Mapping Properties of Bäcklund Transformations and the Asymptotic Stability of Soliton Solutions for the Nonlinear Schrödinger and Modified Korteweg-de-Vries Equation

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Summary

We consider the cubic Nonlinear Schrödinger Equation (NLS) and the Modified Korteweg-de-Vries Equation (mKdV) in the one-dimensional, focusing case. For the mKdV, we also restrict ourselves to the case of real-valued solutions. The Lax formalism for the Nonlinear Schrödinger Hierarchy gives rise to a Bäcklund transformation, which connects the trivial zero solution to the elementary soliton solution for both equations.

Following an approach pioneered by Mizumachi and Pelinovsky, this thesis uses the Bäcklund transformation to prove asymptotic stability of NLS- and mKdV-solitons by showing that known stability properties of the zero solution transfer to the solitons. An important feature of the argument presented here is that it proceeds by relatively elementary techniques, without invoking the Riemann-Hilbert-formalism of inverse scattering theory.

For the essential asymptotic stability of the zero solution, we will invoke results by Ifrim and Tataru and by Harrop-Griffiths, which also provide asymptotic expressions for the potentials under consideration. This will enable us to understand the behaviour of the Jost solutions for the corresponding Lax systems as the time $t \to \infty$. Our asymptotic stability results contain time-dependent position, and in the NLS case, phase shift functions, which we will show to converge to constant values as $t \to \infty$ on the basis of these findings. It is a particularly notable point that we will be able to show quantitative estimates for how fast position and phase will go to their respective limits. In the mKdV case, we will actually be able to show convergence properties of the Jost solutions beyond what is necessary for our stability arguments, which could potentially be useful in applying the methods of this thesis to other special mKdV-solutions.

A large portion of the arguments given should generalize to other equations in the NLS hierarchy, provided a suitable asymptotic stability result for the zero solution is available.

Contents

Chapter 1

Introduction

The family of *Nonlinear Schrödinger Equations* is used to model a wide variety of wave phenomena, ranging from the theory of water waves to nonlinear optics (compare e.g. [2], [10], [17]). In the following we will concern ourselves with the initial value problem for the one-dimensional, cubic, focusing equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{1.1}$$

with $u = u(t, x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ a complex-valued function with $u(0, \cdot) = u_0$[1]. Unless otherwise indicated, we will use the abbreviation "NLS" exclusively to refer to this case.

The formalism of inverse scattering theory, originally developed in [1], gives rise to a map that will send any solution of NLS to another solution of the same equation, called a *Bäcklund transformation.* Bäcklund transformations are intimately connected to a class of particle-like solutions called *solitons.* In [22], Mizumachi and Pelinovsky used the Bäcklund transformation for NLS to show L^2*-stability* of soliton solutions. Their proof was based on reduction of the problem to the L^2-stability of the zero solution (which is known by conversation of L^2-energy). There are also several *asymptotic stability* results for the zero solution available (e.g. [13], [16] and several others). It is therefore a natural question if these, too, transfer to soliton solutions via the Bäcklund transformation, which Mizumachi and Pelinovsky left open at the end of their paper. In [8], Cuccagna and Pelinovsky gave one such proof. Their argument is partially based on an asymptotic stability result for the zero solution of their own, which at the same time provides greater generality in the sense that it applies to all pure radiation solutions. To do so, they rely on an explicit analysis of the *Riemann-Hilbert problem* from inverse scattering theory.

[1] For simplicity, we restrict ourselves to positive times t in the first argument.

Let $H^{0,1}(\mathbb{R})$ be the function space defined by the norm $\|f\|_{H^{0,1}(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} + \|xf\|_{L^2(\mathbb{R})}$. In Theorem 5.2 of this thesis, we will give a proof of asymptotic stability of NLS solitons with initially small $H^{0,1}$-norm that does not rely on analysis of the Riemann-Hilbert problem[2] and thus provides a more elementary alternative to [8]. (It should be emphasized that, while we are going to restrict ourselves to $H^{0,1}(\mathbb{R})$ for simplicity, similar arguments work for all other spaces to which Remark 2.4 applies.)

In order to achieve this (in Theorem 5.2), we will follow in the footsteps of [22] in analyzing the properties of the *Jost solutions*[3] for the spatial part of the *Lax system* from inverse scattering theory (see (3.1), (3.2) and the following discussion in Chapter 3):

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\zeta & u \\ -\overline{u} & i\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.2}$$

where $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) > 0$, $\psi = \psi(x) : \mathbb{R} \to \mathbb{C} \times \mathbb{C}$ (actually $\psi(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C} \times \mathbb{C}$, considered at a fixed timepoint $t \in \mathbb{R}_+$ here), and we are mostly concerned with the case that the potential u is a solution of NLS in an appropriate function space. The Jost solutions behave like free solutions for potential $u = 0$ as $x \to \infty$ or $-\infty$: The *left Jost solution* behaves like $e^{-i\zeta x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $x \to -\infty$, the *right Jost solution* like $e^{i\zeta x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $x \to \infty$.

We will show that in the case of a small L^∞-norm of $u \in L^2(\mathbb{R})$, the "large" component that dominates the Jost solution at $+\infty$ or $-\infty$ dominates everywhere, in a manner controlled by the L^∞-norm. The decisive arguments are in Proposition 5.8 and Claim 5.9, the latter of which plays a similar role to Lemma 4.3 in [8]. While Cuccagna and Pelinovsky employ their detailed analysis of pure radiation NLS solutions, we will use an argument that employs Gronwall's inequality and the boundedness of the Jost solutions on one side to show Claim 5.9. An interesting point to note is that Claim 5.9 only depends on the L^∞-smallness of the potential. Again using Gronwall's inequality, we will utilize this result to understand a suitable sense in which the left and right Jost solution behave like exponential functions (see especially (5.27) and (5.28)).

The arguments outlined in the previous paragraph yield a preliminary result where our limit still includes time-dependent position and phase parameters, which we can characterize in terms of the absolute values of the Jost solutions on the real line. A particularly important feature of Theorem 5.2 is that we show convergence (as $t \to \infty$) of these parameters with quantitative

[2] Another difference to [8] is how the pullback at $t = 0$ is established in Lemma 5.3.

[3] Although Mizumachi and Pelinovsky define the solutions slightly differently than we are going to do in the following. For clarification, see Remark 4.2.

estimates for the rate, given in (5.3) and (5.4). To do so, we will make use of [16], where Tataru and Ifrim derived an asymptotic stability result for solutions with small $H^{0,1}$-norms at $t = 0$ without the aid of inverse scattering theory, and especially, they obtained an asymptotic expression for the solution (we give their findings in Theorem 2.3). This can be used to understand the behaviour of the Jost solutions as $t \to \infty$ (Lemma 5.10) and show the desired convergence properties of position and phase.

Another nonlinear partial differential equation which admits a family of soliton solutions is the modified Korteweg-de-Vries equation (mKdV in the following) which we pose for a real-valued[4] function $u = u(t, x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$:

$$u_t + u_{xxx} + 2(u^3)_x = 0, \tag{1.3}$$

where $u(0, \cdot) = u_0$.

This equation, too, has an associated Lax system, with the spatial part (1.2) the same as in the NLS case. And just like for the NLS, asymptotic stability results for the zero solution are available, see [12], [14], [15]. We will employ the main theorem from [12], in which Harrop-Griffiths showed such a result without relying on inverse scattering theory, including an asymptotic expression. This can be used for an asymptotic stability argument similar to the NLS case, as we will do in Theorem 6.1. While we have to treat a phase and a position shift when showing the asymptotic stability of (1.1), we only have to deal with a time dependent position shift function in the mKdV case. Showing convergence of this function to a constant value as $t \to \infty$ turns out to be easier than in the proof for NLS (once again, note the quantitative estimate for the rate of convergence, in this case given by (6.2)). This is because the center of an mKdV soliton is near $x = t$ on the real line, where Theorem 2.5 provides particularly sharp bounds. Unlike in the NLS result Theorem 5.2, we therefore do not need to show pointwise convergence of the Jost solutions to a "long term profile" as $t \to \infty$. However, such convergence does hold, as will be proved in Proposition 6.2. Some other results concerning the asymptotic stability of mKdV solitons have recently been published by Chen and Liu ([5], [6]). Similarly to [8], they make use of the Riemann-Hilbert formalism and the steepest descent method. It also appears that their proof needs stronger assumptions than the one presented in this thesis.

As we will discuss in Chapter 3, the systems of (1.1), (1.3) are connected with a multitude of other equations via the *NLS hierarchy*, and indeed, a significant part of the argument presented in Chapter 5 (particularly the

[4]The reason we restrict ourselves to the real-valued case is that Theorem 2.5 is only available for real-valued solutions.

proof of Proposition 5.8) would transfer to other NLS hierarchy equations with little modification, provided a suitable stability result for the zero solution is available, and particularly if asymptotic expressions as in Theorems 2.3 and 2.5 are given.

This thesis is organized as follows: The second chapter will introduce the results for asymptotic stability of zero solutions for NLS and mKdV, as well as give a brief introduction to the stability problem for soliton solutions. The third chapter will explain the NLS hierarchy and the Bäcklund transformation for NLS and mKdV, while the fourth chapter will expand on some basic facts about Jost solutions we are going to need, which will immediately put us in a position to understand some useful properties of Bäcklund transformations. Chapter 5 will show asymptotic stability of the NLS soliton solution in the manner sketched above. Chapter 6 will deal with the application of our stability arguments to mKdV and give a proof of Proposition 6.2. The latter might potentially be useful in the application of our methods to other mKdV solutions, such as *breather solutions* which can be constructed via two iterations of a Bäcklund transformation (see [19], Chapter 5).

Chapter 2

The Stability Problem for special NLS and mKdV Solutions

This chapter will provide some basic facts about the stability of *soliton solutions*. Our goal is to analyze the stability of NLS and mKdV on the line by using *Bäcklund transformations*. This is accomplished by linking soliton stability to known stability theorems for the trivial zero solution. As already discussed in the Introduction, we will use the results from [12] and [16], which we will provide at the end of this chapter in Theorems 2.3 and 2.5.

One solution of the NLS equation:

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

is the elementary soliton

$$u(t,x) = e^{it} \operatorname{sech}(x) \tag{2.1}$$

The original motivation for the theory of solitons was the observation by John Scott Russel in the 19th century that certain localized water waves in a narrow channel retain their form for an extended amount of time. Russel dubbed his discovery a "wave of translation", although the term *solitary wave* is more common today. It is called a *soliton* if it additionally retains its form after collision with other solitary waves or radiation (see [24]). In a nonlinear setting, a soliton is generally a solution for which (attractive) nonlinear effects and dispersion of wave solutions cancel out.

Two well-known symmetries of the NLS are under scaling

$$u(t,x) \to ku(k^2 t, kx), k \in \mathbb{R} \tag{2.2}$$

and the Galilei transform

$$u(t,x) \to e^{ivx-iv^2t}u(t,x-2vt), v \in \mathbb{R}, \tag{2.3}$$

i.e. if u is a solution, so are the above transformations of u. These symmetries generate a whole family of soliton solutions from (2.1):

$$Q_{k,v}(t,x) = k\,\mathrm{sech}(k(x-vt))e^{i\frac{vx}{2}+i(k^2-\frac{v^2}{4})t}, k \in \mathbb{R}, v \in \mathbb{R} \tag{2.4}$$

A (not quite sharply defined, see [24]) conjecture or general expectation on soliton solutions is that they exhibit *stability properties*, i.e. "closeness" of a solution to a soliton at $t = 0$ implies closeness at any other $t \in \mathbb{R}_+$ (*orbital stability*) or even convergence in some function norm (*asymptotic stability*) - of course, the definition of "closeness" must be specified. For (2.4), the Bäcklund transform was used in [22] to show orbital stability in the following, particularly strong sense:[1]

Theorem 2.1. (Mizumachi/Pelinovsky) *Let $(k,v) \in \mathbb{R}_+ \times \mathbb{R}$. If a solution $u(t,x) \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^8_{loc}(\mathbb{R}, L^4(\mathbb{R}))$ of the NLS equation (1.1) satisfies $\|u(0,\cdot) - Q_{k,v}(0,\cdot)\|_{L^2(\mathbb{R})} \leq \epsilon$ for sufficiently small ϵ, then for all $t \in \mathbb{R}$, we have constants $(k_0, v_0, t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ such that*

$$\|u(t+t_0, \cdot + x_0) - Q_{k_0,v_0}\|_{L^2} \lesssim \|u(0,\cdot) - Q_{k,v}(0,\cdot)\|_{L^2(\mathbb{R})},$$

and

$$|k_0 - k| + |v - v_0| + |t_0| + |x_0| \lesssim \|u(0,\cdot) - Q_{k,v}(0,\cdot)\|_{L^2(\mathbb{R})} \tag{2.5}$$

holds.

Notice that the soliton that the NLS solution u in Theorem 2.1 remains "close" to is different from the soliton that it is close to at $t = 0$. Instead it has, and needs to have, slightly different parameters k_0, v_0, t_0 and x_0 (which depend on the specific u under consideration), to incorporate the symmetries (2.2) and (2.3). However, we can control the closeness of these parameters to the original soliton parameters by (2.5). While the optimization over small t_0 and x_0 is, strictly speaking, unnecessary in Theorem 2.1, phase and position shifts will be relevant for our treatment of asymptotic stability.

The mKdV (for a real-valued[2] function u)

$$u_t + u_{xxx} + 2(u^3)_x = 0$$

[1] Mizumachi and Pelinovsky formulate Theorem 2.1 specifically for $v = 0$, from which the statement for the full soliton group can easily be obtained by translation invariance of integrals and additivity properties of the Galilei transform.

[2] The more general form for a complex-valued u is

$$u_t + u_{xxx} + 6|u|^2u_x = 0$$

has an elementary soliton solution

$$u(t,x) = \operatorname{sech}(x-t) \tag{2.6}$$

Similar to the NLS case, the symmetry under the scaling

$$u(t,x) \to ku(k^3t, kx), k \in \mathbb{R} \tag{2.7}$$

as well as shifts in position $u(t,x) \to u(t,x+x_0)$, $x_0 \in \mathbb{R}$, generates a family of solitons from (2.6):

$$Q_{k,x_0}(t,x) = k \operatorname{sech}(kx - k^3t + x_0), k \in \mathbb{R}, x_0 \in \mathbb{R}, \tag{2.8}$$

for which, again, we ask the stability question.

It is trivial that $u = 0$ is a solution of both NLS and mKdV. The L^2-norm of any NLS and mKdV solution is conserved, as can be seen formally by differentiating $\int |u|^2 dx$ under the integral sign ([4] and[20] are standard references here), which establishes that the zero solution is orbitally stable in L^2 in a very straightforward sense: If the initial data $u(0,x) = u_0$ are L^2-close to 0, so is $u(t,x)$ at any time $t > 0$. Additionally, various *asymptotic stability* results for NLS exist, i.e. if u_0 is close to zero in certain spaces, its L^∞-norm will decay to zero as $t \to \infty$. The formulation most relevant to this thesis is due to Ifrim and Tataru in [16] (we leave out some of their findings):

Definition 2.2. *The function space $H^{r,s}(\mathbb{R})$ is the closure of $C_0^\infty(\mathbb{R})$ under the norm $\|f\|_{H^{r,s}(\mathbb{R})} = \|(1+|\cdot|^2)^{\frac{r}{2}}\hat{f}(\cdot)\|_{L^2(\mathbb{R})} + \|(1+|\cdot|^2)^{\frac{s}{2}}f(\cdot)\|_{L^2(\mathbb{R})}$*

On $H^{0,1}(\mathbb{R})$, we mostly use the equivalent norm $\|f\|_{H^{0,1}(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} + \|xf\|_{L^2(\mathbb{R})}$ for $f = f(x) \in H^{0,1}(\mathbb{R})$.

Theorem 2.3. (Ifrim/Tataru) *There is $\epsilon > 0$ such that for an initial datum $u_0 \in H^{0,1}(\mathbb{R})$ with $\|u_0\|_{H^{0,1}(\mathbb{R})} \le \epsilon$, we have a unique solution $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ with $e^{-\frac{it}{2}\partial_x^2}u \in C(\mathbb{R}, H^{0,1}(\mathbb{R}))$ of the Nonlinear Schrödinger Equation (1.1) such that $u(0,\cdot) = u_0$. This solution satisfies the estimate*

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim \epsilon |t|^{-\frac{1}{2}}$$

for all $t > 0$. Moreover, there is an asymptotic expression

$$u(t,x) = t^{-\frac{1}{2}} e^{i\frac{x^2}{2t}} W\left(\frac{x}{t}\right) e^{i\log(t)|W(\frac{x}{t})|^2} + err_x(t,x) \tag{2.9}$$

with a complex-valued function $W \in H^{1-C\epsilon^2}(\mathbb{R})$ satisfying $\|W\|_{H^{1-C\epsilon^2}(\mathbb{R})} \lesssim \epsilon$ for a fixed $C > 0$ and $err_x(t,\cdot) \in O_{L^\infty(\mathbb{R})}\Big((1+t)^{-\frac{3}{4}+C\epsilon^2}\Big) \cap O_{L^2(\mathbb{R})}\Big((1+t)^{-1+C\epsilon^2}\Big)$.

Remark 2.4. *As per Remark 1.1 in [16], an analogue to Theorem 2.3 can be shown for initial data in all spaces* $H^{0,s}(\mathbb{R})$ *with* $s \in (\frac{1}{2}, 1]$.

For the mKdV (real-valued case)

$$u_t + u_{xxx} + 2(u^3)_x = 0, \tag{2.10}$$

we have by [12]:

Theorem 2.5. (Harrop-Griffiths) *There is an* $\epsilon > 0$ *such that if an initial datum* $u_0 \in H^{1,1}(\mathbb{R})$ *satisfies* $\|u_0\|_{H^{1,1}(\mathbb{R})} \leq \epsilon$, *there exists a unique global solution* $u(t,x)$ *of* (2.10) *with* $u(0,\cdot) = u_0$ *and*[3]

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim \epsilon t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{-\frac{1}{4}},$$

as well as asymptotics as $t \to \infty$ *(with any real number* $\rho \in [0, \frac{1}{3}(\frac{1}{6} - C\epsilon^2)]$*):*

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}x)^{\frac{3}{4}} u\|_{L^\infty(\Omega_\rho^+)} \lesssim \epsilon \quad \|t^{\frac{1}{6}}(t^{-\frac{1}{3}}x) u\|_{L^2(\Omega_\rho^+)} \lesssim \epsilon \tag{2.11}$$

on $\Omega_\rho^+ = \{x > 0 : t^{-\frac{1}{3}} x \gtrsim t^{2\rho}\}$, *called the* decaying region,

$$u(t,x) = t^{-\frac{1}{3}} Q(t^{-\frac{1}{3}}x) + err(x), \tag{2.12}$$

on the self-similar region $\Omega_\rho^0 = \{x \in \mathbb{R} : t^{-\frac{1}{3}}|x| \lesssim t^{2\rho}\}$, *where* $Q : \mathbb{R} \to \mathbb{R}$ *with* $|Q| \lesssim \epsilon$ *is a solution of the Painlevé II equation* $yQ - Q_{yy} - 3Q^3 = 0$ *and*

$$err(x) \in O_{L^\infty(\Omega_\rho^0)}\left(\epsilon t^{-\frac{1}{2}(\frac{5}{6} - C\epsilon^2)}\right) \cap O_{L^2(\Omega_\rho^0)}\left(\epsilon t^{-\frac{2}{3}(\frac{5}{12} - C\epsilon^2)}\right), \tag{2.13}$$

Finally, on the oscillatory region $\Omega_\rho^- = \{x < 0 : t^{-\frac{1}{3}}|x| \gtrsim t^{2\rho}\}$*:*

$$u(t,x) = \pi^{-\frac{1}{2}} t^{-\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} \cdot \\ \cdot \operatorname{Re}\left(e^{i\alpha(t,x) + \frac{3i\sigma}{4\pi}|W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})|^2 \log(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})} W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})\right) + E(x), \tag{2.14}$$

with $W \in H^{1-C\epsilon^2,1}(\mathbb{R})$ *a complex-valued function on the reals satisfying* $\|W\|_{H^{1-C\epsilon^2,1} \cap L^\infty(\mathbb{R})} \lesssim \epsilon$ *and* $\alpha(t,x) = -\frac{2}{3} t^{-\frac{1}{2}} |x|^{\frac{3}{2}} + \frac{\pi}{4}$. *The error function* E *satisfies the estimates*

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{\frac{3}{8}} E(x)\|_{L^\infty(\Omega_\rho^-)} \lesssim \epsilon, \|t^{\frac{1}{6}}(t^{-\frac{1}{3}}|x|)^{\frac{1}{4}} E(x)\|_{L^2(\Omega_\rho^-)} \lesssim \epsilon$$

The leading terms of the asymptotic expressions in both Theorem 2.3 and Theorem 2.5 relate to analysis of the corresponding linearized equations.

[3] $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$ denotes the usual Japanese brackets.

Chapter 3

The NLS Hierarchy and the Bäcklund Transformation

In the following two chapters, we will review the basic facts associated to Bäcklund transformations for the Nonlinear Schrödinger hierarchy which are of interest in this thesis. This theory is standard in the relevant literature. For reference, see particularly [11], [19], [21] and also e.g. [8], [18] or [22].

Consider the following system of first order PDEs for a function[1] $\psi = \psi(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C} \times \mathbb{C}$, referred to as a *Lax system* in the following:

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\zeta & u \\ -\overline{u} & i\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.1}$$

and

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = i \begin{pmatrix} -2\zeta^2 + |u|^2 & \partial_x u - 2i\zeta u \\ -\partial_x \overline{u} + 2i\zeta\overline{u} & 2\zeta^2 - |u|^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.2}$$

with a parameter $\zeta \in \mathbb{C}$ and a potential $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$. Some other ways of writing this system can be found in the literature, e.g. with parameter $\eta := i\zeta$, or as two equivalent Riccati equations for $\gamma := \frac{\psi_1}{\psi_2}$ (see [22]). Solutions of (3.1) and (3.2), or just solutions of (3.1), considered at a fixed time t, are often referred to as *wave functions*. When we consider (3.1) at a fixed time (or just in isolation as an ODE system for $\psi = \psi(t,x) : \mathbb{R} \to \mathbb{C} \times \mathbb{C}$), we often suppress the t-dependence in our notation.

Let us assume that a sufficiently regular solution ψ exists, so that we have to demand interchangeability of the order of differentiation:

$$\partial_t \partial_x \psi = \partial_x \partial_t \psi. \tag{3.3}$$

[1]As mentioned before, we restrict ourselves to positive times for simplicity and physicality, which is not necessarily done in other literature on the topic.

From this compatibility condition, it can be shown that u must be a solution of the NLS equation. For $\partial_x \psi = A\psi$ and $\partial_t \psi = B\psi$, this is equivalent to the *zero curvature condition*:

$$\partial_x B - \partial_t A - [A, B] = 0, \tag{3.4}$$

where $[\cdot, \cdot]$ denotes the commutator of two operators.

The previous statements hold for mKdV if we replace (3.2) by

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = B(u, \zeta) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.5}$$

with

$$B(u, \zeta) = \begin{pmatrix} -4i\zeta^3 + 2i\zeta|u|^2 - 2i\,\mathrm{Im}(u_x\overline{u}) & 4\zeta^2 u + 2i\zeta u_x - (u_{xx} + 2|u|^2 u) \\ -4\zeta^2\overline{u} + 2i\zeta\overline{u}_x + (\overline{u}_{xx} + 2|u|^2\overline{u}) & 4i\zeta^3 - 2i\zeta|u|^2 + 2i\,\mathrm{Im}(u_x\overline{u}) \end{pmatrix} \tag{3.6}$$

When u is real-valued, (3.6) reduces to:

$$B(u, \zeta) = \begin{pmatrix} -4i\zeta^3 + 2i\zeta u^2 & 4\zeta^2 u + 2i\zeta u_x - (u_{xx} + 2u^3) \\ -4\zeta^2 u + 2i\zeta u_x + (u_{xx} + 2u^3) & 4i\zeta^3 - 2i\zeta u^2 \end{pmatrix} \tag{3.7}$$

This is part of a more general framework known as the *(focusing) NLS hierarchy*. We follow [18], Appendix C in this brief outline: Suppose we have the equation

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\zeta & u \\ -\overline{u} & i\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.8}$$

with potential u and parameter $\zeta \in \mathbb{C}$. We then look for a matrix $B(u, \zeta)$ such that the compatibility condition (3.3) arising from (3.8) and

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = B(u, \zeta) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.9}$$

is a partial differential equation for u. This will be true if we have

$$\left[\partial_t - B(u, \zeta), \partial_x - \begin{pmatrix} -i\zeta & u \\ -\overline{u} & i\zeta \end{pmatrix} \right] = \begin{pmatrix} 0 & \partial_t u - F \\ -(\partial_t u - F) & 0 \end{pmatrix} \overset{!}{=} 0, \tag{3.10}$$

with $[A, B] = AB - BA$ denoting the commutator of two operators A and B and the function F being a polynomial in the derivatives of u and $\overline{u}$. Indeed, when inserting the equations (3.8) and (3.9) into (3.3) it follows quickly that the left-hand side of (3.10) must be zero, so we would obtain

$\partial_t u - F(u) = 0$ as our desired partial differential equation. We make the ansatz $B(\zeta, u) = \beta_k(\zeta, u) = \sum_{j=0}^{k} \zeta^{k-j} Q_j(u)$ for $k \in \mathbb{N}_0$, where each Q_j is a 2×2 matrix in the special unitary group $SU(2)$. By inserting this into the commutator in (3.10), we get

$$\partial_t \begin{pmatrix} -i\zeta & u \\ -\overline{u} & i\zeta \end{pmatrix} - \partial_x \sum_{j=0}^{k} \zeta^{k-j} Q_j(u) + \left[\begin{pmatrix} -i\zeta & u \\ -\overline{u} & i\zeta \end{pmatrix}, \sum_{j=0}^{k} \zeta^{k-j} Q_j(u) \right] \overset{!}{=} 0$$

We want to have the left side depend only on u, not ζ. To achieve this, it is sufficient to demand

$$Q_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

a recursive relation:

$$\left[\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, Q_{j+1} \right] = \partial_x Q_j + \left[Q_j, \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix} \right], 0 \leq j \leq k-1$$

and that

$$\partial_x Q_k + \left[Q_k, \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix} \right]$$

is off-diagonal. Thus, we can write $Q_k = \begin{pmatrix} -ir_k & p_k \\ -\overline{p}_k & ir_k \end{pmatrix}$ with

$$\begin{aligned} p_{k+1} &= \frac{i}{2} p_k' + r_k u \\ r_k' &= i(p_k \overline{u} - \overline{p}_k u) \end{aligned}$$

It can be shown that r can, for any k, be expressed as a polynomial in the derivatives of u and $\overline{u}$. The first few steps of the recursion give $r_0 = 1$, $p_0 = 0$, $p_1 = u$, $r_1 = 0$, $p_2 = \frac{i}{2} u'$, $r_2 = -\frac{1}{2}|u|^2$, $p_3 = -\frac{1}{4}(u'' + 2|u|^2 u)$, $r_3 = -\frac{i}{4}(u'\overline{u} - u\overline{u}')$. The cases $k = 2$ and $k = 3$ correspond to the NLS and mKdV case, respectively[2]. (We get (3.2) by demanding u to be real-valued.) By $\overline{u} \to -\overline{u}$ we would have obtained the *defocusing NLS hierarchy* and by $\overline{u} \to 1$ the *KdV hierarchy*. We note once again that the stability arguments for NLS and mKdV would largely transfer to any equation in the NLS hierarchy, provided one had the analogue of Theorem 2.3 and 2.5 for the zero solution.

[2]Strictly speaking, the versions of NLS and mKdV obtained with these p_k, r_k are rescaled in the time parameter compared to how we have stated these equations. We choose simplicity over consistency in our notation here.

