Geometry of random 3-manifolds

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Gabriele Viaggi

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1 Gutachter: Prof. Dr. Ursula Hamenstädt
2 Gutachter: Prof. Dr. Sebastian Hensel

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SUMMARY

We study *random 3-manifolds*, as introduced by Dunfield and Thurston, from a geometric point of view. Within this framework, work of Maher allows us to equip a typical random 3-manifold with a canonical geometric structure, namely, a *hyperbolic metric*.

By Mostow rigidity, such metric is unique up to isometries and, hence, we can attach to a random 3-manifold geometric invariants such as volume, Laplace and length spectra, diameter.

Our goal is to develop tools to compute these invariants and, in general, to get an effective and explicit description of the hyperbolic structure. More precisely, in this thesis we obtain the following results:

(1) We compute the *coarse growth* rate of volume, diameter and spectral gap for a typical family of random 3-manifolds.
(2) We show that the volumes of random 3-manifolds obey to a *law of large numbers*.
(3) We find an explicit *model manifold* that captures, up to uniform bilipschitz distortion, the geometry of a random 3-manifold.

Contents

INTRODUCTION

Geometry of random 3-manifolds

The purpose of this work is to study random 3-manifolds, as introduced by Dunfield and Thurston [15], from a geometric point of view[1].

The notion of random 3-manifold that we use comes from the observation that certain families of closed orientable 3-manifolds are naturally parametrized by diffeomorphisms of surfaces. We consider two examples: *Heegaard splittings* and *fibered 3-manifolds.*

The first family consists of those 3-manifolds M with a Heegaard decomposition of genus $g \geq 2$. This means that M is diffeomorphic a 3-manifold M_f obtained by gluing together two copies of a handlebody H_g of genus g along a diffeomorphism f of their boundaries $\partial H_g = \Sigma$

$$M_f := H_g \cup_{f:\partial H_g \to \partial H_g} H_g.$$

The second example is the family of 3-manifolds M that fiber over the circle with a fiber Σ of genus $g \geq 2$. In this case, M is diffeomorphic to the mapping torus T_f of a diffeomorphism f of the fiber Σ

$$T_f := \Sigma \times [0,1] \, / \, (x,0) \sim (f(x),1)\,.$$

The diffeomorphism type of the 3-manifolds M_f and T_f only depends on the isotopy class of f, which means that it is well-defined for the *mapping class* $[f] \in \mathrm{Mod}\,(\Sigma) := \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0(\Sigma)$ in the *mapping class group.*

Following [15], we define a family of random Heegaard splittings, or random 3-manifolds, as one of the form $(M_{f_n})_{n\in\mathbb{N}}$ where $(f_n)_{n\in\mathbb{N}}$ is a random walk on the mapping class group driven by some initial probability measure μ whose support is finite and generates $\mathrm{Mod}(\Sigma)$. We denote by $\mathbb{P}_n$ the distribution of the n-th step of the walk f_n and by $\mathbb{P}$ the distribution of the sample path $(f_n)_{n\in\mathbb{N}}$.

Analogously, we can define families of random mapping tori $(T_{f_n})_{n\in\mathbb{N}}$. The reason for comparing mapping tori and Heegaard splittings is that, geometrically, they behave similarly in random families as we will see.

We focus now on random Heegaard splittings.

The original approach to random 3-manifolds by Dunfield and Thurston was mostly topological and group theoretic. However, in their foundational paper [15], they also considered the geometric side and made the following

[1] A list of references is provided at the end of each of the four parts of the thesis. The references for the introduction are on page 21.

Conjecture (Dunfield-Thurston, Conjecture 2.11 in [15]). *A random 3-manifold is hyperbolic and its volume grows linearly in the step of the walk.*

We remark that the problem of finding hyperbolic structures on *most* Heegaard splitting of a fixed genus $g \geq 2$ was originally raised by Thurston (see Problem 24 in [31]). The introduction of the notion of random 3-manifolds allows to make the statement of the problem more precise.

After Perelman's solution of Thurston's geometrization conjecture, the only obstruction to the existence of a hyperbolic metric on M_f can be phrased in topological terms: A closed orientable 3-manifold is hyperbolic if and only if it is irreducible and atoroidal. Mapping classes that are sufficiently complicated in an appropriate sense (see Hempel [17]) give rise to Heegaard splittings that satisfy these properties.

Relying on this criterion, Maher established the existence of a hyperbolic metric on random 3-manifolds

Theorem (Maher [22])**.** *A random 3-manifold is hyperbolic.*

This is the starting point for our work. By Mostow rigidity, such a metric is unique up to isometry, thus it makes sense to refine Dunfield and Thurston question and study the growth of geometric invariants, such as volume, diameter, length spectrum and eigenvalues of the Laplacian in families of random 3-manifolds.

We will work towards this goal and develop a more *constructive* and *effective* approach to the hyperbolization of random 3-manifolds. In particular, we give an answer to Dunfield and Thurston's conjecture interpreting it in a strict way.

We state informally our contribution

Theorem 1. *There is an explicit, Ricci flow free, hyperbolization for random 3-manifolds. Furthermore, the volumes of random 3-manifolds obey to a law of large numbers.*

We will formulate precise statements for the two parts of Theorem 1 only later on (as Theorems 10 and 4).

By explicit Ricci flow free hyperbolization we mean that we construct the hyperbolic metric by assembling simple pieces and that we only use tools from the deformation theory of Kleinian groups. We use the model manifold technology by Minsky [25] and Brock, Canary and Minsky [10], as well as the effective version of Thurston's Hyperbolic Dehn Surgery by Hodgson and Kerckhoff [18] and Brock and Bromberg's Drilling Theorem [8].

We remark that, even though we do not rely on Perelman's geometrization, we do use the main result from Maher [22], namely, the fact that the *Hempel distance* of the Heegaard splittings (see [17]) grows coarsely linearly along the random walk.

Our construction gives new and more refined information than the mere existence of a hyperbolic metric. In fact, we also provide a *model manifold* that captures, up to *uniform bilipschitz distorsion*, the geometry of the random 3-manifold and allows the computation of its geometric invariants.

The additional structure that we get is the one of a ϵ-*model metric*. We describe it in the next pargraph.

A model metric. The existence of a hyperbolic metric does not guarantee by itself much control on the invariants of a random 3-manifold such as volume, Laplace and length spectra or diameter.

Following a strategy started by Minsky, Namazi, Brock and Souto [25], [26], [27], [12], we first construct a much more controlled negatively curved metric (M_f, ρ) which we can handle explicitly and then use it to understand the underlying hyperbolic structure. The requirements we want to impose are the following: There exists $\epsilon < 1/2$ such that (M_f, ρ) decomposes into five pieces $M_f = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2$ satisfying

(1) Topologically, H_1 and H_2 are homeomorphic to genus g handlebodies while Ω_1, Ω_2 and Q are homeomorphic to $\Sigma \times [1, 2]$.
(2) Geometrically, ρ has negative curvature $\sec \in (-1-\epsilon, -1+\epsilon)$, but outside the region $\Omega = \Omega_1 \cup \Omega_2$ the metric is purely hyperbolic.
(3) Volume-wise, we have $\mathrm{vol}(Q) \geq (1-\epsilon)\mathrm{vol}(M)$.
(4) The piece Q is almost isometrically embeddable in a complete hyperbolic 3-manifold diffeomorphic to $\Sigma \times \mathbb{R}$.

We call such a metric a ϵ-*model metric*.

The importance of the last requirement resides in the fact that we understand explicitly hyperbolic 3-manifolds diffeomorphic to $\Sigma \times \mathbb{R}$ thanks to the work of Minsky [25] and Brock, Canary and Minsky [10] which provides a detailed combinatorial description of their internal geometry.

Our first result, which is joint work with Ursula Hamenstädt, is the following:

Theorem 2. *For every* $0 < \epsilon < 1/2$ *we have*

$$\mathbb{P}_n[M_f \text{ admits a } \epsilon\text{-model metric}] \overset{n\to\infty}{\longrightarrow} 1.$$

Observe that Theorem 2 does not immediately provide an explicit relation between the ϵ-model metric and the underlying hyperbolic structure. However, the presence of a ϵ-model metric gives a lot of mileage on the topology and geometry of the 3-manifold as we will explain in a moment.

Before going on, we should mention that, using a result claimed by Tian [32], the mere fact that a metric ρ is a ϵ-model metric and that the region Ω where it is not hyperbolic has uniformly bounded diameter (as follows from the proof of Theorem 2), implies, if $\epsilon > 0$ is sufficiently small, that ρ is uniformly close up to third derivatives to a hyperbolic metric. However, Tian's result is not published and we do not rely on it.

Volumes of random 3-manifolds. Our strategy for the computation of the geometric invariants given the data of a ϵ-model metric is simple to explain. We first compute the invariant for the middle piece Q using the model manifold technology [25], [10]. Then, we argue that the invariant of Q is uniformly comparable to the one of M_f in a random setting.

The first example we provide is a computation of the volume growth rate. It is an immediate consequence of our construction, work of Besson, Courtois and Gallot [4], and work of Brock [7].

Proposition 3. *There exists a constant* $c > 0$ *such that*

$$\mathbb{P}_n[\mathrm{vol}(M_f) \in [n/c, cn]] \stackrel{n\to\infty}{\longrightarrow} 1.$$

The coarsely linear behaviour of the volume of a random Heegaard splitting follows from work by Maher [22] combined with an unpublished work of Brock and Souto. We refer to the introduction of [22] for more details. Here we give a different proof.

In the next result we refine the coarsely linear behaviour to a precise asymptotic. As a consequence, we answer to Dunfield and Thurston volume conjecture (Conjecture 2.11 in [15]) interpreting it in a strict way (see also Conjecture 9.2 in Rivin [28]).

Here we work with a broader notion of random 3-manifolds: We consider not only random Heegaard splittings but also random mapping tori. We remark that, again, a result by Maher [21] combined with Thurston's Hyperbolization Theorem [30] ensures that a random mapping torus is hyperbolic.

Our result is the following *law of large numbers* for the volume of random 3-manifolds: Recall that μ, the probability measure driving the random walk, has a finite support which generates the mapping class group

Theorem 4. *There exists* $v = v(\mu) > 0$ *such that for almost every* $(f_n)_{n\in\mathbb{N}}$ *the following holds*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(X_{f_n})}{n} = v.$$

Here $(X_{f_n})_{n\in\mathbb{N}}$ *is either the family of mapping tori* $(T_{f_n})_{n\in\mathbb{N}}$ *or Heegaard splittings* $(M_{f_n})_{n\in\mathbb{N}}$.

We observe that the asymptotic is the same for both mapping tori and Heegaard splittings.

We also stress the fact that the important part is the existence of an *exact asymptotic* for the volume as the *coarsely linear* behaviour follows from previous work. In the case of mapping tori, it is a consequence of work of Brock [6], who proved that there exists a constant $c(g) > 0$ such that for every pseudo-Anosov f

$$\frac{1}{c(g)} d_{\mathrm{WP}}(f) \leq \mathrm{vol}\,(T_f) \leq c(g) d_{\mathrm{WP}}(f)$$

where $d_{\mathrm{WP}}(f)$ is the Weil-Petersson translation length of f, and the theory of random walks on weakly hyperbolic groups (see for example [24]) which provides a linear asymptotic for $d_{\mathrm{WP}}(f)$. As already mentioned, for the Heegaard splitting case we refer to Maher [22] (or Proposition 3 for another approach).

Theorem 4 will be derived from the more technical Theorem 5 concerning *quasi-fuchsian* manifolds. We recall that a quasi-fuchsian manifold is a hyperbolic 3-manifold Q homeomorphic to $\Sigma \times \mathbb{R}$ that has a *compact* subset, the *convex core* $\mathcal{CC}(Q) \subset Q$, that contains all geodesics of Q joining two of its points. The asymptotic geometry of Q is captured by two conformal classes on Σ, i.e. two points in the Teichmüller space $\mathcal{T} = \mathcal{T}(\Sigma)$. Bers [3] showed that for every ordered pair $X, Y \in \mathcal{T}$ there exists a unique quasi-fuchsian manifold, which we denote by $Q(X, Y)$, realizing those asymptotic data.

Theorem 5. *There exists $v = v(\mu) > 0$ such that for every $o \in \mathcal{T}$ and for almost every $(f_n)_{n\in\mathbb{N}}$ the following limit exists:*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(\mathcal{CC}(Q(o, f_n o)))}{n} = v.$$

The relation between Theorem 4 and Theorem 5 is provided again by a *model manifold* construction. For random 3-manifolds the heuristic picture is the following: The geometry of X_{f_n} largely resembles the geometry of the convex core of $Q(o, f_n o)$, more precisely, as far as the volume is concerned, we have

$$|\mathrm{vol}\,(X_{f_n}) - \mathrm{vol}\,(\mathcal{CC}(Q(o, f_n o)))\,| = o(n).$$

We now describe the basic ideas behind Theorem 3: Suppose that the support of μ equals a finite generating set S and consider $f = s_1 \dots s_n$, a long random word in the generators $s_i \in S$. It corresponds to a quasi-fuchsian manifold $Q(o, fo)$. Fix N large, and assume $n = Nm$ for simplicity. We can split f into smaller blocks of size N

$$f = (s_1 \dots s_N) \cdots (s_{N(m-1)+1} \dots s_{Nm})$$

which we also denote by $h_j := s_{jN+1} \cdots s_{(j+1)N}$. Each block corresponds to a quasi-fuchsian manifold $Q(o, h_j o)$ as well. The main idea is that the geometry of the convex core $\mathcal{CC}(Q(o, fo))$ can be roughly described by juxtaposing, one after the other, the convex cores of the single blocks $\mathcal{CC}(Q(o, h_j o))$. In particular, the volume $\mathrm{vol}(\mathcal{CC}(Q(o, fo)))$ can be well approximated by the *ergodic sum*

$$\sum_{1\le j\le m} \mathrm{vol}\,(\mathcal{CC}(Q(o, h_j o)))$$

which converges in average by the Birkhoff ergodic theorem.

We will make this heuristic picture more accurate. Our three main ingredients are the model manifold, bridging between the geometry of the Teichmüller space $\mathcal{T}$ and the internal geometry of quasi-fuchsian manifolds

[25],[10], a recurrence property for random walks [1] and the method of natural maps from Besson, Courtois and Gallot [4].

As an application of the same techniques, along the way, we give another proof of the following well-known result [19], [9] relating iterations of pseudo-Anosovs, volumes of quasi-fuchsian manifolds and mapping tori

Proposition 6. *Let ϕ be a pseudo-Anosov mapping class. For every $o \in \mathcal{T}$ the following holds:*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(\mathcal{CC}(Q(o,\phi^n o)))}{n} = \mathrm{vol}\,(T_\phi)\,.$$

Small eigenvalues of random 3-manifolds. So far, we did not need to know much about how the ϵ-model metric (M_f,ρ) relates to the underlying hyperbolic structure (M_f,σ). A careful inspection shows that we only needed the fact that it almost computes its volume. The reason is that it is easy to manipulate the volume with the tools provided by Besson, Courtois and Gallot [4]. It is not straigthforward, instead, to use the same method to manipulate other invariants. This is the source of some of the difficulties we have to face next.

We now consider the *spectral gap* of the hyperbolic metric. This is a geometric invariant which is more sensitive of the metric than the volume. We recall that the spectral gap of (M,σ) is the smallest positive eigenvalue $\lambda_1(M,\sigma)>0$ of the Beltrami-Laplace operator Δ_σ on functions.

Theorem 7. *We have the following*

- *Spectral gap: There exists a constant $c=c(g)>0$ such that*

$$\mathbb{P}_n[\lambda_1(M_f)\le c/\mathrm{vol}(M_f)^2] \overset{n\to\infty}{\longrightarrow} 1.$$

- *Injectivity radius: For every $\epsilon>0$ we have*

$$\mathbb{P}_n[\mathrm{inj}(M_f)\le \epsilon] \overset{n\to\infty}{\longrightarrow} 1.$$

Theorem 7 is joint work with Ursula Hamenstädt and can be seen as a direct analogue for Heegaard splittings of a result of Baik, Gekhtman and Hamenstädt [1] for random mapping tori. In fact, we use Theorem 2 to import the strategy of [1] to the Heegaard splitting setting. We remark that the theorem holds also for the ϵ-model metric. In that case the proof can be copied almost word by word from [1].

This brings us to the main difficulty that we have to deal with which is exactly the comparison between the ϵ-model metric and the underlying hyperbolic structure. We develop a comparison technique which is tailored to the random 3-manifold setting. It is based on the method of natural maps introduced by Besson, Courtois and Gallot [4].

Before illustrating our technique, we pause for a second for a couple of comments on the qualitative and quantitative aspects of Theorem 7. The first one is that Theorem 7 identifies the coarse decay rate for the spectral gap: Schoen [29] proved that there exists a universal constant $a > 0$ such that for all closed hyperbolic 3-manifolds M we have

$$\lambda_1(M) \geq a/\mathrm{vol}(M)^2.$$

In the opposite direction, it is not possible to expect any inequality of the same form, that is independent of the genus g of a Heegaard splitting of M. There are examples of sequences of manifolds $(M_n)_{n\in\mathbb{N}}$ with the property that $\lambda_1(M_n) \geq \delta$ for all $n \in \mathbb{N}$ for some $\delta > 0$ while $\mathrm{vol}(M_n) \uparrow \infty$ (see for example [13]).

The second remark is that the decay rate in Theorem 7 is the fastest among all 3-manifolds with a Heegaard splitting of genus at most g as we now explain. Keeping in consideration the Hegaard genus, it is possible to give upper bounds on $\lambda_1(M)$: Combining work of Buser [14] and Lackenby [20], there exists a constant $b(g) > 0$ such that all closed hyperbolic 3-manifolds M with Heegaard genus g also satisfy

$$\lambda_1(M) \leq b(g)/\mathrm{vol}(M).$$

Moreover, the discrepancy between Schoen and the previous inequality is inevitable: There are sequences of hyperbolic 3-manifolds $(M_n)_{n\in\mathbb{N}}$ with a splitting of genus at most g and volume $\mathrm{vol}(M_n) \uparrow \infty$ that roughly saturate both sides of the inequalities. In particular, for every $\epsilon > 0$ there exists g, C and sequences (M_n) for which the Heegaard genus is bounded by g and $\lambda_1(M_n) \geq C/\mathrm{vol}(M_n)^{1+\epsilon}$ (see for example [1]).

The last remark is that a result of White [36] says that, in the presence of a uniform lower bound on the injectivity radius $\mathrm{inj}(M) \geq \epsilon > 0$, the range for the spectral gap behaviour is coarsely the one allowed by Schoen inequality: There exists $b(g,\epsilon) > 0$ such that $\lambda_1(M) \leq b(g,\epsilon)/\mathrm{vol}(M)^2$. The second part of Theorem 7 shows that we cannot hope to apply White's result in our setting: A random 3-manifold develops many thin parts.

We now briefly describe the main tool that we develop to compare the ϵ-model metric (M_f, ρ) to the hyperbolic structure (M_f, σ). Our technique relies on a local analysis of Besson, Courtois and Gallot *natural maps*.

Given two negatively curved metrics ρ and σ on M, one can produce a one-parameter family of natural maps $\{F_c : (M,\rho) \to (M,\sigma)\}_{c\in(a,b)}$ which are homotopic to the identity and enjoy the following geometric properties

(i) They do not increase the volume, meaning that $\mathrm{Jac}(F_c) \leq c$.
(ii) At points where they are almost volume preserving, that is where $\mathrm{Jac}(F_c)$ is large enough, they are also almost isometric, that is dF_c is close to an isometry.
(iii) On uniform neighbourhoods of those points the map is also uniformly Lipschitz.

The range (a, b) in which we can choose c is determined by the curvature of ρ and σ. For an ϵ-model metric and a hyperbolic metric, c can be chosen very close to 1. In particular

$$\mathrm{vol}(M_f,\rho)/\mathrm{vol}(M_f,\sigma) = 1 + o(\epsilon).$$

This implies that, in our case, the natural maps are forced to be almost volume preserving, and hence almost isometric, on large portions on M.

Combining with the explicit description of the ϵ-model metric (M_f,ρ) we are able to deduce some useful information on the geometry of the hyperbolic structure (M_f,σ).

The idea is the following: Fix a compact 3-manifold with boundary U endowed with some fixed metric. We call such an object a *geometric block*. The prototypical example for us will be and a fundamental portion of a hyperbolic mapping torus T_f, that is $U = T_f - \Sigma$ where $\Sigma \subset T_f$ is a standard fiber with controlled geometry. Fix also a small number $\alpha \in (0,1)$.

Suppose that we can find pairwise disjoint isometric copies $U_1,\cdots,U_k$ of U isometrically embedded in the ϵ-model metric (M_f,ρ) so that they eat a definite proportion of the volume

$$\mathrm{vol}\left(\bigsqcup_{j\le k} U_j\right) \ge \alpha \mathrm{vol}(M_f,\rho).$$

Since we also have $\mathrm{vol}(M_f,\rho)/\mathrm{vol}(M_f,\sigma) = 1+o(\epsilon)$, if $\epsilon > 0$ is small enough compared to α, then any natural map $F_c : (M_f,\rho) \to (M_f,\sigma)$ with $c \simeq 1$ must be almost volume preserving and, hence, locally almost isometric at some points in at least half of the components U_j. For each of these components, such local control can be upgraded to an almost isometric embedding of U_j by standard arguments of [4]. So we find many copies of U uniformly embedded also in (M_f,σ).

Ergodicity of a random walk [1] and the model manifold technology [25], [10] are the two main tools for finding such collections of blocks in the ϵ-model metric and hence ensure the presence of such blocks also in the hyperbolic metric. Having enough geometric blocks allows us to use the same arguments of [1] and get the upper bound for the spectral gap $\lambda_1(M_f,\sigma)$.

A particularly careful choice of blocks also gives us the following two consequences: As the middle piece Q has many short curves, we immediately see that the injectivity radius $\mathrm{inj}(M_{f_n})$ drops to 0 almost surely. This provides a proof for the second assertion in Theorem 7.

In the opposite direction, we also see larger and larger geometric blocks where the injectivity radius is uniformly bounded from below. Hence, we can also choose basepoints $x_n \in M_{f_n}$ such that the sequence (M_{f_n},x_n) converges geometrically to a doubly degenerate structure Q_∞ on $\Sigma\times\mathbb{R}$ with $\mathrm{inj}(Q_\infty) > 0$, i.e. Q_∞ has *bounded geometry*.

Commensurability and arithmeticity. The fact that the injectivity radius can be made arbitrarily small and still we can choose basepoints so that the sequence of random 3-manifolds converges to a doubly degenerate structure with bounded geometry has the following consequence:

Proposition 8. *For $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ the following holds*

(1) There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are arithmetic.
(2) There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are in the same commensurability class.

The arguments for the proof of Proposition 8 are mostly borrowed from Biringer and Souto [5].

We observe that Dunfield and Thurston, using a simple homology computation have shown in [15] that their notion of random 3-manifold is not biased towards a certain fixed set of 3-manifolds. This means that for every fixed 3-manifold M, only finitely many elements in the family $(M_{f_n})_{n\in\mathbb{N}}$ can be diffeomorphic to M. Proposition 8 can be seen as a strengthening of their conclusions. It shows that the notion of random 3-manifolds is also not biased towards the class of arithmetic hyperbolic 3-manifolds and to the class of 3-manifolds which are commensurable to a fixed 3-manifold M.

Constuction of model metrics. Having discussed some consequences of Theorem 2, we now illustrate what goes into its proof.

The construction is somehow implicit in the description of a ϵ-model metric (M_f, ρ). Recall that such a Riemannian manifold decomposes as $M_f = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2$ and that its restriction to H_1, Q and H_2 is purely hyperbolic. We think of Ω_1, Ω_2 as the collar structure of the three larger pieces $\mathbb{N}_1 = H_1\cup\Omega_1$, $\mathbb{N}_2 = \Omega_2\cup H_2$ and $\mathbb{Q} = \Omega_1\cup Q\cup\Omega_2$. Topologically, they are two handlebodies and one I-bundle.

The idea is to find, on each of the three topological pieces $\mathbb{N}_1$, $\mathbb{N}_2$ and $\mathbb{Q}$, a Riemannian metric which is purely hyperbolic in the interior and such that the geometry of large collars of $\partial\mathbb{N}_1$ and $\partial\mathbb{N}_2$ almost isometrically match the geometry of the collar of the corresponding boundary component of $\partial\mathbb{Q}$. Moreover, we want to keep the size of the collars and the volumes under control. If we can do so, then we can patch together the Riemannian metrics and obtain a ϵ-model metric on M_f.

Such constructions are available in the literature (see [26], [27], [12], [11]), however we have to deal with one major difficulty, namely the fact that there is no *a priori* control on the thick-thin decomposition of M_f. This piece of information is implicitly assumed in the works mentioned above.

We will describe a route to overcome this issue that follows closely [12].

The geometric building blocks $\mathbb{N}_1, \mathbb{N}_2$ and $\mathbb{Q}$ for the ϵ-model metric on M_f are portions of the convex cores of *convex cocompact* complete hyperbolic structures on H_g and $\Sigma_g \times [1, 2]$ respectively.

Notice that the gluing construction only requires a uniform control on the geometry near the boundaries of the blocks. The model manifold technology developed by Minsky [25] and Brock, Canary and Minsky [10] provides such a control for $\mathbb{Q}$. The same is not available in full generalities for $\mathbb{N}_1, \mathbb{N}_2$ and this is the main challenge in pursuing the gluing strategy that we described.

We supply a uniform model for the collar geometry of $\partial\mathbb{N}_1$ and $\partial\mathbb{N}_2$ by slightly generalizing works of Brock, Minsky, Namazi and Souto [26], [12]. This is our main contribution here. In particular, we find conditions so that we can make sure that those collars are geometrically very close to a quasi-fuchsian hyperbolic structure on $\Sigma \times [1, 2]$.

To this extent, we introduce a condition of relative *R-relative bounded combinatorics* and large height for geometrically finite structures on handlebodies. It differs from the *R-bounded combinatorics* condition of [26] and [12] only because it is a local condition.

Theorem 9. *For every $R > 0$ and $\epsilon > 0$ there exists $h = h(R, \epsilon) > 0$ such that if M_f has relative R-bounded combinatorics and height at least h then the Heegaard splitting M_f admits a ϵ-model metric.*

Compared to [26], [12], the main novelty is that we allow a non trivial thick-thin decomposition and require only a local control.

Ergodicity of the random walk implies that the condition of having R-relative bounded combinatorics is generic, so the previous discussion is applicable to random 3-manifolds.

We also remark that, in the random setting, it follows from the construction that the middle piece $\mathbb{Q}$ closely resembles a large portion of the convex core of the quasi-fuchsian manifold $Q(o, fo)$ where $o \in \mathcal{T}$ is a base point carefully fixed once and for all.

Hyperbolization and uniform models. Up to now, the existence of a hyperbolic structure was guaranteed by Maher's theorem which rests upon the solution of the geometrization conjecture by Perelman.

We now describe a *constructive* proof of Maher's result that bypasses the use of Ricci flow with surgery.

Given the amount of information that can be extracted from the model manifold technology, it is desirable, for random 3-manifolds, to have not only a hyperbolic metric, but also a uniform bilipschitz model for it with the structure of a ϵ-model metric. This is indeed the case: We have the following *effective* version of Theorem 1

Theorem 10. *For every $0 < \epsilon < 1/2$ and $K > 1$ we have*

$$\mathbb{P}_n[M_f \text{ has a hyperbolic metric } K\text{-bilipschitz to a } \epsilon\text{-model metric}] \stackrel{n\to\infty}{\longrightarrow} 1.$$

Our methods follow closely [12] and [11] where uniform ϵ-model metrics are constructed for special classes of 3-manifolds.

The idea is the following: We can obtain a hyperbolic metric on M_f by a hyperbolic cone manifold deformation from a finite volume *drilled* manifold $\mathbb{M}$ which has the following form: Let $\Sigma \times [1, 4]$ be a tubular neighbourhood of $\Sigma \subset M_f$. We consider 3-manifolds

$$\mathbb{M} = M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\})$$

where P_j is a pants decomposition of the surface $\Sigma \times \{j\}$. A finite volume hyperbolic metric on such a manifold can be constructed explicitly by gluing together the convex cores of two *maximally cusped* handlebodies H_1, H_2 and three *maximally cusped* I-bundles Ω_1, Q, Ω_2.

$$\mathbb{M} = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2.$$

Most of our work consists of finding suitable pants decompositions for which the Dehn surgery slopes needed to pass from $\mathbb{M}$ to M_f satisfy the assumptions of the effective Hyperbolic Dehn Surgery Theorem [18]. In order to find them we crucially need two major tools: The work of [16] on the geometry of hyperbolic handlebodies and ergodic properties of the random walks proved by Baik, Gekhtman and Hamenstädt [1].

In order to provide a uniform bilipschitz control we exploit, instead, ergodic properties of the random walk and drilling and filling theorems by Hodgson and Kerckhoff [18] and Brock and Bromberg [8].

In the next paragraphs we present two further geometric applications that use the control given by Theorem 10.

Diameter growth. As a first application we compute the coarse growth rate of the diameter of random 3-manifolds.

Proposition 11. *There exists $c > 0$ such that*

$$\mathbb{P}_n[\mathrm{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The coarsely linear upper bound follows from a theorem by White [35] relating the diameter to the *presentation length* of $\pi_1(M_f)$. It is not difficult to see that the latter grows at most linearly in a family of random 3-manifolds. The lower bound comes from the ϵ-model metric structure.

Geometric limits. We have already established that it is possible to obtain a doubly degenerate structure on $\Sigma \times \mathbb{R}$ with bounded geometry as a limit, in the *pointed geometric topology* (see Chapter E.1 of [2]), of a sequence of random 3-manifolds.

Using Theorem 10, it is possible to show that, in fact, it is possible to deform a sequence of random 3-manifolds towards *any pointed doubly degenerate structure* on $\Sigma \times \mathbb{R}$. This is the content of the next proposition

Proposition 12. *For every ϕ pseudo-Anosov with associated hyperbolic mapping torus T_ϕ and for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ there exists a sequence of base points $x_n \in M_{f_n}$ such that the sequence (M_{f_n}, x_n) converges geometrically to the infinite cyclic covering of T_ϕ.*

The family of infinite cyclic coverings of hyperbolic mapping tori is dense in the space of doubly degenerate structures endowed with the pointed geometric topology. This follows from two facts: The first one is that the subset of $\mathcal{PML} \times \mathcal{PML}$ consisting of the pairs $(\lambda^-(\phi), \lambda^+(\phi))$ of repelling and attracting fixed points of pseudo-Anosov elements ϕ is dense. The second one is Thurston's Double Limit Theorem [30] combined with the Ending Lamination Theorem [25], [10].

A larger class of random walks. We conclude by just adding a couple of words on the assumptions on the probability measure μ that drives the random walk.

We recall that we only considered probability measures whose finite support generates the entire mapping class group. These assumptions can be considerably weakened and still obtain model metrics and convergence results as in Theorems 2, 4 and 5 but we have to distinguish between mapping tori and Heegaard splittings.

In the mapping torus case (and also in Theorem 5), it is enough that the subgroup generated by the support of μ contains two pseudo-Anosov elements that act as independent loxodromics on the curve graph. In the Heegaard splitting case, we further require that the two pseudo-Anosov elements act as independent loxodromics also on the handlebody graph. We refer to Maher and Tiozzo [24] and Maher and Schleimer [23] for more details on random walks on these spaces. In such higher generalities the proofs will be word by word the same, no change is needed.

Outline. This thesis is divided into three parts.

The first part contains the article *Small eigenvalues of random 3-manifolds* [16] in which the existence of an ϵ-model metric is established as in Theorem 2 and Theorem 9 and then used to control the first positive eigenvalue of the Laplacian as stated in Theorem 7. This is joint work with Ursula Hamenstädt.

The second part corresponds to the article *Volumes of random 3-manifolds* [33] where we prove a law of large numbers for the volumes of a family of random 3-manifolds. The results discussed in this part are Theorem 4, Theorem 5 and Proposition 6.

The last chapter is the article *Uniform models for random 3-manifolds* [34]. There, we produce hyperbolic metrics uniformly bilipschitz to ϵ-model metrics on random Heegaard splittings (as in Theorems 1 and 10). As an application, we coarsely identify, in Proposition 11, the coarse growth rate of the diameter of random 3-manifolds. We also prove Proposition 8 about arithmeticity and commensurability classes of random 3-manifold.

The three different parts are presented in their preprint form. As such, they are as self-contained and as independent of each other as possible. Their conclusions and ideas are discussed organically in the introduction.

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SMALL EIGENVALUES OF RANDOM 3-MANIFOLDS

URSULA HAMENSTÄDT AND GABRIELE VIAGGI

ABSTRACT. We show that for every $g \geq 2$ there exists a number $c = c(g) > 0$ such that the smallest positive eigenvalue of a random closed 3-manifold M of Heegaard genus g is at most $c(g)/\mathrm{vol}(M)^2$.

1. INTRODUCTION

By celebrated work of Perelman, any closed oriented aspherical atoroidal 3-manifold admits a hyperbolic metric, and such a metric is unique by Mostow rigidity[1]. In recent years, there was considerable progress in the understanding of the relation between geometric and topological invariants of such a manifold. The program to construct an explicit combinatorial model which describes the geometry up to uniform quasi-isometry turned out to be particularly fruitful [40, 10, 11], but it is far from completed.

The main purpose of this article is obtain an understanding of geometric and topological invariants for *random* hyperbolic 3-manifolds in the sense of Dunfield and Thurston [16]. Namely, fix a genus $g \geq 2$. A closed 3-manifold of Heegaard genus at most g can be obtained by gluing two handlebodies of genus g along the boundary with a diffeomorphism ϕ. The resulting 3-manifold M only depends on the isotopy class of ϕ, and it is aspherical atoroidal and hence hyperbolic if ϕ is sufficiently complicated. Thus hyperbolic 3-manifolds of Heegaard genus g correspond to suitable elements of the mapping class group $\mathrm{Mod}(\Sigma)$ of the boundary surface Σ of a handlebody of genus g (in fact, they correspond to double cosets in this group, see [16]).

Now let us choose a symmetric probability measure on $\mathrm{Mod}(\Sigma)$ whose finite support generates $\mathrm{Mod}(\Sigma)$. This measure generates a random walk on $\mathrm{Mod}(\Sigma)$, and hence it induces a notion of a random 3-manifold, glued from two handlebodies with a random gluing map. A random 3-manifold is hyperbolic [16] and hence we can study the behavior of geometric invariants of such random hyperbolic 3-manifolds M.

Our main technical result (Theorem 6.7) constructs for a 3-manifold obtained from a gluing map with some additional properties a Riemannian metric of sectional curvature close to -1 everywhere and different from

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[1]For the bibliography of this part of the thesis see page 70.

-1 only in geometrically controlled regions where the injectivity radius is bounded from below by a universal constant. These constraints are fulfilled for random gluing maps.

We use this construction to obtain information on the spectrum of the Laplacian of a random hyperbolic 3-manifold M. List the eigenvalues as $0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \lambda_3(M) \leq \dots$, with each eigenvalue repeated according to its multiplicity. By [44] and [21], there exists a universal constant $\chi > 0$ such that

$$\lambda_1(M) \geq \frac{\chi}{\mathrm{vol}(M)^2} \quad \text{ and } \lambda_{\mathrm{vol}(M)/\chi}(M) \geq \chi$$

for every closed hyperbolic 3-manifold M. Manifolds which fibre over the circle provide examples for which these estimates are essentially sharp. We refer to the introduction of [1] for a more comprehensive discussion.

On the other hand, it follows from the work of Buser [12] and Lackenby [26] that there exists a number $b(g) > 0$ such that for a hyperbolic 3-manifold M of Heegaard genus g, there is a bound

$$\lambda_1(M) \leq \frac{b(g)}{\mathrm{vol}(M)}.$$

Hyperbolic 3-manifolds constructed from expander graphs have arbitrarily large volume, yet their smallest positive eigenvalue is bounded from below by a universal constant. Hence in this estimate, the dependence of the constant $b(g)$ on the Heegaard genus g can not be avoided.

Under geometric constraints, one obtains better estimates. White [49] showed that there is a number $a(g) > 0$ such that $\lambda_1(M) \leq a(g)/\mathrm{vol}(M)^2$ if M is of Heegaard genus g and the injectivity radius of M is bounded from below by universal constant. The same holds true for random hyperbolic 3-manifolds which fibre over the circle, with fibre genus g [1].

Using the model metric for random hyperbolic 3-manifolds as our main tool we show.

Theorem 1. *For every $g \geq 2$ there exists a number $c(g) > 1$ such that*

$$\lambda_1(M) \leq \frac{c(g)}{\mathrm{vol}(M)^2} \quad \textit{ and } \lambda_{\mathrm{vol}(M)/c(g)}(M) \leq c(g)$$

for a random hyperbolic 3-manifold of Heegaard genus g.

Here the upper bound for $\lambda_{\mathrm{vol}(M)/c(g)}(M)$ is a straightforward consequence of domain monotonicity with Dirichlet boundary conditions. For the upper bound for $\lambda_1(M)$, we expect that the dependence of the constant $c(g)$ on g can not be avoided.

Strategy of the proof. As mentioned above, our main technical result is Theorem 6.7 which provides of an explicit Riemannian metric of curvature close to -1 on a 3-manifold of Heegaard genus g with some constraints on the gluing map. Constructions of geometrically controlled model metrics appear frequently in the literature, for example as a main tool in [42] and in [41]. For doubly degenerate hyperbolic 3-manifolds whose fundamental group is isomorphic to the fundamental group of a closed surface, there is a completely explicit combinatorial model for the geometry [40, 10]. More recently these results were used to describe explicitly the geometry of hyperbolic 3-manifolds with a lower bound on the injectivity radius and some topological constraints [11].

We can not apply the constructions in [11] as there are no lower bounds for the injectivity radius of a random hyperbolic 3-manifold M. Instead we use properties of the random walk to locate regions in a random 3-manifold which are diffeomorphic to a trivial I-bundle over a closed surface and such that a combinatorial model would predict a uniform lower bound on the injectivity radius in those regions. This is the constraint on the gluing map required in Theorem 6.7. The model metric is then constructed by cutting M open at two such regions and by using information on suitable model metrics for the pieces.

For random hyperbolic 3-manifolds M, we find that the spectrum of the model metric fulfills the properties stated in Theorem 1.

The last step consists in comparing the model metric on M and the hyperbolic metric. A result of Tian [47] implies that the model metric is $\mathcal{C}^2$-close to the hyperbolic metric. As this work is neither published nor available in electronic form, we prove a weak substitute which is sufficient for the proof of Theorem 1. Our argument is based on the methods introduced in [4].

Organization of the article. In Section 2 we collect some properties of the pointed geometric topology for 3-dimensional Riemannian manifolds which are used later on.

In Section 3 we introduce a relative version of bounded combinatorics and set up sufficient conditions for the construction of a model metric. This construction depends on the existence of *large thick collars*, a property which is introduced in Section 4.

Sections 5 and 6 are devoted to the proof of Theorem 6.7 which provides a model metric for a hyperbolic 3-manifold of fixed Heegaard genus with some additional properties. In Section 7 we show that random hyperbolic 3-manifolds have the properties required in Section 6, and in Section 8 we relate the model metric to the hyperbolic metric using tools from [4]. The information on the hyperbolic metric we obtain then leads to Theorem 1.

2. Hyperbolic structures on handlebodies

The goal of this section is to collect some results from the deformation theory of convex cocompact hyperbolic metrics on handlebodies in the form used later on. We also introduce some notations which are used throughout the article.

We begin with making precise what we understand by looking at a convex cocompact hyperbolic handlebody from the point of view of the boundary of the convex core. We give a quantitative description of the notion of a large collar with bounded geometry (a *large-thick collar*). As a preparation for Section 4, we describe some basic general compactness properties of the geometric topology.

Fix, once and for all, a genus $g \geq 2$. Let $\mathcal{H}$ be a handlebody of genus g, with boundary surface $\Sigma := \partial\mathcal{H}$. We fix on $\mathcal{H}$ an orientation, and we coherently orient Σ as the boundary of $\mathcal{H}$.

A *marked hyperbolic structure* on the handlebody $\mathcal{H}$ is a quotient $N = \mathbb{H}^3/\Gamma$ of hyperbolic 3-space by a discrete free subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+\left(\mathbb{H}^3\right)$, together with a homeomorphism (the marking) $\phi : \mathrm{int}\,(\mathcal{H}) \longrightarrow N$. We say that the marked structures $\phi : \mathrm{int}\,(\mathcal{H}) \longrightarrow N$ and $\psi : \mathrm{int}\,(\mathcal{H}) \longrightarrow N'$ are *equivalent* if there exists an isometry $f : N \longrightarrow N'$ such that $f \circ \phi$ is isotopic to ψ.

2.1. **Parametrization of marked convex cocompact structures.** We denote by $\mathcal{T} = \mathcal{T}(\Sigma)$ the *Teichmüller space* of marked hyperbolic metrics on Σ, and by $\mathcal{M} = \mathcal{M}(\Sigma) = \mathrm{Mod}(\Sigma)\backslash\mathcal{T}(\Sigma)$ the *moduli space* of hyperbolic metrics on Σ.

By classical results due to Bers, Kra, Maskit, Sullivan and others, so-called *convex cocompact* hyperbolic structures on the handlebody $\mathcal{H}$ are parametrized by a parameter that lies in the Teichmüller space $\mathcal{T}$ of the boundary surface.

Namely, let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic structure on $\mathcal{H}$. Associated to Γ we have the *limit set* $\Lambda \subset \partial\mathbb{H}^3$ which consists of the points at infinity of a $\Gamma-$orbit closure, and the *domain of discontinuity* $\Omega_\Gamma = \partial\mathbb{H}^3 - \Lambda$, the complement of the limit set. The group Γ, isomorphic to a free group $\mathbb{F}_g$ of rank g, acts freely and properly discontinuously both on the convex hull of the limit set $\mathcal{CH}(\Lambda) \subset \mathbb{H}^3$ and on the domain of discontinuity $\Omega_\Gamma \subset \partial\mathbb{H}^3$. In the remainder of this section we will always assume that $\Omega_\Gamma \neq \emptyset$.

The quotient

$$\mathcal{CC}(N) := \mathcal{CH}(\Lambda)/\Gamma$$

is the *convex core* of N. It is a convex topological submanifold of N, possibly with boundary. The manifold N is called *convex cocompact* if $\mathcal{CC}(N)$ is compact. The complement $N - \mathcal{CC}(N)$ is naturally homeomorphic to $\partial\mathcal{CC}(N) \times (0, \infty)$.

The quotient $\partial_c N := \Omega_\Gamma/\Gamma$ is the (unmarked) *conformal boundary* of N (we can think of it as a point in moduli space). As N is convex cocompact, $\partial_c N$ is homeomorphic to the closed surface $\Sigma = \partial\mathcal{H}$. The conformal boundary is equipped with a natural conformal structure and hence a hyperbolic metric (which we refer to as the Poincaré metric) coming from the fact that

Γ acts via Möbius transformations on $\partial\mathbb{H}^3$. The quotient

$$\overline{N} = \mathbb{H}^3 \cup \Omega_\Gamma/\Gamma = N \cup \partial_c N$$

gives a natural compactification of N.

Using a marking $\phi : \mathrm{int}\,(\mathcal{H}) \longrightarrow N$, the isotopy class of the inclusion of the boundary $\Sigma := \partial\mathcal{H} \hookrightarrow \mathcal{H}$ determines an isotopy class of an embedding $\Sigma \hookrightarrow N$. We use this isotopy class to give a marking to the conformal boundary $\partial_c N$ and to the boundary of the convex core $\partial\mathcal{CC}(N)$.

In this terminology, Bers parametrization can be stated as follows: Equivalence classes of marked convex cocompact structures are parametrized by the marked conformal boundary. Given a marked conformal boundary $X \in \mathcal{T}$, we denote by $\mathcal{H}(X)$ the corresponding marked convex cocompact hyperbolic handlebody.

2.2. **The boundary of the convex core.** As before, let $N = \mathbb{H}/\Gamma$ be a convex cocompact hyperbolic structure on $\mathcal{H}$. Then the boundary $\partial\mathcal{CC}(N) \subset N$ of the convex core is an embedded *pleated surface.*

Definition (Pleated Surface, Thurston [46]). Let M be a hyperbolic 3-manifold and let us fix a homotopy class of maps $j : \Sigma \to M$. A pleated surface in the homotopy class of j consists of the following data:

- A hyperbolic metric σ on Σ.
- A path-isometry $f : (\Sigma, \sigma) \to M$ homotopic to j such that every point $x \in \Sigma$ is contained in a geodesic segment which is mapped to a geodesic in M.

Associated to every pleated map $f : (\Sigma, \sigma) \to M$ there is a geodesic lamination $\lambda \subset \Sigma$, called the pleating locus with the following property. Every leaf of λ is mapped to a geodesic by f, and the restriction of f to every component of $\Sigma - \lambda$, is a locally isometric immersion. We say that f realizes $\lambda \subset \Sigma$ in M within the homotopy class j. For more on laminations and pleated surfaces we refer the reader to Chapter I.5 of [15].

There is a natural *nearest point retraction* (see Chapter II.1.3 of [15]) from the conformal boundary to the boundary of the convex core $r : \partial_c N \longrightarrow \partial\mathcal{CC}(N)$. With respect to the induced markings on the conformal boundary and on the boundary of the convex core, r lies in the homotopy class of the identity.

The following result of Bridgeman and Canary provides control of the boundary of the convex core when we have a good understanding of the geometry of the conformal boundary:

Theorem 2.1 (Bridgeman-Canary, [8]). *There are maps $J, G : (0,\infty) \to (1,\infty)$ such that the following holds: Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a finitely generated, non-elementary, torsion free Kleinian group. Suppose that the length, measured with respect to the Poincaré metric, of every curve in the conformal boundary Ω_Γ/Γ which is compressible in the 3-manifold $\left(\mathbb{H}^3 \cup \Omega_\Gamma\right)/\Gamma$*

is bounded from below by $\rho > 0$. Then the nearest point retraction from the conformal boundary to the boundary of the convex core is $J(\rho)-$Lipschitz and admits a $G(\rho)-$Lipschitz homotopy inverse.

2.3. **Limits of hyperbolic manifolds.** Let us choose for every (marked) convex cocompact structure on the handlebody $\mathcal{H}$ a basepoint $x \in \partial\mathcal{CC}(N)$ on the boundary of the convex core. We then can talk about a marked pointed convex cocompact handlebody.

Definition (Geometric Convergence). A sequence $\{(M_n, m_n)\}_{n\in\mathbb{N}}$ of pointed hyperbolic 3-manifolds is said to *converge in the pointed geometric topology* to a pointed hyperbolic 3-manifold (M_∞, m_∞) if the following conditions are satisfied. For every $R > 0, \xi > 0$ there are numbers $n(R, \xi) > 0$, and for every $n \geq n(R, \xi)$ there exists a map (the *approximating map*) $k : U_n \subset M_\infty \to M_n$ such that

- k is defined on the ball $B_{M_\infty}(m_\infty, R)$ of radius R centered at the basepoint m_∞ of M_∞ and sends this basepoint to the base point of M_n.
- the restriction of k to the ball $B_{M_\infty}(m_\infty, \frac{R}{2})$ is ξ-close to an isometry in the $\mathcal{C}^2$-topology: The metric tensor ρ_∞ of M_∞ and the pullback $k^*\rho_n$ by k of the metric tensor ρ_n of M_n are ξ-close in the $\mathcal{C}^2$-norm on $2-$tensors on the ball $B_{M_\infty}\left(m_\infty, \frac{R}{2}\right)$ which we denote by $||k^*\rho_n - \rho_\infty||_{\mathcal{C}^2, B\,(m_\infty,\,\frac{R}{2})}$.

