Moduli spaces of K3 surfaces and cubic fourfolds

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
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aus
Amsterdam

Bonn, 2019

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 16.10.2019
Erscheinungsjahr: 2019

Abstract

This thesis is concerned with the Hodge-theoretic relation between polarized K3 surfaces of degree d and special cubic fourfolds of discriminant d, as introduced by Hassett.

For half of the d, K3 surfaces associated to cubic fourfolds come naturally in pairs. As our first main result, we prove that if (S, L) and (S^τ, L^τ) form such a pair of polarized K3 surfaces, then S^τ is isomorphic to the moduli space of stable coherent sheaves on S with Mukai vector $(3, L, d/6)$. We also explain for which d the Hilbert schemes $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S^\tau)$ are birational.

Next, we study the more general concept of associated twisted K3 surfaces. Our main contribution here is the construction of moduli spaces of polarized twisted K3 surfaces of fixed degree and order. We strengthen a theorem of Huybrechts about the existence of associated twisted K3 surfaces. We show that like in the untwisted situation, half of the time, associated twisted K3 surfaces come in pairs, and we explain how the elements of such a pair are related to each other.

Contents

List of symbols

In Chapter 2, we sometimes use a different notation.

Symbol	Chapter 2	Meaning
S		K3 surface
$\rho(S)$		Picard number of S
(S, L)		polarized K3 surface
M_d	$\mathcal{M}_d$	moduli space of polarized K3 surfaces of degree d
$\mathrm{Hilb}^n(S)$		Hilbert scheme of n points on S
$M(v)$	$M_S(v)$	moduli space of sheaves on S with Mukai vector v
X		cubic fourfold
$\mathrm{H}^4(X, \mathbb{Z})^-$		middle cohomology of X with intersection product changed by a sign
$\mathrm{H}^{2,2}(X, \mathbb{Z})$	$A(X)$	$\mathrm{H}^4(X, \mathbb{Z}) \cap \mathrm{H}^{2,2}(X)$
$\mathcal{C}$		moduli space of cubic fourfolds
$\mathcal{C}_d$		moduli space of special cubic fourfolds of discriminant d
$F(X)$		Fano variety of lines on X
Λ	Λ_{K3}	lattice isomorphic to $\mathrm{H}^2(S, \mathbb{Z})$
Λ_d		lattice isomorphic to $L^\perp \subset \mathrm{H}^2(S, \mathbb{Z})$ for polarized K3 surface (S, L) of degree d
$\widetilde{\Lambda}$	$\widetilde{\Lambda}_{\mathrm{K3}}$	lattice isomorphic to $\mathrm{H}^*(S, \mathbb{Z})$
Λ_{Muk}		lattice isomorphic to $\mathrm{H}^*(S, \mathbb{Z})$ with Mukai pairing
Γ'	Λ_{cub}	lattice isomorphic to $\mathrm{H}^4(X, \mathbb{Z})$
Γ	Λ^0_{cub}	lattice isomorphic to $h^\perp \subset \mathrm{H}^4(X, \mathbb{Z})$ for h square of hyperplane class on X
$\mathrm{Disc}\, L$		discriminant group of lattice L
$\widetilde{\mathrm{O}}(L)$		$\mathrm{Ker}(\mathrm{O}(L) \to \mathrm{O}(\mathrm{Disc}\, L))$
$\mathcal{D}(L)$		period domain of lattice L
	$\mathcal{QD}(L)$	$\mathcal{D}(L)/\widetilde{\mathrm{O}}(L)$
$\mathrm{D^b}(Y)$		bounded derived category of coherent sheaves on variety Y

Introduction

This thesis revolves around two main objects: K3 surfaces and cubic fourfolds. Both are classical topics that have been studied extensively before. These two a priori unrelated objects have remarkable similarities: for example, their Hodge numbers are essentially the same. Since Hassett's PhD thesis [Has00], it has become clear that there is a direct relation between certain K3 surfaces and cubic fourfolds. It is this relation that we are interested in.

Background

A K3 surface is a (complex or algebraic) surface S, regular and complete, with trivial canonical bundle, satisfying $\mathrm{H}^1(S, \mathcal{O}_S) = 0$. K3 surfaces show up naturally in the classification of surfaces as one of the four types of minimal algebraic surfaces of Kodaira dimension zero. The theory of K3 surfaces is extremely rich: besides for algebraic geometry, they are of interest to many other areas, such as complex, symplectic, and arithmetic geometry.

We mention two results that will be important for us and that demonstrate how well-understood K3 surfaces are. The *global Torelli theorem* tells us that a complex K3 surface S is completely determined by the Hodge structure on $\mathrm{H}^2(S, \mathbb{Z})$. The *derived Torelli theorem* says that the derived category $\mathrm{D}^\mathrm{b}(S)$ is determined by the Hodge structure on $\mathrm{H}^*(S, \mathbb{Z})$.

In this thesis, we will only consider algebraic K3 surfaces over the complex numbers. These admit a polarization, that is, a primitive ample class in $\mathrm{H}^2(S, \mathbb{Z})$, whose square we call the degree of the polarization. Polarized K3 surfaces of a fixed degree d have a 19-dimensional coarse moduli space M_d, which is an irreducible quasi-projective variety.

We will also be interested in the Hilbert scheme of length n subschemes of a K3 surface S, denoted by $\mathrm{Hilb}^n(S)$. This is a hyperkähler variety of dimension $2n$. Besides being interesting in their own right, these schemes form one of the only two series of hyperkähler varieties known, up to deformation equivalence, that exists in every possible dimension. As such, they are important test objects for results on hyperkähler varieties.

Cubic fourfolds are cubic hypersurfaces in $\mathbb{P}^5$, which we will always assume to be smooth. It is a very interesting and very difficult question which cubic fourfolds are birational to $\mathbb{P}^4$. There are examples of rational cubic fourfolds, but in contrast to lower-dimensional cubic hypersurfaces, most cubic fourfolds are not known to be rational or irrational. It is expected that the very general cubic fourfold is not rational, even though we do not know of a single example of a non-rational cubic.

The cubic fourfolds that are known to be rational are all *special*, which means that they admit additional algebraic cycles of codimension two. Special cubics form a countable union of irreducible divisors $\mathcal{C}_d$ in the 20-dimensional moduli space of cubic fourfolds. It turns out that for infinitely many of these divisors, the cubic fourfolds in them can be related to K3 surfaces.

There exist several descriptions of relations between K3 surfaces and cubic fourfolds, and between objects associated with them. The first to describe such a relation was Hassett [Has00]. He gave the following numerical condition on d:

$$(**) \qquad d \text{ is even and not divisible by 4, 9, or any odd prime } p \equiv 2 \bmod 3$$

and proved that a cubic fourfold X is in $\mathcal{C}_d$ for some d satisfying $(**)$ if and only if there exists a polarized K3 surface whose primitive middle cohomology embeds Hodge-isometrically into $\mathrm{H}^4(X, \mathbb{Z})$. We say that S is *associated* to X.

Kuznetsov [Kuz10] described a different relation, in terms of the derived category of X. He studied the subcategory

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp} \subset \mathrm{D^b}(X),$$

called the K3 or Kuznetsov component of X. It is very similar to the bounded derived category of a K3 surface: It has the same Hochschild (co)homology and its Serre functor is the shift by two. In fact, for some cubics X, there exists a K3 surface S such that $\mathcal{A}_X$ is equivalent to $\mathrm{D^b}(S)$. Kuznetsov conjectured that the cubics for which this holds should exactly be the rational ones.

Conjecture (Kuznetsov)**.** *A cubic fourfold X is rational if and only if $\mathcal{A}_X$ is equivalent to $\mathrm{D^b}(S)$ for some K3 surface S.*

It was proven in [AT14, BLM$^+$] that the two relations above are equivalent: a cubic fourfold has a Hodge associated K3 surface if and only if it has an associated K3 surface in the sense of Kuznetsov. For their proof, [AT14] introduced a Hodge structure $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ associated with the category $\mathcal{A}_X$. They reformulated Hassett's relation as follows: X is in $\mathcal{C}_d$ for d satisfying $(**)$ if and only if there exists a K3 surface S such that $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ is Hodge isometric to the full cohomology $\mathrm{H}^*(S, \mathbb{Z})$ of S.

There also exists a geometric version of $(**)$ due to Addington [Add16]. It involves $F(X)$, the Fano variety of lines on a cubic fourfold X, a four-dimensional hyperkähler variety which is deformation equivalent to the Hilbert scheme of two points on a K3 surface. Addington showed that X is in $\mathcal{C}_d$ for d satisfying $(**)$ if and only if there exists a K3 surface S such that $F(X)$ is birational to a moduli space of sheaves on S. Moreover, if we instead ask d to satisfy the stronger condition

$$(***) \qquad a^2 d = 2(n^2 + n + 1) \text{ for some } a, n \in \mathbb{Z},$$

then $F(X)$ is birational to $\mathrm{Hilb}^2(S)$ for some K3 surface S (see also [BD85]) and [Has00, Sec. 6]).

Finally, Huybrechts [Huy17] generalized the above to a relation between cubic fourfolds and *twisted* K3 surfaces: pairs (S, α) where S is a K3 surface and α is an element of the Brauer group $\mathrm{Br}(S)$ of S. The Brauer group of a smooth projective scheme is a higher-degree analogue of the Picard group. For a complex K3 surface S, we have isomorphisms

$$\mathrm{Br}(S) \cong \mathrm{H}^2(S, \mathcal{O}_S^*)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(S)}.$$

The Brauer group has many applications in more arithmetic contexts. For us, the main interest in twisted K3 surfaces comes from the fact that they can be given a generalized Calabi–Yau structure, as introduced by Hitchin [Hit03]. In particular, if (S, α) is a twisted K3 surface, one can use α to define a new Hodge structure $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$ on the full cohomology of S [Huy05]. It is this Hodge structure that Huybrechts used to define associated twisted K3 surfaces. He considered the following condition on $d' \in \mathbb{Z}$:

$$(**') \qquad d' = dr^2 \text{ where } d \text{ satisfies } (**)$$

and showed that a cubic fourfold X is in $\mathcal{C}_{d'}$ with d' satisfying $(**')$ if and only if there exists a twisted K3 surface (S, α) with a Hodge isometry

$$\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}).$$

Like for untwisted K3 surfaces, there exists a derived version of this relation due to [Huy17, BLM$^+$]. It says that a cubic fourfold has an associated twisted K3 surface if and only if for some twisted K3 surface (S, α), the category $\mathcal{A}_X$ is equivalent to the bounded derived category $\mathrm{D^b}(S, \alpha)$ of α-twisted sheaves on S.

Finally, one can prove the twisted analogue of Addington's version of $(**)$, which says that the above holds if and only if the variety $F(X)$ is birational to a moduli space of twisted sheaves on a K3 surface [Huy18].

Overview

We begin with the introductory Chapter 1, which explains the prerequisites necessary to understand the further chapters.

In Chapter 2, which is based on [Bra18], we use the relation between K3 surfaces and cubic fourfolds to define an involution on the moduli space M_d. Hassett's relation induces, for d satisfying $(**)$, a rational map $\gamma\colon \mathrm{M}_d \dashrightarrow \mathcal{C}_d$ which is birational when $d \equiv 2 \bmod 6$, and has degree two when $d \equiv 0 \bmod 6$.

In the second case, the covering involution $\tau\colon \mathrm{M}_d \to \mathrm{M}_d$ of γ is regular and does not depend on the choices made to construct γ (see Proposition 2.3.1). Our first theorem gives a geometric description of τ.

Theorem (see Thm. 2.3.2)**.** *Let $(S^\tau, L^\tau) = \tau(S, L)$. Then S^τ is isomorphic to the moduli space $M_S(v)$ of stable coherent sheaves on S with Mukai vector $v = (3, L, d/6)$.*

The K3 surfaces S and S^τ are derived equivalent. If S has Picard number one, they are not isomorphic. The proof of the result makes use of the derived Torelli theorem; apart from that, it is purely lattice-theoretic.

Next, we assume that $d \equiv 0 \bmod 6$ satisfies Addington's condition $(***)$. We can prove (see Lemma 2.4.3) that there is a choice of the map $\gamma\colon \mathrm{M}_d \dashrightarrow \mathcal{C}_d$ such that if $\gamma(S, L) = X$, then we have

$$\mathrm{Hilb}^2(S) \sim_{\mathrm{bir}} F(X) \sim_{\mathrm{bir}} \mathrm{Hilb}^2(S^\tau).$$

It is a natural question when exactly $\mathrm{Hilb}^2(S)$ and $\mathrm{Hilb}^2(S^\tau)$ are birational. The answer is given by our next result.

Theorem (see Prop. 2.4.5, Cor. 2.4.9)**.** *Let $d \equiv 0 \bmod 6$ satisfy $(**)$. Consider the following statements:*

(i) $\mathrm{Hilb}^2(S)$ *is isomorphic to* $\mathrm{Hilb}^2(S^\tau)$*;*
(ii) $\mathrm{Hilb}^2(S)$ *is birational to* $\mathrm{Hilb}^2(S^\tau)$*;*
(iii) *There exists an integral solution to* $3p^2 - (d/6)q^2 = -1$*;*
(iv) $\mathrm{Hilb}^2(S)$ *has a line bundle of self-intersection 6.*

We have implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv)*. If $\rho(S) = 1$, then these are all equivalent.*

The proof of (ii) $\Longleftrightarrow$ (iii) uses results of [MMY18] which mainly depend on the birational Torelli theorem for manifolds of $\mathrm{K3}^{[n]}$ type [Mar11, Cor. 9.9]. For the equivalence (i) $\Longleftrightarrow$ (ii) when $\rho(S) = 1$, we show that the ample cone of $\mathrm{Hilb}^2(S)$ is equal to the interior of the movable cone, using [DM17, Thm. 5.1].

As an application of this theorem, we find examples of two interesting phenomena:

1. Two Hilbert schemes of two points on K3 surfaces that are derived equivalent but not birational;
2. Two non-isomorphic K3 surfaces whose Hilbert schemes of two points are isomorphic.

In Chapter 3, we switch from usual K3 surfaces to twisted K3 surfaces. First, we construct moduli spaces of these objects. Second, we use a notion of associated twisted K3 surfaces equivalent to the one in [Huy17] to construct the analogue of Hassett's maps $\mathrm{M}_d \dashrightarrow \mathcal{C}_d$ in the twisted situation.

To be precise, the moduli space N_d^r we aim to construct parametrizes polarized twisted K3 surfaces: triples (S, L, α) of a K3 surface S with a polarization L and a Brauer class α. Here we fix $(L)^2 = d$ and $\mathrm{ord}(\alpha) = r$. However, as we show, N_d^r does not exist as a locally Noetherian scheme.

To solve this, we replace $\mathrm{Br}(S)[r]$ by $\mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$. This group has a surjective map to $\mathrm{Br}(S)[r]$, which is an isomorphism if and only if $\rho(S) = 1$.

Theorem (see Def. 3.2.1, Prop. 3.2.4)**.** *There exists a scheme* $\mathrm{M}_d[r]$ *which is a coarse moduli space for triples* (S, L, α) *where* $(S, L) \in \mathrm{M}_d$ *and* $\alpha \in \mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$*. Every connected component of* $\mathrm{M}_d[r]$ *is an irreducible quasi-projective variety with at most finite quotient singularities.*

The (finitely many) connected components of $\mathrm{M}_d[r]$ are quotients of a bounded symmetric domain. Let $\mathrm{M}_d^r \subset \mathrm{M}_d[r]$ be the subspace parametrizing those triples (S, L, α) for which α has order r. This is a union of components of $\mathrm{M}_d[r]$.

Proposition (see Prop. 3.2.5)**.** *The space* M_d^r *has at most* $r \cdot \gcd(r, d)$ *many connected components.*

For each component M_v of M_d^r, we introduce a period domain $\mathcal{D}(T_v)$ and a period map $\mathcal{P}_v$. We explain that $\mathcal{P}_v$ induces an open embedding of M_v into a quotient of $\mathcal{D}(T_v)$.

Next, we determine for which values of d and r the space M_d^r contains twisted K3 surfaces that are associated to cubic fourfolds. When $d' = dr^2$ satisfies $(**')$, Huybrechts' result ensures that there exists a, possibly different, decomposition $d' = d_0 r_0^2$ with d_0 satisfying $(**)$, such that cubics in $\mathcal{C}_{d'}$ have associated twisted K3 surfaces in $\mathrm{M}_{d_0}^{r_0}$. We can prove the following stronger result.

Theorem (see Cor. 3.3.6)**.** *A cubic fourfold* X *is in* $\mathcal{C}_{d'}$ *for some* d' *satisfying* $(**')$ *if and only if for every decomposition* $d' = dr^2$ *as in* $(**')$*,* X *has an associated polarized twisted K3 surface of degree* d *and order* r*.*

As a consequence, we can show that for some connected component M_v of M_d^r and a finite covering $\widetilde{\mathrm{M}}_v \to \mathrm{M}_v$, there exists a rational map

$$\gamma_v \colon \widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$$

like Hassett's map γ. The degree of $\widetilde{\mathrm{M}}_v \to \mathrm{M}_v$ can be bounded, see Proposition 3.4.3. We give an explicit example of the map γ_v, for $d = 2 = r$. This is a particularly interesting case, because there is a geometric description of the (rational) inverse of γ_v [Kuz10].

Like before, γ_v is birational when $d' \equiv 2 \bmod 6$ and has degree two when $d' \equiv 0 \bmod 6$. In the second case, the covering involution $\tau_v \colon \widetilde{\mathrm{M}}_v \to \widetilde{\mathrm{M}}_v$ is regular and independent of the choice of γ_v. Defining S^τ similarly as in the untwisted case, one can show that S^τ is isomorphic to a moduli space of α-*twisted* sheaves on S.

Finally, in Chapter 4, we give a short reflection on the results of this thesis, and suggest some questions for further research. We present a possible arithmetic application of moduli spaces of twisted K3 surfaces, and explain that besides being of interest for K3 surfaces, our results also motivate questions about cubic fourfolds.

Acknowledgements

There are many, many people who in some way contributed to the writing of this thesis.

I should start with my supervisor Daniel Huybrechts, who has been the best Doktorvater I could have hoped for. Thank you for always taking time for me, for many interesting discussions, for answering all my questions, for teaching me what geometric intuition really is and for showing me that it is ok to make mistakes or have naive ideas.

Many thanks go to my past and current fellow PhD students: Ulrike, especially for the help during the last months, and Isabell, Lisa, Thorsten and Denis, for being such nice (though sometimes a little distracting) office mates. Corinne, although you can be loud and messy, you are also a very nice person, and I have many good memories from the time you were in Bonn.

I need to thank some more people for mathematical discussions – just to name a few: Andrey Soldatenkov, Laura Pertusi, Tony Várilly-Alvarado, Emanuel Reinecke, Matthew Dawes. To Georg Oberdieck: It was a whole lot of work to be an assistant for your course, but I did learn a lot, and it is good to see somebody put so much care into teaching.

I am also very grateful to BIGS and the SFB/TR45 for supporting me financially.

Lots of people helped me by making the past few years in Bonn enjoyable. Tim, Julia, Felix, Jason, you are some of the craziest and nicest people I know. Thanks for the great holidays and for teaching me to dance. Eva, Hanna, it was wonderful to sing and hike with you. Some day I will manage to visit you, wherever you are living then.

To everybody I have done partner acrobatics with: thanks for being welcoming, for teaching me cool tricks and for always being in for trying out new things, even though we have no idea how they should work.

There are two more large groups of people I have to thank: the International Choir and the orchestra Camerata Musicale (especially the violists, of course). You have meant a lot to me during my time here. Martin Kirchharz, thank you so much for encouraging me to take part in both, and for being such an inspiring and humorous conductor. Marlene, you are the person I shared my Pult with the longest, and it was a great pleasure.

Most of the space here should be dedicated to the amazing Malvin Gattinger. Thank you for always being there, even though you live far away, for helping me (at least trying to) with everything, from bureaucracy to fixing my bike, for listening to me no matter what language I'm speaking, and for taking me to metal concerts. Besides, please don't underestimate your academic contributions to my PhD: I am very grateful you told me to stop doing large matrix calculations by hand and taught me Haskell instead, for discussions about English grammar and for proofreading my more and more incomprehensible maths gibberish ("sonst klingt alles prima, auch wenn ich nun wirklich nichts mehr verstehe").

Together with Malvin, I want to thank his family for their support and for teaching me more about Germany and its sometimes mysterious habits and inhabitants.

This brings me, at last, to my own family. Mam, pap, Es, ik was er niet zo vaak, maar het is heel fijn om ergens een plaats te hebben waar het altijd precies zo gez... is als vroeger. Oma, ik vind het heel leuk dat je in Bonn op bezoek bent geweest!

Chapter 1

Preliminaries

1.1 Conventions

In this thesis, we will only consider schemes and analytic spaces over the complex numbers. By a variety, we mean a separated reduced scheme of finite type over $\mathbb{C}$.
A property holds for the *very general* point in a variety embedded in a projective space, if and only if it holds for all points outside a countable union of hypersurfaces. It holds for a *generic* point if it is satisfied by all points in a Zariski open dense subset.

1.2 Lattices

We recall some basics on lattice theory, which will be used throughout this thesis. There are many good introductory references, such as [Ser73]. We should also mention Nikulin's influential paper [Nik80]. For an introduction directed towards K3 surfaces, we refer to [Huy16, Ch. 14].

1.2.1 Definitions

A *lattice* is a free abelian group L of finite rank together with a non-degenerate symmetric bilinear form $(\ ,\ )\colon L \times L \to \mathbb{Z}$. After choosing a basis, $(\ ,\ )$ can be represented by a symmetric matrix M. The determinant of M is independent of the choice of basis and is called the *discriminant* of L; we denote it by $\operatorname{disc} L$. The *dual lattice* of L is

$$L^\vee := \operatorname{Hom}(L, \mathbb{Z}) \cong \{x \in L \otimes \mathbb{Q} \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and the *discriminant group* of L is $\operatorname{Disc}(L) := L^\vee/L$. The minimal number of generators of $\operatorname{Disc} L$ is denoted by $\ell(L)$. We say that L is *unimodular* if $\operatorname{Disc} L$ is trivial.

We denote the group of orthogonal transformations of L by $\mathrm{O}(L)$. Every $g \in \mathrm{O}(L)$ induces an automorphism on $\operatorname{Disc}(L)$, which we denote by $\overline{g}$.

When L is *even*, i.e. $(x, x) \in 2\mathbb{Z}$ for all $x \in L$, the product $(\ ,\ )$ induces a quadratic form

$$q_L\colon \operatorname{Disc} L \to \mathbb{Q}/2\mathbb{Z}.$$

We denote by $\mathrm{O}(\operatorname{Disc} L)$ the group of automorphisms preserving q_L. We further define

$$\widetilde{\mathrm{O}}(L) := \operatorname{Ker}(\mathrm{O}(L) \to \mathrm{O}(\operatorname{Disc} L)).$$

The following result by Nikulin will be used several times.

Theorem 1.2.1 ([Nik80, Thm. 1.14.2], see also [Huy16, Thm. 14.2.4])**.** *Let L be an even indefinite lattice satisfying $\ell(L)+2 \leq \operatorname{rk} L$. Then the map $\mathrm{O}(L) \to \mathrm{O}(\operatorname{Disc} L)$ is surjective.*

A second result we will use is Eichler's criterion. For an element x in a lattice L, the *divisibility* $\operatorname{div}(x)$ of x is the positive generator of $(x, L) \subset \mathbb{Z}$. The element $x/\operatorname{div}(x)$ defines an element of $\operatorname{Disc} L$ of order $\operatorname{div}(x)$.

Theorem 1.2.2 ([Eic74, Ch. 10], see also [GHS09, Prop. 3.3])**.** *Let L be a lattice of the form $U^{\oplus 2} \oplus L'$. For $x, y \in L$ primitive, there exists an element of $\widetilde{\mathrm{O}}(L)$ mapping x to y if and only if $(x)^2 = (y)^2$ and $x/\operatorname{div}(x) \equiv y/\operatorname{div}(y) \mod L$.*

Finally, we will need a result that is slightly stronger than [Huy16, Prop. 14.2.6], but is proven in the same way. We give the proof here for completeness.