If $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ is a solution of (3.1) and (3.2), or respectively, (3.5) with parameter ζ and a potential u that is a solution of NLS or mKdV (depending on the matrix in the "time part" of the system), then $\begin{pmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix}$ with

$$\tilde{\psi}_1 = \frac{\overline{\psi}_2}{\|\psi\|^2} \quad \tilde{\psi}_2 = -\frac{\overline{\psi}_1}{\|\psi\|^2} \tag{3.11}$$

is a solution of the analogous system with parameter ζ and potential

$$\tilde{u}(t,x) = u(t,x) + 4\,\mathrm{Im}(\zeta)\frac{\psi_1\overline{\psi}_2}{\|\psi\|^2} \tag{3.12}$$

The mapping $u \to \tilde{u}$ defined by (3.12) is called an *(auto-)Bäcklund transformation* of u. The term "Bäcklund transformation" is used in various contexts to denote mappings between solutions of various partial differential equations - e.g. if v is a solution of the Laplace equation and ϕ the (unique) holomorphic function with $\mathrm{Re}(\phi) = v$, the mapping $v \to \tilde{v} := \mathrm{Im}(\phi)$ is sometimes (e.g. in [23]) referred to as a Bäcklund transformation for the Laplace equation, which $\tilde{v}$ solves. Indeed, since we have shown that the Lax system for NLS (or mKdV, respectively) is only solvable if its potential is a solution of NLS (respectively mKdV) by (3.4). Thus, (3.12) must necessarily map a solution of the partial differential equation arising as a compability condition for the Lax system to another such solution.

Writing the transformation (3.12) with potential u, parameter ζ and solution ψ of (3.1) and (3.2) as $B(u,\zeta,\psi)(t,x)$, we first note the elementary, but useful, property

$$B(u,\zeta,c\psi)(t,x) = B(u,\zeta,\psi)(t,x),\ \forall c \in \mathbb{C}\backslash\{0\} \tag{3.13}$$

i.e. the transformation is invariant under multiplication of the wave function with a constant. Moreover, with $B(u,\zeta,\psi)(t,x) =: \tilde{u}(t,x)$, the Bäcklund transformation satisfies the *reiteration relation*

$$u(t,x) = B(\tilde{u},\zeta,\tilde{\psi})(t,x) \tag{3.14}$$

with $\tilde{\psi}$ defined by (3.11), which allows us to recover the original potential u from $\tilde{u}$. Similarly to what we remarked following (3.1) and (3.2), we may also suppress the t-dependence of the Bäcklund transformation in the following, whenever we consider (3.1) at a fixed time $t \in \mathbb{R}_+$ (or in isolation), or leave the (t,x)-argument out entirely in our notation.

As already discussed, our goal is to use the Bäcklund transformation to transfer stability properties of the trivial zero solution of NLS and mKdV to the soliton solutions. For the NLS system, (3.1) and (3.2) with $u = 0$ and parameter ζ become:

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\zeta & 0 \\ 0 & i\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.15}$$

and

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = i \begin{pmatrix} -2\zeta^2 & 0 \\ 0 & 2\zeta^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.16}$$

Now, (3.15) has a fundamental system of solutions consisting of

$$\begin{pmatrix} e^{-i\zeta x} \\ 0 \end{pmatrix} \tag{3.17}$$

and

$$\begin{pmatrix} 0 \\ e^{i\zeta x} \end{pmatrix}, \tag{3.18}$$

and in particular, $\begin{pmatrix} e^{-i\zeta x} \\ e^{i\zeta x} \end{pmatrix}$ is a solution. For a given initial value $\psi(0, \cdot)$, equation (3.16) is solved by $\psi(t, x) = \begin{pmatrix} e^{-2i\zeta^2 t}\psi_1(0, x) \\ e^{2i\zeta^2 t}\psi_2(0, x) \end{pmatrix}$. Together, this gives one solution:

$$\psi(t, x) = \begin{pmatrix} e^{-2i\zeta^2 t} e^{-i\zeta x} \\ e^{2i\zeta^2 t} e^{i\zeta x} \end{pmatrix}$$

of (3.15) and (3.16). If $\zeta = \frac{i}{2}\Big(k + iv\Big)$ with $k \in \mathbb{R}_+$, $v \in \mathbb{R}$, plugging this ψ and $u = 0$ into (3.12) gives the solution

$$Q_{k,v}(t, x) = k \operatorname{sech}(k(x - vt))e^{i\frac{vx}{2} + i(k^2 - \frac{v^2}{4})t},$$

recovering the soliton group (2.4). Similarly, we obtain the soliton group (2.8) for mKdV from the solution

$$\psi(t, x) = \begin{pmatrix} e^{-4i\zeta^3 t} e^{-i\zeta x} \\ e^{4i\zeta^3 t} e^{i\zeta x} \end{pmatrix}$$

if $-2i\zeta = k \in \mathbb{R}_+$. In particular, notice that for $\zeta = \frac{i}{2}$, we get mappings of the zero function to the elementary soliton solutions (2.1) and (2.6) (and most basically, the hyperbolic secant function $\operatorname{sech}(\cdot)$ at $t = 0$).

As shown in [3], if a parameter ζ is not already an eigenvalue for potential u, it will be for the Bäcklund transformed potential $\tilde{u}$, i.e. the effect of the Bäcklund transformation on the spectrum is that of inserting an eigenvalue. ("Eigenvalue" to be understood in the sense that there is an L^2-solution ψ of (3.1), which would be an ζ-eigenfunction of the operator $i\begin{pmatrix}\partial_x & -u \\ -\overline{u} & -\partial_x\end{pmatrix}$.) For the cases close to a soliton solution that interest us, the existence of L^2 eigenfunctions can be shown more explicitly (cf. [22], Lemma 3.1), and we will elaborate on this in the next chapter.

More generally, if we just consider the spatial part of the Lax system (at a fixed time, suppressing the time variable), a potential $u(\cdot) \in H^1(\mathbb{R})$ would satisfy $\lim_{R\to\infty} \|u\|_{L^\infty(\{|x|>R\})} = 0$, i.e. we might expect the system to behave like (3.15) as the space variable $|x| \to \infty$. This leads one to seek solutions that behave like (3.17) or (3.18) at $+\infty$ or $-\infty$, called *Jost solutions.* restricting ourselves to the case where the complex parameter ζ is in the upper half-space, we give a more rigorous definition:

Definition 3.1. *For* $\mathrm{Im}(\zeta) > 0$ *and a given potential, the left Jost solution is a differentiable function* ψ_l *solving the spatial part* (3.1) *such that* $\lim_{x\to-\infty} e^{i\zeta x}\psi_l(x) = \begin{pmatrix}1\\0\end{pmatrix}$. *The right Jost solution is a differentiable solution of* (3.1) *which, correspondingly, satisfies* $\lim_{x\to\infty} e^{-i\zeta x}\psi_r(x) = \begin{pmatrix}0\\1\end{pmatrix}$.

Thus, the left resp. right Jost solution is defined by exponentially decaying at $-\infty$ resp. $+\infty$. Of course, to justify the above definition, one has to show that such solutions exist and are unique in the first place. We will do so in the next chapter, as well as collect all properties of Jost solutions that are of interest for our purposes.

Chapter 4

Properties of Jost Solutions

Our first task is to show that the left and right Jost solutions exist for $u \in L^2(\mathbb{R})$. If $\|u\|_{L^2(\mathbb{R})}$ is very small, this follows from a standard fixed point argument. In general, we divide the real line into a finite number of intervals and iterate over them. The following statement is similar to [22], Lemma 4.1, but with no restriction on the L^2-norm of the potential.

Lemma 4.1. *If $u \in L^2$ and $\mathrm{Im}(\zeta) > 0$, the spatial part (3.1) of the Lax system for NLS and mKdV possesses unique differentiable solutions ψ_l and ψ_r satisfying Definition 3.1 The left solution ψ_l satisfies*

$$\|e^{i\zeta x}\psi_{l,1} - 1\|_{L^\infty(\mathbb{R})} \leq C(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^2(\mathbb{R})} \tag{4.1}$$

and

$$\|e^{i\zeta x}\psi_{l,2}\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})} \leq C(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^2(\mathbb{R})}, \tag{4.2}$$

and for the right Jost solution ψ_r corresponding estimates

$$\|e^{-i\zeta x}\psi_{r,1}\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})} \leq C(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^2(\mathbb{R})} \tag{4.3}$$

and

$$\|e^{-i\zeta x}\psi_{r,2} - 1\|_{L^\infty(\mathbb{R})} \leq C(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^2(\mathbb{R})} \tag{4.4}$$

hold. Moreover, the derivatives satisfy the estimates

$$\|(e^{i\zeta x}\psi_{l,1})'\|_{L^2(\mathbb{R})} \leq C(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|^2_{L^2(\mathbb{R})}$$

and, assuming $u \in L^\infty(\mathbb{R})$,

$$\|(e^{i\zeta x}\psi_{l,1})'\|_{L^\infty} \leq C(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^2(\mathbb{R})}\|u\|_{L^\infty(\mathbb{R})}$$

as well as

$$\|(e^{i\zeta x}\psi_{l,2})'\|_{L^2(\mathbb{R})} \leq \tilde{C}(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^2(\mathbb{R})}$$

and, again assuming $u \in L^\infty(\mathbb{R})$,

$$\|(e^{i\zeta x}\psi_{l,2})'\|_{L^\infty(\mathbb{R})} \leq \tilde{C}(\zeta, \|u\|_{L^2(\mathbb{R})})\|u\|_{L^\infty(\mathbb{R})}$$

With the role of the first and second component exchanged, analogous estimates on the derivatives hold for the right Jost solution. $C(\zeta, \|u\|_{L^2(\mathbb{R})})$ and $\tilde{C}(\zeta, \|u\|_{L^2(\mathbb{R})})$ denote constants depending on ζ and $\|u\|_{L^2(\mathbb{R})}$, which remain bounded for bounded $\|u\|_{L^2(\mathbb{R})}$.

Proof. We will explicitly treat the left Jost solution, the argument for the right Jost solution is similar. We set $\varphi(x) = \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix} := e^{i\zeta x}\psi_l(x)$ (we suppress t-dependence in our notation), and plugging $\psi = e^{-i\zeta x}\varphi(x)$ into (3.1), we get:

$$\begin{aligned}\varphi_1'(x) &= u(x)\varphi_2(x)\\ \varphi_2'(x) &= -\overline{u}(x)\varphi_1(x) + 2i\zeta\varphi_2(x)\end{aligned} \tag{4.5}$$

We pose the problem on a half-open (or open, in which case the argument would proceed in the same way) interval $(a, b] \subset \mathbb{R}$ with boundary values $\lim_{x\downarrow a} \varphi_1(x) \stackrel{!}{=} k_1$, $\lim_{x\downarrow a} \varphi_2(x) \stackrel{!}{=} k_2$. (4.5) can be transformed into integral equations e.g. by variation of constants (writing (4.5) as $\partial_x\varphi(x) = A(x)\varphi(x) + \begin{pmatrix}u(x)\varphi_2(x)\\ -\overline{u}(x)\varphi_1(x)\end{pmatrix}$). We obtain:

$$\begin{aligned}\varphi_1(x) &= k_1 + \int_a^x u(y)\varphi_2(y)dy\\ \varphi_2(x) &= e^{2i\zeta(x-a)}k_2 - \int_a^x e^{2i\zeta(x-y)}\overline{u}(y)\varphi_1(y)dy\end{aligned} \tag{4.6}$$

for $x \in (a, b]$. Now, $M = \left\{\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in L^\infty \times L^2 \cap L^\infty\Big((a, b]\Big) : \lim_{x\downarrow a}\varphi_1(x) = k_1, \lim_{x\downarrow a}\varphi_2(x) = k_2\right\}$ is a closed subset of $L^\infty \times L^2 \cap L^\infty\Big((a, b]\Big)$. By Hölder's and Young's inequality, we can define $S : M \to M$ by

$$S\left(\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right) = \begin{pmatrix} k_1 + \int_a^x u(y)\varphi_2(y)dy \\ e^{2i\zeta(x-a)}k_2 - \int_a^x e^{2i\zeta(x-y)}\overline{u}(y)\varphi_1(y)dy\end{pmatrix}$$

For φ, $\tilde{\varphi} \in M$, set $S(\varphi) - S(\tilde{\varphi}) = \begin{pmatrix}s_1(x)\\ s_2(x)\end{pmatrix}$. Because the boundary values of all functions in M are the same, we can use Hölder's and Young inequality once again to obtain (recall $\mathrm{Im}(\zeta) > 0$):

$$\|s_1\|_{L^\infty(a,b]} \le \|u\|_{L^2(a,b]}\|\varphi_2 - \tilde{\varphi}_2\|_{L^2(a,b]},$$

and:

$$\begin{aligned}\|s_2\|_{L^2\cap L^\infty(a,b]} &\le \|e^{-2i\zeta\cdot}\|_{L^1\cap L^\infty(\mathbb{R}_-)}\|u\|_{L^2(a,b]}\|\varphi_1 - \tilde{\varphi}\|_{L^\infty(a,b]} \\ &\le c_\zeta\|u\|_{L^2(a,b]}\|\varphi_1 - \tilde{\varphi}\|_{L^\infty(a,b]}\end{aligned}$$

with c_ζ depending only on ζ. Hence, S is a contraction whenever $\|u\|_{L^2(a,b]} < \max(1, c_\zeta^{-1})$ and the Banach fixed point theorem implies there exists a unique solution of (4.6) in M. Therefore, we can establish global existence by partitioning $\mathbb{R}$ into $N \sim \|u\|^2_{L^2(\mathbb{R})}$ intervals $I_1 = (-\infty, x_1]$, $I_2 = (x_1, x_2]$, ..., $I_N = (x_{N-1}, \infty)$ with $\|u\|_{L^2(I_k)} < \max(1, c_\zeta^{-1})$ for all $1 \le k \le N$. We can then proceed in the standard way, applying our argument iteratively on each interval and getting the boundary values for I_k from the previously obtained solution on I_{k-1} (the boundary values $\begin{pmatrix}1\\0\end{pmatrix}$ on I_1, of course, given by the definition of the left Jost solution).

We now turn to the estimates (4.1) and (4.2). Assume we have picked our sequence of $N \sim \|u\|_{L^2(\mathbb{R})}$ intervals I_k from above with $N \ge \sqrt{2c_\zeta}\|u\|_{L^2(\mathbb{R})}$ and consequently $c_\zeta\|u\|^2_{L^2(I_k)} \le \frac{1}{2}$ for every k. By inserting the second equation of (4.6) into the first:

$$\begin{aligned}\|\varphi_1 - 1\|_{L^\infty(I_k)} \le |\varphi_1(x_{k-1}) - 1| + c_\zeta\|u\|_{L^2(I_k)}|\varphi_2(x_{k-1})| + \\ + c_\zeta\|u\|^2_{L^2(I_k)}\|\varphi_1\|_{L^\infty(I_k)}\end{aligned}$$

With $a \to -\infty$ in (4.6), we also get:

$$\|\varphi_2\|_{L^\infty(-\infty,x_{k-1}]} \le c_\zeta\|u\|_{L^2(\mathbb{R})}\|\varphi_1 - 1\|_{L^\infty(-\infty,x_{k-1}]} + c_\zeta\|u\|_{L^2(\mathbb{R})}$$

Taken together:

$$\begin{aligned}\frac{1}{2}\|\varphi_1 - 1\|_{L^\infty(-\infty,x_k]} \le (1 + c_\zeta^2\|u\|^2_{L^2(\mathbb{R})})\|\varphi_1 - 1\|_{L^\infty(-\infty,x_{k-1}]} \\ + c_\zeta(c_\zeta + 1)\|u\|_{L^2(\mathbb{R})}\|u\|_{L^2(I_k)}\end{aligned}$$

Iteratively applying this to I_N, I_{N-1}, ..., I_1, we get:

$$\begin{aligned}\|\varphi_1 - 1\|_{L^\infty(\mathbb{R})} &\le c_\zeta(c_\zeta + 1)\|u\|_{L^2(\mathbb{R})}\sum_{k=1}^{N} 2^{k+1}(1 + c_\zeta^2\|u\|^2_{L^2(\mathbb{R})})^k \\ &\le 2^{N+2}(1 + c_\zeta^2\|u\|^2_{L^2(\mathbb{R})})^{N+1}c_\zeta(c_\zeta + 1)\|u\|_{L^2(\mathbb{R})},\end{aligned}$$

together with our choice of N, this implies (4.1), and (4.2) follows from (4.1) by, again, using (4.6) with $a = -\infty$ and applying Young's inequality. The estimates on the derivatives given in Lemma 4.1 are now an easy consequence of (4.1), (4.2) and (4.5). □

If $\|u\|_{L^2(\mathbb{R})} \leq \epsilon$ for sufficiently small ϵ, iteration over several intervals is not necessary and we get $C(\zeta, \|u\|_{L^2(\mathbb{R})}) = \frac{c_\zeta}{1-c_\zeta\|u\|^2_{L^2(\mathbb{R})}}$ in Lemma 4.1, similar to Lemma 4.1 in [22], which is explicitly proved for $\zeta = \frac{i}{2}$ (or, in their notation, $\eta = \frac{1}{2}$).

Remark 4.2. *Some further comments on the relationship between the solutions derived in [22] and the right and left Jost solution of Lemma 4.1 will serve both to elucidate the relationship between our argument and theirs, as well as clear up the behaviour of $\psi_{l,1}$ as $x \to \infty$ and $\psi_{r,2}$ as $x \to -\infty$. With parameter $\zeta = \frac{i}{2}$, Mizumachi and Pelinovsky give the proof that, under the assumption of small $\|u\|_{L^2(\mathbb{R})}$, solutions ψ_+ and ψ_- to (3.1) exist such that*

$$\lim_{x\to-\infty} e^{\frac{x}{2}}\psi_{+,2}(x) = 0 \qquad \lim_{x\to\infty} e^{-\frac{x}{2}}\psi_{+,1}(x) = 1$$
$$\lim_{x\to-\infty} e^{\frac{x}{2}}\psi_{-,2}(x) = 1 \qquad \lim_{x\to\infty} e^{-\frac{x}{2}}\psi_{-,1}(x) = 0 \tag{4.7}$$

From the integral equation (47) in [22] (which is similar to (4.6)) and the corresponding equation for ψ_-, it is immediate that $\lim_{x\to\infty} e^{-\frac{x}{2}}\psi_{+,2}(x) = 0$ and $\lim_{x\to-\infty} e^{\frac{x}{2}}\psi_{-,1}(x) = 0$, i.e.

$$\psi_-(x) \sim \begin{pmatrix} 0 \\ e^{-\frac{x}{2}} \end{pmatrix}, \quad x \to -\infty \qquad \psi_+(x) \sim \begin{pmatrix} e^{\frac{x}{2}} \\ 0 \end{pmatrix}, \quad x \to \infty,$$

so while ψ_l is the solution with characteristic exponent $+\frac{1}{2}$ at $-\infty$, ψ_- corresponds to the characteristic exponent $-\frac{1}{2}$. In particular, ψ_l and ψ_- are linearly independent and form a basis of the solution space, and the same holds for ψ_r and ψ_+. If we consider the basis representation $e^{\frac{x}{2}}\psi_+ = se^{\frac{x}{2}}\psi_- + te^{\frac{x}{2}}\psi_l$, $s,\ t \in \mathbb{C}$ and let $x \to -\infty$, the behaviour of $e^{\frac{x}{2}}\psi_{+,2}$ in (4.7) enforces $s = 0$, and thus $\psi_+(x) = t\psi_l(x)$, $t \in \mathbb{C}\setminus\{0\}$ and, similarly, $\psi_- = \tilde{t}\psi_r$, $\tilde{t} \in \mathbb{C}\setminus\{0\}$. The Wronskian $det(\psi_+, \psi_-) = det(te^{-\frac{x}{2}}\psi_l, e^{\frac{x}{2}}\psi_-) = det(e^{-\frac{x}{2}}\psi_+, \tilde{t}e^{\frac{x}{2}}\psi_r)$ is constant in x by Abel's theorem, from which we get $t = \tilde{t}$ by taking limits at $\pm\infty$. It now follows from (4.7) that

$$\lim_{x\to\infty} e^{-\frac{x}{2}}\psi_{l,1}(x) = t^{-1} \qquad \lim_{x\to-\infty} e^{\frac{x}{2}}\psi_{r,2}(x) = t^{-1}$$

This argument does, of course, easily apply to other parameters than $\frac{i}{2}$. Indeed, it is well-known in inverse scattering theory that, with $T(\zeta)$ the transmission coefficient *and $a(\zeta) := T^{-1}(\zeta)$ the* inverse transmission coefficient *for (3.1):*

$$\lim_{x\to\infty} e^{i\zeta x}\psi_l(x) = a(\zeta)\begin{pmatrix}1\\0\end{pmatrix} \qquad \lim_{x\to-\infty} e^{-i\zeta x}\psi_r(x) = a(\zeta)\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.8}$$

with ψ_l, ψ_r the left and right Jost solution of (3.1) with parameter $\zeta \in \mathbb{C}$, $u \in L^2(\mathbb{R})$.

Remark 4.3. *With $a = -\infty$ in (4.6), we get the following integral equations for the left Jost solution (with $\varphi_l = e^{i\zeta x}\psi_l$), which we include for further reference:*

$$\varphi_{l,1}(x) = 1 + \int_{-\infty}^{x} u(y)\varphi_{l,2}(y)dy \tag{4.9}$$

and

$$\varphi_{l,2}(x) = -\int_{-\infty}^{x} e^{2i\zeta(x-y)}\overline{u(y)}\varphi_{l,1}(y)dy \tag{4.10}$$

Plugging (4.10) into (4.9), we get an implicit representation for $\varphi_{l,1}$

$$\varphi_{l,1}(x) = 1 - \int_{-\infty}^{x}\int_{-\infty}^{y} u(y)e^{2i\zeta(y-z)}\overline{u(z)}\varphi_{l,1}(z)dzdy, \tag{4.11}$$

Differentiating (4.11) and applying Young's inequality and (4.1), we get another estimate for the derivative beyond what we stated in Lemma 4.1

$$\|\varphi_{l,1}'\| \leq c_\zeta \|u\|^2_{L^\infty(\mathbb{R})} \tag{4.12}$$

Similarly, it is straightforward to obtain for the right Jost solution (with $\varphi_r = e^{-i\zeta x}\psi_r$)

$$\varphi_{r,1}(x) = -\int_{x}^{\infty} e^{-2i\zeta(x-y)}u(y)\varphi_{r,2}(y)dy \tag{4.13}$$

and

$$\varphi_{r,2}(x) = 1 + \int_{x}^{\infty} \overline{u(y)}\varphi_{r,1}(y)dy \tag{4.14}$$

Similarly to (4.11), we get

$$\varphi_{r,2}(x) = 1 - \int_{x}^{\infty}\int_{y}^{\infty} \overline{u(y)}e^{-2i\zeta(y-z)}u(z)\varphi_{r,2}(z)dzdy \tag{4.15}$$

and

$$\|\varphi_{r,2}'\|_{L^\infty(\mathbb{R})} \leq c_\zeta \|u\|^2_{L^\infty(\mathbb{R})} \tag{4.16}$$

Remark 4.4. *If the parameter ζ is an eigenvalue, it is worth noting that it necessarily has a geometric multiplicity of* 1*. That holds true because a geometric multiplicity of* 2 *would be excluded by the existence of one characteristic exponent with negative real part at* $-\infty$*, corresponding to an exponential growth. Thus, there are two possible cases:*

Case 1: *The parameter ζ is an eigenvalue, and there is some constant $c \in \mathbb{C}$ such that $\psi_l = c\psi_r$, and ψ_l is an eigenfunction. By Remark 4.2, this implies $a(\zeta) = 0$, i.e. eigenvalues are zeroes of the inverse transmission coefficient.*

Case 2: *ψ_l and ψ_r form a basis of the solution space.*

Using the instruments introduced so far, one can show:

Corollary 4.5. *Let $u \in L^2(\mathbb{R})$, a ζ-wavefunction ψ be given as a linear combination of left and right Jost solution, and $\tilde{u} = B(u, \zeta, \psi)$ the Bäcklund transformation as in* (3.12)*. Then $\tilde{u} \in L^2(\mathbb{R})$.*

Proof. For $u \in L^2(\mathbb{R})$, $\zeta \in \mathbb{C}$ with $\text{Im}(\zeta) > 0$ and $\psi = c_l\psi_l + c_r\psi_r$ a wave function of the corresponding system (3.1) expressed as a combination of the right and left Jost solution, $c_l, c_r \in \mathbb{C}$. If $c_r = 0$ - which, in particular, we can assume whenever $a(\zeta) = 0$ by Remark 4.4 -, inequalities (4.1) and (4.2) in Lemma 4.1 imply that $|\psi_{l,1}| \lesssim |e^{-i\zeta x}|(1 + \|u\|_{L^2(\mathbb{R})})$ and, as $e^{i\zeta x}\psi_{l,2} \in L^2(\mathbb{R})$,

$$\frac{|\psi_{l,1}\overline{\psi}_{l,2}|}{\|\psi_l\|^2} \leq \frac{|\psi_{l,1}\overline{\psi}_{l,2}|}{|\psi_{l,1}|^2} \lesssim |e^{2i\zeta x}||\psi_{l,1}\overline{\psi}_{l,2}| \in L^2(\mathbb{R})$$

Hence, $B(u, \zeta, \psi) \in L^2(\mathbb{R})$.

For $a(\zeta) \neq 0$, $c_r \neq 0$, we show that $B(u, \zeta, \psi) \in L^2(\mathbb{R}_-)$, the proof that it is in $L^2(\mathbb{R}_+)$ is similar. By (4.8) and the arguments developed in Lemma 4.1, we have:

$$\|e^{i\zeta x}\psi_{r,2}(x) - a(\zeta)\|_{L^\infty((-\infty,-R])} \lesssim \|u\|_{L^2((-\infty,-R])} \tag{4.17}$$

for $R > 0$. If $a(\zeta) \neq 0$ and we set $\|u\|_{L^2((-\infty,-R])} = \alpha \geq 0$, (4.17) and Lemma 4.1 imply (with $\beta := \text{Im}(\zeta) > 0$)

$$\begin{aligned}
|\psi_2(x)|^2 &\geq |c_r|^2|\psi_{l,2}(x)|^2 - 2|c_l||c_r||\psi_{l,2}(x)||\psi_{r,2}(x)| \\
&\geq |c_r|^2|\psi_{l,2}(x)|^2 - 2|c_l||c_r||\psi_{l,2}(x)||\psi_{r,2}(x)| \\
&= |c_r|^2e^{-2\beta x}(|a(\zeta)| - |r_2(x)|)^2 \\
&\qquad - 2|c_l||c_r|e^{\beta x}e^{-\beta x}(1 + |r_1(x)|)(|a(\zeta)| + |r_2(x)|)
\end{aligned}$$

for $x \in (-\infty, -R]$, where $|r_1(x)|+|r_2(x)| \leq 2\alpha$. For sufficiently large R (and hence sufficiently small α), this implies $|\psi(x)|^2 \gtrsim e^{-\beta x}$. Therefore, again for $x \in (-\infty, -R]$,

$$
\begin{aligned}
\left| \frac{\psi_1(x)\overline{\psi}_2(x)}{\|\psi(x)\|^2} \right| &\lesssim e^{2\beta x} |\psi_1(x)\overline{\psi}_2(x)| \\
&\lesssim e^{2\beta x} (|\psi_{l,1}(x)||\psi_{l,2}(x)| + |\psi_{l,1}(x)||\psi_{r,2}(x)| + \\
&\qquad + |\psi_{r,2}(x)||\psi_{l,1}(x)| + |\psi_{r,2}(x)||\psi_{l,2}(x)|),
\end{aligned}
$$

by (4.1)-(4.4), this is an L^2-function. Therefore, we get that $B(u, \zeta, \psi) \in L^2((-\infty, -R])$, and $B(u, \zeta, \psi) \in L^2((-R, 0])$ easily follows from $\left| \frac{\psi_1 \overline{\psi}_2}{\|\psi\|^2} \right| \leq 1$. □

Let $u(x) = \text{sech}(x)$ be the potential in the spatial part of the Lax system (3.1) with parameter $\zeta = \frac{i}{2}$. Then the solution space is spanned by the eigenfunction

$$
\Psi^{(1)}(x) = \frac{1}{2} \begin{pmatrix} -e^{-\frac{x}{2}} \\ e^{\frac{x}{2}} \end{pmatrix} \text{sech}(x) \tag{4.18}
$$

and the unbounded function

$$
\Psi^{(2)}(x) = \frac{1}{2} \begin{pmatrix} e^{\frac{x}{2}}[e^x + 2(1+x)e^{-x}] \\ e^{-\frac{x}{2}}(e^{-x} + 2xe^x) \end{pmatrix} \text{sech}(x)
$$

We use a standard argument to show that if we perturb $\text{sech}(x)$ slightly, another eigenvalue close to $\frac{i}{2}$ with the corresponding eigenfunction close to (4.18) can be found (compare [22], Lemma 3.1., although our technique of proof is slightly different):

Lemma 4.6. *If $\|u - \text{sech}(x)\|_{L^2(\mathbb{R})} \leq \epsilon$ for a sufficiently small constant $\epsilon > 0$, there exists a unique $\zeta \in \mathbb{C}$ such that the spatial part (3.1) of the Lax system with potential u and parameter ζ has a solution $\Psi \in L^2 \times L^2(\mathbb{R})$ and we have, for a fixed constant C:*

$$
\left| \zeta - \frac{i}{2} \right| + \|\Psi - \Psi_1\|_{H^1(\mathbb{R})} \leq C\|u - \text{sech}(x)\|_{L^2(\mathbb{R})}
$$

If u is a real-valued function, we can get the same result with a purely imaginary eigenvalue ζ and a correspondingly real-valued eigenfunction.