To be more precise, let ∇^∞ be the Levi-Civita connection for ρ_∞. Define

$$\begin{aligned} ||k^*\rho_n - \rho_\infty||_{\mathcal{C}^2, B\,(m_\infty,\,\frac{R}{2})} &= ||k^*\rho_n - \rho_\infty||_{\mathcal{C}^0, B\,(m_\infty,\,\frac{R}{2})} \\ + ||\nabla^\infty(k^*\rho_n)||_{\mathcal{C}^0, B\,(m_\infty,\,\frac{R}{2})} &+ ||\nabla^\infty\nabla^\infty(k^*\rho_n)||_{\mathcal{C}^0, B\,(m_\infty,\,\frac{R}{2})}. \end{aligned}$$

We then say that the restriction of k to $B(m_\infty, R/2)$ is ξ-*almost isometric.*

Note that this definition of geometric convergences is slightly more restrictive than what is found in the literature (see e.g. Chapter E of [2]). We shall make use of the following compactness result for geometric convergence (Theorem E.1.10 of [2]).

Theorem 2.2. *Suppose that $\{(M_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence of pointed hyperbolic 3-manifolds such that there is a uniform positive lower bound $\eta > 0$ on the injectivity radius at the base points m_n. Then there exists a subsequence that converges in the geometric topology to a pointed hyperbolic 3-manifold (M_∞, m_∞).*

We observe next that in combination with *Margulis' Lemma* (see e.g. [2]), Theorem 2.1 implies that if the conformal boundary of a convex cocompact hyperbolic structure N on a handlebody $\mathcal{H}$ is ϵ-thick in the Poincaré metric (i.e. its injectivity radius is at least ϵ), then there is a uniform lower bound, only depending on $\epsilon > 0$ and $g = g(\Sigma)$, on the injectivity radius of N at points that are close to the boundary of the convex core $\mathcal{CC}(N)$. This enables us to take geometric limits.

For the formulation of this fact, for sufficiently small $\epsilon > 0$ we denote by $\mathcal{T}_\epsilon \subset \mathcal{T}$ the subset of Teichmüller space of all marked hyperbolic metrics on Σ with injectivity radius at least ϵ, and we let $\mathcal{M}_\epsilon = \mathrm{Mod}(\Sigma)\backslash\mathcal{T}_\epsilon$ be the ϵ-thick part of moduli space. Furthermore, let $\mathrm{inj}_x(N)$ be the injectivity radius of a hyperbolic manifold N at the point x, and let $\mathrm{inj}(N) = \inf_x\{\mathrm{inj}_x(N) \mid x \in N\}$ the global injectivity radius of N.

LEMMA 2.3. *For every $\epsilon > 0$ and $g \geq 2$ there exists $\eta = \eta(\epsilon, g) > 0$ such that the following holds: Let N be a convex cocompact hyperbolic structure on $\mathcal{H}$. If $\partial_c N \in \mathcal{M}_\epsilon$ then* $\inf_{x\in\partial\mathcal{CC}(N)}\{\mathrm{inj}_x(N)\} \geq \eta$.

In the proof and in the sequel we use the following notations:

> **Notation**. If X is a hyperbolic surface and $\gamma : S^1 \to X$ is a smooth closed curve, then we denote by $L(\gamma)$ the length of γ, and by $L_X(\gamma)$ the length of the geodesic representative of γ on X. For a curve γ in a hyperbolic 3-manifold M we use the notation $l(\gamma)$ and $l_M(\gamma)$ for the analogous quantities.

Proof. By Theorem 2.1, we have that every simple closed curve on the boundary of the convex core $\gamma \subset \partial\mathcal{CC}(N)$ has length (with respect to the induced hyperbolic metric)

$$L_{\partial\mathcal{CC}(N)}(\gamma) \geq \frac{1}{G(\epsilon)} L_{\partial_c N}(\gamma) \geq \frac{2\epsilon}{G(\epsilon)}$$

as $\mathrm{inj}(\partial_c N) \geq \epsilon$.

Hyperbolic trigonometry shows that there exists a uniform upper bound for the diameter of $\partial\mathcal{CC}(N)$ in the intrinsic metric, say $\mathrm{diam}\ \partial\mathcal{CC}(N) \leq D$ where $D = D(\epsilon, g)$ only depends on $\epsilon > 0$ and $g \geq 2$. Since the inclusion $\partial\mathcal{CC}(N) \subset N$ is $1-$Lipschitz by definition of the intrinsic path-metric, we have the same control on the diameter when we compute distances in N.

Let $\epsilon_3 > 0$ be a Margulis constant for hyperbolic manifolds in dimension 2 and 3. By Margulis' Lemma, for some small number $\rho < \epsilon_3$ the $\rho-$thin part $N_{(0,\rho]} := N - \{x \in N \mid \mathrm{inj}_x N > \rho\}$ of N is a disjoint union of Margulis tubes, i.e. metric tubular neighbourhoods of simple closed geodesics of length smaller than ρ.

Having bounded diameter and carrying all the information about the fundamental group of $\mathcal{H}$, the surface $\partial\mathcal{CC}(N)$ cannot penetrate deeply into a Margulis tube. Namely, let $\gamma \subset N$ be the core geodesic of the ρ-Margulis tube containing $x \in \partial\mathcal{CC}(N)$. Standard hyperbolic geometry yields that the distance between the boundary of the $\rho-$Margulis tube and the boundary of the ϵ_3-Margulis tube of γ grows to ∞ as ρ approaches 0. In particular, if ρ is sufficiently small then by the diameter bound for $\partial\mathcal{CC}(N)$, this surface is entirely contained in the ϵ_3-Margulis tube. This contradicts the fact that the inclusion $\partial\mathcal{CC}(N) \hookrightarrow N$ is π_1-surjective. Hence the injectivity radius of N at points in $\partial\mathcal{CC}(N)$ is bounded from below by a universal positive constant. $\square$

By Lemma 2.3 and compactness of pleated surfaces (see section I.5.2 of [15], in particular Theorem I.5.2.2), given a sequence of triples

$$\{(N_n, j_n : X_n \subset \mathcal{M} \to \partial\mathcal{CC}(N_n), x_n)\}_{n\in\mathbb{N}}$$

consisting of convex-cocompact hyperbolic structures, corresponding (pleated surface parametrizations of the) boundaries of the convex cores and basepoints such that the conformal boundary is ϵ-thick, we can always extract a subsequence (say the whole sequence) that converges in the geometric topology to a triple $(N_\infty, j_\infty : X_\infty \to N_\infty, x_\infty)$, consisting of a hyperbolic 3-manifold, a pleated surface and a common basepoint, in the following sense:

- The sequence of pointed 3-manifolds (N_n, x_n) converges to (N_∞, x_∞).
- The sequence of pointed hyperbolic surfaces $(\partial\mathcal{CC}(N_n), x_n)$ converges to the pointed hyperbolic surface $(X_\infty, \overline{x}_\infty)$. Observe that, by Theorem 2.1, $\partial\mathcal{CC}(N_n)$ has a uniform lower bound on the injectivity radius and a uniform upper bound on the diameter. In particular, the surfaces X_n are contained in a compact subset of moduli space, and the surface $X_\infty \in \mathcal{M}$ is an accumulation point of the sequence (X_n). In particular, it shares the same uniform bounds on the injectivity radius and the diameter.
- The pleated surface embeddings $\partial\mathcal{CC}(N_n) \hookrightarrow N_n$, which we denote by j_n, converge to a pleated surface $j_\infty : X_\infty \to N_\infty$. The diagram where all the maps respect the basepoints, and the vertical arrows are the approximating maps provided by the geometric convergence,

$$\begin{tikzcd}
\partial\mathcal{CC}(N_n) \arrow[r, "j_n"] & N_n \\
X_\infty \arrow[u, "\phi_n"] \arrow[r, "j_\infty"'] & N_\infty \arrow[u, "k_n"']
\end{tikzcd}$$

  commutes up to local (pointed) homotopies, i.e. those homotopies that respect the base points and take place in small neighbourhoods of the images of the pleated surfaces.
- Lastly, we also recall that $\partial\mathcal{CC}(N_n)$ and X_∞ come together with a marking (as in the definition of pleated surface), i.e. they are isometrically identified with $\left(\Sigma, \sigma_{\partial\mathcal{CC}(N_n)}\right)$ and (Σ, σ_∞). These markings consist of collections of curves whose lengths are uniformly bounded for the induced hyperbolic metrics. We can assume that the composition of the identification $\left(\Sigma, \sigma_{\partial\mathcal{CC}(N_n)}\right) \simeq \partial\mathcal{CC}(N_n)$ with the inclusion in N_n is isotopic to the marking of N_n, and that the marked (and hence parametrized) pleated surfaces converge.

3. Relative bounded combinatorics

Bounded combinatorics is a combinatorial condition which translates into explicit geometric control of the hyperbolic metric on the convex cocompact handlebody near the boundary of its convex core.

Relative versions of bounded combinatorics were introduced in [41], [42] and [11]. We are interested in bounded combinatorics relative to a decorated handlebody $\mathcal{H}$, where the decoration is either a marking μ on $\partial\mathcal{H}$ or a point $X \in \mathcal{T}$, which we can think of as a "fixed convex cocompact structure" on $\mathcal{H}$. We recall that, if $X \in \mathcal{T}$, then the geometric realization $\mathcal{H}(X)$ is the convex cocompact handlebody whose marked conformal boundary equals X.

Following ideas of Minsky [40] and of Brock-Canary-Minsky [10], our goal is to construct for a random 3-manifold M a model metric which is close to the hyperbolic metric and such that on a submanifold M_0 of M whose volume is bigger than a definite proportion of the volume of M, this metric can explicitly be described.

Our construction is a variation of earlier constructions of [41], [42] and [11]. Namely, we glue the model metric from hyperbolic metrics on pieces with large overlap on which the metrics have large injectivity radius and are very close. The condition we need to successfully glue these metrics on the overlap to a metric which is close to a hyperbolic metric can be described as "bounded combinatorics and large height".

Our setup, however, is different from the setup in these earlier work as we can not assume any global control of the injectivity radius, i.e. unbounded geometry appears. The purpose of this section is to provide the control needed in the sequel.

We begin with collecting the essential facts about coarse geometry and Gromov hyperbolic spaces, in particular about the geometry of the *curve graph* and the *disk graph*.

3.1. **Coarse Geometry.** A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is an $(L, C)-$*quasi-isometric embedding* if for every $x, x' \in X$

$$\frac{1}{L} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq L d_X(x, x') + C.$$

A parametrized $(L, C)-$*quasi-geodesic* segment, ray or line is an (L, C)-quasi-isometric embedding of an interval, a half line or the entire real line $\mathbb{R}$. Later on we will have to deal also with *unparametrized* $(L, C)-$quasi-geodesic segments, rays and lines which are maps $f : I \subset \mathbb{R} \longrightarrow (X, d_X)$ such that there exists a homeomorphism ϕ from an interval $I' \subset \mathbb{R}$ onto the interval I with the property that the composition $f \circ \phi : I' \longrightarrow (X, d_X)$ is a $(L, C)-$quasi-isometric embedding (the intervals I, I' can be finite or infinite).

3.2. **Curve and Disk Graphs.** Masur and Minsky proved in [32] that the curve graph $\mathcal{C} := \mathcal{C}(\Sigma)$ of the closed surface Σ of genus $g \geq 2$ is a *Gromov hyperbolic* space of infinite diameter, and Klarreich [24] identified the *Gromov boundary* $\partial_\infty \mathcal{C} = \partial_\infty \mathcal{C}(\Sigma)$ with the space $\mathcal{EL} = \mathcal{EL}(\Sigma)$ of minimal filling unmeasured laminations (see also [19] for a different approach).

Definition (Gromov Product and Convergence). Given $\alpha, \beta, \gamma \in \mathcal{C}$, the quantity

$$(\alpha|\beta)_\gamma := \frac{1}{2}\left[d(\alpha,\gamma) + d(\beta,\gamma) - d(\alpha,\beta)\right]$$

is the Gromov product of α, β based at γ. A sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ *converges at infinity* to a point in $\partial_\infty \mathcal{C}$ if and only if for some base point γ (and hence for any) we have $\liminf_{n,m\to\infty} (\alpha_n|\alpha_m)_\gamma \to \infty$. If $\{\alpha_n\}_{n\in\mathbb{N}}$ converges at infinity and $\{\beta_n\}_{n\in\mathbb{N}}$ satisfies $\liminf_{n,m\to\infty} (\alpha_n|\beta_m) = \infty$, then $\{\beta_n\}_{n\in\mathbb{N}}$ converges to the same point in $\partial_\infty \mathcal{C}$.

For material on Gromov products and Gromov boundaries we refer the reader to Section 3 of Chapter III.H of [9].

The geometry of the curve graph is coarsely tied to the geometry of Teichmüller space. There is a (coarsely well-defined) $\mathrm{Mod}(\Sigma)$-equivariant map

$$\Upsilon : \mathcal{T} \to \mathcal{C},$$

called the *systole map*, that associates to every marked hyperbolic structure $X \in \mathcal{T}$ a shortest geodesic on it $\Upsilon(X)$. It follows from Masur-Minsky [32] that there exist constants $L \geq 1, C \geq 0$ only depending on Σ such that for every Teichmüller geodesic $l : I \to \mathcal{T}$ (here I can be an interval, a half-line or the whole real line) the composition $\Upsilon \circ l : I \to \mathcal{C}$ is an *unparametrized* (L, C)-quasi-geodesic. Moreover, if we restrict our attention to the ϵ-thick part $\mathcal{T}_\epsilon$ of Teichmüller space, then the situation improves: In [20] it is shown that for every $\epsilon > 0$ there exist $L_\epsilon \geq 1, C_\epsilon \geq 0$ such that if l is parametrized by arc length on an interval of length $l(I) \geq L_\epsilon$ and if $l(I) \subset \mathcal{T}_\epsilon$ then $\Upsilon \circ l$ is a *parametrized* (L_ϵ, C_ϵ)-quasi-geodesic.

The *disk graph* $\mathcal{D}$ associated to the identification $\Sigma = \partial \mathcal{H}$ is the subgraph of $\mathcal{C}$ spanned by disk-bounding curves. Masur and Minsky showed in [34] that the disk graph $\mathcal{D}$ is a *quasi-convex* subset of the curve graph $\mathcal{C}$. Being quasi-convex, by hyperbolicity of $\mathcal{C}$, there is a coarsely defined nearest point projection $\pi_\mathcal{D} : \mathcal{C} \longrightarrow \mathcal{D}$.

3.3. **Subsurface projection and Bounded Combinatorics.** An essential tool for describing the geometry of the curve graph is the notion of *subsurface projection* introduced by Masur-Minsky in [33]: For every proper essential subsurface $W \subset \Sigma$ (with some care for annuli and pairs of pants) and every point $\alpha \in \mathcal{C} \cup (\partial_\infty \mathcal{C} = \mathcal{EL})$, there is a subsurface projection $\pi_W(\alpha) \subset \mathcal{C}(W)$ which consists of the (possibly empty) subset of $\mathcal{C}(W)$ of all the possible essential surgeries of $\alpha \cap W$ (see [33] for the details).

We can also define subsurface projections for markings (taking the projection of the marking as a subset of $\mathcal{C}$). All the markings μ on Σ we consider are *complete*, i.e. they are given, as a subset of $\mathcal{C}$, by a *pants decomposition*, called the *base* of the marking, and for every curve α in the base a *transversal* t_α, that is, a simple closed curve which intersects α essentially in the least possible number of points (either one or two points) and does not intersect the other curves in the base. The total geometric intersection number between all curves in a marking is required to be uniformly bounded.

For every $X \in \mathcal{T}$ we denote by μ_X a short marking on the hyperbolic surface X. A short marking μ on X is a marking which is shortest among all markings. If we denote for $X \in \mathcal{T}$ by $L_X(\mu)$ the sum of the geodesic lengths of all curves in the marking μ, then for every $\epsilon > 0$ there exists $B_\epsilon > 0$ such that if $X \in \mathcal{T}_\epsilon$, then $L_X(\mu_X) \leq B_\epsilon$.

Bounded combinatorics means no large subsurface projections:

Definition (Bounded Combinatorics, [39]). Let $R > 0$ be a positive number. Let α, β be either complete markings or unmeasured minimal filling laminations on Σ. The pair α, β has *R-bounded combinatorics* if for every proper essential subsurface $W \subset \Sigma$ we have

$$d_W(\alpha, \beta) := \operatorname{diam}_{\mathcal{C}(W)}(\pi_W(\alpha) \cup \pi_W(\beta)) \leq R.$$

For some $\epsilon > 0$, two points $X, Y \in \mathcal{T}_\epsilon$ have *R-bounded combinatorics* if this holds true for short markings μ_X, μ_Y for X, Y.

Just like ϵ-thickness, R-bounded combinatorics implies nice compactness properties: The following can be found as Proposition 6.2 in [41]

Lemma 3.1. *Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ be a sequence of markings that have R-bounded combinatorics with respect to a marking β. Then either there exists a constant subsequence, or there exists a subsequnce that converges to an unmeasured minimal filling lamination in the Gromov boundary $\lambda \in \partial_\infty \mathcal{C}$. Moreover, λ has $(R+1)$-bounded combinatorics with respect to β.*

3.4. **Bounded combinatorics and Heegaard splittings.** When considering $\partial\mathcal{H} = \Sigma$ one has to take into account the compressibility of the boundary. Denote as before by $\mathcal{D}$ the disk set of $\mathcal{H}$, viewed as a quasi-convex subset of the curve graph $\mathcal{C}$.

Motivated by a construction of Namazi [41], we shall use the following relative version of bounded combinatorics for convex cocompact handlebodies.

Definition (Relative Bounded Combinatorics). We say that an ordered pair (μ, ν) of markings of $\partial\mathcal{H}$ has *relative R-bounded combinatorics* with respect to the handlebody $\mathcal{H}$ if the pair μ, ν has R-bounded combinatorics and the following holds:

$$d_{\mathcal{C}}(\mathcal{D}, \mu) + d_{\mathcal{C}}(\mu, \nu) \leq d_{\mathcal{C}}(\mathcal{D}, \nu) + R. \tag{1}$$

The *height of the pair* (μ, ν) is $d_{\mathcal{C}}(\mu, \nu)$.

For a fixed thickness threshold $\epsilon > 0$, we say that an ordered pair $(Y, X) \in \mathcal{T} \times \mathcal{T}$ has relative R-bounded combinatorics with respect to $\mathcal{H}$ if $Y, X \in \mathcal{T}_\epsilon$ and the pair (μ_Y, μ_X) satisfies the above conditions. The height in this case is $d_{\mathcal{T}}(Y, X)$.

In the definition "μ (or Y) lies between $\mathcal{D}$ and ν (or X)".

The next lemma, which is analogous to Lemma 3.1, provides some compactness in our setting:

LEMMA 3.2. *Fix $g \geq 2$ and $R > 0$. Let $\mathcal{H}$ be a handlebody of genus g. Let $\{(\alpha_n, \beta_n)\}_{n \in \mathbb{N}}$ be a sequence of ordered pairs of markings on $\Sigma := \partial\mathcal{H}$. Suppose that:*

- *The pair (α_n, β_n) has relative R-bounded combinatorics.*
- *The sequence of heights diverges, i.e. $H_n = d_{\mathcal{C}}(\alpha_n, \beta_n) \longrightarrow \infty$.*

Then we have

$$(\delta|\alpha_n)_{\beta_n} \longrightarrow \infty$$

uniformly in $\delta \in \mathcal{D}$. If we renormalize the configuration by translating β_n to a fixed base-point $\beta \in \mathcal{C}$ with $\tau_n \in \mathrm{Mod}\,(\Sigma)$, then the sequence $\{\tau_n \mathcal{D}\}_{n \in \mathbb{N}}$ converges, up to possibly passing to a subsequence, to a point $\lambda \in \partial_\infty \mathcal{C}$. Moreover λ has $(R+1)-$bounded combinatorics with respect to β.

The meaning of the term "uniformly" in the statement is the following: For every $M > 0$ there exists $n_0 > 0$ such that for every $n \geq n_0$ and $\delta \in \mathcal{D}$ we have $(\delta|\alpha_n)_{\beta_n} \geq M$. Informally, Lemma 3.2 says that the disk set $\mathcal{D}$ disappears if we look at it from the point of view of β_n.

Proof. The proof is an easy application of Lemma 3.1 and basic properties of hyperbolic spaces. By definition, the Gromov product is computed by the following formula

$$(\delta|\alpha_n)_{\beta_n} := \frac{1}{2}\left[d_{\mathcal{C}}(\beta_n, \delta) + d_{\mathcal{C}}(\beta_n, \alpha_n) - d_{\mathcal{C}}(\delta, \alpha_n)\right].$$

The Gromov product measures the fellow-traveling of the segments $[\beta_n, \delta]$ and $[\beta_n, \alpha_n]$.

Fix $M > 0$. Consider a disk $\delta \in \mathcal{D}$. We claim that $(\delta|\alpha_n)_{\beta_n} \geq M$ for large n. To show the claim it suffices to analyze the geodesic segment $[\beta_n, \delta]$. Namely, the quasi-convexity of $\mathcal{D}$ and the Condition (1) imply together that $[\beta_n, \alpha_n]$ uniformly fellow-travels $[\beta_n, \delta]$.

By quasi-convexity of $\mathcal{D}$, the segment $[\beta_n, \delta]$ passes uniformly close to the nearest-point projection of β_n to $\mathcal{D}$, which we denote by $\overline{\beta}_n = \pi_{\mathcal{D}}(\beta_n)$. Hence we have:

$$d_{\mathcal{C}}(\beta_n, \delta) \approx d_{\mathcal{C}}(\beta_n, \overline{\beta}_n) + d_{\mathcal{C}}(\overline{\beta}_n, \delta).$$

Here the symbol $\approx$ means "equal up to a uniform additive constant". The same holds for α_n: If we denote by $\overline{\alpha}_n := \pi_{\mathcal{D}}(\alpha_n)$ the projection to the disk set, then we have $d_{\mathcal{C}}(\alpha_n, \delta) \approx d_{\mathcal{C}}(\alpha_n, \overline{\alpha}_n) + d_{\mathcal{C}}(\overline{\alpha}_n, \delta)$.

The conclusion $(\delta|\alpha_n)_{\beta_n} \approx d(\alpha_n,\beta_n) = H_n$ would follow directly from the formula of the Gromov product if we knew that α_n, β_n have coarsely the same projection to $\mathcal{D}$, i.e. $\overline{\alpha}_n \approx \overline{\beta}_n$, and the segment $\left[\beta_n, \overline{\beta}_n\right]$ passes uniformly close to α_n, i.e. $d_{\mathcal{C}}(\beta_n, \overline{\beta}_n) \approx d_{\mathcal{C}}(\beta_n, \alpha_n) + d_{\mathcal{C}}(\alpha_n, \overline{\beta}_n)$. These properties are a consequence of Condition (1) and can be derived from the fact that equality holds in the following chain of inequalities:

$$\begin{aligned} d_{\mathcal{C}}(\beta_n, \overline{\beta}_n) &= d_{\mathcal{C}}(\beta_n, \mathcal{D}) \approx d_{\mathcal{C}}(\beta_n, \alpha_n) + d_{\mathcal{C}}(\alpha_n, \mathcal{D}) \\ &= d_{\mathcal{C}}(\beta_n, \alpha_n) + d_{\mathcal{C}}(\alpha_n, \overline{\alpha}_n) \geq d_{\mathcal{C}}(\beta_n, \overline{\alpha}_n) \geq d_{\mathcal{C}}(\beta_n, \overline{\beta}_n). \end{aligned}$$

Now we normalize the situation by translating β_n to a fixed point $\beta \in \mathcal{C}$ and prove the convergence statement. Let us denote by $\hat{\alpha}_n$ also the translate of α_n and by $\mathcal{D}_n$ the translated disk sets. Since having bounded combinatorics is invariant under $\mathrm{Mod}(\Sigma)$, every $\hat{\alpha}_n$ has R-bounded combinatorics with respect to β.

By Lemma 3.1 the sequence $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, not having any constant subsequence as $d_{\mathcal{C}}(\beta, \hat{\alpha}_n) = H_n \to \infty$, admits a subsequence, say the whole sequence, converging to a minimal filling lamination in the Gromov boundary $\lambda \in \partial_\infty \mathcal{C}$ which furthermore has $(R+1)$-bounded combinatorics with respect to β. We show that $\mathcal{D}_n \to \lambda$ as well.

Consider a sequence of disks $\{\delta_n \in \mathcal{D}_n\}_{n\in\mathbb{N}}$. We identify the Gromov boundary $\partial_\infty \mathcal{C}$ with the space of equivalence classes of diverging sequences $\{x_n\}_{n\in\mathbb{N}}$. The claim follows by showing that $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$ are equivalent, i.e. $(\delta_m|\hat{\alpha}_n)_\beta \to \infty$ $(n, m \to \infty)$.

Fix $M > 0$. By the first part of the proof, there exists $N > 0$ such that for every $n, m \geq N$ we have $(\delta_n|\hat{\alpha}_n)_\beta \geq M$. The claim follows from basic properties of Gromov products:

$$(\delta_m|\hat{\alpha}_n)_\beta \gtrapprox \min\left\{(\delta_m|\hat{\alpha}_m)_\beta\,, (\delta_n|\hat{\alpha}_n)_\beta\right\} \geq M$$

for every $n, m \geq N$. Here $\gtrapprox$ means "greater or equal up to a uniform additive constant". □

3.5. **Gluings with relative bounded combinatorics.** The next definition describes the class of gluings for which we can relate the glued metric to the hyperbolic metric on the glued manifold.

Definition (Gluings with Relative Bounded Combinatorics). Given a gluing map $f \in \mathrm{Mod}(\Sigma)$, a quadruple of markings (μ, ν, ν_f, μ_f) *has relative* (f, R)-*bounded combinatorics* with respect to $\mathcal{H}$ if:

- (μ, ν) has relative R-bounded combinatorics with respect to $\mathcal{D}$.
- (ν_f, μ_f) has relative R-bounded combinatorics with respect to $f\mathcal{D}$.

The *height* of the pair is $\min\{d_{\mathcal{C}}(\mu,\nu), d_{\mathcal{C}}(\mu_f,\nu_f)\}$.

As in the definition of pairs with relative bounded combinatorics, for a fixed a thickness threshold $\epsilon > 0$, there is also a version for quadruples of points $\left(Y, X, \overline{X}, \overline{Y}\right) \in$ in Teichmüller space $\mathcal{T}_\epsilon$.

Remark 3.3. We make the following observations:

(i) The point Y lies between the disk set $\mathcal{D}$ and X. On the other side $\overline{Y}$ lies between $f\mathcal{D}$ and $\overline{X}$.
(ii) The pair $\left(\overline{Y},\overline{X}\right)$ has relative R-bounded combinatorics with respect to the handlebody defined by the disk set $f\mathcal{D}$, i.e. by declaring that the curves in $f\mathcal{D}\subset\mathcal{C}$ are exactly the compressible ones.
(iii) If $\left(Y,X,\overline{X},\overline{Y}\right)$ is a quadruple with (f,R)-relative bounded combinatorics, then also every quadruple $\left(Y_0,X_0,\overline{X}_0,\overline{Y}_0\right)$, where the segments $[Y_0,X_0]\,,\left[\overline{Y}_0,\overline{X}_0\right]\subset\mathcal{T}_\epsilon$ are, respectively, subsegments of $[Y,X]$ and $\left[\overline{Y},\overline{X}\right]$, satisfies the (f,R)-bounded combinatorics condition.

4. Large-Thick Collars

The goal of this section is to prove the following Proposition. Recall that for $X\in\mathcal{T}$ we denote by $\mathcal{H}(X)$ a convex cocompact handlebody with conformal boundary X.

Proposition 4.1. *Let $g\geq 2$ be fixed. For all $R,L,\epsilon>0$ there exists $H=H(R,L,\epsilon)>0$ such that the following holds: If the pair $(Y,X)\in\mathcal{T}_\epsilon^2$ has R-relative bounded combinatorics with respect to $\mathcal{H}$ and height at least H, then the boundary of the convex core of $\mathcal{H}(X)$ has a collar of width at least L and injectivity radius at least $\eta>0$ where η only depends on g and R.*

The strategy is easy to state: We argue by contradiction. Suppose we have a sequence of counterexamples $\mathcal{H}(X_n)$ with relative R-bounded combinatorics and diverging heights, but no large-thick collar. Using the results from Section 2, we can take a geometric limit N_∞ by looking at $\mathcal{H}(X_n)$ from the boundary of the convex core. The main result of this section, Proposition 4.2, states that N_∞ is a singly degenerate structure on $\Sigma\times\mathbb{R}$ with bounded geometry. Once we know this we are done because we can pullback a large thick collar of arbitrary size via the approximating maps, thus obtaining a contradiction.

We begin with collecting some structural facts used in the proof.

4.1. Ends of hyperbolic 3-manifolds.

Let M be a hyperbolic 3-manifold. For a fixed $\epsilon>0$ let us denote by $M_{[\epsilon,\infty)}$ the *ϵ-thick part* of M, the set of points in M where the injectivity radius is greater or equal to ϵ, and by $M_{(0,\epsilon]}$ the *ϵ-thin part* of M, the closure of the complement of $M_{[\epsilon,\infty)}$. There exists a universal constant $\epsilon_3>0$, called a *Margulis constant*, such that for every $\epsilon\leq\epsilon_3$, every connected component of the ϵ-thin part of the *thick-thin decomposition* $M=M_{(0,\epsilon]}\cup M_{[\epsilon,\infty)}$ is of one of the following two types: Margulis tubes, i.e. metric tubular neighbourhood of a closed geodesic γ (the *core* of the tube) of length $l_M(\gamma)\leq 2\epsilon$, rank two cusps (which will not be relevant for us) or rank one cusps, isometric to a quotient of a horoball $\mathcal{O}\subset\mathbb{H}^3$ by an infinite cyclic group of parabolic isometries. A simple closed curve generating the fundamental group of the cusp is called the *core* of the cusp (see Chapter D of [2] for more information).

From now on let us assume that M is homeomorphic to $\Sigma \times \mathbb{R}$. As Σ is closed, $\pi_1(M) \simeq \pi_1(\Sigma)$ does not contain subgroups isomorphic to $\mathbb{Z}^2$ and hence M does not have rank two cusps. Every element $\gamma \in \pi_1(\Sigma)$ is hyperbolic on Σ, but might act as a parabolic motion on the universal covering $\mathbb{H}^3$ of M. In this case we call γ an *accidental parabolic*. Such an element generates the fundamental group of a rank one cusp in M.

Fix a Margulis constant $\epsilon \leq \epsilon_3$ and consider the *non-cuspidal part* of M defined as $M_0 = M - M^{\text{cusp}}_{(0,\epsilon]}$ where $M^{\text{cusp}}_{(0,\epsilon]}$, the *cuspidal part*, is the union of the interiors of the (rank one) cusps in $M_{(0,\epsilon]}$. Scott proved in [45] that there exists a compact submanifold $\mathcal{SC} \subset M$, called a *Scott core*, homeomorphic to $\Sigma \times [0, 1]$, with the following properties.

- The inclusion of $\mathcal{SC}$ into M is a homotopy equivalence.
- The intersection of $\mathcal{SC}$ with the closure of the cuspidal part $M^{\text{cusp}}_{(0,\epsilon]}$ consists of a disjoint union of annuli on $\partial\mathcal{SC}$ whose cores represent the cores of the corresponding rank one cusps.
- The topological *ends* of M_0 (which are *relative ends* for M) are in bijective correspondence with the connected components of $M_0 - \mathcal{SC}$. The closure of every connected component $E \subset M_0 - \mathcal{SC}$ is homeomorphic to $Y \times [0, \infty)$ where Y is the (connected) subsurface of $\partial\mathcal{SC}$ obtained as the intersection $\overline{E} \cap \mathcal{SC}$.

This description of the (relative) compact core uses results of McCullogh [36] and Kulkarni-Shalen [25].

By work of Thurston [46], Bonahon [7], Canary [13] and others, to each relative end $E \simeq Y \times [0, \infty)$ we can associate an *end invariant* which either is a finite type hyperbolic structure (conformal boundary) or a minimal filling lamination (the *ending lamination*) on Y. In the first case we say that the end is *geometrically finite*, while in the second case it is *simply degenerate*. Following Thurston [46], Bonahon [7] and Canary [13], this dichotomy can be characterized as follows:

- The end E is geometrically finite if there is a compact set $K \subset M$ such that $E - K$ does not contain any closed geodesic.
- The end E is simply degenerate if there exists a sequence of simple closed curves $\{\gamma_n \subset Y\}$ with geodesic representatives $\{\gamma_n^* \subset E\}_{n\in\mathbb{N}}$ (equivalently, a sequence of pleated surfaces $\{f_n : (Y, \sigma_n) \to E\}_{n\in\mathbb{N}}$ in the homotopy class of the inclusion $Y \subset E$), that *exit the end*, i.e. γ_n (or $f_n(Y)$) is eventually contained in $E - K$ for any compact set $K \subset M$. The curves γ_n (or the curves $\Upsilon(Y, \sigma_n)$, see Section 3) converge in $\mathcal{C}(Y)$ to the ending lamination $\lambda_E \in \partial_\infty\mathcal{C}(Y)$.

4.2. **Marked hyperbolic structures on I-bundles.** A marked hyperbolic structure on $\Sigma \times \mathbb{R}$, is a hyperbolic 3-manifold $Q = \mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$ together with a homotopy equivalence $\phi : \Sigma \to Q$, the *marking*. We always assume that there are no *accidental parabolics*, i.e. every element of $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ is hyperbolic. Equivalently, there are no cusps in Q.

Every marked hyperbolic structure Q on $\Sigma \times \mathbb{R}$ without accidental parabolics has exactly two relative ends homeomorphic to $\Sigma \times [0, \infty)$. To them we can associate a pair of end invariants (μ^+, μ^-), each of which is either a marked hyperbolic structure or a minimal filling lamination on Σ.

- If both μ^+ and μ^- are marked hyperbolic surfaces we call the manifold *quasi-fuchsian*. Its convex core $\mathcal{CC}(Q)$ is compact and homeomorphic to $\Sigma \times [0, 1]$ (except in the *fuchsian* case $\mu^+ = \mu^-$ where $\mathcal{CC}(Q)$ is a totally geodesic embedded surface). The boundary components are pleated surfaces.
- If μ_+ and μ_- are distinct minimal filling laminations, then we call the manifold Q *doubly degenerate*. In this case the convex core coincides with the whole manifold $\mathcal{CC}(Q) = Q$.
- If one end invariant is a marked hyperbolic structure and the other is a filling lamination, then the manifold is *singly degenerate*. The convex core is homeomorphic to $\mathcal{CC}(Q) = \Sigma \times [0, \infty)$.

By Bers Simultaneous Uniformization [3], marked quasi-fuchsian structures Q on $\Sigma \times \mathbb{R}$ are parametrized by $\mathcal{T} \times \mathcal{T}$ via the map that associates to Q the conformal boundary $\partial_c Q = (\mu^+, \mu^-) \in \mathcal{T} \times \mathcal{T}$. Given a pair $(Y, X) \in \mathcal{T} \times \mathcal{T}$ of conformal structures at infinity, we denote by $Q(Y, X)$ the unique quasi-fuchsian manifold that realizes those boundary data.

The solution of the Ending Lamination Conjecture by Minsky [40] and Brock-Canary-Minsky [10] implies that, as a marked hyperbolic structure, Q is uniquely determined by its end invariants (μ^+, μ^-). We recall that the manifold Q has *bounded geometry* if there is a positive lower bound on the injectivity radius.

The mapping class group $\mathrm{Mod}\,(\Sigma)$ acts on marked hyperbolic structures Q on $\Sigma \times \mathbb{R}$ by precomposition of marking. On the quasi-fuchsian subspace, the action coincides with the diagonal action on $\mathcal{T} \times \mathcal{T}$.

4.3. **Convergence to singly degenerate.** We now prove what we stated at the beginning of the section.

Proposition 4.2. *Let $\left\{\left(Y_n, X_n\right) \in \mathcal{T}_\epsilon^2\right)\right\}_{n\in\mathbb{N}}$ be a sequence where every pair has relative R-bounded combinatorics, and the heights H_n diverge. Then up to passing to a subsequence, the sequence $\{(N_n := \mathcal{H}(X_n), x_n)\}_{n\in\mathbb{N}}$ of pointed convex cocompact handlebodies converges geometrically to a singly degenerate hyperbolic structure N_∞ on $\Sigma \times \mathbb{R}$ with* $\mathrm{inj}(N_\infty) \geq \eta$. *Here η only depends on g and R.*

For convenience, we divide the proof of Proposition 4.2 into several small steps: Lemma 4.3, Lemma 4.4, Lemma 4.5 and its Corollaries.

To begin with, note that by Theorem 2.2 and Lemma 2.3, from the sequence of triples $\{(N_n, j_n : \partial\mathcal{CC}(N_n) \hookrightarrow N_n, x_n)\}_{n\in\mathbb{N}}$ we can extract a subsequence that converges to $(N_\infty, j_\infty : X_\infty \to N_\infty, x_\infty)$ in the sense described in Section 2. We then have a diagram provided by geometric convergence

$$\begin{tikzcd} \partial\mathcal{CC}(N_n) \arrow[r, "j_n"] & N_n \\ X_\infty \arrow[u, "\phi_n"] \arrow[r, "j_\infty"'] & N_\infty \arrow[u, "k_n"'] \end{tikzcd}$$

where the vertical arrows are the approximating maps, where $j_n : \partial\mathcal{CC}(N_n) = \left(\Sigma, \sigma_{\partial\mathcal{CC}(N_n)}\right) \to N_n$ is a pleated surface parametrization of $\partial\mathcal{CC}(N_n)$ in the isotopy class of the marking of N_n, and $X_\infty = (\Sigma, \sigma_\infty) \to N_\infty$ is the "limit" pleated surface in N_∞.

Lemma 4.3. *The map j_∞ is incompressible.*

Proof. For large n, the maps $k_n^{-1} j_n \phi_n$ are defined and they are homotopic to j_∞ within a neighbourhood of $j_\infty(X_\infty)$ of uniformly bounded diameter. Thus j_∞ is compressible if and only if $k_n^{-1} j_n \phi_n$ is compressible. As the diameters of the pleated surfaces $j_n \partial\mathcal{CC}(N_n)$ are uniformly bounded, there exists n_0 such that the map $k_n^{-1} j_n \phi_n$ is an embedding for every $n \geq n_0$. Suppose that j_∞ is compressible. By the Loop Theorem, there exists a simple closed curve $\gamma \subset X_\infty$ such that $k_{n_0}^{-1} j_{n_0} \phi_{n_0}(\gamma)$ bounds an embedded disk D^2 in N_∞.

For large enough $n \geq n_0$ the map k_n is defined on the disk D^2. Since the maps $k_n j_\infty$ and $j_n \phi_n$ are (locally) homotopic we observe that the simple closed curve $\phi_n(\gamma) \subset \partial\mathcal{CC}(N_n)$ whose image under the inclusion j_n is freely homotopic to $k_n j_\infty(\gamma)$, is compressible. As N_n is a handlebody, this means that $\phi_n(\gamma) \in \mathcal{D}$, where, as before, $\mathcal{D} \subset \mathcal{C}$ is the set of diskbounding curves. The length $L_{\partial\mathcal{CC}(N_n)}(\phi_n(\gamma))$ of $\phi_n(\gamma)$ is bounded from above by $\leq 2L_{X_\infty}(\gamma)$ for n large enough.

To obtain a contradiction, it suffices to show that the $\partial\mathcal{CC}(N_n)-$lengths of any sequence of simple closed diskbounding curves $\zeta_n \subset \mathcal{D}$ blow up along the sequence. To this end let as before μ_{X_n} be a short marking for the conformal boundary X_n of N_n, with base the pants decomposition P_n. Denote by $i(\xi, \zeta)$ the geometric intersection number between two simple closed curves ξ, ζ in the boundary surface Σ. Let $B = B(\Sigma, \epsilon)$ be an upper bound for the length of a short marking for a surface $X \in \mathcal{T}_\epsilon$. Then there is a number $C = C(\Sigma, \epsilon) > 0$ such that

$$L_{X_n}(\zeta_n) \geq C \cdot i\,(\zeta_n, P_n) \geq \frac{C}{B} 2^{(d_{\mathcal{C}}(\zeta_n, P_n) - 2)/2} \underset{n\to\infty}{\longrightarrow} \infty.$$

Namely, the first inequality follows from the fact that $L_{X_n}(P_n) \leq B$ and standard hyperbolic geometry as explicitly described in Lemma 4.2 in [17]. The second inequality is a consequence of $d_{\mathcal{C}}(\xi,\zeta) \leq 2\log_2 i(\xi,\zeta)+2$ for all $\xi,\zeta \in \mathcal{C}$. □

Let us consider now the $\pi_1(j_\infty(X_\infty), x_0)-$covering of N_∞ which we denote by $p : \overline{N}_\infty \to N_\infty$. By covering theory, the map $j_\infty : X_\infty \to N_\infty$ lifts to $\overline{N}_\infty$, and any lift $\overline{j}_\infty : X_\infty \to \overline{N}_\infty$ is a homotopy equivalence. We fix once and for all such a lift and denote by $\overline{X}_\infty = \overline{j}_\infty(X_\infty)$ the image of X_∞ under this lift.

By work of Thurston [46] and Bonahon [7], we know that, in this setting, $\overline{N}_\infty$ is homeomorphic to $\Sigma \times \mathbb{R}$.

The next step of the proof consists in showing that $\overline{N}_\infty$ has a "visible" geometrically finite end homeomorphic to $\Sigma \times [0,\infty)$.

LEMMA 4.4. *$\overline{N}_\infty$ has a geometrically finite end homeomorphic to* $\Sigma \times [0,\infty)$.

Proof. For large n consider the nearest point retraction to the convex core $r_n : N_n \to \mathcal{CC}(N_n) \subset N_n$. It is a (base point preserving) 1-Lipschitz projection map, i.e. $r_n^2 = r_n$, with the property that $r_n j_n = j_n$. It also fits into a one-parameter family of projections that gives a deformation of r_n to the identity Id_{N_n}. Namely, for $t \geq 0$ define r_n^t to be the nearest point retraction to the t-neighbourhood of the convex core which is a convex subset $\mathcal{CC}_t(N_n)$. Notice that $r_n^t \overset{t\to\infty}{\longrightarrow} \mathrm{Id}_{N_n}$, uniformly on compact subsets.

The map $h_n : N_n \times [0,\infty) \to N_n$ defined by

$$h_n(x,t) = r_n^t(x)$$

is 1-Lipschitz with respect to the product metric. Hence, by geometric convergence, there is an induced 1-Lipschitz family of projections $h_\infty : N_\infty \times [0,\infty) \to N_\infty$. Let us denote by $r_\infty : N_\infty \to N_\infty$ the map induced by the r_n's, i.e. $r_\infty(\cdot) = h_\infty(\cdot, 0)$. It has the properties $r_\infty^2 = r_\infty$ and $r_\infty j_\infty = j_\infty$. The last property implies, in particular, that h_∞ lifts to a map $\overline{h}_\infty : \overline{N}_\infty \times [0,\infty) \to \overline{N}_\infty$ with $\overline{h}_\infty(\cdot,0) = \overline{r}_\infty$, the lift of r_∞ satisfying $\overline{r}_\infty \overline{j}_\infty = \overline{j}_\infty$.

Since $h_\infty(\cdot,t) \to \mathrm{Id}_{N_\infty}$ $(t\to\infty)$ we also get $\overline{h}_\infty(\cdot,t) \to \mathrm{Id}_{\overline{N}_\infty}$ $(t \to \infty)$. From the existence of such a map we can immediately conclude that the image $\overline{r}_\infty\left(\overline{N}_\infty\right)$ is a convex subset of $\overline{N}_\infty$ containing the convex core $\mathcal{CC}\left(\overline{N}_\infty\right)$. To this end let $x,y \in \overline{r}_\infty(\overline{N}_\infty)$ be a pair of points. Let α be a geodesic joining them. Then $\overline{r}_\infty(\alpha)$ is homotopic, relative to the endpoints, to α and has length $l\left(\overline{r}_\infty(\alpha)\right) \leq l(\alpha)$. Since α is the unique length minimizer in its homotopy class with fixed endpoints we conclude that $\overline{r}_\infty(\alpha) = \alpha$. To summarize, $\bar{r}_\infty(\bar{N}_\infty)$ is convex.

Now the boundary $\overline{X}_\infty$ of $\bar{r}_\infty(\bar{N}_\infty)$ is a closed embedded (by convexity) incompressible pleated surface, and such a surface is contained in the convex core of $\bar{N}_\infty$. To be more precise, the preimage $\tilde{X}_\infty$ of $\overline{X}_\infty$ in $\mathbb{H}^3$ is a pleated surface which bounds a convex $\pi_1(\bar{N}_\infty)$- half-space V. Furthermore, the group $\pi_1(\bar{N}_\infty)$ acts properly and cocompactly on $\tilde{X}_\infty$, equipped with the intrinsic path metric. If $\tilde{\lambda}$ is a pleating lamination for $\tilde{X}_\infty$, then every leaf of $\tilde{\lambda}$ connects two points in the limit set of $\pi_1(\overline{N}_\infty)$ by invariance under the action of $\pi_1(\bar{N}_\infty)$. But $\tilde{X}_\infty$ is contained in the convex hull of $\tilde{\lambda}$, whence $X_\infty \subset \partial\mathcal{CC}(\bar{N}_\infty)$. This shows that $\overline{X}_\infty$ bounds a geometrically finite end in $\bar{N}_\infty$. □

Up to now we have not fully used the condition of relative bounded combinatorics. We do it now by observing the following

LEMMA 4.5. *There exists a minimal filling lamination $\lambda \in \partial_\infty\mathcal{C}(X_\infty)$ such that for any sequence of diskbounding curves $\{\delta_n \in \mathcal{C}(\partial\mathcal{CC}(N_n))\}_{n\in\mathbb{N}}$, the sequence of simple closed curves $\{\phi_n^{-1}\delta_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(X_\infty)$ converges (up to passing to subsequences) to λ.*

Proof. Let μ_{X_∞} be a short marking on X_∞. Since ϕ_n is almost an isometry, the marking $\phi_n\mu_{X_\infty}$ is short also for $\partial\mathcal{CC}(N_n)$, i.e. we can choose $\mu_{\partial\mathcal{CC}(N_n)} = \phi_n\mu_{X_\infty}$. By assumption, we also find markings ν_n on $\partial\mathcal{CC}(N_n)$ that satisfy the relative bounded combinatorics conditions.

By Lemma 3.2, we have $(\phi_n^{-1}\delta_n|\phi_n^{-1}\nu_n)_{\mu_{X_\infty}} = (\delta_n|\nu_n)_{\phi_n\mu_{X_\infty}} \to \infty$. Moreover, μ_{X_∞} and $\phi_n^{-1}\nu_n$ have R-bounded combinatorics and their distances in $\mathcal{C}(X_\infty)$, being equal to the distances between $\mu_{\partial\mathcal{CC}(N_n)}$ and ν_n, diverge. In particular $\phi_n^{-1}\nu_n$ converges (up to passing to subsequences) to a minimal filling lamination $\lambda \in \partial_\infty\mathcal{C}(X_\infty)$ and so does $\phi_n^{-1}\delta_n$ (see the proof of Lemma 3.2). □

As an easy consequence, we obtain

COROLLARY 4.6. *The lamination $\overline{j}_\infty(\lambda) \subset \overline{X}_\infty$ is not realized in $\overline{N}_\infty$.*

Proof. Suppose we can realize $\overline{j}_\infty(\lambda)$. Then by composition with the covering projection, the lamination $\lambda \subset X_\infty$ can be realized in N_∞. Let $Y_\lambda \subset N_\infty$ be a pleated surface realizing λ.

Lemma 4.5, combined with the "long-branches-small-switch-angles" train-track argument due to Bonahon [7] (see the proof of Proposition 3.2 in [42] for a nice exposition), tells us that for sufficiently large n and a diskbounding simple closed curve δ_n on $X_n \subset N_n$, we could also realize $\phi_n^{-1}\delta_n$ as a closed geodesic, in a bounded neighbourhood of Y_λ in N_∞.

By geometric convergence, this implies that for large n, we can represent the curve $j_n\delta_n$ in N_n as a curve with very small geodesic curvature. Such a curve is not nullhomotopic in N_n, But this is absurd as δ_n is, by definition, compressible in N_n. □

The next corollary is certainly well known. As we were not able to locate it in the literature in the form we need, we include a proof in the Appendix. The main issue here is the possible presence of accidential parabolics.

