular, the functor $\Gamma^\times \to \mathrm{Fin}_*$ is the Cartesian fibration arising as the Grothendieck construction of

$$\mathbb{F}\mathrm{in}^* \to \mathrm{Cat}; \quad S \mapsto \mathcal{P}(S^\circ)^{\mathrm{op}}.$$

For an ∞-category $\mathcal{D}$ with enough colimits, the functor $\mathbb{F}\mathrm{in}_*^{\amalg} \to \mathbb{F}\mathrm{in}_*$ can therefore be used to construct a coCartesian fibration $\mathcal{D}^{\amalg} \to \mathbb{F}\mathrm{in}_*$ modeling the coCartesian symmetric monoidal structure on $\mathcal{D}$. ▮

Construction 1.4.3. The functor cut : $\Delta \to \mathbb{F}\mathrm{in}_*^{\mathrm{op}}$ yields a functor $\Delta^{\amalg} \to (\mathbb{F}\mathrm{in}_*^{\amalg})^{\mathrm{op}}$. To see this, we first note that for $\{i,j\} \subset [n]$ in $\Delta^{\amalg}$, we have $\mathbb{O}(\{i \leq j\}) \subset \mathbb{O}([n])$. On objects we therefore define $\{i,j\} \subset [n] \mapsto (\mathbb{O}([n]), \mathbb{O}(\{i \leq j\}))$

Given a morphism $f : ([n], \{i,j\})$ to $([m], \{k, \ell\})$ in $\Delta^{\amalg}$, we get a morphism $\mathbb{O}(f) : \mathbb{O}([m]) \to \mathbb{O}([n])$. Moreover, the condition that $f(i) \leq k \leq \ell \leq f(j)$ ensures that $\mathbb{O}(f)\,(\mathbb{O}(\{k \leq \ell\})) \subset \mathbb{O}(\{i \leq j\})$. ▮

Construction 1.4.4 (Cartesian monoidal structures)**.** Given an ∞-category $\mathcal{C}$ with finite products, we can associate two Cartesian fibrations to $\mathcal{C}$ as follows.

We define a functor of ∞-categories $\overline{\mathcal{C}^{\boxtimes}} \to \Delta$ via the universal property

$$\mathrm{Hom}_{\Delta}(K, \overline{\mathcal{C}^{\boxtimes}}) \cong \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(K \times_{\Delta} \Delta^{\amalg}, \mathcal{C}).$$

Similarly, we define a functor $\overline{\mathcal{C}^{\times}} \to \mathbb{F}$ via the universal property

$$\mathrm{Hom}_{\mathbb{F}}(K, \overline{\mathcal{C}^{\boxtimes}}) \cong \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(K \times_{\mathbb{F}} \mathbb{F}^{\amalg}, \mathcal{C}).$$

Both of these are Cartesian fibrations by dint of [32, p. 3.2.2.13].

We now let $\mathcal{C}^{\boxtimes} \subset \overline{\mathcal{C}^{\boxtimes}}$ be the full subcategory on those objects $G : I_{[n]}^{\mathrm{op}} \to \mathcal{C}$ for which G displays $G(\{i \leq j\})$ as a product over $G(\{k \leq k+1\})$ for $i \leq k < j$.

Similarly, we let $\mathcal{C}^{\times} \subset \overline{\mathcal{C}^{\times}}$ be the full subcategory on those objects $G : \mathcal{P}(S^{\circ})^{\mathrm{op}} \to \mathcal{C}$ for which G displays $G(S)$ as a product over $G(i)$ for $i \in S$. ∎

Proposition 1.4.5. *The functor $\mathcal{C}^{\boxtimes} \to \Delta$ is a Cartesian fibration exhibiting the Cartesian monoidal structure on $\mathcal{C}$.*

Proof. This is [15, Prop. 10.3.8]. □

Proposition 1.4.6. *The functor $\mathcal{C}^{\times} \to \mathbb{F}$ is a Cartesian fibration exhibiting the Cartesian symmetric monoidal structure on $\mathcal{C}$.*

Proof. The proof of this statement is, *mutatis mutandis*, the same as the proof of [31, Proposition 2.4.1.5]. □

1.4.1.2 ∞-Categories of Spans

We will briefly recall here the requisite constructions and definitions for ∞-categories of spans. For a fuller exposition, see [15, Chapter 10]. Throughout this section, we will assume that $\mathcal{C}$ is now an ∞-category with small limits.

Definition 1.4.7. Let S be a linearly ordered set. We define I_S to be the poset of non-empty sub-intervals $\{i \leq j\} \subset S$.

Let Δ^n be the standard n-simplex. We define the *spine* $\mathcal{J}^n \subset \Delta^n$ to be

$$\mathcal{J}^n := \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}.$$

∎

Construction 1.4.8 (Categories of Spans)**.** We define the functor $\mathrm{Tw} : \Delta \to \mathrm{Set}_{\Delta}$ by

$$[n] \mapsto N(I_{[n]})^{\mathrm{op}}.$$

By left Kan extension along the Yoneda embedding and restriction, we get an adjunction, which we will also denote by

$$\mathrm{Tw} : \mathrm{Set}_{\Delta} \leftrightarrow \mathrm{Set}_{\Delta} : \overline{\mathrm{Span}}. \tag{1.2}$$

For an ∞-category $\mathcal{D}$, the simplicial set $\mathrm{Tw}(\mathcal{D})$ is an ∞-category, which we will call the *twisted arrow ∞-category* of $\mathcal{D}$. Note that $\mathrm{Tw}(\mathcal{D})$ comes with a canonical projection $\eta_{\mathcal{D}} : \mathrm{Tw}(\mathcal{D}) \to \mathcal{D}$. If $\mathcal{D}$ is the nerve of a 1-category D, $\mathrm{Tw}(\mathcal{D})$ can be identified with the nerve of the 1-category $\mathrm{Tw}(D)$ whose objects are morphisms $f : a \to b$ in $\mathcal{D}$ and whose morphisms $f \to g$ are commutative diagrams

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \uparrow \\ c & \xrightarrow[g]{} & d \end{array}$$

in D, i.e. factorizations $f = h \circ g \circ \ell$.

Given $X \in \mathrm{Set}_\Delta$, we can extend the adjunction 1.2 to an adjunction

$$\mathrm{Tw}_X : (\mathrm{Set}_\Delta)_{/X} \leftrightarrow (\mathrm{Set}_\Delta)_{/X} : \overline{\mathrm{Span}}_X$$

by setting $\mathrm{Tw}_X(S \to X)$ to be the composite

$$\mathrm{Tw}(S) \to \mathrm{Tw}(X) \overset{\eta_X}{\to} X$$

and by setting $\overline{\mathrm{Span}}_X(S \to X)$ to be the left-hand column of the pullback

$$\begin{array}{ccc} \overline{\mathrm{Span}}_X(S) & \longrightarrow & \overline{\mathrm{Span}}(S) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \overline{\mathrm{Span}}(X) \end{array}$$

in Set_Δ.

Let $p : S \to X$ be a map of simplicial sets. We call an n-simplex in $\overline{\mathrm{Span}}_X(S)$ represented by a map $\sigma : \mathrm{Tw}(\Delta^n) \to S$ a *Segal simplex* if, for every $\Delta^k \subset \Delta^n$, the composite diagram

$$\{0,k\} \star \mathrm{Tw}(\mathcal{J}^k) \subset \mathrm{Tw}(\Delta^k) \subset \mathrm{Tw}(\Delta^n) \overset{\sigma}{\to} S$$

is a p-limit diagram. We denote by $\mathrm{Span}_X(S) \subset \overline{\mathrm{Span}}_X(S)$ the simplicial subset consisting of the Segal simplices. ∎

Remark 1.4.9. There are many (related) constructions of ∞-categories of spans to be found in the literature. See, for example [6], [7], [25], and [35]. The construction we follow is that of [15, Ch. 10], which, in its full generality, provides an $(\infty, 2)$-category of *bispans*. ∎

Proposition 1.4.10. *Let $p : \mathcal{C}^{\otimes} \to N(\Delta)$ be a Cartesian fibration exhibiting a monoidal structure on $\mathcal{C}^{\otimes}_{[1]}$ such that p admits relative pullbacks. Then* $\mathrm{Span}_{\Delta}(\mathcal{C}^{\otimes}) \to N(\Delta)$ *is a Cartesian fibration exhibiting a monoidal structure on* $\mathrm{Span}_{*}(\mathcal{C}^{\otimes}_{[1]})$.

Proof. This is [15, Prop. 10.2.31]. □

Corollary 1.4.11. *Let $p : \mathcal{C}^{\otimes} \to N(\Gamma)$ be a Cartesian fibration exhibiting a symmetric monoidal structure on $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ such that p admits relative pullbacks. Then* $\mathrm{Span}_{\Gamma}(\mathcal{C}^{\otimes}) \to N(\Gamma)$ *is a Cartesian fibration exhibiting a symmetric monoidal structure on* $\mathrm{Span}_{*}(\mathcal{C}^{\otimes})$.

Corollary 1.4.12. *Let $\mathcal{C}$ be an ∞-category that admits small limits. Then the functors*

$$\mathrm{Span}_{\Delta}(\mathcal{C}^{\boxtimes}) \to N(\Delta)$$
$$\mathrm{Span}_{\mathbb{\Gamma}}(\mathcal{C}^{\times}) \to N(\mathbb{\Gamma})$$

are Cartesian fibrations exhibiting, respectively, a monoidal or a symmetric monoidal structure on $\mathrm{Span}_{*}(\mathcal{C})$.

Remark 1.4.13. The monoidal structures from Corollary 1.4.12 can be seen as 'pointwise cartesian' monoidal structures, with monoidal product given by the product in $\mathcal{C}$. ∎

Lemma 1.4.14. *For any ∞-category $\mathcal{D}$ with enough limits, there is an isomorphism of simplicial sets*

$$\mathrm{Span}_{*}(\mathcal{D}) \cong \mathrm{Span}_{*}(\mathcal{D})^{\mathrm{op}}.$$

Proof. There is a natural isomorphism $\gamma : I_{[n]} \cong I_{[n]^{\mathrm{op}}}$ given by sending an interval $\{i, j\} \mapsto \{j, i\}$. This establishes a natural isomorphism

$$F : \overline{\mathrm{Span}}_{*}(\mathcal{D}) \cong \overline{\mathrm{Span}}_{*}(\mathcal{D})^{\mathrm{op}}$$

. Moreover, $N(\gamma)$ sends $\mathcal{J}^n \subset \mathrm{Tw}(\Delta^n)$ to $\mathcal{J}^{\backslash} \subset \mathrm{Tw}((\Delta^n)^{\mathrm{op}})$, and preserves Segal cones. Therefore, $\Delta^n \subset \overline{\mathrm{Span}}_{*}(\mathcal{D})$ is a Segal simplex if and only if $F(\Delta) \subset \mathrm{Span}_{*}(\mathcal{D})^{\mathrm{op}}$ is a Segal simplex, proving the lemma. □

Corollary 1.4.15. *Let $\mathcal{C}$ be an ∞-category that admits small limits. Then the functors*

$$\mathrm{Span}_{\Delta}(\mathcal{C}^{\boxtimes})^{\mathrm{op}} \to N(\Delta^{\mathrm{op}})$$
$$\mathrm{Span}_{\mathbb{\Gamma}}(\mathcal{C}^{\times})^{\mathrm{op}} \to N(\mathbb{F}\mathrm{in}_{*})$$

are coCartesian fibrations exhibiting, respectively, a monoidal or a symmetric monoidal structure on $\mathrm{Span}_*(\mathcal{C})$*.*

Remark 1.4.16. Given an ∞-category $\mathcal{D}$ with enough limits, denote by $\mathrm{Span}_*^{\mathrm{triv}}(\mathcal{D})$ the largest subcategory of $\mathrm{Span}_*(\mathcal{D})$ containing all the objects, and only those 1-simplices

$$x \xleftarrow{\simeq} z \longrightarrow y.$$

Then $\mathrm{Span}_*^{\mathrm{triv}}(\mathcal{D}) \simeq \mathcal{D}$ (see, e.g. [7]). Moreover, there is an equivalence of maximal Kan complexes $\mathrm{Span}_*(\mathcal{D})^{\simeq} \simeq \mathcal{D}^{\simeq}$. ∎

1.4.2 Calabi-Yau algebras

Throughout the following section, we take $\mathcal{C}^{\otimes} \to \mathbb{F}\mathrm{in}_*$ to be a symmetric monoidal ∞-category with monoidal unit $\mathbb{1}$ and tensor product $\otimes$.

Construction 1.4.17. There is a functor $B : \mathbb{\Lambda} \to \mathcal{A}\mathrm{ss}$ defined as follows. On objects, send each $S \in \mathbb{\Lambda}$ to $S \amalg \{*\}$, forgetting the cyclic order. On morphisms, send $f : S \to T$ to its underlying map of sets. Define a linear order on the fibers of f by choosing embeddings of S and T into S^1 compatible with the cyclic order, and representing f as a commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\tilde{f}} & S^1 \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ S & \xrightarrow{f} & T \end{array}$$

where $\tilde{f}$ is monotone of degree 1. For $i \in T$, the preimage of $\beta(i)$ under $\tilde{f}$ is an interval, and $\beta(f^{-1}(i)) \subset \tilde{f}^{-1}(\beta(i))$. The orientation of S^1 induces an orientation of $\tilde{f}^{-1}(\beta(i))$, and hence a linear order on $f^{-1}(i)$. ∎

Definition 1.4.18. The *cyclic bar object* of an algebra object $X : \mathcal{A}\mathrm{ss} \to \mathcal{C}^{\otimes}$ is the composition $B^*(X)$. A *cyclic trace* on X is a natural transformation η from $B^*(X)$ to the constant cyclic object on $\mathbb{1} \in \mathcal{C}$. We call a pair (X, η) consisting of an algebra object in $\mathcal{C}^{\otimes}$ and a cyclic trace a *trace algebra*. ∎

Remark 1.4.19. A natural transformation to a constant cyclic object may be modeled as a functor from the category $\mathbb{\Lambda}_{\diamond}$ obtained from $\mathbb{\Lambda}$ by formally adjoining a terminal object. We denote the terminal object of $\mathbb{\Lambda}_{\diamond}$ by $\diamond$. ∎

Definition 1.4.20. A morphism $\gamma : X \otimes X \to \mathbb{1}$ in $\mathcal{C}$ is called *non-degenerate* if there exists a morphism $\eta : \mathbb{1} \to X \otimes X$ such that

- The composite

$$X \xrightarrow{\simeq} X \otimes \mathbb{1} \xrightarrow{\eta \otimes \mathrm{id}_{\mathbb{1}}} X \otimes X \otimes X \xrightarrow{\mathrm{id}_{\mathbb{1}} \otimes \gamma} \mathbb{1} \otimes X \xrightarrow{\simeq} X$$

  is homotopic to the identity.

- The composite

$$X \xrightarrow{\simeq} \mathbb{1} \otimes X \xrightarrow{\mathrm{id}_{\mathbb{1}} \otimes \eta} X \otimes X \otimes X \xrightarrow{\gamma \otimes \mathrm{id}_{\mathbb{1}}} X \otimes \mathbb{1} \xrightarrow{\simeq} X$$

  is homotopic to the identity.

∎

Definition 1.4.21. Let (X, η) be a trace algebra in $\mathcal{C}$, and let $\eta_2 : X \otimes X \to \mathbb{1}$ be the map induced by $\langle 2 \rangle \to \diamond$ in $\mathbb{A}_\diamond$ under η. We call (X, η) a *Calabi-Yau algebra* in $\mathcal{C}$ if η_2 is non-degenerate. ∎

Remark 1.4.22. The definition above is precisely that of [33, Example 4.2.8]. When Hochschild homology is defined, the map $\eta : B^*(X) \to \mathbb{1}$ is equivalently an S^1-equivariant trace

$$\int_{S^1} X \to \mathbb{1}.$$

∎

Definition 1.4.23. Let $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ be the category with

- Objects $\mathrm{ob}(\mathcal{A}\mathrm{ss}) \amalg \{\diamond\}$.

- Morphisms between $S, T \in \mathcal{A}\mathrm{ss}$

$$\mathrm{Hom}_{\mathcal{A}\mathrm{ss}_{\mathrm{CY}}}(S, T) := \mathrm{Hom}_{\mathcal{A}\mathrm{ss}}(S, T).$$

- For $S \in \mathcal{A}\mathrm{ss}$,

$$\mathrm{Hom}_{\mathcal{A}\mathrm{ss}_{\mathrm{CY}}}(\diamond, S) := \varnothing$$

  and a morphism $S \to \diamond$ is a choice of a subset $T \subset S^\circ$ and a cyclic order on T.

- For $S, T \in \mathcal{A}\mathrm{ss}$, and morphisms $\phi : S \to T$ and $\psi : T \to \diamond$, the composite $\psi \circ \phi$ is given by the induced cyclic order

Note that $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ comes equipped with a functor $\mathcal{A}\mathrm{ss}_{\mathrm{CY}} \to \mathbb{F}\mathrm{in}_*$ sending $\diamond \mapsto \langle 1 \rangle$. ∎

Construction 1.4.24. Let $\mathbb{\Lambda} \to \mathbb{\Lambda}_\diamond$ and $\mathcal{A}\mathrm{ss} \to \mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ be the inclusions. Define a functor $F : \mathbb{\Lambda}_\diamond \to \mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ by setting $F = B$ on $\mathbb{\Lambda} \subset \mathbb{\Lambda}_\diamond$, and sending $\diamond \mapsto \diamond$. By definition, the diagram

$$\begin{CD} \mathbb{\Lambda} @>>> \mathbb{\Lambda}_\diamond \\ @VBVV @VVFV \\ \mathcal{A}\mathrm{ss} @>>> \mathcal{A}\mathrm{ss}_{\mathrm{CY}} \end{CD} \tag{1.3}$$

commutes. ∎

Definition 1.4.25. We take $\mathfrak{P}$ to be the categorical pattern of [31, Proposition 2.1.4.6]. In the following proof, we will freely make reference to this proposition, and Appendix B of [31]. ∎

Lemma 1.4.26. *The diagram*

$$\begin{CD} N(\mathbb{\Lambda}) @>>> N(\mathbb{\Lambda}_\diamond) \\ @VBVV @VVFV \\ N(\mathcal{A}\mathrm{ss}) @>>> N(\mathcal{A}\mathrm{ss}_C\,Y) \end{CD}$$

induces an $\mathfrak{P}$-anodyne morphism of ∞-categories

$$\theta : N(\mathcal{A}\mathrm{ss}) \coprod_{N(\mathbb{\Lambda})} N(\mathbb{\Lambda}_\diamond) \to N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$$

over $\mathbb{F}\mathrm{in}_*$, where the non-degenerate marked simplices are precisely the inert morphisms of $\mathcal{A}\mathrm{ss}$.

Proof. An n-simplex of $N(\mathcal{A}\mathrm{ss}) \coprod_{N(\mathbb{\Lambda})} N(\mathbb{\Lambda}_\diamond)$ is an equivalence class in $N(\mathcal{A}\mathrm{ss}) \amalg N(\mathbb{\Lambda}_\diamond)$ under the relation that

$$\underbrace{(S_0 \to S_1 \to \cdots \to S_n)}_{\in N(\mathcal{A}\mathrm{ss})_n} \sim \underbrace{(T_0 \to T_1 \to \cdots \to T_n)}_{\in N(\mathbb{\Lambda})_n}$$

if and only if

$$B(T_0 \to T_1 \to \cdots \to T_n) = (S_0 \to S_1 \to \cdots \to S_n).$$

In particular, θ is injective, and a bijection on 0-simplices.

We proceed by induction. For ease of notation, we set $Q = N(\mathcal{A}\mathrm{ss}) \coprod_{N(\mathbb{A})} N(\mathbb{A}_\diamond)$.

1. Suppose $f : S \to \diamond$ is a 1-simplex not contained in the image of θ. Then S is determined by $T \subsetneq S^\circ$ and a cyclic order on S. Adding a basepoint to T to get $T_f \in \mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ we get a factorization of f as

$$\begin{array}{ccccc} & & T_f & & \\ & \overset{\beta}{\nearrow} & & \overset{\alpha}{\searrow} & \\ S & & \xrightarrow{\quad f \quad} & & \diamond \end{array}$$

in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$. Taking such a 2-simplex σ_f for every such f, we can form the pushout

$$\begin{array}{ccc} \coprod_{\{f\}} (\Lambda^2_1)^\flat & \longrightarrow & Q_0 \\ \downarrow & & \downarrow \\ \coprod_{\{f\}} (\Delta^2)^\flat & \longrightarrow & Q_1 \end{array}$$

The morphism on the left is of type (C_1) from [31, B.1.1], so we get a factorization

$$Q_0 \xrightarrow{\tau_1} Q_1 \xrightarrow{\theta_1} N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}), \qquad \theta = \theta_1 \circ \tau_1$$

where τ_1 is $\mathfrak{P}$-anodyne, and θ_1 is bijective on 1-simplices.

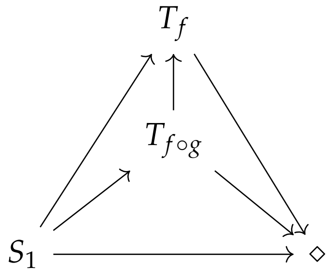
2. Now suppose that $\sigma : \Delta^2 \to \mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ is a 2-simplex not in the image of θ_1. Then σ must be given by a sequence

$$S_1 \xrightarrow{g} S_2 \xrightarrow{f} \diamond$$

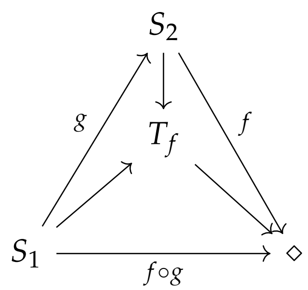
(if σ does not contain $\diamond$, it is the image of a simplex in $\mathcal{A}\mathrm{ss}$, if it contains two copies of $\diamond$, it is degenerate). Consequently, we get two 2-simplices, $\sigma_{f\circ g}$ and σ_g in the image of θ_1. Moreover, g restricts to a morphism

$$g : T_{f\circ g} \to T_f,$$

and we get a 2-simplex $S_1 \to S_2 \to T_f$. We then note that the Λ^3_1 horn



can be filled to a 2-simplex $S_1 \to T_f \to \diamond$ via a horn of type (C_1). Finally, we get a Λ^3_2-horn



of type (C_1). This gives us a factorization of θ as $Q_0 \xrightarrow{\tau_2} Q_2 \xrightarrow{\theta_2} N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$ where τ_1 is $\mathfrak{P}$-anodyne and θ_2 is bijective on simplices of dimension ≤ 2.

3. Now suppose inductively that we have obtained a factorization through $\theta_{n-1} : Q_n \to N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$ such that

   - θ_{n-1} is bijective on k-simplices for $k \leq n-1$.
   - The image of θ_{n-1} contains all n-simplices of the form

   $$S_0 \to S_1 \to \cdots \to S_{n-1} \to \diamond$$

   where $S_{n-1} \to \diamond$ is a 1-simplex in the image of $\Lambda_\diamond$.

   Suppose given an n-simplex σ not in the image of θ_{n-1}. Then, by similar reasoning to that above, σ must be of the form

   $$S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \to S_{n-1} \xrightarrow{\phi_n} \diamond$$

with $S_{n+1} \to \diamond$ not in the image of $\Lambda_\diamond$. Define $\psi_k := \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_{n-k}$, we then get n-simplices in the image of θ_{n-1}

$$S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \to \widehat{S_k} \to S_{n-1} \to T_{\phi_n} \to \diamond$$

and an n-simplex in the image of θ_{n-1}

$$S_0 \to S_1 \to \cdots \to S_{n-1} \to T_f.$$

These n n-simplices form a Λ_n^{n+1}-horn in $N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$ which, once again, can be filled by a pushout of type (C_1).

We therefore get a factorization

$$Q_0 \to Q_1 \to \cdots \to N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$$

which exhausts $N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$. Each morphism in this sequence is $\mathfrak{P}$-anodyne, and so the transfinite composition $Q_0 \to N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}})$ is $\mathfrak{P}$-anodyne. □

Corollary 1.4.27. *The ∞-category of trace algebras in $\mathcal{C}$ is equivalent to the full subcategory of* $\mathrm{Map}^\sharp_{\mathbb{F}\mathrm{in}_*}(N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}), \mathcal{C}^\otimes)$ *sending $\diamond$ to $\mathbb{1}$.*

Definition 1.4.28. We define the *∞-category of Calabi-Yau algebras in $\mathcal{C}$* to be the full subcategory of $\mathrm{Map}^\sharp_{\mathbb{F}\mathrm{in}_*}(N(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}), \mathcal{C}^\otimes)$ on those objects which

1. send $\diamond$ to $\mathbb{1}$, and
2. send the morphism $\langle 2 \rangle \to \diamond$ in $\mathbb{A}_\diamond$ to a non-degenerate morphism $X \otimes X \to \mathbb{1}$.

▮

2 1-Categorical field theories

In this chapter, we provide a generalization of the well-known result relating open topological field theories to Frobenius algebras by making use of the formalism of crossed simplicial groups and structure graphs described in sections 1.1 through 1.3.

2.1 Cobordism categories

We begin by providing a combinatorial characterization of the cobordism category in terms of the structured graphs of Section 1.3.

2.1.1 G-structured cobordisms

Definition 2.1.1. For a planar Lie group G, the *G-structured interval* $\mathscr{I}$ consists of the interval $I = [0,1]$ together with the 'reduction of structure group'

$$\overline{G} \times I \to GL(2,\mathbb{R}) \times I \cong \mathrm{Fr}(TI \oplus \mathbb{R})$$

where the isomorphism on the left is given by trivializing the $GL(2,\mathbb{R})$-torsor $\mathrm{Fr}(TI \oplus \mathbb{R})$ using the frame $(\partial_x, 1)$.

A *structured boundary embedding of $\mathscr{I}$* into a G-structured surface (S,F) is a smooth embedding

$$f : I \hookrightarrow \partial S \subset S$$

together with a smooth extension $\hat{f} : TI \oplus \mathbb{R} \xrightarrow{\cong} TS$ of df, and a morphism of principal $\overline{G}$-bundles $\tilde{f} : \overline{G} \times I \to F$ covering $\hat{f}$.

We say that two G-structured embeddings $(f, \hat{f}, \tilde{f})$ and $(g, \hat{g}, \tilde{g})$ are equivalent if there exists a G-structured diffeomorphism $(\phi, \tilde{\phi}) : (S,F) \to$

(S, F) such that the diagram

$$\begin{array}{ccc} & & (S,F) \\ & \overset{(f,\hat{f},\tilde{f})}{\nearrow} & \\ \mathscr{I} & & \downarrow (\phi,\tilde{\phi}) \\ & \underset{(g,\hat{g},\tilde{g})}{\searrow} & \\ & & (S,F) \end{array}$$

commutes. ∎

Construction 2.1.2. Given a G-structured boundary embedding, $(f, \hat{f}, \tilde{f}) : \mathscr{I} \to (S, F)$, we can identify $\hat{f}$ with a germ of embeddings $I \times [0, \epsilon) \to S$ restricting to f on $I \times \{0\}$. This identification induces an identification $\psi_f : TI \oplus \mathbb{R} \cong T(I \times [0, \epsilon))$ on $I \times \{0\}$. Write x for the coordinate on I and y for the coordinate on $[0, \epsilon)$. We say that $(f, \hat{f}, \tilde{f})$ is *incoming* if $\psi_f(1) = a\partial_y$ with $a \in \mathbb{R}_{>0}$, and *outgoing* if $\psi_f(1) = -a\partial_y$ with $a \in \mathbb{R}_{>0}$. ∎

Remark 2.1.3. Note that if $\overline{G} \to GL(2, \mathbb{R})$ factors through $GL_+(2, \mathbb{R})$, then a G-structured boundary embedding can only be incoming if the positive normal to $f(I)$ in S points inwards. Similarly, a G-structured boundary embedding can only be outgoing if the positive normal to $f(I)$ in S points outwards. ∎

Definition 2.1.4. A G-structured *cobordism* $(S, B_{\text{in}}, B_{\text{out}})$ consists of

1. A G-structured surface S with non-empty boundary ∂S.

2. A pair of finite disjoint unions of G-structured intervals

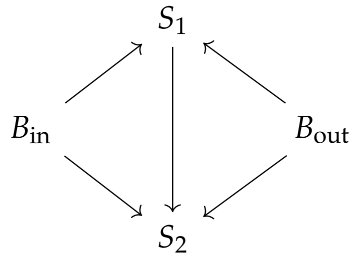
$$B_{\text{in}} := \coprod_{j \in L_{\text{in}}} \mathscr{I}, \quad B_{\text{out}} := \coprod_{j \in L_{\text{out}}} \mathscr{I}.$$

3. Structured boundary inclusions

$$(f, \hat{f}, \tilde{f}) : B_{\text{in}} \to \partial S \leftarrow B_{\text{out}} : (g, \hat{g}, \tilde{g})$$

such that the images of all of the underlying intervals are pairwise disjoint.

An *equivalence of G-structured cobordisms* is a G structured diffeomorphism $(\phi, \tilde{\phi}) : S_1 \to S_2$ such that the diagram



commutes. ∎

Remark 2.1.5. Note that we do not require that the images of B_{in} and B_{out} meet every boundary component. We call those boundary components not hit by either the *free boundary components* of the cobordism. ∎

Definition 2.1.6. The *G-structured open cobordism category* has objects given by finite disjoint unions of copies of $\mathscr{I}$ and morphisms given by equivalence classes of G-structured cobordisms. Composition is given by the gluing of cobordisms, and the disjoint union provides a symmetric monoidal structure. We denote this category by Cob^G. ∎

2.1.2 Combinatorial cobordisms

Definition 2.1.7. Let $\Delta\mathfrak{G}$ be a balanced crossed simplicial group. A $\Delta\mathfrak{G}$-*structured cobordism from* $\coprod_{j=1}^{n}[1]$ *to* $\coprod_{k=1}^{m}[1]$ consists of:

1. An augmented $\Delta\mathfrak{G}$-structured graph Γ.

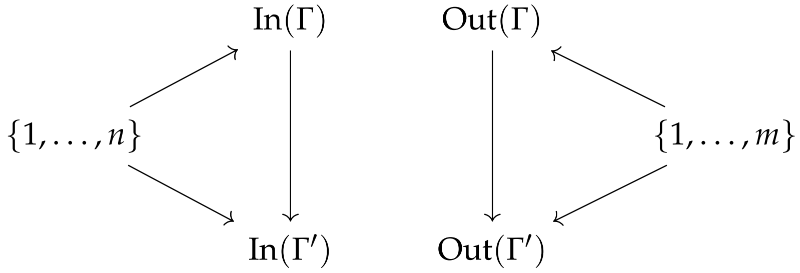
2. Bijections

$$f : \{1, \ldots, n\} \xrightarrow{\cong} \mathrm{In}(\Gamma)$$

and

$$g : \{1, \ldots, m\} \xrightarrow{\cong} \mathrm{Out}(\Gamma)$$

An *equivalence* of $\Delta\mathfrak{G}$-structured coborisms is an equivalence of augmented structured graphs $\Gamma \to \Gamma'$ such that the induced diagrams



commute. We say that Γ_1 and Γ_2 are *equivalent* $\Delta\mathfrak{G}$-structured cobordisms if there is a zig-zag of equivalences between them. ∎

Construction 2.1.8. Suppose given two $\Delta\mathfrak{G}$-structured cobordisms $\Gamma_1 = (H_1, V_1, s_1, \eta_1)$ and $\Gamma_2 = (H_2, V_2, s_2, \eta_2)$, with boundary bijections

$$f_1 : \{1, \ldots, n\} \to \mathrm{In}(\Gamma_1), \quad g_1 : \{1, \ldots, m\} \to \mathrm{Out}(\Gamma_1)$$

and

$$f_2 : \{1, \ldots, m\} \to \mathrm{In}(\Gamma_2), \quad g_2 : \{1, \ldots, k\} \to \mathrm{Out}(\Gamma_2)$$

We can define a new graph $\Gamma = \Gamma_2 \circ \Gamma_1$, which we call the *concatenation* of Γ_2 and Γ_1 as follows:

1. We set $H = H_1 \amalg H_2$, and $V = V_1 \amalg V_2$.

2. We define the map $s : H \to V$ to be

$$s(h) = \begin{cases} s_1(h) & h \in H_1 \\ s_2(h) & h \in H_2. \end{cases}$$

3. We define the map η to agree with η_1 on the incoming half-edges and internal edges of Γ_1, and to agree with η_2 on the outgoing half-edges and internal edges of Γ_2. On the outgoing half-edges of Γ_1, we define

$$\eta(h) := f_2^{-1} \circ g_1(h)$$

and on the incoming half-edges of Γ_2, we define

$$\eta(h) := g_1^{-1} \circ f_2(h).$$

The concatenation Γ becomes a $\Delta\mathfrak{G}$-structured cobordism as follows. For every vertex of Γ, and every internal edge of Γ_1 or Γ_2, the functor $\widetilde{AI}(\Gamma)$ is already defined. Suppose h is an outgoing half-edge of Γ_1 attached to a vertex $v_1 \in V_1$, and suppose $\eta(h) = h'$ is attached to a vertex $v_2 \in V_2$. Then we identify $[1]$ with the set $\{h, h'\}$ by sending $0 \mapsto h'$, the augmentation maps at h and h' thereby provide morphisms

$$(H(v_1), \mathcal{O}_{v_1}) \to (\{h, h'\}, \mathfrak{G}_1) \leftarrow (H(v_2), \mathcal{O}_{v_2}).$$

Performing this procedure for every new internal edge gives Γ a $\Delta\mathfrak{G}$-structure. The augmentations and boundary bijections are then inherited from Γ_1 (for the incoming) and Γ_2 (for the outgoing) ▮

Proposition 2.1.9. *Let Γ_1 and Γ_2 be $\Delta\mathfrak{G}$-structured cobordisms as in Construction 2.1.8, and let Γ_1' and Γ_2' be $\Delta\mathfrak{G}$-structured cobordisms equivalent to Γ_1 and Γ_2, respectively. Then $\Gamma_2 \circ \Gamma_1$ is equivalent to $\Gamma_2' \circ \Gamma_1'$.*

Proof. Immediate from the definitions. □

Definition 2.1.10. Let $\Delta\mathfrak{G}$ be a balanced crossed simplicial group. We define a symmetric monoidal category $\mathfrak{G}$ -Bord as follows:

1. The objects of $\mathfrak{G}$ -Bord are disjoint unions of copies of $[1] \in \Delta\mathfrak{G}$.

2. Morphisms from $\coprod_{j=1}^{n}[1]$ to $\coprod_{k=1}^{m}[1]$ are equivalence classes of $\Delta\mathfrak{G}$-structured cobordisms from $\coprod_{j=1}^{n}[1]$ to $\coprod_{k=1}^{m}[1]$.

3. Composition is induced by the concatenation of augmented $\Delta\mathfrak{G}$-structured graphs.

4. The symmetric monoidal structure is induced by the disjoint union of graphs.

▮

2.1.3 Drilling and patching

Lemma 2.1.11. *Let $A := \{R_1 \leq |x| \leq R_2\} \subset \mathbb{R}^2$ and let $D_* := \{0 < |x| \leq R_2\} \subset \mathbb{R}$. Suppose given a diffeomorphism $\phi : A \to A$ Then there is a diffeomorphism $\hat{\phi} : D_* \to D_*$ extending A.*

Proof. By Whitney's extension theorem, we can extend ϕ to a smooth map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$. Since ϕ is a diffeomorphism, $d\psi$ is bijective on ∂A, and thus restricts to a diffeomorphism on some ϵ-neighborhood of A. Rescaling to the punctured disc then gives the desired diffeomorphism. □

Corollary 2.1.12. *Let A and D_* be as in Lemma 2.1.11 and let F_1 and F_2 be two G-structures on D_* restricting to E_1 and E_2 on A, respectively. Then we can extend any G-structured diffeomorphism $(\phi, \tilde{\phi}) : A \to A$ to a G-structured diffeomorphism $(\psi, \hat{\psi}) : D_* \to D_*$.*

Proof. This follows directly from Lemma 2.1.11 and the fact that the inclusion $A \hookrightarrow D_*$ is a homotopy equivalence. □

Definition 2.1.13. Let $\mathscr{U}$ denote the restriction of the structure on $\mathscr{I}$ to $(0,1) \subset [0,1]$. Given a structured marked surface (S, M) with boundary, a *open boundary trivialization* on (S, M) is a triple $(f, \hat{f}, \tilde{f})$ with $f : (0,1) \to \partial S \setminus M$ a diffeomorphism onto a connected component, $\hat{f} : T(0,1) \oplus \mathbb{R} \to TS$ an isomorphism extending df, and $\tilde{f} : \overline{G} \times (0,1) \to F$ a morphism of principal $\overline{G}$-bundles covering $\hat{f}$.

We say that (S, M) has *trivialized boundary* if there is a collection of open boundary trivializations

$$(f, \hat{f}, \tilde{f}) : \coprod_{i=1}^{n} \mathscr{U} \to S$$

inducing a diffeomorphism

$$f : \coprod_{i=1}^{n} (0,1) \to \partial S \setminus M.$$

A *G-structured diffeomorphism* of marked G-structured surfaces with trivialized boundary is a structured diffeomorphism of the underlying marked surfaces which commutes with the boundary parameterization. ∎

Proposition 2.1.14. *For every G-structured marked surface (S, M) with trivialized boundary $(f, \hat{f}, \tilde{f})$, there is a G-structured cobordism $(\Sigma, B_{\text{in}}, B_{\text{out}}$ and an embedding $e : \Sigma \to S$ inducing the G-structure and structured boundary embeddings for Σ such that*

1. *e is a homotopy equivalence.*

2. *Let (S', M') be another such structured marked surface with trivialized boundary, and let Σ' be the cobordism corresponding to it. If there is a G-structured diffeomorphism $(S, M) \to (S', M')$, then there is an equivalence of cobordisms $\Sigma \to \Sigma'$.*

3. *Let $\Sigma \to \Sigma'$ be an equivalence of cobordisms. Then there is a G-structured diffeomorphism $(S, M) \to (S', M')$.*

Proof. For $m \in M^\circ$, we choose a small open ball U_m around m in S such that the pairwise intersections of the U_m are empty. We then define Σ to be $S \setminus \coprod U_m$. Restricting the boundary trivialization to $[1/3, 2/3] \subset (0, 1)$ yields the desired structured boundary intervals for the cobordism.