Proof. For $v \in L^2(\mathbb{R})$, define $A(v) : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ by:

$$
A(v) := i \begin{pmatrix} \partial_x - \frac{1}{2} & -(\text{sech}(x) + v(x)) \\ -(\text{sech}(x) + v(x)) & -\partial_x - \frac{1}{2} \end{pmatrix}
$$

As in [22], Lemma 3.1, we can see that the unperturbed $A(0)$ is a Fredholm operator: D defined by

$$D := i \begin{pmatrix} \partial_x - \frac{1}{2} & 0 \\ 0 & -\partial_x - \frac{1}{2} \end{pmatrix}$$

is a closed operator with domain $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and its inverse is given by

$$D^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = -i \begin{pmatrix} \int\limits_{-\infty}^{x} e^{-\frac{x-y}{2}} g(y) \\ \int\limits_{x}^{\infty} e^{\frac{x-y}{2}} f(y) dy \end{pmatrix} \tag{4.19}$$

Thus, we have:

$$A(0) = \left[I - \begin{pmatrix} 0 & \operatorname{sech}(x) \\ \operatorname{sech}(x) & 0 \end{pmatrix} D^{-1} \right] D =: (I - K)D, \tag{4.20}$$

and by (4.19), K is a Hilbert-Schmidt integral operator and hence compact. As D is closed and $I - K$ a compact perturbation of the identity, $A(0)$ is indeed Fredholm with index 0.

We now consider the function (with L^2, H^1 to be understood as $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$)

$$F : L^2 \times \mathbb{C} \times C(\mathbb{R}, H^1 \times H^1) \to C(\mathbb{R}, L^2 \times L^2) \times \mathbb{C}$$

$$(v, \lambda, \psi) \to \begin{pmatrix} (A(v) - \lambda)\psi \\ \langle \psi, \Theta \rangle_{L^2} - 1 \end{pmatrix},$$

where $\Theta = \begin{pmatrix} -e^{\frac{x}{2}} \operatorname{sech}(x) \\ e^{\frac{x}{2}} \operatorname{sech}(x) \end{pmatrix}$ spans $\ker(A(0)^*)$. In $(0, \frac{i}{2}, \Psi^{(1)})$, differentiation by the second and third argument gives

$$\partial_{(\lambda,\psi)} F(0, \lambda, \psi) = \begin{pmatrix} -\Psi^{(1)} & A(0) \\ 0 & \langle \Theta | \end{pmatrix}, \tag{4.21}$$

where $\langle \Theta |$ is to be understood in the sense of the bra-ket notation for the scalar product in L^2. If we now consider the equation

$$\begin{pmatrix} -\Psi^{(1)} & A(0) \\ 0 & \langle \Theta | \end{pmatrix} \begin{pmatrix} \lambda \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ \alpha \end{pmatrix}$$

with $f \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, $\alpha \in \mathbb{C}$. Because $A(0)$ is a Fredholm operator, it has closed range, so $\operatorname{ran}(A(0)) = \overline{\operatorname{ran}(A(0))} = \ker(A(0)^*)^\perp$ and a solution $\psi \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ of

$$-\lambda \Psi^{(1)} + A(0)\psi = f \tag{4.22}$$

exists if and only if $f + \lambda\Psi^{(1)} \in \ker(A(0)^*)^\perp$, i.e. $\lambda\langle\Theta, \Psi^{(1)}\rangle \stackrel{!}{=} -\langle\Theta, f\rangle$. Because $\langle\Theta, \Psi^{(1)}\rangle = 1 \neq 0$, we can always find λ such that this condition is satisfied. Moreover, if ψ_0 is a solution of (4.22), so is $\psi := \psi_0 + s\Psi^{(1)}$ for any $s \in \mathbb{C}$. There is a unique s so that

$$\langle\Theta, \psi\rangle = \langle\Theta, \psi_0\rangle + s \stackrel{!}{=} \alpha$$

is satisfied. It follows that (4.21) is an invertible operator and we can apply extended versions of the implicit function theorem such as [9], Corollary 15.1. to find small $r > 0$ and a unique $T : B_r^{L^2(\mathbb{R})}(0) \to \mathbb{C} \times (L^2(\mathbb{R}, \mathbb{C}^2))$ such that $F(v, T(v)) = 0$ for $\|v\|_{L^2(\mathbb{R})} < r$ and $T \in C^1$, from which the lemma follows. Our argument can obviously be restricted to the real-valued case. □

Finally, we have now discussed all necessary preliminaries to show a continuity property of the Bäcklund transform near the $\operatorname{sech}(x)$-potential:

Lemma 4.7. *Let $\tilde{u} \in B_r^{L^2(\mathbb{R})}(\operatorname{sech}(\cdot))$ be a potential for (3.1), where $r > 0$ such that it satisfies the assumptions of Lemma 4.6.*
Let

- *ζ be the corresponding eigenvalue which exists and is unique by Lemma 4.6 and $\tilde{\psi} = c\tilde{\psi}_l$ a ζ-eigenfunction, given as the left Jost solution multiplied by a normalization constant $c \in \mathbb{C}$*
- *$u := B(\tilde{u}, \zeta, \tilde{\psi}_l)$, with B the Bäcklund transform of $\tilde{u}$ with parameter ζ and wave function $\tilde{\psi}_l$ (the constant c does not matter for the result by (3.13))*
- *$\alpha_l \in \mathbb{C}$ be defined via the reiteration relation (3.14), by which the system (3.1) with potential u has a wave function ψ for parameter ζ given by (3.11). With ψ_l and ψ_r the right and left Jost solution of (3.1) with potential u and parameter ζ, there are unique complex numbers α_l, α_r such that*

$$\psi = \alpha_l\psi_l + \alpha_r\psi_r, \tag{4.23}$$

and we choose the normalization constant c such that $\alpha_r = 1$.

The mapping

$$\begin{aligned} \mathcal{B} : B_r(\operatorname{sech}(\cdot)) &\to \mathbb{C} \times \mathbb{C} \times L^2(\mathbb{R}) \\ \tilde{u} &\to (\zeta, \alpha_l, B(\tilde{u}, \zeta, \tilde{\psi}_l)) \end{aligned} \tag{4.24}$$

is continuous and injective. The inverse mapping $\mathcal{B}^{-1}$ which we can define on $\operatorname{ran}(\mathcal{B})$ is also continuous.

Proof. By uniqueness of the initial value problem (3.1) (compare the proof of Lemma 4.1), we know that $\tilde{\psi}_l(x) \neq 0 \; \forall x \in \mathbb{R}$. Since it is also continuous, this implies that, for $R > 0$ to be chosen later

$$\|\tilde{\psi}_l\|^{-2} \leq C \tag{4.25}$$

on a bounded interval $[-R, R]$. We also know from the proof of Lemma 4.6 that the eigenfunction mapping $\tilde{u} \to \tilde{\psi}(\tilde{u})$ is continuous as a $B_r^{L^2(\mathbb{R})}(\text{sech}(\cdot)) \to L^2(\mathbb{R})$-map, and hence, so is the Bäcklund transform $\tilde{u} \to B(\tilde{u}, \zeta, \tilde{\psi}_l)$ (again as a function from $B_r^{L^2(\mathbb{R})}(\text{sech}(\cdot)) \to L^2(\mathbb{R})$). Indeed, similarly to the proof we sketched for Corollary 4.5, we can choose sufficiently large $R > 0$ with $\|u\|_{L^2(\{|x|>R\})} \leq \epsilon \ll 1$. To show that

$$\begin{aligned} &\left| \frac{\tilde{\psi}_{l,1}(u_1)\overline{\tilde{\psi}_{l,2}(u_1)}}{|\tilde{\psi}_l(u_1)|^2} - \frac{\tilde{\psi}_{l,1}(u_2)\overline{\tilde{\psi}_{l,2}(u_2)}}{|\tilde{\psi}_l(u_2)|^2} \right| \\ &= |\tilde{\psi}_l(u_1)|^{-2}|\tilde{\psi}_l(u_2)|^{-2}[|\tilde{\psi}_l(u_2)|^2\tilde{\psi}_{l,1}(u_1)\overline{\tilde{\psi}_{l,2}(u_1)} - |\tilde{\psi}_l(u_1)|^2\tilde{\psi}_{l,1}(u_2)\overline{\tilde{\psi}_{l,2}(u_2)}] \end{aligned} \tag{4.26}$$

converges to zero in $L^2([-R, R])$ as $\|u_1 - u_2\|_{L^2([-R,R])} \to 0$, we can use (4.25) and the fact that $\psi_l \sim \psi_r \in L^2 \cap L^\infty([-R, R])$. For $(-\infty, -R]$, we utilize

$$\tilde{\psi}_l = \begin{pmatrix} e^{-i\zeta x}(1 + O(\epsilon)) \\ e^{-i\zeta x}O(\epsilon) \end{pmatrix},$$

which follows from the choice of R and the proof of Lemma 4.1. This means $|\tilde{\psi}_l(u_1)|^{-2}|\tilde{\psi}_l(u_2)|^{-2} \lesssim |e^{4i\zeta x}|$ in (4.26). Continuity in $L^2((-\infty, -R])$ now follows from (4.9) and (4.10), and similarly for $L^2([R, \infty))$. It is also immediate from the proof of Lemma 4.6 that the eigenvalue mapping $\tilde{u} \to \zeta(\tilde{u})$ defined in the obvious manner from (4.24) is continuous.

Concerning $\tilde{u} \to \alpha_l(\tilde{u})$ (defined as implicit in (4.24)), we have established that $\tilde{u} \to u = B(\tilde{u}, \zeta, \tilde{\psi}_l)$ is continuous, and arguing similarly as for the Bäcklund transform in the previous paragraph, (3.14) implies that ψ in (4.23), considered as an L^2-function on any bounded interval, continuously depends on $\tilde{u} \in B_r^{L^2(\mathbb{R})}(\text{sech}(\cdot))$. Therefore, writing $\psi(\tilde{u}) = \psi(u(\tilde{u}))$ and with a sequence $\tilde{u}_n$ such that $\tilde{u}_n \to \tilde{u}$ in $L^2(\mathbb{R})$ and a bounded interval (a, b), we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|\psi(\tilde{u}) - \psi(\tilde{u}_n)\|_{L^2(a,b)} \\ &= \lim_{n\to\infty} \|\alpha_l(\tilde{u})\psi_l(\tilde{u}) + \psi_r(\tilde{u}) - \alpha_l(\tilde{u}_n)\psi_l(\tilde{u}_n) - \psi_r(\tilde{u}_n)\|_{L^2(a,b)} \end{aligned} \tag{4.27}$$

Equations (4.9)-(4.10) give us that, considered as L^2-functions on bounded intervals, the right and left Jost solution of the corresponding system (3.1) continuously depend on its potential $u \in L^2(\mathbb{R})$, so (4.27) implies

$$0 = \lim_{n\to\infty} \|(\alpha_l(\tilde{u}) - \alpha_l(\tilde{u}_n))\psi_l(\tilde{u}) + \psi_r(\tilde{u})\|_{L^2(a,b)}$$

Because ψ_l and ψ_r are linearly independent, this implies we have $\lim_{n\to\infty}(\alpha_l(\tilde{u}) - \alpha_l(\tilde{u}_n)) = 0$ and hence continuity of $\alpha_l(\tilde{u})$.

Finally, the reiteration relation (3.14) gives us an inverse mapping $\mathcal{B}^{-1}$: $\operatorname{ran}(\mathcal{B}) \to B_r^{L^2(\mathbb{R})}(\operatorname{sech}(\cdot))$ and hence the injectivity of $\mathcal{B}$. We obtain continuity as above. $\square$

Chapter 5

Asymptotic Stability of the NLS Soliton Solutions

As in [8] and [22], we use the following well-posedness result due to Tsutsumi, originally shown in [25]:

Theorem 5.1. *For $u_0 \in L^2(\mathbb{R})$, there is a unique solution $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ with $u \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^4_{loc}(\mathbb{R}, L^\infty(\mathbb{R}))$ of*

$$u(t,\cdot) = e^{it\partial_x^2} u_0(t,\cdot) + 2i \int_0^t e^{i(t-s)\partial_x^2} |u(s,\cdot)|^2 u(s) ds, \tag{5.1}$$

(i.e. the integral equation formulation of the NLS (1.1)).
This solution satisfies energy conservation $\|u(t,\cdot)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ $\forall t \in \mathbb{R}$. If there is a sequence $u_{0n} \in L^2(\mathbb{R})$ $\forall n \in \mathbb{N}$ with $u_{0n} \to u_0$ in $L^2(\mathbb{R})$ and $u_n(t,\cdot)$ denote the unique solutions of (5.1) with initial data u_{0n}, respectively, we have $u_n(t,\cdot) \to u(t,\cdot)$ $\forall t \in \mathbb{R}$ in the L^2-sense.

When we refer to the unique solution of the NLS (1.1) with initial datum $u_0 \in L^2$ in this chapter, it is to be understood in the sense of Theorem 5.1. Of course, if the assumptions of Theorem 2.3 are satisfied, the solutions of Theorem 2.3 coincide with those of Theorem 5.1.

We now want to prove the following asymptotic stability result:

Theorem 5.2. *Given $\tilde{u}_0 \in H^{0,1}(\mathbb{R})$ with $\|\tilde{u}_0 - \operatorname{sech}(x)\|_{H^{0,1}(\mathbb{R})} = \epsilon$ for a sufficiently small ϵ, let $\tilde{u} = \tilde{u}(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ be a solution of the focusing NLS (1.1) in one dimension with $\tilde{u}(0,\cdot) = \tilde{u}_0(\cdot)$ (as in Theorem 5.1). Then we have asymptotic stability in the sense that for any large enough $t \in \mathbb{R}_+$, there are k, $v \in \mathbb{R}$ and functions $\tilde{x}_0 : \mathbb{R}_+ \to \mathbb{R}$, $\tilde{\theta} : \mathbb{R}_+ \to \mathbb{R}$:*

$$\|e^{i\frac{vx}{2} + i(k^2 - \frac{v^2}{4})t} k\tilde{u}(k^2 t, k(\cdot - vt)) - e^{i(t+\tilde{\theta}(t))} \operatorname{sech}(\cdot - \tilde{x}_0(t))\|_{L^\infty(\mathbb{R})} \lesssim \epsilon t^{-\frac{1}{2}} \tag{5.2}$$

And, for all such t, $|\tilde{\theta}(t)| \lesssim \epsilon$ and $|\tilde{x}_0(t)| \lesssim \epsilon$ and $|k-1| \lesssim \epsilon$, $|v| \lesssim \epsilon$. The functions $\tilde{\theta}(t)$ and $\tilde{x}_0(t)$ satisfy $\lim_{t\to\infty} \tilde{\theta}_0(t) = \theta_0 \in \mathbb{R}$ and $\lim_{t\to\infty} \tilde{x}_0(t) = x_0 \in \mathbb{R}$. Quantitatively,

$$|\tilde{x}_0(t) - x_0| \lesssim \epsilon^2 \log(t) t^{-1+C\epsilon^2} \tag{5.3}$$

and

$$|\tilde{\theta}_0(t) - \theta_0| \lesssim \epsilon^2 \log(t) t^{-1+C\epsilon^2}, \tag{5.4}$$

with C as in Theorem 2.3, give us estimates for the rate of convergence.

For the sake of simplicity, we only explicitly formulate Theorem 5.2 for the elementary soliton. Using the symmetries of NLS (2.2) and (2.3) it is, however, relatively straightforward to see how it generalizes to a similar statement for the entire soliton group. As mentioned before, the restriction to $H^{0,1}(\mathbb{R})$ is another simplification, and the following argument can be extended to all spaces covered by Remark 2.4 with little change.

For the moment, we assume that $\tilde{u}_0 \in H^3(\mathbb{R})$. We will use an approximation argument to generalize later, similar to [22]. This assumption and Lemma 5.3 will ensure that the system (3.2) is well-defined and Lemma 5.6 and 5.7 which we will state and prove below hold.

We give a rough sketch of the essential steps of the following proof for Theorem 5.2. The overall structure is close to [22]: Pull initial data close to $\operatorname{sech}(\cdot)$ back to initial data in a neighbourhood of the zero solution via the Bäcklund transformation, evolve in time, recover the original NLS solution, again via the Bäcklund transformation, and show that it transfers the stability properties of the zero solution.

1. *Fixing the eigenvalue:*
   Starting with our $\tilde{u}_0 \sim \operatorname{sech}(\cdot)$, we exploit the symmetries of the NLS equation to generate a "modified" potential such that the eigenvalue of the corresponding system (3.1) (the spatial part of the Lax system) at $t = 0$ is set to $\zeta = \frac{i}{2}$. This transformation is where the parameters k, v in (5.2) come in. By a slight abuse of notation, we will continue to write the "new" potential (given in (5.5)) as $\tilde{u}_0$.

2. *Pullback to a neighbourhood of the zero solution:*
   Now, we are in a position to "pull back" $\tilde{u}_0$ from a $H^{0,1}$-neighbourhood of $\operatorname{sech}(\cdot)$ to a potential $u_0 \sim 0$ in an $H^{0,1}$-neighbourhood of the zero solution (Lemma 5.3) via the Bäcklund transformation (3.12). In the case $\zeta = \frac{i}{2}$, it is given by

$$u_0 = B(\tilde{u}_0, \frac{i}{2}, \tilde{\psi}_0) = \tilde{u}_0 + 2\frac{\tilde{\psi}_{0,1}\overline{\tilde{\psi}_{0,2}}}{\|\tilde{\psi}_0\|^2}$$

($\tilde{\psi}_0$ as in Lemma 5.3.)

3. *The wave function at $t = 0$:*
A solution ψ_0 for the system (3.1) with potential u_0 and parameter $\zeta = \frac{i}{2}$ can be generated from the relation (3.11), and be represented in terms of the left and right Jost solutions. For a potential of (3.1) L^2-close to zero and with parameter $\frac{i}{2}$, the left Jost solution is "close" (compare Lemma 4.1) to $\begin{pmatrix} e^{\frac{x}{2}} \\ 0 \end{pmatrix}$ and the right solution to $\begin{pmatrix} 0 \\ e^{-\frac{x}{2}} \end{pmatrix}$. Because of the reiteration relation (3.14) and (by assumption) $\tilde{u}_0 \sim \operatorname{sech}(\cdot)$, this implies that $\psi_0 \approx c \begin{pmatrix} e^{\frac{x}{2}} \\ e^{-\frac{x}{2}} \end{pmatrix}$ with $c \in \mathbb{C}$, as we will see in Lemma 5.5.

4. *Time evolution of the wave function:*
By Theorem 2.3, the initial datum u_0 can be evolved into an NLS-solution $u(t, \cdot)$ with $\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \lesssim \epsilon t^{-\frac{1}{2}}$ for sufficiently large $t > 0$. In Lemma 5.6 and 5.7, we will give the corresponding time evolution of ψ_0 into a simultaneous solution ψ of (3.1) and (3.2) with potential $u(t, x)$ and parameter $\frac{i}{2}$. We represent this ψ in terms of the left and right Jost solution of (3.1) at time $t > 0$ (with the corresponding potential $u(t, \cdot)$). Using the previous step, we will have $\psi \approx c \begin{pmatrix} e^{\frac{it}{2}} e^{\frac{x}{2}} \\ e^{-\frac{it}{2}} e^{-\frac{x}{2}} \end{pmatrix}$.
Recall from the discussion following (3.15) and (3.16) that this is precisely the wave function that is mapped to the elementary soliton solution via an appropriate Bäcklund transformation (the constant $c \in \mathbb{C}$ does not change this by (3.13)).

5. *Recovering the original NLS solution via the Bäcklund transformation:*
The reiteration relation (3.14) gives us that at $t = 0$, $\tilde{u}_0 = B(u_0, \frac{i}{2}, \psi_0)$. By the defining property of the Bäcklund transformation (discussed in the text following (3.12)), $(t, x) \to B(u, \frac{i}{2}, \psi)(t, x)$ is thus an NLS-solution with initial datum $\tilde{u}_0$. By uniqueness of solutions, this implies $\tilde{u}(t, x) = B(u, \frac{i}{2}, \psi)(t, x)$, and by the previously sketched step we would, indeed, expect $\tilde{u}(t, x) \approx e^{it} \operatorname{sech}(x)$ for all (or sufficiently large) $t > 0$.

6. L^∞*-estimates* The decisive point in showing the stability results in [22] and the present thesis is to bound the error in this approximation in terms of the appropriate function norm of u (the L^2-norm in [22] and the L^∞-norm here), transferring stability properties from the zero solution to the soliton solution. This is achieved in Proposition 5.8. An essential step in the proof of this proposition is marked by (5.27),

(5.28), by which, if we choose a "reference point" $\tilde{x}_0 = \tilde{x}_0(t) \in \mathbb{R}$, we have $|\psi_{l,1}(t,x)| = |\psi_{l,1}(t,\tilde{x}_0)|e^{\frac{x-\tilde{x}_0}{2}+r(x)}$ with $|r(x)| \lesssim \|u\|^2_{L^\infty(\mathbb{R})}|x-\tilde{x}_0|$, and similarly for $|\psi_{r,2}(t,x)|$. An appropriate choice of $\tilde{x}_0(t)$ for Proposition 5.8 to hold, depending on the left and right Jost solution at time t, gives the position shift function in (5.2), and also gives rise to the phase shift $\tilde{\theta}(t)$.

7. *Convergence of Jost solutions:*
To understand the behaviour of $\tilde{x}_0(t)$ and $\tilde{\theta}_0(t)$ as $t \to \infty$, we want to exploit the fact that both are characterized in terms of the right and left Jost solution of (3.1) (still with potential u and parameter $\frac{i}{2}$) at time t. A necessary preliminary to do so is Proposition 5.10, where we will utilize the asymptotic expression (2.9) from Theorem 2.3 to establish convergence properties of the Jost solutions as $t \to \infty$, and give quantitative estimates for the rates of convergence.

8. *Proof of Theorem 5.2: Approximation of lower-regularity solutions and convergence of position and phase shift:*
At the end of the chapter, we will finally be in a position to prove Theorem 5.2. The two things that we still have to do at this point are a) provide an approximation argument which extends our results (including the characterization of $\tilde{x}_0$ and $\tilde{\theta}_0$) to NLS solutions whose initial value has a lower regularity than $H^3(\mathbb{R})$ but still satisfy the assumptions of Theorem 5.2 and b) show that Proposition 5.10 does, indeed, imply the convergence of $\tilde{x}_0$ and $\tilde{\theta}_0$, and we have the quantitative estimates (5.3) and (5.4) for the rate of convergence. We will use Lemma 4.7 to accomplish the former task, and utilize (5.27) and (5.28) for the latter, finishing the proof.

We now give the details of the proof outlined above. With the exception of the fifth, which we will repeat near the end of the chapter, we will follow the order in which these steps are given above:

Fixing the eigenvalue

By Lemma 4.6, the spatial part (3.1) of the Lax system with potential $\tilde{u}_0$ has an eigenvalue $\zeta \in \mathbb{C}$ close to $\frac{i}{2}$. We can transform the potential to fix this eigenvalue at $\zeta = \frac{i}{2}$. This can be achieved by performing the same change of variables as in [22], Remark 3.2: If the potential $\tilde{u}_0(\cdot)$, the parameter $\zeta = \frac{i}{2}(\tilde{k} + i\tilde{v})$ and the function $\begin{pmatrix}\psi_1(\cdot)\\ \psi_2(\cdot)\end{pmatrix}$ satisfy

$$\partial_x \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix}-i\zeta & \tilde{u}_0\\ -\bar{\tilde{u}}_0 & i\zeta\end{pmatrix} \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$$

then for

$$\begin{aligned}\tilde{\psi}_1(x) &= e^{-\frac{i}{2}\tilde{v}\tilde{k}^{-1}x}\psi_1(\tilde{k}^{-1}x)\\ \tilde{\psi}_2(x) &= e^{\frac{i}{2}\tilde{v}\tilde{k}^{-1}x}\psi_2(\tilde{k}^{-1}x)\\ v_0(x) &= \tilde{k}^{-1}e^{-\frac{i}{2}\tilde{v}\tilde{k}^{-1}x}\tilde{u}_0(\tilde{k}^{-1}x),\end{aligned} \tag{5.5}$$

we have

$$\partial_x\begin{pmatrix}\tilde{\psi}_1(x)\\ \tilde{\psi}_2(x)\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & v_0\\ -\overline{v}_0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\tilde{\psi}_1(x)\\ \tilde{\psi}_2(x)\end{pmatrix}$$

Scaling (2.2) and Galilei transform (2.3) of the NLS solution with initial value $\tilde{u}_0$ gives an NLS solution with initial value v_0. This use of the NLS symmetries is what gives rise to the parameters $k = \tilde{k}^{-1}$ and $v = -\tilde{v}\tilde{k}^{-1}$ in Theorem 5.2, and the estimates on these parameters are a consequence of Lemma 4.6. From now on, it suffices to treat $\tilde{u}_0$ such that $\zeta = \frac{i}{2}$.

Pullback to a neighbourhood of the zero solution

Next, we use a Bäcklund transformation employing the $\frac{i}{2}$-eigensolutions to pull back our $H^{0,1}$-neighbourhood of $\operatorname{sech}(\cdot)$ to a $H^{0,1}$-neighbourhood of 0. It is possible to adapt the argument from [22] to this purpose, but we will use a variation-of-constants approach instead:

Lemma 5.3. *Let $\tilde{u}_0 \in H^{0,1}(\mathbb{R})$ satisfy $\|\tilde{u}_0 - \operatorname{sech}(\cdot)\|_{H^{0,1}(\mathbb{R})} \le \epsilon$ for some small $\epsilon > 0$. Assume that the spatial part (3.1) of the Lax system has an eigenvalue in $\zeta = \frac{i}{2}$ and $\tilde{\psi}_0$ is an associated eigenfunction of (3.1) (which can be normalized to be both the right and left Jost solution by Remark 4.4). Then $B(\tilde{u}_0, \frac{i}{2}, \tilde{\psi}_0) =: u_0$ satisfies $\|u_0\|_{H^{0,1}(\mathbb{R})} \lesssim \epsilon$. Moreover, if $\tilde{u}_0 \in H^3(\mathbb{R})$, then $u_0 \in H^3(\mathbb{R})$.*

Proof. Write $\tilde{u}_0(x) = \operatorname{sech}(x) + v(x)$ with $\|v\|_{H^{0,1}(\mathbb{R})} \le \epsilon$ for ϵ small enough. Consider the (spatial part (3.1) of the) Lax system,

$$\partial_x\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \tilde{u}_0\\ -\overline{\tilde{u}_0} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

with parameter $\frac{i}{2}$. For $v = 0$, i.e., the unperturbed system, this has an L^2-, and, in fact, $H^{0,1}$-solution[1]

$$\psi^{(0)} = \frac{1}{2}\operatorname{sech}(x)\begin{pmatrix}e^{-\frac{x}{2}}\\ -e^{\frac{x}{2}}\end{pmatrix} \tag{5.6}$$

We normalized (5.6) to be the left Jost solution in the unperturbed case.