Corollary 4.7. *$\overline{N}_\infty$ has a simply degenerate end homeomorphic to $\Sigma \times [0,\infty)$ with ending lamination λ.*

To conclude, we found that $\overline{N}_\infty$ is a hyperbolic structure on $\Sigma \times \mathbb{R}$ for which one of the end invariants is a minimal filling lamination, and the second is a marked conformal structure on Σ. Since there is no room for other ends, we see that $\overline{N}_\infty$ is singly degenerate. Our final step consists in showing that the covering $p : \overline{N}_\infty \to N_\infty$ is trivial. This step concludes the proof of Proposition 4.2 with the exception of the bounded geometry condition.

Lemma 4.8. *The covering $p : \overline{N}_\infty \to N_\infty$ is trivial.*

Proof. By Canary's Covering Theorem [14], $p : \overline{N}_\infty \to N_\infty$ is finite-to-one, N_∞ is homeomorphic to $\Omega \times \mathbb{R}$ where Ω is a closed surface, and the covering is induced by the inclusion $\pi_1(\Sigma) < \pi_1(\Omega)$. However, by Lemma 4.3, we also have an incompressible embedding of Σ into $N_\infty \simeq \Omega \times \mathbb{R}$. We conclude by evoking the standard fact of 3-manifold topology that the closed incompressible surfaces in $\Omega \times \mathbb{R}$ are all isotopic to the standard embedding $\Omega \hookrightarrow \Omega \times \{0\}$ (see Proposition 9.3.18 of [30]). □

We are left with the observation that the injectivity radius of $\overline{N}_\infty$ is bounded from below by a universal positive constant. As there are no accidential parabolics and the ending lamination λ has $\overline{R}-$bounded combinatorics with respect to X_∞ ($\overline{R}$ might be bigger than the initial R, but it is still uniformly bounded), this is an immediate consequence of Minsky's theorem on Bounded Geometry for Kleinian Surface Groups [39].

5. Cut and Glue construction

Recall that our goal is to produce a model metric on a Heegaard splitting $H_1 \cup_f H_2$ and that we want to achieve it by gluing together some simple building blocks which are pieces of hyperbolic manifolds we understand better. In our case the building blocks are two convex cocompact handlebodies interpolated by a quasi-fuchsian manifold.

This section describes how the gluing works. We use a standard procedure, summarized in Lemma 5.1. Suppose we are given the following data:

(1) A pair $(N, \partial N)$, $(M, \partial M)$ of Riemannian 3-manifolds with boundary, with metrics ρ_N, ρ_M.
(2) A pair of collars U, V of $\partial N, \partial M$ with a smooth diffeomorphism $k : U \to V$ between them.
(3) A smooth bump function $\theta : U \to [0,1]$ which takes value 1 on ∂N and 0 on the other boundary $\partial U \setminus \partial N$.

Then we can form the Riemannian 3-manifold $N \cup_{k:U\to V} M$ where the metrics ρ_N and ρ_M are replaced on U by the convex combination $(1-\theta)\rho_N +$

$\theta k^* \rho_M$. A crucial feature that can be implemented is a control on the *sectional curvatures*: If the diffeomorphism is almost isometric and the bump function has uniformly bounded derivatives, then the sectional curvatures of the gluing will be comparable to those of the pieces.

Let us now describe the case of a convex cocompact handlebody N and quasi-fuchsian manifold Q. The first item of the list is provided by small smooth neighbourhoods of their convex cores $\mathcal{CC}(N)$ and $\mathcal{CC}(Q)$.

The second item requires us to control a pair of collars of $\partial\mathcal{CC}(N)$ and $\partial\mathcal{CC}(Q)$, and this is provided by Proposition 4.1. We also have to produce a nice diffeomorphism between them, but this will be the object of the next section (Proposition 6.1). We anticipate here that, in general, it will not be possible to use exactly the collars of $\partial\mathcal{CC}(N)$ and $\partial\mathcal{CC}(Q)$, but one has to allow a more flexible notion of *product region* which we define below.

As for the last item, Lemma 5.2 will produce for us uniformly controlled bump functions on product regions with bounded geometry.

Now we describe the details. Let us start with a definition:

Definition (Product Region). Let $(M, \partial M, j : \Sigma \to \partial_0 M)$ be a compact oriented 3-manifold with boundary ∂M and a distinguished parametrized (or simply, marked) boundary component $j : \Sigma \to \partial_0 M \subset \partial M$. A *product region* $U \subset M$ relative to $\partial_0 M \subset \partial M$ is a codimension 0 submanifold homeomorphic to $\Sigma \times [-1, 1]$ which is isotopic to a collar of $\partial_0 M$.

Using the product structure of U we can define a *top boundary* $\partial_+ U$, the one that faces $\partial_0 M$, and a *bottom boundary* $\partial_- U$. We denote by M_- and M_+ the parts of M that lie *below* $\partial_+ U$ and *above* $\partial_- U$ respectively. In particular, U is a collar of a boundary component of both M_-, M_+.

We will be interested in essentially three parameters of a product region: The injectivity radius $\mathrm{inj}\, U := \inf\{\mathrm{inj}_x M \,|\, x \in U\}$, the diameter $\mathrm{diam}\, U$ and the *width* $\mathrm{width}\, U := \inf\{d(x, y) \,|\, x \in \partial_+ U, y \in \partial_- U\}$. When the injectivity radius, the width and the diameter are uniformly bounded we say that the product region has *bounded geometry.*

Finally, we also observe that the marking $j : \Sigma \to \partial_0 M$ can be isotoped into U producing a *marking* of U.

One can cut and glue 3-manifolds along product regions. Namely, suppose we have a pair $(M, \partial M, j : \Sigma \to \partial_0 M)$ and $(N, \partial N, i : \Sigma \to \partial_0 N)$ of compact 3-manifolds with boundary together with distinguished parametrized boundary components. Let $U \subset M$ and $V \subset N$ be product regions relative to $\partial_0 M, \partial_0 N$ respectively. Let $k : U \to V$ be an orientation preserving diffeomorphism between them. Then we can form the 3-manifold

$$\mathbb{X} := M_- \cup_{k:U \to V} N_+.$$

Up to homeomorphism, the result only depends on the homotopy class of k, which we are going to define. Denote by $j_U, j_V : \Sigma \to U, V$ the induced markings on the product regions. The composition $f := j_V^{-1} \circ k \circ j_U$ is well-defined up to isotopy, that is, it does not depend on the choice of induced markings. It is called the *homotopy class of k with respect to the markings j, i*. The manifold $\mathbb{X}$ is homeomorphic to

$$M \cup_{\phi:\partial_0 M \to \partial_0 N} N$$

where $\phi := i \circ f \circ j^{-1}$.

Now we turn to the Riemannian part of the construction. The following observation, which we state as a lemma, is the main conclusion of the cut and glue construction. The proof is straightforward, and we omit it.

LEMMA 5.1. *Let $(M, \partial M, j : \Sigma \to \partial_0 M)$, $(N, \partial N, i : \Sigma \to \partial_0 N)$ be marked hyperbolic structures on M and N with distinguished boundary components $\partial_0 M, \partial_0 N$. Denote by ρ_M, ρ_N the Riemannian metrics of M, N. Suppose we have product regions $U \subset M, V \subset N$ relative to $\partial_0 M, \partial_0 N$ and an orientation preserving diffeomorphism $k : U \to V$. Suppose also that $\theta : U \to [0,1]$ is a smooth function with $\theta|_{\partial_- U, \partial_+ U} \equiv 0, 1$. Then we can form the 3-manifold*

$$\mathbb{X} = M_- \cup_{k:U \to V} N_+$$

and endow it with the Riemannian metric

$$\rho := \begin{cases} \rho_M & \text{on } M_- \setminus U \\ (1-\theta)\rho_M + \theta k^* \rho_N & \text{on } U \\ \rho_N & \text{on } N_+ \setminus V. \end{cases}$$

If k is ξ-almost isometric for some $\xi < 1$, then we have the following sectional curvature bound on $U \subset \mathbb{X}$

$$|1 + \sec_{\mathbb{X}}| \leq c_3 \, ||\theta||_{\mathcal{C}^2} \cdot ||\rho_M - k^* \rho_N||_{\mathcal{C}^2} \,,$$

where $c_3 > 0$ is some universal constant. If k lies in the homotopy class of the identity, then compositions of the inclusions $M_- \subset \mathbb{X}, N_+ \subset \mathbb{X}$ with the (induced) markings $j_U, i_V : \Sigma \to M, N$ are homotopic.

Once we fix the size of a product region we can produce a uniform bump function $\theta : U \to [0,1]$ on it.

LEMMA 5.2. *For all $\eta, D > 0$ there exists $B > 0$ such that the following holds: Let $U \simeq \Sigma \times [0,1]$ be a product region with* $\operatorname{inj} U \geq \eta$, $\operatorname{diam} U \leq 2D$, $\operatorname{width} U \geq D$. *Then there exists a smooth function $\theta : U \to [0,1]$ with the following properties:*

- *Near the boundaries it is constant: $\theta|_{\partial_- U} \equiv 0$ and $\theta|_{\partial_+ U} \equiv 1$.*
- *Uniformly bounded $\mathcal{C}^2-$norm: $||\theta||_{\mathcal{C}^2} \leq B$.*

Proof. For every $D, \eta > 0$, the space of pointed hyperbolic 3-manifolds

$$\mathcal{S}(\eta, 2D, D) = \left\{ (M, \star \in U) \;\middle|\; \begin{array}{c} U \simeq \Sigma \times [-1,1] \text{ is a product,} \\ \operatorname{inj} U > \eta, \operatorname{diam} U < 2D, \operatorname{width} U > D \end{array} \right\}$$

is relatively compact in the geometric topology. For any $\delta > 0$, the accumulation points not in the space are still contained in $\mathcal{S}(\eta - \delta, 2D + \delta, D - \delta)$. $\square$

6. Almost-isometric embeddings and Gluing

If we want to apply the cut-and-glue construction, we have to understand when we can find almost isometric product regions in a convex cocompact handlebody and in a quasi-fuchsian manifold. The following technical proposition provides the control we need.

Proposition 6.1. *Fix bounded combinatorics parameters $R, \epsilon > 0$ and an almost-isometry parameter $\xi > 0$. There exists $L_0 = L_0(R, \epsilon) > 0$ such that for every $L \geq L_0$ there exists a height $H = H(L, R, \epsilon, \xi) > 0$ such that the following holds: Let $(Y, X) \in \mathcal{T}_\epsilon \times \mathcal{T}_\epsilon$ be a pair with relative R-bounded combinatorics and height at least H. Let $Z \in \mathcal{T}_\epsilon$ be any other point such that the Teichmüller geodesic $[Y, Z]$ contains $[Y, X]$ as a subsegment. Consider the convex cocompact handlebody $N = \mathcal{H}(X)$ and the quasi-fuchsian manifold $Q = Q(Y, Z)$. Then there exist product regions $U \subset N$ and $V \subset Q$ and an orientation preserving diffeomorphism $k : U \to V$ such that*

1. *Bounded geometry:* $\operatorname{inj} U \geq \eta = \eta(R, \epsilon) > 0$, $\operatorname{diam} U \leq 2L$ *and* $\operatorname{width} U \geq L$.
2. *Almost isometry: $||\rho_N - k^*\rho_Q||_{\mathcal{C}^2} < \xi$.*
3. *Homotopy class: k lies in the homotopy class of the identity with respect to the markings.*

Moreover, U contains the geodesic representative of $\alpha \in \mathcal{C}$, a curve which has moderate length for both N and some hyperbolic surface $T \in [Y, X]$, i.e. $l_N(\alpha), L_T(\alpha) \leq B = B(g, \epsilon)$.

We call the point $Z \in \mathcal{T}_\epsilon$ the *free boundary* of $Q(Y, Z)$.

Conditions (1)-(3) guarantee that we can uniformly glue $\mathcal{H}(X)$ to $Q(Y, Z)$ using the cut and glue construction. The application to Heegaard splitting is given in Theorem 6.7 at the end of the section.

A few words on the proof: We have seen that the boundary of the convex core of a convex cocompact handlebody with relative bounded combinatorics and large height has a large-thick collar. This means that, to some extent, we can treat it as if it was a hyperbolic structure on $\Sigma \times \mathbb{R}$.

Then the strategy is to reduce the problem to the following statement, which solves the analogue question of finding almost isometric embeddings of product regions in hyperbolic manifolds homeomorphic to $\Sigma \times \mathbb{R}$

Proposition 6.2. *For every $\epsilon, \xi, \delta, L > 0$ there exists $H = H(\epsilon, \xi, \delta, L) > 0$ such that the following holds: Let Q_1, Q_2 be marked hyperbolic structures on $\Sigma \times \mathbb{R}$ without accidental parabolics with associated Teichmüller geodesics $l_i : I_i \subseteq \mathbb{R} \to \mathcal{T}$ with $i = 1, 2$. Suppose that l_1, l_2 δ-fellow-travel on a subsegment J of length at least $20H$ and entirely contained in the ϵ-thick part $\mathcal{T}_\epsilon$. Then there exist product regions $U_i \subset Q_i$ with* $\operatorname{diam}(U_i) \leq 2L$, $\operatorname{width}(U_i) \geq L$ *and a $\xi-$almost isometric embedding $k : U_1 \to U_2$ in the homotopy class of the identity with respect to the markings. Moreover, U_i contains the geodesic* *representative of $\alpha \in \mathcal{C}$, a curve which has moderate length for both Q_i and $T \in J$ the midpoint of the segment, i.e. $l_{Q_i}(\alpha), L_T(\alpha) \leq B = B(\epsilon, g)$.*

6.1. **Lipschitz model.** Proposition 6.2 is a direct consequence of the Lipschitz Model by Minsky [40]. We use the following statement:

THEOREM 6.3 (Minsky [40]). *Fix $\epsilon > 0$. Let Q be a marked hyperbolic structure on $\Sigma \times \mathbb{R}$ without accidental parabolics. Let $l : I \to \mathcal{T}$ be the corresponding Teichmüller geodesics. There exists $H_0 = H_0(g, \epsilon) > 0$ and $B = B(g, \epsilon) > 0$ such that the following holds: Suppose that I contains a subsegment $[a - H_0, b + H_0]$ such that $b - a \geq 20H_0$ and $l\,[a - H_0, b + H_0] \subset \mathcal{T}_\epsilon$. Denote by J the subsegment $[a, b]$. Then for every $X \in J$ and any curve $\alpha \in \mathcal{C}$ with $L_X(\alpha) \leq B$ there exists a pleated surface $i_X : \hat{X} \to Q$ realizing α and such that $d_{\mathcal{T}}(X, \hat{X}) \leq H_0$. In particular $l_Q(\alpha) \leq Be^{2H_0}$.*

We remark the following crucial consequence of the Margulis Lemma:

LEMMA 6.4. *There exists $\eta = \eta(g, \epsilon) > 0$ such that $\mathrm{inj}_x(Q) > \eta$ for every $x \in i_X(\hat{X})$ and every $X \in J$.*

Proof. The surface $i_X(\hat{X})$ is π_1-surjective and has uniformly bounded diameter ($1-$Lipschitz image of $\hat{X} \in \mathcal{T}_{\epsilon'(\epsilon,H)}$). Such a surface cannot enter any very thin part of Q (see Lemma 2.3). □

We prove Proposition 6.2.

Proof of Proposition 6.2. We argue by contradiction. Suppose we have a sequence of structures Q_1^n, Q_2^n that satisfy the assumptions, but do not satisfy the conclusions. Denote by J_n the δ-fellow-traveling ϵ-thick segment for their Teichmüller geodesics $l_i^n : I_i^n \to \mathcal{T}$ with $i = 1, 2$. Let $c_n \in J_n$ the midpoint. Up to the action of the mapping class group we can assume that c_n lies in a fixed compact set of $\mathcal{T}$. Let us parametrize J_n by $j_n : [-a_n, a_n] \to J_n$ so that $j_n(0) = c_n$. After the renormalization, we can extract a subsequence that converges uniformly on compact sets to a bi-infinite Teichmüller geodesic $j_\infty : \mathbb{R} \to J_\infty$ entirely contained in $\mathcal{T}_\epsilon$ with distinct uniquely ergodic minimal filling endpoints $\lambda^-, \lambda^+ \in \mathcal{PML}$.

By the Double Limit Theorem and the Ending Lamination Theorem, the geodesic j_∞ defines a unique doubly degenerate structure on $\Sigma \times \mathbb{R}$ whose ending laminations are λ^+, λ^-. We show that we can choose basepoints $x_j^n \in Q_j^n$ so that the sequence of pointed manifolds (Q_j^n, x_j^n) converges geometrically to Q_∞ and derive a contradiction.

Let us focus on Q_1^n. Parametrize l_1^n so that when we restrict it to $[-a_n + \delta, a_n - \delta]$ it δ-fellow-travels $j_n : [-a_n + \delta, a_n - \delta] \to J_n$. Observe that the geodesic l_1^n converges uniformly on compact sets to some bi-infinite geodesic J_1^∞ which is entirely contained in $\mathcal{T}_\epsilon$ and has uniquely ergodic minimal filling endpoints λ_1^+, λ_1^-. Since l_1^n and j_n are δ-fellow-travelers, we have

$\lambda_1^{\pm} = \lambda_\infty^{\pm}$. Therefore $J_1^\infty = J^\infty$, as cobounded geodesics in Teichmüller space are uniquely determined by their endpoints.

The points $X_1^n := l_1^n(0)$ converge to $X_\infty = j_\infty(0)$, hence, for n large enough, by Theorem 6.3, we can find pleated surfaces $i_{X_1^n} : \hat{X}_1^n \to Q_1^n$ that realize any short curve $\gamma \in \Upsilon(X_\infty)$ for X_∞. We choose a basepoint x_1^n on $i_{X_1^n}(\hat{X}_1^n)$. Lemma 6.4 tells us that we can take a geometric limit of the sequence triples $(Q_1^n, x_1^n, i_{X_1^n} : \hat{X}_1^n \to Q_1^n)$. The limit is a triple $(Q_1^\infty, x_1^\infty, i_{X_\infty} : \hat{X}_1^\infty \to Q_1^\infty)$ where i_{X_∞} is a pleated surface realizing γ, the short curve on $\Upsilon(X_\infty)$ as above, in Q_1^∞.

The proof can now proceed with the standard arguments of Section 4 with less complications. We only give a sketch. The map i_{X_∞} is incompressible and determines a covering of Q_1^∞ homeomorphic to $\Sigma \times \mathbb{R}$. We claim that it is doubly degenerate. It suffices to check that λ^+ and λ^- are not realized. Choose a diverging sequence of times $t_n \in [0, a_n]$ such that $l_1^n(t_n)$ is very close to J_∞. Theorem 6.3 implies that a short curve α_n^+ for $l_1^n(t_n)$ has moderate length for Q_1^n. As $t_n \uparrow \infty$ we have $\alpha_n^+ \to \lambda^+$. Suppose that λ^+ is realized. The "long-branches-small-switch-angles" train track argument implies that we can also realize α_n^+ in a fixed compact set and hence $l_{Q_1^\infty}(\alpha_n^+)$ must necessarily diverge. By geometric convergence, the same must happen in Q_1^n, but α_n^+ has always moderate length in Q_1^n, a contradiction. The argument for λ^- is the same. By the Ending Lamination Theorem, the $i_{X_\infty}-$covering is isometric to Q_∞. The Covering Theorem [14] implies that the covering is trivial. The uniform bound on the injectivity radius is guaranteed by the fact that λ^+, λ^- have uniformly bounded combinatorics [39].

In conclusion, for any fixed size L and almost isometric parameter ξ, we can pull-back a product region of that size from Q_∞ to Q_1^n in a $\xi-$almost isometric fashion. Moreover, we can also assume that the product region lies uniformly close to the basepoint and that, if the size L is sufficiently large, it also contains the geodesic representative of a short curve γ on $l_1^n(0)$, the midpoint of J_n. This contradicts the initial assumptions. $\square$

We are now ready to prove Proposition 6.1

6.2. **Proof of Proposition 6.1.** We argue again by contradiction.

Suppose we have a sequence of examples $\{(Y_n, X_n, Z_n)\}_{n\in\mathbb{N}}$ with relative R-bounded combinatorics and diverging heights, but $N_n := \mathcal{H}(X_n)$ and $Q_n := Q(Y_n, Z_n)$ do not satisfy the conclusion of the proposition. As a first step, we reduce the problem to the realm of hyperbolic structures on $\Sigma \times \mathbb{R}$. If we fix basepoints $x_n \in \partial\mathcal{CC}(N_n)$, we know that we can take, up to passing to subsequences, a geometric limit of the sequence

$$\{(N_n, x_n, j_n : (\Sigma, \sigma_n) \to \partial\mathcal{CC}(N_n))\}_{n\in\mathbb{N}}.$$

The limit is a triple $(N_\infty, x_\infty, j_\infty : (\Sigma, \sigma_\infty) \to N_\infty)$ where N_∞ is a singly degenerate hyperbolic structure on $\Sigma \times \mathbb{R}$ with injectivity radius bounded

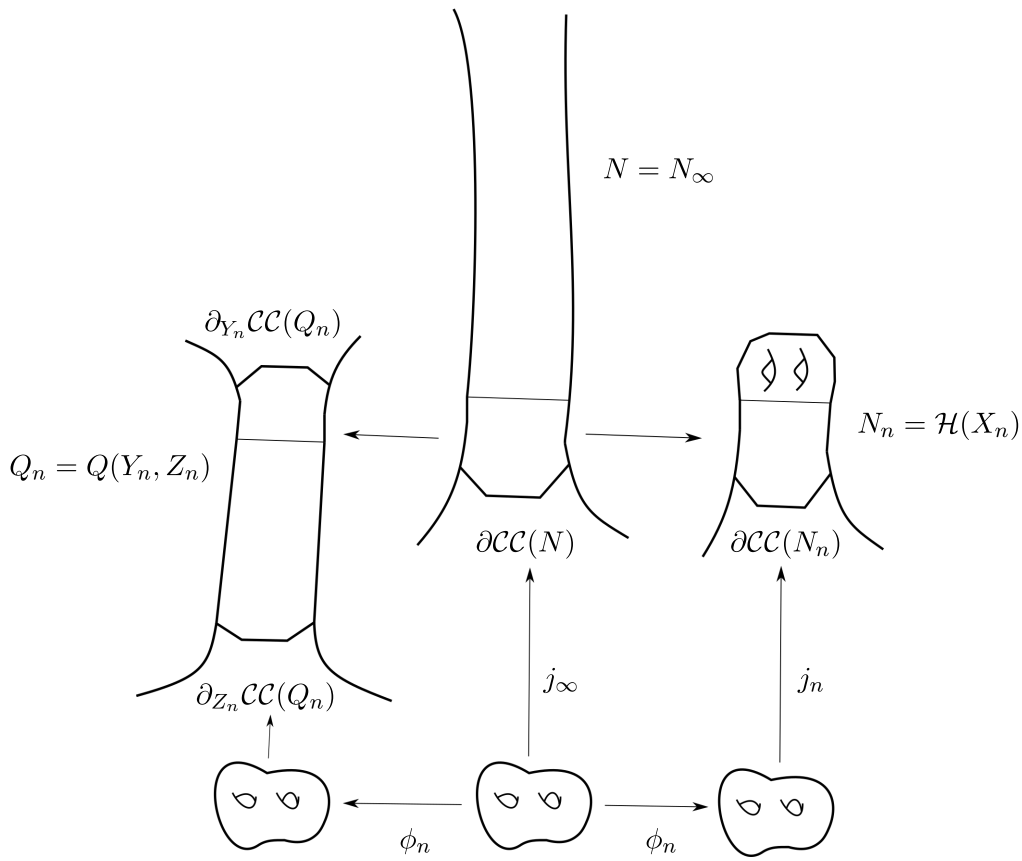

FIGURE 1. Almost-isometric embeddings of product regions.

from below by $\eta = \eta(R, \epsilon) > 0$ and j_∞ is a pleated surface marking. The approximating maps $k_n : N_\infty \to N_n$ and the change of marking $\phi_n : \Sigma \to \Sigma$ provided by the pleated surface convergence fit into the following diagram that commutes up to local homotopies

$$\begin{tikzcd}
(\Sigma, \sigma_n) \arrow[r, "j_n"] & N_n \\
(\Sigma, \sigma_\infty) \arrow[u, "\phi_n"] \arrow[r, "j_\infty"'] & N_\infty . \arrow[u, "k_n"']
\end{tikzcd}$$

The ending lamination λ is the support of a unique projective measured lamination that can be characterized as the limit in $\mathcal{T} \cup \mathcal{PML}$ of the sequence of remarked surfaces $Y_n' := \phi_n^{-1} Y_n$, i.e. $\lambda = \lim_{n\to\infty} Y_n' \in \mathcal{PML}$ (see [24]).

We fully renormalize the picture by introducing $X_n' := \phi_n^{-1} X_n$, $Z_n' := \phi_n^{-1} Z_n$ and the quasi-fuchsian manifolds $Q_n' := Q(Y_n', Z_n')$. Observe that the points $X_n' \in [Y_n', Z_n']$ lie in a fixed compact set. In fact, by the main theorem of [8], X_n' lies uniformly close to $\phi_n^{-1}(\Sigma, \sigma_n)$ which is converging to (Σ, σ_∞), the hyperbolic structure of the limit pleated surface. Hence, up to subsequences, the segments $[Y_n', X_n']$ converge uniformly on compact sets to the Teichmüller geodesic ray $(\lambda, X_\infty]$.

The next lemma is used to determine a product region of N_∞ which can be embedded in all N_n and Q_n' with n large via Proposition 6.2 (see Figure 1). Let $l : (-\infty, 0] \to \mathcal{T}$ the Teichmüller ray corresponding to N_∞ with $l(-\infty) = \lambda$. As N_∞ has bounded geometry, it is entirely contained in a ϵ'-thick part (by [39], [43]) for some ϵ' only depending on R, ϵ. Moreover, by Theorem A of [37], it passes uniformly close to (Σ, σ_∞) and hence to X_∞.

Lemma 6.5. *There exists $\delta, b > 0$ such that for every $H > 0$, if n is sufficiently large, then there is a point $W_n' \in [Y_n', X_n']$ such that $[W_n', X_n']$, parametrized in this order, δ-fellow-travels the restriction of $l : (-\infty, 0] \to \mathcal{T}$ to $J_H = [-b - 200H, -b]$.*

Proof. The geodesics $(\lambda, X_\infty]$ and l have the same endpoint at infinity, the uniquely ergodic projective measured lamination λ. By Masur [31], these geodesics stay at a uniformly bounded distance $\delta/2$. As $[Y_n', X_n']$ converges uniformly on compact sets to $(\lambda, X_\infty]$, if n is large enough, we find $W_n' \in [Y_n', X_n']$ which is δ-close to l and with $d_{\mathcal{T}}(W_n', X_n')$ as large as we want. □

By Lemma 6.5, for any fixed H, we can apply Proposition 6.2 to the δ-fellow-traveling ϵ'-thick geodesics $[Z_n', Y_n']$ and l along $l(J_H)$ if n is sufficiently large. We choose H to be larger than $H(\epsilon', \xi, \delta, L)$. For n large enough, we get a product region $U_n \subset N_\infty$ of L-bounded geometry and a $\xi-$almost isometric embedding $h_n : U_n \to Q_n'$. Moreover the product region U_n contains a geodesic α_n^* of uniformly bounded length that represents a short curve α_n for the midpoint T_n of $[W_n', X_n']$. The curve α_n has also moderate length for the midpoint T of $l(J_H)$ as $L_T(\alpha_n) \leq L_{T_n}(\alpha_n)e^{2\delta}$. As there is only a finite number of curves of moderate length on T, we can assume that $\alpha_n = \alpha$ is fixed.

Consider a sufficiently large collar U of $\partial\mathcal{CC}(N_\infty)$ containing the $2L$-neighbourhood of the geodesic representative of $\alpha \in \Upsilon(T)$. If n is sufficiently large, the approximating map $N_\infty \to N_n$ is defined and $\xi-$almost isometric on U. The product region U_n, containing α^* and having size comparable with L, is contained in U. This is a contradiction.

The proof of Proposition 6.1 is now complete.

6.3. **Position of the product regions.** As we have already pointed out, Proposition 6.1 guarantees that we can uniformly glue $\mathcal{H}(X)$ to $Q(Y, Z)$ using the cut and glue construction. However, for the model metric on $H_1 \cup_f H_2$, we need a more quantitative control: If we want to glue a pair of convex cocompact handlebodies to a single quasi-fuchsian manifold on top and on bottom, we have to make sure that the gluing regions appear in the *right order* along the quasi-fuchsian manifold. We control the order using the distance from the boundaries of the convex core: In the notations of Proposition 6.1 we have

Lemma 6.6. *There exists some function $A : (0, \infty) \to (0, \infty)$ such that*

$$d_N\left(U, \partial\mathcal{CC}(N)\right) \leq A(d_{\mathcal{T}}(X, Y)),$$
$$d_Q\left(k(U), \partial_Y\mathcal{CC}(Q)\right) \leq A(d_{\mathcal{T}}(X, Y)).$$

Here $\partial_Y\mathcal{CC}(Q = Q(Y, Z))$ denotes the boundary component of the convex core that faces the conformal boundary Y.

Proof. Let α be a the curve which is of moderate length for some $T \in [Y, X]$ and whose geodesic representative has length $l_Q(\alpha) \in [\eta, B]$ and lies in U as in Proposition 6.1. We have

$$d_N\left(U, \partial\mathcal{CC}(N)\right) \leq d_N\left(\alpha^*, \partial\mathcal{CC}(N)\right),$$
$$d_Q\left(k(U), \partial_Y\mathcal{CC}(Q)\right) \leq d_Q\left(k(\alpha^*), \partial_Y\mathcal{CC}(Q)\right).$$

Since k is $\xi-$almost isometric, the curve $k(\alpha^*)$ has uniformly bounded geodesic curvature, hence its lift to $\mathbb{H}^3$ is a uniform quasi-geodesic and lies uniformly close to its geodesic representative by the Morse Lemma. The length of the geodesic representative for $k(\alpha^*)$ is uniformly comparable with the one of α^*, in particular it is uniformly bounded away from 0 and ∞. By basic hyperbolic geometry

$$\cosh\left(d_Q\left(\alpha^*, \partial\mathcal{CC}(N)\right)\right) \leq L_{\partial\mathcal{CC}(N)}(\alpha)/l_N(\alpha),$$
$$\cosh\left(d_Q\left(k(\alpha)^*, \partial_Y\mathcal{CC}(Q)\right)\right) \leq L_{\partial_Y\mathcal{CC}(Q)}(\alpha)/l_Q(\alpha).$$

Thus, it is enough to show that the numerators are uniformly bounded:

$$L_{\partial\mathcal{CC}(N)}(\alpha) \simeq L_X(\alpha) \leq L_T(\alpha)e^{2d_{\mathcal{T}}(X,T)} \leq Be^{2d_{\mathcal{T}}(X,Y)},$$
$$L_{\partial_Y\mathcal{CC}(Q)}(\alpha) \simeq L_Y(\alpha) \leq L_T(\alpha)e^{2d_{\mathcal{T}}(Y,T)} \leq Be^{2d_{\mathcal{T}}(X,Y)}.$$

The inequalities are applications of Theorem 2.1 and Wolpert's inequality $L_R(\alpha) \leq L_S(\alpha)e^{2d_{\mathcal{T}}(R,S)}$. □

6.4. **The gluing.** The following theorem is the main technical result of this article. Recall that we denote by M_f the closed 3-manifold obtained by gluing two handlebodies with boundary Σ with a map $f \in \mathrm{Mod}(\Sigma)$.

Theorem 6.7. *Let $R, \epsilon, \xi > 0$ be fixed. There exists $H_{\mathrm{gluing}}(R, \epsilon, \xi) > 0$ such that for every $H \geq H_{\mathrm{gluing}}$ the following holds: Let $f \in \mathrm{Mod}(\Sigma)$ be a gluing map. Suppose that $\left(Y, X, \overline{X}, \overline{Y}\right) \in \mathcal{T}_\epsilon^4$ is a quadruple with relative $(f, R)-$bounded combinatorics and height in $[H, 2H]$. Then there exists a metric g on $M_f = H_1 \cup_f H_2$ with the following properties.*

1. *The sectional curvature of the metric is contained in the interval $sec \in (-1 - \xi, -1 + \xi)$.*
2. *The curvature of g is constant outside the union Ω of two disjoint regions of uniformly bounded diameter and uniform lower bound on the injectivity radius diffeomorphic to $\Sigma \times [0, 1]$.*

(3) $M_f - \Omega = H_1 \cup H_2 \cup Q$ *where* Q *is isometric to the complement in* $Q(Y, \overline{Y})$ *of a collar neighborhood of* $\partial \mathcal{CC}(Q(Y, \overline{Y})$ *of uniformly bounded radius (depending on* H*), and where* H_1, H_2 *are isometric to the complement in* $\mathcal{H}(X_0), \mathcal{H}(\overline{X}_0)$ *of a collar neighborhood of* $\partial\mathcal{CC}(\mathcal{H}(X_0))$*,* $\partial\mathcal{CC}(\mathcal{H}(\overline{X}_0))$ *of uniformly bounded diameter (where* $X_0, \overline{X}_0$ *are points on* $[Y, X]$*,* $\left[\overline{X}, \overline{Y}\right]$*).*

Proof. Let L_0 be as in Proposition 6.1. Let $B_0 := B_0(L_0, \eta)$ be the $\mathcal{C}^2-$bound produced by Lemma 5.2. Let c_3 be as in Lemma 5.1. Consider the height $H_0 := H_0(2L_0, R, \epsilon, \xi/c_3B_0)$ provided by Proposition 6.1.

We choose $X_0 \in [Y, X]$ (resp. $\overline{X}_0 \in \left[\overline{X}, \overline{Y}\right]$) so that $d_{\mathcal{T}}(Y, X_0) = H_0$ (resp. $d_{\mathcal{T}}(\overline{X}_0, \overline{Y}) = H_0$). This is possible if the height is sufficiently large. By Remark 3.3 the $(f, R)-$relative bounded combinatorics condition is still satisfied by $(Y, X_0, \overline{X}_0, \overline{Y})$. We use Proposition 6.1 twice in order to produce product regions:

- When applied to the pair (Y, X_0) using $\overline{Y}$ as a free boundary: A ξ/B_0-almost-isometric embedding in the homotopy class of the identity of a product region (with $2L_0-$bounded geometry and injectivity radius bounded by η) $k_U : U \subset \mathcal{H}(X_0) \to Q(Y, \overline{Y})$.
- When applied to the pair $\left(\overline{Y}, \overline{X}_0\right)$ using Y as a free boundary: A ξ/B_0-almost-isometric embedding in the homotopy class of f of a product region with ($2L_0-$bounded geometry and injectivity radius bounded by η) $k_V : V \subset \mathcal{H}(\overline{X}_0) \to Q(\overline{Y}, Y)$.
- Observe that the manifolds $Q := Q(Y, \overline{Y})$ and $Q(\overline{Y}, Y)$ are isometric via an orientation reversing isometry.

By Lemma 6.6 we have

$$d_Q\left(k_U(U), \partial_Y\mathcal{CC}(Q)\right) \le A(2H_0) \text{ and } d_Q\left(k_V(V), \partial_{\overline{Y}}\mathcal{CC}(Q)\right) \le A(2H_0).$$

In particular, if $d_Q\left(\partial_Y\mathcal{CC}(Q), \partial_{\overline{Y}}\mathcal{CC}(Q)\right)$ is much bigger than $A(2H_0)$ then $k_U(U), k_V(V)$ are disjoint and appear in the correct order along Q. Proposition 4.1 (or Proposition 6.2) implies that there exists $H_{\text{gluing}} \ge H_0$ such that this condition is satisfied (we can choose H_{gluing} to be the height that implies the presence of a large-thick collar of $\partial_Y\mathcal{CC}(Q)$ of width at least $20A(2H_0)$).

Lemma 5.2 gives us uniform bump functions $\theta_U, \theta_V : U, V \to [0, 1]$ on U, V whose $\mathcal{C}^2-$norm is bounded by B_0. Finally, we apply Lemma 5.1 twice and glue $\mathcal{H}(X_0), Q, \mathcal{H}(\overline{X}_0)$ along $k_U : U \to Q$ and $k_V : V \to Q$ using uniform bump functions θ_U, θ_V. The sectional curvatures of the resulting manifold satisfy $|\sec + 1| \le c_3B_0 \cdot \xi/c_3B_0 = \xi$. The requirements (2) and (3) follow from the cut and glue construction and Lemma 6.6. $\square$

7. Random Heegaard splittings

The goal of this section is to establish some geometric control on random 3-manifolds. We begin with defining the type of control we need.

Definition. For $\delta \in (0, 1/2), b > 1$ and $g \geq 2$, a *(b, δ)-product region of genus g* in a Riemannian 3-manifold M is a closed subset V of M with the following properties.

(1) V is diffeomorphic to $\Sigma \times [0, 1]$ where Σ is a closed surface of genus g, and V *separates* M, i.e. $M - \mathrm{int}(V)$ consists of two connected components with boundary $\Sigma \times \{0\}, \Sigma \times \{1\}$, respectively.
(2) The injectivity radius of M at points in V is contained in the interval $[\delta, 1/\delta]$, and the diameters of the surfaces $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ are at most $1/\delta$.
(3) The restriction of the metric of M to V is of constant curvature -1.
(4) The distance between the boundary components $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ equals at least b.

Note that as $b > 1$, the volume of an (b, δ)-product region is bounded from below by a universal constant which can be chosen to be the volume of a ball of radius δ in hyperbolic 3-space, and up to a universal additive constant, its diameter is bounded from above by the distance between the boundary surfaces.

Example. Let M be a doubly degenerate hyperbolic 3-manifold which is homeomorphic to $\Sigma \times \mathbb{R}$ for a closed surface Σ of genus $g \geq 2$ and whose injectivity radius is at least δ. Then the injectivity radius is also bounded from above by a universal constant (see [40] for details), and for any $b > 1$, any sufficiently large metric ball in M contains a (b, δ)-product region of genus g.

By definition, a (b, δ)-product region $V \subset M$ separates M. In particular, if $V' \subset M$ is another such region which is disjoint from V, then it is contained in one of the two components of $M - V$. Thus if $\mathcal{V} \subset M$ is a disjoint union of $k \geq 1$ (b, δ)-product regions in M, then the dual graph whose vertices are the components of $M - \mathcal{V}$ and where two such components are connected by an edge if their closures intersect the same component of $\mathcal{V}$ is a tree. We say that the components of $\mathcal{V}$ are *linearly aligned* if this tree is just a line segment.

We shall show that for there exists a number $C_1 > 0$, and for given numbers $b > 1, \delta > 0$ there exist a number $C_2 = C_2(b, \delta) > 0$ such that for any $\epsilon > 0$, a random 3-manifold admits a negatively curved metric as described in Theorem 6.7 with the following additional properties.

(a) The gluing control parameter ξ is smaller than ϵ.
(b) $\mathrm{vol}(H_1 \cup H_2 \cup \Omega) \leq \epsilon n$ where n is the step of the walk.
(c) $\mathrm{vol}(Q) \geq C_1 n$ where n is the step of the walk.

(d) The set Q contains a subset Q' which is a disjoint union of linearly aligned (h,δ)-product regions of genus g and cardinality at least $C_2 n$.

We call such a gluing a *gluing with* (ϵ, b, δ)*-controlled geometry.* The point here is that $C_1 > 0$ is a universal constant, and the constant $C_2(b,\delta)$ only depends on b,δ and, in particular, is independent of ϵ.

We begin with describing the basic setup of random 3-manifolds.

Definition (Random Walk). Let us fix a symmetric probability measure μ on $\mathrm{Mod}(\Sigma)$ whose support is a finite generating set S. Let $\{S_j\}_{j\in\mathbb{N}}$ be a sequence of independent, $\mu-$distributed random variables with values in $\mathrm{Mod}(\Sigma)$. The $n-$th step of the random walk is the random variable $\omega_n := S_1 \cdots S_n$ (with $\omega_0 := \mathrm{Id}_\Sigma$). The random walk is the discrete process $(\omega_n)_{n\in\mathbb{N}}$.

Let $\mathcal{P}$ be a property of mapping classes or 3-manifolds. We say that $\mathcal{P}$ holds for a *random mapping class* (resp. for a *random* 3*-manifold*) if

$$\mathbb{P}_n\left[f \in \mathrm{Mod}(\Sigma) \mid f \text{ (resp. } M_f) \text{ has } \mathcal{P}\right] \overset{n\to\infty}{\longrightarrow} 1$$

where $\mathbb{P}_n$ is the distribution of the $n-$th step of the random walk ω_n and coincides with the n-th convolution of μ with itself.

The following is the main result of this section. The constants $C_1 > 0, C_2(b,\delta) > 0$ appearing implicitly in its statement depend on the probability measure μ and will be determined in the course of the proof.

Proposition 7.1. *Let $g \geq 2$ and $\epsilon > 0, b > 0, \delta > 0$ be fixed. Let μ be a symmetric probability measure on* $\mathrm{Mod}(\Sigma)$ *whose support is a finite symmetric generating set. We have*

$$\mathbb{P}_n\left[f \in \mathrm{Mod}(\Sigma) \,|\, M_f \textit{ has gluing with } (\epsilon, b, \delta)\textit{-controlled geometry }\right] \underset{n\to\infty}{\longrightarrow} 1.$$

We first recall some facts about random walks on $\mathrm{Mod}(\Sigma)$.

7.1. **Random walks on the mapping class group.** Much of the material we present here is also contained in higher generality and with more details in Section 6 of [1].

In the sequel we always consider a symmetric probability measure μ on $\mathrm{Mod}(\Sigma)$ whose support S is a finite generating set. Associated to the random walk generated by μ is a *space of sample paths* $(\Omega, \mathcal{E}, \theta)$ where $\Omega = \mathrm{Mod}(\Sigma)^{\mathbb{N}}$ is endowed with the product topology, $\mathcal{E}$ is the σ-algebra of Borel sets and $\mathbb{P}$ is the push-forward of the product measure $\mu^{\otimes\mathbb{N}}$ under the measurable map

$$T : \Omega \to \Omega, \quad \text{ defined by } (T(s_i)_j = s_1 \cdots s_j = \omega_j).$$

We have

Theorem 7.2 (Maher [28]).

$$\mathbb{P}_n\left[f \in \mathrm{Mod}(\Sigma) \mid f \textit{ is pseudo-Anosov }\right] \overset{n\to\infty}{\longrightarrow} 1.$$

We will use a geometric statement for the action of random mapping classes on Teichmüller space. The following result is due to Tiozzo.

THEOREM 7.3 (Tiozzo, Theorem 1 of [48]). *Fix some X in the Teichmüller space $\mathcal{T}$ of Σ. Then there exists $L_T > 0$ such that for almost all sample paths (ω_n) there exists a Teichmüller geodesic ray $\gamma : [0,\infty) \to \mathcal{T}$ with $\gamma(0) = X$ and such that*

$$\lim_{n\to\infty} \frac{d_{\mathcal{T}}(\omega_n X, \gamma(L_T n))}{n} \to 0.$$

In particular, the drift for the action of the random walk on Teichmüller space with the Teichmüller metric is positive.

There also is a statement concerning the action of the random walk on the curve graph $(\mathcal{C}, d_{\mathcal{C}})$ of Σ which is due to Maher and Tiozzo [29].

THEOREM 7.4 (Maher-Tiozzo, Theorem 1.2 and Theorem 1.3 of [29]). *Let $\alpha \in \mathcal{C}$ be a basepoint. Then there exists a constant $L_{\mathcal{C}} > 0$ such that for almost every sample path (ω_n) we have*

$$\lim_{n\to\infty} \frac{d_{\mathcal{C}}(\alpha, \omega_n\alpha)}{n} = L_{\mathcal{C}} > 0.$$

Moreover, there is a uniform quasigeodesic ray γ which tracks the sample path sublinearly, i.e.

$$\lim_{n\to\infty} \frac{d_{\mathcal{C}}(\omega_n\alpha, \gamma)}{n} = 0 \textit{ almost surely }.$$

As an application of Theorem 7.3 and Theorem 7.4, we obtain the following result which was first shown by Kaimanovich and Masur [22]. For its formulation, recall that a point in $\partial_\infty\mathcal{C}$ is an unmeasured filling geodesic lamination on Σ.

THEOREM 7.5 (Kaimanovich-Masur [22], Maher-Tiozzo [29]). *For $\mathbb{P}$-almost every sample path $\omega = (\omega_n)_{n\in\mathbb{N}} \in \mathrm{Mod}(\Sigma)^{\mathbb{N}}$, the following holds true.*

(1) For every base-point $\alpha \in \mathcal{C}$, the sequence $\{\omega_n\alpha\}_{n\in\mathbb{N}} \subset \mathcal{C}$ converges to a point $\mathrm{bnd}(\omega) \in \partial_\infty\mathcal{C}$ in the Gromov boundary which is independent of α.

(2) The point $\mathrm{bnd}(\omega)$ supports a unique transverse invariant measure up to scale, and the Teichmüller ray $\tau_{X,\mathrm{bnd}(\omega)}$ issuing from a fixed basepoint $X \in \mathcal{T}$ which determined by $\mathrm{bnd}(\omega)$, equipped with this transverse invariant measure, has the sublinear tracking property from Theorem 7.3-

Furthermore, the map $\mathrm{bnd} : \mathrm{Mod}(\Sigma)^{\mathbb{N}} \to \partial_\infty\mathcal{C}$ is measurable with respect to the σ-algebra of Borel subsets of $\partial_\infty\mathcal{C}$.

By Theorem 7.5, we may view the map bnd as both a map with values in the boundary $\partial_\infty\mathcal{C}$ of the curve graph as well as a map with values in the space $\mathcal{PML}$ of projective measured laminations. We will not distinguish between the two viewpoints in the sequel to keep the notations simple.

Theorem 7.5 leads to the next definition.

Definition (Harmonic Measure). The measure $\nu := (\mathrm{bnd})_* \mathbb{P}$ on $\partial_\infty \mathcal{C}$ (or on $\mathcal{PML}$) is called the *harmonic measure* associated to the random walk (or to the distribution μ).

The next statement is Proposition 6.10 of [1]. It can be viewed as a statement about the harmonic measure on $\mathcal{PML}$.

Proposition 7.6. *Let $W \subset \mathcal{T}$ be a* $\mathrm{Mod}(\Sigma)$*-invariant open subset that contains an axis of a pseudo-Anosov mapping class. Then for all $H > 0$ there exists a $\hat{c} = \hat{c}(W, H) > 0$ such that for almost every sample path ω, we have*

$$\liminf \frac{1}{T} |\{t \in [0, T] \mid \tau_{X,\mathrm{bnd}(\omega)}[t - H, t + H] \subset W\}| > \hat{c}.$$

The $\mathrm{Mod}(\Sigma)$-invariants sets W we are going to use in the sequel are the sets $\mathcal{T}_\delta$ for some suitably chosen numbers $\delta > 0$.

7.2. **Random handlebodies.** From now on we fix a handlebody $\mathcal{H}$ and a marking of the boundary surface Σ. The disk set $\mathcal{D}$ of $\mathcal{H}$ defines a subset $\partial \mathcal{D}$ of $\partial_\infty \mathcal{C}$ by taking its closure in $\mathcal{C} \cup \partial_\infty \mathcal{C}$ and intersecting with the boundary, i.e. $\partial_\infty \mathcal{D} := \overline{\mathcal{D}} \cap \partial_\infty \mathcal{C}$. Maher, exploiting work of Kerckhoff [23] (Kerckhoff's proof contained a small gap that has been settled by Gadre in [18]), showed that $\partial_\infty \mathcal{D}$ has measure zero with respect to the harmonic measure ν.