Let L_1 be a primitive sublattice of a unimodular lattice L and let $L_2 \subset L$ be its orthogonal complement. Then $L_1 \oplus L_2$ is a finite index sublattice of L. We have an inclusion $L^\vee \subset (L_1 \oplus L_2)^\vee$ which induces

$$L^\vee/(L_1 \oplus L_2) \hookrightarrow \operatorname{Disc}(L_1 \oplus L_2) \cong \operatorname{Disc}(L_1) \oplus \operatorname{Disc}(L_2).$$

The projection morphisms $p_i\colon L^\vee/(L_1 \oplus L_2) \to \operatorname{Disc}(L_i)$ are isomorphisms, by unimodularity of L and primitivity of L_1 and L_2. This gives an isomorphism $\varphi\colon \operatorname{Disc}(L_1) \to \operatorname{Disc}(L_2)$, sending an element $\overline{x} \in \operatorname{Disc}(L_1)$ to the unique class $\overline{y} \in \operatorname{Disc}(L_2)$ such that $x+y \in L_1^\vee \oplus L_2^\vee$ is in L.

Proposition 1.2.3. *Let L be a unimodular lattice, let $L_1 \subset L$ be a primitive sublattice and let $L_2 \subset L$ be its orthogonal complement. Let $g_1 \in \mathrm{O}(L_1)$ and $g_2 \in \mathrm{O}(L_2)$. Then $g_1 \oplus g_2\colon L_1 \oplus L_2 \to L_1 \oplus L_2$ extends to an automorphism of L if and only if $\overline{g}_1 = \overline{g}_2$ under the identification $\operatorname{Aut}(\operatorname{Disc} L_1) \cong \operatorname{Aut}(\operatorname{Disc} L_2)$.*

Proof. The map $g_1 \oplus g_2$ extends to L if and only if for all $x_1 \in L_1^\vee$, $x_2 \in L_2^\vee$ with $x_1 + x_2 \in L$, the element $g_1^\vee(x_1) + g_2^\vee(x_2)$ also lies in L. We have $x_1 + x_2 \in L$ if and only if $\varphi(\overline{x}_1) = \overline{x}_2$. So $g_1^\vee(x_1) + g_2^\vee(x_2)$ is in L if and only if $\varphi(\overline{g}_1(\overline{x}_1))$ equals $\overline{g}_2(\overline{x}_2) = \overline{g}_2(\varphi(\overline{x}_1))$. This holds for all x_1, x_2 if and only if $\varphi \circ \overline{g}_1 = \overline{g}_2 \circ \varphi$. □

1.2.2 Examples

We fix notation for some standard lattices that will be used in Chapters 2 and 3.

1. The *K3 lattice* Λ is defined as

$$\begin{aligned}\Lambda &:= E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}\\ &= E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3\end{aligned}$$

which is the unique even, unimodular lattice of signature $(3, 19)$. Here E_8 is the lattice associated to the Dynkin diagram E_8 and $U_i = U$ is the hyperbolic plane. We denote the standard basis of U_i by (e_i, f_i).

2. We fix the following primitive element of square d in Λ (which is unique up to the action of $\mathrm{O}(\Lambda)$):

$$l_d := e_3 + \tfrac{d}{2} f_3 \in U_3.$$

Let Λ_d be its orthogonal complement in Λ. Then we have

$$\Lambda_d \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-d)$$

where $\mathbb{Z}(-d)$ is generated by $l'_d := e_3 - \frac{d}{2} f_3 \in U_3$.

3. The *extended K3 lattice* is

$$\begin{aligned} \widetilde{\Lambda} &:= \Lambda \oplus U \\ &= E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4. \end{aligned}$$

It is isomorphic to the *Mukai lattice*

$$\Lambda_{\mathrm{Muk}} := \Lambda \oplus U_4(-1)$$

via $f_4 \mapsto -f_4$. In Chapter 2 it will be helpful to distinguish the two.

4. Inside the lattice

$$\Gamma' := E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3},$$

let Γ be the orthogonal complement of $h = (1, 1, 1) \in \mathbb{Z}(-1)^{\oplus 3}$. Then we have

$$\Gamma \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1)$$

where

$$A_2 = \left(\mathbb{Z}^{\oplus 2}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right).$$

The lattice Γ has rank 22 and signature (2,20).

1.2.3 Period domain

For references, see Sections 4.3 of [CMSP03] and 6.1 of [Huy16]. We define the *period domain* of a lattice L of signature (n_+, n_-) with $n_+ \geq 2$ as follows:

$$\mathcal{D}(L) = \{x \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0; (x, \bar{x}) > 0\}.$$

This is an open manifold of dimension $\mathrm{rk}(L) - 2$, which is connected when $n_+ > 2$ and has two connected components when $n_+ = 2$. In the second case, the components are interchanged by complex conjugation.

A Hodge structure on L of weight two is of *K3 type* if it satisfies

$$\dim_{\mathbb{C}}(L^{2,0}) = 1;\ L^{p,q} = 0 \text{ when } |p-q| > 2.$$

The period domain parametrizes Hodge structures on L of K3 type such that for all non-zero $(2,0)$-classes, the following holds:

(i) $(\sigma)^2 = 0$;
(ii) $(\sigma, \overline{\sigma}) > 0$;
(iii) $L^{1,1} \perp \sigma$.

The space $\mathcal{D}(L)$ has a natural action by $\mathrm{O}(L)$. We will only be interested in the action by the subgroup $\widetilde{\mathrm{O}}(L)$. This is an arithmetic group; if $n_+ = 2$, then by [Sat80, BB66], the quotient

$$\mathcal{D}(L)/\widetilde{\mathrm{O}}(L)$$

is a quasi-projective variety with at most finite quotient singularities.

1.3 K3 surfaces

Our main reference for this section is [Huy16].

An algebraic *K3 surface* is a two-dimensional smooth projective variety S satisfying $\Omega_S^2 \cong \mathcal{O}_S$ and $\mathrm{H}^1(S, \mathcal{O}_S) = 0$. The Hodge diamond of a K3 surface has the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \end{array}$$

The group $\mathrm{H}^2(S, \mathbb{Z})$, together with the intersection product, is a lattice of signature $(3, 19)$ which is isomorphic to the K3 lattice Λ.

There exists an analytic fine moduli space $\mathrm{M}^{\mathrm{mar}}$ of *marked* K3 surfaces, i.e. K3 surfaces S together with an isomorphism $\varphi\colon \mathrm{H}^2(S, \mathbb{Z}) \to \Lambda$. The *period map* for K3 surfaces is given by

$$\mathrm{M}^{\mathrm{mar}} \to \mathcal{D}(\Lambda),\ (S, \varphi) \mapsto [\varphi(\mathrm{H}^{2,0}(S)].$$

It is a holomorphic map.

Theorem 1.3.1 ([Tod80])**.** *The period map for K3 surfaces is surjective.*

Thus, for every Hodge structure on Λ, there exists a K3 surface whose middle cohomology is Hodge isometric to Λ.

One uses the period map to prove the global Torelli theorem:

Theorem 1.3.2 (Global Torelli, [PŠŠ71])**.** *Two K3 surfaces S and S' are isomorphic if and only if there exists a Hodge isometry* $\mathrm{H}^2(S, \mathbb{Z}) \cong \mathrm{H}^2(S', \mathbb{Z})$.

There also exists a derived analogue of global Torelli, which uses the full cohomology of S. The *Mukai pairing* $\langle\ ,\ \rangle$ on the full cohomology of S is defined as follows: for $v = (v_0, v_2, v_4), w = (w_0, w_2, w_4) \in (\mathrm{H}^0 \oplus \mathrm{H}^2 \oplus \mathrm{H}^4)(S, \mathbb{Z})$, we set

$$\langle v, w\rangle = (v_2, w_2) - (v_0, w_4) - (v_4, w_0)$$

where $(\ ,\ )$ denotes the usual intersection pairing. We denote the lattice $(\mathrm{H}^*(S, \mathbb{Z}), \langle\ ,\ \rangle)$ with the given Hodge structure (but suppressing the grading) by $\widetilde{\mathrm{H}}(S, \mathbb{Z})$. As a lattice, it is isomorphic to Λ_{Muk}.

This is motivated by the following: If E, F are coherent sheaves on S, the Euler characteristic of the pair (E, F) is

$$\chi(E, F) = \sum (-1)^i \dim \mathrm{Ext}^i(E, F).$$

Denoting by $v(E)$ and $v(F)$ the Mukai vectors (see 1.3.3) of the respective sheaves, one has

$$\chi(E, F) = -\langle v(E), v(F)\rangle.$$

Theorem 1.3.3 (Derived Torelli, [Orl97]). *Two K3 surfaces S and S' are derived equivalent if and only if there exists a Hodge isometry $\widetilde{\mathrm{H}}(S, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S', \mathbb{Z})$.*

The numbers of Fourier–Mukai partners of any K3 surface is finite. It can, however, be arbitrarily large. For the very general K3 surface, there is a precise formula:

Proposition 1.3.4 ([Ogu02]). *Let S be a K3 surface of Picard number 1 and degree d. The number of Fourier–Mukai partners of S is $2^{\tau(d/2)} - 1$, where $\tau(d/2)$ denotes the number of prime factors of $d/2$.*

1.3.1 Polarized K3 surfaces

A *polarization* on a K3 surface S is a primitive ample class

$$L \in \mathrm{H}^2(S, \mathbb{Z}).$$

Its *degree* d is the number $(L)^2 \in 2\mathbb{Z}$. The primitive cohomology

$$\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}} = \langle L\rangle^{\perp} \subset \mathrm{H}^2(S, \mathbb{Z})$$

of (S, L) is isomorphic to the lattice Λ_d. By definition, two polarized K3 surfaces (S, L) and (S', L') are isomorphic if there exists an isomorphism $f\colon S \to S'$ such that $f^*L' = L$.

Polarized K3 surfaces of fixed degree d have a 19-dimensional coarse moduli space which can be constructed using GIT, or as an open subvariety of the quotient $\mathcal{D}(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$. We give a sketch of the second construction, which will come back in Chapter 3.

Just like for unpolarized K3 surfaces, there exists an analytic fine moduli space $\mathrm{M}_d^{\mathrm{mar}}$ of marked polarized K3 surfaces: triples (S, L, φ) consisting of a polarized K3 surface (S, L) with an isomorphism $\varphi\colon \mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}} \cong \Lambda_d$. The period map

$$\mathrm{M}_d^{\mathrm{mar}} \to \mathcal{D}(\Lambda_d),\ (S, \varphi) \mapsto [\varphi(\mathrm{H}^{2,0}(S))]$$

is holomorphic and induces an open immersion

$$\mathrm{M}_d := \mathrm{M}_d^{\mathrm{mar}}/\widetilde{\mathrm{O}}(\Lambda_d) \hookrightarrow \mathcal{D}(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d).$$

Using a theorem of Borel [Bor72], one shows that it is an algebraic morphism, see [Huy16, Sec. 6.4]. It follows that M_d is a quasi-projective variety with at most finite quotient singularities.

The image of the period map is the complement of the set $\bigcup_{\delta\in\Delta(\Lambda_d)} \delta^{\perp}$, where $\Delta(\Lambda_d)$ is the set of (-2)-classes in Λ_d. This implies that $\mathcal{D}(\Lambda_d)$ parametrizes periods of *quasi-polarized* K3 surfaces: pairs (S, L) of a K3 surface with the class of a line bundle that is nef and big. Therefore, one may view the space $\mathcal{D}(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$ a moduli space of quasi-polarized K3 surfaces (however, the corresponding moduli stack is not separated, see [Huy16, Rem. 6.4.5]).

1.3.2 Hilbert scheme of points

The *Hilbert scheme of n points* $\mathrm{Hilb}^n(S)$ on a K3 surface S parametrizes subschemes of S of length n. It is a smooth projective variety of dimension $2n$. It can be constructed as a resolution of singularities of the n-th symmetric product of S. For references, see [Bea83] and [GHJ03, Chapter 3].

The space $\mathrm{Hilb}^2(S)$ is a *hyperkähler variety*, that is, a simply connected projective compact Kähler manifold Y such that $\mathrm{H}^0(Y, \Omega^2_X)$ is generated by a nowhere degenerate 2-form. There are only four types of hyperkähler varieties known up to deformation equivalence. If a hyperkähler variety is deformation equivalent to a Hilbert scheme of n points on a K3 surface, we say it is *of* $\mathrm{K3}^{[n]}$ *type*.

A hyperkähler variety Y admits a non-degenerate integral quadratic form q on the second cohomology $\mathrm{H}^2(Y, \mathbb{Z})$. It is called the Beauville–Bogomolov form (or Beauville–Bogomolov–Fujiki form). For the scheme $\mathrm{Hilb}^n(S)$, Beauville showed that there is a Hodge isometry

$$\mathrm{H}^2(\mathrm{Hilb}^n(S), \mathbb{Z}) \cong \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where δ is a $(1,1)$-class whose square is $-2(n-1)$. In particular, $\mathrm{H}^2(\mathrm{Hilb}^n(S), \mathbb{Z})$ can be embedded Hodge isometrically into $\widetilde{\mathrm{H}}(S, \mathbb{Z})$ as the orthogonal complement of the vector $(1, 0, 1-n)$, see also (1.2).

It was shown more generally by Markman [Mar11] that for any variety Y of $\mathrm{K3}^{[n]}$ type, there exists an extension

$$\mathrm{H}^2(Y, \mathbb{Z}) \subset \widetilde{\Lambda}_Y \tag{1.1}$$

of lattices and Hodge structures of weight 2, such that as a lattice, $\widetilde{\Lambda}_Y$ is isomorphic to Λ_{Muk}. Using this, he proved a birational Torelli theorem for hyperkähler varieties of $\mathrm{K3}^{[n]}$ type.

Theorem 1.3.5 ([Mar11])**.** *Two hyperkähler varieties Y_1, Y_2 of* $\mathrm{K3}^{[n]}$ *type are birational if and only if there exists a Hodge isometry $\widetilde{\Lambda}_{Y_1} \cong \widetilde{\Lambda}_{Y_2}$ which maps* $\mathrm{H}^2(S, Y_1)$ *to* $\mathrm{H}^2(S, Y_2)$.

1.3.3 Moduli spaces of sheaves

As a generalization of $\mathrm{Hilb}^n(S)$, we consider moduli spaces of Gieseker stable sheaves on S. For details, see [HL10, Huy16].

The *Mukai vector* of a coherent sheaf F on any smooth projective variety Y is defined as

$$v(F) := ch(F).\sqrt{td(S)} \in \mathrm{H}^{2*}(X, \mathbb{Q}).$$

When Y is a K3 surface S, one has

$$v(F) = (\mathrm{rk}(F), c_1(F), c_1(F)^2/2 - c_2(F) - \mathrm{rk}\, F) \in (\mathrm{H}^0 \oplus \mathrm{H}^2 \oplus \mathrm{H}^4)(S, \mathbb{Z}).$$

Fix

$$v \in (\mathrm{H}^0 \oplus \mathrm{H}^2 \oplus \mathrm{H}^4)(S, \mathbb{Z})$$

and let H be a polarization on S. There exists a quasi-projective scheme $M_H(v)^s$ which is a coarse moduli space for coherent sheaves on S that are Gieseker stable with respect to H and have Mukai vector v. We will usually omit the subscript H.

Theorem 1.3.6. *Suppose $v = (r, l, s) \in (\mathrm{H}^0 \oplus \mathrm{H}^2 \oplus \mathrm{H}^4)(S, \mathbb{Z})$ is primitive and satisfies*

1. *$(v, v) \geq -2$;*
2. *If $r = 0$, then $(l)^2 \geq -2$ and $(l, H) > 0$.*

Then for H generic, the space $M(v)^s$ is a smooth projective variety of dimension $2 + (v, v)$. Moreover, there exists a natural 2-form

$$\sigma \in \mathrm{H}^0(M(v)^s, \Omega^2_{M(v)^s})$$

which is symplectic and makes $M(v)^s$ into a hyperkähler variety, deformation equivalent to $\mathrm{Hilb}^n(S)$, where $n = (2 + (v, v))/2$.

In the situation of the theorem, we denote $M(v)^s$ by $M(v)$.

One example of a moduli space of sheaves is the Hilbert scheme itself:

Proposition 1.3.7. *There is an isomorphism*

$$\mathrm{Hilb}^n(S) \to M(1, 0, 1 - n)$$

which sends a length n subscheme $Z \subset S$ to its ideal sheaf.

When we take v as in Theorem 1.3.6 such that $(v)^2 = 0$, we find that $M(v)$ is a K3 surface. In that case, there exists a natural Hodge isometry

$$\mathrm{H}^2(M(v), \mathbb{Z}) \cong v^\perp / \mathbb{Z}v$$

where the orthogonal complement is taken in $\widetilde{\mathrm{H}}(S, \mathbb{Z})$. When $(v)^2 > 0$, there is a Hodge isometry

$$\mathrm{H}^2(M(v), \mathbb{Z}) \cong v^\perp \subset \widetilde{\mathrm{H}}(S, \mathbb{Z}). \tag{1.2}$$

In fact, this extension is the same as the one given by Markman (1.1). For moduli spaces of sheaves, the birational Torelli theorem thus translates to

Corollary 1.3.8. *Let v and v' satisfy the conditions of Theorem 1.3.6. Then $M(v)$ and $M(v')$ are birational if and only if there exists a Hodge isometry $\widetilde{\mathrm{H}}(S,\mathbb{Z}) \cong \widetilde{\mathrm{H}}(S,\mathbb{Z})$ that sends v to v'.*

Finally, we add one more condition on v:

3. *There exists an element $w \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ such that $\langle v, w\rangle = 1$.*

Then $M(v)$ is a fine moduli space, so there exists a universal family $\mathcal{E}$ on $M(v) \times S$. The associated Fourier–Mukai transform

$$\Phi_{\mathcal{E}}\colon\ \mathrm{D^b}(M(v)) \to \mathrm{D^b}(S)$$

is an equivalence of categories. Conversely, it follows from the proof of Derived Torelli that any Fourier–Mukai partner of S is isomorphic to a moduli space of sheaves on S.

1.4 Cubic fourfolds

A *cubic fourfold* is a cubic hypersurface in $\mathbb{P}^5_{\mathbb{C}}$, which we always assume to be smooth and denote by X. The Hodge numbers of X are

$$h^{p,q}(X) = \begin{cases} 1 & \text{if } p = q \in \{0,1,3,4\} \text{ or } \{p,q\} = \{1,3\} \\ 21 & \text{if } p = q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathrm{H}^4(X,\mathbb{Z})^-$ the middle cohomology of X with the intersection product changed by a sign. Then $\mathrm{H}^4(X,\mathbb{Z})$ is isomorphic, as a lattice, to Γ'. Similarly, let $\mathrm{H}^4(X,\mathbb{Z})^-_{\mathrm{pr}}$ be the orthogonal complement to the square of the hyperplane class, with the intersection product changed by a sign. Then $\mathrm{H}^4(X,\mathbb{Z})^-_{\mathrm{pr}}$ is isomorphic to the lattice Γ.

The coarse moduli space of smooth cubic fourfolds is the GIT quotient

$$\mathcal{C} = |\mathcal{O}_{\mathbb{P}^5}(3)|_{\mathrm{smooth}} /\!\!/ \mathrm{SL}(6).$$

It is a 20-dimensional, quasi-projective variety. The period map induces a holomorphic map $\mathcal{C} \to \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$.

Theorem 1.4.1 (Torelli theorem for cubic fourfolds, [Voi86])**.** *The period map from $\mathcal{C}$ to $\mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$ is an open immersion of analytic spaces.*

Like for polarized K3 surfaces, it follows from [Bor72] that $\mathcal{C} \hookrightarrow \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$ is an algebraic map, see [Has00, Prop. 2.2.2].

1.4.1 Special cubic fourfolds

For a very general cubic fourfold X, the lattice

$$\mathrm{H}^{2,2}(X,\mathbb{Z}) := \mathrm{H}^4(X,\mathbb{C}) \cap \mathrm{H}^{2,2}(X)$$

is generated by the square of the hyperplane class. We call X *special* if $\mathrm{H}^{2,2}(X,\mathbb{Z})$ has rank at least two. By the Hodge conjecture for cubic fourfolds [Zuc77], this is the case if and only if X contains a surface which is not homologous to a complete intersection.

If X is special, then $\mathrm{H}^{2,2}(X,\mathbb{Z})$ contains a primitive sublattice K of rank two. Define

$$\mathcal{C}_d := \{X \in \mathcal{C} \mid \exists K \subset \mathrm{H}^{2,2}(X,\mathbb{Z}) \text{ primitive}, \ h_X \in K, \ \operatorname{rk} K = 2, \ \operatorname{disc} K = d\}.$$

Then the set of special cubic fourfolds in $\mathcal{C}$ is the union of all $\mathcal{C}_d$.

Theorem 1.4.2 ([Has00]). *The set $\mathcal{C}_d$ is either empty or an irreducible divisor in $\mathcal{C}$. It is non-empty if and only if d satisfies*

$$(*) \qquad d > 6 \text{ and } d \equiv 0, 2 \text{ mod } 6.$$

For many small d, one can describe what kind of surfaces are contained in cubic fourfolds in $\mathcal{C}_d$. For example, the divisor $\mathcal{D}_8$ consists of those cubics which contain a plane. We will consider these in Section 3.4.2.

Inside the period domain $\mathcal{D}(\Gamma)$, we can identify those periods coming from special cubic fourfolds. Note that if $K \subset \mathrm{H}^{2,2}(X,\mathbb{Z})$ is as above, then $K \otimes \mathbb{C}$ is contained in $\mathrm{H}^{3,1}(X)^{\perp} \subset \mathrm{H}^4(X,\mathbb{C})^{-}$. On the level of the period domain, this means the following: After choosing a marking $\mathrm{H}^4(X,\mathbb{Z})^{-}_{\mathrm{pr}} \xrightarrow{\cong} \Gamma$, the period of X lands in

$$\{x \in \mathcal{D}(\Gamma) \mid (K \cap \Gamma)_{\mathbb{C}} \subset x^{\perp}\}$$

for some primitive, negative definite sublattice $K \subset \Gamma'$ of rank two containing h. Let $K^{\perp} \subset \Gamma'$ be its orthogonal complement. Then the set above is the divisor $\mathcal{D}(K^{\perp}) \subset \mathcal{D}(\Gamma)$.

We fix one sublattice $K_d \subset \Gamma'$ as above, with discriminant d. Let

$$\overline{\mathcal{C}}_d \subset \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$$

be the image of $\mathcal{D}(K_d^{\perp})$ under the quotient map $\mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$. As the embedding of $K_d^{\perp}$ into Γ' is unique up to $\widetilde{\mathrm{O}}(\Gamma)$, the space $\overline{\mathcal{C}}_d$ does not depend on the choice of K_d. The morphism $\mathcal{C} \hookrightarrow \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$ maps $\mathcal{C}_d$ into $\overline{\mathcal{C}}_d$. To be precise, we have $\mathcal{C}_d = \mathcal{C} \cap \overline{\mathcal{C}}_d$.

1.4.2 Associated K3 surfaces

We will only be concerned with Hodge associated K3 surfaces as in [Has00]. Consider the following condition on d:

$$(**) \qquad d \text{ is even and not divisible by 4, 9, or any odd prime } p \equiv 2 \text{ mod } 3.$$

This implies that $d \equiv 0, 2$ mod 6. Hassett proved the following statement:

Proposition 1.4.3 ([Has00]). *The number d satisfies $(**)$ if and only if there is an isomorphism $\Lambda_d \cong K_d^\perp$.*

As a consequence, for every cubic fourfold $X \in \mathcal{C}_d$, there exists a polarized K3 surface (S, L) of degree d such that $K_d^\perp \subset \mathrm{H}^4(X, \mathbb{Z})$ is Hodge isometric to $\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}$, up to a sign and a Tate twist. We say that (S, L) is *associated* to X.

In general, (S, L) is not the only K3 surface associated to X, because the isomorphism $\Lambda_d \cong K_d^\perp$ can be precomposed with elements of $\mathrm{O}(\Lambda_d)$. To be precise, the associated K3 surfaces are the ones in the orbit of the period of (S, L) under the action of

$$\mathrm{Im}\big(\mathrm{O}(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d) \to \mathrm{Aut}(\mathcal{D}(\Lambda_d))\big).$$

It follows [HLOY03, Prop. 3.2] that if $\rho(S) = 1$, then $(S', L') \in \mathrm{M}_d$ is also associated to X if and only if S' and S are derived equivalent.