Step 1: Our first goal is to show the following claim:

[1]The superscript notation here is not to be confused with the superscript notation used later in this chapter to indicate dependency of (3.1)-solutions on the potential.

Claim 5.4. *The left Jost solution $\psi_l(x)$ for $x < 0$ satisfies $\psi_l(x) = \psi^{(0)}(x) + \phi(x)$ with $\psi^{(0)}$ as in (5.6) and*

$$\|(1 + |\cdot|)e^{-\frac{\cdot}{2}}\phi(\cdot)\|_{L^\infty(\mathbb{R}_-)\times L^2\cap L^\infty(\mathbb{R}_-)} \lesssim \|v\|_{H^{0,1}(\mathbb{R})} \tag{5.7}$$

On $\mathbb{R}_+$, with the corresponding right Jost solution $\psi_r(x) = -\psi^{(0)}(x) + \tilde{\phi}(x)$, we have a similar estimate to (5.7) for $(1 + |\cdot|)e^{+\frac{\cdot}{2}}\tilde{\phi}(\cdot)$.

As we will see (5.7) and its analogue for the right solution on $\mathbb{R}_+$ suffice to show that the Bäcklund transform maps to a potential that is $H^{0,1}$-close to 0.

Proof of Claim. We only explicitly show the estimate (5.7) for the left Jost solution. Rewrite the system:

$$\begin{aligned}\partial_x \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= \begin{pmatrix}\frac{1}{2} & \operatorname{sech}(x)\\ -\operatorname{sech}(x) & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + \begin{pmatrix}0 & v\\ -\overline{v} & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\ &=: \begin{pmatrix}\frac{1}{2} & \operatorname{sech}(x)\\ -\operatorname{sech}(x) & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + \begin{pmatrix}f_1\\ f_2\end{pmatrix}\end{aligned} \tag{5.8}$$

For $v = 0$, there is a fundamental solution matrix:

$$U(x) = -\frac{1}{2}\operatorname{sech}(x)\begin{pmatrix}-e^{-\frac{x}{2}} & e^{\frac{x}{2}}[e^x + 2(1+x)e^{-x}]\\ e^{\frac{x}{2}} & e^{-\frac{x}{2}}(e^{-x} - 2xe^x)\end{pmatrix}$$

with inverse

$$U(y)^{-1} = \frac{1}{2}\operatorname{sech}(y)\begin{pmatrix}e^{\frac{y}{2}}(e^{-2y} - 2y) & e^{-\frac{y}{2}}(e^{2y} + 2y + 2)\\ e^{\frac{y}{2}} & -e^{-\frac{y}{2}}\end{pmatrix}$$

By the Lemma 4.1, the left Jost solution

$$\psi_l(y) = \psi^{(0)}(y) + \phi(y) \tag{5.9}$$

satisfies the estimate

$$\|e^{-\frac{y}{2}}\psi_l(y)\|_{L^\infty\times L^2\cap L^\infty(\mathbb{R})} \leq C_{\|\tilde{u}_0\|_{L^2(\mathbb{R})}} \tag{5.10}$$

where $C_{\|\tilde{u}_0\|_{L^2(\mathbb{R})}}$ is bounded for bounded $\|\tilde{u}_0\|_{L^2}$ (compare Lemma 4.1). Moreover, $\lim_{y\to-\infty} e^{-\frac{y}{2}}\phi(y) = 0$, and in particular, $\lim_{y\to-\infty}\phi(y) = 0$ by the definition of ψ_l. Plugging (5.9) into (5.8) and using that $\psi^{(0)}$ is a solution for $v = 0$, we get:

$$\partial_x \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \operatorname{sech}(x)\\ -\operatorname{sech}(x) & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \begin{pmatrix}f_1\\ f_2\end{pmatrix} \tag{5.11}$$

This yields a variation of constants formula:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \int\limits_{-\infty}^{x} U(x)U(y)^{-1}\begin{pmatrix}f_1(y)\\ f_2(y)\end{pmatrix} dy$$

Calculate:

$$U(x)U(y)^{-1} = -\frac{1}{4}\operatorname{sech}(x)\operatorname{sech}(y)\begin{pmatrix}u_{11}(x,y) & u_{12}(x,y)\\ u_{21}(x,y) & u_{22}(x,y)\end{pmatrix}$$

with

$$u_{11}(x,y) = -e^{\frac{y-x}{2}}(e^{-2y} - 2y) + e^{\frac{x+y}{2}}[e^{x} + 2(1+x)e^{-x}]$$

$$\begin{aligned}
u_{12}(x,y) &= -e^{-\frac{x+y}{2}}(e^{2y} + 2y + 2) - e^{\frac{x-y}{2}}[e^{x} + 2(1+x)e^{-x}]\\
u_{21}(x,y) &= e^{\frac{x+y}{2}}(e^{-2y} - 2y) + e^{\frac{y-x}{2}}(e^{-x} - 2xe^{x})\\
u_{22}(x,y) &= e^{\frac{x-y}{2}}(e^{2y} + 2y + 2) - e^{-\frac{x+y}{2}}(e^{-x} - 2xe^{x})
\end{aligned}$$

For two matrices A and B, write $|A| \leq B$ whenever the absolute value of every matrix entry of A is smaller or equal than the corresponding entry of B. restricting ourselves to the case $y \leq x \leq 0$ and dropping all but the largest terms from the above matrix:

$$\begin{aligned}
e^{-\frac{x}{2}}|U(x)U(y)^{-1}| &\lesssim e^{-\frac{x}{2}}e^{x}e^{y}\begin{pmatrix}e^{-\frac{3}{2}y}e^{-\frac{x}{2}} & (1+|y|)e^{-\frac{x+y}{2}}\\ e^{\frac{x}{2}}e^{-\frac{3}{2}y} & (1+|y|)e^{\frac{x-y}{2}} + e^{-\frac{3}{2}x}e^{-\frac{y}{2}}\end{pmatrix}\\
&= \begin{pmatrix}e^{-\frac{y}{2}} & (1+|y|)e^{\frac{y}{2}}\\ e^{x}e^{-\frac{y}{2}} & (1+|y|)e^{x}e^{\frac{y}{2}} + e^{-x}e^{\frac{y}{2}}\end{pmatrix}
\end{aligned}$$

Moreover, in the same notation, whenever $y \leq 0$:

$$\begin{aligned}
\left|\begin{pmatrix}f_1(y)\\ f_2(y)\end{pmatrix}\right| &= \left|\begin{pmatrix}v(y)\psi_2^{(0)}(y) + v(y)\phi_2(y)\\ -\overline{v}(y)\psi_1^{(0)}(y) - \overline{v}(y)\phi_1(y)\end{pmatrix}\right|\\
&\lesssim \begin{pmatrix}|v(y)|e^{\frac{3}{2}y} + |v(y)||\phi_2(y)|\\ |v(y)|e^{\frac{y}{2}} + |v(y)||\phi_1(y)|\end{pmatrix}
\end{aligned}$$

Thus, with

$$\begin{aligned}
w_1(y) &= e^{-\frac{y}{2}}(|v(y)|e^{\frac{3}{2}y} + |v(y)||\phi_2(y)|)\\
&\qquad + (1+|y|)e^{\frac{y}{2}}(|v(y)|e^{\frac{y}{2}} + |v(y)||\phi_1(y)|)\\
&= |v(y)|\Big[e^{y} + e^{-\frac{y}{2}}|\phi_2(y)| + (1+|y|)e^{y}(1 + e^{-\frac{y}{2}}|\phi_1(y)|)\Big]
\end{aligned}$$

and

$$w_2(y) = e^x e^{-\frac{y}{2}}(|v(y)|e^{\frac{3}{2}y} + |v(y)||\phi_2(y)|) \\ + [(1+|y|)e^x e^{\frac{y}{2}} + e^{-x}e^{\frac{y}{2}}](|v(y)|e^{\frac{y}{2}} + |v(y)||\phi_1(y)|) \\ = |v(y)|\Big[e^x[e^y + e^{-\frac{y}{2}}|\phi_2(y)| + (1+|y|)e^y(1+e^{-\frac{y}{2}}|\phi_1(y)|)] \\ + e^{-x}e^y(1+e^{-\frac{y}{2}}|\phi_1(y)|)\Big]$$

we have

$$e^{-\frac{x}{2}}|U(x)U(y)^{-1}f(y)| \lesssim \begin{pmatrix} w_1(y) \\ w_2(y) \end{pmatrix} \\ \lesssim |v(y)| \begin{pmatrix} e^y + e^{-\frac{y}{2}}|\phi_2(y)| + (1+|y|)e^y \\ e^x[e^y + e^{-\frac{y}{2}}|\phi_2(y)| + (1+|y|)e^y] \end{pmatrix} + |v(y)| \begin{pmatrix} 0 \\ e^{y-x} \end{pmatrix},$$

We used (5.10) (and (5.9)) for the last inequality. Again by (5.10), we have thus shown the existence of $L^2(\mathbb{R}_-)$-functions g, h, with $L^2(\mathbb{R}_-)$-norms below a uniform bound, such that:

$$e^{-\frac{x}{2}}|U(x)U(y)^{-1}f(y)| \lesssim |v(y)| \begin{pmatrix} |g(y)| \\ e^x|h(y)| \end{pmatrix} + |v(y)| \begin{pmatrix} 0 \\ e^{y-x} \end{pmatrix}$$

Hence, whenever $x \leq 0$

$$e^{-\frac{x}{2}}|\phi(x)| = \left| e^{-\frac{x}{2}} \int_{-\infty}^{x} U(x)U(y)^{-1} \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} dy \right| \\ \lesssim \int_{-\infty}^{x} |v(y)| \begin{pmatrix} |g(y)| \\ e^x|h(y)| \end{pmatrix} dy + \int_{-\infty}^{x} |v(y)| \begin{pmatrix} 0 \\ e^{y-x} \end{pmatrix} dy$$

Hölder's inequality (for the first summand) and Young's inequality (for the second) give us $\|e^{-\frac{\cdot}{2}}\phi(\cdot)\|_{L^\infty(\mathbb{R}_-)\times L^2\cap L^\infty(\mathbb{R}_-)} \lesssim \|v\|_{L^2(\mathbb{R})}$. Using the estimate $|x|\int_{-\infty}^{x}|r(y)|dy \leq \int_{-\infty}^{x}|y||r(y)|dy$ for $x < 0$, we can finally get the desired $H^{0,1}$-result (5.7). □

Step 2: With Claim 5.4 established, it is now straightforward to show that, for $x \leq 0$, we have

$$\frac{\psi_{l,1}(x)\overline{\psi_{l,2}(x)}}{\|\psi_l(x)\|^2} = -\frac{1}{2}\operatorname{sech}(x) + R(x),$$

with $\|R\|_{H^{0,1}(\mathbb{R})} \leq \|v\|_{H^{0,1}(\mathbb{R})}$. Because we have

$$u_0 = B(\tilde{u}_0, \frac{i}{2}, \tilde{\psi}_0) = \tilde{u}_0 + 2\frac{\tilde{\psi}_{0,1}\overline{\tilde{\psi}_{0,2}}}{\|\tilde{\psi}_0\|^2}$$

in our situation, and $\tilde{\psi}_0 \sim \psi_l$ by Remark 4.4, this suffices to establish the $H^{0,1}$-bound on u_0 of Lemma 5.3 on $\mathbb{R}_-$. Notice first that we can use

$$-\frac{1}{2}\operatorname{sech}(x) = \frac{\psi_1^{(0)}(x)\overline{\psi_2^{(0)}}(x)}{\|\psi^{(0)}(x)\|^2}$$

to get:

$$\begin{aligned}
R(x) =& \frac{[\psi_1^{(0)}(x) + \phi_1(x)][\psi_2^{(0)}(x) + \overline{\phi}_2(x)]}{[\psi_1^{(0)}(x) + \phi_1(x)]^2 + [\psi_2^{(0)}(x) + \phi_2(x)]^2} - \frac{\psi_1^{(0)}(x)\overline{\psi_2^{(0)}}(x)}{\|\psi^{(0)}(x)\|^2} \\
=& \frac{[\psi_1^{(0)}(x) + \phi_1(x)][\psi_2^{(0)}(x) + \overline{\phi}_2(x)] - \psi_1^{(0)}\psi_2^0}{[\psi_1^{(0)}(x) + \phi_1(x)]^2 + [\psi_2^{(0)}(x) + \phi_2(x)]^2} \\
& + \psi_1^{(0)}\psi_2^{(0)}\Big[[(\psi_1^{(0)}(x) + \phi_1(x))^2 + (\psi_2^{(0)}(x) + \phi_2(x))^2]^{-1} \\
& \qquad - [\psi_1^{(0)}(x)^2 + \psi_2^{(0)}(x)^2]^{-1}\Big]
\end{aligned} \tag{5.12}$$

Now, $\psi_1^{(0)}(x) + \phi_1(x) = e^{\frac{x}{2}}[e^{-x}\operatorname{sech}(x) + e^{-\frac{x}{2}}\phi_1(x)]$, so the first summand is bounded by

$$\frac{e^{-x}}{(1 + \|v\|_{H^{0,1}(\mathbb{R})})^2}(|\psi_1^{(0)}\overline{\phi}_2| + |\psi_2^{(0)}\phi_1| + |\phi_1\overline{\phi}_2|)$$

while the absolute value of the second can similarly be bounded by

$$\frac{e^{-2x}\operatorname{sech}^2(x)}{(1 + \|v\|_{H^{0,1}(\mathbb{R})})^2}(2|\psi_1^{(0)}\phi_1| + |\phi_1|^2 + 2|\psi_2^{(0)}\phi_2| + |\phi_2|^2)$$

and by (5.7), both can be estimated against $\|v\|_{H^{0,1}(\mathbb{R})}$ for $x \le 0$. The argument for $x > 0$ proceeds similarly. Finally, if $\tilde{u}_0 \in H^3(\mathbb{R})$ in Lemma 5.3, (5.11) and (5.12) imply $u_0 \in H^3(\mathbb{R})$ by standard arguments. □

The wave function at $t = 0$

For $\tilde{u}_0 \sim \operatorname{sech}(\cdot)$ the initial value of NLS in Theorem 5.2, let $\tilde{\psi}_0$ and $u_0 \sim 0$ be as in Lemma 5.3. By (3.11), a solution ψ_0 of the spatial part (3.1) of the Lax system with parameter $\frac{i}{2}$ and potential u_0 is given by

$$\psi_1(x) = \frac{\overline{\tilde{\psi}_2}}{|\tilde{\psi}|^2} \quad \psi_2(x) = -\frac{\overline{\tilde{\psi}_1}}{|\tilde{\psi}|^2}$$

We can represent

$$\psi_0 = c_l \psi_l^{(u_0)} + c_r \psi_r^{(u_0)} \tag{5.13}$$

with ψ_l and ψ_r the left and right Jost functions of the system (3.1) with potential u_0 and parameter $\zeta = \frac{i}{2}$. We indicate the dependency of the Jost solutions on the potential by a superscript. We next use the reiteration relation (3.14) to show

Lemma 5.5. *For the coefficients $c_l \in \mathbb{C}$ and $c_r \in \mathbb{C}$ from (5.13), we have $\left|\frac{c_l}{c_r} - 1\right| \lesssim \epsilon$.*

Proof. We momentarily drop the superscript for ease of notation. Moreover, we can assume $c_l = c$, $c_r = 1$ by (3.13). We also assume that $|c| > 1$, and a similar argument to the following can be made for $|c| < 1$ by exchanging the role of the left and right Jost solution:

We first consider that on $1 \leq x \leq 8$:

$$\begin{aligned}\left|\operatorname{sech}(x) - \frac{2c}{|c|^2 e^x + e^{-x}}\right| &= 2\left|\frac{(|c|^2 - c)e^x - (c-1)e^{-x}}{(e^x + e^{-x})(|c|^2 e^x + e^{-x})}\right| \\ &\geq \frac{1}{2}\frac{|c-1|}{|c|}e^{-x} - \frac{|c-1|}{|c|^2}e^{-3x} \\ &\geq \frac{1}{2}\frac{|c-1|}{|c|}e^{-x} - \frac{1}{2}\frac{|c-1|}{|c|}e^{-2x},\end{aligned}$$

which implies

$$\left\|\operatorname{sech}(x) - \frac{2c}{|c|^2 e^x + e^{-x}}\right\|_{L^2([1,8])} \geq \frac{1}{20}\frac{|c-1|}{|c|}$$

We have, by Lemma 4.1:

$$\|e^{-\frac{x}{2}}\psi_{l,1}^{(u_0)} - 1\|_{L^\infty(\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})} \qquad \|e^{-\frac{x}{2}}\psi_{l,2}^{(u_0)}\|_{L^2\cap L^\infty(\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})} \tag{5.14}$$

$$\|e^{\frac{x}{2}}\psi_{r,1}^{(u_0)}\|_{L^2\cap L^\infty(\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})} \qquad \|e^{\frac{x}{2}}\psi_{r,2}^{(u_0)} - 1\|_{L^\infty(\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})} \tag{5.15}$$

We use this to estimate

$$\left|\frac{c}{\|\psi_0\|^2} - \frac{c}{|c|^2 e^x + e^{-x}}\right| = |c|\left|\frac{(|c|^2 e^x + e^{-x}) - \|\psi_0\|^2}{\|\psi_0\|^2(|c|^2 e^x + e^{-x})}\right|$$

We have $\|\psi_0\|^2 = |c\psi_{l,1}^{(u_0)} + \psi_{r,1}^{(u_0)}|^2 + |c\psi_{l,2}^{(u_0)} + \psi_{r,2}^{(u_0)}|^2$ and for sufficiently small ϵ,

$$0 < |c|^2 r(\epsilon)e^x + r(\epsilon)e^{-x} - 4|c|\epsilon(1+\epsilon) \leq \|\psi_0\|^2 \leq \\ \leq |c|^2 r(\epsilon)e^x + r(\epsilon)e^{-x} + 4|c|\epsilon(1+\epsilon),$$

where $r(\epsilon) = \epsilon^2 + (1+\epsilon)^2$. Hence, by (5.14) and (5.15), we have, for $x > 0$,

$$\left|(|c|^2 e^x + e^{-x}) - \|\psi_0\|^2\right| \lesssim |c|^2 \epsilon e^x + \epsilon e^{-x} + |c|\epsilon,$$

as well as:

$$\|\psi_0\|^2 (|c|^2 e^x + e^{-x}) \gtrsim |c|^4 e^{2x}$$

Because $|c| > 1$, we have

$$\left\| \frac{c}{\|\psi_0\|^2} - \frac{c}{|c|^2 e^x + e^{-x}} \right\|_{L^2([1,8])} \lesssim \epsilon \tag{5.16}$$

Finally, by similar arguments:

$$\left| \frac{(\psi_0)_1 \overline{(\psi_0)_2}}{\|\psi_0\|^2} - \frac{c}{\|\psi_0\|^2} \right| \lesssim \frac{|c||\psi_{l,1}^{(u_0)} \overline{\psi_{r,2}^{(u_0)}} - 1| + |c|^2 \epsilon e^x + |c|\epsilon^2 + \epsilon^2 e^{-x}}{|c|^2 e^x} \\ \lesssim \epsilon,$$

and

$$\left\| \frac{(\psi_0)_1 \overline{(\psi_0)_2}}{\|\psi_0\|^2} - \frac{c}{\|\psi_0\|^2} \right\|_{L^2([1,8])} \lesssim 7\epsilon \tag{5.17}$$

If ϵ small enough, (5.16) and (5.17) and the assumption from Lemma 5.3 give us:

$$\epsilon \gtrsim \left\| \operatorname{sech}(x) - \frac{2(\psi_0)_1 \overline{(\psi_0)_2}}{\|\psi_0\|^2} \right\|_{L^2(\mathbb{R})} \geq \left\| \operatorname{sech}(x) - \frac{2(\psi_0)_1 \overline{(\psi_0)_2}}{\|\psi_0\|^2} \right\|_{L^2([1,8])} \\ \geq \left\| \operatorname{sech}(x) - \frac{2c}{|c|^2 e^x + e^{-x}} \right\|_{L^2([1,8])} - \left\| \frac{2(\psi_0)_1 \overline{(\psi_0)_2}}{\|\psi_0\|^2} - \frac{2c}{\|\psi_0\|^2} \right\|_{L^2([1,8])} \\ - \left\| \frac{2c}{\|\psi_0\|^2} - \frac{2c}{|c|^2 e^x + e^{-x}} \right\|_{L^2([1,8])} \\ \gtrsim \frac{|c-1|}{|c|} - \epsilon$$

Thus, we obtain $\frac{|c-1|}{|c|} \lesssim \epsilon$, which implies $|c-1| \lesssim \epsilon$. □

Time evolution of the wave function

Because $\|u_0\|_{H^{0,1}(\mathbb{R})} \lesssim \epsilon$, by Theorem 2.3 we have a unique NLS solution $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ with initial data $u(0, \cdot) = u_0(\cdot)$ which satisfies $\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \lesssim \epsilon |t|^{-\frac{1}{2}}$. Because we showed $u_0 \in H^3(\mathbb{R})$ in Lemma 5.3, by standard theory of NLS (compare [4] and [20], Chapter 5) we have $u \in C(\mathbb{R}_+, H^3(\mathbb{R}))$. We can now consider the full Lax system (3.1) and (3.2) with potential u. We follow [22] in the proof of the following two lemmas:

Lemma 5.6. *There are solutions $\psi_l(t, x)$ and $\psi_r(t, x)$ of the full Lax system (3.1) and (3.2) (spatial and temporal part) with potential u and parameter $\zeta = \frac{i}{2}$ such that $\psi_l(0, x) = \psi_l^{(u_0)}$ and $\psi_r(0, x) = \psi_r^{(u_0)}$. Thus, we obtain*

$$\psi(t, x) = c_l \psi_l(t, x) + c_r \psi_r(t, x) \tag{5.18}$$

as a simultaneous solution of (3.1) and (3.2) with initial value $\psi(0, \cdot) = \psi_0(\cdot)$, where ψ_0 as in (5.13).

Sketch of Proof. Let $\psi(t, x)$ be a solution of the temporal part (3.2) of the Lax system that satisfies the spatial part at $t = 0$. By the assumption on regularity of the potential and bootstrapping arguments, the mixed derivatives $\partial_t \partial_x \psi(t, x)$ and $\partial_x \partial_t \psi(t, x)$ exist and are continuous, and thus equal. Because the potential u satisfies the NLS, if we write the spatial part (3.1) of our Lax system as $\partial_x \psi = A\psi$ and $\partial_t \psi = B\psi$ for the temporal part (3.2), the zero curvature condition (3.4) discussed in Chapter 3

$$\partial_x B - \partial_t A - [A, B] = 0$$

holds. If we now consider the function $F(t, x) = \partial_x \psi(t, x) - A(t, x)\psi(t, x)$, we get

$$\begin{aligned}
\partial_t F(t, x) &= \partial_t \partial_x \psi - (\partial_t A)\psi - A\partial_t \psi = \partial_x \partial_t \psi - (\partial_t A)\psi - AB\psi \\
&= \partial_x (B\psi) - (\partial_t A)\psi - AB\psi \\
&= (\partial_x B - \partial_t A - [A, B])\psi - BA\psi + B\partial_x \psi = BF
\end{aligned}$$

Because the potential is in $C(\mathbb{R}, H^3(\mathbb{R}))$, this implies $|\partial_t F(t)| \leq k(t)|F(0)|$ with $k(t)$ bounded on bounded (in t) intervals. Since $F(0) = 0$, Gronwall's equality now yields $F(t) = 0 \; \forall t > 0$ and thus the claim. □

Lemma 5.7. *For $\psi_l = \psi_l(t, x)$ and $\psi_r = \psi_r(t, x)$ as in Lemma 5.6, we have*

$$\lim_{x \to -\infty} e^{-\frac{x}{2}} \psi_l(t, x) = e^{\frac{it}{2}} \tag{5.19}$$

and

$$\lim_{x \to \infty} e^{\frac{x}{2}} \psi_r(t, x) = e^{-\frac{it}{2}}, \tag{5.20}$$

implying that $\psi_l(t, x) = e^{\frac{it}{2}} \psi_l^{(u(t,x))}$ and $\psi_l(t, x) = e^{-\frac{it}{2}} \psi_r^{(u(t,x))}$, with the superscript notation as introduced before Lemma 5.5.

Proof. Let

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the third Pauli matrix. The solution for the temporal system (3.2) can be implicitly represented by the following integral equation:

$$\psi_l(t,x) = e^{i\frac{\sigma}{2}t}\psi_l(0,x) + \int_0^t e^{i\frac{\sigma(t-s)}{2}}(B(s,x) - i\frac{\sigma}{2})\psi_l(s,x)ds \tag{5.21}$$

We have $|\partial_t(e^{-x}\|\psi_l\|^2)| = 4|\operatorname{Im}(e^{-x}q\overline{\psi_{l,1}}\psi_{l,2})| \lesssim \|q(t,\cdot)\|_{L^\infty(\mathbb{R})}e^{-x}\|\psi_l\|^2$, so Gronwall's inequality and the fact that $\psi(0,\cdot) \in L^\infty \times L^\infty(\mathbb{R})$ shows that $e^{-\frac{x}{2}}\psi_l \in C([0,t], L^\infty \times L^\infty(\mathbb{R}))$ for any $t > 0$. The regularity assumption that $q \in C(\mathbb{R}, H^3(\mathbb{R}))$ also implies that B has bounded operator norm on bounded t-intervals. Lebesgue's bounded convergence theorem therefore implies

$$\lim_{x\to-\infty}(e^{-\frac{x}{2}}\psi_l(t,x)) = e^{i\frac{\sigma}{2}t}\lim_{x\to-\infty}(e^{-\frac{x}{2}}\psi_l(0,x)) = e^{i\frac{\sigma}{2}t}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

by the definition of the left Jost solution, this yields (5.19), and for ψ_r, we can obtain (5.20) in the same way. □

L^∞-estimates

The following Proposition 5.8 is decisive for showing our asymptotic stability result Theorem 5.2. Nevertheless, we now temporarily switch notation and denote the potential of (3.1) by ϕ to emphasize its universality and independence from any other assumptions except L^∞-smallness of ϕ and existence of the left and right Jost solution. By Lemma 4.1, finiteness of the L^2-norm is sufficient for the latter assumption to hold, smallness of $\|\phi\|_{L^2(\mathbb{R})}$ is not needed. Moreover, Proposition 5.8 only depends on properties of the spatial part (3.1) of the Lax system and can thus be immediately employed in the soliton stability analysis for other equations in the NLS hierarchy (see Chapter 3), unlike Proposition 5.10, which depends on the asymptotic expression given in Theorem 2.3.

Proposition 5.8. *Let $\phi \in L^2(\mathbb{R})$, so that the left and right Jost solutions of the Lax system (3.1) with potential ϕ and parameter $\frac{i}{2}$ exist by Lemma 4.1, and let $\|\phi\|_{L^\infty(\mathbb{R})}$ be sufficiently small. Let a_l, $a_r \in \mathbb{C}$. Then there exists and is unique a point x_0 where $|a_l\psi_{l,1}^{(\phi)}(x_0)| = |a_r\psi_{r,2}^{(\phi)}(x_0)|$. If θ is the phase of $a_l\bar{a}_r\psi_{l,1}^{(\phi)}\overline{\psi_{r,2}^{(\phi)}}$ in that point, i.e. $a_l\bar{a}_r\psi_{l,1}^{(\phi)}\overline{\psi_{r,2}^{(\phi)}}(x_0) = e^{i\theta}|a_l\bar{a}_r\psi_{l,1}^{(\phi)}\overline{\psi_{r,2}^{(\phi)}}(x_0)|$,*

$$\|B(\phi, \frac{i}{2}, a_l\psi_l^{(\phi)} + a_r\psi_r^{(\phi)}) - e^{i\theta}\operatorname{sech}(x - x_0)\|_{L^\infty(\mathbb{R})} \lesssim \|\phi\|_{L^\infty(\mathbb{R})} \tag{5.22}$$

holds.