Theorem 7.7 (Maher, [27]). *The harmonic measure of the boundary of the disk set vanishes, i.e. $\nu(\partial_\infty \mathcal{D}) = 0$. Moreover, the Hempel distance increases linearly along the random walk, i.e. there exists a constant $K > 1$ such that*

$$\mathbb{P}_n \left[(\omega_n) \in \Omega \mid d_{\mathcal{C}}(\mathcal{D}, \omega_n \mathcal{D}) \in \left[\frac{1}{K} n, K n \right] \right] \overset{n \to \infty}{\longrightarrow} 1.$$

Maher's theorem has a few immediate consequences. First of all, for a random mapping class f, the 3-manifold M_f is hyperbolic (see Dunfield and Thurston [16]). Furthermore, let us choose once and for all a basepoint $X \in \mathcal{T}_\epsilon$ contained in the ϵ-thick part of Teichmüller space for a suitably chosen number $\epsilon > 0$. We select X so that it admits a short marking whose base is a pants decomposition made of diskbounding curves for $\mathcal{H}$. By Theorem 7.4, the distance in the curve graph between $\Upsilon(X)$ and $\Upsilon(\omega_n X)$ makes linear progress in n, and by Theorem 7.7, it makes linear progress away from the diskbounding curves. Here as before, $\Upsilon : \mathcal{T} \to \mathcal{C}$ denotes the systole map.

This property, however, is not sufficient to conclude that for a random element $f \in \mathrm{Mod}(\Sigma)$, the manifold M_f satisfies the assumptions in Proposition 7.1. As additional properties, we have to control the transition of the Teichmüller geodesic segment $\tau_{X,\omega_n X}$ connecting X to $\omega_n X$ through the thick part of Teichmüller space while controlling the rate of divergence of its trace from the disk set. We next establish this control.

Thus let $f \in \mathrm{Mod}(\Sigma)$ be a random mapping class. By Theorem 7.2 we know that f is p-A (pseudo-Anosov).

Quantitatively, a possible measure for the fellow-travelling of the disk set is given by the size of the nearest point projection of the disk set $\mathcal{D}$ to the uniformly quasi-convex subset

$$G_f := \Upsilon(\tau_{X,fX})$$

of the curve graph. We denote this nearest point projection by π_{G_f}. As G_f is a uniform unparametrized quasi-geodesic in the curve graph, hyperbolicity of $\mathcal{C}$ yields that the projection $\pi_{G_f}(\mathcal{D})$ is a quasi-convex subset of G_f. Let $|\pi_{G_f}(\mathcal{D})|$ be its diameter. Our next goal is to prove that as the step length tends to infinity, this diameter is arbitrarily small compared to the diameter $|G_f|$ of G_f.

PROPOSITION 7.8. *Let $g \geq 2$ and $\epsilon > 0$ be fixed. Let μ be a symmetric probability measure on* $\mathrm{Mod}(\Sigma)$ *whose support is a finite generating set. We have*

$$\mathbb{P}_n\left[f \in \mathrm{Mod}(\Sigma)\,\middle|\,f \text{ is p-A}, \left|\pi_{G_f}(\mathcal{D})\right|/|G_f| \leq \epsilon\right] \overset{n\to\infty}{\longrightarrow} 1.$$

Proof. Let $\epsilon > 0$ be arbitrary. Let $K > 0$ be the constant from Theorem 7.7, let $L_{\mathcal{C}}$ be the constant from Theorem 7.4 and assume without loss of generality that $L_{\mathcal{C}}\epsilon < 1/2K$.

Let $\alpha = \Upsilon(X) \in \mathcal{C}$. We may assume that α is diskbounding in the handlebody $\mathcal{H}$.

For $n_0 > 0$ let $\Omega_{n_0} \subset \Omega$ be the set of all sample paths $\omega = (\omega_n)$ such that for all $n \geq n_0$ the following properties are fulfilled.

(1) $L_{\mathcal{C}}(1-\epsilon/2)n \leq d_{\mathcal{C}}(\alpha, \omega_n\alpha) \leq L_{\mathcal{C}}(1+\epsilon/2)n$.
(2) Let γ be a uniform quasigeodesic ray in $\mathcal{C}$ connecting $\gamma(0) = \alpha$ to $\gamma(\infty) = \mathrm{bnd}(\omega)$; then $d_{\mathcal{C}}(\gamma, \omega_n(\alpha)) \leq L_{\mathcal{C}}\epsilon n/2$.
(3) $d_{\mathcal{C}}(\mathcal{D}, \omega_n\mathcal{D}) \geq n/2K$.

Note that we have $\Omega_{n_1} \supset \Omega_{n_0}$ for all $n_1 \geq n_0$. By Theorem 7.4 and Theorem 7.7, for every $\rho > 0$ there exists a number $n_0 = n_0(\rho) > 0$ so that $\mathbb{P}_n(\Omega_{n_0}) \geq 1 - \rho$.

The disk set $\mathcal{D} \subset \mathcal{C}$ is quasi-convex. Thus by hyperbolicity of $\mathcal{C}$, there exists a number $A > 0$ with the following property. Let $\zeta : [0,\infty) \to \mathcal{C}$ be a uniform quasi-geodesic ray beginning at $\zeta(0) = \alpha \in \mathcal{D}$; if $t > 0$ is such that $d_{\mathcal{C}}(\zeta(t), \mathcal{D}) > A$ and if $\beta \in \mathcal{C}$ is such that $\zeta(t)$ equals a shortest distance projection of β into ζ , then a shortest geodesic connecting β to $\mathcal{D}$ passes through a uniformly bounded neighborhood of $\zeta(t)$. In particular, up to increasing A, we have $\zeta(t) \notin \pi_{G_f}(\mathcal{D})$.

Assume from now on that $n_0/4K > A$. Let $(\omega_n) \in \Omega_{n_0}$ and let $n \geq n_0$. Denote by γ the quasi-geodesic ray in $\mathcal{C}$ as in property (2) above. Then on the one hand, we have

$$L_{\mathcal{C}}(1-\epsilon/2)n \leq d_{\mathcal{C}}(\alpha, \omega_n(\alpha)) \leq L_{\mathcal{C}}(1+\epsilon/2)n,$$

on the other hand also $d_{\mathcal{C}}(\gamma, \omega_n(\alpha)) \leq L_{\mathcal{C}}\epsilon n/2$. In particular, by property (3) above, the nearest point projection q_n of $\omega_n(\alpha)$ into γ is of distance at least $n/2K - L_{\mathcal{C}}\epsilon n/2 \geq n/4K > A$ from $\mathcal{D}$. This implies that a geodesic in $\mathcal{C}$ which connects $\omega_n(\alpha)$ to a shortest distance projection into $\mathcal{D}$ passes through a uniformly bounded neighborhood of q_n. Using again uniform quasi-convexity of $\mathcal{D}$ and the fact that $\alpha \in \mathcal{D}$ we conclude that the diameter of the shortest distance projection of $\mathcal{D}$ into the geodesic $\tau_{X,\omega_n X}$ does not exceed the distance between α and q_{n_0} which is at most $L_{\mathcal{C}}(1+\epsilon)n_0$, independent of $n \geq n_0$ and $\omega \in \Omega_{n_0}$.

Let now $n_1 > 0$ be sufficiently large that $L_{\mathcal{C}}(1+\epsilon)n_0 \leq \epsilon L_{\mathcal{C}}(1-\epsilon)n_1$. Then for $\omega \in \Omega_{n_0}$ and for $n \geq n_1$, the distance between $\omega_n(\alpha)$ and α is at least $L_{\mathcal{C}}(1-\epsilon/2)n$, while the diameter of the projection of $\mathcal{D}$ into $\tau_{X,\omega_n X}$ does not exceed $L_{\mathcal{C}}(1+\epsilon)n_0$. By the choice of n_1, this means that the properties required in the proposition are fulfilled for this n_1, i.e. we have $|\pi_{G_{\omega_n}}(\mathcal{D})| \leq \epsilon |G_{\omega_n}|$ as claimed.

As $\rho > 0$ was arbitrary, the proposition follows. □

7.3. **Good gluing regions.** The goal of this subsection is to show Proposition 7.1. The argument is very similar to the argument in the proof of Proposition 7.8. We begin with a volume control for convex cocompact hyperbolic structures on handlebodies. To this end choose as before once and for all a marking η for the boundary Σ of the handlebody $\mathcal{H}$ so that the base pants decomposition consists of diskbounding curves. The following proposition is well known in various settings. As we did not find a directly quotable statement in the literature, we sketch a proof.

PROPOSITION 7.9. *Let $\epsilon > 0$ be a fixed number and let ν be any marking on Σ of Hempel distance at least three to η. Suppose that $\mathcal{H}$ is equipped with a convex cocompact hyperbolic structure $\mathcal{H}(X)$ with conformal boundary $X \in \mathcal{T}_\epsilon$ such that ν is short for X. Then the volume of the convex core of $\mathcal{H}(X)$ is bounded from above by a fixed multiple of the distance between η, ν in the marking graph.*

Proof. The volume of any simplex with totally geodesic sides in a hyperbolic 3-manifold M is bounded from above by a universal constant. Furthermore, any abstract simplex, i.e. an embedded subset of M which is the image of an embedding $\Delta \to M$ where Δ is the standard 3-simplex, can be *straightened* in a unique way to a simplex with the same vertex set and with totally geodesic sides [2], and this construction is compatible with the side relation.

On the other hand, as by Theorem 2.1 the diameter of the boundary of the convex core of the handlebody $\mathcal{H}(X)$ is uniformly bounded, the volume of a uniformly bounded neighborhood of this boundary is uniformly bounded as well. Thus for the purpose of the proposition, it suffices to show that the complement in $\mathcal{CC}(\mathcal{H}(X))$ of a neighborhood of the boundary of uniformly bounded radius admits a triangulation by simplices with totally geodesic sides whose number does not exceed a fixed multiple of the distance between η and ν in the marking graph.

The strategy now is to construct for each marking of Σ a triangulation of Σ and control these triangulations as we move through the marking graph. We begin with noting that a marking decomposes the surface into a uniformly bounded number of polygonal disks. This means that the intersection points between the curves from the marking determine a collection of marked points on the boundaries of these disks. Subdivide each disk into triangles in such a way that the marked points are precisely the vertices of these triangles. Note that this procedure is by no means unique, but there are only finitely many combinatorial possibilities.

If we apply this procedure to the marking η, then we can extend this (topological) triangulation of the boundary of $\mathcal{CC}(\mathcal{H}(X))$ to a topological triangulation with uniformly few simplices. This is true because the base of η consists of diskbounding curves, and the disks with boundary in the base of η decompose $\mathcal{H}$ into balls.

Now let us assume that η' is obtained from η by a Dehn twist about the pants curves (i.e the base) of η. Let T be a triangulation of Σ defined by η and let T' be its image under the Dehn twist. Then there exists a triangulation τ of $\Sigma \times [0,1]$ which restricts to T, T' on the boundary. As up to the action of the mapping class group there are only finitely many combinatorial possibilities for this situation, we can find such a triangulation of $\Sigma \times [0,1]$ with a uniformly bounded number of simplices.

The same argument holds true for the move which replaces a pants curve by a marking curve and clears intersections. In a number of such steps whose number does not exceed a fixed multiple of the distance between η and ν in the marking graph, we obtain a triangulation of $\mathcal{CC}(\mathcal{H}(X))$. By the diameter bound for the boundary of $\mathcal{CC}(\mathcal{H}(X))$ and the assumption that ν is short for X and hence by Theorem 2.1), ν is short for the boundary of the convex core, straightening this triangulation then yields a triangulation of a subset of $\mathcal{CC}(\mathcal{H}(X))$ whose complement is contained in a uniformly bounded neighborhood of the boundary and hence has uniformly bounded volume. This yields the proposition. $\square$

Using Proposition 7.9 we are now ready to complete the proof of Proposition 7.1.

Proof of Proposition 7.1. Let X be a point in the thick part of Teichmüller space for which a fixed marking η on Σ with pants curves consisting of diskbounding curves is short. The strategy is to isolated a region on the Teichmüller geodesic connecting X to its image under a random pseudo-Anosov mapping class which fulfills the assumptions in Theorem 6.7. Furthermore, this region should be contained in the initial subsegment of the geodesic of length at most ϵ times the total length. We also isolate a region with similar properties near the end of the segment.

Using Proposition 7.9 we then argue that the sum of the volumes of the convex cocompact handlebodies corresponding to this initial and terminal segment of the geodesic is small compared to the volume of the center piece and that the center piece contains linearly aligned product regions as predicted in the proposition.

Let as before $A > 0$ be sufficiently large that the following holds true. Let $\gamma : [0, \infty) \to \mathcal{C}$ be a uniform quasi-geodesic beginning at the diskbounding curve $\gamma(0) = \alpha$ (this should mean that we choose once and for all a quasi-geodesic constant so that any two distinct points in $\mathcal{C} \cup \partial_\infty \mathcal{C}$ can be connected by a quasi-geodesic for this constant). We require that whenever $\beta \in \mathcal{C}$ is such that a shortest distance projection $\gamma(t)$ of β into γ has distance at least A from $\mathcal{D}$, then a shortest geodesic connecting β to $\mathcal{D}$ passes through a uniformly bounded neighborhood of $\gamma(t)$.

Let μ be a finitely supported probability measure on $\mathrm{Mod}(\Sigma)$ which induces the probability measure $\mathbb{P}$ on Ω. Let $\epsilon > 0, H > 0, \delta > 0$ be arbitrarily fixed. We require that $\delta > 0$ is small enough that the conditions in Proposition 7.6 are fulfilled for $W = \mathcal{T}_{2\delta}$.

Let $f \in \mathrm{Mod}(\Sigma)$ and consider as before the Teichmüller geodesic $\tau_{X,fX}$ connecting X to fX. We say that M_f admits a (H, δ, ϵ)-*good gluing region* if the following holds true. Let $\ell(\tau_{X,fX})$ be the length of the geodesic segment $\tau_{X,fX}$; then there exists an initial subsegment $\tau_{X,fX}[0, \rho]$ of length $\rho \leq \epsilon\ell(\tau_{X,fX})$ such that the distance between $\Upsilon(\tau_{X,fX}[\rho - 2H, \rho])$ and $\mathcal{D}$ is at least A and that $\tau_{X,fX}[\rho - 2H, \rho] \subset \mathcal{T}_\delta$. We claim that

$$\mathbb{P}_n[f \in \mathrm{Mod}(\Sigma) \mid f \text{ is p.A. and } f \text{ has a } (H, \delta, \epsilon) \text{ good gluing region }] \to 1.$$

Note that a (H, δ, ϵ)-good gluing region is related but a priori different from a gluing with controlled geometry.

To show the claim let $\sigma > 0$. By Theorem 7.4 and Proposition 7.8, we can find a number $n_0 = n_0(\sigma) > 0$ with the following property.

Let $\Omega_{n_0} \subset \Omega$ be the set of all sample paths (ω_n) so that

$$d_{\mathcal{C}}(\alpha, w_n\alpha) \in [(1 - \epsilon)L_{\mathcal{C}}, (1 + \epsilon)L_{\mathcal{C}})]$$

for all $n \geq n_0$ and that furthermore $|\pi_{G_{\omega_n}}(\mathcal{D})/|G_{\omega_n}| \leq \epsilon/2$ for all $n \geq n_0$; then $\mathbb{P}(\Omega_{n_0}) \geq 1 - \sigma$.

Note that by convexity and hyperbolicity, if $(\omega_n) \in \Omega_{n_0}$ then the diameter of the projection $\pi_{G_{\omega_n}}(\mathcal{D})$ is at most $n_0 q$ for some fixed number q, independent of n (see the proof of Proposition 7.8). Thus by Proposition 7.6, for a fixed number $A > 0$ there is a number $T > 0$ with the following property. Let $\Omega' \subset \Omega_{n_0}$ be the set of all $\omega = (\omega_n) \in \Omega_{n_0}$ such that the geodesic segment $\tau_{\mathrm{bnd}(\omega)}[0, T]$ contains a subsegment of length at least $2H + 2A$ entirely contained in $\mathcal{T}_{2\delta}$; then $\mathbb{P}_n(\Omega_{n_1}) \geq 1 - 2\sigma$ for all $n \geq n_1$.

By Theorem 7.3, the orbit of X under the random path ω tracks the geodesic ray $\tau_{\mathrm{bnd}(\omega)}$ sublinearly. This implies the following. For $n > 0$ let

$s(n) \geq 0$ be such that $\tau_{X,\mathrm{bnd}(\omega)}(s(n))$ is the shortest distance projection of $\omega_n(X)$ into $\tau_{X,\mathrm{bnd}(\omega)}$; then $s(n) \to \infty$ as $n \to \infty$.

Minsky [38] showed that for a given number $\delta > 0$, there exists a number $A > 0$ so that the following holds true. Consider for the moment two Teichmüller geodesics $\gamma, \zeta : [0, \infty) \to \mathcal{T}$ with the same starting point $\gamma(0) = \zeta(0)$. Suppose that for some $T > A$ and some $H \geq 0$ the segment $\gamma[0, T]$ contains a subsegment β of length $H + 2A$ entirely contained in $\mathcal{T}_{2\delta}$. Suppose furthermore that for some large m the shortest distance projection of $\zeta(m)$ into γ is contained in $\gamma - \gamma[0, T]$; then ζ contains a subsegment of length at least H which is contained in a uniformly bounded neighborhood of β.

Together with the above discussion, this implies that there exists a number $n_1 > n_0$ such that the set $\Omega_{n_1} \subset \Omega_{n_0}$ of all $(\omega_n) \in \Omega_{n_0}$ with the property that $\tau_{0,\omega_n(o)}$ contains a subsegment of length $2H$ entirely contained in $\mathcal{T}_\delta$ satisfies $\mathbb{P}_n(\Omega_{n_1}) \geq 1 - 2\sigma$ for all $n \geq n_1$.

Now following the reasoning in the proof of Proposition 7.8, we conclude that there exists a number $n_2 > 0$ so that for $n \geq n_2$ the proportion of the length of the smallest initial subsegment of the geodesic $\tau_{X,\omega_n X}$ which contains the above segment of length $2H$ with respect to the total length of $\tau_{X,\omega_n X}$ is at most ϵ. In particular, for $n > n_2$ we have $\mathbb{P}_n(\Omega_{n_1}) \geq 1 - 2\sigma$ and, furthermore, if $(\omega_n) \in \Omega_{n_1}$ and if $n \geq n_2$ then ω_n is p.A. and admits a (H, δ, ϵ)-good gluing region.

On the other hand, for a fixed (sufficiently small) number $\delta > 0$ and a given number $H > 0$, Theorem 7.6 shows that there exists a number $\hat{c} = \hat{c}(H, \delta)$ such that for almost every sample path ω, we have

$$\liminf \frac{1}{T} |\{t \in [0, T] \mid \tau_{X,\mathrm{bnd}(\omega)}[t - 2h, t + 2h] \subset T_{2\delta}\}| > \hat{c}.$$

This implies that for this number $\hat{c}$ and for $\sigma > 0$ as before, there exists a number $n_3 > n_2$ such that for $n \geq n_3$ we have

$$\mathbb{P}_n\{(\omega_n) \in \Omega_{n_0} \mid \tau_{X,\omega_n X} \text{ contains } \hat{c}(1 - \epsilon)/2H \text{ pairwise disjoint segments of length at least } 2H \text{ and contained in } \mathcal{T}_{2\delta}\} > 1 - 3\rho.$$

However, if $n \geq n_3$ and if we consider the quasifuchsian manifold defined by the (H, δ, ϵ)-good gluing region and the Teichmüller segment $\tau_{X,\omega_n X}$ then Minsky's model theorem shows that this quasifuchsian manifold satisfies property (4) in the definition of a gluing with (ϵ, b, δ)-controlled geometry where the constant $H > 0$ as above depends on the choice of the a priori prescribed number $b > 1$.

Now the direction of the walk can be reversed and hence we can transfer statements about initial segments of the walk to statements about terminal segments. As $\sigma > 0$ was arbitrary, together this then yields the proposition. □

8. Geometric control of random hyperbolic 3-manifolds

In Section 7 we established that a random hyperbolic 3-manifold of Heegaard genus g admits a Riemannian metric of sectional curvature close to -1 with some specific geometric properties. Furthermore, for any given numbers $b > 1, \delta > 0$, a definitive proportion of the volume for this metric is contained in a union of pairwise disjoint linearly aligned (b, δ)-product regions. Here the proportionality constant depends on the numbers b, δ.

The main goal of this section is to show that this property carries over to the hyperbolic metric on a random 3-manifold. The following lemma shows that this suffices for the proof of Theorem 1 from the introduction.

Lemma 8.1. *For fixed $g \geq 2, \delta > 0$ and sufficiently large $b > 1$, there exists a number $C = C(g, b, \delta) > 0$ with the following property. Let M be a hyperbolic 3-manifold, and suppose that M contains $n \geq 1$ pairwise disjoint linearly aligned (b, δ)-product regions of genus g; then $\lambda_1(M) \leq C/n^2$ and $\lambda_n(M) \leq 1/C$.*

Proof. Let M be as in the lemma. Denote by $\mathcal{V} \subset M$ the union of the n linearly aligned (b, δ)-product regions of genus g whose existence is assumed in the statement of the lemma.

For each component $A = \Sigma \times [0, 1]$ of $\mathcal{V}$ there is an L-Lipschitz function $\psi_A : A \to [0, 1]$ for some $L > 0$ only depending on b so that $\psi_A(\Sigma \times \{0\}) = 0$ and $\psi_A(\Sigma \times \{1\}) = 1$. Since the components of $\mathcal{V}$ are linearly aligned, functions of the form $\psi_A + b_A$ or of the form $1 - \psi_A + b_A$ for a constant b_A can be pasted together to a function on M which is constant on the components of $M - \mathcal{V}$ and whose Rayleigh quotient is bounded from above by a universal multiple of $1/n^2$. We refer to [1] for details.

This shows the upper bound for $\lambda_1(M)$, and the upper bound for $\lambda_n(M)$ follows from the fact that the first eigenvalue of a (b, δ)-product region with Dirichlet boundary conditions is bounded from above by a universal constant together with domain monotonicity of eigenvalues with vanishing Dirichlet data. □

Theorem 1 from the introduction now follows from Proposition 7.1, Lemma 8.1 and the following statement which is the main result of this section. Recall that by hyperbolization, a closed 3-manifold M which admits a Riemannian metric of sectional curvature contained in $[-1-\epsilon, -1+\epsilon]$ for some $\epsilon < 1/2$ admits a hyperbolic metric, unique up to isometry by Mostow rigidity.

Theorem 8.2. *For every $g \geq 2, a \in (0, 1), b > 4, \delta > 0$ there exist numbers $\epsilon = \epsilon(g, a, b, \delta) > 0, a' = a'(g, a, b, \delta) \in (0, 1)$ with the following property. Let M be a closed aspherical atoroidal 3-manifold of Heegaard genus g, and let ρ be a Riemannian metric on M of curvature contained in $(-1-\epsilon, -1+\epsilon)$. Assume that (M, ρ) contains a linearly aligned collection $\mathcal{V}$ of pairwise disjoint (b, δ)-product regions of genus g whose total volume is at least $a\mathrm{vol}(M, \rho)$.*

Let ρ_0 be the hyperbolic metric on M. Then (M,ρ_0) contains a linearly aligned collection $\mathcal{W}$ of pairwise disjoint $(b-1,\delta/2)$-product regions of volume at least $a'\mathrm{vol}(M,\rho_0)$.

By Proposition 7.1, for a fixed choice of a number $b>4$ and sufficiently small $\delta>0$, a random 3-manifold M of Heegaard genus g admits a Riemannian metric ρ which fulfills the assumption in Theorem 8.2 for some number $a\in(0,1)$. Note that b,δ are independent of M, and the number $a\in(0,1)$ depends on the random walk. Thus Theorem 1 is an immediate consequence of Theorem 8.2 and Lemma 8.1.

We are left with the proof of Theorem 8.2 which is carried out in the remainder of this section. We use a construction of [4], [5]. The following is a special case of the main result of [5].

THEOREM 8.3. *Let (M,ρ) and (M_0,ρ_0) be closed oriented Riemannian manifolds of dimension 3 and suppose that for some constant $b\geq 1$*

$$\mathrm{Ric}_\rho\geq -2,\ \text{and}\quad -b^2\leq K_{\rho_0}\leq -1.$$

If there exists a map $f:M\to M_0$ of degree one then

$$\mathrm{vol}(M,\rho)\geq \mathrm{vol}(M_0,\rho_0),$$

with equality if and only if $(M,\rho),(M_0,\rho_0)$ are isometric and hyperbolic.

Here Ric_ρ and K_{ρ_0} are the Ricci curvature and the sectional curvature of ρ and ρ_0.

COROLLARY 8.4. *For $\epsilon<1/2$ let ρ be a Riemannian metric on the closed 3-manifold M of curvature contained in $(-1-\epsilon,-1+\epsilon)$ and let ρ_0 be the hyperbolic metric on M. Then*

$$\mathrm{vol}(M,\rho)/\mathrm{vol}(M,\rho_0)\in[(1-\epsilon)^{3/2},(1+\epsilon)^{3/2}].$$

Proof. Rescaling the metric ρ with the factor $(1-\epsilon)^{-1}$ yields a new metric on M whose volume is $(1-\epsilon)^{-3/2}\mathrm{vol}(M,\rho)$ and whose sectional curvature is bounded from below by -1. In particular, the Ricci curvature of this metric is at least -2. An application of Theorem 8.3 then implies that $\mathrm{vol}(M,\rho)\geq(1-\epsilon)^{3/2}\mathrm{vol}(M,\rho_0)$.

Similarly, rescaling the metric ρ on M with the factor $(1+\epsilon)^{-1}$ yields a metric whose sectional curvature is bounded from above by -1 and whose volume equals $(1+\epsilon)^{-3/2}\mathrm{vol}(M,\rho)$. Another application of Theorem 8.3, with the roles of (M,ρ) and (M,ρ_0) exchanged, shows that $\mathrm{vol}(M,\rho_0)\geq(1+\epsilon)^{-3/2}\mathrm{vol}(M,\rho)$. Together the corollary follows. $\square$

The *volume entropy* $h(\rho)$ of a negatively curved metric ρ on M is the asymptotic growth rate of the volume of balls in its universal covering. The volume entropy of a hyperbolic metric equals 2, and the volume entropy of a metric whose sectional curvature is bounded from below by $-b^2$ for some $b>0$ is at most $2b$.

For $c > h(\rho)$ there exists a smooth *natural map* $F_c : (M, \rho) \to (M, \rho_0)$ [4]. The following statement summarizes some of the results from Section 7 of [4]. Part of the statement is only implicitly contained in [4], but an explicit version can be found in Theorem 2.1 of [6]. We always assume that the constant ϵ which controls the curvature of M is smaller than $1/2$ and that the number $c > h(\rho)$ is bounded from above by 4 to make all constants uniform.

PROPOSITION 8.5. *Let $c > h(\rho)$ and let $F_c : (M, \rho) \to (M, \rho_0)$ be the natural map.*

(1) F_c is of degree one, and its Jacobian satisfies

$$|\mathrm{Jac}(F_c)| \leq \Big(\frac{c}{2}\Big)^3$$

pointwise.

(2) There are $\kappa > 0, r \in (0,1)$ and $L > 1$ not depending on (M, ρ) with the following property. If $x \in (M, \rho)$ is such that $|\mathrm{Jac}(F_c)(x)| \geq (1 - \kappa)(\frac{c}{2})^3$ then the restriction of the map F_c to the ball $B(x, r)$ of radius r about x in (M, ρ) is L-Lipschitz.

(3) For all $\theta > 0$ and $x \in M$ there exists $\beta > 0$ such that if $|\mathrm{Jac}(F_c)(x)| \geq (1 - \beta)(\frac{c}{2})^3$ then

$$(1 - \theta)\Big(\frac{c}{2}\Big)^3 < |d_x F_c(v)| < (1 + \theta)\Big(\frac{c}{2}\Big)^3$$

for all unit tangent vectors $v \in T_x M$.

The strategy is now as follows. Given $a \in (0,1)$ and $b > 4L$ where $L > 1$ is as in Proposition 8.5, for a manifold (M, ρ) which fulfills the assumption in Theorem 8.2 for sufficiently small $\epsilon > 0$, we find a union $\mathcal{W} \subset \mathcal{V}$ of components of the collection $\mathcal{V}$ of (b, δ)-product regions in (M, ρ) whose total measure is large and such that the restriction to this set of the natural map $F_c : (M, \rho) \to (M, \rho_0)$ for a suitably chosen $c > h(\rho)$ has large Jacobian outside of a subset which does not contain any ball of radius r where $r > 0$ is as in the second part of Proposition 8.5. Proposition 8.5 then yields that the map F_c is uniformly Lipschitz on $\mathcal{W}$. We then argue that the image under F_c of a (b, δ)-product region in $\mathcal{W}$ contains a (b', δ')-product region in (M, ρ_0) where b' is close to b and δ' is close to δ. The geometric control on the image of the map F_c is then used to show that suitably chosen sub-regions of these image product regions of controlled total volume are pairwise disjoint and linearly aligned.

The following lemma establishes a first volume control. In its formulation, the numbers $r > 0, L > 1$ are as in Proposition 8.5.

LEMMA 8.6. *Let $a \in (0,1), b > \max\{10r, 4\}, \delta > 0$ and $\xi > 0$. There exists a number $\epsilon_0 = \epsilon_0(a, b, \delta, \beta) > 0$ with the following property. Let (M, ρ) be as in Theorem 8.2, with sectional curvature contained in $(-1 - \epsilon_0, -1 + \epsilon_0)$. Then for $c > h(\rho)$ sufficiently close to $h(\rho)$, there is a subset $\mathcal{W} \subset \mathcal{V}$ with the following properties.*

(1) $\mathcal{W}$ is a union of components of $\mathcal{V}$, and its total volume is at least $a\mathrm{vol}(M,\rho)/2$.
(2) The restriction of F_c to each component of $\mathcal{W}$ is L-Lipschitz, and its image is contained in the σ-thick part of (M,ρ_0) for a universal constant $\sigma > 0$.
(3) If V is any component of $\mathcal{W}$ then $\mathrm{vol}(F_c(V)) \geq (1-\xi)\mathrm{vol}(V)$, and there exists a subset A of V with $\mathrm{vol}(A) \geq a\mathrm{vol}(V)$ such that $F_c^{-1}(F_c(x)) \subset V$ for all $x \in A$.

Proof. Let $r > 0$ be as in the second part of Proposition 8.5. Assume without loss of generality that $r < 1$. For $x \in (M,\rho)$ let $B(x,r)$ be the open ball of radius r about x. Let $\mathcal{V}$ be a union of (b,δ)-product regions as in the statement of Theorem 8.2. Since the components of $\mathcal{V}$ are linearly aligned and $b > 4$, any ball $B(y,r)$ in (M,ρ) intersects at most two different components of $\mathcal{V}$.

Let us consider a point $x \in V$. The injectivity radius of (M,ρ) at x is at least δ. Therefore by comparison, the volume of the ball $B(x,r)$ is bounded from below by a universal constant $\alpha > 0$. On the other hand, as the diameters of the boundary surfaces of a component V of $\mathcal{V}$ are uniformly bounded, the volume of the r-neighborhood $N_r(V)$ of any component V of $\mathcal{V}$ is bounded from above by a universal constant $\beta > 0$. Thus if $x \in V$ then the ratio $\mathrm{vol}(B(x,r))/\mathrm{vol}(N_r(V))$ is bounded from below by a universal constant α/β.

Let $\xi > 0$. Define

$$Z = \{x \in M \mid |\mathrm{Jac}(F_c)(x)| \geq (1-\xi)(\frac{c}{2})^3\}.$$

By Corollary 8.4 and the first part of Proposition 8.5, for sufficiently small $\epsilon > 0$ and for $c > h(\rho)$ sufficiently close to $h(\rho)$, the volume of the union $\mathcal{W}$ of all components V of $\mathcal{V}$ with the property that $N_r(V) - Z$ does not contain a ball of radius r centered at a point $x \in V$ is at least $3a\mathrm{vol}(M)/4$. Namely, if $V_1,\dots,V_k$ are the components of $\mathcal{V}-\mathcal{W}$ and if $x_i \in V_i$ is such that $B(x_i,r) \subset M - Z$, then by the above discussion, any of the balls $B(x_i,r)$ intersects at most one other ball $B(x_j,r)$ for $j \neq 1$. In particular, at least $k/2$ of the balls $B(x_i,r)$ are pairwise disjoint and hence

$$\mathrm{vol}(\cup_i B(x_i,r)) \geq k\alpha/2.$$

Thus if $\mathrm{vol}(\mathcal{V}-\mathcal{W}) \geq a\mathrm{vol}(M,\rho)/4$ then $\mathrm{vol}(\cup_i B(x_i,r)) \geq \alpha a\mathrm{vol}(M,\rho)/8\beta$. But the restriction of F_c to $\cup_i B(x_i,r)$ decreases the volume by a definitive factor. For $\epsilon > 0$ sufficiently close to 0 and $c - h(\rho) > 0$ sufficiently small, this violates Corollary 8.4.

By the second part of Proposition 8.5, the restriction of F_c to any component V of $\mathcal{W}$ is L-Lipschitz where $L > 1$ is a universal constant. In particular, if γ is a closed loop entirely contained in V, then the length of its image $F_c(\gamma)$ is at most L times the length of γ.

By the definition of a (b,δ)-product region, for an arbitrary point $x \in V$ the subgroup of $\pi_1(M)$ generated by the homotopy classes of uniformly short loops at x which are entirely contained in V is not virtually abelian. But this implies that for any point $y \in F_c(V)$, there are closed loops of uniformly bounded length passing though y which generated a non-solvable subgroup of $\pi_1(M)$. As a consequence, the set $F_c(V)$ is contained in the σ-thick part of (M,ρ_0) for a universal constant $\sigma > 0$. Together this shows the first and second part of the lemma.

Now if V is a component of $\mathcal{W}$ and if $B = \{x \in V \mid |F_c^{-1}(F(x)) \not\subset V\}$ then the volume of (M,ρ_0) equals the volume of $F_c(M - B)$. Thus as $\epsilon \to 0$ and $c - h(\rho) \to 0$, by volume comparison the proportion of the volume of $\mathcal{W}$ contained in the union of those components of $\mathcal{W}$ which violate the conditions in the third part of the lemma has to tend to zero. This then implies the third part of the lemma. □

For a number $\xi > 0$ we say that a map F between two metric spaces X, Y is a *ξ-coarse isometry* if $|d(Fx,Fy) - d(x,y)| \le \xi$ for all x, y.

LEMMA 8.7. *For $b' < b, \delta' < \delta$ and $\xi > 0$ there exists a number $\epsilon_0 = \epsilon_0(b',\delta')$ with the following property. Let (M,ρ) be as in Lemma 8.6 and let V be a component of $\mathcal{W}$ where $\mathcal{W}$ is as in Lemma 8.6; then the restriction of F_c to V is a ξ-coarse isometry whose image contains a (b',δ')-product region of genus g.*

Proof. We argue by contradiction and we assume that a number $\epsilon_0 > 0$ as in the lemma does not exist. Then there exists a sequence of closed 3-manifolds (M_i,ρ) which fulfill the assumptions in Theorem 8.2 for a sequence $\epsilon_i \to 0$ and fixed numbers $g \ge 2, a > 0, b > 4, \delta > 0$ and such that for each i, there is a component V_i of the collection $\mathcal{W}_i$ as in Lemma 8.6 whose image under the natural map $F_i : (M_i,\rho) \to (M_i,\rho_0)$ does not contain a (b',δ') product region where $b' < b$ and $\delta' < \delta$ are fixed constants. Note that in contrast to similar statements in the literature, we do not assume the existence of a bound on the diameters of the manifolds (M_i,ρ). Let as before ρ_0 be the hyperbolic metric on the manifold M_i.

Let h_i be the volume entropy of M_i. We know that $h_i \to 2$ $(i \to \infty)$. Choose a sequence $\chi_i \to 0$ such that $h_i < 2 + \chi_i$. For each i consider the natural map $F_i : (M_i,\rho) \to (M_i,\rho_0)$ for the parameter $c_i = 2 + \chi_i$. By the choice of $\mathcal{W}_i$ and the second part of Lemma 8.6, we know that the restriction of F_i to V_i is L-Lipschitz where $L > 1$ does not depend on i. Furthermore, for each $\beta > 0$, the measure of the set of all points $z \in V_i$ so that $|\mathrm{Jac}(F_i)(x)| \le (1-\beta)(\frac{c_i}{2})^3$ tends to zero as $i \to \infty$. By the third part of Proposition 8.5, as $i \to \infty$, on subset of the component V_i of $\mathcal{W}_i$ containing a larger and larger proportion of the volume of V_i, the differential of F_i is close to an isometry.

For each i let $x_i \in V_i$. The set $F_i(V_i)$ is contained in the σ-thick part of (M_i, ρ_0) where σ does not depend on i. Thus by passing to a subsequence, we may assume that the pointed manifolds (M_i, x_i, ρ) converge in the geometric topology to a pointed hyperbolic manifold (M, x) and that the pointed hyperbolic manifolds $(M_i, F_i(x_i), \rho_0)$ converge in the geometric topology to a pointed hyperbolic manifold (N, y).

Let (V, x) be the geometric limit of the pointed (b, δ)-product regions (V_i, x_i). Then V is a (b, δ)-product region in M containing the basepoint x. Furthermore, as the restriction of F_i to V_i is L-Lipschitz for a universal constant $L > 1$, up to passing to another subsequence we may assume that $F_i|V_i$ converges to an L-Lipschitz map $F : (V, x) \to (N, y)$.

By the definition of geometric convergence, for large enough i there exists a $(1 + \xi_i)$-bilipschitz homeomorphism ϕ_i of a neighborhood U of V in M onto a neighborhood U_i of V_i in M_i where $\xi_i \to 0$ $(i \to \infty)$. We use ϕ_i to identify U with U_i.

As $i \to \infty$ and by the choice of the sets V_i, the Jacobians of the restriction of F_i to V_i converge to one almost surely. We now follow the reasoning in the proof of Lemma 7.5 of [4]. Namely, using the map ϕ_i^{-1} we can think of U_i as a neighborhood of V in M. Egoroff's theorem then implies that for each n there exists a subset $K_n \subset V$ with $\mathrm{vol}(V - K_n) < 1/n$ and such that on K_n the differentials dF_i converge to an isometry uniformly. By Lemma 7.7 and Lemma 7.8 of [4], the map $F|V$ is one-Lipschitz. Its differential exists almost everywhere and is an isometry. It then follows from Appendix B that $F : V \to N$ is an isometric embedding. In particular, $F(V)$ is a (h, δ)-product region in N, and for sufficiently large i the map F_i is a ξ-coarse isometry.

Geometric convergence now implies that for large enough i, the image of V_i under F_i is a (b', δ')-product region in (M_i, ρ_0). This is a contradiction to the assumption on the sets V_i. $\square$

Proof of Theorem 8.2. We showed so far that for sufficiently small $\epsilon_0 > 0$, if (M, ρ) is as in Theorem 8.2, of sectional curvature contained in $(1-\epsilon_0, 1+\epsilon_0)$, then (M, ρ_0) contains a union of (b', δ')-product region for some b' close to b, δ' close to δ which cover a fixed proportion of the volume of (M, ρ). These product regions are the images under a suitably chosen natural map F_c of a subcollection $\mathcal{W} \subset \mathcal{V}$ of the family $\mathcal{V}$ of (b, δ)-product regions whose existence is assumed for (M, ρ). Furthermore, the volume of $\mathcal{W}$ is at least $a\mathrm{vol}(M, \rho)$ for some fixed number $a > 0$ (with a slight abuse of notation). The restriction of F_c to $\mathcal{W}$ is L-Lipschitz and a $1/4$-coarse isometry, and $\mathrm{vol}(F_c(\mathcal{W}))/\mathrm{vol}(\mathcal{W})$ is very close to one.

Let $\hat{b} < b - 2$ and $\hat{\delta} < \delta$ be such that each component V of $\mathcal{W}$ contains a $(\hat{b}, \hat{\delta})$-product region $\hat{V}$ in its interior whose one-neighborhood is entirely contained in V. The volume of $\hat{V}$ is at least $b\mathrm{vol}(V)$ for a universal constant $b > 0$.

Our goal is to show that there is a subcollection $\mathcal{Z}$ of $\mathcal{W}$ of volume at least $a\mathrm{vol}(M,\rho)/2$ with the additional property that whenever $V \neq W \in \mathcal{Z}$ then $\hat{V} \cap \hat{W} = \emptyset$.

To this end let us assume that for $V \neq W \in \mathcal{W}$ we have $F_c(\hat{V}) \cap F_c(\hat{W}) \neq \emptyset$. As the restriction of the map F_c is L-Lipschitz and a 1/4-coarse isometry, this implies that there are balls $B_1 \subset V, B_2 \subset W$ of radius $1/2L$ such that $F_c(B_1) \subset F_c(W)$ and $F_c(B_2) \subset F_c(V)$. Namely, for all $z \in \hat{V}$ the ball of radius 1/2 about $F_c(z)$ is contained in $F_c(V)$, furthermore F_c is L-Lipschitz.

Let $2\sigma > 0$ be a lower bound for the volume of a ball of radius $1/2L$ entirely contained in an (b,δ)-product region. Such a number exists since the injectivity radius in such a region is at least δ. Then the volume of $F_c(V \cup W)$ is at most $(\frac{c}{2})^3(\mathrm{vol}(V) + \mathrm{vol}(W) - 2\sigma)$. In particular, the contribution of $F_c(V)$ to the volume of $\mathcal{W}$ does not exceed $(\frac{c}{2})^3(\mathrm{vol}(V) - \sigma)$.

Since $\sigma > 0$ is independent of all choices and for c sufficiently close to 2 the restriction of the map F_c to $\mathcal{W}$ is very close to being volume preserving, we deduce that for c sufficiently close to 2 the union $\mathcal{Z}$ of all product regions $\hat{V}$ with $V \in \mathcal{W}$ and such that the sets from $\mathcal{Z}$ are mapped disjointly by F_c covers a fixed proportion of the volume of (M, ρ_0). Furthermore, the image of each of the components in $\mathcal{Z}$ contains a (b', δ')-product region for some fixed $b' < \hat{b}$ and some δ' close to $\hat{\delta}$. Thus we found a collection of pairwise disjoint product regions in (M, ρ_0) as claimed in the theorem.

We are left with showing that the regions $F_c(\hat{V})$ for $\hat{V} \in \mathcal{Z}$ are linearly aligned. However, F_c is a homotopy equivalence. If $\hat{V} \in \mathcal{Z}$ then as the restriction of F_c to $\hat{V}$ is a homeomorphism, for a fixed choice of an embedded surface $\Sigma \subset V$ which decomposes M into two handlebodies, the image surface $F_c(\Sigma)$ separates (M, ρ_0) into two components. The restriction of F_c to the closure of a component of $M - \Sigma$ is a generator of the relative homology group $H_3(M, M - F_c(\Sigma))$. But this homology group also is generated by the inclusion of a component of $M - F_c(\Sigma)$ and hence each component A of $M - \Sigma$ determines uniquely a component $\mathcal{F}(A)$ of $M - F_c(\Sigma)$ with the additional property that $F_c(A) \supset \mathcal{F}(A)$.

Now let $\hat{V} \neq \hat{W} \in \mathcal{Z}$; as the components of $\mathcal{Z}$ are pairwise disjoint, the component $\hat{W}$ is entirely contained in a component of $M - \hat{V}$, say the component A. Furthermore, as $F_c(\hat{V}), F_c(\hat{W})$ are disjoint, the component $F_c(\hat{W})$ is contained in a component Z of $M - F_c(\hat{V})$. We claim that $Z = \mathcal{F}(A)$.

Namely, let B be the component of $M - \hat{W}$ entirely contained in A. If $Z \neq \mathcal{F}(A)$ then we have $F_c(\hat{V}) \subset \mathcal{F}(B)$. But the restriction of F_c to B maps B to a subset that contains $\mathcal{F}(B)$. In particular, we have $F_c(\hat{V}) \subset F_c(M - \hat{V})$ which violates property (3) in Lemma 8.6.

But this just means that the components of $F_c(\mathcal{W})$ are linearly aligned. This completes the proof of the theorem. $\square$

Appendix A. Computation of end invariants

We give a proof, which is certainly well-known but hard to find in the literature, of Corollary 4.7.

Lemma A.1. *Let M be a marked hyperbolic structure on $\Sigma \times \mathbb{R}$. Let $\lambda \in \mathcal{PML}$ be a minimal filling measured lamination that is not realized in M. Then λ is the ending lamination of a relative end of M.*

Proof. Let us fix a pleated surface $f_0 : (\Sigma, \sigma_0) \to M$ whose hyperbolic metric we will use to parametrize $\mathcal{ML}$ and $\mathcal{PML}$. Recall that the length function $L_{\sigma_0}(\bullet)$ and the intersection form $i(\bullet, \bullet)$ extends continuously to all measured laminations $\mathcal{ML}$.

Let us pick a representative $\lambda \in \mathcal{ML}$. Consider a sequence of weights $a_n >$ such that the sequence of weighted curves $\{a_n\alpha_n\}_{n\in\mathbb{N}} \subset \mathcal{ML}$ converges to $\lambda \in \mathcal{ML}$. By continuity of the intersection form on $\mathcal{ML}$ and the fact that λ is filling, we have:

$$a_n i(\alpha_n, \gamma) \longrightarrow i(\lambda, \gamma) > 0$$

for every curve $\gamma \in \mathcal{C}$.

Denote by $l_M : \mathcal{C} \longrightarrow [0, \infty)$ the length function associated to the 3-manifold M, i.e. $l_M(\alpha) = l_M(f_0(\alpha)^*)$ where $f_0(\alpha)^*$ is the geodesic representative of $f_0(\alpha)$. Since f_0 is a path-isometry, we have

$$L_{\sigma_0}(\alpha) = l(f_0(\alpha)) \geq l(f_0(\alpha)^*) = l_M(\alpha)$$

for every curve $\alpha \in \mathcal{C}$.

Let $f : (\Sigma, \sigma) \longrightarrow M$ be a pleated surface realizing α in the homotopy class of f_0. By work of Thurston, there is a Margulis constant $\epsilon_1 > 0$ such that only the ϵ-thin part of (Σ, σ) may enter the ϵ_1-thin part of M.

Let $\gamma \in \mathcal{C}$ represent a cusp of M. Suppose that α intersects γ. By standard hyperbolic geometry, namely, by the Collar Lemma, we have

$$l_M(\alpha) = L_\sigma(\alpha) \geq i(\alpha, \gamma) \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{L_\sigma(\gamma)}{2}\right)}\right).$$

In conclusion, putting together the previous observations, we get that the following holds: For the sequence of geodesics representatives of α_n realized in M by the pleated surfaces $f_n : (\Sigma, \sigma_n) \longrightarrow M$ we have

$$\begin{aligned} L_{\sigma_0}(\lambda) &\simeq a_n L_{\sigma_0}(\alpha_n) \geq a_n l_M(\alpha_n) = a_n L_{\sigma_n}(\alpha_n) \\ &\geq a_n i(\alpha_n, \gamma) \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{L_{\sigma_n}(\gamma)}{2}\right)}\right) \simeq i(\lambda, \gamma) \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{L_{\sigma_n}(\gamma)}{2}\right)}\right). \end{aligned}$$

As a consequence $L_{\sigma_n}(\gamma)$ is bounded from below, say by $\eta_\gamma > 0$. Let η be much smaller than $\min\{\eta_\gamma \mid \gamma \text{ cusp }\}$ and Thurston constant ϵ. We have that $f_n(\Sigma) \cap M^{\mathrm{cusp}}_{(0,\eta]} = \emptyset$, i.e. $f_n(\Sigma) \subset M_0 = M \setminus M^{\mathrm{cusp}}_{(0,\eta)}$ in the notation of Section 4.

We are now able to prove that λ is an ending laminaiton of M. Observe that, in this setting, i.e. for sequences $f_n : (\Sigma, \sigma_n) \to M_0$, we have Compactness for Pleated Surfaces (see the proof of Lemma 6.13 in [35], note that, there, it is the hypothesis that the pleated surfaces are *type-preserving* that allows to conclude that $f_n(\Sigma) \subset M_0$).

The proof can be completed as in Corollary 6.14 in [35]. Since λ is not realized, the pleated surfaces f_n have to leave every compact set in M_0 and, in particular, they will miss the Scott core $\mathcal{SC} \subset M_0$ and exit at least one end for n sufficiently large. By the criterion for computing the end invariants of Section 4 we can conclude that λ is an ending lamination of a simply degenerate end of M. $\square$

Appendix B. Local control of one-Lipschitz maps

The goal of this appendix is to show (compare Appendix C of [4] for a different variation)

Proposition B.1. *Let U be a domain in a hyperbolic 3-manifold and let $F : U \to N$ be a volume preserving one-Lipschitz map into a hyperbolic 3-manifold N. Then F is an isometric embedding.*

Proof. As F is volume preserving, all we need to show that F is a local isometry.