For completeness, we also state the derived version of associated K3 surfaces [Kuz10]. Define the *Kuznetsov component* (or *K3 component*) of a cubic fourfold X as the right semi-orthogonal complement

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^\perp \subset \mathrm{D}^{\mathrm{b}}(X).$$

It was shown by [AT14, BLM$^+$] that X lies in $\mathcal{C}_d$ for some d satisfying $(**)$ if and only if there exists a K3 surface S such that $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$.

1.4.3 Fano variety of lines

The *Fano variety of lines* $F(X)$ of a cubic fourfold X is the subscheme of the Grassmannian $\mathbb{G}(1, \mathbb{P}^5)$ parametrizing lines in $\mathbb{P}^5$ that are contained in X. It is a four-dimensional smooth projective variety. Let us stress that $F(X)$ is *not* a Fano variety in the sense that the canonical bundle is anti-ample. On the contrary, $F(X)$ is a hyperkähler variety of $K3^{[n]}$ type [BD85]; in particular, the canonical bundle is trivial.

The cohomology of $F(X)$ can be related to the cohomology of X via the *Abel–Jacobi map*

$$\alpha\colon\ \mathrm{H}^4(X, \mathbb{Z}) \to \mathrm{H}^2(F(X), \mathbb{Z})$$

which is defined as follows: let $Z \subset F(X) \times X$ be the universal family of lines, i.e. the variety of pairs (l, x) such that $x \in l$. Let $p\colon Z \to F(X)$ and $q\colon Z \to X$ be the projections to the first and second factor, respectively. Then $\alpha = p_* \circ q^*$.

Let $\mathrm{H}^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ be the orthogonal complement, with respect to the Beauville–Bogomolov form, to the polarization coming from the Plücker embedding. It was shown in [BD85] that α induces a Hodge isometry

$$\mathrm{H}^4(X, \mathbb{Z})_{\mathrm{pr}} \cong \mathrm{H}^2(F(X), \mathbb{Z})_{\mathrm{pr}}$$

up to a sign and a Tate twist.

Chapter 2

Two polarized K3 surfaces associated to the same cubic fourfold

This chapter is based on [Bra18].

Special cubic fourfolds were first studied by Hassett [Has00]. They are distinguished by the property that they carry additional algebraic cycles. They arise in countably many families, parametrized by irreducible divisors $\mathcal{C}_d$ in the moduli space of cubic fourfolds. For infinitely many d, the cubic fourfolds in $\mathcal{C}_d$ are related to polarized K3 surfaces of degree d via their Hodge structures. For half of the d, K3 surfaces associated to generic cubics in $\mathcal{C}_d$ come in pairs. The goal of this chapter is to explain how two such K3 surfaces are related.

More precisely, denote by $\mathcal{M}_d$ the moduli space of polarized K3 surfaces of degree d. Hassett constructed, for admissible d, a surjective rational map $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$ sending a K3 surface to a cubic fourfold it is associated to. This map is of degree two when $d \equiv 0 \bmod 6$ and generically injective otherwise. In the former case, its (regular) covering involution $\tau \colon \mathcal{M}_d \to \mathcal{M}_d$ does not depend on the choices made to construct $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$. We prove the following geometric description of τ.

Theorem 2.1 (see Thm. 2.3.2)**.** *Let $(S^\tau, L^\tau) = \tau(S, L)$. Then S^τ is isomorphic to the moduli space $M_S(v)$ of stable coherent sheaves on S with Mukai vector $v = (3, L, d/6)$.*

In particular, S and S^τ are Fourier–Mukai partners. For general $(S, L) \in \mathcal{M}_d$, this also follows from the fact that the bounded derived categories of S and S^τ are both exact equivalent to the Kuznetsov category of the image cubic fourfold [AT14]. If $\rho(S) = 1$, then S is not isomorphic to S^τ (as unpolarized K3 surfaces). The number of Fourier–Mukai partners of S, which depends on d, can be arbitrarily high [Ogu02]. The above gives a natural way of selecting one of them for each $(S, L) \in \mathcal{M}_d$.

We also explain when the Hilbert schemes of n points $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S^\tau)$ are birational. Our main result is the following.

Theorem 2.2 (see Prop. 2.4.5, Cor. 2.4.9)**.** *Let $d \equiv 0 \bmod 6$ satisfy* $(**)$*. Consider the following statements:*

(i) $\mathrm{Hilb}^2(S)$ *is isomorphic to* $\mathrm{Hilb}^2(S^\tau)$*;*
(ii) $\mathrm{Hilb}^2(S)$ *is birational to* $\mathrm{Hilb}^2(S^\tau)$*;*
(iii) *There exists an integral solution to* $3p^2 - (d/6)q^2 = -1$*;*
(iv) $\mathrm{Hilb}^2(S)$ *has a line bundle of self-intersection 6.*

We have implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv)*. If* $\rho(S) = 1$*, then these are all equivalent.*

We will see that this condition is satisfied for infinitely many d but not for all of them. As an application, we obtain an example of derived equivalent Hilbert schemes of two points on K3 surfaces which are not birational.

2.1 Lattices

In this section we set up the notation for the lattice theory that will be needed, see [Huy16, Ch. 14] for references.

The first type of lattices that we use comes from K3 surfaces. The middle cohomology $\mathrm{H}^2(S,\mathbb{Z})$ of a K3 surface S (with the usual intersection pairing) is isomorphic to the *K3 lattice*

$$\Lambda_{\mathrm{K3}} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3.$$

We denote the standard basis of U_i by e_i, f_i. On the full cohomology $\mathrm{H}^*(S,\mathbb{Z})$ of S we consider the *Mukai pairing*, given by $\big((x_0,x_2,x_4),(x_0',x_2',x_4')\big) = x_2x_2' - x_0x_4' - x_0'x_4$ for $x_i, x_i' \in \mathrm{H}^i(S,\mathbb{Z})$. With this pairing, $\mathrm{H}^*(S,\mathbb{Z})$ becomes isomorphic to the *Mukai lattice*

$$\Lambda_{\mathrm{Muk}} := \Lambda_{\mathrm{K3}} \oplus U(-1) = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4(-1).$$

As $U \cong U(-1)$, the Mukai lattice is isomorphic to $\Lambda_{\mathrm{K3}} \oplus U$. To avoid confusion, we denote the latter by $\widetilde{\Lambda}_{\mathrm{K3}}$, and fix an isomorphism $\widetilde{\Lambda}_{\mathrm{K3}} \xrightarrow{\sim} \Lambda_{\mathrm{Muk}}$ by sending f_4 to $-f_4$.

We fix $\ell_d = e_3 + \frac{d}{2} f_3 \in U_3 \subset \Lambda_{\mathrm{K3}}$ and let $\Lambda_d := \ell_d^\perp \subset \Lambda_{\mathrm{K3}}$ be its orthogonal complement in Λ_{K3}. Then

$$\Lambda_d \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-d)$$

is isomorphic to the primitive cohomology $L^\perp \subset \mathrm{H}^2(S,\mathbb{Z})$ of any polarized K3 surface (S,L) of degree d. We will need the following subgroup

$$\widetilde{\mathrm{O}}(\Lambda_d) := \{f \in \mathrm{O}(\Lambda_d) \mid \overline{f} = \mathrm{id}_{\mathrm{Disc}\,\Lambda_d}\}$$

of $\mathrm{O}(\Lambda_d)$, which, by Lemma 1.2.3, is isomorphic to $\{f \in \mathrm{O}(\Lambda_{\mathrm{K3}}) \mid f(\ell_d) = \ell_d\}$.

Next, we define some lattices related to cubic fourfolds. Fix a primitive embedding of the lattice $A_2 = \langle \lambda_1, \lambda_2 \rangle = \big(\mathbb{Z}^{\oplus 2}, \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)\big)$ into $U_3 \oplus U_4 \subset \widetilde{\Lambda}_{\mathrm{K3}}$, for instance by $\lambda_1 \mapsto e_3 + f_3$ and $\lambda_2 \mapsto e_4 + f_4 - e_3$. This embedding is unique up to composition with elements of $\mathrm{O}(\widetilde{\Lambda}_{\mathrm{K3}})$. We are mostly interested in the complement $A_2^\perp \subset \widetilde{\Lambda}_{\mathrm{K3}}$ of A_2:

$$A_2^\perp \cong E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus A_2(-1).$$

Denote by $\mathrm{H}^4(X,\mathbb{Z})^-$ the middle cohomology of a cubic fourfold X, with the intersection product changed by a sign. This lattice is isomorphic to

$$\Lambda_{\mathrm{cub}} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}.$$

Let $h = (1,1,1) \in \mathbb{Z}(-1)^{\oplus 3} \subset \Lambda_{\mathrm{cub}}$. The primitive cohomology $\mathrm{H}^4(X,\mathbb{Z})^-_{\mathrm{pr}} \subset \mathrm{H}^4(X,\mathbb{Z})^-$ of X is isomorphic to $\Lambda^0_{\mathrm{cub}} := h^\perp \subset \Lambda_{\mathrm{cub}}$. An easy computation shows that

$$\Lambda^0_{\mathrm{cub}} \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1),$$

so $\Lambda^0_{\mathrm{cub}} \cong A_2^\perp \subset \widetilde{\Lambda}_{\mathrm{K3}}$. As for Λ_d, we will consider the subgroup

$$\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}) := \{f \in \mathrm{O}(\Lambda^0_{\mathrm{cub}}) \mid \overline{f} = \mathrm{id}_{\mathrm{Disc}\,\Lambda^0_{\mathrm{cub}}}\} \cong \{f \in \mathrm{O}(\Lambda_{\mathrm{cub}}) \mid f(h) = h\}$$

of $\mathrm{O}(\Lambda^0_{\mathrm{cub}})$ acting on Λ^0_{cub}.

2.2 Hassett's construction

We summarize Hassett's construction of the rational maps $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$, explaining those proofs that we need for our results. For details, see [Has00].

2.2.1 Special cubic fourfolds

As, above, we denote by $\mathrm{H}^4(X,\mathbb{Z})^-$ the middle cohomology lattice of a cubic fourfold X with the intersection form changed by a sign. Inside it, we consider the lattice $A(X)$ of Hodge classes:

$$A(X) = \mathrm{H}^4(X,\mathbb{Z})^- \cap \mathrm{H}^{2,2}(X)$$

which is negative definite by the Hodge–Riemann bilinear relations. We also fix the notation $h_X \in \mathrm{H}^4(X,\mathbb{Z})$ for the square of a hyperplane class on X. For X general, the lattice $A(X)$ has rank one and is generated by h_X. We call X *special* if $\mathrm{rk}\,A(X) \geq 2$. By the Hodge conjecture for cubic fourfolds [Zuc77], X is special if and only if X contains a surface that is not homologous to a complete intersection.

If X is special, then $A(X)$ contains a primitive sublattice K of rank two. Hassett proved that fixing the discriminant of such K gives divisors in the moduli space $\mathcal{C}$ of smooth cubic fourfolds. Namely, define

$$\mathcal{C}_d := \{X \in \mathcal{C} \mid \exists K \subset A(X),\ h_X \in K,\ \mathrm{rk}\,K = 2,\ \mathrm{disc}\,K = d\}.$$

Then the set of special cubic fourfolds in $\mathcal{C}$ is the union of all $\mathcal{C}_d$.

Theorem 2.2.1. [Has00, Thm. 1.0.1] *The set $\mathcal{C}_d$ is either empty or an irreducible divisor in $\mathcal{C}$. It is non-empty if and only if d satisfies*

$$(*) \qquad d > 6 \text{ and } d \equiv 0, 2 \text{ mod } 6.$$

Inside the period domain $\mathcal{D}(\Lambda^0_{\mathrm{cub}})$ for cubic fourfolds, periods of special cubics of discriminant d are parametrized by sets of the form $\mathcal{D}(K^\perp) \subset \mathcal{D}(\Lambda^0_{\mathrm{cub}})$, where $K \subset \Lambda_{\mathrm{cub}}$ is a primitive, negative definite sublattice of rank two and discriminant d containing h. Such a sublattice is unique up to $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$. Let $K_d \subset \Lambda_{\mathrm{cub}}$ be one of these sublattices, and let

$$\overline{\mathcal{C}}_d \subset \mathcal{QD}(\Lambda^0_{\mathrm{cub}})$$

be the image of $\mathcal{D}(K_d^\perp) \subset \mathcal{D}(\Lambda^0_{\mathrm{cub}})$ under the quotient map $\mathcal{D}(\Lambda^0_{\mathrm{cub}}) \to \mathcal{QD}(\Lambda^0_{\mathrm{cub}})$. Then the immersion $\mathcal{C} \hookrightarrow \mathcal{QD}(\Lambda^0_{\mathrm{cub}})$ maps $\mathcal{C}_d$ into $\overline{\mathcal{C}}_d$. In fact, we have $\mathcal{C}_d = \mathcal{C} \cap \overline{\mathcal{C}}_d$.

2.2.2 Associated K3 surfaces

Consider the following condition on d:

(∗∗) d is even and not divisible by 4, 9, or any odd prime $p \equiv 2 \bmod 3$.

This implies that $d \equiv 0, 2 \bmod 6$. Hassett proved the following statement:

Proposition 2.2.2 ([Has00, Prop. 5.1.4])**.** *The number d satisfies* (∗∗) *if and only if there is an isomorphism $\Lambda_d \cong K_d^\perp$.*

So when d satisfies (∗∗), there is an isomorphism of period domains $\mathcal{D}(\Lambda_d) \cong \mathcal{D}(K_d^\perp)$. Under the identification $\Lambda_d \cong K_d^\perp$, the group $\widetilde{\mathrm{O}}(\Lambda_d)$ forms a subgroup of $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$, see Proposition 2.2.5 below, so we also get a surjective map $\mathcal{QD}(\Lambda_d) = \mathrm{O}(\Lambda_d)\backslash\mathcal{D}(\Lambda_d) \to \overline{\mathcal{C}}_d$. This gives us the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}(\Lambda_d) \arrow[r, "\sim"] \arrow[d, two heads] & \mathcal{D}(K_d^\perp) \arrow[r, hook] \arrow[d, two heads] & \mathcal{D}(\Lambda^0_{\mathrm{cub}}) \arrow[d, two heads] \\
\mathcal{QD}(\Lambda_d) \arrow[r, two heads] & \overline{\mathcal{C}}_d \arrow[r, hook] & \mathcal{QD}(\Lambda^0_{\mathrm{cub}}) \\
\mathcal{M}_d \arrow[u, hook] \arrow[r, dashed, "\varphi"] & \mathcal{C}_d \arrow[u, hook] \arrow[r, hook] & \mathcal{C} \arrow[u, hook]
\end{tikzcd}$$

It can be shown that the rational map $\varphi\colon \mathcal{M}_d \dashrightarrow \mathcal{C}_d$ is regular on an open subset which maps surjectively to $\mathcal{C}_d$, see [Has00, p. 14]. Note that φ depends on the choice of an isomorphism $\Lambda_d \cong K_d^\perp$, thus it is only unique up to $\mathrm{O}(\Lambda_d)/\widetilde{\mathrm{O}}(\Lambda_d)$.

If φ sends $(S, L) \in \mathcal{M}_d$ to X then there exists, up to a Tate twist, an isometry of Hodge structures

$$\mathrm{H}^4(X, \mathbb{Z})^- \supset K^\perp \cong L^\perp \subset \mathrm{H}^2(S, \mathbb{Z})$$

for some primitive sublattice $K \subset A(X)$ of rank two and discriminant d containing h_X. Conversely, if such a Hodge isometry exists, this induces a lattice isomorphism

$$\Lambda_d \cong L^\perp \cong K^\perp \cong K_d^\perp$$

such that the induced map $\varphi\colon \mathcal{M}_d \dashrightarrow \mathcal{C}_d$ sends (S, L) to X. This motivates the following definition.

Definition 2.2.3. Let $X \in \mathcal{C}_d$. A polarized K3 surface $(S, L) \in \mathcal{M}_d$ is *associated* to X if there exists a Hodge isometry

$$\mathrm{H}^4(X, \mathbb{Z})^-(1) \supset K^\perp \cong L^\perp \subset \mathrm{H}^2(S, \mathbb{Z})$$

for some primitive sublattice $K \subset A(X)$ of rank two and discriminant d containing h_X.

For the rest of this section, we fix one choice of the rational map φ. We suppress the Tate twist and view $\mathrm{H}^4(X, \mathbb{Z})^-$ as a Hodge structure of weight two.

Remark 2.2.4. The complement of the image of the inclusion $\mathcal{C} \hookrightarrow \mathcal{QD}(\Lambda^0_{\mathrm{cub}})$ is $\overline{\mathcal{C}}_2 \cup \overline{\mathcal{C}}_6$, see [Laz10, Loo09]. Therefore, φ is defined on $(S, L) \in \mathcal{M}_d$ if and only if its image under $\mathcal{QD}(\Lambda_d) \to \overline{\mathcal{C}}_d$ is contained in $\overline{\mathcal{C}}_d \backslash \left(\overline{\mathcal{C}}_2 \cup \overline{\mathcal{C}}_6\right)$. In particular, this holds when $\rho(S) = 1$.

To describe the map $\mathcal{QD}(\Lambda_d) \to \overline{\mathcal{C}}_d$, we define two subgroups of $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$. First, consider the group of isomorphisms of Λ_{cub} that are the identity on K_d. Elements of $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$ are in this group if and only if they fix a generator v_d of $K_d \cap \Lambda^0_{\mathrm{cub}}$, which is unique up to a sign. We denote the group by $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)$:

$$\begin{aligned}\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d) &= \{f \in \mathrm{O}(\Lambda_{\mathrm{cub}}) \mid f|_{K_d} = \mathrm{id}_{K_d}\} \\ &= \{f \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}) \mid f(v_d) = v_d\}.\end{aligned}$$

The next statement is part of [Has00, Thm. 5.2.2]. It follows directly from Proposition 1.2.3.

Proposition 2.2.5. *Suppose that d satisfies $(**)$. Under the isomorphism $\Lambda_d \cong K_d^\perp$, the group $\widetilde{\mathrm{O}}(\Lambda_d)$ is identified with $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)$.*

In particular, there is an isomorphism $\mathcal{QD}(\Lambda_d) \cong \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d) \backslash \mathcal{D}(K_d^\perp)$.

The second group we consider consists of the elements $f \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$ that preserve K_d. These are exactly the ones satisfying $f(v_d) \in \{v_d, -v_d\}$. We therefore denote this group by $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle)$:

$$\begin{aligned}\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle) &= \{f \in \mathrm{O}(\Lambda_{\mathrm{cub}}) \mid f(h) = h \text{ and } f(K_d) = K_d\} \\ &= \{f \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}) \mid f(v_d) = \pm v_d\}.\end{aligned}$$

Again by Baily–Borel, the quotient $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp)$ is a normal quasi-projective variety. In fact, the map $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \twoheadrightarrow \overline{\mathcal{C}}_d$ is the normalization of $\overline{\mathcal{C}}_d$.[1] Sum-

[1] A very general special cubic fourfold X satisfies $\mathrm{rk}\, A(X) = 2$ [Zar90, Sec. 5.1], so there is only one sublattice $K_d \subset A(X)$. It follows that the map $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \twoheadrightarrow \overline{\mathcal{C}}_d$ is generically injective. To see that it is proper, note that the action of $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$ on $\mathcal{D}(\Lambda^0_{\mathrm{cub}})$ is properly discontinuous [Huy16, Rem. 6.1.10]. Hence the map $\mathcal{D}(\Lambda^0_{\mathrm{cub}}) \to \mathcal{QD}(\Lambda^0_{\mathrm{cub}})$ is closed, as is its restriction $\mathcal{D}(K_d^\perp) \twoheadrightarrow \overline{\mathcal{C}}_d$ to the closed subset $\mathcal{D}(K_d^\perp) \subset \mathcal{D}(\Lambda^0_{\mathrm{cub}})$. Since this factors as

$$\mathcal{D}(K_d^\perp) \twoheadrightarrow \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \twoheadrightarrow \overline{\mathcal{C}}_d,$$

the map $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d \rangle) \backslash \mathcal{D}(K_d^\perp) \twoheadrightarrow \overline{\mathcal{C}}_d$ is closed as well. Moreover, it has finite fibres, so it is proper.

marizing, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{QD}(\Lambda_d) \cong \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)\backslash\mathcal{D}(K_d^{\perp}) \arrow[r, "\overline{\gamma}"] & \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)\backslash\mathcal{D}(K_d^{\perp}) \cong \overline{\mathcal{C}}_d^{\mathrm{norm}} \arrow[r, two heads] & \overline{\mathcal{C}}_d \\ \mathcal{M}_d \arrow[u, hook] \arrow[r, dashed, "\gamma"] & \mathcal{C}_d^{\mathrm{norm}} \arrow[r, two heads] & \mathcal{C}_d \arrow[u, hook] \end{tikzcd}$$

The spaces $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)\backslash\mathcal{D}(K_d^{\perp})$ and $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)\backslash\mathcal{D}(K_d^{\perp})$ can be seen as period domains of *marked* and *labelled* cubic fourfolds, respectively, see [Has00, Sec. 3.1, 5.2].

The following describes the generic fibre of the quotient map

$$\overline{\gamma}\colon \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)\backslash\mathcal{D}(K_d^{\perp}) \twoheadrightarrow \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)\backslash\mathcal{D}(K_d^{\perp}).$$

Proposition 2.2.6 ([Has00, Prop. 5.2.1])**.** *There is an isomorphism*

$$\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)/\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d) \cong \begin{cases} \{0\} & \text{if } d \equiv 2 \bmod 6 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } d \equiv 0 \bmod 6. \end{cases}$$

As a consequence, $\overline{\gamma}$ is an isomorphism when $d \equiv 2 \bmod 6$ and has degree two when $d \equiv 0 \bmod 6$. In the latter case, the covering involution of $\overline{\gamma}$ is induced by an automorphism $g \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)$ whose class modulo $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)$ generates $\mathbb{Z}/2\mathbb{Z}$. In the proof of the proposition for $d \equiv 0 \bmod 6$, which we will explain below, an explicit such g is constructed.

Fix a primitive negative definite sublattice $K_d \subset \Lambda_{\mathrm{cub}}$ containing h of rank two and discriminant d, where $3|d$. Let v_d be a generator of $K_d \cap \Lambda^0_{\mathrm{cub}}$. Suppose $T \in K_d$ is a primitive element such that h and T generate K_d. Then (h,T) is also divisible by 3. We can write $v_d = \frac{1}{3}(h,T)h - T$, which has square $-d/3$. It follows that $K_d = \mathbb{Z}h \oplus \mathbb{Z}v_d$.

Lemma 2.2.7. *Let $x \in \Lambda^0_{\mathrm{cub}}$ be primitive with $(x,x) \neq 0$ and $3 \not| (x,x)$. There exists an $f \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$ such that $f(x) = e_2 + \frac{(x,x)}{2} f_2$.*

Proof. As $\mathrm{Disc}\,\Lambda^0_{\mathrm{cub}} = \mathbb{Z}/3\mathbb{Z}$, the integer n defined by $\{(x,y) \mid y \in \Lambda^0_{\mathrm{cub}}\} = n\mathbb{Z}$ is either 1 or 3. Since $3 \not| (x,x)$, it must be 1. Therefore the class $\overline{x} \in \mathrm{Disc}\,\Lambda^0_{\mathrm{cub}}$ is trivial. By Eichler's criterion [GHS09, Prop. 3.3] there is an $f \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$ sending x to $e_2 + \frac{(x,x)}{2} f_2$. □

Proof of Proposition 2.2.6. Any $g \in \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)$ whose class modulo $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)$ is non-zero satisfies $g(v_d) = -v_d$. We construct an explicit such g. By Lemma 2.2.7, we can assume

$$v_d = e_2 - \tfrac{d}{6} f_2 \in \Lambda_{\mathrm{cub}} = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus \mathbb{Z}(-1)^{\oplus 3}.$$

We extend $v_d \mapsto -v_d$ to an element of $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}})$ by taking multiplication with -1 on $U_1 \oplus U_2$ and the identity on the rest of Λ_{cub}. This is unique up to composition with elements of $\widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, v_d)$. □

We will denote by g the automorphism of Λ_{cub} given by

$$g|_{E_8(-1)^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}} = \mathrm{id};\; g|_{U_1 \oplus U_2} = -\,\mathrm{id}\,.$$

2.3 The involution on $\mathcal{M}_d$

We will now prove Theorem 2.1. In the previous section, we explained that the map

$$\overline{\gamma}\colon \mathcal{QD}(\Lambda_d) \twoheadrightarrow \widetilde{\mathrm{O}}(\Lambda^0_{\mathrm{cub}}, \langle v_d\rangle)\backslash\mathcal{D}(K_d^{\perp})$$

is an isomorphism if $d \equiv 2 \bmod 6$ and has degree two if $d \equiv 0 \bmod 6$. In the second case, define $\tau\colon \mathcal{QD}(\Lambda_d) \to \mathcal{QD}(\Lambda_d)$ to be the covering involution of $\overline{\gamma}$. Note that τ maps $\mathcal{M}_d$ to itself: As explained in e.g. [Huy16, Rem. 6.3.7], the complement of $\mathcal{M}_d$ in $\mathcal{QD}(\Lambda_d)$ is

$$\bigcup_{\delta\in\Lambda_d,\, \delta^2=-2} \delta^{\perp}$$

and this set is clearly preserved under g.