Given a G-structured diffeomorphism $(\phi, \tilde{\phi}) : S \to S'$ respecting boundary trivializations, there is an induced homotopy equivalence $\psi : \Sigma \xrightarrow{\simeq} \Sigma'$ agreeing with ϕ on a neighborhood of the boundary of S. By the Dehn-Nielsen theorem, ψ is homotopic relative to the boundaries to a diffeomorphism $\gamma : \Sigma \to \Sigma'$. Since the G-structures on Σ and Σ' are induced by the emebeddings e and e', the $\overline{G}$-bundle on Σ induced by pullback along γ is isomorphic over the identity to the G-structure on Σ. Therefore, Γ can be lifted to a G-structured diffeomorphism, which agrees with $(\phi, \tilde{\phi})$ on a neighborhood of the boundary and thus preserves embedded boundary intervals.

In the other direction, we simply take neighborhood $m \in U_m \subset V_m$ together with charts sending m to the origin of $\mathbb{R}^2$ and ∂U_m and ∂V_m to concentric circles about the orgin. The result then follows from Corollary 2.1.12. □

Corollary 2.1.15. *Let $(\Sigma, B_{\text{in}}, B_{\text{out}})$ be a G-structured cobordism, then there is a G-structured marked surface (S, M) with trivialized boundary and an embedding $e : \Sigma \to S$ satisfying the conditions of Proposition 2.1.14.*

Proof. We can construct (S, M) by gluing punctured discs into the free boundaries of Σ and extending the G-structure and the trivializations by homotopy equivalence. Therefore, $(\Sigma, B_{\text{in}}, B_{\text{out}})$ can be extracted from (S, M) by the procedure outlined in the proof of Proposition 2.1.14. □

Definition 2.1.16. We call the procedure of removing the U_m *drilling* and the procedure of gluing in discs *patching*. We will call a G-structured cobordism Σ *stable* if the marked G-structured surface (S, M) defined by patching Σ is stable. ∎

2.1.4 Stable and exceptional graphs

Construction 2.1.17. Given a connected augmented $\Delta\mathfrak{G}$-structured graph, Γ, we can contract internal edges until there are at most two 1-valent vertices. Similarly, we can contract internal edges until there is at most one 2-valent vertex. Denote the structured augmented graph resulting from this procedure by Θ. We will call Γ *stable* if Θ contains no 1-valent and 2-valent vertices, and *exceptional* otherwise. There are four exceptional cases, which we will consider in two sub-cases:

1. Suppose Θ contains a 1-valent vertex x. We then have two cases
   a) The half-edge h attached to x is external, in which case Θ is comprised solely of x and h.
   b) The half-edge h is part of an edge e attached to another vertex v. If v is not 1-valent then we could have contracted the edge e, meaning that v must be 1-valent. Therefore, Θ is comprised precisely of x, v, and e.
2. Suppose Θ contatins a 2-valent vertex x attached to half-edges h and g. We again have two cases:
   a) Both h and g are external. In this case, Θ is comprised solely of x, h, and g. See also the discussion in [14, p. IV.3] on structured intervals.
   b) h is part of an edge e attached to a vertex v. If $v \neq x$, then the edge e can be contracted, contradicting the minimality of Θ. Therefore Θ consists of the vertex x and the single edge $e := \{g, h\}$.

As we will see in the sequel, equivalence classes of stable, G-structured graphs are related to the connected components of $\mathfrak{G}$ -Graph. ∎

Definition 2.1.18. We denote by $\mathfrak{G}$ -Aug^s the full subcategory of $\mathfrak{G}$ -Aug on augmented graphs without 1- or 2-valent vertices. We will denote by $\mathfrak{G}$ -WkAug^s the full subcategory of $\mathfrak{G}$ -WkAug on the same.

We denote by $\mathfrak{G}$ -Stab the category of stable, augmented $\Delta\mathfrak{G}$-structured graphs with weak morphisms. ∎

Proposition 2.1.19. *There is an equivalence of categories*

$$\mathfrak{G}\text{-WkAug}^s \cong \mathfrak{G}\text{-Graph}\,.$$

Proof. Given a structured graph $\Gamma \in \mathfrak{G}$-Graph, we obtain a graph $T(\Gamma)$, which we will call the *truncation of* Γ by forgetting the 1-valent vertices. At each resulting external half edge, h, choosing an identification $(\{h, \eta(h)\}, \mathcal{O}_e) \cong \epsilon_{\mathfrak{G}}[1]$ yields an augmented $\Delta\mathfrak{G}$-structure on $T(\Gamma)$. The resulting graph has neither 1- nor 2-valent vertices, and it is immediate that this construction is functorial.

Given a structured graph $\Gamma \in \mathfrak{G}$-WkAug^s, we define a structured graph $E(\Gamma)$ as follows:

1. Attach a 1-valent vertex v to each external half edge h (via a new half-edge $\eta(h)$).
2. Denote by x the vertex $s(h)$ and by $\phi_h : (H(x), \mathcal{O}_x) \to \epsilon_{\mathfrak{G}}[1]$ the corresponding augmentation map. Choose an identification $\psi\{0,1\} \cong \{h, \eta(h)\}$ such that $\psi \circ \phi_h(h) = \eta(h)$.
3. Choose a $\Delta\mathfrak{G}$-structure $(H(v), \mathcal{O}_v)$ (unique up to isomorphism over the identity) and a morphism

$$\phi_v : (H(v), \mathcal{O}_v) \to \epsilon_{\mathfrak{G}}[1]$$

such that $\psi \circ \phi_v(\eta(h)) = h$.

To see that this is functorial, simply note that by canonical factorization, if we have a diagram

$$\begin{array}{ccc} (H(v), \mathcal{O}_v) & \xrightarrow{\phi_v} & \epsilon_{\mathfrak{G}}[1] \\ & & \downarrow f \\ (H(v), \mathcal{O}_v) & \xrightarrow[\phi_v]{} & \epsilon_{\mathfrak{G}}[1] \end{array}$$

such that f acts as the identity on underlying sets, then there is a unique automorphism $g : (H(v), \mathcal{O}_v) \to (H(v), \mathcal{O}_v)$ such that the diagram

$$\begin{array}{ccc} (H(v), \mathcal{O}_v) & \xrightarrow{\phi_v} & \epsilon_{\mathfrak{G}}[1] \\ g \downarrow & & \downarrow f \\ (H(v), \mathcal{O}_v) & \xrightarrow[\phi_v]{} & \epsilon_{\mathfrak{G}}[1] \end{array}$$

commutes. This gives us a canonical extension of a morphism $\Gamma_1 \to \Gamma_2$ to a morphism $E(\Gamma_1) \to E(\Gamma_2)$. The functors E and T are weakly inverse to one another, yielding the desired equivalence. $\square$

Corollary 2.1.20. *There is an equivalence of categories*

$$\coprod_{(S,M)} \mathrm{Mod}^G(S,M) \ltimes P(S,M) \simeq \mathfrak{G}\text{-WkAug}^s$$

where the (S,M) ranges over all topological types of stable, marked G-structured surfaces.

Proof. This follows from the proof of [14, Thm. IV.12]. □

Proposition 2.1.21. *There is a bijection*

$$\pi_0(\mathfrak{G}\text{-WkAug}^s) \cong \pi_0(\mathfrak{G}\text{-Stab})$$

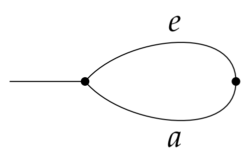
Proof. We need to show that Construction 2.1.17 defines a map $\psi : \pi_0(\mathfrak{G}\text{-Stab}) \to \pi_0(\mathfrak{G}\text{-WkAug}^s)$, i.e. that equivalent stable augmented $\Delta\mathfrak{G}$-structured graphs yield equivalent augmented graphs.

Note that Γ, given a univalent vertex v attached by an edge e to a non-univalent vertex w, and given any edge a in Γ which can be contracted, we get a commutative diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma/_a \\ \downarrow & & \downarrow \\ \Gamma/_e & \longrightarrow & \Gamma/_{\{a,e\}} \end{array}$$

in $\mathfrak{G}$-Graph. It therefore follows that a zig-zag of augmented stable $\Delta\mathfrak{G}$-structured graphs yields a zig-zag of augmented stable $\Delta\mathfrak{G}$-structured graphs without 1-valent vertices.

A similar argument holds for bivalent vertices, excepting a single special case, where both edges attached to the bivalent vertex v are attached to a single vertex w, as depicted below.



In this case, we can contract either e or a, but not both. However, since the morphisms along which we form the pullback are isomorphisms, the two possible contractions will yield *isomorphic* $\Delta\mathfrak{G}$-structures on the same underlying graph.

We therefore have a well-defined map ψ, which is obviously injective. Since every pre-image $\psi^{-1}(\Gamma)$ contains the structure graph Γ itself, ψ is a bijection. □

2.1.5 The functor Spine

Definition 2.1.22. Let $(S, B_{\text{in}}, B_{\text{out}})$ be a G-structured cobordism. A *spanning graph* for $(S, B_{\text{in}}, B_{\text{out}})$ is an embedded graph $\gamma : |\Gamma| \to S$ such that

1. $\gamma(\partial\Gamma) \subset B_{\text{in}} \cup B_{\text{out}}$, and this inclusion is a homotopy equivalence.

2. γ is a homotopy equivalence.

We denote by $Q(S, B_{\text{in}}, B_{\text{out}})$ the poset of spanning graphs for $(S, B_{\text{in}}, B_{\text{out}})$.

Let (S, M) be a marked G-structured surface. A *spanning graph* for (S, M) is an embedded graph $\gamma : |\Gamma| \to S$ such that

1. $\gamma(\partial\Gamma) \subset \partial S \setminus M$ and this inclusion is a homotopy equivalence.

2. γ is a homotopy equivalence.

We denote by $P(S, M)$ the set of spanning graphs of (S, M) under contractions of edges not connected to a 1-valent vertex. ∎

Construction 2.1.23. Given a graph Γ embedded in a G-structured surface S (via $\gamma : |I(\Gamma)| \to S$, let x be a vertex of Γ. We define $C(T_xS) := (T_xS \setminus \{0\})/_{\mathbb{R}_{>0}}$ to be the *circle of directions at x*. There is a commutative diagram of topological groups

$$\begin{array}{ccc} \overline{G} & \longrightarrow & \mathrm{Homeo}^G(S^1) \\ {\scriptstyle p}\downarrow & & \downarrow \\ GL(2,\mathbb{R}) & \xrightarrow[\ell]{} & \mathrm{Homeo}(S^1) \end{array} \tag{2.1}$$

and we have the p-equivariant map of torsors

$$F|_x \to \mathrm{Fr}(T_xS) \cong \mathrm{Isom}(\mathbb{R}^2, T_xS).$$

We can then define the corresponding coinduced torsors

$$\begin{aligned} F_{\mathrm{Homeo}} &:= (F|_x) \times_{\overline{G}} \mathrm{Homeo}^G(S^1) \\ \mathrm{Homeo}(S^1, C(T_xS)) &\cong \mathrm{Fr}(T_xS) \times_{GL(2,\mathbb{R})} \mathrm{Homeo}(S^1). \end{aligned}$$

Passing to these coinduced torsors gives us a right an equivariant map

$$\pi : F_{\mathrm{Homeo}} \to \mathrm{Homeo}(S^1, C(T_xS))$$

Fitting into a commutative diagram (equivariant with respect to (2.1))

$$\begin{array}{ccc} F|_x & \longrightarrow & F_{\mathrm{Homeo}} \\ \downarrow & & \downarrow \pi \\ \mathrm{Fr}(T_xS) & \longrightarrow & \mathrm{Homeo}(S^1, C(T_xS)) \end{array}$$

In particular, given a trivialization of $F|_x$ as a $\overline{G}$-torsor, we get a trivialization of F_{Homeo} as a $\mathrm{Homeo}^G(S^1)$-torsor. ∎

Proposition 2.1.24. *Let $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$ be a G-structured cobordism, and $\gamma : |\Gamma| \to S$ a spanning graph for $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$. Then Γ inherits an augmented $\Delta\mathfrak{G}$-structure from the G-structure and boundary intervals. Moreover, the embedding γ induces bijections $\mathrm{In}(\Gamma) \cong \pi_0(B_{\mathrm{in}})$ and $\mathrm{Out}(\Gamma) \cong \pi_0(B_{\mathrm{out}})$, defining the structure of a $\Delta\mathfrak{G}$-structured cobordism on Γ.*

Proof. Clearly we get an induced $\Delta\mathfrak{G}$-structure, so we need only construct the augmentations. At one boundary interval $(f, \hat{f}, \tilde{f}) : \mathscr{I} \to S$, meeting $\gamma(h)$ for an external half-edge h at a point x, denote by v the tangent direction to $\gamma(h)$ in $C(T_xS)$. Further let z be the vertex to which h is attached. Then, by the procedure from [14, Prop. IV.8], we get a morphism of structured sets

$$\phi : (H(z), \mathcal{O}_z) \to (\{-v, v\}, \mathcal{O}_x)$$

such that $\phi(h) = v$ and $\phi^{-1}(\phi(h)(h)) = \{h\}$.

Moreover, the isomorphism $\tilde{f} : \overline{G} \to F|_x$ provides, via Construction 2.1.23, a canonical isomorphism $\psi : \epsilon_{\mathfrak{G}}[1] \cong (\{-v, v\}, \mathcal{O}_x)$. Tracing through the definitions, we see that $\psi(0) = -v$ if the boundary interval is incoming and $\phi(0) = v$ if the boundary interval is outgoing. We therefore can define an augmentation map staisfying the desired properties by $\phi_h := \psi^{-1} \circ \phi$. □

Proposition 2.1.25. *Suppose given two composable cobordisms $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$ and $(T, D_{\mathrm{in}}, D_{\mathrm{out}})$, together with two spanning graphs $\gamma : |\Gamma| \to S$ and $\theta : \Theta \to T$.*

1. *There is an induced embedding $\theta \circ \gamma : |\Theta \circ \Gamma| \to T \circ S$ restricting to the isotopy classes of θ and γ.*

2. *The augmented $\Delta\mathfrak{G}$-structure on Γ induced by $\theta \circ \gamma$ is isomorphic to that induced by the concatenation of $\Delta\mathfrak{G}$-structured graphs.*

Proof. Part 1 follows by isotopic γ and θ so that the image of $\mathrm{Out}(\Gamma)$ and $\mathrm{In}(\Theta)$ in the outgoing boundaries of S and incoming boundaries of T, respectively, agree. Part 2 follows directly from the definitions. □

Construction 2.1.26. Let S be a C^∞ surface, and let $\gamma : |\Gamma| \to S$ an embedded graph. Suppose e is an internal edge of Γ between two *different* vertices v_1 and v_2.

Choose a contractible open neighborhood with $U \subset S$ such that $\gamma(e) \subset U$, and U contains no object of $I(\Gamma)$ other than v_1, v_2, and e. Choose an open subneighborhood V with $\overline{V} \subset U$ which has a local coordinate chart $\phi : V \to (a,b) \times (c,d)$ such that $\gamma(e)$ is given by the coordinates $[-1/2, 1/2] \times \{0\} \subset (a,b) \times (c,d)$.

By taking a partition of unity, we may write down a smooth map $f : U \to U$ which is the identity on $U \setminus V$, and collapses $[-1/2,1/2] \times [-\epsilon, \epsilon]$ to $\{0\} \times [-\epsilon, \epsilon]$. In particular, $f(\gamma(e)) = \{\mathrm{pt}\}$, so that $f \circ \gamma : \gamma^{-1}(U) \to U$ factors through a morphism $\nu : \gamma^{-1}(U)/_e \to U$.

Therefore, we can define an embedding $\gamma_e : |\Gamma/_e| \to S$ by setting $\gamma_e = \gamma$ on $S \setminus U$, and $\gamma_e = \nu$ on U.

Denote by v the vertex is $\Gamma/_e$ which is the image of v_1 and v_2. Note that the images of the half-edges e under γ provide canonical paths h_1 from v to v_1 and h_2 from v to v_2 respectively. By the procedure described in the proof of [14, Prop. IV.8], we get an induced pullback diagram

$$\begin{array}{ccc} (H(v), \mathcal{O}_v) & \longrightarrow & (H(v_1), \mathcal{O}_{v_1}) \\ \downarrow & & \downarrow \\ (H(v_2), \mathcal{O}_{v_2}) & \longrightarrow & (\{h_1, h_2\}, \mathcal{O}_e) \end{array} \tag{2.2}$$

giving a morphism $\Gamma \to \Gamma/_e$ of $\Delta\mathfrak{G}$-structured graphs. ▮

Proposition 2.1.27. *Let (S, M) be a marked G-structured surface with trivialized boundary, and let $(\phi, \tilde{\phi})$ be a structured self-diffeomorphism of (S, M). Suppose $\gamma : |\Gamma| \to S$ is a spanning graph for (S, M) with its induced $\Delta\mathfrak{G}$-structure. Then $(\phi, \tilde{\phi})$ is isotopic relative to the marked points to a structured self-diffeomorphism preserving the boundary trivializations if and only if the induced weak morphism μ of augmented structured graphs is a strict morphism of augmented structured graphs.*

Proof. It is clear that ϕ preserves the labels of the open boundary intervals if and only if μ preserves the labels of the augmentations. Since this is the case, we may assume (by taking an isotopy) that ϕ fixes the boundary pointwise.

By construction, if $\tilde{\phi}$ preserves the trivialization of the G-structure over the boundary, μ will act as the identity on the augmentations. To see the converse, note that, if μ acts as the identity on the augmentations, $\tilde{\phi}$ will preserve the trivialization of the G-structure over a chosen point in each boundary interval. Since the boundary intervals are contractible, this implies that $\tilde{\phi}$ preserves the boundary trivializations in their entirety. □

Construction 2.1.28. We define an assignment Spine : $\mathrm{Cob}^G \to \mathfrak{G}$-Bord on objects and morphisms as follows:

1. On objects, we send $\coprod_{j=1}^{n} \mathscr{I}$ to $\coprod_{j=1}^{n} [1]$.

2. Let $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$ be a G-structured cobordism. Choose a spanning graph G, and denote by $[\Gamma] \in \pi_0(\mathfrak{G}\text{-Aug})$ be the equivalence class of the induced augmented $\Delta\mathfrak{G}$-structure on G. We then send $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$ to $[\Gamma]$.

Note that, as it currently stands, the assignment Spine is dependent on the choice of the spanning graph Γ. ∎

Notation 2.1.29. We denote by SC^G the set of equivalence classes of stable, G-structured cobordisms. ∎

Proposition 2.1.30. *There is a bijection*

$$SC^G \cong \pi_0(\mathfrak{G}\text{-Stab})$$

sending an equivalence class of cobordisms to the equivalence class of a $\Delta\mathfrak{G}$-structured spanning graph.

Proof. By Proposition 2.1.14 and Corollary 2.1.15, we may consider equivalence classes of stable G-structured marked surfaces with boundary trivialization instead of equivalence classes of cobordisms. Then the proposition follows by Proposition 2.1.27, Proposition 2.1.21, and Corollary 2.1.20. □

Remark 2.1.31. To show that Spine is a functor, it now only remains to check well-definedness in the unstable cases (since composability is established by Proposition 2.1.25). The proof to Proposition 2.1.32 below will establish a similar bijection to that of Proposition 2.1.30 in the unstable cases, establishing both that Spine is a functor and that it is fully faithful. ∎

Proposition 2.1.32. *The assignment* $\mathrm{Spine} : \mathrm{Cob}^G \to \mathfrak{G}$-Bord *is an equivalence of categories.*

Proof. We need only show that Spine is fully faithful.

By Proposition 2.1.30, we know that Spine induces a bijection between equivalence classes of stable G-structured cobordisms and equivalence classes of stable $\Delta\mathfrak{G}$-structured cobordisms. It therefore only remains to check the non-stable cases. These fall into two categories, orientable, and non-orientable, which we will deal with separately.

- The orientable cases are as follows

  1. D^2 with a single incoming embedded boundary interval. In this case, equivalence classes of $\Delta\mathfrak{G}$-structured spanning graphs are in bijection with isomorphism classes of augmented $\Delta\mathfrak{G}$-structures on the graph with a single vertex and single external half-edge with incoming augmentation map. Any two such graphs are isomorphic, so we are left with a single combinatorial bordism. However, since D^2 retracts onto the boundary interval in question, any two such G-structured cobordisms are equivalent.
  2. D^2 with a single outgoing embedded boundary interval. This case is effectively the same as the first.
  3. D^2 with two embedded boundary intervals, one incoming, and one outgoing. In this case, equivalence classes of G-structured cobordisms are in bijection with $\mathrm{Ker}(\overline{G} \to GL(2,\mathbb{R}))$, and so by [14, Prop. III.1], they are in bijection with elements of $\mathfrak{G}_0$.

     Equivalence classes of augmented $\Delta\mathfrak{G}$-structures on spanning graphs for D^2 are in bijection with isomorphism classes of $\Delta\mathfrak{G}$-structures on the graph Γ with one bivalent vertex and two external half-edges. Such equivalence classes are in bijection with automorphisms $f : \epsilon_{\mathfrak{G}}[1] \to \epsilon_{\mathfrak{G}}[1]$ covering the identity, i.e. with elements of $\mathrm{Stab}_1(0) \cong \mathfrak{G}_0$.
  4. The cases of D^2 with two incoming (or two outgoing) boundary intervals follow in much the same way as the previous case.

5. D^2 with no embedded boundary intervals. In this case, equivalence classes of $\Delta\mathfrak{G}$-structured spanning graphs are in bijection with isomorphism classes of augmented $\Delta\mathfrak{G}$-structures on the (compact) graph with two univalent vertices and a single edge between them. All such $\Delta\mathfrak{G}$-structures are isomorphic, yielding a single isomorphism class. Similarly, D^2 is contractible, so there is a single diffeomorphism class of G-structured cobordisms.
6. The annulus A with no embedded boundary intervals. In this case, equivalence classes of $\Delta\mathfrak{G}$-structures on spanning graphs are in bijection with isomorphism classes of $\Delta\mathfrak{G}$-structures on the graph Γ with one bivalent vertex v and one edge e. This case breaks into two sub-cases, depending on whether or not G preserves orientation or not
   a) If G preserves orientation, then structured diffeomorphism classes of G-structures on the annulus are in bijection with elements of $\mathrm{Ker}(\overline{G} \to GL(2,\mathbb{R}))$ (since the clutching construction along the identity yields the tangent bundle). Consequently, by [14, Prop. III.1], we see that they are in bijection with elements of $\mathfrak{G}_0$.
   The isomorphism classes of $\Delta\mathfrak{G}$-structures on Γ are in bijection with the set of morphisms $\epsilon_{\mathfrak{G}}[1] \to \epsilon_{\mathfrak{G}}[1]$ covering the identity, i.e. with $\mathrm{Stab}_1(0) \cong \mathfrak{G}_0$.
   b) If G does not preserve orientation, then structured diffeomorphism classes of G-structures on the annulus are in bijection with elements of $\mathrm{Ker}(\overline{G} \to GL(2,\mathbb{R}))$ (since the clutching construction along the identity yields the tangent bundle). Consequently, by [14, Prop. III.1], we see that they are in bijection with elements of $\mathrm{Ker}(\mathfrak{G}_0 \to \mathbb{Z}/2)$. As before, the isomorphism classes of $\Delta\mathfrak{G}$-structures on Γ are in bijection with the set of automorphisms $f : \epsilon_{\mathfrak{G}}[1] \to \epsilon_{\mathfrak{G}}[1]$ covering the identity, i.e. with $\mathrm{Stab}_1(0) \cong \mathfrak{G}_0$. However, those structures which result from an embedding into the annulus must have the additional property that the image of f in Ξ must be in the image of $\Lambda \hookrightarrow \Xi$. These are precisely the elements of $\mathrm{Ker}(\mathfrak{G}_0 \to \mathbb{Z}/2)$.

- There is only a single unstable non-orientable case: that of a Möbius band with no embedded boundary intervals. In this case, equivalence classes of $\Delta\mathfrak{G}$-structures on spanning graphs are in bijection with isomorphism classes of $\Delta\mathfrak{G}$-structures on the graph Γ from case 6 above.

Structured diffeomorphism classes of G-structures on the Möbius band are in bijection with elements of the preimage of -1 under the map $\overline{G} \to GL(2, \mathbb{R})$ (since the clutching construction along -1 yields the tangent bundle to the Möbius band). Consequently, by [14, Prop. III.1], we see that they are in bijection with elements in the preimage of -1 under the map $\mathfrak{G}_0 \to \mathbb{Z}/2$.

Similarly to the above, the isomorphism classes of $\Delta\mathfrak{G}$-structures on Γ arising from an embedding into the Möbius band are in bijection with elements $f \in \mathrm{Stab}_1(0) \cong \mathfrak{G}_0$ such that the image of f in Ξ is *not* in the image of $\Lambda \hookrightarrow \Xi$. These are precisely the preimages of -1 under the map $\mathfrak{G}_0 \to \mathbb{Z}/2$.

To conclude the proof, we then need only note that the cases listed above also exhaust the unstable $\Delta\mathfrak{G}$-structured cobordisms. □

2.2 Classifying field theories

Having now established that $\mathfrak{G}$-Bord is a combinatorial model for the cobordism category, we proceed to the classification of open structured topological field theories. We will fix a balanced crossed simplicial group $\Delta\mathfrak{G}$ and a symmetric monoidal category $\mathscr{C}$ with monoidal unit I, and we will consider symmetric monoidal functors $Z : \mathfrak{G}\text{-Bord} \to \mathscr{C}$.

2.2.1 Operads and $\Delta\mathfrak{G}$-Frobenius algebras

Definition 2.2.1. We call a graph Γ a *tree* if $|\Gamma|$ is simply connected and has at least one external half-edge.

Denote by $P_{\Delta\mathfrak{G}}(n) \subset \mathrm{Hom}_{\mathrm{Cob}^G}(\coprod_{i=1}^{n}[1], [1])$ the subset on morphisms whose underlying graphs are connected trees. We call the elements of $P_{\Delta\mathfrak{G}}(1)$ $\Delta\mathfrak{G}$-*structured intervals*. Note that the concatenation of augmented structured graphs gives us operations

$$P_{\Delta\mathfrak{G}}(n) \times P_{\Delta\mathfrak{G}}(k_1) \times \cdots \times P_{\Delta\mathfrak{G}}(k_n) \to P_{\Delta\mathfrak{G}}(k_1)(\sum_{i=1}^{n} k_i).$$

▮

Proposition 2.2.2. *For a balanced crossed simplicial group $\Delta\mathfrak{G}$,*

1. *The monoid $P_{\Delta\mathfrak{G}}(1)$ is canonically isomorphic to the group $\mathfrak{G}_0$.*
2. *The action of the symmetric groups S_n on $P_{\Delta\mathfrak{G}}(n)$ by relabeling incoming augmentations makes $P_{\Delta\mathfrak{G}}$ into an operad in the category of sets.*

Proof. This summarizes [14, Cor. IV.17] and [14, Prop. IV.18]. □

Remark 2.2.3. By definition, we see that any symmetric monoidal functor

$$F : \mathfrak{G}\text{-Bord} \to \mathscr{C}$$

will, in particular, exhibit $F([1])$ as an algebra over the operad $P_{\Delta\mathfrak{G}}$. ∎

Proposition 2.2.4. *For any balanced crossed simplicial group $\Delta\mathfrak{G}$, there is a homomorphism*

$$\chi_2 : \mathfrak{G}_0 \to S_n \wr \mathfrak{G}_0$$

such that an algebra A over the operad $P_{\Delta\mathfrak{G}}$ is precisely a monoid (A, m, e) equipped with an action of $\mathfrak{G}_0$ such that

$$g \cdot m(a, b) = m(\chi_2(g) \cdot (a, b))$$

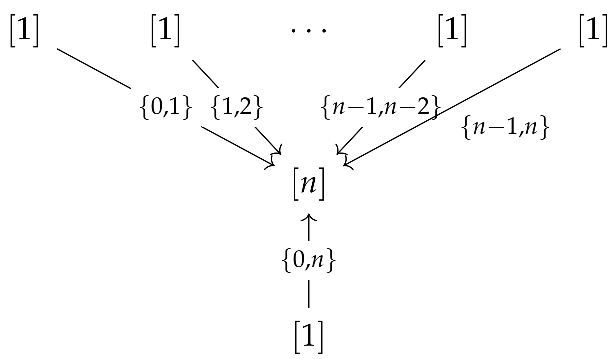
for all $a, b \in A$ and $g \in \mathfrak{G}_0$.

Proof. We will follow the strategy of [14, Prop. IV.18]. For any equivalence class in $P_{\Delta\mathfrak{G}}(n)$, we can choose a unique standard representative such that the outgoing augmentation map is the unique augmentation morphism $\phi \in \Delta$ sending 0 to 0. Since the set of possible incoming augmentation maps forms a torsor under the operadic action of $\mathfrak{G}_0$, we get that $P_{\Delta\mathfrak{G}}(n)$ forms a torsor under

$$H := S_n \wr \mathfrak{G}_n$$

Trivializing this torsor will allow us to find a copy of $\mathcal{A}$ss in $P_{\Delta\mathfrak{G}}$.

We consider the diagram in Δ



Applying the duality $D_{\mathfrak{G}}$, we then obtain an augmented structured corolla, which we take as our trivialization of the torsor. To see that this choice respects composition, we simply compute that the pushout of a diagram in Δ of the form:

$$\begin{array}{ccc} [1] & \xrightarrow{\{0,n\}} & [n] \\ {\scriptstyle \{i-1,i\}}\downarrow & & \\ [m] & & \end{array}$$

is given by the map $\gamma : [n] \to [n+m-1]$ with $\gamma(j) = i + j$ and the map $\epsilon : [m] \to [n+m-1]$ given by

$$\epsilon(j) = \begin{cases} j & j \leq i \\ j+n & j > i \end{cases}$$

So that composing two such diagrams and taking a pushout gives us a diagram of the same form. Therefore, the trivializations given by the duality are closed under concatenation and contraction. This identifies a copy of the associative operad $\mathcal{A}$ss in $P_{\Delta\mathfrak{G}}$.

Let $\mathfrak{m} \in P_{\Delta\mathfrak{G}}(n)$. If we compose $h \in \mathfrak{G}_0 \cong P_{\Delta\mathfrak{G}}(1)$ with $\mathfrak{m}$, we can pull back h along the outgoing augmentation map of $\mathfrak{m}$ by canonical factorization, so that the outgoing augmentation map of the tree representing $h \circ \mathfrak{m}$ is gicen by $\phi \circ g$, with $g \in \mathfrak{G}_n$, and $\phi = D_{\mathfrak{G}}(\{0,n\})$.

Since two 1-vertex trees define the same object of $P_{\Delta\mathfrak{G}}(n)$ if and only if they are related by an automorphism of the central vertex, we can act by g^{-1} to the central vertex of our representative to obtain another representative of $h \circ \mathfrak{m}$. This changes the incoming augmentation maps by precomposing with g^{-1}. However, the original augmentation maps induce isomorphisms $\mathfrak{G}_0 \cong \mathrm{Stab}(k)$ for all $k \neq 0$ so that we can represent the new augmentation maps permuting the old ones according to $\lambda_n(g^{-1})$ and postcomposing with elements of $\mathfrak{G}_0$. That is, we can find an element $\chi_n(g) \in H$ whose action on the incoming half-edges of $\mathfrak{m}$ gives the equivalence class of $\mathfrak{m} \circ h$. By construction this procedure is compatible with composition, and so defines a group homomorphism $\chi_n : \mathfrak{G}_0 \to H$ under which the n-fold multiplication must be equivariant. Applying the composability conditions for $\mathcal{A}$ss in $P_{\Delta\mathfrak{G}}$, it is sufficient to require that the multiplication $m_2 : A \otimes A \to A$ be equivariant under χ_2 to get equivariance under all of the χ_n.

To see that these conditions are sufficient to give a well-defined algebra over $P_{\Delta\mathfrak{G}}$, we need only note that any automorphism of the central vertex relating two representatives of the same $\mathfrak{m} \in P_{\Delta\mathfrak{G}}(n)$ must fix the outgoing half-edge, and therefore in $\mathrm{Stab}_n(0) \cong \mathfrak{G}_0$. But by Lemma 1.2.28 this corresponds to acting by g and $\chi_n(g)^{-1}$, so that χ_n-equivariance is sufficient to guarantee our algebra is well-defined. □

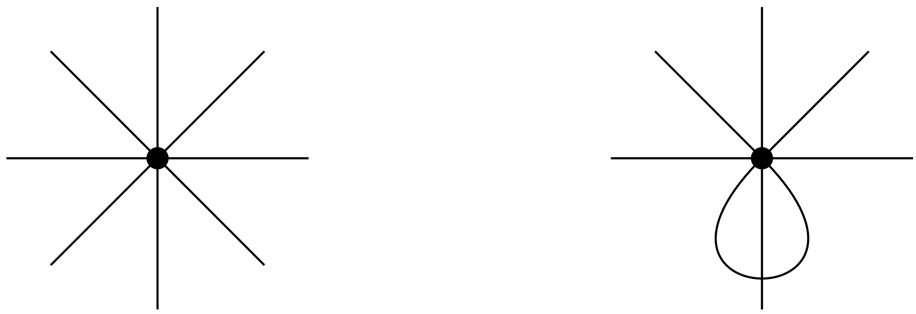
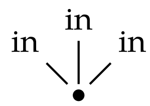
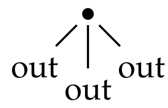
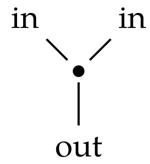
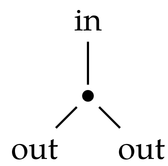


Figure 2.1: A corolla (left) with 8 half edges and a rose with 8 half-edges and one loop.

Remark 2.2.5. For clarity of notation, we will usually use lower-case fraktur characters to denote morphisms in $\mathfrak{G}$ -Bord. Where possible, we will often use the corresponding lower-case greek character to denote the corresponding morphism in $\mathscr{C}$. ▮

Definition 2.2.6. A *rose* is an augmented $\Delta\mathfrak{G}$-structured graph with only one vertex. A *corolla* is a rose with no loops (cf. Fig. 2.1). We will fix some further terminology for special types of corollas:

Name	Definition	Picture
n-Trace	A corolla with n half-edges all labeled "in" under the augmentation.	
n-Cotrace	A corolla with n half-edges all labeled "out" under the augmentation.	
Muliplication	Elements of $P_{\Delta\mathfrak{G}}(n)$ for $n \geq 1$	
Comultiplication	Trees with precisely one incoming half-edge.	

We will also sometimes refer to the (unique) 1-cotrace as the *unit*, and denote it by $\mathfrak{p}_1$. We will similarly denote the (unique) 1-trace by $\mathfrak{b}_1$. ∎

Definition 2.2.7. We define $\mathfrak{G}$-Gen to be the symmetric monoidal subcategory of $\mathfrak{G}$-Bord on all objects with morphisms given by (disjoint unions of) traces, multiplications, and the unit. ∎

Remark 2.2.8. A symmetric monoidal functor $Z : \mathfrak{G}\text{-Gen} \to \mathcal{C}$ is determined by its values on $P_{\Delta\mathfrak{G}}$, $\mathfrak{b}_1$, and $\mathfrak{p}_1$. ∎

Notation 2.2.9. We fix a copy of $\mathcal{A}ss$ in $P_{\Delta\mathfrak{G}}$, and a trivialization of $\mathcal{A}ss(n)$ as an S_n-torsor given by multiplications $\mathrm{id} = \mathfrak{m}_1, \mathfrak{m}_2, \ldots$. Composing with $\mathfrak{b}_1$, we get traces, which we will denote by

$$\mathfrak{b}_n := \mathfrak{b}_1 \circ \mathfrak{m}_n.$$

We will denote by $\mathfrak{p}_2$ the unique 2-cotrace such that

$$(\mathfrak{b}_2 \amalg \mathrm{id})(\mathrm{id} \amalg \mathfrak{p}_2)$$

as depicted graphically in Fig. 2.2 ∎

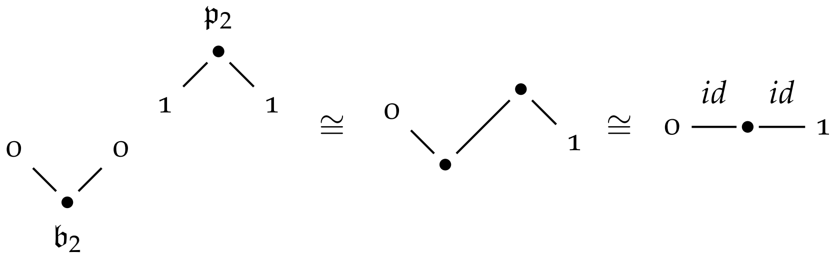


Figure 2.2: Pictorial representation of a composition of $\mathfrak{p}_2$ and $\mathfrak{b}_2$

Proposition 2.2.10. *Let $\mathfrak{n}$ be any representative of the n-trace $\mathfrak{b}_n$. Then there is a homomorphism*

$$\eta_n : \mathfrak{G}_n \to S_n \wr \mathfrak{G}_0$$

such that the representative isomorphic to $\mathfrak{n}$ by applying $g \in \mathfrak{G}_n$ to the central vertex is given by the operadic action of η_n on the incoming half-edges of $\mathfrak{n}$.