To this end let $x \in U$ and let $r_0 > 0$ be such that the closed balls $B(x, r_0)$, $B(F(x), r_0)$ of radius r_0 about x and $F(x)$ are isometric to the closed ball of the same radius in hyperbolic 3-space. Since F is one-Lipschitz we know that $F(B(x, r_0)) \subset B(F(x), r_0)$. Furthermore, as F is continuous and $B(x, r_0)$ is compact, $F(B(x, r_0))$ is a closed subset of $B(F(x), r_0)$ and hence coincides with $B(F(x), r_0)$ as F is volume preserving.

Using once more the fact that F is volume preserving, the differential of F exists almost everywhere and is an isometry. Furthermore, the set of all points $x \in U$ such that $F^{-1}(F(x)) = \{x\}$ has full measure.

Let x be such a point. We saw above that there is a closed subset A of the distance sphere of radius r_0 about x which is mapped by F onto the distance sphere of radius r_0 about $F(x)$. If $y \in A$ then using once more that F is a contraction, the geodesic γ_y connecting x to y is mapped by F to the geodesic γ_{Fy} connecting $F(x)$ to $F(y)$. As F is differentiable at x and $dF(x)$ is an isometry, we have $dF(\gamma'_y(0)) = \gamma'_{Fy}(0)$. In particular, if exp denotes the exponential map at x then $F(\exp(s \exp^{-1}(z))) = \exp(sdF(\exp^{-1}(z))$ for all $z \in A$. On the other hand, $F(A) = \partial B(F(y), r_0)$ and hence $A = \partial B(x, r_0)$ and the restriction of F to $B(x, r_0)$ is an isometry.

As x was a point from a subset of U of full measure, F is indeed a local isometry and hence an isometry. $\square$

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VOLUMES OF RANDOM 3-MANIFOLDS

GABRIELE VIAGGI

Abstract. We prove a law of large numbers for the volumes of families of random hyperbolic mapping tori and Heegaard splittings providing a sharp answer to a conjecture of Dunfield and Thurston.

1. Introduction

Every orientation preserving diffeomorphism $f \in \mathrm{Diff}^+(\Sigma)$ of a closed orientable surface $\Sigma = \Sigma_g$ of genus $g \geq 2$ can be used to define 3-manifolds in two natural ways: We can construct the *mapping torus*

$$T_f := \Sigma \times [0,1] \,/\, (x,0) \sim (f(x),1)\,,$$

and we can form the *Heegaard splitting*

$$M_f := H_g \cup_{f:\partial H_g \to \partial H_g} H_g.$$

The latter is obtained by gluing together two copies of the handlebody H_g of genus g along the boundary $\partial H_g = \Sigma$. In both cases the diffeomorphism type of the 3-manifold only depends on the isotopy class of f, which means that it is well-defined for the *mapping class* $[f] \in \mathrm{Mod}\,(\Sigma) := \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^+_0(\Sigma)$ in the *mapping class group*. We use X_f to denote either T_f or M_f[1].

Invariants of the 3-manifold X_f give rise to well-defined invariants of the mapping class $[f]$. For example, if X_f supports a *hyperbolic metric*, then we can use the geometry to define invariants of $[f]$: By Mostow rigidity, if such hyperbolic metric exists, then it is unique up to isometry.

After Perelman's solution of Thurston's geometrization conjecture, the only obstruction to the existence of a hyperbolic metric on X_f can be phrased in topological terms: A closed orientable 3-manifold is hyperbolic if and only if it is aspherical and atoroidal. Mapping classes that are sufficiently complicated in an appropriate sense (see Thurston [35] and Hempel [17]) give rise to manifolds that satisfy these properties.

For a closed hyperbolic 3-manifold X_f, a good measure of its complexity is provided by the *volume* $\mathrm{vol}(X_f)$. According to a celebrated theorem by Gromov and Thurston, it equals a universal multiple of the *simplicial volume* of X_f, a topologically defined invariant (see for example Chapter C of

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[1]For the bibliography of this part of the thesis see page 101.

[2]). As X_f is not always hyperbolic, in general we define $\mathrm{vol}(X_f)$ to be its simplicial volume, a quantity that always makes sense.

The purpose of this article is to study the *growth* of the volume for families of *random 3-manifolds* or, equivalently, *random mapping classes*.

A random mapping class is the result of a *random walk* generated by a probability measure on the mapping class group, and a random 3-manifold is one of the form X_f where f is a random mapping class. Such notion of random 3-manifolds has been introduced in the foundational work by Dunfield and Thurston [12]. They conjectured that a random 3-manifold is hyperbolic and that its volume grows linearly with the step length of the random walk (Conjecture 2.11 of [12]).

The existence of a hyperbolic metric has been settled by Maher for both mapping tori [21] and Heegaard splittings [22].

Here we answer to Dunfield and Thurston volume conjecture interpreting it in a strict way (see also Conjecture 9.2 in Rivin [32]). Our main result is the following *law of large numbers*: Let μ be a probability measure on $\mathrm{Mod}\,(\Sigma)$ whose support is a finite symmetric generating set. Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the associated random walk

Theorem 1. *There exists* $v = v(\mu) > 0$ *such that for almost every* $\omega = (\omega_n)_{n\in\mathbb{N}}$ *the following holds*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(X_{\omega_n})}{n} = v.$$

Here $(X_{\omega_n})_{n\in\mathbb{N}}$ *is either the family of mapping tori or Heegaard splittings.*

We observe that the asymptotic is the same for both mapping tori and Heegaard splittings. We also remark that the important part is the existence of an *exact asymptotic* for the volume as the *coarsely linear* behaviour follows from previous work. In the case of mapping tori, it is a consequence of work of Brock [6], who proved that there exists a constant $c(g) > 0$ such that for every pseudo-Anosov f

$$\frac{1}{c(g)} d_{\mathrm{WP}}(f) \leq \mathrm{vol}\,(T_f) \leq c(g) d_{\mathrm{WP}}(f)$$

where $d_{\mathrm{WP}}(f)$ is the Weil-Petersson translation length of f, and the theory of random walks on weakly hyperbolic groups (see for example [24]) which provides a linear asymptotic for $d_{\mathrm{WP}}(f)$.

The coarsely linear behaviour for the volume of a random Heegaard splitting follows from results by Maher [22] combined with an unpublished work of Brock and Souto. We refer to the introduction of [22] for more details.

Theorem 1 will be derived from the more technical Theorem 2 concerning *quasi-fuchsian* manifolds. We recall that a quasi-fuchsian manifold is a hyperbolic 3-manifold Q homeomorphic to $\Sigma \times \mathbb{R}$ that has a *compact* subset, the *convex core* $\mathcal{CC}(Q) \subset Q$, that contains all geodesics of Q joining two of its points. The asymptotic geometry of Q is captured by two conformal classes on Σ, i.e. two points in the Teichmüller space $\mathcal{T} = \mathcal{T}(\Sigma)$. Bers [3] showed that for every ordered pair $X, Y \in \mathcal{T}$ there exists a unique quasi-fuchsian manifold, which we denote by $Q(X, Y)$, realizing those asymptotic data.

Theorem 2. *There exists $v = v(\mu) > 0$ such that for every $o \in \mathcal{T}$ and for almost every $\omega = (\omega_n)_{n\in\mathbb{N}}$ the following limit exists:*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(\mathcal{CC}(Q(o, \omega_n o)))}{n} = v.$$

We remark that $v(\mu)$ is the same as in Theorem 1. Once again, the coarsely linear behaviour of the quantity in Theorem 2 was known before: The technology developed around the solution of the ending lamination conjecture by Minsky [28] and Brock-Canary-Minsky [8], with fundamental contributions by Masur-Minsky [25], [26], gives a combinatorial description of the internal geometry of the convex core of a quasi-fuchsian manifold. This combinatorial picture is a key ingredient in Brock's proof [5] of the following coarse estimate: There exists a constant $k(g) > 0$ such that

$$\frac{1}{k(g)} d_{\mathrm{WP}}(X, Y) - k(g) \le \mathrm{vol}\,(\mathcal{CC}(Q(X, Y))) \le k(g) d_{\mathrm{WP}}(X, Y) + k(g).$$

This link between volumes of hyperbolic 3-manifolds and the Weil-Petersson geometry of Teichmüller space, as in the case of random mapping tori, leads to the coarsely linear behaviour for the volume of the convex cores of $Q(o, \omega_n o)$, but does not give, by itself, a law of large numbers. The main novelty in this paper is that we work directly with the geometry of the quasi-fuchsian manifolds rather than their combinatorial counterparts which allows us to get exact asymptotics rather than coarse ones.

The relation between Theorem 1 and Theorem 2 is provided by a *model manifold* construction similar to Namazi [29], Namazi-Souto [30], Brock-Minsky-Namazi-Souto [9]. In the case of random 3-manifolds the heuristic picture is the following: The geometry of X_{ω_n} largely resembles the geometry of the convex core of $Q(o, \omega_n o)$, more precisely, as far as the volume is concerned, we have

$$|\mathrm{vol}\,(X_{\omega_n}) - \mathrm{vol}\,(\mathcal{CC}(Q(o, \omega_n o)))\,| = o(n).$$

We now describe the basic ideas behind Theorem 2: Suppose that the support of μ equals a finite generating set S and consider $f = s_1 \dots s_n$, a long random word in the generators $s_i \in S$. It corresponds to a quasi-fuchsian manifold $Q(o, fo)$. Fix N large, and assume $n = Nm$ for simplicity. We can split f into smaller blocks of size N

$$f = (s_1 \dots s_N) \cdots (s_{N(m-1)+1} \dots s_{Nm})$$

which we also denote by $f_j := s_{jN+1} \cdots s_{(j+1)N}$. Each block corresponds to a quasi-fuchsian manifold $Q(o, f_j o)$ as well. The main idea is that the geometry of the convex core $\mathcal{CC}(Q(o, fo))$ can be roughly described by juxtaposing, one after the other, the convex cores of the single blocks $\mathcal{CC}(Q(o, f_j o))$. In particular, the volume $\mathrm{vol}(\mathcal{CC}(Q(o, fo)))$ can be well approximated by the *ergodic sum*

$$\sum_{1 \le j \le m} \mathrm{vol}\left(\mathcal{CC}(Q(o, f_j o))\right)$$

which converges in average by the Birkhoff ergodic theorem.

In the paper, we will make this heuristic picture more accurate. Our three main ingredients are the model manifold, bridging between the geometry of the Teichmüller space $\mathcal{T}$ and the internal geometry of quasi-fuchsian manifolds [28],[8], a recurrence property for random walks [1] and the method of natural maps from Besson-Courtois-Gallot [4]. They correspond respectively to Proposition 3.9, Proposition 4.3 and Proposition 3.10. Proposition 3.9 and Proposition 4.3 are used to construct a geometric object, i.e. a negatively curved model for T_f, associated to the ergodic sum written above. Proposition 3.10 let us compare this model to the underlying hyperbolic structure.

As an application of the same techniques, along the way, we give another proof of the following well-known result [19], [7] relating iterations of pseudo-Anosovs, volumes of quasi-fuchsian manifolds and mapping tori

Proposition 3. *Let ϕ be a pseudo-Anosov mapping class. For every $o \in \mathcal{T}$ the following holds:*

$$\lim_{n \to \infty} \frac{\mathrm{vol}\left(\mathcal{CC}(Q(o, \phi^n o))\right)}{n} = \mathrm{vol}\left(T_\phi\right).$$

Outline. The paper is organized as follows.

In Section 2 we introduce quasi-fuchsian manifolds. They are the building blocks for the cut-and-glue construction of Section 3. We prove that, under suitable assumptions, we can glue together a family of quasi-fuchsian manifolds in a geometrically controlled way. The geometric control on the glued manifold is good enough for the application of volume comparison results.

As an application of the cut-and-glue construction we show that the volume of a random gluing is essentially the volume of a quasi-fuchsian manifold (Proposition 3 follows from this fact). As a consequence, in Section 5, we deduce Theorem 1 from Theorem 2 whose proof is carried out shortly after.

In Section 4 we discuss random walks on the mapping class group and on Teichmüller space. The goal is to describe the picture of a random Teichmüller ray and state the main recurrence property.

In the last section, Section 6, we formulate some questions related to the study of *growth in random families* of 3-manifolds.

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2. Quasi-Fuchsian manifolds

We start by introducing *quasi-fuchsian* manifolds and their geometry.

2.1. **Marked hyperbolic 3-manifolds.** Let M be a compact, connected, oriented 3-manifold. A marked hyperbolic structure on M is a complete Riemannian metric on $\mathrm{int}(M)$ of constant sectional curvature $\sec \equiv -1$. We regard two Riemannian metrics as equivalent if they are isometric via a diffeomorphism homotopic to the identity.

Every marked hyperbolic structure corresponds to a quotient $\mathbb{H}^3/\Gamma$ of the hyperbolic 3-space $\mathbb{H}^3$ by a discrete and torsion free group of isometries $\Gamma < \mathrm{Isom}^+\left(\mathbb{H}^3\right) = \mathrm{PSL}_2(\mathbb{C})$ together with an identification of $\pi_1(M)$ with Γ, called the *holonomy representation* $\rho : \pi_1(M) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$.

We are mostly interested in the cases where $M = \Sigma \times [-1, 1]$ is a trivial I-bundle over a surface and when M is closed. By Mostow Rigidity, if M is closed and admits a hyperbolic metric, then the metric is unique up to isometries. In this case we denote by $\mathrm{vol}(M)$ the volume of such a metric.

2.2. **Quasi-fuchsian manifolds.** A particularly flexible class of structures is provided by the so-called *quasi-fuchsian* manifolds

Definition (Quasi-Fuchsian). A marked hyperbolic structure Q on $\Sigma \times [-1, 1]$ is *quasi-fuchsian* if $\mathbb{H}^3/\rho(\pi_1(\Sigma))$ contains a *compact* subset which is *convex*, that is, containing every geodesic joining a pair of points in it. The smallest convex subset is called the *convex core* and is denoted by $\mathcal{CC}(Q)$.

The convex core $\mathcal{CC}(Q)$ is always a topological submanifold. If it has codimension 1 then it is a totally geodesic surface and we are in the *fuchsian* case, the group $\Gamma < \mathrm{Isom}^+\left(\mathbb{H}^3\right)$ stabilizes a totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$. In the generic case it has codimension 0 and is homeomorphic to $\Sigma \times [-1, 1]$. The inclusion $\mathcal{CC}(Q) \subset Q$ is always a homotopy equivalence.

We denote by

$$\mathrm{vol}(Q) := \mathrm{vol}(\mathcal{CC}(Q)) \in [0, \infty)$$

the volume of the convex core of the quasi-fuchsian manifold Q.

2.3. **Deformation space.** We denote by $\mathcal{T}$ the Teichmüller space of Σ, that is, the space of marked hyperbolic structures on Σ up to isometries homotopic to the identity. We equip $\mathcal{T}$ with the Teichmüller metric $d_{\mathcal{T}}$.

To every quasi-fuchsian manifold Q one can associate the *conformal boundary* $\partial_c Q$ in the following way: The surface group $\pi_1(\Sigma)$ acts on $\mathbb{H}^3$ by isometries and on $\mathbb{CP}^1 = \partial\mathbb{H}^3$ by Möbius transformations. It also preserves a convex set, the lift of $\mathcal{CC}(Q)$ to the universal cover, on which it acts co-compactly. By Milnor-Švarc, for any fixed basepoint $o \in \mathbb{H}^3$, the orbit map

$\gamma \in \pi_1(\Sigma) \to \gamma o \in \mathbb{H}^3$ is a quasi-isometric embedding and extends to a topological embedding on the boundary $\partial\pi_1(\Sigma) \hookrightarrow \mathbb{CP}^1$. The image is a topological circle Λ, called the *limit set*, that divides the Riemann sphere $\mathbb{CP}^1$ into a union of two topological disks $\Omega = \mathbb{CP}^1 \backslash \Lambda$. The action $\pi_1(\Sigma) \curvearrowright \Omega$ preserves the connected components, and is free, properly discontinuous and conformal. The quotient $\partial_c Q = \Omega/\pi_1(\Sigma) = X \sqcup Y$ is a disjoint union of two marked oriented Riemann surfaces, homeomorphic to Σ, and it is called the *conformal boundary* of Q. The quotient $\bar{Q} := (\mathbb{H}^3 \cup \Omega)/\Gamma$ compactifies Q.

THEOREM 2.1 (Double Uniformization, Bers [3]). *For every ordered pair of marked hyperbolic surfaces $(X, Y) \in \mathcal{T} \times \mathcal{T}$ there exists a unique equivalence class of quasi-fuchsian manifolds, denoted by $Q(X, Y)$, realizing the conformal boundary $\partial_c Q(X, Y) = X \sqcup Y$.*

The mapping class group $\mathrm{Mod}\,(\Sigma)$ acts on quasi-fuchsian manifolds by precomposition with the marking. In Bers coordinates it plainly translates into $\phi Q(X, Y) = Q\,(\phi X, \phi Y)$.

2.4. **Teichmüller geometry and volumes.** Later, it will be very important for us to quantify the price we have to pay in terms of volume if we want to replace a quasi-fuchsian manifold Q with another one Q'. We would like to express $|\mathrm{vol}\,(Q) - \mathrm{vol}\,(Q')|$ in terms of the geometry of the conformal boundary.

Despite the fact that Weil-Petersson geometry is more natural when considering questions about volumes, we will mainly use the Teichmüller metric $d_{\mathcal{T}}$. The reason is that we are mostly concerned with *upper bounds* for the volumes of the convex cores. It is a classical result of Linch [20] that the Teichmüller distance is bigger than the Weil-Petersson distance $d_{\mathrm{WP}} \le \sqrt{2\pi|\chi(\Sigma)|}\,d_{\mathcal{T}}$. The following is our main tool:

PROPOSITION 2.2 (Proposition 2.7 in Kojima-McShane [19], see also Schlenker [33]). *There exists $\kappa = \kappa(\Sigma) > 0$ such that*

$$\left|\mathrm{vol}\,(Q(X, Y)) - \mathrm{vol}\left(Q(X', Y')\right)\right| \le \kappa \left(d_{\mathcal{T}}(X, X') + d_{\mathcal{T}}(Y, Y')\right) + \kappa.$$

This formulation is not literally Proposition 2.7 of [19] so we spend a couple of words to explain the two diffenrences. Firstly, the estimate in Proposition 2.7 of [19] concerns the *renormalized volume* and not volume of the convex core. However, the two quantities only differ by a uniform additive constant (see Theorem 1.1 in [33]). Secondly, their statement is limited to the case where $X = X' = Y'$, but their proof exteds word by word to the more general setting: It suffices to apply their argument to the one parameter families $Q(X, Y_t)$ and $Q(X_t, Y')$, where X_t and Y_t are the Teichmüller geodesics joining X to X' and Y to Y'.

2.5. **Geometry of the convex core.** We associate to the quasi-fuchsian manifold $Q = Q(X, Y)$ the Teichmüller geodesic $l : [0, d] \to \mathcal{T}$ joining X to Y where $d = d_{\mathcal{T}}(X, Y)$. Work of Minsky [28] and Brock-Canary-Minsky [8] relates the geometry of the Teichmüller geodesic l to the internal geometry of $\mathcal{CC}(Q)$. In the next section we will use this information to glue together convex cores of quasi-fuchsian manifolds in a controlled way.

As a preparation, we start with a description of the boundary $\partial\mathcal{CC}(Q)$ and introduce some useful notation. We recall that, topologically, $\mathcal{CC}(Q) \simeq \Sigma \times [-1, 1]$. The convex core separates $\bar{Q} = Q \cup \partial_c Q$ into two connected components, containing, respectively, X and Y. We denote by $\partial_X\mathcal{CC}(Q)$ and $\partial_Y\mathcal{CC}(Q)$ the components of $\partial\mathcal{CC}(Q)$ that face, respectively, X and Y. As observed by Thurston, the surfaces $\partial_X\mathcal{CC}(Q)$ and $\partial_Y\mathcal{CC}(Q)$, equipped with the induced path metric, are hyperbolic. By a result of Sullivan, they are also uniformly bilipschitz equivalent X and Y (see Chapter II.2 of [10]).

3. Gluing and Volume

This section describes a gluing construction (Proposition 3.9) which is a major technical tool in the article. It allows us to cut and glue together quasi-fuchsian manifolds in a sufficiently controlled way. The control on the *models* obtained with this procedure is then exploited to get volume comparisons via the method of natural maps (Proposition 3.10 as in [4]) which is the second major tool of the section.

Along the way we recover a well-known result (Proposition 3) relating iterations of pseudo-Anosov maps and volumes of quasi-fuchsian manifolds.

3.1. **Product regions and Cut and Glue construction.**

The cut and glue construction we are going to describe is a standard way to glue Riemannian 3-manifolds. Here we import the discussion and some of the observations of Section 5 of [16] and adapt them to our special setting. We start with a pair of definitions.

Definition (Product Region). Let Q be a quasi-fuchsian manifold. A *product region* $U \subset Q$ is a codimension 0 submanifold homeomorphic to $\Sigma \times [0, 1]$ whose inclusion in Q is a homotopy equivalence.

Using the orientation and product structure of Q we can define a *top boundary* ∂_+U and a *bottom boundary* ∂_-U. We denote by Q_- and Q_+ the parts of Q that lie *below* ∂_+U and *above* ∂_-U respectively.

A product region comes together with a *marking*, an identification $j_U : \pi_1(\Sigma) \xrightarrow{\sim} \pi_1(U)$, defined as follows: The data of a marked hyperbolic structure Q gives us an identification $\pi_1(\Sigma) \simeq \pi_1(Q)$ and the inclusion $U \subset Q$, being a homotopy equivalence, gives $\pi_1(Q) \simeq \pi_1(U)$. The marking allows us to talk about the homotopy class of a map between product regions.

Any homotopy equivalence $k : U \to V$ induces a well-defined mapping class $[k] \in \mathrm{Mod}\,(\Sigma) \simeq \mathrm{Out}^+(\pi_1(\Sigma))$ (Dehn-Nielsen-Baer, Theorem 8.1 in [13]), namely, the one corresponding to the outer automorphism

$$\pi_1(\Sigma) \overset{j_U}{\simeq} \pi_1(U) \overset{k}{\simeq} \pi_1(V) \overset{j_V}{\simeq} \pi_1(\Sigma).$$

We also want to quantify the geometric quality of a map between product regions. Since we want to keep the curvature tensor under control, a good measurement for us is provided by the $\mathcal{C}^2$-norm.

Definition (Almost-Isometric). Let $k : (U, \rho_U) \to (V, \rho_V)$ be a smooth embedding between Riemannian manifolds. Denote by ∇_U, ∇_V the Levi-Civita connections. Consider the $\mathcal{C}^2$-norm

$$||\rho_U - k^*\rho_V||_{\mathcal{C}^2} := ||\rho_U - k^*\rho_V||_{\mathcal{C}^0} + ||\nabla_U k^*\rho_V||_{\mathcal{C}^0} + ||\nabla_U \nabla_U k^*\rho_V||_{\mathcal{C}^0}.$$

For $\xi > 0$ we say that k is ξ-almost isometric if $||\rho_U - k^*\rho_V||_{\mathcal{C}^2} < \xi$.

The following lemma is what we refer to as the cut-and-glue construction.

Lemma 3.1. *Let Q, Q' be quasi-fuchsian manifolds. Denote by $\rho_Q, \rho_{Q'}$ their Riemannian metrics. Suppose we have product regions $U \subset Q$, $U' \subset Q'$ and a diffeomorphism $k : U \to U'$ between them. Suppose also that $\theta : U \to [0,1]$ is a smooth function with $\theta|_{\partial_- U, \partial_+ U} \equiv 0, 1$. Then we can form the 3-manifold*

$$Q'' = Q_- \cup_{k:U\to U'} Q'_+$$

and endow it with the Riemannian metric

$$\rho := \begin{cases} \rho_Q & \text{on } Q_- \setminus U \\ (1-\theta)\rho_Q + \theta k^*\rho_{Q'} & \text{on } U \\ \rho_{Q'} & \text{on } Q'_+ \setminus U'. \end{cases}$$

If k is ξ-almost isometric for some $\xi < 1$, then, on $U \subset Q''$, we have the following sectional curvature and diameter bounds

$$\left|1 + \sec_{Q''}\right| \le c_3 \, ||\theta||_{\mathcal{C}^2} \cdot \left|\left|\rho_Q - k^*\rho_{Q'}\right|\right|_{\mathcal{C}^2}$$

for some universal constant c_3 and

$$\mathrm{diam}_\rho(U) \le (1+\xi)\mathrm{diam}_{\rho_U}(U).$$

In particular, if $\mathrm{diam}_{\rho_U}(U)$ is uniformly bounded, the same is true for $\mathrm{vol}_\rho(U)$.

We associate two parameters to a product region, *diameter* and *width*

$$\mathrm{diam}(U) := \sup\{d_Q(x,y)\,|x,y \in U\},$$
$$\mathrm{width}(U) := \inf\{d_Q(x,y)\,|x \in \partial_+ U, y \in \partial_- U\}.$$

If a product region has width at least D and diameter at most $2D$ we say that it has *size* D. The Margulis Lemma implies that the injectivity radius of a product region of size D, defined as

$$\mathrm{inj}(U) := \inf_{x\in U}\{\mathrm{inj}_x(Q)\},$$

is bounded from below in terms of D

Lemma 3.2. *For every $D > 0$ there exists $\epsilon_0(D,g) > 0$ such that a product region U of size D has $\mathrm{inj}(U) > \epsilon_0$.*

Proof. The inclusion of U in Q is π_1-surjective. Having diameter bounded by $2D$, the region U cannot intersect too deeply any very thin Margulis tube $\mathbb{T}_\gamma$ otherwise $\pi_1(U) \to \pi_1(Q)$ would factor through $\pi_1(U) \to \pi_1(\mathbb{T}_\gamma)$. $\square$

In particular, a compactness argument with the geometric topology on pointed hyperbolic manifolds gives us the following property: Once we fix the size of a product region we can produce a uniform bump function on it.

LEMMA 3.3 (Lemma 5.2 of [16]). *For every $D > 0$ there exists $K > 0$ such that the following holds: Let $U \simeq \Sigma \times [0, 1]$ be a product region of size D. Then there exists a smooth function $\theta : U \to [0, 1]$ with the following properties:*

- *Near the boundaries it is constant: $\theta|_{\partial_- U} \equiv 0$ and $\theta|_{\partial_+ U} \equiv 1$.*
- *Uniformly bounded $\mathcal{C}^2$-norm $||\theta||_{\mathcal{C}^2} \leq K$.*

3.2. **Almost-isometric embeddings of product regions.** For a fixed $\eta > 0$ we denote by $\mathcal{T}_\eta$ the *η-thick part* of Teichmüller space consisting of those hyperbolic structures with no geodesic shorter than η.

The following is a consequence of the *model manifold* technology developed by Minsky [28] around the solution of the Ending Lamination Conjecture (completed then in Brock-Canary-Minsky [8]).

PROPOSITION 3.4 (see Proposition 6.2 [16]). *For every $\eta, \xi, \delta, D > 0$ there exists $D_0(\eta, g)$ and $h = h(\eta, \xi, \delta, D) > 0$ such that the following holds: Let Q_1, Q_2 be quasi-fuchsian manifolds with associated Teichmüller geodesics $l_i : I_i \subseteq \mathbb{R} \to \mathcal{T}$ with $i = 1, 2$. Suppose that l_1, l_2 δ-fellow travel on a subsegment J of length at least h and entirely contained in $\mathcal{T}_\eta$. Then there exist product regions $U_i \subset \mathcal{CC}(Q_i)$ of size D and a ξ-almost isometric embedding $k : U_1 \to U_2$ in the homotopy class of the identity. Moreover, if $D \geq D_0$ we can assume that U_i contains the geodesic representative of α, a curve which has moderate length for both Q_i and $T \in J$ the midpoint of the segment, i.e. $l_{Q_i}(\alpha)$, $L_T(\alpha) \leq D_0$.*

In the statement and in the next section we use the following notation:

> **Notation**. If $\alpha : S^1 \to Q$ is a closed loop in a hyperbolic 3-manifold, we denote by $l(\alpha)$ its length and by $l_Q(\alpha)$ the length of the unique geodesic representative in the homotopy class. If the target instead is a hyperbolic surface $\alpha : S^1 \to X$, we use the notations $L(\alpha)$ and $L_X(\alpha)$.

For a proof we refer to [16]. The geodesic α is used to locate the product regions inside the convex cores. We explain that in the following section. For now we remark the following immediate consequence:

DEFINITION (η-Height). Let $l : I \to \mathcal{T}$ be a Teichmüller geodesic. The *η-height* of l is the length of the maximal connected subsegment of I whose image is entirely contained in $\mathcal{T}_\eta$.

COROLLARY 3.5. *Fix $\eta > 0$. There exists a function $\rho : (0, \infty) \to (0, \infty)$ with $\rho(h) \uparrow \infty$ as $h \uparrow \infty$ and the following property: Let $Q = Q(X, Y)$ be a quasi-fuchsian manifold with associated geodesic $l : I \to \mathcal{T}$. Suppose that* *the η-height is at least h then*

$$d_Q\left(\partial_X \mathcal{CC}(Q), \partial_Y \mathcal{CC}(Q)\right) \geq \rho(h).$$

3.3. **Position of the product region.** From now on we fix once and for all a sufficiently large size $D_1 \geq D_0$ for the product regions we consider.

Let $\alpha : S^1 \to Q$ be a non-trivial closed curve in a hyperbolic 3-manifold Q that has a geodesic representative $\alpha^* \subset Q$. By basic hyperbolic geometry

$$\cosh\left(d_Q(\alpha, \alpha^*)\right) l_Q(\alpha) \leq l(\alpha).$$

Suppose that $Q = Q(X, Y)$ is a quasi-fuchsian manifold. Let $U \subset Q$ be a product region of size D_1 containing a closed geodesic α. By the assumption on the size of U and Lemma 3.2 we have $l_Q(\alpha) \geq 2\epsilon_0(D_1, g)$. Recall that $\partial_X \mathcal{CC}(Q)$ denotes the boundary of the convex core that faces the conformal boundary X. By a Theorem due to Sullivan (see Chapter II.2 and in particular Theorem II.2.3.1 in [10]), there exists a universal constant K such that $\partial_X \mathcal{CC}(Q)$ and X are K-bilipschitz equivalent via a homeomorphism in the homotopy class of the identity. We have

$$d_Q(\partial_X \mathcal{CC}(Q), \alpha) \leq \operatorname{arccosh}\left(\frac{L_{\partial_X \mathcal{CC}(Q)}(\alpha)}{l_Q(\alpha)}\right) \leq \operatorname{arccosh}\left(\frac{K L_X(\alpha)}{2\epsilon_0(D_1, g)}\right).$$

Let $T \in \mathcal{T}$ be a hyperbolic structure for which $L_T(\alpha) \leq D_0(\eta, g)$. Wolpert's inequality $L_X(\alpha) \leq L_T(\alpha) e^{2d_{\mathcal{T}}(X,T)}$ (see Lemma 12.5 in [13]) allows us to continue the chain of inequalities to the following:

$$d_Q(\partial_X \mathcal{CC}(Q), \alpha) \leq \operatorname{arccosh}\left(\frac{K D_0(\eta, g)}{2\epsilon_0(D_1, g)} e^{2d_{\mathcal{T}}(X,T)}\right).$$

Let us introduce the function $F : (0, \infty) \to (0, \infty)$ defined by

$$F(t) = \operatorname{arccosh}\left(\frac{K D_0(\eta, g)}{2\epsilon_0(D_1, g)} e^{2t}\right).$$

With this notation we have

Lemma 3.6. *Let $U \subset Q(X, Y)$ be a product region of size D_1 containing a closed geodesic $\alpha \subset U$. Let $T \in \mathcal{T}$ be a surface such that $L_T(\alpha) \leq D_0$. Then*

$$d_Q(\partial_X \mathcal{CC}(Q), U) \leq F(d_{\mathcal{T}}(X, T)).$$

Combining Corollary 3.5 and Lemma 3.6 we can ensure that a pair of product regions is well separated. To this extent we introduce the function $G : (0, \infty) \to (0, \infty)$ defined by

$$G(t) = \inf_{t \in \mathbb{R}} \{\text{for every } s > t \text{ we have } \rho(s) > 2F(t) + 4D_1\}.$$

Lemma 3.7. *Let U^-, U^+ be product regions of size D_1 in $Q = Q(X^-, X^+)$. Suppose they contain, respectively, closed geodesics α^-, α^+. Let $T^-, T^+ \in \mathcal{T}$ be surfaces such that $L_{T^-}(\alpha^-)$, $L_{T^+}(\alpha^+) \leq D_0$. Consider*

$$d := \max\left\{d_{\mathcal{T}}(X^-, T^-), d_{\mathcal{T}}(X^+, T^+)\right\}.$$

If the η-height h of Q is at least $h \geq G(d)$ then the product regions are disjoint and cobound a codimension 0 submanifold $Q^0 \subset Q$ homeomorphic to $\Sigma \times [0,1]$ for which U^-, U^+ are collars of the boundary.

Proof. We have $d_Q(\partial_{X^-}\mathcal{CC}(Q), \partial_{X^+}\mathcal{CC}(Q)) \geq \rho(h)$ and $d_Q(\partial_{X^\pm}\mathcal{CC}(Q), U^\pm) \leq F(d)$. If $\rho(h) - F(d) - 2D_1 \geq F(d) + 2D_1$, the product regions U^-, U^+ are separated. By definition of G, if $h > G(d)$, the previous inequality holds. $\square$

Finally we take care of the volume.

LEMMA 3.8. *If $X^-, X^+ \in \mathcal{T}_\eta$, then there exists $V_0(D_1, \eta, d)$ such that*

$$\left|\mathrm{vol}\,(Q) - \mathrm{vol}\left(Q^0\right)\right| \leq V_0.$$

Proof. There is a uniform upper bound on the diameter of a η-thick hyperbolic surface. By Sullivan, the same holds for every component of $\partial\mathcal{CC}(Q)$. It follows that the diameter of the region enclosed by U^- and $\partial_{X^-}\mathcal{CC}(Q)$ is uniformly bounded in terms of D_1, η. As an upper bound for its volume we can take the volume of a ball with the same radius in $\mathbb{H}^3$. $\square$

3.4. **A gluing theorem.** Recall that we fixed $D_1 > 0$ sufficiently large once and for all. The following is our first crucial technical tool.

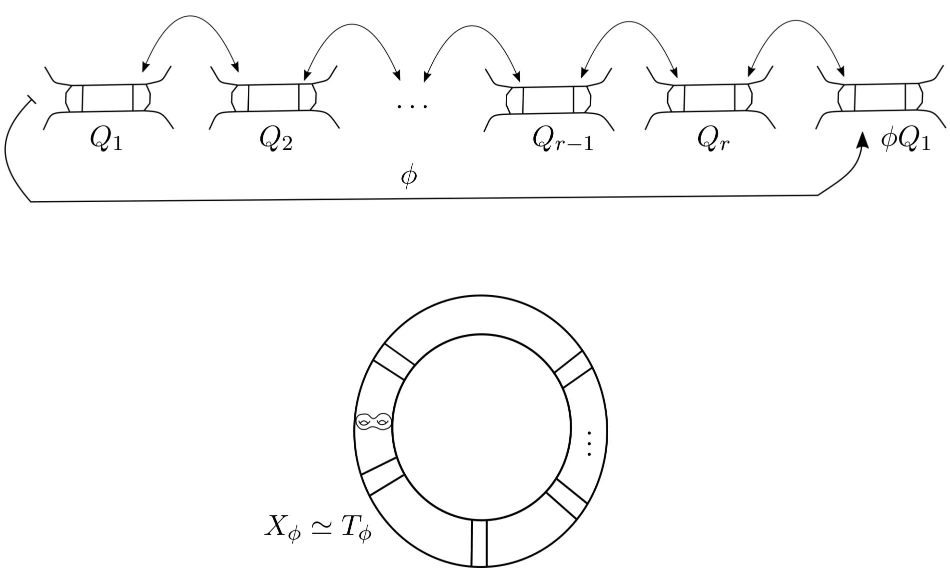

FIGURE 1. Gluing.

PROPOSITION 3.9. *Fix $\eta, \delta > 0$ and $\xi \in (0,1)$. There exists $h_0(\eta, \xi, \delta) > 0$ such that the following holds: Let $\left\{Q_i = Q(X_i^-, X_i^+)\right\}_{i=1}^r$ be a family of quasi-fuchsian manifolds. Let $\{l_i : I_i \to \mathcal{T}\}_{i=1}^r$ be the corresponding Teichmüller geodesics. Suppose that the following holds:*

- *For every $i < r$, the geodesics l_i, l_{i+1} δ-fellow travel when restricted to $J_i^+ \subset I_i$ and $J_{i+1}^- \subset I_{i+1}$. The segments J_i^+ and J_{i+1}^- are respectively terminal and initial, have length $\left|J_i^-\right|, \left|J_i^+\right| \in [h_0, 2h_0]$ and are entirely contained in $\mathcal{T}_\eta$.*
- *The η-height of l_i is at least $G(2h_0)$ for all $i \leq r$.*

Let $k_i : U_i^+ \subset Q_i \to U_{i+1}^- \subset Q_{i+1}$ be the ξ-almost isometric embedding of product regions in the homotopy class of the identity for $i < r$ corresponding to the segments J_i^+, J_{i+1}^- as in Proposition 3.4. The product regions have size D_1 and are disjoint as in Lemma 3.7. Let Q_i^0 be the region of Q_i bounded by $\partial^- U_i^-$ and $\partial^+ U_i^+$ for which U_i^-, U_i^+ are collars of the boundaries as in Lemma 3.7. Then we can form

$$X := Q_1^0 \cup_{k_1 : U_1^+ \to U_2^-} Q_2^0 \cup \cdots \cup Q_{r-1}^0 \cup_{k_{r-1} : U_{r-1}^+ \to U_r} Q_r^0$$

using the cut and glue construction Lemma 3.1. The compact 3-manifold X has the following properties:

- *Curvature: $|1 + \sec_X| \leq K\xi$ where $K = K(D_1)$ is as in Lemma 3.3.*
- *The inclusions $Q_i^0 \setminus \left(U_i^- \cup U_i^+\right) \subset X$ are isometric.*
- *Volume: There exists $V_0 = V_0(\eta, \xi, D_1, h_0)$ such that*

$$\left| \mathrm{vol}\,(X) - \sum_{i<r} \mathrm{vol}\,(Q_i) \right| \leq rV_0.$$

Furthermore, let $\phi \in \mathrm{Mod}\,(\Sigma)$ be a mapping class. Suppose it has the property that ϕl_1 and l_r δ-fellow travel along $J_1^- \subset I_1$ and $J_r^+ \subset I_r$. Then there is a ξ-almost isometric embedding $k_r : U_r^+ \subset Q_r \to U_1^- \subset Q_1$ in the homotopy class of ϕ and we can form the manifold

$$X_\phi = X / \left(k_r : U_r^+ \subset Q_r^0 \to U_1^- \subset Q_1^0\right).$$

Topologically X_ϕ is diffeomorphic to the mapping torus of ϕ.

The ξ-almost isometric embedding k_r is obtained as the composition of the one provided by Proposition 3.4 for the fellow traveling of l_r, ϕl_1 and the isometric remarking $\phi Q_1 \to Q_1$ in the isotopy class of ϕ (see Figure 1).

Proposition 3.9 follows directly from several applications of Proposition 3.4 and the cut and glue construction Lemma 3.1 once we can ensure that the product regions are well separated as in Lemma 3.7. Separation and volume bounds follow from the discussion in the previous section.

We remark that, by a celebrated theorem of Thurston [35], if ϕ is a *pseudo-Anosov* mapping class, then the mapping torus T_ϕ admits a hyperbolic metric. A pseudo-Anosov element ϕ is one that acts as a hyperbolic isometry of Teichmüller space: It preserves a unique Teichmüller geodesic $l : \mathbb{R} \to \mathcal{T}$ on which it acts by translations $\phi l(t) = l(t + L(\phi))$. The quantity $L(\phi) > 0$ is called the *translation length* of ϕ (see Chapter 13 of [13]).

3.5. **Comparing the volume.** The second fundamental ingredient is a volume comparison result. If we have two Riemannian metrics g_0 and g on the same 3-manifold M we can compare their volume using the method of *natural maps* introduced by Besson, Courtois and Gallot. We mainly refer to their work [4] as we use some consequences of it. Given a map $f : N \to M$ between Riemannian manifolds satisfying certain curvature conditions, the method produces families of natural maps $F : N \to M$ homotopic to f and with Jacobian bounded in terms of the *volume entropies* of the manifolds. We need the following result:

THEOREM 3.10 (Besson-Courtois-Gallot [4]). *Let (M, g) and (M_0, g_0) be closed orientable Riemannian 3-manifolds such that there exists:*

- *A lower bound for the Ricci curvature of the source* $\mathrm{Ric}_g \geq -2g$.
- *A uniform bound for the sectional curvatures of the target* $-k \leq \sec_{g_0} \leq -1$ *for some* $k \geq 1$.

Then for every continuous map $f : M \longrightarrow M_0$ we have

$$\mathrm{vol}\,(M) \geq |\deg(f)|\,\mathrm{vol}\,(M_0)\,.$$

We now describe some applications.

The first one is to the models constructed in Proposition 3.9:

COROLLARY 3.11. *If ϕ is a pseudo-Anosov mapping class, X_ϕ is as in Proposition 3.9 and $K\xi < 1$ then*

$$(1 - K\xi)^{-3/2}\mathrm{vol}\,(X_\phi) \leq \mathrm{vol}\,(M_\phi) \leq (1 + K\xi)^{3/2}\mathrm{vol}\,(X_\phi)\,.$$

Proof. The mapping torus of ϕ admits a purely hyperbolic Riemannian metric and the metric X_ϕ with $\sec_{X_\phi} \in (-1 - K\xi, -1 + K\xi)$. We apply Theorem 3.10 to the identity map in both directions after suitably rescaling the metric on X_ϕ so that it fulfills the Ricci and sectional curvature bounds. □

The second application is a construction of a very peculiar model of a mapping torus T_ϕ of a pseudo-Anosov diffeomorphism ϕ. Recall that ϕ acts on its axis by translating points by $L(\phi)$.

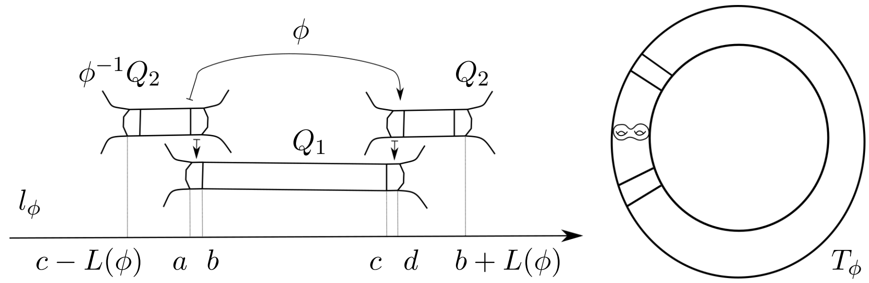

FIGURE 2. Model for a mapping torus.

Corollary 3.12. *Fix $\eta > 0$ and $\xi \in (0,1)$. There exists $h(\xi,\eta) > 0$ such that the following holds: Let ϕ be a pseudo-Anosov with axis $l : \mathbb{R} \to \mathcal{T}$. Suppose that there are disjoint intervals $I = [a,b]$ and $J = [c,d]$ with $a < b < c < d < a + L(\phi)$ such that $l(I), l(J) \subset \mathcal{T}_\eta$ and $|I|,\ |J| \geq h$. Then*

$$|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q(l(a),l(d)))| \leq \kappa(L(\phi) + b - c) + \xi\kappa(d-a) + \mathrm{const}$$

where const *depends only on η, ξ, h, D_1.*

Proof. Let $h_0(\eta,\xi,0)$ be as in Proposition 3.9. If $h \geq \max\{h_0, G(2h_0)\}$ is large enough, then the quasi-fuchsian manifolds (see Figure 2)

$$\{Q_1 = Q(l(a),l(d)), Q_2 = Q(l(c), l(L(\phi)+b))\}$$

satisfy the assumption of Proposition 3.9. Moreover $\phi Q_1 = Q(l(a+L(\phi)), d+L(\phi))$ and the segments $[l(c), l(b+L(\phi))]$ and $[l(a+L(\phi)), l(d+L(\phi))]$ overlap along $[l(a+L(\phi)), l(b+L(\phi))] = \phi\,[l(a), l(b)]$. The upper bound for the volume is just an application of Proposition 2.2

$$\begin{aligned}
&|\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q(l(a),l(d)))| \\
&\leq \mathrm{vol}\,(Q(l(c), l(L(\phi)+b))) + 2V_0 + \xi\mathrm{vol}\,(Q(l(a),l(d))) \\
&\leq \kappa(L(\phi)+b-c) + \xi\kappa(d-a) + 2V_0 + 2\kappa.
\end{aligned}$$

□

Using this estimates we recover the following well-known result (see for example [7], [19]):

Proposition 3. *Let ϕ be a pseudo-Anosov mapping class. Then for every $o \in \mathcal{T}$ we have*

$$\lim \frac{\mathrm{vol}\,(Q(o,\phi^n o))}{n} = \mathrm{vol}\,(T_\phi)\,.$$

Proof. There exists $\eta_\phi > 0$ such that $l_\phi : \mathbb{R} \to \mathcal{T}$, the Teichmüller axis of ϕ, lies in $\mathcal{T}_{\eta_\phi}$. Fix $\xi > 0$ and consider $h = h(\eta_\phi, \xi)$. For n large enough the intervals $I = [0,h]$ and $J = [nL(\phi) - h, nL(\phi)]$ fulfill the assumption of Corollary 3.12 with respect to ϕ^n. Hence, for all large n, $|\mathrm{vol}\,(Q(l_\phi(0), l_\phi(nL(\phi)))) - n\mathrm{vol}\,(T_\phi)| \leq \kappa 2h + \xi\kappa n L(\phi) + \mathrm{const}$. Observe that $l_\phi(nL(\phi)) = \phi^n l_\phi(0)$. Denote $l_\phi(0)$ by o_1. Dividing by $n\mathrm{vol}\,(T_\phi)$ and passing to the limit we get

$$1-\xi\kappa L(\phi) \leq \liminf \frac{\mathrm{vol}\,(Q(o_1,\phi^n o_1))}{n\mathrm{vol}\,(T_\phi)} \leq \limsup \frac{\mathrm{vol}\,(Q(o_1,\phi^n o_1))}{n\mathrm{vol}\,(T_\phi)} \leq 1+\xi\kappa L(\phi).$$

As ξ is arbitrary, the claim for o_1 follows. For a general o, it suffices to notice that, by Proposition 2.2, the difference $|\mathrm{vol}\,(Q(o,\phi^n o)) - \mathrm{vol}\,(Q(o_1,\phi^n o_1))|$ is uniformly bounded by $\kappa(d_{\mathcal{T}}(o,o_1) + d_{\mathcal{T}}(\phi^n o, \phi^n o_1)) + \kappa = 2\kappa d_{\mathcal{T}}(o,o_1) + \kappa$. □

We remark that the results mentioned above [7], [19] prove something stronger, that is $|2n\mathrm{vol}\,(T_\phi) - \mathrm{vol}\,(Q(\phi^{-n}o, \phi^n o))| = O(1)$.

4. Random Walks

We start talking about random walks on the mapping class group. We set up terminology, notations and first observations. The goal of the section is to introduce the third and last major technical tool of the paper which is a recurrence property (Proposition 4.3).

4.1. **Random walks on the mapping class group.**

We will work in the following generalities:

> **Standing assumption**. Let $S \subset \mathrm{Mod}\,(\Sigma)$ be a finite symmetric set $S = S^{-1}$ generating the group $G = \langle S \rangle$. Let μ be a probability measure whose support equals S. We only consider random walks driven by probability measures arising this way with $G = \mathrm{Mod}(\Sigma)$.