Proposition 2.3.1. *The morphism τ does not depend on the choice of $\Lambda_d \cong K_d^{\perp}$.*

Proof. Precomposing the isomorphism $\Lambda_d \cong K_d^{\perp}$ with $f \in \mathrm{O}(\Lambda_d)$ changes g on Λ_d to $f^{-1}\circ g\circ f$. Note that this has the same action on $\mathrm{Disc}(\Lambda_d) \cong \mathbb{Z}/d\mathbb{Z}$ as g, thus it induces the same action on $\mathcal{QD}(\Lambda_d) = \widetilde{\mathrm{O}}(\Lambda_d)\backslash\mathcal{D}(\Lambda_d)$. □

For a K3 surface S, we denote by $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ the full cohomology of S with the Mukai pairing and the Hodge structure of weight two defined by $\widetilde{\mathrm{H}}^{2,0}(S) := \mathrm{H}^{2,0}(S)$. For a primitive vector $v = (r, \ell, s) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$, denote by $M_S(v)$ the moduli space of stable coherent sheaves on S with Mukai vector v, with respect to a generic polarization. Recall [Huy16, Ch. 10] that if there exists a w in $\widetilde{\mathrm{H}}^{1,1}(S,\mathbb{Z}) = \mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^{1,1}(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$ with $(v,w) = 1$, then this is a fine moduli space. If $(v)^2 = 0$ and $r > 0$, then $M_S(v)$ is a K3 surface.

Theorem 2.3.2. *Let $(S^\tau, L^\tau) = \tau(S,L)$. Then S^τ is isomorphic to the moduli space $M_S(v)$ of stable coherent sheaves on S with Mukai vector $v = (3, L, d/6)$. Under the natural identification $\mathrm{H}^2(S^\tau,\mathbb{Z}) \cong v^{\perp}/\mathbb{Z}v \subset \widetilde{\mathrm{H}}(S,\mathbb{Z})$, we have $L^\tau = (d, (d/3-1)L, d/3(d/6-1))$.*

Let us describe the strategy of the proof. The restriction of $g \in \mathrm{O}(\Lambda_{\mathrm{cub}})$ to $K_d^{\perp}$ can be viewed as an involution on Λ_d. Because g does not induce the identity on $\mathrm{Disc}\, K_d^{\perp}$, this involution does not extend to an automorphism on Λ_{K3}. However, we will show that g extends to $\widetilde{g} \in \mathrm{O}(\widetilde{\Lambda}_{\mathrm{K3}})$.

Using $\widetilde{g}$, we find S^τ as follows. Let x be a representative of the period of (S,L) in $\Lambda_{d,\mathbb{C}} \subset \widetilde{\Lambda}_{\mathrm{K3},\mathbb{C}}$. The K3 surface S^τ is the one whose period can be represented by $g(x) \in \Lambda_{d,\mathbb{C}}$. The map $\widetilde{g}$ induces a Hodge isometry $\widetilde{\mathrm{H}}(S,\mathbb{Z}) \cong \widetilde{\mathrm{H}}(S^\tau,\mathbb{Z})$, so by the derived Torelli theorem ([Orl97], see also [Huy16, Prop. 16.3.5]), S and S^τ are Fourier–Mukai partners.

More precisely, denote by $\widetilde{g}_{\mathrm{Muk}}$ the morphism $\widetilde{g}$, seen as an automorphism of Λ_{Muk}. Let $v = (r, \ell, s) := \widetilde{g}^{-1}_{\mathrm{Muk}}(0,0,1)$. We will see that $r > 0$, so there exists a universal sheaf $\mathcal{E}$ on $S \times M_S(v)$. Let $\Phi^H_{\mathcal{E}}\colon \widetilde{\mathrm{H}}(M_S(v),\mathbb{Z}) \to \widetilde{\mathrm{H}}(S,\mathbb{Z})$ be the induced cohomological Fourier–Mukai transform. Then $(\Phi^H_{\mathcal{E}})^{-1} \circ \widetilde{g}^{-1}_{\mathrm{Muk}}$ sends $\mathrm{H}^2(S^\tau,\mathbb{Z})$ to $\mathrm{H}^2(M_S(v),\mathbb{Z})$, which shows that S^τ is isomorphic to $M_S(v)$.

To describe L^τ, note that $\Phi_{\mathcal{E}}^H$ induces an isomorphism $\mathrm{H}^2(M_S(v),\mathbb{Z}) \cong v^\perp/\mathbb{Z}v$, where $v^\perp \subset \widetilde{\mathrm{H}}(S,\mathbb{Z})$ (this is a result by Mukai, see [Huy16, Rem. 10.3.7]). Thus, $\widetilde{g}_{\mathrm{Muk}}^{-1}$ restricts to an isomorphism $\mathrm{H}^2(S^\tau,\mathbb{Z}) \cong v^\perp/\mathbb{Z}v$. Under this identification, the polarization L^τ is mapped to $\widetilde{g}_{\mathrm{Muk}}^{-1}(\ell_d)$.

Remark 2.3.3. The extension $\widetilde{g}$ of g is not unique. But if $\widetilde{g}'$ is another extension, then $\widetilde{g}_{\mathrm{Muk}}^{-1} \circ \widetilde{g}'_{\mathrm{Muk}}$ is an orthogonal transformation of Λ_{Muk} sending $v' = (\widetilde{g}'_{\mathrm{Muk}})^{-1}(0,0,1)$ to v. This induces a Hodge isometry $\mathrm{H}^2(M_S(v'),\mathbb{Z}) \cong \mathrm{H}^2(M_S(v),\mathbb{Z})$, so $M_S(v')$ and $M_S(v)$ are isomorphic.

Remark 2.3.4. The space $\mathcal{QD}(\Lambda_d)$ can be interpreted as the moduli space of *quasi-polarized* K3 surfaces, i.e. pairs (S,L) with L the class of a big and nef line bundle, see [HP13, Sec. 5]. For such pairs the theorem is still valid.

Remark 2.3.5. For $d \equiv 0 \bmod 6$, the ramification locus of $\overline{\gamma}$ over $\mathcal{M}_d$ consists of those (S,L) which are polarized isomorphic to (S^τ, L^τ). It follows from [HP13, Sec. 8] that $\overline{\gamma}$ is unramified over $\{(S,L) \in \mathcal{M}_d \mid \rho(S) = 1\}$.

2.3.1 Proof of Theorem 2.1

We will first compute the action of g on $\operatorname{Disc} K_d^\perp \cong \operatorname{Disc}\Lambda_d$. We have seen that $K_d \cong \mathbb{Z}h \oplus \mathbb{Z}v_d$, so

$$\operatorname{Disc} K_d^\perp \cong \operatorname{Disc} K_d \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/\tfrac{d}{3}\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z},$$

since $9 \not| \, d$.

Lemma 2.3.6. *The action of g on* $\operatorname{Disc} K_d^\perp \cong \mathbb{Z}/d\mathbb{Z}$ *is given by $x \mapsto (d/3-1)x$.*

Proof. On $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/\frac{d}{3}\mathbb{Z}$, the map $\overline{g}$ is given by $(1,0) \mapsto (1,0)$ and $(0,1) \mapsto (0,-1)$ modulo d. Let α be such that $\overline{g}$ acts on $\mathbb{Z}/d\mathbb{Z}$ by $x \mapsto \alpha x$. We can compute α as follows: let (s,t) be any generator of $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/\frac{d}{3}\mathbb{Z}$. Then

$$((\alpha+1)s, (\alpha+1)t) = \overline{g}(s,t) + (s,t) = (2s, 0)$$

has order 3. Since $9 \not| \, d$, this means that $t(\alpha+1) \equiv 0 \bmod d/3$, and thus $\alpha \equiv -1 \bmod d/3$. So α is in $\{-1, d/3-1, 2d/3-1\}$.

We directly see that $\alpha = -1$ is not possible, since then $\alpha(s,t) + (s,t) = 0 \neq 2s$. Next, we should have $q_{K_d^\perp}(\overline{g}(s,t)) = q_{K_d^\perp}(s,t) \in \mathbb{Q}/2\mathbb{Z}$. Now $q_{K_d^\perp}(s,t) = n/d$ for some $n \in \mathbb{Z}$ with $\gcd(n,d) = 1$. Suppose that $\alpha = 2d/3 - 1$. Then

$$q_{K_d^\perp}(\alpha(s,t)) = (2d/3-1)^2 q_{K_d^\perp}(s,t) = 4n/3(d/3-1) + q_{K_d^\perp}(s,t).$$

Since n is not divisible by 3 and $d/3 \equiv 2 \bmod 3$, the number $4n/3(d/3-1)$ is not an integer, so $q_{K_d^\perp}(\alpha(s,t)) \neq q_{K_d^\perp}(s,t)$. We conclude that $\alpha = d/3 - 1$. $\square$

To extend g on Λ_d to $\widetilde{\Lambda}_{\mathrm{K3}}$ it thus suffices, by Proposition 1.2.3, to find an orthogonal transformation of $\Lambda_d^\perp = \mathbb{Z}\ell_d \oplus U_4$ acting on the discriminant group by $x \mapsto (d/3-1)x$. Consider $u \in \mathrm{O}(\mathbb{Z}\ell_d \oplus U_4)$ defined by

$$\begin{aligned} e_4 &\mapsto -\tfrac{d}{6}e_4 - \tfrac{1}{3}\left(\tfrac{d}{6}-1\right)\ell_d + \tfrac{1}{3}\left(\tfrac{d}{6}-1\right)^2 f_4 \\ f_4 &\mapsto 3e_4 + \ell_d - \tfrac{d}{6}f_4 \\ \ell_d &\mapsto de_4 + \left(\tfrac{d}{3}-1\right)\ell_d - \tfrac{d}{3}\left(\tfrac{d}{6}-1\right)f_4. \end{aligned}$$

One computes that this is an involution. The discriminant group $\mathrm{Disc}(\mathbb{Z}\ell_d \oplus U_4) \cong \mathbb{Z}/d\mathbb{Z}$ is generated by the class $\overline{\frac{1}{d}\ell_d}$, which $\overline{u}$ multiplies by $d/3-1$. So the action of $\overline{u}$ on $\mathbb{Z}/d\mathbb{Z}$ is given by $x \mapsto (d/3-1)x$.

It follows that $g \oplus u \in \mathrm{O}(\Lambda_d \oplus \mathbb{Z}\ell_d \oplus U_4)$ extends to $\widetilde{g} \in \mathrm{O}(\widetilde{\Lambda}_{\mathrm{K3}})$. Since $\widetilde{g}$ is an involution, we have $\widetilde{g}^{-1}(f_4) = \widetilde{g}(f_4) = 3e_4 + \ell_d - \frac{d}{6}f_4$. As an element of the Mukai lattice, this is

$$v = (3, \ell_d, d/6) \in \Lambda_{\mathrm{Muk}} = \mathbb{Z}e_4 \oplus \Lambda_{\mathrm{K3}} \oplus \mathbb{Z}(-f_4).$$

The polarization $L^\tau = \widetilde{g}^{-1}(\ell_d)$, seen as an element of $v^\perp/\mathbb{Z}v \subset \widetilde{\mathrm{H}}(S,\mathbb{Z})$, is

$$L^\tau = (d, (d/3-1)L, d/3(d/6-1)).$$

This finishes the proof of Theorem 2.1.

Remark 2.3.7. As shown in the proof of Proposition 2.3.1, we can replace g by any element of $\mathrm{O}(\Lambda_d)$ with the same action on $\mathrm{Disc}\,\Lambda_d$. For instance, we can take the automorphism given by the identity on $E_8(-1)^{\oplus 2} \oplus U_1$ and by the map u on $U_2 \oplus \mathbb{Z}(e_3 - \frac{d}{2}f_3) \cong (\mathbb{Z}\ell_d \oplus U_4)(-1)$. This allows us to define τ on $\mathcal{M}_d$ for all $d \equiv 0 \bmod 6$ with $d/6 \equiv 1 \bmod 3$.

2.4 Birationality of Hilbert schemes

In this section we study the Hilbert schemes of n points $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S^\tau)$ of our K3 surfaces S and S^τ. Corollary 2.4.9 and the results in Section 2.4.2 hold for all d such that $d/6 \equiv 1 \bmod 3$, using Remark 2.3.7.

2.4.1 Hilbert schemes of two points

For a cubic fourfold X we denote by $F(X)$ the Fano variety of lines on X, a four-dimensional hyperkähler variety of $\mathrm{K3}^{[2]}$ type. Hassett proved the following:

Theorem 2.4.1 ([Has00, Thm. 6.1.4]). *Assume that d satisfies*

$$d = 2(n^2+n+1)$$

for some integer $n \geq 2$. Let X be a generic cubic fourfold in $\mathcal{C}_d$. Then $F(X)$ is isomorphic to $\mathrm{Hilb}^2(S)$, where $(S,L) \in \mathcal{M}_d$ is associated to X.

If also $d \equiv 0 \bmod 6$, that $F(X)$ is isomorphic to both $\mathrm{Hilb}^2(S)$ and $\mathrm{Hilb}^2(S^\tau)$ (Hassett calls $F(X)$ *ambiguous*). Since birationality specializes in families of hyperkähler manifolds, it follows that $\mathrm{Hilb}^2(S)$ is birational to $\mathrm{Hilb}^2(S^\tau)$ for all K3 surfaces S of degree d.

We can generalize this using the following result by Addington. See also Remark 2.4.6.

Theorem 2.4.2 ([Add16, Thm. 2])**.** *A cubic fourfold X lies in $\mathcal{C}_d$ for some d satisfying*

$$(***) \qquad a^2 d = 2(n^2 + n + 1)$$

if and only if $F(X)$ is birational to $\mathrm{Hilb}^2(S)$ *for some K3 surface S.*

Note that $(***)$ implies $(**)$.

Lemma 2.4.3. *Suppose that d satisfies $(***)$. Then there exists a choice of the rational map $\varphi\colon \mathcal{M}_d \dashrightarrow \mathcal{C}_d$ such that if $(S, L) \in \mathcal{M}_d$ is associated to $X \in \mathcal{C}_d$ via φ, then* $\mathrm{Hilb}^2(S)$ *and $F(X)$ are birational.*

Proof. Consider the primitive sublattices

$$K_d^\perp \oplus T \subset \widetilde{\Lambda}_{\mathrm{K3}} \supset \Lambda_d \oplus \mathbb{Z}\ell_d \oplus U_4,$$

where $T \supset A_2 = \langle \lambda_1, \lambda_2 \rangle$ is the orthogonal complement of $K_d^\perp$ in $\widetilde{\Lambda}_{\mathrm{K3}}$. Then d satisfies $(**)$ if and only if $T \cong \mathbb{Z}\ell_d \oplus U_4$. Addington showed that $(***)$ holds if and only if $\psi\colon T \to \mathbb{Z}\ell_d \oplus U_4$ can be chosen such that $\psi(\lambda_1) = e_4 + f_4$. Extend ψ to an element of $\mathrm{O}(\widetilde{\Lambda}_{\mathrm{K3}})$ (use Proposition 1.2.3 and [Huy16, Thm. 14.2.4]) and let φ be the induced map $\mathcal{M}_d \dashrightarrow \mathcal{C}_d$.

Assume that $(S, L) \in \mathcal{M}_d$ is associated to $X \in \mathcal{C}_d$ via φ. Choose an isomorphism $\mathrm{H}^2(S, \mathbb{Z}) \cong U_4^\perp \subset \widetilde{\Lambda}_{\mathrm{K3}}$ sending L to ℓ_d, and consider the induced Hodge structure on $\widetilde{\Lambda}_{\mathrm{K3}}$. There are isometries of sub-Hodge structures

$$\mathrm{H}^2(F(X), \mathbb{Z}) \cong \lambda_1^\perp \cong \psi(\lambda_1)^\perp = (e_4 + f_4)^\perp \cong \mathrm{H}^2(M_S(1, 0, -1), \mathbb{Z}),$$

where the sign in $M_S(1, 0, -1)$ appears because we view the Mukai vector as an element of Λ_{Muk}. By Markman's birational Torelli theorem for manifolds of $\mathrm{K3}^{[n]}$ type, [Mar11, Cor. 9.9], $F(X)$ is birational to $M_S(1, 0, -1) \cong \mathrm{Hilb}^2(S)$. $\square$

Corollary 2.4.4. *When $d \equiv 0 \bmod 6$ satisfies $(***)$, then* $\mathrm{Hilb}^2(S) \sim_{\mathrm{bir}} \mathrm{Hilb}^2(S^\tau)$ *for any K3 surface $(S, L) \in \mathcal{M}_d$.*

The following proposition shows that we have more than just birationality: if d is such that $\mathrm{Hilb}^2(S) \sim_{\mathrm{bir}} \mathrm{Hilb}^2(S^\tau)$, then for S generic, $\mathrm{Hilb}^2(S)$ and $\mathrm{Hilb}^2(S^\tau)$ are isomorphic.

Proposition 2.4.5. *Let (S, L) be a polarized K3 surface of degree d with* $\mathrm{Pic}(S) = \mathbb{Z}L$ *and $3|d$. Then* $\mathrm{Hilb}^2(S)$ *has only one birational model.*

Proof. By [DM17, Thm. 5.1], the walls of the ample cone of $\mathrm{Hilb}^2(S)$ in the interior of the movable cone are given by the hypersurfaces $x^\perp \subset \mathrm{NS}(\mathrm{Hilb}^2(S)) \otimes \mathbb{R}$ for all $x \in \mathrm{NS}(\mathrm{Hilb}^2(S))$ of square -10 and divisibility two. We will show that there are no such x.

There is an isomorphism

$$\mathrm{NS}(\mathrm{Hilb}^2(S)) \cong \mathrm{NS}(S) \oplus \mathbb{Z}\delta = \mathbb{Z}L \oplus \mathbb{Z}\delta,$$

where δ is a (-2)-class orthogonal to L [Bea83]. So any class in $\mathrm{NS}(\mathrm{Hilb}^2(S))$ is given by $aL + b\delta$ for some $a, b \in \mathbb{Z}$, and its square is $a^2d - 2b^2$. Setting this equal to -10 gives the Pell equation $b^2 - a^2d/2 = 5$ which, after reducing modulo 3, gives $b^2 \equiv 2 \bmod 3$. This is not possible.

It follows that under any birational map $\mathrm{Hilb}^2(S) \dashrightarrow Y$ the pullback of an ample class is ample, thus the map is an isomorphism [Fuj81]. $\square$

Remark 2.4.6. This also implies that when d satisfies $(***)$ and $3|d$, then for a generic cubic fourfold X of discriminant d, $F(X)$ is actually *isomorphic* to $\mathrm{Hilb}^2(S)$ for a K3 surface S associated to X. It would be interesting to find out what happens when $3 \nmid d$. In that case there can be divisor classes of square -10 (for instance, when $d = 62$), but one would have to check the divisibility to find out whether $\mathrm{Hilb}^2(S)$ has more than one birational model.

It is natural to ask for the exact conditions on d for $\mathrm{Hilb}^2(S)$ to be birational to $\mathrm{Hilb}^2(S^\tau)$, for all S of degree d. It turns out that $(***)$ is too strong. We use the following two results by [MMY18]. Note that parts of the statements hold without the assumption $\rho(S) = 1$.

Proposition 2.4.7 ([MMY18, Prop. 2.1])**.** *Let S be a K3 surface and let $L \in \mathrm{H}^2(S, \mathbb{Z})$ be a $(1,1)$-class. Let $v = (x, cL, y)$ be a primitive isotropic Mukai vector. Then we have $v = (p^2r, pqL, q^2s)$ for some integers p, r, q, s with $\gcd(pr, qs) = 1$ and $(L)^2 = 2rs$. When $\mathrm{Pic}(S) = \mathbb{Z}L$ and $M_S(v)$ is a fine moduli space, then there is an isomorphism $M_S(v) \cong M_S(r, L, s)$. Moreover, $M_S(r, L, s)$ is isomorphic to $M_S(r', L, s')$ if and only if $\{r, s\} = \{r', s'\}$.*

Proposition 2.4.8 ([MMY18, Thm. 2.2])**.** *Let S_1 and S_2 be two derived equivalent K3 surfaces. Then $\mathrm{Hilb}^n(S_1)$ and $hilb^n(S_2)$ are birational if and only if $S_2 \cong M_{S_1}(p^2r, pqL, q^2s)$ for some p, q with $p^2r(n-1) - q^2s = \pm 1$ and $L \in \mathrm{H}^2(S, \mathbb{Z}) \cap \mathrm{H}^{1,1}(S)$. If X and Y have Picard number one, then $\{r, s\}$ is uniquely determined by S_2.*

Note that $p^2r(n-1) - q^2t = \pm 1$ is equivalent to $\big((1, 0, 1-n), (p^2r, pqL, q^2t)\big) = \pm 1$. So when (p^2r, pqL, q^2s) is primitive then $M_{S_1}(p^2r, pqL, q^2s)$ is a fine moduli space, isomorphic to $M_{S_1}(r, L, s)$ by Proposition 2.4.7.

Our description of τ gave us $S^\tau = M_S(3, L, d/6)$, so $r = 3$ and $s = d/6$. Thus, Proposition 2.4.8 tells us that $\mathrm{Hilb}^2(S) \sim_{\mathrm{bir}} \mathrm{Hilb}^2(S^\tau)$ if and only if there exist non-zero integers p, q such that $3p^2 - (d/6)q^2 = \pm 1$. Note that $3p^2 - (d/6)q^2 = 1$ does not happen in our case: since $d/6 \equiv 1 \bmod 3$, reducing modulo 3 gives $q^2 \equiv 2 \bmod 3$ which is not possible.

Corollary 2.4.9. *The schemes* $\mathrm{Hilb}^2(S)$ *and* $\mathrm{Hilb}^2(S^\tau)$ *are birational if and only if there exists an integral solution to the equation*

$$F : 3p^2 - (d/6)q^2 = -1.$$

If $\rho(S) = 1$*, this is equivalent to the existence of a line bundle on* $\mathrm{Hilb}^2(S)$ *of self-intersection 6.*

Proof. A class $aL + b\delta$ in $\mathrm{NS}(\mathrm{Hilb}^2(S)) = \mathbb{Z}L \oplus \mathbb{Z}\delta$ has square $a^2d - 2b^2 = 6$, in particular $b = 3b_0$ for some b_0, if and only if $3b_0^2 - (d/6)a^2 = -1$. □

Condition $(***)$ implies that F is solvable. Namely, assume we have $a^2d/2 = n^2+n+1$. Multiplying with 4 gives $(2a)^2d/2 = (2n + 1)^2 + 3$. As d is divisible by 3, so is $2n + 1$, and we find that $3(\frac{2n+1}{3})^2 - (2a)^2d/6 = -1$.

In fact, $(***)$ is equivalent to the existence of a solution to F with p odd and q even. One can show that such a solution always exists when $d/6$ is a prime $m \equiv 3 \bmod 4$. The following example shows that there exist d for which F is solvable but $(***)$ does not hold.

Example 2.4.10. Let $d = 78$, which satisfies $(**)$ but not $(***)$ (see [Add16]). Equation F holds with $p = 2$ and $q = 1$. In particular, $\mathrm{Hilb}^2(S)$ is birational to $\mathrm{Hilb}^2(S^\tau)$ for any S of degree 78.