Proof. We first note that, given such a $g \in \mathfrak{G}_n$, the relabelling of the half-edges is given by $\lambda_n(g)$. The condition that we want, expressed in terms of the augmentation maps ϕ_i of $\mathfrak{n}$, is that

$$\phi_j \circ g = h_i \circ \phi_{\sigma(i)}$$

for a unique $h_i \in \mathfrak{G}_1^0$. More precisely, if we let

$$\mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1])^i = \left\{\psi \in \mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1]) \mid \psi(i) = 1 \text{ and } \psi^{-1}(1) = i\right\}$$

then we want that the action of $\mathfrak{G}_1^0$ on $\mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1])^i$ by postcomposition is simply transitive. However, choosing a representative $\gamma \in \mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1])^i$, we see that this is the same as saying that the subgroup $\gamma^*(\mathfrak{G}_1^0)$ acts transitively by precomposition.

Without loss of generality, we can reduce this to the case where $i = n$, since every morphism in $\mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1])^i$ is given by a composition of a morphism in $\mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1])^n$ with an element of $\mathfrak{G}_n$. Reducing to this case, we see that the elements of $\mathfrak{G}_n$ that act on $\mathrm{Hom}_{\Delta\mathfrak{G}}([n],[1])^n$ are precisely the members of $\mathrm{Stab}(n) \subset \mathfrak{G}_n$. Since the action of $\mathfrak{G}_n$ on $\mathrm{Hom}_{\Delta\mathfrak{G}}([n],[0])$ is simply transitive, it suffices to show that $\mathfrak{G}_0$ is isomorphic to $\mathrm{Stab}(n)$ via the homomorphism induced by pullback. However, this is precisely the statement of Lemma 1.2.28, so the proposition is proved. □

Remark 2.2.11. Given a representative $\mathfrak{n}$ as in the proposition with augmentation morphisms given by $\phi_i = \psi \circ g_i$, we can compute the form of the homomorphism η_n. Allowing n to represent the morphism in $\mathrm{Hom}_\Delta([0],[n])$ with target n, we can write:

$$\begin{aligned} \eta_n : \mathfrak{G}_n &\to \mathfrak{G}_0 \wr \Sigma_{n+1} && (2.3) \\ g &\mapsto \left(n^*(g_0 \circ g \circ g^{-1}_{\sigma^{-1}(0)}), \ldots, n^*(g_n \circ g \circ g^{-1}_{\sigma^{-1}(n)}), \sigma\right) && (2.4) \end{aligned}$$

where $\sigma := \lambda_n(g)$. We can thus deduce that the image of η_n is independent of the choice of representative $\mathfrak{n}$. We will denote this image by $\mathfrak{I}_n$. ∎

Lemma 2.2.12. *Suppose that $h \in S_n \wr \mathfrak{G}_n$, and $\mathfrak{b}_n \circ h = \mathfrak{b}_\mathfrak{n}$. Then $h \in \mathfrak{I}_n$.*

Proof. Let $h = (\sigma, h_1, \ldots, h_n)$. Choosing a representative $\mathfrak{n}$ for $\mathfrak{b}_n$ with augmentation maps ϕ_i, our hypothesis implies that there is an automorphism $g \in \mathfrak{G}_n$ of the $\Delta\mathfrak{G}$-structured set assigned to the central vertex such that

$$\phi_j \circ g = h_i \circ \phi_{\sigma(i)}$$

However, this implies precisely that $h = \eta_n(g) \in \mathfrak{I}_n$. □

Definition 2.2.13. A $\Delta\mathfrak{G}$-*trace algebra* in $\mathscr{C}$ consists of

1. A unital associative algebra (A, μ_2, e) in $\mathscr{C}$ equipped with an action of $\mathfrak{G}_0$ such that μ_2 is χ_2-equivariant.

2. A morphism $\beta_1 \to I$ such that

$$\beta_n := \beta_1 \circ \mu_n : A^{\otimes n} \to I$$

   is invariant under the action of $\mathfrak{I}_n$.

We call a $\Delta\mathfrak{G}$-trace algebra a $\Delta\mathfrak{G}$-*Frobenius algebra* if, in addition, $\beta_2 : A^{\otimes 2} \to I$ is non-degenerate.

We denote by $\Delta\mathfrak{G}$ -Trace$_\mathscr{C}$ the category whose objects are $\Delta\mathfrak{G}$-trace algebras (A, μ_2, e, β_1) and whose morphisms $(A, \mu_2, e, \beta_1) \to (B, \zeta_2, e, \xi_1)$ are $\mathfrak{G}_0$-equivariant algebra homomorphisms $f : A \to B$ such that

$$\xi_1 \circ f = \beta_1.$$

We denote by $\Delta\mathfrak{G}$ -Frob$_\mathscr{C}$ the full subcategory of $\Delta\mathfrak{G}$ -Trace$_\mathscr{C}$ on the $\Delta\mathfrak{G}$-Frobenius algebras. ∎

Proposition 2.2.14. *A symmetric monoidal functor*

$$Z : \mathfrak{G}\text{-Gen} \to \mathscr{C}$$

can be reconstructed from its underlying $\Delta\mathfrak{G}$*-trace algebra. This construction induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathfrak{G}\text{-Gen}, \mathscr{C}) \simeq \Delta\mathfrak{G}\text{-Trace}_{\mathscr{C}}$$

Proof. By Proposition 2.2.4, we see that a $\Delta\mathfrak{G}$-trace algebra yields a well-defined algebra over the operad $P_{\Delta\mathfrak{G}}$. Moreover, we can represent any n-trace $\mathfrak{r}$ as a composite

$$\mathfrak{r} = \mathfrak{b}_n \circ h$$

for $h \in S_n \wr \mathfrak{G}_0$. Suppose given two such different decompositions of $\mathfrak{r}$ given by elements $h_1, h_2 \in S_n \wr \mathfrak{G}_0$. Then, in particular, we see that there is $g \in \mathfrak{I}_n$ with $gh_1 = h_2$. We therefore see that, because of the $\mathfrak{I}_n$-invariance condition on β_n, the assignment

$$Z(\mathfrak{r}) = \beta_n \circ h$$

yields a well-defined functor on all of $\mathfrak{G}$-Gen, regardless of the choice of decomposition $\mathfrak{r} = \mathfrak{b}_n \circ h$.

This construction means that we have a faithful, essentially surjective functor

$$\mathrm{Fun}^{\otimes}(\mathfrak{G}\text{-Gen}, \mathscr{C}) \to \Delta\mathfrak{G}\text{-Trace}_{\mathscr{C}}$$

Moreover, via the decomposition $Z(\mathfrak{r}) = \beta_n \circ h$, we see that, given a morphism $f : A \to B$ of trace algebras, the induced morphisms $f^{\otimes n} : A^{\otimes n} \to B^{\otimes n}$ commute not only with the morphisms in $P_{\Delta\mathfrak{G}}$, but also with any n-trace in $\mathfrak{G}$-Gen. Thus any morphism of $\Delta\mathfrak{G}$-trace algebras extends to a natural transformation of functors, yielding the desired equivalence of categories. □

Remark 2.2.15. Note that, if Z is the restriction of a symmetric monoidal functor $\mathfrak{G}\text{-Bord} \to \mathscr{C}$, the existence of the cotrace $\mathfrak{p}_2$ from Notation 2.2.9 means that the trace algebra corresponding to Z is, in fact, a $\Delta\mathfrak{G}$-Frobenius algebra. ∎

2.2.2 Examples

Example 2.2.16. The simplest case is the cyclic case $\Delta\mathfrak{G} = \Lambda$, which corresponds to $GL^{+}(2, \mathbb{R})$. In this case, we see that $P_{\Lambda} = \mathcal{A}\mathrm{ss}$, and that $\mathfrak{G}_0$

is trivial. As a result, the action of an element of $g \in \mathfrak{G}_n$ on an n-trace given in terms of the operadic action on half-edges is just the action of $\lambda_n(g) \in \mathbb{Z}/(n+1)$. As a result, we can simplify the condition of Proposition 2.2.4 to require simply that traces be invariant under cyclic permutation of inputs, as a result, we see that a $\Delta\mathfrak{G}$-Frobenius algebra is precisely a Frobenius algebra.

Example 2.2.17. The next simple case is the dihedral case $\Delta\mathfrak{G} = \Xi$. Here, the corresponding additional datum on surfaces of a reduction of the structure group is trivial. Since we can find a copy of Λ in Ξ, we can take the copy of $\mathcal{A}$ss $\subset P_\Xi$ given by Λ-structured trees. In this case, we can compute χ_2, and we see that an algebra over P_Ξ is an algebra A with an anti-automorphism $*$. More precisely, if we pull back the non-trivial element $f \in \mathfrak{G}_1^0$ (the element which simply reverses orientation), we see that it pulls back to the reflection of the center circle fixing the outgoing marked point, so that it switches the inputs. Pushing out along the incoming augmentation maps, we see that it amounts to reversing orientation in each case, so that we get

$$\chi_2(f) = (f, f; (1,2))$$

ie, that f acts as an anti-automorphism of the algebra in question. We then see that Ξ-Frobenius algebras are precisely Frobenius algebras $(A, *, \beta_1)$ with involution $(-)^*$ such that $\beta_1(a^*) = \beta_1(a)$.

Example 2.2.18. In the N-cyclic case, where $\Delta\mathfrak{G} = \Lambda_N$, corresponding to $\mathrm{Spin}_N(2)$-structured surfaces, we have to be a little more careful. To find a copy of $\mathcal{A}$ss, we need a particular characterization of Λ_N, given in [14, Ex. I.24]. Let C be the unit circle in $\mathbb{C}$, and let C_n be C equipped with $n+1$ marked points $\{0, 1, \ldots, n\}$ included into C via the map

$$k \mapsto \exp\left(\frac{2\pi i k}{n+1}\right)$$

Fixing an N-sheeted cover $\tilde{C} \to C$, we can then describe Λ_N in the following way: Its objects are $\langle n \rangle$ for all n. A morphism $\langle m \rangle \to \langle n \rangle$ is given by a homotopy class of monotone maps $C_m \to C_n$ preserving the marked points together with a lift to $\tilde{C}$.

Using this, we can define elements $\tau_i^n \in \mathfrak{G}_n$ which will allow us to choose a copy of $\mathcal{A}$ss consisting of multiplications $\mathfrak{m}_i$. Let t_i^n be the automorphism of $[n]$ in Λ sending 0 to i, represented as a homotopy class of monotone maps $C_n \to C_n$ preserving the marked points. Then there is a lift τ_i^n of t_i^n to $\tilde{C}$ which is homotopy equivalent to the smallest positive rotation of $\tilde{C}_{Nn}$ covering t_i. If we define, for each n, a multiplication $\mathfrak{m}_n$ with n incoming half-edges via the augmentation maps $(\psi_n \circ \tau_2^n, \ldots, \psi_n \circ \tau_n^n, \psi \circ \tau_0^n, \phi_n)$, where ψ is the map in $\mathrm{Hom}_\Delta([n],[1])$ sending n to 1 and everything else to 0, and ϕ_n is the map in $\mathrm{Hom}_\Delta([n],[1])$ sending 0 to 0 and everything else to 1, it is trivial to verify that $\{\mathfrak{m}_n\}$ forms a system of multiplications. Moreover, we can see that the traces $\mathfrak{b}_n := \beta_1 \circ \mathfrak{m}_n$ can be represented by the maps $\{\psi_{n-1} \circ \tau_{n-1}^i\}_{i=0}^{n-1}$. However, since by construction, for any map $i \in \mathrm{Hom}_\Delta([0],[n])$, we have $i^*(\tau_j^n) = id_{[0]}$, we see that the homomorphisms η_n are precisely the homomorphisms L_n from the proof of [14, Thm. I.37].

We can also calculate χ_2. If we pull back an element $f \in \mathfrak{G}_1^0$, which can be represented by a rotation of $\tilde{C}$ by $2k$ markings, along ϕ_2, we get a rotation by $3k$ markings of $\tilde{C}$. Pushing this out along the incoming augmentation maps, we again get a rotation of $\tilde{C}$ by $2k$ markings. That is,

$$\chi_2(f) = (f, f; id)$$

or, more usefully: $f \circ \mathfrak{m} = \mathfrak{m} \circ (f \sqcup f)$. Hence, the elements of $\mathfrak{G}_0$ act on A by automorphisms.

Since this is the case, our definition of a Λ_N-Frobenius algebra simplifies to the one from [14], and so we have the following characterization.

A Λ_N-Frobenius algebra is a finite-dimensional unital associative algebra A together with a linear function $\beta_1 : A \to k$ such that

- The form $\beta_2(a,b) = \beta_1(ab)$ is a (non necessarily symmetric) non-degenerate bilinear form on A.
- The Nakayama automorphism F of β_2 is an algebra automorphism of A such that $F^N = id_A$

Example 2.2.19. In the N-dihedral case $\Delta\mathfrak{G} = \Xi_N$, we can use a similar characterization to the one for Λ_N (this time allowing both orientation preserving and reversing circle maps). With this characterization we immediately find a copy of Λ_N in Ξ_N, and using the construction from example Example 2.2.18, we can again reduce our notion of a $\Delta\mathfrak{G}$-Frobenius algebra to that of [14]. In this case, we find that a Ξ_N-Frobenius algebra is a Λ_N-Frobenius algebra equipped with a trace-preserving involution.

Example 2.2.20. In the paracyclic case $\Delta\mathfrak{G} = \Lambda_\infty$, we can again use a characterization with circle maps $C_n \to C_m$, this time using a lift to a chosen universal cover $\mathbb{R} \to C$. The construction in example Example 2.2.18 generalizes, and we find that a Λ_∞-frobenius algebra is a Λ_N-frobenius algebra A in which we no longer require that $F^N = id_A$.

As in example Example 2.2.19, we can carry our argument over to the paradihedral case $\Delta\mathfrak{G} = \Xi_\infty$. In this case, we find that a Ξ_N-frobenius algebra is a Λ_∞-frobenius algebra with a trace-preserving involution.

2.3 Reconstructing Field Theories

As in the previous section we fix a balanced crossed simplicial group $\Delta\mathfrak{G}$ and a symmetric monoidal category $\mathscr{C}$ with monoidal unit I.

Lemma 2.3.1. *Any corola $\mathfrak{u}$ can be expressed as a concatenation of disjoint unions of the identity, $\mathfrak{p}_2$, permutations, $\mathfrak{b}_n$, and elements of $P_{\Delta\mathfrak{G}}$.*

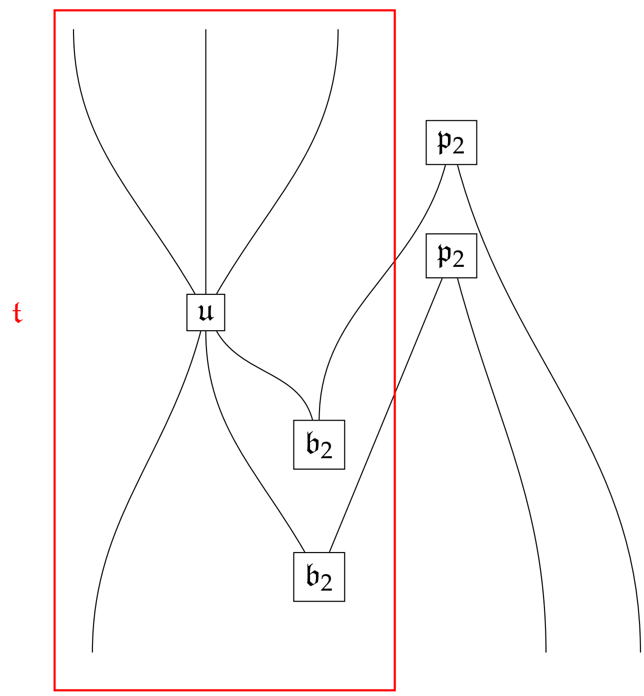


Figure 2.3: A string diagram depiction of the decomposition of a corolla into copies of $\mathfrak{p}_2$ and elements of $\mathfrak{G}$-Gen.

Proof. If $|\operatorname{Out}(\mathfrak{u})| \leq 1$, then $\mathfrak{u}$ is either in $P_{\Delta\mathfrak{G}}$ or is an n-trace, so the lemma is trivial.

In the case where $|\operatorname{Out}(\mathfrak{u})| = k > 1$, we choose one of the outgoing half-edges, labeled i, and choose a permutation $\sigma \in S_k$ sending i to 1. We can then consider the composite

$$\begin{aligned} \mathfrak{t} = (\mathrm{id} \amalg \mathfrak{b}_2) \circ (\mathrm{id}^{\amalg 2} \amalg \mathfrak{b}_2 \amalg \mathrm{id}) \cdots \circ (\mathrm{id}^{\amalg(k-1)} \mathfrak{b}_2 \amalg \mathrm{id}^{\amalg(k-2)}) \\ \circ(\sigma \amalg \mathrm{id}^{\amalg(k-1)}) \circ (\mathfrak{u} \amalg \mathrm{id}^{\amalg(k-1)}) \end{aligned}$$

which is, by definition, a tree. We then have that the composite (see Fig. 2.3 for a pictorial example)

$$(\sigma^{-1}) \circ (\mathfrak{t} \amalg \mathrm{id}^{\amalg(k-1)}) \circ (\mathrm{id}^{\amalg m} \amalg \mathfrak{p}_2 \amalg \mathrm{id}^{\amalg(k-2)}) \circ \cdots \circ (\mathrm{id}^{\amalg m} \amalg \mathfrak{p}_2)$$

gives us $\mathfrak{u}$, proving the lemma. □

Construction 2.3.2. Let $(A, \{\mu_n\}, e, \{\beta_n\})$ be a $\Delta\mathfrak{G}$-Frobenius algebra in $\mathcal{C}$. Per Proposition 2.2.14, there is a corresponding symmetric monoidal functor $Z : \mathfrak{G}\text{-Gen} \to \mathcal{C}$. Moreover, by non-degeneracy, we have a unique

$$\rho : I \to A \otimes A$$

exhibiting the non-degeneracy of $\beta_2 := Z(\mathfrak{b}_2)$. We construct an assignment Z of morphisms in $\mathcal{C}$ to corollas as follows

Given a corolla $\mathfrak{u}$, fix a decomposition

$$\mathfrak{u} = (\sigma^{-1}) \circ (\mathfrak{t} \amalg \mathrm{id}^{\amalg(k-1)}) \circ (\mathrm{id}^{\amalg m} \amalg \mathfrak{p}_2 \amalg \mathrm{id}^{\amalg(k-2)}) \circ \cdots \circ (\mathrm{id}^{\amalg m} \amalg \mathfrak{p}_2)$$

as in Lemma 2.3.1. Then set

$$Z(\mathfrak{u}) = (\sigma^{-1}) \circ (Z(\mathfrak{t}) \otimes \mathrm{id}^{\otimes(k-1)}) \circ (\mathrm{id}^{\otimes m} \otimes \rho_2 \amalg \mathrm{id}^{\otimes(k-2)}) \circ \cdots \circ (\mathrm{id}^{\otimes m} \otimes \rho_2)$$

to define a morphism in $\mathcal{C}$ on general corollas. ∎

Proposition 2.3.3. *For a corolla represented by an augmented structured graph $\mathfrak{u}$, $Z(\mathfrak{u})$ does not depend on the choices made.*

Proof. We made two choices in our construction of the decomposition of Lemma 2.3.1: the chosen half-edge i, and the permutation $\sigma \in S_k$. Given two choices of half-edges i and j, a permutation σ sending i to 1 and a permutation γ sending j to 1, denote by $\mathfrak{t}_\sigma$ and $\mathfrak{t}_\gamma$ the two trees obtained from Lemma 2.3.1. If we denote $\mathfrak{f}_\sigma := \mathfrak{b}_2 \circ (\mathfrak{t}_\sigma \amalg \mathrm{id})$, and $\mathfrak{f}_\gamma := \mathfrak{b}_2 \circ (\mathfrak{t}_\gamma \amalg \mathrm{id})$, then we have

$$\mathfrak{f}_\gamma = \mathfrak{f}_\sigma \circ (\mathrm{id}^{\amalg m} \amalg (\gamma \circ \sigma^{-1}))$$

meaning that, applying Construction 2.3.2 yields

$$\begin{aligned} Z(\mathfrak{u}_\gamma) &= (\gamma^{-1}) \circ (Z(f_\gamma) \otimes \mathrm{id}^{\otimes(k)}) \circ \cdots \\ &= \gamma^{-1} \circ \gamma \circ \sigma^{-1} \circ (Z(f_\sigma) \otimes \mathrm{id}^{\otimes(k)}) \circ \cdots \\ &= Z(\mathfrak{u}_\sigma), \end{aligned}$$

showing the desired independence. □

Corollary 2.3.4. *Let* $\mathfrak{u}$ *and* $\mathfrak{v}$ *be two corollas representing the same cobordism. Then* $Z(\mathfrak{u}) = Z(\mathfrak{v})$.

Proof. Since $\mathfrak{v}$ and $\mathfrak{u}$ represent the same cobordism. They are related by an automorphism of their central vertex. Therefore, by the universal property of the pullback by which we contract the composed edge, the corollas comprising the decompositions of $\mathfrak{u}$ and $\mathfrak{v}$ must also be related by automorphisms of their central vertices. □

Construction 2.3.5. Given an augmented $\Delta\mathfrak{G}$-structured graph Γ, we decompose it as follows:

Given an internal edge e, whose $\Delta\mathfrak{G}$-structure is represented by a diagram

$$(H_v, \mathcal{O}_v) \xrightarrow{\phi} (\{h, h'\}, \mathcal{O}_e) \xleftarrow{\psi} (H(w), \mathcal{O}_w)$$

of structured sets, we can expand this edge to a new vertex c to yield an equivalent graph Γ', with local $\Delta\mathfrak{G}$-structure represented[1] by

$$\begin{array}{ccccc} v & & c & & w \\ (H_v, \mathcal{O}_v) \xrightarrow{\phi} (\{h, h'\}, \mathcal{O}_e) & \xleftarrow{\mathrm{id}} & (\{h, h'\}, \mathcal{O}_e) & \xrightarrow{\mathrm{id}} & (\{h, h'\}, \mathcal{O}_e) \xleftarrow{\psi} (H(w), \mathcal{O}_w) \end{array}$$

We may then choose two identifications $f, g : (\{h, h'\}, \mathcal{O}_e) \cong \epsilon_{\mathfrak{G}}[1]$ so that the structured graph given by

$$\epsilon_{\mathfrak{G}}[1] \xleftarrow{f} (\{h, h'\}, \mathcal{O}_e) \xrightarrow{g} \epsilon_{\mathfrak{G}}[1]$$

[1]Note that there is a slight abuse of notation here, as the underlying set of the $\Delta\mathfrak{G}$-structure set assigned to the central vertex is not the underlying set assigned by the incidence diagram. This is not a problem, however, since there is a canonical relabeling yielding the correct structured graph. We will neglect such concerns in the sequel.

represents $\mathfrak{b}_2$.

We can construct a new $\Delta\mathfrak{G}$-structured graph Γ'' from Γ prime by removing the vertex c and the half-edges attached to it, yielding two new external half-edges. The identifications f and g then give outgoing augmentations at this new half edges via the maps

$$(H_v, \mathcal{O}_v) \xrightarrow{f^{-1}\circ\phi} \epsilon_{\mathfrak{G}}[1]$$

$$(H(w), \mathcal{O}_w) \xrightarrow{g^{-1}\circ\psi} \epsilon_{\mathfrak{G}}[1].$$

Applying this procedure at every internal edge of Γ allows us to write

$$[\Gamma] = (\mathrm{id}^{\amalg n} \amalg \mathfrak{b}_2^{\amalg m}) \circ (\coprod_{i=1}^{k} \mathfrak{u}_i) \tag{2.5}$$

Given a $\Delta\mathfrak{G}$-Frobenius algebra A, with notation fixed as in Construction 2.3.2, we then define an assignment on structured cobordisms by choosing a decomposition (2.5) and setting

$$Z_A(\Gamma) = (\mathrm{id}_A^{\otimes n} \otimes \beta_2^{\otimes m}) \circ \left(\bigotimes_{i=1}^{k} Z(\mathfrak{u}_i) \right).$$

We will denote the resulting assignment on the morphisms of $\mathfrak{G}$ -Bord by Z_A. ∎

Proposition 2.3.6. *Given a $\Delta\mathfrak{G}$-Frobenius algebra A, the assignment Z_A is well-defined on isomorphism classes of $\Delta\mathfrak{G}$-structured augmented graphs.*

Proof. We fix a structured graph Γ and an edge e as in Construction 2.3.5. There are two implicit choices we made in Construction 2.3.5: We (implicitly) chose an order of the half-edges attached to the new vertex c, and we choose the identifications f and g.

However, by Proposition 2.2.10, we see that, since a different choice of the order and the identifications f and g must yield a representative of $\mathfrak{b}_2$, changing these choices amounts to inserting an element $h \in \mathfrak{I}_2$ and its inverse into the computation. These choices, therefore, do not change the final value of $Z_A(\Gamma)$. □

Proposition 2.3.7. *Let Γ be an augmented $\Delta\mathfrak{G}$-structured graph with a single edge e such that $\Gamma/_e$ is a corolla. Then $Z_A(\Gamma) = Z(\Gamma_e)$.*

Proof. The $\Delta\mathfrak{G}$-structure on the edge e is represented by a diagram

$$(H_v, \mathcal{O}_v) \xrightarrow{\phi} (\{h,h'\}, \mathcal{O}_e) \xleftarrow{\psi} (H(w), \mathcal{O}_w)$$

of $\Delta\mathfrak{G}$-structured sets. We fix a choice of indentifications $f, g : (\{h,h'\}, \mathcal{O}_e) \cong \epsilon_{\mathfrak{G}}[1]$ as in Construction 2.3.5, to obtain two augmented corollas Θ and Ω with central vertices v and w respectively. By definition, we then have that

$$Z_A(\Gamma) = (\mathrm{id}^{\otimes n} \otimes \beta_2) \circ (Z(\Theta) \otimes Z(\Omega)).$$

If Γ has no outgoing half-edges (i.e. if $n = 0$), then the latter is equal to $Z(\Gamma/_e)$ by the functoriality of Z on $\mathfrak{G}$-Gen.

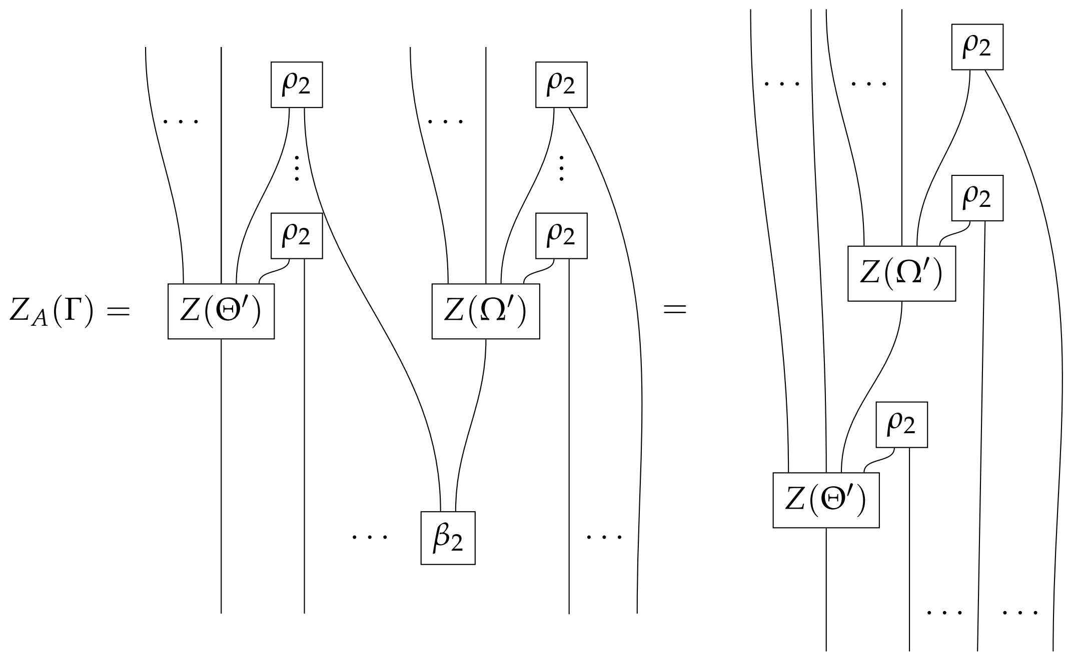
In the case that Γ has outgoing half-edges, assume without loss of generality that the one of these labeled 1 is attached to v. Then, following Lemma 2.3.1, we have

$$Z(\Theta) = (Z(\Theta') \otimes \mathrm{id}^{\otimes k-1}) \circ (\mathrm{id}^{\otimes \ell} \otimes \rho_2 \otimes \mathrm{id}^{\otimes k-2}) \cdots \circ (\mathrm{id}^{\otimes \ell} \otimes \rho_2)$$

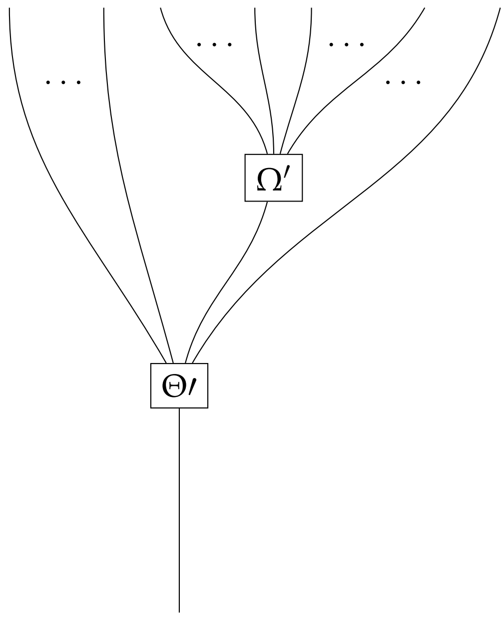
where Θ' is a tree. Similarly, fixing the halfedge forming e, applying Lemma 2.3.1 yields

$$Z(\Omega) = (Z(\Omega') \otimes \mathrm{id}^{\otimes s-1}) \circ (\mathrm{id}^{\otimes t} \otimes \rho_2 \otimes \mathrm{id}^{\otimes s-2}) \cdots \circ (\mathrm{id}^{\otimes t} \otimes \rho_2)$$

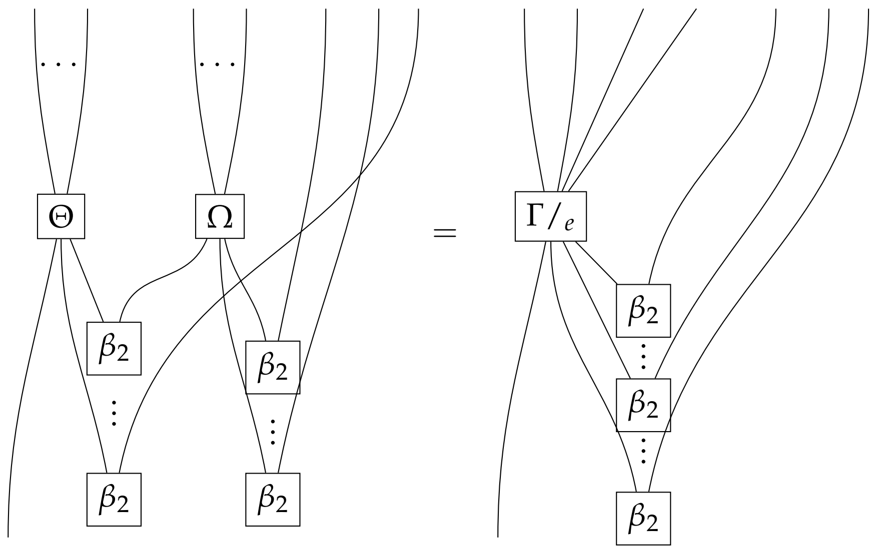
where Ω' is a tree. We then compute (using string diagrams for ease of notation)



On the level of graphs, it will therefore be enough to show that the structured graph Ψ given by contracting the unique internal edge in the concatenation



is obtained by the procedure in Lemma 2.3.1. This follows by construction, since



Meaning that $Z_A(\Gamma) = Z(\Gamma/_e)$ as desired. □

Corollary 2.3.8. *Let Γ and Γ' be two $\Delta\mathfrak{G}$-structured cobordisms such that Γ' is obtained, together with its $\Delta\mathfrak{G}$-structure, from Γ by contracting an edge e. Then $Z_A(\Gamma) = Z_A(\Gamma')$.*

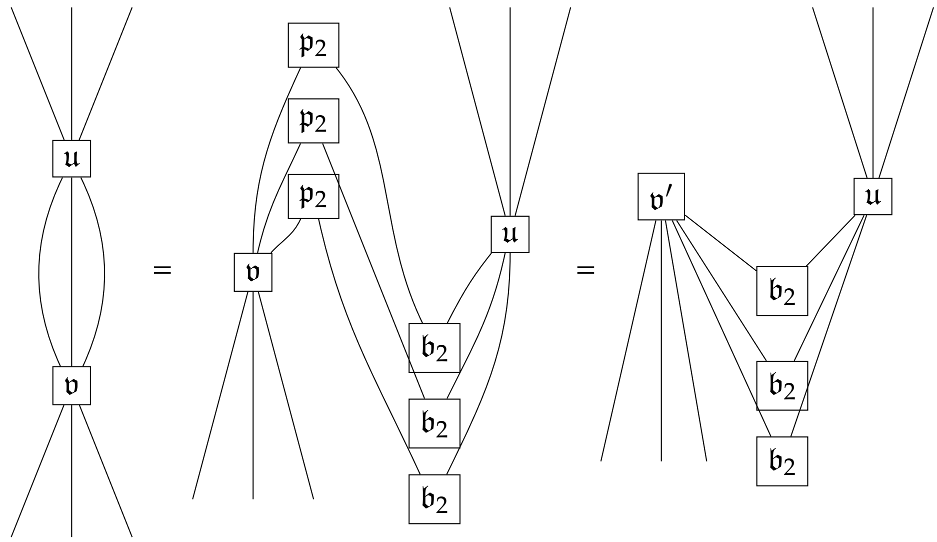
Proof. This follows from applying Proposition 2.3.7 to the two vertices joined by e. □

Corollary 2.3.9. *The assignment Z_A is well-defined on morphisms of* $\mathfrak{G}$ -Bord.

Proposition 2.3.10. *The assignment Z_A defines a functor*

$$Z_A : \mathfrak{G}\text{-Bord} \to \mathscr{C}.$$

Proof. We need only show composability. Let $\mathfrak{u}$ and $\mathfrak{v}$ be two composable cobordisms. We may insert copies of $\mathfrak{b}_2$ and $\mathfrak{p}_2$ into their composition without changing the value of the composition as shown in the following string diagram



Since a corolla representative for $\mathfrak{p}_2$ consists of a pair of isomorphisms between $\Delta\mathfrak{G}$-structured sets of cardinality 2, we can represent $\mathfrak{v}'$ by the same structured graph as $\mathfrak{v}$, but with the incoming augmentations composed with an automorphism of $\epsilon_{\mathfrak{G}}[1]$. This gives us a canonical decomposition of the kind constructed in Construction 2.3.5 in terms of the decompositions for $\mathfrak{u}$ and $\mathfrak{v}$ respectively. More precisely, if we write

$$Z_A(\mathfrak{u}) = (\mathrm{id}_A^{\otimes n} \otimes \beta_2^{\otimes m}) \circ \left(\bigotimes_{i=1}^{k} Z(\mathfrak{u}_i) \right)$$

and

$$Z_A(\mathfrak{v}) = (\mathrm{id}_A^{\otimes s} \otimes \beta_2^{\otimes t}) \circ \left(\bigotimes_{i=1}^{\ell} Z(\mathfrak{v}_i) \right),$$

and we denote by $\mathfrak{v}_i'$ the corollas corresponding to the vertices of $\mathfrak{v}'$, then we have

$$Z_A(\mathfrak{v} \circ \mathfrak{u}) = \left(\mathrm{id}_A^{\otimes s} \otimes \beta_2^{\otimes t} \otimes \beta_2^n \otimes \beta_2^{\otimes m}\right) \circ \left(\bigotimes_{i=1}^{\ell} Z(\mathfrak{v}_i') \otimes \bigotimes_{j=1}^{k} Z(\mathfrak{u}_j) \right).$$

By construction $\mathfrak{v}_i = \mathfrak{v}_i'$ whenever the vertex i is not attached to any incoming half-edges. If i is attached to p incoming half-edges, we have that

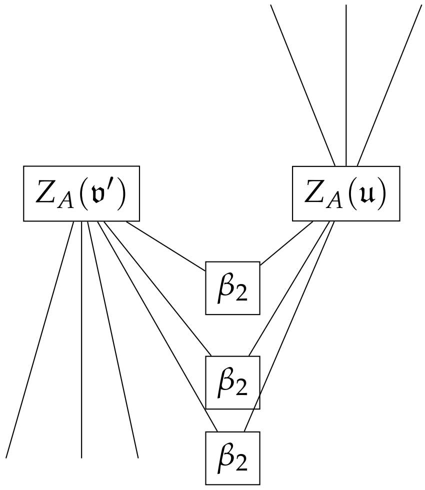
$$\mathfrak{v}_i' = (\mathfrak{v}_i \otimes \mathrm{id}^{\otimes p}) \circ (\mathfrak{p}_2^{\otimes p}).$$

It therefore suffices to show that for the corolla $\mathfrak{v}_i$,

$$Z(\mathfrak{v}_i') = (Z(\mathfrak{v}_i) \otimes \mathrm{id}^{\otimes p}) \circ (\rho_2^{\otimes p}), \tag{2.6}$$

since we would then have

$Z_A(\mathfrak{v} \circ \mathfrak{u}) =$



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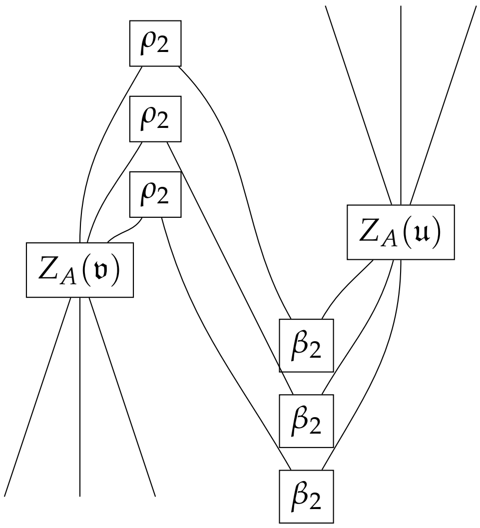

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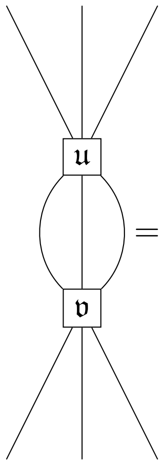


$= Z_A(\mathfrak{v}) \circ Z_A(\mathfrak{u})$

However, (2.6) is immediate from the construction of Lemma 2.3.1, completing the proof. □

Theorem 2.3.11. *The functor*

$$F : \mathrm{Fun}^{\otimes}(\mathfrak{G}\text{-Bord}, \mathscr{C}) \to \Delta\mathfrak{G}\text{-Frob}_{\mathscr{C}}$$

defined by restricting to $\mathfrak{G}$-Gen *is an equivalence of categories.*

Proof. By definition F is faithful, and Corollary 2.3.9 shows that F is essentially surjective. Moreover, the construction of Z_A from $A : \mathfrak{G}\text{-Gen} \to \mathscr{C}$ immediately implies that every natural transformation $A \Rightarrow B$ defines a natural transformation $Z_A \Rightarrow Z_B$, completing the proof. □

Remark 2.3.12. Theorem 2.3.11 retrieves two well-known results, namely the folkloric equivalence between open oriented 2d topological field theories and Frobenius algebras (in the case $\Delta\mathfrak{G} = \Lambda$), and the equivalence between open unoriented 2d topological field theories and Frobenius algebras with trace-preserving anti-involution proved in [8] (in the case $\Delta\mathfrak{G} = \Xi$). Also of interest is the relation with [38], where a state-sum construction of 2d *closed* N-spin topological field theories is given in terms of a Λ_N-Frobenius algebra satisfying the additional property that its window element is invertible. ∎

2.4 Example: equivariant topological field theories

Definition 2.4.1. Let G be a planar lie group, and let H be a discrete group. We define a *GH-cobordism* to be a G-structured cobordism $(S, B_{\text{in}}, B_{\text{out}})$ together with a homotopy class

$$[f_S] \in \pi_0 \operatorname{Map}\left((S, B), (|BH|, *)\right)$$

where $B = B_{\text{in}} \cup B_{\text{out}}$. An *equivalence of GH-cobordisms* is an equivalence $(\tilde{\phi}, \phi) : (S, B_{\text{in}}, B_{\text{out}}) \to (T, B_{\text{in}}, B_{\text{out}})$ such that the induced diagram

$$\begin{array}{ccc} (S,B) & & \\ & \searrow^{f_S} & \\ \phi \downarrow & & (|BH|, *) \\ & \nearrow_{f_T} & \\ (T,B) & & \end{array}$$

commutes up to homotopy relative to B.