For ease of calculation, we can rewrite[2] the spatial part of the Lax system (3.1) with potential ϕ and parameter $\frac{i}{2}$:

$$\frac{d}{dx}(e^{-\frac{x}{2}}\psi_1(x)) = \phi(x)e^{-\frac{x}{2}}\psi_2(x) \tag{5.23}$$

$$\frac{d}{dx}(e^{\frac{x}{2}}\psi_2(x)) = -\overline{\phi}(x)e^{\frac{x}{2}}\psi_1(x) \tag{5.24}$$

To show Proposition 5.8, we first prove the following Claim 5.9, which will put us in a position to appropriately control the "smaller" components of the Jost solutions in terms of the "larger" ones (equations (5.25) and (5.26)). This will enable us to approximate $\psi_1\overline{\psi}_2$ by $\psi_{l,1}\overline{\psi}_{r,2}$ (compare (5.29)). Via Gronwall's inequality, (5.25) and (5.26) also imply (5.27) and (5.28), which already suggest that the absolute values of the Jost solutions' large components behave like "recentered" exponential functions, with an appropriately chosen center $\tilde{x}_0$. In (5.30), we will basically split up the task of proving (5.22) into two subtasks: Bounding the deviation of the phase (which concerns the first summand on the right side of (5.30)) and bounding the deviation of the absolute value (which concerns the second). The former will be accomplished by another application of Gronwall's inequality (see (5.31) and (5.32)), the latter by suitable bounds on the norm of ψ in terms of the sech$(\cdot)$-function (see (5.34)).[3]

Claim 5.9. *Let $\psi(x) = (\psi_1(x), \psi_2(x))^T$ be a solution of (5.23), (5.24) that is absolutely continuous and satisfies $\psi \in L^\infty((-\infty, r])$ for some $r \in \mathbb{R}$. Then for any real number $m > 2$, we have the pointwise estimate*

$$\frac{|\psi_2(x)|}{|\psi_1(x)|} \leq 2\|\phi\|_{L^\infty(\mathbb{R})} \; \forall x \in \mathbb{R}$$

whenever $\|\phi\|_{L^\infty(\mathbb{R})} < \frac{1}{4}$. A similar statement, with the roles of the first and second component of ψ exchanged, holds when $\psi \in L^\infty([r, \infty))$, $r \in \mathbb{R}$.

Proof. The case $\|\phi\|_{L^\infty(\mathbb{R})} = 0$ is clear (see Chapter 4). For $\|\phi\|_{L^\infty(\mathbb{R})} > 0$, pick any real number $m > 2$ and assume there was some u and ψ so that $\|\phi\|_{L^\infty(\mathbb{R})} < \frac{1-m^{-1}}{m}$ and $|\psi_2(x_1)| > m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x_1)|$ for some $x_1 < r$. Our first step is to prove by contradiction that in this case, $|\psi_2(x)| \geq m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x)|$ for all $x \leq x_1$. In a second (easy) step, we will see that this leads to a contradiction, establishing that such an x_1 can not exist:

Step 1: Suppose that in the above situation, there was some $x_0 < x_1$ such that $|\psi_2(x_0)| < m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x_0)|$ and consider

$$\tilde{x} = \inf\{x \in [x_0, x_1] : |\psi_2(x)| > m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x)|\}$$

[2] Using this form of the equations is inspired by [21].

[3] Also recall for the following proof that the absolute value of an absolutely continuous function is absolutely continuous and hence differentiable almost everywhere.

By continuity, we have $\tilde{x} > x_0$ and $|\psi_2(\tilde{x})| = m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(\tilde{x})|$. Moreover, $|\psi_2(x)| \leq m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x)|$ on $[x_0, \tilde{x}]$. Therefore, if $\psi_2(\tilde{x}) = \psi_1(\tilde{x}) = 0$, (5.23) and Gronwall's inequality would imply that $\psi_1(x) = 0$ for any $x \in [x_0, \tilde{x}]$, and together with (5.24) this would mean $\psi_2(x_0) = 0$. However, this possibility is excluded by the assumption that $|\psi_2(x_0)|$ ist strictly smaller than $m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x_0)|$.

By picking a sufficiently small neighbourhood of $\tilde{x}$, we can thus obtain an interval $I := [a, b]$ such that:

a) $0 < \frac{\psi_2(x)}{\psi_1(x)} < \infty$ is absolutely continuous on $[a, b]$.

b) $(m-\delta)\|\phi\|_{L^\infty(\mathbb{R})} < \frac{|\psi_2(x)|}{|\psi_1(x)|} < (m+\delta)\|\phi\|_{L^\infty(\mathbb{R})}$ for some small $\delta > 0$ (which can be made arbitrarily small).

c) $\frac{|\psi_2(a)|}{|\psi_1(a)|} \leq m\|\phi\|_{L^\infty(\mathbb{R})}$ and $\frac{|\psi_2(b)|}{|\psi_1(b)|} > m\|\phi\|_{L^\infty(\mathbb{R})}$. (Notice that this is not necessarily true for *all* $b > \tilde{x}$ close to $\tilde{x}$, but there are such real points arbitrarily close to $\tilde{x}$ for which it is.)

For ease of notation, assume $a = -\epsilon$, $b = 0$.

From (5.24) and property b) of I, we get $\frac{d}{dx}(e^{\frac{x}{2}}|\psi_2(x)|) \leq \|\phi\|_{L^\infty(\mathbb{R})}e^{\frac{x}{2}}|\psi_1(x)| \leq (m-\delta)^{-1}e^{\frac{x}{2}}|\psi_2(x)|$ on $[-\epsilon, 0]$. Gronwall's inequality now yields $e^0|\psi_2(0)| \leq e^{(m-\delta)^{-1}\epsilon}e^{-\frac{\epsilon}{2}}|\psi_2(-\epsilon)|$ or

$$|\psi_2(-\epsilon)| \geq e^{(\frac{1}{2}-(m-\delta)^{-1})\epsilon}|\psi_2(0)|$$

Similarly, $\left|\frac{d}{dx}(e^{-\frac{x}{2}}|\psi_1(x)|)\right| \leq (m+\delta)\|\phi\|^2_{L^\infty(\mathbb{R})}e^{-\frac{x}{2}}|\psi_1(x)|$ on $[-\epsilon, 0]$, which we obtain using (5.23), implies that $e^{\frac{\epsilon}{2}}|\psi_1(-\epsilon)| \leq e^{(m+\delta)\|\phi\|^2_{L^\infty(\mathbb{R})}\epsilon}|\psi_1(0)|$ or

$$|\psi_1(-\epsilon)| \leq e^{[(m+\delta)\|\phi\|^2_{L^\infty(\mathbb{R})}-\frac{1}{2}]\epsilon}|\psi_1(0)|$$

Together, these estimates give us:

$$\begin{aligned}\frac{|\psi_2(-\epsilon)|}{|\psi_1(-\epsilon)|} &\geq \frac{|\psi_2(0)|}{|\psi_1(0)|} \exp\left[\left[\frac{1}{2}-(m-\delta)^{-1}\right]\epsilon - \left[(m+\delta)\|\phi\|^2_{L^\infty(\mathbb{R})} - \frac{1}{2}\right]\epsilon\right] \\ &= \frac{|\psi_2(0)|}{|\psi_1(0)|} \exp((1-(m-\delta)^{-1}-(m+\delta)\|\phi\|^2_{L^\infty(\mathbb{R})})\epsilon)\end{aligned}$$

Now, if $1-(m-\delta)^{-1}-(m+\delta)\|\phi\|^2_{L^\infty(\mathbb{R})} > 0$, which is equivalent to $\|\phi\|_{L^\infty(\mathbb{R})} < \sqrt{\frac{1-(m-\delta)^{-1}}{m+\delta}}$, this yields $\frac{|\psi_2(-\epsilon)|}{|\psi_1(-\epsilon)|} > \frac{|\psi_2(0)|}{|\psi_1(0)|}$, in contradiction to property c) of our interval. As δ can be made arbitrarily small, we can obtain this contradiction whenever $m > 1$, $\|\phi\|^2_{L^\infty(\mathbb{R})} < \frac{1-m^{-1}}{m}$.

Step 2: Now that we have established $|\psi_2(x)| \geq m\|\phi\|_{L^\infty(\mathbb{R})}|\psi_1(x)|$ on $(-\infty, x_1]$, assume for simplicity that $x_1 = 0$. By (5.24), we know that $\frac{d}{dx}(e^{\frac{x}{2}}|\psi_2(x)|) \leq m^{-1}e^{\frac{x}{2}}|\psi_2(x)|$ for any $x < 0$, and Gronwall's inequality gives us $|\psi_2(x)| \geq e^{-\frac{x}{2}+m^{-1}x}|\psi_2(0)|$, leading to a contradiction with $\psi_2 \in L^\infty(\mathbb{R})$ whenever $m > 2$. Because m can be chosen arbitrarily close to 2, we are done. □

Proof of Proposition 5.8: First, it is relatively easy to see that x_0 as defined in Proposition 5.8 exists, because the "large" first component of the left Jost solution goes to infinity as $x \to \infty$, and the "large" second component of the right Jost solution tends to infinity as $x \to -\infty$.

Moreover, since r in the statement of Claim 5.9 is arbitrary, the left Jost solution of (5.23), (5.24) (which is equivalent to (3.1) in our situation) satisfies[4]

$$\frac{|\psi_{l,2}(x)|}{|\psi_{l,1}(x)|} \leq 2\|\phi\|_{L^\infty(\mathbb{R})} \tag{5.25}$$

and a similar argument for the right Jost solution gives us

$$\frac{|\psi_{r,1}(x)|}{|\psi_{r,2}(x)|} \leq 2\|\phi\|_{L^\infty(\mathbb{R})} \tag{5.26}$$

whenever $\|\phi\|_{L^\infty(\mathbb{R})}$ small enough. In particular, because the left side of (5.25) and the inverse of the left side of (5.26) would otherwise have to be related by a constant factor, the spatial part (5.23) and (5.24) of the Lax system does not have an eigenvalue in $\frac{i}{2}$.

We consider $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = a_l\psi_l + a_r\psi_r$. Because the Bäcklund transformation (3.12) is, in this case, given by

$$B(\phi, \frac{i}{2}, \psi) = \phi + \frac{2\psi_1\overline{\psi_2}}{\|\psi\|^2}$$

we need to show that

$$\left|\frac{2\psi_1\overline{\psi_2}}{\|\psi\|^2} - e^{i\theta}\operatorname{sech}(x - x_0)\right| \lesssim \|\phi\|_{L^\infty(\mathbb{R})}$$

By (5.23), (5.24), (5.25) and (5.26):

$$\left|\frac{d}{dx}|e^{-\frac{x}{2}}\psi_{l,1}(x)|\right| \leq \|\phi\|_{L^\infty(\mathbb{R})}e^{-\frac{x}{2}}|\psi_{l,2}(x)| \leq 2\|\phi\|^2_{L^\infty(\mathbb{R})}e^{-\frac{x}{2}}|\psi_{l,1}(x)|,$$

[4]In the context of this proof, we omit the superscript ϕ. It is used for emphasis when we apply Proposition 5.8 in the proof of Theorem 5.2.

and

$$\left|\frac{d}{dx}|e^{\frac{x}{2}}\psi_{r,2}(x)|\right| \le \|\phi\|_{L^\infty(\mathbb{R})}e^{\frac{x}{2}}|\psi_{r,1}(x)| \le 2\|\phi\|^2_{L^\infty(\mathbb{R})}e^{\frac{x}{2}}|\psi_{r,2}(x)|.$$

Using Gronwall's inequality, we get:

$$\frac{e^{-\frac{y}{2}}|\psi_{l,1}(y)|}{e^{-\frac{\tilde{y}}{2}}|\psi_{l,1}(\tilde{y})|} \le e^{2\|\phi\|^2_{L^\infty(\mathbb{R})}|y-\tilde{y}|} \tag{5.27}$$

and

$$\frac{e^{\frac{y}{2}}|\psi_{r,2}(y)|}{e^{\frac{\tilde{y}}{2}}|\psi_{r,2}(\tilde{y})|} \le e^{2\|\phi\|^2_{L^\infty(\mathbb{R})}|y-\tilde{y}|} \tag{5.28}$$

for any y, $\tilde{y} \in \mathbb{R}$. Notice that as a consequence of (5.27) and (5.28), $|\psi_l|$ is strictly monotone increasing and $|\psi_r|$ is strictly monotone decreasing. This implies the uniqueness of x_0.

By (5.25) and (5.26), we can see that $|\psi_1\overline{\psi}_2 - a_l\bar{a}_r\psi_{l,1}\overline{\psi}_{r,2}| \lesssim \|\phi\|_{L^\infty(\mathbb{R})}\|\psi\|^2$. Indeed,

$$\begin{aligned}\max\{|a_l\psi_{l,1}|, |a_r\psi_{r,2}|\} &\le (1-\|\phi\|_{L^\infty(\mathbb{R})})^{-1}[|a_l||\psi_{l,1}| - |a_r||\psi_{r,1}| + \\ &\qquad\qquad + |a_r||\psi_{r,2}| - |a_l||\psi_{l,2}|] \\ &\le (1-\|\phi\|_{L^\infty(\mathbb{R})})^{-1}[|\psi_1| + |\psi_2|] \\ &\le (1-||\phi||_{L^\infty})^{-1}\sqrt{2}\|\psi\|\end{aligned}$$

from which we get the desired estimate because

$$\begin{aligned}|\psi_1\overline{\psi}_2 - a_l\bar{a}_r\psi_{l,1}\overline{\psi}_{r,2}| &= \left||a_l|^2\psi_{l,1}\overline{\psi}_{l,2} + a_r\bar{a}_l\psi_{r,1}\overline{\psi}_{l,2} + |a_r|^2\psi_{r,1}\overline{\psi}_{r,2}\right| \le \\ &\le |a_l|^2\|\phi\|_{L^\infty(\mathbb{R})}|\psi_{l,1}|^2 + |a_r\bar{a}_l|\|\phi\|_{L^\infty(\mathbb{R})}|\psi_{r,2}||\overline{\psi}_{l,1}| \\ &\qquad + |a_r|^2\|\phi\|_{L^\infty(\mathbb{R})}|\psi_{r,2}|^2 \\ &\le \frac{2}{(1-\|\phi\|_{L^\infty(\mathbb{R})})^2}\|\phi\|_{L^\infty(\mathbb{R})}\|\psi\|^2\end{aligned}$$

Thus, we have, with x_0, θ as specified in Proposition 5.8,

$$\begin{aligned}&\left|\frac{2\psi_1\overline{\psi_2}}{\|\psi\|^2} - e^{i\theta}\operatorname{sech}(x-x_0)\right| \\ &\qquad \lesssim \|\phi\|_{L^\infty(\mathbb{R})} + \left|\frac{2a_l\bar{a}_r\psi_{l,1}\overline{\psi_{r,2}}}{\|\psi\|^2} - e^{i\theta}\operatorname{sech}(x-x_0)\right|,\end{aligned} \tag{5.29}$$

In the following, we only treat the estimate in the case $x \geq x_0$, as $x < x_0$ is similar. We employ the definition of x_0 to get:

$$\left|\frac{2a_l\bar{a}_r\psi_{l,1}\overline{\psi_{r,2}}}{\|\psi\|^2} - \frac{e^{i\theta}}{\cosh(x-x_0)}\right|$$

$$\leq \left|\frac{2a_l\bar{a}_r\psi_{l,1}\overline{\psi_{r,2}}}{\|\psi\|^2} - \frac{2e^{i\theta}|a_l||\bar{a}_r||\psi_{l,1}(x_0)||\psi_{r,2}(x_0)|}{\|\psi\|^2}\right| + \left|\frac{2|a_l|^2|\psi_{l,1}(x_0)|^2}{\|\psi\|^2} - \frac{1}{\cosh(x-x_0)}\right| \tag{5.30}$$

By (5.23) and (5.24), we have

$$\begin{aligned}\frac{d}{dx}(\psi_{l,1}\overline{\psi}_{r,2} - \psi_{l,1}(x_0)\overline{\psi}_{r,2}(x_0)) &= \frac{d}{dx}(\psi_{l,1}\overline{\psi}_{r,2}) \\ &= \phi(x)[\psi_{l,2}(x)\overline{\psi}_{r,2}(x) - \psi_{l,1}(x)\overline{\psi}_{r,1}(x)]\end{aligned}$$

Using (5.25) and (5.26), we get:

$$\begin{aligned}&\left|\frac{d}{dx}|\psi_{l,1}\overline{\psi}_{r,2} - \psi_{l,1}(x_0)\overline{\psi}_{r,2}(x_0)|\right| \\ &\quad\lesssim \|\phi\|^2_{L^\infty(\mathbb{R})}|\psi_{l,1}\overline{\psi}_{r,2}| \\ &\quad\leq \|\phi\|^2_{L^\infty(\mathbb{R})}|\psi_{l,1}\overline{\psi}_{r,2} - \psi_{l,1}(x_0)\overline{\psi}_{r,2}(x_0)| + \|\phi\|^2_{L^\infty(\mathbb{R})}|\psi_{l,1}(x_0)||\psi_{r,2}(x_0)|\end{aligned} \tag{5.31}$$

With Gronwall's inequality and the definitions of θ and x_0, this gives us:

$$\begin{aligned}&|a_l\bar{a}_r\psi_{l,1}\overline{\psi}_{r,2}(x) - e^{i\theta}|a_l\bar{a}_r|\psi_{l,1}(x_0)||\psi_{r,2}(x_0)| \\ &\quad= |a_l\bar{a}_r\psi_{l,1}\overline{\psi}_{r,2}(x) - a_l\bar{a}_r\psi_{l,1}(x_0)\overline{\psi}_{r,2}(x_0)| \\ &\quad= |a_l\bar{a}_r||\psi_{l,1}\overline{\psi}_{r,2}(x) - \psi_{l,1}(x_0)\overline{\psi}_{r,2}(x_0)| \\ &\quad\leq \|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}|a_l\bar{a}_r||\psi_{l,1}(x_0)||\psi_{r,2}(x_0)| \\ &\quad= \|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}|a_l|^2|\psi_{l,1}(x_0)|^2\end{aligned} \tag{5.32}$$

With (5.25)-(5.28), we also get:

$$\begin{aligned}\|\psi\|^2 &\geq (1-\|\phi\|_{L^\infty(\mathbb{R})})^2(|a_l|^2|\psi_{l,1}|^2 + |a_r|^2|\psi_{r,2}|^2) \\ &\geq (1-\|\phi\|_{L^\infty(\mathbb{R})})^2|a_l|^2|\psi_{l,1}(x_0)|^2(e^{x-x_0} + e^{-(x-x_0)})e^{-\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}\end{aligned} \tag{5.33}$$

From these estimates, it is immediate that the first summand on the right side of (5.30) is $\lesssim \|\phi\|_{L^\infty(\mathbb{R})}$.

For the second term, consider that besides (5.33), (5.25)-(5.28) also imply an upper bound for $\|\psi\|^2$, and thus

$$
\begin{aligned}
(1-\|\phi\|_{L^\infty(\mathbb{R})})^2|a_l|^2|\psi_{l,1}(x_0)|^2(e^{x-x_0}+e^{-(x-x_0)})e^{-\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)} \leq \|\psi\|^2 \leq \\
\leq (1+\|\phi\|_{L^\infty(\mathbb{R})})^2|a_l|^2|\psi_{l,1}(x_0)|^2(e^{x-x_0}+e^{-(x-x_0)})e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}
\end{aligned}
\tag{5.34}
$$

Therefore:

$$
\begin{aligned}
&\left|\frac{2|a_l|^2|\psi_{l,1}(x_0)|^2}{\|\psi\|^2} - \frac{1}{\cosh(x-x_0)}\right| \\
&\lesssim \|\phi\|_{L^\infty(\mathbb{R})} \\
&\qquad + \max\{(e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}-1),(1-e^{-\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)})\}\,\mathrm{sech}((x-x_0)) \\
&\leq \|\phi\|_{L^\infty(\mathbb{R})} + \max\{(e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}-1),(1-e^{-\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)})\}e^{-(x-x_0)}
\end{aligned}
$$

By differentiating and evaluating at $a=0$ the function $a \to \epsilon^2 a e^{\epsilon^2 a} - e^{\epsilon^2 a}$, we get the inequality $e^{\epsilon^2 a}-1 \leq \epsilon^2 a e^{\epsilon^2 a}$ for $a \geq 0$. For $x \geq x_0$, this gives

$$
\begin{aligned}
&\max\{(e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}-1),(1-e^{-\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)})\} \\
&\qquad \leq \max\{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)e^{\|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)}, \|\phi\|^2_{L^\infty(\mathbb{R})}(x-x_0)\},
\end{aligned}
$$

from which the desired estimate on (5.30) follows. □

Convergence of Jost solutions

On its own, Proposition 5.8 would not suffice to show the convergence of the position and phase shift functions $\tilde{x}_0(t)$ and $\tilde{\theta}_0(t)$. In order to do so, we show convergence of the left, respectively right, Jost solution's "large" components to a monotone increasing, respectively decreasing, function as $t \to \infty$. More precisely, we will establish uniform convergence for rescaled versions of ψ_l and ψ_r, quantitatively controlled by (5.35), from which pointwise convergence of the original Jost functions follows, quantitatively controlled by (5.37). If W in Theorem 2.3 was slightly more regular, partial integration would, in fact, suffice to prove Proposition 5.10 (see discussion after (5.42)), and we generalize to the lower regularity actually given in Theorem 2.3:

Proposition 5.10. *Let $u \in H^{0,1}(\mathbb{R})$ with $\|u(0,\cdot)\|_{H^{0,1}(\mathbb{R})} \leq \epsilon$ be a solution of the NLS (1.1) as in Theorem 2.3. Let $\psi_l^{(u(t,\cdot))} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ and $\psi_r^{(u(t,\cdot))} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ be the left and right Jost solutions of the spatial part (3.1) of the corresponding Lax system with parameter $\zeta = \frac{i}{2}$ at time t, using the* *same superscript notation as above. If we set* $\varphi_l(t,x) := e^{-\frac{x}{2}}\psi_l^{(u(t,\cdot))}(t,x)$ *and* $\tilde{\varphi}(t,x) := \varphi_{l,1}(t,tx)$*, we have, for all sufficiently large* $t > 0$*,*

$$|\tilde{\varphi}(t,x) - P_l(x)| \lesssim \epsilon^2 \log(t) t^{-1+C\epsilon^2}, \tag{5.35}$$

uniformly as $t \to \infty$*, where*

$$P_l(x) := \exp\left(- \int_{-\infty}^{x} \frac{|W(y)|^2}{1-iy} dy \right), \tag{5.36}$$

with $W \in H^{1-C\epsilon^2}(\mathbb{R})$*,* $C > 0$ *as in Theorem 2.3.*
In particular, this implies the pointwise convergence of $\varphi_{l,1}(t,x) \to P_l(0)$ *as* $t \to \infty$ *with an estimate*

$$|\varphi_{l,1}(t,x) - P_l(0)| \lesssim \epsilon^2 |x| t^{-1} + \epsilon^2 \log(t) t^{-1+C\epsilon^2} \tag{5.37}$$

Similar statements hold for the second component of the right Jost solution $\psi_r(t,x) =: e^{-\frac{x}{2}}\varphi_r(t,x)$*, with* $\tilde{\varphi}$ *replaced by* $\tilde{\phi}(t,x) := \varphi_{r,2}(t,tx)$ *and* P_l *by*

$$P_r(x) := \exp\left(- \int_{x}^{\infty} \frac{|W(y)|^2}{1-iy} dy \right) \tag{5.38}$$

Proof. By Theorem 2.3, an asymptotic expression (2.9) for an NLS-potential $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ with small $\|u(0,\cdot)\|_{H^{0,1}(\mathbb{R})} \leq \epsilon$ is given by

$$u(t,x) = t^{-\frac{1}{2}} e^{i\frac{x^2}{2t}} W\left(\frac{x}{t}\right) e^{i\log(t)|W(\frac{x}{t})|} + err_x, \tag{5.39}$$

where $W \in H^{1-C\epsilon^2}(\mathbb{R})$ and

$$err_x \in \epsilon O_{L^\infty(\mathbb{R})}((1+t)^{-\frac{3}{4}+C\epsilon^2}) \cap O_{L^2(\mathbb{R})}((1+t)^{-1+C\epsilon^2}) \tag{5.40}$$

For ease of notation we will modify ϵ to set $C = 1$.

We use the formula (4.11) from Remark 4.3. Then an easy change of variables gives us, with $w(t,x) = e^{i\log(t)|W(x)|}W(x)$,

$$\tilde{\varphi}(t,x) = 1 - \int_{-\infty}^{x} w(t,y) \int_{-\infty}^{y} te^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \tilde{\varphi}(t,z) dz dy + e(t,x), \tag{5.41}$$

where we set $e(t,x) := e_1(t,x) + e_2(t,x) + e_3(t,x)$ with

$$e_1(t,x) = \int_{-\infty}^{tx} \int_{-\infty}^{y} t^{-\frac{1}{2}} e^{i\frac{y^2}{2t}} W\left(\frac{y}{t}\right) e^{i\log(t)|W(\frac{y}{t})|} e^{-(y-z)} \overline{err_x(z)} \varphi_1(t,z) dz dy$$

and

$$e_2(t,x) = \int\limits_{-\infty}^{tx} \int\limits_{-\infty}^{y} err_x(y) e^{-(y-z)} t^{-\frac{1}{2}} e^{-i\frac{z^2}{2t}} \overline{W}\Big(\frac{z}{t}\Big) e^{-i\log(t)|W(\frac{z}{t})|} \varphi_1(t,z) dz dy,$$

as well as

$$e_3(x) = \int\limits_{-\infty}^{tx} \int\limits_{-\infty}^{y} err_x(y) e^{-(y-z)} \overline{err_x(z)} \varphi_1(t,z) dz dy$$

Young's inequality and (5.40) immmediately give:

$$\begin{aligned} \|e_1\|_{L^\infty(\mathbb{R})} &\lesssim \|t^{-\frac{1}{2}} W(t^{-1}\cdot)\|_{L^2(\mathbb{R})} \|e^{-t\cdot}\|_{L^1(\mathbb{R}_+)} \|err_x\|_{L^2(\mathbb{R})} \\ &= \|W\|_{L^2(\mathbb{R})} \|err_x\|_{L^2(\mathbb{R})} \lesssim \epsilon^2 (1+t)^{-1+\epsilon^2}, \end{aligned}$$

and similarly

$$\|e_2\|_{L^\infty(\mathbb{R})} \lesssim \epsilon^2 (1+t)^{-1+\epsilon^2},$$

as well as

$$\|e_3\|_{L^\infty(\mathbb{R})} \leq \|e^{-t\cdot}\|_{L^1(\mathbb{R}_+)} \|err_x\|^2_{L^2(\mathbb{R})} \lesssim \epsilon^2 (1+t)^{-2+2\epsilon^2}$$

Hence, showing that (5.41) does, indeed, behave like (5.35) depends on the main term

$$\int\limits_{-\infty}^{x} w(t,y) \int\limits_{-\infty}^{y} t e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \tilde{\varphi}(t,z) dz dy \tag{5.42}$$

Applying a partial integration argument to (5.42) under the stronger assumption that $W \in H^1(\mathbb{R})$ with

$$\|W\|_{H^1(\mathbb{R})} \lesssim \epsilon \tag{5.43}$$

(and, for simplicity, also that $|W|$ is differentiable everywhere) does, as we claimed, suggest that $\lim\limits_{t\to\infty} \tilde{\varphi}(t,x) = \exp\Big(-\int\limits_{-\infty}^{x} \frac{|W(y)|^2}{1-iy} dy\Big)$ uniformly at the desired rate. Indeed, set the function

$$\phi(t,z) := \frac{\tilde{\varphi}(t,z)}{1-iz} \tag{5.44}$$

Using $e^{-t(y-z)} = t^{-1}\partial_z e^{-t(y-z)}$, we get

$$\begin{aligned} &\int\limits_{-\infty}^{x} w(t,y) \int\limits_{-\infty}^{y} t e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \phi(t,z) dz dy \\ &= \int\limits_{-\infty}^{x} w(t,y) \int\limits_{-\infty}^{y} \partial_z e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \phi(t,z) dz dy \end{aligned}$$

$$= \int_{-\infty}^{x} w(t,y) \left[e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \phi(t,z) \right]_{z=-\infty}^{y} dy$$
$$+ \int_{-\infty}^{x} w(t,y) \int_{-\infty}^{y} e^{-t(y-z)} itze^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \phi(t,z) dzdy$$
$$+ \int_{-\infty}^{x} w(t,y) \int_{-\infty}^{y} e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \partial_z \overline{w(t,z)} \phi(t,z) dzdy$$
$$+ \int_{-\infty}^{x} w(t,y) \int_{-\infty}^{y} e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \partial_z \phi(t,z) dzdy, \tag{5.45}$$

and, using $|w| = |W|$,

$$w(t,y) \left[e^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \phi(t,z) \right]_{z=-\infty}^{y} = \frac{|W(y)|^2}{1-iz} \tilde{\varphi}(t,y),$$

as well as $(1-iz)\phi(t,z) = \varphi(t,z)$ by (5.44). Moreover, $|\partial_z w(t,z)| \lesssim \log(t)|W'(z)|$ for sufficiently large $t > 0$, as well as $|\partial_z \phi(t,z)| \lesssim t|\partial_2 \varphi_{l,1}(t,tz)| + |\varphi_{l,1}(t,tz)|$. The boundedness of $t|\partial_2 \varphi_{l,1}(t,tz)|$ is a consequence of (4.12) and Theorem 2.3 (compare text after (5.56)), and the boundedness of $|\varphi_{l,1}(t,z)|$ is, of course, known by Lemma 4.1. Using Young's inequality and (5.43), we can therefore bound the last two summands in (6.31) by $\epsilon^2 t^{-1} \log(t)$, and thus

$$\tilde{\varphi}(t,x) = 1 - \int_{-\infty}^{x} \frac{|W(y)|^2}{1-iy} \tilde{\varphi}(t,y) dy + O(\epsilon^2 \log(t) t^{-1}) \tag{5.46}$$

It seems natural that, for a slightly lower regularity than $W \in H^1(\mathbb{R})$, we get a loss in the rate of convergence of (5.46) as given in (5.35).