More interesting is the next example, where F is not solvable. Because S and S^τ are derived equivalent, so are $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S^\tau)$ for all $n \geq 1$ [Plo07, Prop. 8]. Therefore, we obtain two derived equivalent Hilbert schemes of two points on K3 surfaces which are not birational. The first example of this phenomenon was given in [MMY18, Ex. 2.5]. Note the similarity between S^τ and the K3 surface Y in [MMY18, Prop. 1.2].

Example 2.4.11. Consider $d = 6 \cdot 73$. This again satisfies $(**)$ but not $(***)$. Note that F holds if and only if $(3p)^2 - (d/2)q^2 = -3$, which is equivalent to $x^2 - (d/2)y^2 = -3$ when 3 divides d. This is a usual Pell type equation and one can easily check (using e.g. [AAC10, Thm. 4.2.7]) that it has no solution for $d = 6 \cdot 73$. So for S generic of degree $6 \cdot 73$, $\mathrm{Hilb}^2(S)$ is not birational to $\mathrm{Hilb}^2(S^\tau)$.

2.4.2 Higher-dimensional Hilbert schemes

For $n \geq 2$, Proposition 2.4.8 tells us that $\mathrm{Hilb}^n(S)$ and $\mathrm{Hilb}^n(S^\tau)$ are birational if and only if there is a solution to

$$F_1 : 3p^2(n - 1) - (d/6)q^2 = -1$$

or to

$$F_2 : 3p^2 - (d/6)q^2(n - 1) = -1.$$

We give some examples for low n.

$n = 3$. The lowest d satisfying $(**)$ and $6 \mid d$ is $d = 42$. Equation F_1 with $n = 3$ reads $6p^2 - 7q^2 = -1$, which is solved by $p = q = 1$. In general, one can show that if $d/6$ is a prime $m \equiv 5, 7 \bmod 8$, then $\mathrm{Hilb}^3(S) \sim_{\mathrm{bir}} \mathrm{Hilb}^3(S^\tau)$.

$n = 4$. In this case, only F_1 is solvable and reads $(3p)^2 - (d/6)q^2 = -1$. This is always solvable when $d/6$ is a prime $m \equiv 1 \bmod 4$. Namely, note that when $m > 2$ is prime, $x^2 - my^2 = -1$ has a solution if and only if $m \equiv 1 \bmod 4$. Reducing this modulo 3 gives $x^2 - y^2 \equiv -1 \bmod 3$. This implies that $x^2 \equiv 0 \bmod 3$. Writing $x = 3x'$ gives $9(x')^2 - (d/6)y^2 = -1$, i.e. F_1 with $n = 4$.

$n = 5$. Equations F_1 and F_2 are given by

$$F_1 :\ 3(2p)^2 - (d/6)q^2 = -1$$
$$F_2 :\ 3p^2 - (d/6)(2q)^2 = -1$$

which are both solutions for F. Since in $3x^2 - (d/6)y^2 = -1$ one of x, y has to be even, the existence of a solution for F also implies the existence of a solution for F_1 or F_2. This shows that $\mathrm{Hilb}^2(S) \sim_{\mathrm{bir}} \mathrm{Hilb}^2(S^\tau)$ if and only if $\mathrm{Hilb}^5(S) \sim_{\mathrm{bir}} \mathrm{Hilb}^5(S^\tau)$.

Chapter 3

Moduli spaces of twisted K3 surfaces and cubic fourfolds

A twisted K3 surface is a pair (S, α) consisting of a K3 surface S and a Brauer class α on S. Using the isomorphism $\mathrm{Br}(S) \cong \mathrm{H}^2(S, \mathcal{O}_X^*)_{\mathrm{tors}}$, twisted K3 surfaces can be seen as a degree two version of polarized K3 surfaces. We may also view them from the perspective of Hitchin's generalized K3 surfaces [Hit03], using α to change the volume form on S. This gives us a generalized Calabi–Yau structure, to which we associate a Hodge structure $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$ of K3 type on the full cohomology of S [HS05]. In this way, we can view (S, α) as a geometric realization of a point in the extended period domain for K3 surfaces.

This chapter is concerned with polarized twisted K3 surfaces, that is, K3 surfaces together with a Brauer class and a primitive ample class in $\mathrm{H}^2(X, \mathbb{Z})$. Our first goal is to construct a moduli space of these objects, fixing the degree of the polarization and the order of the Brauer class. This can be done up to the following concession: when $\rho(S) > 1$, one parametrizes lifts of Brauer classes to $\mathrm{H}^2(S, \mathbb{Q})$, which gives a strictly bigger group than $\mathrm{Br}(S)$.

Theorem 3.1 (see Def. 3.2.1, Prop. 3.2.4)**.** *There exists a scheme $\mathrm{M}_d[r]$ which is a coarse moduli space for triples (S, L, α) where S is a K3 surface, $L \in \mathrm{H}^2(S, \mathbb{Z})$ is a polarization of degree $(L)^2 = d$ and α is an element of $\mathrm{Hom}(\mathrm{H}^2(S, Z)_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$. This group has a surjection to $\mathrm{Br}(S)[r]$, which is an isomorphism if and only if $\rho(S) = 1$.*

We prove this by mimicking the construction of the moduli space of (untwisted) polarized K3 surfaces via the period domain. In particular, $\mathrm{M}_d[r]$ is a quasi-projective variety with at most finite quotient singularities, whose number of connected components is at most $r \cdot \gcd(r, d)$ (Proposition 3.2.5).

In the second part of the chapter, we will concentrate on a Hodge-theoretic relation between twisted K3 surfaces and special cubic fourfolds. For untwisted K3 surfaces, this relation was first studied by Hassett [Has00]. In particular, he constructed, for d satisfying a numerical condition $(**)$, rational maps

$$\mathrm{M}_d \dashrightarrow \mathcal{C}_d$$

from the moduli space of polarized K3 surfaces of degree d to the moduli space of special cubic fourfolds of discriminant d, sending a K3 surface to the cubic it is associated to.

Associated twisted K3 surfaces were studied in [Huy17], extending results of [AT14]. The numerical condition $(**')$ on the discriminant given by Huybrechts can be formulated as follows:

$$(**') \qquad d' = dr^2, \text{ where } d \text{ satisfies } (**).$$

We give a full generalization of Hassett's results to the setting of twisted K3 surfaces.

Theorem 3.2 (see Cor. 3.3.6)**.** *A cubic fourfold X is in $\mathcal{C}_{d'}$ for some d' satisfying $(**')$ if and only if for every decomposition $d' = dr^2$ with d satisfying $(**)$, X has an associated polarized twisted K3 surface of degree d and order r.*

We also give the analogous construction of Hassett's rational maps to $\mathcal{C}_d$. Just like for untwisted K3 surfaces, these maps are either birational or of degree two. We end with a discussion of the covering involution in the degree two case, relating this chapter to Chapter 2.

Notation

- Λ is the lattice isomorphic to the second cohomology $\mathrm{H}^2(S,\mathbb{Z})$ of a K3 surface S.
- $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ is the full cohomology of S with the Mukai pairing, viewed as an ungraded module.
- $\widetilde{\Lambda}$ is the lattice isomorphic to $\widetilde{\mathrm{H}}(S,\mathbb{Z})$. There is an isomorphism $\widetilde{\Lambda} \cong \Lambda \oplus U$.
- $\Lambda_d \subset \Lambda$ is the orthogonal complement of a primitive element $\ell_d \in \Lambda$ of square d, which is unique up to $\mathrm{O}(\Lambda)$.
- $\Lambda_{d,r}^\vee := \mathrm{Hom}(\Lambda_d, \mathbb{Z}/r\mathbb{Z}) \cong \Lambda_d^\vee \otimes \mathbb{Z}/r\mathbb{Z}$.
- $\widetilde{\mathrm{O}}(\Lambda_d) := \mathrm{Ker}(\mathrm{O}(\Lambda_d) \to \mathrm{O}(\mathrm{Disc}\,\Lambda_d))$. This group acts naturally on $\Lambda_{d,r}^\vee$.
- For a lattice isomorphism $\varphi\colon L \to L'$, φ_r is the induced map $L^\vee \otimes \mathbb{Z}/r\mathbb{Z} \to (L')^\vee \otimes \mathbb{Z}/r\mathbb{Z}$.
- $G[r]$ is the r-torsion subgroup of an abelian group G.
- Cohomology with coefficients in $\mathbb{G}_m$ means étale cohomology. Otherwise we always use the analytic topology.

Remark 3.0.1. By a moduli functor $\mathcal{M}$, we will mean a functor on the category of schemes of finite type over $\mathrm{Spec}\,\mathbb{C}$. A coarse moduli space for $\mathcal{M}$ is a scheme M with a morphism $\xi\colon \mathcal{M} \to \mathrm{M}$ such that $\xi(\mathbb{C})$ is a bijection, and we have factorization over M of morphisms $\mathcal{M} \to T$ for T a $\mathbb{C}$-scheme of finite type.

3.1 Twisted K3 surfaces

3.1.1 Definitions

For references, see [Huy09], [Huy05]. Recall that the *Brauer group* $\mathrm{Br}(X)$ of a scheme X is the group of sheaves of Azumaya algebras modulo Morita equivalence, with multiplication given by the tensor product. If X is quasi-compact and separated and has an ample line bundle, then $\mathrm{Br}(X)$ is isomorphic to the *cohomological Brauer group*

$$\mathrm{Br}(X)' := \mathrm{H}^2(X, \mathbb{G}_m)_{\mathrm{tors}},$$

which equals $\mathrm{H}^2(X, \mathbb{G}_m)$ when X is regular and integral. If X is a complex K3 surface, one can moreover show that

$$\mathrm{Br}(X) \cong \mathrm{H}^2(X, \mathcal{O}_X^*)_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}.$$

A *twisted K3 surface* is a pair (S, α) where S is a K3 surface and $\alpha \in \mathrm{Br}(S)$. Two twisted K3 surfaces (S, α) and (S', α') are *isomorphic* if there exists an isomorphism $f\colon S \to S'$ such that $f^*\alpha' = \alpha$.

One sees from the exponential sequence on S that any Brauer class $\alpha \in \mathrm{H}^2(S, \mathcal{O}_X^*)_{\mathrm{tors}}$ can be written as $\exp(B^{0,2})$ for some $B \in \mathrm{H}^2(S, \mathbb{Q})$, which is unique up to $\mathrm{H}^2(S, \mathbb{Z})$ and $\mathrm{NS}(S) \otimes \mathbb{Q}$. Thus, denoting by $T(S)$ the transcendental lattice of S, intersecting with B gives a linear map $f_\alpha = (B, -)\colon T(S) \to \mathbb{Q}/\mathbb{Z}$ which only depends on α. One can show that $\alpha \mapsto f_\alpha$ yields an isomorphism $\mathrm{Br}(S) \cong \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$.

Given a lift $B \in \mathrm{H}^2(S, \mathbb{Q})$ of α, we define a weight two Hodge structure of K3 type $\widetilde{\mathrm{H}}(S, B, \mathbb{Z})$ on the full cohomology of S by

$$\widetilde{\mathrm{H}}^{2,0}(S, B) := \mathbb{C}[\exp(B)\sigma] \subset \widetilde{\mathrm{H}}(S, \mathbb{C}),$$

where σ is a nowhere degenerate holomorphic 2-form on S and $\exp(B)\sigma := \sigma + B \wedge \sigma$. This does not depend on our choice of B (up to non-canonical isomorphism), so we can define

$$\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) := \widetilde{\mathrm{H}}(S, B, \mathbb{Z})$$

for any $B \in \mathrm{H}^2(X, \mathbb{Q})$ with $\exp(B^{0,2}) = \alpha$.

The *Picard group* of (S, α) is defined as $\widetilde{\mathrm{H}}^{1,1}(S, \alpha) \cap \widetilde{\mathrm{H}}(S, \mathbb{Z})$, so

$$\mathrm{Pic}(S, \alpha) = \{\delta \mid (\delta, \exp(B)\sigma) = 0\} \subset \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$$

for $B \in \mathrm{H}^2(S, \mathbb{Q})$ lifting α. If α is trivial, then $\mathrm{Pic}(S, \alpha) = \mathrm{H}^0(S, \mathbb{Z}) \oplus \mathrm{Pic}(S) \oplus \mathrm{H}^4(S, \mathbb{Z})$. The *transcendental lattice* $T(S, \alpha)$ is defined as the orthogonal complement of $\mathrm{Pic}(S, \alpha)$ in $\mathrm{H}^*(S, \alpha, \mathbb{Z})$. If α is trivial, then $T(S, \alpha)$ is the transcendental lattice $T(S)$ of S. One can show that

$$T(S, \alpha) \cong \mathrm{Ker}(f_\alpha\colon T(S) \to \mathbb{Q}/\mathbb{Z}) = \{x \in T(S) \mid (B, x) \in \mathbb{Z}\}.$$

Definition 3.1.1. A *polarized twisted K3 surface* is a triple (S, L, α), where S is a K3 surface, L is a primitive ample line bundle on S and $\alpha \in \mathrm{Br}(S)$ Two twisted polarized K3 surfaces (S, L, α) and (S', L', α') are isomorphic if there exists an isomorphism $f\colon S \to S'$ such that $f^*L' = L$ and $f^*\alpha' = \alpha$.

We define two invariants of (S, L, α): its *degree* $d = (L)^2$ and its *order* $r = \mathrm{ord}(\alpha)$ (also known as its *period*).

3.1.2 A non-existence result for moduli spaces

Ideally, one would like to find a (coarse) moduli space $\mathrm{N}_d[r]$ for the following functor:

$$\mathcal{N}_d[r]\colon (Sch/\mathbb{C})^o \to (Sets), T \mapsto \{(f\colon S \to T, L, \alpha)\}.$$

Here, $(f : S \to T, L \in \mathrm{H}^0(T, R^1 f_*\mathbb{G}_m))$ is a smooth proper family of polarized K3 surfaces of degree d and $\alpha \in \mathrm{H}^0(T, R^2 f_*\mathbb{G}_m)$ such that for any closed point $x \in T$, base change gives a Brauer class $\alpha_x \in \mathrm{H}^2(S_x, \mathbb{G}_m)[r]$.

It is, however, not difficult to show that $N_d[r]$ does not exist as a locally Noetherian scheme. Namely, suppose $\Psi\colon \mathcal{N}_d[r] \to \mathrm{N}_d[r]$ exists. Consider the natural transformation $\xi\colon \mathcal{N}_d[r] \to \mathcal{M}_d$ which at a scheme T is defined by $(S \to T, L, \alpha) \mapsto (S \to T, L)$. By the properties of a coarse moduli space, there exists a unique morphism $\pi\colon \mathrm{N}_d[r] \to \mathrm{M}_d$ which makes the following diagram commute:

$$\begin{tikzcd} \mathcal{N}_d[r] \arrow[r, "\Psi"] \arrow[d, "\xi"'] & \mathrm{N}_d[r] \arrow[d, dashed, "\exists\pi"] \\ \mathcal{M}_d \arrow[r, "\Phi"'] & \mathrm{M}_d\,. \end{tikzcd}$$

For a closed point $y \in \mathrm{N}_d[r]$ corresponding to a tuple (S, L, α), the image $\pi(y)$ should be the point x of M_d corresponding to (S, L). So the fibre of π over x is

$$(\mathrm{N}_d[r])_x = \{(S, L, \alpha) \mid \alpha \in \mathrm{Br}(S),\ \operatorname{ord}\alpha = r\}/\operatorname{Aut}(S, L).$$

For $d > 2$, let $U \subset \mathrm{M}_d$ be the open subset where $\mathrm{Aut}(S, L)$ is trivial. Over U, we have $(\mathrm{N}_d[r])_x \cong \mathrm{Br}(S)[r] \cong (Z/r\mathbb{Z})^{22-\rho(S)}$. In particular, $\pi|_{\mathrm{N}_d[r]\times_{\mathrm{M}_d} U}$ is ramified exactly over the locus where $\rho(S) > 1$. However, this set is dense in U, thus not closed, giving a contradiction.

For $d = 2$, let $U \subset \mathrm{M}_2$ be the open subset where $\mathrm{Aut}(S, L) \cong \mathbb{Z}/2\mathbb{Z}$. Then over U, we have $2^{21-\rho(S)} \leq |(\mathrm{N}_2[r])_x| \leq 2^{22-\rho(S)}$. So $\pi|_{\mathrm{N}_2^r \times_{\mathrm{M}_2} U}$ is ramified (at least) over the locus where $\rho(S) > 2$, again a dense set in U, which leads to a contradiction.

If we require that $\mathrm{ord}(\alpha)$ equals r, we can prove non-existence in a similar way. We obtain a morphism π to M_d such that over an open subset $U \subset \mathrm{M}_d$, the cardinality of the fibre of π over $(S, L) \in U$ is the number of elements of order r in $(\mathbb{Z}/r\mathbb{Z})^{22-\rho(S)}$ (or half this number when $d = 2$). Again, $\pi|_{\pi^{-1}(U)}$ is ramified exactly over the locus where $\rho(S) > 1$ (at least over the locus where $\rho(S) > 2$ when $d = 2$), a contradiction.

3.2 Moduli spaces of polarized twisted K3 surfaces

We will construct a slightly different moduli space $\mathrm{M}_d[r]$ mapping to M_d, whose fibre over $(S, L) \in \mathrm{M}_d$ parametrizes triples (S, L, α) with $\alpha \in \mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$. There is a surjective homomorphism from this group to $\mathrm{Br}(S)[r]$, which is an isomorphism if and only if $\rho(S) = 1$.

3.2.1 Definition of the moduli functor

Note that the Kummer sequence

$$0 \to \mu_r \to \mathbb{G}_m \overset{(\cdot)^r}{\to} \mathbb{G}_m \to 0$$

induces a short exact sequence

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Z}/r\mathbb{Z} \to \mathrm{H}^2(S, \mu_r) \to \mathrm{Br}(S)[r] \to 0.$$

If $L \in \mathrm{H}^2(S, \mathbb{Z})$ is a polarization, we have injections

$$\mathbb{Z}/r\mathbb{Z} \cdot L \hookrightarrow \mathrm{Pic}\, S \otimes \mathbb{Z}/r\mathbb{Z} \hookrightarrow \mathrm{H}^2(S, \mathbb{Z}/r\mathbb{Z}).$$

Hence, we get a surjective map

$$\begin{array}{ccc} \mathrm{H}^2(S, \mathbb{Z}/r\mathbb{Z})/(\mathbb{Z}/r\mathbb{Z} \cdot L) & \twoheadrightarrow & \mathrm{H}^2(S, \mathbb{Z}/r\mathbb{Z})/(\mathrm{Pic}\, S \otimes \mathbb{Z}/r\mathbb{Z}) \cong \mathrm{Br}(S)[r] \\ \| & & \| \\ \mathrm{H}^2(S, \mathbb{Z})^{\vee}_{\mathrm{pr}} \otimes \mathbb{Z}/r\mathbb{Z} & & T(S)^{\vee} \otimes \mathbb{Z}/r\mathbb{Z} \end{array}$$

which is an isomorphism if and only if $\rho(S) = 1$.

We define a relative version of $\mathrm{H}^2(S, \mathbb{Z})^{\vee}_{\mathrm{pr}} \otimes \mathbb{Z}/r\mathbb{Z} \cong \mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$ as follows. For a smooth proper family $(f\colon S \to T, L)$ of polarized K3 surfaces, set

$$R^2_{\mathrm{pr}} f_* \mathbb{Z} := \mathrm{Ker}\left(R^2 f_* \mathbb{Z} \xrightarrow{\cdot c_1(L)} R^4 f_* \mathbb{Z}\right)$$

where $c_1(L)$ is the image of L under $\mathrm{H}^0(T, R^1 f_* \mathbb{G}_m) \to \mathrm{H}^0(T, R^2 f_* \mathbb{Z})$. Let $\mathcal{F}[r]$ be the following local system:

$$\mathcal{F}[r] := \mathscr{H}om(R^2_{\mathrm{pr}} f_* \mathbb{Z}, \underline{\mathbb{Z}/r\mathbb{Z}}).$$

Definition 3.2.1. The moduli functor $\mathcal{M}_d[r]$ is defined as

$$\mathcal{M}_d[r]\colon (Sch/\mathbb{C})^o \to (Sets),\ T \mapsto \{(f\colon S \to T, L, \alpha)\}/_{\cong}$$

where $\big(f : S \to T, L \in \mathrm{H}^0(T, R^1 f_* \mathbb{G}_m)\big)$ is a smooth proper family of polarized K3 surfaces of degree d and $\alpha \in \mathrm{H}^0(T, \mathcal{F}[r])$. We define

$$\mathcal{M}_d^r\colon (Sch/\mathbb{C})^o \to (Sets)$$

to be the subfunctor sending a scheme T to the set of those tuples (f, L, α) for which α has order r.

We will construct coarse moduli spaces for $\mathcal{M}_d[r]$ and $\mathcal{M}_d^r$.

3.2.2 Construction of the moduli space

Recall the construction of M_d as a quotient of a bounded symmetric domain, see e.g. [Huy16]. The moduli functor $\mathcal{M}_d^{\mathrm{mar}}$ of marked polarized K3 surfaces of degree d is given by

$$\mathcal{M}_d^{\mathrm{mar}}(T) = \{(f\colon S \to T, L \in \mathrm{H}^0(T, R^1 f_*\mathbb{G}_m), \varphi\colon R^2_{\mathrm{pr}} f_*\mathbb{Z} \cong \underline{\Lambda_d})\}.$$

It has an analytic fine moduli space $\mathrm{M}_d^{\mathrm{mar}}$, which can be constructed as an open submanifold of the period domain $\mathcal{D}(\Lambda_d)$ of Λ_d. In particular, there exists a universal family

$$(f\colon S^{\mathrm{mar}} \to \mathrm{M}_d^{\mathrm{mar}}, L^{\mathrm{mar}}, \varphi^{\mathrm{mar}}).$$

We denote the morphism $\mathcal{M}_d^{\mathrm{mar}} \to \mathrm{M}_d^{\mathrm{mar}}$ by Φ^{mar}. The moduli space M_d is obtained from $\mathrm{M}_d^{\mathrm{mar}}$ by taking the quotient under the action of $\widetilde{\mathrm{O}}(\Lambda_d)$.

Note that φ^{mar} induces an isomorphism $\varphi_r^{\mathrm{mar}}\colon \mathcal{F}[r] \cong \underline{\Lambda_{d,r}^\vee}$, thus, using that $\mathrm{M}_d^{\mathrm{mar}}$ is irreducible, $\mathcal{F}[r]$ is the sheaf of sections of the trivial finite cover

$$\begin{aligned} \mathrm{M}_d^{\mathrm{mar}}[r] &:= \underline{\mathrm{Spec}}\, \mathscr{H}\!om(\underline{\Lambda_{d,r}^\vee}, \mathcal{O}_{\mathrm{M}_d^{\mathrm{mar}}}) \\ &= \mathrm{M}_d^{\mathrm{mar}} \times \Lambda_{d,r}^\vee. \end{aligned}$$

The space $\mathrm{M}_d^{\mathrm{mar}}[r]$ is a coarse moduli space for the functor

$$\mathcal{M}_d^{\mathrm{mar}}[r]\colon (Sch/\mathbb{C})^o \to (Sets),\ T \mapsto \{(f\colon S \to T, L, \varphi, \alpha)\},$$

where $(f\colon S \to T, L, \varphi) \in \mathrm{M}_d^{\mathrm{mar}}(T)$ and $\alpha \in \mathrm{H}^0(T, \mathcal{F}[r])$, and the morphism

$$\Phi^{\mathrm{mar}}[r]\colon \mathcal{M}_d^{\mathrm{mar}}[r] \to \mathrm{M}_d^{\mathrm{mar}}[r]$$

is defined over a scheme T by

$$(S \to T, L, \varphi, \alpha) \mapsto (\Phi^{\mathrm{mar}}(S \to T, L, \varphi), \varphi_r(\alpha)).$$

So we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}_d^{\mathrm{mar}} \arrow[r, "\Phi^{\mathrm{mar}}"] & \mathrm{M}_d^{\mathrm{mar}} \\ \mathcal{M}_d^{\mathrm{mar}}[r] \arrow[u] \arrow[r, "\Phi^{\mathrm{mar}}[r]"'] & \mathrm{M}_d^{\mathrm{mar}}[r] \arrow[u] \end{tikzcd}$$

The action of $\widetilde{\mathrm{O}}(\Lambda_d)$ on $\mathrm{M}_d^{\mathrm{mar}}$ lifts to $\mathrm{M}_d^{\mathrm{mar}}[r]$ via

$$g(S, L, \varphi, \alpha) = \left(S, L, g \circ \varphi, \varphi_r^{-1} g \varphi_r(\alpha)\right).$$

Under $\mathrm{M}_d^{\mathrm{mar}}[r] \hookrightarrow \mathcal{D}(\Lambda_d) \times \Lambda_{d,r}^\vee$, this is the restriction of the natural action of $\widetilde{\mathrm{O}}(\Lambda_d)$ on $\mathcal{D}(\Lambda_d) \times \Lambda_{d,r}^\vee$. This action is properly discontinuous: it is on $\mathcal{D}(\Lambda_d)$, see [Huy16, Rem. 6.1.10], so also on the product with the finite group $\Lambda^\vee_{d,r}$. Hence, its quotient exists as a complex analytic space. In particular, the quotient

$$\mathrm{M}_d[r] := \mathrm{M}_d^{\mathrm{mar}}[r]/\widetilde{\mathrm{O}}(\Lambda_d)$$

exists as a complex space. Similarly, let $\mathrm{M}_d^{\mathrm{mar},r} \subset \mathrm{M}_d^{\mathrm{mar}}[r]$ be the union of those connected components corresponding to elements of $\Lambda^\vee_{d,r}$ of order r. Then the quotient

$$\mathrm{M}_d^r := \mathrm{M}_d^{\mathrm{mar},r}/\widetilde{\mathrm{O}}(\Lambda_d)$$

exists as a complex space.