Let us start with the most basic definition:

Definition (Random Walk). A *random walk on G driven by μ* is given by the following data: Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of random variables with values in S which are independent and have the same distribution μ. The *n-th step of the random walk* is the random variable $\omega_n := s_1 \dots s_n$. The random walk is the process $\omega := (\omega_n)_{n\in\mathbb{N}}$.

> **Notation**. We will always denote by $s = (s_n)_{n\in\mathbb{N}}$ the sequence of labels and by $\omega = (\omega_n := s_1 \dots s_n)_{n\in\mathbb{N}}$ the path traced by the sequence of labels.

The distribution of the n-th step of the random walk coincides with the n-th fold convolution $\mathbb{P}_n$ of μ with itself. It is given inductively by:

$$\mathbb{P}_n\left[E\right] := \sum_{s\in S} \mu(s)\mathbb{P}_{n-1}\left[s^{-1}E\right].$$

Let $\mathcal{P}$ be a property of mapping classes $f \in \mathrm{Mod}\,(\Sigma)$. We call it *typical* if it is very likely that a random mapping class has it, that is

$$\mathbb{P}_n\left[f \in \mathrm{Mod}\,(\Sigma) \mid f \text{ has } \mathcal{P}\right] \overset{n\to\infty}{\longrightarrow} 1.$$

The starting point of our discussion are two results by Maher [21], [22] that ensure that the property "X_f is hyperbolic" is typical and hence it makes sense to consider the hyperbolic volume of X_f.

Definition (Sample Paths). The *space of sample paths* is the measurable space $(\Omega, \mathcal{E})$ where $\Omega := G^{\mathbb{N}}$ and $\mathcal{E}$ is the σ-algebra generated by the *cylinder sets.* Given a probability measure μ on G, we get a *probability measure* $\mathbb{P}$ on Ω induced by the random walk driven by μ. It is the push-forward $\mathbb{P} := T_*\mu^{\mathbb{N}}$ of the product measure $\mu^{\mathbb{N}}$ under the following measurable transformation:

$$T : \Omega \to \Omega \quad \text{defined by} \quad T(s) = \omega.$$

Definition (Shift Operator). On the space of sample paths Ω there is a natural *shift operator* $\sigma : \Omega \to \Omega$ defined by

$$\left(\sigma\left(s_i\right)_{i\in\mathbb{N}}\right)_j = s_{j+1}.$$

If $\omega = T(s) = (\omega_n = s_1 \dots s_n)_{n\in\mathbb{N}} \in \Omega$ is the path traced by a random walk, then we can write $(\sigma^i\omega)_j = \omega_i^{-1}\omega_{i+j}$. It is a standard computation on cylinder sets to check that σ preserves $\mu^{\mathbb{N}}$ and that $(\Omega, \mu^{\mathbb{N}}, \sigma)$ is mixing and hence *ergodic*.

4.2. **Linear drift and sublinear tracking.** Consider the action on Teichmüller space $G \curvearrowright \mathcal{T}$ and fix a basepoint $o \in \mathcal{T}$. Every random walk $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ traces an orbit $\{\omega_n o\}_{n\in\mathbb{N}} \subset \mathcal{T}$.

It follows from the triangle inequality that the random variables $d_{\mathcal{T}}(o, \omega_n o)$ are subadditive with respect to the shift map σ. By Kingman's subadditive ergodic theorem and ergodicity of $(\Omega, \mathbb{P}, \sigma)$, there exists a constant $L_{\mathcal{T}} \geq 0$, called the *drift* of the random walk on Teichmüller space, such that for $\mathbb{P}$-almost every sample path $\omega \in \Omega$ we have

$$\frac{d_{\mathcal{T}}\left(o, \omega_n o\right)}{n} \overset{n\to\infty}{\longrightarrow} L_{\mathcal{T}}.$$

It is natural to ask whether the orbit $\{\omega_n o\}_{n\in\mathbb{N}}$ converges to some point on the Thurston compactification of Teichmüller space $\mathcal{PML}$. This property was first established by Kaimanovich-Masur [18].

Theorem 4.1 (Kaimanovich-Masur [18]). *We have $L_{\mathcal{T}} > 0$. For $\mathbb{P}$-almost every sample path $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and for every basepoint $o \in \mathcal{T}$, the sequence $\{\omega_n o\}_{n\in\mathbb{N}}$ converges to a point $\mathrm{bnd}(\omega) \in \mathcal{PML}$ which is independent of $o \in \mathcal{T}$. The map $\mathrm{bnd} : \Omega \to \mathcal{PML}$ is Borel measurable. Moreover, $\mathbb{P}$-almost surely, the point $\mathrm{bnd}(\omega)$ is uniquely ergodic, minimal and filling.*

Moreover, Tiozzo [36] showed that the orbit $\{\omega_n o\}_{n\in\mathbb{N}}$ can also be *tracked* by a Teichmüller ray in the following sense:

Theorem 4.2 (Tiozzo [36]). *For $\mathbb{P}$-almost every sample path $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and for every basepoint $o \in \mathcal{T}$, there exists a unit speed Teichmüller ray $\tau : [0, +\infty)$ starting at $\tau(0) = o$ and ending at $\tau(\infty) = \mathrm{bnd}(\omega)$ such that*

$$\lim_{n\to\infty} \frac{d_{\mathcal{T}}\left(\omega_n o, \tau(L_{\mathcal{T}} n)\right)}{n} = 0.$$

4.3. **Recurrence.** Now we can present our last fundamental ingredient which is the following recurrence property:

Theorem 4.3 (Baik-Gekhtman-Hamenstädt, Propositions 6.9 and 6.11 of [1]). *Let $o \in \mathcal{T}$ be a basepoint and τ_ω the tracking ray for ω. Then:*

- Recurrence*: For every $\eta > 0$ sufficiently small, for every $0 < a < b$ and $h > 0$, for $\mathbb{P}$-almost every ω with tracking ray τ_ω there exists $N = N(\omega) > 0$ such that for every $n \geq N$ the segment $\tau_\omega[an, bn]$ has a connected subsegment of length h entirely contained in $\mathcal{T}_\eta$.*

- Fellow-Traveling*: There exists $\delta > 0$ such that for every $\epsilon > 0$ and for $\mathbb{P}$-almost every sample path ω there exists $N = N(\omega) > 0$ such that for every $n \geq N$, the element ω_n is pseudo-Anosov with translation length $L(\omega_n) \in [(1-\epsilon)L_{\mathcal{T}}n, (1+\epsilon)L_{\mathcal{T}}n]$. Its axis l_n δ-fellow-travels the tracking ray τ_ω on $[\epsilon L_{\mathcal{T}}n, (1-\epsilon)L_{\mathcal{T}}n]$, i.e. for every $t \in [\epsilon L_{\mathcal{T}}n, (1-\epsilon)L_{\mathcal{T}}n]$ we have $d_{\mathcal{T}}(\tau_\omega(t), l_n) < \delta$.*

For the convergence $L(\omega_n)/n \to L_{\mathcal{T}}$ see also Dahmani-Horbez [11].

4.4. **A larger class of random walks.** As stated at the beginning of the section, in this paper we only work with probability measures μ with finite support S that generates the full mapping class group $G = \mathrm{Mod}(\Sigma)$. This allows us to keep the statements uniform and to avoid distinguishing between different families of random 3-manifolds.

However, at the price of making a distinction between mapping tori, quasi-fuchsian manifolds and Heegaard splittings, the assumptions on μ can be considerably relaxed and still obtain the convergence results in Theorems 1 and 2. We briefly describe, without details, two larger classes of random walks to which our results can be extended.

For mapping tori and quasi-fuchsian manifolds it is enough that S, the finite support of μ, generates a subgroup G containing two pseudo-Anosov elements that act as independent loxodromics on the *curve graph* (see [24] for the definitions). All the theorems in this section hold in these generalities.

For Heegaard splittings, we further require that the two pseudo-Anosov elements also act as independent loxodromics on the *handlebody graph* (see [23] for a definition). Crucially, the condition implies, by work of Maher-Schleimer [23] and Maher-Tiozzo [24], that random walks on G have a *positive drift* on the handlebody graph. This ensures that a random Heegaard splitting is hyperbolic and plays a role also in the construction of the model metric from [16] used in the next section.

With these caveats, the proofs can be extended by following word-by-word the same lines, no change is needed.

5. A Law of Large Numbers for the Volume

We are ready to prove the law of large numbers for the volumes of random 3-manifolds.

Theorem 1. *$\mathbb{P}$-almost surely the limit following limit exists*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(X_{\omega_n})}{n} = v.$$

The family of 3-manifold $\{X_{\omega_n}\}_{n\in\mathbb{N}}$ can denote either the mapping tori or the Heegaard splittings defined by ω_n.

We will deduce it from the following analogue concerning quasi-fuchsian manifolds. The idea is that, according to the geometric models, the volume of a random 3-manifold is always captured by a quasi-fucshian manifold.

Theorem 2. *For every $o \in \mathcal{T}$ and for $\mathbb{P}$-almost every $\omega \in \Omega$ the following limit exists:*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(Q(o,\omega_n o))}{n} = v.$$

Let us remark again that $v = v(\mu) > 0$ is the same as in Theorem 1.

5.1. **Mapping tori and Heegaard splittings.** Let us assume Theorem 2 and prove the result for random 3-manifolds:

Proof of Theorem 1. Fix $\epsilon > 0$. Let $\tau_\omega : [0,\infty) \to \mathcal{T}$ be the ray connecting o to $\mathrm{bnd}(\omega)$.

Mapping tori. We use the model for T_{ω_n} coming from Corollary 3.12 (see also Figure 2): By Proposition 4.3, if n is large enough, we can find on τ_ω four points $x_n < y_n < z_n < w_n < x_n + L(\omega_n)$ such that the intervals $[x_n, y_n]$ and $[z_n, w_n]$ satisfy the hypotheses of Corollary 3.12: They are contained in $[\epsilon L_{\mathcal{T}} n, 2\epsilon L_{\mathcal{T}} n]$ and $[(1-2\epsilon)L_{\mathcal{T}} n, (1-\epsilon)L_{\mathcal{T}} n]$ respectively. Their length is at least h and their image is η-thick. The restriction of τ_ω to $[x_n, w_n]$ δ-fellow travels the Teichmüller axis $l_n : \mathbb{R} \to \mathcal{T}$ of ω_n whose translation length is roughly $(1-\epsilon)L_{\mathcal{T}} n \le L(\omega_n) \le (1+\epsilon)L_{\mathcal{T}} n$. Applying Corollary 3.12 we get:

Lemma 5.1. *For $\mathbb{P}$-almost every ω and every large enough $n \ge n_\omega$ we have*

$$|\mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n))) - \mathrm{vol}\,(T_{\omega_n})| \le \epsilon n.$$

and

$$|\mathrm{vol}\,(Q_{\omega_n}) - \mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \le \epsilon n.$$

Proof of Lemma 5.1. By Corollary 3.12 we have

$$\begin{aligned}
&|\mathrm{vol}\,(T_{\omega_n}) - \mathrm{vol}\,(Q(l_n(x_n), l_n(w_n)))| \\
&\le \kappa(L(\omega_n) + y_n - z_n) + \xi\kappa(w_n - x_n) + \mathrm{const} \\
&\le \kappa 4\epsilon L_{\mathcal{T}} n + \xi\kappa(1-2\epsilon)L_{\mathcal{T}} n + \mathrm{const}.
\end{aligned}$$

Up to a uniform additive constant we can also replace $Q(l_n(x_n), l_n(w_n))$ with $Q(\tau_\omega(x_n), \tau_\omega(w_n))$. If n is large enough we can improve the last quantity to ϵn. Instead, from Proposition 2.2

$$\begin{aligned}
&|\mathrm{vol}\,(Q(o,\omega_n o)) - \mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \\
&\le \kappa(d_{\mathcal{T}}(o, \tau_\omega(x_n)) + d_{\mathcal{T}}(\tau_\omega(w_n), \omega_n o)) + \kappa \\
&\le \kappa(d_{\mathcal{T}}(o, \tau_\omega(x_n)) + d_{\mathcal{T}}(\tau_\omega(w_n), \tau_\omega(L_{\mathcal{T}} n)) + d_{\mathcal{T}}(\tau_\omega(L_{\mathcal{T}} n), \omega_n o)) + \kappa.
\end{aligned}$$

By our choice of x_n, w_n and Tiozzo's sublinear tracking (Theorem 4.2), if n is large enough, we can bound the last quantity by ϵn. $\square$

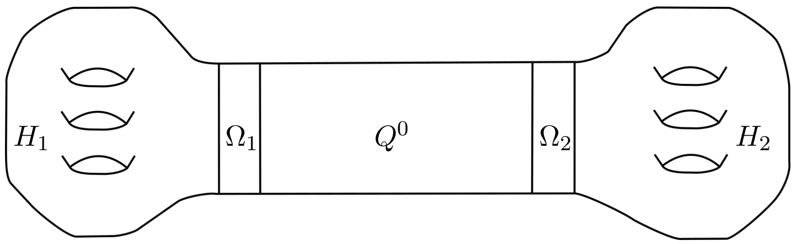

FIGURE 3. Model for a random Heegaard splitting.

Lemma 5.1 and Theorem 2 imply that $|\mathrm{vol}(T_{\omega_n}) - nv| = o(n)$ which concludes the proof for mapping tori.

Heegaard splittings. The argument is completely analogous to the previous one, but the model is different. We use the one constructed in [16], in particular Proposition 7.1. For convenience of the reader we give a brief description of it: Recall that $\epsilon > 0$ is fixed. A random Heegaard splitting M_{ω_n} admits a negatively curved Riemannian metric ρ with the following properties (see Figure 3): It is purely hyperbolic outside two regions $\Omega := \Omega_1 \sqcup \Omega_2$ which have uniformly bounded diameter and where the sectional curvatures lie in the interval $(-1-\epsilon, -1+\epsilon)$. The complement $M_{\omega_n} - \Omega$ decomposes into three connected pieces $H_1 \sqcup Q^0 \sqcup H_2$. The pieces H_1, H_2 are homeomorphic to handlebodies and have small volume $\mathrm{vol}(H_1 \sqcup H_2 \sqcup \Omega) \leq \epsilon n$. The middle piece Q^0 embeds isometrically in the convex core of $Q(o, \omega_n o)$, moreover $\mathrm{vol}(Q(o, \omega_n o)) - \mathrm{vol}(Q^0) \leq \epsilon n$. Hence we can apply again Theorem 3.10 and Theorem 2. □

We now proceed with the proof of Theorem 2.

5.2. **Strategy overview.** Denote by Q_ϕ the manifold $Q(o, \phi o)$.

We want to show that for $\mathbb{P}$-almost every ω the sequence $\mathrm{vol}(Q_{\omega_n})/n$ converges. Suppose this is not the case. Then there exists a set Ω_{bad} with positive measure $\mathbb{P}[\Omega_{\mathrm{bad}}] > 0$ on which

$$\limsup_{n\to\infty} \frac{\mathrm{vol}(Q_{\omega_n})}{n} - \liminf_{n\to\infty} \frac{\mathrm{vol}(Q_{\omega_n})}{n} > 0.$$

We can as well assume that there is a small $\epsilon_0 > 0$ and a set $\Omega_{\mathrm{bad}}^{\epsilon_0}$ with positive measure $\zeta_0 := \mathbb{P}\left[\Omega_{\mathrm{bad}}^{\epsilon_0}\right] > 0$ on which the difference is at least $\epsilon_0 > 0$. Hence, in order to get a contradiction, it is enough to prove that for every $\epsilon, \zeta > 0$ there exists a set $\Omega_{\epsilon,\zeta}$ with measure $\mathbb{P}[\Omega_{\epsilon,\zeta}] \geq 1 - \zeta$ on which the difference between limsup and liminf is smaller than ϵ.

First we observe that we can exploit a *neighbour approximation property* of the volumes (Lemma 5.3). It allows a convenient technical reduction: We can make the random walk *faster* and still keep under control the asymptotic behaviour (Lemma 5.4). The faster we make the random walk the more properties we can prescribe, a feature that will be important in Proposition 5.5. The central step of the proof consists of finding a set on which the variables $\mathrm{vol}(Q_{\omega_{nN}})$ and the *ergodic sum* $\sum_{j<n}\mathrm{vol}(Q_{\sigma^{jN}(\omega)_N})$ are comparable (Proposition 5.5). Finally, we use the *ergodic theorem* to conclude the proof.

5.3. **A faster random walk.** For every $N\in\mathbb{N}$ we can replace the random walk ω with $(\omega_{jN})_{j\in\mathbb{N}}$ and the shift map σ with σ^N. The dynamical system $(\Omega,\mu^{\mathbb{N}},\sigma^N)$ is still ergodic. As we wish to apply the ergodic theorem, we discuss the integrability condition of the volume function and the relations between the asymptotics of the faster random walk and the original one. Recall that S, the support of μ, is symmetric and generates $G=\mathrm{Mod}(\Sigma)$.

Lemma 5.2. *There exists $C>0$ such that for every $\phi\in G$ we have $\mathrm{vol}\,(Q_\phi)\leq C\,|\phi|_S+C$ where $|\phi|_S$ is the word length in the generating set S.*

Proof. Let $\phi=s_1\dots s_n$ with $s_i\in S$. By Proposition 2.2 we have $\mathrm{vol}\,(Q_\phi)\leq \kappa d_{\mathcal{T}}(o,\phi o)+\kappa$. By the triangle inequality $d_{\mathcal{T}}(o,s_1\dots s_n o)\leq\sum_{j<n}d_{\mathcal{T}}(o,s_j o)$ $\leq \max_{s\in S}\{d_{\mathcal{T}}(o,so)\}\,n$. □

In particular, for any fixed $n\in\mathbb{N}$, the function $\mathrm{vol}\,(Q_{\omega_n})$ is integrable on $(\Omega,\mathcal{E},\mathbb{P})$ and we can apply the Birkhoff ergodic theorem. Moreover, we have the following neighbour approximation property.

Lemma 5.3. *For $\mathbb{P}$-almost every sample path $\omega\in\Omega$, for every n,m we have*

$$\left|\mathrm{vol}\left(Q_{\omega_{n+m}}\right)-\mathrm{vol}\left(Q_{\omega_n}\right)\right|\leq Cm+C.$$

Proof. By Proposition 2.2 $\left|\mathrm{vol}\left(Q_{\omega_{n+m}}\right)-\mathrm{vol}\left(Q_{\omega_n}\right)\right|\leq\kappa d_{\mathcal{T}}(\omega_n o,\omega_{n+m}o)+\kappa$. From the triangle inequality $d_{\mathcal{T}}(\omega_n o,\omega_{n+m}o)\leq C\left|\omega_n^{-1}\omega_{n+m}\right|_S\leq Cm$. □

The next completely elementary lemma illustrates why the neighbour approximation property allows to speed up the random walk without loosing control on the asymptotic behaviour.

Lemma 5.4. *Consider a sequence $\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ and an integer $N\in\mathbb{N}$. Suppose that the sequence satisfies $|a_{n+m}-a_n|\leq Cm+C$ for every n,m. Assume that $A:=\limsup_{j\to\infty}\frac{a_{jN}}{jN}$ and $a:=\liminf_{j\to\infty}\frac{a_{jN}}{jN}$ are finite. Then*

$$a\leq\liminf_{n\to\infty}\frac{a_n}{n}\leq\limsup_{n\to\infty}\frac{a_n}{n}\leq A.$$

5.4. **Comparison with ergodic sums.** The following is our main estimate

Proposition 5.5. *Fix $\epsilon,\zeta>0$. There exists $N(\epsilon,\zeta)>0$ and a set $\Omega_{\epsilon,\zeta,N}$ with $\mathbb{P}\left[\Omega_{\epsilon,\zeta,N}\right]\geq 1-\zeta$ such that for every $\omega\in\Omega_{\epsilon,\zeta,N}$ and $n\in\mathbb{N}$ large enough we have*

$$\left|\mathrm{vol}\left(Q_{\omega_{nN}}\right)-\sum_{0\leq j<n}\mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)\right|\leq\mathrm{const}\cdot\epsilon nN$$

for some uniform const > 0.

We will show that, for a suitably chosen N, both families $\{Q_{\omega_{nN}}\}$ and $\{Q_{(\sigma^{jN}\omega)_N}\}_{j<n}$ can be *refined* to construct *models*, via Proposition 3.9, for the hyperbolic mapping torus $T_{\omega_{nN}}$. The central property of the models is that they *nearly compute the volume* $\mathrm{vol}\,(T_{\omega_{nN}})$. This suffices to conclude.

Proof. Let $\delta > 0$ be the fellow traveling constant of Proposition 4.3 and h a large height. Since the value of $L_{\mathcal{T}} > 0$ is irrelevant and only complicates some formulas below by affecting the value of some constants, we are going to assume $L_{\mathcal{T}} = 1$. In the course of the proof, specifically in the inequalities (1)-(13), we will get several uniform constants which depend on previous steps and whose explicit expressions are irrelevant for the argument. In order to simplify the exposition we will always denote these different constants by const > 0.

For every N denote by $\Omega_{\epsilon,N}$ the set of paths satisfying the following properties

(1) ω_n is pseudo-Anosov and $L(\omega_n)/n \in (1-\epsilon, 1+\epsilon)$ for every $n \geq N$.
(2) l_n, the axis of ω_n, δ-fellow travels $\tau_\omega[\epsilon n, (1-\epsilon)n]$ for every $n \geq N$.
(3) $\omega_n \tau_\omega[\epsilon n, \infty]$ δ-fellow travels $\tau_\omega[(1+\epsilon)n, \infty]$ for every $n \geq N$.
(4) $\tau_\omega[\epsilon n, 2\epsilon n]$ and $\tau_\omega[(1 \pm \epsilon)n, (1 \pm 2\epsilon)n]$ contain η-thick subsegments of length at least h for every $n \geq N$.
(5) The conclusions of Lemma 5.1 hold for every $n \geq N$.
(6) $d_{\mathcal{T}}(o, \omega_n o)/n \in (1-\epsilon, 1+\epsilon)$ for all $n \geq N$.

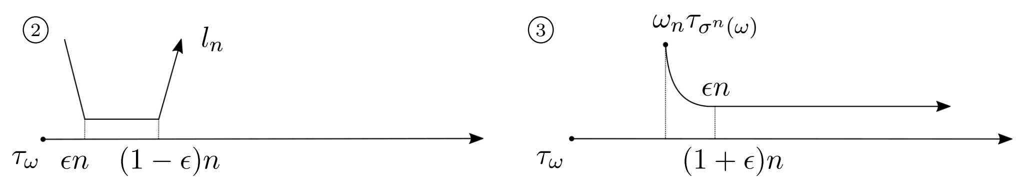

FIGURE 4. Properties 2 and 3.

Observe that if $N_1 \geq N_2$ then $\Omega_{\epsilon,N_2} \subseteq \Omega_{\epsilon,N_1}$, if we enlarge N the set can only get bigger. We reserve ourselves the right to determine later suitably modified constants δ, h, N. Since all the properties are satisfied asymptotically with probability one, for fixed $\epsilon, \zeta > 0$ there exists some $N(\epsilon, \zeta, h)$ such that $\Omega_{\epsilon,N}$ has measure at least $1-\zeta$. Fix N larger than this threshold and speed up the random walk, that is replace ω with $(\omega_{jN})_{j\in\mathbb{N}}$ and σ with σ^N.

By ergodicity of $(\Omega, \mu^{\mathbb{N}}, \sigma^N)$, the orbits $\{\sigma^{jN}\omega\}_{j\in\mathbb{N}}$ will visit the set $\Omega_{\epsilon,N}$ very often, the number of hitting times being proportional to the measure of the set $\geq 1-\zeta$. We record the hitting times by subdividing the interval $[n] = \{0, \ldots, n\}$ into a disjoint union of consecutive intervals $[n] = I_1 \sqcup J_1 \sqcup \cdots \sqcup I_k \sqcup J_k$ where the I_i's contain the indices j for which $\sigma^{jN}\omega \in \Omega_{\epsilon,N}$, whereas the J_i's are the bad indices (J_k might be empty). By the ergodic theorem the total length of the bad intervals is controlled by

$$\frac{1}{n}\sum_{j<n}\mathbb{1}_{\Omega\setminus\Omega_{\epsilon,N}}(\sigma^{jN}\omega)=\frac{1}{n}\sum_{i\leq k}|J_i|\overset{n\to\infty}{\longrightarrow}\mathbb{P}\left[\Omega\setminus\Omega_{\epsilon,N}\right]\leq\zeta.$$

Basic case. We start by proving the proposition assuming that all indices are good. Since our considerations are all independent of the past, we will also get a "local version" of the proposition for every good interval I_j.

We are going to define two families of quasi-fuchsian manifolds that satisfy the hypotheses of Proposition 3.9 and can be glued to form a model for $T_{\omega_{nN}}$ that nearly computes its volume. The two families consist of:

I Quasi-fuchsian manifolds related to $Q_{\sigma^{jN}(\omega)_N}$ for every $j\in[n]$.
II A single quasi-fuchsian manifold related to $Q_{\omega_{nN}}$ as in Lemma 5.1.

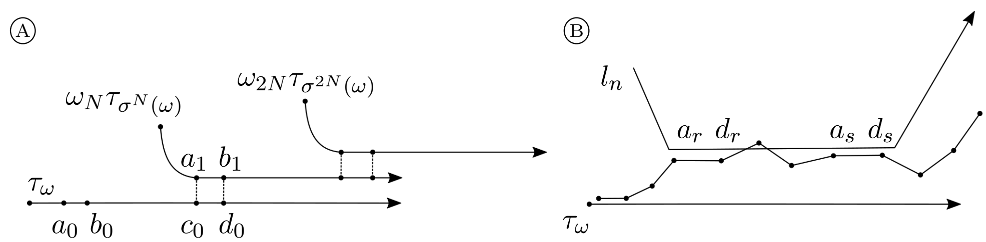

Figure 5. Basic case.

Family I. Proceed inductively. Begin with $i=0$ and the two Teichmüller rays τ_ω and $\omega_N\tau_{\sigma^N(\omega)}$. The restrictions $\omega_N\tau_{\sigma^N(\omega)}|_{[\epsilon N,\infty)}$ and $\tau_\omega|_{[(1+\epsilon)N,\infty)}$ are δ-fellow travelers. The ray τ_ω contains four points $a_0<b_0<c_0<d_0$ such that $[a_0,b_0]\subset[\epsilon N,2\epsilon N]$ and $[c_0,d_0]\subset[(1+\epsilon)N,(1+2\epsilon)N]$, their image is η-thick and their length is at least h (see Figure 5 A). The segment $[c_0,d_0]$ determines $[a_1,b_1]$ by the condition that $\omega_N\tau_{\sigma^N(\omega)}[a_1,b_1]$ δ-fellow travels $\tau_\omega[a_0,b_0]$ and $[a_1,b_1]\subset[\epsilon N,2\epsilon N]$. As $1\in[n]$ is good, we can go on and find $[c_1,d_1]\subset[(1+\epsilon)N,(1+2\epsilon)N]$ of length at least h and with $\tau_{\sigma^N(\omega)}$-image in $\mathcal{T}_\eta$. Inductively we determine $a_i<b_i<c_i<d_i$ for every $i\leq n$. Before going on, let us simplify a little the notation by introducing

$$\begin{aligned}A_i&=\omega_{iN}\tau_{\sigma^{iN}(\omega)}(a_i), & B_i&=\omega_{iN}\tau_{\sigma^{iN}(\omega)}(b_i),\\ C_i&=\omega_{iN}\tau_{\sigma^{iN}(\omega)}(c_i), & D_i&=\omega_{iN}\tau_{\sigma^{iN}(\omega)}(d_i).\end{aligned}$$

We associate to the index $i\leq n$ the quasi-fuchsian manifold $Q(A_i,D_i)$. Informally, we renormalized the picture by placing ourselves at the iN-th point of the orbit $O_i=\omega_{iN}o$. From there we see the segment $[A_i,D_i]$ that δ-fellow travels $[O_i,\mathrm{bnd}(\omega)]$. Observe that, by Proposition 2.2, we have

$$\begin{aligned}&\left|\mathrm{vol}\left(Q(A_i,D_i)\right)-\mathrm{vol}\left(Q_{\sigma^{iN}(\omega)_N}\right)\right| \\ &\leq\kappa(d_{\mathcal{T}}(O_i,A_i)+d_{\mathcal{T}}(D_i,O_{i+1}))+\kappa\leq\kappa4\epsilon N+\mathrm{const}.\end{aligned}\tag{1}$$

Sequences of consecutive good indices are geometrically controlled:

LEMMA 5.6. *The segment* $[A_i, D_i]$ *uniformly fellow travels* $[O, O_n]$.

Proof. Let $\mathcal{C}$ be the curve graph of Σ. Consider the shortest curve projection $\Upsilon : \mathcal{T} \to \mathcal{C}$. By Masur-Minsky [25] we have the following: The curve graph $\mathcal{C}$ is hyperbolic and the projection is uniformly coarsely Lipschitz and sends Teichmüller geodesics to *unparametrized* uniform quasi geodesics. In particular, by stability of quasi geodesics, $\Upsilon[A_i, D_i]$ is uniformly Hausdorff close to the geodesic segment $[\Upsilon(A_i), \Upsilon(C_i)]$. The same holds true for $\Upsilon[O, O_n]$ and $[\Upsilon(O), \Upsilon(O_n)]$.

Since the composition of Υ with a parametrized, η-thick and sufficiently long Teichmüller geodesic is a uniform *parametrized* quasi geodesic (see [15]), we also have the following: If the δ-fellow traveling h between $[C_{i-1}, D_{i-1}]$ and $[A_i, B_i]$ is sufficiently long, then the geodesics $[\Upsilon(A_{i-1}), \Upsilon(D_{i-1})]$ and $[\Upsilon(A_i), \Upsilon(D_i)]$ uniformly fellow travel along a segment, terminal for the first and initial for the second, which is as long as we wish.

In particular this implies that, if h is large enough, then the concatenation of the geodesic segments

$$[\Upsilon(O), \Upsilon(C_0)] \cup [\Upsilon(A_1), \Upsilon(C_1)] \cup \cdots \cup [\Upsilon(A_{n-1}), \Upsilon(C_{n-1})] \cup [\Upsilon(A_n), \Upsilon(O_n)]$$

is a uniform $(1, K)$ local quasi geodesic. By the stability of uniform local quasi geodesics in hyperbolic spaces, we conclude that every segment $[\Upsilon(A_i), \Upsilon(D_i)]$ lies uniformly Hausdorff close to $[\Upsilon(O), \Upsilon(O_n)]$.

In particular, there are points $P_i, Q_i \in [O, O_n]$ for which the projection is uniformly close to the projections of $[A_i, B_i]$ and $[C_i, D_i]$. As Teichmüller geodesics in the thick part are uniformly contracting (by [27] and [15]) it follows that P_i, Q_i are uniformly close to the thick subsegments of $[A_i, B_i]$, $[C_i, D_i]$. Therefore, by [31], $[P_i, Q_i]$ uniformly fellow travels $[A_i, D_i]$ provided that the height h is sufficiently large. □

Observe that, by property (2), the segment $[O, O_n]$ uniformly fellow-travels the axis l_n of the pseudo-Anosov ω_{nN} along the subsegment $[\epsilon Nn, (1-\epsilon)Nn]$. By Lemma 5.6, there is a subsegment $[r, s] \subset [n]$ of size $s - r \geq (1-\epsilon)n$, obtained by discarding an initial and a terminal subsegment of length proportional to ϵn, such that for all $r \leq i \leq s$ $[A_i, D_i]$ uniformly fellow travels l_n (see Figure 5 B). We add to the collection the quasi-fuchsian manifold $Q(C_s, \omega_{nN} B_r)$. Using Proposition 2.2 we see that

$$\text{vol}\,(Q(C_s, \omega_{nN} B_r)) \leq \kappa d_{\mathcal{T}}(C_s, \omega_{nN} B_r) + \kappa \leq \text{const} \cdot \epsilon n N. \tag{2}$$

In fact, on the one hand, the points B_r, C_s are, respectively, uniformly close to points $l_n(t_r), l_n(t_s)$ so their distance is roughly $t_s - t_r$ and $d_{\mathcal{T}}(C_s, \omega_{nN} B_r)$ can be bounded by $L(\omega_{nN}) - (t_s - t_r)$. On the other hand, combining property (6) and $s - r \geq (1-\epsilon)n$, their distance, up to an error of ϵN, is also given by $d_{\mathcal{T}}(O_r, O_s) \geq (1-\epsilon)(s-r)N$. By property (1) we have $L(\omega_{nN}) \leq (1+\epsilon)nN$ so $L(\omega_{nN}) - (t_s - t_r) \simeq (1+\epsilon)nN - (1-\epsilon)^2 nN$ whence inequality (2).

Moreover, by Lemma 5.2 and the fact that $|[n] \setminus [r,s]| \le \epsilon n$, we have

$$\sum_{j \notin [r,s]} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right) \le \sum_{j \notin [r,s]} CN + C \le \mathrm{const} \cdot \epsilon n N. \tag{3}$$

By construction, the family $\{Q(A_i, D_i)\}_{r \le i \le s} \sqcup \{Q(C_s, \omega_{nN} B_r)\}$ satisfies the gluing conditions of Proposition 3.9 provided that h is very large. As a result

$$\left|\mathrm{vol}\,(T_{\omega_{nN}}) - \sum_{i \in [r,s]} \mathrm{vol}\,(Q(A_i, D_i)) - \mathrm{vol}\,(Q(C_s, \omega_{nN} B_r))\right| \le nV_0 + \mathrm{const} \cdot \epsilon n N \tag{4}$$

where $V_0 = V_0(\eta, \xi, h, D_1)$ is as in Proposition 3.9.

Family II. By property (5) and Lemma 5.1, we can find on τ_ω a pair of points $x_n \in [\epsilon nN, 2\epsilon nN]$ and $w_n \in [(1-2\epsilon)nN, (1-\epsilon)nN]$ which define a quasi-fuchsian manifold whose volume approximate simultaneously the volume of the mapping torus $T_{\omega_{nN}}$ and the volume of the quasi-fuchsian manifold $Q_{\omega_{nN}}$

$$|\mathrm{vol}\,(T_{\omega_{nN}}) - \mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \le \mathrm{const} \cdot \epsilon n N \tag{5}$$

and

$$|\mathrm{vol}\,(Q_{\omega_{nN}}) - \mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n)))| \le \mathrm{const} \cdot \epsilon n N. \tag{6}$$

Notice that inequalities (5) and (6) hold also in the presence of bad intervals as we only used property (5). We will use them in the general case as well.

Putting together the previous estimates (1)-(5) we get

$$\left|\mathrm{vol}\,(Q(\tau_\omega(x_n), \tau_\omega(w_n))) - \sum_{j \in [n]} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)\right| \le \mathrm{const} \cdot \epsilon n N$$

Together with (6) this settles the basic case.

General case. We now allow the presence of bad intervals. First, let us observe that the argument of the basic case, being independent of the past, immediately implies that if $I = [i,t] \subset [n]$ is an interval consisting entirely of good indices then we can find along $\tau_{\sigma^{iN}(\omega)}$ a pair of points $\epsilon|I|N < x < 2\epsilon|I|N$ and $(1-2\epsilon)|I|N < w < (1-\epsilon)|I|N$ such that

$$\left|\mathrm{vol}\left(Q(\tau_{\sigma^{iN}(\omega)}(x), \tau_{\sigma^{iN}(\omega)}(w))\right) - \sum_{j \in I} \mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)\right| \le \mathrm{const} \cdot \epsilon |I| N. \tag{7}$$

Inequality (7) means, in words, that we can represent the ergodic sum over a good interval by a quasi-fuchsian manifold whose geodesic lies on the tracking ray of the interval. The idea of the general case is to proceed as in the basic case but with different building blocks.

The presence of bad intervals brings in some issues, whose nature is related to the way the the random walk deviates from the tracking ray, that we have to address. However, no new ingredients are needed, only a more careful choice of the interval subdivision.

The problem can be summarized as follows: Consider a good interval I_j and the adjacent bad interval J_j. Look at the deviation from the tracking ray of I_j introduced by J_j. It might happen that the quasi-fuchsian manifold associated to the good interval I_{j+1} is too small compared to the deviation and we are uncertain whether or not to include it in the gluing family. In order to get around the issue, we wait until the first time when the fellow traveling between the tracking rays of I_j and I_{j+1} is restored, discard all the good small intervals in between and replace the quasi-fuchsian manifold associated to I_j. So we start by refining the interval subdivision.

Refinement of the interval subdivision. Denote by $i_j < t_j$ the initial and the terminal indices in the j-th good interval $I_j = [i_j, t_j]$. We proceed inductively. Start with $I_1 = [i_1 = 0, t_1]$ and $J_1 = [t_1 + 1, i_2 - 1]$. Consider $I_2 = [i_2, t_2]$. We determine a new $i_3^{\rm new}$ by the following condition

$$i_3^{\rm new} := \min \{ i > t_2 + \epsilon(|I_1| + |J_1|) \text{ and } i \text{ is good} \} .$$

This requirement restores, by property (3), the fellow traveling between $\omega_{i_1 N}\tau_{\sigma^{i_1 N}(\omega)}$ and $\omega_{i_2 N}\tau_{\sigma^{i_2 N}(\omega)}$. That is $\omega_{i_1 N}\tau_{\sigma^{i_1 N}(\omega)}[(1+\epsilon)(|I_1|+|J_1|)N, \infty)$ and $\omega_{i_2 N}\tau_{\sigma^{i_2 N}(\omega)}[\epsilon(|I_1|+|J_1|)N, \infty)$ are δ-fellow travelers (property (3)). The index $i_3^{\rm new}$ lies in some good interval I_{j_3}. We make the following replacement

$$\begin{aligned} I_3 \longrightarrow & I_3^{\rm new} := [i_3^{\rm new}, t_{j_3}] \\ J_2 \longrightarrow & J_2^{\rm new} := [t_2 + 1, i_3^{\rm new} - 1] \\ & \quad = J_2^{\rm old} \sqcup I_3 \sqcup \cdots \sqcup J_{j_3 - 1} \sqcup [i_{j_3}, i_3^{\rm new} - 1]. \end{aligned}$$

By our choice, if $j_3 > 3$, then the sum of the lengths $|J_2^{\rm old}| + |I_3| + \cdots + |I_{j_3-1}|$ and $i_3^{\rm new} - i_{j_3}$ are controlled by $\epsilon(|I_1| + |J_1|)$. The length of $|J_{j_3-1}|$ can be, instead, arbitrarily long. Furthermore $|I_3^{\rm new}| \le |I_{j_3}|$. Observe that, for the new J_2 we have $|J_2^{\rm new}| = i_3^{\rm new} - t_2 \le \epsilon(|I_1|+|J_1|) + |J_{j_3-1}|$. We leave untouched all the intervals after I_{j_3}, but we shift back the remaining indices $j \to 3 + j - j_3$ for all $j > j_3$. We repeat the process and get inductively the new set of indices

$$i_r^{\rm new} := \min \left\{ i > t_{r-1}^{\rm new} + \epsilon(|I_{r-2}^{\rm new}| + |J_{r-2}^{\rm new}|) \text{ and } i \text{ is good} \right\}$$

and intervals

$$\begin{aligned} I_r \longrightarrow & I_r^{\rm new} := [i_r^{\rm new}, t_{j_r}] \\ J_{r-1} \longrightarrow & J_{r-1}^{\rm new} := [t_{r-1} + 1, i_r^{\rm new} - 1] \end{aligned}$$

that satisfy $|J_r^{\rm new}| \le \epsilon(|I_{r-2}^{\rm new}| + |J_{r-2}^{\rm new}|) + |J_{j_{r+1}-1}|$. We end up with a new subdivision $[n] = I_1^{\rm new} \sqcup J_1^{\rm new} \sqcup \cdots \sqcup I_{k'}^{\rm new} \sqcup J_{k'}^{\rm new}$ that still has the property

$$\sum_{t \le k'} |J_t^{\rm new}| \le \sum_{t \le k'} \epsilon(|I_{t-2}^{\rm new}| + |J_{t-2}^{\rm new}|) + |J_{j_{t+1}-1}^{\rm old}| \le \epsilon \sum_{t \le k'} |J_{t-2}^{\rm new}| + \epsilon n + \zeta n.$$

Hence $\sum_{t\leq k'}|J_t^{\rm new}|\leq(\epsilon n+\zeta n)/(1-\epsilon)\leq 4\epsilon n$ if $\zeta<\epsilon<1/2$. In particular the volumes corresponding to the new bad indices still add up to a small amount. In fact, by Lemma 5.2, we have

$$(8)\qquad \sum_{i\in\bigsqcup J_j^{\rm new}}\mathrm{vol}\left(Q_{\sigma^{iN}(\omega)_N}\right)\leq(CN+C)\sum_{i<k'}|J_i^{\rm new}|<\mathrm{const}\cdot\epsilon nN.$$

For the sake of simplicity, after the refinement, we return to the previous notation $i_j:=i_j^{\rm new}$, $t_j:=t_j^{\rm new}$ and $I_j:=I_j^{\rm new}$, $J_j:=J_j^{\rm new}$, but assume the new properties.

Family III. The proof can now proceed parallel to the basic case, so we only sketch the arguments. We define a family of quasi-fuchsian manifolds, one for every pair of adjacent intervals $I_j\sqcup J_j$, that can be glued to form a model for $T_{\omega_{nN}}$ that nearly computes its volume.

Proceed inductively. Start with $I_1\sqcup J_1=[0,t_1=|I_1|-1]\sqcup[t_1+1,i_2-1=|I_1|+|J_1|]$. Since τ_ω is a good ray, we can find segments $[a_1,b_1]\subset[\epsilon|I_1|N,2\epsilon|I_1|N]$ and $[c_1,d_1]\subset[(1+\epsilon)(|I_1|+|J_1|)N,(1+2\epsilon)(|I_1|+|J_1|)N]$ which are η-thick and have length at least h. Now consider $I_j\sqcup J_j$ for $j>1$. As in the basic case, we single out a pair of segments $[a_j,b_j]$, $[c_j,d_j]$ on the tracking ray of $\sigma^{i_jN}(\omega)$ normalized so that it starts at O_{i_j}. The first one, $[a_j,b_j]$, is determined by the condition that it is a δ-fellow traveler of $[c_{j-1},d_{j-1}]$ contained in $[\epsilon(|I_j|+|J_j|)N,2\epsilon(|I_j|+|J_j|)N]$ (see Figure 5 A). Here we are using in an essential way the properties of the refined interval and property (3) of good rays. The second one, $[c_j,d_j]$, is a η-thick h-long subsegment of $[(1+\epsilon)(|I_j|+|J_j|)N,(1+2\epsilon)(|I_j|+|J_j|)N]$. We simplify the notation by introducing

$$\begin{aligned} A_j&=\omega_{i_jN}\tau_{\sigma^{i_jN}(\omega)}(a_j), & B_j&=\omega_{i_jN}\tau_{\sigma^{i_jN}(\omega)}(b_j),\\ C_j&=\omega_{i_jN}\tau_{\sigma^{i_jN}(\omega)}(c_j), & D_i&=\omega_{i_jN}\tau_{\sigma^{i_jN}(\omega)}(d_j).\end{aligned}$$

We associate to $I_j\sqcup J_j$ the manifold $Q(A_j,D_j)$.

The analogue of Lemma 5.6 holds word by word if we replace the old segments with the new ones, that is $[A_i,D_i]$ uniformly fellow travels $[O,O_n]$.

By property (2), the latter uniformly fellow travels l_n, the axis of ω_{nN}, along $\tau_\omega[\epsilon nN,(1-\epsilon)nN]$. In particular we can find $0<r<s<n$ such that $[A_r,D_r]$ and $[A_s,D_s]$ are, respectively, the first and the last segments that fellow travel $\tau_\omega[\epsilon nN,(1-\epsilon)nN]$ along some subsegments, which is terminal for the first and initial for the second.

Up to discarding an initial (resp. terminal) segment of $[A_r,D_r]$ (resp. $[A_s,D_s]$) of length smaller than $\epsilon|A_rD_r|$ (resp. $\epsilon|A_sD_s|$) we can assume that $[A_r,D_r]$ (resp. $[A_s,D_s]$) uniformly fellow travels subsegments of $\tau_\omega[\epsilon nN,(1-\epsilon)nN]$ and l_n (as in Figure 5 B). The volumes of the associated quasi-fuchsian manifolds change at most by $\mathrm{const}\cdot\epsilon nN$ according to Proposition 2.2.

We can also assume, by recurrence, that $[A_r,D_r]$ (resp. $[A_s,D_s]$) contains an initial (resp. terminal) η-thick subsegment $[A_r,B_r]$ (resp. $[C_s,D_s]$) of length at least h. We add the quasi-fuchsian manifold $Q(C_s, \omega_{nN} B_r)$ to the family. As in the basic case we have

$$\mathrm{vol}\,(Q(C_s, \omega_{nN} B_r)) \leq \mathrm{const} \cdot \epsilon n N. \tag{9}$$

Applying Proposition 3.9 to the family $\{Q(A_j, D_j)\}_{j \in [r,s]} \sqcup \{Q(C_s, \omega_{nN} B_r)\}$ we can perform the cut and glue construction and get a manifold diffeomorphic to $T_{\omega_{nN}}$ with volume

$$\begin{aligned}\left| \mathrm{vol}\,(T_{\omega_{nN}}) - \sum_{i \in [r,s]} \mathrm{vol}\,(Q(A_i, D_i)) - \mathrm{vol}\,(Q(C_s, \omega_{nN} B_r)) \right| \\ \leq n V_0 + \mathrm{const} \cdot \epsilon n N.\end{aligned} \tag{10}$$

The fellow traveling property of $\bigsqcup_{i<r} [A_i, D_i]$ (resp. $\bigsqcup_{i>s} [A_i, D_i]$) with $\tau_\omega[0, 2\epsilon n N]$ (resp. $[\tau_\omega((1-\epsilon) n N), O_n]$) implies that $\sum_{i \notin [r,s]} d_{\mathcal{T}}(A_i, D_i) \leq 2\epsilon n N$ and, by Lemma 2.2,

$$\sum_{i \notin [r,s]} \mathrm{vol}\,(Q(A_i, D_i)) \leq \mathrm{const} \cdot \epsilon n N. \tag{11}$$

We compare now the volume of $Q(A_i, D_i)$ with the ergodic sum over the good interval I_i. Since the interval I_j is good, we find on $\tau_{\sigma^{i_j N}(\omega)}$ two points $\epsilon |I_j| N < x_j < 2\epsilon |I_j| N$ and $(1 - 2\epsilon)|I_j| N < w_j < (1-\epsilon)|I_j| N$ such that inequality (7) holds for $I = I_j$. Before going on, let us relax the notation, by introducing $X_j = \omega_{i_j N} \tau_{\sigma^{i_j N}(\omega)}(x_j)$ and $W_j = \omega_{i_j N} \tau_{\sigma^{i_j N}(\omega)}(w_j)$. We have

$$\left| \mathrm{vol}\,(Q(X_j, W_j)) - \sum_{i \in I_j} \mathrm{vol}\left(Q_{\sigma^{iN}(\omega)_N} \right) \right| \leq \mathrm{const} \cdot \epsilon |I_j|. \tag{12}$$

By Proposition 2.2, we have

$$|\mathrm{vol}\,(Q(A_j, D_j)) - \mathrm{vol}\,(Q(X_j, W_j))| \leq \kappa (d_{\mathcal{T}}(A_j, X_j) + d_{\mathcal{T}}(D_j, W_j)) + \kappa.$$

As $a_j, x_j \in [0, \epsilon(|I_j| + |J_j|) N]$ and $d_j, w_j \in [(1-\epsilon)|I_j| N, (1+2\epsilon)(|I_j| + |J_j|) N]$ we can continue the chain of inequalities with

$$\leq \mathrm{const} \cdot \epsilon |I_j| N + \mathrm{const} \cdot |J_j| N.$$

Adding all the contributions we get

$$\begin{aligned}\left| \sum_{j \leq k} \mathrm{vol}\,(Q(A_j, D_j)) - \sum_{j \leq k} \mathrm{vol}\,(Q(X_j, W_j)) \right| \\ \leq N \sum_{j \leq k} \mathrm{const} \cdot \epsilon |I_j| + \mathrm{const} \cdot |J_j| \leq \mathrm{const} \cdot \epsilon n N + \mathrm{const} \cdot \zeta n N.\end{aligned} \tag{13}$$

Putting together inequalities (10)-(13) and (5), (6) concludes the proof. $\square$

Theorem 2 is now reduced to an application of the ergodic theorem which says that for $\mathbb{P}$-almost every ω the following limit exists finite

$$\lim_{n\to\infty}\frac{1}{nN}\sum_{j<n}\mathrm{vol}\left(Q_{(\sigma^{jN}\omega)_N}\right)=v_N.$$

If N and $\Omega_{\epsilon,\zeta,N}$ are as in Proposition 5.5 then

$$\limsup_{j\to\infty}\frac{\mathrm{vol}\left(Q_{\omega_{jN}}\right)}{jN}-\liminf_{j\to\infty}\frac{\mathrm{vol}\left(Q_{\omega_{jN}}\right)}{jN}\leq\epsilon$$

on $\Omega_{\epsilon,\zeta,N}$ which has measure at least $1-\zeta$. Applying Lemma 5.4 we get

$$\limsup_{n\to\infty}\frac{\mathrm{vol}\left(Q_{\omega_n}\right)}{n}-\liminf_{n\to\infty}\frac{\mathrm{vol}\left(Q_{\omega_n}\right)}{n}\leq\epsilon.$$

This concludes the proof of Theorem 2.