Given two GH-cobordisms $(S, B_{\text{in}}, B_{\text{out}})$ and $(S, D_{\text{in}}, D_{\text{out}})$ with $D_{\text{in}} = B_{\text{out}}$, note that, for any choice of representatives, f_S and f_T agree on $D_{\text{in}} = B_{\text{out}}$, so we get a unique homotopy class

$$[f_{T\circ S}] \in \pi_0 \operatorname{Map}((T \circ S, D_{\text{out}} \circ B_{\text{in}}), (|BH|, *)),$$

defining a composite GH-cobordism. We define $H\operatorname{Cob}^G$ to be the symmetric monoidal category with objects disjoint unions of G-structured intervals, and morphisms equivalence classes of GH-cobordisms. We will call symmetric monoidal functors $H\operatorname{Cob}^G \to \mathscr{C}$ *H-equivariant G-structured topological field theories.* ∎

Remark 2.4.2. Equivalently, we could have defined $H\,\mathrm{Cob}^G$ in terms of G-structured bordisms equipped with the additional structure of an H-principal bundle with chosen trivializations over the boundary intervals. ∎

Definition 2.4.3. For a discrete group H, we define the groupoid H-Tor whose objects are H-torsors (in Set) and whose morphisms are H-equivariant maps. Note that there is an equivalence of categories $BH \simeq H\text{-Tor}$. ∎

Construction 2.4.4. Recall from Construction 1.2.31 that, given a balanced crossed simplicial group $\Delta\mathfrak{G}$ and a discrete group H, we can form a new balanced crossed simplicial group $\Delta\mathfrak{G}\mathfrak{H} = \Delta\mathfrak{G} \times BH$. This new crossed simplicial group admits an obvious morphism of crossed simplicial groups $\Delta\mathfrak{G}\mathfrak{H} \to \Delta\mathfrak{G}$ such that the diagram

$$\begin{array}{ccccc} & & \Delta\mathfrak{G} & & \\ & \nearrow & & \searrow^{\lambda_{\mathfrak{G}}} & \\ \Delta\mathfrak{G}\mathfrak{H} & & \xrightarrow{\quad\lambda_{\mathfrak{G}\mathfrak{H}}\quad} & & \mathbb{F}\mathrm{in} \end{array}$$

commutes.

The corresponding category $\mathcal{GH}$ of $\Delta\mathfrak{G}\mathfrak{H}$-structured sets admits forgetful functors

$$\begin{array}{ccc} & & H\text{-Tor} \\ & \nearrow & \\ \mathcal{GH} & & \\ & \searrow & \\ & & \mathcal{G} \end{array}$$

Given by the formulas

$$(S, \mathcal{O}_S) \mapsto \mathcal{O}_S/\mathfrak{G}_{|S|-1}$$
$$(S, \mathcal{O}_S) \mapsto (S, \mathcal{O}_S/H)$$

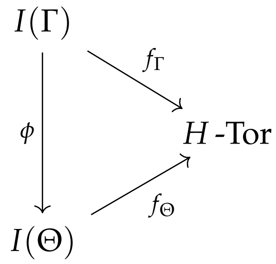
respectively. These functors define an equivalence $\mathcal{GH} \simeq H\text{-Tor} \times \mathcal{G}$.

Consequently an augmented $\Delta\mathfrak{G}\mathfrak{H}$-structure on a graph Γ can be viewed as a pair $(\widetilde{AI}_\Gamma, f_\Gamma)$ consisting of:

1. An augmented $\Delta\mathfrak{G}$-structure $\widetilde{AI}_\Gamma : I(\Gamma) \to \mathcal{G}$.

2. A functor $f_\Gamma : I(\Gamma) \to H\text{-Tor}$. such that $f_\Gamma(\partial|\Gamma|) = H$ (where H is the canonically trivialized H-torsor).

An equivalence $(\widetilde{AI}_\Gamma, f_\Gamma) \to (\widetilde{AI}_\Theta, f_\Theta)$ of $\Delta\mathfrak{G}\mathfrak{H}$-structure cobordisms consists of a morphism (ϕ, μ) of the underlying $\Delta\mathfrak{G}$-structured cobordisms such that the diagram



commutes up to a natural isomorphism which is the identity on $\partial\Gamma$. ∎

Proposition 2.4.5. *Let G be a planar lie group corresponding to a crossed simplicial group $\Delta\mathfrak{G}$, and let H be a discrete group. Then there is an equivalence of categories*

$$\Delta\mathfrak{G}\mathfrak{H}\text{-Bord} \simeq H\,\mathrm{Cob}^G .$$

Proof. We will use $|H\text{-Tor}\,|$ as our model for the classifying space, with chosen basepoint H as an H-torsor. Given a GH-bordism $(S, B_{\text{in}}, B_{\text{out}})$, the image of the underlying G-structured bordism in $\mathfrak{G}$ -Bord is given by taking the equivalence class $[\Gamma]$, where Γ is a spanning graph of $(S, B_{\text{in}}, B_{\text{out}})$ equipped with the induced augmented $\Delta\mathfrak{G}$-structure. Additionally, however, the homotopy class of maps $[f_S] : (S, B) \to (|H\text{-Tor}\,|, H)$, together with the embedding of $\gamma : |\Gamma| \to S$ gives rise to a homotopy class of maps

$$[f_S \circ \gamma] : (|\Gamma|, \partial|\Gamma|) \to (|H\text{-Tor}\,|, H).$$

By cellular approximation[2], this yields a functor

$$g_S : I(\Gamma) \to H\text{-Tor}$$

which is unique up to natural isomorphism. Since γ is a homotopy equivalence, this construction yields a bijection between homotopy classes of maps $(S, B) \to (|H\text{-Tor}\,|, H)$ and natural isomorphism classes of functors $I(\Gamma) \to H\text{-Tor}$ sending $\partial\Gamma$ to H.

To see that this construction defines the desired equivalence of categories, we need to show that equivalence classes of maps $[f_S] : (S, B) \to$

[2]Note that this step only follows because H -Tor is a groupoid.

$(|H\text{-Tor}|, H)$ under both homotopy and the action of the structured mapping class group of $(S, B_{\text{in}}, B_{\text{out}})$ are in one-to-one correspondence with equivalence classes of bordisms in $\Delta\mathfrak{G}\mathfrak{H}$-Bord. In the case where $(S, B_{\text{in}}, B_{\text{out}})$ is stable, this follows immediately from Corollary 2.1.20 and Proposition 2.1.27. We then treat the unstable bordisms case-by-case:

1. D^2 with a single incoming or outgoing embedded boundary interval. In either case, the action of structured mapping class group is trivial, and the bijection follows immediately from the above discussion.

2. D^2 with two embedded boundary intervals, one incoming, and one outgoing. In this case, as with the previous, the action structured mapping class group is trivial, and the bijection follows.

3. D^2 with no embedded boundary intervals. In this case, since D^2 is contractible (as is any corresponding graph), the morphism to $|BH|$ carries no additional information.

4. The annulus A with no embedded boundary intervals. This case splits in two, depending on whether or not G preserves orientation.

   a) If G preserves orientation, then the action of the structured mapping class group is trivial.

   b) If G does not preserve orientation, then elements of the structured mapping class group either act by the identity, or by the reflection. The latter case occurs if and only if the element $a \in \mathfrak{G}_0$ classifying the G-structure (described in Proposition 2.1.32 6.b) is self-inverse. Similarly, there is an automophism of the structured graph with one loop and one vertex which switches the half-edges if and only if the element $a \in \mathfrak{G}_0$ classifying the $\Delta\mathfrak{G}$-structure is invertible.

5. The case of the Möbius band is virtually identical to that from part (b).

Consequently, out construction yields a fully-faithful, essentially surjective functor $\Delta\mathfrak{G}\mathfrak{H}\text{-Bord} \to H\,\mathrm{Cob}^G$. □

Remark 2.4.6. Thinking of the morphisms of $H\,\mathrm{Cob}^G$ as G-cobordisms with the additional datum of an H-principal bundle as in Remark 2.4.2

provides a different way of looking at Proposition 2.4.5. We can construct a bundle over from a $\Delta\mathfrak{G}\mathfrak{H}$-structured graph embedded into a G-structured cobordism $(S, B_{\text{in}}, B_{\text{out}})$ by covering S with contractible opens each of which contain the image of a single object in $I(\Gamma)$, as shown in Fig. 2.4. We then define the principal bundle over each open to be the H-torsor associated to the given object, with transition functions given by the maps of torsors assigned to the half-edges. ∎

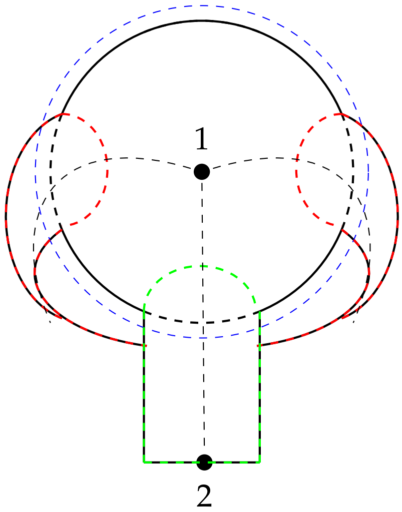


Figure 2.4: A cobordism $(S, B_{\text{in}}, B_{\text{out}})$ with spanning graph Γ, covered by contractible sets.

Corollary 2.4.7. *An H-equivariant G-structured topological field theory in $\mathscr{C}$ is determined up to natural isomorphism by a $\Delta\mathfrak{G}$-Frobenius algebra equipped with an action of H, i.e. by a functor $BH \to \mathfrak{G}\text{-Frob}_{\mathscr{C}}$.*

Proof. A $\Delta\mathfrak{G}\mathfrak{H}$-Frobenius algebra in $\mathscr{C}$ consists of an object $A \in \mathscr{C}$ with an action of $\mathfrak{G}_0 \times H$, a multiplication $\mu_2 : A \otimes A \to A$, a unit $e : I \to A$, and a morphism $\beta_1 : \to I$ satisfying the invariance conditions of Definition 2.2.13. Write $\chi_2^{\mathfrak{G}}$ and $\mathfrak{I}_n^{\mathfrak{G}}$ for the map and subgroup defining the invariance conditions for $\Delta\mathfrak{G}$-Frobenius algebras, and write χ_2 and $\mathfrak{I}_n$ for those of $\Delta\mathfrak{G}\mathfrak{H}$-Frobenius algebras. We then have

$$\chi_2(g, h) = \chi_2^{\mathfrak{G}}(g) \times (h, h).$$

Similarly, $\mathfrak{I}_n \cong \mathfrak{I}_n^{\mathfrak{G}} \times H$, where H is considered as a subgroup of $H^{\times n}$ via the diagonal map. We therefore have the structure of a $\Delta\mathfrak{G}$-Frobenius algebra on A, and H acts by automorphisms in the category $\mathfrak{G}\text{-Frob}_{\mathscr{C}}$. □

Remark 2.4.8. In light of [11], which shows that open oriented topological conformal field theories are equivalent to Calabi-Yau A_∞-categories, one can view Corollary 2.4.7 as a sort of decategorified analogue of [18, Thm. 1.1]. The latter classifies open H-equivariant topological conformal field theories in terms Calabi-Yau A_∞-categories with an H-action. ∎

3 Calabi-Yau algebras in spans

In this chapter, we present the main results of this thesis: the equivalences between 2-Segal cyclic (resp. simplicial) objects in $\mathcal{C}$, and Calabi-Yau algebras (resp. associative algebras) in $\mathrm{Span}(\mathcal{C})$. The proofs of these two propositions parallel one another, and are presented in sections 3.1 and 3.2, respectively. The final section, 3.3, is devoted to exploring the consequences of these theorems for topological field theories, and exposing an example — that of the cyclic Čech nerve. Throughout the chapter, we will draw on the background established in sections 1.1 and 1.4, and section 3.3 makes additional use of the structured graph formalism exposed in sections 1.2 and 1.3.

3.1 Algebras in Spans

Throughout this section, we set $\Theta := \mathrm{Tw}(\Delta) \times_{\Delta} \Delta^{\amalg}$. Morphisms in Θ will be represented as diagrams

$$\begin{array}{ccccc} \{i,j\} & \subseteq & [n] & \xrightarrow{f} & [m] \\ & & {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle \overline{g}} \\ \{i',j'\} & \subseteq & [n'] & \xrightarrow[f']{} & [m'] \end{array}$$

in Δ. In this section and the next, $\mathcal{C}$ will denote an ∞-category with small limits. We will, on occasion, denote an object $\{i,j\} \subset [n] \xrightarrow{f} [m]$ in Θ by the pair $(f, \{i,j\})$.

3.1.1 Conditions on functors

We fix a functor $G : \Theta \to \mathcal{C}$, which corresponds to a functor

$$\tilde{G} : \mathrm{Tw}(\Delta) \to \mathcal{C}^{\boxtimes}$$

over Δ.

Proposition 3.1.1. *The functor G defines a functor $\overline{G} : \Delta \to \mathrm{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$ if and only if, for every simplex $[n_0] \xrightarrow{\phi_1} [n_1] \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} [n_k]$ in Δ and every interval $\{i, j\} \subset [n_0]$, the corresponding diagram*

$$\begin{array}{ccccccc}
 & & & G(\phi_n \circ \cdots \circ \phi_1, \{i,j\}) & & & \\
 & & \swarrow & & \searrow & & \\
 & G(\phi_1, \{i,j\}) & & \cdots & & G(\phi_k, \{\psi_{k-1}(i), \psi_{k-1}(j)\}) & \\
\swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \\
G([n_0], \{i,j\}) & & G([n_1], \{\psi_1(i), \psi_1(j)\}) & \cdots & G([n_{k-1}], \{\psi_{k-1}(i), \psi_{k-1}(j)\}) & & G([n_k], \{\psi_k(i), \psi_k(j)\})
\end{array} \tag{3.1}$$

where $\psi_i := \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$, is a limit diagram in $\mathcal{C}$.

Proof. By definition, G defines a functor

$$\overline{G} : \Delta \to \mathrm{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$$

if and only if every restriction of $\tilde{G}$ to $\mathrm{Tw}(\Delta^n) \subset \mathrm{Tw}(\Delta)$ is a Segal simplex in $\mathcal{C}^{\boxtimes}$.

Let $\Delta^k \hookrightarrow \Delta$ be the simplex

$$[n_0] \xrightarrow{\phi_1} [n_1] \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} [n_k].$$

Then by [15, Lemma 10.2.13], there is a functor

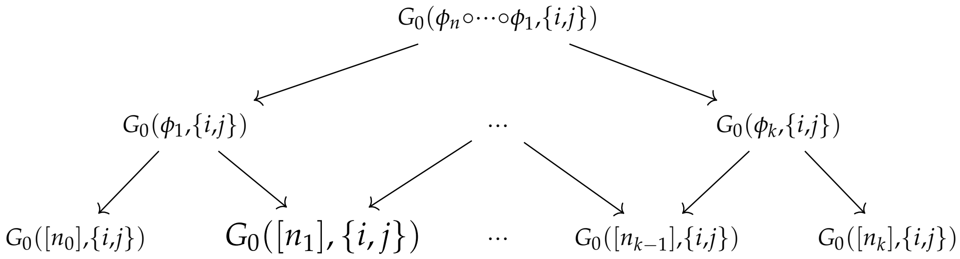
$$H : \left(\Delta^1 \times \mathrm{Tw}(\Delta^k)\right) \times_{\Delta} \Delta^{\mathrm{II}} \to \mathcal{C}$$

representing a homotopy

$$\tilde{H} : \Delta^1 \times \mathrm{Tw}(\Delta^k) \to \mathcal{C}^{\boxtimes}.$$

This homotopy has components that are Cartesian morphisms, and the component $\tilde{G}_0 := \tilde{H}|_{\{0\} \times \mathrm{Tw}(\Delta^k)}$ has image contained in $\mathcal{C}^{\boxtimes}_{[n_0]}$. Since this is the case, the condition that $\tilde{G}$ is a p-limit diagram when restricted to the Segal cone is equivalent to the condition that $\tilde{G}_0$ is a limit diagram in $\mathcal{C}^{\boxtimes}_{[n_0]}$ when restricted to the Segal cone. This can be checked componentwise, using one component for each subinterval of $[n_0]$.

Fix one such subinterval, $\{i, j\}$. Then the coresponding Segal cone diagram in $\mathcal{C}$ will be



Since the homotopy has Cartesian components, H will restrict to a natural equivalence between this diagram and the diagram (1). Therefore, a simplex is Segal if and only if all such diagrams are limit diagrams. □

3.1.1.1 Cartesian morphisms and equivalences

Construction 3.1.2. Suppose G represents a coalgebra object. Given an inert morphism $\Delta^1 \xrightarrow{\{\phi\}} \Delta$ ($\phi : [n] \to [m]$), G must send ϕ to a Cartesian morphism in $\mathrm{Span}_\Delta(\mathcal{C}^{\boxtimes})$. This means that the adjoint map

$$\mathrm{Tw}(\Delta^1) \to \mathcal{C}^{\boxtimes}$$

is comprised only of Cartesian morphisms. Therefore:

- For the source map $\phi \to [n]$ in $\mathrm{Tw}(\Delta)$, and for any $\{i, j\} \in [n]$, the induced morphism

$$G(\phi, \{i, j\}) \to G([n], \{i, j\})$$

is an equivalence.

- For the target map $\phi \to [m]$ in $\mathrm{Tw}(\Delta)$, and for any $\{i, j\} \in [n]$ The induced morphism

$$G(\phi, \{i, j\}) \to G([m], \{\phi(i), \phi(j)\})$$

is an equivalence.

We will write $\phi_{i,j} : [i, \ldots, j] \to [n]$ for the inert morphism which includes the interval $[i, \ldots, j]$ into $[n]$. ∎

Proposition 3.1.3. *Suppose G represents a coalgebra object. Let $f : [n] \to [m]$ be a morphism in Δ, viewed as an object in* $\mathrm{Tw}(\Delta)$.

1. *Let $f|_{\{i,j\}} : [i, \ldots, j] \to [m]$ be the restriction of f to $[i, \ldots, j] \subset [n]$. Then the induced morphism*
$$G(f|_{\{i,j\}}, \{i,j\}) \to G(f, \{i,j\})$$
is an equivalence.

2. *Let $\tilde{f} : [n] \to [i, \ldots, j] \subset [m]$ be a morphism such that composing with the inert morphism $\phi_{i,j} : [i, \ldots, j] \to [m]$ yields f. Then the induced morphism*
$$G(f, \{i,j\}) \to G(\tilde{f}, \{i,j\})$$
is an equivalence.

Proof. Applying our conclusion from above, we find that in case (1), the diagram

$$\begin{array}{ccccc} & & G(f|_{\{i,j\}}, \{i,j\}) & & \\ & \swarrow & & \searrow & \\ G(\phi_{i,j}, \{i,j\}) & & & & G(f, \{i,j\}) \\ & \searrow & & \swarrow & \\ & & G(id_{[n]}, \{i,j\}) & & \end{array}$$

Must be pullback. Therefore, since $G(\phi_{i,j}, \{i,j\}) \to G(\mathrm{id}_{[n]}, \{i,j\})$ must be an equivalence, so must $G(f|_{\{i,j\}}, \{i,j\}) \to G(f, \{i,j\})$.

Similarly, in case (2), the diagram

$$\begin{array}{ccccc} & & G(f, \{i,j\}) & & \\ & \swarrow & & \searrow & \\ G(\tilde{f}, \{i,j\}) & & & & G(\phi_{i,j}, \{i,j\}) \\ & \searrow & & \swarrow & \\ & & G(id_{[i,\ldots,j]}, \{i,j\}) & & \end{array}$$

must be pullback. Therefore, since $G(\phi_{i,j}, \{i,j\}) \to G(\mathrm{id}_{[i,\dots,j]}, \{i,j\})$ must be an equivalence, so must $G(f, \{i,j\}) \to G(\tilde{f}, \{i,j\})$. □

Lemma 3.1.4. *Suppose G sends the morphisms from Proposition 3.1.3 to equivalences. Let*

$$\mu := \left\{ \begin{array}{ccccc} [k] \cong \{i,j\} & \subseteq & [n] & \xrightarrow{f} & [m] \\ & & \downarrow g & & \uparrow \overline{g} \\ [k] \cong \{i',j'\} & \subseteq & [n'] & \xrightarrow{f'} & [m'] \end{array} \right\}$$

be a morphism such that g restricts to an isomorphism $[i,\dots,j] \xrightarrow{\cong} [i',\dots,j']$ and $\overline{g}$ restricts to an isomorphism $[f'(i'), f'(i')+1, \dots, f'(j')] \xrightarrow{\cong} [f(i), f(i)+1, \dots, f(j)]$. Then G sends μ to an equivalence.

Proof. We first note that, under the given hypotheses, G will send morphisms of the form

$$\nu := \left\{ \begin{array}{ccccc} \{0,k\} & \subseteq & [k] & \xrightarrow{s} & [m] \\ & & \mathrm{id}_{[k]} \downarrow & & \uparrow h \\ \{0,k\} & \subseteq & [k] & \xrightarrow[s']{} & [m'] \end{array} \right\}$$

to equivalences, where h sends $[s'(i'), s'(i')+1, \dots, s'(j')]$ isomorphically to $[m]$. This follows from composing

$$\begin{array}{ccccc} \{0,k\} & \subseteq & [k] & \xrightarrow{s} & [m] \\ & & \mathrm{id}_{[k]} \downarrow & & \uparrow h \\ \{0,k\} & \subseteq & [k] & \xrightarrow{s'} & [m'] \\ & & \mathrm{id}_{[k]} \downarrow & & \uparrow \psi \\ \{0,k\} & \subseteq & [k] & \xrightarrow[s]{} & [m] \end{array}$$

Where ψ is the inclusion of the interval $[f(0), \dots, f(k)]$. The lower morphism is then one of the morphisms of type (2) from Proposition 3.1.3 and the two morphisms compose to the identity. So, by 2-out-of-3, ν must be sent to an equivalence.

Now write $[\ell] := [f(i), f(i)+1, \ldots, f(j)]$, and consider the composition

$$\begin{array}{ccccc}
\{0,k\} & \subseteq & [k] & \xrightarrow{s} & [\ell] \\
 & & \big\downarrow{\scriptstyle \phi_{i,j}} & & {\scriptstyle h}\big\uparrow \\
\{i,j\} & \subseteq & [n] & \xrightarrow{f} & [m] \\
 & & \big\downarrow{\scriptstyle g} & & {\scriptstyle \overline{g}}\big\uparrow \\
[k] \cong \{i',j'\} & \subseteq & [n'] & \xrightarrow{f'} & [m']
\end{array}$$

Where h sends $[\ell]$ isomorphically to itself. The upper morphism is the composite of a morphism of type (1) from Proposition 3.1.3 and a morphism of the same kind as ν. Moreover, the composite

$$\begin{array}{ccccc}
\{0,k\} & \subseteq & [k] & \xrightarrow{s} & [\ell] \\
 & & {\scriptstyle \phi_{i',j'}}\big\downarrow & & \big\uparrow{\scriptstyle h'} \\
[k] \cong \{i',j'\} & \subseteq & [n'] & \xrightarrow[f']{} & [m']
\end{array}$$

is also the composite of a morphism of type (1) from Proposition 3.1.3 and a morphism of the same kind as ν. Therefore, by the 2-out-of-3 property, μ must be sent to an equivalence. □

Notation 3.1.5. We define E to be the set of all morphisms of the form from Lemma 3.1.4. Note that E is closed under composition. ▮

Corollary 3.1.6. *A functor $G : \Theta \to \mathcal{C}$ defines a coalgebra object if and only if*

1. *G sends degenerate intervals to the terminal object.*
2. *G sends $(\{i,j\} \subset [n] \xrightarrow{f} [m])$ together with its projections to sub-intervals to a product diagram.*
3. *G sends the morphisms in E to equivalences.*
4. *G sends all diagrams of the form (3.1) to limit diagrams.*

3.1.1.2 Forgetting degenerate intervals

Definition 3.1.7. We denote by $\mathrm{Alg}_{\mathrm{Sp}}(\mathcal{C})$ the full sub-∞-category of $\mathrm{Fun}(\Theta, \mathcal{C})$ on those functors satisfying conditions (1)-(4) from the corollary. We denote by $\mathrm{Fun}^*(\Theta, \mathcal{C})$ the full sub-∞-category of functors sending every degenerate interval to a terminal object in $\mathcal{C}$ (i.e., those functors satisfying condition (1) from the corollary).

Let Ω be the full subcategory of Θ on those objects $\{i,j\} \subset [n] \xrightarrow{f} [m]$ such that the interval $\{i,j\}$ is not degenerate (i.e. $i \neq j$). Pulling back along the inclusion $\Omega \to \Theta$ induces a functor $S : \mathrm{Fun}^*(\Theta, \mathcal{C}) \to \mathrm{Fun}(\Omega, \mathcal{C})$. ∎

Definition 3.1.8. Given a 1-category D, call an object $d \in D$ *attracting* if, for all $a \in D$,

$$\mathrm{Hom}_D(a,d) \neq \emptyset, \quad \text{and} \quad \mathrm{Hom}_D(d,a) = \emptyset.$$

∎

Lemma 3.1.9. *Let $d \in D$ be an attracting object, denote by* $\mathrm{Fun}^*(D, \mathcal{C})$ *the full sub-∞-category on those functors sending d to the terminal object, and denote by D° the full subcategory on all objects other than d. Then the functor*

$$\mathrm{Fun}^*(D, \mathcal{C}) \to \mathrm{Fun}(D^\circ, \mathcal{C})$$

is an equivalence.

Proof. Without loss of generality, we assume that $\mathcal{C}$ has a unique terminal object. when f sends d to the terminal object. Denote by $\mathcal{C}' \subset \mathcal{C}$ the largest subcategory not containing morphisms from the terminal object to any other object, and denote by $\mathcal{C}^\circ$ the full subcategory on non-terminal objects. Then we have an equivalence $\mathcal{C}' \simeq (\mathcal{C}^\circ)^{\triangleright}$ since the hom-spaces to the terminal object are all contractible. Any simplex in $\mathrm{Fun}^*(D, \mathcal{C})$ factors through $\mathrm{Fun}^*(D, \mathcal{C}')$, so it will suffice to show that

$$\mathrm{Fun}^*(D, (\mathcal{C}^\circ)^{\triangleright}) \to (D^\circ, \mathcal{C})$$

is a trivial Kan fibration.

Unwinding the definitions, this amounts to solving the extension problem

$$\begin{array}{ccc} (\partial\Delta^n \times D) \coprod_{\partial\Delta^n \times D^\circ} \Delta^n \times D^\circ & \xrightarrow{f} & (\mathcal{C}^\circ)^{\triangleright} \\ \downarrow & \nearrow & \\ \Delta^n \times D & & \end{array}$$

where f sends $\partial\Delta^n \times D$ to the cone point. However, this implies that f factors through $(\Delta^n \times D^\circ)^{\triangleright}$. Pulling back along $\Delta^n \times D \to (\Delta^n \times D^\circ)^{\triangleright}$ then gives the desired extension.

□

Corollary 3.1.10. *The functor* $S : \mathrm{Fun}^*(\Theta, \mathcal{C}) \to \mathrm{Fun}(\Omega, \mathcal{C})$ *is an equivalence of ∞-categories.*

Proof. We again assume that $\mathcal{C}$ has a unique terminal object. Let Θ^{deg} be the full subcategory on only the degenerate intervals. We can write $\mathrm{Fun}^*(\Theta, \mathcal{C})$ as a pullback in Set_Δ

$$\begin{array}{ccc} \mathrm{Fun}^*(\Theta, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\Theta, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Theta^{\mathrm{deg}}, *) & \longrightarrow & \mathrm{Fun}(\Theta^{\mathrm{deg}}, \mathcal{C}) \end{array}$$

There is a natural transformation of diagrams to the pullback diagram

$$\begin{array}{ccc} \mathrm{Fun}^*(\Theta \coprod_{\Theta^{\mathrm{deg}}} *, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\Theta, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^*(*, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\Theta^{\mathrm{deg}}, \mathcal{C}) \end{array}$$

Since this natural transformation is an isomorphism on the bottom three objects, the universal property of the pullback gives us an isomorphism $\mathrm{Fun}^*(\Theta, \mathcal{C}) \cong \mathrm{Fun}^*(\Theta \coprod_{\Theta^{\mathrm{deg}}} *, \mathcal{C})$. $* \in \Theta \coprod_{\Theta^{\mathrm{deg}}} *$ is an attracting object, and so Lemma 3.1.9 yields the desired result. □

3.1.2 The localization

Construction 3.1.11. Let $\phi : ([n], \{i, j\}) \to ([m], \{k, \ell\})$ be a morphism in $\Delta^{\amalg}$, and write $\{i \le j\}$ for the linearly ordered set $\{i, i+1, \ldots, j\}$. Applying $\mathbb{O}$ to ϕ, we obtain a diagram

$$\begin{array}{ccc} \mathbb{O}([m]) & \xrightarrow{\mathbb{O}(\phi)} & \mathbb{O}([n]) \\ \cup\!\mid & & \cup\!\mid \\ \mathbb{O}(\{k \le \ell\}) & & \mathbb{O}(\{i \le j\}) \\ \cup\!\mid & & \cup\!\mid \\ \mathbb{I}(\{k \le \ell\}) & & \mathbb{I}(\{i \le j\}) \end{array}$$

Since, $\phi(i) \leq k \leq \ell \leq \phi(j)$, we see that, for every $a \in \{k \leq \ell\}$, there exists a $b \in \{i \leq j\}$ such that $\phi(b) \leq a \leq a+1 \leq \phi(b+1)$. That is, $\mathbb{O}(\phi)$ descends uniquely to a map

$$\operatorname{res}(\phi) : \mathbb{I}(\{k \leq \ell\}) \to \mathbb{I}(\{i \leq j\}).$$

Note that we here apply the convention that $\mathbb{I}([0]) = \emptyset$. We therefore obtain a functor

$$\operatorname{res} : \Delta^{\amalg} \to \Delta_{+}^{op}$$

which sends all non-degenerate intervals into $\Delta \subset \Delta_{+}$. ▮

Definition 3.1.12. Define a category $\Delta^{\star}$ to have objects finite (non-empty) ordered tuples of elements in Δ. The morphisms of $\Delta^{\star}$ from $([n_0], \dots, [n_k])$ to $([m_0], \dots, [m_\ell])$ consist of

1. A morphism $\phi : [\ell] \to [k]$ in Δ.
2. For each $i \in \{0, 1, \dots k\}$, with $\phi^{-1}(i) = (j_1, \dots, j_r)$, a morphism

$$f_i : [m_{j_1}] \star [m_{j_2}] \star \cdots \star [m_{j_r}] \to [n_i]$$

   in Δ.

Satisfying the conditions that

1. If there is a $p \in \langle \ell \rangle^{\circ}$ with $p > \max_{j \in \phi^{-1}(i)}(j)$, then f_i hits $n_i \in [n_i]$.
2. If there is a $p \in \langle \ell \rangle^{\circ}$ with $p < \min_{j \in \phi^{-1}(i)}(j)$, then f_i hits $0 \in [n_i]$.

▮

Remark 3.1.13. We could equivalently define the morphisms to be

1. A morphism $\phi : [\ell] \to [k]$ in Δ.
2. A morphism

$$f : [m_1] \star [m_2] \star \cdots \star [m_\ell] \to [n_1] \star [n_2] \star \cdots \star [n_k]$$

   in Δ.

Satisfying the condition that, for any $i \in [k]$ with $\phi^{-1}(i) = (j_1, \dots, j_r)$, the restriction

$$f_i : [m_{j_1}] \star [m_{j_2}] \star \cdots \star [m_{j_r}] \to [n_1] \star [n_2] \star \cdots \star [n_k]$$

has image contained in $[n_i]$. ▮

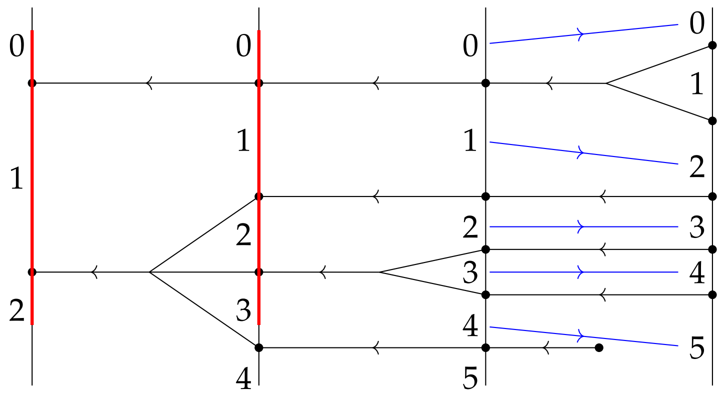


Figure 3.1: A pictorial representation of a morphism μ in Ω, viewed as a triple of composable morphisms $[n] \xrightarrow{g} [n'] \xrightarrow{f'} [m'] \xrightarrow{\overline{g}} [m]$ in Δ. The dual forest is drawn in black, the chosen subintervals of $[n]$ and $[n']$ marked in red, and the induced morphism $\mathcal{L}(\mu)$ is drawn in blue. Note that that source of $\mathcal{L}(\mu)$ is the imbrication of the sets $\{f'(i), f'(i)+1, \ldots f'(i+1)\}$.

Construction 3.1.14. We now define a functor $\mathcal{L} : \Omega \to \Delta^{\star}$. On objects it is given by

$$\{i,j\} \subset [n] \xrightarrow{f} [m] \mapsto (\{f(i) \leq f(i+1)\}, \ldots, \{f(j-1) \leq f(j)\})$$

where $\{f(k) \leq f(k+1)\} := \{f(k), f(k)+1), \ldots, f(k+1)\}$ are considered to be ordered via the order on $[m]$. Note that the indexing set of $\mathcal{L}(\{i,j\}, f)$ is precisely $\mathbb{I}(\{i \leq j\})$

On morphisms, $\mathcal{L}$ is more complicated. A morphism in Ω is given by a commutative diagram of the form

$$\mu = \left\{ \begin{array}{ccccc} [k] := \{i,j\} & \subset & [n] & \xrightarrow{f} & [m] \\ & & {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle \overline{g}} \\ [k'] := \{i',j'\} & \subset & [n'] & \xrightarrow[f']{} & [m'] \end{array} \right\}$$

where $g(i) \leq i' \leq j' \leq g(j)$. We define $\mathcal{L}(\mu)$ to be a pair (ϕ_f, ψ_f). We then write $\phi_f := \mathrm{res}(g) : \mathbb{I}(\{i' \leq j'\}) \to \mathbb{I}(\{i \leq j\})$.