Thus, we know proceed with the general case of Theorem 2.3 that $W \in H^{1-\epsilon^2}(\mathbb{R})$. We can now show (5.35) by proving that

$$\int_{-\infty}^{x} \left| \frac{|W(y)|^2}{1-iy} \tilde{\varphi}(t,y) - \right.$$
$$\left. - w(t,y) \int_{-\infty}^{y} te^{-t(y-z)} e^{\frac{it}{2}(y^2-z^2)} \overline{w(t,z)} \tilde{\varphi}(t,z) dz \right| dy \lesssim \epsilon^2 \log(t) t^{-1+\epsilon^2} \tag{5.47}$$

Before we show (5.47), we briefly establish that it does, indeed, imply (5.35). For, differentiating (5.41), we get

$$\partial_x \tilde{\varphi}(t,x) = -\frac{|W(x)|^2}{1-ix}\tilde{\varphi}(t,x) + \alpha(t,x), \tag{5.48}$$

with

$$\begin{aligned}\alpha(t,x) =& \frac{|W(x)|^2}{1-ix}\tilde{\varphi}(t,x) - \\ & - w(t,x)\int_{-\infty}^{x} te^{-t(x-z)}e^{\frac{it}{2}(x^2-z^2)}\overline{w(t,z)}\tilde{\varphi}(t,z)dzdy + \partial_x e(t,x),\end{aligned}$$

so if (5.47) holds, we have $\int_{-\infty}^{x} |\alpha(t,y)|dy \lesssim \epsilon \log(t) t^{-1+\epsilon^2}$. By variation of constants, (5.48) has the general solution

$$C\exp\left(-\int_{-\infty}^{x}\frac{|W(y)|^2}{1-iy}dy\right) + \int_{-\infty}^{x}\exp\left(-\int_{y}^{x}\frac{|W(y)|^2}{1-iy}dy\right)\alpha(t,y)dy$$

Because $\exp\left(-\int_{y}^{x}\frac{|W(y)|^2}{1-iy}dy\right) \leq \exp(\|W\|^2_{L^2(\mathbb{R})})$ is bounded by a finite value, the second summand is $\lesssim \epsilon \log(t) t^{-1+\epsilon^2}$. The definition of the left Jost solution ψ_l enforces $C = 1$, and hence we get (5.35).

To prove (5.47), notice that a partial integration of $\int_{-\infty}^{x} te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}dz$ gives us

$$\int_{-\infty}^{x} t\Big(1-iz\Big)e^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}dz = 1 \tag{5.49}$$

Inserting (5.49) into the left side of (5.47) gives us that it is equal to

$$\begin{aligned}&\int_{-\infty}^{x}\Bigg||w(t,y)|^2\tilde{\varphi}(t,y)\int_{-\infty}^{y}\frac{1-iz}{1-iy}te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}dz \\ &\qquad\qquad - w(t,y)\int_{-\infty}^{y} te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}\overline{w(t,z)}\tilde{\varphi}(t,z)dz\Bigg|dy \\ &= \int_{-\infty}^{x}\Bigg||w(t,y)|^2\tilde{\varphi}(t,y)\int_{-\infty}^{y}\frac{1-iz}{1-iy}te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}dz -\end{aligned}$$

$$- w(t,y)\int_{-\infty}^{y} te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}\overline{w(t,z)}\tilde{\varphi}(t,y)dz$$

$$+ w(t,y)\int_{-\infty}^{y} te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}\overline{w(t,z)}(\tilde{\varphi}(t,y)-\tilde{\varphi}(t,z))dz\Bigg|dy,$$

which is bounded by the following term:

$$\int_{-\infty}^{x} |w(t,y)|^2|\tilde{\varphi}(t,y)|\int_{-\infty}^{y}\left|\frac{i(y-z)}{1-iy}te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}\right|dzdy$$

$$+\int_{-\infty}^{x}|w(t,y)|\int_{-\infty}^{y}\left|\frac{1-iy}{1-iy}te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}(\overline{w(t,y)}-\overline{w(t,z)})\tilde{\varphi}(t,y)\right|dzdy$$

$$+\int_{-\infty}^{x}|w(t,y)|\int_{-\infty}^{y}\left|te^{-t(y-z)}e^{\frac{it}{2}(y^2-z^2)}\overline{w(t,z)}(\tilde{\varphi}(t,y)-\tilde{\varphi}(t,z))\right|dzdy$$

$$=: (I)+(II)+(III) \tag{5.50}$$

Using (5.14), Young's inequality and $|w| = |W|$ (we suppress the explicit t-dependence of w in the following estimates):

$$(I) \lesssim \int_{-\infty}^{x}|w(y)|^2\int_{-\infty}^{y}t(y-z)e^{-t(y-z)}dzdy = \frac{1}{t}\int_{-\infty}^{x}|w(y)|^2\int_{0}^{\infty}ze^{-z}dzdy$$

$$\lesssim \frac{1}{t}\|w\|_{L^2}^2 = \frac{1}{t}\|W\|_{L^2}^2 \tag{5.51}$$

Moreover:

$$(II) \le \int_{-\infty}^{x}|w(y)||\tilde{\varphi}(t,y)|\int_{-\infty}^{y}te^{-t(y-z)}|w(y)-w(z)|dzdy$$

$$\lesssim \int_{-\infty}^{x}|w(y)|\int_{0}^{\infty}e^{-z}\left|w(y)-w\Big(y-\frac{z}{t}\Big)\right|dzdy$$

$$\le \|w\|_{L^2(\mathbb{R})}\left\|\int_{0}^{\infty}e^{-z}\left|w(y)-w\Big(y-\frac{z}{t}\Big)\right|dz\right\|_{L^2(\mathbb{R},dy)} \tag{5.52}$$

Notice that the derivative of $\alpha \to e^{i\log(t)\alpha}$ has its absolute value bounded by $\log(t)$, so for y, $\tilde{y}\in\mathbb{R}$, we have

$$|w(y)-w(\tilde{y})| \le (1+log(t))|W(y)-W(\tilde{y})| \tag{5.53}$$

by the definition of w. Furthermore, $\mu(z) = e^{-z}dz$, we get $\mu(\mathbb{R}_+) = 1$, so by Jensen's inequality:

$$
\begin{aligned}
\left\| \int_0^\infty e^{-z} \left| W(y) - W\left(y - \frac{z}{t}\right) \right| dz \right\|_{L^2(dy)}^2 &\leq \int_{\mathbb{R}} \int_0^\infty e^{-z} \left| W(y) - W\left(y - \frac{z}{t}\right) \right|^2 dz dy \\
&= \int_0^\infty \int_{\mathbb{R}} e^{-z} \left| W(y) - W\left(y - \frac{z}{t}\right) \right|^2 dy dz \\
&= \int_0^\infty e^{-z} \left\| W(y) - W\left(y - \frac{z}{t}\right) \right\|_{L^2(dy)}^2 dz \\
&\leq \int_0^\infty e^{-z} \left(\frac{z}{t}\right)^{2(1-\epsilon^2)} \|W\|_{H^{1-\epsilon^2}(\mathbb{R})}^2 dz \\
&\lesssim t^{2(\epsilon^2-1)} \|W\|_{H^{1-\epsilon^2}(\mathbb{R})}^2, \qquad (5.54)
\end{aligned}
$$

by a standard estimate on H^s-functions. Combined, (5.52), (5.53) and (5.54) give us that

$$
|(II)| \lesssim \log(t) t^{-1+\epsilon^2} \|W\|_{H^{1-\epsilon^2}(\mathbb{R})} \|W\|_{L^2(\mathbb{R})} \qquad (5.55)
$$

As for (III), by (4.12), we have

$$
\|\partial_x \varphi_{l,1}(t,x)\|_{L^\infty} \lesssim \|u\|_{L^\infty(\mathbb{R})}^2 \qquad (5.56)
$$

And by (5.56), the definition of $\tilde{\varphi}$, and the asymptotic stability of Theorem 2.3:

$$
\begin{aligned}
|\tilde{\varphi}(t,y) - \tilde{\varphi}(t,z)| &\leq t \|\partial_x \varphi_{l,1}\|_{L^\infty(\mathbb{R})} |y - z| \\
&\lesssim t\epsilon^2 t^{-1} |y - z| = \epsilon^2 |y - z|
\end{aligned}
$$

Similarly to (I), this implies that we can control (III) by the estimate

$$
|(III)| \lesssim \frac{1}{t} \epsilon^2 \|w\|_{L^2(\mathbb{R})}^2 = \frac{1}{t} \epsilon^2 \|W\|_{L^2(\mathbb{R})}^2 \qquad (5.57)
$$

Together, (5.51), (5.55) and (5.57) give us (5.47), as desired. The pointwise convergence estimate (5.37) is now an easy consequence of (5.35), the fact that $\tilde{\varphi}(t,0) = \varphi_{l,1}(t,0)$ $\forall t > 0$ by definition, and the bound $\|\partial_x \varphi_{l,1}\|_{L^\infty(\mathbb{R})} \lesssim \epsilon^2 t^{-1}$ which holds by (5.56) and Theorem 2.3. □

Remark 5.11. *Like Lemma 5.6 and 5.7 and Proposition 5.8, Proposition 5.10 is formulated specifically for $\zeta = \frac{i}{2}$ here, but it is easy to see that similar statements hold for any other $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) > 0$. In this case, it might be of particular interest to explicitly give the version of (5.36) and*

(5.38) *for general* ζ *in the upper halfspace. In that case,* (5.49) *is replaced by* $\int\limits_{-\infty}^{y} t(-2i\zeta - iz)e^{\frac{it}{2}(y^2-z^2)}e^{2i\zeta t(y-z)}dz = 1$*, and hence we would have*

$$P_l(x) = \exp\left(\int\limits_{-\infty}^{x} \frac{|W(y)|^2}{2i\zeta + iy}dy\right)$$

and

$$P_r(x) = \exp\left(\int\limits_{x}^{\infty} \frac{|W(y)|^2}{2i\zeta + iy}dy\right),$$

for which similar estimates as in Proposition 5.10 hold, with constants in the estimates depending on $\mathrm{Im}(\zeta)$*.*

Proof of Theorem 5.2: Approximation of lower-regularity solutions and convergence of position and phase shift

We can now turn to the proof of our main theorem:

Proof of Theorem 5.2: We start with the assumption $\tilde{u} \in C(\mathbb{R}, H^3(\mathbb{R}))$. Our previous considerations imply that, for sufficiently large $t > 0$,

$$\left\| B\Big(u(t,\cdot), \frac{i}{2}, \psi(t,x)\Big) - e^{i(t+\theta(t))}\operatorname{sech}(x - x_0(t)) \right\|_{L^\infty(\mathbb{R})} \lesssim \epsilon |t|^{-\frac{1}{2}}, \quad (5.58)$$

for ψ as in (5.18), $\tilde{x}_0(t)$ the point where $|c_l\psi_{l,1}^{(u(t,x))}(t,x)| = |c_r\psi_{r,2}^{(u(t,x))}(t,x)|$ and $\tilde{\theta}(t)$ the phase of $c_l\bar{c}_r\psi_{l,1}^{(u(t,x))}(t,\tilde{x}_0(t))\overline{\psi_{r,2}^{(u(t,x))}(t,\tilde{x}_0(t))}$. Indeed, by Lemma 5.6 and 5.7 and the asymptotic stability result for the zero solution Theorem 2.3, estimate (5.58) follows as a special case of Proposition 5.8, with $\phi = u$, and

$$a_l = e^{\frac{it}{2}}c_l \qquad a_r = e^{-\frac{it}{2}}c_r \quad (5.59)$$

for all large enough $t \in \mathbb{R}_+$. By well-posedness of the NLS, $B(u(0,\cdot), \frac{i}{2}, \psi(0,x)) = \tilde{u}(0,\cdot)$ implies that we have $B(u(t,\cdot), \frac{i}{2}, \psi(t,x)) = \tilde{u}(t,\cdot)\ \forall t > 0$, so (5.2) holds for the original NLS solution $\tilde{u}$.

We now drop the assumption $\tilde{u}_0 \in H^3(\mathbb{R})$[5]. Let $\tilde{u}_0 \in H^{0,1}(\mathbb{R})$ but $\notin H^3(\mathbb{R})$ satisfy the assumptions of Theorem 5.2 and approximate it in $L^2(\mathbb{R})$ by a

[5] Alternatively to the following argument, it is also possible to proceed similarly as in [8] at the end of Sections 3.1.2 and 4 here.

sequence of $H^3(\mathbb{R})$-functions $\tilde{u}_{0n} \in B_\epsilon^{H^{0,1}(\mathbb{R})}(\operatorname{sech}(\cdot))$. By Theorem 5.1, at any time $t \geq 0$

$$\tilde{u}_n(t,\cdot) \to \tilde{u}(t,\cdot)$$

in the L^2-sense, with $\tilde{u}_n$, $\tilde{u}$ the NLS-solutions with initial data $\tilde{u}_{0n}$ and $\tilde{u}_0$.

Let $n \in \mathbb{N}$, the coefficients $a_l^{(n)}(t)$ and $a_r^{(n)}(t)$ defined in the obvious manner from (5.59) and the mapping $\mathcal{B}$ as in Lemma 4.7. Remember that by the remarks surrounding (5.5), we assume that the eigenvalue of the Lax system (3.1) with potential $\tilde{u}_{0n}$ is fixed at $\frac{i}{2}$. Moreover, the proof of Lemma 4.7 established that $\mathcal{B}^{-1}$ is precisely given as the Bäcklund transformation arising from the reiteration relation (3.14) - the same Bäcklund transformation we used to analyse the regular case. Our previous analysis for $\tilde{u}_0 \in H^3(\mathbb{R})$ thus gives us that for

$$\mathcal{B}(\tilde{u}_{0n}) = \Big(\frac{i}{2}, a_r^{(n)}(0)^{-1} a_l^{(n)}(0), u_{0n}\Big)$$

and $u_n(t,\cdot)$ the unique NLS solution of Theorem 2.3 with $u_n(0,\cdot) = u_{0n}$, we have

$$\mathcal{B}(\tilde{u}_n(t,\cdot)) = \Big(\frac{i}{2}, a_r^{(n)}(t)^{-1} a_l^{(n)}(t), u_n(t,\cdot)\Big)$$

(To understand the second component, compare (3.13) and the statement of Lemma 4.7.) Continuity of $\mathcal{B}$ now gives us that, as $n \to \infty$,

$$\mathcal{B}(\tilde{u}_{0n}) = \Big(\frac{i}{2}, a_r^{(n)}(0)^{-1} a_l^{(n)}(0), u_{0n}\Big) \to \mathcal{B}(\tilde{u}_0) = \Big(\frac{i}{2}, \alpha_l(0), u_0\Big)$$

and, for $t > 0$,

$$\mathcal{B}(\tilde{u}_n(t,\cdot)) = \Big(\frac{i}{2}, a_r^{(n)}(t)^{-1} a_l^{(n)}(t), u_n(t,\cdot)\Big) \to \mathcal{B}(\tilde{u}(t,\cdot)) = \Big(\frac{i}{2}, \alpha_l(t), u(t,\cdot)\Big) \tag{5.60}$$

in $\mathbb{C} \times \mathbb{C} \times L^2(\mathbb{R})$. Recall here that $a_r^{(n)}(t) \neq 0$ for all $t > 0$ as a consequence of Lemma 5.5, so all terms are well-defined.

Since the $\tilde{u}_n$ are regular, Lemma 5.6 and 5.7 apply, i.e. $a_l^{(n)}(t) = a_l^{(n)}(0)e^{\frac{i}{2}t}$ and $a_r^{(n)}(t) = a_r^{(n)}(0)e^{-\frac{i}{2}t}$. In particular, (5.60) therefore implies

$$\alpha_l(t) = \lim_{n\to\infty} a_r^{(n)}(0)^{-1} e^{\frac{i}{2}t} \cdot a_l^{(n)}(0) e^{\frac{i}{2}t} = e^{it} a_r^{(n)}(0)^{-1} a_l^{(n)}(0) = e^{it}\alpha_l(0)$$

Because

$$\tilde{u}(t,\cdot) = \mathcal{B}^{-1}\Big(\frac{i}{2}, \alpha_l(t), u(t,\cdot)\Big)(t,\cdot),$$

we know, again by Lemma 4.7 and (3.13), that $\tilde{u}$ is given as a Bäcklund transformation $\tilde{u}(t,\cdot) = B\Big(u(t,\cdot), \frac{i}{2}, c_l e^{\frac{i}{2}t}\psi_l^{(u(t,\cdot))} + c_r e^{-\frac{i}{2}t}\psi_r^{(u(t,\cdot))}\Big)$ with $c_l = \alpha_l(0)$ and $c_r = 1$. Notice that by(5.60)

$$\frac{c_l}{c_r} \sim 1 \tag{5.61}$$

still holds in the sense of Lemma 5.5. By Proposition 5.8, for sufficiently large $t > 0$:

$$\|\tilde{u}(t,\cdot) - e^{i(t+\tilde{\theta}(t))}\operatorname{sech}(\cdot + \tilde{x}_0(t))\|_{L^\infty(\mathbb{R})} \lesssim \epsilon t^{-\frac{1}{2}},$$

where the position and phase shift functions are still characterized by

$$|c_l\psi_l^{(u(t,\cdot))}(\tilde{x}_0(t))| = |c_r\psi_r^{(u(t,\cdot))}(\tilde{x}_0(t))|$$

and

$$\tilde{\theta}(t) = \arg(c_l\psi_l^{(u(t,\cdot))}(\tilde{x}_0(t))\overline{c_r\psi_r^{(u(t,\cdot))}}(\tilde{x}_0(t)))$$

It now remains to prove the boundedness and the convergence (5.3) and (5.4) for the phase and position shift functions $\tilde{\theta}$, $\tilde{x}_0$. To do so, first consider the position shift $\tilde{x}_0(t)$ for a fixed $t > 0$. The estimates (5.14) and (5.15) imply

$$(1-\epsilon)e^{\frac{x}{2}} \le |\psi_{l,1}(x)| \le (1+\epsilon)e^{\frac{x}{2}} \tag{5.62}$$

and

$$(1-\epsilon)e^{-\frac{x}{2}} \le |\psi_{r,2}(x)| \le (1+\epsilon)e^{-\frac{x}{2}} \tag{5.63}$$

With this and c_l, c_r as above, we get that $|c_l||\psi_{l,1}(\tilde{x}_0(t))| = |c_r||\psi_{r,2}(\tilde{x}_0(t))|$ implies $log\Big(\frac{|c_r|}{|c_l|}\frac{1-\epsilon}{1+\epsilon}\Big) \le \tilde{x}_0(t) \le log\Big(\frac{|c_l|}{|c_r|}\frac{1+\epsilon}{1-\epsilon}\Big)$. By (5.61), $\frac{|c_l|}{|c_r|} \sim 1$, so $|\tilde{x}_0(t)| \lesssim \epsilon$ holds.

For $\tilde{\theta}$, consider that for all $x \in \mathbb{R}$, (5.14) and (5.15) give us:

$$\begin{aligned}
|\psi_{l,1}(x)\overline{\psi}_{r,2}(x) - 1| &= |(\psi_{l,1}(x) - e^{\frac{x}{2}})\overline{\psi}_{r,2}(x) + e^{\frac{x}{2}}\overline{\psi}_{r,2}(x) - 1| \\
&\le \|(e^{-\frac{x}{2}}\psi_{l,1}(x) - 1)\|_{L^\infty(\mathbb{R})}\|e^{\frac{x}{2}}\psi_{r,2}(x)\|_{L^\infty(\mathbb{R})} \\
&\qquad\qquad + \|e^{\frac{x}{2}}\psi_{r,2}(x) - 1\|_{L^\infty(\mathbb{R})} \\
&\le (1+\epsilon)\epsilon + \epsilon
\end{aligned}$$

Moreover, Lemma 5.5 implies that $\arg(c_l\bar{c}_r) = \arg(c_l c_r^{-1}) \lesssim \epsilon$. Together, this gives us $\arg(c_l\bar{c}_r\psi_{l,1}(x)\overline{\psi}_{r,2}(x)) \lesssim \epsilon$ for all $x \in \mathbb{R}$, and thus in particular, $|\tilde{\theta}(t)| \lesssim \epsilon$.

For the convergence of position and phase $\tilde{x}_0(t)$, $\tilde{\theta}(t)$ as $t \to \infty$, we know by Proposition 5.10 that there are constants $P_l(0)$ and $P_r(0) \in \mathbb{C}$ such that

$$|\psi_{l,1}(t,x) - e^{\frac{x}{2}} P_l(0)| \lesssim \epsilon^2 |x| e^{\frac{x}{2}} t^{-1} + \epsilon^2 e^{\frac{x}{2}} \log(t) t^{-1+C\epsilon^2} \tag{5.64}$$

and

$$|\psi_{r,2}(t,x) - e^{-\frac{x}{2}} P_r(0)| \lesssim \epsilon^2 |x| e^{-\frac{x}{2}} t^{-1} + \epsilon^2 e^{-\frac{x}{2}} \log(t) t^{-1+C\epsilon^2} \tag{5.65}$$

We have, of course,

$$|P_l(0) - 1| \lesssim \epsilon \tag{5.66}$$

and

$$|P_r(0) - 1| \lesssim \epsilon \tag{5.67}$$

Define x_0 by $|c_l||e^{\frac{x_0}{2}} P_l(0)| = |c_r||e^{-\frac{x_0}{2}} P_r(0)|$. By Lemma 5.5, (5.66) and (5.67), we have $|x_0| \lesssim \epsilon$. Setting $y = \tilde{x}_0(t)$ and $\tilde{y} = x_0$ in (5.27) and (5.28), we obtain

$$\frac{|\psi_{l,1}(\tilde{x}_0(t))|}{|\psi_{l,1}(x_0)|} \geq e^{\frac{\tilde{x}_0(t)-x_0}{2} - 2\epsilon^2 t^{-1} |x_0 - \tilde{x}_0(t)|} \tag{5.68}$$

and

$$\frac{|\psi_{r,2}(x_0)|}{|\psi_{r,2}(\tilde{x}_0(t))|} \geq e^{\frac{\tilde{x}_0(t)-x_0}{2} - 2\epsilon^2 t^{-1} |x_0 - \tilde{x}_0(t)|} \tag{5.69}$$

Similarly:

$$\frac{|\psi_{l,1}(x_0)|}{|\psi_{l,1}(\tilde{x}_0(t))|} \geq e^{\frac{x_0-\tilde{x}_0(t)}{2} - 2\epsilon^2 t^{-1} |x_0 - \tilde{x}_0(t)|} \tag{5.70}$$

and

$$\frac{|\psi_{r,2}(\tilde{x}_0(t))|}{|\psi_{r,2}(x_0)|} \geq e^{\frac{x_0-\tilde{x}_0(t)}{2} - 2\epsilon^2 t^{-1} |x_0 - \tilde{x}_0(t)|} \tag{5.71}$$

Multiplying first the inequalities (5.68) and (5.69), letting $t \to \infty$, and then doing the same for (5.70) and (5.71) gives, for sufficiently large t,

$$\begin{aligned} |\tilde{x}_0(t) - x_0| - 4\epsilon^2 t^{-1} |\tilde{x}_0(t) - x_0| &\lesssim \left| \log\left(\frac{|\psi_{l,1}(x_0)|}{e^{\frac{x_0}{2}} |P_l(0)|} \right) - \log\left(\frac{e^{-\frac{x_0}{2}} |P_r(0)|}{|\psi_{r,2}(x_0)|} \right) \right| \\ \leq \log\left(1 + \frac{|\psi_{l,1}(x_0) - e^{\frac{x_0}{2}} P_l(0)|}{e^{\frac{x_0}{2}} |P_l(0)|}\right) &+ \log\left(1 + \frac{|\psi_{r,2}(x_0) - e^{-\frac{x_0}{2}} P_r(0)|}{e^{-\frac{x_0}{2}} |P_r(0)|}\right), \end{aligned} \tag{5.72}$$

Because of (5.66), (5.67) and $|x_0| \lesssim \epsilon$, the right side of (5.72) is $\lesssim |\psi_{l,1}(x_0) - e^{\frac{x_0}{2}} P_l(0)| + |\psi_{r,2}(x_0) - e^{-\frac{x_0}{2}} P_r(0)|$. By (5.64), (5.65) and, again, $|x_0| \lesssim \epsilon$, this implies

$$\begin{aligned}
|\tilde{x}_0(t) - x_0| &\lesssim \epsilon^2 |x_0| e^{x_0} t^{-1} + \epsilon^2 e^{\frac{x_0}{2}} \log(t) t^{-1+C\epsilon^2} \\
&\lesssim \epsilon^3 e^{\epsilon} t^{-1} + \epsilon^2 e^{\frac{\epsilon}{2}} \log(t) t^{-1+C\epsilon^2} \\
&\lesssim \epsilon^2 e^{\frac{\epsilon}{2}} \log(t) t^{-1+C\epsilon^2},
\end{aligned}$$

for ϵ small and t large enough, proving (5.3).