6. Some questions

We conclude with four questions.

Question 6.1. *What about other geometric invariants (e.g. diameter, systole, Laplace spectrum)? That is, given a geometric invariant $G(\bullet)$ of hyperbolic 3-manifolds, is there a function $f_G:\mathbb{N}\to\mathbb{R}$ such that $G(X_{\omega_n})/f_G(n)$ approaches a positive constant for almost every ω? More specifically:*

- *Does $\frac{1}{n}\cdot\mathrm{diam}(X_{\omega_n})$ converge?*
- *Does $\log(n)^2\cdot\mathrm{systole}(T_{\omega_n})$ converge (see also* [34]*)?*
- *Does $n^2\cdot\lambda_1(X_{\omega_n})$ converge (see also* [1], [16]*)?*

The strategy pursued in this article can be applied to the study of the asymptotic for other geometric invariants. The control one needs consists essentially of two parts:

(i) A comparison theorem for the geometric invariant computed for the negatively curved models and the underlying hyperbolic metric.
(ii) An understanding of the behaviour of the function that computes the geometric invariant for quasi-fuchsian manifolds.

In the next question we consider a different notion of randomness: Observe that, up to conjugacy, there is only a finite number of mapping classes with translation length at most L. Hence, for every fixed L, it makes sense to sample at random and uniformly a conjugacy class ω_L of a mapping class with translation length at most L.

Question 6.2. *Does $\mathrm{vol}(T_{\omega_L})/L$ converge almost surely for $L\to\infty$?*

A companion question for quasi-fuchsian manifolds is the following:

Question 6.3. *For which Teichmüller rays $\tau:[0,\infty)\to\mathcal{T}$ does the mean value $\mathrm{vol}(Q(\tau(0),\tau(t)))/t$ converge for $t\to\infty$?*

For pseudo-Anosov axes l_ϕ the limit exists and is equal to $\mathrm{vol}\,(T_\phi)\,/L(\phi)$ [19], [7]. Theorem 2 implies that it exists for every point and almost every ray with respect to exit measures of random walks. What about the Lebesgue measure on $\mathcal{PML}$ which is singular with respect to the exit measures [14]?

The last question concerns the relation between hyperbolic volume and Teichmüller data: We know that $\mathrm{vol}(T_f)/d_{\mathrm{WP}}(f) \in [1/k(g), k(g)]$ (see [5], [6], [33]). If we consider random walks, both numerator and denominator have a linear asymptotic $\mathrm{vol}(T_{\omega_n})/n \to v > 0$ and $d_{\mathrm{WP}}(\omega_n)/n \to d > 0$.

QUESTION 6.4. *How does v/d distribute? Does the ratio v/d display an extremal behaviour?*

One can ask the same for the Teichmüller translation lengths.

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UNIFORM MODELS FOR RANDOM 3-MANIFOLDS

GABRIELE VIAGGI

ABSTRACT. We construct hyperbolic metrics on random Heegaard splittings and provide uniform bilipschitz models for them.

1. INTRODUCTION

Every closed orientable 3-manifold M can be presented as a *Heegaard splitting*[1]. This means that M is diffeomorphic to a 3-manifold M_f obtained by gluing together two handlebodies of the same genus H_g along an orientation preserving diffeomorphism f of their boundaries $\Sigma := \partial H_g$

$$M_f = H_g \cup_{f:\partial H_g \to \partial H_g} H_g.$$

The problem of finding hyperbolic structures on *most* 3-manifolds with a splitting of a fixed *genus* $g \geq 2$ was originally raised by Thurston (as Problem 24 in [30]) and made more precise by Dunfield and Thurston (see Conjecture 2.11 of [13]) via the introduction of the notion of *random Heegaard splittings*.

Such notion is based on the observation that the diffeomorphism type of M_f only depends on the isotopy class of the gluing map f, so it is well-defined for elements in the *mapping class group* $[f] \in \mathrm{Mod}(\Sigma)$. Therefore, Heegaard splittings of genus $g \geq 2$ are naturally parametrized by mapping classes $[f] \in \mathrm{Mod}(\Sigma)$.

A family $(M_n)_{n\in\mathbb{N}}$ of random Heegaard splittings of genus $g \geq 2$, or *random 3-manifolds*, is one of the form $M_n = M_{f_n}$ where $(f_n)_{n\in\mathbb{N}}$ is a random walk on the mapping class group $\mathrm{Mod}(\Sigma)$ driven by some initial probability measure μ with a finite support that generates $\mathrm{Mod}(\Sigma)$. If $(f_n)_{n\in\mathbb{N}}$ is such a random walk, we will denote by $\mathbb{P}_n$ the distribution of the n-th step f_n and by $\mathbb{P}$ the distribution of the path $(f_n)_{n\in\mathbb{N}}$.

Exploiting work of Hempel [14] and the solution of the geometrization conjecture by Perelman, Maher showed in [20] that a random Heegaard splitting M_f of genus $g \geq 2$ admits a hyperbolic metric, thus answering Dunfield and Thurston's conjecture.

Theorem (Maher [20])**.** *A random 3-manifold is hyperbolic.*

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[1]For the bibliography of this part of the thesis see page 132.

The main goal of this article is to provide a *constructive* and *effective* approach to the hyperbolization of random 3-manifolds.

Our first contribution is a constructive proof of Maher's result

Theorem 1. *There is a Ricci flow free hyperbolization for random 3-manifolds.*

By Ricci flow free hyperbolization we mean that we construct explicitly the hyperbolic metric only using tools from the deformation theory of Kleinian groups. We use the model manifold technology by Minsky [25] and Brock, Canary and Minsky [7], as well as the effective version of Thurston's Hyperbolic Dehn Surgery by Hodgson and Kerckhoff [16] and Brock and Bromberg's Drilling Theorem [5].

We remark that, even though we do not rely on Perelman's solution of the geometrization conjecture, we do use the main result from Maher [20], namely, the fact that the *Hempel distance* of the Heegaard splittings (see Hempel [14]) grows coarsely linearly along the random walk.

Our construction gives new and more refined information than the mere existence of a hyperbolic metric. In fact, we also provide a *model manifold* that captures, up to *uniform bilipschitz distortion*, the geometry of the random 3-manifold and allows the computation of its geometric invariants.

The notion of model manifold that we use is similar to the ones considered by Brock, Minsky, Namazi and Souto in [26], [27], [9] and is depicted in the following definition of *ϵ-model metric*: A Riemannian metric (M_f, ρ) is a ϵ-model metric for $\epsilon < 1/2$ if there is a decomposition into five pieces $M_f = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2$ satisfying the three requirements

(1) Topologically, H_1 and H_2 are homeomorphic to genus g handlebodies, while Q, Ω_1 and Ω_2 are homeomorphic to $\Sigma \times [0, 1]$.
(2) Geometrically, ρ has negative curvature $\sec \in (-1-\epsilon, -1+\epsilon)$, but outside the region $\Omega = \Omega_1 \cup \Omega_2$ the metric is purely hyperbolic.
(3) The piece Q is almost isometrically embeddable in a complete hyperbolic 3-manifold diffeomorphic to $\Sigma \times \mathbb{R}$.

The importance of the last requirement resides in the fact that we understand explicitly hyperbolic 3-manifolds diffeomorphic to $\Sigma \times \mathbb{R}$ thanks to the work of Minsky [25] and Brock, Canary and Minsky [7] which provides a detailed combinatorial description of their internal geometry.

The following is our more precise version of Theorem 1

Theorem 2. *For every $\epsilon > 0$ and $K > 1$ we have*

$$\mathbb{P}_n[M_f \text{ has a hyperbolic metric } K\text{-bilipschitz to a } \epsilon\text{-model metric}] \stackrel{n\to\infty}{\longrightarrow} 1.$$

We remark that ϵ-model metrics on random Heegaard splittings, similar to the ones that we build here, are constructed in [15]. There, the existence of a underlying hyperbolic metric is guaranteed by Maher's result and it is unclear whether the ϵ-model metrics are uniformly bilipschitz to it.

However, we should also mention that, using a result claimed by Tian [31], the mere fact that a metric ρ is a ϵ-model metric and that the regions Ω_1, Ω_2 where it is not hyperbolic have uniformly bounded diameter (as follows from [15]), implies, if $\epsilon > 0$ is sufficiently small, that ρ is uniformly close up to third derivatives to a hyperbolic metric. However, Tian's result is not published. In order to provide a uniform bilipschitz control we exploit, instead, ergodic properties of the random walk and drilling and filling theorems by Hodgson and Kerckhoff [16] and Brock and Bromberg [5].

Our methods follow closely [9] and [8] where uniform ϵ-model metrics are constructed for special classes of 3-manifolds.

The idea is the following: We can obtain a hyperbolic metric on M_f by a hyperbolic cone manifold deformation from a finite volume metric on a *drilled* manifold $\mathbb{M}$ which has the following form: Let $\Sigma \times [1,4]$ be a tubular neighbourhood of $\Sigma \subset M_f$. We consider 3-manifolds diffeomorphic to

$$\mathbb{M} = M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\})$$

where P_j is a pants decomposition of the surface $\Sigma \times \{j\}$. A finite volume hyperbolic metric on such a manifold can be constructed explicitly by gluing together the convex cores of two *maximally cusped handlebodies* H_1, H_2 and three *maximally cusped I-bundles* Ω_1, Q, Ω_2.

$$\mathbb{M} = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2.$$

Most of our work consists of finding suitable pants decompositions for which the Dehn surgery slopes needed to pass from $\mathbb{M}$ to M_f satisfy the assumptions of the effective Hyperbolic Dehn Surgery Theorem [16]. In order to find them we crucially need two major tools: The work of [15] on the geometry of hyperbolic handlebodies and ergodic properties of the random walks proved by Baik, Gekhtman and Hamenstädt [1].

We stress the fact that, for both Theorem 1 and Theorem 2, we assume that the support of μ is finite and generates the *entire* mapping class group.

We describe now some consequences of Theorem 2.

We start with a geometric application: We exploit the geometric control given by the ϵ-model metric to compute the coarse growth or decay rate of the geometric invariants along the family $(M_{f_n})_{n\in\mathbb{N}}$.

The general strategy is very simple: We use the model manifold technology [25], [7] and compute the geometric invariants for the middle piece Q. Then, we argue that the invariants of Q are uniformly comparable with those of M_f.

For example, combined with a result of Brock [4], Theorem 2 allows the computation of the coarse growth rate of the volume, which is well-known to be linear as explained in [20] (see also [15]). Combined with results of Baik, Gekhtman and Hamenstädt [1] it shows that the smallest positive eigenvalue of the Laplacian behaves like $1/n^2$ as computed in [15]. We notice that Theorem 2 allows a uniform approach to those result.

Here we do not carry out those computations because they are already well established. Instead, we have chosen to consider the diameter growth rate, which appears to be not available in the literature

Proposition 3. *There exists $c > 0$ such that*

$$\mathbb{P}_n[\mathrm{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The ingredients of the proof are Theorem 2 and a result by White [33].

In a completely different direction we use Theorem 2 to prove the following

Proposition 4. *For $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ the following holds*

(1) There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are arithmetic.
(2) There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are in the same commensurability class.

The proof combines a study of geometric limits of random 3-manifolds, Proposition 5.1, with arguments from Biringer and Souto [3].

Overview. In Section 2 we outline the construction of the ϵ-model metric. In Section 3 we develop the two main technical tools that we need and use them to build many examples to which the model metric construction applies. In Section 4 we prove Theorem 2 by showing that the examples of Section 3 are generic from the point of view of a random walk. Lastly, in Section 5 we prove Proposition 3 and Proposition 4.

Acknowledgements. This work builds upon the constructions of [15] and [13]. The possibility to adapt them to construct uniform models for random 3-manifolds has also been suggested by a conversation with Peter Feller.

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2. A gluing scheme

Here we outline a construction for the ϵ-model metric which follows closely ideas of Brock and Dunfield [8] and Brock, Minsky, Namazi and Souto [9]. At the end of the discussion we formulate a criterion of applicability.

2.1. **Assembling simple pieces.** The construction is somehow implicit in the description of a ϵ-model metric. It has two steps. We start with five building blocks H_1, H_2 and Q, Ω_1, Ω_2 which are the *convex cores* of *geometrically finite maximally cusped* complete hyperbolic structures on H_g and $\Sigma \times [1,2]$ respectively. The pieces Ω_1 and Ω_2 will play the role of the collars of the other structures as we are going to explain later on.

For convenience of the reader, we briefly describe the geometry of H_1, H_2 and Q, Ω_1, Ω_2. The convex core Q of a geometrically finite maximally cusped structure on $\Sigma \times [1,2]$ is diffeomorphic to the drilled product

$$Q \simeq \Sigma \times [1,2] - (P_1 \times \{1\} \cup P_2 \times \{2\})$$

where P_1, P_2 are pants decompositions of Σ such that no curve in P_1 is isotopic to a curve in P_2. The drilled product is endowed with a complete finite volume hyperbolic metric with totally geodesic boundary

$$\partial Q = \partial_1 Q \sqcup \partial_2 Q = (\Sigma \times \{1\} - P_1 \times \{1\}) \sqcup (\Sigma \times \{2\} - P_2 \times \{2\}).$$

and rank one cusps at $P_1 \cup P_2$. If we fix the isotopy class of the identification of Q with the drilled product, there exists a unique maximally cusped structure with cusp data $P_1 \cup P_2$. We denote it by $Q(P_1, P_2)$.

Analogously, the convex core H of a geometrically finite maximally cusped structure on H_g is diffeomorphic to the drilled handlebody

$$H \simeq H_g - P$$

where P is a pants decomposition of $\partial H_g = \Sigma$ (throughout this article we keep this identification fixed) with the property that every curve in P is not compressible and no two curves in P are isotopic within H_g. Again, H is endowed with a complete finite volume hyperbolic metric with totally geodesic boundary

$$\partial H \simeq \partial H_g - P$$

and rank one cusps at P. If we keep track of the isotopy class of the identification between H and the drilled handlebody, there exists a unique maximally cusped structure with cusp data P. We denote it by $H(P)$.

Each component of the boundaries $\partial Q, \partial H$ is a three punctured sphere. It inherits a complete finite area hyperbolic metric. Such a structure is unique up to isometries isotopic to the identity. Hence, once we decided a pairing of the components of ∂H_1, ∂H_2 with $\partial_1\Omega_1$, $\partial_2\Omega_2$ and of $\partial_2\Omega_1$, $\partial_1\Omega_2$ with $\partial_1 Q$, $\partial_2 Q$, there is no ambiguity in implementing it to an isometric diffeomorphism. Gluing the pieces together along such a diffeomorphism we get a 3-manifold

$$\mathbb{M} := H_1 \cup_{\partial H_1 \simeq \partial_1 \Omega_1} \Omega_1 \cup_{\partial_2 \Omega_1 \simeq \partial_1 Q} Q \cup_{\partial_2 Q \simeq \partial_1 \Omega_2} \Omega_2 \cup_{\partial_2 \Omega_2 \simeq \partial H_2} H_2$$

which is non-compact and has a naturally defined complete finite volume hyperbolic structure.

In our case the pairing is natural as our structures are of the form

$$\begin{aligned} H_1 &= H(P_1), \\ \Omega_1 &= Q(P_1, P_2), \\ Q &= Q(P_2, P_3), \\ \Omega_2 &= Q(P_3, P_4), \\ H_2 &= H(f^{-1}P_4). \end{aligned}$$

We think of Ω_1 and Ω_2 as the collar structures of the boundaries of the three larger pieces $\mathbb{N}_1 = H_1 \cup \Omega_1$, $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ and $\mathbb{N}_2 = \Omega_2 \cup H_2$.

Topologically, $\mathbb{M}$ is diffeomorphic to a drilled M_f, namely, let $\Sigma \times [1,4]$ denote a tubular neighbourhood of the Heegaard surface $\Sigma \subset M_f$, then

$$\mathbb{M} \simeq M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\}).$$

The pieces Ω_1, Q, Ω_2 are identified with

$$\begin{aligned} \Omega_1 &= \Sigma \times [1,2] - (P_1 \times \{1\} \sqcup P_2 \times \{2\}), \\ Q &= \Sigma \times [2,3] - (P_2 \times \{2\} \sqcup P_3 \times \{3\}), \\ \Omega_2 &= \Sigma \times [3,4] - (P_3 \times \{3\} \sqcup P_4 \times \{4\}). \end{aligned}$$

The curves in $P_1 \cup P_2 \cup P_3 \cup P_4$ represent the rank two cusps of $\mathbb{M}$.

In order to pass from $\mathbb{M}$ to the closed 3-manifold M_f we have to perform Dehn fillings on each cusp. This is the second step of the construction. The filling slopes are completely determined by the identification of $\mathbb{M}$ with the drilled M_f: They are the meridians γ of small tubular neigbourhoods of the curves in $\alpha \times \{j\} \subset P_j \times \{j\}$ inside $\Sigma \times [1,4]$.

Under such circumstances, the Hodgson and Kerckhoff effective version [16] of Thurston's Hyperbolic Dehn Surgery Theorem gives us sufficient conditions to guarantee that M_f has a hyperbolic metric obtained via a hyperbolic *cone manifold deformation* of the metric $\mathbb{M}$.

The condition is as follows: For every cusp of $\mathbb{M}$ we fix a torus horosection $\mathbb{T} \subset \mathbb{M}$ on the boundary of the η_M-thin part where $\eta_M > 0$ is some fixed Margulis constant. On each such horosection we have the slope $\gamma \subset \mathbb{T}$, determined by the gluing. We represent it as a simple closed geodesic for the intrinsic flat metric of $\mathbb{T}$. Hodgson and Kerckhoff deformation theory requires that the flat geodesic γ has sufficiently large *normalized length*, a quantity defined by

$$\mathrm{nl}(\gamma) := l(\gamma)/\sqrt{\mathrm{Area}(\mathbb{T})}.$$

We have

THEOREM 2.1 (Hodgson-Kerckhoff [16]). *Let $\mathbb{M}$ be a complete finite volume hyperbolic 3-manifold with n cusps. Let γ_j be flat geodesic slopes on torus horosections of the cusps. Suppose that the normalized length of each γ_j is at least $\mathrm{nl}_{HK} = 10.6273$. Then, there is a family $(M_t)_{t\in[0,2\pi]}$ of hyperbolic cone manifold structures on the Dehn filled manifold M whose singular loci are the core curves of the added tori and such that the cone angles of M_t equal t. The final hyperbolic cone manifold $M_{2\pi}$ is non singular. Moreover, the length of the core geodesic α_j is controlled by $l_{M_{2\pi}}(\alpha_j) \leq a/\mathrm{nl}(\gamma_j)^2$ for some universal constant $a > 0$.*

We want to guarantee that these conditions are fulfilled. This is where most of our work lies.

Once we know that $\mathbb{M}$ and M_f are connected by a family of hyperbolic cone manifolds, an application of Brock and Bromberg's Drilling Theorem [5] ensures that $\mathbb{M}$ is K-bilipschitz to M_f away from its cusps. The constant

K only depends on the length $l_{M_f}(\alpha_j)$ which, by Theorem 2.1, is again controlled by the normalized length $\mathrm{nl}(\gamma_j)^2$.

THEOREM 2.2 (Brock-Bromberg [5]). *Let $\eta_M > 0$ be a Margulis constant. For every $n > 0$ and $\xi > 0$ there exists $0 < \eta_B(\xi) < \eta_M$ such that the following holds: Let M be a geometrically finite hyperbolic 3-manifold. Let $\Gamma = \alpha_1 \sqcup \cdots \sqcup \alpha_n \subset M$ be a collection of simple closed geodesics of length $l_M(\alpha_j) < \eta_B(\xi)$ for all $j \le n$. Let N be the unique geometrically finite hyperbolic structure on $M - \Gamma$ with the same conformal boundary as M. Then, there exists a $(1+\xi)$-bilipschitz diffeomorphism*

$$\left(N - \bigsqcup_{j\le n} \mathbb{T}_{\eta_M}(\alpha_j), \bigsqcup_{j\le n} \partial\mathbb{T}_{\eta_M}(\alpha_j)\right) \longrightarrow \left(M - \bigsqcup_{j\le n} \mathbb{T}_{\eta_M}(\alpha_j), \bigsqcup_{j\le n} \partial\mathbb{T}_{\eta_M}(\alpha_j)\right)$$

where $\mathbb{T}_{\eta_M}(\alpha)$ denotes a standard η_M-Margulis neighbourhood for α.

2.2. **Two criteria for Dehn filling with long slopes.** Certifying that the filling slopes have large normalized length is the main point that we have to address. We now discuss two criteria to check this condition.

The argument branches in two cases: We consider separately the filling slopes in $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ and the ones in $\mathbb{N}_j = H_j \cup \Omega_j$. The two cases are similar in spirit, but the second one is technically more involved than the first one. However, the ideas are the same, so we will explain them with more details in the easier setting.

The I-bundle case. Consider first the hyperbolic manifold

$$\begin{aligned}\mathbb{Q} &= \Omega_1 \cup Q \cup \Omega_2 \\ &= Q(P_1, P_2) \cup Q(P_2, P_3) \cup Q(P_3, P_4).\end{aligned}$$

Topologically it is diffeomorphic to

$$\Sigma \times [1,4] - (P_1 \times \{1\} \sqcup P_2 \times \{2\} \sqcup P_3 \times \{3\} \sqcup P_4 \times \{4\})$$

The curves in P_1 and P_4 represent rank one cusps on $\partial\mathbb{Q}$ while the curves in P_2 and P_3 represent rank two cusps. We now try to understand what happens when we Dehn fill only the *rank two* cusps.

The filling slopes are chosen such that after the Dehn surgery, the natural inclusions $\Sigma \hookrightarrow \Omega_1, Q, \Omega_2$ become isotopic in the filled manifold so that it is naturally identified with

$$\mathbb{Q}^{\mathrm{fill}} \simeq \Sigma \times [1,4] - (P_1 \times \{1\} \sqcup P_4 \times \{4\}).$$

We observe that there exists a unique marked maximally cusped structure on $\mathbb{Q}^{\mathrm{fill}}$ where the rank one cusps are precisely given by $P_1 \times \{1\} \sqcup P_4 \times \{4\}$ (we assume that no curve in P_i is isotopic to a curve in P_j if $i \neq j$). We denote such a structure by $Q(P_1, P_4)$.

We are now ready to explain the main idea. Recall that our goal is to show that the filling slopes we singled out on the rank two cusps of $\mathbb{Q}$ have very large normalized length. This can be checked also in $\mathbb{Q}^{\mathrm{fill}}$ once we know that $\mathbb{Q}$ uniformly bilipschitz embeds in $\mathbb{Q}^{\mathrm{fill}}$ away from standard cusp neighbourhoods.

The strategy is as follows: Consider the maximally cusped structure $Q(P_1, P_4)$ and denote by Γ the collection of geodesic representatives of P_2 and P_3. Suppose that the collection Γ consists of extremely short simple closed geodesics, say of length at most $\eta < \eta_B(1/2)$, and that it is isotopic to $P_2 \times \{2\} \cup P_3 \times \{3\}$ under the identification with the drilled product.

Under such assumptions, we have the following.

Topologically, since the diffeomorphism type of $Q(P_1, P_4) - \Gamma$ only depends on the isotopy class of Γ, the manifold $Q(P_1, P_4) - \Gamma$ is diffeomorphic to $\mathbb{Q}$.

Geometrically, by Theorem 2.2, we can replace, up to 3/2-bilipschitz distortion away from standard Margulis neighbourhoods of Γ, the hyperbolic metric on $Q(P_1, P_4) - \Gamma$ with the unique geometrically finite structure with the same conformal boundary and rank two cusps instead of Γ. By uniqueness, such a geometrically finite structure structure on $Q(P_1, P_4) - \Gamma$ is precisely our initial manifold $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$.

In conclusion, $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ uniformly bilipschitz embeds in $Q(P_1, P_4) - \Gamma$ and the filling slopes are mapped to meridians of large Margulis tubes. Comparing the normalized length of a filling slope γ in the two metrics we deduce that it must be very large because in $Q(P_1, P_4)$ the curve γ is the meridian on the boundary of a very large Margulis tube. In fact

LEMMA 2.3. *Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube of radius R around a simple closed geodesic α of length $l(\alpha) < \eta_M$. Let γ be the flat geodesic representing the meridian on $\partial\mathbb{T}_{\eta_M}(\alpha)$. Then the normalized length is*

$$\mathrm{nl}(\gamma) = \sqrt{\frac{2\pi \tanh(R)}{l(\alpha)}}.$$

In particular $\mathrm{nl}(\gamma) \to \infty$ *as* $l(\alpha) \to 0$ *independently of the radius* R.

For example, there exists $\eta > 0$ such that if $l(\alpha) < \eta$ then $\mathrm{nl}(\gamma)$ is much bigger than nl_{HK}, the Hodgson-Kerckhoff constant.

Proof. The metric on $\mathbb{T}_{\eta_M}(\alpha)$ can be written in Fermi coordinates as

$$ds^2 = dr^2 + \cosh(r)^2 dl^2 + \sinh(r)^2 d\theta^2$$

where $(r, l, \theta) \in [0, R] \times [0, l(\gamma)] \times [0, 2\pi]$ are, respectively, the distance from α, the length along α and the angle around α parameters. The flat torus on the boundary has area $\mathrm{Area}(\partial\mathbb{T}_{\eta_M}(\alpha)) = 2\pi l(\alpha)\cosh(R)\sinh(R)$. The flat meridian $\gamma \subset \partial\mathbb{T}_{\eta_M}(\alpha)$ is represented by the curve $\theta \to (R, 0, \theta)$ of length $l(\gamma) = 2\pi\sinh(R)$. Hence the formula for the normalized length.

Notice that $\tanh(R)$ is roughly 1 when R is very large so that, in this case, the normalized length is approximately $\mathrm{nl}(\gamma) \approx l(\alpha)^{-1/2}$. It follows from work of Brooks and Matelski [10] that the radius of the Margulis tube

$\mathbb{T}_{\eta_M}(\alpha)$ is at least $R \geq \frac{1}{2}\log(\eta_M/l(\alpha)) - R_0$ where $R_0 > 0$ is some universal constant. Hence the second claim of the lemma when $l(\alpha)$ is very small. □

Applying Lemma 2.3 to the previous situation, we can conclude the following criterion

> **Criterion for I-bundles**: Fix $\mathrm{nl}_0 > \mathrm{nl}_{HK}$. The normalized length of the filling slopes corresponding to P_2 and P_3 is at least nl_0 provided that the collection of geodesic representatives in $Q(P_1, P_4)$ of the curves in $P_2 \cup P_3$ consists of simple geodesics of length at most η, where η only depends on nl_0, and is isotopic to $P_2 \times \{2\} \cup P_3 \times \{3\}$.

This concludes the I-bundle case.

The handlebody case. The second part consists of the same analysis for $\mathbb{N}_j = H_j \cup \Omega_j$ and $j = 1, 2$. The strategy is exactly the same. We only consider $\mathbb{N}_1 = H_1 \cup \Omega_1$ as the case of $\mathbb{N}_2 = \Omega_2 \cup H_2$ is completely analogous.

Parametrize a collar neighbourhood of $\Sigma = \partial H_g$ in H_g as $\Sigma \times [1, 2]$ with $\partial H_g = \Sigma \times \{2\}$. Topologically we have

$$\mathbb{N}_1 = H_g - (P_1 \times \{1\} \sqcup P_2 \times \{2\}).$$

Geometrically, the curves in P_2 correspond to rank one cusps while the one in P_1 correspond to rank two cusps. We are interested in filling in the rank two cusps. As before, the filling slopes are determined by the gluing.

After filling we have

$$\mathbb{N}_1^{\mathrm{fill}} = H_g - P_2.$$

Again, there is a unique maximally cusped structure on $\mathbb{N}_1^{\mathrm{fill}}$ whose cusps are given by P_2. We denote it by $H(P_2)$. We argue as before and assume that the collection Γ of geodesic representatives of P_1 consists of very short curves and is isotopic to $P_1 \times \{1\}$. Using the Drilling Theorem we compare the normalized length in $\mathbb{N}_1$ and $H(P_2)$.

Again, relying on Lemma 2.3, we will use the following criterion.

> **Criterion for handlebodies**: Fix $\mathrm{nl}_0 > \mathrm{nl}_{HK}$. The normalized length of the filling slopes corresponding to P_1 is at least nl_0 provided that the collection of the geodesic representatives in $H(P_2)$ of the curves in P_1 consists of simple closed geodesic of length at most η, where η only depends on nl_0, and is isotopic to $P_1 \times \{1\}$.

When considering $\mathbb{N}_2 = \Omega_2 \cup H_2 = Q(P_3, P_4) \cup H(f^{-1}P_4)$, we ask the same requirements replacing P_1 with $f^{-1}P_4$ and P_2 with $f^{-1}P_3$.

This concludes the handlebody case.

Thus, from the previous discussion we established the following

PROPOSITION 2.4. *Fix $K \in (1, 2)$. Suppose that there are four pants decompositions P_1, P_2, P_3, P_4 such that the* I-bundle *and the* handlebody criteria *are satisfied with parameter η sufficiently small only depending on K. Then,*

M_f admits a hyperbolic metric and a model metric $\mathbb{M}$. Furthermore, $\mathbb{M}$ and M_f can be connected by a family of hyperbolic cone manifolds and we have a K-bilipschitz diffeomorphism

$$\left(\mathbb{M} - \bigsqcup_{\alpha\in P_1\cup P_2\cup P_3\cup P_4} \mathbb{T}_{\eta_M}(\alpha)\right) \simeq \left(M_f - \bigsqcup_{\alpha\in P_1\cup P_2\cup P_3\cup P_4} \mathbb{T}_{\eta_M}(\alpha)\right).$$

We conclude with a small remark. The model manifold technology of Minsky [25] and Brock, Canary and Minsky [7], provides several tools to locate and measure the length of the geodesic representatives of P_2 and P_3 in $Q(P_1, P_4)$. However, the same technology is not available for handlebodies. This is the place where the difficulties arise.

3. A family of examples

In this section we construct many examples satisfying the I-bundle and handlebody criteria. Later, in the next section, we will show that this family is *generic* from the point of view of random walks.

We need two ingredients: The first one is a model for a collar of the boundary of a maximally cusped handlebody H or I-bundle Q. Following [15], we have that, in certain cases, it is possible to force a H and Q to look exactly like a maximally cusped I-bundle Ω near the boundary ∂H and $\partial_1 Q$ or $\partial_2 Q$. This is roughly the content of Propositions 3.1 and 3.2.

The second ingredient is a family of hyperbolic mapping tori T_ψ on which we want to model the collars Ω. These mapping tori have a distinguished fiber $\Sigma \subset T_\psi$ with a pants decomposition P consisting of extremely short geodesics. The collars Ω will look like a large portion of the infinite cyclic covering of T_ψ. See Theorem 3.3 and its corollaries, in particular Corollary 3.7.

In the end we will be able to detect whether M_f can be described as one of the examples we constructed simply by staring at the geometry of the Teichmüller segment $[o, fo]$ where $o \in \mathcal{T}$ is some base point that we will carefully fix once and for all. This is the content of Proposition 3.9.

3.1. **The geometry of the collars.**

We discuss now the first main tool, that is, Propositions 3.1 and 3.2. For the statements we need to introduce some terminology and facts from the deformation theory of geometrically finite structures on handlebodies and I-bundles. We also need a suitable definition of collars for the boundary of such structures which is not just purely topological, but also geometrically significant.

We start by describing the deformation spaces of geometrically finite metrics. Even if we are mainly interested in maximal cusps, we begin with the more flexible class of *convex cocompact* structures.

A convex cocompact hyperbolic metric on a handlebody H_g or an I-bundle $\Sigma\times[1,2]$ is a complete hyperbolic metric on the interior, $\mathrm{int}(H_g)$ or $\Sigma\times(1,2)$, that has a compact subset which is convex in a strong sense. This means that it contains all the geodesics joining two of its points. The minimal such subset is called the *convex core*. It is always a topological submanifold homeomorphic to the ambient manifold (except in the *fuchsian* case which we ignore). Its boundary is parallel to the boundary of the ambient manifold.

The Ahlfors-Bers theory associates to each convex cocompact metric a conformal structure on each boundary component. The deformation spaces of such metrics are parametrized by those conformal structures. Hence, they are identified with the Teichmüller space of the boundary. For each $Y \in \mathcal{T}(\partial H_g)$ and $(X, Y) \in \mathcal{T}(\Sigma \times \{1\}) \times \mathcal{T}(\Sigma \times \{2\})$ there are convex cocompact structures $H(Y)$ on H_g and $Q(X, Y)$ on $\Sigma \times [1, 2]$, unique up to isometries isotopic to the identity, realizing those boundary data.

Geometrically finite *maximally cusped* hyperbolic structures on H_g or $\Sigma \times [1, 2]$ can be thought as lying on the boundary of the deformation spaces. For every pair of pants P on ∂H_g such that no curve in P is compressible and no two curves in P are isotopic in H_g there exists a unique maximally cusped handlebody $H(P)$ with rank one cusps at P. Similarly, for every pants decomposition $P_1 \cup P_2$ of $\Sigma \times \{1\} \cup \Sigma \times \{2\}$ such that no curve in P_1 is isotopic to a curve in P_2, there exists a unique maximally cusped structure $Q(P_1, P_2)$ on $\Sigma \times [1, 2]$ realizing those cusp data.

With a slight abuse of notations, sometimes we will denote both the complete convex cocompact or maximally cusped structure and the corresponding convex core in the same way. However, it will be clear from the context which one we are using.

The internal geometry of the convex cores of geometrically finite I-bundles has a rich structure. It is captured by the combinatorics and geometry of the *curve graph* $\mathcal{C} = \mathcal{C}(\Sigma)$ by the groundbreaking work of Minsky [25] and Brock, Canary and Minsky [7] with fundamental contributions by Masur and Minsky [22], [23].

This is the second piece of deformation theory that we need, it goes under the name of *model manifold technology*. Our use of this technology will not be heavy as we only need a few concepts and consequences, but we mostly hide the relation between the two. We briefly explain what we need.

The starting point is the following: To every convex cocompact structure Q on $\Sigma \times [1, 2]$ we have an associated pair of curve graph invariants P_1 and P_2. They are pants decompositions on $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$ that are the shortest for the conformal structure on the boundary. They might not be uniquely defined, in such case we just pick two. Similarly, for a maximally cusped structure Q we associate to it the cusp data P_1 and P_2. We think of these pants decompositions as subsets of the curve graph $\mathcal{C}$.

Recall now that for every proper essential subsurface $W \subsetneq \Sigma$ which is not a three punctured sphere there is a *subsurface projection*, as defined by Masur and Minsky in [23]. It associates to each curve $\alpha \in \mathcal{C}$ the subset

$\pi_W(\alpha)$ (possibly empty) of the curve graph $\mathcal{C}(W)$ of all possible essential surgeries of $\alpha \cap W$. The definition is a bit different for annuli. We associate to the curve graph invariants P_1 and P_2 the collection of coefficients

$$\{d_W(P_1, P_2) = \operatorname{diam}_{\mathcal{C}(W)}(\pi_W(P_1) \cup \pi_W(P_2))\}_{W \subsetneq \Sigma}.$$

As established by Minsky [25], the pants decompositions P_1 and P_2 together with the list $\{d_W(P_1, P_2)\}_{W \subsetneq \Sigma}$ allow to determine and locate the collection of *short curves* in Q. A special case, which is important for Propositions 3.1 and 3.2, is when the subsurface coefficients are all uniformly bounded. It corresponds to the situation where the only possible very short curves are the geodesic representatives of P_1 and P_2. For each other closed geodesic there is a positive uniform lower bound for the length.

The following notion was introduced by Minsky in [24] (see also [9]).

DEFINITION (Bounded Combinatorics and Height). We say that two pants decompositions P_1, P_2 of Σ have *R-bounded combinatorics* if for every proper subsurface $W \subsetneq \Sigma$ we have $d_W(P_1, P_2) \leq R$. We say that they have *height* at least h if we have $d_{\mathcal{C}}(P_1, P_2) \geq h$.

As for the internal geometry of a geometrically finite handlebody the situation is more complicated as the compressibility of the boundary brings in several issues. We will restrict our attention to the geometry of some *collars* of the the boundary of the convex core.

We still choose for every convex cocompact structure on H_g a curve graph invariant, namely, a pants decomposition P on $\Sigma = \partial H_g$ which is the shortest when measured with the conformal boundary. In a similar way we associate to every maximally cusped structure the cusp data P.

DEFINITION (Disk Set). The *disk set* $\mathcal{D}$ associated to the handlebody H_g is the subset of the curve graph $\mathcal{C}$ of the boundary $\Sigma = \partial H_g$ defined by

$$\mathcal{D} = \{\delta \in \mathcal{C} \mid \delta \text{ compressible in } H_g\}.$$

In order to construct a model for the collar of a geometrically finite handlebody we will have to keep track of how the curve graph invariant P of the geometrically finite structure interacts with the disk set $\mathcal{D}$.

The idea is the following: If P is far away from $\mathcal{D}$ then a large collar of the boundary of the convex core looks like a geometrically finite I-bundle.

We are almost ready for the statements of Propositions 3.1 and 3.2, we only need one last definition, the one of a geometrically controlled collar of the boundary of a geometrically finite structure. For convenient technical simplifications, it will be better for us to work with *quasi collars* (see below for the definition) instead of using directly collars. The reason is that we might wish to allow ourselves to throw away a uniform initial piece from a collar and still call the result a collar.

Let $M = H$ or Q denote the convex core of either a convex cocompact or a maximally cusped structure on either H_g or $\Sigma \times [1,2]$. Consider

$$M^{\mathrm{nc}} = M - \bigcup_{\alpha \in \mathrm{cusp}(M)} \mathbb{T}_{\eta_M}(\alpha).$$

the *non cuspidal part* of M. As before, $\mathbb{T}_{\eta_M}(\alpha)$ denotes a standard η_M-Margulis neighbourhood of the cusp α. We have that M^{nc} is homeomorphic to M. Its boundary ∂M^{nc} is parallel to the boundary of the ambient manifold, that is ∂H_g or $\Sigma \times \{1\} \cup \Sigma \times \{2\}$. Hence, it is naturally identified with it, up to isotopy. In particular, each component of ∂M^{nc} is always naturally identified with Σ.

The definition of quasi collar is analogous to the one of *product region* given in [15]: Consider a component Σ_0 of ∂M^{nc} and identify it with $\Sigma_0 \simeq \Sigma$ as above.

Definition (Quasi Collar). A *quasi collar* of *size* (D, W, K) of the component $\Sigma_0 \subset \partial M^{\mathrm{nc}}$, denoted by

$$\mathrm{collar}_{D,W,K}(\Sigma_0),$$

is a subset of a topological collar of Σ_0 in M^{nc}, denoted by $\mathrm{collar}(\Sigma_0)$. We require the following additional geometric properties: There exists a parametrization $\mathrm{collar}(\Sigma_0) = \Sigma \times [0,3]$ such that Σ_0 is identified with $\Sigma \times \{0\}$ and $\mathrm{collar}_{D,W,K}(\Sigma_0)$ corresponds to $\Sigma \times [1,2]$. Furthermore we have

- The diameter of $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$, measured with the intrinsic metric, is at most D.
- The *width* of $\mathrm{collar}_{D,W,K}(\Sigma_0)$, that is the distance between $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$, is at least W and at most $2W + 2D$.
- The distance of $\Sigma \times \{1\}$ from the distinguished boundary $\Sigma \times \{0\} = \Sigma_0$ is at least K and at most $2K + 2D$.

Notice that each quasi collar $\mathrm{collar}_{D,W,K}(\Sigma_0)$ is *marked* with the isotopy class of an inclusion of Σ. Using this marking we can associate to every homotopy equivalence f between quasi collars a homotopy class $[f] \in \mathrm{Mod}(\Sigma)$.

We are ready to state Propositions 3.1 and 3.2.

Proposition 3.1 (Propositions 4.1 and 6.1 of [15]). *For every $R, \epsilon, \xi > 0$ there exist $D_0 = D_0(R) > 0$ and $K_0, W_0 > 0$ such that for every $W \geq W_0$ there exists $h = h(\epsilon, R, \xi, W) > 0$ such that the following holds: Consider $(Y, Z) \in \mathcal{T} \times \mathcal{T}$ and $X \in \mathcal{T}$. Suppose that $X, Y \in \mathcal{T}_\epsilon$. Let P_X, P_Y and P_Z be short pants decompositions for X, Y and Z respectively. Consider the convex cores of the convex cocompact structures $Q(X, Y)$ and $H(Y), Q(Z, Y)$. Suppose that*

- *P_X, P_Y have R-*bounded combinatorics *and* height *at least h.*
- *In the handlebody case H_g:*

$$d_{\mathcal{C}}(P_Y, \mathcal{D}) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, \mathcal{D}) - R.$$

- *In the I-bundle case* $\Sigma \times [1,2]$*:*
$$d_{\mathcal{C}}(P_Y, P_Z) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, P_Z) - R.$$

Then, there exist $(1+\xi)$*-bilipschitz diffeomorphisms of quasi collars*
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(Y))$$
and
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_2 Q(Z,Y))$$
for some slightly perturbed parameters D_1, W_1, K_1*. The diffeomorphisms are in the homotopy class of the identity with respect to the natural markings.*

Proposition 3.1 follows from [15].

We will use the following mild variation for maximally cusped structures.

PROPOSITION 3.2. *For every* R, ξ *there exist* $D_0 = D_0(R) > 0$ *and* $W_0, K_0 > 0$ *such that for every* $W \geq W_0$ *there exists* $h = h(\xi, R, W) > 0$ *such that the following holds: Consider pants decompositions* P_Y*,* P_X *and* P_Z *of* Σ*. Consider the convex cores of the maximally cusped structures* $Q(P_X, P_Y)$ *and* $H(P_Y), Q(P_Z, P_Y)$*. Suppose that*

- P_X, P_Y *have* R-bounded combinatorics *and* height *at least* h*.*
- *In the handlebody case* H_g*:*
$$d_{\mathcal{C}}(P_Y, \mathcal{D}) \geq d_{\mathcal{C}}(P_X, P_Y) + d_{\mathcal{C}}(P_X, \mathcal{D}) - R.$$
- *In the I-bundle case* $\Sigma \times [1,2]$*:*
$$d_{\mathcal{C}}(P_Y, P_Z) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, P_Z) - R.$$

Then, there exist $(1+\xi)$*-bilipschitz diffeomorphism between quasi collars*
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(P_X,P_Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(P_Y))$$
and
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(P_X,P_Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_2 Q(P_Z,P_Y))$$
for some slightly perturbed parameters D_1, W_1, K_1*. The diffeomorphisms are in the homotopy class of the identity with respect to the natural markings.*

Sketch of proof. Using Theorem 2.2 it is possible to quickly reduce Proposition 3.2 to the previous one. We only sketch the proof. We only treat the handlebody case as the I-bundle case is completely analogous.

First, we approximate P_X, P_Y with hyperbolic surfaces X, Y on which the pair of pants decompositions consist of very short geodesics, say of length contained in the interval $[\epsilon, 2\epsilon]$ with ϵ much smaller than a Margulis constant. Such surfaces are contained in $\mathcal{T}_\epsilon$.

By results of Canary [12] and Otal [28], the collections of geodesic representatives $\Gamma_X \cup \Gamma_Y$ and Γ_Y of the curves $P_X \cup P_Y$ and P_Y in $Q(X,Y)$ and $H(Y)$ have length $O(\epsilon)$ and are isotopic to $P_X \cup P_Y \subset \partial_1 Q(X,Y) \cup \partial_2 Q(X,Y)$ and $P_Y \subset \partial H(Y)$. Hence, by Theorem 2.2, if ϵ is small enough, we have $(1+\xi)$-bilipschitz embeddings of the non cuspidal part of the convex core of the maximally cusped structures in the corresponding complete convex cocompact hyperbolic 3-manifolds

$$\phi_Q : \left(Q(P_X, P_Y) - \bigcup_{\alpha \in P_X \cup P_Y} \mathbb{T}_{\eta_M}(\alpha) \right) \to \left(Q(X, Y) - \bigcup_{\alpha \in P_X \cup P_Y} \mathbb{T}_{\eta_M}(\alpha) \right)$$

and

$$\phi_H : \left(H(P_Y) - \bigcup_{\alpha \in P_Y} \mathbb{T}_{\eta_M}(\alpha) \right) \to \left(H(Y) - \bigcup_{\alpha \in P_Y} \mathbb{T}_{\eta_M}(\alpha) \right).$$

Now, if h is large enough only depending on ϵ, ξ, R, and W, we can apply Proposition 3.1 and find a $(1+\xi)$-bilipschitz diffeomorphism

$$g : \mathrm{collar}_{D_0,W,K}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(Y)).$$

If K, W are sufficiently large, then both quasi collars will be contained in the images of ϕ_Q and ϕ_H. We just compose those with g, that is $f := \phi_H g \phi_Q^{-1}$. □

3.2. **Models for the collars.** As anticipated, we use Proposition 3.2 to construct a very particular class of maximally cusped handlebodies with a simple collar structure. This is our second main ingredient.

Recall that our goal is to construct examples that satisfy the criteria for handlebodies and for I-bundles. Also, recall that these criteria require to control the length and the isotopy class of the collection of geodesic representatives of a pants decomposition of the boundary. The examples we are going to describe are exactly tailored for that goal.

The idea is as follows: We first construct maximally cusped I-bundles Ω for which the length and isotopy class conditions are satisfied almost by definition for many pants decompositions. If a maximally cusped structure on H_g or $\Sigma \times [1,2]$ has a collar that is geometrically very close to Ω, then it also satisfies the criteria.

We now develop the strategy in more details.

The structure of the collar Ω will be modelled on the geometry of a *hyperbolic mapping torus*, or pseudo-Anosov mapping class, *with a short pants decomposition*. Such object is one that is obtained from the following procedure: Let P be a pants decomposition of Σ. Let $\phi \in \mathrm{Mod}(\Sigma)$ be a mapping class such that no curve in P is isotopic to a curve in ϕP. For example, a large power of any pseudo-Anosov suffices. Consider the convex core Q of the maximally cusped structure $Q(P, \phi P)$. The boundary ∂Q consists of totally geodesic hyperbolic three punctured spheres that are paired according to ϕ. We glue them together isometrically as prescribed by the pairing. The glued manifold is a finite volume hyperbolic 3-manifold diffeomorphic to

$$T_\phi - P \times \{0\} = (\Sigma \times [0,1]/(x,0) \sim (\phi x, 1)) - P \times \{0\}.$$

The curves in $P \times \{0\}$ represent rank two cusps. By Thurston's Hyperbolic Dehn Surgery (see Chapter E.6 of [2]) we can do Dehn surgery on the cusps so that the resulting manifold still carries a hyperbolic metric for which the core curves of the added solid tori are very short geodesics.