We claim that $\mathrm{M}_d[r]$ and M_d^r are in fact quasi-projective varieties. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{M}_d^{\mathrm{mar}}[r] = \mathrm{M}_d^{\mathrm{mar}} \times \Lambda^\vee_{d,r} \arrow[r, "\pi"] \arrow[d] & \Lambda^\vee_{d,r} \arrow[d] \\
\mathrm{M}_d[r] \arrow[r, "\bar\pi"] & \Lambda^\vee_{d,r}/\widetilde{\mathrm{O}}(\Lambda_d).
\end{tikzcd}$$

Giving the sets on the right side the discrete topology, all these maps are continuous. So under $\bar\pi$, each connected component of $\mathrm{M}_d[r]$ is mapped to a point. Vice versa, given $[v] \in \Lambda^\vee_{d,r}/\widetilde{\mathrm{O}}(\Lambda_d)$, its inverse image under $\bar\pi$ is

$$\mathrm{M}_v := \left(\mathrm{M}_d^{\mathrm{mar}} \times \widetilde{\mathrm{O}}(\Lambda_d)\cdot v\right)/\widetilde{\mathrm{O}}(\Lambda_d) \cong \left(\mathrm{M}_d^{\mathrm{mar}} \times \{v\}\right)/\operatorname{Stab}(v)$$

where $\operatorname{Stab}(v) \subset \widetilde{\mathrm{O}}(\Lambda_d)$ is the stabilizer of $v \in \Lambda^\vee_{d,r}$ under the action of $\widetilde{\mathrm{O}}(\Lambda_d)$ on $\Lambda^\vee_{d,r}$. Now $\operatorname{Stab} v$ contains the reflection s_δ for an element $\delta \in \Lambda_d$ of square -2 orthogonal to v and ℓ'_d, which interchanges the two connected components of $\mathrm{M}_d^{\mathrm{mar}}$. Hence, M_v is connected, even irreducible. Thus, the components of $\mathrm{M}_d[r]$ are in one-to-one correspondence with $\Lambda^\vee_{d,r}/\widetilde{\mathrm{O}}(\Lambda_d)$. Each component M_v parametrizes triples (S, L, α) that admit a marking φ with $\varphi_r(\alpha) = v$. The components belonging to M_d^r are those M_v for which v has order r.

Corollary 3.2.2. *Every connected component of* $\mathrm{M}_d[r]$ *(and therefore of* M_d^r*) is an irreducible, quasi-projective variety with at most finite quotient singularities.*

Proof. The finite index subgroup $\operatorname{Stab}(v) \subset \widetilde{\mathrm{O}}(\Lambda_d)$ being arithmetic, the quotient space $\mathcal{D}(\Lambda_d)/\operatorname{Stab}(v)$ is a quasi-projective variety with finite quotient singularities by [BB66] and [Sat80, Lemma IV.7.2]. To show that $\mathrm{M}_v = \left(\mathrm{M}_d^{\mathrm{mar}} \times \{v\}\right)/\operatorname{Stab}(v)$ is a Zariski open subset of it, we use the same argument as for the algebraicity of the moduli space of untwisted polarized K3 surfaces, see e.g. [Huy16, Sec. 6.4.1].

Let ℓ be a large enough multiple of r such that there exists a fine moduli space $\mathrm{M}_d^{\mathrm{lev}}$ of polarized K3 surfaces with a $\Lambda/\ell\Lambda$-level structure, see Remark 3.2.3. For the universal family $\pi\colon S^{\mathrm{lev}} \to \mathrm{M}_d^{\mathrm{lev}}$, there exists a marking $R^2_{\mathrm{pr}}\pi_*\mathbb{Z} \otimes \underline{\mathbb{Z}/\ell\mathbb{Z}} \cong \underline{\Lambda_d \otimes \mathbb{Z}/\ell\mathbb{Z}}$. This induces a holomorphic map $\mathrm{M}_d^{\mathrm{lev}} \to \mathcal{D}(\Lambda_d)/\Gamma_\ell$, where

$$\Gamma_\ell = \{g \in \widetilde{\mathrm{O}}(\Lambda_d) \mid g \equiv \mathrm{id} \bmod \ell\} \subset \operatorname{Stab}(v).$$

The image of this map is $\mathrm{M}_d^{\mathrm{mar}}/\Gamma_\ell$. Dividing out further by $\mathrm{Stab}(v)$ yields a holomorphic map

$$\mathrm{M}_d^{\mathrm{lev}} \twoheadrightarrow \mathrm{M}_d^{\mathrm{mar}}/\,\mathrm{Stab}(v) \subset \mathcal{D}(\Lambda_d)/\,\mathrm{Stab}(v).$$

By a theorem of Borel [Bor72] (and also [Sat80, Lemma IV.7.2]), this map is algebraic, and therefore the image $\mathrm{M}_d^{\mathrm{mar}}/\,\mathrm{Stab}(v)$ is constructible. Since it is also analytically open in $\mathcal{D}(\Lambda_d)/\,\mathrm{Stab}(v)$, it is Zariski open. □

Remark 3.2.3. Recall (see e.g. [Huy16, Sec. 6.4.2]) that for ℓ large enough, there exists a fine moduli space $\mathrm{M}_d^{\mathrm{lev}}$ of polarized K3 surfaces (S, L) of degree d with a $\Lambda/\ell\Lambda$-*level structure*, i.e. an isometry $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} \otimes \mathbb{Z}/\ell\mathbb{Z} \cong \Lambda_d \otimes \mathbb{Z}/\ell\mathbb{Z}$. The space $\mathrm{M}_d^{\mathrm{lev}}$ is a smooth quasi-projective variety which is a finite cover of M_d. We could have constructed $\mathrm{M}_d[r]$ as a quotient of $\mathrm{M}_d^{\mathrm{lev}}[r] := \mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee}$ instead, choosing ℓ to be a large enough multiple of r.

One constructs a morphism $\Psi\colon \mathcal{M}_d[r] \to \mathrm{M}_d[r]$ in the following way. Consider a point $(f\colon S \to T, L, \alpha)$ in $\mathcal{M}_d[r](T)$. Proceeding as for untwisted polarized K3 surfaces, we pass to the (infinite) étale covering

$$\eta\colon \widetilde{T} := \mathrm{Isom}(R^2_{\mathrm{pr}} f_*\mathbb{Z}, \underline{\Lambda_d}) \to T,$$

which has a natural $\widetilde{\mathrm{O}}(\Lambda_d)$-action, satisfying $\widetilde{T}/\widetilde{\mathrm{O}}(\Lambda_d) \cong T$. Write $\widetilde{f}\colon \widetilde{S} \to \widetilde{T}$ for the pullback family. The local system $R^2_{\mathrm{pr}}\widetilde{f}_*\mathbb{Z}$ is trivial: there exists a canonical isomorphism $\varphi\colon R^2_{\mathrm{pr}}\widetilde{f}_*\mathbb{Z} \cong \underline{\Lambda_d}$. Now $\Phi^{\mathrm{mar}}[r]\big(\widetilde{S}, \eta^*L, \varphi, \eta^*\alpha\big)$ is an element of $\mathrm{M}_d^{\mathrm{mar}}[r](\widetilde{T})$, i.e. a holomorphic map $\widetilde{T} \to \mathrm{M}_d^{\mathrm{mar}}[r]$. This map is $\widetilde{\mathrm{O}}(\Lambda_d)$-equivariant, thus descends to a map $T \to \mathrm{M}_d[r]$. This map is algebraic by Borel's theorem, thus defines a point in $\mathrm{M}_d[r](T)$. We let $\Psi(S \to T, L, \alpha)$ be this point.

Proposition 3.2.4. *The space* $\mathrm{M}_d[r]$ *is a coarse moduli space for the functor* $\mathcal{M}_d[r]$.

Proof. By definition, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_d^{\mathrm{mar}}[r] \arrow[r, "\Phi^{\mathrm{mar}}[r]"] \arrow[d, "F"'] & \mathrm{M}_d^{\mathrm{mar}}[r] \arrow[d, "q"] \\
\mathcal{M}_d[r] \arrow[r, "\Psi"'] & \mathrm{M}_d[r]
\end{tikzcd}$$

where the map F forgets the marking and q is the quotient map. We need to show that $\Psi(\mathbb{C})\colon \mathcal{M}_d[r](\mathbb{C}) \to \mathrm{M}_d[r](\mathbb{C})$ is a bijection. For $x \in \mathrm{M}_d[r](\mathbb{C})$, let $y \in \mathrm{M}_d^{\mathrm{mar}}[r](\mathbb{C})$ such that $q(y) = x$. Set $\Psi(\mathbb{C})^{-1}(x) := F(\Phi^{\mathrm{mar}}[r](\mathbb{C})^{-1}(y))$; note that this does not depend on the choice of y. One checks that $\Psi(\mathbb{C})^{-1}$ defines a set-theoretic inverse to $\Psi(\mathbb{C})$.

For the universal property of Ψ, let $s\colon \mathcal{M}_d[r] \to T$ be a morphism to a scheme T. Then $s \circ F$ is a map from $\mathcal{M}_d^{\mathrm{mar}}[r]$ to T; since $\mathcal{M}_d^{\mathrm{mar}}[r] \to \mathrm{M}_d^{\mathrm{mar}}[r]$ is a coarse moduli space, this induces a unique holomorphic map $t\colon \mathrm{M}_d^{\mathrm{mar}}[r] \to T$ such that $t \circ \Phi^{\mathrm{mar}}[r] = s \circ F$. It follows from the uniqueness that t is equivariant, thus factors over a holomorphic map $\mathrm{M}_d[r] \to T$. We will show that this map is algebraic.

Like before, let ℓ be a large enough multiple of r such that there exists a fine moduli space $\mathrm{M}_d^{\mathrm{lev}}$ of K3 surfaces with a $\Lambda/\ell\Lambda$-level structure. Then the map $\mathrm{M}_d^{\mathrm{mar}}[r] \to T$ factors as

$$\mathrm{M}_d^{\mathrm{mar}}[r] \to \mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee} \to \mathrm{M}_d[r] \to T$$

(see Remark 3.2.3). The map $\mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee} \to T$ is algebraic and equivariant under the algebraic action of $\widetilde{\mathrm{O}}(\Lambda_d)$. The induced algebraic morphism on $(\mathrm{M}_d^{\mathrm{lev}} \times \Lambda_{d,r}^{\vee})/\widetilde{\mathrm{O}}(\Lambda_d)$ is the given map $\mathrm{M}_d[r] \to T$. $\square$

The proof that M_d^r is a coarse moduli space for $\mathcal{M}_d^r$ is similar.

Proposition 3.2.5. *The moduli space* M_d^r *has at most* $r \cdot \gcd(r,d)$ *many connected components.*

This follows directly from the following lemma. Denote $\Lambda = E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus U_3$. Let $\{e_i, f_i\}$ be the standard basis for the i'th copy of U. Fix $\ell_d := e_3 + \frac{d}{2} f_3$ and $\ell_d' := e_3 - \frac{d}{2} f_3$, so $\Lambda_d^{\vee} \cong E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2 \oplus \mathbb{Z}(\frac{1}{d}\ell_d')$.

Lemma 3.2.6. *Every element of order* r *in* $\Lambda_{d,r}^{\vee}$ *is equivalent under the action of* $\widetilde{\mathrm{O}}(\Lambda_d)$ *to*

$$w_{n,k} := e_1 + n f_1 + \tfrac{k}{d}\ell_d'$$

for some $n, k \in \mathbb{Z}$. *Moreover, if* $n \equiv n' \bmod r$ *and* $k \equiv k' \bmod \gcd(r,d)$, *then* $w_{n,k}$ *and* $w_{n',k'}$ *are equivalent.*

Proof. Elements in $\Lambda_{d,r}^{\vee}$ of order r are of the form $m[w]$ where $\gcd(m,r) = 1$ and $w \in \Lambda_d^{\vee}$ is primitive, so $w = sx + \frac{t}{d}\ell_d'$ for some $x \in E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2$ primitive and $\gcd(s,t) = 1$. Write $d = d_0 \cdot \gcd(d,t)$ and $t = t_0 \cdot \gcd(d,t)$. Then $d_0 w = d_0 s x + t_0 \ell_d' \in \Lambda_d$ is primitive and

$$\begin{aligned} (d_0 w, \Lambda_d) &= \gcd\big((d_0 s x, E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2), (t_0 \ell_d', \mathbb{Z}\ell_d')\big) \\ &= \gcd(d_0 s, d t_0) \\ &= d_0. \end{aligned}$$

By Eichler's criterion [GHS09, Prop. 3.3], $d_0 w$ is equivalent under $\widetilde{\mathrm{O}}(\Lambda_d)$ to $d_0(e_1 + n f_1) + t_0 \ell_d'$ for some n. So w is equivalent to $e_1 + n f_1 + \frac{t}{d}\ell_d' = w_{n,t}$.

Now $mw \equiv m(w_{n,t})$ is equivalent modulo r to $m e_1 + (mn + r) f_1 + \frac{mt}{d}\ell_d'$. Since we assumed that $\gcd(r,m) = 1$, this element is primitive, thus by the above, it is equivalent under $\widetilde{\mathrm{O}}(\Lambda_d)$ to some $w_{n',t'}$.

Next, note that if $t' \equiv t \bmod d$, then $w_{n,t}$ is equivalent to $w_{n',t'}$ for some n' (by Eichler's criterion). Write $\gcd(r,d) = pr + qd$. Then

$$w_{n,\gcd(r,d)+t} = e_1 + n f_1 + (pr + qd + t)\tfrac{1}{d}\ell_d' \equiv e_1 + n' f_1 + \tfrac{t}{d}\ell_d'$$

for some n'. We see that every $w_{n,k}$ is equivalent to some $w_{n',k'}$ with $0 \le n' < r$ and $0 \le k' < \gcd(r,d)$. $\square$

3.3 Twisted K3 surfaces and cubic fourfolds

Recall that a smooth cubic fourfold X is *special* if the lattice $\mathrm{H}^{2,2}(X) \cap \mathrm{H}^4(X, \mathbb{Z})$ has rank at least two. Special cubic fourfolds form a countably infinite union of irreducible divisors $\mathcal{C}_d$ in the moduli space of cubic fourfolds. Here $\mathcal{C}_d \neq \emptyset$ if and only if $d > 6$ and $d \equiv 0, 2 \mod 6$. Hassett [Has00] showed that X is in $\mathcal{C}_d$ with d satisfying

(∗∗) d is even and not divisible by 4, 9, or any odd prime $p \equiv 2 \mod 3$

if and only if there exists a polarized K3 surface (S, L) of degree d whose primitive cohomology $\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}$ can be embedded Hodge-isometrically into $\mathrm{H}^4(X, \mathbb{Z})$, up to a sign and a Tate twist.

We will generalize this result to twisted K3 surfaces in Theorem 3.3.5 and Corollary 3.3.6. An important ingredient will be the period map for polarized twisted K3 surfaces.

3.3.1 Period maps for twisted K3 surfaces

We have seen that the connected components of M_d^r are of the form

$$\mathrm{M}_v = (\mathrm{M}_d^{\mathrm{mar}} \times \{v\}) / \operatorname{Stab}(v)$$

for $[v] \in \Lambda_{d,r}^\vee / \widetilde{\mathrm{O}}(\Lambda_d)$. We will construct a period map from $\mathrm{M}_d^{\mathrm{mar}} \times \{v\}$ to the period domain $\mathcal{D}(T_v)$ of the lattice

$$T_v := \operatorname{Ker}(v \colon \Lambda_d \to \mathbb{Z}/r\mathbb{Z}).$$

Recall that for an even lattice Γ and $B \in \Gamma$, the *B-field shift* $\exp(B) \in \mathrm{O}(\Gamma \oplus U)$ is defined by

$$z \mapsto z + (B.z)f, \;\; e \mapsto e - B + \tfrac{(B)^2}{2} f, \;\; f \mapsto f$$

for $z \in \Gamma$, where $\{e, f\}$ is the standard basis of the hyperbolic plane U. For $B \in \Gamma_{\mathbb{Q}}$, we define $\exp(B) \in \mathrm{O}((\Gamma \oplus U)_{\mathbb{Q}})$ by linear extension.

Let $(S, L, \varphi, v) \in \mathrm{M}_d^{\mathrm{mar}} \times \{v\}$. The corresponding twisted Hodge structure $\widetilde{\mathrm{H}}(S, v, \mathbb{Z})$ on S is given as follows. Let $x \in \Lambda_d^\vee$ be a representative of v. Let

$$x' = (\varphi^\vee)^{-1}(x) \in \mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}^\vee \subset \mathrm{H}^2(S, \mathbb{Q}).$$

Then $\widetilde{\mathrm{H}}^{2,0}(S, v)$ is $\mathbb{C}[\sigma + x' \wedge \sigma]$, where σ is a non-degenerate holomorphic 2-form on S, and the Hodge structure does not depend on the choice of x (up to non-canonical isomorphism).

Let $\widetilde{\Lambda} = \Lambda \oplus U_4$ be the extended K3 lattice. We can extend φ to an isomorphism $\widetilde{\varphi} \colon \widetilde{\mathrm{H}}(S, \mathbb{Z}) \to \widetilde{\Lambda}$ by sending $1 \in \mathrm{H}^0(S, \mathbb{Z})$ to $e_4 \in U_4$ and $1 \in \mathrm{H}^4(S, \mathbb{Z})$ to $f_4 \in U_4$. Then

$$\widetilde{\varphi}(\sigma + x' \wedge \sigma) = \varphi(\sigma) + (x, \varphi(\sigma))f_4 = \exp(x)\varphi(\sigma).$$

We thus obtain a map

$$\mathcal{P}_x\colon \mathrm{M}_d^{\mathrm{mar}} \times \{v\} \to \mathcal{D}\big((\exp(x)\Lambda_d) \cap \widetilde{\Lambda}\big)$$

sending (S, L, φ, v) to $[\widetilde{\varphi}(\mathrm{H}^{2,0}(S, v))]$.

We can get rid of the choice of x in the following way. First, the lattice T_v is a finite index sublattice of Λ_d, so we have $\mathcal{D}(T_v) = \mathcal{D}(\Lambda_d)$. Second, note that the map $\exp(v)$ gives an isomorphism $T_v \cong (\exp(v)\Lambda_d) \cap \widetilde{\Lambda}$. We see that $\mathcal{P}_x$ factors over the usual period map $\mathcal{P}$ for $\mathrm{M}_d^{\mathrm{mar}}$: the diagram

$$\begin{tikzcd}
\mathrm{M}_d^{\mathrm{mar}} \times \{v\} \cong \mathrm{M}_d^{\mathrm{mar}} \arrow[r, "\mathcal{P}"] \arrow[d, "\mathcal{P}_x"'] & \mathcal{D}(\Lambda_d) \arrow[d, "="] \\
\mathcal{D}\big((\exp(x)\Lambda_d) \cap \widetilde{\Lambda}\big) \arrow[r, "\cong"'] & \mathcal{D}(T_v)
\end{tikzcd}$$

commutes. Denote by $\mathcal{P}_v$ the composition from $\mathrm{M}_d^{\mathrm{mar}} \times \{v\}$ to $\mathcal{D}(T_v)$. It follows from the above diagram that $\mathcal{P}_v$ is holomorphic and injective.

The group $\mathrm{Stab}_{\widetilde{\mathrm{O}}(\Lambda_d)}(v) \subset \widetilde{\mathrm{O}}(\Lambda_d)$ is an arithmetic group acting holomorphically on $\mathcal{D}(T_v)$, so by Baily–Borel, the quotient $M_v = (\mathrm{M}_d^{\mathrm{mar}} \times \{v\}) / \mathrm{Stab}(v)$ is a quasi-projective variety. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{M}_d^{\mathrm{mar}} \times \{v\} \arrow[r, hook, "\mathcal{P}_v"] \arrow[d] & \mathcal{D}(T_v) \arrow[d] \\
M_v \arrow[r, hook, "\overline{\mathcal{P}}_v"] & \mathcal{D}(T_v) / \mathrm{Stab}(v)
\end{tikzcd}$$

where $\overline{\mathcal{P}}_v$ is algebraic by the same argument as in Corollary 3.2.2 (note that when ℓ is a multiple of $r^2 d$, the group $\Gamma_\ell = \{g \in \widetilde{\mathrm{O}}(\Lambda_d) \mid g \equiv \mathrm{id} \bmod \ell\}$ is contained in $\mathrm{Stab}(v)$).

Recall (see e.g. [Huy16, Rem. 6.4.5]) that $\mathcal{D}(T_v) \setminus \operatorname{Im} \mathcal{P}_v = \mathcal{D}(\Lambda_d) \setminus \operatorname{Im} \mathcal{P}$ is a union of hyperplanes $\bigcup_{\delta \in \Delta(\Lambda_d)} \delta^\perp$, where $\Delta(\Lambda_d)$ is the set of (-2)-classes in Λ_d. It follows that $\mathcal{D}(T_v)$ parametrizes periods of twisted *quasi-polarized* K3 surfaces, i.e. twisted K3 surfaces with a line bundle that is nef and big (however, the corresponding moduli stack is not separated). Hence, the quotient $\mathcal{D}(T_v) / \mathrm{Stab}(v)$ be viewed as a moduli space of quasi-polarized twisted K3 surfaces.

3.3.2 The lattice T_v

Let us denote by $\mathrm{H}^4(X, \mathbb{Z})^-$ the middle cohomology of a cubic fourfold X with the intersection product changed by a sign. This cubic lies in the divisor $\mathcal{C}_d$ if and only if there exists a primitive sublattice

$$K_d \subset \mathrm{H}^{2,2}(X) \cap \mathrm{H}^4(X, \mathbb{Z})^-$$

of rank two and discriminant d containing the square of the hyperplane class. The orthogonal complement $K_d^\perp \subset \mathrm{H}^4(X,\mathbb{Z})$ has an induced Hodge structure which determines X when $X \in \mathcal{C}_d$ is very general. As a lattice, $K_d^\perp$ only depends on d. Theorem 3.3.5 tells us that for certain d and r, and $v \in \mathrm{Hom}(\Lambda_d, \mathbb{Q}/\mathbb{Z})$ of order r, the lattice T_v is isomorphic to $K_{dr^2}^\perp$.