Finally, the estimates (5.14) and (5.15) already establish that $\psi_{l,1}$ and $\psi_{r,2}$ are both Lipschitz continuous. Since we have $\tilde{x}_0(t) \to x_0$, it follows that

$$a_l \overline{a_r} \psi_{l,1}(\tilde{x}_0(t)) \overline{\psi}_{r,2}(\tilde{x}_0(t)) - a_l \overline{a_r} \psi_{l,1}(x_0) \overline{\psi}_{r,2}(x_0) \to 0,$$

where, as before, $a_l = e^{\frac{it}{2}} c_l$ and $a_r = e^{-\frac{it}{2}} c_r$. This yields convergence of the phase shift $\tilde{\theta}(t)$ to $\theta_0 = \arg(c_l \overline{c}_r P_l(x_0) P_r(x_0))$, with a rate of convergence at least as good as $x_0(t)$, which shows (5.4). $\square$

Chapter 6

The mKdV Case

The spatial part (3.1) of the Lax system for mKdV is the same as for NLS, and as discussed in Chapter 2, we can use the Bäcklund transformation to map the zero solution to solitons similarly to the NLS case. It is thus natural to apply our methods from the previous chapter to the asymptotic stability of mKdV solitons. The well-posedness of mKdV in H^1 follows, e.g., from Theorem 3 in [7], which will take the place of Theorem 5.1 in the following. Similarly to Chapter 5, the solutions given by this result trivially have to coincide with the solutions of Theorem 2.5 whenever its assumptions are satisfied.

Theorem 6.1. *Given a real-valued $\tilde{u}_0 \in H^{1,1}(\mathbb{R})$ with $\|\tilde{u}_0 - \operatorname{sech}(x)\|_{H^{1,1}(\mathbb{R})} = \epsilon$ for a sufficiently small $\epsilon > 0$, let $\tilde{u} = \tilde{u}(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a real-valued solution of the focusing mKdV (1.3) in one dimension with $\tilde{u}(0,\cdot) = \tilde{u}_0(\cdot)$. Then we have asymptotic stability in the sense that there is a constant $\lambda \in \mathbb{R}$ such that for large enough $t \in \mathbb{R}_+$:*

$$\|\lambda \tilde{u}(\lambda^3 t, \lambda \cdot) - \operatorname{sech}(\cdot - \tilde{x}_0(t))\|_{L^\infty(\mathbb{R})} \lesssim \epsilon |t|^{-\frac{1}{3}}, \tag{6.1}$$

where $|\lambda - 1| \lesssim \epsilon$ and $|\tilde{x}_0(t) - t| \lesssim \epsilon$ for all $t > 0$. Moreover, the function $\tilde{x}_0(t)$ satisfies $\lim_{t\to\infty} \tilde{x}_0(t) - t = x_0 \in \mathbb{R}$ with

$$|\tilde{x}_0(t) - t - x_0| \lesssim \epsilon^2 t^{-\frac{5}{3}} \tag{6.2}$$

holding.

As with Theorem 5.2, while we only state Theorem 6.1 for the elementary soliton to keep our argument simple, it generalizes to a similar statement for the entire mKdV soliton group (2.8) via (2.7).

Proof. The Lax system for mKdV has the same spatial part as the NLS system, so Lemma 5.3 and Proposition 5.8 still apply. Furthermore, if the potential is a solution of mKdV, the zero curvature condition (3.4) holds in the mKdV case, too, i.e. $\partial_x B - \partial_t A - [A, B] = 0$, where A is the matrix for the spatial part (3.1) and B for the temporal part (3.5) as given in (3.7). Thus, most arguments from the previous chapter can be repeated in a similar manner for the present proof, with the following modifications:

- The spatial part (3.1) with the real-valued potential $\tilde{u}_0$ has a purely imaginary eigenvalue ζ close to $\frac{i}{2}$ by Lemma 4.6. Because at $t = 0$ the scaling symmetry for mKdV (2.7) is the same as the scaling symmetry for the NLS (2.2) at $t = 0$, we can fix the eigenvalue at $\zeta = \frac{i}{2}$ by employing the change of variables (5.5) with $v = 0$ (and, still, $k = 2\,\mathrm{Im}(\zeta)$).

- For the time evolution (5.21) for the left Jost solution at $t = 0$, which is now given by

$$\psi_l(t,x) = e^{-\frac{\sigma}{2}t}\psi_l(0,x) + \int_0^t e^{-\frac{\sigma(t-s)}{2}}(B(s,x) + \frac{\sigma}{2})\psi_l(s,x)ds,$$

which, arguing as in Lemma 5.6 and 5.7, yields $\psi_l(t,x) = e^{-\frac{t}{2}}\psi_l^{(u(t,x))}(x)$ for the time evolution of the left and $\psi_r(t,x) = e^{\frac{t}{2}}\psi_r^{(u(t,x))}(x)$ for the right Jost solution. This gives rise to the time evolution of mKdV solitons (given as $\operatorname{sech}(x-t)$ for the elementary soliton) rather than NLS solitons (given by $e^{\frac{it}{2}}\operatorname{sech}(x)$).

What remains to be shown is the convergence of the position function $\tilde{x}_0(t)$. Just as we used the asymptotic expression in Theorem 2.3 to this end for the NLS system, we will use the asymptotics in Theorem 2.5 for mKdV:

By Lemma 5.8, $\tilde{x}_0 = \tilde{x}_0(t)$ is characterized as the point where (with c_l and c_r as in the previous chapter):

$$|c_l|e^{-\frac{t}{2}}|\psi_{l,1}^{(u(t))}(\tilde{x}_0)| = |c_r|e^{\frac{t}{2}}|\psi_{r,2}^{(u(t))}(\tilde{x}_0)|, \tag{6.3}$$

We can write[1] (compare (4.15) from Remark 4.2 and (4.8), from which the second equation can be obtained in a similar way to (4.11) and (4.15)):

$$e^{\frac{x}{2}}\psi_{r,2}^{(u(t))}(x) =: \varphi_r(x) = 1 - \int_x^\infty \int_y^\infty u(y)e^{-(z-y)}u(z)\varphi_r(z)dzdy$$

$$e^{-\frac{x}{2}}\psi_{l,1}^{(u(t))}(x) =: \varphi_l(x) = a\Big(\frac{i}{2}\Big) + \int_x^\infty \int_{-\infty}^y u(y)e^{-(y-z)}u(z)\varphi_l(z)dzdy, \tag{6.4}$$

[1]Notice that $\overline{u} = u$ because we are considering real-valued solutions.

where a denotes the inverse transmission coefficient. As in (5.62) and (5.63), $\varphi_l(x) \in [(1-\epsilon)e^{\frac{x}{2}}, (1+\epsilon)e^{\frac{x}{2}}]$ and $\varphi_r(x) \in [(1-\epsilon)e^{-\frac{x}{2}}, (1+\epsilon)e^{-\frac{x}{2}}]$, which entails $|\tilde{x}_0(t) - t| \lesssim \epsilon$. Hence, for ϵ small enough, $\frac{1}{2}\tilde{x}_0$ will be in the decaying region Ω_0^+ of Theorem 2.5, assume e.g. $\tilde{x}_0 \geq (1-\epsilon)t$. By the same result, $\|t^{\frac{1}{6}}(t^{-\frac{1}{3}}x)u\|_{L^2(\Omega_0^+)} \lesssim \epsilon$, so by Young's inequality

$$|\varphi_r(\tilde{x}_0) - 1| = \int_{\tilde{x}_0}^{\infty}\int_{y}^{\infty} u(y)e^{-(z-y)}u(z)\varphi_r(z)dzdy \lesssim \|u\|^2_{L^2(\Omega_0^+)}$$

$$\lesssim \epsilon^2(t^{-\frac{1}{6}}(t^{-\frac{1}{3}}t)^{-1})^2 = \epsilon^2 t^{-\frac{5}{3}}$$

and (remember $\tilde{x}_0 > 0$ by assumption):

$$|\varphi_l(x) - a\Big(\frac{i}{2}\Big)| = \left|\int_{\tilde{x}_0}^{\infty}\int_{-\infty}^{y} u(y)e^{-(y-z)}u(z)\varphi_l(z)dzdy\right| \leq$$

$$\leq \left|\int_{\tilde{x}_0}^{\infty}\int_{-\infty}^{\frac{y}{2}} u(y)e^{-(y-z)}u(z)\varphi_l(z)dzdy\right| + \left|\int_{\tilde{x}_0}^{\infty}\int_{\frac{y}{2}}^{y} u(y)e^{-(y-z)}u(z)\varphi_l(z)dzdy\right|$$

$$\lesssim e^{-ct}\left|\int_{\tilde{x}_0}^{\infty}\int_{-\infty}^{\frac{y}{2}} u(y)e^{-\frac{1}{2}(y-z)}u(z)\varphi_l(z)dzdy\right| + \|u\|^2_{L^2(\Omega_0^+)}$$

$$\lesssim e^{-ct}\|u\|^2_{L^2(\mathbb{R})} + \epsilon^2 t^{-\frac{5}{3}},$$

with $c > 0$. Thus, in (6.3) we equate

$$|c_l|e^{-\frac{t}{2}}e^{\frac{\tilde{x}_0}{2}}\left|a\Big(\frac{i}{2}\Big) + O(\epsilon^2 t^{-\frac{5}{3}})\right| = |c_r|e^{\frac{t}{2}}e^{-\frac{\tilde{x}_0}{2}}\left|1 + O(\epsilon^2 t^{-\frac{5}{3}})\right|$$

From this, we get

$$\tilde{x}_0(t) - t \to \log\left(\frac{|c_r|}{|c_l|}a^{-1}\Big(\frac{i}{2}\Big)\right) = \log\left(\frac{|c_r|}{|c_l|}T\Big(\frac{i}{2}\Big)\right) =: x_0,$$

and

$$|\tilde{x}_0(t) - t - x_0| \lesssim \epsilon^2 t^{-\frac{5}{3}}$$

as claimed in (6.2). □

Unlike in the previous chapter, we did not establish any analogue of Proposition 5.10 in the above proof. This was not necessary, as the center of the soliton is near $+t$ and thus in the decaying region. Still, it remains an interesting question if some kind of convergence of the right and left

Jost solution does hold as $t \to \infty$. We will show so in the following. As discussed in Chapter 1, this result might be useful in applying the methods of this thesis to other special mKdV solutions that can be constructed via the Bäcklund transformation:

Proposition 6.2. *Let $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ with $\|u(0,\cdot)\|_{H^{1,1}(\mathbb{R})} \leq \epsilon$, $\epsilon > 0$ small, be a real-valued solution of the mKdV (1.3) that satisfies the assumptions of Theorem 2.5 and let $\psi_l^{(u(t,x))}$ and $\psi_r^{(u(t,\cdot))}$ be the corresponding right and left Jost solutions for the Lax system with parameter ζ such that $\operatorname{Im}(\zeta) > 0$ (and the usual superscript notation). Let $\varphi_l(t,x) := e^{-i\zeta x}\psi_l^{(u(t,x))}(x)$, and writing the region Ω_0^- defined in Theorem 2.5 as $\Omega_0^- = (-\infty, -Kt^{\frac{1}{3}}]$, $K > 0$, define*

$$\begin{aligned}&\tilde{\varphi} : \mathbb{R}_+ \times [K^{\frac{1}{2}}t^{-\frac{1}{3}}, \infty) \to \mathbb{C}\\ &\tilde{\varphi}(t,x) = \varphi_{l,1}(t, -tx^2)\end{aligned}$$

Then

$$\|\tilde{\varphi}(t,x) - P_l(x)\|_{L^\infty([K^{\frac{1}{2}}t^{-\frac{1}{3}},\infty))} \lesssim \epsilon t^{-\frac{1}{6}}, \tag{6.5}$$

where (with W as in Theorem 2.5) $P_l : [K^{\frac{1}{2}}t^{-\frac{1}{3}}, \infty) \to \mathbb{R}$ is given by

$$P_l(x) = \exp\Bigg(-2\operatorname{Im}(\zeta)\pi^{-1} \int_x^\infty \frac{|W(y)|^2}{4|\zeta|^2 + 4\operatorname{Re}(\zeta)y + y^2}dy\Bigg),$$

as well as

$$|\varphi_{l,1}(t,x) - P_l(t^{-\frac{1}{3}})| \lesssim \epsilon t^{-\frac{1}{6}} \tag{6.6}$$

for any $x \notin \Omega_0^-$.
For $\varphi_r(t,x) := e^{i\zeta x}\psi_l^{u(t,x)}(x)$, let

$$\begin{aligned}&\tilde{\phi} : \mathbb{R}_+ \times [K^{\frac{1}{2}}t^{-\frac{1}{3}}, \infty) \to \mathbb{C}\\ &\tilde{\phi}(t,x) = \varphi_{r,2}(t, -tx^2)\end{aligned}$$

Then we have

$$\|\tilde{\phi}(t,x) - P_r(x)\|_{L^\infty([K^{\frac{1}{2}}t^{-\frac{1}{3}},\infty))} \lesssim \epsilon t^{-\frac{1}{6}} \tag{6.7}$$

with $P_r : [K^{\frac{1}{2}}t^{-\frac{1}{3}}, \infty) \to \mathbb{R}$ given by

$$P_r(x) = \exp\Bigg(-2\operatorname{Im}(\zeta)\pi^{-1} \int_{t^{-\frac{1}{3}}}^x \frac{|W(y)|^2}{4|\zeta|^2 + 4\operatorname{Re}(\zeta)y + y^2}dy\Bigg),$$

as well as

$$|\varphi_{r,2}(t,x)-1| \lesssim \epsilon^2 t^{-\frac{1}{3}} \tag{6.8}$$

for any $x \notin \Omega_0^-$.

The basic structure of our proof of Proposition 6.2 is as follows:

- It is relatively easy to show (6.8) and that, provided (6.5) holds, (6.6) is also true. We will do so in Step 1 and Step 2 of the proof.
- In Step 3, we will show (6.5), which requires more work: Similar to the proof of Proposition 5.10, we will use the asymptotic expression on the oscillatory region of Theorem 2.5 to give the implicit representation of $\tilde{\varphi}$ as the sum of a main term (6.12) and an error term which is under control by Theorem 2.5. We will then further decompose (6.12) into four summands, which we will dub "$a\overline{b}$-part", "ab-part", "$\overline{a}b$-part" and "$\overline{a}\overline{b}$-part" (Step 3a)). As suggested by these names, the latter two parts are given as complex conjugates of the former, so it suffices to understand the behaviour of just the "$a\overline{b}$-part" and "ab-part". In Step 3b), we will treat the "$a\overline{b}$-part", which is closest to the analysis undertaken in Proposition 5.10: Partial integration under the assumption of slightly higher regularity suggests convergence to a time-independent integral as $t \to \infty$, and we will estimate the difference to this putative limit in the general case. In Step 3c), applying these methods to the "ab-part" will give us that it behaves like an oscillatory integral (6.30), which, as we will show, converges to zero with an appropriate rate as $t \to \infty$. Step 3d) will summarize these results.
- Finally, we will sketch how a similar treatment of $\tilde{\phi}$ gives us (6.7) in Step 4.

Proof. For ease of notation, we will set $\rho = 0$ and use a "toy version" of the regions introduced in Theorem 2.5, where we replace "$\lesssim$" in the definition of $\Omega_0^{\{+,0,-\}}$ by $\leq$, e.g. $\Omega_0^0 = \{x \in \mathbb{R} : |x| \leq t^{\frac{1}{3}}\}$. For the same reason, we restrict ourselves to $\zeta = \frac{i}{2}$, the general case is a relatively straightforward extension (compare Remark 5.11, (6.18) and (6.39)). Our final simplification will be to modify ϵ so that $C = 1$ in Theorem 2.5, i.e. $W \in H^{1-\epsilon^2,1} \cap L^\infty(\mathbb{R})$ with $\|W\|_{H^{1-\epsilon^2,1}\cap L^\infty(\mathbb{R})} \lesssim \epsilon$.

Step 1: Proof of (6.8)
The estimate (6.8) is the simplest to show: We have

$$\varphi_{r,2}(t,x) := e^{\frac{x}{2}}\psi_{r,2}(t,x) = 1 - \int_x^\infty \int_y^\infty u(t,y)e^{-(z-y)}u(t,z)\varphi_{r,2}(t,z)dzdy,$$

and we consider the case that $[x,\infty) \subset \Omega_0^0 \cup \Omega_0^+$. Now, by (2.11), (2.12) and (2.13), Theorem 2.5 entails that

$$\begin{aligned} \|u\|_{L^2(\Omega_0^+ \cup \Omega_0^0)} &\lesssim \epsilon t^{-\frac{1}{6}} + \|Q(t^{-\frac{1}{3}} \cdot)\|_{L^2(\Omega_0^0)} t^{-\frac{1}{3}} + \epsilon t^{-\frac{5}{18} + \frac{2}{3} C\epsilon^2} \\ &\lesssim \epsilon t^{-\frac{1}{6}} + \|\epsilon\|_{L^2(\Omega_0^0)} t^{-\frac{1}{3}} + \epsilon t^{-\frac{5}{18} + \frac{2}{3} C\epsilon^2} \\ &\lesssim \epsilon t^{-\frac{1}{6}}, \end{aligned} \tag{6.9}$$

where Q is the L^∞-function on $\mathbb{R}$ specified in the same theorem, and, by definition, $|\Omega_0^0| \sim t^{\frac{1}{3}}$. Young's inequality yields:

$$\begin{aligned} |\varphi_{r,2}(t,x) - 1| &\le \left| \int\limits_{\Omega_0^+ \cup \Omega_0^0} \int\limits_y^\infty u(t,y) e^{-(z-y)} u(t,z) \varphi_{r,2}(t,z) dz dy \right| \\ &\lesssim \|u\|^2_{L^2(\Omega_0^+ \cup \Omega_0^0)} \lesssim \epsilon^2 t^{-\frac{1}{3}} \end{aligned}$$

for any $x \in \Omega_0^0 \cup \Omega_0^+$.

Step 2: (6.5) implies (6.6)
Before we show the estimate (6.5), we will first establish that (6.6) follows from (6.5). We consider

$$\varphi_{l,1}(t,x) := e^{-\frac{x}{2}} \psi_{l,1}(t,x) = 1 - \int\limits_{-\infty}^x \int\limits_{-\infty}^y u(t,y) e^{-(y-z)} u(t,z) \varphi_{l,1}(t,z) dz dy, \tag{6.10}$$

for $x \notin \Omega_0^-$. We need to show that the contribution of $\Omega_0^0 \cup \Omega_0^+ \times \mathbb{R}$ to the above integral goes to zero as $t \to \infty$, with a rate of convergence that is $\lesssim \epsilon t^{-\frac{1}{6}}$, from which (6.6) will follow. In particular, $\Omega_0^- \times \Omega_0^-$ is thus only region to make a nonzero contribution to the limit of $\varphi_{l,1}$ as $t \to \infty$. Indeed, this is an easy implication of (6.9), which gives us

$$\begin{aligned} &\left| \int\limits_{(-\infty,x] \cap (\Omega_0^0 \cup \Omega_0^+)} \int\limits_{-\infty}^y u(t,y) e^{-(y-z)} u(t,z) \varphi_{l,1}(t,z) dz dy \right| \\ &\lesssim \|u\|_{L^2(\Omega_0^0 \cup \Omega_0^+)} \|u\|_{L^2(\mathbb{R})} \lesssim \epsilon^2 t^{-\frac{1}{6}} \end{aligned}$$

with Young's inequality, and hence, for

$$\begin{aligned} \varphi_{l,1}(t,x) &= 1 - \int\limits_{-\infty}^x \int\limits_{-\infty}^y u(t,y) e^{-(y-z)} u(t,z) \varphi_{l,1}(t,z) dz dy \\ &= \varphi_{l,1}(t, -t^{\frac{1}{3}}) - \int\limits_{-t^{\frac{1}{3}}}^x \int\limits_{-\infty}^y u(t,y) e^{-(y-z)} u(t,z) \varphi_{l,1}(t,z) dz dy \end{aligned}$$

$$=: \tilde{\varphi}(t, t^{-\frac{1}{3}}) + I(x)$$

we have $|I(x)| \lesssim \epsilon^2 t^{-\frac{1}{6}}$. If (6.5) holds, this implies (6.6). All other rates of convergence we are going to establish in this proof are at least as fast as $\epsilon^2 t^{-\frac{1}{6}}$.

Step 3: Proof of (6.5)
To show (6.5), consider

$$\tilde{\varphi}(t, x) = \varphi_{l,1}(t, -tx^2) = 1 - \int_{(-\infty, -tx^2]} \int_{(-\infty, y]} u(y) e^{-(y-z)} u(z) dz dy, \quad (6.11)$$

for $x \geq t^{-\frac{1}{3}}$ (which is equivalent to $(-\infty, -tx^2] \in \Omega_0^-$). On $\Omega_0^- \times \Omega_0^-$, we know by (2.14) that the potential is given by:

$$u(t, x) = \pi^{-\frac{1}{2}} t^{-\frac{1}{3}} (t^{-\frac{1}{3}} |x|)^{-\frac{1}{4}} \cdot \\ \cdot \operatorname{Re}\left(e^{\left[i\alpha(t,x) - \frac{3i}{4\pi} |W(t^{-\frac{1}{2}} |x|^{\frac{1}{2}})|^2 \log(t^{-\frac{1}{2}} |x|^{\frac{1}{2}}) \right]} W(t^{-\frac{1}{2}} |x|^{\frac{1}{2}}) \right) + E(x),$$

where $\alpha(t, x) = -\frac{2}{3} t^{-\frac{1}{2}} |x|^{\frac{3}{2}} + \frac{\pi}{4}$, $W : \mathbb{R} \to \mathbb{C}$ with $\|W\|_{H^{1-\epsilon,1} \cap L^\infty(\mathbb{R})} \lesssim \epsilon$ and the error term E as in Theorem 2.5.

Step 3a): Decomposition of (6.11)
We first estimate the contribution of all terms containing E to (6.11). By Theorem 2.5, we have $\|E\|_{L^2(\Omega_0^-)} \lesssim \epsilon t^{-\frac{1}{6}}$, so Young's inequality gives, for $x^* \in \Omega_0^-$

$$\int_{-\infty}^{x^*} \int_{-\infty}^{y} |E(y)| e^{-(y-z)} |u(z)| dz dy + \int_{-\infty}^{x^*} \int_{-\infty}^{y} |u(y)| e^{-(y-z)} |E(z)| dz dy \lesssim \epsilon^2 t^{-\frac{1}{6}},$$

as well as

$$\int_{-\infty}^{x^*} \int_{-\infty}^{y} |E(y)| e^{-(y-z)} |E(z)| dz dy \lesssim \epsilon^2 t^{-\frac{1}{3}}$$

Thus we are, similarly to the proof of Proposition 5.10, left to consider the main term given at the end of the following calculation:

$$\pi(1 - \varphi_{l,1}(-tx^2)) = \\ = \int_{tx^2}^{\infty} \int_{y}^{\infty} t^{-\frac{1}{2}} y^{-\frac{1}{4}} z^{-\frac{1}{4}} e^{y-z} \varphi_{l,1}(-z) \cdot \\ \cdot \operatorname{Re}\left[e^{i\alpha(t,y) - \frac{3i}{4\pi} \log(t^{-\frac{1}{2}} y^{\frac{3}{2}}) |W(t^{-\frac{1}{2}} y^{\frac{1}{2}})|} W(t^{-\frac{1}{2}} y^{\frac{1}{2}}) \right] \cdot$$

$$\cdot \operatorname{Re}\left[e^{i\alpha(t,z)-\frac{3i}{4\pi}\log(t^{-\frac{1}{2}}z^{\frac{3}{2}})|W(t^{-\frac{1}{2}}z^{\frac{1}{2}})|}W(t^{-\frac{1}{2}}z^{\frac{1}{2}})\right]dzdy$$

$$+O(\epsilon^2 t^{-\frac{1}{6}})$$

$$\approx 4\int_x^\infty\int_y^\infty ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}\tilde{\varphi}(z)\operatorname{Re}\left[e^{i\alpha(ty^2)-\frac{3i}{4\pi}\log(ty^3)|W(y)|^2}W(y)\right]\cdot$$

$$\cdot \operatorname{Re}\left[e^{i\alpha(tz^2)-\frac{3i}{4\pi}\log(tz^3)|W(z)|^2}W(z)\right]dzdy, \tag{6.12}$$

using symmetry in the first equality and then a change of variables. As complex numbers a and b generally satisfy

$$\operatorname{Re}(ab)=\frac{1}{4}(a\bar{b}+ab+\bar{a}b+\bar{a}\bar{b}), \tag{6.13}$$

we can split the last integral (6.12) into four parts: An "$a\bar{b}$-part"

$$\int_x^\infty\int_y^\infty ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}\tilde{\varphi}(z)e^{i\alpha(ty^2)-i\alpha(tz^2)+\frac{3i}{4\pi}\log(tz^3)|W(z)|^2-\frac{3i}{4\pi}\log(ty^3)|W(y)|^2}\cdot$$

$$\cdot W(y)\overline{W(z)}dzdy, \tag{6.14}$$

an "ab-part"

$$\int_x^\infty\int_y^\infty ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}\tilde{\varphi}(z)e^{i\alpha(ty^2)+i\alpha(tz^2)-\frac{3i}{4\pi}\log(ty^3)|W(y)|^2-\frac{3i}{4\pi}\log(tz^3)|W(z)|^2}\cdot$$

$$\cdot W(y)W(z)dzdy, \tag{6.15}$$

and an "$\bar{a}\bar{b}$"- and "$\bar{a}b$"-part of (6.12) defined in the obvious way. It is, of course, sufficient to understand the behaviour of (6.15) and (6.14), the other two summands just being the complex conjugates.

We will show that (6.14) (and hence its conjugate) goes to an integral of $\tilde{\varphi}$ multiplied by a time-independent function, while (6.15) (and its conjugate) goes to 0, all with the rate of convergence appropriately bounded by $\epsilon^2 t^{-\frac{1}{6}}$.

Step 3b): The "$a\bar{b}$"-part
First, we consider the "$a\bar{b}$"-part (6.14), which we will show goes to

$$\int_x^\infty \frac{|W(y)|^2}{2(1-iy)}\tilde{\varphi}(t,y)dy \tag{6.16}$$

To this end, we will proceed similarly to the NLS case in the last chapter. The difference of the $a\bar{b}$-part from its putative limit (6.16) is bounded by

$$\int_x^\infty \left| \frac{|W(y)|^2}{2(1-iy)}\tilde{\varphi}(y) - \tilde{w}(t,y)\int_y^\infty ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}\overline{\tilde{w}(z)}\tilde{\varphi}(z)dz \right| dy \tag{6.17}$$

with $\tilde{w}(t,x) := e^{-\frac{3i}{4\pi}\log(tx^3)|W(x)|^2}W(x)$. Now note that by partial integration

$$\int_y^\infty ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}dz = \frac{1}{2(1-iy)} + m(t,y) + n(t,y), \tag{6.18}$$

where

$$m(t,y) = -\frac{1}{4}\int_y^\infty y^{\frac{1}{2}}z^{-\frac{3}{2}}e^{-t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}dz \tag{6.19}$$

and

$$n(t,y) = \int_y^\infty ity^{\frac{1}{2}}z^{\frac{1}{2}}(z-y)e^{-t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}dz \tag{6.20}$$

By definition, we have $z \geq y \geq t^{-\frac{1}{3}}$ on the domain of integration, and $|z^2-y^2| = |(z+y)(z-y)| \geq 2t^{-\frac{1}{3}}|z-y|$, which we can utilize to show that the $L^\infty([t^{-\frac{1}{3}},\infty))$ norm of $m(t,\cdot)$ goes to zero as $t\to\infty$ with a rate of $t^{-\frac{1}{3}}$. For (6.20), consider that, if $y \geq t^{-\frac{1}{3}}$,

$$\begin{aligned} \int_y^\infty ty^{\frac{1}{2}}z^{\frac{1}{2}}(z-y)e^{-t(z^2-y^2)}dz &= \int_0^\infty ty^{\frac{1}{2}}(z+y)^{\frac{1}{2}}ze^{-t(z+2y)z}dz \\ &\lesssim \int_0^y tyze^{-2tyz}dz + \int_y^\infty tz^2e^{-tz^2}dz \\ &\leq t^{-1}\int_0^\infty ze^{-2z}dz + te^{-\frac{1}{2}t^{\frac{1}{3}}}\int_y^\infty z^2e^{-\frac{1}{2}tz^2}dz, \end{aligned} \tag{6.21}$$

so (6.20) is $\lesssim t^{-1}$.