Furthermore, we can restrict ourselves to Dehn fillings for which the filled manifold still fibers over S^1 in a way compatible with the restriction of the fibering of T_ϕ to $T_\phi - P \times \{0\}$. In fact, observe that for each $\alpha \in P$ corresponding to a boundary torus $\mathbb{T}_\alpha$ we have a preferred meridian m_α and longitude l_α coming from the fibering of T_ϕ. If we perform Dehn surgeries with slopes $m_\alpha + kl_\alpha$, the filled manifold will be diffeomorphic to the mapping torus T_ψ where $\psi = \phi\delta_P^k$ and $\delta_P \in \mathrm{Mod}(\Sigma)$ is a Dehn twist about the pants decomposition P.

Consider the infinite cyclic covering $\hat{T}_\psi$ of T_ψ. Topologically, we can identify it with $\Sigma \times \mathbb{R}$ where the level sets $\Sigma_n := \Sigma \times \{n\}$ correspond to all the lifts of the fiber $\Sigma \times \{0\} \subset T_\psi$ and in such a way that the curves in

$$\bigcup_{n \in \mathbb{Z}} \psi^n P \times \{n\} \subset \bigcup_{n \in \mathbb{Z}} \Sigma \times \{n\}$$

are very short geodesics. A fundamental domain for the deck group action on $\hat{T}_\psi$ is given by the submanifold $[\Sigma_0, \Sigma_1]$ bounded by Σ_0 and Σ_1. The region $[\Sigma_n, \Sigma_m]$ bounded by Σ_n and Σ_m with $n < m$ is a stack of $m - n$ isometric copies of $[\Sigma_0, \Sigma_1]$.

We now approximate $\hat{T}_\psi$ with a maximally cusped I-bundle $Q(P, \psi^n P)$. Our collars will be of the form $\Omega = Q(P, \psi^m P)$ for some suitably chosen m.

We will use the following from [8], see also Figure 3.7 of the same article.

Theorem 3.3 (Theorem 3.5 of [8]). *Let ψ be a mapping class with a short pants decomposition P. For every $\xi > 0$ there exist $k > 0$ and $d > 0$ such that for every $n > 0$ sufficiently large the non-cuspidal part of $Q_n = Q(P, \psi^n P)$ admits a decomposition*

$$Q_n^{\mathrm{nc}} = A_n \cup B_n \cup C_n$$

where A_n and C_n have diameter bounded by d while B_n is the image of a $(1+\xi)$-bilipschitz embedding with a quasi collar image

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n^{\mathrm{nc}}.$$

The embedding f is in the homotopy class of the identity with respect to the natural markings. Moreover, we can parametrize Q_n^{nc} as $\Sigma \times [0,3]$ in such a way that A_n, B_n and C_n correspond respectively to $\Sigma \times [0,1], \Sigma \times [1,2]$ and $\Sigma \times [2,3]$.

Observe that in the maximally cusped I-bundles $Q(P, \psi^n P)$ we have, by default, many pants decompositions whose length and isotopy class are well controlled.

In order to be able to exploit such control to check I-bundles and handlebody criteria we will need three consequences of Theorem 3.3. For them, we use the following fact proved in the Appendix A.

LEMMA 3.4. *For every $\eta < \eta_M/2$ there exists $\xi > 0$ such that the following holds: Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube with core geodesic α of length $l(\alpha) \in [\eta, \eta_M/2]$. Suppose that there exists a $(1+\xi)$-bilipschitz embedding of the tube in a hyperbolic 3-manifold $f : \mathbb{T}_{\eta_M}(\alpha) \to M$. Then $f(\alpha)$ is homotopically non-trivial and it is isotopic to its geodesic representative within $f(\mathbb{T}_{\eta_M}(\alpha))$.*

Consider the $(1+\xi)$-bilipschitz embedding given by Theorem 3.3

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n^{\mathrm{nc}}.$$

Recall that $[\Sigma_k, \Sigma_{n-k}] = \Sigma \times [k, n-k]$ and that the curves $\psi^j P \times \{j\} \subset \Sigma_j = \Sigma \times \{j\}$ are short geodesics in the infinite cyclic covering and have length in the interval $[\eta, \eta_M]$. Denote by Γ_j the collection of geodesic representatives of $f(\psi^j P \times \{j\})$ in Q_n. By Lemma 3.4, if ξ is small compared to η, we get

COROLLARY 3.5. *The collection $\Gamma = \Gamma_k \cup \cdots \cup \Gamma_{n-k}$ is isotopic to*

$$\bigcup_{k<j<n-k} f(\psi^j P \times \{j\}) \subset \bigcup_{k<j<n-k} f(\Sigma_j).$$

via an isotopy supported on $\bigsqcup_{\alpha \in \psi^{k+1}P \cup \cdots \cup \psi^{n-k-1}P} f(\mathbb{T}_{\eta_M}(\alpha))$.

We now locate suitable quasi collars inside $Q(P, \psi^n P)$. First, notice that

$$[\Sigma_k, \Sigma_{n-k}] = \bigcup_{k<j\leq n-k} [\Sigma_{j-1}, \Sigma_j]$$

and each $[\Sigma_{j-1}, \Sigma_j]$ is an isometric copy of the fundamental domain $[\Sigma_0, \Sigma_1]$. Each $f[\Sigma_{j-1}, \Sigma_{j+1}] \subset Q$ is a quasi collar for $\partial_1 Q^{\mathrm{nc}}$ for every $k < j < n-k$. We now estimate the quasi collar size (D, W, K).

By Theorem 3.3, we also have that each component of

$$Q^{\mathrm{nc}} - f[\Sigma_k, \Sigma_{n-k}]$$

has diameter bounded by $d = d(\psi, \xi)$. Denote by $w = w(\psi) > 0$ the width of the fundamental domain $[\Sigma_0, \Sigma_1]$. Denote, instead, by $a = a(\psi) > 0$ the intrinsic diameter of the isometric surfaces Σ_j. Notice that, up to replacing ψ with a power (a change that does not seriously affect any of the arguments), we can as well assume that $2a$ is much smaller than w. Since f is $(1+\xi)$-bilipschitz, up to increasing a little and uniformly a and w, those are also the diameter and width parameters for each $f[\Sigma_{j-1}, \Sigma_j]$. We have for $j \geq k$

$$w(j-k) \leq d_Q(f(\Sigma_j), \partial_1 Q^{\mathrm{nc}}) \leq (w+a)(j-k) + d.$$

Therefore the size of the quasi collar $f[\Sigma_{j-1}, \Sigma_{j+1}]$ can be chosen to be

$$\begin{aligned} D &= a, \\ W &= 2w, \\ K_j &= (w+a)(j-k) + d + 2w. \end{aligned}$$

Analogous estimates hold for $\partial_2 Q^{\mathrm{nc}}$. Hence

COROLLARY 3.6. *There exists $w, a > 0$ and only depending on ψ such that for every $k < j < n-k$ the surface $f(\Sigma_j)$ is contained in*

$$\mathrm{collar}_{a,2w,K_j}(\partial_1 Q)$$

and, similarly, the surface $f(\Sigma_{n-j})$ is contained in

$$\mathrm{collar}_{a,2w,K_j}(\partial_2 Q).$$

The Corollaries 3.5 and 3.6 combined with Proposition 3.2 help us in checking that the handlebody and I-bundle criteria are satisfied. In fact, we have the following: With the same notation as before, consider again the $(1+\xi)$-bilipschitz embedding as a quasi collar

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n = Q(P, \psi^n P).$$

The bilipschitz parameter ξ can be chosen to be arbitrarily small provided that n is sufficiently large.

COROLLARY 3.7. *Let ψ be a mapping class with a short pants decomposition P of length η. Consider $Q_n = Q(P, \psi^n P)$. Let Σ_0 be a component of ∂M^{nc} where M is a maximally cusped handlebody or I-bundle. Suppose that we have a $(1+\xi)$-bilipschitz diffeomorphism*

$$g : \mathrm{collar}_{a,2w,K_j}(\partial_1 Q_n) \to \mathrm{collar}_{D,W,K}(\Sigma_0)$$

or

$$g : \mathrm{collar}_{a,2w,K_j}(\partial_2 Q_n) \to \mathrm{collar}_{D,W,K}(\Sigma_0)$$

for some D, W, K. If ξ is small enough (only depending on ψ), n is large enough (only depending on ψ and ξ) and $k < j < n-k$, then the collection of geodesic representatives of

$$gf(\psi^j P \times \{j\}) \subset gf(\Sigma_j)$$

or

$$gf(\psi^{n-j} P \times \{n-j\}) \subset gf(\Sigma_{n-j})$$

has length $O(\eta)$, is contained in the image of g, and is isotopic within it to $gf(\psi^j P \times \{j\})$ or $gf(\psi^{n-j} P \times \{n-j\})$.

Proof. We only treat the first case, the other one is analogous. By Corollary 3.5 and Corollary 3.6, we can assume that the geodesic representatives Γ_j of $\psi^j P$ in Q_n are contained in $\mathrm{collar}_{a,2w,K_j}(\partial_1 Q_n)$ and isotopic within it to $f(\psi^j P \times \{j\}) \subset f(\Sigma_j)$. Their length is $O(\eta)$. Since g is a $(1+\xi)$-bilipschitz diffeomorphism, if ξ is small enough compared to η, by Lemma 3.4, we can assume that the geodesic representatives of $g(\Gamma_j)$ in M are contained in $\mathrm{collar}_{D,W,K}(\partial H)$ and isotopic within it to $g(\Gamma_j)$ which, in turn, is isotopic to $gf(\psi^j P \times \{j\}) \subset gf(\Sigma_j)$. $\square$

3.3. **Criteria for I-bundles and handlebodies revised.** We are now ready to give a more manageable version of the criteria for I-bundles and handlebodies and construct many example that satisfy those conditions. This is the goal of Proposition 3.8.

PROPOSITION 3.8. *Let ψ, ϕ be mapping classes with short pants decompositions P, P' of length in $[\eta, \eta_M/2]$. There exists $j = j(\psi, \phi)$ such that the following holds: Consider*

$$(P_1, P_2) = (\psi^{-j}P, P) \text{ and } (P_3, P_4) = (P', \phi^j P').$$

Suppose that for some n very large we have respectively

(1) $d_{\mathcal{C}}(P, \mathcal{D}) \geq d_{\mathcal{C}}(P, \psi^{-n}P) + d_{\mathcal{C}}(\psi^{-n}P, \mathcal{D}) - R$,
(2) $d_{\mathcal{C}}(\psi^{-j}P, \phi^j P') \geq d_{\mathcal{C}}(\psi^{-j}P, \psi^n P) + d_{\mathcal{C}}(\psi^n P, \phi^j P') - R$,

and

(3) $d_{\mathcal{C}}(P', f\mathcal{D}) \geq d_{\mathcal{C}}(P', \phi^n P') + d_{\mathcal{C}}(\phi^n P', f\mathcal{D}) - R$,
(4) $d_{\mathcal{C}}(\psi^{-j}P, \phi^j P') \geq d_{\mathcal{C}}(\phi^j P', \phi^{-n}P') + d_{\mathcal{C}}(\phi^{-n}P', \psi^{-j}P) - R$.

Then P_1, P_2, P_3, P_4 satisfies the I-bundle *and* handlebody criteria *with parameter $O(\eta)$.*

Proof. We have to check two handlebody and one I-bundle criteria. The arguments for the three different cases follow the same lines. In order to avoid repetitions, we only prove in details that there exists $j = j(\phi, \psi)$ such that the pair $(\psi^{-j}P, P)$ satisfies the handlebody criterion if n is large enough. The other cases are completely analogous and require no new ideas. In the end of the proof, we briefly discuss the adjustments needed for the I-bundle criterion.

The handlebody criterion. In order to check the handlebody criterion for $(\psi^{-j}P, P)$, by Corollary 3.7, we just need to get a $(1 + \xi)$-bilipschitz diffeomorphism

$$g : \text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P)) \to \text{collar}_{D,W,K}(\partial H(P))$$

in the homotopy class of the identity. Such a diffeomorphism will be provided by Proposition 3.2. Notice at this point that $Q(\psi^{-n}P, P)$ and $Q(P, \psi^n P)$ are isometric as they only differ by the marking.

In order to apply Proposition 3.2, observe that, by work of Minsky [24], the pairs $(\psi^n P, \psi^m P)$ and $(\phi^m P', \phi^n P')$ satisfy for all n, $m \in \mathbb{Z}$ the R-bounded combinatorics condition for some $R = R(\psi, \phi) > 0$. Furthermore, for any fixed h, if $|n - m|$ is large, again, depending only on ψ, ϕ and h, they also satisfy the large height assumption as pseudo-Anosov elements act as loxodromic motions on the curve graph by Masur and Minsky [22]. Property (1) from our assumptions is exactly the last one needed to guarantee that Proposition 3.2 can be applied.

Before applying Proposition 3.2 we have to be a bit careful with the various constants and their dependence. We pause for a moment and discuss this delicate point. The mapping classes ψ and ϕ determine η and R and also the parameters a and w of Corollary 3.7 and D_0 of Proposition 3.2. Furthermore, the mapping classes together with the choice of ξ determine k and d in Proposition 3.3. In turn, k determines the allowable range $k < j < n - k$.

So, we want to choose ξ much smaller than the one, only depending on the mapping classes, required by Corollary 3.7 to hold. Once this is fixed we have a collection of potential candidates for the quasi collars

$$\mathrm{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P))$$

with $k < j < n - k$ for any n very large.

Once we fixed ξ, we have also fixed $K_0, W_0 > 0$ of Proposition 3.2. So, for every $W \geq W_0$ we have a $(1+\xi)$-bilipschitz diffeomorphism of quasi collars

$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(\psi^{-n}P, P)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(P)).$$

In order to get the desired embeddings, we just have to choose $k < j < n-k$ and W so that one of our candidate quasi-collars is contained in the domain of definition of f

$$\mathrm{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P)) \subset \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(\psi^{-n}P, P)).$$

It suffices to do the following: We first choose j so that $K_j - a > K_0 + D_0$. This determines a minimal $j = j(\psi, \phi)$ as required by the statement of Proposition 3.8. Then, we choose W so that $K_0 + W - D_0 > 2K_j + 4a + 2w$. This finally determines a final threshold for h and n.

The I-bundle criterion. The proof is word by word the same as in the handlebody case, one only has to replace the collar of $\partial H(P)$ with $\partial_1 Q(\psi^{-j}P, \phi^j P')$ and $\partial_2 Q(\psi^{-j}P, \phi^j P')$. Again, if n is very large, Proposition 3.2 furnishes $(1+\xi)$-bilipschitz diffeomorphisms

$$\mathrm{collar}_{a,2w,K_j}(\partial_1 Q(\psi^{-j}P, \psi^{n-j}P)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_1 Q(\psi^{-j}P, \phi^j P'))$$

and

$$\mathrm{collar}_{a,2w,K_j}(\partial_2 Q(\phi^{j-n}P, \phi^j P)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_2 Q(\psi^{-j}P, \phi^j P')).$$

Notice that the constant $j = j(\phi, \psi)$ is the same as before. Corollary 3.7 together with a careful bookkeping of the markings concludes the proof. $\square$

3.4. **From the curve graph to Teichmüller space.** We now translate the curve graph conditions (1) - (4) in terms of Teichmüller geometry in such a way that it will not be hard to check them for a random segment $[o, fo]$.

It is convenient to recall now a few facts due to Masur and Minsky [22], [23] about the relation between Teichmüller space $\mathcal{T}$ endowed with the Teichmüller metric $d_{\mathcal{T}}$ and the curve graph $\mathcal{C}$.

The connection is established via the *shortest curves projection* $\Upsilon : \mathcal{T} \to \mathcal{C}$, a coarsely defined map that associates to every marked hyperbolic surface

$X \in \mathcal{T}$ a shortest geodesic pants decomposition on it $\Upsilon(X)$. We choose $o \in \mathcal{T}$ with the following property

> **Standing assumption**: The base point $o \in \mathcal{T}$ is chosen so that its projection to the curve graph lies on the disk set of the handlebody $\Upsilon(o) \in \mathcal{D}$.

From a geometric point of view, Masur and Minsky proved that the curve graph $\mathcal{C}$ is a Gromov hyperbolic space (see [22]) and the disk set $\mathcal{D} \subset \mathcal{C}$ is a uniformly quasi-convex subspace (see [21]).

It also follows from [22] that Υ is uniformly Lipschitz and sends Teichmüller geodesics to uniform *unparametrized* quasi-geodesics. This means that there is a constant B only depending on Σ such that $d_{\mathcal{C}}(\Upsilon(Y), \Upsilon(Z)) \geq d_{\mathcal{C}}(\Upsilon(Y), \Upsilon(X)) + d_{\mathcal{C}}(\Upsilon(X), \Upsilon(Z)) - B$ for every $Z < X < Y$ aligned on a Teichmüller geodesic. In particular, by hyperbolicity of $\mathcal{C}$, the image $\Upsilon[Z, Y] \subset \mathcal{C}$ is a uniformly quasi-convex subset.

We have the following:

Proposition 3.9. *Let ψ, ϕ be pseudo-Anosov mapping classes with short pants decompositions P, P'. Let $l_\psi, l_\phi : \mathbb{R} \to \mathcal{T}$ be their Teichmüller geodesics. For every $\delta > 0$ there exists $h > 0$ such that the following holds: Suppose that on the segment $[o, fo]$ there are four points $o < S_1 < S_2 < S_3 < S_4 < fo$ with the following properties*

(i) *$[S_1, S_2]$ and $[S_3, S_4]$ have length at least h and δ-fellow travel l_ψ and l_ϕ respectively.*
(ii) *We have $d_{\mathcal{C}}(\Upsilon[S_1, S_4], \mathcal{D}) \geq h$ and $d_{\mathcal{C}}(\Upsilon[S_1, S_4], f\mathcal{D}) \geq h$.*

Then, up to perhaps replacing P, P' with $\psi^r P, \phi^{r'} P'$, (1) - (4) *hold.*

The proof uses the arguments from Proposition 7.1 of [15].

Proof. We have to show that (i) and (ii) imply (1) - (4).

We start with Properties (1) and (3). Let us only consider (1) as (3) is completely analogous. Recall that we fixed o so that $\Upsilon(o) \in \mathcal{D}$ and hence $\Upsilon(fo) = f\Upsilon(o) \in f\mathcal{D}$.

As $\Upsilon[o, fo]$ is a uniformly quasi-convex subset of the Gromov hyperbolic space $\mathcal{C}$, there is a coarsely well defined nearest point projection $\pi : \mathcal{C} \to \Upsilon[o, fo]$. Since $\mathcal{D}, f\mathcal{D}$ are also uniformly quasi-convex and $\Upsilon[S_1, S_4]$ is very far from both while the endpoints satisfy $\Upsilon(o) \in \mathcal{D}$ and $f\Upsilon(o) \in f\mathcal{D}$, we conclude that $\pi(\mathcal{D})$ and $\pi(f\mathcal{D})$ lie on opposite sides of $\Upsilon[S_1, S_4]$ and are far from it.

Consider $S_1 \leq a < b \leq S_4$ and the projections $\alpha := \Upsilon(a)$ and $\beta := \Upsilon(b)$. By hyperbolicity of $\mathcal{C}$ and uniform quasi-convexity of $\Upsilon[o, fo]$, any geodesic joining $\delta_0 \in \mathcal{D}$ to β can be broken into two subsegments $[\delta_0, \beta_0] \cup [\beta_0, \beta]$ where β_0 is uniformly close to $\pi(\delta_0)$, the nearest point projection of δ_0, which has the form $\Upsilon(t)$ for some $o < t < S_1$. By the uniform unparametrized quasi-geodesic image property of Υ, the segment $[\beta_0, \beta]$ passes uniformly close to

α. Therefore, we have

$$d_{\mathcal{C}}(\beta,\delta_0)\geq d_{\mathcal{C}}(\beta,\alpha)+d_{\mathcal{C}}(\alpha,\delta_0)-R_0$$

for some uniform R_0. Taking the minimum over all $\delta_0\in\mathcal{D}$ we get

$$d_{\mathcal{C}}(\Upsilon(b),\mathcal{D})\geq d_{\mathcal{C}}(\Upsilon(a),\Upsilon(b))+d_{\mathcal{C}}(\Upsilon(a),\mathcal{D})-R_0.$$

The last ingredient that we need is the fact that the sequence of curves $\{\psi^n P\}_{n\in\mathbb{Z}}$ lie uniformly close, only depending on ψ, to the uniform quasi-axis of ψ given by the composition Υl_ψ. This follows from work of Minsky [24]. Notice that Υl_ψ lies uniformly close to $\Upsilon[S_1,S_2]$ by fellow traveling assumption. In particular, there are $\psi^{r+n}P$, $\psi^r P$ and $\psi^{r-n}P$ that lie uniformly close, only depending on ψ, to $\Upsilon(S_1)$, $\Upsilon(S_\psi)$ and $\Upsilon(S_2)$ where $S_1<S_\psi<S_2$. The difference $(r+n)-(r-n)=2n$ is bounded from below by some linear function of h of the form $\kappa h-\kappa$ with $\kappa>0$ only depending on ψ.

Therefore, as $S_1<S_\psi$, we have

$$d_{\mathcal{C}}(\psi^r P,\mathcal{D})\geq d_{\mathcal{C}}(\psi^r P,\psi^{r-n}P)+d_{\mathcal{C}}(\psi^{r-n}P,\mathcal{D})-R$$

for some uniform R, only depending on ψ.

For simplicity, we replace P with $\psi^r P$ and still denote it by P.

We now move on to Properties (2) and (4). Again, we consider only (2) as (4) uses the same arguments. Property (2) just follows from the uniform unparametrized quasi-geodesic property of $\Upsilon[S_1,S_4]$.

In more details we proceed as follows: As before, up to replacing P and P' with $\psi^r P$ and $\phi^{r'}P'$, we can assume that $\psi^{-n}P,\psi^n P$ (resp. $\phi^{-n'}P',\phi^{n'}P'$) are uniformly close, only depending on ψ (resp. ϕ), to $\Upsilon(S_1),\Upsilon(S_2)$ (resp. $\Upsilon(S_3),\Upsilon(S_4)$).

Recall that P_1 and P_4 are of the form $P_1=\psi^{-j}P$ and $P_4=\phi^j P'$ for some uniform $j=j(\phi,\psi)$. So, up to a uniform error, we can replace them with P and P'. For them we have

$$d_{\mathcal{C}}(\Upsilon(S_\psi),\Upsilon(S_\phi))\geq d_{\mathcal{C}}(\Upsilon(S_\psi),\Upsilon(S_2))+d_{\mathcal{C}}(\Upsilon(S_2),\Upsilon(S_\phi))-R.$$

□

4. The proof of Theorem 1

In this section we prove

Theorem 2. *For every $\epsilon>0$ and $K>1$ we have*

$$\mathbb{P}_n[M_f \text{ has a hyperbolic metric } K\text{-bilipschitz to a } \epsilon\text{-model metric}]\overset{n\to\infty}{\longrightarrow}1.$$

The proof of Theorem 2 does not use 3-dimensional hyperbolic geometry anymore. Rather, via Proposition 3.9, we will only have to work with the dynamics of a random walk on Teichmüller space and the curve graph.

Thank to the work done in the previous sections, namely Proposition 2.4, Proposition 3.8 and Proposition 3.9, we only need to check that the

Teichmüller segment $[o, fo]$ contains four points $o < S_1 < S_2 < S_3 < S_4 < fo$ satisfying the conditions (i) and (ii) of Proposition 3.9.

In order to find them we will exploit a useful ergodic property of random walks on Teichmüller space due to Baik, Gekhtman and Hamenstädt [1].

4.1. **Random walks.** We start by recalling some background material on random walks on the mapping class group. We crucially consider only random walks driven by probability measures μ whose support S is a finite symmetric generating set for the *full* mapping class group.

Definition (Random Walk). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with values in S and distribution μ. The *n-th step of the random walk* is the random variable $f_n := s_1 \dots s_n$. We denote by $\mathbb{P}_n$ its distribution. The *random walk driven by* μ is the process $(f_n = s_1 \dots s_n)_{n\in\mathbb{N}} \in \mathrm{Mod}(\Sigma)^{\mathbb{N}}$. It has a distribution which we denote by $\mathbb{P}$.

The mapping class group acts on Teichmüller space $\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{T}$. If we fix a base point $o \in \mathcal{T}$ we can associate to every random walk $(f_n)_{n\in\mathbb{N}}$ an orbit $\{f_n o\}_{n\in\mathbb{N}} \subset \mathcal{T}$. This sequence of jumps in Teichmüller space has two important properties, *positive* linear drift and *sublinear* geodesic tracking, which we now explain.

It is a standard consequence of the subadditive ergodic theorem that there exists a constant $L \geq 0$, called the *drift* of the random walk on Teichmüller space, such that for $\mathbb{P}$-almost every sample path $(f_n)_{n\in\mathbb{N}}$ we have

$$\frac{d_{\mathcal{T}}(o, f_n o)}{n} \overset{n\to\infty}{\longrightarrow} L.$$

In general, however, the drift might vanish $L = 0$. It has been established by Kaimanovich and Masur [17] that, in our case, $L > 0$ and that the orbit $\{f_n o\}_{n\in\mathbb{N}}$ converges $\mathbb{P}$-almost surely to some point on the Thurston compactification of Teichmüller space $\mathcal{PML}$. Tiozzo [32] then showed that this convergence happens by sublinearly tracking a Teichmüller ray:

Theorem 4.1 (Tiozzo [32]). *For $\mathbb{P}$-almost every sample path $(f_n)_{n\in\mathbb{N}}$ and for every base point $o \in \mathcal{T}$, there exists a unit speed Teichmüller ray $\tau : [0, +\infty)$ starting at $\tau(0) = o$ and converging to $\mathcal{PML}$ such that*

$$\lim_{n\to\infty} \frac{d_{\mathcal{T}}(f_n o, \tau(Ln))}{n} = 0.$$

More precisely, the sublinear tracking property can be implemented to the following (see, for example, the proof of Proposition 6.11 of [1]): There exists $\delta > 0$ such that for every $\epsilon > 0$ and $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with tracking ray τ we have that the segment $[o, f_n o]$ δ-fellow travels $\tau[0, (1-\epsilon)Ln]$ for every n sufficiently large.

4.2. **Random Teichmüller rays.** We can now state the result from Baik, Gekhtman and Hamenstädt [1] that we need

THEOREM 4.2 (Proposition 6.9 of [1]). *Let $W \subset \mathcal{T}$ be a $\mathrm{Mod}(\Sigma)$-invariant open subset containing the axis of a pseudo-Anosov element. For every $h > 0$ and every $0 < a < b$ and for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with tracking ray τ there exists $n_0 > 0$ such that for every $n \geq n_0$ we have that $\tau[an, bn] \cap W$ contains a segment of length at least h.*

The statement of Theorem 4.2 slightly differs from the one of Proposition 6.9 of [1]. In fact, there it is only considered the case where W is a large metric neighbourghood of the mapping class group orbit of the base point. However, their arguments apply also to the more general setting.

A relevant example of an invariant open set, which is also the smallest possible, is the following: Let ϕ be a pseudo-Anosov mapping class. It determines a closed geodesic in moduli space $\gamma_\phi \subset \mathcal{M} := \mathcal{T}/\mathrm{Mod}(\Sigma)$ and an axis in Teichmüller space $l_\phi \subset \mathcal{T}$.

Fix $\delta > 0$ very small and consider a tiny δ-metric neighbourhood $V_{\phi,\delta}$ of γ_ϕ. Its preimage in Teichmüller space $W_{\phi,\delta} \subset \mathcal{T}$ consists of the union of tiny neighbourhoods of the Teichmüller axes gl_ϕ of the pseudo-Anosov elements $g\phi g^{-1}$ as g varies in the mapping class group.

The set $W_{\phi,\delta}$ has the following useful property

LEMMA 4.3. *For every ϕ and h there exists $\delta = \delta(\phi, h) > 0$ such that if $l : [a, b] \to \mathcal{T}$ is a Teichmüller geodesic segment of length $b - a \geq h$ whose image is entirely contained in $W_{\phi,\delta}$ then it 1-fellow travels one of the lines $gl_\phi \subset W_{\phi,\delta}$.*

Proof. This can be proved by contradiction. Suppose this is not the case and consider a sequence $\delta_n \downarrow 0$ and Teichmüller segments $l_n : [0, h_n] \to W_{\phi,\delta_n}$ of length $h_n \geq h$. Up to the action of the mapping class group, we can assume that $l_n(0)$ lies in a fixed compact set. Up to subsequences, we can then arrange that $l_n[0, h]$ converges uniformly to a Teichmüller geodesic of length h entirely contained in $\bigcap_{n\in\mathbb{N}} W_{\phi,\delta_n}$ which is nothing but the union of the axes gl_ϕ. Hence, l_∞ is a subsegment of one of the axes gl_ϕ and, therefore, at a certain point l_n must have 1-fellow traveled this axis, a contradiction. $\square$

4.3. **The proof of Theorem 2.** Consider the Teichmüller segment $[o, fo]$. Fix $\delta > 0$ large enough. We need to find two pseudo Anosov mapping classes ψ and ψ' with short pants decompositions and four surfaces $o < S_1 < S_2 < S_3 < S_4 < fo$ such that

(i) $[S_1, S_2]$, $[S_3, S_4]$ have length at least h, depending only on δ, ψ, ψ', and δ-fellow travel $l_\psi, l_{\psi'}$.
(ii) $d_{\mathcal{C}}(\Upsilon[S_1, S_4], \mathcal{D}) \geq h$ and $d_{\mathcal{C}}(\Upsilon[S_1, S_4], f\mathcal{D}) \geq h$

We first prove the second property: Recall that we chose $o \in \mathcal{T}$ so that $\delta := \Upsilon(o) \in \mathcal{D}$ (and hence $f\delta = \Upsilon(fo) \in f\mathcal{D}$).

Claim: For every $h > 0$ and $\epsilon > 0$, for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with associated tracking ray τ, there exists n_0 and such that for every $n > n_0$

both $d_{\mathcal{C}}(\Upsilon\tau[\epsilon Ln,(1-\epsilon)Ln],\mathcal{D})$ and $d_{\mathcal{C}}(\Upsilon\tau[\epsilon Ln,(1-\epsilon)Ln],f_n\mathcal{D})$ are greater than h.

Proof of the claim. Denote by $\ell = \lim d_{\mathcal{C}}(\delta, f_n\delta)/n > 0$ the drift of the random walk on the curve graph. It is positive by a result of Maher [19]. The claim is a consequence of

THEOREM 4.4 (Maher [20]). *For every $\epsilon > 0$ we have*

$$\mathbb{P}_n\left[d_{\mathcal{C}}(\mathcal{D}, f\mathcal{D}) \in [(\ell-\epsilon)n,(\ell+\epsilon)n]\right] \xrightarrow{n\to\infty} 1.$$

Choose $\epsilon_2 > \epsilon_1 > 0$ much smaller than ϵ. By Theorem 4.4 we have $d_{\mathcal{C}}(\mathcal{D}, f_n\delta) \geq (\ell-\epsilon_1)n$ for n large.

By Theorem 4.1, the random orbit $f_n o$ has a positive drift $d_{\mathcal{T}}(o, f_n o)/n \to L > 0$ and is sublinearly tracked by a geodesic ray $\tau : [0,\infty) \to \mathcal{T}$ with $\tau(0) = o$. This means that $d_{\mathcal{T}}(f_n o, \tau(Ln)) = o(n)$. We assume $d_{\mathcal{T}}(f_n o, \tau(Ln)) < \epsilon_1 n$ and $\ell - \epsilon_1 < d_{\mathcal{C}}(\delta, f_n\delta) < \ell + \epsilon_1$ for every n large.

Consider $m \in [\epsilon_1 n, (1-\epsilon_2)n]$.

We have the following estimate on the distance from $\mathcal{D}$: Let $B > 0$ be the Lipschitz constant of $\Upsilon : \mathcal{T} \to \mathcal{C}$

$$\begin{aligned} d_{\mathcal{C}}(\Upsilon\tau(Lm),\mathcal{D}) &\geq d_{\mathcal{C}}(f_m\delta,\mathcal{D}) - d_{\mathcal{C}}(f_m\delta,\Upsilon\tau(Lm)) \\ &\geq (\ell-\epsilon_1)m - B\epsilon_1 m \\ &\geq (\ell-\epsilon_1-B\epsilon_1)\epsilon_1 n. \end{aligned}$$

Notice that if ϵ_1 is small enough, the right hand side increases linearly in n with uniform constants.

As for the other disk set $f_n\mathcal{D}$, we also get

$$\begin{aligned} d_{\mathcal{C}}(\Upsilon\tau(Lm), f_n\mathcal{D}) &\geq d_{\mathcal{C}}(\delta, f_n\mathcal{D}) - d_{\mathcal{C}}(\delta, f_m\delta) - d_{\mathcal{C}}(f_m\delta,\Upsilon\tau(Lm)) \\ &\geq (\ell-\epsilon_1)n - (\ell+\epsilon_1)m - B\epsilon_1 m \\ &\geq [(\ell-\epsilon_1) - (\ell+\epsilon_1)(1-\epsilon_2) - B\epsilon_1(1-\epsilon_2)]n. \end{aligned}$$

As before, if ϵ_1 is very small compared to ϵ_2, the right hand side increases linearly in n with uniform constants. In conclusion, if ϵ_1 is small enough and n is large enough, the claim holds as $[\epsilon Ln,(1-\epsilon)Ln] \subset [\epsilon_1 n,(1-\epsilon_2)n]$.

This settles the proof of property (ii) for the segment $\tau[\epsilon Ln,(1-\epsilon)Ln]$. Observe that any subsegment $[S_1^n, S_4^n]$ will enjoy the same property.

We now take care of (i).

Claim: Let ϕ pseudo-Anosov element with a short pants decomposition. Let $l_\phi : \mathbb{R} \to \mathcal{T}$ be its axis. For every $\epsilon > 0$, for every $h > 0$, for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with tracking ray $\tau : [0,\infty) \to \mathcal{T}$ there exists n_0 such that for every $n \geq n_0$ the Teichmüller segments $\tau[\epsilon n, 2\epsilon n]$ and $\tau[(1-2\epsilon)n,(1-\epsilon)n]$ 1-fellow travel along subsegments $\tau[t_1^n, t_2^n]$ and $\tau[t_3^n, t_4^n]$ of length at least h some translates $\psi = g_n l_\phi$ and $\psi' = g_n' l_\phi$ of the axis l_ϕ.

Proof of the claim. Let $\delta = \delta(\phi, h)$ and $W = W_{\phi,\delta}$ be as in Lemma 4.3. We just need to apply Theorem 4.2 to W with parameters $0 < a < b$ given by $0 < L\epsilon < 2\epsilon L$ and $0 < (1-2\epsilon)L < (1-\epsilon)L$ respectively.

Conclusion of the proof: For every fixed $\epsilon > 0$ the Teichmüller segment $[o, f_n o]$ uniformly fellow travels $\tau[\epsilon Ln, (1-\epsilon)Ln]$ we define $o < S_1^n < S_2^n < S_3^n < S_4^n < f_n o$ to be the four surfaces that fellow travel $\tau(t_1^n) < \tau(t_2^n) < \tau(t_3^n) < \tau(t_4^n)$ as given by the second claim.

By construction they satisfy the properties (i) and (ii) of Proposition 3.9 and hence can be used in the model metric construction of Proposition 2.4.

This concludes the proof of Theorem 2.

5. Three applications

We describe three applications of Theorem 2.

5.1. **The model metric.**

For convenience of the reader we recall again the description of the model metric $\mathbb{M}_n = H_1^n \cup \Omega_1^n \cup Q_n \cup \Omega_2^n \cup H_2^n$ that comes from the proof of Theorem 2.

For the applications of this section we focus only on the maximally cusped structure $Q_n = Q(P_2^n, P_3^n)$ and recall that it bilipschitz embeds, away from its cusps, into M_{f_n} with bilipschitz constant arbitrarily close to 1 as n goes to ∞.

A careful inspection shows that P_2^n and P_3^n can be chosen to be very short pants decompositions on some hyperbolic surfaces S_2^n and S_3^n that are located in Teichmüller space uniformly close to the segments $\tau[\epsilon Ln, 2\epsilon Ln]$ and $\tau[(1-2\epsilon)Ln, (1-\epsilon)Ln]$ where $\tau : [0, \infty) \to \mathcal{T}$ is the tracking ray of the random walk and ϵ is an arbitrarily small constant.

Theorem 2.2 allows us to replace, up to uniformly bilipschitz distortion $Q(P_2^n, P_3^n)$ with $Q(S_2^n, S_3^n)$ away from the cusps. It will be convenient to work also with $Q(S_2^n, S_3^n)$ instead of Q_n. They are both geometrically very close to M_{f_n}.

5.2. **Geometric limits of random 3-manifolds.**

As a first application, we exploit the model metric structure to establish the existence of certain geometric limits (see Chapter E.1 of [2] for the definition of the pointed geometric topology) for families of random 3-manifolds.

Proposition 5.1. *For every pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(\Sigma)$ and $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ there exists a sequence of base points $x_n \in M_{f_n}$ such that the sequence of pointed manifolds (M_{f_n}, x_n) converges to the infinite cyclic covering of T_ϕ in the pointed geometric topology.*

Proof. As in the proof of the second claim of Theorem 2, Proposition 4.2 and Lemma 4.3 imply that the segment $\tau[3\epsilon Ln, (1-3\epsilon)Ln]$, and hence $[S_2^n, S_3^n]$, 1-fellow travels a translate $g_n l_\phi$ of the Teichmüller axis l_ϕ along an arbitrarily long subsegment. Therefore, up to remarking $[S_2^n, S_3^n]$, an operation that does not change the isometry type of $Q(S_2^n, S_3^n)$, we can assume that $[S_2^n, S_3^n]$ 1-fellow travels l_ϕ along the subsegment $l_\phi[-a_n, a_n]$ with $a_n \uparrow \infty$. In particular, the sequence of Teichmüller segments $[S_2^n, S_3^n]$ is converging uniformly on compact subsets to l_ϕ. By Thurston's Double Limit Theorem [29] and the solution of the Ending Lamination Conjecture [25], [7], this implies that, if we take suitable base points, the sequence of convex cocompact manifolds $Q(S_2^n, S_3^n)$ converges in the geometric topology to $\hat{T}_\psi$. As $Q(S_2^n, S_3^n)$ becomes geometrically arbitrarily close to M_{f_n}, the claim follows. □

5.3. **Commensurability and arithmeticity.** Dunfield and Thurston, using a simple homology computation, have shown in [13] that their notion of random 3-manifold is not biased towards a certain fixed set of 3-manifolds. This means that for every fixed 3-manifold M, only finitely many elements in the family $(M_{f_n})_{n\in\mathbb{N}}$ can be diffeomorphic to M.

Using geometric tools it is possible to strengthen this conclusions and show that Dunfield and Thurston's notion of random 3-manifolds is also transverse, in a sense made precise in the proposition below, to the class of arithmetic hyperbolic 3-manifolds and to the class of 3-manifolds which are commensurable to a fixed 3-manifold M.

Proposition 4. *For $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ the following holds*

(1) There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are arithmetic.

(2) There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are in the same commensurability class.

Proof. The argument is mostly borrowed from Biringer-Souto [3].

The proof of both points starts from the following observation: Each M_{f_n} finitely covers a maximal orbifold $M_{f_n} \to \mathcal{O}_n$. By Proposition 5.1 we can choose base points $x_n \in M_{f_n}$ so that the sequence (M_{f_n}, x_n) converges geometrically to (Q_∞, x_∞) where Q_∞ is a doubly degenerate structure on $\Sigma \times \mathbb{R}$ with $\mathrm{inj}(Q_\infty) > 0$.

Suppose that infinitely many M_{f_n} are arithmetic, say all of them. In this case, the orbifolds $\mathcal{O}_n$ are congruence and have $\lambda_1(\mathcal{O}_n) \geq 3/4$ (see [11] or Theorem 7.1 in [3]). By Proposition 4.3 of [3], the orbifolds $\mathcal{O}_n$ cannot be all different, hence we can assume that they are fixed all the time $\mathcal{O}_n = \mathcal{O}$. We get a contradiction by observing that $\mathcal{O}$ is covered by closed 3-manifolds M_{f_n} with arbitrarily small injectivity radius.

Suppose that infinitely many M_{f_n} are commensurable. By the first part we can also assume that they are non-arithmetic. Commensurability and non-arithmeticity imply together that $\mathcal{O}_n = \mathcal{O}$ is fixed all the time: It is the orbifold corresponding to the commensurator $\mathrm{Comm}(\pi_1(M_{f_n}))$, which is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$ by Margulis (see Theorem 10.3.5 in [18]) and is an invariant of the commensurability class. We conclude with the same argument as before. □

5.4. **Diameter growth.** As a more geometric application, we compute the coarse growth rate for the diameter of random 3-manifolds.

Proposition 3. *There exists $c > 0$ such that*

$$\mathbb{P}_n[\operatorname{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The proof of Proposition 3 has two different arguments, one for the coarse upper bound and one for the coarse lower bound. The upper bound comes from a result by White [33] that relates the diameter to the *presentation length* of the fundamental group, a topological and algebraic invariant. Of a different nature is the coarse lower bound where we heavily use the ϵ-model metric structure of Theorem 2 and the relation with the model manifold.

We start with the upper bound. We need the following definition

Definition (Presentation Length). Let G be a finitely presented group. The length of a finite presentation $G = \langle F \,|R \rangle$ is given by

$$l(F, R) = \sum_{r \in R} |r|_F - 2$$

where $|r|_F$ denotes the word length of the relator $r \in R$ with respect to the generating set F. The presentation length of G is defined to be

$$l(G) := \min \{ l(F, R) \mid G = \langle F \,|R \rangle \text{ finite presentation} \} .$$

We also recall that a relator $r \in R$ is triangular if $|r|_F \leq 3$.

Theorem 5.2 (White [33]). *There exists $c > 0$ such that for every closed hyperbolic 3-manifold M we have*

$$\operatorname{diam}(M) \leq c \cdot l(\pi_1 M).$$

Let $S \subset \operatorname{Mod}(\Sigma)$ be the finite support of the probability measure μ.

Lemma 5.3. *There exists $C(S) > 0$ such that for every $f \in \operatorname{Mod}(\Sigma)$ we have*

$$l(\pi_1(M_f)) \leq C|f|_S.$$

In particular $\operatorname{diam}(M_f) \leq K|f|_S$ *where* $K = c \cdot C$.

Proof. The 3-manifold M_f admits a triangulation T with a number of simplices uniformly proportional, depending on S, to the word length $|f|_S$. We have $\pi_1(M_f) = \pi_1(T_2)$ where T_2 denotes the 2-skeleton of T. By van Kampen, the fundamental group of a 2-dimensional connected simplicial complex X admits a presentation $\pi_1(X) = \langle F \,|R \rangle$ where every relation is triangular and the number of relations $|R|$ is roughly the number of 2-simplices. □

As a corollary, we get

$$\operatorname{diam}(M_{f_n}) \leq K|f_n = s_1 \dots s_n|_S \leq Kn$$

thus proving the upper bound in Proposition 3.

The coarse lower bound follows from the structure of the model metric and the following estimate that comes from the model manifold technology of Minsky [25].

Proposition 5.4 (Theorem 7.16 of [6]). *For every $L > 0$ there exists $A > 0$ such that the following holds: Let Q be a marked hyperbolic structure on $\Sigma \times \mathbb{R}$. Suppose that $\alpha, \beta \in \mathcal{C}$ have length bounded by $l_Q(\alpha), l_Q(\beta) \leq L$. Then*

$$d_Q(\mathbb{T}_{\eta_M}(\alpha), \mathbb{T}_{\eta_M}(\beta)) \geq A d_{\mathcal{C}}(\alpha, \beta) - A.$$

In particular, if $Q = Q(P_2, P_3)$ is a maximal cusp then the distance between the boundary components of its non-cuspidal part Q^{nc} is at least $A d_{\mathcal{C}}(P_2, P_3) - A$. In the case of random 3-manifolds we have

$$\begin{aligned} d_{\mathcal{C}}(P_2^n, P_3^n) &\simeq d_{\mathcal{C}}(\Upsilon(S_2^n), \Upsilon(S_3^n)) \\ &\geq d(o, f_n o) - d_{\mathcal{C}}(\Upsilon(o), \Upsilon(S_2^n)) - d_{\mathcal{C}}(\Upsilon(S_3^n), f_n \Upsilon(o)) \\ &\simeq \ell n - o(n). \end{aligned}$$

Appendix A. Isotopies of Margulis tubes

We prove the following

Lemma. *For every $\eta < \eta_M/2$ there exists $\xi > 0$ such that the following holds: Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube with core geodesic α of length $l(\alpha) \in [\eta, \eta_M/2]$. Suppose that there exists a $(1+\xi)$-bilipschitz embedding of the tube in a hyperbolic 3-manifold $f : \mathbb{T}_{\eta_M}(\alpha) \to M$. Then $f(\alpha)$ is homotopically non-trivial and it is isotopic to its geodesic representative within $f(\mathbb{T}_{\eta_M}(\alpha))$.*

Proof. The universal cover of $\mathbb{T}_{\eta_M}(\alpha)$ is a a-neighbourhood $N_a(l)$ of a geodesic $l \subset \mathbb{H}^3$. Denote by $F : N_a(l) \to \mathbb{H}^3$ the lift of f to the universal coverings.

By basic hyperbolic geometry, we have that for every subsegment $[p, q] \subset l$ of length $l([p, q]) \leq \eta$, the image $F[p, q]$ is contained in the ϵ-neighbourhood of the geodesic $[F(p), F(q)]$ with $\epsilon = O(\xi)$. This implies, if ξ is sufficiently small, that F restricted to l is a uniform quasi-geodesic. As a consequence f is π_1-injective and $f(\alpha)$ is homotopic to its geodesic representative β within $N_\epsilon(\beta)$ with $\epsilon = O(\xi)$. We want to show that $f(\alpha)$ is actually isotopic to β.

The proof can now be concluded using topological tools.

Up to a very small isotopy we can assume that $f(\alpha)$ is disjoint from β and still contained in $N_\epsilon(\beta)$. For safety, we assume that an entire metric tubular neighbourhood of $f(\alpha)$ of the form $f(N_\delta(\alpha))$ for some tiny δ is disjoint from β and contained in $N_\epsilon(\beta)$.

Since the radius of the tube $f(\mathbb{T}_{\eta_M}(\alpha))$ is large, we can assume that a metric tubular neighbourhood of β of the form $N_r(\beta)$ with $r > \epsilon$ is contained in $f(\mathbb{T}_{\eta_M}(\alpha))$. Denote by $T_\beta = \partial N_r(\beta)$ its boundary and observe that $T_\beta \subset f(\mathbb{T}_{\eta_M}(\alpha)) - f(N_\delta(\alpha))$. The complementary region $f(\mathbb{T}_{\eta_M}(\alpha)) - f(N_\delta(\alpha))$ is diffeomorphic to $T_\alpha \times [0, 1]$ where T_α is a 2-dimensional torus.

Notice that T_β is incompressible in $T_\alpha \times [0,1]$. In fact, the only possible compressible curve on T_β is the boundary ∂D_β of the compressing disk D_β of the tubular neighbourhood of $N_r(\beta)$. Every other simple closed curve is homotopic in $f(\mathbb{T}_{\eta_M}(\alpha))$ to a multiple of $\beta \simeq f(\alpha)$ and hence it is not trivial. However, the curve ∂D_β cannot be compressible in $T_\alpha \times [0,1]$ otherwise it would bound a disk D'_β with interior disjoint from D_β and together they would give a 2-sphere $S^2 \simeq D_\beta \cup D'_\beta$ intersecting once β. Such a sphere is homologically non trivial in $f(\mathbb{T}_{\eta_M}(\alpha))$, but a solid torus does not contain such an object.

By standard 3-dimensional topology, incompressibility implies that T_β is parallel to $T_\alpha \times \{1\} = f(\partial \mathbb{T}_{\eta_M}(\alpha))$. Therefore, β is the core curve $\beta \simeq 0 \times S^1$ for another product structure $f(\mathbb{T}_{\eta_M}(\alpha)) \simeq D^2 \times S^1$ or, in other words, there exists an orientaion preserving self diffeomorphism of $f(\mathbb{T}_{\eta_M}(\alpha))$ that sends $f(\alpha)$ to β. Such a diffeomorphism is isotopic to a power of the Dehn twist along the meridian disk of the solid torus, hence it does not change the isotopy class of the core curve.

This concludes the proof. $\square$

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