Since the diagram commutes, for each pair $\{p, p+1\} \subset \{i,j\} \subset [n]$, we have that $\overline{g}(f(g(p))) = p$ and $\overline{g}(f(g(p+1))) = p+1$, so that $\overline{g}$ descends to a map of ordered sets

$$\overline{g}_p : \{f'(g(p) \leq f'(g(p)+1)\} \star \cdots \star \{f'(g(p+1)-1) \leq f'(g(p+1))\} \\ \to \{f(p) \leq f(p+1)\}$$

It is easy to verify that conditions (1) and (2) from the definition of $\Delta^\star$ are satisfied by the $\overline{g}_p$. On morphisms, therefore, we define

$$\mathcal{L}(\mu) := \left(\phi(g), \left\{\overline{g}_p\right\}_{i \leq p < j}\right).$$

This is functorial via the functoriality of res and the restriction of $\overline{g}$. ∎

3.1.2.1 Decomposing morphisms

Construction 3.1.15. Given a morphism

$$f : [m] \to [n]$$

in Δ, we can uniquely decompose it as follows: Let $[1] =: [1_i] \subset [m]$ be the interval $\{i-1 \leq i\}$, and let $[n_i] \subset [n]$ be the interval $\{f(i-1) \leq f(i)\}$. Moreover, let $[n_{left}]$ and $[n_{right}]$ be the intervals $\{0 \leq f(0)\}$ and $\{f(m) \leq n\}$ in $[n]$ respectively. Then f is completely determined by the decomposition of $[n]$, since, given such a decomposition, we can reconstruct f by defining $f_i : [1_i] \to [n_i]$ to be the unique map preserving maximal and minimal elements, so that f is the imbrication $f_1 \star \cdots \star f_m$

$$f : [1_1] \star \cdots \star [1_m] \to [n_1] \star \cdots [n_m] \hookrightarrow [n_{left}] \star [n_1] \star \cdots [n_m] \star [n_{right}].$$

We can clarify the indexing of the decomposition of $[n]$ by noting that the pairs $(i-1, i)$ considered above are precisely the inner interstices of $[m]$. Hence, we have decomposed f as a morphism

$$\bigstar_{(i-1,i) \in \mathbb{I}([m])} \{i-1, i\} \to \bigstar_{(i-1,i) \in \mathbb{I}([m])} [n_i].$$

∎

Definition 3.1.16. Given a morphism $\gamma : [n] \to [m]$ in Δ, we can uniquely factor γ as

$$[n] \xrightarrow{\gamma_1} [m_\gamma] \xhookrightarrow{\gamma_2} [m]$$

where $[m] = [k] \oplus [m_\gamma] \oplus [\ell]$. Applying O, we get

$$O([m]) \to O([m_\gamma]) \to O([n]).$$

Where $O([m]) \to O([m_\gamma])$ acts as projection onto a sub-interval. We call $O([m_\gamma])$ the *minimal interval* of γ. ∎

Lemma 3.1.17. *Given an interval $\{i,j\} \subset [n]$ and a morphism $\eta : (\{i,j\} \subset [n]) \to (\{r, r+k\} \subset [m])$ in Δ^{II}, let $[p,\dots,q]$ be the minimal interval of $\gamma := \operatorname{res}(\eta)$. Then $\eta|_{[p+1,\dots,q-1]} = O(\gamma)|_{[p+1,\dots,q-1]}$.*

Proof. If $[p+1,\dots,q-1]$ is empty, the statement is vacuously true. Otherwise, note that for $s \in [p+1,\dots,q-1]$, the requirement that $\operatorname{res}(\eta) = \gamma$ means that $\gamma(\eta(s)) \leq s < \gamma(\eta(s)+1)$. Such an $\eta(s)$ always exists, and this inequality uniquely determines $\eta(s)$. (Note that, for p or q in $[p,\dots,q]$, we only have one-half of the inequality so that uniqueness need not hold.) □

3.1.2.2 Constructing morphisms

In what follows, we will be interested in the weak fibers of the functor $\mathcal{L} : \Omega \to \Delta^\star$. We first note that, given an object $M = ([m_1],\dots,[m_k]) \in \Delta^\star$, the fiber Ω_M is non-empty. We can explicitly build an object

$$\{0,k\} \subset [k] \xrightarrow{f_M} [m_1] \star [m_2] \star \cdots \star [m_k] =: [m]$$

in the fiber over M, given by

$$f_M(i) = \begin{cases} 0 \in [m_{i+1}] & i < k \\ m_k \in [m_k] & i = k. \end{cases}$$

Definition 3.1.18. For $M = ([m_1],\dots,[m_k]) \in \Delta^\star$, we define a subcategory $\Omega^E_M \subset \Omega_M$ as follows. The objects of Ω^E_M are the same as those of Ω_M, but the morphisms are only those in E. ∎

Lemma 3.1.19. *The object $\{0,k\} \subset [k] \xrightarrow{f_M} [m]$ is an initial object in Ω^E_M.*

Proof. Given another object

$$\{i,j\} \subset [n] \xrightarrow{f} [m']$$

in Ω^E_M, and a morphism

$$\begin{array}{ccccc} \{0,k\} & \subset & [k] & \xrightarrow{f_M} & [m] \\ & & {\scriptstyle\phi}\big\downarrow & & \big\uparrow{\scriptstyle h} \\ \{i,j\} & \subset & [n] & \xrightarrow[f]{} & [m'] \end{array}$$

ϕ must be the inclusion of $[i, \ldots, j]$, since any such morphism in E will induce an isomorphism $[k] \to [i, \ldots, j]$. Moreover, h is clearly uniquely determined by the condition that it maps $[f(i), f(i+1), \ldots, f(j)]$ isomorphically to $[m]$. $\square$

Notation 3.1.20. Suppose given an object

$$Z := \left\{ \{i, j\} \subset [n] \xrightarrow{f} [\ell] \right\}$$

in Ω whose image under $\mathcal{L}$ is $([\ell_{i+1}], \ldots, [\ell_j])$, and a morphism

$$g : ([\ell_{i+1}], \ldots, [\ell_j]) \to ([m_0], \ldots, [m_{k-1}])$$

in $\Delta^\star$. Write $\gamma : [k-1] \to [i+1, \ldots, j] \in \Delta$ and $\overline{g} : [m_0] \star \cdots \star [m_{k-1}] \to [\ell_{i+1}] \star \cdots \star [\ell_j]$ for the morphisms defining g. Denote by

$$[n_c] := [p, \ldots, q] \subset \{i, j\} \subset [n]$$

the minimal interval of γ and by $\psi : [i, \ldots, j] \to [n_c]$ the projection as above, and let $\{0, k\} \subset [k] \xrightarrow{f_M} [m] := [m_0] \star \cdots \star [m_{k-1}]$ be the minimal object in Ω representing the target.

Note that, by definition, the morphism $\overline{g}$ has image contained in $[\ell_{p+1}] \star \cdots \star [\ell_q] =: [\ell_c]$. We introduce some notation for specific decompositions:

$$[n] = [n_\ell] \star [n_c] \star [n_r]$$

$$[\ell] = [\ell_\ell] \star [\ell_c] \star [\ell_r]$$

for use in the upcoming argumentation. ∎

Lemma 3.1.21. *There is a morphism in Ω*

$$\begin{array}{ccccc} \{p, q\} & \subset & [p, \ldots, q] & \xrightarrow{f|_{\{p,q\}}} & [\ell_c] \\ & & \downarrow{\scriptstyle \nu} & & \uparrow{\scriptstyle \overline{g}'} \\ \{0, k\} & \subset & [1] \star [k] \star [1] & \xrightarrow[f'_M]{} & [\ell^1] \star [m] \star [\ell^2] \end{array}$$

which extends to a morphism $\mu_{Z,M}$ in Ω covering g

$$\mu_{Z,M} :=$$

$$\left\{ \begin{array}{ccccc} Z = \{i,j\} & \subset & [n_\ell] \star [n_c] \star [n_r] & \longrightarrow & [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ & & \downarrow & & \uparrow \\ Z_M := \{0,k\} & \subset & [n_\ell] \star [1] \star [k] \star [1] \star [n_r] & \to & [\ell_\ell] \star [\ell^1] \star [m] \star [\ell^2] \star [\ell_r] \end{array} \right\}$$

Moreover, given any other morphism $Z \to X$ covering g, there is a unique morphism $Z_M \to X$ in E such that the diagram

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ Z_M & \longrightarrow & X \end{array}$$

commutes.

Proof. In the first diagram, we define the map ν on $[p+1,\dots,q-1]$ to be the unique map from Lemma 3.1.17 dual to γ under res, and send the endpoints to the endpoints of $[1] \star [k] \star [1]$. Then we write

$$[\ell_c] = [\ell^1] \star [\ell_c^m] \star [\ell^2],$$

where $[\ell_c^m]$ is the minimal interval containing the image of $\overline{g} : [m] \to [\ell]$. Note that $\overline{g} : [m] \to [\ell_c^m]$ hits both endpoints. We then define

$$\overline{g}' := \mathrm{id}_{[\ell^1]} \star \overline{g} \star \mathrm{id}_{[\ell^2]} : [\ell^1] \star [m] \star [\ell^2] \to [\ell_c]$$

(which then, by definition, hits both endpoints), and

$$f'_M : [1] \star [k] \star [1] \to [\ell^1] \star [m] \star [\ell^2]$$

to be f_M on $[k]$, and to send endpoints to endpoints. Then we can decompose the diagram as

$$\begin{array}{ccccc} \{p,q\} & \subset & [1_{p+1}] \star \cdots \star [1_q] & \xrightarrow{f|_{\{p,q\}}} & [\ell_{p+1}] \star \cdots \star [\ell_q] \\ & & \nu \downarrow & & \uparrow \overline{g}' \\ \{0,k\} & \subset & [1] \star [k_{p+1}] \star \cdots \star [k_q] \star [1] & \xrightarrow[f'_M]{} & [\ell^1] \star [m_{p+1}] \star \cdots \star [m_q] \star [\ell^2] \end{array}$$

by decomposing the morphisms ν, $f|_{\{p,q\}}$, and $f'_M \circ \nu$. The condition that the diagram commute is then equivalent to the conditions that, (1) for each $r \in \{p+2,\dots,q-1\}$, the endpoints of $[m_r]$ are sent to the endpoints of $[\ell_r]$ by $\overline{g}$, and (2) that $\overline{g}$ sends the endpoints of $[\ell^1] \star [m_{p+1}]$ and $[m_q] \star [\ell^2]$ to the endpoints of $[\ell_{p+1}]$ and $[\ell_q]$, respectively. Since

$$[m_r] = \bigstar_{a \in \mathbb{I}([k_r])} [f'_M(a-1), f'_M(a-1)+1, \dots, f'_M(a)]$$

we see that case (1) is true by the definition of $\Delta^\star$. Case (2) is true by construction.

This diagram is defined so that the maps ν, f'_M, $\overline{g}'$, and $f|_{\{p,q\}}$ preserve endpoints. Therefore, we can take the appropriate star products with the morphisms $\mathrm{id}_{[n_\ell]}$, $\mathrm{id}_{[n_r]}$, $\mathrm{id}_{[\ell_\ell]}$, $\mathrm{id}_{[\ell_r]}$, $f|_{[n_\ell]} : [n_\ell] \to [\ell_\ell]$, and $f|_{[n_r]} : [n_r] \to [\ell_r]$ to get a commutative diagram

$$\begin{array}{ccccc} Z = \{i,j\} & \subset & [n_\ell] \star [n_c] \star [n_r] & \longrightarrow & [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ & & \downarrow & & \uparrow \\ Z_M := \{0,k\} & \subset & [n_\ell] \star [1] \star [k] \star [1] \star [n_r] & \longrightarrow & [\ell_\ell] \star [\ell^1] \star [m] \star [\ell^2] \star [\ell_r] \end{array}$$

By construction, the morphism $\mathrm{res}(\nu) : [k-1] \to \langle i+1, \dots, j\rangle$ is γ, and the morphism $\overline{g}'$ restricts to $\overline{g}$ on $[m]$, so this diagram determines a morphism in Ω covering g. Call this morphism $\mu_{Z,M} : Z \to Z_M$.

Now suppose we are given a morphism

$$\begin{array}{ccccc} Z = \{i,j\} & \subset & [n] & \xrightarrow{f} & [\ell] \\ & & \rho \downarrow & & \uparrow w \\ X := \{0,k\} & \subset & [a] & \xrightarrow[h]{} & [b] \end{array}$$

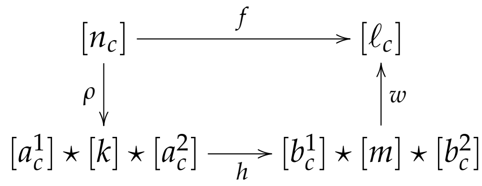
covering g. We can decompose this into

$$\begin{array}{ccccc} Z = \{i,j\} & \subset & [n_\ell] \star [n_c] \star [n_r] & \xrightarrow{f} & [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ & & \rho \downarrow & & \uparrow w \\ Z_M := \{0,k\} & \subset & [a_\ell] \star [a_c] \star [a_r] & \xrightarrow[h]{} & [b_\ell] \star [b_c] \star [b_r] \end{array}$$

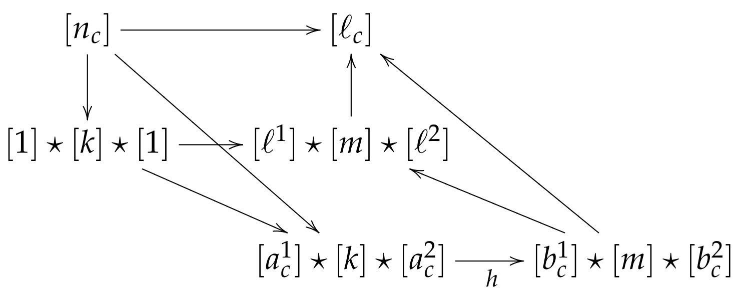
Where $\{0,k\} \subset [a_c]$. By Lemma 3.1.17, we know that ρ is uniquely determined on all of $[n_c]$ except the endpoints. This allows us to further decompose the diagram

$$\begin{array}{ccc} [n_c] & \xrightarrow{f} & [\ell_c] \\ \rho \downarrow & & \uparrow w \\ [a_c] & \xrightarrow[h]{} & [b_c] \end{array}$$

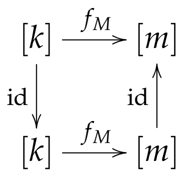
as a diagram where the bottom map is a star product with f_M.



If there is morphism $Z_M \to X$ in E commuting with the morphisms $Z \to X$ and $\mu_{Z,M}$, it must, in particular, restrict to a commutative diagram



Moreover, since the morphism is in E, the bottom square must restrict to the commutative diagram



As a result, the component morphism $[1] \star [k] \star [1] \to [a_c^1] \star [k] \star [a_c^2]$ is uniquely determined by the commutativity of the left-hand triangle. Additionally, since $w : [b] \to [\ell]$ must restrict to $\overline{g}$ on $[m]$, we can decompose w as a star product

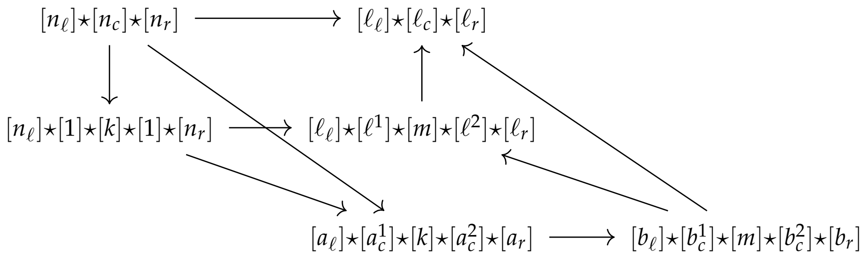
$$w = w^1 \star \overline{g} \star w^2 : [b_c^1] \star [m] \star [b_c^2] \to [\ell^1] \star [\ell_c^m] \star [\ell^2]$$

Therefore, the component morphism

$$[b_c^1] \star [m] \star [b_c^2] \to [\ell^1] \star [m] \star [\ell^2]$$

is uniquely determined, and must be $w^1 \star \mathrm{id}_{[m]} \star w^2$.

We now extend back to the full diagram



and note that, since the vertical components of the back square restrict to identities on $[n_\ell]$, $[n_r]$, $[\ell_\ell]$, and $[\ell_r]$, the bottom square is uniquely determined by the morphisms $[n_\ell] \to [a_\ell]$, $[n_r] \to [a_r]$, $b_\ell] \to [\ell_\ell]$, and $[b_r] \to [\ell_r]$. So there is a unique morphism $Z_M \to X$ in Ω with the desired properties. □

Proposition 3.1.22. *The functor $\mathcal{L} : \Omega \to \Delta^\star$ is an ∞-categorical localization at the morphisms in E.*

Proof. Consider the inclusion $\iota_M : \Omega_M^E \subset \Omega_M \hookrightarrow \Omega_{/M}$, and

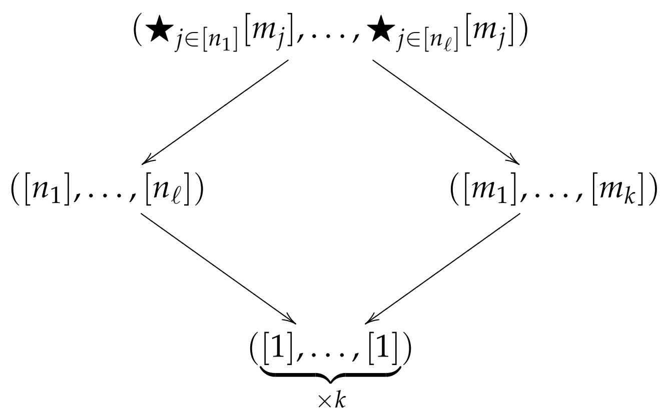
$$Z := \begin{cases} \{i, j\} \subset [n] \xrightarrow{f_Z} [\ell] & + \\ g_Z : ([\ell_i], \ldots, [\ell_j]) \to ([m_1], \ldots, [m_k]) & \text{in } \Delta^\star \end{cases}$$

in $\Omega_{/M}$. Denote the overcategory $\left(\Omega_M^E\right)_{Z/} := \Omega_M^E \times_{\Omega_{/M}} (\Omega_{/M})_{Z/}$. Lemma 3.1.21 tells us that $\left(\Omega_M^E\right)_{Z/}$ is non-empty, and that the object Z_M constructed in the lemma is an initial object. Moreover, by Lemma 3.1.19, Ω_M^E has an initial object. Therefore, by [41, Lemma 3.1.1], $\mathcal{L}$ is a localization at the morphisms of E. □

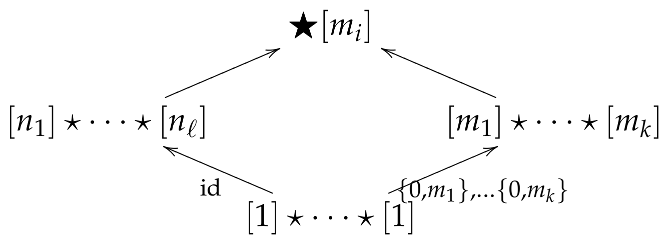
3.1.2.3 Algebra conditions

Notation 3.1.23. Denote by $\mathrm{Fun}^{\mathrm{alg}}(\Delta^\star, \mathcal{C})$ the full sub-∞-category of functors f which

(A) send the diagrams

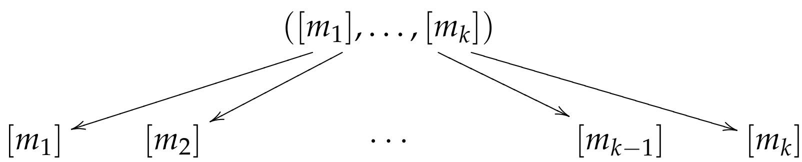


opposite the diagrams



to pullback diagrams.

(B) send the diagrams



to product diagrams.

∎

Proposition 3.1.24. *There is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathrm{Sp}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{alg}}(\Delta^{\star},\mathcal{C}).$$

Proof. It is clear that condition (B) corresponds to condition (2) from Corollary 3.1.6. For condition (A), first consider a 3-simplex $[n_0] \xrightarrow{\phi_1} [n_1] \xrightarrow{\phi_2}$

$[n_2] \xrightarrow{\phi_3} [n_3]$ in Δ. The corresponding limit diagram (3.1) can be written as

$$\begin{array}{ccccccc} & & & G(\phi_2 \circ \phi_1, \{i,j\}) & & & \\ & \swarrow & & \downarrow & & \searrow & \\ G(\phi_1, \{i,j\}) & & & G(\phi_2, \{\phi_1(i), \phi_1(j)\}) & & & G(\phi_3, \{\psi_2(i), \psi_2(j)\}) \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & G([n_1], \{\psi_1(i), \psi_1(j)\}) & & & & G([n_2], \{\psi_2(i), \psi_2(j)\}) & \end{array}$$

However, by (the dual of) [32, Proposition 4.4.2.2], this diagram is a limit if and only if the induced diagram

$$\begin{array}{ccc} & G(\phi_2 \circ \phi_1, \{i,j\}) & \\ \swarrow & & \searrow \\ G([n_1], \{\psi_1(i), \psi_1(j)\}) & & G([n_2], \{\psi_2(i), \psi_2(j)\}) \\ \searrow & & \swarrow \\ & G(\phi_3, \{\phi_1(i), \phi_1(j)\}) & \end{array}$$

is pullback. However, combining these two diagrams, we get

$$\begin{array}{ccccccc} & & & G(\phi_2 \circ \phi_1, \{i,j\}) & & & \\ & & \swarrow & & \searrow & & \\ & G([n_1], \{\psi_1(i), \psi_1(j)\}) & & & & G([n_2], \{\psi_2(i), \psi_2(j)\}) & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ G(\phi_1, \{i,j\}) & & & G(\phi_3, \{\phi_1(i), \phi_1(j)\}) & & & G(\phi_k, \{\psi_2(i), \psi_2(j)\}) \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & G([n_1], \{\psi_1(i), \psi_1(j)\}) & & & & G([n_2], \{\psi_2(i), \psi_2(j)\}) & \end{array}$$

By the pasting property for pullback diagrams, we thus see that it is sufficient to require that each of the diagrams corresponding to the sub-2-simplices of our simplex is pullback. Iterating this argument, we find that property (4) of corollary 3.1.6 is satisfied if and only if it is satisfied on 2-simplices. Since condition (A) is the image of this 2-simplex condition under $\mathcal{L}$, this proves the proposition. □

Lemma 3.1.25. *A functor $f \in \mathrm{Fun}(\Delta^\star, \mathcal{C})$ satisfies condition (A) if and only if it satisfies condition (A) for collections where all but one of the $[m_i]$ are equal to $[1]$.*

Proof. This follows from applying the pasting law to diagrams of the form

$$\begin{array}{ccccccc}
([n_1],\ldots,[n_\ell]) & \longleftarrow & ([m_1]\star[1]\star\cdots\star[1],[1],\ldots,[1]) & \longleftarrow & ([m_1]\star[m_2]\star\cdots\star[1],[1],\ldots,[1]) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
([1],\ldots,[1]) & \longleftarrow & ([m_1],[1],\ldots,[1]) & \longleftarrow & ([m_1],[m_2],\ldots,[1]) & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & ([1],\ldots,[1]) & \longleftarrow & ([1],\ldots,[1],[m_2],[1],\ldots,[1]) & \longleftarrow & \cdots
\end{array}$$

If condition (A) is satisfied for squares where all but one of the $[m_i]$ are equal to $[1]$, then the bottom right square and the right-hand rectangle are all pullback. Therefore, the top right square is pullback. Since our restricted version of condition (A) also implies that the top left square is pullback, the top rectangle is pullback. Iterating this argument then yields the lemma. □

3.1.3 Extension and restriction

Considering the full subcategory of $\Delta^\star$ on the objects $([n])$ for $n \geq 0$ we get

$$\iota : \Delta^{op} \to \Delta^\star.$$

Taking restriction and right Kan extension gives us an adjunction of infinity categories

$$\iota_* : \mathrm{Fun}(\Delta^\star, \mathcal{C}) \leftrightarrow \mathrm{Fun}(\Delta^{op}, \mathcal{C}) : \iota_!$$

Notation 3.1.26. Denote by $\mathrm{Fun}^\times(\Delta^\star, \mathcal{C})$ the full sub-∞-category that sends each diagram

$$\begin{array}{ccccc}
 & & ([m_1],\ldots,[m_k]) & & \\
\swarrow & \swarrow & & \searrow & \searrow \\
[m_1] & [m_2] & \cdots & [m_{k-1}] & [m_k]
\end{array}$$

to a limit diagram. ▮

Proposition 3.1.27. *The adjunction* $\iota_* : \mathrm{Fun}(\Delta^\star, \mathcal{C}) \leftrightarrow \mathrm{Fun}(\Delta^{op}, \mathcal{C}) : \iota_!$ *descends to an equivalence of ∞-categories*

$$\mathrm{Fun}^\times(\Delta^\star, \mathcal{C}) \simeq \mathrm{Fun}(\Delta^{op}, \mathcal{C}).$$

Proof. We compute the overcategory $(\Delta^{op})_{([m_1],\ldots,[m_k])/}$. An object in the overcategory will consist of a choice of $i \in \{1, 2, \ldots, k\}$ and a morphism $[n] \to [m_i]$. A morphism $(i, [n] \to [m_i]) \to (j, [\ell] \to [m_j])$ only exists if $i = j$, and in this case is given by a commutative diagram

$$\begin{array}{ccc} [\ell] & \longrightarrow & [n] \\ & \searrow \qquad \swarrow & \\ & [m_i] & \end{array}$$

consequently, we find that the induced diagram

$$\begin{array}{ccccc} & & (\Delta^{op})_{([m_1],\ldots,[m_k])/} & & \\ & \nearrow \nearrow & & \nwarrow \nwarrow & \\ (\Delta^{op})_{([m_1])/} \quad (\Delta^{op})_{([m_2])/} & & \cdots & & (\Delta^{op})_{([m_{k-1}])/} \quad (\Delta^{op})_{([m_k])/} \end{array}$$

displays $(\Delta^{op})_{([m_1],\ldots,[m_k])/}$ as a coproduct, and, hence, for any $f \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$, the diagram

$$\begin{array}{ccccc} & & \iota_! f\,(([m_1], \ldots, [m_k])) & & \\ & \swarrow \swarrow & & \searrow \searrow & \\ \iota_! f\,([m_1]) \quad \iota_! f\,([m_2]) & & \cdots & & \iota_! f\,([m_{k-1}]) \quad \iota_! f\,([m_k]) \end{array} \tag{3.2}$$

displays $\iota_! f\,(([m_1], \ldots, [m_k]))$ as a product. Consequently, the adjunction descends to an adjunction $\iota_* : \mathrm{Fun}^{\times}(\Delta^{\star}, \mathcal{C}) \leftrightarrow \mathrm{Fun}(\Delta^{op}, \mathcal{C}) : \iota_!$.

Since this is a right Kan extension from a full subcategory, the counit is an equivalence. Moreover, the components of the unit are equivalences on the objects of Δ^{op}. However, for every object $([m_1], \ldots, [m_k])$, the unit induces a natural transformation of limit diagrams of the form in diagram (3.2). Therefore, we see that the components of the unit are equivalences for all objects, and thus, the unit is also an equivalence. □

Proposition 3.1.28. *Denote by* $2\text{-}\mathrm{Seg}_{\Delta}(\mathcal{C})$ *the full subcategory of* $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ *on 2-Segal objects. Then the equivalence of the previous proposition descends to an equivalence of* ∞*-categories*

$$\mathrm{Fun}^{\mathrm{alg}}(\Delta^{\star}, \mathcal{C}) \simeq 2\text{-}\mathrm{Seg}_{\Delta}(\mathcal{C}).$$

Proof. Let $G \in \mathrm{Fun}^{\mathrm{alg}}(\Delta^\star, \mathcal{C})$, and consider the diagram

$$\begin{array}{ccc} [n] & \longleftarrow & [n+m-1] \\ \downarrow & & \downarrow \\ (\{0,1\},\{1,2\},\ldots,\{n-1,n\}) & \longleftarrow & ([1],\ldots,\overbrace{[m]}^{j^{th}},\ldots,[1]) \end{array}$$

in $\Delta^\star$. We can expand this diagram to

$$\begin{array}{ccc} [n] & \longleftarrow & [n+m-1] \\ \downarrow & & \downarrow \\ (\{0,1\},\{1,2\},\ldots,\{n-1,n\}) & \longleftarrow & ([1],\ldots,\overbrace{[m]}^{j^{th}},\ldots,[1]) \\ \downarrow & & \downarrow \\ \{j-1,j\} & \longleftarrow & [m] \end{array}$$

Since the two vertical morphisms in the lower square are sent to projections onto factors of a product, the lower square is sent to a pullback diagram under G. We therefore see that the exterior square is sent to a pullback if and only if the upper square is sent to a pullback. However, the exterior square is opposite to the diagram

$$\begin{array}{ccc} [n] & \longrightarrow & [n+m-1] \\ {\scriptstyle \{j-1,j\}}\uparrow & & \uparrow \\ [1] & \underset{\{0,m\}}{\longrightarrow} & [m] \end{array}$$

in Δ, which is precisely the diagram for the 2-Segal conditions when $[m] \neq [0]$, and is the diagram for the unitality condition when $[m] = [0]$. Therefore, we see that $G \in \mathrm{Fun}^\times(\Delta^\star, \mathcal{C})$ is in $\mathrm{Fun}^{\mathrm{alg}}(\Delta^\star, \mathcal{C})$ if and only if the underlying simplicial object is 2-Segal. □

We can summarize our results in the following theorem.

Theorem 3.1.29. *There is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathrm{Sp}}(\mathcal{C}) \simeq 2\text{-}\mathrm{Seg}_\Delta(\mathcal{C}).$$

3.2 Calabi-Yau algebras in Spans

We now extend the results of the previous section to Calabi-Yau algebras. Throughout this section we set $\Theta : \mathrm{Tw}(\mathcal{A}\mathrm{ss}^{\mathrm{op}}_{\mathrm{CY}}) \times_{\mathbb{F}} \mathbb{F}^{\amalg}$. We will represent morphisms in Θ diagrammatically as

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{\;f\;} & T \\ & & {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \overline{g}} \\ P & \subseteq & S' & \xleftarrow[\;f'\;]{} & T' \end{array}$$

where f, f', g, and $\overline{g}$ are morphisms in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ (*not* $\mathcal{A}\mathrm{ss}^{\mathrm{op}}_{\mathrm{CY}}$).

Notation 3.2.1. In general, for a morphism $\diamond \xleftarrow{f} T$ in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$, we will denote the two possible subsets of the image of $\diamond$ in $\mathbb{F}\mathrm{in}_*$ by $\emptyset$ and $\{1\}$. ∎

3.2.1 Conditions on functors

We fix a functor

$$G : \Theta \to \mathcal{C}$$

corresponding to a functor $\mathrm{Tw}(\mathcal{A}\mathrm{ss}^{\mathrm{op}}_{\mathrm{CY}}) \to \mathcal{C}^{\times}$ over $\mathbb{F}$.

Proposition 3.2.2. *The functor G defines a functor $\overline{G} : \mathcal{A}\mathrm{ss}^{\mathrm{op}}_{\mathrm{CY}} \to \mathrm{Span}_{\mathbb{F}}(\mathcal{C}^{\times})$ if and only if for every simplex $S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} S_2 \to \cdots \to S_n$ in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$, and every subset $P \subset S_n^{\circ}$ the corresponding diagram*

$$\begin{array}{ccccccc} & & & G(\psi_{n-1},P) & & & \\ & & \swarrow & & \searrow & & \\ & G(\phi_n,P) & & \cdots & & G(\phi_1,\psi_{n-2}^{-1}(P)) & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \\ G(S_n,P) & & G(S_{n-1},\phi_n^{-1}(P)) & \cdots & G(S_1,\psi_{n-2}^{-1}(P)) & & G(S_0,\psi_{n-1}^{-1}(P)) \end{array} \qquad (3.3)$$

is a limit diagram in $\mathcal{C}$, where $\psi_k := \phi_n \circ \phi_{n-1} \cdots \circ \phi_{n-k}$.

Proof. This is, *mutatis mutandis*, the same as the proof of Proposition 3.1.1. Note that if $S_k = \diamond$, then $S_j = \diamond$ for all $j \geq k$. □

3.2.1.1 Equivalences

Construction 3.2.3. Suppose that $G : \Theta \to \mathcal{C}$ represents a co-Calabi-Yau algebra. This means that, for every inert morphism $\phi : S \to T$ in $\mathcal{A}\mathrm{ss} \subset \mathcal{A}\mathrm{ss}_{\mathrm{CY}}$, and every $P \subset T^\circ$,

- For the source map $\phi \to S$ in $\mathrm{Tw}(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}})$, the induced morphism
$$G(\phi, P) \to G(S, \phi^{-1}(P))$$
is an equivalence
- For the target map $\phi \to T$ in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}}$, the induced morphism
$$G(\phi, P) \to G(T, P)$$
is an equivalence.

∎

Lemma 3.2.4. *Suppose G represents a co-Calabi-Yau algebra object. Let $\phi : S \to T$ be a morphism in $\mathcal{A}\mathrm{ss}$ viewed as an object in $\mathrm{Tw}(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}})$ and let $P \subset T$.*

1. *Let $\psi_2 : T \to P$ be the inert morphism is $\mathcal{A}\mathrm{ss}$ that acts as the identity on P and sends all other elements to the basepoint. Then the induced morphism*
$$G(\psi_2 \circ \phi, P) \to G(\phi, P)$$
is an equivalence.
2. *Let $\psi_1 : \phi^{-1}(P) \to S$ be morphism in $\mathcal{A}\mathrm{ss}$ defined via the inclusion. Then the induced morphism*
$$G(\psi_2 \circ \phi \circ \psi_1, P) \to G(\phi, P)$$
is an equivalence.

Proof. By Proposition 3.2.2, the diagrams

$$\begin{array}{ccccc} & & G(\psi_2 \circ \phi, P) & & \\ & \swarrow & & \searrow & \\ G(\psi_2, P) & & & & G(\phi, P) \\ & \searrow & & \swarrow & \\ & & G(T, P) & & \end{array}$$

is a pullback diagram. Since ψ_2 is inert in $\mathcal{A}\mathrm{ss}$, the morphism

$$G(\psi_2, P) \to G(T, P)$$

is an equivalence. Therefore, the morphism

$$G(\psi_2 \circ \phi, P) \to G(\phi, P)$$

is an equivalence.

We now note that the morphism $(\psi_2 \circ \phi \circ \chi, P) \to (\phi, P)$ can be factored as

$$(\psi_2 \circ \phi \circ \chi, P) \to (\psi_2 \circ \phi, P) \to (\phi, P).$$

Since the second of these morphisms is an equivalence, we need only show that the first is as well. To do this, we write down a composite

$$\begin{array}{ccccc}
P & \subseteq & P & \xleftarrow{\psi_2 \circ \phi \circ \psi_1} & \phi^{-1}(P) \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \psi_1} \\
P & \subseteq & P & \xleftarrow[\psi_2 \circ \phi]{} & S \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\
P & \subseteq & P & \xleftarrow{\psi_2 \circ \phi \circ \psi_1} & \phi^{-1}(P)
\end{array}$$

in $\mathrm{Tw}(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}}) \times_{\Gamma} \Gamma^{\amalg}$, where π is the inert morphism projecting S onto the subset $\phi^{-1}(S)$. Since the composite is the identity, it will suffice to show that the bottom square is sent to an equivalence under G.

Denote by ν the morphism defined by the bottom square. By Proposition 3.2.2, we can write down a pullback square

$$\begin{array}{ccccc}
 & & G(\psi_2 \circ \phi, P) & & \\
 & \overset{\nu}{\swarrow} & & \searrow & \\
G(\psi_2 \circ \phi \circ \psi_1, P) & & & & G(\pi, \phi^{-1}(P)) \\
 & \searrow & & \swarrow & \\
 & & G(\phi^{-1}(P), \phi^{-1}(P)) & &
\end{array}$$

The bottom right morphism is the source map of an inert morphism, and thus is an equivalence. Therefore, ν is also an equivalence. □

Proposition 3.2.5. *Suppose that G sends the morphisms from Lemma 3.2.4 to equivalences. Let μ be a morphism*

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{\;f\;} & T \\ & & {\scriptstyle g}\big\uparrow & & \big\downarrow{\scriptstyle \overline{g}} \\ P & \subseteq & U & \xleftarrow[\;h\;]{} & V \end{array}$$

such that $g|_P : P \to Q$ is an isomorphism, $P = g^{-1}(Q)$, and $\overline{g}|_{f^{-1}(Q)} : f^{-1}(Q) \to h^{-1}(P)$ is an isomorphism. Then $G(\mu)$ is an equivalence.