Because $\|W\|_{L^2(\mathbb{R})} \lesssim \epsilon$, inserting (6.18) into (6.17), these estimates give us that the latter differs by $O(\epsilon^2 t^{-\frac{1}{3}})$ from

$$\int_x^\infty \Bigg| \tilde{w}(t,y)\overline{\tilde{w}(t,y)}\tilde{\varphi}(y) \int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} e^{\frac{2}{3}it(z^3-y^3)} dz \\ - \tilde{w}(t,y) \int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} e^{\frac{2}{3}it(z^3-y^3)} \overline{\tilde{w}(t,z)} \tilde{\varphi}(z) dz \Bigg| dy, \tag{6.22}$$

(Recall the definition of α here.) To get an appropriate bound for (6.22), notice that, for $z > y > 0$, by (4.12) and the chain rule (compare the estimate of term (III) from (5.50) in the last chapter),

$$|\tilde{\varphi}(t,y) - \tilde{\varphi}(t,z)| \lesssim \epsilon^2 t^{\frac{1}{3}} z(z-y) \tag{6.23}$$

Moreover, utilizing the derivatives of the functions $z \to e^{iC\log(tz^3)}$ and $|W(y)| \to e^{iC\log(tz^3)|W(y)|^2}$, we get[2]:

$$|\tilde{w}(t,y) - \tilde{w}(t,z)| \lesssim \log(t) \Big| |W(t,y)|^2 - |W(t,z)|^2 \Big| + |W(y) - W(z)| \\ + \log(z) \Big| |W(t,y)|^2 - |W(t,z)|^2 \Big| + \epsilon^2 t^{\frac{1}{3}} (z-y), \tag{6.24}$$

for large t (and, again, $z > y > 0$). By (6.23) and (6.24) (and, of course, L^∞-boundedness of $\tilde{\varphi}$), (6.22) is smaller or equal to a constant times the sum of

$$\int_x^\infty |W(y)| \int_y^\infty \log(t) t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} \Big| |W(y)|^2 - |W(z)|^2 \Big| dz dy \tag{6.25}$$

and

$$\int_x^\infty |W(y)| \int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} |W(y) - W(z)| dz dy \tag{6.26}$$

and

$$\int_x^\infty |W(y)| \int_y^\infty \log(z) t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} \Big| |W(y)|^2 - |W(z)|^2 \Big| dz dy \tag{6.27}$$

[2] E.g., the last summand in (6.24) bounds the contribution of $\Big| e^{\frac{3i}{4\pi}\log(ty^3)|W(y)|^2} W(y) - e^{\frac{3i}{4\pi}\log(tz^3)|W(y)|^2} W(y) \Big|$, since $\Big|\partial_z \log(tz^3)\Big| = \Big|\frac{3tz^2}{tz^3}\Big| \lesssim t^{\frac{1}{3}}$ for $z \geq t^{-\frac{1}{3}}$.

and

$$\epsilon^2 \int_x^\infty |W(y)| \int_y^\infty t^{\frac{4}{3}} y^{\frac{1}{2}} z^{\frac{1}{2}} (1+z)(z-y) e^{-t(z^2-y^2)} dz dy \tag{6.28}$$

Because W is both bounded and Hölder continuous with exponent $\frac{1-\epsilon^2}{2}$, (6.25) can be estimated by

$$\int_x^\infty |W(y)| \int_y^\infty \log(t) t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} |W(y) - W(z)| dz dy$$
$$\lesssim \epsilon \int_x^\infty |W(y)| \int_y^\infty \log(t) t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} |y-z|^{\frac{1-\epsilon^2}{2}} dz dy,$$

from which we can proceed with a similar change of variables and calculation to (6.21):

$$= \epsilon \int_x^\infty |W(y)| \int_0^\infty \log(t) t y^{\frac{1}{2}} (z+y)^{\frac{1}{2}} e^{-t(z+2y)z} z^{\frac{1-\epsilon^2}{2}} dz dy$$
$$\lesssim \epsilon \int_x^\infty |W(y)| \int_0^y \log(t) t y e^{-2tyz} z^{\frac{1-\epsilon^2}{2}} dz dy +$$
$$+ \epsilon \int_x^\infty |W(y)| \int_y^\infty t y^{\frac{1}{2}} e^{-tz^2} z^{\frac{3-\epsilon^2}{2}} dz dy$$
$$= \epsilon \int_x^\infty |W(y)| \int_0^{ty^2} \log(t) t y e^{-2z} \left(\frac{z}{ty}\right)^{\frac{1-\epsilon^2}{2}} \frac{1}{ty} dz dy +$$
$$+ \epsilon \int_x^\infty |W(y)| \int_y^\infty \log(t) t y^{\frac{1}{2}} e^{-tz^2} z^{\frac{3-\epsilon^2}{2}} dz dy$$
$$\lesssim \epsilon \log(t) t^{-\frac{1-\epsilon^2}{2}} \|W\|_{L^1(\mathbb{R})} +$$
$$+ \epsilon \log(t) t e^{-\frac{1}{3}tx^2} \int_x^\infty y^{\frac{1}{2}} e^{-\frac{1}{3}ty^2} |W(y)| \int_y^\infty e^{-\frac{1}{3}tz^2} z^{\frac{3-\epsilon^2}{2}} dz dy$$
$$\lesssim \epsilon \log(t) (t^{-\frac{1-\epsilon^2}{2}} + t e^{-\frac{1}{3}t^{\frac{1}{3}}}) \|W\|_{L^1(\mathbb{R})} \tag{6.29}$$

Notice that $\|W\|_{L^1(\mathbb{R})} \lesssim \|W\|_{H^{0,1}(\mathbb{R})} \lesssim \epsilon$, so the right-hand side of (6.29) is $\lesssim \epsilon^2 \log(t) t^{-\frac{1-\epsilon^2}{2}}$. The integral (6.26) can be shown to be $\lesssim \epsilon^2 t^{-\frac{1-\epsilon^2}{2}}$ by a similar, but simpler calculation. Now,

$$\int_x^\infty \log(y)|W(y)|dy = \int_x^\infty \frac{\log(y)}{y}|y||W(y)|dy \lesssim \|yW\|_{L^2(\mathbb{R})}\left\|\frac{\log(y)}{y}\right\|_{L^2(\mathbb{R})} \lesssim \|W\|_{H^{0,1}(\mathbb{R})},$$

so a minor modification of the argument for (6.25) gives us for (6.27):

$$\begin{aligned}
&\int_x^\infty |W(y)| \int_y^\infty \log(z)ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}\Big||W(y)|^2 - |W(z)|^2\Big|dzdy \\
&\lesssim \epsilon \int_x^\infty |W(y)| \int_0^\infty \log(z+y)ty^{\frac{1}{2}}(z+y)^{\frac{1}{2}}e^{-t(z+2y)z}z^{\frac{1-\epsilon^2}{2}}dzdy \\
&\lesssim \epsilon \int_x^\infty |W(y)| \int_0^y \log(y)tye^{-z}\Big(\frac{z}{2ty}\Big)^{\frac{1-\epsilon^2}{2}}\frac{1}{2ty}dzdy \\
&\qquad + \epsilon \int_x^\infty |W(y)| \int_y^\infty \log(z)ty^{\frac{1}{2}}e^{-tz^2}z^{\frac{3-\epsilon^2}{2}}dzdy \\
&\lesssim \epsilon t^{-\frac{1-\epsilon^2}{2}}\|W\|_{H^{0,1}(\mathbb{R})} + \epsilon te^{-\frac{1}{3}t^{\frac{1}{3}}}\|W\|_{L^1(\mathbb{R})}
\end{aligned}$$

Finally, for (6.28),

$$\epsilon^2\left|\int_x^\infty |W(y)| \int_y^\infty t^{\frac{4}{3}}y^{\frac{1}{2}}z^{\frac{1}{2}}(1+z)(z-y)e^{-t(z^2-y^2)}dzdy\right| \lesssim \epsilon^2 t^{-\frac{1}{3}},$$

also follows by spliting the integral and changing variables as in (6.21) and (6.29), for we get

$$\int_y^\infty t^{\frac{4}{3}}y^{\frac{1}{2}}z^{\frac{1}{2}}(1+z)(z-y)e^{-t(z^2-y^2)}dz \lesssim \int_0^{t^{-1}y^{-1}y} t^{\frac{4}{3}}y(1+y)\frac{z}{ty}\frac{1}{ty}e^{-2z}dz + \int_y^\infty t^{\frac{4}{3}}z^2(1+z)e^{-tz^2}dz,$$

which, for $y \geq t^{-\frac{1}{3}}$, gives the desired estimate in the same way as before.

It follows that (6.22), and hence (6.17) does indeed go to zero with a rate of convergence $\lesssim \epsilon^2 t^{-\frac{1}{3}}$.

Step 3c): The "ab"-part
Let us now deal with the second summand indicated by (6.13), i.e. the "ab-part" (6.15):

$$\int_x^\infty \int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} W(y) W(z) \tilde{\varphi}(z) \cdot e^{i(\alpha(ty^2)+\alpha(tz^2)) - \frac{3i}{4\pi}(\log(ty^3)|W(y)|^2 + \log(tz^3)|W(z)|^2)} dz dy$$

We are going to show that the absolute value of (6.15) is $\lesssim \epsilon^2 t^{-\frac{1}{3}+2\epsilon^2}$. In order to do so, we first show that for large t, this behaves like

$$\int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} W(y)^2 e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy. \tag{6.30}$$

We use:

$$\int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} e^{-\frac{2}{3}it(y^3+z^3)+i\frac{\pi}{2}} dz = \frac{i}{2}\frac{1}{1+iy} e^{-\frac{4}{3}ity^3} + r(t,y)$$

where our same considerations for (6.19), (6.20) apply to r. Thus, the difference of (6.15) to (6.30) ($\tilde{w}$ as in (6.17))

$$\Bigg| \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} W(y)^2 e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy - \int_x^\infty \int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} \tilde{w}(y)\tilde{w}(z)\tilde{\varphi}(z) e^{\frac{2}{3}it(y^3+z^3)} dz dy \Bigg|$$

is, except for an error of $O(\epsilon^2 t^{-\frac{1}{3}})$, given by

$$\left| \int_x^\infty \int_y^\infty t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} e^{\frac{2}{3}it(y^3+z^3)} \tilde{w}(t,y)[\tilde{\varphi}(y)\tilde{w}(t,y) - \tilde{\varphi}(z)\tilde{w}(t,z)] dz dy \right|$$

This can be shown to be $\lesssim \epsilon^2 t^{-\frac{1}{3}}$ by the same estimates we used for (6.22). (Notice that these estimates only used the absolute values of the integrands, and $|e^{\frac{2}{3}it(y^3+z^3)}| = |e^{\frac{2}{3}it(y^3-z^3)}| = 1$.)

Now, we want to show that (6.30) goes to 0 with a sufficiently fast rate as $t \to \infty$. Let us first work with the stronger assumption $W \in H^1(\mathbb{R})$ and

$\|W\|_{H^1} \lesssim \epsilon$, so that partial integration yields:

$$\int_x^\infty \frac{-i\tilde{\varphi}(y)W(y)^2}{2(1+iy)} e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy =$$

$$= \int_x^\infty \frac{\tilde{\varphi}(y)W(y)^2}{2(1+iy)} \frac{\partial_y e^{-\frac{4}{3}ity^3}}{4ty^2} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$= \frac{\tilde{\varphi}(x)W(x)^2}{2(1+ix)} \frac{e^{-\frac{4}{3}itx^3}}{4tx^2} e^{-\frac{3i}{2\pi}\log(tx^3)|W(x)|^2}$$

$$- \int_x^\infty \frac{\partial_y\tilde{\varphi}(y)W(y)^2}{2(1+iy)} \frac{e^{-\frac{4}{3}ity^3}}{4ty^2} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$- \int_x^\infty \tilde{\varphi}(y)W(y)^2 \partial_y \left[\frac{1}{8y^2(1+iy)}\right] \frac{e^{-\frac{4}{3}ity^3}}{t} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$- \int_x^\infty 2W(y)\partial_y W(y) \frac{\tilde{\varphi}(y)}{2(1+iy)} \frac{e^{-\frac{4}{3}ity^3}}{4ty^2} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$+ \int_x^\infty \frac{\tilde{\varphi}(y)W(y)^2}{2(1+iy)} \frac{e^{-\frac{4}{3}ity^3}}{4ty^2} \frac{3i}{2\pi}\frac{3}{y}|W(y)|^2 e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$+ \int_x^\infty \frac{\tilde{\varphi}(y)W(y)^2}{2(1+iy)} \frac{e^{-\frac{4}{3}ity^3}}{4ty^2} |W(y)|^2 \frac{3i}{\pi}\log(ty^3)\cdot$$

$$\cdot \operatorname{Re}\left[W(y)\overline{\partial_y W(y)}\right] e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy \qquad (6.31)$$

Because $x \geq t^{-\frac{1}{3}}$, $\|W\|_{L^2\cap L^\infty(\mathbb{R})} \leq \epsilon$, $\|\partial_y W(y)\|_{L^2(\mathbb{R})}$ and (as can be seen from (6.23)) $|\partial_y \tilde{\varphi}(y)| \lesssim \epsilon |y| t^{-\frac{1}{3}}$, each of these summands can be bounded by a term that is $\lesssim \epsilon^2 t^{-\frac{1}{3}}$.

Finally, we use an approximation argument to prove that (6.30) is $\lesssim \epsilon^2 t^{-\frac{1}{3}+2\epsilon^2}$ in the general case of Theorem 2.5, i.e. just $\|W\|_{H^{1-\epsilon^2,1}\cap L^\infty(\mathbb{R})} \lesssim \epsilon$. Recall that $H^s(\mathbb{R}^n)$ is an algebra with respect to multiplication of functions whenever $s \geq \frac{n}{2}$, and $\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$ in that case. Therefore, $W \in H^{1-\epsilon^2}(\mathbb{R})$ implies $V := |W|^2 \in H^{1-\epsilon}(\mathbb{R})$ for ϵ small enough, and $\|V\|_{H^{1-\epsilon^2}(\mathbb{R})} \lesssim \|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}$. Pick a smooth cutoff function $\phi : \mathbb{R} \to \mathbb{R}$ with

Fourier transformation $\hat{\phi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\phi}(\xi) = 0$ for $|\xi| \geq 2$. In that case, with $\delta = \delta(t) > 0$ to be chosen later:

$$\left\| \mathcal{F}^{-1}\Big[\hat{\phi}\Big(\frac{\cdot}{\delta}\Big)\hat{V}\Big]\right\|_{H^1}^2 \sim \int_{\mathbb{R}} (1+|\xi|^2)\Big|\hat{\phi}\Big(\frac{\xi}{\delta}\Big)\Big|^2 |\hat{V}(\xi)|^2 d\xi$$
$$\lesssim (1+|\delta|^2)^{\epsilon^2} \|V\|_{H^{1-\epsilon^2}(\mathbb{R})}^2 \tag{6.32}$$

and

$$\left\| \mathcal{F}^{-1}\Big[\Big(1-\phi\Big(\frac{\cdot}{\delta}\Big)\Big)\hat{V}\Big]\right\|_{H^{\frac{3}{4}}}^2 \sim \int_{\mathbb{R}} (1+|\xi|^2)^{\frac{3}{4}}\Big|1-\hat{\phi}\Big(\frac{\xi}{\delta}\Big)\Big|^2 |\hat{V}(\xi)|^2 d\xi$$
$$\leq (1+|\delta|^2)^{-\frac{1}{4}+\epsilon^2} \|V\|_{H^{1-\epsilon^2}(\mathbb{R})}^2 \tag{6.33}$$

Thus, we can write $|W(y)|^2 = G(y)+H(y)$ with $G \in C_0^\infty(\mathbb{R})$ and its H^1-norm controlled by (6.32), while H's $H^{\frac{3}{4}}$-norm and thus its L^∞-norm is bounded by (6.33). We can construct a similar decomposition $W^2(y) = M(y)+N(y)$. We obtain:

$$\int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} W(y)^2 e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$
$$= \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} M(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$
$$+ \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} N(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$
$$= \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} M(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)[G(y)+H(y)]} dy$$
$$+ \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} N(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$= \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} M(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)G(y)} dy$$
$$+ \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} M(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)G(y)} (e^{-\frac{3i}{2\pi}\log(ty^3)H(y)} - 1) dy$$
$$+ \int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)} N(y) e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$
$$=: (A)+(B)+(C) \tag{6.34}$$

The obvious modification of the partial integration (6.31) for more regular W, combined with (6.32), gives us for the first term:

$$|(A)| \lesssim \|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}(1+\delta(t)^2)^{\frac{\epsilon^2}{2}} \cdot t^{-\frac{1}{3}}, \tag{6.35}$$

while the analogue to (6.33) for the function N shows

$$|(C)| \lesssim \|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}(1+\delta(t)^2)^{-\frac{1}{8}+\frac{\epsilon^2}{2}} \tag{6.36}$$

For the second term (B), we have

$$\begin{aligned}
&\left|\int_x^\infty \frac{-i\tilde{\varphi}(y)}{2(1+iy)}M(y)e^{-\frac{4}{3}ity^3}e^{-\frac{3i}{2\pi}\log(ty^3)G(y)}(e^{-\frac{3i}{2\pi}\log(ty^3)H(y)}-1)dy\right| \lesssim \\
&\lesssim \int_x^\infty \frac{1}{|1+iy|}|M(y)||\log(ty^3)H(y)|dy \\
&\lesssim \log(t)(1+|\delta|^2)^{-\frac{1}{8}+\frac{\epsilon^2}{2}}\int_x^\infty \frac{1}{|1+iy|}|M(y)|dy \\
&\qquad\qquad + (1+|\delta|^2)^{-\frac{1}{8}+\frac{\epsilon^2}{2}}\int_x^\infty \frac{\log(y)}{|1+iy|}|M(y)|dy
\end{aligned} \tag{6.37}$$

Because $M \in L^2(\mathbb{R})$ with $\|M\|_{L^2(\mathbb{R})} \lesssim (1+\delta(t)^2)^{\frac{\epsilon^2}{2}}\|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}$ (similar to (6.32)), choosing

$$\delta(t) \sim t^2 \tag{6.38}$$

with appropriate $\alpha > \frac{1}{8-\epsilon^2}$ therefore gives us:

$$|(B)| \lesssim \|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}\log(t)t^{-\frac{1}{2}+4\epsilon^2},$$

while (6.35) and (6.36) become:

$$|(A)| \lesssim \|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}t^{-\frac{1}{3}+2\epsilon^2}$$

and

$$|(C)| \lesssim \|W\|^2_{H^{1-\epsilon^2}(\mathbb{R})}t^{-\frac{1}{2}+2\epsilon^2}$$

which, if ϵ is small enough, shows that (6.34) is $\lesssim \epsilon^2 t^{-\frac{1}{3}+2\epsilon}$.

Step 3d): Result
We can now apply our results for (6.14) and (6.15) to their complex conjugates "$\overline{a}b$"-part and "$\overline{a}\overline{b}$"-part of (6.13). Putting all of this together gives us, with (6.16) and

$$\frac{1}{2(1+iy)} + \frac{1}{2(1-iy)} = \frac{1}{1+y^2} \tag{6.39}$$

that

$$\tilde{\varphi}(t,x) = 1 - \int\limits_{\max\{t^{-\frac{1}{3}},x\}}^{\infty} \frac{|W(y)|^2}{1+y^2}\tilde{\varphi}(t,y)dy + R(t,x)$$

where $|R(t,x)| \lesssim \epsilon^2 t^{-\frac{1}{6}}$. Arguing as we did in the previous chapter after (5.48), (6.5) follows.

Step 4: Sketch of proof of (6.7)
For the proof of (6.7), we can repeat many arguments from the proof of (6.5). For the right Jost solution, we consider

$$\tilde{\phi}(t,x) = \varphi_2^{(r)}(t,-tx^2) = 1 + \int\limits_{-tx^2}^{\infty}\int\limits_{y}^{\infty} u(t,y)e^{-(z-y)}u(t,z)\varphi_2^{(r)}(t,z)dzdy$$

for $x \in [t^{-\frac{1}{3}},\infty)$ (and hence $-tx^2 \in \Omega_0^-$ by our setting $C = 1$). We can control

$$\int\limits_{-t^{\frac{1}{3}}}^{\infty}\int\limits_{y}^{\infty} u(t,y)e^{-(z-y)}u(t,z)\varphi_{r,2}(t,z)dzdy$$

and

$$\int\limits_{-tx^2}^{\infty}\int\limits_{-t^{\frac{1}{3}}}^{\infty} u(t,y)e^{-(z-y)}u(t,z)\varphi_{r,2}(t,z)dzdy$$

by $\epsilon^2 t^{-\frac{1}{6}}$ because of (6.9) and Young's inequality. Similarly as in (6.12), we obtain

$$\tilde{\phi}(t,x) = O(\epsilon^2 t^{-\frac{1}{6}}) + \\ + 4\pi^{-1}\int\limits_{t^{-\frac{1}{3}}}^{x}\int\limits_{t^{-\frac{1}{3}}}^{y} ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{-t(z^2-y^2)}\tilde{\varphi}(z)\,\mathrm{Re}\left[e^{i\alpha(ty^2)-\frac{3i}{4\pi}\log(ty^3)|W(y)|^2}W(y)\right].$$

$$\cdot \operatorname{Re}\left[e^{i\alpha(tz^2)-\frac{3i}{4\pi}\log(tz^3)|W(z)|^2}W(z)\right]dzdy, \tag{6.40}$$

with α as in Theorem 2.5. We can, again, use (6.13) to decompose the last integral into four summands. The analogue of the "$a\bar{b}$"-part (6.14) is

$$\int_{t^{-\frac{1}{3}}}^{x}\int_{t^{-\frac{1}{3}}}^{y} ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}\tilde{w}(t,y)\overline{\tilde{w}(z)}\tilde{\varphi}(z)dzdy \tag{6.41}$$

with $\tilde{w}(t,x) = e^{-\frac{3i}{4\pi}\log(tx^3)|W(x)|^2}W(x)$, as in Step 3. Partial integration gives

$$\begin{aligned}&(1+iy)\int_{t^{-\frac{1}{3}}}^{y} ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}dz\\ &= \frac{1}{2}-\frac{1}{2}y^{\frac{1}{2}}t^{\frac{1}{6}}e^{t(t^{-\frac{2}{3}}-y^2)}e^{\frac{2}{3}it(t^{-1}-y^3)}+\tilde{m}(t,y)+\tilde{n}(t,y)\end{aligned} \tag{6.42}$$

with

$$\tilde{m}(t,y) = \frac{1}{4}\int_{t^{-\frac{1}{3}}}^{y} y^{\frac{1}{2}}z^{-\frac{3}{2}}e^{t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}dz \tag{6.43}$$

and

$$\tilde{n}(t,y) = \int_{t^{-\frac{1}{3}}}^{y} ity^{\frac{1}{2}}z^{\frac{1}{2}}(y-z)e^{t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}dz, \tag{6.44}$$

Equation (6.42) plays the same role here as (6.18) did for the left Jost solution, and (6.43) and (6.44) can be bounded in the same way as (6.19) and (6.20). Moreover

$$\begin{aligned}\int_{t^{-\frac{1}{3}}}^{x}\left|\frac{1}{2}y^{\frac{1}{2}}t^{\frac{1}{6}}e^{t(t^{-\frac{2}{3}}-y^2)}e^{\frac{2}{3}it(t^{-1}-y^3)}W(y)^2\right|dy &\le \frac{1}{2}\int_{t^{-\frac{1}{3}}}^{x}|y|^{\frac{1}{2}}t^{\frac{1}{6}}e^{t^{\frac{2}{3}}(t^{-\frac{1}{3}}-y)}|W(y)|^2dy\\ &\le \epsilon t^{\frac{1}{6}}\|e^{-t^{\frac{2}{3}}\cdot}\|_{L^1(\mathbb{R}_+)}\|W(y)\|_{H^{0,1}(\mathbb{R})}\\ &\le \epsilon^2 t^{-\frac{1}{2}}\end{aligned}$$

So we get, similarly to our treatment of (6.17)

$$\int_{t^{-\frac{1}{3}}}^{x}\left|\frac{|W(y)|^2}{2(1+iy)}\tilde{\varphi}(y)-\tilde{w}(t,y)\int_{t^{-\frac{1}{3}}}^{y} ty^{\frac{1}{2}}z^{\frac{1}{2}}e^{t(z^2-y^2)}e^{\frac{2}{3}it(z^3-y^3)}\overline{\tilde{w}(z)}\tilde{\varphi}(z)dz\right|dy$$

$$= \int_{t^{-\frac{1}{3}}}^{x} \Bigg| \tilde{w}(t,y)\overline{\tilde{w}(t,y)}\tilde{\varphi}(y) \int_{t^{-\frac{1}{3}}}^{y} ty^{\frac{1}{2}} z^{\frac{1}{2}} e^{t(z^2-y^2)} e^{\frac{2}{3}it(z^3-y^3)} dz$$

$$- \tilde{w}(t,y) \int_{t^{-\frac{1}{3}}}^{y} ty^{\frac{1}{2}} z^{\frac{1}{2}} e^{t(z^2-y^2)} e^{\frac{2}{3}it(z^3-y^3)} \overline{\tilde{w}(t,z)}\tilde{\varphi}(z) dz \Bigg| dy$$

$$+ O(\epsilon^2 t^{-\frac{1}{3}}),$$

on which similar estimates as for (6.22) can be used to show that (6.41) converges at the desired rate. E.g., in place of (6.25), we get

$$\int_{t^{-\frac{1}{3}}}^{x} |W(y)| \int_{t^{-\frac{1}{3}}}^{y} \log(t) t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} \Big| |W(y)|^2 - |W(z)|^2 \Big| dz dy$$

$$\lesssim \int_{t^{-\frac{1}{3}}}^{x} |W(y)| \int_{t^{-\frac{1}{3}}}^{y} \log(t) t y^{\frac{1}{2}} z^{\frac{1}{2}} e^{-t(z^2-y^2)} |y-z|^{\frac{1-\epsilon}{2}} dz dy$$

$$= \int_{t^{-\frac{1}{3}}}^{x} |W(y)| \int_{0}^{y-t^{-\frac{1}{3}}} t \log(t) y^{\frac{1}{2}} (y-z)^{\frac{1}{2}} e^{-t(2y-z)z} z^{\frac{1-\epsilon}{2}} dz dy$$

$$\le \int_{t^{-\frac{1}{3}}}^{x} |W(y)| \int_{0}^{y-t^{-\frac{1}{3}}} t \log(t) y e^{-tyz} z^{\frac{1-\epsilon}{2}} dz dy,$$

and this is $\lesssim \epsilon^2 \log(t) t^{-\frac{1-\epsilon}{2}}$ by the calculation (6.29). Terms (6.26), (6.27) and (6.28) can be bounded in a very similar manner. The analogue of the "*ab*-part" can also be treated similarly as we did (6.15) in Step 3b). In particular, notice that in place of (6.30), we get

$$\int_{t^{-\frac{1}{3}}}^{x} \frac{-i\tilde{\varphi}(y)}{2(1+iy)} W(y)^2 e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy$$

$$= \int_{t^{-\frac{1}{3}}}^{\infty} \frac{-i\tilde{\varphi}(y)}{2(1+iy)} W(y)^2 e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy -$$

$$- \int_{x}^{\infty} \frac{-i\tilde{\varphi}(y)}{2(1+iy)} W(y)^2 e^{-\frac{4}{3}ity^3} e^{-\frac{3i}{2\pi}\log(ty^3)|W(y)|^2} dy,$$

so our estimates for (6.30) also cover this case. □

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