We will see that the discriminant group of T_v can always be generated by three elements. As T_v has rank 21, it follows that T_v is determined by its discriminant group and the quadratic form on it, see e.g. [Huy16, Thm. 14.1.5]. To prove Theorem 3.3.5, it thus suffices to determine when

$$\left(\operatorname{Disc} T_v, q_{T_v}\right) \cong \left(\operatorname{Disc} K_{dr^2}^\perp, q_{K_{dr^2}^\perp}\right).$$

Write $d' = dr^2$. We will use the following result by Hassett (using our sign convention):

Proposition 3.3.1. [Has00, Prop. 3.2.5] *When $d' \equiv 0 \bmod 6$, then $\operatorname{Disc}(K_{d'}^\perp)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/\frac{d'}{3}\mathbb{Z}$, which is cyclic unless 9 divides d'. One can choose generators $(1,0)$ and $(0,1)$ such that the quadratic form $q_{K_{d'}^\perp}$ satisfies $q_{K_{d'}^\perp}(1,0) = 3/d' \bmod 2\mathbb{Z}$ and $q_{K_{d'}^\perp}(1,0) = -2/3 \bmod 2\mathbb{Z}$.*

When $d' \equiv 2 \bmod 6$, then we have $\operatorname{Disc}(K_{d'}^\perp) \cong \mathbb{Z}/d'\mathbb{Z}$. One can choose a generator u such that $q_{K_{d'}^\perp}(u) = (1-2d')/3d' \bmod 2\mathbb{Z}$.

Discriminant group of T_v

Let $v \in \Lambda_d^\vee$ such that $[v] \in \Lambda_{d,r}^\vee$ has order r. We will describe the group $\operatorname{Disc} T_v$ and the quadratic form on it. Note that if $g \in \widetilde{\mathrm{O}}(\Lambda_d)$, then g induces an isomorphism $T_v \cong T_{g(v)}$. So by Lemma 3.2.6, we can assume that

$$v = w_{n,k} = e_1 + nf_1 + \tfrac{k}{d}\ell'_d$$

for some n, k. Then $T_v = E_8(-1)^{\oplus 2} \oplus U_2 \oplus T_0$, where

$$T_0 = \{y \in U_1 \oplus \mathbb{Z}\ell'_d \mid (y,v) \in r\mathbb{Z}\} = \langle e_1 - nf_1, rf_1, kf_1 + \ell'_d\rangle.$$

Since $E_8(-1)^{\oplus 2} \oplus U_2$ is unimodular, $\operatorname{Disc} T_v$ is isomorphic to $\operatorname{Disc} T_0$. The intersection matrix of T_0 is (compare [Rei19, Lemma 2.12])

$$\begin{pmatrix} -2n & r & k \\ r & 0 & 0 \\ k & 0 & -d \end{pmatrix}$$

The discriminant group $\operatorname{Disc} T_0$ can be expressed in the invariants

$$g_1 = \gcd(2n, r, k, d),\ g_2 = \gcd(r^2, kr, rd, 2nd - k^2)/g_1,\ g_3 = dr^2/g_1g_2;$$

we have

$$\operatorname{Disc} T_0 = \mathbb{Z}/g_1\mathbb{Z} \times \mathbb{Z}/g_2\mathbb{Z} \times \mathbb{Z}/g_3\mathbb{Z}.$$

We obtain

Proposition 3.3.2. *Let $v = w_{n,k} \in \Lambda_d^\vee$.*

(i) *The group* $\operatorname{Disc} T_v$ *is cyclic if and only if* $\gcd(r, 2nd - k^2) = 1$.

(ii) *We have*

$$\operatorname{Disc} T_v \cong \mathbb{Z}/(r^2 d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

if and only if $\gcd(r, 2nd - k^2) = 3$, *and if* $3|d$ *then* $9 \nmid nd$.

In order to determine the quadratic form on $\operatorname{Disc} T_v$, we write down explicit generators. Consider the following elements of $\operatorname{Disc} T_0$:

$$\begin{aligned}[][f_1] &= [\tfrac{1}{r}(rf_1)] \\ [\ell'_d/d] &= [\tfrac{1}{d}(kf_1 + \ell'_d) - \tfrac{k}{rd}(rf_1)] \\ [v] &= [\tfrac{1}{r}(e_1 - nf_1) + \tfrac{2nd-k^2}{r^2 d}(rf_1) + \tfrac{k}{rd}(kf_1 + \ell'_d)].\end{aligned}$$

Note that

$$\operatorname{ord}[v] = \operatorname{lcm}\left(\tfrac{r^2 d}{\gcd(r^2 d, 2nd-k^2)}, \tfrac{rd}{\gcd(rd,k)}, r\right).$$

One checks that the group generated $[f_1]$, $[\ell'_d/d]$ and $[v]$ has $r^2 d$ many elements, so it is the whole discriminant group:

$$\operatorname{Disc} T_v = \langle [f_1], [\ell'_d/d], [v] \rangle.$$

Lemma 3.3.3. *If* $\gcd(d, k, r) = s$, *then there is an integer* p *such that* $\gcd(d, k + pr) = s$.

First assume $\operatorname{Disc} T_v$ is cyclic, so $\gcd(r, 2nd - k^2) = 1$. In particular, this implies $\gcd(r, d, k) = 1$. By Lemma 3.3.3 there exists a p such that $\gcd(d, k + pr) = 1$. Since $T_{w_{n,k}} \cong T_{w_{n,k+pr}}$, we can replace k by $k + pr$. Then we have $\gcd(r^2 d, 2nd - k^2) = 1$; hence, $[v]$ generates $\operatorname{Disc} T_v$. So the quadratic form q_{T_v} on $\operatorname{Disc} T_v$ is determined by

$$q_{T_v}([v]) = [(v)^2] = \tfrac{1}{r^2 d}(2nd - k^2) \bmod 2\mathbb{Z}.$$

Next, assume $\operatorname{Disc} T_v$ is isomorphic to $\mathbb{Z}/(r^2 d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. If 3 divides d, then we have $\gcd(r, 2nd - k^2) = 3$, and $9 \nmid nd$ implies $9 \nmid 2nd - k^2$. It follows that $\gcd(r^2, 2nd - k^2) = 3$, so $[v]$ generates $\mathbb{Z}/(dr^2/3)\mathbb{Z}$. As a generator of the factor $\mathbb{Z}/3\mathbb{Z}$, we take the element

$$u := \tfrac{k}{3}[f_1] - \tfrac{d}{3}[\ell'_d/d] = \tfrac{1}{3}[kf_1 + \ell'_d].$$

We have $(u)^2 = -\frac{d}{9} \bmod 2\mathbb{Z}$.

If 3 does not divide d, we may have $9 | 2nd - k^2$, but this implies $9 \nmid r$. As above, we may replace n by $n + r$ and obtain $9 \nmid 2nd - k^2$. This gives $\gcd(r^2, 2nd - k^2) = 3$, so $[v]$ generates $\mathbb{Z}/(dr^2/3)\mathbb{Z}$. For a generator of the factor $\mathbb{Z}/3\mathbb{Z}$, consider

$$u' := \tfrac{rd}{3}[v] - \tfrac{2nd-k^2}{3}[f_1] = \tfrac{1}{3}([d(e_1 - nf_1) + k(kf_1 + \ell'_d)]).$$

We have $(u')^2 = \frac{d}{9}(k^2 - 2nd) \bmod 2\mathbb{Z}$.

Corollary 3.3.4. *Let $v = w_{n,k} \in \Lambda_d^\vee$.*

(i) *If* $\operatorname{Disc} T_v$ *is cyclic, there exists a generator t such that*

$$q_{T_v}(t) = \tfrac{1}{r^2 d}(2nd - k^2) \bmod 2\mathbb{Z}.$$

(ii) *If* $\operatorname{Disc} T_v \cong \mathbb{Z}/(r^2d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *and* $3 \nmid d$*, there exist generators* $(1,0)$ *and* $(0,1)$ *such that*

$$q_{T_v}(1,0) = \tfrac{1}{r^2 d}(2nd - k^2) \bmod 2\mathbb{Z}$$

and

$$q_{T_v}(0,1) = \tfrac{d}{9}(k^2 - 2nd) \bmod 2\mathbb{Z}.$$

(iii) *If* $\operatorname{Disc} T_v \cong \mathbb{Z}/(r^2d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *and* $3|d$*, there exist generators* $(1,0)$ *and* $(0,1)$ *such that*

$$q_{T_v}(1,0) = \tfrac{1}{r^2 d}(2nd - k^2) \bmod 2\mathbb{Z}$$

and

$$q_{T_v}(0,1) = -\tfrac{d}{9} \bmod 2\mathbb{Z}.$$

3.3.3 Existence of associated twisted K3 surfaces

Consider the following condition on $d' \in \mathbb{Z}$.

$$(**') \qquad d' = dr^2, \text{ where } d \text{ satisfies } (**).$$

The results of Section 3.3.2 allow us to prove the following.

Theorem 3.3.5. *The number d' satisfies $(**')$ if and only if for every decomposition $d' = dr^2$ with d satisfying $(**)$, there exists an element $v \in \operatorname{Hom}(\Lambda_d, \mathbb{Q}/\mathbb{Z})$ of order r such that $K_{d'}^\perp$ is isomorphic to* $\operatorname{Ker} v$.

For a cubic fourfold $X \in \mathcal{C}_{d'}$, the inclusion $K_{d'}^\perp \subset \mathrm{H}^4(X,\mathbb{Z})$ gives an induced Hodge structure of K3 type on $K_{d'}^\perp$ and thus on $T_v = \operatorname{Ker} v$, yielding a point x in the period domain $\mathcal{D}(T_v)$. In [Voi86], it was shown that for a smooth cubic fourfold X, there are no classes in $\mathrm{H}^4(X,\mathbb{Z})_{\mathrm{pr}} \cap \mathrm{H}^{2,2}(X)$ of square 2. It follows that the class of x in $\mathcal{D}(T_v)/\operatorname{Stab} v$ lies in the image of the period map $\mathcal{P}_v$. As a consequence, we obtain

Corollary 3.3.6. *A cubic fourfold X is in $\mathcal{C}_{d'}$ for some d' satisfying $(**')$ if and only if for every decomposition $d' = dr^2$ with d satisfying $(**)$, there exists a polarized K3 surface (S, L) of degree d and an element $\alpha \in \operatorname{Hom}(\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}, \mathbb{Q}/\mathbb{Z})$ of order r such that $K_{d'}^\perp$ is Hodge isometric to* $\operatorname{Ker} \alpha$*, up to a sign and a Tate twist.*

Remark 3.3.7. We will say that the twisted K3 surface in Corollary 3.3.6 is *associated* to X. One can show that this notion is the same as the one given by Huybrechts [Huy17]. He relates the full cohomology $\widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z})$ to the Hodge structure $\widetilde{\mathrm{H}}(\mathcal{A}_X,\mathbb{Z})$ of K3 type associated to the K3 category $\mathcal{A}_X \subset \mathrm{D}^{\mathrm{b}}(X)$, which was introduced in [AT14].

To be precise, a cubic fourfold X is in $\mathcal{C}_{d'}$ for some d' satisfying $(**')$ if and only if there is a twisted K3 surface (S,α) such that $\widetilde{\mathrm{H}}(\mathcal{A}_X,\mathbb{Z})$ is Hodge isometric to $\widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z})$. When S has Picard number one, it follows that $K_d^\perp \subset \mathrm{H}^4(X,\mathbb{Z})$ is Hodge isometric to $\mathrm{Ker}(\alpha\colon \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} \to \mathbb{Q}/\mathbb{Z})$ (these are the transcendental parts of $\widetilde{\mathrm{H}}(\mathcal{A}_X,\mathbb{Z})$ and $\widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z})$). When $\rho(S)>1$, the same holds for some lift of $\alpha\in\mathrm{Br}(S)$ to $\mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}},\mathbb{Q}/\mathbb{Z})$. Vice versa, one can show that a Hodge isometry

$$K_{d'}^\perp \cong \mathrm{Ker}(\alpha\colon\ \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} \to \mathbb{Q}/\mathbb{Z})$$

always extends to $\widetilde{\mathrm{H}}(\mathcal{A}_X,\mathbb{Z}) \cong \widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z})$, see Proposition 3.4.6.

In particular, one can use Huybrechts' result to prove a weaker version of Corollary 3.3.6, replacing "every decomposition" with "some decomposition".

We prove Theorem 3.3.5 by comparing the quadratic forms on $\mathrm{Disc}\, K_{d'}^\perp$ and $\mathrm{Disc}\, T_v$. We distinguish the cases when the groups are cyclic and non-cyclic. We will use the following statements, which follow from quadratic reciprocity, see also [Has00, proof of Prop. 5.1.4].

Lemma 3.3.8. *When $d\equiv 2 \bmod 6$, then d satisfies $(**)$ if and only if -3 is a square modulo $2d$. When $d\equiv 0 \bmod 6$, write $d=6t$. Then d satisfies $(**)$ if and only if -3 is a square modulo $4t$ and $4t$ is a square modulo 3.*

Proof of Theorem 3.3.5, cyclic case

Assuming that $\mathrm{Disc}\, K_{d'}^\perp$ is cyclic, we will show that $d' = dr^2$ with d satisfying $(**')$ if and only if there exists a $v\in\Lambda_{d,r}^\vee$ of order r such that $K_{d'}^\perp \cong T_v$. The proof consists of Propositions 3.3.9 and 3.3.10.

Proposition 3.3.9. *Assume that $\mathrm{Disc}\, K_{d'}^\perp$ is cyclic. If there is a $v\in\Lambda_{d,r}^\vee$ of order r such that $K_{d'}^\perp\cong T_v$, in particular $d'=dr^2$, then d satisfies $(**)$.*

Proof. First assume that 3 does not divide d. By Proposition 3.3.1 and Corollary 3.3.4, we have $K_{d'}^\perp \cong T_{w_{n,k}}$ if and only if there is an x such that

$$\tfrac{x^2}{r^2d}(k^2-2nd) \equiv \tfrac{2dr^2-1}{3dr^2} \bmod 2.$$

Multiplying by $3dr^2$ gives

$$3x^2(k^2-2nd) \equiv 2dr^2-1 \bmod 6dr^2$$

which is equivalent to

$$3x^2(k^2-2nd) \equiv -1 \bmod 2dr^2. \tag{3.1}$$

It follows that -3 is a square modulo $2d$, so by Lemma 3.3.8, d satisfies $(**)$.

Next we assume $3|d$. By Proposition 3.3.1, and Corollary 3.3.4, we have $K_{d'}^\perp\cong T_{w_{n,k}}$ if and only if there is an x such that

$$\tfrac{x^2}{r^2d}(k^2-2nd) \equiv \tfrac{2}{3} - \tfrac{3}{dr^2} \bmod 2.$$

Writing $d = 6t$ and multiplying by dr^2 gives

$$x^2(k^2 - 12nt) \equiv 4tr^2 - 3 \bmod 12tr^2. \tag{3.2}$$

In particular, -3 is a square modulo $4t$, and we have $4tr^2 \equiv x^2k^2 \bmod 3$. Since 3 does not divide r, this implies that $4t$ is a square modulo 3. It follows from Lemma 3.3.8 that d satisfies $(**)$. □

Write $r = 2^s q r_0$ where q consists of all prime factors of r which are 1 modulo 3, and r_0 consists of all odd prime factors of r which are 2 modulo 3. In particular, dq^2 still satisfies $(**)$, and we have $\gcd(r_0, dq^2) = 1$.

Proposition 3.3.10. *There exists an n such that for $v = w_{nq^2,r_0} \in \Lambda^\vee_{d,r}$, we have $K_{d'}^\perp \cong T_v$.*

Proof. We first assume $3 \nmid d$. By (3.1) we have to show that for some x and some n,

$$f_n(x) := 3x^2(r_0^2 - 2ndq^2) + 1 \equiv 0 \bmod m \tag{3.3}$$

where $m = 2dr^2$.

Since dq^2 satisfies $(**)$, the number -3 is a square modulo $2dq^2$. As $3r_0$ is invertible in $\mathbb{Z}/2dq^2\mathbb{Z}$, we have $-3 \equiv (3r_0x)^2 \bmod 2dq^2$ for some $x \in \mathbb{Z}$. This yields $3x^2r_0^2 + 1 \equiv 0 \bmod 2dq^2$, which shows that (3.3) has a solution modulo $m = 2dq^2$, for any n. In particular, it has solutions modulo $dq^2/2$ and modulo 4.

It follows that the equation $(f_n/2)(x) = 0$ has a solution modulo 2. Besides, $(f_n/2)'(x) = 3x(r_0^2 - 2ndq^2)$ is always odd. By Hensel's lemma, $(f_n/2)(x) = 0$ has a solution modulo 2^l for any $l \geq 1$. It follows that (3.3) has a solution modulo 2^l for any $l \geq 2$.

By the Chinese remainder theorem, there exists a solution x for (3.3) modulo $m = 2d(2^sq)^2$. We can assume $\gcd(x, r_0) = 1$: Otherwise, write $ar_0 + b \cdot 2d(2^sq)^2 = 1$ and replace x by $x + b \cdot 2d(2^sq)^2(1 - x) = 1 + ar_0(x - 1)$.

Now we have $\gcd(r_0^2, 6x^2dq^2) = 1$, so there exist a and b such that $ar_0^2 - b \cdot 6x^2dq^2 = 1$. In particular, r_0^2 divides $3x^2 \cdot 2bdq^2 + 1$. We see that for $n = b$, there is a solution to (3.3) modulo $m = r_0^2$. By the Chinese remainder theorem, there exists a solution modulo $2dr^2$.

Next, assume $3|d$. Write $d = 6t$. By (3.2) we have to show that for some x and n,

$$g_n(x) := x^2(r_0^2 - 12ntq^2) - 4tr^2 + 3 \equiv 0 \bmod m \tag{3.4}$$

where $m = 12tr^2$.

Since dq^2 satisfies $(**)$, first, $4tq^2$ is a square modulo 3, so also $4tr^2 = 4t(2^sqr_0)^2$ is a square modulo 3. Second, -3 is a square modulo $4tq^2$. Since 3 does not divide $4tq^2$, it follows that $4tr^2 - 3$ is a square modulo $12tq^2$.

Now r_0 is invertible in $\mathbb{Z}/12tq^2\mathbb{Z}$, which implies that $4tr^2 - 3 \equiv (xr_0)^2 \bmod 12tq^2$ for some x. So $x^2r_0^2 - 4tr^2 + 3$ is divisible by $12tq^2$, which shows that (3.4) has a solution modulo $m = 12tq^2$, for any n. In particular, there exist solutions modulo $3tq^2$ and modulo

4. As before, it follows from Hensel's lemma that (3.4) has a solution modulo 2^l for any $l \geq 2$.

By the Chinese remainder theorem, there exists a solution x for (3.4) modulo $12t(2^s q)^2$. Like before, if $\gcd(x, r_0) \neq 1$, take a and b such that $ar_0 + b \cdot 12t(2^s q)^2 = 1$ and replace x by $x + b \cdot 12t(2^s q)^2 \cdot (1 - x) = 1 + ar_0(x - 1)$.

Now we have $\gcd(r_0^2, 4tx^2q^2) = 1$, so we can write $3ar_0^2 + bx^2 \cdot 12tx^2q^2 = 3$ for some a and b. So for $n = b$, we find that r_0^2 divides $x^2 \cdot -12ntq^2 + 3$, hence (3.4) has a solution modulo $m = r_0^2$. By the Chinese remainder theorem, it has a solution modulo $2dr^2$. □

Proof of Theorem 3.3.5, non-cyclic case

We now assume $\mathrm{Disc}(K_{d'}^{\perp}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/\frac{d'}{3}\mathbb{Z}$, and we again show that $d' = dr^2$ with d satisfying $(**')$ if and only if there exists a $v \in \Lambda_{d,r}^{\vee}$ of order r such that $K_{d'}^{\perp} \cong T_v$. The proof consists of Propositions 3.3.11 and 3.3.12.

Proposition 3.3.11. *Assume that* $\mathrm{Disc}\, K_{d'}^{\perp} \cong \mathbb{Z}/(r^2d/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. *If there is a* $v \in \Lambda_{d,r}^{\vee}$ *of order* r *such that* $K_{d'}^{\perp} \cong T_v$, *in particular* $d' = dr^2$, *then* d *satisfies* $(**)$.

Proof. By Proposition 3.3.1, there exist generators $(1, 0)$ and $(0, 1)$ of $\mathrm{Disc}\, K_{d'}^{\perp}$ such that the quadratic form $q_{K_{d'}^{\perp}}$ satisfies $q_{K_{d'}^{\perp}}(1, 0) = 3/d' \bmod 2\mathbb{Z}$ and $q_{K_{d'}^{\perp}}(1, 0) = -2/3 \bmod 2\mathbb{Z}$. Assume that $K_{d'}^{\perp}$ is isomorphic to $T_{w_{n,k}}$. Then in particular, considering the factor $\mathbb{Z}/\frac{d'}{3}\mathbb{Z}$, there exists an x such that $x^2(w_{n,k})^2 = x^2\frac{2nd-k^2}{r^2d}$ is congruent to $\frac{3}{dr^2}$ modulo 2. Multiplying both expressions with $-dr^2$ gives

$$x^2(k^2 - 2nd) \equiv -3 \bmod 2dr^2. \tag{3.5}$$

We see that -3 is a square modulo $2d$, which implies that d satisfies $(**)$. □

Write $r = 2^s q r_0$, where q consists of all prime factors of r which are congruent to 1 modulo 3, and r_0 consists of all other odd prime factors of r. In particular, dq^2 still satisfies $(**)$, and we have $\gcd(r_0, dq^2) = 1$. Besides, note that 3 divides r_0.

Proposition 3.3.12. *There exists an* n *such that for* $v = w_{nq^2,r_0} \in \Lambda_{d,r}^{\vee}$, *we have* $K_{d'}^{\perp} \cong T_v$.

Proof. Again, first assume that 3 does not divide d. Write $d = 2t$. By (3.5) we need that there exists n and x such that

$$x^2(r_0^2 - 4ntq^2) + 3 \equiv 0 \bmod 4tr^2,$$

which is equivalent to

$$x^2(r_0^2 - 4ntq^2) - 4tr^2 + 3 \equiv 0 \bmod 4tr^2. \tag{3.6}$$

In the proof of Proposition 3.3.10, the case $3|d$ (note that we did not use $3 \nmid r$), we saw that for $m = 12tr^2$, we have

$$x^2(r_0^2 - 12btq^2) - 4tr^2 + 3 \equiv 0 \bmod m$$

for some x, and b defined by $ar_0^2 + b\cdot 2dx^2q^2 = 1$. In particular, this holds for $m = 4tr^2$. Taking $n = 3b$ gives us a solution for (3.6).

We still need to check that for the generator u of $\mathbb{Z}/3\mathbb{Z} \subset \operatorname{Disc} T_v$, there exists $y \in \mathbb{Z}$ such that $y^2(u)^2 = y^2 \cdot \frac{d}{9}(r_0^2 - 2nq^2d)$ is congruent to $-2/3$ modulo 2. Multiplying both expressions by 3/2 gives

$$y^2 \tfrac{d}{2} \tfrac{r_0^2 - 2nq^2d}{3} \equiv -1 \bmod 3.$$

Now note that $d/2 \equiv 1 \bmod 3$, so taking y such that 3 does not divide y, we have

$$y^2 \cdot \tfrac{d}{2} \cdot \tfrac{r_0^2 - 2nq^2d}{3} \equiv \tfrac{r_0^2 - 2nq^2d}{3} \bmod 3.$$

The element on the right hand side is

$$3(r_0/3)^2 - 2bq^2d \equiv -b \bmod 3.$$

Now reducing $ar_0^2 + b\cdot 2dx^2(2^sq)^2 = 1$ modulo 3, we indeed find that $b \equiv 1 \bmod 3$.

We are left with the case $3|d$. Again, by (3.5) we need n and x such that

$$x^2(r_0^2 - 2nq^2d) + 3 \equiv 0 \bmod 2dr^2,$$

or equivalently,

$$3x^2((r_0/3)^2 - 2nq^2d/3) + 1 \equiv 0 \bmod 2dr^2/3.$$

We found such n and x in the proof of Proposition 3.3.10, the case $3 \nmid d$ (note that we did not use $3 \nmid r$ there).

Finally, we need to check that for the generator u' of $\mathbb{Z}/3\mathbb{Z} \subset \operatorname{Disc} T_v$, there exists a y such that $y^2(u')^2 = -y^2d/9$ is congruent to $-2/3$ modulo 2. Multiplying by $-3/2$, we get

$$y^2d/6 \equiv 1 \bmod 3$$

which is true whenever 3 does not divide y. $\square$

3.4 Rational maps to $\mathcal{C}_{d'}$

For untwisted K3 surfaces, an isomorphism $\Lambda_d \cong K_d^\perp$ can be used to construct a rational map $\mathrm{M}_d \dashrightarrow \mathcal{C}_d$. We will generalize these maps to the situation of twisted K3 surfaces.