Proof. Consider the diagram

$$\begin{array}{ccccc} Q & \subseteq & Q & \longleftarrow & f^{-1}(Q) \\ & & {\scriptstyle \mathrm{proj}}\big\uparrow & & \big\downarrow{\scriptstyle \subset} \\ Q & \subseteq & S & \xleftarrow{\;f\;} & T \\ & & {\scriptstyle g}\big\uparrow & & \big\downarrow{\scriptstyle \overline{g}} \\ P & \subseteq & U & \xleftarrow[\;h\;]{} & V \end{array}$$

The top square is a morphism from Lemma 3.2.4, and hence is sent to an equivalence. Moreover, the composite morphism can be decomposed as

$$\begin{array}{ccccc} Q & \subseteq & Q & \longleftarrow & f^{-1}(P) \\ & & {\scriptstyle \cong}\big\uparrow & & \big\downarrow{\scriptstyle \cong} \\ P & \subseteq & P & \longleftarrow & h^{-1}(P) \\ & & {\scriptstyle \mathrm{proj}}\big\uparrow & & \big\downarrow{\scriptstyle \subset} \\ P & \subseteq & U & \xleftarrow[\;h\;]{} & V \end{array}$$

Since the lower square is sent to an equivalence by Lemma 3.2.4 and the upper square is an isomorphism, this composite is sent to an equivalence. Therefore, by the 2-out-of-3 property, $G(\mu)$ is an equivalence. □

Proposition 3.2.6. *Suppose that G represents a co-Calabi-Yau algebra. Let μ be a morphism*

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{\;f\;} & T \\ & & {\scriptstyle \mathrm{id}}\big\uparrow & & \big\downarrow{\scriptstyle \overline{g}} \\ \{1\} & \subseteq & \diamond & \xleftarrow[\;h\;]{} & S \end{array}$$

such that $\overline{g}|_{f^{-1}(\diamond)} : f^{-1}(\diamond) \to g^{-1}(\diamond)$ *is an isomorphism. Then* $G(\mu)$ *is an equivalence.*

Proof. This is, *mutatis mutandis*, the same as the proof of Lemma 3.2.4 part (2). □

Notation 3.2.7. We define the set E of morphisms in Θ to be the set of all morphisms from Proposition 3.2.5 and Proposition 3.2.6. ▮

3.2.1.2 Non-degeneracy

We now consider a morphism γ in $\mathrm{Span}_\Gamma(\mathcal{C}^\times)$ represented by

$$X \times X \overset{(\gamma_1,\gamma_2)}{\longleftarrow} Y \to *,$$

and explore when it is non-degenerate in the sense of Definition 1.4.20

Lemma 3.2.8. *The morphism* γ *is non-degenerate if and only if* γ_1 *and* γ_2 *are equivalences.*

Proof. If γ_1 and γ_2 are equivalences, we can define a morphism

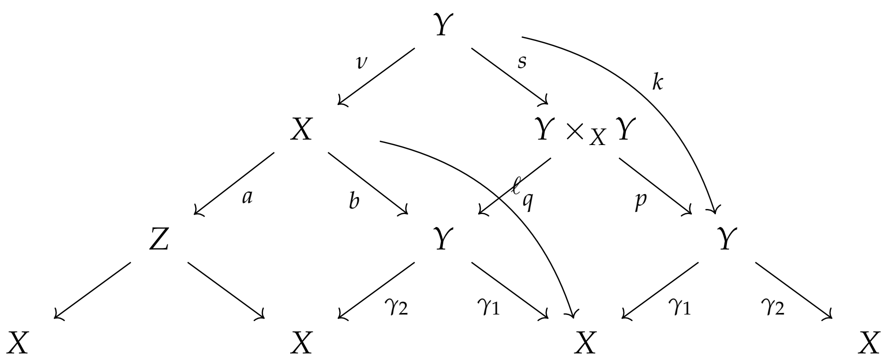
$$* \leftarrow Y \overset{(\gamma_1,\gamma_2)}{\longrightarrow} X \times X$$

which displays the non-degeneracy of γ.

Now suppose that γ is non-degenerate, and let $\eta := (\eta_1, \eta_2)$ be a morphism

$$* \leftarrow Z \overset{(\eta_1,\eta_2)}{\longrightarrow} X \times X$$

displaying the non-degeneracy of γ. Then we have the diagram



where every square is pullback. The left hand pullback must define an equivalence in $\mathrm{Span}(\mathcal{C})$, and therefore, the morphism ℓ is an equivalence.

We thus see that γ_1 must have a left inverse up to homotopy. Similarly, we see that the morphism k must be an equivalence. By the symmetry of the left-hand pullback square, $q \circ s$ must be an equivalence, and thus , $b \circ \nu$ is an equivalence. However, ν is a pullback of γ_1 along an equivalence, and therefore is homotopic to γ_1. Therefore, we see that γ_1 has a right inverse up to homotopy, and so, γ_1 is an equivalence. A similar argument shows that γ_2 is an equivalence. □

Construction 3.2.9. Let $G : \Theta \to \mathcal{C}$ be a functor representing a trace co-algebra in $\mathrm{Span}_{\Gamma}(\mathcal{C}^{\times})$. In particular, we have the object

$$Y := G(\{1\} \subset \diamond \leftarrow \langle 2 \rangle)$$

and the object

$$X_n := G(\{2\} \subset \langle 2 \rangle \xleftarrow{f} \langle n+1 \rangle)$$

where $f(1) = 1$ and $f(i) = 2$ for all $i \neq 1$. Finally, we have the object

$$Z_n := G(\{1\} \subset \diamond \leftarrow \langle n \rangle$$

By 3.2.2, we get a pullback diagram

$$\begin{array}{ccccc} & & Z_n & & \\ & \swarrow & & \searrow & \\ X_n & & & & Y \\ & \searrow & & \swarrow & \\ & & (\langle 2 \rangle, \{2\}) & & \end{array}$$

By 3.2.8, we know that the trace is non-degenerate if and only if the bottom right morphism is an equivalence. From the structure of the pullback diagram, we see that this is equivalent to requiring that the morphism $Z_n \to Z_n$ is an equivalence for all n. ∎

Corollary 3.2.10. *A functor $G : \Theta \to \mathcal{C}$ defines a Calabi-Yau co-algebra in* $\mathrm{Span}_{\Gamma}(\mathcal{C}^{\times})$ *if and only if it satisfies the following conditions:*

1. *G sends empty subsets to the terminal object.*

2. *G sends $P \subset S \leftarrow T$ together with its projections to $\{i\} \subset S \leftarrow T$ for $i \in P$ to a product diagram.*

3. *G sends the morphisms in E to equivalences.*

4. *G sends all diagrams of the form (3.3) to limit diagrams.*

5. *G sends the morphisms $Z_n \to X_n$ from 3.2.9 to equivalences.*

Definition 3.2.11. We define $\mathrm{Alg}^{\mathrm{CY}}_{\mathrm{Sp}}(\mathcal{C})$ to be the full ∞-subcategory of $\mathrm{Fun}(\Theta, \mathcal{C})$ satisfying the conditions of Corollary 3.2.10. ∎

3.2.2 The localization

Definition 3.2.12. Let $\Lambda^\star$ be the category with objects

- finite collections $\{[m_i]\}_{i \in S}$ in Δ indexed by $S \in \mathbb{F}\mathrm{in}$, and
- $\langle n \rangle$ in Λ,

and morphisms given by:

1. Morphisms $\{[m_i]\}_{i \in S} \to \{[n_j]\}_{j \in T}$ given by
   - a morphism $\phi : T \to S$ in $\mathbb{F}\mathrm{in}$, with a chosen linear order on each fiber, and
   - for each $i \in S$, a morphism

$$\bigoplus_{j \in \phi^{-1}(i)} [n_j] \to [m_i]$$

2. Morphisms $\langle n \rangle \to \{[m_i]\}_{i \in S}$ given by
   - a cyclic order on S, and
   - a morphism
   $$\bigcup^S [m_i] \to \langle n \rangle$$
   in Λ.

3. Empty homsets $\{[m_i]\} \to \langle n \rangle$.

Composition is defined by taking lexicographic linear and cyclic orders. It is well-defined by Lemma 1.1.13. ∎

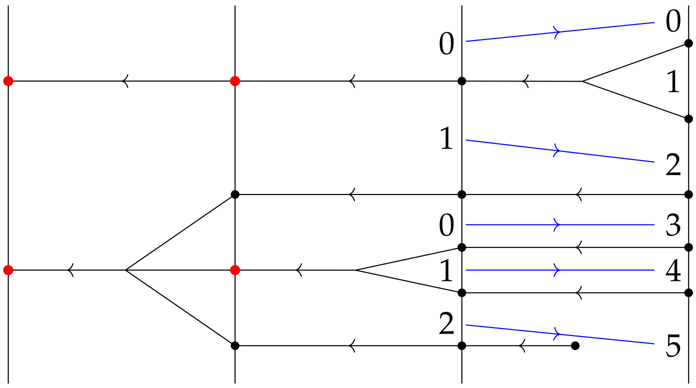


Figure 3.2: A pictorial representation of a morphism μ in Ω, considered as a sequence $T \xrightarrow{\overline{g}} V \xrightarrow{h} U \xrightarrow{g} S$ of morphisms in $\mathcal{A}ss$. The chosen subsets $Q \subset S$ and $P \subset U$ are marked in red, and the induced morphism $\mathcal{L}(\mu)$ is drawn in blue. Note that, unlike in the analogous Fig. 3.1, the source of $\mathcal{L}(\mu)$ is the ordinal sum $\bigoplus_{i \in P} O(f^{-1}(i))$, owing to the presence interstitial trees with roots not in P.

Definition 3.2.13. As in the case of algebra objects, we define a version of Θ on non-degenerate subsets. Let Ω be the full subcategory of Θ on those objects

$$Q \subset S \xleftarrow{f} T$$

such that $Q \neq \emptyset$ and $f : T \to S$ is not $\mathrm{id}_\diamond$. ∎

Lemma 3.2.14. *The is an equivalence of ∞-categories*

$$\mathrm{Fun}^*(\Theta, \mathcal{C}) \simeq \mathrm{Fun}(\Omega, \mathcal{C})$$

Where $\mathrm{Fun}^*(\Theta, \mathcal{C})$ denotes the full subcategory on those functors which send empty subsets to the terminal object of $\mathcal{C}$.

Proof. This is, *mutatis mutandis*, the same proof as that of Lemma 3.1.9. □

Construction 3.2.15. We define a functor $\mathcal{L} : \Omega \to \Lambda^\star$ as follows. Let

$$P \subset S \xleftarrow{f} T$$

be an object in Ω with f a morphism in $\mathcal{A}ss$. We send this object to the collection

$$\left\{ O\left(f^{-1}(i)\right)\right\}_{i \in P}.$$

Let

$$\{\diamond\} \subset \diamond \xleftarrow{f} S$$

be an object in Ω. Then we send this object to

$$D(f^{-1}(\diamond)) \in \Lambda.$$

To define $\mathcal{L}$ on morphisms, we proceed by cases:

1. Suppose we have a diagram

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f} & T \\ & & {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \overline{g}} \\ P & \subseteq & U & \xleftarrow[h]{} & V \end{array}$$

representing a morphism μ in Ω, where all of the objects are in $\mathcal{A}\mathrm{ss} \subset \mathcal{A}\mathrm{ss}_{\mathrm{CY}}$. $\mathcal{L}(\mu)$ will be given by a morphism ϕ_μ in Fin_* and a set of morphisms $\{\psi_i\}_{i\in Q}$ in Δ. The morphism ϕ_μ we take to be the restriction of g to $P \subset U^\circ$. Fixing $i \in Q$, we see that $\overline{g}$ restricts to a morphism $\overline{g}_i : f^{-1}(i) \to h^{-1}(g^{-1}(i))$ of linearly ordered sets. This can be rewritten as

$$\overline{g}_i : f^{-1}(i) \to \bigoplus_{j\in g^{-1}(i)} h^{-1}(j)$$

It therefore induces a morphism

$$\bigstar_{j\in\overline{g}^{-1}(i)} O(h^{-1}(i)) \to O(f^{-1}(i))$$

We then define ψ_i to be the composite

$$\bigoplus_{j\in\overline{g}^{-1}(i)\cap P} O(h^{-1}(i)) \to \bigstar_{j\in\overline{g}^{-1}(i)} O(h^{-1}(i)) \to O(f^{-1}(i))$$

See Fig. 3.2 for a pictorial representation.

2. Suppose we have a diagram

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f} & T \\ & & {\scriptstyle \mathrm{id}}\uparrow & & \downarrow{\scriptstyle \overline{g}} \\ \{1\} & \subseteq & \diamond & \xleftarrow[h]{} & V \end{array}$$

representing a morphism μ in Ω with $T, U \in \mathcal{A}\text{ss}$. Then $\mathcal{L}(\mu)$ will be given by a morphism $\psi : D(h^{-1}(\diamond)) \to D(f^{-1}(\diamond))$. The morphism $\overline{g}$ restricts to a morphism of cyclically ordered sets

$$\overline{g}_\diamond : f^{-1}(\diamond) \to h^{-1}(\diamond)$$

we therefore define ψ to be $D(\overline{g}_\diamond)$.

3. Suppose we have a diagram

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f} & T \\ & & \uparrow g & & \downarrow \overline{g} \\ P & \subseteq & U & \xleftarrow[h]{} & V \end{array}$$

representing a morphism μ in Ω, where all objects except $\diamond$ are in $\mathcal{A}\text{ss}$. The morphism $\mathcal{L}(\mu)$ will be given by a cyclic order on P and a morphism $\psi : \bigcup^S O(f^{-1}(i)) \to D(f^{-1}(\diamond))$. The cyclic order on P is induced by the cyclic order on $g^{-1}(\diamond) \supset P$. The morphism $\overline{g}$ restricts to a morphism

$$\overline{g}_\diamond : f^{-1}(\diamond) \to (g \circ h)^{-1}(\diamond)$$

of cyclically ordered sets. Passing through D gives a morphism

$$D(\overline{g}_\diamond) : D((g \circ h)^{-1}(\diamond)) \to D(f^{-1}(\diamond)).$$

Choosing any linear order on $g^{-1}(\diamond)$ compatible with the cyclic order we can write $D(\overline{g}_\diamond)$ as

$$C(O(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i))) = D(K(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i)) \to D(f^{-1}(\diamond))$$

We then have the canonical morphism

$$K(\bigoplus_{i \in g^{-1}(\diamond)} O(h^{-1}(i))) \to C(\bigstar_{i \in g^{-1}(\diamond)} O(h^{-1}(i))) = C(O(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i)))$$

And so we define ψ to be the composite

$$K(\bigoplus_{i \in P} O(h^{-1}(i))) \to K(\bigoplus_{i \in g^{-1}(\diamond)} O(h^{-1}(i))) \to C(O(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i))) \to D(f^{-1}(\diamond))$$

See Fig. 3.3 for a pictorial representation.

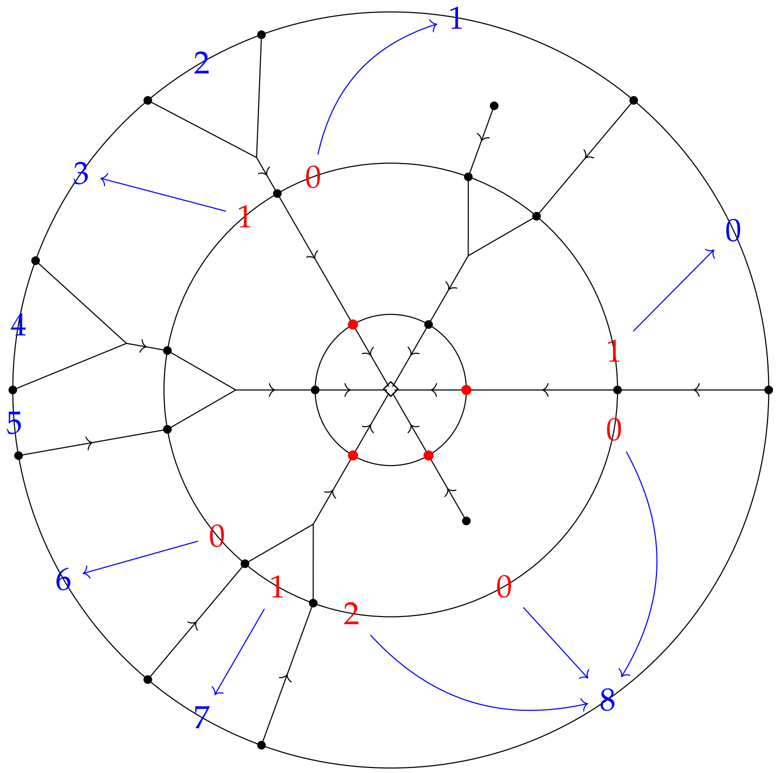


Figure 3.3: A morphism in Ω represented as a composite of three morphisms in $\mathcal{A}ss_{\mathrm{CY}}$, $T \xrightarrow{\overline{g}} V \xrightarrow{h} U \xrightarrow{g} \diamond$. The chosen subset $P \subset U$ is marked by red points. The corresponding interstice sets $I(h^{-1}(i))$ are written in red numbers, and the set $D(f^{-1}(\diamond))$ in blue numbers. The induced morphism $\mathcal{L}(\mu) : \bigcup^P I(h^{-1}(i)) \to D(f^{-1}(\diamond))$ is drawn in blue. Note that the unmarked points in U are the reason that we do not necessarily get a morphism $C(\bigstar_{i \in P} I(h^{-1}(i)) \to D(f^{-1}(\diamond))$.

∎

Definition 3.2.16. Let $M \in \Lambda^\star$. We denote by Ω_M^E the subcategory of the weak fiber Ω_M whose morphisms are morphisms in E. ∎

Proposition 3.2.17. *For every M in $\Lambda^\star$, there is an initial element in Ω_M^E.*

Proof. We will complete the proof in two cases:

Suppose first that $M = \{[m_i]\}_{i \in P}$. Then the weak fiber only involves morphisms in $\mathcal{A}ss \subset \mathcal{A}ss_{\mathrm{CY}}$. We define a set

$$T := \coprod_{i \in P} \mathbb{I}([m_i])$$

and a morphism $f_M : T \to P$ by setting $f_M(\mathbb{I}([m_i])) = i$. The canonical isomorphisms

$$\eta_i : O(\mathbb{I}([m_i])) \cong [m_i]$$

equip $P \subset P \xleftarrow{f_M} T$ with the structure of an object of Ω^E_M. Given an element

$$P \subset U \xleftarrow{f} V$$

and an isomorphism $\phi_i : O(f^{-1}(i)) \cong [m_i]$, we define a unique morphism μ in Ω^E_M given by

$$\begin{array}{ccccc} P & \subseteq & P & \xleftarrow{f_M} & T \\ & & g\uparrow & & \downarrow\overline{g} \\ P & \subseteq & U & \xleftarrow[f]{} & V \end{array}$$

as follows. Since this must be a morphism in E, we see that g must map P identically to P, and send $U^\circ \setminus P$ to the basepoint. On fibers, we consider the isomorphisms

$$\eta_i^{-1} \circ \phi_i : O(f^{-1}(i)) \to O(I[m_i])$$

Since O is fully faithful, this lifts to a unique isomorphism $I(\phi_i) : I([m_i]) \cong f^{-1}(i)$. We therefore see that $\overline{g}$ must be the coproduct of these morphisms if μ is to be a morphism in the weak fiber. It is immediate that this does, indeed, define a morphism in Ω^E_M.

Now suppose instead $M = \langle m \rangle$. We define $f_M : D(\langle m \rangle) \to \diamond$ to be the morphism with $f_M^{-1}(\diamond) = D(\langle m \rangle)$. Since D is an equivalence, we choose the isomorphism

$$\eta : D^2(\langle m \rangle) \cong \langle m \rangle$$

Suppose given another element

$$\{1\} \subset \diamond \xleftarrow{f} T$$

with $\phi : D(f^{-1}(\diamond)) \cong \langle n \rangle$ in the weak fiber. We define a unique morphism $\mu \in \Omega^E_M$ given by

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f_M} & D(\langle m \rangle) \\ & & g\uparrow & & \downarrow\overline{g} \\ \{1\} & \subseteq & \diamond & \xleftarrow[f]{} & T \end{array}$$

as follows. The morphism g must be the identity, so we need only define $\overline{g}$. The condition that μ be in the weak fiber implies that $\eta_i \circ D(\overline{g}|_{D(\langle m \rangle)}) = \phi_i$, i.e. $D(\overline{g}|_{D(\langle m \rangle)}) = \eta_i^{-1} \circ \phi_i$. However, since D is fully faithful, this condition defines a unique isomorphism $D(\langle m \rangle) \cong f^{-1}(\diamond)$, determining $\overline{g}$, and thus μ, uniquely. □

Proposition 3.2.18. *Suppose given an object $M = \{[m_i]\}_{i \in P}$ in $\Lambda^\star$, an object*

$$Z := \left\{ Q \subset S \overset{f_Z}{\leftarrow} T \right\}$$

in Ω, and a morphism

$$(\phi, \{\gamma_i\}_{i \in Q}) : \mathcal{L}(Z) \to M$$

in $\Lambda^\star$. Then there is an element $X_{M,Z}$ in Ω_M^E and a morphism $\Phi : Z \to X_{M,Z}$ in Ω covering $(\phi, \{\gamma_i\}_{i \in Q})$ such that, for any other morphism $\Psi : Z \to X$ covering $(\phi, \{\gamma_i\}_{i \in Q})$, there is a unique morphism $\tau : X_{M,Z} \to X$ which makes the diagram

$$\begin{array}{ccc} & Z & \\ {}^{\Phi}\swarrow & & \searrow^{\Psi} \\ X_{M,Z} & \xrightarrow[\tau]{} & X \end{array}$$

commute.

Proof. There are two cases to consider, corresponding to whether or not $S = \diamond$.

Case 1: First suppose $S \in \mathcal{A}\mathrm{ss}$. In this case, we construct $X_{M,Z}$ as follows. Let

$$P \subset P \overset{f_M}{\longleftarrow} U$$

be the object constructed in Proposition 3.2.18. Then, in particular, $\phi : P \to Q \subset S$.

For each $i \in Q$, we have a morphism

$$\gamma_i : \bigoplus_{j \in \phi^{-1}(i)} [m_i] \to O(f_Z^{-1}(i))$$

For each $j \in \phi^{-1}(i)$ denote by $\gamma_i([m_j])$ the smallest subinterval of $O(f_Z^{-1}(i))$ containing the image of $[m_j]$ under γ_i. Then $\gamma_i|_{[m_j]} \to \gamma_i([m_j])$ preserves boundary, and thus corresponds to a map $\overline{g}_j : I(\gamma_i([m_j])) \to I([m_j])$ of linearly ordered sets. Moreover, $\overline{g}_j$ fits into a commutative diagram

$$\begin{array}{ccc} S & \xleftarrow{f_Z} & I(\gamma_i([m_j])) \\ {\scriptstyle \phi}\uparrow & & \downarrow{\scriptstyle \overline{g}_j} \\ P & \xleftarrow{f_M} & I([m_j]) \end{array}$$

in $\mathcal{A}$ss. We here use the identification of $I(\gamma_i([m_j]))$ with a subset of T.

Since, by definition, $U = \coprod_{j\in P} I([m_j])$, we can then write down a commutative diagram

$$\begin{array}{ccc} S & \xleftarrow{f_Z} & \coprod_{i,j} I(\gamma_i([m_j])) \\ {\scriptstyle \phi}\uparrow & & \downarrow{\scriptstyle \coprod_{i,j} \overline{g}_j} \\ P & \xleftarrow{f_M} & \coprod_{j\in P} I([m_j]) \end{array} \tag{3.4}$$

in $\mathcal{A}$ss.

For each $i \in Q$, this restricts to a diagram of ordered sets

$$\begin{array}{ccc} \{i\} & \xleftarrow{f_Z} & \coprod_j I(\gamma_i([m_j])) \\ {\scriptstyle \phi}\uparrow & & \downarrow{\scriptstyle \coprod_j \overline{g}_j} \\ \phi^{-1}(i) & \xleftarrow{f_M} & \coprod_{j\in\phi^{-1}(i)} I([m_j]) \end{array}$$

We denote $L_i := f_Z^{-1}(i) \setminus \coprod_{j\in\phi^{-1}(i)} I(\gamma_i([m_j]))$, and proceed as follows.

- For $p, p+1$ in $\phi^{-1}(i)$, if there is at least one $k \in L_i$ such that

  $$I(\gamma_i([m_p])) < k < I(\gamma_i([m_{p+1}]))$$

  we define a new element r_p and append it to $\phi^{-1}(i)$ between p and $p+1$.

- If there exists $k \in L_i$ such that

  $$k < I(\gamma_i([m_p]))$$

  for all $p \in \phi^{-1}(i)$, then we append a new minimal element r_{min} to $\phi^{-1}(i)$.

- If there exists $k \in L_i$ such that
$$I(\gamma_i([m_p])) < k$$
for all $p \in \phi^{-1}(i)$, then we append a new maximal element to $\phi^{-1}(i)$.

Call the resulting set $W_i \supset \phi^{-1}(i)$. We then set

$$R_i := U \amalg L_i$$

and define $f_i : R_i \to W_i$ to act as f_M on U and on L_i to send

- $k \mapsto r_p$ if
$$I(\gamma_i([m_p])) < k < I(\gamma_i([m_{p+1}]))$$
- $k \mapsto r_{min}$ if
$$k < I(\gamma_i([m_p]))$$
for all $p \in \phi^{-1}(i)$
- $k \mapsto r_{max}$ if
$$I(\gamma_i([m_p])) < k$$
for all $p \in \phi^{-1}(i)$

We make f_i into a morphism in $\mathcal{A}$ss by taking the linear order induced by L_i on the fibers over the r_p, r_{min} and r_{max}. We then define

$$\overline{g}^i : f_Z^{-1}(i) \to R_i$$

to act as $\coprod_{j \in \phi^{-1}(i)} \overline{g}_j$ on $\coprod_{j \in \phi^{-1}(i)} I(\gamma_i([m_j]))$ and as the identity on L_i. We further define $\phi_i : W_i \to \{i\}$ to send every element to i. We thus have a commutative diagram

$$\begin{array}{ccccc}
\{i\} & \subseteq & \{i\} & \xleftarrow{f_Z} & \coprod_j I(\gamma_i([m_j])) \\
 & & \uparrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \overline{g}^i} \\
P \cap \phi^{-1}(i) & \subseteq & W_i & \xleftarrow{f_i} & R_i
\end{array}$$

in $\mathcal{A}$ss, which covers the morphism $\gamma_i : \bigoplus_{j \in \phi^{-1}(i)} [m_j] \to O(f_Z^{-1}(i))$. Taking the coproduct over $i \in \mathrm{Im}(\phi)$ gives us a morphism

$$\begin{array}{ccccc} \mathrm{Im}(\phi) & \subseteq & \mathrm{Im}(\phi) & \xleftarrow{f_Z} & f_Z^{-1}(\mathrm{Im}(\phi)) \\ & & {\scriptstyle \coprod_i \phi_i}\uparrow & & \downarrow{\scriptstyle \coprod_i \overline{g}^i} \\ P \cap \phi^{-1}(i) & \subseteq & \coprod_i W_i & \xleftarrow{\coprod_i f_i} & \coprod_i R_i \end{array}$$

Finally, we set

$$W = (\coprod_{i \in \mathrm{Im}(\phi)} W_i) \amalg (S \setminus \mathrm{Im}(\phi))$$

and

$$R = (\coprod_{i \in \mathrm{Im}(\phi)} R_i) \amalg (T \setminus f_Z^{-1}(\mathrm{Im}(\phi)))$$

We then define morphisms:

- $g : W \to S$ to act as $\coprod_{i \in \mathrm{Im}(\phi)} \phi_i$ on $\coprod_{i \in \mathrm{Im}(\phi)} W_i$ and as the identity otherwise.
- $f_{M,Z} : R \to W$ to act as f_i on R_i and as f_Z on $T \setminus f_Z^{-1}(\mathrm{Im}(\phi))$.
- $\overline{g} : T \to R$ to act as $\coprod_{i \in \mathrm{Im}(\phi)} \overline{g}^i$ on $f_Z^{-1}(\mathrm{Im}(\phi))$ and the identity elsewhere.

By construction, this defines a commutative diagram

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f_Z} & T \\ & & {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \coprod_i \overline{g}} \\ P & \subseteq & W & \xleftarrow{f_{M,Z}} & R \end{array} \tag{3.5}$$

in $\mathcal{A}$ss, covering $(\phi, \{\gamma_i\})$, and the bottom row is in Ω_M. We therefore define $X_{M,Z}$ to be the bottom row, and Φ to be the morphism defined by the diagram (3.5).

To check the remaining universal property, we let

$$P \subset A \overset{f_X}{\leftarrow} B$$

and $\beta_i : O(f_X^{-1}(i)) \cong [m_i]$ be another element in Ω^E_M, and let ν be a morphism

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f_Z} & T \\ & & \rho\uparrow & & \downarrow\overline{\rho} \\ P & \subseteq & A & \xleftarrow[f_X]{} & B \end{array}$$

covering $(\phi, \{\gamma_i\}_{i\in Q})$.

For each $i \in \mathrm{Im}(\phi)$, the identity on P and the condition nothing be sent to the basepoint uniquely determines a map of ordered sets

$$\zeta_i : \rho^{-1}(i) \to W_i.$$

Moreover, the ζ_i together with the restriction of ρ to $A \setminus \rho^{-1}(\mathrm{Im}(\phi))$ uniquely determines a map

$$\zeta : A \to W$$

such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & S \\ & {}_{\zeta}\searrow \quad \nearrow_{g} & \\ & W & \end{array}$$

commutes. Note that $\zeta|_P$ induces the identity $P \to P$.

Moreover, for each $i \in \mathrm{Im}(\phi)$ the isomorphisms $I(\beta_i)$ on $I([m_j])$ and restriction $\overline{\rho}_i : L_i \to f_X^{-1}(\rho^{-1}(i))$ uniquely determine a map

$$\overline{\zeta}_i : R_i \to f_X^{-1}(\rho^{-1}(i)).$$

These, together with the restriction of $\overline{\rho}$ to $T \setminus f_Z^{-1}(\mathrm{Im}(\phi))$ uniquely determine a morphism

$$\overline{\zeta} R \to B$$

such that the diagram

$$\begin{array}{ccc} B & \xleftarrow{\overline{\rho}} & T \\ & {}_{\overline{\zeta}}\nwarrow \quad \swarrow_{\overline{g}} & \\ & R & \end{array}$$

commutes, and the restriction of $\overline{\zeta}$ to $f_X^{-1}(P)$ is the isomorphism $\coprod_i I(\beta_i)$.

We therefore have constructed a unique morphism

$$(\zeta, \bar{\zeta}) : X_{Z,M} \to X$$

in Ω_M^E such that the diagram

$$\begin{array}{ccc} & Z & \\ {}^{\Phi}\swarrow & & \searrow^{\Psi} \\ X_{M,Z} & \xrightarrow[(\zeta,\bar{\zeta})]{} & X \end{array}$$

commutes.

Case 2: Now suppose that $S = \diamond$. Then ϕ is completely determined by a cyclic order on P, and γ is a morphism

$$\gamma : \bigcup^{S} [m_i] \to D(f_Z^{-1}(\diamond)).$$

We note that, given any morphism

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{p} & V \\ & & {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \bar{g}} \\ A & \subseteq & B & \xleftarrow[\ell]{} & C \end{array}$$

a choice of linear order on $g^{-1}(\diamond)$ compatible with the cyclic order uniquely determines a factorization

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{p} & V \\ & & \uparrow & & \downarrow{\scriptstyle \mathrm{id}_V} \\ \{1\} & \subseteq & \{1\} & \xleftarrow{p} & V \\ & & {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \bar{g}} \\ A & \subseteq & B & \xleftarrow[\ell]{} & C \end{array}$$

Similarly, given a morphism $(\psi, \eta) : \langle n \rangle \to \{[n_i]\}_{i \in S}$, a choice of linear order on S compatible with the cyclic order uniquely determines a factorization

$$\langle n \rangle \to \{[n]\} \to \{[n_i]\}.$$

We can therefore choose a linear order on P and define Y to be the object

$$\{1\} \subset \{1\} \xleftarrow{f_Z} T.$$

Then take (ϕ_Y, γ_Y) to be the unique morphism yielding a factorization

$$\mathcal{L}(\phi, \gamma) : D(f_Z^{-1}(\diamond)) \to K(f_Z^{-1}(\diamond)) \overset{(\phi_Y, \gamma_Y)}{\longrightarrow} \{[m_i]\}_{i \in P}$$

We can then construct $X_{M,Y}$ as in case 1. It is immediate that

$$\begin{array}{ccccc}
\{1\} & \subseteq & \diamond & \xleftarrow{f_Z} & T \\
 & & \uparrow & & \downarrow{\scriptstyle \mathrm{id}_T} \\
\{1\} & \subseteq & \{1\} & \xleftarrow{f_Z} & T \\
 & & {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \overline{g}} \\
P & \subseteq & W_Y & \xleftarrow[f_{M,Y}]{} & R_Y
\end{array}$$

defines a morphism Φ in Ω covering (ϕ, γ).

Now suppose given any other morphism $\Psi = (\psi, \overline{\psi}) : Z \to X$ covering (ϕ, γ). A choice of linear order on $\psi^{-1}(\diamond)$ compatible with the chosen linear order on P uniquely factors Ψ through Y. We therefore get a morphism $\tau : X_{M,Y} \to X$ such that the diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & {\scriptstyle \Phi}\swarrow & & \searrow{\scriptstyle \Psi} & \\
X_{M,Y} & & \xrightarrow[\tau]{} & & X
\end{array}$$

commutes.

To see that this morphism is unique, suppose that $(\xi, \overline{\xi}), (\zeta, \overline{\zeta}) : X_{M,Y} \to X$ are two such morphisms. Then, choosing a linear order on $\psi^{-1}(\diamond)$ compatible with the chose linear order on P uniquely factors the diagram as

$$\begin{array}{ccccc}
 & & Z & & \\
 & {\scriptstyle \Phi}\swarrow & \downarrow & \searrow{\scriptstyle \Psi} & \\
 & & Y & & \\
 & \swarrow & & \searrow & \\
X_{M,Y} & & \underset{(\xi,\overline{\xi}),(\zeta,\overline{\zeta})}{\rightrightarrows} & & X
\end{array}$$

But, by case 1, there is a unique morphism making the bottom triangle commute. Therefore, $(\xi, \overline{\xi}) = (\zeta, \overline{\zeta})$, proving the proposition. $\square$

Proposition 3.2.19. *Suppose given an object $M = \langle m \rangle$ in $\Lambda^\star$, an object*

$$Z := \left\{ Q \subset S \overset{f_Z}{\leftarrow} T \right\}$$

in Ω, and a morphism

$$(\phi, \{\gamma_i\}_{i \in Q}) : \mathcal{L}(Z) \to M$$

in $\Lambda^\star$. Then there is an element $X_{M,Z}$ in Ω^E_M and a morphism $\Phi : Z \to X_{M,Z}$ in Ω covering $(\phi, \{\gamma_i\}_{i \in Q})$ such that, for any other morphism $\Psi : Z \to X$ covering $(\phi, \{\gamma_i\}_{i \in Q})$, there is a unique morphism $\tau : X_{M,Z} \to X$ which makes the diagram

$$\begin{array}{ccc} & Z & \\ {}^{\Phi}\swarrow & & \searrow^{\Psi} \\ X_{M,Z} & \xrightarrow[\tau]{\qquad} & X \end{array}$$

commute.

Proof. We first note that $S = \diamond$, since otherwise no such morphism $(\phi, \{\gamma_i\}_{i \in Q})$ can exist. Consequently, $\phi = \mathrm{id}_\diamond$, and γ is a morphism of cyclically ordered sets $\langle m \rangle \to D(f_Z^{-1}(\diamond))$. We can therefore take $X_{Z,M}$ to be the object

$$\{1\} \subset \diamond \overset{f_M}{\leftarrow} D(\langle m \rangle)$$

constructed in the proof of Proposition 3.2.17. We then get a commutative diagram

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{\qquad f_Z \qquad} & T \\ & & {\scriptstyle \mathrm{id}}\uparrow & & \downarrow{\scriptstyle \overline{g}} \\ \{1\} & \subseteq & \diamond & \xleftarrow[f_M]{} & D(\langle m \rangle) \amalg (T \setminus f_Z^{-1}(\diamond)) \end{array}$$

where $\overline{g}$ acts as $D(\gamma)$ on $f_Z^{-1}(\diamond)$ and the identity on $T \setminus f_Z^{-1}(\diamond)$. This morphism in Ω clearly covers (id, γ).

Given $X \in \Omega^E_M$ and $\Psi : Z \to X$, represented by a diagram

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f_Z} & T \\ & & {\scriptstyle \mathrm{id}}\uparrow & & \downarrow{\scriptstyle \overline{\ell}} \\ \{1\} & \subseteq & \diamond & \xleftarrow[f_X]{} & A \end{array}$$

by 3.2.17 that there is a unique morphism

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f_M} & D(\langle m \rangle) \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \overline{h}} \\ \{1\} & \subseteq & \diamond & \xleftarrow[f_X]{} & A \end{array}$$

in Ω_M^E. Via the restriction of $\overline{\ell}$ to $T \setminus f_Z^{-1}(\diamond)$, this extends to a morphism

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f_M} & D(\langle m \rangle) \amalg (T \setminus f_Z^{-1}(\diamond)) \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \overline{\xi}} \\ \{1\} & \subseteq & \diamond & \xleftarrow[f_X]{} & A \end{array}$$

in Ω_M^E.

Since all of the left-hand vertical morphisms are required to be identities, we only need to check that $\overline{\xi} \circ \overline{g} = \overline{\ell}$, which is true by construction. The requirement that $\overline{\xi}$ define a morphism in Ω_M^E uniquely determines $\overline{\xi}$ on $D(\langle m \rangle)$ and the requirement that $\overline{\xi} \circ \overline{g} = \overline{\ell}$ uniquely determines $\overline{\xi}$ on $T \setminus f_Z^{-1}(\diamond)$. □

Corollary 3.2.20. *The functor $\mathcal{L}$ is an ∞-categorical localization of Ω at the morphisms of E.*

Proof. This follows again from [41, Lemma 3.1.1]. Proposition 3.2.17 shows that the weak fibers Ω_M^E have initial objects, and Proposition 3.2.18 and Proposition 3.2.19 show that the inclusions

$$\Omega_M^E \subset \Omega_{/M}$$

are cofinal. □

We now rephrase the conditions from Corollary 3.2.10 in terms of functors from $\Lambda^\star$. Note that by forgetting degerate intervals and localizing along $\mathcal{L}$, we have already dealt with conditions 1 and 3.

Construction 3.2.21. Given $\langle n \rangle$ in $\Lambda^\star$, we define a morphism

$$\sigma_n : \langle n \rangle \to \{[1]_{(i,i+1)}\}_{(i,i+1) \in D(\langle n \rangle)}$$

in $\Lambda^\star$ as follows. Take the canonical cyclic order on $D(\langle n\rangle)$, and define

$$\bigcup^{D(\langle n\rangle)}[1]_{(i,i+1)} \to \langle n\rangle$$

sending

$$\begin{aligned} 0 \in [1]_{(i,i+1)} &\mapsto i \\ 1 \in [1]_{(i,i+1)} &\mapsto i+1. \end{aligned}$$

Note that given an object $X \in \Omega_{\langle n\rangle}$ in the fiber over $\langle n\rangle$, σ_n is simply the image of the source morphism in Ω.

Similarly, given an object $\{[m_i]\}_{i\in S}$ in $\Lambda^\star$, define two morphisms

$$\begin{aligned} t_{\{m_i\}} &: \{[m_i]\}_{i\in S} \to \{[1]_i\}_{i\in S} \\ s_{\{m_i\}} &: \{[m_i]\}_{i\in S} \to \{[1]_{(j,j+1)}\}_{(j,j+1)\in\bigoplus_{i\in S} I([m_i])} \end{aligned}$$

in $\Lambda^\star$ as follows. We define $t_{\{m_i\}} := (\mathrm{id}_S, \{f_i\})$ where $f_i : [1]_i \to [m_i]$ is given by the formula

$$\begin{aligned} f_i(0) &= 0 \\ f_i(1) &= m_i \end{aligned}$$

We define $s_i := (\phi, \{g_i\})$, where

$$\phi : \bigoplus_{i\in S} I([m_i]) \to S$$

sends $I([m_i])$ to i, and the morphism

$$g_i : \bigoplus_{(j,j+1)\in\bigoplus_{i\in S} I([m_i])} [1]_{(j,j+1)} \to [m_i]$$

is given by

$$\begin{aligned} g_i(0 \in [1]_{(j,j+1)}) &= j \\ g_i(1 \in [1]_{(j,j+1)}) &= j+1 \end{aligned}$$

Note that, given an object $X \in \Omega_M$ in the fiber over $M := \{[m_i]\}_{i\in S}$, the morphisms s_M and t_M are simply the images under $\mathcal{L}$ of the source and target morphisms, respectively. ∎

Lemma 3.2.22. *Given a functor $G : \Lambda^\star \to \mathcal{C}$, $G \circ \mathcal{L}$ satisfies condition 4 if and only if the following two conditions on G are satisfied:*

1. *For any $\{[m_i]\}_{i\in S}$, and any $\{[n_{(j,j+1)}]\}_{(j,j+1)\in\bigoplus_{i\in S} I([m_i])}$ the diagram*

$$\begin{array}{ccccc} & & \{\bigstar_{(j,j+1)\in I([m])}[n_{(j,j+1)}]\}_{i\in S} & & \\ & \swarrow & & \searrow & \\ \{[n_{(j,j+1)}]\}_{(j,j+1)\in\bigoplus_{i\in S}} & & & & \{[m_i]\}_{i\in S} \\ & \underset{t_N}{\searrow} & & \underset{s_M}{\swarrow} & \\ & & \{[1]_{(j,j+1)}\}_{(j,j+1)\in\bigoplus_{i\in S} I([m_i])} & & \end{array}$$

   is sent to a pullback under G.

2. *For $\langle n\rangle$ and any $\{[m_{(j,j+1)}\}_{(j,j+1)\in D(\langle n\rangle)}$ the diagram*

$$\begin{array}{ccccc} & & C(\bigstar_{(j,j+1)\in D(\langle n\rangle)}[m_{(j,j+1)}]) & & \\ & \swarrow & & \searrow & \\ \{[m_{(j,j+1)}\}_{(j,j+1)\in D(\langle n\rangle)} & & & & \langle n\rangle \\ & \underset{t_N}{\searrow} & & \underset{\sigma_M}{\swarrow} & \\ & & \langle n\rangle I([m_i]) & & \end{array}$$

   is sent to a pullback diagram under G.

Proof. Using the same technique as in the proof of Proposition 3.1.24, we can reduce condition 4 to a statement about pullback squares along source and target maps. The diagrams of the lemma are then the images under $\mathcal{L}$ of the requisite pullback diagrams. □

Definition 3.2.23. We denote by $\mathrm{Fun}^{\mathrm{alg}}(\Lambda^\star, \mathcal{C})$ the full sub-category on those functors which

1. Send $\{[m_i\}_{i\in S}$ together with the projections to $[m_i]$ to product diagrams.
2. Send the diagrams from 3.2.22 to pullback diagrams.
3. Send the morphisms $\{[n]\} \to \langle n\rangle$ to equivalences.

▮

Corollary 3.2.24. *There is an equivalence of ∞-categories*

$$\mathrm{Alg}^{\mathrm{CY}}_{\mathcal{S}\mathrm{p}}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{alg}}(\Lambda^\star, \mathcal{C}).$$

3.2.3 Extension and restriction

Definition 3.2.25. We define a category Λ_Δ to be the Grothendieck construction of the functor

$$\Delta^1 \overset{\{K\}}{\to} \mathrm{Cat}\,.$$

explicitly, $\mathrm{ob}(\Lambda_\Delta) = \mathrm{ob}(\Lambda) \amalg \mathrm{ob}(\Delta)$, with morphisms

- $f : [n] \to [m]$ morphism in Δ
- $f : \langle n \rangle \to \langle m \rangle$ morphism in Λ
- $f : [n] \to \langle m \rangle$ given by a morphism $f : K([n]) \to \langle m \rangle$ in Λ.

The category $(\Lambda_\Delta)^{\mathrm{op}}$ can be identified with the full subcategory of $\Lambda^\star$ on the objects $\{[m]\}$ and $\langle n \rangle$. ▮

Construction 3.2.26. By taking restriction and right Kan extension along the inclusion $\Lambda_\Delta^{\mathrm{op}} \subset \Lambda^\star$, we get an adjunction

$$\iota_* \,\mathrm{Fun}(\Lambda^\star, \mathcal{C}) \leftrightarrow \mathrm{Fun}(\Lambda_\Delta^{\mathrm{op}}, \mathcal{C}) : \iota_!$$

of ∞-categories. ▮

Definition 3.2.27. Denote by $\mathrm{Fun}^\times(\Lambda^\star, \mathcal{C})$ the full ∞-subcategory of $\mathrm{Fun}(\Lambda^\star, \mathcal{C})$ on those functors which satisfy Item 1 from Definition 3.2.23. ▮

Proposition 3.2.28. *The adjunction of Construction 3.2.26 restricts to an equivalence of ∞-categories*

$$\mathrm{Fun}^\times(\Lambda^\star, \mathcal{C}) \simeq \mathrm{Fun}(\Lambda_\Delta, \mathcal{C})$$

Proof. Since there are no morphisms $\{[m_i]\}_{i \in S} \to \langle n \rangle$ in $\Lambda^\star$, this is, *mutatis mutandis*, the same as the proof of 3.1.27. □

Construction 3.2.29. We have a full subcategory $F : \Lambda \subset \Lambda_\Delta$. We can similarly define a functor

$$H : \Lambda_\Delta \to \Lambda$$

by acting as K on Δ and as the identity on all other objects and morphisms. This defines an adjunction

$$F : \Lambda \leftrightarrow \Lambda_\Delta : H$$

It is easy to see that H is a reflective localization at the morphisms $[n] \to \langle n \rangle$ given by isomorphisms $K([n]) \cong \langle n \rangle$. ∎

Proposition 3.2.30. *There is an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{alg}}(\Lambda^\star, \mathcal{C}) \simeq 2\text{-}\mathrm{Seg}_\Lambda(\mathcal{C}).$$

Proof. Proposition 3.2.28 and Construction 3.2.29 show that $\mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{C})$ is equivalent, as an ∞-category, to the full subcategory of $\mathrm{Fun}^\times(\Lambda^\star, \mathcal{C})$ satisfying 1 and 3 from Definition 3.2.23. The relation between the 2-Segal condition and condition 2 from Definition 3.2.23 follows from the proof of Proposition 3.1.28. □

We can then summarize our results in the following theorem:

Theorem 3.2.31. *There is an equivalence of ∞-categories*

$$\mathrm{Alg}^{\mathrm{CY}}_{\mathrm{Sp}}(\mathcal{C}) \simeq 2\text{-}\mathrm{Seg}_\Lambda(\mathcal{C}).$$

3.3 Constructions and examples

We now apply Theorem 3.2.31 to the study of open 2-dimensional ∞-categorical topological field theories, making use of the following theorem, which appears as [33, Thm. 4.2.14].

Theorem 3.3.1. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the following types of data are equivalent:*

1. *Open, oriented topological field theories valued in $\mathcal{C}$.*
2. *Calabi-Yau Algebras in $\mathcal{C}$.*

This equivalence is implemented by carrying a topological field theory Z to the Calabi-Yau algebra $Z([0,1])$.

3.3.1 Topological field theories

Given a 2-Segal cyclic object X in $\mathcal{C}$, Theorem 3.2.31 and Theorem 3.3.1 together imply the existence of a topological field theory $Z_X : \mathrm{Bord}_2^{\mathrm{nc}} \to \mathrm{Span}^{\times}(\mathcal{C})$. We now aim to identify the invariants that Z_X associates to surfaces up to equivalence.

Proposition 3.3.2. *Let $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal cyclic object in $\mathcal{C}$. Denote by $C_X : \mathcal{A}\mathrm{ss}_{\mathrm{CY}} \to \mathrm{Span}(\mathcal{C}^{\boxtimes})$ the image of X under the equivalence of Theorem 3.2.31.*

1. *Let $\psi_n : \langle n \rangle \to \langle 1 \rangle$ be the active morphism in $\mathcal{A}\mathrm{ss}$ corresponding to the linear order $\{1 < 2 < \ldots < n\}$. Then $C_X(\psi_n)$ is equivalent to*
$$X_1^n \xleftarrow{\{0,1\},\ldots,\{n-1,n\}} X_n \xrightarrow{\{0,n\}} X_1$$
2. *Let $\beta_{n+1} : \langle n+1 \rangle \to \diamond$ be the morphism in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ given by the cyclic closure of $\{1 < 2 < \ldots < n\}$. Then $C_X(\beta_{n+1})$ is equivalent to*
$$X_1^{n+1} \xleftarrow{\{0,1\},\ldots,\{n-1,n\},\{n,0\}} X_n \longrightarrow *$$

Proof. To show (1), we note that the span to which ψ_n is sent corresponds under adjunction to the pair of morphisms in Ω represented by the diagram

$$\begin{array}{ccccc}
\langle n\rangle^\circ & \subseteq & \langle n\rangle & \xleftarrow{\mathrm{id}_{\langle n\rangle}} & \langle n\rangle \\
 & & \downarrow{\scriptstyle \psi_n} & & \uparrow{\scriptstyle \mathrm{id}_{\langle n\rangle}} \\
\{1\} & \subseteq & \langle 1\rangle & \xleftarrow{\psi_n} & \langle n\rangle \\
 & & \uparrow{\scriptstyle \mathrm{id}_{\langle 1\rangle}} & & \downarrow{\scriptstyle \psi_n} \\
\{1\} & \subseteq & \langle 1\rangle & \xleftarrow[\mathrm{id}_{\langle 1\rangle}]{} & \langle 1\rangle
\end{array}$$

in $\mathcal{A}\mathrm{ss}$. The image of this diagram under $\mathcal{L}$ is then

$$\{[1]\}_{i\in\langle n\rangle^\circ} \xleftarrow{s} \{[n]\} \xrightarrow{t} \{[1]\}.$$

The morphism s is specified by $O(\bar{s})$, where

$$\bar{s} : \langle n\rangle^\circ \xrightarrow{\cong} \bigoplus_{i\in\langle n\rangle^\circ} [0]$$

is the unique isomorphism of linearly ordered sets, so s is the morphism

$$\bigoplus_{i\in\langle n\rangle^\circ}[1] \xrightarrow{\bigoplus_{i\in\langle n\rangle^\circ}\{i-1,i\}} [n]$$

in Δ. Under the Kan extension and localization of Section 3.2.3, this morphism is therefore sent to the morphism

$$X_n \xrightarrow{\{0,1\},\ldots,\{n-1,n\}} X_1^n$$

as desired. The identification of the image of t proceeds similarly.

To show (2), we now note that the span to which β_n is sent corresponds to the pair of morphisms in Θ represented by the diagram

$$\begin{array}{ccccc}
\langle n+1\rangle^\circ & \subseteq & \langle n+1\rangle & \xleftarrow{\mathrm{id}_{\langle n+1\rangle}} & \langle n+1\rangle \\
 & & \beta_{n+1}\Big\downarrow & & \Big\uparrow \mathrm{id}_{\langle n+1\rangle} \\
\{1\} & \subseteq & \diamond & \xleftarrow{\beta_{no+1}} & \langle n+1\rangle \\
 & & \mathrm{id}_{\langle 1\rangle}\Big\uparrow & & \Big\downarrow \beta_{no+1} \\
\{1\} & \subseteq & \diamond & \xleftarrow[\mathrm{id}_\diamond]{} & \diamond
\end{array}$$

in $\mathcal{A}ss_{\mathrm{CY}}$. The bottom object must be sent to a terminal object in $\mathcal{C}$, meaning that the bottom morphism is specified up to contractible choice. The image of the top morphism under $\mathcal{L}$ is

$$\{[1]\}_{i\in\langle n+1\rangle^\circ} \xleftarrow{\ s\ } \{[n]\},$$

where the morphism s is given by the composite of the canonical morphisms.

$$K\left(\bigoplus_{i\in\langle n+1\rangle} O([0])\right) \to C\left(\bigstar_{i\in\langle n+1\rangle^\circ} O([0])\right) \to C\left(O\left(\bigoplus_{i\in\langle n+1\rangle^\circ}[0]\right)\right)$$

which can be rewritten as

$$K\left(\bigoplus_{i\in\langle n+1\rangle}[1]\right) \xrightarrow{\{0,1\},\ldots,\{n-1,n\},\{n,n+1\}} C([n+1]) \xrightarrow{\ =\ } [n].$$

under the identification $n+1\sim 0$ in the definition of $C([n+1])$, the inclusion $\{n,n+1\}:[1]\to[n+1]$ becomes $\{n,0\}:[1]\to[n]$. Consequently, under the Kan extension and localization of Section 3.2.3, this is sent to the morphism

$$X_n \xrightarrow{\{0,1\},\dots,\{n-1,n\},\{n,0\}} X_1^n$$

in $\mathcal{C}$, as desired. □

Lemma 3.3.3. *Let X be a 2-Segal cyclic object in $\mathcal{C}$. Denote by B the cobordism defined by the unit disk with two embedded outgoing boundary intervals. Then $Z_X(B)$ is equivalent to the morphism*

$$* \longleftarrow X_1 \xrightarrow{\{0,1\},\{1,0\}} X_1^2$$

of $\mathrm{Span}(\mathcal{C})$.

Proof. The morphism $Z_X(B)$ must display the non-degeneracy of $C_X(\beta_2)$. By [31, Lem. 4.6.1.10], such a morphism must be unique up to contractible choice. Finally, the proof of Lemma 3.2.8 shows that the above morphism displays the non-degeneracy of $C_X(\beta_2)$. □

Remark 3.3.4. Note that, for an oriented bordism B given by a disk with either

1. n incoming boundary intervals and one outgoing boundary interval,
2. n incoming boundary intervals and no outgoing boundary intervals, or
3. 2 outgoing boundary intervals and no incoming boundary intervals

and Γ a cyclically structured spanning graph for B, Proposition 3.3.2 and Lemma 3.3.3 identify $Z_X(B)$ with the state sum

$$X_1^{\mathrm{In}(\Gamma)} \leftarrow X(\Gamma) \rightarrow X_1^{\mathrm{Out}(\Gamma)}$$

of Construction 1.3.36 up to equivalence. ▮

Proposition 3.3.5. *Let Γ_1 and Γ_2 be Λ-structured cobordisms with boundary bijections $f_1 : \{1,\dots,n\} \to \mathrm{In}(\Gamma_1)$, $g_1 : \{1,\dots,\ell\} \to \mathrm{Out}(\Gamma_1)$, $f_2 : \{1,\dots,m\} \to \mathrm{In}(\Gamma_2)$, and $g_2 : \{1,\dots,n\} \to \mathrm{Out}(\Gamma_2)$. Denote by $\Gamma_1 \circ \Gamma_2$*

their concatenation. Let $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object. Then the diagram of state sums

$$\begin{array}{ccccccc} & & & X(\Gamma_1 \circ \Gamma_2) & & & \\ & & \swarrow & & \searrow & & \\ & X(\Gamma_2) & & & & X(\Gamma_1) & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X_1^{\mathrm{In}(\Gamma_2)} & & & X_1^{\mathrm{Out}(\Gamma_2)} & & & X_1^{\mathrm{Out}(\Gamma_1)} \end{array}$$

is pullback.

Proof. We need only note that

$$N(I(\Gamma_1 \circ \Gamma_2)) \cong N(I(\Gamma_1)) \coprod_{\mathrm{Out}(\Gamma_2)} N(I(\Gamma_2))$$

whence the proposition follows from (the dual of) [32, Prop. 4.4.2.2]. □

Proposition 3.3.6. *Let $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$ be an oriented cobordism, let Γ be a cyclically structured spanning graph for $(S, B_{\mathrm{in}}, B_{\mathrm{out}})$, and let $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$ be a 2-Segal object. Then $Z_X(S, B_{\mathrm{in}}, B_{\mathrm{out}})$ is equivalent to the state sum*

$$X_1^{\mathrm{In}(\Gamma)} \leftarrow X(\Gamma) \rightarrow X_1^{\mathrm{Out}}$$

of X over Γ.

Proof. By Construction 2.3.5, every oriented cobordism can be written, up to equivalence, as a composition of the cobordisms from Remark 3.3.4. By Proposition 3.3.2 and Lemma 3.3.3, the proposition holds for these cobordisms. However, by Proposition 3.3.5, the state sum of a concatenation of graphs is equivalent to the composition of the state sums in $\mathrm{Span}(\mathcal{C})$, completing the proof. □

Remark 3.3.7. A number of interesting examples of the state sum of cyclic objects over cyclically structured graphs have appeared in the literature. See, e.g. [16] and [14, Sec. V.3]. By Proposition 3.3.6, these invariants arise from ∞-categorical topological field theories. ▮

Example 3.3.8. Per [16], the Waldhausen S-construction gives rise to many cyclic 2-Segal spaces. An interesting special case is discussed in [16, 13, 14], where versions of topological Fukaya categories are constructed as invariants $X(S, M)$ associated to 2-Segal objects arising from the Waldhausen S-construction.

3.3.2 The Čech nerve

Definition 3.3.9. Let $\Delta_+^{\leq}$ denote the full subcategory of Δ_+ on the objects $\emptyset$ and $[0]$, and denote by j the inclusion

$$j : \Delta_+^{\leq} \hookrightarrow \Delta_+.$$

Denoting by $\mathcal{S}$ the ∞-category of spaces, we get a Kan extension functor

$$j_* : \mathrm{Fun}\left(N(\Delta_+^{\leq})^{\mathrm{op}}, \mathcal{S}\right) \to \mathrm{Fun}\left(N(\Delta_+)^{\mathrm{op}}, \mathcal{S}\right).$$

Given $f : X \to Y$ in $\mathcal{S}$, considered as an object in $\mathrm{Fun}\left(N(\Delta_+^{\leq})^{\mathrm{op}}, \mathcal{S}\right)$ the *Čech nerve of f* is the simplicial space $j_*(f)$. ∎

Remark 3.3.10. Per [32, Cor. 6.1.3.20], a simplicial object $U : N(\Delta^{\mathrm{op}}) \to \mathcal{S}$ in the ∞-category of spaces is a Čech nerve if and only if it satisfies the 1-Segal conditions, i.e., for every n and every decomposition $[n] = S \cup S'$ into ordered sets with $S \cap S' = \{s\}$, the induced diagram

$$\begin{array}{ccc} U([n]) & \longrightarrow & U(S) \\ \downarrow & & \downarrow \\ U(S') & \longrightarrow & U(\{s\}) \end{array}$$

is pullback in $\mathcal{S}$. ∎

Construction 3.3.11. Consider the colimit of the constant diagram

$$N(\Delta^{\mathrm{op}}) \to \mathcal{S}$$

whose value is a terminal object $*$. Since Δ^{op} is contractible, we then have

$$\underset{N(\Delta^{\mathrm{op}})}{\mathrm{colim}} * \simeq |N(\Delta^{\mathrm{op}})| \simeq *.$$

So that we can take our colimit diagram

$$f : N(\Delta^{\mathrm{op}})^{\triangleright} \to \mathcal{S}$$

to be the constant diagram on $*$. Consequently, we can identify $N(\Delta_+)^{\mathrm{op}}$ with the relative nerve $N_f((\Delta^{\mathrm{op}})^{\triangleright})$.

Analogously, given a crossed simplicial group $\Delta\mathfrak{G}$, We can form a colimit diagram

$$f : N(\Delta\mathfrak{G})^{\triangleright} \to \mathcal{S}$$

for the constant functor with value $*$. We will call the relative nerve $N_f(\Delta\mathfrak{G}^{\triangleright})$ the *augmented crossed simplicial group* $\Delta\mathfrak{G}_+$. Note that the fiber of $\Delta\mathfrak{G}_+$ over the cone point of $N(\Delta\mathfrak{G}^{\triangleright})$ is the realization $|\Delta\mathfrak{G}|$. In particular, the *augmented cyclic category* Λ_+ has BS^1 as the fiber over the cone point. We choose a model of BS^1 with a single 0-simplex, and denote by $*_{BS^1}$ the corresponding 0-simplex lying over the cone point.

The inclusion $\Delta^{\mathrm{op}} \to \Lambda$ induces an inclusion

$$N((\Delta_+^{\mathrm{op}})) \cong N(\Delta^{\mathrm{op}})^{\triangleright} \to N(\Lambda)^{\triangleright}. \tag{3.6}$$

We denote by $\Lambda_+^{\leq}$ the pullback

$$\begin{array}{ccc} \Lambda_+^{\leq} & \longrightarrow & \Lambda_+ \\ \downarrow & & \downarrow \\ N((\Delta_+^{\leq})^{\mathrm{op}}) & \longrightarrow & N(\Lambda)^{\triangleright} \end{array}$$

and note that the inclusion (3.6) induces a diagram of ∞-categories

$$\begin{array}{ccc} \Lambda_+^{\leq} & \xrightarrow{\;i\;} & \Lambda_+ \\ \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle q} \\ N(\Delta_+^{\leq})^{\mathrm{op}} & \xrightarrow{\;j\;} & N(\Delta_+)^{\mathrm{op}} \end{array} \tag{3.7}$$

commutative up to equivalence. ∎

Proposition 3.3.12. *The diagram*

$$\begin{array}{ccc} \mathrm{Fun}(\Lambda_+^{\leq}, \mathcal{S}) & \xrightarrow{\;i_*\;} & \mathrm{Fun}(\Lambda_+, \mathcal{S}) \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ \mathrm{Fun}\left(N(\Delta_+^{\leq})^{\mathrm{op}}, \mathcal{S}\right) & \xrightarrow[j_*]{} & \mathrm{Fun}\left(N(\Delta_+)^{\mathrm{op}}, \mathcal{S}\right) \end{array}$$

induced by (3.7) commutes up to equivalence.

Proof. Passing to adjoints yields

$$\begin{array}{ccc} \mathrm{Fun}(\Lambda_+^{\leq}, \mathcal{S}) & \xleftarrow{\;i^*\;} & \mathrm{Fun}(\Lambda_+, \mathcal{S}) \\ {\scriptstyle p_!}\uparrow & & \uparrow{\scriptstyle q_!} \\ \mathrm{Fun}\left(N(\Delta_+^{\leq})^{\mathrm{op}}, \mathcal{S}\right) & \xleftarrow[j^*]{} & \mathrm{Fun}\left(N(\Delta_+)^{\mathrm{op}}, \mathcal{S}\right) \end{array}$$

so it suffices to check that this diagram commutes up to equivalence.

To see this, note first that the functor $j : N(\Delta_+^{\leq})^{\mathrm{op}} \hookrightarrow N(\Delta_+)^{\mathrm{op}}$ is cofinal, since for any object c, the ∞-category $N(\Delta_+^{\leq})^{\mathrm{op}} \times N(\Delta_+)^{\mathrm{op}}_{c/}$ has a final object $c \to \emptyset$ and is thus contractible.

The left fibration $\pi : \Lambda_+ \to N(\Lambda)^{\triangleright}$ yields, for every 0-simplex in $c \in \Lambda_+$ a left fibration

$$\Lambda_{+/c} \to \Lambda_+ \times_{N(\Lambda)^{\triangleright}} N(\Lambda)^{\triangleright}_{/\pi(c)}.$$

Since $N(\Lambda_+)^{\triangleright}_{/*_{BS^1}} \cong N(\Lambda_+)^{\triangleright}$, we then find that

$$(\Lambda_+)_{/*_{BS^1}} \to \Lambda_+$$

is a left fibration. In particular, it is smooth. Since j is a cofinal inclusion of simplicial sets, it is right anodyne, and so by [32, p. 4.1.2.8], the map

$$N(\Delta_+^{\leq})^{\mathrm{op}} \times_{\Lambda_+} (\Lambda_+)_{/*_{BS^1}} \to N(\Delta_+)^{\mathrm{op}} \times_{\Lambda_+} (\Lambda_+)_{/*_{BS^1}}$$

is cofinal.

Similarly, since $(\Lambda_+)_{/[0]} \cong \Lambda_{/[0]}$, we see that

$$N(\Delta_+^{\leq})^{op} \times_{\Lambda_+} \Lambda_{+/[0]} \to N(\Delta_+)^{op} \times_{\Lambda_+} \Lambda_{+/[0]}$$

is cofinal. Hence, left Kan extensions along p and q agree for the vertices $\emptyset$ and $[0]$, showing the commutativity of the diagram above. □

Definition 3.3.13. The *cyclic Čech nerve functor* is the composite

$$\check{C}_{\mathrm{cyc}} : \mathrm{Fun}((\Lambda_+^{\leq}), \mathcal{S}) \xrightarrow{i_*} \mathrm{Fun}\left(\Lambda_+, \mathcal{S}\right) \xrightarrow{f^*} \mathrm{Fun}\left(N(\Lambda), \mathcal{S}\right),$$

where f denotes the inclusion $f : N(\Lambda) \hookrightarrow \Lambda_+$. ∎

3.3.2.1 Representables and circle bundles

Construction 3.3.14. We denote by

$$\Lambda^n := \mathrm{Hom}_{\Lambda}(-, [n])|_{\Delta^{\mathrm{op}}} \in \mathrm{Set}_{\Delta}$$

the simplicial sets obtained from the representable functors on Λ. Their realizations are $|\Lambda^n| \cong |\Delta^n| \times S^1 \simeq S^1$, and can thus be viewed as trivial circle bundles on the standard n-simplices $|\Delta^n|$. If we denote by $\iota : \Delta \to \Lambda$

the inclusion, then $N(\Delta_{/\iota[n]})$ is the barycentric subdivision of Λ^n, and all the maps

$$S^1 \simeq N(\Delta_{/\iota[n]}) \to N(\Delta_{/\iota[m]}) \simeq S^1$$

induced by $f : \iota[n] \to \iota[m]$ are homotopy equivalences. Quillen's Theorem B therefore implies that there is a homotopy cartesian square

$$\begin{array}{ccccccc} S^1 & \simeq & N(\Lambda_{/[n]}) & \longrightarrow & N(\Delta) & \simeq & ES^1 \\ & & \downarrow & & \downarrow & & \\ & & * & \longrightarrow & N(\Lambda) & \simeq & BS^1 \end{array}$$

of simplicial sets.

Let $\psi : \Lambda^{\mathrm{op}} \to \mathrm{Set}_\Delta$ be the functor resulting from applying Lurie's straightening to $N(\iota) : N(\Delta) \to N(\Lambda)$. Since the Grothendieck construction computes the colimit in spaces, the fiber sequence above thus becomes

$$\begin{array}{ccccc} S^1 & \longrightarrow & \mathrm{colim}_\Lambda\, \psi(-) & \simeq & ES^1 \\ \downarrow & & \downarrow & & \\ * & \longrightarrow & N(\Lambda) & \simeq & BS^1 \end{array}$$

∎

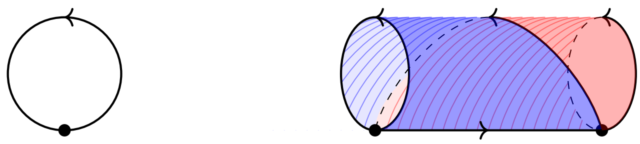


Figure 3.4: The non-degenerate simplices of the representables Λ^n for $n = 0, 1$.

Lemma 3.3.15. *The straightening ψ of the functor $N(\iota) : N(\Delta) \to N(\Lambda)$ is equivalent to the functor*

$$\Lambda \to \mathrm{Set}_\Lambda \to \mathrm{Set}_\Delta$$

given by the projection of the Yoneda embedding on Λ to Set_Δ.

Proof. Since the 'rigidification' functor $\mathfrak{C}$ is left adjoint to the homotopy coherent nerve, it preserves colimits. Thus, the category

$$\mathcal{M} := \mathfrak{C}[N(\Delta)^{\triangleright}] \coprod_{\mathfrak{C}[N(\Delta)]} \mathfrak{C}[N(\Lambda)]$$

is, in fact, the image under $\mathfrak{C}$ of

$$\underline{M} := N(\Delta)^{\triangleright} \coprod_{N(\Delta)} N(\Lambda).$$

We then see that the Hom-sets $\mathfrak{C}[\underline{M}]([n], *)$ are given by the nerves of the undercategories

$$\Delta_{\iota[n]/}$$

Which, applying the duality on Λ, can be identified with the opposites of the overcategories above. As a result, the functor given by Lurie's straightening is equivalent to the functor sending $[n] \mapsto \Lambda^n$. □

Construction 3.3.16. Let I be a category, and $G : I \to \Lambda$ a functor. Then we get a composite

$$I \xrightarrow{G} \Lambda \xrightarrow{\psi} \mathrm{Set}_{\Delta}.$$

The functoriality of the Grothendieck construction yields a homotopy pullback square

$$\begin{array}{ccc} |\operatorname{colim}_I \psi \circ G| & \longrightarrow & |\operatorname{colim}_\Lambda \psi| \\ \downarrow & & \downarrow \\ |N(I)| & \longrightarrow & |N(\Lambda)| \end{array}$$

i.e. a circle bundle over $|N(I)|$ classifying by the map $|G|$. ▮

Definition 3.3.17. Given a functor $G : I \to \Lambda$, the associated S^1-bundle

$$|\operatorname{colim}_I \psi \circ G| \to |N(I)|$$

will be denoted by $T(G)$. ▮

Proposition 3.3.18. *The functor*

$$\mathrm{Fun}\left((\Lambda_+^{\leq}), \mathcal{S}\right) \xrightarrow{\check{C}_{\mathrm{cyc}}} \mathrm{Fun}\left(N(\Lambda), \mathcal{S}\right) \xrightarrow{\mathrm{ev}_n} \mathcal{S}$$

that sends g to $\check{C}_{\mathrm{cyc}}(g)_n$ is representable.

Proof. The pullback $(f^* \circ i_*)^*$ has a left adjoint

$$((f_!)^{op} \circ (i^*)^{op})$$

Therefore, the diagram

$$\begin{array}{ccccc} \mathcal{P}\left((\Lambda_{+}^{\leq})^{\mathrm{op}}\right) & \xrightarrow{i_*} & \mathcal{P}\left(\Lambda_{+}^{\mathrm{op}}\right) & \xrightarrow{f^*} & \mathcal{P}\left(N(\Lambda)^{\mathrm{op}}\right) \\ {\scriptstyle J}\downarrow & & & & \downarrow{\scriptstyle J} \\ \mathcal{P}\left(\mathcal{P}\left((\Lambda_{+}^{\leq})^{\mathrm{op}}\right)\right) & & \xrightarrow[((f_!)^{\mathrm{op}}\circ(i^*)^{\mathrm{op}})]{} & & \mathcal{P}\left(\mathcal{P}\left(N(\Lambda)^{\mathrm{op}}\right)\right) \end{array}$$

commutes up to homotopy by [32, Prop. 5.2.6.3].

Moreover, by [32, Thm. 5.2.6.5],

$$\left(J_{N(\Lambda)^{\mathrm{op}}}^{\mathrm{op}}\right)^* \circ J_{\mathcal{P}(N(\Lambda)^{\mathrm{op}})^{\mathrm{op}}} \simeq \mathbf{id}_{\mathcal{P}(N(\Lambda)^{\mathrm{op}})}.$$

This means that

$$F := \left(J_{N(\Lambda)^{\mathrm{op}}}^{\mathrm{op}}\right)^* \circ ((f_!)^{\mathrm{op}} \circ (i^*)^{\mathrm{op}})^* \circ J_{\mathcal{P}\left((\Lambda_{+}^{\leq})^{\mathrm{op}}\right)}$$

is equivalent to the Čech nerve functor.

To compute F, we consider the functor G given by the composite

$$N(\Lambda) \overset{J_{(N(\Lambda)^{\mathrm{op}})}^{\mathrm{op}}}{\to} \mathcal{P}\left(N(\Lambda)^{\mathrm{op}}\right)^{\mathrm{op}} \overset{(f_!)^{\mathrm{op}}}{\to} \mathcal{P}\left(\Lambda_{+}^{\mathrm{op}}\right)^{\mathrm{op}} \overset{(i^*)^{\mathrm{op}}}{\to} \mathcal{P}\left((\Lambda_{+}^{\leq})^{\mathrm{op}}\right)^{\mathrm{op}}$$

G^{op} sends

$$[n] \mapsto \operatorname{Sing}\operatorname{Hom}_{\Lambda}([n], -)$$

and then computes the colimit

$$\operatorname*{colim}_{N(\Lambda)_{/*_{BS^1}}} \operatorname{Sing}\operatorname{Hom}_{\Lambda}([n], -) \simeq \Lambda^n.$$

So, the composite G^{op} sends

$$[n] \mapsto \gamma^n = \begin{cases} *_{BS^1} \mapsto & \Lambda^n \\ [n] \mapsto & \operatorname{Sing}\operatorname{Hom}_{\Lambda}([n],[n]) \end{cases}.$$

Indeed, postcomposing G^{op} with the evaluation on $*_{BS^1}$, we retrieve the functor

$$N(\Lambda) \to \mathcal{S}$$

given by the realization of the Yoneda embedding.

As a result, the cyclic Čech nerve is given by

$$\hat{C}(g)_n \simeq \operatorname{Map}_{\mathcal{P}\left((\Lambda_{+}^{\leq})^{op}\right)}\left(\gamma^n, g\right).$$

In particular, it is representable. □

Proposition 3.3.19. *Given* $f \in \mathcal{P}\left((\Lambda_{+}^{\leq})^{\mathrm{op}}\right)$ *corresponding to a morphism* $X \to Y \circlearrowright S^1$ *and a functor*

$$g : N(I) \to N(\Lambda),$$

the limit of $\check{C}_{\mathrm{cyc}}(f) \circ g$ *over* I *is equivalent to the space of* S^1*-equivariant* Y*-local systems on* $T(f)$ *equipped with reduction of the structure group to* X *over the* 0*-simplices of* $T(g)$.

Proof. Because the Yoneda embedding commutes with limits,

$$\lim_I \check{C}_{\mathrm{cyc}}(f) \circ g \simeq \mathrm{Map}_{\mathcal{P}\left((\Lambda_{+}^{\leq})^{op}\right)}\left(\operatorname*{colim}_I \gamma^n, f\right).$$

But colimits in a functor category can be computed objectwise. Considering the evaluation maps for $*_{BS^1}$ and $[0]$ respectively, we see that $\mathrm{colim}_I\, \gamma^n$ assigns $T(g)$ to $*_{BS^1}$ and the set of 0-simplices to $[0]$. □

3.3.2.2 The circle bundle of a ribbon graph

Construction 3.3.20. Let (S, M) be a stable, oriented marked surface, and let $\mathcal{T}$ be a triangulation of S with respect to the marked points. The dual graph Γ to $\mathcal{T}$ is a spanning graph for S, and so comes equipped with a canonical cyclic structure

$$I_\Gamma : I(\Gamma) \to \mathrm{Cyc}$$

where Cyc denotes the category of Λ-structured sets.

We now aim to define a category $J(\Gamma)$ with the properties that

1. There is a homotopy equivalence $|N(J(\Gamma))| \to S$,
2. $I(\Gamma)$ is a full subcategory of $J(\Gamma)$, and
3. The embedding $|N(I(\Gamma))| \to S$ factors through $|N(J(\Gamma))|$.