3.4.1 Construction

The middle cohomology $\mathrm{H}^4(X,\mathbb{Z})$ of a cubic fourfold X is isomorphic, up to a sign, to the lattice

$$\Gamma' := E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}.$$

The isomorphism can be chosen such that the square of the hyperplane class on X is mapped to $h := (1,1,1) \in \mathbb{Z}(-1)^{\oplus 3}$. We denote the orthogonal complement to h by Γ, so Γ is isomorphic to $\mathrm{H}^4(X,\mathbb{Z})_{\mathrm{pr}}$ (again, up to a sign).

A primitive sublattice $K \subset \Gamma'$ of rank two and discriminant d' containing h is unique up to the action of the stable orthogonal group $\widetilde{\mathrm{O}}(\Gamma) = \mathrm{Ker}(\mathrm{O}(\Gamma) \to \mathrm{O}(\mathrm{Disc}\,\Gamma))$. We fix one such K for each discriminant d' and denote it by $K_{d'}$. Its orthogonal complement $K_{d'}^{\perp}$ is contained in Γ.

Note that the group $\widetilde{\mathrm{O}}(K_{d'}^{\perp})$ can be viewed as a subgroup of $\widetilde{\mathrm{O}}(\Gamma)$: By Lemma 1.2.3, any element f in $\widetilde{\mathrm{O}}(K_{d'}^{\perp})$ can be extended to an orthogonal transformation $\widetilde{f}$ of the unimodular lattice Γ' such that $\widetilde{f}|_{K_{d'}}$ is the identity. Then restrict to Γ to get an element of $\widetilde{\mathrm{O}}(\Gamma)$.

The group $\widetilde{\mathrm{O}}(T_v)$ is an arithmetic group which acts holomorphically on $\mathcal{D}(T_v)$, so the quotient $\mathcal{D}(T_v)/\widetilde{\mathrm{O}}(T_v)$ is a quasi-projective variety by Baily–Borel. On the level of the period domain, the above gives us a commutative diagram

$$\begin{tikzcd}
\mathcal{D}(T_v) \arrow[r, "\cong"] \arrow[d] & \mathcal{D}(K_{d'}^{\perp}) \arrow[rr, hook] \arrow[d] & & \mathcal{D}(\Gamma) \arrow[d] \\
\mathcal{D}(T_v)/\widetilde{\mathrm{O}}(T_v) \arrow[r, "\cong"] & \mathcal{D}(K_{d'}^{\perp})/\widetilde{\mathrm{O}}(K_{d'}^{\perp}) \arrow[r] & \overline{\mathcal{C}}_{d'} \arrow[r, hook] & \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)
\end{tikzcd}$$

Here $\overline{\mathcal{C}}_{d'}$ is the image of $\mathcal{D}(K_{d'}^{\perp})$ under $\mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$. Embedding the moduli space $\mathcal{C}$ of smooth cubic fourfolds into $\mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$ via the period map, one shows that $\overline{\mathcal{C}}_{d'}$ is the closure of $\mathcal{C}_{d'} \subset \mathcal{C}$ in $\mathcal{D}(\Gamma)/\widetilde{\mathrm{O}}(\Gamma)$.

Lemma 3.4.1. *The group* $\widetilde{\mathrm{O}}(T_v)$ *is a subgroup of* $\mathrm{Stab}(v)$.

Proof. Let $g \in \widetilde{\mathrm{O}}(T_v)$. By assumption, $g^{\vee}$ sends any $x \in T_v^{\vee}$ to $x+y$ for some $y \in T_v \subset \Lambda_d$. In particular, this holds for $x \in \Lambda_d \subset T_v^{\vee}$, which shows that $g^{\vee}$ preserves Λ_d. Moreover, it induces the identity on $\mathrm{Disc}\,\Lambda_d$, so $g^{\vee}|_{\Lambda_d}$ is an element of $\widetilde{\mathrm{O}}(\Lambda_d)$.

Now if $x \in \Lambda_d^{\vee}$ is a representative of v, then $\frac{1}{r}x$ lies in $T_v^{\vee}$. Then $g^{\vee}(\frac{1}{r}x) = \frac{1}{r}x + y$ for some $y \in T_v$ implies that $g^{\vee}(x) - x$ is in $rT_v \subset r\Lambda_d^{\vee}$. It follows that $g^{\vee}|_{\Lambda_d}$ is in $\mathrm{Stab}(v)$. $\square$

The period map $\mathcal{P}_v\colon \mathrm{M}_d^{\mathrm{mar}} \times \{v\} \to \mathcal{D}(T_v)$ induces an embedding of

$$\widetilde{\mathrm{M}}_v := (\mathrm{M}_d^{\mathrm{mar}} \times \{v\})/\widetilde{\mathrm{O}}(T_v)$$

into $\mathcal{D}(T_v)/\widetilde{\mathrm{O}}(T_v)$. This map is algebraic, which is shown similarly as for the embedding $\mathrm{M}_v \hookrightarrow \mathcal{D}(T_v)/\mathrm{Stab}(v)$. The space $\widetilde{\mathrm{M}}_v$ parametrizes tuples (S, L, α, f) where (S, L, α) is in M_v and f is an isomorphism from $\mathrm{Ker}(\alpha\colon \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} \to \mathbb{Z}/r\mathbb{Z})$ to T_v. The composition

$$\widetilde{\mathrm{M}}_v \to \mathcal{D}(T_v)/\widetilde{\mathrm{O}}(T_v) \to \overline{\mathcal{C}}_{d'}$$

induces a rational map $\widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$, which is regular on an open subset that maps surjectively (by Corollary 3.3.6) to $\mathcal{C}_{d'}$. Hassett showed that $\mathcal{D}(K_{d'}^{\perp})/\widetilde{\mathrm{O}}(K_{d'}^{\perp}) \to \overline{\mathcal{C}}_{d'}$ generically has degree one when $d' \equiv 2 \bmod 6$, and degree two when $d' \equiv 0 \bmod 6$, see also Section 3.4.3. Hence, $\widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$ is birational in the first case and has degree two in the second case.

The map $\gamma\colon \widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$ is in general not unique: it depends on the choice of an isomorphism $T_v \cong K_{d'}^{\perp}$. To be precise, let $\iota\colon\ \mathrm{O}(T_v) \to \mathrm{Aut}(\mathcal{D}(T_v))$ send an isometry of T_v to the induced action on the period domain. Then γ is unique up to $\iota(\mathrm{O}(T_v))/\iota(\widetilde{\mathrm{O}}(T_v))$. We can compute this group as in [HLOY03, Lemma 3.1]: there is a short exact sequence

$$0 \to \widetilde{\mathrm{O}}(T_v) \to \mathrm{O}(T_v) \to \mathrm{O}(\mathrm{Disc}\, T_v) \to 0.$$

Using $\iota(\widetilde{\mathrm{O}}(T_v)) \cong \widetilde{\mathrm{O}}(T_v)$ and $\iota(\mathrm{O}(T_v)) \cong \mathrm{O}(T_v)/\pm\mathrm{id}$, we find that

$$\iota(\mathrm{O}(T_v))/\iota(\widetilde{\mathrm{O}}(T_v)) \cong \mathrm{O}(\mathrm{Disc}\, T_v)/\pm\mathrm{id}\,.$$

Corollary 3.4.2. *The map $\widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$ is unique up to elements of* $\mathrm{O}(\mathrm{Disc}\, T_v)/\pm\mathrm{id}$.

When $\mathrm{Disc}\, T_v \cong \mathbb{Z}/d'\mathbb{Z}$, this group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus\tau(d'/2)-1}$, where $\tau(d'/2)$ is the number of prime factors of $d'/2$.

We have seen that there is a difference to the untwisted situation: the rational map to $\mathcal{C}_{d'}$ can only be defined after taking a finite covering $\pi\colon \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$. We give an upper bound for the degree of this covering.

Corollary 3.4.3. *The degree of the quotient map $\pi\colon \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$ is at most*

$$I = |\,\mathrm{O}(\mathrm{Disc}\, T_v)/\pm\mathrm{id}\,|.$$

If $\mathrm{Disc}\, T_v$ *is cyclic, then* $I = 2^{\tau(d'/2)-1}$.

Proof. The degree of π is the index of $\iota(\widetilde{\mathrm{O}}(T_v))$ in $\iota(\mathrm{Stab}\, v)$. This is at most the index I of $\iota(\widetilde{\mathrm{O}}(T_v))$ in $\iota(\mathrm{O}(T_v))$. □

3.4.2 Example

We consider the case $d = r = 2$, so $d' = 8$. Kuznetsov [Kuz10] has given a geometric construction to obtain, for a generic cubic fourfold of discriminant 8, a twisted K3 surface of degree 2 and order 2 associated to it.

By Lemma 3.2.6, the moduli space M_2^2 has at most four connected components, corresponding to the vectors $w_{n,k} = e_1 + nf_1 + \frac{k}{2}\ell_2'$ with $n, k \in \{0, 1\}$. Now by Eichler's criterion, e_1 is equivalent to $e_1 + f_1 + \ell_2'$ under $\widetilde{\mathrm{O}}(\Lambda_2)$, and this is equivalent to $e_1 + f_1$ modulo $r\Lambda_2^\vee = 2\Lambda_2^\vee$. Thus, M_2^2 has only (at most) three connected components.

The discriminant group of $K_8^\perp$ is cyclic, and one can choose a generator u such that $q_{K_8^\perp}(u) = -\frac{5}{8} \bmod 2\mathbb{Z}$. By Proposition 3.3.2, the discriminant group of $T_{w_{n,k}}$ is cyclic if and only if $k = 1$. Then we also have $\gcd(8, 4n - k^2) = 1$, so $[\frac{1}{2}w_{n,1}]$ is a generator of $\mathrm{Disc}\, T_{w_{n,1}}$. Hence, $T_{w_{n,1}}$ is isomorphic to $K_8^\perp$ if and only if there exists an $x \in \mathbb{Z}$ such that $(\frac{x}{2}w_{n,1})^2 \equiv -\frac{5}{8} \bmod 2$. For $n = 0$, we compute

$$(\tfrac{x}{2}w_{0,1})^2 = \tfrac{x^2}{4} \cdot -\tfrac{1}{2} = -\tfrac{x^2}{8}$$

which is never equivalent to $-\frac{5}{8}$ modulo 2. For $n = 1$, we have

$$(\tfrac{x}{2} w_{1,1})^2 = \tfrac{x^2}{4}(2 - \tfrac{1}{2}) = \tfrac{3x^2}{8}$$

which is equivalent to $-\frac{5}{8}$ modulo 2 when $x = 3$.

We see that for $v = w_{1,1}$, there exists a rational map $\widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$ as above. As $d'/2 = 4$ has only one prime factor, it follows from Corollary 3.4.3 that the map $\pi\colon \widetilde{\mathrm{M}}_{w_{1,1}} \to \mathrm{M}_{w_{1,1}}$ has degree one. Hence, we obtain a rational map

$$\mathrm{M}_{w_{1,1}} \dashrightarrow \mathcal{C}_8$$

which gives an inverse to Kuznetsov's construction over the locus where $\rho(S) = 1$.

Remark 3.4.4. The three types of Brauer classes occurring in this example have been studied before by Van Geemen [vG05] (see also [MSTVA17, Sec. 2]). He relates the twisted K3 surfaces in the components $T_{w_{0,0}}$ and $T_{w_{0,1}}$ to certain double covers of $\mathbb{P}^2 \times \mathbb{P}^2$ and to complete intersections of three quartics in $\mathbb{P}^4$, respectively.

Remark 3.4.5. In general, the component $\mathrm{M}_v \subset \mathrm{M}_d^r$ for which a rational map $\widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$ exists is not unique. For instance, let $d = 14$ and $r = 7$, so $\operatorname{Disc} K_{d'}^{\perp}$ is cyclic. Since r divides d, [MSTVA17, Thm. 9] tells us that for $v \in \Lambda_{d,r}^{\vee}$ of order r, there is only one isomorphism class of lattices T_v with cyclic discriminant group. By Theorem 3.3.5, these T_v are all isomorphic to $K_{d'}^{\perp}$.

Consider $w_{0,1} = e_1 + \ell'_{14}/14$ and $w_{1,3} = e_1 + f_1 + 3\ell'_{14}/14$. By Proposition 3.3.2, $\operatorname{Disc} T_{w_{0,1}}$ and $\operatorname{Disc} T_{w_{1,3}}$ are both cyclic. The components $M_{w_{0,1}}$ and $M_{w_{1,3}}$ are the same if and only if $w_{1,3}$ lies in the orbit $\widetilde{\mathrm{O}}(\Lambda_{14}) \cdot w_{0,1} \subset \Lambda_{14}^{\vee} \otimes \mathbb{Z}/7\mathbb{Z}$. This means that there exists $z \in \Lambda_{14}^{\vee}$ such that $f^{\vee}(w_{0,1}) = w_{1,3} + 7z$ for some $f \in \widetilde{\mathrm{O}}(\Lambda_{14})$, that is, $f(14w_{0,1}) = 14w_{1,3} + 14 \cdot 7z$. Write $z = z_0 + \frac{t}{14}\ell'_{14}$ for some $z_0 \in E_8(-1)^{\oplus 2} \oplus U_1 \oplus U_2$ and $t \in \mathbb{Z}$, so

$$14w_{1,3} + 14 \cdot 7z = 14(e_1 + f_1) + 14 \cdot 7z_0 + (3 + 7t)\ell'_{14}.$$

Comparing the square of the right hand side with $(14w_{0,1})^2 = -14$, we find

$$-14 = 2 \cdot 14^2 + 14^3(e_1 + f_1, z_0) + (14 \cdot 7)^2(z_0)^2 - 14(9 + 6 \cdot 7t + (7t)^2)$$

which simplifies to

$$8 = 2 \cdot 14 + 14^2(e_1 + f_1, z_0) + 14 \cdot 7^2(z_0)^2 - (6 \cdot 7t + (7t)^2).$$

Reducing modulo 7, one sees that this is not possible. So $M_{w_{0,1}}$ and $M_{w_{1,3}}$ are two different components of M_{14}^7 that are both related to cubics in $\mathcal{C}_{14 \cdot 7^2}$.

3.4.3 Pairs of associated twisted K3 surfaces

In [Bra18], we studied the covering involution of Hassett's rational map $\mathrm{M}_d \dashrightarrow \mathcal{C}_d$ in the case this has degree two. To be precise, we showed that if $(S, L) \in \mathrm{M}_d$ is mapped to (S^τ, L^τ) under this involution, then S^τ is isomorphic to a moduli space of stable sheaves on S with Mukai vector $(3, L, d/6)$. In this section, we discuss the analogous twisted situation.

We will identify elements in $\mathrm{Hom}(\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})$ with their images in $\mathrm{Br}(S)$. Likewise, any B-field $\beta \in \mathrm{H}^2(S, \mathbb{Q})$ will be identified with $\exp(\beta^{0,2}) \in \mathrm{Br}(S)$. We denote the bounded derived category of coherent sheaves on S by $\mathrm{D^b}(S)$, and of α-twisted sheaves by $\mathrm{D^b}(S, \alpha)$.

Assume that 3 divides $d' = dr^2$. Hassett showed (see also [Bra18]) that

$$\mathcal{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp) \to \overline{\mathcal{C}}_d$$

is a composition $\nu \circ f$, where ν is the normalization of $\overline{\mathcal{C}}_d$ and f is generically of degree two, induced by an element in $\mathrm{O}(K_d^\perp)$ of order two. The corresponding element $g \in \mathrm{O}(T_v)$ induces a covering involution

$$\tau \colon \mathcal{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp) \to \mathcal{D}(K_{d'}^\perp)/\widetilde{\mathrm{O}}(K_{d'}^\perp)$$

that preserves $\widetilde{\mathrm{M}}_v$. We claim that g extends to an orthogonal transformation of $\widetilde{\Lambda}$. This follows from [Nik80, Cor. 1.5.2] and the following statement.

Proposition 3.4.6. *Let $S_v := T_v^\perp \subset \widetilde{\Lambda}$. The map*

$$\mathrm{O}(S_v) \to \mathrm{O}(\mathrm{Disc}\, S_v)$$

is surjective.

Proof. The lattice S_v has rank three. When $\mathrm{Disc}\, T_v \cong \mathrm{Disc}\, S_v$ is cyclic, the statement follows from [Nik80, Thm. 1.14.2]. When $\mathrm{Disc}\, T_v$ is $\mathbb{Z}/(d'/3)\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, it follows from Corollary VIII.7.3 in [MM09]. □

This implies that when τ maps $(S, L, \alpha, f) \in \widetilde{\mathrm{M}}_v$ to (S', L', α', f'), then there is a Hodge isometry

$$\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S', \alpha', \mathbb{Z}).$$

This map might not preserve to orientation of the four positive directions. However, by [Huy17, Lemma 2.3], there exists an orientation reversing Hodge isometry in $\mathrm{O}(\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}))$. By composing with it, we see that there exists a Hodge isometry $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) \cong \widetilde{\mathrm{H}}(S', \alpha', \mathbb{Z})$ which is orientation preserving. By [HS06], there is an equivalence

$$\Phi \colon \mathrm{D^b}(S, \alpha) \to \mathrm{D^b}(S', \alpha')$$

between the bounded categories of α-twisted (α'-twisted) sheaves on S and S', which is of Fourier–Mukai type: there exists $\mathcal{E} \in \mathrm{D^b}(S \times S', \alpha^{-1} \boxtimes \alpha')$ and an isomorphism of functors $\Phi \cong \Phi_{\mathcal{E}}$. Let $\Phi^H_{\mathcal{E}} \colon \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) \to \widetilde{\mathrm{H}}(S', \alpha', \mathbb{Z})$ be the associated cohomological Fourier–Mukai transform. It follows that S' is a moduli space of complexes of α-twisted sheaves on S with Mukai vector

$$v = (\Phi^H_{\mathcal{E}})^{-1}(v(k(x))) = (\Phi^H_{\mathcal{E}})^{-1}(0,0,1),$$

where x is any closed point in S'. It is a coarse moduli space: the universal family on $S \times S'$ exists as an $\alpha^{-1} \boxtimes \alpha'$-twisted sheaf, which is an untwisted sheaf if and only if α' is trivial.

In fact, one can show that S' is isomorphic to a moduli space of stable α-twisted sheaves on S. Namely, by [Yos06] (see also [HS06]), there exists a (coarse) moduli space $M(v)$ of stable α-twisted sheaves on S with Mukai vector v. By precomposing $\Phi_{\mathcal{E}}$ with autoequivalences of $\mathrm{D^b}(S, \alpha)$, we may assume $M(v)$ is non-empty, see [HS06, Sec. 2]. Hence, as $(v)^2 = 0$, the space $M(v)$ is a K3 surface.

For some B-field $\beta \in \mathrm{H}^2(M(v), \mathbb{Q})$, there exists a universal family $\mathcal{E}_v$ on $S \times M(v)$ which is an $\alpha^{-1} \boxtimes \beta$-twisted sheaf. It induces an equivalence $\Phi_{\mathcal{E}_v} \colon \mathrm{D^b}(S, \alpha) \to \mathrm{D^b}(M(v), \beta)$ whose associated cohomological Fourier–Mukai transform $\Phi^H_{\mathcal{E}_v}$ sends v to the element $(0,0,1) \in \widetilde{\mathrm{H}}(M(v), \beta, \mathbb{Z})$. The composition

$$\Phi^H_{\mathcal{E}_v} \circ (\Phi^H_{\mathcal{E}})^{-1} \colon \widetilde{\mathrm{H}}(S', \alpha', \mathbb{Z}) \to \widetilde{\mathrm{H}}(M(v), \beta, \mathbb{Z})$$

is an orientation preserving Hodge isometry (since both $\Phi^H_{\mathcal{E}_v}$ and $\Phi^H_{\mathcal{E}}$ are – for $\Phi^H_{\mathcal{E}_v}$, see [HS05]) that sends $(0,0,1)$ to $(0,0,1)$. It follows from [HS06, Sec. 2] that S' is isomorphic to $M(v)$.

Chapter 4

Conclusion and future work

We have seen that cubic fourfolds can be used to obtain interesting information about K3 surfaces. We have found natural pairs of polarized (twisted) K3 surfaces, coming from involutions on moduli spaces of special cubic fourfolds. In the untwisted case, we described one of the K3 surfaces as a moduli space of sheaves on the other with an explicit Mukai vector, and deduced information about the relation between the associated Hilbert schemes of points. In the twisted case, the main input from cubic fourfolds is that it inspired us to construct moduli spaces of twisted K3 surfaces.

Now that we have defined these moduli spaces, it is desirable to find out more about their geometry. For example, one can compare M_d^r to the untwisted moduli space M_d via the "forgetful map" $\mathrm{M}_d^r \to \mathrm{M}_d$.

Question 1. What is the degree of this map?

Question 2. Is M_d^r of general type when d (or also r) is large, just like M_d [GHS07]?

These questions are interesting in light of a certain arithmetic application of moduli spaces of twisted K3 surfaces, as described in [VA16, Sec. 4]. The ultimate goal would be to prove an analogue for K3 surfaces of Merel's theorem:

Theorem ([Mer96])**.** *Let $d \geq 1$. There is a constant $c = c(d)$ such that*

$$|E(k)_{\mathrm{tors}}| < c$$

for all elliptic curves E/k, where k is any number field satisfying $[k : \mathbb{Q}] = d$.

In the case of K3 surfaces, one replaces the group $E(k)_{\mathrm{tors}}$ by

$$\mathrm{Br}(S)/\mathrm{Br}_0(S),$$

where $\mathrm{Br}_0(S) = \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(S))$ is the group of so-called constant Brauer classes. Subsets of $\mathrm{Br}(S)/\mathrm{Br}_0(S)$ are used to construct Brauer–Manin obstructions to the Hasse principle (or to weak approximation).

The group $\operatorname{Br}(S)/\operatorname{Br}_0(S)$ is finite, and the hope is to prove some uniform boundedness for its size, depending on, for example, the degree of the K3 surface. Here moduli spaces of twisted K3 surfaces should play the role of the modular curve $X_1(N)$ ($N \in \mathbb{N}$), which is crucial for the proof of Merel's theorem.

With this application in mind, it is also natural to try to define our moduli spaces over other fields, notably number fields. It should be possible to do this using level structures on K3 surfaces as in [Riz06] (compare also Remark 3.2.3).

Besides, it would be useful to study variations of the spaces M_d^r. For instance, one could look at *lattice polarized* (twisted) K3 surfaces, which means that one adds the datum of an isomorphism of $\operatorname{Pic} S$ with a fixed lattice L. Moduli spaces of these objects should be used to prove Conjectures 4.5 and 4.6 of [VA16].

Another option is to try to parametrize not the elements $v \in \operatorname{Hom}(\Lambda_d, \mathbb{Z}/r\mathbb{Z})$ themselves, but the associated lattices $T_v = \operatorname{Ker}(v) \subset \Lambda_d$, up to isomorphism. For r prime, these lattices have been classified in [vG05, MSTVA17]. We have seen (see Section 3.4.3) that when

$$(S, L, \alpha \in \operatorname{Hom}(\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z})) \text{ and } (S', L', \alpha' \in \operatorname{Hom}(\mathrm{H}^2(S',\mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z}))$$

satisfy $\operatorname{Ker}\alpha \cong \operatorname{Ker}\alpha'$, then there is an equivalence of categories $\mathrm{D}^{\mathrm{b}}(S,\alpha) \cong \mathrm{D}^{\mathrm{b}}(S',\alpha')$. Hence, in this case, a moduli functor would identify two twisted K3 surfaces if and only if they are twisted derived equivalent (at least, when the Picard number is one).

Finally, we would like to mention two follow-up questions for cubic fourfolds. For K3 surfaces of Picard number one, we know exactly what it means to be associated to the same cubic fourfold. It would be interesting to also study the non-uniqueness on the cubic fourfold side of Hassett's relation. Let us phrase it in terms of derived categories.

Question 3. Suppose X and X' satisfy $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S) \cong \mathcal{A}_{X'}$. What can be said about the geometry of X and X'? For example, is X rational if and only if X' is?

One can also generalize this to twisted K3 surfaces, asking that X and X' have the same associated K3 surface up to a twist.

Question 4. Suppose that $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$ and $\mathcal{A}_{X'} \cong \mathrm{D}^{\mathrm{b}}(S,\alpha)$. Can X' be related geometrically to X?

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