Let $\mathrm{Bary}(\mathcal{T})$ be the triangulation of S given by barycentric subdivision of $\mathcal{T}$. The Δ-complex $\mathrm{Bary}(\mathcal{T})$ is an ordered Δ-complex (in the terminology of [24, App. A]) in a canonical way, induced by taking the orders given by the poset of non-degenerate simplices. Since $\mathrm{Bary}(\mathcal{T})$ is a 2-dimensional Δ-complex, this implies that it defines a 2-truncated simplicial set, and hence a 2-skeletal simplicial set, which we will denote by $\mathcal{T}_B$. Since $\mathcal{T}_B$ is 2-skeletal, we have, in particular, that it is at least 3-coskeletal (cf. eg. [26, Thm. 3.19]). ▮

Proposition 3.3.21. *For $\mathcal{T}$ a triangulation of a stable, oriented marked surface (S, M), the simplicial set $\mathcal{T}_B$ is the nerve of a* 1*-category.*

Proof. Note that, since (S, M) is stable, $\mathrm{Bary}(\mathcal{T})$ has the property that every non-degenerate 2-simplex is completely determined by its 0-simplices (and every degerate 2-simplex σ is completely determined by its 0-simplices, σ_{01}, and σ_{12}). This implies that $\mathcal{T}_B$ has unique inner horn fillers in dimension 2. Moreover, any 1-simplex from x to x is degenerate. We need to check the existence and uniqueness of horn fillers in degree $2 < k \leq 4$. We proceed by cases:

1. Let $\sigma : \Lambda^3_i \to \mathcal{T}_B$ be a horn and assume all 2-simplices of σ are non-degenerate. Then $\sigma(0), \sigma(1)$, $\sigma(2)$, and $(\sigma(3))$ must correspond to distinct 0-simplices of $\mathcal{T}_B$. However, if $f : x \to y$ is a 1-simplex of $\mathcal{T}_B$, then, for the corresponding simplices of $\mathcal{T}$ we have $\dim(x) > \dim(y)$. Since $\mathcal{T}$ is a 2-dimensional complex, this means that all 3-horns contain at least one degenerate 2-simplex.

2. Let $\sigma : \Lambda^3_1 \to \mathcal{T}_B$ be a horn with at least one non-degenerate 2-face τ. We proceed by cases:

   a) $\tau = \sigma_{012}$. There are three possibilities to consider. If σ_{03} is degenerate, then $\sigma_3 = \sigma_0$, and hence $\sigma_0 = \sigma_1$, contradicting the non-degeneracy of τ. If σ_{13} is degenerate, then $\sigma_3 = \sigma_1$, If σ_{23} is degenerate, then $\sigma_{012} = \sigma_{013}$, and the horn can be uniquely filled by the 3-simplex $s_2(\sigma_{012})$.

   b) $\tau = \sigma_{013}$. There are again three possibilities. If σ_{02} is degenerate then $\sigma_0 = \sigma_2 = \sigma_1$ contradicting the non-degeneracy of σ_{013}. If σ_{12} is degenerate, then the horn can be uniquely filled by the 3-simplex $s_1(\sigma_{012})$. If σ_{23} is degenerate, then $\sigma_{012} = \sigma_{013}$ and the horn can be uniquely filled by the 3-simplex $s_2(\sigma_{012})$.

   c) $\tau = \sigma_{123}$. There are once more three possibilities. If σ_{03} is degenerate, then $\sigma_0 = \sigma_3 = \sigma_1$, contradicting the non-degeneracy of τ. If σ_{01} is degenerate, then the horn is uniquely filled by $s_0(\sigma_{123})$. If σ_{02} is degenerate, then $\sigma_0 = \sigma_1 = \sigma_2$, contradicting the non-degeneracy of τ.

3. Let $\sigma : \Lambda^3_2 \to \mathcal{T}_B$ be a horn with at least one non-degenerate 2-face τ. We proceed by cases, each with three short subcases:

a) $\tau = \sigma_{012}$. If σ_{03} is degenerate, then $\sigma_0 = \sigma_2 = \sigma_3$, contradicting the non-degeneracy of τ. If σ_{23} is degenerate, then the horn is uniquely filled by $s_2(\sigma_{012})$. If σ_{13} is degenerate, then $\sigma_1 = \sigma_2 = \sigma_3$, contradicting the non-degeneracy of τ.

b) $\tau = \sigma_{023}$. If σ_{01} is degenerate, then $\sigma_{023} = \sigma_{123}$, and $s_0(\sigma_{023})$ uniquely fills the horn. If σ_{12} is degenerate, then the horn is uniquely filled by $s_1(\sigma_{023})$. If σ_{13} is degenerate, then $\sigma_1 = \sigma_2 = \sigma_3$, contradicting the non-degeneracy of τ.

c) $\tau = \sigma_{123}$. If σ_{01} is degenerate, then $\sigma_{023} = \sigma_{123}$, so the horn is uniquely filled by $s_0(\sigma_{123})$. If σ_{02} is degenerate, then $\sigma_0 = \sigma_1 = \sigma_2$, contradicting the non-degeneracy of τ. If σ_{03} is degenerate, then $\sigma_0 = \sigma_2 = \sigma_3$, contradicting the non-degeneracy of τ.

4. Any map $\partial\Delta^3 \to \mathcal{T}_B$ will be the unique extension of the corresponding Λ^3_2-horn. Therefore, $\mathcal{T}_B$ is, in fact, 2-coskeletal.

Consequently $\mathcal{T}_B$ is the nerve of its fundamental category. □

Definition 3.3.22. Given an oriented, stable marked surface (S, M) and a triangulation $\mathcal{T}$ of S with respect to M, dual to a graph Γ embedded in S, we define $J(\Gamma)$ to be the fundamental category of $\mathcal{T}_B$. ▮

Remark 3.3.23. We can give a more explicit characterization of $J(\Gamma)$. The objects are the simplices of $\mathcal{T}$, and the morphisms are the 1-simplices of $\mathcal{T}_B$. We can also, equivalently, write $\mathrm{Obj}(J(\Gamma)) = \mathrm{Obj}(I(\Gamma)) \cup M$. It is worth noting that $J(\Gamma)$ can be seen as a kind of 'non-poset' version of the category of simplices of $\mathcal{T}$, which remembers the difference between topologically distinct paths.

We also note that since $|\mathcal{T}_B| \cong |\mathrm{Bary}(\mathcal{T})| \cong |\mathcal{T}|$, the induced map $|N(J(\Gamma))| \to S$ is a homeomorphism. Moreover, $I(\Gamma)$ is, by construction, the full subcategory of $J(\Gamma)$ on $\mathrm{Obj}(J(\Gamma)) \setminus M$, the composite

$$|N(I(\Gamma))| \to |N(J(\Gamma))| \xrightarrow{\cong} S$$

is the embedding of the dual graph of $\mathcal{T}$. Therefore, $J(\Gamma)$ fulfills the conditions of Construction 3.3.20. ▮

Definition 3.3.24. Let Γ be a graph with cyclic structure given by $\tilde{I}_\Gamma : I(\Gamma) \to \mathrm{Cyc}$. An *oriented edge* $\vec{e}$ of Γ consists of an edge $e = \{h, \eta(h)\}$, together with a chosen linear order on the set e. We will write $h_{\mathrm{in}}(\vec{e})$ for the first half-edge of e under this order and h_{out} for the second.

An *ordered loop* is an ordered sequence $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_k)$ of oriented edges of Γ satisfying the following conditions.

1. For every $1 \leq i < k$, $h_{\text{out}}(\vec{e}_i)$ and $h_{\text{in}}(\vec{e}_{i+1})$ are attached to the same vertex v_i.

2. Let $\tau_v : (H(v), \mathcal{O}_v) \to (H(v), \mathcal{O}_v)$ be the cyclic shift. Then, for every $1 \leq i < k$ $\tau_{v_i}(h_{\text{out}}(\vec{e}_i)) = h_{\text{in}}(\vec{e}_{i+1})$.

3. $\vec{e}_1 = \vec{e}_k$.

4. For all $1 < i < k$, $\vec{e}_i \neq \vec{e}_1$.

Given an ordered loop $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_k)$ in Γ, we can form new ordered loops $(\vec{e}_2, \vec{e}_3, \ldots, \vec{e}_k, \vec{e}_2)$ and $(\vec{e}_{k-1}, \vec{e}_1, \ldots, \vec{e}_{k-1})$. These two procedures are inverse to one another, and define an equivalence relation on the set of ordered loops in Γ. We will call an equivalence class of ordered loops under this relation a *loop*. We will denote the set of loops of Γ by $L(\Gamma)$. ∎

Remark 3.3.25. Equivalently, we can view a loop as a cyclically ordered set of distinct ordered edges $\{\vec{e}_i\}$ satisfying the properties 1. and 2. ∎

Construction 3.3.26. Let (S, M) be an oriented marked surface, $\mathcal{T}$ a triangulation of S with respect to M, and Γ the dual graph of $\mathcal{T}$. Given a point $m \in M$, we can define a loop ℓ_m in Γ as follows. (See Fig. 3.5 for a schematic representation.)

Let W_m be the collection of 2-simplices of $\text{Bary}(\mathcal{T})$ which have m as a vertex. Such 2-simplices will have ordered set of vertices (σ, e, m), where σ is a 2-simplex of $\mathcal{T}$ and e is a 1-simplex of $\mathcal{T}$ incident to m. We will denote elements of W_m as pairs (σ, p), leaving the m implicit. Note that, by the definition of the dual graph, each e with $(\sigma, e) \in W_m$ corresponds to an edge in Γ (which, in an abuse of notation, we will also denote by e). Moreover, the pair (σ, e) specifies a half-edge $h_{(\sigma,e)}$ of the edge e.

The orientation on S induces a cyclic order on the elements of W_m (or, more precisely, their germs at m). Taking the opposite of this cyclic order, we get a cyclically ordered set of half-edges of Γ $(h_1, \ldots, h_n)$. These satisfy two conditions:

1. If h_i and h_{i+1} are connected to the same vertex σ of $\text{Bary}(\mathcal{T})$ corresponding to a 2-simplex in $\mathcal{T}$, then $\tau_\sigma(h_i) = h_{i+1}$.

2. If h_i and h_{i+1} are connected to the same vertex e of $\text{Bary}(\mathcal{T})$ corresponding to a 1-simplex in $\mathcal{T}$, then $\eta(h_i) = h_{i+1}$.

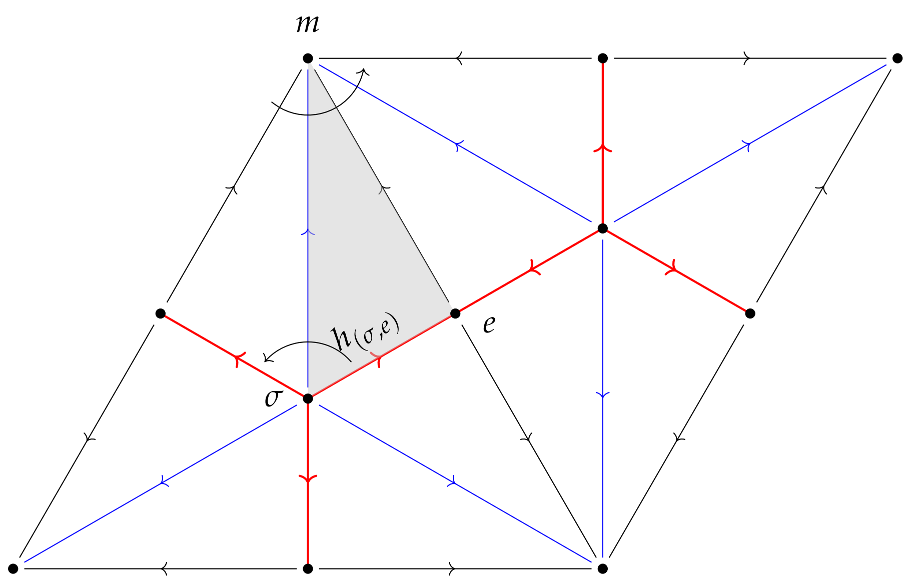


Figure 3.5: Two triangles of a triangulation $\mathcal{T}$ (black), the dual graph of $\mathcal{T}$ (red), and a triangle in W_m (shaded). The cyclic orders induced by the blackbord orientation are marked at the vertices σ and m.

Without loss of generality, we may relabel so that $\eta(h_1) = h_2$. Grouping the half-edges into their corresponding edges then yields a loop $\ell_m := ((h_1, h_2), (h_3, h_4), \dots (h_{n-1}, h_n), (h_1, h_2))$. ▮

Proposition 3.3.27. *Let (S, M) be an oriented, stable marked surface, $\mathcal{T}$ be a triangulation of S with respect to M, and Γ the dual graph to $\mathcal{T}$. Then the map*

$$M \to L(\Gamma), \quad m \mapsto \ell_m$$

is a bijection.

Proof. We construct an inverse map. Let $\ell \in L(\Gamma)$. Note that ℓ defines a continuous map $\gamma_\ell : S^1 \to S$ which is locally an embedding. At each edge e in the loop ℓ, there are two non-degenerate 1-simplices σ_1 and σ_2 of $\mathcal{T}_B$ starting at the vertex corresponding to e, each of which connect to a vertex in M. Denote by $\sigma_i'(e)$ the germ of σ_i at e. Then there is precisely one σ_i such that $(\gamma_\ell'(e), \sigma_i'(e))$ is an oriented basis of $T_e S$. We then associate to the loop ℓ the endpoint m_ℓ of σ_i. It is immediate that this is inverse to the map $m \mapsto \ell_m$. □

Notation 3.3.28. Given a loop ℓ_m and an oriented edge $\vec{e}$ in ℓ_m, we denote the canonical 1-simplex σ from e to m constructed in the proof of Proposition 3.3.27 by the symbol $\sigma_m^{\vec{e}}$. Note that this does, in fact depend on $\vec{e}$ and not just on e. ∎

Proposition 3.3.29. *Let (S, M) be an oriented, stable marked surface, let $\mathcal{T}$ be a triangulation of S with respect to M, and let $\tilde{I}_\Gamma : I(\Gamma) \to \mathrm{Cyc}$ be the induced cyclic structure on the dual graph Γ of $\mathcal{T}$. Then there is a canonical functor*

$$\tilde{J}_\Gamma : J(\Gamma) \to \mathrm{Cyc}$$

such that

1. *On the objects $m \in M$, $\tilde{J}(m) = \epsilon_\Lambda[1]$.*

2. *The restriction of $\tilde{J}_\Gamma$ to $I(\Gamma)$ is $\tilde{I}_\Gamma$.*

Proof. We extend the cyclic structure $\tilde{I}_\Gamma$ to the marked points. Let $m \in M$. We can represent the Cyc-structure on the loop ℓ_m as a diagram

$$\tilde{I}_\Gamma(v_1) \overset{\phi_1}{\to} \tilde{I}_\Gamma(e_1) \overset{\psi_1}{\leftarrow} \tilde{I}_\Gamma(v_2) \overset{\phi_2}{\to} \tilde{I}_\Gamma(e_2) \cdots \tilde{I}_\Gamma(e_{k-1}) \overset{\psi_{k-1}}{\leftarrow} \tilde{I}_\Gamma(v_k) = \tilde{I}_\Gamma(v_1)$$

in Cyc. Here, the e_i are the edges comprising ℓ_m, written in a linear order compatible with the cyclic order defined by ℓ_m, and v_i is the vertex at which e_{i-1} and e_i intersect in ℓ. Note that the e_i are not necessarily distinct as *un*oriented edges, though they are as oriented edges.

To extend $\tilde{I}_\Gamma$ to the desired functor $\tilde{J}_\Gamma$, amounts to choosing, for every $m \in M$, an extension of this diagram to a commutative diagram

$$\begin{array}{ccccccccccccc} \tilde{I}_\Gamma(v_1) & \overset{\phi_1}{\to} & \tilde{I}_\Gamma(e_1) & \overset{\psi_1}{\leftarrow} & \tilde{I}_\Gamma(v_2) & \overset{\phi_2}{\to} & \tilde{I}_\Gamma(e_2) & \cdots & \tilde{I}_\Gamma(e_{k-1}) & \overset{\psi_{k-1}}{\leftarrow} & \tilde{I}_\Gamma(v_k) & = & \tilde{I}_\Gamma(v_1) \\ \downarrow{\xi_1} & & \downarrow{\zeta_1} & & \downarrow{\xi_2} & & \downarrow{\zeta_2} & & \downarrow{\zeta_{k-1}} & & \downarrow{\xi_k} & & \downarrow{\xi_1} \\ [0] & = & [0] & = & [0] & = & [0] & \cdots & [0] & = & [0] & = & [0] \end{array}$$

Since the cyclic structure on each $\tilde{I}_\Gamma(v_i)$ and each $\tilde{I}_\Gamma(e_i)$ is defined by the oriented tangent circle at v_i (resp. e_i) in S, we can think of the ξ_i and ζ_i (via cyclic duality) as picking an interstice in this structured marked circle. However, for every oriented edge $\vec{e}$ in ℓ_m, $\sigma_m^{\vec{e}}$ defines a germ in the tangent circle to e, and thus provides a canonical choice or interstice. This provides us with the maps ζ_i, and therefore, by precomposing with

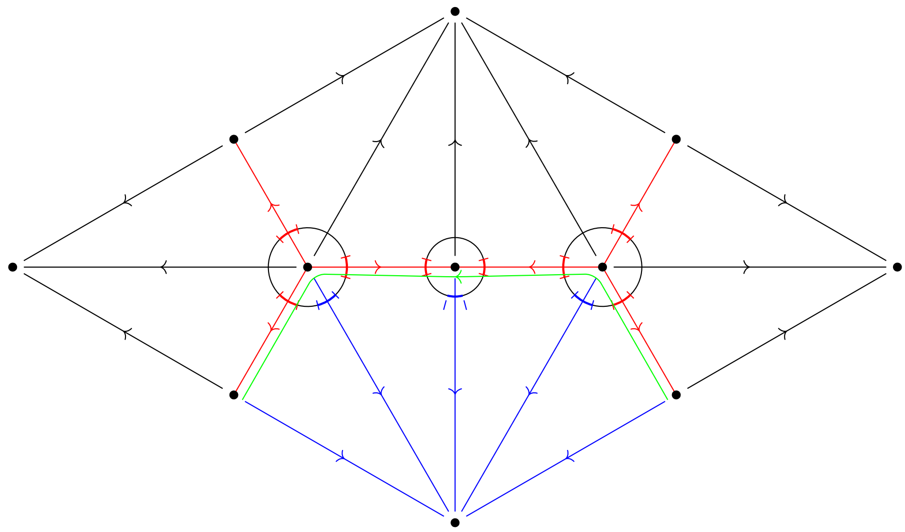


Figure 3.6: The barycentric subdivision of two simplices in a triangulation $\mathcal{T}$ which meet at a marked point m , represented by a loop ℓ_m (green). The dual graph in marked in red, as are the marked intervals in the tangent circles at the vertices and edges of ℓ_m. The 1-simplices $\sigma_m^{\vec{e}}$ and $\eta_m^{v_i}$ are pictured in blue, as are the corresponding interstices in the tangent circles.

the maps ϕ_i and ψ_i provides us pairs of candidates ξ_i, ξ_i' for the maps $\tilde{I}_\Gamma(v_i) \to [0]$.

As in the proof of Proposition 3.3.27, the loop ℓ_m defines a continuous map $\gamma_m : S^1 \to S$ from the oriented circle, which is locally an embedding. In a fashion identical to that from the proof, for every vertex v_i appearing between two oriented edges $\vec{e}_{i-1}$ and $\vec{e}_i$ in ℓ_m, the orientations on S^1 and S give rise to a canonical choice of 1-simplex $\eta_m^{v_i}$ in $\mathcal{T}_B$ from v_i to m. Working locally on simplices of the subdivision $\mathcal{T}_B$ (as pictured in Fig. 3.6) we see that each of the composites ξ_i, ξ_i' must select the interstice corresponding to the germ of $\eta_m^{v_i}$ in the tangent circle to v_i. □

Construction 3.3.30. Let (S, M) be a stable oriented marked surface, $\mathcal{T}$ a triangulation of S with respect to M, and Γ the dual graph to $\mathcal{T}$. Denote by $\mathbf{3}$ the category corresponding to the poset $\{0 < 1 < 2\}$. Define a functor

$$d : J(\Gamma) \to \mathbf{3}$$

by setting $d(x)$ to be the dimension of the simplex of $\mathcal{T}$ represented by x. Further define wide subcategories

$$J_+(\Gamma) := \mathrm{sk}_0(J(\Gamma))$$
$$J_-(\Gamma) := J(\Gamma).$$

Note that non-identity morphisms in $J_-(\Gamma)$ correspond to non-degenerate 1-simplices in $\mathcal{T}_B$. Consequently, all such morphisms lower the value of d, and we see that the triple $(J(\Gamma), J_+(\Gamma), J_-(\Gamma))$ is a Reedy category. ∎

Proposition 3.3.31. *Let (S, M) be a stable oriented marked surface, $\mathcal{T}$ a triangulation of S with respect to M, and Γ the dual graph to $\mathcal{T}$. Let $\Lambda^{(-)} : \Lambda^{\mathrm{op}} \to \mathrm{Set}_\Delta$ be the composite of the forgetful functor $\mathrm{Set}_\Lambda \to \mathrm{Set}_\Delta$ with the Yoneda embedding, and let $\tilde{F}_\Gamma : J(\Gamma) \to \Lambda$ be a factorization of $\tilde{J}_\Gamma$ through the equivalence $\Lambda \to \mathrm{Cyc}$. Then the functor*

$$Z := \Lambda^{(-)} \circ \tilde{F}_\Gamma^{\mathrm{op}} : J(\Gamma)^{\mathrm{op}} \to \mathrm{Set}_\Delta$$

is Reedy cofibrant with respect to the classical model structure on Set_Δ and the Reedy category structure $(J(\Gamma)^{\mathrm{op}}, J_-(\Gamma)^{\mathrm{op}}, J_+(\Gamma)^{\mathrm{op}})$.

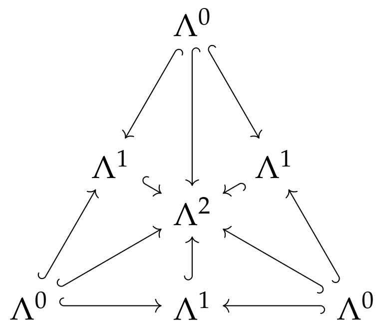
Proof. We compute the latching objects $L_x(Z)$associated to objects $x \in J(\Gamma)^{\mathrm{op}}$ in three cases. Throughout the proof we will use the word "colimit" to mean strict (1-categorical) colimit.

1. $x \in M$. In this case, the category $(J_-(\Gamma)^{\mathrm{op}})_{/x}$ consists only of the object $x \xrightarrow{\mathrm{id}} x$, and its identity morphism. Therefore, the natural map
$$L_x(Z) \to \Lambda^0$$
is an isomorphism, and thus a cofibration.

2. x corresponds to a 1-simplex of $\mathcal{T}$. This 1-simplex connects to two (not necessarily distinct) elements $m_1, m_2 \in M$. In $\mathcal{T}_B$, there are then precisely two distinct 1-simplices $\sigma_1 : x \to m_1$ and $\sigma_2 : x \to m_2$ with source x. The category $(J_-(\Gamma)^{\mathrm{op}})_{/x}$ is therefore the discrete category on the two objects σ_1 and σ_2. The latching object is therefore given by
$$L_x(Z) \cong \Lambda^0 \amalg \Lambda^0.$$
By the construction from the proof of Proposition 3.3.29, the map $L_x(Z) \to Z(x) \cong \Lambda^1$ is then given by the pair of face maps
$$\Lambda^0 \amalg \Lambda^0 \xrightarrow{(s_0^*, s_1^*)} \Lambda^1$$

which is a cofibration.

3. x corresponds to a 2-simplex σ in $\mathcal{T}$. The only objects of $J_{-}(\Gamma)^{\mathrm{op}}$ admitting morphisms to x are those corresponding to the 0- and 1-simplices of σ. Therefore, in this case, the overcategory $(J_{-}(\Gamma)^{\mathrm{op}})_{/x}$ corresponds to the barycentric subdivision of $\partial\sigma$. The canonical cone diagram



exhibits each copy of Λ^0 as the intersection in Λ^2 of the two adjacent copies of Λ^1. The latching object $L_x(Z)$ can therefore be identified with the union of the three copies of Λ^1 in Λ^2, so the map $L_x(Z) \to Z(x) \cong \Lambda^2$ is a cofibration.

□

Remark 3.3.32. Note that putting an analogous Reedy model structure on $I(\Gamma)$ does *not* yield a Reedy cofibrant diagram as in Proposition 3.3.31. Indeed, in the third case, from the proof above, the latching object for $I(\Gamma)$ can be shown to be $\Lambda^1 \amalg \Lambda^1 \amalg \Lambda^1$, which cannot admit a cofibration to Λ^2, even on the level of 0-simplices. See Fig. 3.7 for a pictorial depiction of the diagram in this case. ∎

Corollary 3.3.33. *Using the notation of Proposition 3.3.31, the homotopy colimit of Z over $J(\Gamma)^{\mathrm{op}}$ is homotopy equivalent to the strict colimit of Z over $J(\Gamma)^{\mathrm{op}}$.*

Definition 3.3.34. Let $\mathcal{T}$ be a triangulation of an oriented marked surface (S, M). The orientation of S induces an orientation on each 2-simplex of $\mathcal{T}$. For each 2-simplex, we can then choose a labeling of its one-simplices by $\{0, 1, 2\}$ compatible with the induced orientation. For such a labeling L of $\mathcal{T}$, we call the pair $(\mathcal{T}, L)$ an *oriented triangulation*. A oriented

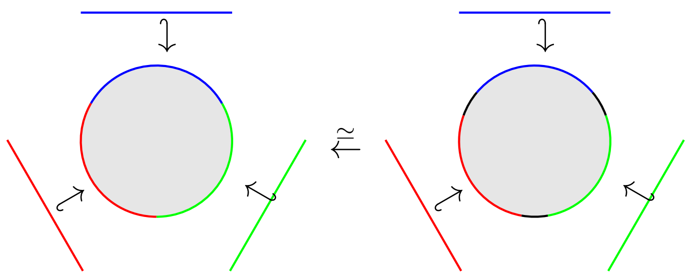


Figure 3.7: A heuristic depiction of the latching object for cyclic representables associated to $I(\Gamma)$, as well as a possible cofibrant replacement.

triangulation $(\mathcal{T}, L)$ yields a factorization of $\tilde{J}_\Gamma$ through the equivalence $\Lambda \simeq \text{Cyc}$. ∎

Notation 3.3.35. In the presence of a factorization

$$\begin{array}{ccc} J(\Gamma) & \xrightarrow{\tilde{J}_\Gamma} & \text{Cyc} \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\simeq} & \\ & \Lambda & \end{array}$$

we will abuse notation and denote $T(\psi)$ by $T(\tilde{J}_\Gamma)$. ∎

Remark 3.3.36. A labeling of a 2-simplex σ induces an orientation on each of the one-simplices in $\partial\sigma$ via the face maps. Note that in our definition of an oriented triangulation of (S, M), we do *not* require that the two induced orientations on each 1-simplex coincide. ∎

Construction 3.3.37. Considering the representables Λ^n, we see that, given two labeled 2-simplices meeting along one edge e, there are two cases we might encounter in the strict colimit over $J(\Gamma)^{\text{op}}$:

1. The induced orientations on e coincide. In this case, the colimit of the resulting diagram
$$\Lambda^2 \leftarrow \Lambda^1 \rightarrow \Lambda^2$$
is given on the level of topological spaces by gluing $S^1 \times \Delta^2$ to $S^2 \times \Delta^2$ via the identity map on the boundary cylinder $S^1 \times \Delta^1$.

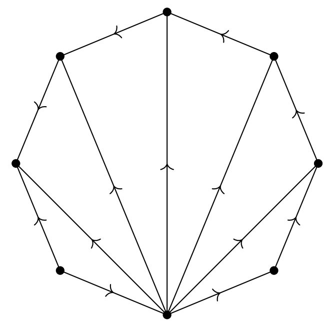
2. The induced orientations on e are opposite. In this case, the colimit of
$$\Lambda^2 \leftarrow \Lambda^1 \rightarrow \Lambda^2$$

is still given by gluing $S^1 \times \Delta^2$ to $S^2 \times \Delta^2$ via an automorphism of the cylinder. Now, however, that automorphism is given by a Dehn twist (specified by the difference between the two one-simplices in Λ^2).

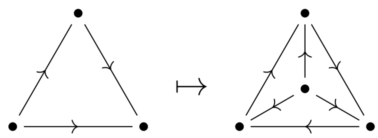
∎

Lemma 3.3.38. *For a marked surface (S, M), there is an oriented triangulation of (S, M) such that there are precisely $2g - 2 + |M|$ 1-simplices such that the two induced orientations differ.*

Proof. When $g \geq 1$ and $|M| = 1$, take the triangulation of the fundamental polygon P given by choosing a vertex of P, and drawing a line to every other vertex (and taking the boundary of P, of course) as shown below.



The labeling from the diagram gives us precisely $2g - 1$ 1-simplices with differing induced orientations (all along the edge). Inductively, we can use the rule



To flip one of the orientations previously defined, adding one conflicting orientation. □

Proposition 3.3.39. *For S an orientable surface of genus $g \neq 0$ and $|M| \geq 1$, the degree of the circle bundle $T(\tilde{J}_\Gamma)$ is $2 - 2g - |M|$.*

Proof. In consequence of Corollary 3.3.33, we the circle bundle in question is given by a strict colimit. As a result of Lemma 3.3.38, this colimit is given by cutting the trivial bundle along $2g - 2 + |M|$ arcs in S, and then gluing the resulting boundary components along a Dehn twist of the cylinder. For each such gluing, the degree of the bundle $T(\tilde{J}_\Gamma)$ changes by -1. □

Remark 3.3.40. A consequence of the proposition is that, in the presence of a holomorphic structure on S, we can associate $T(\tilde{J}_\Gamma)$ with the circle bundle associated to the logarithmic tangent bundle $T_{\log(M)}S$ of S with respect to the marked points. [16, Rmk 3.4.2] gives a related observation.

The identification of $T(\tilde{J}_\Gamma)$ with $T_{\log(M)}S$ also gives us a canonical identification of $T(\tilde{I}_\Gamma)$ with the circle bundle associated to the tangent bundle of $S \setminus M$ by including $T(\tilde{I}_\Gamma) \hookrightarrow T(\tilde{J}_\Gamma)$.

3.3.2.3 The case of a discrete group

We now specialize to a 1-categorical context. Throughout this subsection, we fix a discrete group G and a subgroup H, and apply the machinery of the previous subsections to the morphism $BH \to BG$. The ∞-categorical limits of the previous sections now manifest themselves as 2-limits of groupoids.

Lemma 3.3.41. *Specifying a strict S^1-action on BG is equivalent to specifying an element z in the center of G.*

Proof. We can view $BS^1 \simeq BB\mathbb{Z}$ as the 2-groupoid with one object, one 1-morphism, and $\mathbb{Z}$ as the group of 2-morphisms. A strict S^1-action on BG is thus a strict 2-functor $BB\mathbb{Z} \to \mathrm{Grpd}$, which is uniquely specified by an automorphism of the identity on BG, i.e. a central element. □

Construction 3.3.42. Given such a central element $z \in G$, we can now compute explicitly the cyclic Čech nerve of $BH \to BG$. The n-cells will be given by the n-fold 2-pullback

$$BH \times^{(2)}_{BG} BH \times^{(2)}_{BG} \cdots \times^{(2)}_{BG} BH.$$

Performing this computation, we find that the n-fold 2-pullback is $\mathcal{C}_k(G,H) := G^k//H^k+1$. The objects of this groupoid are sequences of composable morphisms in BG, with morphisms given by commutative diagrams

$$\begin{array}{ccccccccc}
* & \xrightarrow{g_1} & * & \xrightarrow{g_2} & \cdots & & * & \xrightarrow{g_k} & * \\
\downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & & & \downarrow{\scriptstyle h_{k-1}} & & \downarrow{\scriptstyle h_k} \\
* & \xrightarrow[g_1']{} & * & \xrightarrow[g_2']{} & \cdots & & * & \xrightarrow[g_k']{} & *
\end{array}$$

This formulation of the simplicial groupoid is useful because it makes it easy to compute the action the cyclic shift explicitly. Namely, the diagram above is sent to

$$\begin{array}{ccccccc} * & \xrightarrow{z(\prod_i g_i)^{-1}} & * & \xrightarrow{g_1} \cdots & * & \xrightarrow{g_{k-1}} & * \\ {\scriptstyle h_k}\downarrow & & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_{k-2}}\downarrow & & {\scriptstyle h_{k-1}}\downarrow \\ * & \xrightarrow[z(\prod_i g'_i)^{-1}]{} & * & \xrightarrow[g'_2]{} \cdots & * & \xrightarrow[g'_k]{} & * \end{array} \tag{3.8}$$

However, there are two other models for $\check{C}_{cyc}(BH \to BG)$ that are of interest. These are the *Hecke-Waldhausen construction* of [15]. Setting $E := G/H$, the first is

$$\mathcal{S}_n(G, E) := G\backslash\backslash E^{n+1}.$$

The second is the equivalent simplicial groupoid whose n^{th} component is

$$\mathcal{Z}_n(G, E) := H\backslash\backslash E^n.$$

∎

Lemma 3.3.43. *The simplicial groupoids $\mathcal{C}_\bullet(G, H)$, $\mathcal{S}_\bullet(G, E)$ and $\mathcal{Z}_\bullet(G, E)$ are equivalent.*

Proof. Fix a set $\{\gamma_\alpha\}_\alpha$ of coset representatives for G/H. We will write down an explicit equivalence $\mathcal{C}_\bullet(G, H) \simeq \mathcal{Z}_\bullet(G, E)$ (the other equivalence is clear).

In one direction, send

$$\begin{array}{ccccccc} * & \xrightarrow{g_1} & * & \xrightarrow{g_2} \cdots & * & \xrightarrow{g_n} & * \\ {\scriptstyle h_0}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_{n-1}}\downarrow & & {\scriptstyle h_n}\downarrow \\ * & \xrightarrow[g'_1]{} & * & \xrightarrow[g'_2]{} \cdots & * & \xrightarrow[g'_n]{} & * \end{array} \longmapsto \begin{array}{cccc} [g_n \cdots g_1] & [g_n \cdots g_2] & \cdots & [g_n] \\ {\scriptstyle h_n}\downarrow & {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_n} \\ [g'_n \cdots g'_1] & [g'_n \cdots g'_2] & \cdots & [g'_n] \end{array}$$

In the other,

$$\begin{array}{cccc} [\gamma_1] & [\gamma_2] & \cdots & [\gamma_n] \\ {\scriptstyle h}\downarrow & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ [\gamma'_1] & [\gamma'_2] & \cdots & [\gamma'_n] \end{array} \longmapsto \begin{array}{ccccccc} * & \xrightarrow{\gamma_2^{-1}\gamma_1} & * & \xrightarrow{\gamma_3^{-1}\gamma_2} \cdots & * & \xrightarrow{\gamma_n} & * \\ {\scriptstyle h_0}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_{n-1}}\downarrow & & {\scriptstyle h}\downarrow \\ * & \xrightarrow[(\gamma'_2)^{-1}\gamma'_1]{} & * & \xrightarrow[(\gamma'_3)^{-1}\gamma'_2]{} \cdots & * & \xrightarrow[\gamma'_n]{} & * \end{array}$$

It is easily seen that this is an equivalence. □

Remark 3.3.44. We can transport the cyclic shift map (3.8) via our equivalences to obtain an explicit cyclic shift map on the Hecke-Waldhausen construction. In the case of $\mathcal{S}_\bullet(G, E)$, this is

$$\begin{array}{ccccc} [\gamma_0] & [\gamma_1] & \cdots & [\gamma_n] \\ \downarrow g & \downarrow g & & \downarrow g \\ [\gamma'_0] & [\gamma'_1] & \cdots & [\gamma'_n] \end{array} \longmapsto \begin{array}{ccccc} [z\gamma_n] & [\gamma_0] & \cdots & [\gamma_{n-1}] \\ \downarrow g & \downarrow g & & \downarrow g \\ [z\gamma'_n] & [\gamma'_0] & \cdots & [\gamma'_{n-1}] \end{array}$$

We now wish to expand on the characterization in proposition 3.3.19 in this special case.

Proposition 3.3.45. *Let $f : BH \to BG$ and $z \in G$ be as above. Given an oriented marked surface (S, M) corresponding to a cyclic graph $\tilde{I}_\Gamma : I(\Gamma) \to \Lambda$, the limit*

$$\lim_{I(\Gamma)} \check{C}_{cyc}(f) \circ \tilde{I}_\Gamma$$

computes the groupoid of G-local systems on $C(TS)$ with monodromy z around the tangent circles, and with reduction of structure group over the marked points.

Proof. Proposition 3.3.19 gives us that the limit above is equivalent to the groupoid whose points are diagrams

$$\begin{array}{ccc} T(\tilde{I}_\Gamma) & \longrightarrow & BG \\ \uparrow & & \uparrow \\ M & \longrightarrow & BH \end{array}$$

where the top morphism is S^1-equivariant, together with a natural isomorphism making the diagram commute.

Since BG and BH are 1-truncated objects of $\mathcal{S}$, the datum of a morphism $T(\tilde{I}_\Gamma) \to BG$ is, in fact, equivalent to a functor

$$\Pi_1(T(I_\Gamma)) \to BG,$$

that is, a G-local system.

From the S^1-equivariance of this functor, we see that we get a commuting square of spaces

$$\begin{array}{ccc} T(I_\Gamma) & \longrightarrow & BG \\ \downarrow & & \downarrow \\ |I(\Gamma)| & \longrightarrow & BG/z \end{array}$$

by passing to S^1-quotients. Passing to homotopy fibers (and trucating appropriately) yields a map

$$\pi_1(S^1) \to B\langle z \rangle,$$

which shows that the monodromy around a fiber in the circle bundle $T(I_\Gamma)$ is z. □

Remark 3.3.46. The kinds of local systems which appear in our identification of the surface invariant Proposition 3.3.45 have been studied from the perspective of algebraic geometry by Fock and Goncharov in [20]. Called *twisted local systems* in [20], their moduli spaces provide generalizations of the classical Teichmüller spaces. Indeed, in the specific case where $G = \mathrm{SL}(2, \mathbb{Z})$, H is the subgroup of upper unitriangular matrices, and $z = -1$, the moduli spaces of [20] retrieve the decorated Teichmuller space of [34].

These moduli spaces also encode combinatorial data related to cluster algebras. This combinatorial data manifests itself in *flips* — analogues to our structured graph contractions. ∎

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Bibliography

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